

Amplified reflection, transmission, and self-oscillation in real-time holography

Baruch Fischer, Mark Cronin-Golomb, Jeffrey O. White, and Amnon Yariv

California Institute of Technology, Pasadena, California 91125

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A theory of phase conjugation in asymmetric materials that allow a phase shift between the grating and the light-interference pattern is developed. We find that when this phase is nonzero, maximum phase-conjugate reflectivity occurs for unequal pump intensities. The conditions for self-oscillation are studied.

The application by Yariv and Pepper¹ of the formalism of nonlinear optics to the study of four-wave mixing in phase-conjugate optics led to the prediction of a variety of phenomena, including amplified (phase-conjugate) reflection, amplified transmission, and mirrorless self-oscillation. These phenomena were later observed by a number of investigators.^{2,3} Also, the kinematic similarity between phase-conjugate optics and real-time holography has been noted,⁴ and a large number of experiments have been performed recently in photorefractive crystals.^{5,6}

There exists a fundamental difference between four-wave mixing in holographic, say photorefractive, media⁷ and in media, such as atomic vapors and anisotropic molecular liquids, used in phase-conjugate optics; in the second case the nonlinearity that gives rise to four-wave interaction is due to a local atomic, or molecular (nonlinear), electronic response. A complex susceptibility $\chi_{ijkl}(-\omega, \omega, \omega, -\omega)$ in this case reflects a temporal displacement between the induced polarization and the product of optical fields that drive that polarization. This displacement is large when the frequency ω is near that of an atomic or molecular transition.⁸ In the holographic case the mixing is due to spatial holograms written by the four interacting waves. Here the complex coupling constant reflects a spatial displacement between the interference pattern of a pair of beams and the resultant holographic grating. The physically distinct origin of the temporal phase and the spatial phase leads to fundamental and hence qualitative differences in the mechanisms of energy exchange between the interacting waves in two cases.

In this Letter we formulate the nonlinear four-wave coupled-mode equations of real-time holography in a manner that is close to the spirit of phase-conjugate optics.¹ We then proceed to solve these equations in a number of important special cases and obtain expressions for oscillation thresholds, amplified reflection, and amplified transmission.

The basic interaction geometry is illustrated in Fig. 1. Four waves of equal frequency ω and, for simplicity, of the same polarization, are propagating through the nonlinear medium. Let the electric-field amplitude associated with the j th beam be

$$E_j = A_j(\mathbf{r}) \exp[i(\mathbf{k}_j \cdot \mathbf{r} - \omega t)] + \text{c.c.} \quad (1)$$

We solve the problem in steady state so that the A_j may

be taken to be time independent. The propagation directions come in two oppositely directed pairs, $\mathbf{k}_1 = -\mathbf{k}_2$ and $\mathbf{k}_3 = -\mathbf{k}_4$, whereas the relative direction of \mathbf{k}_3 and \mathbf{k}_1 is arbitrary.

It is the fringes in the time-independent part of the light intensity that generate the hologram, whose fringes have the same periodicity as the light-interference pattern. In general, the holographic fringes of refractive index will have a spatial phase shift with respect to the light-interference pattern,⁷ so we can write the fundamental components of the intensity-induced grating as

$$\begin{aligned} n = n_0 &+ \frac{n_{\text{I}} e^{i\phi_{\text{I}}} (A_1^* A_4 + A_2 A_3^*)}{2 I_0} \exp(i\mathbf{k}_{\text{I}} \cdot \mathbf{r}) + \text{c.c.} \\ &+ \frac{n_{\text{II}} e^{i\phi_{\text{II}}} (A_1 A_3^* + A_2^* A_4)}{2 I_0} \exp(i\mathbf{k}_{\text{II}} \cdot \mathbf{r}) + \text{c.c.} \\ &+ \frac{n_{\text{III}} e^{i\phi_{\text{III}}} (A_1 A_2^*)}{2 I_0} \exp(i\mathbf{k}_{\text{III}} \cdot \mathbf{r}) + \text{c.c.} \\ &+ \frac{n_{\text{IV}} e^{i\phi_{\text{IV}}} (A_3^* A_4)}{2 I_0} \exp(i\mathbf{k}_{\text{IV}} \cdot \mathbf{r}) + \text{c.c.}, \end{aligned} \quad (2)$$

where

$$I_0 = \sum_{j=1}^4 |A_j|^2, \quad (3)$$

$\phi_{\text{I}}, \phi_{\text{II}}, \phi_{\text{III}},$ and ϕ_{IV} are real, $n_{\text{I}}, n_{\text{II}}, n_{\text{III}},$ and n_{IV} are real and positive, $\mathbf{k}_{\text{I}} = \mathbf{k}_4 - \mathbf{k}_1 = \mathbf{k}_2 - \mathbf{k}_3$, $\mathbf{k}_{\text{II}} = \mathbf{k}_1 - \mathbf{k}_3 = \mathbf{k}_4 - \mathbf{k}_2$, $\mathbf{k}_{\text{III}} = 2\mathbf{k}_1$, and $\mathbf{k}_{\text{IV}} = 2\mathbf{k}_4$. The complex constant $n_{\text{I}} e^{i\phi_{\text{I}}}$, as an example, characterizes the spatial hologram written by the stationary intensity-interference pattern

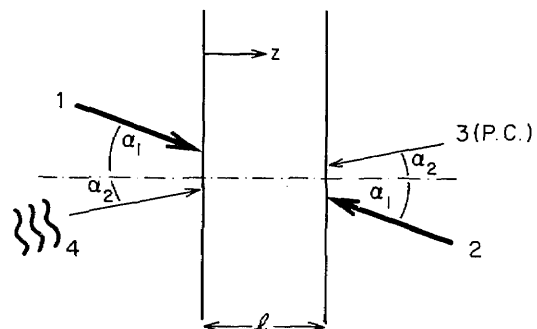


Fig. 1. Scheme of the four beams involved in phase conjugation.

of beams 1 and 4 and also that of beams 3 and 2. These two pairs of waves are characterized by the same constant because $\mathbf{k}_4 - \mathbf{k}_1 = \mathbf{k}_2 - \mathbf{k}_3$.

The expressions for $n_{\text{I}}e^{i\phi_{\text{I}}}$, $n_{\text{II}}e^{i\phi_{\text{II}}}$, $n_{\text{III}}e^{i\phi_{\text{III}}}$, and $n_{\text{IV}}e^{i\phi_{\text{IV}}}$ are obtained by solving the specific physical process responsible for the hologram formation. In the case of photorefractive crystals, such as $\text{Bi}_{12}\text{SiO}_{20}$ and BaTiO_3 as examples, the hologram is due to a refractive-index modulation produced by trapped charges that are excited by the intensity-interference pattern.

This charge distribution gives rise to a spatially alternating electric field, which in turn spatially modulates the index through the electro-optic effect. Expressions for n_{I} and ϕ_{I} , for example, derived from two theories of the process are^{9,10}

$$n_{\text{I}} = -r_{\text{eff}} n_0^3 E_p \left[\frac{(E_0^2 + E_d^2)}{E_0^2 + (E_d + E_p)^2} \right]^{1/2}, \quad (4)$$

$$\tan \phi_{\text{I}} = \frac{E_d(E_d + E_p) + E_0^2}{E_0 E_p}, \quad (5)$$

where r_{eff} is the relevant electro-optic coefficient, E_0 is an applied electric dc field directed along \mathbf{k}_1 , and E_d and E_p are electric fields characteristic of diffusion and maximum space charge, respectively.

$E_d = k_B T k_1 / e$ and $E_p = ep_d / (\epsilon k_1)$, where p_d is the density of traps in the material, k_B is Boltzmann's constant, T is the temperature, e is the electron charge, and ϵ is the permittivity of the material. Now, by using expression (2) for n and the scalar-wave equation, we can derive, by the standard slowly varying field approximation,¹¹ the following four coupled-wave equations:

$$\begin{aligned} \frac{2c}{\omega} \cos \alpha_1 \frac{dA_1}{dz} = & -\frac{in_{\text{I}}e^{i\phi_{\text{I}}}}{I_0} (A_1 A_4^* + A_2^* A_3) A_4 \\ & -\frac{in_{\text{II}}e^{i\phi_{\text{II}}}}{I_0} (A_1 A_3^* + A_2^* A_4) A_3 \\ & -\frac{in_{\text{III}}e^{i\phi_{\text{III}}}}{I_0} (A_1 A_2^*) A_2, \end{aligned} \quad (6)$$

$$\begin{aligned} \frac{2c}{\omega} \cos \alpha_1 \frac{dA_2}{dz} = & \frac{in_{\text{I}}e^{-i\phi_{\text{I}}}}{I_0} (A_1^* A_4 + A_2 A_3^*) A_3 \\ & + \frac{in_{\text{II}}e^{-i\phi_{\text{II}}}}{I_0} (A_1^* A_3 + A_2 A_4^*) A_4 \\ & + \frac{in_{\text{III}}e^{-i\phi_{\text{III}}}}{I_0} (A_1^* A_2) A_1, \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{2c}{\omega} \cos \alpha_2 \frac{dA_3}{dz} = & \frac{in_{\text{I}}e^{i\phi_{\text{I}}}}{I_0} (A_1 A_4^* + A_2^* A_3) A_2 \\ & + \frac{in_{\text{II}}e^{-i\phi_{\text{II}}}}{I_0} (A_1^* A_3 + A_2 A_4^*) A_1 \\ & + \frac{in_{\text{IV}}e^{-i\phi_{\text{IV}}}}{I_0} (A_3 A_4^*) A_4, \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{2c}{\omega} \cos \alpha_2 \frac{dA_4}{dz} = & -\frac{in_{\text{I}}e^{-i\phi_{\text{I}}}}{I_0} (A_1^* A_4 + A_2 A_3^*) A_1 \\ & -\frac{in_{\text{II}}e^{i\phi_{\text{II}}}}{I_0} (A_1 A_3^* + A_2^* A_4) A_2 \\ & -\frac{in_{\text{IV}}e^{i\phi_{\text{IV}}}}{I_0} (A_3^* A_4) A_3. \end{aligned} \quad (9)$$

When A_3 and A_2 are taken to be zero in the above equations, we recover the well-known and analytically soluble theory of holographic two-beam coupling.⁷ There too, the spatial phase difference ϕ_{I} between the light interference pattern and the grating plays an important role. Its sign determines the direction of energy transfer from one beam to the other. The effect of the phase is to shift the spatial pattern of refractive index toward one beam and away from the other. It introduces an asymmetry that allows one beam to accept and the other to donate power. In the present analysis of phase conjugation, we will show that this leads to an asymmetry between the roles of the counterpropagating pump beams A_1 and A_2 . The problem may be simplified by making two assumptions. First we take only $n_{\text{I}} \neq 0$. That is, we consider a holographic system whose spatial-frequency response is such that of all the gratings present in the system, only one grating, in this case the one created by the interference of beams 1 and 4 and 2 and 3, which is characterized by $n_{\text{I}} \exp(i\phi_{\text{I}})$, gives rise to strong beam coupling. This predominance of one grating is common in many practical situations and is due to the directions, polarization, and coherence relationships of the four beams relative to the crystal axes and to the application, in some cases, of an electric field that enhances certain gratings.

Second, we use the nondepleted pump approximation, in which $|A_1|^2, |A_2|^2 \gg |A_3|^2, |A_4|^2$. In this case, the derivatives in Eqs. (6) and (7) are of the order of A_4^2 or $A_3 A_4$, and the approximation $(dA_1/dz) = (dA_2/dz) = 0$ becomes reasonable. Moreover, as can be seen from its definition, Eq. (3), the normalization factor I_0 becomes constant. With these assumptions, the equations reduce to

$$\frac{2c}{\omega} \cos \alpha_2 \frac{dA_3}{dz} = \frac{in_{\text{I}}e^{i\phi_{\text{I}}}}{I_0} [|A_2|^2 A_3 + (A_1 A_2) A_4^*], \quad (10)$$

$$\frac{2c}{\omega} \cos \alpha_2 \frac{dA_4^*}{dz} = \frac{in_{\text{I}}e^{i\phi_{\text{I}}}}{I_0} [|A_1|^2 A_4^* + (A_1 A_2)^* A_3]. \quad (11)$$

With the boundary conditions $A_3(l) = 0$ and $A_4^*(0)$, the solutions of Eqs. (10) and (11) are

$$A_3(z) = A_4^*(0) \frac{A_1/A_2^*}{r^{-1}e^{-\gamma l} + 1} (e^{\gamma(z-l)} - 1), \quad (12)$$

$$A_4^*(z) = A_4^*(0) \frac{1}{r^{-1}e^{-\gamma l} + 1} (r^{-1}e^{\gamma(z-l)} + 1), \quad (13)$$

where r is the pump-beam intensity ratio,

$$r = \frac{A_2 A_2^*}{A_1 A_1^*} \equiv \frac{I_2}{I_1}, \quad (14)$$

$$\gamma = \frac{i\omega n_{\text{I}} e^{i\phi_{\text{I}}}}{2c \cos \alpha_2} \equiv iae^{i\phi}, \quad (15)$$

with a positive and ϕ real. The phase-conjugate reflectivity is thus

$$R = \left| \frac{A_3(0)}{A_4^*(0)} \right|^2 = \left| \frac{\sinh\left(\frac{\gamma l}{2}\right)}{\cosh\left(\frac{\gamma l}{2} + \frac{\ln r}{2}\right)} \right|^2, \quad (16)$$

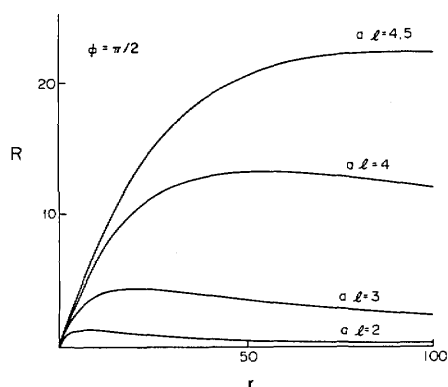


Fig. 2. Phase-conjugate reflectivity as a function of pump-intensity ratio for $\phi = \pi/2$ and different coupling strengths al .

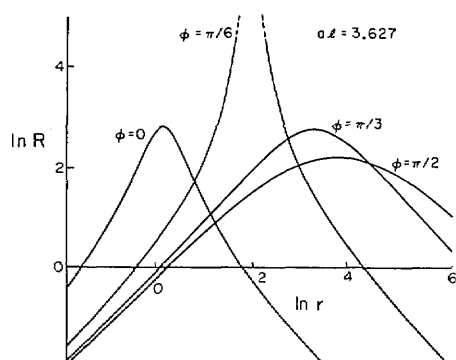


Fig. 3. Phase-conjugate reflectivity for coupling strength $al = 3.627$ and, peaking from left to right, $\phi = 0, \pi/6, \pi/3,$ and $\pi/2$. Oscillation here occurs for $\phi = \pi/6$ and $r = 6.13$.

the transmittivity of A_4 is

$$T = \left| \frac{A_4(l)}{A_4^*(0)} \right|^2 = \left| \frac{e^{\gamma l/2} \cosh\left(\frac{\ln r}{2}\right)}{\cosh\left(\frac{\gamma l}{2} + \frac{\ln r}{2}\right)} \right|^2, \quad (17)$$

and we see that only the intensity-independent quantity γl and the pump-intensity ratio r enter the expressions for R and T . The coupling constant in the second coupled-wave equation is the same as that in the first; it is not its complex conjugate, as is the case with the coupled-wave equations associated with four-wave mixing by third-order nonlinear susceptibility.⁴ This is because the complex coupling constant in the holographic case represents a *spatial phase shift*, whereas, in the case of a third-order nonlinearity, a complex coefficient is due to absorption; that is, it represents a *temporal phase shift*.

We find that the maximum phase-conjugate reflectivity occurs when $r = \exp(al \sin \phi)$:

$$R_{\max} = \left| \frac{\sinh\left(\frac{ial}{2} e^{i\phi}\right)}{\cos\left(\frac{al}{2} \cos \phi\right)} \right|^2. \quad (18)$$

If $al \sin \phi$ is not small compared to unity, the maximum reflectivity, Eq. (25) occurs at a value of a pumping ratio r that differs from unity by a large factor. The case in which $\phi = \pi/2$ is of special interest, for it occurs in photorefractive materials that operate by diffusion only⁷ ($E_0 = 0$). If we use symmetric pumps, then the reflectivity $\tanh^2(al/2)$ never exceeds unity. Large reflectivities may, however, be reached simply by using asymmetric pumps. The maximum reflectivity obtains with $r = \exp(al)$ and is equal to $\sinh^2(al/2)$. In Fig. 2 we show plots of reflectivity versus r for various values of the coupling constant. These values are large but nevertheless obtainable in certain crystals with large electro-optic coefficients, such as BaTiO_3 .¹²

The threshold conditions for self-oscillation ($R = \infty$) in holographic phase conjugation may be obtained from Eq. (16). We see that the phase-conjugate reflectance becomes infinite for

$$al \cos \phi = \pi, \quad r = \exp(al \sin \phi). \quad (19)$$

In Fig. 3, we plot, for a given value of al , the phase-conjugate reflectivity as a function of r for several values of ϕ . We find, for the value of al chosen, that self-oscillation occurs for $\phi = \pi/6$ and $r \approx 6$.

In order to achieve self-oscillation, it is necessary according to Eq. (18) that $\cos \phi \neq 0$. In particular, we find that our model predicts that self-oscillation is impossible for $\phi = \pi/2$ (the pure diffusion case), no matter how strong the coupling constant is.

When an ordinary mirror of amplitude reflectivity ρ is used to reflect the output $A_4(l)$ in the direction of A_3 , we find that the oscillation threshold is simply $R|\rho|^2 = 1$.

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