Computing GPS Satellite Velocity and Acceleration from the Broadcast Navigation Message

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ABSTRACT

We present an extension to the Global Positioning System (GPS) broadcast navigation message user equations for computing GPS space vehicle (SV) velocity and acceleration. Although similar extensions have been published (e.g., Remondi,[1] Zhang J.,[2] Zhang W.[3]), the extension presented herein includes a distinct kinematic method for computing SV acceleration which significantly reduces the complexity of the equations and improves the mean magnitude results by approximately one order of magnitude by including oblate Earth perturbation effects. Additionally, detailed analyses and validation results using multiple days of precise ephemeris data and multiple broadcast navigation messages are presented. Improvements in the equations for computing SV position are also included, removing ambiguity and redundancy in the existing user equations. The recommended changes make the user equations more complete and more suitable for implementation in a wide variety of programming languages employed by GPS users. Furthermore, relativistic SV clock error rate computation is enabled by the recommended equations. A complete, stand-alone table of the equations in the format and notation of the GPS interface specification[4] is provided, along with benchmark test cases to simplify implementation and verification.

1 | INTRODUCTION

Basic positioning of a Global Positioning System (GPS) receiver requires accurate modeling of the location of the antenna phase center of four or more orbiting space vehicles (SV) in view. The SV orbit is nominally determined and predicted in four-hour arcs, with two hours of overlap, by the Master Control Station (MCS) at Schriever Air Force Base, Colorado. The predicted orbit is parameterized and, along with other information, becomes the *broadcast navigation message*. The message is periodically uploaded to each SV where it is modulated onto a carrier signal, along with a unique pseudorandom noise (PRN) navigation code, and broadcast to GPS user segment receivers. Users can then apply equations like those prescribed in Table 20-IV of the GPS interface specification (IS)[4] to accurately compute the position of each SV antenna phase center in the WGS-84 earth-centered earth-fixed (ECEF) rotating coordinate system.[5] (For brevity, we refer to the position, velocity, and acceleration of the SV antenna phase center as simply the position, velocity, and acceleration of the SV.) Accuracy estimates of SV position computed from the broadcast navigation message are on the order of 1.5 meters rms.[6]

The published broadcast navigation user equations were formulated for computing SV position in near real time. Users may also require SV velocity and acceleration for more complex, near real-time navigation purposes such as receiver velocity determination, GPS/INS (inertial navigation system) integration, etc. SV velocity and acceleration can be computed by extension of the broadcast navigation user equations. This has been done by several researchers including Remondi,[1] Zhang J.,[2] Zhang W.[3] and others, undoubtedly. We present a comparable extension for SV velocity with thorough validation against multiple days of precise ephemeris data and multiple broadcast navigation messages for vehicle PRN 11, which, at the time of this writing, has the greatest off-nominal eccentricity and inclination of all SV in the GPS constellation. (SV number and PRN number are generally not the same, but we follow the common practice of using the PRN number to uniquely identify a particular SV at a particular time.) For SV acceleration we use a kinematic approach that significantly reduces the complexity of the equations and simplifies inclusion of oblate Earth (J_2) gravity perturbations. Including J_2 perturbations reduces the mean acceleration magnitude difference by approximately one order of magnitude compared to the more commonly used derivative method. The acceleration equations were also validated against multiple days of precise ephemeris and multiple broadcast navigation messages for PRN 11. The effects of other potential error sources were also analyzed including polar motion, Earth precession, and higher-order gravity.

In the sections that follow, we present improvements to the SV position equations and the derivation of the SV velocity and acceleration equations. Detailed validation results are presented along with a complete table of the recommended user equations in the format and notation of the public interface specification. Benchmark test cases are provided to assist with implementation and verification of the entire set of equations.

2 | SV POSITION – ECCENTRIC AND TRUE ANOMALY

The broadcast navigation user equation tables correctly state that Kepler's equation $(M = E - e \sin E)$ can be solved for eccentric anomaly (E) by iteration. There are many known methods of various complexity for doing this,[7,8] but no particular method is specified or recommended by the IS.[4] Because SV orbits are near-circular (maximum valid eccentricity $e = 0.03$, according to Table 20-III in the IS), simple methods requiring a limited number of iterations can be used. We evaluated two such methods for use with the broadcast navigation user equations: 1) *successive substitutions*,[9] and 2) *Newton iteration*.[10] For successive substitutions, the initial estimate of eccentric anomaly is set equal to the mean anomaly (M), and the final value of E is converged upon by iteration of a simple variation of

Kepler's equation:

$$E_0 = M$$
$$E_j = M + e \sin E_{j-1} \tag{1}$$

where j is the iteration index. The Newton iteration method begins the same way. The iterative equation for E is slightly more complex, but no more complex than other broadcast navigation user equations.

$$E_0 = M$$
$$E_j = E_{j-1} + \frac{M - E_{j-1} + e \sin E_{j-1}}{1 - e \cos E_{j-1}} \tag{2}$$

At the nominal GPS altitude ($r \approx 26,560$ km), an angular variation of 1×10^{-10} radians corresponds to an in-track positional difference on the order of 3mm, well within the expected positional accuracy of the broadcast navigation equations (≈ 1.5 meters rms[6]). Fig. 1 shows the number of iterations required for both methods of solving Kepler's equation to converge on eccentric anomaly to a precision of 1×10^{-10} radians, for the maximum valid eccentricity of $e = 0.03$. The figure shows that iterating three times on eq. (2) for any value of mean anomaly will guarantee

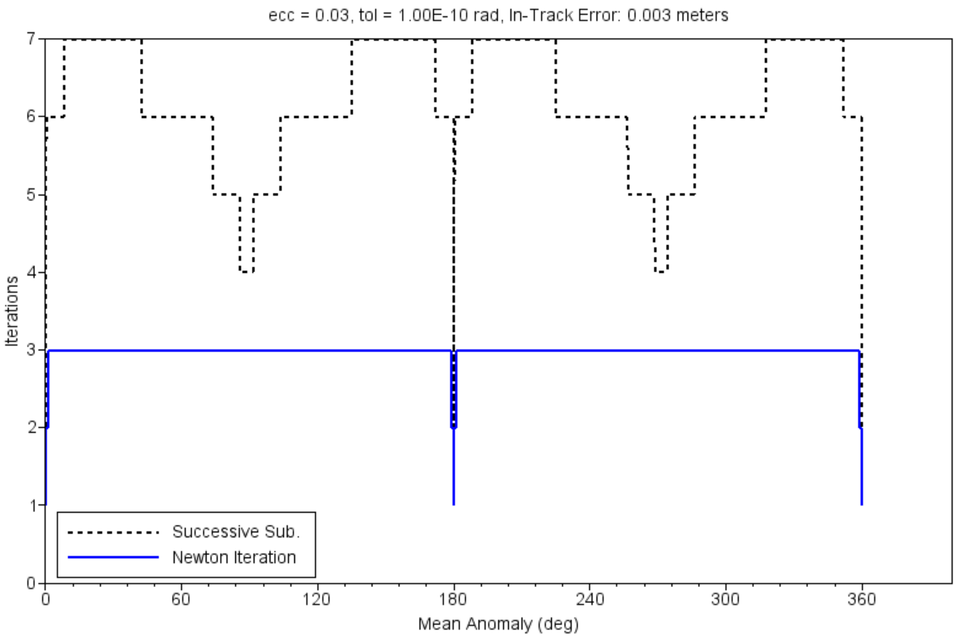

Figure 1. Iterations required for two methods of solving Kepler's equation for eccentric anomaly ($e = 0.03$).

convergence within $\approx$ 3mm in-track (or the level of precision supported by the computer being used) of the broadcast navigation message for any valid eccentricity. The Newton iteration method of eq. (2) is simple to implement in a wide range of programming languages, and guaranteed to quickly converge for eccentricity $e < 0.03$. Including it explicitly in the IS tables, along with the recommended number of iterations (three), enables users less familiar with astrodynamics and Kepler's equation to directly apply the equations without the need to find a suitable solution by consulting an external reference, thus saving time and effort and reducing risk.

True anomaly (ν) is computed from eccentric anomaly and orbital eccentricity using inverse trigonometry functions. The equations currently listed in the IS result in quadrant ambiguity. No information is given for resolving the quadrant, requiring the user to invoke a function such as *atan2* (which may not be available in all programming languages) or to resolve the ambiguity by some other, unspecified means. We recommend deleting from the tables the current, ambiguous equations for true anomaly and replacing them with the unambiguous form derived directly from the geometric relationship between the eccentric and true anomalies:[9–12]

$$\nu = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) \tag{3}$$

We note that eq. (3) results in an unambiguous value of true anomaly in the range $-\pi \leq \nu \leq \pi$, which is different from the more customary range $0 \leq \nu < 2\pi$. However, this poses no issue for the user equations because of the periodic nature of the trigonometric functions that use true anomaly as the independent variable, either directly or indirectly (see Table 3).

We further recommend deleting the equation for eccentric anomaly that appears after the equations for true anomaly in the IS user equation table. Eccentric anomaly is computed prior to computing true anomaly. The redundant equation for eccentric anomaly is unnecessary and could lead to confusion when the equations are being implemented by a programmer with little or no knowledge of astrodynamics.

The upper panel of Fig. 2 shows the WGS-84 position coordinates and RSS (root sum squared) magnitude of an example 72-hour series (approximately six GPS orbital periods) of precise ephemeris for PRN 11. The precise ephemeris data are publicly available from the National Geospatial-Intelligence Agency (NGA).[13] The lower panel shows the position coordinates and magnitude differences between the precise ephemeris and the broadcast navigation message results using eqs. (2) and (3) over the same time period. The differences are shown in radial, in-track, and cross-track (RIC) coordinates in the upper panel of Fig. 3. The relatively consistent differences in the radial direction and magnitude are indicative of the orbit prediction efforts of the Master Control Station. The lower panel of Fig. 3 shows the position differences without fully converging on eccentric anomaly. As expected, large positioning errors arise, primarily in the in-track direction, by simply not allowing proper convergence of Kepler's equation. We therefore recommend the minimum number of iterations (three) be specified explicitly along with the user equations in the IS (e.g., see Table 3). Again, experienced users would not be required to use these equations or number of iterations. They are simply baseline recommendations for users and programmers with limited knowledge of astrodynamics.

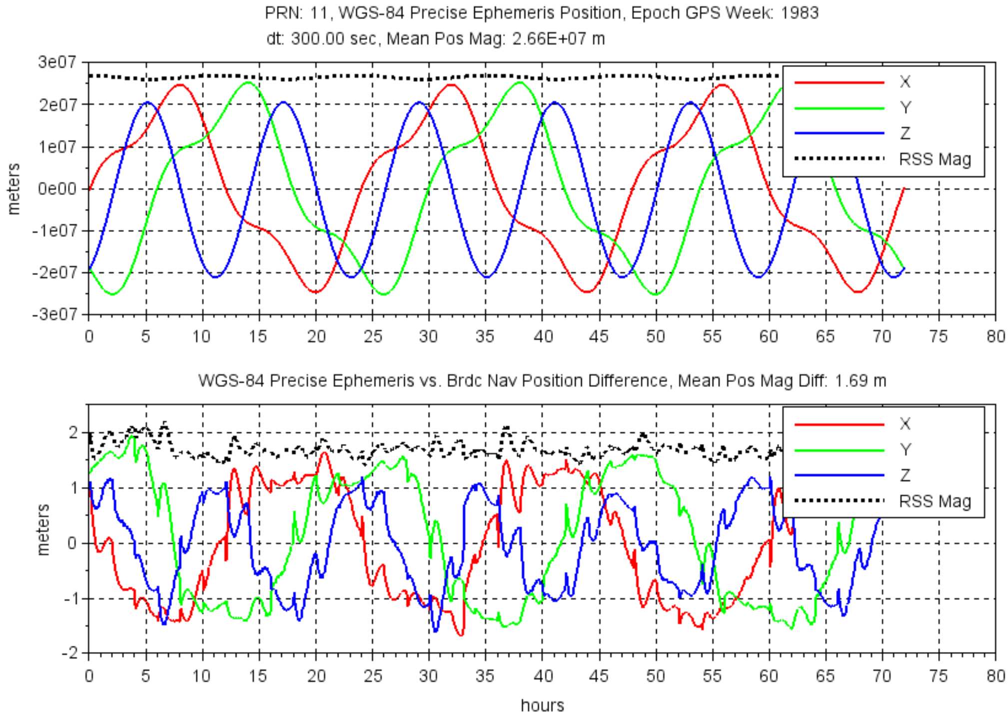

Figure 2. WGS-84 precise ephemeris position (upper panel) and differences with broadcast navigation messages (lower panel).

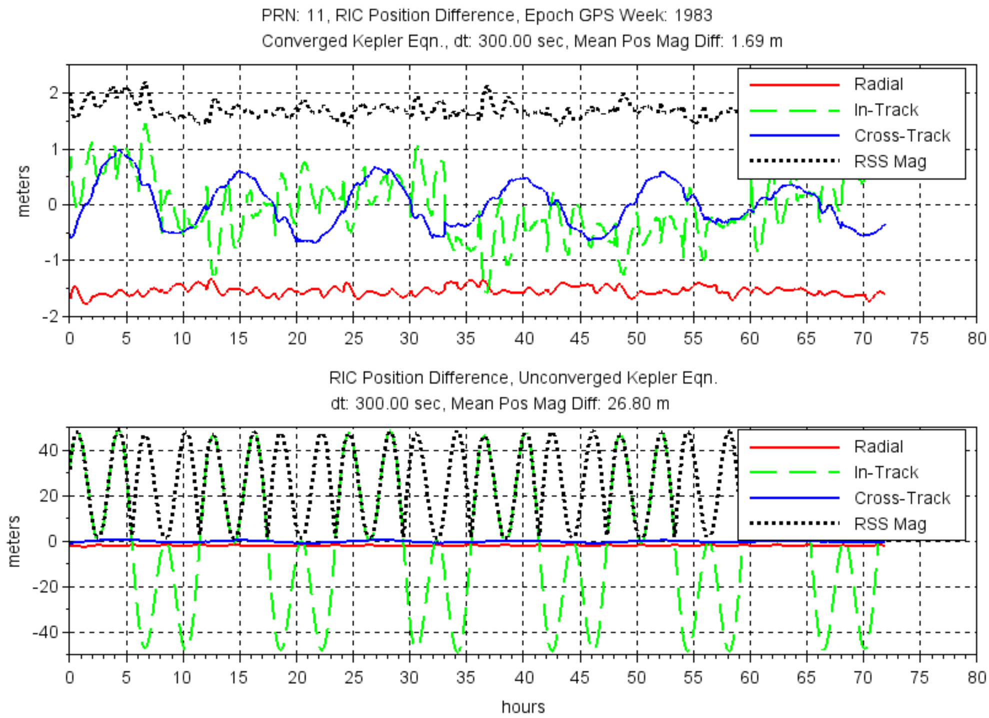

Figure 3. RIC position differences using fully converged Kepler equation (upper panel) and partially converged Kepler equation (lower panel).

3 | SV VELOCITY

The ECEF velocity of the SV can be computed by taking the time-derivative of the SV position equations in the interface specification.[4] The resulting ECEF velocity equations are:

$$
\begin{aligned}
\dot{x} &= -x'\dot{\Omega}_k \sin\Omega - y'(\dot{\Omega}_k \cos\Omega \cos i - (di/dt)\sin\Omega \sin i) + \dot{x}'\cos\Omega - \dot{y}'\sin\Omega\cos i \\
\dot{y} &= \ \ x'\dot{\Omega}_k \cos\Omega - y'(\dot{\Omega}_k \sin\Omega \cos i + (di/dt)\cos\Omega \sin i) + \dot{x}'\sin\Omega + \dot{y}'\cos\Omega\cos i \\
\dot{z} &= y'(di/dt)\cos i + \dot{y}'\sin i
\end{aligned}
\tag{4}
$$

where x' and y' are the orbit-plane position coordinates prior to transformation to the ECEF system, and $\dot{\Omega}_k$ is the rate of the *corrected* longitude of the ascending node. To fully implement these equations, several additional derivatives are required (Table 1). These derivatives are functions of the parameters of the broadcast navigation message and parameters computed by the SV position equations. Note that the term (di/dt) is not the same as $IDOT$. The term $IDOT$ is the "Rate of Inclination Angle" from the broadcast navigation message (not to be confused with $IODE$ or $IODC$), while (di/dt) is the inclination rate *corrected* by the equation in Table 1. The $\dot{E}$ equation results from taking

Table 1. Velocity ancillary equations.

$\dot{E} = n/(1 - e\cos E)$
$\dot{\nu} = \dot{E}\sqrt{1-e^2}/(1 - e\cos E)$
$(di/dt) = (IDOT) + 2\dot{\nu}(c_{is}\cos 2\Phi - c_{ic}\sin 2\Phi)$
$\dot{u} = \dot{\nu} + 2\dot{\nu}(c_{us}\cos 2\Phi - c_{uc}\sin 2\Phi)$
$\dot{r} = eA\dot{E}\sin E + 2\dot{\nu}(c_{rs}\cos 2\Phi - c_{rc}\sin 2\Phi)$
$\dot{\Omega}_k = \dot{\Omega} - \dot{\Omega}_e$
$\dot{x}' = \dot{r}\cos u - r\dot{u}\sin u$
$\dot{y}' = \dot{r}\sin u + r\dot{u}\cos u$

the time-derivative of Kepler's equation ($M = E - e\sin E$) and substituting mean motion (n) for the derivative of mean anomaly[6] (i.e., $\dot{M} = n$). The rate of true anomaly ($\dot{\nu}$) results from taking the time-derivative of an alternate form of the true anomaly equation[10]

$$\sin\nu = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E} \tag{5}$$

and substituting for $\cos\nu$ from

$$\cos\nu = \frac{\cos E - e}{1 - e\cos E} \tag{6}$$

Furthermore, the argument of perigee is treated as constant (i.e., $\dot{\omega} = 0$) for the derivation of the equations in Table 1.

Fig. 4 (upper panel) shows 72-hours of precise ephemeris velocity magnitude along with the absolute value of SV latitude. As expected, the inverse correlation between velocity magnitude and SV latitude (absolute value) is apparent

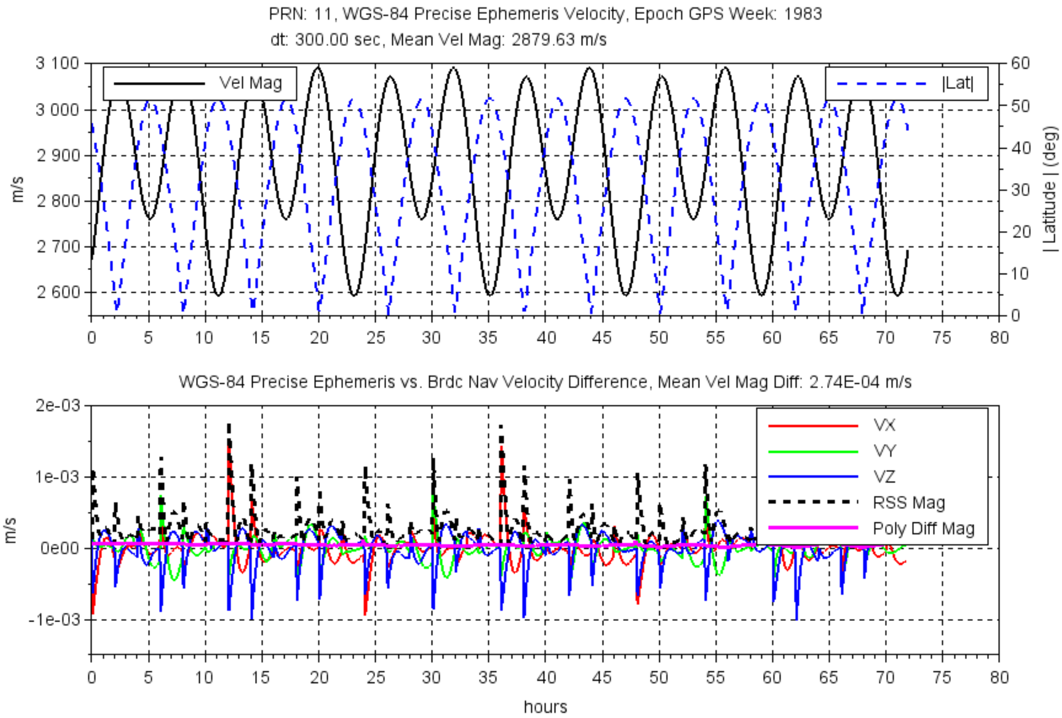

Figure 4. WGS-84 precision ephemeris velocity magnitude and absolute value SV latitude (upper panel) and velocity differences with broadcast navigation message and polynomial derivative (lower panel).

in the figure and indicates that the periodic variations in velocity magnitude are due primarily to the motion of the SV in its inclined orbit relative to the WGS-84 coordinate system. The differences between the precise ephemeris velocity and velocity computed by eqs. (4) are shown in the lower panel by component and RSS magnitude. Also shown in Fig. 4 are the velocity magnitude differences of the time-derivative of a 12th order Chebyshev polynomial fit to the precision position data. This was included to show that the achievable accuracy of the polynomial derivative is commensurate with the differences between the precise and computed velocity. Because precise acceleration data are not publicly available from the NGA, 12th order Chebyshev polynomials[14] were fit to the precise position and velocity data in three-hour segments (approximately 1/4 GPS orbital period) with constrained end points. To assess the accuracy of computing precise acceleration by this method (next section), the polynomial derived velocity was compared to the precise ephemeris velocity in Fig. 4.

4 | SV ACCELERATION

SV acceleration can be computed by taking the time-derivative of the velocity equations, as shown by Zhang J.[2] and Zhang W.[3] However, with the WGS-84 position and velocity known (i.e., computed previously), Newton's second law of motion (for constant mass) and the kinematic equation of acceleration[10] can be employed instead, resulting in a much simpler formulation.

$$\mathbf{f} = \mathbf{a}_o + \mathbf{a}_R + 2(\boldsymbol{\omega} \times \mathbf{v}_R) + (\dot{\boldsymbol{\omega}} \times \mathbf{r}_R) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_R) \tag{7}$$

All vectors are in the ECEF (WGS-84) coordinate system – the earth-centered inertial (ECI) system is not used. The vectors in eq. (7) are defined below:

$\mathbf{f}$	specific force, i.e. net external force per (constant) mass
$\mathbf{a}_o$	acceleration of the origin of the rotating frame
$\mathbf{a}_R$	acceleration with respect to the rotating frame
$\boldsymbol{\omega}$	angular velocity of the rotating frame
$\mathbf{r}_R$	position with respect to the rotating frame
$\mathbf{v}_R$	velocity with respect to the rotating frame

Note that $\mathbf{a}_o = \mathbf{0}$ (null vector) because the origin of WGS-84 is fixed at the center of mass of Earth, co-located with origin of the inertial frame. Let $\boldsymbol{\omega} = [0, 0, \dot{\Omega}_e]^T$ and $\dot{\boldsymbol{\omega}} = \mathbf{0}$, where $\dot{\Omega}_e$ is the WGS-84 Earth rotation rate. Solve for $\mathbf{a}_R$, the desired acceleration vector with respect to the ECEF frame,

$$\mathbf{a}_R = \mathbf{f} - 2(\boldsymbol{\omega} \times \mathbf{v}_R) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_R) \tag{8}$$

Let $\mathbf{f}$ be the two-body gravitational force-per-mass expressed in ECEF coordinates. Carrying out the vector cross products in eq. (8), the equations for ECEF acceleration in component form become

$$\begin{aligned} a_x &= -\frac{\mu x}{r^3} + 2\dot{y}\dot{\Omega}_e + x\dot{\Omega}_e^2 \\ a_y &= -\frac{\mu y}{r^3} - 2\dot{x}\dot{\Omega}_e + y\dot{\Omega}_e^2 \\ a_z &= -\frac{\mu z}{r^3} \end{aligned} \tag{9}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, and μ is the Earth gravitational parameter ($\mu = GM$). These equations are much less complex than those of the derivative method and produce the same level of accuracy. For comparison and completeness, the derivative method equations of acceleration are included here without derivation.

$$\begin{aligned} \ddot{x}' &= -\frac{\mu x}{r^3} \\ \ddot{y}' &= -\frac{\mu y}{r^3} \end{aligned} \tag{10}$$

$$
\begin{aligned}
\ddot{x} = & - x'\dot{\Omega}^2 \cos\Omega_k + \ddot{x}' \cos\Omega_k - \ddot{y}' \sin\Omega_k \cos i_k \\
& + y' \left((\dot{\Omega}_k^2 + (di_k/dt)^2) \sin\Omega_k \cos i_k + 2\dot{\Omega}_k (di_k/dt) \cos\Omega_k \sin i_k \right) \\
& - 2\dot{x}'\dot{\Omega}_k \sin\Omega_k - 2\dot{y}'(\dot{\Omega}_k \cos\Omega_k \cos i_k - (di_k/dt) \sin\Omega_k \sin i_k)
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\ddot{y} = & - x'\dot{\Omega}^2 \sin\Omega_k + \ddot{x}' \sin\Omega_k + \ddot{y}' \cos\Omega_k \cos i_k \\
& - y' \left((\dot{\Omega}_k^2 + (di_k/dt)^2) \cos\Omega_k \cos i_k - 2\dot{\Omega}_k (di_k/dt) \sin\Omega_k \sin i_k \right) \\
& + 2\dot{x}'\dot{\Omega}_k \cos\Omega_k - 2\dot{y}'(\dot{\Omega}_k \sin\Omega_k \cos i_k + (di_k/dt) \cos\Omega_k \sin i_k)
\end{aligned} \tag{12}
$$

$$
\ddot{z} = -y'(di_k/dt)^2 \sin i_k + 2\dot{y}'(di_k/dt) \cos i_k + \ddot{y}' \sin i_k \tag{13}
$$

Note the difference in complexity between eqs. (9) and (10) - (13). Also note that eq. (10) is the two-body gravitational acceleration in the orbit-plane coordinate system of the broadcast navigation position equations (see Table 3), where x' points to the ascending node of the SV orbit, and y' points to $90°$ in-track in the orbital plane. (There is no z' position component because the SV position vector lies entirely in the orbital plane, $z' = 0$ always). The final three broadcast navigation position equations are, in fact, the orbit-plane SV position coordinates transformed to ECEF by an orthogonal transformation comprising orbital inclination (i_k) and longitude of the ascending node (Ω_k). Incorporating non-spherical Earth gravity effects in eq. (10) would first require transforming the gravitational acceleration from ECEF to orbit-plane coordinates. This would also introduce a non-zero $\ddot{z}'$ acceleration component. Including non-spherical gravity in the derivative eqs. (10) - (13) is possible, but unnecessarily complex. It is much simpler to modify the kinematic eqs. (9) to include non-spherical gravity, as shown below in eq. (14). Earth oblateness (J_2) is the dominant perturbing force acting on the SV orbits. The third-body gravity of the Sun and Moon are approximately one order of magnitude less than J_2, and solar radiation pressure is approximately two orders of magnitude less.[6,8] The broadcast navigation orbit model includes correction terms to account for J_2 and other perturbations, nominally over the span of four hours. It follows that computing acceleration with the dominant J_2 effects is commensurate with the accuracy of the broadcast navigation position and velocity.

An example 72-hour acceleration comparison is shown in Fig. 5. The upper panel shows acceleration magnitude and SV latitude (absolute value), revealing strong correlation due to the motion of the SV in the WGS-84 coordinate frame. The lower panel shows the RSS magnitude differences compared to precise values (i.e., time-derivative of the Chebyshev velocity polynomials) for the derivative method and the kinematic method. Also shown in the lower panel is the effect of including oblate Earth gravity (J_2) in the kinematic method, which improves the mean acceleration magnitude difference by approximately one order of magnitude. The numerical value for J_2 was derived from the fully-normalized coefficient $\bar{C}_{20}$ of the EGM-2008 gravity model[15] (by convention: $J_2 = -\sqrt{5}\bar{C}_{20}$). The kinematic

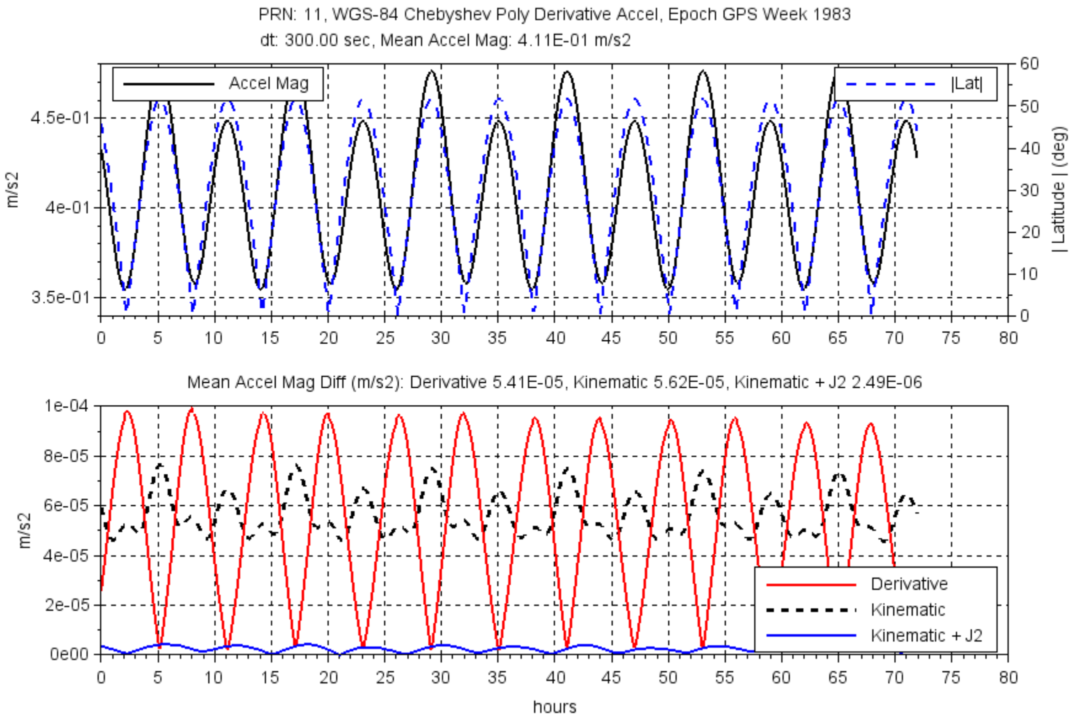

Figure 5. WGS-84 acceleration magnitude and absolute value SV latitude (upper panel) and acceleration differences computed using various methods (lower panel).

acceleration equations including J_2 perturbations are[16]

$$\begin{aligned}\ddot{x} &= -\frac{\mu x}{r^3} + F\left(1 - 5(z/r)^2\right)(x/r) + 2\dot{y}\dot{\Omega}_e + x\dot{\Omega}_e^2 \\ \ddot{y} &= -\frac{\mu y}{r^3} + F\left(1 - 5(z/r)^2\right)(y/r) - 2\dot{x}\dot{\Omega}_e + y\dot{\Omega}_e^2 \\ \ddot{z} &= -\frac{\mu z}{r^3} + F\left(3 - 5(z/r)^2\right)(z/r)\end{aligned} \tag{14}$$

where $F = -(3/2)J_2(\mu/r^2)(R_E/r)^2$ and R_E is the equatorial radius of Earth. With the J_2 terms included, these equations are still less complex than the less accurate, two-body form of the derivative-based equations (10) - (13), and the mean magnitude difference results are improved by approximately one order of magnitude.

Errors introduced by simplifying assumptions (i.e., $\boldsymbol{\omega} = [0, 0, \dot{\Omega}_e]^T$ and $\dot{\boldsymbol{\omega}} = \mathbf{0}$) and other effects on the acceleration equations were analyzed, including the difference between the geodetic constants of the WGS-84 system and the EGM-2008 gravity field (Table 2). Because EGM-2008 is a tide-free gravity field, the effect of the solid earth *permanent tide* should be included in the oblate gravity term for high precision applications.[17]

$$< \Delta J_2 >= - < \Delta \bar{C}_{20} > \sqrt{5} = 9.3324 \times 10^{-9} \tag{15}$$

Including the permanent tide effect on J_2 was found to have no significant impact on the accuracy of the kinematic

Table 2. Geodetic parameters. The WGS-84 value for μ comes from the broadcast navigation equations in the IS, not the separately published value.

Parameter	WGS-84	EGM-2008
μ $(\mathrm{m}^3/\mathrm{s}^2)$	3.986005×10^{14}	$3.986004415 \times 10^{14}$
R_E (meters)	6378137.0	6378136.3
$\bar{C}_{2,0}$		$-0.484165143790815 \times 10^{-3}$
$\bar{C}_{2,1}$		$-0.206615509074176 \times 10^{-9}$
$\bar{S}_{2,1}$		$+0.138441389137979 \times 10^{-8}$

acceleration equations (see Fig. 6). The final value of J_2 used for evaluation was $J_2 = 0.0010826262$.

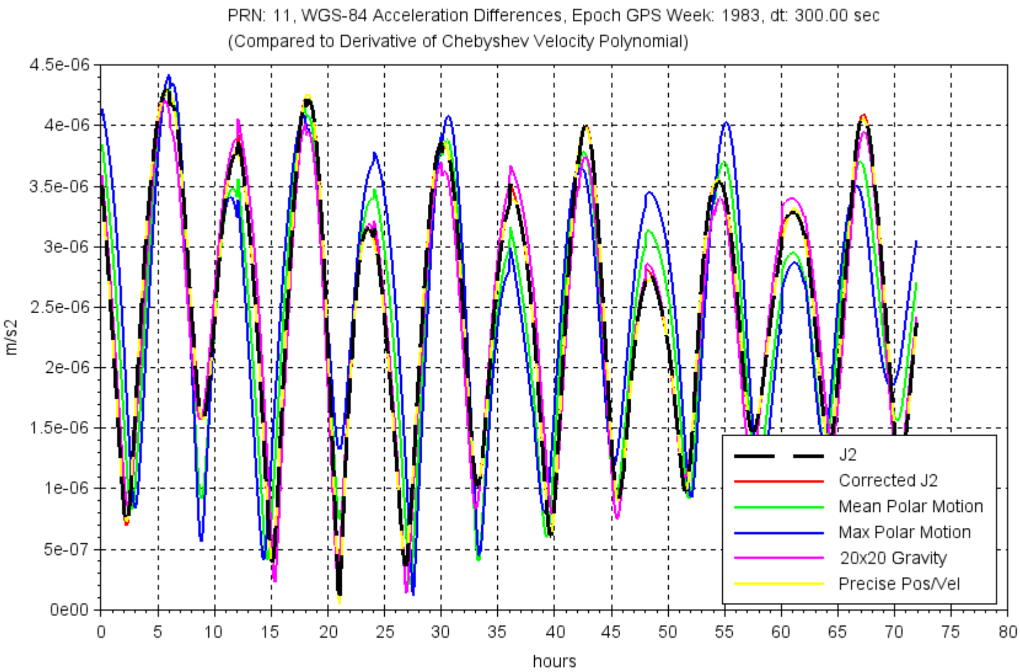

Figure 6. WGS-84 acceleration differences with various corrections applied for comparison.

Also considered were the effects of *polar motion* – the location of the true Earth rotation axis with respect to the conventional pole. Because the J_2 oblate gravity term is dominant over the higher order terms, the approximate mean pole location, or Earth Orientation Parameters (EOP), can be derived from the gravity field coefficients,[17]

$$\begin{aligned} \bar{x}_p &\approx \frac{\bar{C}_{2,1}}{\sqrt{3}\bar{C}_{2,0}} \frac{6.48 \times 10^5}{\pi} = 0.051 \quad \text{arcsec} \\ \bar{y}_p &\approx \frac{-\bar{S}_{2,1}}{\sqrt{3}\bar{C}_{2,0}} \frac{6.48 \times 10^5}{\pi} = 0.341 \quad \text{arcsec} \end{aligned} \tag{16}$$

which include the conversion from radians to arc-seconds. Polar motion EOP data from the International Earth Rotation and Reference Systems Service (IERS)[18] from 1962 to 2018 are plotted in Fig. 7, the mean values from eqs.

(16) indicated by dashed lines. From the figure, the "worst case" maximum polar motion EOP for this ≈ 60 year data set is seen to be approximately $x_p = 0.05$ arcsec and $y_p = 0.6$ arcsec. The mean and maximum polar motion had no significant effect on the computed SV acceleration (Fig. 6). The effects of omitting higher order gravity perturbations (up to spherical harmonic degree and order twenty – 20x20) were also found to be insignificant at the precision level of the kinematic acceleration equations, as shown by Fig. 6. Using the precise ephemeris values for position and velocity in lieu of the less accurate computed values in the kinematic acceleration equations was also found to have insignificant impact. Finally, Earth precession (≈ 50 arcsec per year) and polar motion rate were analyzed and found to have no significant effect on SV acceleration. This analysis shows that the accuracy of the kinematic equations for SV acceleration is not significantly affected by the simplifying assumptions and omission of perturbations when deriving the equations.

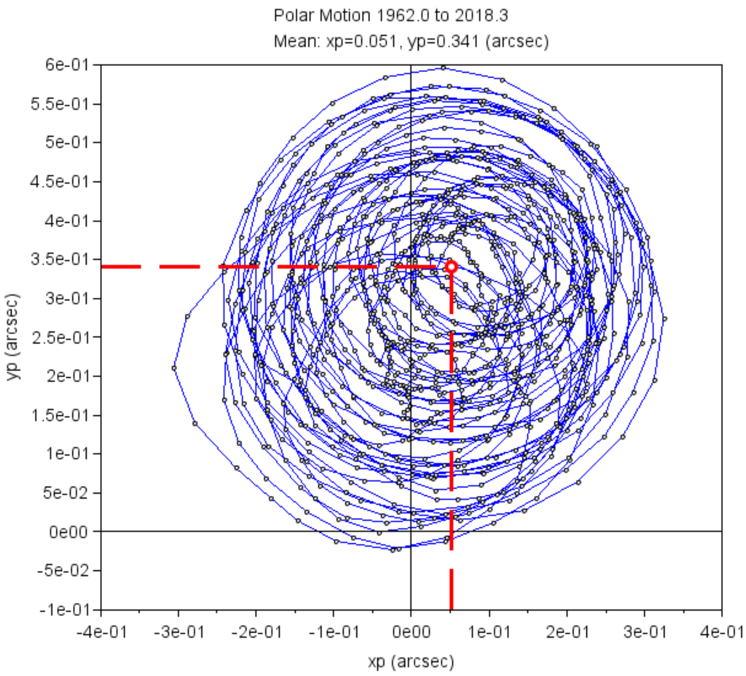

Figure 7. Polar motion Earth Orientation Parameters (EOP), 1962 - 2018. Mean values indicated by dashed lines.

5 | SV CLOCK RELATIVISTIC CORRECTION

The SV clock error (or offset) relative to GPS Time at time t is modeled as

$$\Delta t_{SV} = a_{f0} + a_{f1}(t - t_{oc}) + a_{f2}(t - t_{oc})^2 + \Delta t_r \tag{17}$$

where t_{oc} is the clock data reference time, and a_{f0}, a_{f1}, and a_{f2} are polynomial coefficients unique to each SV clock and sent in the broadcast navigation message. The last term, Δt_r, is the unique relativistic correction arising from the orbital altitude and speed of each SV. The interface specification[4] gives two different equations for computing Δt_r, the first being

$$\Delta t_r = Fe\sqrt{A}\sin E \quad \text{seconds} \tag{18}$$

where e, $\sqrt{A}$, and E are from the broadcast navigation message or computed by the user equations, and

$$F = \frac{-2\sqrt{\mu}}{c^2} = -4.442807633 \times 10^{-10} \quad \frac{\text{s}}{\sqrt{\text{m}}} \tag{19}$$

Equation (18) can be included in the broadcast navigation user equations. The second equation for Δt_r is

$$\Delta t_r = -\frac{2\mathbf{r}\cdot\mathbf{v}}{c^2} \quad \text{seconds} \tag{20}$$

where $\mathbf{r}$ and $\mathbf{v}$ are the ECEF position and velocity vectors computed using the broadcast navigation user equations, i.e., $\mathbf{r}_R$ and $\mathbf{v}_R$ in eqs. (8). Equation (20) is utilized by the GPS Control Segment,[4] but could alternatively be adopted by GPS users after computing $\mathbf{r}$ and $\mathbf{v}$ from the enhanced broadcast user equations.

The time-derivative of the relativistic clock correction is

$$\frac{d}{dt}\left(\Delta t_r\right) = Fe\sqrt{A}\dot{E}\cos E = -\frac{2}{c^2}\left(|\mathbf{v}|^2 + \mathbf{r}\cdot\mathbf{a}\right) \tag{21}$$

where $\dot{E}$ and the acceleration vector $\mathbf{a}$ (acceleration with respect to ECEF) are computed by the recommended user equations (see Table 3). The top panel in Fig. 8 shows 72 hours of clock error for PRN 28 computed from the broadcast navigation messages using eqs. (17) and (18) and a series of 12th order Chebyshev polynomials fit to the clock error tabulated in the NGA precise ephemeris, corrected using eq. (20). The two data sets are practically indistinguishable at the scale of the figure, the maximum difference over the 72-hour period being 0.00903 microseconds. The bottom panel shows the time-derivative of the clock error including the relativistic correction computed by both forms in eq. (21). Again, the data are practically indistinguishable, the maximum difference being 0.0738×10^{-6} microsec/sec. Similar results occurred for all SV tested. The results in Fig. 8 validate eqs. (20) and (21) for use with the newly recommended user equations.

6 | RECOMMENDED EQUATIONS

The recommended equations are shown as part of the complete set of user equations in Table 3, in the form and notation of Table 20-IV in the interface specification.[4] Use of these equations would continue to be optional for all GPS users, as specified in section *20.3.3.4.3 User Algorithm for Ephemeris Determination* of the IS: "The user shall compute the ECEF coordinates of position for the phase center of the SVs antennas utilizing a variation of the equations shown in Table 20-IV." Note that we also suggest a more appropriate title for Table 20-IV. These changes

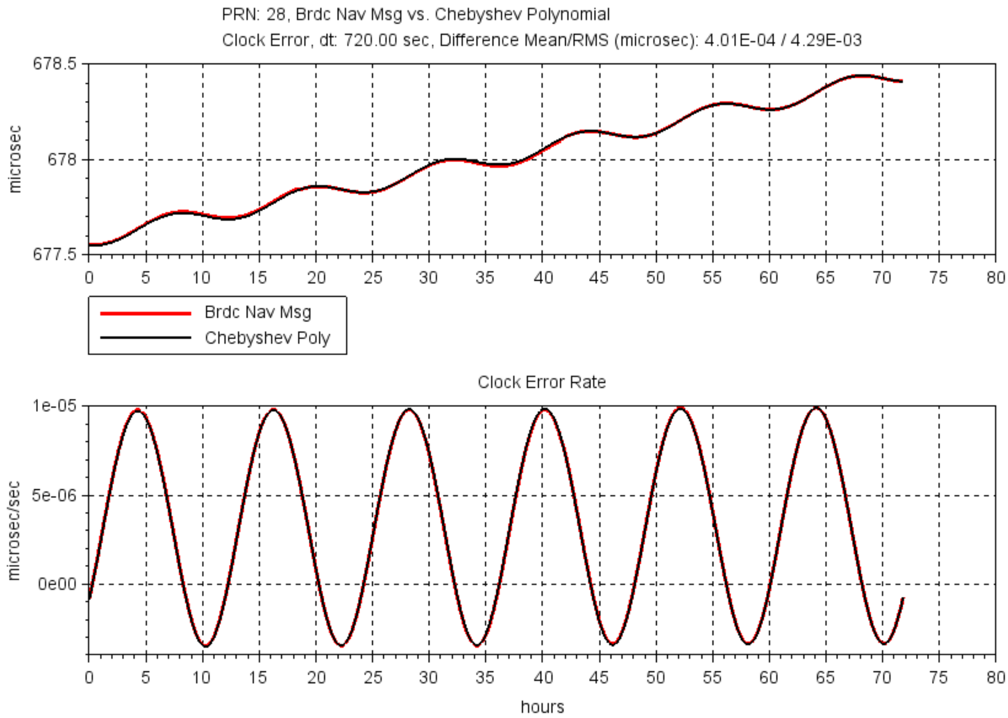

Figure 8. SV clock error: broadcast navigation message vs. precise ephemeris polynomial (top panel), and clock drift rate (bottom panel).

and improvements are also applicable to Table 30-II in the IS,[4] as well as Table 20-II in IS-GPS-705E,[19] Table 3.5-2 in IS-GPS-800E,[20] and other similar documents.

Table 3: Recommended updated Table 20-IV for Interface Specification IS-GPS-200J.

~~Table 20-IV. Elements of Coordinate Systems~~	
Table 20-IV. Broadcast Navigation User Equations	
$\mu = 3.986005 \times 10^{14}$ meters^3/sec^2	WGS 84 value of the earth's gravitational constant for GPS user
$\dot{\Omega}_e = 7.2921151467 \times 10^{-5}$ rad/sec	WGS 84 value of the earth's rotation rate
$A = \left(\sqrt{A}\right)^2$	Semi-major axis
$n_0 = \sqrt{\frac{\mu}{A^3}}$	Computed mean motion (rad/sec)
$t_k = t - t^*_{oe}$	Time from ephemeris reference epoch
$n = n_0 + \Delta n$	Corrected mean motion
(table continues)	

Table 3 (continued)

$M_k = M_0 + nt_k$	Mean anomaly
~~$M_k = E_k - e\sin E_k$~~	~~Kepler's Equation for Eccentric Anomaly (may be solved by iteration) (radians)~~
	Kepler's equation $(M_k = E_k - e\sin E_k)$ solved for eccentric anomaly (E_k) by iteration:
$E_0 = M_k$	- Initial value (radians)
$E_j = E_{j-1} + \frac{M_k - E_{j-1} + e\sin E_{j-1}}{1 - e\cos E_{j-1}}$	- Refined value, three iterations, $(j = 1, 2, 3)$
$E_k = E_3$	- Final value
~~$\nu_k = \tan^{-1}\left\{\frac{\sin\nu_k}{\cos\nu_k}\right\}$~~	~~True Anomaly~~
~~$= \tan^{-1}\left\{\frac{\sqrt{1-e^2}\sin E_k/(1-e\cos E_k)}{(\cos E_k - e)/(1-e\cos E_k)}\right\}$~~	
$\nu_k = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E_k}{2}\right)$	True anomaly (unambiguous quadrant)
~~$E_k = \cos^{-1}\left\{\frac{e+\cos\nu_k}{1+e\cos\nu_k}\right\}$~~	~~Eccentric Anomaly~~
$\Phi_k = \nu_k + \omega$	Argument of Latitude
$\delta u_k = c_{us}\sin 2\Phi_k + c_{uc}\cos 2\Phi_k$	Argument of Latitude Correction
$\delta r_k = c_{rs}\sin 2\Phi_k + c_{rc}\cos 2\Phi_k$	Radius Correction
$\delta i_k = c_{is}\sin 2\Phi_k + c_{ic}\cos 2\Phi_k$	Inclination Correction
$u_k = \Phi_k + \delta u_k$	Corrected Argument of Latitude
$r_k = A(1 - e\cos E_k) + \delta r_k$	Corrected Radius
$i_k = i_0 + \delta i_k + (\mathrm{IDOT})t_k$	Corrected Inclination
$x'_k = r_k\cos u_k$	Positions in Orbital Plane
$y'_k = r_k\sin u_k$	
$\Omega_k = \Omega_0 + (\dot{\Omega} - \dot{\Omega}_e)t_k - \dot{\Omega}_e t_{oe}$	Corrected longitude of ascending node
$x_k = x'_k\cos\Omega_k - y'_k\cos i_k\sin\Omega_k$	Earth-fixed coordinates
$y_k = x'_k\sin\Omega_k + y'_k\cos i_k\cos\Omega_k$	
$z_k = y'_k\sin i_k$	

* t is GPS system time at time of transmission, i.e., GPS time corrected for transit time (range/speed of light). Furthermore, t_k shall be the actual total time difference between the time t and the epoch time t_{oe}, and must account for beginning or end of week crossovers. That is, if t_k is greater than 302,400 seconds, subtract 604,800 seconds from t_k. If t_k is less than -302,400 seconds, add 604,800 seconds to t_k.

(table continues)

Table 3 (continued)

SV Velocity	
$\dot{E}_k = n/(1 - e\cos E_k)$	Eccentric anomaly rate
$\dot{\nu}_k = \dot{E}_k\sqrt{1-e^2}/(1 - e\cos E_k)$	True anomaly rate
$(di_k/dt) = \text{(IDOT)} + 2\dot{\nu}_k(c_{is}\cos 2\Phi_k - c_{ic}\sin 2\Phi_k)$	Corrected Inclination rate
$\dot{u}_k = \dot{\nu}_k + 2\dot{\nu}_k(c_{us}\cos 2\Phi_k - c_{uc}\sin 2\Phi_k)$	Corrected Argument of Latitude rate
$\dot{r}_k = eA\dot{E}_k\sin E_k + 2\dot{\nu}_k(c_{rs}\cos 2\Phi_k - c_{rc}\sin 2\Phi_k)$	Corrected Radius rate
$\dot{\Omega}_k = \dot{\Omega} - \dot{\Omega}_e$	Longitude of ascending node rate
$\dot{x}'_k = \dot{r}_k\cos u_k - r_k\dot{u}_k\sin u_k$	In-plane x velocity
$\dot{y}'_k = \dot{r}_k\sin u_k + r_k\dot{u}_k\cos u_k$	In-plane y velocity
$\dot{x}_k = -x'_k\dot{\Omega}_k\sin\Omega_k + \dot{x}'_k\cos\Omega_k - \dot{y}'_k\sin\Omega_k\cos i_k$ $-y'_k(\dot{\Omega}_k\cos\Omega_k\cos i_k - (di_k/dt)\sin\Omega_k\sin i_k)$	Earth-fixed x velocity (m/s)
$\dot{y}_k = x'_k\dot{\Omega}_k\cos\Omega_k + \dot{x}'_k\sin\Omega_k + \dot{y}'_k\cos\Omega_k\cos i_k$ $-y'_k(\dot{\Omega}_k\sin\Omega_k\cos i_k + (di_k/dt)\cos\Omega_k\sin i_k)$	Earth-fixed y velocity (m/s)
$\dot{z}_k = y'_k(di_k/dt)\cos i_k + \dot{y}'_k\sin i_k$	Earth-fixed z velocity (m/s)
SV Acceleration	
$R_E = 6378137.0$ meters	WGS 84 Earth equatorial radius
$J_2 = 0.0010826262$	Oblate Earth gravity coefficient
$F = -(3/2)J_2(\mu/r_k^2)(R_E/r_k)^2$	Oblate Earth acceleration factor
$\ddot{x}_k = -\mu\frac{x_k}{r_k^3} + F\left[(1 - 5(z_k/r_k)^2)(x_k/r_k)\right]$ $+2\dot{y}_k\dot{\Omega}_e + x_k\dot{\Omega}_e^2$	Earth-fixed x acceleration (m/s^2)
$\ddot{y}_k = -\mu\frac{y_k}{r_k^3} + F\left[(1 - 5(z_k/r_k)^2)(y_k/r_k)\right]$ $-2\dot{x}_k\dot{\Omega}_e + y_k\dot{\Omega}_e^2$	Earth-fixed y acceleration (m/s^2)
$\ddot{z}_k = -\mu\frac{z_k}{r_k^3} + F\left[(3 - 5(z_k/r_k)^2)(z_k/r_k)\right]$	Earth-fixed z acceleration (m/s^2)

7 | IMPLEMENTATION AND BENCHMARK TESTS

To assist with implementation and verification of the recommended equations, we have included benchmark test results for position, velocity, and acceleration. The associated broadcast navigation message parameters are shown in Table 4. The test results of the user equations are shown in Table 5 for two different times. The values of the precise ephemeris and broadcast navigation user equations are shown in component form along with the RSS magnitude. Note that for acceleration the precise values were computed from the derivatives of Chebyshev polynomials fit to the precise ephemeris velocity data, as described earlier.

Table 4. Broadcast navigation message parameters used for benchmark tests: GPST 7 Jan 2018, 00:00:00.0

Parameter	Value	Parameter	Value
PRN	11	c_{ic}	$0.199303030968E-06$
c_{rs}	$-0.965625000000E+01$	Ω_0	$-0.657960408566E+00$
Δn	$0.583845748090E-08$	c_{is}	$0.173225998878E-06$
M_0	$-0.286954703389E+01$	i_0	$0.903782727230E+00$
c_{uc}	$-0.379979610443E-06$	c_{rc}	$0.293218750000E+03$
e	$0.167867515702E-01$	Ω	$0.173129682312E+01$
c_{us}	$0.277347862720E-05$	$\dot{\Omega}$	$-0.868929051526E-08$
$\sqrt{A}$	$0.515375480270E+04$	$IDOT$	$0.789318592573E-10$
t_{oe}	$0.000000000000E+00$	GPS Week	$0.198300000000E+04$

Table 5. Benchmark test results.

GPST 7 Jan 2018	00:35:00.0			
Position (m)	x_k	y_k	z_k	RSS
Broadcast Nav	3166192.017	-21511945.818	-15899623.697	26936715.065
Precise Ephemeris	3166191.446	-21511947.161	-15899624.824	26936716.607
Velocity (m/s)	$\dot{x}_k$	$\dot{y}_k$	$\dot{z}_k$	RSS
Broadcast Nav	1533.973749	-1209.904136	2000.871636	2796.503314
Precise Ephemeris	1533.973891	-1209.904144	2000.871617	2796.503382
Acceleration (m/s^2)	$\ddot{x}_k$	$\ddot{y}_k$	$\ddot{z}_k$	RSS
Broadcast Nav	-0.224186	0.100579	0.324295	0.406870
Precise Ephem. Poly.	-0.224188	0.100577	0.324296	0.406871
GPST 7 Jan 2018	01:50:00.0			
Position (m)	x_k	y_k	z_k	RSS
Broadcast Nav	7847635.362	-25169173.996	-4315772.358	26715137.871
Precise Ephemeris	7847635.584	-25169175.522	-4315773.249	26715139.518
Velocity (m/s)	$\dot{x}_k$	$\dot{y}_k$	$\dot{z}_k$	RSS
Broadcast Nav	595.709009	-259.303963	2970.973426	3041.182478
Precise Ephemeris	595.708923	-259.304060	2970.973219	3041.182268
Acceleration (m/s^2)	$\ddot{x}_k$	$\ddot{y}_k$	$\ddot{z}_k$	RSS
Broadcast Nav	-0.160162	0.305506	0.090248	0.356554
Precise Ephem. Poly.	-0.160162	0.305506	0.090249	0.356554

8 | SUMMARY

Accurate SV position modeling is required for basic GPS navigation (positioning). SV velocity and acceleration modeling are required for more advanced navigation and other purposes such as receiver velocity determination and GPS/INS integration. The current interface specification includes equations for computing SV position only. We present equations for extending the broadcast navigation user equations to compute SV velocity and acceleration. Previous work in this area has been published, but the methods herein are validated over a longer period of time using multiple broadcast navigation messages. Additionally, the kinematic form of the acceleration equations are less complex and improve mean magnitude differences by approximately one order of magnitude when oblate Earth gravity effects are included. The new methods were validated using three days (6 GPS orbit periods) of precise ephemeris data from the NGA. For acceleration data, the derivative of a 12th order Chebyshev polynomial fit to the velocity data was used. Also analyzed were the effects of the permanent tide on the J_2 gravity coefficient, mean and maximum polar motion effects, polar motion rate, precession, and higher order gravity. The recommended equations are provided in the format and notation of the existing user equations table in the interface specification, and benchmark test results are provided to ease with implementation and validation.

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DISCLAIMER

The views expressed in this paper are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government.

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