XX. *A Dynamical Theory of the Electric and Luminiferous Medium.*—Part II. *Theory of Electrons.*

By Joseph Larmor, *F.R.S., Fellow of St. John's College, Cambridge.*

Received May 16,—Read June 20, 1895.

1. In a previous paper the concrete representation of electrical and optical phenomena by means of a rotationally elastic fluid æther has been discussed.* In an Appendix it has been shown that whenever there is direct question of interaction between the molecules of matter and the æther, whether it be in the phenomena of magnetism or in the optical phenomena of dispersion and moving material media, the consideration of groups of electrons or permanent strain-centres in the æther, which form a part of, or possibly the whole of, the constitution of the atoms of matter, suffices to lead to a correlation of the various modes of activity; while this scheme seems to be free from the chief difficulties which have pressed on other methods of representation.

The present paper is chiefly concerned with the further development of the molecular aspect of this theory. As a preliminary, it is maintained that a dynamical theory of electric currents, based on the ordinary conception of a current-element, must lead to expressions for the electrodynamic forces which are at variance with the facts. On the other hand, a theory which considers moving electrons to be the essential elements of the true currents in material media, gives a definite account of the genesis and the mutual relations of both types of forcive, the electromotive and the ponderomotive, and gives formulæ for them, which correspond in the main with those originally deduced by Maxwell from consideration of the properties of his concrete model of the electric field, though they are not substantiated by his later abstract theory based on current-elements. That theory is held to be defective, in the first place on account of the discrepancy with experiment above mentioned, and in the second place, because it is not competent to describe the mode of genesis of a conduction current by electrical separation produced in the element of volume of the conductor under the influence of the field of force; it is thus an incomplete formulation of the phenomena.

The application of the method of electrons to vibrational phenomena leads to formulæ for optical dispersion, and optical propagation in metals, which are in general

* 'Phil. Trans.,' 1894, A, pp. 719–822.

agreement with experimental knowledge. These subjects have already been treated from a similar point of view by VON HELMHOLTZ in 1892, but his equations, which are arrived at in a more abstract manner, are fundamentally at variance with those of the present theory. They lead, however, to the same type of dispersion formula when the medium is transparent, a type which has been generally accepted as in good agreement with exact measurements of dispersion.

The application to the optical properties of moving media leads to FRESNEL'S well-known formula, as had already been shown in the previous paper. If the theory of the constitution of matter which is suggested in that paper is allowed, it also leads to an explanation of the null result of the well-known second-order experiment of MICHELSON and MORLEY, of which previous theoretical discussions have quite failed to take cognizance.

As the statistical processes connected with molecular theory are of a very delicate character, and subject to suspicion especially when pushed to the second order of small quantities, the formulæ are derived by two independent methods, which supplement and illuminate each other but are analytically of very different types. One of these methods, which lies nearer to recent procedure in electrodynamics, and has been used by LORENTZ in a comprehensive discussion of the subject of moving media, published at the beginning of the present year, is to adapt the fluxes and forces occurring in the fundamental circuital equations of the free æther, so as to make similar equations apply to the æther in ponderable media. The other method is to investigate the forcive acting on a single electron, and then to build up the electric and optical phenomena out of the various types of movements of electrons that can occur in dielectric and conducting media. It is only in this latter way that a rational detailed account of the ponderomotive forces in media which are transmitting currents, or are electrically or magnetically polarized, can be derived ; as a special example, a formula for the ponderomotive pressure due to radiation is obtained.

The discussion of the topics above mentioned, with the exception of the MICHELSON-MORLEY experiment, is quite independent of any speculation as to the nature of electrons or the relation of æther to matter. It may be founded directly on MAXWELL'S equations of the electric field in free æther, as now experimentally confirmed.

But when we go further, it seems to be a strong argument in favour of a rotational æther, or at any rate of an æther whose actual properties are represented with close fidelity by the scheme of a rotationally elastic fluid medium, that in this way we derive an actual physical structure for an electron, and that by the orbital motions of electrons in the atom we derive a representation of an atom as a fluid vortex, an idea which has always been present to physical speculation from LEUCIPPUS, through DESCARTES, down to the recent definite dynamical conceptions of vortices. To justify tentative adhesion to such a view, it is not necessary to be able to produce an explanation of the fundamental properties of mass and gravitation in matter : in particular

these are not in necessary connection with the electrical and optical phenomena. But it is easy to see that a rotational æther is not inadequate to including such properties among its relations: if the nuclei of the electrons are supposed small enough, the inertia of matter would be definitely represented by the electric inertia of the electrons; and as the electrons may then have vacuous nuclei and have each a free period of radial vibration in the fluid æther, which is not subject to damping by radiation (but subject, however, to a certain instability unless the free spherical form of surface is fortified by some kind of constraint), the gravitation between them may be represented or illustrated by the hydrodynamical pulsatory theory of BJERKNES and HICKS.

Examination of Theories involving the Electrodynamic Potential Function.

2. Every scheme for reducing the phenomena of electric currents to a purely dynamical basis must start ultimately from the formula for the electrodynamic potential of a system of ordinary currents, discovered and first extensively applied by F. E. NEUMANN.

In a preliminary tentative discussion of this formula, the only course open appears to be to take the current flowing in an element of volume, and the coordinates of the element, as the independent variables of a continuous analysis extended over the field of currents.

If the function is treated merely as a potential of the forces of the field, we may employ the principle of NEUMANN, afterwards elaborated and extended by VON HELMHOLTZ, but still hypothetical in its dynamical aspect, that a variation of the potential, due to alteration of the positions of the conductors, but without any variation of the strength of the current flowing along any linear element of a tube of flow, leads to the ponderomotive forces tending to alter their positions: while a variation due to alteration of the strengths of the currents, the conductors being fixed or moving, leads to electromotive forces tending to change the strengths of the said currents, but not to alter the motion of the bodies.

The dynamical theory of MAXWELL aims at going further: the potential function with sign changed is assumed to be the kinetic energy residing in the latent or impalpable active medium which is associated with the current system. The induced electric forces are now taken to be reversed kinetic reactions corresponding to the currents considered as generalized electric velocity components, in accordance with the Lagrangian formula $-\left(\frac{\delta}{dt}\frac{d\mathrm{T}}{d\dot{\theta}} - \frac{d\mathrm{T}}{d\theta}\right)$; and these together with the applied electromotive forces make up the total ones that drive the current in conformity with OHM's law. In the same way the ponderomotive forces acting on the conductors are the reversed kinetic reactions corresponding to change of position of the material system, which require to be compensated by equal and opposite applied

forces if mechanical equilibrium of the conductors is to subsist. It was one of the discoveries of MAXWELL that, for a system of *circuital* conduction currents, the potential function can actually be formally represented as the kinetic energy of a latent moving system, coupled with the palpable conductors and thereby influencing them—of which system the currents are to be treated as generalized velocity components corresponding to electric coordinates which do not appear themselves in the function—without thereby introducing any discrepancy into the general scheme, as already experimentally determined, of the equations of the electric field.

Whichever of these methods of development of the potential function is essayed, it is a necessary preliminary with a view to a complete analysis to take the strength of the current flowing in each element of volume, or it may be in a linear element of a tube of flow, as dynamically a separate electric entity. The object of the following discussion is to consider how far this is consistent with a more detailed examination of the forcives thus derivable, particularly with the nature of that part of the internal stress in a conductor carrying a current which tends to alter its shape but not to produce motion of translation of the conductor as a whole.

3. According then to the type of theory which considers a current system to be built up of physical current-elements of the form $(u, v, w)\,\delta\tau$, the energy associated with an element of volume $\delta\tau$, as existing in the surrounding field and controlled by the element, is

$$\mathrm{T} = (\mathrm{F}u + \mathrm{G}v + \mathrm{H}w)\,\delta\tau.$$

The ponderomotive force acting on the element will be derived from a potential energy function $-\mathrm{T}$, by varying the coordinates of the material framework: it must in fact consist, per unit volume, of a force

$$\left(u\frac{d\mathrm{F}}{dx} + v\frac{d\mathrm{G}}{dx} + w\frac{d\mathrm{H}}{dx},\ u\frac{d\mathrm{F}}{dy} + v\frac{d\mathrm{G}}{dy} + w\frac{d\mathrm{H}}{dy},\ u\frac{d\mathrm{F}}{dz} + v\frac{d\mathrm{G}}{dz} + w\frac{d\mathrm{H}}{dz}\right),$$

and a couple

$$(v\mathrm{H} - w\mathrm{G},\ w\mathrm{F} - u\mathrm{H},\ u\mathrm{G} - v\mathrm{F}),$$

the former being derived from a translational, the latter from a rotational virtual displacement of the element.* We may simplify these expressions by taking the axis of z parallel to the current in the element $\delta\tau$, so that u and v become null; then we have

$$\text{a force}\left(w\frac{d\mathrm{H}}{dx},\ w\frac{d\mathrm{H}}{dy},\ w\frac{d\mathrm{H}}{dz}\right)\text{ and a couple }(-w\mathrm{G},\ w\mathrm{F},\ 0).$$

According to the AMPÈRE-MAXWELL formula, there should be simply a force at right angles to the current, specified by the general formula

$$(vc - wb,\ wa - uc,\ ub - va),$$

* In the previous paper, § 120, the couple was omitted.

which becomes for the present special axes of coordinates

$$\left\{-w\left(\frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{dx}\right),\ w\left(\frac{d\mathrm{H}}{dy}-\frac{d\mathrm{G}}{dz}\right),\ 0\right\}.$$

The forcive at which we have here arrived thus differs from the AMPÈRE-MAXWELL one by

$$\text{a force}\left(u\frac{d\mathrm{F}}{dz},\ v\frac{d\mathrm{G}}{dz},\ w\frac{d\mathrm{H}}{dz}\right)\text{ and a couple }(-w\mathrm{G},\ w\mathrm{F},\ 0):$$

these are equivalent to forces acting on the ends of each linear current element, equal at each end numerically to (wF, wG, wH) per unit of cross section, positive at the front end and negative at the rear end. They are thus of the nature of an internal stress in the medium, and are self-equilibrating for each circuital current and so do not disturb the resultant forcive on the conductor as a whole due to the field in which it is situated. From MAXWELL'S stress standpoint they would form an equilibrating addition to the stress-specification in the conductor which is the formal equivalent of the electrodynamic forcive.

According to the AMPÈRE-MAXWELL formula, the forcive on an element of a linear conductor carrying a current is at right angles to it, so that the tension along the conductor is constant so far as that forcive is concerned. The traction in the direction of the current, arising from the above additional stress, would introduce an additional tension, equal to the current multiplied by the component of the vector potential in its direction, which is not usually constant along the circuit, and so may be made the subject of experimental test with liquid conductors, as it would introduce differences of fluid pressure. There will also be an additional transverse shearing stress which should reveal itself in experiments on solid conductors with sliding contacts.

In particular these additional forces should reveal themselves in the space surrounding a closed magnetic circuit, where the ordinary Amperean force vanishes because the magnetic field is null; in that case (F, G, H) may be interpreted as the total impulsive electric force induced at any point by the making of the circuit. Professor G. F. FITZGERALD has devised an experiment in which the behaviour of a thread of mercury carrying a strong current and linked with a complete magnetic circuit was closely observed when the circuit was made and broken. No movement was detected, whereas, when the magnetic circuit was incomplete, the ordinary Amperean forces were very prominent. According to the above analysis, the two types of forcive should be of the same order of magnitude in such a case: the result of the experiment is therefore against this theory. A like negative result has also attended an experiment by Professor O. J. LODGE, in which he proposed to detect minute changes of level along the upper surface of a uniform mercury thread by an interference arrangement on the principle of NEWTON'S rings: when the current was

turned on, the section of the thread became more nearly circular owing to the mutual attractions of the different filaments of the current, but there was no alteration in the direction of its length.

This experimental evidence, combined with the fact (*infra*) that a theory of moving electrons, which are certainly independent physical entities, leads simply to the AMPÈRE-MAXWELL forcive, seems to justify the conclusion that either the above analysis is wrong, or else the ordinary treatment of electrodynamics in terms of a specification by current elements is physically untenable. As tending to the exclusion of the first alternative, and also as of independent critical interest, the following discussion by aid of the more usual analytical method employed by AMPÈRE, NEUMANN, and VON HELMHOLTZ is given.

4. Assuming that current elements serve as a sufficient physical specification, it is known that the mutual energy (kinetic) of two such elements, $\iota\delta s$ and $\iota'\delta s'$, must be

$$\iota\delta s \,.\, \iota'\delta s' \left(\frac{1}{r}\frac{dr}{ds}\frac{dr}{ds'} + \frac{d^2\phi(r)}{ds\,ds'}\right),$$

where $\phi(r)$ is some function of their distance apart.

The variation of this energy, due to mutual displacement of the elements, is

$$\iota\delta s \,.\, \iota'\delta s' \left(-\frac{\delta r}{r^2}\frac{dr}{ds}\frac{dr}{ds'} + \frac{1}{r}\frac{d\delta r}{ds}\frac{dr}{ds'} + \frac{1}{r}\frac{dr}{ds}\frac{d\delta r}{ds'} + \frac{d^2\delta\phi(r)}{ds\,ds'}\right).$$

On substituting

$$\frac{d}{ds}\left(\frac{1}{r}\frac{dr}{ds'}\delta r\right) - \frac{d}{ds}\left(\frac{1}{r}\frac{dr}{ds'}\right)\delta r \text{ for } \frac{1}{r}\frac{d\delta r}{ds}\frac{dr}{ds'},$$

and similarly for the following term, the variation becomes

$$\iota'\delta s' \,.\, \iota\delta s \frac{d}{ds}\left(\frac{1}{r}\frac{dr}{ds'}\delta r\right) + \iota\delta s \,.\, \iota'\delta s' \frac{d}{ds'}\left(\frac{1}{r}\frac{dr}{ds}\delta r\right) + \iota\delta s \,.\, \iota'\delta s' \frac{d^2\delta\phi(r)}{ds\,ds'}$$
$$+ \iota\delta s \,.\, \iota'\delta s' \left(\frac{1}{r^2}\frac{dr}{ds}\frac{dr}{ds'} - \frac{2}{r}\frac{d^2r}{ds\,ds'}\right)\delta r.$$

If we take the aggregate for a series of elements $\iota'\delta s'$ which form a circuital current, the second and third terms vanish ; and the forcive on the element $\iota\delta s$, due to a circuital current system, is thus compounded of

$$\iota\delta s \,.\, \iota'\delta s' \left(\frac{1}{r^2}\frac{dr}{ds}\frac{dr}{ds'} - \frac{2}{r}\frac{d^2r}{ds\,ds'}\right)$$

towards each element $\iota'\delta s'$ of the influencing current system, which is the ordinary forcive of AMPÈRE, together with a force acting on each end of $\iota\delta s$, positive on the forward end and negative on the rearward end, and for each end compounded of

$\iota'\delta s' \cdot \iota r^{-1} dr/ds'$ acting towards each element $\iota'\delta s'$ of the influencing system. The components of the aggregate of this latter force, by which the present result differs from the Amperean one, are

$$\iota \int \iota' \frac{x}{r^2} \frac{dr}{ds'} ds', \quad \iota \int \iota' \frac{y}{r^2} \frac{dr}{ds'} ds', \quad \iota \int \iota' \frac{z}{r^2} \frac{dr}{ds'} ds',$$

where (x, y, z) are the coordinates of the element $\iota'\delta s'$ with reference to an origin situated at the element under consideration $\iota\delta s$. On integration by parts round the complete current circuit these become

$$\iota \int \frac{\iota'}{r} \frac{d}{dx}\left(r \frac{dr}{ds'}\right) ds', \quad \iota \int \frac{\iota'}{r} \frac{d}{dy}\left(r \frac{dr}{ds'}\right) ds', \quad \iota \int \frac{\iota'}{r} \frac{d}{dz}\left(r \frac{dr}{ds'}\right) ds',$$

that is, by definition,

$$\iota \mathrm{F}, \quad \iota \mathrm{G}, \quad \iota \mathrm{H},$$

which is the same expression as we had previously arrived at.

Apart from detailed explanation, the previous mode of variation may be cast into the briefer analytical form

$$\delta \iota (\mathrm{F}\, dx + \mathrm{G}\, dy + \mathrm{H}\, dz)$$

$$= \iota \left(\frac{d\mathrm{F}}{dx} dx + \frac{d\mathrm{G}}{dx} dy + \frac{d\mathrm{H}}{dz} dz\right) \delta x + \ldots + \ldots + \iota (\mathrm{F}\, d\,\delta x + \mathrm{G}\, d\,\delta y + \mathrm{H}\, d\,\delta z)$$

of which the last term

$$= \iota \{\mathrm{F}\, \delta x + \mathrm{G}\, \delta y + \mathrm{H}\, \delta z\} - \iota\, \delta x \left(\frac{d\mathrm{F}}{dx} dx + \frac{d\mathrm{F}}{dy} dy + \frac{d\mathrm{F}}{dz} dz\right) - \ldots - \ldots,$$

so that the whole

$$= \left\{\left(\frac{d\mathrm{G}}{dx} - \frac{d\mathrm{F}}{dy}\right) \iota\, dy - \left(\frac{d\mathrm{F}}{dz} - \frac{d\mathrm{H}}{dx}\right) \iota dx\right\} \delta x + \ldots + \ldots + \{\iota \mathrm{F}\, \delta x + \iota \mathrm{G}\, \delta y + \iota \mathrm{H}\, \delta z\},$$

which represents the work of the Amperean forcive on the linear element together with a traction $\iota (\mathrm{F}, \mathrm{G}, \mathrm{H})$ on its ends.

Discrimination between Velocities and Momenta in Generalized Dynamics.

5. In the dynamics of systems whose internal connexions are only partially known, it is essential to have a clear view of the circumstances which determine whether the various quantities which enter into the specification of the energy are to be classed as coordinates, velocities, or momenta. For example, in determining the forcives between cores in problems of cyclic motion, the circulations must be treated as generalized momenta, while in the Maxwellian electrodynamics of complete circuits

the currents flowing in them are rather to be classed as velocity components. The basis of the distinction between these two classes of quantities is of course fundamental; and it is to be found in the way in which they occur in the Hamiltonian analysis. The essential property of a velocity is that it is a perfect differential coefficient, with respect to the time; any function involving rate of change of configuration, which enjoys this property so that its time-integral is a function of position only, may be taken to be a velocity, provided we, if need be, contemplate also a corresponding forcive. On the other hand, any such function of the rate of change of configuration, even though it be a perfect differential with respect to the time, must be treated as a momentum if it is known to remain constant with the time, while no applied forcive controls it; for if it were a velocity, linked up with other velocities, its constancy in the free motion could not usually fit in with the analytical theory.

In the theory of cyclic fluid motion, the circulations, being constant, must thus be taken as momenta; and when the energy is expressed in terms of them, it must be modified before the forcives can be derived from it in the manner of LAGRANGE and HAMILTON. In the theory of electrodynamics the currents are not unalterable with the time; and as they are differential coefficients with respect to the time of definite physical quantities, the charges of electricity, they may be taken as velocities, provided we recognize the play of corresponding forcives. In the electrodynamics of complete circuits, after MAXWELL, there is no reason, in that theory taken by itself, why the functions designated as the electrokinetic momenta should not be taken as velocities instead, if so desired; for they satisfy all the above conditions, though of course the corresponding forcives would be of quite different types from the usual ones. This remark is in illustration of the fact that the distinction between momenta and velocities is to a certain extent one of convenience.

A theory which would regard the current flowing in a complete circuit as a generalized momentum of the latent æthereal motion labours under many difficulties. Such a hypothesis seems tempting at first sight, and, for example, lends itself easily to the inclusion of permanent magnetism in a dynamical system: it has accordingly been developed, amongst others, by VON HELMHOLTZ, who formulated the equations on this basis in his Memoir on Least Action. The electrodynamic potential function would then represent the kinetic energy modified in ROUTH'S manner by the substitution of electric momenta for velocities, which in the present case simply changes its sign: and this would seem to require that the kinetic energy of the electric field should in ordinary circumstances be negative, which could hardly be correct.

In the analogous problem of cyclic irrotational motion in fluids, with the circulation guided by rigid ring-cores, the circulations which correspond formally to the electric currents remain constant in the free motion, and so must be analytically momenta, and not velocities. The energy in this vortex problem is given by the same formula as in the electric problem, but it must be modified by change of its

sign before the forcives are derived from it; and thus the circumstance, first emphasized by Lord KELVIN, that the forcives between such rigid ring-cores are equal but opposite to the forcives between the analogous rigid conductors carrying electric currents, must find its explanation in the fact that the electric currents are of the type of velocities, while the fluid circulations are of the type of momenta.

6. In the dynamics of the circulation of a fluid through ring-shaped solids, the expression for the energy in terms of vortex filaments,

$$T = \tfrac{1}{2}\Sigma\iint\sigma_1\sigma_2 r^{-1}\cos\epsilon\, ds_1\, ds_2,$$

represents its proper distribution with respect to each solid as a whole, but does not represent its distribution as regards each element of a vortex filament. Thus, when the solids are treated as rigid bodies, and are referred to the appropriate number of coordinates (six for each), this expression for the energy is competent to determine the aggregate forcive on each solid. But it is not competent to express the actual forcive on an element of the solid, because, in its formation by integration from the actual distribution of kinetic energy throughout the elements of volume of the fluid, a process of integration round the apertures of the solids has been employed, which supposes that each element of a vortex filament is connected with other elements so as to be part of a complete circuit.

In any such case the distribution of the resultant forcive throughout the solid has an undetermined part which is of the nature of an internal stress. In the example of cyclic fluid motion there is however only needed, in addition to this expression for the energy, the property that the forcive on the solids is distributed as a fluid pressure over their surfaces, in order to obtain from this form of the energy a quite definite result. The distribution of forcive on a flexible core is thus at each point at right angles to its axis, and the tension in such a core is therefore constant all along it.

If we attempt to deduce electrodynamic forcives, in cases of circuital electric flow, from the cognate expression for the energy, with current in place of vorticity but with different sign, the applicability of the method is limited in like manner. An additional hypothesis, or experimental principle, that the electrodynamic forcive on an element of a conductor is at right angles to the current which it carries, will, however, make the problem definite.

But if we were to remove the restriction that the flow of true electricity (*i.e.*, of electrons) must be circuital, and to treat for each element of the conductor the strength of the current as dynamically an electric velocity corresponding to a single definite electric coordinate which does not explicitly appear, the energy function would now be adequate by itself to determine the distribution of the forcive, and the result would be as given above, § 3. The discrepancy between this result and experiment thus seems to show that it is not legitimate to consider the current as a generalized

velocity distributed over the elements of the conductor and independent for each of them. Nor is this in any way surprising, when we bear in mind that the velocity of a *single* electron is certainly of the type required for a generalized dynamical velocity component, and when we infer, as below (§ 23), that the forcive on a current considered as made up statistically from the moving electrons, is different from what would be obtained by taking the current-element itself as a generalized velocity component. Our conclusion is, then, that the only proper basis for the dynamical analysis of the phenomena of currents flowing in conductors, in fact, of all cases of the flow of true electricity, is to treat the currents as the statistical aggregates of the movements of the electrons.

Two different Methods of Analysis.

7. With a view to the analytical formulation of the properties of an æther pervaded by electrons, it is necessary in the first place to take a survey of the various ways in which these electrons are distributed. In statical circumstances the great bulk of them are grouped together into polar molecules, which may be either in totally irregular orientation as in an unpolarized dielectric, or may possess features of regularity which can be represented in a statistical manner by the type of theory first developed by POISSON with respect to induced magnetism. If the element of volume possesses an electric charge, it must also contain free electrons not so grouped; but their number is excessively small compared with the former class.

Proceeding now to kinetics, the motions of these various classes of electrons constitute true currents of various types. A drift of free electrons constitutes an ordinary conduction current, and FARADAY'S fundamental law of electrolysis shows that when there is an electrolyte (or a dielectric) in the circuit of a steady current, the current must be made up half by a drift of positive electrons, and half by a drift of negative ones. The simplicity of the relations of a steady circuital current is due to the fact that it involves circulation, but not strain, in the surrounding æther. As a sub-class there is the so-called convection current, due to the transfer of a charge of electrons along with a material body, in its motion through the æther. There is also the current of displacement when the polar dielectric is excited, consisting of a drift within small range of the opposed positive and negative electrons in each molecule, owing to its orientation by the electric field. There is also a current due to convection of an excited dielectric, which in the case of uniform velocity may be represented as formally equivalent to a magnetization of the medium. Then again there is the molecular current, due to the orbital motions of the electrons in the molecule; so long as we avoid the dynamics of molecular structure we can only consider the time average of this current, and that is most conveniently represented as the magnetic polarity of the molecule; the displacement current in a dielectric above mentioned involves a similar process of averaging with respect to

time. There is also the current of convection of electrons which rearranges the electric charge of a conductor. Of this last practically nothing is known ; the subject has wholly eluded experimental enquiry, except in the case of electrolytes, where the modern theories of solution and electrolysis, in the hands of HITTORF, KOHLRAUSCH, ARRHENIUS, and NERNST, have yielded at any rate the first approximation to an analytical theory. As a matter of fact, the empirical and variable character of conductivity as depending on physical state, and its overwhelming importance as compared with the other phenomena in conductors of metallic type, restricts within narrow limits the problems in which a knowledge of their relations of convection and polarization would be of practical import.

Finally, there is one case in which an experimentally recognized result in the medium involves the dynamics of the molecule ; when the magnetic field is increased the orbital motions in the molecule are altered so as to oppose the change, and this induced alteration in the molecular current constitutes diamagnetism.

8. In the face of all these various types of motion of electrons that have to be taken into account, there are two distinct modes of procedure open to us ; and they have both been used by MAXWELL, without perhaps that sharp demarcation between essentially different methods which is necessary to the perfect grasp of either of them.

We may start from the dynamical equations of the free æther, which are known, and which apply exactly to a sub-element of volume which does not contain any electrons ; we can make use of the various forces and fluxes which have proved useful in forming a representation of electrical phenomena, and by their help we can replace these dynamical equations, which are of the second order in differentiations, by the two circuital relations which are each of the first order. The advantages, both for analysis and for explanation, of a transformation of this kind have long been recognized in pure dynamics. Now we can pass, by integration of these relations, from the sub-element of volume which contains no electrons, to the effective element of volume which contains a number of electrons large enough to enable us to smooth out their individual peculiarities and so retain a continuous differential analysis. The transformation is most conveniently effected so that the circuital relations shall remain unaltered in form, while the meanings of the various quantities that enter into them are modified by taking account of the presence of the distribution of electrons in the enlarged element of volume. This method of smoothing out the molecular discreteness of the æther is the one that has been most in favour in recent years ;* but when we

* [It appears to have escaped notice that the expression of the equations of the electric field in the form of the two circuital relations, together with linear equations (involving the constitution of the medium) which connect the corresponding forces and fluxes in these relations, had been given by MAXWELL himself. At the end of the Memoir "On a Method of Making a Direct Comparison of Electrostatic with Electromagnetic Force . . . ;" 'Phil. Trans.,' 1868, he alludes to the difficulty that had been felt in grasping the basis on which the electric theory of light is founded ; and he proceeds to attempt to diminish this difficulty by setting down the formally simplest foundation of hypothesis which

come to the consideration of moving media and convection currents, and particularly of ponderomotive forces, it will be necessary to supplement it by the more direct procedure now to be described.

The other method of analysis involves the utilization of the ideas of the electrodynamic potential in an amended form. It is concerned primarily with the dynamics of a single electron in an electric field given as regards motion and strain, or kinetic and potential energy; it forms the kinetic energy T, and the potential energy W, of the electron, and thence its fundamental Lagrangian function or electrodynamic potential T − W. The relations between the motion and the forcives of the electron are now to be deduced by dynamical methods; and there only remains a process of summation in order to determine the equations that are appropriate to the various modes of coordinated groupings and movements of electrons that have been enumerated above. This process has a fundamental theoretical superiority over the previous one, in that it includes in its scope the ponderomotive forces which act on the bodies in the field. The various discussions in MAXWELL'S "Treatise," which proceed by use of the vector potential of electrodynamic induction, are to be classed as essentially related to this point of view.

Method of averaged Forces and Fluxes: the Circuital Relations: Optical Dispersion.

9. The æther is to be regarded as containing a distribution of electrons, that is of intrinsic centres or nuclei from each of which a configuration of rotational strain spreads out into the surrounding space. An additional strain may be considered as imparted to this medium in two distinct ways, (i) as the result of tractions applied over an outer boundary surface, the electrons being supposed held fixed, and (ii) as the result of movement of the electrons, each of which will carry its atmosphere of strain along with it, practically without alteration unless the velocity of the electron is so great as to approximate to the velocity of radiation. A rotational strain imparted in the first manner is circuital from its nature, as depending on absolute rotation of the element of æther. But the circuital character of a strain of the second kind is vitiated by the existence of the nuclei or intrinsic singularities that belong to it; the surface integral of the rotation, taken over any interface, remains constant and therefore keeps its initial zero value only so long as no electrons cross that interface. But we can retain the circuital property for this kind of strain also, if we associate an ideal rotational displacement with the movement of each electron, in such manner that the integral of this rotation taken (vectorially) throughout a small volume including the initial and final positions of the electron is equal to the strength of the electron multiplied by its linear displacement. This is expressed electrically by saying

will lead to the theory at which he had previously arrived in a more dynamical and inductive manner. The form which he thus gives to the theory is precisely the one into which its equations have more recently been thrown by HEAVISIDE and HERTZ. Aug. 25.]

that a moving electron constitutes a convection current, which must be added to the flux of elastic rotation in the æther in order to obtain a total current which shall possess the circuital property. But this convection current is, so far, only a kinematic fiction; it allows us to retain the circuital relation between the total electric current and the velocity of the æther, but it is not to be counted in estimating the stress in the æther, from which the static dynamical effects result.

In ordinary electrodynamics, what we may call the *true current*, whether of convection or conduction, is measured in this way. per unit volume, by the aggregate directed drift of the electrons in the element of volume. It is completed or rendered circuital by the *displacement current*, or rate of change of rotational strain, in the æther; but in most cases of ordinary occurrence the magnitude of this displacement current is so very small compared with that of the true current, that it is allowable to neglect the former, and therefore make the true current flow in complete circuits. The *total current* of MAXWELL, which consists of both parts taken together, is exactly circuital in all cases.

In circumstances of conduction, though the electric displacement (*i.e.*, rotational strain) in the medium is absolutely negligible, yet the drift of the electrons which constitutes the true current causes an irrotational flow of the medium (the magnetic field) which is related to the current in precisely the same way as the flow in a perfect fluid is related to the vortex filaments which suffice to specify it; the energy of the current system is thus the kinetic energy of this irrotational flow, the rotational flow arising from strain being in comparison inappreciable. As here primarily introduced, the true current was a fiction, so far as elastic stress in the medium is concerned; but it has now acquired an objective meaning as the mathematical quantity that serves to completely specify the energy of the flow of the medium which is associated with the movement of electrons in bulk, that is, the energy of the magnetic field. The ordinary electrodynamics of conduction currents is a dynamical problem of the æther in which the kinetic energy is a function of the true current, and the potential energy, when there is such in the field, is a function of the rotational strain of the medium, that is, of the æthereal displacement current only.

10. We proceed to apply these ideas to the comparatively simple circumstances of the mode of transmission of regular vibrations by a medium thus constituted. Using the ordinary vector notation for brevity, let $\mathfrak{D}$ denote the actual rotational strain in the æther, so that $d\mathfrak{D}/dt$ is the displacement current; let $d\mathfrak{D}'/dt$ denote the true current due to movement of electrons, and let $\mathfrak{B}$ denote the velocity of the flow of the æther, that is, the magnetic induction. Each of these quantities is supposed to have its averaged value per unit volume, the irregularity of distribution due to the presence of electrons in the element of volume under consideration having been smoothed out in the analysis. The kinematic relation, which the introduction of the true current was here primarily intended to conserve, is

$$\text{curl } \mathfrak{H} = 4\pi \frac{d}{dt} (\mathfrak{D} + \mathfrak{D}').$$

Again, as the electric force at any point in the æther is $4\pi c^2 \mathfrak{D}$, c being the velocity of radiation, the ordinary dynamical equation of the free æther,

$$-\frac{d\mathfrak{H}}{dt} = 4\pi c^2 \text{ curl } \mathfrak{D},$$

applies exactly throughout any sub-element of volume which does not contain electrons. On summing up for all the sub-elements which go to make up the ordinary element with its contained electrons, we derive the same equation in which $\mathfrak{H}$ and $\mathfrak{D}$ are now defined as averages taken over the volume of the element, as in the kinematic equation above.

It remains to specify the relation between $\mathfrak{D}$ and $\mathfrak{D}'$. For the case of a dielectric medium in which the positive and negative electrons are combined into polar molecules so that in each molecule they exactly compensate each other, the statical effect of an applied electric force, $4\pi\mathfrak{D}$, is to induce a polarity, $\mathfrak{D}'$, so that the total circuital electric displacement is $\mathfrak{D} + \mathfrak{D}'$; thus

$$\mathfrak{D} + \mathfrak{D}' = \mathrm{K}/4\pi . 4\pi\mathfrak{D} = \mathrm{K}\mathfrak{D},$$

where K is a quantity which enters in electrostatics as the inductivity of the complex medium; we take it to be a constant independent of $\mathfrak{D}$, for sufficient reason *à priori*, at any rate for small values of $\mathfrak{D}$, such as occur in ordinary electric vibrations. If we suppose that the periods of the free oscillations of the electric polarity of a molecule are very high compared with the periods of the vibrations of the medium that are under consideration, this equation will still hold for the problem of a vibrating medium, as an equilibrium theory will apply to the molecules. But when the medium shows dispersive quality, its vibrations must excite sensibly the independent vibrations of the molecules, and the equation of equilibrium for the induced polarity of the molecules must involve the kinetic reaction of their vibrations. Thus

$$\mathfrak{D} - \frac{\mathfrak{D} + \mathfrak{D}'}{\mathrm{K}} = \frac{\mathrm{i}}{\mathrm{K}} \frac{d^2\mathfrak{D}'}{dt^2},$$

where i/K is a coefficient of inertia, of a kind which we do not need at present to further particularize. As the element of volume resists rotation, the uncompensated statical forcive tends to produce absolute rotation,—not relative strain as in the case of an elastic solid,—and is used up in accelerating it.

This is the general form of the relation between $\mathfrak{D}$ and $\mathfrak{D}'$ when vibrations are transmitted across a dielectric medium which contains a large number of molecules to the wave length; the actual number for light-waves, about 10^3 for solids and liquids

and 10^2 for gases, is amply sufficient to justify the process of averaging which has been employed. To retain the present simple form, the analysis must however be restricted to molecules with a single efficient free period. And the hypothesis implicitly involved, that no portions of free æther (supposed perfectly continuous, *i.e.*, not itself molecular) are so effectively enclosed by surrounding molecules as to have efficient free periods of their own, involves an assumption that the volume occupied by the nuclei of electrons is small compared with the whole space. If this were not the case, the relation connecting $\mathfrak{D}$ and $\mathfrak{D}'$ would contain second differential coefficients of $\mathfrak{D}$ with respect to time, as well as of $\mathfrak{D}'$, so that in the formula for K′ in § 11 (*infra*), the first term would no longer be unity. First differential coefficients of $\mathfrak{D}$ are always excluded because they would imply viscosity in the free æther.

As yet the true current is a current of dielectric molecular displacement which is perfectly reversible, so there is no dissipation of energy such as would cause absorption of the vibrations in the medium and consequent opacity. This will, however, occur should the paired electrons of the polar molecules occasionally get separated from each other when under the action of the electric force; such electrons would then travel different ways across the field and thus give rise to a conduction current. It is definitely known that this is what happens in the case of electrolytes under experimental conditions; and it is very probable (§ 23 *infra*) that the transfer of electricity by metals is of a similar nature. It is therefore necessary that we should include a similar agency in the case of light propagation; only it must be borne in mind that under a force which reverses with such tremendous rapidity, there are not the same opportunities for diffusion of momentum by action between contiguous molecules as there are in cases of steady force, so that the coefficient of opacity must be expected to be very much smaller than the conductivity of the substance as ordinarily measured by aid of OHM's law.

Propagation in Metals.

11. In circumstances of statical strain, the relation between the total strain $\mathfrak{D}$ communicated to a given portion of the æther from its surroundings, and the polarization $\mathfrak{D}'$ of the molecules that are excited by it, is $\mathfrak{D} + \mathfrak{D}' = \mathrm{K}\mathfrak{D}$.

Under vibrational circumstances there will be a kinetic reaction, $i d^2\mathfrak{D}'/dt^2$, which will take part in this equilibrium, and the corresponding equation is

$$i \frac{d^2 \mathfrak{D}'}{dt^2} + \mathfrak{D}' = (\mathrm{K} - 1)\, \mathfrak{D}.$$

When there are free electrons in the medium, as well as polar molecules, there will also be a conduction current; if m be the translational inertia of a free electron e, the electric force being $\mathrm{C}^2\mathfrak{D}/4\pi$, its equation of motion will be $m\ddot{x} + \sigma'\dot{x} = e.4\pi\mathrm{C}^2\mathfrak{D}$, not

indeed in the literal sense that there is a frictional force $\sigma' \dot{x}$ acting on each electron, but in the sense that on the average of a large number of electrons the part of the electric force that is not used up in accelerating their motion, is spent in maintaining their steady drift, σ' being thus a coefficient of resistance to migration through the medium, whose value may however be dependent on the period; the conduction current $\mathfrak{C}$ will therefore be of the type

$$\mathfrak{C} = k(md/dt + \sigma')^{-1} \mathfrak{D}.$$

The circuital relations which express the properties of the æther will be

$$\frac{1}{4\pi} \operatorname{curl} \mathfrak{H} = \frac{d}{dt}(\mathfrak{D} + \mathfrak{D}') + \mathfrak{C}$$

$$-\frac{d\mathfrak{H}}{dt} = 4\pi c^2 \operatorname{curl} \mathfrak{D}.$$

It is thus only the first of these relations that is modified; but we may reduce it back again to the standard form which obtains for free æther. Eliminating $\mathfrak{D}'$ from it, and writing $-p^2$ for d^2/dt^2, so that $2\pi/p$ represents the period of the vibrations, we have

$$\frac{1}{4\pi} \operatorname{curl} \mathfrak{H} = \frac{d}{dt}\left(1 + \frac{K - 1}{-ip^2 + 1}\right)\mathfrak{D} + k\,\frac{-md/dt + \sigma'}{m^2p^2 + \sigma'^2}\,\mathfrak{D},$$

which is now of the standard form

$$\frac{1}{4\pi} \operatorname{curl} \mathfrak{H} = K' \frac{d\mathfrak{D}}{dt},$$

provided

$$K' = 1 + \frac{K - 1}{1 - ip^2} - \frac{km}{m^2p^2 + \sigma'^2} - \frac{kp^{-1}\sigma' i}{m^2p^2 + \sigma'^2},$$

being thus a complex quantity.

The interfacial conditions are, as usual, that the tangential components of $\mathfrak{D}$ and $\mathfrak{H}$ are continuous (the medium being non-magnetic), so that this theory differs from the one applicable to transparent media simply in the complex value of K'.

When the medium is perfectly transparent the square of the refractive index assumes the form $1 + A/(\beta^2 - p^2)$, which agrees with a formula given by VON HELMHOLTZ and generally recognized to be a good representation of the dispersive properties of ordinary media; as that author has remarked, it may be helped out of any outstanding discrepancy by assuming slight non-selective opacity due to ordinary conduction, such as would not sensibly affect FRESNEL'S laws of reflexion. In this theory there is only one absorption band in the ultra violet, corresponding to $p = \beta$; if however the molecule has several free periods, the square of the index must still,

irrespective of special theory, be a rational function of p^2, and therefore when expressed in partial fractions must assume the form $1 + \Sigma A_r/(\beta_r^2 - p^2)$; while in the case of opaque media the constants typified by A will be complex, and other partial fractions not corresponding to absorption bands will also enter.

In the case of opaque media, a peculiarity of this analysis, as compared with that of VON HELMHOLTZ* and others, is that the conduction current is derived from the electric displacement by the inverse operator $k\,(md/dt + \sigma')^{-1}$ instead of a direct operator involving a coefficient of conductivity in place of the coefficient of resistance σ'. Notwithstanding that there are as many as four adjustable constants, it is possible, to some extent, to crucially compare the resulting formula for K′ with the experimental facts for metallic media. It is known that the broad type of formal theory which simply assigns to the metals a complex index of refraction $K'^{\frac{1}{2}}$ is confirmed by the general agreement between results deduced from calculations relating to reflexion experiments by BEER, VOIGT, DRUDE and others, and those obtained directly from deviation experiments with thin metallic prisms by KUNDT.† It turns out however that the real part of K′ is invariably negative for metallic media; and this is a fundamental difficulty in ordinary elastic theories, as it implies instability of the optical medium. On the present theory it implies that k is sufficiently large to allow the third term of K′ to outweigh the first two terms. It is also found (by both methods) that if $K'^{\frac{1}{2}} = n\,(1 - \iota\kappa)$, then for the better conducting metals, silver, gold, copper, n is less than unity, involving velocity of propagation greater than in a vacuum, while κ is a considerable number;‡ this implies that the fourth term in the formula for K′ is in these cases small compared with the third, which is just what is to be expected from the smaller value of the resistance coefficient σ'. This point is

* VON HELMHOLTZ, "Electromagnetische Theorie der Farbenzerstreuung," 'Wied. Ann.,' xlviii., 1893. The analysis of VON HELMHOLTZ consists, as usual with him, in a tentative process of fitting known electric laws into a minimum theorem which is an extended form of the Principle of Least Action. For this purpose he uses two sets of variables to represent what we have here called the true current and the displacement current of the free æther; and he varies them independently of each other. There is no distinction drawn between the polarization and the conduction parts of the true current; so that the current of conduction appears in the potential energy function, thus being assumed to imply elastic strain of the medium, in opposition to the views that have been here set forth. It has been shown by REIFF ('Wied. Ann.,' 1, 1893, p. 361) that the theory of VON HELMHOLTZ does not lead to FRESNEL'S formula for the influence of moving media on the velocity of propagation, unless the æther is supposed to partially partake of the motion of the material medium; but that when certain terms are omitted from his potential energy function, it is not necessary to assume that the æther is moved with the matter. But it does not appear that any reason is assigned for such modification of the theory, which, as already remarked, seems to be intrinsically at variance with the view here taken.

† KUNDT, 'Phil. Mag.,' 1888 (2). It had been already shown by VOIGT ('Wied. Ann.,' xxiv., 1885), that the ordinary optical formulæ for prismatic deviation apply when the metallic prism is of very minute angle.

‡ *Cf.* the numbers quoted from DRUDE in Professor J. J. THOMSON'S "Recent Researches . . . . " § 355.

confirmatory of the present scheme; for if the conduction current were derived from the displacement $\mathfrak{D}$ by a direct operator involving conductivity instead of an inverse one involving resistance as here, the opposite effect would be indicated.

When the third and fourth terms in K′ are predominant, the dispersion will usually be in the abnormal direction, as it is known to be for the great majority of metals, a few of the more conducting ones being however exceptions; and the value of κ will usually increase with the frequency, which is also in accordance with fact.

[Aug. 25.—The introduction of effective inertia into the equation of conduction, as above, thus appears to be essential to the theory. For the actual negative value of the real part of K′ in metals cannot arise from dielectric polarity of the material medium; that, as has often been remarked, would imply instability and consequent destruction of such polar structure. To make the other agency which is at work, namely conduction, effective for that purpose, its equation must involve inertia: and the influence of this inertia appears in fact also in other ways; thus the work done by the electric force on the ions in an electrolyte must be used up proximately in accelerating their velocities, while the increased average velocity reveals itself as Joulean heat. If there were a large number of dissociated ions along a wave-length, which is indeed the condition that the above continuous analysis be literally applicable, it is easy to see by consideration of molecular magnitudes that the effective inertia of an ion would have to be very much greater than its actual mass, or else the effect would be excessive. But this difficulty is only apparent. When the ions are more thinly scattered through the medium, there will be two sets of waves propagated; the waves of free æther modified somewhat by the presence of the ions, but not extremely different from what they would be if the ions were held fixed in the medium; and much slower waves propagated from ion to ion with the intervening æther nearly in an equilibrium condition at each instant. The former class alone would be sensibly excited by optical means: it may be formally represented by the above scheme of equations with the inertia coefficient large, and it is wide enough to include the phenomena both of transparent dielectrics and of opaque metallic media. It is found that, for the wave-lengths of luminous radiation, the real and imaginary parts of the square of the index of refraction are of about the same order of magnitude for all metals. This possibly indicates that the depth to which the light can penetrate in the metal, and therefore also the coefficient of absorption, depends essentially on the ratio of molecular magnitudes to wave-length, which would be about the same for all.]

Refraction distinct from Dispersion.

12. In the former paper, a physical foundation was assigned for MacCullagh's theory of dispersion. That theory being a statical one, must rest on the discreteness of the medium being comparable with the wave-length of the radiation; but conside-

rations similar to those given in that paper (§ 123) in connection with CAUCHY'S theory, show that the number of molecules in the wave-length is too great to allow this cause to account for the magnitude of the actual dispersion. Thus by far the greater part of ordinary dispersion, as distinct from refraction, is to be assigned to the sympathetic vibrations of the molecules as here discussed.

[Aug. 25.—For waves of long period and therefore great length, p approaches the value zero; thus K′ approaches the limiting value K and a statical theory represents the phenomena, the molecules being at each instant in the equilibrium position corresponding to the strained state of the surrounding æther. In any case, we may conveniently designate this constant part of K′, the square of the index, by the name of the *refraction*, and the variable part, which depends on the period and ultimately vanishes when the period is long, by the name of the *dispersion*. In the very wide class of media for which the specific inductive capacity, that is the square of the index of refraction for very long waves, is nearly equal to the square of the index for ordinary light-waves, the dispersion is thus small compared with the refraction. In all such cases a statical theory of refraction, which, according to the argument of the previous paper, must be MACCULLAGH'S theory, will certainly be correct; and there is ground for making this conclusion general. Thus, in particular, MACCULLAGH'S theory of the double refraction in crystalline media will hold good as the first approximation. But just like ordinary refraction, this crystalline refraction is subject to dispersive variation when the wave-length of the light is altered. This dispersion of the optic axes in crystals is usually small compared with the double refraction itself, which justifies the present mode of treatment of it as a subsidiary effect to be joined to the main part of the double refraction. In order to obtain equations of propagation in which it shall be included, we have only to add to the effective coefficient of inertia of the æther in MACCULLAGH'S equations a subsidiary æolotropic part. This part may be complex instead of real when the medium is not perfectly transparent, thus including the effect of conduction arising from the presence of free ions; the general equations of absorbing doubly refracting media may in fact be formulated without difficulty on the lines of § 11.

For transparent media, this generalization of MACCULLAGH'S theory preserves the wave-surface, corresponding to any given period of the light, exactly FRESNEL'S. The character of the laws of crystalline reflexion also remains unaltered, but the constants that are involved in them are no longer exactly the same constants that occur in the equation of the wave-surface.

The asymmetric refraction of higher order, which evidences itself by rotation of the plane of polarization, is obviously of a structural kind, and so is correctly represented by MACCULLAGH'S terms. It is itself a highly dispersive phenomenon, on account of the higher differential coefficients on which it depends; thus the dispersion due to variation of its constants with the wave-length may usually be neglected in comparison.]

Influence of Motion of the Medium on Light-Propagation.

13. To deduce FRESNEL's law for moving media directly from these principles we have to remember that a movement of the material dielectric through the æther with velocity v parallel to the axis of x produces an additional displacement current at any point fixed in the æther, which (§ 31) is equal to $v d\mathfrak{D}'/dx$. Thus in AMPÈRE's circuital relation $d\mathfrak{D}'/dt$ must be replaced by $(d/dt + v d/dx)\mathfrak{D}'$. Again, from the mode in which the other circuital relation appears in the dynamical theory of the medium, $d\mathfrak{B}/dt$ must mean the total acceleration of velocity of the æther, due in part to change of time and in part to movement of the material dielectric; thus this d/dt also, when it operates on $\mathfrak{D}'$, must be replaced by $d/dt + v d/dx$. In the present connection we may neglect dispersional phenomena and so take $\mathfrak{D} + \mathfrak{D}' = \mathrm{K}\mathfrak{D}$. Thus finally

$$\mathrm{c}^2\nabla^2\mathfrak{B} = \frac{d^2\mathfrak{B}}{dt^2} + (\mathrm{K}-1)\left(\frac{d}{dt} + v\frac{d}{dx}\right)^2\mathfrak{B},$$

which leads directly to FRESNEL's formula as in § 124 of the previous paper.

14. A clear appreciation of what is involved in the unexplained null result of MICHELSON and MORLEY may be obtained by transferring this equation of propagation to coordinates fixed in the moving material system. Let us assume new coordinates x', t' given by

$$x' = x - vt, \quad t' = t + \alpha x,$$

while y and z are unaltered. Then

$$\frac{d}{dx} = \frac{d}{dx'} + \alpha\frac{d}{dt'}, \qquad \frac{d}{dt} = \frac{d}{dt'} - v\frac{d}{dx'}.$$

Thus the equation of propagation becomes, on dropping the accents,

$$\left(\frac{d}{dt} - v\frac{d}{dx}\right)^2\mathfrak{B} + (\mathrm{K}-1)(1+\alpha v)^2\frac{d^2}{dt^2}\mathfrak{B} = \mathrm{c}^2\left\{\frac{d^2}{dy^2} + \frac{d^2}{dz^2} + \left(\frac{d}{dx} + \alpha\frac{d}{dt}\right)^2\mathfrak{B}\right\}.$$

This will be again of the type of an equation of simple wave-propagation if the coefficient of $d^2\mathfrak{B}/dx\,dt$ vanishes, that is if $v = -\mathrm{c}^2\alpha$; it then becomes

$$\left(1 - \frac{v^2}{\mathrm{c}^2}\right)\left\{1 + \left(1 - \mathrm{K}^{-1}\right)\frac{v^2}{\mathrm{c}^2}\right\}\mathrm{K}\frac{d^2\mathfrak{B}}{dt^2} = \mathrm{c}^2\left\{\frac{d^2}{dy^2} + \frac{d^2}{dz^2} + \left(1 - \frac{v^2}{\mathrm{c}^2}\right)\frac{d^2}{dx^2}\right\}\mathfrak{B}.$$

This means that, referred to a standard of time that varies from point to point, corresponding times at different points being those for which $t - vx/\mathrm{c}^2$ is the same, the propagation relative to the moving material homogeneous isotropic medium is

precisely the same as if that medium were at rest with respect to the æther,* provided it be taken slightly æolotropic, in the sense of being less elastic in the direction of the velocity v than in other directions, the difference depending on v^2/c^2, and so being excessively minute. This different reckoning of time at different points does not affect the paths of the rays, which are determined by FERMAT'S principle of minimum time of transit; thus the paths of the rays relative to the moving material medium are precisely the same as if that medium were fixed, and at the same time made very slightly æolotropic. But so long as this æolotropy is not compensated by some other second-order correction, the time of passage of a ray from one point to another and back again will depend on the direction as well as the length of the line joining these points, which is in opposition to the experimental result.

Following up a conjecture already thrown out by G. F. FITZGERALD and H. A. LORENTZ,† we can find this compensating second-order correction in a change of the dimensions of a material body produced by motion through the æther. For this purpose we must assume, in conformity with the considerations already given in the previous paper (§ 115), that material systems are built up solely out of singular points in the æther which we have called electrons and that atoms are simply very stable collocations of revolving electrons. Now the equation already given is the one which governs phenomena, when the axes of coordinates as well as the material system are moving through the æther with velocity v parallel to the axis of x. If we write $x' = x\,(1 - v^2/c^2)^{-\frac{1}{2}}$, it assumes the form

$$\left(1 - \frac{v^2}{c^2}\right)\left\{1 + \left(1 - K^{-1}\right)\frac{v^2}{c^2}\right\} K \frac{d^2\mathfrak{B}}{dt^2} = c^2\left(\frac{d^2}{dx'^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\mathfrak{B};$$

and this is now of the same type as if the axes of coordinates were not moving. That is, if we know an actual configuration, steady or varying, of matter at rest in the æther, we derive from it a configuration of matter moving through the æther with velocity v by compressing it in the direction of this velocity in the ratio of $(1 - v^2/c^2)^{\frac{1}{2}}$ to unity, and at the same time adopting a standard of time which varies from point to point so that each point has its own origin from which time is reckoned. Now the movable framework of MICHELSON'S experiments forms a steady configuration of matter when it is at rest, the element of time only entering in connection with molecular motions; hence when it is moving, the condition of continued steadiness requires that it should change its steady dimensions in such way that dx becomes dx'. Thus when linear measurements are estimated with respect to this moving

* FRESNEL'S law for the effect of motion of the medium is involved in this statement, as has been shown by LORENTZ, *loc. cit.*, § 339 *infra*.

† *Cf.* especially LORENTZ, *loc. cit.*, § 92, who suggests a possible explanation which comes to the same as the one here advanced. It would necessitate an annual inequality of excessively minute amount, in the length of the sidereal day, owing to the Earth's orbital motion.

framework the equation which governs the relative phenomena is of precisely isotropic type; and the negative result of MICHELSON is explained. This transformation is independent of K altogether; it will therefore hold good equally if the atoms are supposed to have intrinsic inertia in addition to the electric inertia which arises directly from the relations of the electrons to the surrounding æther.

If this argument be allowed, it follows (§ 80, previous paper) that movements of perfectly transparent matter, if without rotation, cannot affect the course of a train of light-waves which passes across that matter, in conformity with the result previously deduced from the law of entropy. It is to be noted, however, that if the radiation exerted a resultant mechanical forcive on the moving matter so as to do work on it in its motion, the argument from entropy would break down. This illustrates the limitations of that kind of argument; but in the present case we shall find (§ 28) that the argument is justified, the resultant forcive in question being null when the matter is perfectly transparent.

Method of Separate Electrons.

15. In the Appendix to the previous paper, the dynamics of the mutual actions of electrons was briefly sketched, under the restriction that the velocities of the individual electrons did not approach the velocity of radiation. If these velocities do not exceed say one per cent. of the velocity of light, the strain in the æther will at each moment be practically in the steady equilibrium state which it would have exactly were the velocity of radiation infinite, and the whole circumstances of the field will depend solely on the coordinates which specify the positions of the electrons. In an approximate theory of this kind the phenomena of radiation cannot, of course, themselves be included.

To obtain an exact theory, free from this restriction, it will be necessary to express the distribution of energy in the æther in terms of coordinates of a kind which will allow us to include among them the position coordinates of the various electrons; we must, therefore, transform the kinetic energy into an expression involving change of the rotational strain (f, g, h) in the æther instead of its translational velocity $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$. This procedure will be of fundamental importance, because the relation of matter to æther must resolve itself into the relation of electrons to æther, and the various forcives between æther and matter must be derivable in that way.

If, for facility of interpretation, we adhere to the ordinary electromagnetic system of units, we have

$$\mathrm{T} = \frac{1}{8\pi}\int(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)\,d\tau$$

$$= \frac{1}{8\pi}\int\left\{\left(\frac{d\mathrm{H}}{dy} - \frac{d\mathrm{G}}{dz}\right)\dot{\xi} + \left(\frac{d\mathrm{F}}{dz} - \frac{d\mathrm{H}}{dx}\right)\dot{\eta} + \left(\frac{d\mathrm{G}}{dx} - \frac{d\mathrm{F}}{dy}\right)\dot{\zeta}\right\}d\tau$$

$$= \frac{1}{8\pi}\int\left\{\mathrm{F}\left(\frac{d\dot{\zeta}}{dy} - \frac{d\dot{\eta}}{dz}\right) + \mathrm{G}\left(\frac{d\dot{\xi}}{dz} - \frac{d\dot{\zeta}}{dx}\right) + \mathrm{H}\left(\frac{d\dot{\eta}}{dx} - \frac{d\dot{\xi}}{dy}\right)\right\}d\tau,$$

on integration by parts, a surface integral over the infinite sphere being neglected, as usual, because the energy is all considered to be in a finite region. Now, in free æther, the curl of the velocity of the æther $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$, divided by 4π, is equal to the displacement current; in ponderable media it is, on the other hand, equal to the total current composed of this displacement current and the drift of electrons, that total current being circuital in accordance with this relation. If the total current be (u, v, w), we have, therefore,

$$\mathrm{T} = \tfrac{1}{2}\int(\mathrm{F}u + \mathrm{G}v + \mathrm{H}w)\,d\tau.$$

In these formulæ $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ is the curl of the vector (F, G, H), so that

$$\nabla^2\mathrm{F} - \frac{d}{dx}\left(\frac{d\mathrm{F}}{dx} + \frac{d\mathrm{G}}{dy} + \frac{d\mathrm{H}}{dz}\right) = -4\pi u,$$

with two similar equations. The unique solution of these equations satisfying the condition $d\mathrm{F}/dx + d\mathrm{G}/dy + d\mathrm{H}/dz$ null is

$$(\mathrm{F}, \mathrm{G}, \mathrm{H}) = \int(u, v, w)r^{-1}\,d\tau.$$

Thus (F, G, H) is the vector potential of the total current, including, when necessary, Amperean molecular currents; and

$$\mathrm{T} = \Sigma\Sigma\,(uu' + vv' + ww')r^{-1}\,\delta\tau\,\delta\tau'.$$

If an electron e moving with velocity v, say $(\dot{x}, \dot{y}, \dot{z})$, constitute a part of the total current, we have, setting out at length all the part of the energy depending on e,

$$\mathrm{T} = \tfrac{1}{2}\mathrm{L}e^2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + (e\dot{x}\mathrm{F} + e\dot{y}\mathrm{G} + e\dot{z}\mathrm{H}) + \ldots$$

Hence the motion provides a kinetic reaction which must be balanced by stress in the æther, just in the same way as the reversed mass-acceleration of a moving body balances the applied forcive by D'Alembert's principle; and the component of this

kinetic reaction on the electron which has reference to translation along x, is in this case, by LAGRANGE'S dynamical equation,

$$P_1' = -\left(\frac{\delta}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx}\right)$$
$$= -Le^2\ddot{x} - e\left(\frac{\delta F}{dt} + \dot{x}\frac{dF}{dx} + \dot{y}\frac{dG}{dx} + \dot{z}\frac{dH}{dx}\right),$$

in which δ/dt implies total rate of variation, due both to the change of time and to change of position of the electron in that time, so that

$$\frac{\delta}{dt} = \frac{d}{dt} + \dot{x}\frac{d}{dx} + \dot{y}\frac{d}{dy} + \dot{z}\frac{d}{dz}.$$

Hence finally,

$$P_1' = -Le^2\ddot{x} + e\left(c\dot{y} - b\dot{z} - \frac{dF}{dt}\right),$$

where

$$(a, b, c) = \text{curl}\,(F, G, H).$$

It is simplest to split this up into two parts, and to say that the motional forcive (P_1, Q_1, R_1) typified by

$$P_1 = e\left(c\dot{y} - b\dot{z} - \frac{dF}{dt}\right),$$

will go on increasing the acceleration of the electron, of effective mass Le^2, until the motion has brought the system of electrons into such a configuration that this forcive is compensated by traction due to the elastic strain of the surrounding æther, or in case of currents of conduction by Ohmic resistance to diffusion.

16. The potential energy of the system resides wholly in the æther and is

$$W = 2\pi C^2 \int (f^2 + g^2 + h^2)\, d\tau.$$

The variation of W will give the statical forcive which would work in altering (f, g, h): but to this forcive must be added a constraining forcive required to maintain the condition of constraint between f, g, and h, which is represented by

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = \rho,$$

where ρ is the density of free electrons. This constraint will be included if we conduct the variation of the equivalent expression

$$W' = 2\pi C^2 \int (f^2 + g^2 + h^2)\, d\tau - \int \Psi\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} - \rho\right) d\tau,$$

and afterwards determine suitably this function of position Ψ. Now

$$\delta W' = \int \left\{ \left(4\pi c^2 f + \frac{d\Psi}{dx} \right) \delta f + \ldots + \ldots \right\} d\tau + \int \Psi \, \delta\rho \, d\tau - \int \Psi \, (lf + mg + nh) \, dS,$$

after the usual integration by parts. Hence we have the forcive on free æther

$$(P, Q, R) = \left(4\pi c^2 f + \frac{d\Psi}{dx}, \quad 4\pi c^2 g + \frac{d\Psi}{dy}, \quad 4\pi c^2 h + \frac{d\Psi}{dz} \right).$$

And we also observe, from the form of $\delta W'$, that Ψ represents the potential of the distribution of electrons in the sense that a change of density in the volume $\delta\tau$ increases the statical energy by $\Psi \, \delta\rho \, \delta\tau$; so that the movement of an electron e is assisted by a force

$$- e \left(\frac{d}{dx}, \quad \frac{d}{dy}, \quad \frac{d}{dz} \right) \Psi.$$

To understand how this latter forcive works, let us consider an electron, in the electric field represented by a rotational strain (f, g, h) in free æther. The energy from which the traction is derived is

$$W = 2\pi c^2 \int (f^2 + g^2 + h^2) \, d\tau,$$

where in the immediate neighbourhood of the electron

$$(f, g, h) = \left(- \frac{e}{4\pi} \frac{dr^{-1}}{dx} + f', \; - \frac{e}{4\pi} \frac{dr^{-1}}{dy} + g', \; - \frac{e}{4\pi} \frac{dr^{-1}}{dz} + h' \right),$$

the first terms representing the part of the field due to the electron itself. Thus expressing only the terms which represent the interaction between the electron and the extraneous part of the field (f', g', h'),

$$\begin{aligned} W &= - c^2 e \int \left(f' \frac{d}{dx} + g' \frac{d}{dy} + h' \frac{d}{dz} \right) r^{-1} \, d\tau + \ldots \\ &= - c^2 e \int (lf' + mg' + nh') \, r^{-1} \, dS + c^2 e \int \left(\frac{df'}{dx} + \frac{dg'}{dy} + \frac{dh'}{dz} \right) r^{-1} \, d\tau + \ldots \\ &= c^2 e \Sigma \frac{e'}{r} + \ldots, \end{aligned}$$

provided the nucleus of the electron sensibly maintains its spherical form. The resultant traction of the medium over the surface of the electron is obtained by varying W, and is therefore a forcive

$$-e\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\Psi, \quad \text{where } \Psi = c^2\Sigma\frac{e'}{r},$$

which helps towards compensating the kinetic reaction aforesaid.

This result has been obtained by separating away the part of the stress which increases indefinitely as the electron under consideration is approached, and which therefore represents the field of that electron itself in the neighbouring æther. It is noteworthy that the resultants of the stresses between the electrons at finite distances apart are represented simply by their electrostatic attractions, whether the field of æther is in equilibrium or is disturbed in any manner whatever; a result which depends on the smallness of the nuclei, and the consequent intensity of the permanent strain near their surfaces.

The forcive thus derived, partly kinetic and partly static, is named the electric force, because in the collocation of polarized molecules and free electrons forming an unelectrified body, it tends to produce electric separation by driving the positive electrons one way and the negative ones the opposite way, while it has no tendency to move the element of volume as a whole. But if there is an excess of one kind of electrons over the other in an element of volume of the body, so that the element has a charge q, it also provides a force, equal to q multiplied by its intensity, acting on the element of volume of the charged body. This mechanical forcive may in certain cases be represented as the unbalanced part of a stress in the manner of MAXWELL'S stress in the medium; but in our present order of ideas it is the force itself that is the reality, and the stress is only a mathematical mode of representing it, which has no physical significance as it does not represent the actual stress either in the æther or in the material medium.

In one case this analysis must go deeper, namely in deducing the traction on an element of surface of the vacuous core of an electron. One mode of procedure would be to imagine the æther to be continuous throughout the core, but unstrained inside it, thus avoiding internal boundaries,—a method which has been already applied (previous paper, §51) in more intricate cases. We must then imagine the electric charge as freely distributed over the surface of the core in order to maintain this state of equilibrium, and the traction will simply be the usual forcive on this electric charge, namely, a normal pull equal to $\sigma^2/8\pi$ per unit area, where σ is its surface-density, as above. The same result would be more directly derived by varying the energy in the æther with respect to a displacement of the surface of the core.

17. It remains to find the electrokinetic part of the forcive tending to increase the electric displacement (f, g, h) in an element $\delta\tau$ of free æther. The kinetic energy associated with the element is

$$T_2 = (\dot{f}F + \dot{g}G + \dot{h}H)\,\delta\tau;$$

therefore

$$P_2 = -\frac{\delta}{dt}\frac{dT_2}{d\dot{f}} = -\frac{dF}{dt}.$$

No difference is here introduced by motion of the æther such as represents magnetic induction: in fact we have seen that the kinetic reaction on an electron e moving along with the æther is $-e d\mathrm{F}/dt$ simply; and to a doublet moving in this way the electric displacement in the element of æther itself may be assimilated. This forcive $(\mathrm{P}_2, \mathrm{Q}_2, \mathrm{R}_2)$ strains the æther, and is compensated jointly by the stress which it thus calls forth and the kinetic reaction of the motion involved in change of strain.

18. We have now to formulate the intensity (u, v, w) of the total circuital current in an extended body, as made up of conduction current, polarization current, æthereal displacement current, convection currents of free electrons and of polarized molecules, and the current which produces redistributions of electric charges in a conductor. If the vector $\mathfrak{D}'$ denote the aggregate result of the polarization in the body, made up of orientation of the electrons with the æthereal displacement bound to them, and $\mathfrak{D}$ the free displacement in the æther which excites it, the Poisson-Mossotti theory of polarization will give a relation $\mathfrak{D} + \mathfrak{D}' = \mathrm{K}\mathfrak{D}$, where the constant K denotes specific inductive capacity; thus of the total displacement $\mathfrak{D} + \mathfrak{D}'$, the fraction $1 - \mathrm{K}^{-1}$ is bound to the polarization of the molecules and only the fraction K^{-1} is free elastic displacement.

When an element of volume of a conductor moves across a magnetic field with velocity (p, q, r), the force (P, Q, R) acting on its contained electrons, supposed at rest in it, is of the type

$$\mathrm{P} = cq - br - \frac{d\mathrm{F}}{dt} - \frac{d\Psi}{dx},$$

acting on the positive electrons one way, on the negative ones the reverse way. When the electrons are in motion forming a current, it might at first sight appear that the ones that drift in the same direction as the conductor moves would experience the greater force, and so carry most of the current. But no difficulty of this kind occurs; for it is only the component of the velocity of the conductor at right angles to the direction of the current that is effective, and this acts equally on both sets of electrons. When a material body is in motion, the force tending to produce electric separation of its electrons, that is the electric force exerted in it, is given by this formula whether it carries a current or not. Strictly, the velocity (p, q, r) which occurs in this formula is relative to the æther, so that it will involve the velocity of the æther itself which constitutes the magnetic field: but it follows from optical experiments that the latter is extremely minute in comparison with ordinary velocities of material bodies.

On the other hand, the force tending to produce free rotational displacement in the æther is $(\mathrm{P}', \mathrm{Q}', \mathrm{R}')$, where

$$\mathrm{P}' = -\frac{d\mathrm{F}}{dt} - \frac{d\Psi}{dx}.$$

The former force acts on $\mathfrak{D}'$, this one on $\mathfrak{D}$. Under steady circumstances the corresponding parts of the current must have for x component

$$\frac{\mathrm{K}-1}{4\pi\mathrm{C}^2}\frac{\delta \mathrm{P}}{dt}+\frac{1}{4\pi\mathrm{C}^2}\frac{d\mathrm{P}'}{dt};$$

in which δ/dt represents $d/dt + p\,d/dx + q\,d/dy + r\,d/dz$, as appears from the fact that mere convection of a steady polarization by motion of the matter through the æther itself constitutes a current.

The total current is thus (u, v, w) given by

$$u=\sigma \mathrm{P}+\frac{\mathrm{K}-1}{4\pi\mathrm{C}^2}\frac{\delta \mathrm{P}}{dt}+\frac{1}{4\pi\mathrm{C}^2}\frac{d\mathrm{P}'}{dt}+p\rho+u_0,$$

where

$$4\pi\rho=\frac{d\mathrm{P}'}{dx}+\frac{d\mathrm{Q}'}{dy}+\frac{d\mathrm{R}'}{dz}=-\nabla^2\Psi;$$

this is on the hypothesis that the material polarization is all induced and therefore circuital, and so adds nothing to the convergence of $(\mathrm{P}', \mathrm{Q}', \mathrm{R}')$. If the velocity of the material medium is supposed uniform, we have

$$\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{Q}}{dy}+\frac{d\mathrm{R}}{dz}=-4\pi(pu+qv+rw)-\nabla^2\Psi;$$

hence, when the material medium is at rest, the condition of circuitality of the current (u, v, w) is

$$0=\left(4\pi\sigma+\mathrm{K}c^{-2}\frac{d}{dt}\right)\rho-\frac{d\rho}{dt};$$

so that, if initially there is any volume density of electrification, it at once diffuses on to the interfaces in the case of conducting media.

The only part of this scheme of equations that is incomplete is the specification of (u_0, v_0, w_0), that portion of the current which redistributes free electrifications. In an ordinary conductor this current is of the same order of magnitude as the polarization current, which is itself wholly masked by the current of conduction, except in the case of optical phenomena. In a dielectric this current does not exist at all. In an electrolyte we may form a provisional scheme of it on the lines of Nernst's theory of migration of the ions, by putting $(u_0, v_0, w_0) = k\rho$ (P, Q, R), on the supposition that the medium is uniform, where k is analogous to a coefficient of ionic mobility: but we shall thereby destroy the linearity of the system of equations. The fact that free electrons act on each other simply by their electrostatic attractions,

however the æther between them be disturbed, shows that they will tend to drive each other to the surfaces of the conductors, so that the current (u_0, v_0, w_0) will usually be a transient phase at the beginning of the settling of the disturbance, in agreement with the above, if it ever exist at all; and once the free charges are on the surfaces of the conductors they will remain there and be redistributed by ordinary conduction currents. This part of the current may therefore possibly be left out of account; that is, the electric density in a good conductor at rest may be taken to be always null.

19. It is the migration of the positive and negative ions through the unelectrified material medium in opposite directions that constitutes the conduction current: movement of the medium itself carries as many positive as negative electrons along with it, and so adds nothing to the current: the medium in a sense moves through the current, without carrying it along,—in opposition to the assumption usually implied in the notion of a current element. This is true irrespective of the manner in which the current is distributed between flux of positive electrons one way, and flux of negative electrons the opposite way. In all cases, however, in which there is an electrolyte in the circuit, the law of FARADAY shows, not that the current is equally divided between positive and negative ions, but that the numbers of these ions crossing in opposite directions any section of the steady current are equal,—although this necessitates extensive changes of concentration in the electrolytic solution after the current has become steady, when the ions have different rates of migration, as has been demonstrated by HITTORF. If we were to assume that such changes of concentration of ionic electrons are not important in the metallic portion of the circuit, the interface between two metallic media not being sensibly polarizable, it would follow that the velocities of migration of the positive and negative electrons in such conductors would be sensibly equal; and this equality would not be altered by the presence of a magnetic field. (*Cf.* however § 23.)

The conduction current does not involve elastic displacement; if it flows in a complete circuit so that electrons are not allowed to accumulate and exert a back electric force, it will go on permanently, a limit being set to it only by the *quasi*-frictional resistance to the motions of the ions through the medium in the sense of the kinetic theory of gases, which is expressed by the law of OHM.

The relation to the principle of energy of the force (P, Q, R) which tends to produce electric separation in a conductor is expressed by the formula that, if there is a drift of positive electrons one way and negative the opposite way, so as to form a *true* current (u', v', w'), the time-rate at which the potential energy of the field is thereby exhausted is $u'\mathrm{P} + v'\mathrm{Q} + w'\mathrm{R}$ per unit volume. The electric force as here introduced therefore agrees with its formulation in the ordinary elementary theory of true currents, in connection with OHM'S law.

20. The system of electromotive equations may now be collected together. The dynamical equations in the free æther are of type

$$4\pi C^2 f + \frac{d\Psi}{dx} = -\frac{dF}{dt}, \text{ where } F = \int \frac{u}{r}\, d\tau + \int \left(B \frac{d}{dz} - C \frac{d}{dy} \right) \frac{1}{r}\, d\tau.$$

The force acting on an electron e, moving with velocity $(\dot{x}, \dot{y}, \dot{z})$, is of type

$$eP = e\dot{y}c - e\dot{z}b - e\frac{dF}{dt} - e\frac{d\Psi}{dx}, \text{ where } (a, b, c) = \text{curl } (F, G, H);$$

and its coefficient of electric inertia is Le^2, where L depends on the radius of its nucleus.

The total current in an element of matter moving with velocity (p, q, r) through the æther is of type

$$u = \sigma P + \frac{K-1}{4\pi C^2} \frac{\delta P}{dt} + \frac{df}{dt} + p\rho + u_0,$$

where

$$P = qc - rb - \frac{dF}{dt} - \frac{d\Psi}{dx},$$

and δ/dt represents $d/dt + p\, d/dx + q\, d/dy + r\, d/dx$; while

$$\rho = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$$

where

$$f = \frac{1}{4\pi C^2} P' = -\frac{1}{4\pi C^2} \left(\frac{dF}{dt} + \frac{d\Psi}{dx} \right).$$

21. When the material system is moving through the æther with uniform velocity p parallel to the axes of x, the equations become

$$(P, Q, R) = \left(\frac{dF}{dt} - \frac{d\Psi}{dx}, \quad -\frac{dG}{dt} - \frac{d\Psi}{dy} - pc, \quad -\frac{dH}{dt} - \frac{d\Psi}{dz} + pb \right).$$

Thus

$$\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} = p\nabla^2 F - \nabla^2 \Psi; \text{ also, in all cases } \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0.$$

As the total current (u, v, w) is circuital, we must have

$$0 = \left(\sigma + \frac{K-1}{4\pi C^2} \frac{\delta}{dt} \right) (p\nabla^2 F - \nabla^2 \Psi) - \frac{1}{4\pi C^2} \frac{d}{dt} \nabla^2 \Psi + \left(\frac{\delta\rho}{dt} - \frac{d\rho}{dt} \right) - \frac{\delta\rho}{dt}.$$

It will suffice at present to confine ourselves to a non-conducting medium, so that σ, ρ, and (u_0, v_0, w_0) are null; then this condition of circuitation gives

$$\left\{ \frac{d}{dt} + (1 - K^{-1}) p \frac{d}{dx} \right\} \Psi = (1 - K^{-1}) p \left(\frac{d}{dt} + p \frac{d}{dx} \right) F,$$

or neglecting cubes of p/c,

$$\Psi = (1 - K^{-1})\, p \left(1 + K^{-1}\, p\, \frac{d/dx}{d/dt}\right) F.$$

We have also, when all the magnetism is induced,

$$\nabla^2 (F, G, H) = - 4\pi\mu\, (u, v, w);$$

thus

$$\frac{c^2}{4\pi\mu}\, \nabla^2 F = - \frac{K - 1}{4\pi} \left(\frac{d}{dt} + p\, \frac{d}{dx}\right) \left(- \frac{dF}{dt} - \frac{d\Psi}{dx}\right) - \frac{1}{4\pi}\, \frac{d}{dt} \left(- \frac{dF}{dt} - \frac{d\Psi}{dx}\right)$$

$$= \frac{K}{4\pi}\, \frac{d^2F}{dt^2} + \frac{K - 1}{4\pi} \left\{\frac{d^2\Psi}{dx\, dt} + p\, \frac{d^2F}{dx\, dt} + p\, \frac{d^2\Psi}{dx^2}\right\} + \frac{1}{4\pi}\, \frac{d^2\Psi}{dx\, dt},$$

so that

$$(K\mu)^{-1} c^2 \nabla^2 F = \frac{d^2F}{dt^2} + 2\, (1 - K^{-1})\, p\, \frac{d^2F}{dx\, dt} + (1 - K^{-1})\, p^2\, \frac{d^2F}{dx^2}.$$

Again

$$\frac{c^2}{4\pi\mu}\, \nabla^2 G = - \frac{K - 1}{4\pi} \left(\frac{d}{dt} + p\, \frac{d}{dx}\right) \left\{- \frac{dG}{dt} - \frac{d\Psi}{dy} - p \left(\frac{dG}{dx} - \frac{dF}{dy}\right)\right\}$$

$$- \frac{1}{4\pi}\, \frac{d}{dt} \left(- \frac{dG}{dt} - \frac{d\Psi}{dy}\right);$$

so that

$$(K\mu)^{-1} c^2\, \nabla^2 G = \frac{d^2G}{dt^2} + 2\, (1 - K^{-1})\, p\, \frac{d^2G}{dx\, dt} + (1 - K^{-1})\, p^2\, \frac{d^2G}{dx^2},$$

and a similar equation holds for H.

Thus F, G, H all satisfy equations of precisely the same type up to and including terms of the second order in p/c; so that if the vector $\mathfrak{A}$ represent the vector potential, we have

$$(K\mu)^{-1} c^2\, \nabla^2 \mathfrak{A} = \frac{d^2\mathfrak{A}}{dt^2} + 2\, (1 - K^{-1})\, p\, \frac{d^2\mathfrak{A}}{dx\, dt} + (1 - K^{-1})\, p^2\, \frac{d^2\mathfrak{A}}{dx^2},$$

which is the same equation (with p substituted for v) that was found (§ 13) for the magnetic induction $\mathfrak{B}$ by the method of averaged fluxes.

22. Hitherto we have omitted the complication which arises in optical applications owing to dispersion. In the case when the material medium is at rest, it is however easy to include the effects of dispersion and opacity. The total current is made up, as before, according to the specification

$$\mathfrak{C} + \mathfrak{D} + \mathfrak{D}' + (u_0, v_0, w_0),$$

where, as in § 11,

$$\mathfrak{C} = k\, (m\, \frac{d}{dt} + \sigma')\, (P, Q, R), \quad P = - \frac{dF}{dt} - \frac{d\Psi}{dx},$$

$$\mathfrak{D} + \mathfrak{D}' = K\mathfrak{D} - Ki\, \frac{d^2\mathfrak{D}'}{dt^2}$$

$$\nabla^2 \Psi = - 4\pi\rho, \quad \frac{du_0}{dx} + \frac{dv_0}{dy} + \frac{dw_0}{dz} = - \frac{d\rho}{dt}$$

The condition of circuitation shows, as in § 18, that a volume-density ρ subsides in a conducting medium in a non-vibrational manner, so that Ψ and ρ may be left out of account. Then $(P, Q, R) = -d/dt\,(F, G, H)$, and we arrive at the same equations as previously in § 11.

Ponderomotive Forces.

23. When a conductor carrying a current is in a magnetic field, the positive and negative electrons* are urged equally by the magnetic force in a common direction perpendicular to the current and to the field, when they are themselves drifting with equal velocities in opposite directions. A difference in their velocities of migration, such as has actually been found to exist in electrolytes, would lead to a greater force on one kind than the other, and so produce a small transverse electric force representing the HALL effect. When the flow in the magnetic field has become steady, the distribution of drifting free positive and negative electrons across the section will not be quite uniform, thus involving also change of resistance.

In any case the aggregate of the transverse forcives acting on the free electrons will constitute a mechanical electrodynamic forcive per unit volume,

$$(X, Y, Z) = (v'c - w'b,\ w'a - u'c,\ u'b - v'a),$$

acting on the conductor, where however (u', v', w') is the *true* current, namely $(u - \dot{f},\ v - \dot{g},\ w - \dot{h})$. There will in addition be a mechanical forcive $\rho\,(P, Q, R)$, if the element of volume contain an excess of one kind of electrons, including as a limiting case, a normal traction $\frac{1}{2}\sigma N$ over the surface of each charged conductor; and there will be the forcives acting on the magnetic and electric polarization of the element, derived respectively from potential functions $\frac{1}{2}\kappa(\alpha^2 + \beta^2 + \gamma^2)$ and $(K - 1)/8\pi\,.\,(P^2 + Q^2 + R^2)$, leaving out of account the part which merely produces molecular stress; while, if part of the magnetic or electric polarity is permanent, there will also be couples as in §§ 33, 35.

24. We have thus attained to a complete scheme of equations of the electric field, simply on the assumption that a material medium contains electrons, as many positive as negative in the element of volume unless it is electrified; that these electrons are in part combined into systems which are, or belong to, the molecules, some of these molecules being neutral, and some having an excess of positive or negative electrons, being therefore ions. The only other assumption is that the nuclei of the electrons occupy a negligibly small part of the whole space. As regards the manner in which the electromotive and the ponderomotive forcives in the electric field are accounted for, a similarity with WEBER'S molecular theory may be remarked.† In that theory electric molecules act on one another directly at a distance according to a law of force

* The argument in these sections is equally applicable, whether the current in metallic conductors is supposed to be carried by material ions, or by electrons considered as immaterial.

† W. WEBER, "Electrodynamische Maasbestimmungen;" MAXWELL, "Treatise," §§ 846–860.

which involves their relative velocity; on the present theory actions are transmitted from one moving electron to another solely by the intervention of the æther. The Weberian theory has been subjected to destructive criticism by VON HELMHOLTZ, on the ground that it implies the possibility of a perpetual motion; the mode of genesis of the present theory obviates such a criticism. The features of the Weberian theory above mentioned were characterized by MAXWELL as "eminently successful" ("Treatise," § 856), although this commendation is afterwards limited by the assertion that these features are necessarily connected by the principle of energy. It is now, however, recognized that this principle cannot by itself furnish more than one relation between the various quantities that enter into the problem of electrodynamic induction, so that the fact that any theory, not otherwise discredited, accounts for the two types of forcive in the electric field, ought to weigh strongly in its favour. The range of a theory of moving electrons of the present type, with its underlying æther, is of course much wider than that covered by WEBER.

When the mechanical forcive is exerted only on the discrete electrons contained in the element of volume, it clearly cannot involve directly in its constitution an equilibrating internal stress, such as we found must be included if we take the current element as a connected physical entity; it is for this reason that WEBER'S theory, though in other respects different from the present one, agrees with it in giving the AMPÈRE-MAXWELL ponderomotive force, involving the *true* current however, not the total current as in MAXWELL'S formula.

Unipolar Induction.

25. The phenomenon of unipolar induction, in which a current is induced when a magnet revolves round its axis of symmetry through its own field of force, is deprived of all difficulty or ambiguity when it is considered under the present point of view. The electrons in the magnet, as they are moved across the magnetic field of the æther, are each subject to a forcive which is proportional to the component magnetic intensity in the meridian plane, and which produces electric separation by drifting the positive ions towards the axis and in the direction of the length of the magnet one way, and negative ions the opposite way. This constitutes an electromotive force along the revolving magnet.

It follows for instance that a magnet symmetrical around its principal axis will, on rotation round that axis in its own field, acquire an electrification of excessively minute amount when the circuit is incomplete, but still sufficient to compensate the electric force induced by the motion. We can utilize MAXWELL'S equations of electric force, modified so as to refer to a system of axes moving through the æther ("Treatise," § 600, or as in 'Phil. Mag.,' Jan., 1884, p. 12), to infer at once that for a *solid* magnet of any form, in motion of any type, the induced electric force is derived from a potential $-(Fp + Gq + Hr)$, where (p, q, r) is the velocity through the æther of the element of the magnet at the point considered; so that it can at each instant be

compensated by the static force due to a minute induced electrification. The maximum difference of potential to be thus compensated, between the axial and circumferential parts of the rotating magnet, is of the order 10^{-4} volt in the case of the Earth. If the force were not thus derivable from a potential, a magnet in motion would induce currents in itself even when there is no extraneous field, and the energy of the absolute motion through the æther, of the Earth and all other magnets, would gradually be converted into heat owing to this cause ; as things are, only the energy of relative motion of magnets is subject to dissipation in this way.

[Aug. 3, 1895.—26. The case of a solid conductor spinning steadily round an axis of symmetry, in a magnetic field which is also symmetrical round that axis, has an important bearing on theory on account of the simplicity of the conditions. If it were legitimate to specify the electrodynamic energy of the system in terms of current elements and their mutual configurations, then in this case the energy belonging to a current element associated with an element of volume of the conductor would remain invariable, on account of the steady configuration of the motion ; therefore there would be no electric force induced in such an element. Thus, for a conductor spinning in this manner there could then be no differences of electric potential caused by the motion. And, moreover, if a so-called unipolar circuit is completed by means of a fixed conducting wire, attached to the spinning conductor by sliding contacts at its equator and one of its poles, there could be no electric force induced in this wire either; and the electromotive force of the current which actually flows round this circuit would have to be sought for wholly in the sliding contacts, which, considering the definiteness of this electromotive force and the great variety of types of contact that are possible, seems to be an untenable alternative.

On the other hand, the formulæ of the present paper, which considers an electrification to be made up of discrete elements each surrounded by free æther, make the induced electromotive force along any open line of material particles consist of two parts, (i) a part due to motion of this line of particles with respect to the quiescent æther, and equal to the time-rate at which it cuts across the tubes of magnetic induction, themselves supposed to be stationary in this computation, and (ii) a part equal to the line integral of $-d/dt$ (F, G, H), to be computed by the integral formula for (F, G, H) given above, which represents the effect of change in the inducing system. When the electromotive force is taken round a complete circuit, all theories are of course in agreement. In a case of steady motion, such as is now under consideration, the second of these parts is null, and the electromotive force induced even in an open circuit is given by FARADAY'S original rule, in terms of the number of tubes of magnetic induction which cut across the current.

In the present order of ideas, a distinction has to be observed between (i) the electric stress in the æther, which is the tangential shear derived from its potential energy and represents the whole forcive acting on free æther, and (ii) the electric force which acts on electrons and moves them through the æther, thus polarizing

dielectric media and producing true electric currents in conductors. In the hands of FARADAY and MAXWELL, the current of conduction was completed or rendered circuital by an effect across dielectrics, which was equivalent to a current, and there was no difference contemplated in this effect depending on whether the dielectric was a material substance or free æther, both being considered to be merely polarizable; on the present more complete view this dielectric action has to be divided into the true polarization current in the material dielectric, which is excited by the electric force orientating its polar molecules, and the rotational strain in the æther which is regulated by the laws of elasticity of that medium. Thus, the question whether there is electric force induced in the free æther itself is, on our present view, nugatory, there being no electrons on which it could operate.

The calculations given in 'Phil. Mag.,' Jan. 1884, of the differences of potential, and the consequent electrification, induced in a sphere or other conductor of revolution, rotating in a symmetrical magnetic field, will thus, on the present view, be absolutely correct when the conductor is moving in a vacuum. When it is rotating in a gaseous dielectric, like air, there will be aerial flow produced owing to viscosity, after the manner of the action of a fan, and there will therefore be electric force induced in the moving portions of the air; but the capacity of air for dielectric polarization is so small that the polarity thereby induced will not sensibly affect the state of electrification of the system. If, on the other hand, the conductor were rotating in a liquid dielectric, the polarity induced in the moving parts of the liquid would depend in part on its motion, which would thus very materially influence the distribution of electrification in a manner which it would not be difficult to calculate if it were necessary to do so. The unipolar current obtained on completing the conducting circuit will, of course, in any case, unless the conductivity is almost evanescent, be practically independent of the nature of the surrounding dielectric.

The analytical theory of MAXWELL'S "Treatise" being based on current elements, that theory should, when correctly developed, give in the rotating conductor the null electrification of the beginning of this section, instead of, as here, an electrification based on FARADAY'S rule. And this is easily verified if to MAXWELL'S original equations of electric force are added, in accordance with VON HELMHOLTZ'S correction, terms derived from a potential function which is the scalar product of the vector potential and the velocity of the material medium; in computing the radial force, which is in cases of symmetrical rotation the total force, these terms exactly cancel the FARADAY part.

The discrepancy between these two theories, is put to a test in a classical experiment already made by VON HELMHOLTZ.* He found that when a conductor was

* 'Berlin Monatsber.,' 1875; 'Ges. Abhandl.,' I., p. 783. MAXWELL had very early considered the question whether in his total kinetic energy as specified in terms of the current elements and the velocities of the conductors, there are any terms which involve products of these quantities, and he had drawn a negative conclusion from experiment; 'Treatise,' II., chap. vi., secs. 568–577. The

spun in a magnetic field symmetrical around the axis of rotation, there was a difference of potential induced between the axial and circumferential parts as evidenced by resulting electrification, which agreed with FARADAY'S rule within three per cent., a quantity well inside the limits of uncertainty of measurement. He drew the conclusion that an electrodynamic potential theory of the NEUMANN type is thus proved inadmissible unless it recognizes polarization currents in the dielectric. The considerations stated at the beginning of this section seem to show that it could not even thereby be helped out; and that the true inference from the experiment must be a wider one, namely the abandonment of a theory of currents ultimately continuous in favour of one which regards them as made up of discrete electrons separated from each other by free æther.

27. The electrification induced in a conductor by rotation in a symmetrical magnetic field has just been examined; we pass on naturally to the conjugate problem of the magnetic field induced by a rotating electrified conductor, where similar considerations must crop up. The difficulties involved in the interpretation of the results of ROWLAND'S experiment, on the hypothesis that an electric charge in a conductor is a continuous distribution of electricity, and an electric current a continuous flux, have been already considered in the previous paper,* and proved surmountable only on the extremely precarious assumption that the rotating gilded glass discs of the experiments were divided into mutually insulated segments by the scratches which were intended to prevent FOUCAULT currents. The translation of an isolated electric charge carries on its own surrounding electric field and so alters the electric intensity at each point in the æther; and can therefore certainly induce a magnetic field with consequent reaction on the moving charge. But the steady rotation round its axis of a charged conductor of revolution in no respect affects the field of electric strain in the surrounding æther; that remains steady, and therefore no magnetic, that is kinetic, energy can be locally generated anywhere in it. And there is also the related difficulty previously enforced, that if a charged conductor connotes merely a field of self-locked electric strain in the surrounding dielectric, the elasticity breaking down when the surface of the conductor is reached, the rotation of the conductor round its axis of symmetry could exert no grip on this field of strain, which would therefore not be affected at all. These difficulties will not vanish unless the electric charge on the conductor is made up of discrete portions, separated by dielectric spaces however narrow. The circumstances will then be no longer symmetrical with respect to the axis of rotation so far as these spaces are concerned; the magnetic field induced, at a place at finite distance from the conductor, will depend not only on the change of electric intensity at that place, which is null as before, but also on the surface conditions which obtain along the boundaries of the dielectric region.

electrification in VON HELMHOLTZ'S crucial experiment might, however, be formally expressed on that system as due to energy terms of this mixed type, but, of course, extremely minute.

* 'Phil. Trans.,' A, 1894, p. 764.

The phenomena are thus accounted for on the hypothesis that the electric charge on a conductor consists of a distribution of discrete electrons over its surface. It is true that the facts of electric vibrations on conductors show that these electrons must be extremely mobile and sensitive to electric force: this is because of the very strong charges involved in them, but it does not imply that when the conductor is rotated, the electrons will slip backward over its surface and remain where they were, owing to the electric inertia. They will not take up the motion of the conductor just at once; but there is no electric force which would tend to prevent that result, and the same steady viscous agencies that produce electric resistance when a steady current of electrons is flowing in a fixed conductor, will ultimately make them move with the rotating conductor, after a time relatively long but probably absolutely very short.

It does not, in fact, appear that there are any of the hitherto outstanding difficulties of pure electrodynamic theory that are not removed by the hypothesis of moving electrons, to which, from the consideration of several distinct classes of phenomena, and apart altogether from electrochemical theory, we have been compelled to resort. This hypothesis, in its wider aspect involving the nature of matter itself, seems also to have a philosophical necessity; for the location of causes of disturbance of the uniform all-pervading medium in permanent discrete singularities or nuclei of strain or motion, that belong to it and can move about through it, is the only way of avoiding the introduction into theory of either direct distance actions, or else those assumptions of independent media superposed in the same space and discharging different functions, which violate the maxims of modern physics. Without a precise conception of the causes which produce disturbances in it and form one side of the play of action and reaction, a theory of the æther can be merely descriptive; while any assumed causes that are not of the nature of singularities arising from the constitution of the medium itself, must introduce a foreign element and so deprive the theory of its interconnection and self-contained character.]

Mechanical Pressure of Radiation.

28. An application of the expression for the ponderomotive forcive will be to the examination of MAXWELL's mechanical pressure of radiation ("Treatise," §§ 792, 793). Let us consider a train of plane waves moving along the axis of x, with their magnetic induction c along z, and their current v along y. The circuital relations give

$$4\pi\mu v = -\frac{dc}{dx}, \qquad \frac{dQ}{dx} = -\frac{dc}{dt};$$

while the equation connecting electric force with current is

$$v = \sigma Q + \frac{K}{4\pi c^2}\frac{dQ}{dt}.$$

Hence

$$\frac{d^2c}{dx^2} = K\mu c^{-2}\frac{d^2c}{dt^2} + 4\pi\sigma\mu\frac{dc}{dt};$$

leading to

$$c = c_0 e^{-px}\cos(nt - qx),$$

where

$$(p + iq)^2 = K\mu c^{-2}n^2 - 4\pi\sigma\mu n i.$$

Of the total current v, the part $v' = v - \frac{1}{4\pi c^2}\frac{dQ}{dt}$ is the true current, derived from motion of electrons and not including the æthereal rotation, c being the velocity of radiation in free æther. The mechanical forcive per unit volume of the material medium is thus $X = v'c$; hence

$$\int X\,dx = -\frac{c^2}{8\pi\mu} - \int\frac{c}{4\pi c^2}\frac{dQ}{dt}\,dx.$$

Let us first take the case of a transparent medium, for which

$$\frac{dQ}{dt} = \frac{4\pi c^2}{K}v = -\frac{c^2}{\mu K}\frac{dc}{dx};$$

so that

$$\int X dx = -\frac{c^2}{8\pi\mu}\left(1 - \frac{1}{K}\right) = -\frac{c^2}{8\pi\mu}\left(1 - m^{-2}\right),$$

where m is the index of refraction.

The value of this indefinite integral, taken over an exact number of half wave lengths in a homogeneous medium, gives a null result for the total forcive; but if an interface between two media is included in the volume of integration, there will also be terms in it which may be represented by averaged tractions $\frac{1}{2}\frac{c_0^2}{8\pi\mu}(1 - m^{-2})$ exerted by this interface on its two sides, or what is the same thing, equal pressures acting on the interface. In air this pressure practically vanishes. This is the result which replaces MAXWELL'S formula $c_0^2/8\pi\mu$ for the mechanical pressure produced by radiation falling on the surface separating two media.

29 When the train of waves falls on an absorbing medium, the circumstances are different. From the value of c above given, it follows that

$$\frac{dQ}{dt} = n^2 c_0\frac{e^{-px}}{(p^2 + q^2)^{\frac{1}{2}}}\cos(nt - qx - \epsilon'), \text{ when } \tan\epsilon' = \frac{q}{p};$$

so that

$$c\frac{dQ}{dt} = \tfrac{1}{2}n^2c_0^2\,\frac{e^{-2px}}{(p^2+q^2)^{\frac{1}{2}}}\{\cos(2nt - 2qx - \epsilon') + \cos\epsilon'\},$$

and

$$\int\frac{c}{4\pi C^2}\frac{dQ}{dt}\,dx = \frac{n^2c_0^2}{8\pi C^2}\,e^{-2px}\left\{\frac{\cos(2nt - 2qx - \epsilon')}{2(p^2+q^2)} - \frac{\cos\epsilon'}{2p(p^2+q^2)^{\frac{1}{2}}}\right\}.$$

Hence

$$\int X\,dx = \frac{c_0^2}{8\pi\mu}\cos^2 nt + \frac{n^2c_0^2}{8\pi C^2}\,e^{-2px}\{\ldots\},$$

in which the lower limit is at the interface, and the upper one at a place where the radiation has been practically extinguished. Thus here there is an average pressure $\frac{1}{2}\frac{c_0^2}{8\pi\mu}$ acting on the interface towards the opaque material medium, which is half of MAXWELL'S result.

Interfacial Conditions.

30. It is important to emphasize the dynamical distinction between electric force and electric displacement in a dielectric. The electric force is that vector which occurs in the rotational stress in the æther, and which, divided by the proper elastic coefficient $C^2/8\pi$, gives the rotational strain or electric displacement in it. This is not circuital when there are moving electrons present, but by adding a fictitious electric displacement representing the drift of the electrons, a circuital total displacement is obtained. This fictitious displacement or true current represents, however, the rotation in a hydrodynamical flow of the æther which the motion of the electrons sets up, which belongs to the kinetic energy and in ordinary electrodynamical applications constitutes practically the whole of it. At an interface separating media of different dielectric qualities, the electric stress in the æther must be continuous, and as this is tangential, equal to the tangential component of the electric force but at right angles to its direction, it follows that the tangential components of the electric force must be continuous.

Again, in a magnetic medium, the magnetic induction represents the smoothed-out velocity of flow due partly to motions communicated by distant disturbances, and partly to the circulatory motions of the magnetic vortices that are caused by the very rapid orbital rotations of the electrons in the molecule. The magnetic induction is therefore always circuital, so that its normal component is continuous at an interface. But as regards the tangential component, we must divide the whole induction into two parts, one of them representing the effect of the vortices in the immediate neighbourhood of the point under consideration, and the other including all the rest of the flow. The latter part is the magnetic force, in the ordinary phraseology, and it is clearly only this part whose tangential component must be continuous, when we cross an interface into a region in which the distribution of the magnetic molecular vortices is different.

It is a check on the averaged form of the dynamical equations for a material medium that these six interfacial conditions regarding force and flux should be consistent with the incompressibility of the æther; and in fact the relations given above show that only four of the six are independent. It is on this account, for example, that the general requirements of the problem of physical optics are satisfied.

Molecular Current Systems replaced by Equivalent Continuous Currents.

31. To secure a perfectly homogeneous specification of the electrodynamic field in a dynamical theory, there is no alternative but to reduce magnetism to molecular electric currents. As, however, our analytical equations for an extended medium involve (u, v, w), the current according to a volume specification, while the molecular currents are minute whirls not involving continuous flow in any direction, it is necessary to determine the volume specification of currents that shall be their equivalent when the element of volume is so great as to contain a large number of whirls of which the average effect only is required. Such a specification is clearly possible; and when we bear in mind that the volumes occupied by the cores of the whirling electrons form only an excessively small part of the total space, it is also clear that the type of magnetic force which represents the velocity of the incompressible æther must be a circuital one, that is, the magnetic induction of MAXWELL. There is no difficulty in verifying by direct analysis that the magnetic force at a point due to a system of molecular currents is the same as the magnetic induction due to the equivalent magnet; in fact it is shown (previous paper, § 120) that the said force is equal to the curl of a vector potential

$$\int (\mathrm{B}d/dz - \mathrm{C}d/dy,\ \mathrm{C}d/dx - \mathrm{A}d/dz,\ \mathrm{A}d/dy - \mathrm{B}d/dx)\, r^{-1}\, d\tau,$$

the same vector potential from which the magnetic induction of the equivalent magnet is derived.

Now let us consider the projections of the molecular circuits parallel to the plane of xy: let these projections swell out in area until they come into contact filling up the whole plane, the currents round them being reduced in the same ratio as their areas are increased. Along each edge common to two circuits there will be a differential current flowing, and by making the enlarged circuits rectangular of the form $\delta x\, \delta y$, it becomes clear that the aggregate of these differential currents make up a volume distribution of currents $(-d\mathrm{A}/dy,\ d\mathrm{B}/dx,\ 0)$ together with a current flowing round the boundary. Hence, adding up for the projections on all three coordinate planes, we obtain the distribution represented by $-$ curl (A, B, C) together with a current sheet on the external bounding surface equal to $(\mathrm{B}n - \mathrm{C}m,\ \mathrm{C}l - \mathrm{A}n,\ \mathrm{A}m - \mathrm{B}l)$ per unit area. The validity of this substitution is verified by the analytical transformation

$$F = \int \frac{u}{r}\, d\tau + \int \left(B \frac{d}{dz} - C \frac{d}{dy} \right) \frac{1}{r}\, d\tau = \int \frac{u}{r}\, d\tau + \int (Bn - Cm) \frac{1}{r}\, d\tau - \int \left(\frac{dB}{dz} - \frac{dC}{dy} \right) \frac{1}{r}\, d\tau.$$

Thus if the magnetism is wholly induced, so that $(A, B, C) = \kappa (\alpha, \beta, \gamma)$, the equivalent volume distribution of currents is $4\pi\kappa (u, v, w)$, provided κ is constant throughout the field; hence the induced magnetism may be ignored if in estimating the induction we multiply the current system by $1 + 4\pi\kappa$ or μ.

More generally, if a part (A_0, B_0, C_0) of the magnetism is permanent, we have in a region of constant μ

$$\nabla^2 F = -4\pi\mu u + 4\pi \left(\frac{dC_0}{dy} - \frac{dB_0}{dz} \right);$$

of which the last term vanishes when there is no permanent magnetism, or when it is of lamellar type. At the interface between two different media (F, G, H) must be continuous. These relations are sufficient to determine (F, G, H) completely in terms of (u, v, w).

Again, a steady electric polarization (f', g', h') in a dielectric, moving across the æther with velocity (p, q, r), is equivalent in its kinetic effects (previous paper, § 125) to a molecular current system or magnetization $(rg' - qh', ph' - rf', qf' - pg')$ together with a current sheet over the bounding surface, *only provided* the velocity is uniform. In any case, however, it is equivalent to the current system

$$\left(p \frac{d}{dx} + q \frac{d}{dy} + r \frac{d}{dz} \right) (f', g', h').$$

Mechanical Forcives acting on Magnetically and Electrically Polarized Media.

32. We have still to calculate the steady forcive on a molecular current. The force on a revolving electron e has been shown to be $e(P, Q, R)$ where

$$P = c\dot{y} - b\dot{z} - \frac{dF}{dt} - \frac{d\Psi}{dx},$$

$(\dot{x}, \dot{y}, \dot{z})$ being its velocity, and Ψ the potential of the static part of the electric field due to such free charges as may exist in the material media. The first two terms, involving the velocity of the electron, will give rise to magnetic forcive, the remaining part will make up into the electric forcive, in a polarized medium.

The averaged value of the first two terms, over the orbit of a single electron, gives $\iota \int (c\, dy - b\, dz)$, where ι the equivalent current is equal to e multiplied by its velocity v and divided by the length of the orbit; this is the same as $\iota \int (l\, da/dx + m\, db/dx + n\, dc/dx)\, dS$ over a surface bounded by the orbit, by STOKES' theorem in conjunction with the circuital character of (a, b, c). Summing up for all

the orbits of electrons, this would give for the translational magnetic forcive per unit volume (X′, Y′, Z′), where

$$X' = A\frac{da}{dx} + B\frac{db}{dx} + C\frac{dc}{dx},$$

and (A, B, C) is the magnetization of the medium. Now the actual magnetic forcive must certainly, from the most cursory observation of its intensity for iron, involve the magnetic force (α, β, γ) and not the induction (a, b, c); whence then this discrepancy? The forcive on an electron involves correctly (a, b, c), for the velocity of the incompressible æther is circuital. But it must be borne in mind that the molecule contains as many negative revolving electrons as positive, and therefore that the forcive on the molecule is a differential one, positive and negative electrons pulling on the average opposite ways; so that in treating of single molecules we cannot take the actual velocity of the æther to be an averaged function of position such as (a, b, c).

33. To find the forcive on a molecule, we must in the first place divide this velocity into two parts, one independent of the immediate surroundings of the point considered and depending only on the general character of the field, the other representing the effect of local configuration. The former is the magnetic force as usually represented by (α, β, γ); it is made up of the force due to the distribution of ideal magnetic matter introduced by Poisson into the theory, together with that due to the distribution of electric flow. The former element in it is independent of local peculiarities by the ordinary theory of the gravitation potential; the latter also has never any term involving the flow at the point considered, for similar reasons. The purely local portion of the velocity of the æther consists itself of two parts, a steady one, and one of varying type with very rapid alternation characteristic of the orbits of the electrons; the part last mentioned averages to nothing, while the former part adds on to the magnetic force to produce the total averaged velocity of flow of the æther, that is the magnetic induction (a, b, c).

Now even if instead of the actual velocity of the æther, rapidly varying from point to point, we substituted this average velocity (a, b, c), we should still obtain both a forcive which depends on the general character of the field, and one (though only the regular part of it) which depends on the interaction of the molecule with its immediate surroundings. The latter forcive, which cannot be completely expressed in terms of the quantities of the present theory, belongs to an intermolecular stress of cohesive type; it is self-equilibrating and contributes nothing to the forcive on the material medium as a whole, though, as we shall see later, it produces deformation of its parts. If we agree to consider it as a separate molecular forcive, we have for the magnetic forcive proper (X′, Y′, Z′) the formula

$$X' = A\frac{d\alpha}{dx} + B\frac{d\beta}{dx} + C\frac{d\gamma}{dx}.$$

There will also be a magnetic couple (L', M', N') per unit volume, where $L' = B\gamma - C\beta$; but there will clearly be no additional molecular couple acting in the element of volume.

34. The steady magnetic force and couple thus obtained may be provisionally considered as derived from an energy function (kinetic) $A\alpha + B\beta + C\gamma$ per unit volume, in which (A, B, C) is unvaried (*cf.* MAXWELL, "Treatise," § 639); the regular part of the molecular forcive may be considered as involving an internal energy function $2\pi (A^2 + B^2 + C^2)$ in which (A, B, C) is varied.

More fundamentally, if we divide (α, β, γ) into a part $(\alpha', \beta', \gamma')$ due to the magnet itself and a part $(\alpha_0, \beta_0, \gamma_0)$ due to the inducing field, the actual energy associated with magnetic force is

$$\tfrac{1}{2}(A\alpha' + B\beta' + C\gamma') + A\alpha_0 + B\beta_0 + C\gamma_0,$$

in which (A, B, C) is now to be varied; and this, when there is no permanent magnetism, leads to the same mechanical forcive as an energy function $\frac{1}{2}\kappa(\alpha^2 + \beta^2 + \gamma^2)$, where κ is the coefficient of magnetization, the part of it not thus compensated being connected with the internal work of orientation of the molecules. This energy function is not the whole kinetic energy of the æther; that would be in the present units the space-integral of $(a^2 + b^2 + c^2)/8\pi$ together with a molecular part, that is the space-integral of $(a\alpha + b\beta + c\gamma)/8\pi + \frac{1}{2}\kappa(\alpha^2 + \beta^2 + \gamma^2)$, of which the latter term is exactly compensated by the above mechanical forces acting on the magnetized body, while the other term which remains over goes to produce electrodynamic effects, and in part to represent intrinsic energy of magnetization, and is in fact the electrodynamic energy formulated in MAXWELL'S scheme.

It is to be remarked that the magnetic vector potential, obtained in the previous paper as the vector potential of the Amperean currents in the form

$$F = \int (Bd/dz - Cd/dy) r^{-1} d\tau,$$

implies that the orbital velocities of the electrons do not approximate very closely to the velocity of radiation, a condition which in reality is sufficiently satisfied: if that were not so, this expression would require correction for an æthereal displacement part of the molecular current.

35. The magnetic forcive (X', Y', Z') acting on the actual magnetization I is different from the forcive that would act on the three components of the magnetization A, B, C, in case there are currents flowing in the magnet; so that in this connection it is not permissible to treat the magnetization as a vector and resolve it into components. The reason here is that we have actually to do with molecular currents; and if we replace a molecular circuit by its three components we thereby alter the character of its linking with the lines of flow of the current flowing across the medium. But there is a consideration of a less special kind which shows that

this cannot be done: if we were to resolve the magnetization, we should obtain results for the forcive which would be physically different according to the system of coordinate axes that is adopted. If, however, we sum up forcives on separate magnetic poles, instead of calculating forcives on resolved magnetic moments, we arrive again at the correct result, independent of the coordinate system: as will be illustrated immediately in connection with the forcive of electric origin.

36. The ponderomotive forcive of electric origin, acting on a dielectrically polarized medium, is made up of a bodily force (X, Y, Z) and a bodily couple (L, M, N), where

$$X = f' \frac{dP}{dx} + g' \frac{dQ}{dx} + h' \frac{dR}{dx}, \quad L = g'R - h'Q;$$

in these formulæ (f', g', h') is the total polarization of the material medium, consisting of the ordinary induced polarization $(K - 1)/4\pi \,.\, (P, Q, R)$ and, under circumstances of residual charge, also a permanent part. When the magnetic field is varying, so that (P, Q, R) is not derived from a potential, there will be delicate considerations concerning this force, similar to those discussed above in connection with magnetization. The formula here given for X is however correct, because it is what is directly obtained by grouping the actual electrons of the neutral molecule in pairs to form polar doublets; this is a legitimate procedure although the resolution of the averaged electric moment of an element of volume into three components proves not to be such. Thus the forcive on a doublet δD lying along the axis of x is a force $(\delta D \,.\, dP/dx,\ 0,\ 0)$ and a couple $(0, -\delta D \,.\, R, \delta D \,.\, Q)$; and when the axes of coordinates are changed so that δD becomes $\delta (f', g', h')$ these expressions change into the ones given above, because $P\,\delta f' + Q\,\delta g' + R\,\delta h'$ is an invariant function as regards change of axes.*

37. The stress of molecular type, produced by magnetization, offers some points of interest. It has been remarked by VON HELMHOLTZ† long ago that in a polarized medium there exists a material tension along the lines of polarization and a pressure at right angles to them. Each of these is, however, proportional to the square of the susceptibility‡ of the medium, and they are not necessarily equal; for media

* [The circumstances of electrified and magnetized media are not parallel, in the sense that K corresponds to μ. In a magnetized medium the circuital vector, namely the induction (a, b, c), is the smoothed out velocity of the æther; and the magnetic force (α, β, γ) is that part of it which is independent of the immediately adjacent vortices or Amperean currents. In a polarized dielectric (f, g, h), $= (P, Q, R)/4\pi$, is the smoothed out *total* elastic rotation or electric displacement in the æther; (f', g', h') is the time-integral of the polarization current of the material medium which is constituted of movements of orientation of the electrons; while it is the sum of these vectors that is now circuital, representing the total electric current arising from æthereal strain and movement of electrons combined.—August 25.]

† 'Berlin Monatsber.,' Feb., 1881; 'Wissen. Abhandl.,' 1, p. 779.

‡ 'Proc. Roy. Soc.,' April, 1892, p. 63.

of very high susceptibility they are thus far more intense than MAXWELL'S hypothetical stress which depends on its first power. Their origin is very conveniently exhibited in a chain of iron nails hanging end to end from a pole of a magnet: the nails hold together longitudinally but repel each other transversely. Now consider a longitudinally magnetized bar: this straining together of opposite polarities in neighbouring molecular groups will produce an internal stress in the bar proportional to I^2, which will usually tend to shorten it, but may conceivably in some cases do the reverse. Furthermore the orientation of the molecular groups due to magnetization will directly alter the length to an extent depending on the first power of I. Hence the whole increase of length will be $AI + BI^2$, in which A and B may each be either positive or negative. When A and B have opposite signs there will be a value of I at which the total effect will reverse its sign by passing through a null value. It appears from experiment that for iron A is positive and B negative; for nickel, A is negative and B negative; for cobalt, A is negative and B positive.*

[Aug. 25.—When A and B have opposite signs there will be an intensity of magnetization, $I = -A/2B$, which corresponds to maximum or minimum elongation. Near this intensity a small change in the magnetization will not affect the length of the bar; and therefore conversely a change of length produced by tension will have no influence on the magnetization, any changes in the two being independent of each other. It follows that the magnetization at which the influence of tension vanishes is only half the VILLARI critical magnetization at which the elongation of the bar is null. This is, of course, on the assumption that the phenomenon is a regular and reversible one, and that there is no sensible term in the elongation involving I^3.]

There must be similar strain effects connected with the polarization of a dielectric in an electric field, though of course they would be far more difficult to detect.

It seems probable that the greater part of the phenomena of electrostriction and magnetostriction is of this character, and that only a small part is due to strain of the material by the direct effect of electric and magnetic attractions between finite portions of the medium.

General Considerations.

38. It has been one aim of the present analysis to examine how far it is possible to identify, in the play of kinetic and potential energy between the electrons, the main phenomena of matter. In order to comply with the requirements of negative optical experiments by LODGE and others, it is necessary to assume the inertia of the æther to be at least comparable numerically with the inertia of derivative character which belongs to dense matter; and the velocities of its movements are thus extremely slow. On the other hand, in the descriptive electric theories that are now commonly held, it is usual to consider the æther to be like ordinary matter as regards

* *Cf.* EWING, "Magnetic Induction . . .," chapter 9.

its inertia, and also as regards capacity for electric and magnetic polarizations. Its density, for astronomical and other reasons, must then be excessively small; and therefore small forces, if unbalanced, will set it into very rapid motion. It has been recently shown by VON HELMHOLTZ* that it is not possible for an electrically polarizable medium of small inertia so to adjust itself by finite motions, that the electric and magnetic forcives on it shall be in internal equilibrium and thereby avoid producing very intense movements in it,—unless there exists finite slip at the surfaces of the moving bodies which set up the surrounding electric field; and it seems difficult to see how this slip could be allowed, or what circumstances would regulate its magnitude.

On the present view, the inertia of matter is different in kind from that of the æther, and is possibly to be found in the electric inertia which is possessed by electrons. The well-known considerations advanced by Lord KELVIN, which find in the magnetic rotation of light evidence of a rotatory motion round the lines of magnetic force, still obtain a place here, but in a modified form; it is not the æther itself which is in rotation round the lines of force, but the electrons of the ponderable medium; rotation of the free æther would not here affect the elastic propagation except convectively. We should thus expect the magnetization of the *material* medium to rotate the plane of polarization without altering to any corresponding extent the mean velocity of radiation, just as in Lord KELVIN's theory: and the present theory is not open to objection on the score that the FARADAY rotation is easy to observe, while convection of the radiation by a magnetic field has hitherto been sought for in vain.

It is to be observed that a primary condition for the permanence of material phenomena according to a scheme such as has here been sketched, is that the æther should be absolutely devoid of friction. If there were the slightest amount of dissipation, the motions by whose stability the system hangs together would gradually diminish, and finally the positive and negative electrons would fall into each other and thus suffer complete extinction; the whole material universe would in fact gradually vanish, and leave no trace behind.

All formulæ with regard to the conservation, under certain circumstances, of linear or angular momentum, or of momenta of more general type as in the case of cyclic fluid motions, are bound up essentially with this absence of friction in the æthereal medium. If friction were present, the relative motions of parts of the system would always be transferring momentum as well as energy out of the system into the æther, and nothing could remain absolutely steady or permanent.

39. In a recent memoir by H. A. LORENTZ,† on electrical and optical phenomena in moving bodies, the author arrives, starting from the ordinary equations of the electric

* "Folgerungen aus MAXWELL'S Theorie über die Bewegungen des reinen Aethers," 'Sitz. Berl. Acad.,' July, 1893, 'Wiss. Abhandl.,' 3, p. 526.

† "Versuch einer Theorie . . . in bewegten Körpern," Leiden, 1895.

field, at fundamental equations of the same type as are given by the present theory. The method adopted by him is, as here, to find the cause of electrical phenomena in the motion of ions. He considers them merely as volume distributions of electricity confined to limited spaces, and with this guiding idea determines the extension of MAXWELL'S fundamental equations that will most conveniently cover the extension of the problem to include their movements. By transforming, as in § 14, to axes of co-ordinates which partake of the uniform motion of the material system, various results relating to the independence between electrodynamic phenomena and such motion are deduced, and FRESNEL'S formula for the effect on the velocity of radiation is shown to be involved. The conception of a molecule, electrically polar by reason of its positive and negative electrons, of its magnetism due to the orbital motions of these electrons, and of its diamagnetism due to changes in the orbits, and the conception of conduction by a *quasi*-electrolytic separation, do not however occur: and in consequence of the absence of recognition of the fact that in an unelectrified element of volume there are present equal numbers of positive and negative ions, the relation between electromotive force and ponderomotive force is not attained. Professor LORENTZ'S conclusions from his system of equations as to the untenability of MAXWELL'S scheme of stress are in agreement with the results of the present theory; but from the indications given, it is not clear that he would admit the form here deduced for the mechanical forcive exerted by radiation. As this memoir of LORENTZ only came to hand after the greater part of the present paper had been developed, the agreement, so far as it goes, may be taken as additional evidence in favour of the validity of the method of investigation that has been employed.

According to the present views, there is nothing paradoxical in the existence of unsaturated atoms or ions in which the two kinds of electrons do not compensate each other; and the properties of these ions will be very different from those of neutral molecules, as it is well known that they are. The fact that a neutral molecule, say HCl, dissociates so that the same element H is always positive, may possibly be ascribed to the catalytic action of the solvent without whose aid the splitting up would not occur. When the solvent is water, or it may be for a gaseous body merely traces of moisture, the decomposition is produced by the formation of an aggregate with the water-molecules which splits easily in only one way; but when the solvent is changed, the characters of the charges of the ions might be inverted, as in fact sometimes occurs.

40. If we consider a molecule to be made up of, or to involve, a steady configuration of revolving electrons, it will follow that every disturbance of this steady motion will involve radiation and consequent loss of energy. The only steady configuration of which we can assert that it will remain permanent under all circumstances is, therefore, that one which possesses least total energy. Thus, by extending the conception of absolute stability, which already plays a part in the theories of material systems which lose sensible energy by viscosity, to systems which lose energy by radiation, we

obtain a possible mode of accounting for the uniqueness of the atomic configuration and the invariability of its spectral lines. Such a view would also account for diffuse lines or bands being sharp at one edge and hazy at the other.*

It is easy to calculate the configuration of minimum energy for the simple but exceptional system of two electrons of opposite sign, considered in § 118 of the previous paper; in that case, however, the distance between their centres comes out to be rather less than the diameter of one of them, so that they will gradually fall into each other owing to loss of their orbital energy. But this need not be the case if the system contained several electrons of the same sign; for example, if it were roughly of the type of a ring of positive electrons revolving round an inner ring of negative ones. In the special case of that investigation, also, an estimate of the diamagnetic coefficient may be obtained. It is easy to see that the result of imposing a magnetic field H at right angles to the circular orbit of an electron e is the same as if the effective inertia were increased from Le^2 to $Le^2 + e\mathrm{H}/\omega$. This effect, though very minute, will be cumulative for all the electrons of the atom, whereas the change of orientation, which represents paramagnetism, will be a differential one as between the positive and negative orbits.

By a representation like the present, the origin of the vast number of spectral lines of some molecules may be simply illustrated. Thus in the analogous astronomical system of the Sun, Earth, and Moon, which has only nine coordinates, three for each body, there exist the much larger number of periodic inequalities or oscillations that are discussed in the Lunar Theory. The fact is that the oscillations of the coordinates of the system are not themselves harmonic or even exactly periodic; it is only when they are analyzed by the mathematical processes suitable to vibrations, or by a physical instrument such as a spectroscope which yields the same results, that the lines of the spectrum come into existence. The radiation traverses the free æther as a homogeneous whole which is a function only of the coordinates of the vibrating molecule: it is only under the influence of dispersion or diffraction by material bodies that it is broken up into harmonic constituents, which are not independent elements of it in case they exceed in number the internal coordinates of the molecule.

* In this connection a suggestive result has recently been obtained by Mr. H. C. Pocklington, 'Roy. Soc. Proc.,' May, 1895. As the energy associated with a simple vortex ring in perfect fluid increases, its velocity diminishes, so that a theory of gases in which the molecules are simple vortex rings fails to represent the facts. In the previous paper (§ 87), a theory of simple vortex atoms in a rotational æther, with the appropriate molecular electric charges, was noticed. It is now found that the effect of the electrification may be to reverse the above peculiarity, and, in particular, when the ring has nearly the configuration of minimum energy for its given vorticity and given charge, the total energy varies very nearly as the square of its velocity, that is as the temperature. But this configuration is precisely the one into which such a ring would settle down under the influence of radiation. Similar statements apply in a general way to the effective vortex made up of a system of revolving electrons, which is considered in the text, and which is not subject to the instability and the other difficulties that would beset an ordinary electrified vortex with hollow core.

As in connection with the previous paper on the present subject, I am under great obligation to Professor G. F. FitzGerald, and also to Professor O. J. Lodge, for the readiness with which they have allowed me to consult and profit by their opinions on some of the questions here treated, and for their kindness in communicating the results of the experimental tests described in § 3 of this paper.