ELECTROMAGNETIC CAVITY RESONANCES IN ROTATING SYSTEMS

BARBAROS CELIKKOL

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by

BARBAROS CELIKKOL

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John F. Dawson, Asst. Prof. of Physics

Robert H. Lambert, Prof. of Physics

Lyman Mower, Prof. of Physics

Harvey Shepard, Asst. Prof. of Physics

Asim Yildiz, Prof. of Mechanics

Date

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ABSTRACT

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Recent and classical experimental and theoretical developments of the electromagnetic theory in accelerated systems (ETAS) is reviewed. The theoretical approaches are delineated according to their mathematical and physical features and their relation to experiments discussed.

Using the Post, Yildiz and Tang (PYT) formulation of ETAS, constitutive equations for the electromagnetic field variables in a rotating system (linear, isotropic, homogeneous and non-dispersive) are obtained. The ETAS as formulated by PYT is found to resolve the Sommerfeld paradox and explain the Kennard and Pegram effects. Next, the appropriate wave equation for a rotating system is derived and the fringe shifts or frequency splitting are calculated and compared to experiments.

The formalism is then extended to rotating wave guides and macroscopically dispersive media corotating with the wave guide or the ring laser.

Furthermore, the frequency locking phenomenon in ring lasers is studied systematically. The modified wave equation is analyzed classically while the density matrix formalism is employed for the source terms.

I. INTRODUCTION

Experimental and theoretical studies of the propagation of electromagnetic waves in a rotating system have been revived in recent years. The invention of the laser has provided new precision and modified the original Sagnac[59,60] ring interferometer into a self-oscillating device where the fringe shift observations are now replaced by a more sensitive device which observes beat frequencies between the cavity resonances.

A considerable amount of literature exists concerning earlier efforts on interferometer experiments for rotating mirror systems. Michelson[44] recognized as early as 1897 that electromagnetic resonances of optical cavities might be affected by accelerated motion. He considered a rectangular loop interferometer and calculated on the basis of the ether theory the fringe shift that would arise by rotation between a clockwise and counterclockwise beam. His results, when translated in terms of frequency expression, showed that the single resonance frequency of the stationary loop would split into a doublet by rotation. Michelson's experiment was followed by similar experiments by Harress[16], Sagnac and Pogany.[50,51] Michelson[45,46] in the meantime repeated his experiments and further refined his results. The theoretical discussions of these experiments were carried through by Harzer[17], Einstein[14], von Laue[71], and Langevin[30,31] using geometrical optics arguments. A few years later Gordon[15] attempted a solution based on physical optics.

More recently, modifications of the Harress-Sagnac experiment were performed by Dufour-Prunier[11,12] and Kantor.[23] In the case of the Dufour-Prunier experiment the medium is stationary, while the mirror system in the interferometer is rotating. Kantor's experiment considers the case in which the cavity is stationary while the medium is rotating.

Two nonoptical experiments performed by Kennard[24] and Pegram[48] also give insight to the electromagnetic theory of noninertial systems. These experiments involve measurements of charges and voltages in rotating capacitors.

The theoretical work of the last decade has been the study of electromagnetic wave propagation in accelerated material media. Heer discussed resonance frequencies of a rotating electromagnetic cavity on the basis of the formalism of general relativity. In his work, the Maxwell equations as well as the constitutive equations are affected by a transformation to an accelerated frame of reference. The covariant rendering of the Maxwell equations requires a linear connection given by the Christoffel coefficients.[63]

The work of Irvine[22] in fluid dynamics, and later extension by Mo[47] to electromagnetic theory involves transforming Maxwell equations. The constitutive equations are not affected by transforming to an accelerated system. The covariant rendering of the Maxwell equations requires a linear connection given by the Ricci coefficients.[63]

Finally, in the works of Post Yildiz, and Tang[53-56,74,75], (PYT), and later that of Anderson and Ryon[2], the constitutive relations are transformed to an accelerated system; however, the Maxwell equations are not affected. The covariant rendering of the Maxwell equations does not require a linear connection. Hence, neither Christoffel nor Ricci coefficients have to be calculated. Thus the publications in the last decade, on the subject of electromagnetic theory in accelerated systems, choose a variety of approaches and present a number of experimentally varifiable results.

The first objective of this study is to delineate these different approaches according to their mathematical and physical features and to discuss their relation to experiment. The most coherent and efficient study of covariant formulation of the classical electromagnetic field in general linear media appears to be the PYT formulation. The PYT covariant formulation stems from the introduction of a space-time constitutive relation which was first proposed by van Dantzig.[70] The PYT formulation gives a coherent account of all the reciprocal and non-reciprocal effects encountered in the general linear electromagnetic medium, and furnishes a unified description of the following effects: the Harress-Sagnac effect, the Fresnel-Fizeau effect[35], and it will be shown to explain Pegram-Kennard effect and Sommerfeld's paradox.[57,64]

The second objective of this study is to extend the PYT formulation to waveguide-like structures and ring lasers with macroscopic dispersion. In this case, the optical path is characterized by an index of refraction n, which is a function of the wave number $\vec{k}$. The index of refraction comes about either due to the presence of an optical medium in the beam path as in ring laser, or through the presence of a guided wave structure which gives the propagation dispersive properties, or a combination of both.

Furthermore, the frequency locking phenomenon in ring lasers is studied systematically starting with the PYT constitutive relations which result in a rotation modified wave equation. Assuming asymmetric damping and taking into account the rotation, frequency locking regions are investigated in the presence or absence of a magnetic field. The modified wave equation is analyzed classically while the density matrix formalism is employed for the source terms.

The results obtained in the dispersion and the frequency locking studies are improvements over previous theoretical work both in methods of calculation and results, and lead to a better understanding of rotating ring lasers.

II. HISTORICAL DEVELOPMENTS

1. Review of Experimental Developments

In 1897, Michelson attempted to observe an indication of the rotational motion of the earth with respect to the ether. Unfortunately, this experiment did not give conclusive data and the problem was further complicated by a calculational error. In 1925, Michelson[46] together with Gale reperformed the experiment and corrected his calculational error. To obtain the required sensitivity they had to choose a large size for the surface area (approximately 0.08 square mile) enclosed by the optical path. Since the rate of rotation could not be changed, they varied the surface area enclosed by the beam (see Fig. 2). It was shown that the fringe shift δz with respect to the fringe position for the stationary interferometer which was observed obeyed the formula $\delta z = 2\vec{\Omega}.A/\lambda_0 c$, where Ω is the angular velocity of rotation, A the area enclosed by the optical loop, λ_0 the free space wavelength of light and c the free space speed of light.

In the meantime, laboratory experiments were being performed by Harress[16] and Sagnac[59] independently (see Fig. 1). Both experiments were attempts, although not realized by Harress, to measure fringe shifts when the observer and the medium were coaccelerated. Both involved light beams circulating in the clockwise and counterclockwise directions in an optical loop and then reunited at a point so that interference fringes could be observed. When the whole interferometer, with light source and fringe detector, is set in rotation with an angular rate of $\vec{\Omega}$ rad/sec, the fringe shifts observed again obeyed the above formula.

It seems that Harress' data was much more precise than Sagnac's data. A great deal of theoretical development took place based on Harress' observations.

A Sagnac experiment of great precision was subsequently performed by Pogany[50]. He reproduced Sagnac's data to within 2% of the theoretically expected fringe shift. Two years later[51], he repeated the experiment, this time with two glass rods in the path of the light beam. He came within 1% of the theoretically expected fringe shift.

Dufour and Prunier[11] confirmed that the fringe shift does not depend on whether the observations are made on the rotating system. Depending on the experimental arrangement, one would expect a slight shift due to a possible Doppler shift in wavelength between a stationary point of observation and the point on the disk where clockwise and counterclockwise beams reemerge reunited. However, Doppler shift is a second order quantity and in their experiment was too small to measure. Doppler shift can occur also if a stationary, instead of a comoving light source is used.

Dufour and Prunier[12] performed another experiment where the light traverses a stationary optical medium while the interferometer is rotating. Their experiment indicated that the observed fringe shift increases with the presence of a stationary medium in the beam path. The effect of the medium vanishes only if the medium rotates with the interferometer.

It is natural next to look for a Fresnel-Fizeau type of experiment where the medium would rotate while the interferometer would stay stationary. A rotational version of this experiment has only been performed recently. An experiment that came close to the rotational version of Fresnel-Fizeau was reported by Kantor[23]; however, in Kantor's experiment the length of the medium through which the light travels is negligible. Massey[36] has very recently developed a spherical Fresnel-Fizeau ring laser.

There are two other experiments that should be mentioned which complement the optical work. These are the experiments of Kennard and Pegram on the phenomenon of unipolar induction. Both experiments utilize a coaxial rotating capacitor. A strong axial magnetic field is generated by an energized coil coaxial with the capacitor. In the Kennard experiment, one measures a potential difference between the plates of the capacitor when the capacitor is rotating. In the Pegram experiment, one measures a charge on the capacitor when the capacitor is being shorted by a corotating short during the rotation. For both experiments, it seems to be immaterial whether the coil generating the B field is stationary or corotating with the capacitor. The two effects thus depend solely on the rotation of the capacitor with respect to inertial space.

The modern version of the Sagnac experiment was developed by Macek and Davis[33,34]. It has been called the ring laser, laser gyroscope, or ring generator. A schematic diagram of the original experimental apparatus is shown in Fig. 3.

2. Sommerfeld's Paradox

Besides the seemingly disconnected optical and unipolar induction experiments to be explained by a unified theory, there are also theoretical problems in the electromagnetic theory that must be clarified. On the last page of his celebrated text on the theory of the electromagnetic field, Sommerfeld[64] introduces and suggests a solution to a problem in-

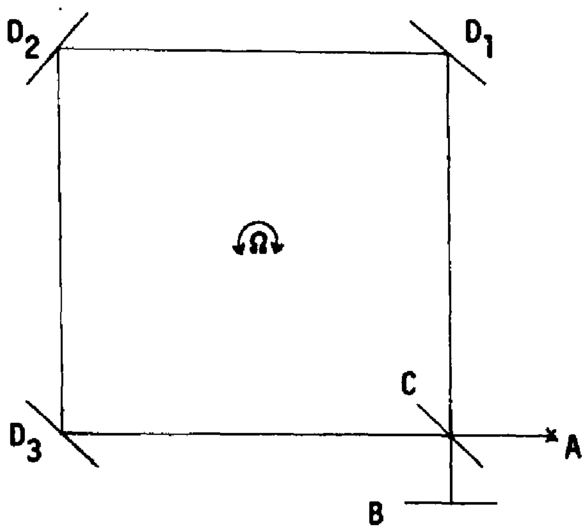

FIGURE 1.

Fig. 1. Idealized Sagnac interferometer. A = light source; B = observer; C = beam splitter (half-silvered mirror); D_1, D_2, and D_3 are corner mirrors.

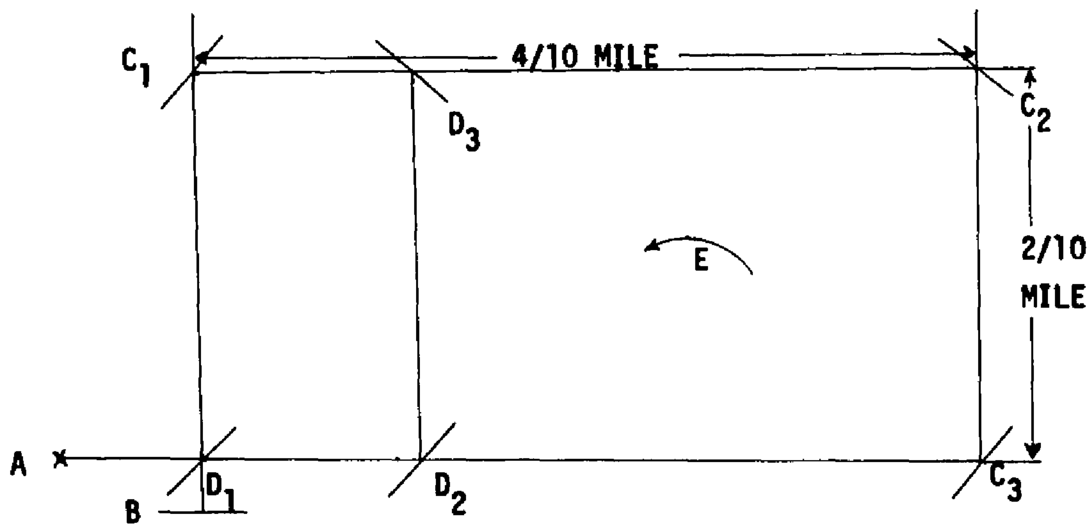

FIGURE 2.

Fig. 2. Michelson-Gale interferometer with calibration circuit. A = light source; B = interferometer; C_1, C_2, C_3 are all reflecting mirrors; D_1, D_2, D_3 are half-silvered mirrors; E = Earth rotation.

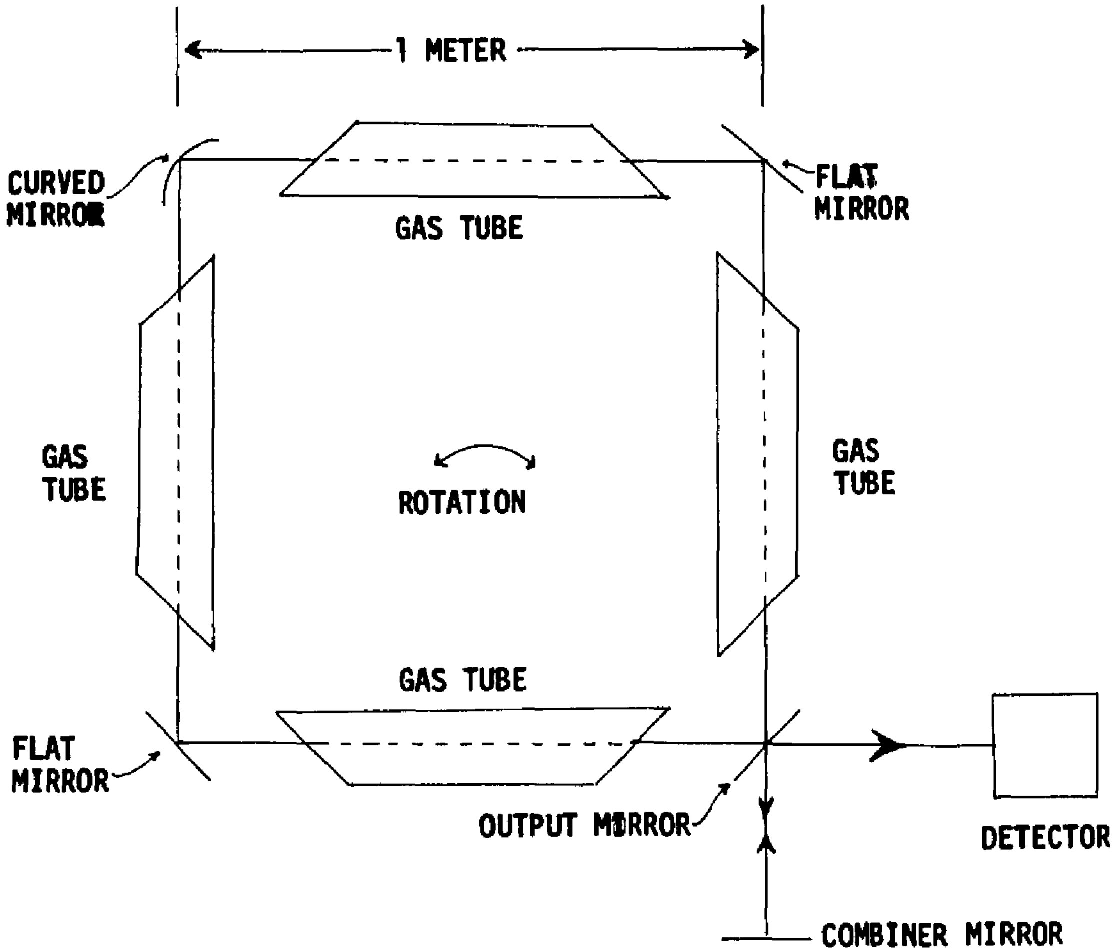

FIGURE 3

Fig. 3. The ring laser (Macek and Davis, 1963).

volving noninertial frame of reference. To understand the paradox, consider an inertial frame of reference in a charge and matter free region of space. Assume a uniform magnetic field $\vec{B}$ in the z direction. If one now rotates the frame of reference around z with angular velocity Ω, then one will observe in the rotating frame an electric field $\vec{E} = \vec{v} \times \vec{B}$, where $\vec{v} = \vec{\Omega} \times \vec{r}$, and r is the position vector from the point of observation. The relation for $\vec{E}$ is obtained by simply performing a Lorentz transformation on the fields.

Now if one considers the divergence of $\vec{E} = (\vec{\Omega} \times \vec{r}) \times \vec{B}$, one observes that it does not vanish. In face, $\nabla . \vec{E} = 2\vec{\Omega} \cdot \vec{B}$, and therefore the charge density is non zero in matter and charge free space due to rotation solely. In a region of space filled with conducting matter, such an observation may be explained away by the inducement of currents. However, in completely matter and charge free region of space, the creation of a charge density is a disturbing defect of the theory. Yet, Maxwell's equations and the special theory of relativity are so well established that we choose to question the applicability of standard electromagnetic theory rather than its structure. We must therefore search for the appropriate theory and the transformation techniques so that they become applicable to noninertial frames. Not unexpectedly, geometrical optics approach gives no insight to the problem.

3. Geometrical Optics Approach

Historically, geometrical optics was employed first to explain optical phenomena. The condition for geometrical optics to be applicable is that the wavelength λ should be small when compared to the characteristic dimension L of the medium through which the light beam is propagating. The relation between physical and geometrical optics is that for $\lambda << L$, any quantity which describes the electromagnetic wave is given by the expression $\phi = \phi_0 e^{i\Psi}$, where ϕ_0 is a slowly varying function of the space and time coordinates.[29] The phase Ψ is a large quantity which is in general nonlinear in the space and time coordinates. Ψ is also called the Eikonal. The Eikonal is linear in time and coordinates if there is no macroscopic dispersion in the medium. Then the time derivative of Ψ gives the frequency ω of the wave

$$\frac{\partial\Psi}{\partial t} = -\omega$$

and the space derivative gives the wave vector $\vec{k}$

$$\vec{\nabla}\Psi = \vec{k}.$$

Hence, the total derivative of Ψ reads

$$d\Psi = \vec{\nabla}\cdot d\vec{r} + \frac{\partial\Psi}{\partial t}dt$$

$$d\Psi = \vec{k}\cdot d\vec{r} - \omega\, dt.$$

Then the phase expression for the light beam after one circulation in a closed loop of arbitrary shape is

$$2\pi = \oint \vec{k}\cdot d\vec{r} - \int_0^\tau \omega\, dt$$

$$\Psi = \frac{1}{2\pi} \oint \vec{k}\cdot d\vec{r} - \frac{1}{2\pi} \int_0^\tau \omega\, dt \qquad (1)$$

where τ is the circulation time for the stationary loop.[54]

The first integral counts the number of wavelengths z in the closed spatial path, the second integral gives the number of radians T over which the signal advances during the circulation time.

Now consider the variation $\delta\Psi$. Suppose that the interferometer is subjected to a small time-dependent displacement $\vec{q}$, generated by the velocity $\vec{v}$. The Eikonal Ψ is a general invariant and should not be affected by this displacement provided the variation in the boundaries is properly accounted for. Hence, one obtains

$$\delta\Psi = \delta z - \delta T = 0$$

$$2\pi\delta = \delta \oint \vec{k}\cdot d\vec{r} = \oint_c \delta\vec{k}\cdot d\vec{r} + \oint_{c^1} \vec{k}(\vec{r})\cdot d\vec{r}^{\,1} - \oint_c \vec{k}(\vec{r})\cdot d\vec{r}$$

where c^1 is the path one obtains from c by a displacement $\vec{q}$, and

$$\vec{k}(\vec{r}) = \vec{k}(\vec{r}) + (\vec{q}\cdot\vec{\nabla})\vec{k}$$

$$d\vec{r}^{\,1} = d\vec{r} + (d\vec{r}\cdot\vec{\nabla})\vec{q}.$$

We can now write the variation in the form

$$\delta\oint\vec{k}\cdot d\vec{r} = \oint[\delta\vec{k} + (\vec{q}\cdot\vec{\nabla})\vec{k}]\cdot d\vec{r} + \oint \vec{k}\cdot(d\vec{r}\cdot\vec{\nabla})\vec{q}$$

Using well-known vector identities one can show that

$$\vec{k}\cdot(d\vec{r}\cdot\nabla)\vec{q} = [\nabla(\vec{q}\cdot\vec{k}) - \vec{q} \times (\vec{\nabla} \times \vec{k}) - (\vec{q}\cdot\vec{\nabla})\vec{k}]\cdot d\vec{r}$$

which when substituted into the expression for δz gives the expression

$$2\pi\delta z = \oint[\delta\vec{k} - \vec{q} \times (\vec{\nabla}\times\vec{k}) + \vec{\nabla}(\vec{q}\cdot\vec{k})] \cdot d\vec{r} \qquad (2)$$

where δz is a measure of the fringe shift associated with a comparison of the accelerated and the inertial system. Since we are dealing with a situation for which geometric optical conditions prevail, we may assume $\vec{k}$ to be irrotational[49] (Sommerfeld-Runge law), that is $\vec{\nabla}\times\vec{k} = 0$.

Thus one obtains for δz

$$2\pi\delta z = \oint \delta\vec{k}\cdot d\vec{r} + (\vec{k}\cdot\vec{q})_2 - (\vec{k}\cdot\vec{q})_1. \qquad (3)$$

The subscribts (1) and (2) refer to the values of the scalar product $(\vec{k}\cdot\vec{q})$ at the beginning and the end of one circulation around the loop. The scalar product $(\vec{k}\cdot\vec{q})$ is not single valued since q acquires a different value at the end of one circulation due to the time dependence of the deformation. The change δz is the fringe shift associated with the velocity field $\vec{v}$. The first part gives the intrinsic change of the wave vector due to $\vec{v}$ and the last part is the contribution due to the change of the boundaries of the integral.

The variation of the time part of the space-time line integral reads:

$$2\pi\delta T = \delta\int_0^\tau \omega\ dt$$

$$= \int_0^\tau (\delta\omega\ dt + \omega\delta\ dt)$$

$$= \int_0^\tau \delta\omega\ dt + \int_0^\tau \omega\ d(\delta t)$$

Since ω is not a function of the time, the second integral becomes

$$\int_0^\tau \omega d(\delta t) = \omega\delta\tau .$$

Hence,

$$2\pi\delta T = \int_0^\tau \delta\omega\ dt + \omega\delta\tau .$$

We may now apply the appropriate conditions to the formula for δz, and evaluate the fringe shift for the ring interferometer and the frequency shift for the ring laser.

i. The Moving Ring Interferometer with Comoving Medium

The external light source determines ω so that

$$\delta\omega = 0$$

Since, $\omega = ku$, where u is the phase velocity,

$$\delta\omega = \delta\ ku + k\ \delta u = 0$$

$$\frac{\delta k}{k} = -\frac{\delta U}{u}$$

where δu is the change in the effective propagation velocity in the moving medium as seen by the stationary observer.

We may assume that the change in the phase velocity is brought about by the presence of the velocity field $\vec{v}$; hence,

$$\delta u = \alpha v_s ,$$

where v_s is the component of the velocity field along the beam path, and α is the coefficient of drag.

We may now write,

$$\delta u = \alpha \, \vec{v} \cdot \frac{d\vec{r}}{ds}$$

and

$$\frac{\delta k}{k} = - \frac{\alpha}{u} \vec{v} \cdot \frac{d\vec{r}}{ds}$$

$$\delta z = - \frac{1}{2\pi} \oint \alpha \frac{\vec{k}}{u} v \cdot d\vec{r} + \frac{1}{2\pi} \{ (\vec{k}\cdot\vec{q})_2 - (\vec{k}\cdot\vec{q})_1 \} .$$

The last two terms can be expressed in terms of the velocity field $\vec{v}$, if one considers that $(\vec{k}\cdot\vec{q})$ changes over the interval of time dt by the amount $\vec{k}\cdot\vec{v}$ dt. Hence, going around the complete loop one obtains

$$(\vec{k}\cdot\vec{q})_2 - (\vec{k}\cdot\vec{q})_1 + \int_o^\tau \vec{k}\cdot\vec{v} \, dt$$

since, $u = ds/dt$, and $\vec{k}$ and $d\vec{r}$ have the same direction:

$$\Delta(\vec{k}\cdot\vec{q}) = \oint \vec{k}\cdot\vec{v} \frac{ds}{u} = \oint \frac{k}{u} \vec{v}\cdot d\vec{r} .$$

Finally one arrives at,

$$\delta z = \frac{1}{2\pi} \oint \frac{k}{u}(1 - \alpha)\vec{v}\cdot d\vec{r} .$$

The actual observed fringe shift is twice δz

$$\Delta z = \frac{2}{2\pi} \oint \frac{k}{u}(1 - \alpha)\vec{v}\cdot d\vec{r}$$

which can be put in the form

$$\Delta z = \frac{2}{\lambda_o c} \oint n^2(1 - \alpha)\vec{v}\cdot d\vec{r}. \tag{4}$$

Consider now,

$$\delta z - \delta T = 0$$

$$\delta z = \delta T = \frac{\omega}{2\pi} \delta\tau$$

or,

$$2\pi\Delta z = \omega\Delta\tau$$

$$\Delta\tau = \frac{2}{\omega} \frac{2}{\lambda_o c} \oint n^2(1 - \alpha)\vec{v}\cdot d\vec{r}$$

but,

$$\frac{2\pi}{\omega\lambda_o} = \frac{1}{\frac{\omega}{2\pi}\lambda_o} = \frac{1}{\nu\lambda_o} = \frac{1}{c}$$

hence,

$$\Delta\tau = \frac{2}{c^2} \oint n^2(1 - \alpha)\vec{v}\cdot d\vec{r}, \tag{5}$$

where $\Delta\tau$ is the difference in circulation time between clockwise and counterclockwise circulation in the moving interferometer with comoving optical medium.

ii. The Moving Ring Laser with Comoving Medium

If one assumes no mode jumping as a result of the motion of the ring laser, then the number of wavelengths in the loop should remain constant, that is

$$\delta z = 0$$

which is to say, $\delta T = 0$,

or,

$$\int_0^\tau \delta\omega \, dt + \omega\delta\tau = 0$$

In the case of motion with constant speed, $\delta\omega$ is constant; hence,

$$\frac{\delta\omega}{\omega} = \frac{\delta\tau}{\tau}$$

or,

$$\frac{\Delta\omega}{\omega} = -\frac{\Delta\tau}{\tau}$$

$$\left|\frac{\Delta\omega}{\omega}\right| = \frac{\Delta\tau}{\tau}$$

$$c\tau = \oint nds$$

The difference in the circulation times between the clockwise and counterclockwise beams does not depend on whether the optical circuit is being used as an interferometer or as a ring laser. Hence, the equation developed for $\Delta\tau$ may be used here, and

$$\frac{\Delta\omega}{\omega} = \frac{\frac{2}{c^2}\oint n^2(1 - \alpha)\vec{v}\cdot d\vec{r}}{\frac{1}{c}\oint nds} = \frac{2 \oint n^2(1 - \alpha)\vec{v}\cdot d\vec{r}}{c \oint nds} \tag{6}$$

iii. The Moving Interferometer and Ring Laser with a Stationary Medium in the Beam Path

Again the fringe shift expression is

$$2\pi \ \delta z = \oint \delta\vec{k}\cdot d\vec{r} + (\vec{k}\cdot\vec{q})\Big|_1^2$$

and

$$\frac{\delta k}{k} = -\frac{\delta u}{u}$$

but $\delta u = 0$ when one observes the propagation velocity in the medium from a frame of reference in which the medium rests. Hence, $\delta k = 0$, and

$$2\pi \ \delta z = (\vec{k}\cdot\vec{q})\Big|_1^2 = \oint (k/u)\vec{v}\cdot d\vec{r}$$

Since, $k = \frac{2\pi}{\lambda_o}$, and $u = c/n$, one obtains for Δz

$$\Delta z = \frac{2}{\lambda_o c} \oint n^2 \, v \cdot d\vec{r}. \tag{7}$$

For the self-oscillating case one obtains

$$\frac{\Delta\omega}{\omega} = \frac{2}{c} \frac{\oint n^2 \, \vec{v} \cdot d\vec{r}}{\oint n ds} . \tag{8}$$

iv. The Stationary Ring Interferometer and Ring Laser with a Moving Medium in the Beam Path

The fringe shift is now due solely to the phenomenon of drag. The fringe shift is therefore

$$\Delta z = \frac{2}{c\lambda_o} \oint n^2 \, \alpha \, \vec{v} \cdot d\vec{r} \tag{9}$$

and the corresponding beat frequency for the ring laser

$$\frac{\Delta\omega}{\omega} = \frac{2}{c}\left(\oint n^2 \, \alpha \, \vec{v} \cdot d\vec{r} / \oint n ds\right) \tag{10}$$

The geometrical optics approach gives no insight to Pegram and Kennard effects, and as expected does not resolve the Sommerfeld paradox. Hence, the need for a theory that explains electromagnetic phenomenon in noninertial systems becomes evident.

III. MODERN METHODS OF ELECTROMAGNETIC THEORY IN ACCELERATED SYSTEMS

In the last decade, there have been numerous publications on the subject of electromagnetic theory in accelerated systems. These publications show a variety of approaches. We have selected for this discussion, publications which properly represent different trends of thinking about the fundamental features of the theory.

These different methods of approach may be divided into three categories according to the effect a transformation to an accelerated frame has on the Maxwell field equations, and on the constitutive equations which describe the properties of a specific medium at hand.

Method I.

The Maxwell equations as well as the constitutive equations are affected by a transformation to an accelerated frame of reference. Heer[19] introduced the method and reopened the study of the electromagnetic theory of noninertial systems.

The procedure is holonomic. The covariant rendering of the Maxwell equations requires a linear connection given by the so-called Christoffel coefficients.

Method II.

The constitutive equations are not affected by transforming to an accelerated system, the Maxwell equations are affected. This procedure is anholonomic. The covariant rendering of the Maxwell equations requires a linear connection given by the so-called Ricci coefficients. Mo[47] applied this method to the electromagnetic theory of noninertial systems which was originally developed for fluids by Irvine.[22]

Method III.

The constitutive equations are affected by transforming to an accelerated system, the Maxwell equations are not affected.

The procedure is holonomic. The covariant rendering of the Maxwell equations does not require a linear connection. Hence, neither Christoffel nor Ricci coefficients have to be calculated. PYT, and Anderson and Ryon are the main proponents of this approach.

1. Mathematical Comparison

The methods I and III are both holonomic, by this one means that tensorial quantities are referred to a natural vector-basis that is associated with the accelerated frame of reference. These natural basis vectors are in an integrable fashion related to the coordinate system from which they arise.

Method II, by contrast, represents an anholonomic procedure. Instead of the natural vector basis of the coordinate system, one takes in every space-time point a local inertial frame that moves uniformly with a velocity that the accelerated frame has at that time, at that o point. The method is an outgrowth of a procedure common in fluid mechanics.

Method II can also be considered as a natural extension from three to four dimensions of the standard procedure for introducing curvilinear coordinates. In the latter case, one chooses local orthogonal triads of unit vectors aligned with the orthonormal coordinates to be used. The local orthogonal triads are used as a physically "easy" reference for vectorial and tensorial quantities; it obviates the distinctions between covariant and contravariant components of vectorial and tensorial quantities.

Method II is accordingly called the method of local inertial tetrads. The tetrads give a physically "easy" reference to inertial frames. Neither the unit vectors of the triad method, nor the unit vectors of the tetrad method are in an integrable fashion related to the underlying coordinate system. One can define coordinate infinitesimals along the legs of the triads and tetrads. They cannot in general be extended to an integrated finite value. In analytical dynamics, Whittaker calls these infinitesimals "quasi" coordinates. The local tetrads have the calculational advantage that one retains the constitutive relations of an inertial frame while working with an accelerated system. Mo adds extra detail to the simple "hydrodynamic" procedure followed by Irvine.

Having established the rather deep distinction between methods I and II, let us now turn to the much minor distinction between Methods I and III.

One may say that method III emerges from Method I simply by using a (physically) permissable modification of defining field quantities so that the fundamental laws of the electromagnetic theory can be related to differential form expressions, a possibility first explicitly noted by Cartan.[8]

The treatment of differential forms does not require a linear connection. Hence, it saves the effort required to calculate the Christoffel or Ricci coefficients induced by the transformation to an accelerated frame. The so-called covariant which plays a prominent role in method I reduces for method III to the equivalent of an exterior derivate. Method III is called the method of natural invariance.

2. Physical Comparison

Having established that the mathematical procedures I, II, and III can be equivalent, let us now turn to a comparison of the physical premises that go into these different treatments. We may do well by first mentioning the points of agreement.

All approaches mentioned agree about the basic space-time transformations that are to be used for describing a rigid rotation. Most experimental results relate to rotating equipment: rotating interferometers, rotating ring lasers, Mossbauer resonance experiments of samples exposed to different centrifugal accelerations, probes measuring electric and magnetic fields in rotating systems or their corresponding choice of rotating experimental arrangements are obvious.

Noting the agreement about the space-time transformations to be used, it may be surprising that all references cling to the notion of a classical rigid rotation on the basis of an absolute time. The asymptotic relation of this transformation to a Lorentz transformation has been discussed by Post. It is possible on the basis of this asymptotic relation to include the feature of a centrifugal red-shift. It is then also possible to formally distinguish whether one transforms from an inertial to a noninertial frame or vice versa; the absolute time Galilei rotation misses this feature.

It is obvious that the absolute time rotation can only have a local meaning. It is not possible to extend its radius arbitrarily. One finds as far as the induced transformation of the field quantities is concerned, that the added feature of including the acceleration red-shift is inconsequential for all first order effects.

One might have expected considerable disagreement about the idea that a space-time transformation based on absolute time should be regarded as fundamental for rotational motion. It seems that the Galilei rotation is much closer to physical reality than the Galilei translation. Most authors have accepted this as a fair and admissable basis for work.

A second point of more or less universal agreement is that most authors, except Mo, borrow from relativity the existence of a space-time manifold that has a (nondefinite) metric. At times, method III has been called a nonmetric procedure. The truth, of course, is that the metric in method III plays a more indirect role. The distinction with method I is that the metric in method III can be introduced at a later stage of the game so that it is easier to identify its physical role. When not dealing with gravitation, a premature introduction of the metric ought to be avoided.

Presently, we are arriving at the points of disagreement. It is so that proponents of method III take position against method I, not because of basic inadequacies of method I, but rather for the manner in which the constitutive relations are obtained. Yet, method I deserves strong credit for clearly and explicitly distinguishing between constitutive relations for inertial and noninertial frames.

Then, method II in a sense disagrees with all other references on the basis of a point of epistomology. Method II holds out for a policy that all observations should be referred to local inertial frames and that the effect of acceleration should be exclusively described by the linear connection interrelating this family of local inertial frames. The fact is that observations are made in noninertial frames, and by adhering too strictly to this extreme stand, one runs a very serious risk of an unsatisfactory relation to experiment.

Method III is clearly superior to the other methods. However, there is disagreement among its proponents. PYT and Anderson and Ryon approaches take issue with each other. Let us first review the basic premises of the PYT formulation.

1. The constitutive behaviour of a linear nondispersive medium, be it material or free-space, is given by a tensor χ of valence 4.

2. If a material is absent, the tensor χ reduces to a tensor χ_o describing the constitutive behaviour of free-space. The tensor χ_o is the same for all inertial frames (Lorentz invariance). It is, however, affected by a change when going from an inertial to a noninertial frame.

3. The constitutive tensor of a material medium permits an invariant decomposition $\chi = \chi_o + \chi_m$, where χ_m is the contribution of the material part of the medium, describing its electric and magnetic susceptibilities. Anderson and Ryon question the validity of this statement, specifically the invariance of the decomposition.

The following physical arguments are necessary to see the need for an in general separate transformational procedure when obtaining χ_o and χ_m on any given accelerated frame.

a. One can not "move", or "accelerate", the free-space component of the medium. Hence, when observing from an accelerated frame, one transforms χ_o from its known inertial form to the required noninertial form.

b. By contrast, one can move the material component of the medium in free-space and one can accelerate the material component with respect to free-space. The material part χ_m is not invariant under a change of reference, be it uniform motion or acceleration. The tensor

χ_m is normally known on a rest-frame, which may be inertial or noninertial.

One can now make the following assumption regarding being on either an inertial rest-frame or a noninertial rest-frame. The assumption is valid for media with a reasonably stiff mechanism of electric and magnetic polarization.

c. The influence of acceleration forces on the electric and magnetic mechanisms of polarization may be neglected for mildly accelerated media. Hence, (χ_m inertial rest-frame) = (χ_m noninertial rest-frame).

No electromagnetic theory of accelerated systems can circumvent an assumption of this nature. Its validity is a problem of solid state physics and cannot be ascertained by means of a general transformation theory. The latter only applies to a change of reference for a position of the observer.

Here we meet with a crucial distinction between accelerated and uniformly moving systems. In the latter χ_o is invariant and χ_m is exactly the same on all inertial rest-frames. This face expresses the principle of relativity for uniform motion. There is no such principle for accelerated motion. Accelerated can be detected through changes of χ_o as well as of χ_m, but mostly χ_o. Assumption c excludes the detection of acceleration through χ_m as a feasible competitor of χ_o.

The practice of applying this procedure is now conceptually straight forward. The algebra required is that of transforming a tensor of valence 4 so that χ_o and χ_m are referred to one and the same frame to make the addition $\chi = \chi_o + \chi_m$ permissible. One can have arbitrary mutual motion of observer and medium. The following three cases, however, are the ones that one may encounter in actual experimentation:

Case I. Observer at rest in inertial frame, material medium rotating.

χ_o, in the inertial frame, and χ_m, in the noninertial rest-frame, are transformed to the inertial frame of observer. Example: a Fizeau experiment with circular symmetry.

Case II: Observer rotating, material medium at rest in inertial frame.

χ_o, in the inertial frame, is transformed to rotating frame; χ_m, in the inertial rest-frame, is transformed to rotating frame. Example: Prunier-Dufour experiment.

Case III. Observer and material medium both rotating.

χ_o, in the inertial frame, is transformed to rotating frame; χ_m in the noninertial rest-frame, is not transformed. Example: Pogany experiment.

Note that Case I and Case II merge into the single classical experiment of Fizeau if one replaces the rotation by uniform translation. The reason is that χ_o is invariant under a Lorentz transformation.

Anderson and Ryon question the independent tensorial character of χ_o and χ_m. The objection can be dealt with in a simple physical argument. Suppose χ_o were nontensorial (inhomogeneous transformation), that an acceleration could give a contribution to χ_m. It is inconceivable that an acceleration would create a material contribution.

They also critize the here presented procedure on the grounds that it would violate the relativistic addition theorem of velocities. The classical Fizeau experiment constitutes a confirmation of the relativistic velocity addition. In the PYT formulation, Case I and Case II merge as mentioned above. Hence, the theory in fact satisfies the relativistic addition theorem.

We shall now briefly review the Anderson and Ryon approach to the electromagnetic theory of noninertial systems. Their method of approach depends on the use of the transformation properties of the local four-velocity of the accelerated system. The field vectors are therefore defined in terms of the local four-velocity of the medium similar to the definitions employed by Landau and Lifshitz[29] in treating inertial electrodynamics. Hence, they set $E_\mu = F_{\mu\nu}\, u^\nu$, $B^\sigma = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu}\, u_\rho$, $D^\mu = G^{\mu\nu}\, u_\nu$, $H_\sigma = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\, G^{\mu\nu}\, u^\rho$ and use the constitutive relations $D^\mu = \varepsilon^{\mu\nu}\, E_\nu$, and $H_\mu = \mu^{-1}_{\mu\nu}\, B^\nu$ to derive $G^{\mu\nu} = \frac{1}{2}\, \chi^{\mu\nu\rho\sigma} F_{\rho\sigma}$.

To calculate the constitutive relations corresponding to the three experimental configurations discussed, the proper transformation of the local four-velocities must be prescribed. A point fixed in the noninertial system has a world line given by $\vec{x}' =$ constant. Hence, the four-velocity when the observer and the medium are coaccelerated can be calculated as follows. Let the position vector in the comoving medium be defined as

$$x^{\mu'} = (\tau, \vec{x}'),$$

where we assume that x' is not a function of τ. Then,

$$\frac{d\chi^{\mu'}}{d\tau} = (1,0,0,0)$$

further,

$$\frac{dx^{\mu'}}{d\tau} = \frac{dx^{\mu'}}{ds'}\ \frac{ds'}{d\tau}$$

where,

$$ds'^2 = g_{\mu\nu}\, dx^{\mu'}\, dx^{\nu'}$$

and

$$\left(\frac{ds}{dx^{0'}}\right)^2 = g_{00}.$$

Hence,

$$u^{\mu'} = \frac{dx^{\mu'}}{ds'} = (g_{00})^{-\frac{1}{2}}\ (1,0,0,0).$$

To calculate the local four-velocity for the case where the observer is inertial and the medium is accelerated all that is needed is the transformation of $u^{\mu'}$ to the inertial axis, that is,

$$u^{\nu} = A^{\nu}_{\mu'}\ u^{\mu'} = A^{\nu}_{0'}\ u^{0'}$$

$$u^{\nu} = (g_{00})^{-\frac{1}{2}}\ A^{\nu}_{0'}.$$

Calculation for the case of the observer accelerated and the medium stationary requires the transformation of the local four-velocity of the stationary frame to the accelerated frame where the observer is stationed. A point fixed in the inertial frame has a world line $\vec{x}$ = constant and $u^{\mu} = (1,0,0,0)$ since $g_{00} = 1$. Now the transformation, $u^{\mu'} = A^{\mu'}_{\nu}\ u^{\nu}$ yields

$$u^{\mu'} = A^{\mu'}_{o}.$$

The constitutive equations derived by Anderson and Ryon require dyadic expressions and have not been put into simple vector form because of the employment of anholonomic definitions of the fields although the theory itself is holonomic. They have also carried the theory to second order by defining relative velocities as done in the special theory of relativity. Their definitions of relative velocity are not clear and within the context of Galilean transformation used, their second order terms are questionable.

3. Experimental Testability

Almost all of the references mentioned relate their theoretical work to experiments that have been performed on rotating interferometers and on rotating ring lasers. A detailed review discussion of these experiments can be found in a review article by Post.

For all practical purposes, the theoretical contributions of the three approaches confirm the experimental results, as well as an earlier kinematic analysis of the problem given by von Laue.

Nevertheless, PYT formulation arrives at interferometer and ring laser results that differ slightly from the conclusions obtained by Heer and Anderson and Ryon, including the kinematic analysis of von Laue. This discrepancy with the kinematic analysis weighs heavy because of the austerity of physical assumptions that enter into such a kinematic analysis.

The discrepancy between Heer and Anderson and Ryon methods versus that of PYT only shows up for media with $\mu_r \neq 1$. Where all known optical media have an effective permeability $\mu_r = 1$ down to the fifth and sixth decimal places, the chances of experimentally resolving this issue through optical experimentation seems than slim.

However, if we turn our attention to the constitutive relations that are produced, Heer and Anderson and Ryon methods, versus that of PYT then there seems to be a realistic chance of further resolving this matter.

In fact, free-space experiments that may be considered as a direct check on free-space constitutive behaviour in accelerated frames have already been performed. The Kennard and Pegram experiments may be considered as brother and sister to the ring laser effect. Of course, these free-space constitutive experiments cannot resolve the discrepancy associated with a $\mu_r \neq 1$. It is so, however, that a constitutive experiment with a medium having $\mu_r = 1$ would be more promising and easier to perform than any of the above mentioned optical experiments.

Mo's method, because of its extreme epistomological view point, denies the existence of these constitutive equations and rejects accordingly any further recourse to experiment.

There exists another method that does not belong to any of the three mentioned methods. Gordon[15] in an early publication has attempted an analysis of the ring-interferometer experiment through the use of a modified Riemann metric. This method may be regarded as a (non unique) physical optical extension of von Laue's geometrical optical approach. Gordon's results align with those of von Laue.

Yildiz[74] has recently revived Gordon's method by separating the free-space part and the material part of the modified Reimann metric, which enables him to discuss also situations in which medium and observer are not at rest in the same frame. This corresponds to the experiments of Prunier-Dufour and Fizeau.

The Gordon-Yildiz approach does not permit an independent introduction of ε_r and μ_r such as required by the complete electromagnetic approach. The method can therefore not be related to a detailed constitutive behaviour and accordingly does not partake in the possibility of permitting an experimental check through electrostatic and magnetostatic experiments.

IV. METHOD OF NATURAL INVARIANCE

1. Vector Form of Maxwell's Equation

Let us consider Maxwell's equation in the MKS units:

$$\text{curl}\ \vec{E} = -\frac{\partial}{\partial t}\vec{B} \tag{11a}$$

$$\text{div}\ \vec{B} = 0 \tag{11b}$$

$$\text{curl}\ \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \tag{11c}$$

$$\text{div}\ \vec{D} = \rho \tag{11d}$$

where $\vec{E}$ and $\vec{D}$ are the electric field and electric displacement, while $\vec{H}$ and $\vec{B}$ are the magnetic field and magnetic induction. We have chosen a coherent system of units (MKS) for the Maxwell equations. The choice of a coherent system of units makes a formal distinction between E, D, and $\vec{H}$, $\vec{B}$ in empty space, as well as in material medium. Use of mixed system of units avoids this distinction in free space; however, for noninertial frames of reference this distinction may become necessary.

The constitutive equations corresponding to Maxwell's equation are, empty space

$$\vec{D} = \varepsilon_o \vec{E} \qquad \vec{B} = \mu_o \vec{H},$$

and for linear, and stationary, isotropic, material medium are

$$\vec{D} = \varepsilon_r \varepsilon_o \vec{E} \qquad \vec{B} = \mu_r \mu_o \vec{H},$$

where,

$$\varepsilon_o = \frac{10^7}{4\pi c^2} \qquad \mu_o = \frac{4\pi}{10^7}.$$

Maxwell was among the first physicists to make a distinction between the field variables. In his text, A Treatise on Electricity and Magnetism[37], Maxwell proposes that there exist four different vector species in three space. His arguments, although mathematical, were motivated by physical needs.

Maxwell introduced the names, force and flux vectors to correspond to the motions of line vector and surface vector. Each of these vectors have the property of being polar or axial. A pairwise combination of these properties leads to four basic vector species in three-space as shown in the following diagram.

SPACE VECTORS	POLAR	AXIAL
FORCE	$\vec{E}$	$\vec{H}$
FLUX	$\vec{D}$	$\vec{B}$

As can be seen from the integral forms that lead to Maxwell's differential forms, the force vectors are associated with line integrals and flux vectors with surface integrals. The labels polar and axial refer to longitudinal and rotational symmetry propties of the space vectors.

By choosing an ad-hoc system of mixed units, one can simplify the relations between the field variables to E = D, and H = B. However, if we wish to address ourselves to the solution of "noninertial" problems, then we must maintain Maxwell's distinction.

Furthermore, we can obtain a naturally invariant representation of the Maxwell equations for three dimensions valid in any spatial curvilinear coordinate system if we make the following identifications[53]:

$$\vec{E} \to E_i,\ \vec{D} \to E^i,\ \vec{J} \to j,\ \vec{B} \to B_{ij},\ \vec{H} \to G^{ij},\ P \to P$$

We can then write Maxwell equations as,

$$2\ \partial_{[\,i}E_{j]} = -\ \partial_t\ B_{ij} \tag{12a}$$

$$\partial_{\,[i}B_{jk]} = 0 \tag{12b}$$

$$\partial_j\ G^{ij} = \partial_t\ D^i + j^i \tag{12c}$$

$$\partial_i\ D^i = \rho \tag{12d}$$

where $G^{ij} = -\ G^{ji}$; and $B_{ij} = -\ B_{ji}$; $i,j = 1,2,3.$

The invariance is realized with the following identification in the Cartesian frame

$$\vec{E} = (E_1,\ E_2,\ E_3) = (E_x,\ E_y,\ E_z)$$

$$\vec{B} = (B_{23},\ B_{31},\ B_{12}) = (B_x,\ B_y,\ B_z)$$

$$\vec{D} = (D^1,\ D^2,\ D^3) = (D_x,\ D_y,\ D_z)$$

$$\vec{H} = (G^{23},\ G^{31},\ G^{12}) = (H_x,\ H_y,\ H_z)$$

$$\vec{J} = (j^1,\ j^2,\ j^3) = (j_x,\ j_y,\ j_z)$$

$$\rho = \rho$$

and with the following transformation rules for a change in coordinates for the field variables,

$$E_{i'} = A^i_{j'}\ E_i\ ,\ D^{i'} = |\Delta|^{-1}\ A^{i'}_i\ D^i,\ j^{i'} = |\Delta|^{-1}\ A^{i'}_i\ j^i$$

$$B_{i'j'} = A^i_{i'}A^j_{j'}\ B_{ij},\ G^{i'j'} = |\Delta|^{-1}\ A^{i'}_i\ A^{j'}_j\ G^{ij},\ \rho(\kappa') = |\Delta|^{-1}\ \rho(\kappa)$$

with

$$A^i_{i'} = \frac{\partial x^i}{\partial x^{i'}}, \; A^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}, \; \Delta = \det A^i_{i'}$$

The constitutive relations may be similarly written for linear, anisotropic, nondispersive medium as

$$D^i = \varepsilon^{ij} E_j$$

$$B_{ij} = \frac{1}{2}\mu_{ijkl} G^{kl}$$

$$j^i = \sigma^{ij} E_j.$$

where ε^{ij}, μ_{ijkl}, σ^{ij} are the dielectric permittivity, magnetic permeability and the conductivity tensors respectively. The transformations of ε, μ, and σ follow from the transformations imposed on the field variables. Hence, their transformation behaviour may be expressed as,

$$\varepsilon^{i'j'} = |\Delta|^{-1} A^{i'}_i A^{j'}_j \varepsilon^{ij}$$

$$\mu_{i'j'k'l'} = |\Delta| A^i_{i'} A^j_{j'} A^k_{k'} A^l_{l'} \mu_{ijkl}$$

$$\sigma^{i'j'} = |\Delta|^{-1} A^{i'}_i A^{j'}_j \sigma^{ij}.$$

We shall now proceed to a less cumbersome method by generalizing the Maxwell equations to a four-dimensional form. The four-dimensional form is necessary, not only for its mathematical simplicity, but also because it provides a better physical ground for the electromagnetic theory in noninertial frames of reference.

2. Covariant Form of Maxwell's Equation

After having distinguished the field variables of electromagnetic theory, we search for a method of approach which is able to exploit this distinction to our best advantage. We may conjecture that the problem may be resolved within the realm of the theory of relativity.

The special theory of relativity describes physical phenomena with respect to inertial frames only. It does not describe events as seen by an observer on a noninertial frame of reference. Hence, special theory of relativity is not suited to handle the problem at hand.

The general theory of relativity relates the physical phenomena of gravitation to the non-Euclidian structure of the space-time manifold. Thus, the general theory of relativity appears irrelevant. We are left still searching for a theory that treats the description of physical phenomena as seen from noninertial frames.

However, gravitational and kinematic acceleration is not distinguishable locally. Hence, we can invoke the principle of local equivalence. The observational indistinguishability holds if one is restricted from exploring the neighborhood of the point of acceleration. For, then one can distinguish the two fields due to the presence of the coriolis force if the acceleration is due to the rotation and in the case of rectilinear acceleration due to the convergence of the lines of force at infinity only.

In fact, the general theory of relativity is relevant. The mathematical formalism that accommodates gravitation also accomodates accelerated systems of reference. However, the description of kinematic acceleration does not require the validity of the gravitational field equations.

The mathematical implementation of the principle of equivalence draws on the principle of general covariance. We must therefore seek a formulation which utilizes general covariance in order to provide an answer to problems associated with nonuniform space time systems. On the basis of general covariance, the electromagnetic field theory may be formulated such that the field equations and the constitutive equations are functionally separated. This method of approach is called the method of natural invariance. The invariance comes about independently of the metric.

Minkowski[53] discovered that electric and magnetic vectors can be pairwise combined into four-dimensional skew symmetric tensors according to the scheme:

$$\left.\begin{matrix}\vec{E}\\ \vec{B}\end{matrix}\right\} \rightarrow F_{\lambda\nu} = - F_{\lambda\nu} \qquad \text{and,} \quad \left.\begin{matrix}\vec{D}\\ \vec{H}\end{matrix}\right\} \rightarrow G^{\lambda\nu} = - G^{\lambda\nu}$$

$$\vec{j} \rightarrow j^{\lambda} \qquad \text{where, } \lambda,\nu = 0,1,2,3$$

and,

$$\begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} \\ F_{10} & 0 & F_{12} & F_{13} \\ F_{20} & F_{21} & 0 & F_{23} \\ F_{30} & F_{31} & F_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & G^{01} & G^{02} & G^{03} \\ G^{10} & 0 & G^{12} & G^{13} \\ G^{20} & G^{21} & 0 & G^{23} \\ G^{30} & G^{31} & G^{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{pmatrix}$$

$$j^{\mu} = (\rho,\vec{j}) \text{ and } x^{\mu} = (t,\vec{x}).$$

The natural covariant form of the Maxwell-Minkowski equations is thus expressed by the equations:

$$\partial_{[\kappa}F_{\lambda\nu]} = 0, \tag{13a}$$

$$\partial_{\nu}\, G^{\lambda\nu} = j^{\lambda} \tag{13b}$$

where $\kappa,\lambda,\nu = 0,1,2,3$, and the brackets denote an alternation over the indices dived by the factorial of the number of indices.

The Maxwell-Minkowski equations retain their form for any general nonlinear holonomic coordinate transformations, if one agrees that the fields transform according to

$$F_{\lambda'\nu'} = A^{\lambda}_{\lambda'}\, A^{\nu}_{\nu'}\, F_{\lambda\nu} \tag{14a}$$

$$G^{\lambda'\nu'} = |\Delta|^{-1}\, A^{\lambda'}_{\lambda}\, A^{\nu'}_{\nu}\, G^{\lambda\nu} \tag{14b}$$

where,

$\Delta = \det A^{\lambda'}_{\lambda}$, and $A^{\lambda'}_{\lambda} = \dfrac{\partial x^{\lambda'}}{\partial x^{\lambda}}$ is an element of the Jacobian matrix of the transformation. The condition of hononomity is expressed as $\partial_{[\nu}A^{\lambda'}_{\lambda]} = 0$.

The constitutive equation which provides the necessary connection between the two tensors $F_{\sigma\kappa}$ and $G^{\sigma\kappa}$ may be written as

$$G^{\lambda\nu} = \frac{1}{2}\chi^{\lambda\nu\sigma\kappa}F_{\sigma\kappa}.$$

This is the most general linear, instantaneous and local relation between $G^{\lambda\nu}$ and $F_{\lambda\nu}$. One must be able to recover from the constitutive tensor χ the customary linear constitutive relations for an isotropic medium with relative permittivity ε_r, and relative permeability μ_r and at rest in an

$\chi^{\lambda\nu\sigma\kappa}$	$\vec{E}$	$\vec{B}$
$-\vec{D}$	$-\varepsilon_o\ \varepsilon_r$	0
$\vec{H}$	0	$\frac{1}{(\mu_o\ \mu_r)}$.

The constitutive relation is capable of describing the general linear, nondispersive medium and therefore both the frame of reference and the medium may have arbitrary motion.

It is instructive to study a more explicit form of the constitutive tensor $\chi^{\lambda\nu\sigma\kappa}$. If one considers the situation where $\chi^{\lambda\nu\sigma\kappa}_{(o)}$ has a degeneracy, so that it can be constructed from a tensor of lower valence, a convenient choice for the lower rank tensor would be the metric $g^{\lambda\nu}$. It is known that the elements of the tensor $g^{\lambda\nu}$ or its inverse $g_{\lambda\nu}$ play a role in the interpretation of the gravitational action. The gravitational action may be considered as a cause of nonuniformity in matter free space. Hence, it is plausible to construct $\chi^{\lambda\nu\sigma\kappa}$ from the elements of a $g^{\lambda\nu}$. The empty space constitutive relations can be investigated if we assume a constitutive relation of the form

$$G^{\lambda\nu} = Y_o\ g^{\frac{1}{2}}\ g^{\lambda\nu}\ g^{\nu\kappa}\ F_{\sigma\kappa} \tag{15}$$

where Y_o is some universal constant and g is the absolute value of the determinant of $g_{\mu\nu}$. Using the relation $F_{\sigma\kappa} = -F_{\kappa\sigma}$ and then interchanging the indices σ and κ, one obtains

$$G^{\lambda\nu} = -\ Y_o\ g^{\frac{1}{2}}\ g^{\lambda\kappa}\ g^{\nu\sigma}\ F_{\sigma\kappa}$$

which then subtracted from the previous expression yields

$$G^{\lambda\nu} = \frac{1}{2}\ Y_o\ g^{\frac{1}{2}}\ (g^{\lambda\sigma}\ g^{\nu\kappa} - g^{\lambda\kappa}\ g^{\nu\sigma})\ F_{\sigma\kappa}\ .$$

Hence one obtains,

$$\chi_{(o)}^{\lambda\nu\sigma\kappa} = \frac{1}{2} \gamma_o \, g^{\frac{1}{2}} \, (g^{\lambda\sigma} \, g^{\nu\kappa} - g^{\lambda\kappa} \, g^{\nu\sigma}), \tag{16}$$

which is the expression for the constitutive tensor for free space. This form of the constitutive tensor was first proposed by Einstein and later applied to rotating systems by Gordon.

In order to ensure the consistency of the formulation, the constitutive tensor χ must transform as a contravariant tensor density of weight +1,

$$\chi^{\lambda'\nu'\sigma'\kappa'} = |\Delta|^{-1} \, A_{\lambda}^{\lambda'} \, A_{\nu}^{\nu'} \, A_{\sigma}^{\sigma'} \, A_{\kappa}^{\kappa'} \, \chi^{\lambda\nu\sigma\kappa}. \tag{17}$$

The constitutive tensor χ also must obey the following symmetry relations

$$\chi^{\lambda\nu\sigma\kappa} = - \chi^{\nu\lambda\sigma\kappa} = - \chi^{\lambda\nu\kappa\sigma} = \chi^{\sigma\kappa\lambda\nu}$$

and $\chi^{[\lambda\nu\sigma\kappa]} = 0$.

These properties can be derived from the symmetry properties of $F_{\mu\nu}$ and $G^{\mu\nu}$ and the constitutive relation between the two.

3. Transformation of the Constitutive Tensor due to Rotation

If we wish to carry on our calculations for a rotating frame where the boundary conditions retain their time-independent form, then we must evaluate the elements of the constitutive tensor for rotating frame of reference, and for rotating media. This is quite a departure from the theory of uniformly translating systems. When a system is in nonuniform motion, one must take the distinction between the motion of the medium and the motion of the observer, while this is not required for uniformly moving systems. Thus, Maxwell equations retain their form on accelerated frames, provided they are expressed in terms of four distinct field variables $\vec{E}$, $\vec{B}$, and $\vec{D}$, $\vec{H}$, and whether or not a system is accelerated depends on the constitutive relations between $\vec{E}$, $\vec{B}$, and $\vec{D}$, $\vec{H}$.

Furthermore, a nonuniform motion affects the accelerated medium and accelerated observer in different fashion. While the medium in motion may undergo real and intrinsic physical change, the motion of the observer produces only a difference in observational viewpoint. We may therefore write the constitutive tensor as a superposition of a free-space part $\chi_{(o)}$ and a medium part $\chi_{(m)}$. Hence we have

$$\chi^{\lambda\nu\sigma\kappa} = \chi^{\lambda\nu\sigma\kappa}_{(o)} + \chi^{\lambda\nu\sigma\kappa}_{(m)} \tag{18}$$

where $\chi_{(o)}$ and $\chi_{(m)}$ obey the same transformation as well as symmetry operations that apply to the total χ.

The optical experiments mentioned previously can be categorized according to the three types of observation possible in a noninertial frame of reference.

Case a.

Medium inertial, observer accelerated.

Case b.

Medium and observer coaccelerated.

Case c.

Medium accelerated, observer inertial.

These three types of experiments further require the application of different transformations to the free-space and medium parts of the constitutive tensor. This procedure is dictated by the fact that the frames in which the free-space part $\chi_{(o)}$ and the medium part $\chi_{(m)}$ of χ are supposed to be known are in different states of motion with respect to the frame in which one desires to obtain the total χ. Hence, the physics of the experiment to be analyzed determines in which frame the total χ is to be calculated.

Besides the optical experiments, this procedure is further justified if we consider experiments performed by Barnett[4] (1915) and the Oppenheimer paradox.[53] In Barnett's experiment a rotating magnetizable bar becomes magnetized in its axial direction, while the rotation of the frame of reference causes no such effect. Oppenheimer paradox, simply stated, is that the rotation of the condenser produces an external magnetic field while the rotation of the frame of reference does not. Going back to the optical experiments, we can see from the geometrical optics discussion that Fresnel-Fizeau type of experiment is not physically equivalent to Dufour-Prunier experiment.

A parallel approach has recently been employed by Yildiz. He develops the metric analog of this method by separating the metric into space and material components. Employing transformations on $g^{\nu\lambda} = g^{\nu\lambda}_{(o)} + g^{\nu\lambda}_{(m)}$, similar to the ones to be discussed, he obtains the same results for the three types of optical experimentation.

We are now ready to write the constitutive tensor for the three types of experiments considered. We will denote the rotating coordinate system by primed indices, while the inertial frame will be denoted by unprimed indices.

Case a.

Dufour-Prunier experiment: In this case, the observer is accelerated, while the medium is stationary. He sees both the free-space and the medium from an accelerated frame of reference, thus

$$\chi^{\lambda'\nu'\sigma'\kappa'} = |\Delta|^{-1} A^{\lambda'\nu'\sigma'\kappa'}_{\lambda\nu\sigma\kappa} (\chi^{\lambda\nu\sigma\kappa}_{(o)} + \chi^{\lambda\nu\sigma\kappa}_{(m)}. \tag{19}$$

Case b.

Harress-Pogany experiment: In this type of experiment, both the medium and the observer are accelerated. Hence, while the observer sees the free-space part of the constitutive tensor from an accelerated frame, the medium part may be taken the same as in an inertial frame, thus

$$\chi^{\lambda'\nu'\sigma'\kappa'} = |\Delta|^{-1} A^{\lambda'\nu'\sigma'\kappa'}_{\lambda\nu\sigma\kappa} \chi^{\lambda\nu\sigma\kappa}_{(o)} + \chi^{\lambda'\nu'\sigma'\kappa'}_{(m)} \tag{20}$$

Case c.

Fresnel-Fizeau type experiment: In this case, observer is stationary, while the medium is accelerated. Hence, it represents a transformation duality with respect to the previous experiment,

$$\chi^{\lambda\nu\sigma\kappa} = \chi^{\lambda\nu\sigma\kappa}_{(o)} + |\Delta|^{-1} A^{\lambda\nu\sigma\kappa}_{\lambda'\nu'\sigma'\nu'} \chi^{\lambda'\nu'\sigma'\kappa'}_{(m)} \tag{21}$$

Having established the recipe for constructing the constitutive tensor χ for all possible types of experimentation in rotating frames of reference, we are still left with the task of selecting the proper space time coordinate transformation which in turn defines the A^{μ}_{ν}. If we consider the time dilation, then we can relate the space-time coordinates for a rotation about the z axis with angular velocity Ω as

$$dt = \frac{dt'}{[1 - (\Omega R/c)^2]^{\frac{1}{2}}} \tag{22a}$$

$$dr = dr' \tag{22b}$$

$$d\phi = d\phi' - \frac{\Omega dt'}{[1 - (\Omega R/c)^2]^{\frac{1}{2}}} \tag{22c}$$

where $R\,\Omega < c$. However, performable experiments are in the range where $R\,\Omega \ll c$, thus the affect of time dilation is of second order, while in the same experiments first order effects due to rotation are observed. That these effects are first order can be experimentally verified when compared to Doppler effect which is well known to be of second order. Hence, neglecting the time dilation, we obtain

$$dt = dt' \tag{23a}$$

$$dr = dr' \tag{23b}$$

$$d\phi = d\phi' - \Omega\, dt' \tag{23c}$$

which is the well known Galilean rotation. While the Lorentz transformation is an improvement over Galilei translation, there seems to be no analogous improvement over Galilei rotation.

Using the recipe outlined above, we can now write down the constitutive relations in conventional vector notation, as

$$\vec{D} = \varepsilon_o\, \varepsilon_r\, \vec{E} + \varepsilon' \,[(\vec{\Omega} \times \vec{r}) \times \vec{B}] \tag{24a}$$

$$\vec{H} = \frac{1}{\mu_o\, \mu_r}\, \vec{B} + \varepsilon' \,[(\vec{\Omega} \times \vec{r}) \times \vec{E}] \tag{24b}$$

where for,

Case a. $\varepsilon' = \varepsilon_o\, \varepsilon_r$ (25a)

Case b. $\varepsilon' = \varepsilon_o$ (25b)

Case c. $\varepsilon' = \varepsilon_o\, (\varepsilon_r - 1)$ (25c)

We can now make the following observations about these constitutive equations:

1. They reduce to the familiar relations $\vec{D} = \varepsilon_o\, \varepsilon_r\, \vec{E}$, and $\vec{B} = \mu_o\, \mu_r\, \vec{H}$ for an inertial frame ($\vec{\Omega} = 0$).

2. The second term in the right hand of the first constitutive equation resembles the induction field that led to the Sommerfeld paradox. The total displacement $\vec{D}$ is generated by the sum of two electric fields: a source related field and an electromotive force.

3. The existence of the second term in the second constitutive equation can be inferred from the assumption that the Lagrangian should be a total differential in the field variables $\vec{E}$ and $\vec{B}$. The term has the characteristics of an H field generated by a convection current.

4. The constitutive equations bear a resemblance to the constitutive equations of a uniformly translating material medium. However, the extra terms in the latter vanish if the product of relative permittivity ε_r and relative permeability μ_r approach unity.

V. SOMMERFELD's PARADOX AND THE THEORY OF KENNARD AND PEGRAM EFFECTS

The constitutive equations for Case b. take the form

$$\vec{D} = \varepsilon_r \, \varepsilon_o \, \vec{E} + \varepsilon_o \, (\vec{\Omega} \times \vec{r}) \times \vec{B}$$

$$\vec{H} = \frac{1}{\mu_o \, \mu_r} \, \vec{B} + \varepsilon_o \, (\vec{\Omega} \times \vec{r}) \times \vec{E}$$

Now if one considers the first equation for the case $\varepsilon_r = 1$ (free space) in cylindrical coordinates, then

$$D_r = \varepsilon_o \, E_r + \varepsilon_o \, \Omega_r \, B_z \qquad (26)$$

where Ω is a rotation about the z axis as shown in Fig. 4. Maxwell's equation gives us the recipe for calculating the charge density which should be zero in this case, that is

$$\text{div} \, \vec{D} = 0$$

or

$$\frac{1}{r} \frac{\partial}{\partial r} r \, D_r = 0$$

If in fact the div $\vec{D}$ vanishes, then the Sommerfeld paradox is resolved.

Let us now go back to Kennard and Pegram effects and make the first constitutive equation and div $\vec{D} = 0$ the cornerstone of our discussion.[57] The solution for D_r is

$$D_r = \frac{A}{r},$$

where A is the constant of integration.

In the ideal Kennard case, the open circuit implies $\vec{D} = 0$, and therefore, $A = 0$. It then follows from the constitutive equations that

$$E_r = - \Omega \, r \, B_z .$$

The Kennard potential V between the plates of the coaxial capacitor becomes, if r_1 and r_2 are the radii of the inner and outer conductor,

$$V = \int_{r_1}^{r_2} E_r \, d_r = - \Omega \, B_z \, \frac{(r_2^2 - r_1^2)}{2}$$

In the ideal Pegram case due to the short between the conductors $D \neq 0$ but the potential V between the capacitor-plates is zero. It then follows from the constitutive equations and the expression for D_r that

$$E_r = \frac{A}{\varepsilon_o r} - \Omega \, rB_z$$

$$V = \int_{r_1}^{r_2} E_r \, d_r = \frac{A}{\varepsilon_o} \ln \frac{r_2}{r_1} - \Omega \, B_z \left(\frac{r_2^2 - r_1^2}{2}\right) = 0$$

$$A = \frac{1}{2} \, \varepsilon_o \, \Omega \, B_z \, \frac{(r_2^2 - r_1^2)}{\ln \left(\frac{r_2}{r_1}\right)} \tag{27}$$

We can now calculate the total convected charge on the capacitor by integrating D_r over the surface of the cylinder of length L and radius r_o enclosing the inner surface of the capacitor, that is, $r_1 < r_o < r_2$:

$$Q = \int D_r \, dS = D_r \, 2 \, \pi \, r_o L,$$

where $D_r = \frac{A}{r_o}$ on this surface. Hence,

$$Q = A \, 2\pi L$$

$$Q = \frac{\varepsilon_o \, \pi \, \Omega L \, B_z \, (r_2^2 - r_1^2)}{\ln \left(\frac{r_2}{r_1}\right)} \tag{28}$$

The capacitance C of a tubular cyclindrical of length L is given by the expression

$$C = \frac{2\pi \ \varepsilon_o \ L}{\ln \ (\frac{r_2}{r_1})} . \tag{29}$$

The absolute value of the Pegram charge to Kennard potential still reproduce the conventional capacitance of a cylindrical capacitor.[65]

Hence, the constitutive equations developed resolve the Sommerfeld paradox, and in turn, the vanishing of the space charge explains the Kennard and Pegram effects. However, this is not the final word, for, if we repeat the above analyses for a corotating dielectric medium of relative permittivity ε_r (see Fig. 5), then we see that our procedure gives

$$V = - \frac{\Omega B_z}{\varepsilon_r} \ \frac{(r_2^2 - r_1^2)}{2} \tag{30}$$

while the total Pegram charge Q is independent of ε_r. For an electromotive force based interpretation, one would obtain a Kennard potential independent of ε_r, while the Pegram charge Q would be directly proportional to ε_r. Repetition of Pegram and Kennard experiments with corotating dielectric is highly desirable.

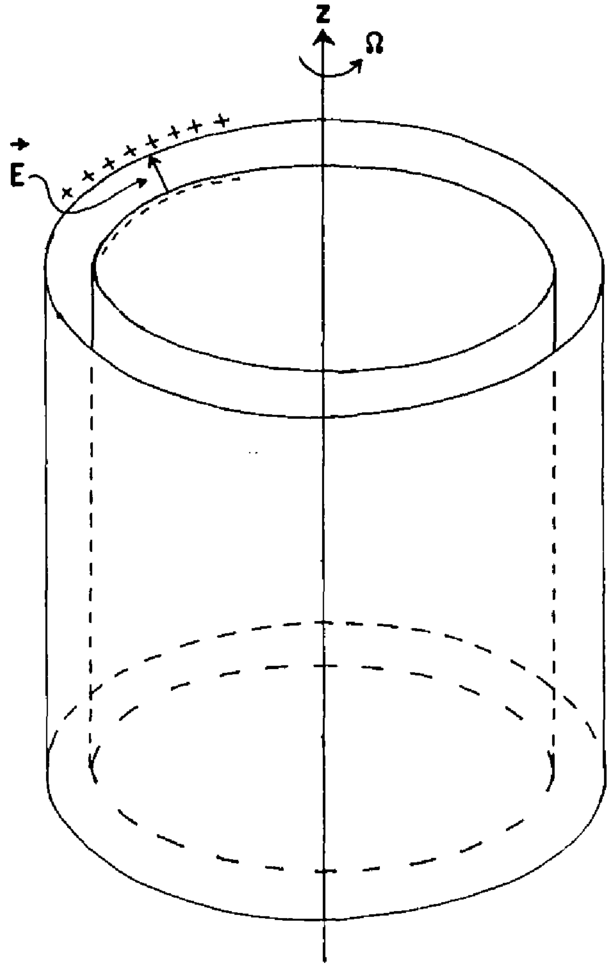

FIGURE 4

Fig. 4. Rotating charged coaxial condensor.

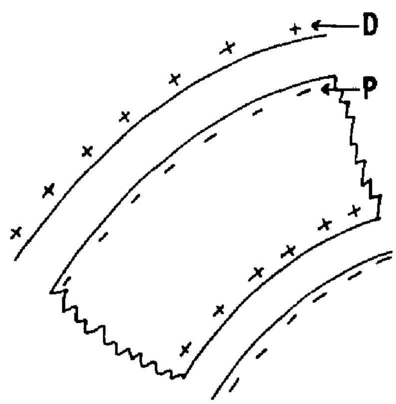

FIGURE 5

Fig. 5. Convected charges in a coaxial condensor filled with a corotating dielectric.

VI. APPLICATION OF THE METHOD OF NATURAL INVARIANCE TO RING LASERS AND WAVE GUIDES

1. The Frequency Splitting Phenomena in Ring Lasers

We shall now derive the wave equation corresponding to the three cases of experimentation discussed from Maxwell-Minkowski equations. The first of the two Maxwell-Minkowski equations

$$\partial_{[\kappa} F_{\lambda\nu]} = 0$$

$$\partial_{\nu} G^{\lambda\nu} = j^{\lambda}$$

implies that $F_{\lambda\nu}$ can be derived from a vector potential A_{ν} according to

$$F_{\lambda\nu} = 2\,\partial_{[\lambda} A_{\nu]} \quad . \tag{31}$$

Substituting $F_{\lambda\nu}$ into the constitutive equation $G^{\lambda\nu} = \frac{1}{2}\chi^{\lambda\nu\sigma\kappa}F_{\sigma\kappa}$ reads $G^{\lambda\nu} = \chi^{\lambda\nu\sigma\kappa}\,\partial_{[\sigma} A_{\kappa]}$. Now using $\partial_{\nu} G^{\lambda\nu} = j^{\lambda}$ we get

$$\partial_{\nu}\,\chi^{\lambda\nu\sigma\kappa}\,\partial_{[\sigma} A_{\kappa]} = j^{\lambda}$$

$$\frac{1}{2!}\,\partial_{\nu}\,\chi^{\lambda\nu\sigma\kappa}(\partial_{\sigma} A_{\kappa} - \partial_{\kappa} A_{\sigma}) = j^{\lambda}$$

$$\frac{1}{2!}\,\partial_{\nu}\,(\chi^{\lambda\nu\sigma\kappa}\,\partial_{\sigma}A_{\kappa} - \chi^{\lambda\nu\sigma\kappa}\,\partial_{\sigma} A_{\kappa}) = j^{\lambda}$$

and using $\chi^{\lambda\nu\sigma\kappa} = -\chi^{\lambda\nu\sigma\kappa}$ yields the generally invariant vector d'Alembertian or the wave equation

$$\partial_{\nu}\,\chi^{\lambda\nu\sigma\kappa}\,\partial_{\sigma} A_{\kappa} = j^{\lambda}. \tag{32}$$

This expression is valid for any curvilinear system if we transform χ as a tensor density of +1. Then, the derivatives are ordinary partial derivatives.

Let us represent the already derived constitutive equations in

$\chi^{\lambda\nu\sigma\kappa}$	$-E_r$	$-E_\phi$	$-E_z$	B_r	B_ϕ	B_z
D_r	$-\varepsilon r$	0	0	0	0	$\varepsilon' r\Omega$
D_ϕ	0	$\frac{-\varepsilon}{r}$	0	0	0	0
D_z	0	0	$-\varepsilon r$	$-\varepsilon' r\Omega$	0	0
H_r	0	0	$-\varepsilon' r\Omega$	$\frac{1}{\mu_r}$	0	0
H_ϕ	0	0	0	0	$\frac{r}{\mu}$	0
H_z	$\varepsilon' r\Omega$	0	0	0	0	$\frac{1}{\mu_r}$

where $\varepsilon = \varepsilon_r\ \varepsilon_o$, $\mu = \mu_r\ \mu_o$, and ε' takes on the values

Case a. $\varepsilon' = \varepsilon_o\ \varepsilon_r$

Case b. $\varepsilon' = \varepsilon_o$

Case c. $\varepsilon' = \varepsilon_o\ (\varepsilon_r - 1)$

It should be noted that diagonal terms are affected by coefficients r and $\frac{1}{r}$, which is the result of introducing cylindrical coordinates by using a holonomic definition of the components of the field quantities $\vec{E}$, $\vec{B}$, $\vec{D}$, and $\vec{H}$. The relation between the conventional and the holonomic definition are the following:

Holonomic		Conventional	Holonomic		Conventional
E_r	=	E_r	D_r	=	rD_r
E_ϕ	=	rE_ϕ	D_ϕ	=	D_ϕ
E_z	=	E_z	D_z	=	zD_z

The vectors $\vec{A}$ and $\vec{H}$ follow the same rules as vector $\vec{E}$, and vector $\vec{B}$ follows the same rules as vector $\vec{D}$.

Consider now the wave function for zero current, that is,

$$\partial_\nu \chi^{\lambda\nu\sigma\kappa} \partial_\sigma A_\kappa = 0.$$

The nonzero components of this equation can be expressed in the form

$\lambda = 0$

$$\partial_r(-r\varepsilon)(\partial_t A_r - \partial_r A_t) + \partial_r(r\Omega\varepsilon')(\partial_r A_\phi - \partial_\phi A_r) + \partial_\phi(\tfrac{\varepsilon}{r})(\partial_t A_\phi - \partial_\phi A_t) +$$
$$\partial_z(-\varepsilon r)(\partial_t A_z - \partial_z A_t) + \partial_z(-r\varepsilon'\Omega)(\partial_\phi A_z - \partial_z A_\phi) = 0$$

$\lambda = 1$

$$\partial_t(-r\varepsilon)(\partial_r A_t - \partial_t A_r) + \partial_t(-\Omega\varepsilon' r)(\partial_r A_\phi - \partial_\phi A_r) + \partial_\phi(\Omega\varepsilon' r)(\partial_t A_r - \partial_r A_t) +$$
$$\partial_\phi(\tfrac{1}{\mu_r})(\partial_r A_\phi - \partial_\phi A_r) + \partial_z(\tfrac{r}{\mu})(\partial_r A_z - \partial_z A_r) = 0$$

$\lambda = 2$

$$\partial_t(\tfrac{\varepsilon}{r})(\partial_\phi A_t - \partial_t A_\phi) + \partial_\phi(\tfrac{1}{\mu_r})(\partial_\phi A_r - \partial_r A_\phi) + \partial_r(-\Omega\varepsilon' r)(\partial_t A_r - \partial_r A_t) +$$
$$(\partial_t A_r - \partial_r A_t) + \partial_z(-\Omega\varepsilon' r)(\partial_t A_z - \partial_z A_t) + \partial_z(\tfrac{1}{\mu_r})(\partial_\phi A_z - \partial_z A_\phi) = 0$$

$\lambda = 3$

$$\partial_t(-\varepsilon)\ (\partial_z A_t - \partial_t A_z) + \partial_t(\Omega\varepsilon' r)(\partial_\phi A_z - \partial_z A_\phi) + \partial_r(\tfrac{r}{\mu})(\partial_z A_r - \partial_r A_z) +$$
$$\partial_\phi(\tfrac{1}{\mu_r})(\partial_z A_\phi - \partial_\phi A_z) + \partial_\phi(\Omega\varepsilon' r)\ (\partial_t A_z - \partial_z A_t) = 0.$$

For the case of a circulating light beam $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial z} \to 0$, and for the idealized circular arrangement shown in Fig. 6, and $r - r_o$, and the equations are simplified to the following:

$\lambda = 0$

$$\partial_\phi \partial_t A_\phi - \partial_\phi \partial_\phi A_t = 0$$

$\lambda = 1$

$$\varepsilon r_o \partial_t \partial_t A_r + \Omega \varepsilon' r_o (\partial_t \partial_\phi \partial A_r + \partial_\phi \partial_t A_r) - \frac{1}{\mu r_o} \partial_\phi \partial_\phi A_r = 0$$

$\lambda = 2$

$$\partial_t \partial_\phi A_t - \partial_t \partial_t A_\phi = 0$$

$\lambda = 3$

$$\varepsilon r_o \partial_t \partial_t A_z + \Omega \varepsilon' r_o (\partial_t \partial_\phi A_z + \partial_\phi \partial_t A_z) - \frac{1}{\mu r_o} \partial_\phi \partial_\phi A_z = 0$$

We observe that the equations for $\lambda = 1$, and $\lambda = 3$ are identical wave equations in the components A_r and A_z, while the other two are guage restrictions. Hence, we have a complete separation of the components, and we may consider the single wave equation

$$\varepsilon r_o \partial_t^2 \Psi + 2 \Omega \varepsilon' r_o \partial_t \partial_\phi \Psi - \frac{1}{\mu r_o} \partial_\phi^2 \Psi = 0$$

or letting $r_o \phi = s$, and multiplying by μ, we obtain

$$\varepsilon \mu \partial_t^2 \Psi + 2 \Omega \varepsilon' \mu r_o \partial_t \partial_s \Psi - \partial_s^2 \Psi = 0 \tag{33}$$

This is the familiar wave equation with the exception of the nonreciprocal term $2 \Omega \varepsilon' \mu r_o \partial_t \partial_s \Psi$, which vanishes as $\Omega \to 0$, and the conventional wave equation is then recovered.

To obtain a solution for the frequency splitting in a ring laser, we may assume

$$\psi = e^{i(\omega t + ks)} \tag{34}$$

which yields

$$\varepsilon\mu\omega^2 + 2\ \Omega\varepsilon'\mu r_o\omega k - k^2 = 0,$$

and has the solutions

$$\omega_{1,2} = -\Omega\frac{\varepsilon'}{\varepsilon}\, r_o\ k \pm \frac{1}{\sqrt{\varepsilon\mu}}\ \sqrt{1 + \frac{\varepsilon^{12}}{\varepsilon}\ \mu\ r_o^2\ \Omega^2} \tag{35}$$

or neglecting the second term under the root sign which is of the order $(r_o\Omega/c)^2$, we obtain

$$\omega_{1,2} = -\ \Omega\ \frac{\varepsilon'}{\ }\ r_o\ k \pm \frac{1}{\sqrt{\varepsilon\mu}}\,.$$

The difference between the absolute value of ω_1 and ω_2 is

$$\Delta\omega = 2\ r_o\ \Omega\ k\ \frac{\varepsilon'}{\varepsilon}.$$

For the stationary loop, we may write $k = \omega_o n/c$, where $n = \sqrt{\varepsilon_r\mu_r}$ is the index of refraction of the medium and ω_o is the single resonant frequency of the stationary loop, thus

$$\frac{\Delta\omega}{\omega} = \frac{2\ r_o\Omega}{c\varepsilon_o}\ \frac{\mu_r}{\varepsilon_r}\ \varepsilon'. \tag{36}$$

For the previously mentioned three types of experiments, we obtain

Case a.

$$\frac{\Delta\omega}{\omega_o} = \frac{2\ r_o\Omega}{c}\sqrt{\varepsilon_r\mu_r} \tag{37a}$$

Case b.

$$\frac{\Delta\omega}{\omega_o} = \frac{2\ r_o\Omega}{c}\ \sqrt{\mu_r/\varepsilon_r} \tag{37b}$$

Case c.

$$\frac{\Delta\omega}{\omega} = \frac{2\ r_o\Omega}{c}\sqrt{\mu_r/\varepsilon_r}(\varepsilon_r - 1) \tag{37c}$$

where c is the speed of light in vacuum

One can obtain the fringe shift expression by noting that

$$\frac{\Delta\omega}{\omega} = \frac{\Delta s}{s}$$

where s is the path length of the circular loop and Δs the path length difference clockwise and counterclockwise beam. Hence, the fringe shift Δz may be expressed as

$$\Delta z = \frac{\Delta s}{\lambda_o/n} = \frac{\Delta\omega}{\omega}\frac{s}{\lambda_o/n}$$

where $_o$ is the wavelength in vacuum.

Finally we obtain

$$\Delta x = 4\Omega\ \pi r_o^2\ \varepsilon'(\frac{\mu}{\varepsilon})^{\frac{1}{2}}\ (\frac{1}{\lambda_o})\ \ (\varepsilon_r\mu_r)^{\frac{1}{2}} \tag{38}$$

and corresponding to each case, we write.

Case a.

$$\Delta z = \frac{4\Omega\ A}{c\lambda_o}\ \ \mu_r\varepsilon_r \tag{39a}$$

Case b.

$$\Delta z = \frac{4\Omega A}{c\lambda_o}\mu_r \tag{39b}$$

Case c.

$$\Delta z = \frac{4\Omega A}{c\lambda_o}(\varepsilon_r - 1)(\frac{\mu_r}{\varepsilon_r})^{\frac{1}{2}} \tag{39c}$$

Both the frequency and fringe shift expressions can be derived from geometrical optical analysis for the case $\mu_r = 1$. Results derived above agree with the expressions derived from geometrical optics for the free-space case, also.

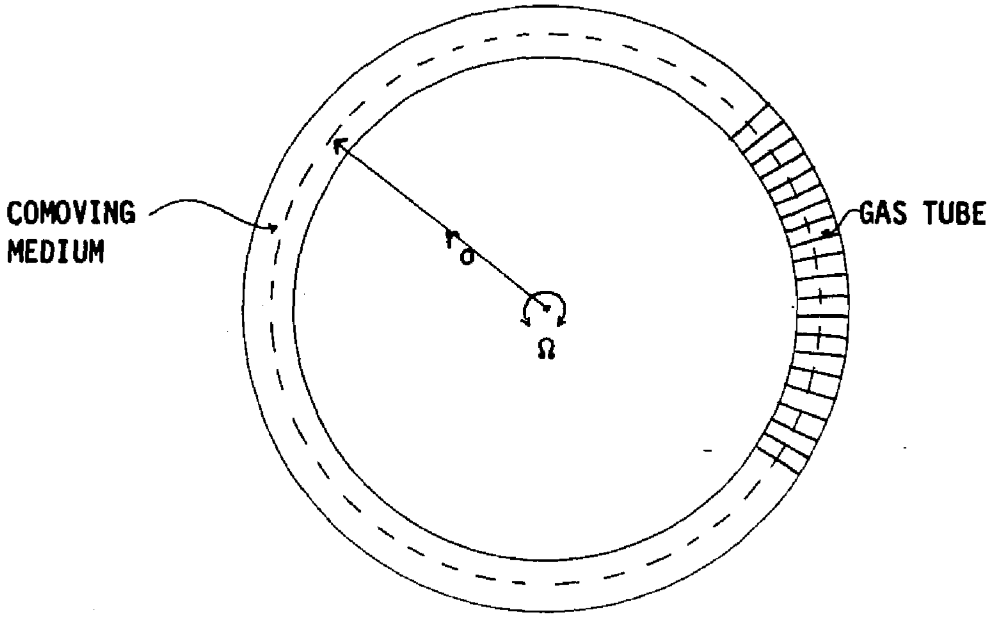

FIGURE 6

Fig. 6. Idealized ring laser configuration.

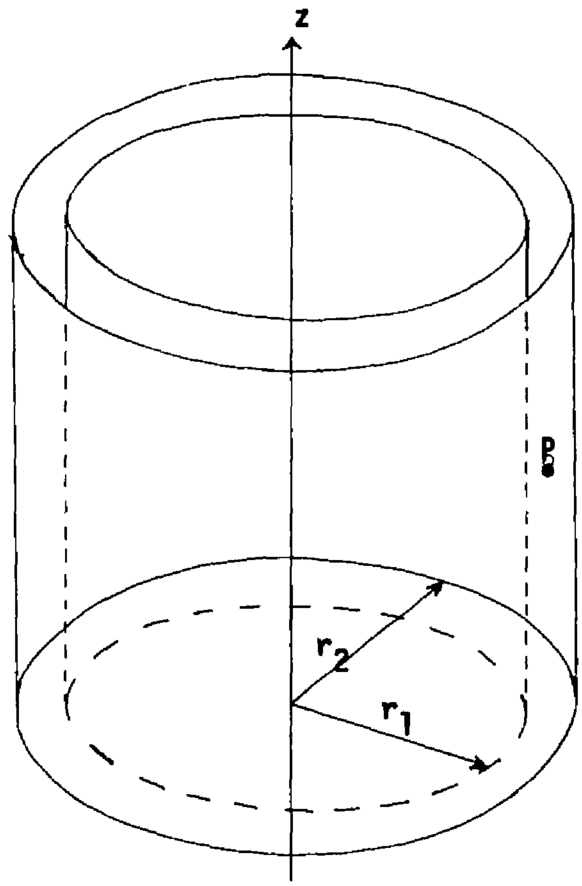

FIGURE 7

Fig. 7. Cylindrical wave duct with wave launcher.

2. Mode Independence of Rotating Wave Guide-Like Structure

In the classical and modern experiments the light beam is guided by the reflection from the mirrors placed at the vertices of the optical path. Such an arrangement has obvious disadvantages associated with the stability, geometry, and the quality of the mirrors. A possible alternative is to replace the mirrors by a wave guide-like structure which can guide the light beam in the desired direction. One such possibility is to use optical fibers which are thin compared to the dimensions of the area enclosed by the beam path. In such an arrangement geometrical optic analysis should still be valid. However, now one has the material medium instead of the vacuum between the mirrors to contend with. We must, therefore, check the dependence of the electromagnetic waves on the medium and verify if in fact the geometrical optics still hold. Since the waves are guided about the z-axis, one can also make the approximation $\frac{\partial}{\partial_z} \to 0$ in the wave equation.

We assume that the model for the optical fiber is a wave guide formed by two concentric cylinders about the z-axis of inner radius r_1 and outer radius r_2. At some point in the cylindrical structure is a device where a wave can be launched and detected after completing one circulation as shown in Fig. 7. Suppose that the wave guide rotates about the z-axis with constant angular velocity Ω. Then the wave equation describing the electromagnetic waves in the circular duct rotating with a comoving material medium having an index of refraction n, may be expressed as

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} - \frac{2\Omega}{c^2}\frac{\partial^2}{\partial t \partial \phi} - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right] A_z = 0 \qquad (40)$$

which differs from the familiar equation for a stationary cylindrical wave guide only through the term $-2\Omega/c^2\ (\partial^2/\partial t\partial\phi)\ A_z$. Again, the non-reciprocal term may be interpreted as describing the drag of the wave motion in the direction of the rotation. Assuming a solution of the wave equation in the form

$$A_z = A_o\, e^{i(\omega t - Z\phi)}\, R(r) \tag{41}$$

one finds for R(r) the Bessel equation

$$\left(\frac{1}{r}\frac{d}{dr}\, r\, \frac{d}{dr} + n^2 \frac{\omega^2}{c^2} + \frac{2\omega\Omega Z}{c^2} - \frac{Z^2}{r^2}\right) R(r) = 0 \tag{42}$$

with the boundary condition for the TEM mode being

$$R(r_1) = R(r_2) = 0 \tag{43}$$

for determining the azimuthal wave number Z governing the circulation of the wave motion in the duct. Defining

$$\alpha^2 = n^2 \frac{\omega^2}{c^2} + \frac{2\omega\Omega Z}{c^2} \tag{44}$$

the wave equation may be rearranged to the form

$$\left[r^2 \frac{d^2}{dr^2} + r\frac{d}{dr} + (r^2 \alpha^2 - Z^2)\right] R(r) = 0 \tag{45}$$

which has the solution

$$R(r) = C_1\, J_z\, (\alpha r) + C_2\, Y_z\, (\alpha r), \tag{46}$$

where $J_z\ (\alpha r)$, and $Y_z\ (\alpha r)$ are the Bessel and the Neumann functions respectively. Applying the boundary conditions, one finds the determining equation for and Z that is allowed to propagate in the wave guide

$$J_z\, (\alpha r_1)\, Y_Z\, (\alpha r_2) - J_z\, (\alpha r_2)\, Y_z\, (\alpha r_1) = 0. \tag{47}$$

The above equation along with the definition for α determines the possible azimuthal wave numbers Z. An asymtotic expansion for large positive zeros, that is, a high mode approximation gives the following quadratic expression for Z

$$Z^2 - 2\, r_1 r_2\, k_o \frac{\Omega}{c} Z + \left(\frac{r_1 r_2}{a} m^2 \pi^2 - r_1 r_2\, k^2 - \frac{1}{4}\right) + 0\left(\frac{1}{m^2}\right) + \cdots = 0$$

where $k_o = \frac{\omega}{c}$ is the free-space wave number, $k = nk_o$ and, $a = r_2 - r_1$. Neglecting terms of order $1/m^2$ and higher, one obtains for Z

$$Z_{1,2} = r_1 r_2\, k_o \frac{\Omega}{c} \pm (r_1 r_2)^{\frac{1}{2}}\, k \left(1 - \frac{m^2 \pi^2}{k^2 a^2} + \frac{r_1 r_2\, \Omega^2}{k^2\, r_1 r_2}\right)^{\frac{1}{2}}.$$

The last two terms under the square root sign may be neglected

$$Z_{1,2} = r_1 r_2\, k_o \frac{\Omega}{c} \pm (r_1 r_2)^{\frac{1}{2}}\, k \sqrt{1 - (m^2 \pi^2)/(k^2 a^2)}. \tag{48}$$

For a non-rotating wave guide, the roots are

$$Z_{o\,1,2} = \pm (r_1 r_2)^{\frac{1}{2}}\, k \sqrt{1 - (m^2 \pi^2)/(k^2 a^2)}. \tag{49}$$

Comparison of $Z_{1,2}$ to $Z_{o\,1,2}$ shows that the correction term $r_1 r_2\, k_o \frac{\Omega}{c}$ is independent of the mode number m. The number of phase reversals gained or lost due to rotation is given by

$$\delta Z = |Z_1 - Z_{o1}| = |Z_2 - Z_{o2}| = r_1 r_2 \frac{\omega \Omega}{c^2} \tag{50}$$

which is again independent of the mode number m.

To get an idea of what happens at low mode numbers let us assume that the dimensions of the wave guide are such that

$$r_2 - r_1 = a \ll r_o = \frac{1}{2}(r_1 + r_2). \tag{51}$$

Then, the radial equation is approximately

$$\left(\frac{d}{dr^2} + \gamma^2\right) R(r) = 0 \tag{52}$$

where,

$$\gamma^2 = n^2 \frac{\omega^2}{c^2} + \frac{2\omega\Omega}{c^2} Z - \frac{Z^2}{r_o^2} . \tag{53}$$

Now we have the solution

$$R(r) = c_1 \sin \gamma r + c_2 \cos \gamma r. \tag{54}$$

Application of the boundary conditions yields

$$\sin \gamma r_2 \cos \gamma r_1 - \sin \gamma r_1 \cos \gamma r_2 = 0, \tag{55}$$

or,

$$\sin \gamma (r_2 - r_1) = \sin \gamma a = 0,$$

where it is necessary that

$$\gamma = \frac{m\pi}{a} \quad \text{for } m = 0,1,2,... \tag{56}$$

Substitution of the expression for γ into the expression for γ^2 gives the following quadratic equation for Z

$$Z^2 - \frac{2\omega\Omega\ r_o^2}{c^2} Z - n^2 \frac{\omega^2}{c^2} r^2 + \frac{m^2\pi^2 r^2}{a^2} = 0$$

Hence, writing $k_o = \frac{\omega}{c}$ for the free-space wave number, and $k = n \frac{\omega}{c}$ for the material medium wave number, one obtains the roots

$$Z_{1,2} = \frac{\omega\ \Omega r_o^2}{c^2} \pm r_o k \sqrt{1 - (\pi^2 m^2)/(a^2k^2) + (\Omega^2 r^2)/(n^2c^2)}, \tag{57}$$

while for an inertial ($\Omega = 0$) wave guide, the solution leads to the roots

$$Z_{o\,1,2} = \pm\ r_o k \sqrt{1 - (\pi^2 m^2)/(a^2k^2)}. \tag{58}$$

A comparison of the equations for $Z_{1,2}$ and $Z_{o1,2}$ shows that the correction terms involving Ω again do not depend on the mode number m. In fact, the velocity ratio $(r_o\Omega/c)^2$ may be neglected as a second order contribution.

The gain or loss of phase reversals due to the rate of rotation is given by δZ

$$\delta Z = |Z_1 - Z_{\delta 1}| = |Z_2 - Z_{\delta 2}| = \frac{\omega r_o^2 \Omega}{c^2} . \tag{59}$$

Again the phase reversal expression due to rotation is independent of the mode number m. The fact that δZ appears to be independent of the mode of propagation in a wave guide structure is a conclusion which is relevant to optical fibers which may be used in interferometry. The result agrees exactly with the geometric optical calculation, where

$$\delta Z = \frac{1}{\lambda_o c} \oint \vec{v}\cdot d\vec{r} = \frac{1}{\lambda_o c} \iint_A \text{curl } \vec{v}\cdot d\vec{S} = \frac{\omega r_o^2 \Omega}{c^2} . \tag{60}$$

3. Dispersion in Rotating Cavities with a Comoving Medium

The beat frequency of a ring laser depends on the optical properties of the comoving medium in the beam path. Therefore, the beat frequency of a ring laser must also depend on the dispersion.[20] The dispersion that affects the beat frequency must further be rotation induced if we consider only the dispersion outside the anamoly range. On the other hand, we have shown that in rotating wave guides the fringe shift δZ is independent of the mode number m and the optical properties of the path traversed by the beam. Therefore, δZ should also be independent of dispersion.

Index of refraction for a rotating system also becomes angular velocity dependent, and is different for clockwise and counterclockwise circulation of the light beam. The constitutive equations

$$\vec{D} = \varepsilon\vec{E} + \varepsilon' \ (\vec{\Omega} \times \vec{r}) \times \vec{B}$$

$$\vec{H} = \frac{1}{\mu}\vec{B} + \varepsilon' \ (\vec{\Omega} \times \vec{r}) \times \vec{E}$$

must be solved along with Maxwell's equations for a plane wave $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$,

$$\vec{D} = \vec{H} \times \frac{\vec{n}}{c} \tag{61a}$$

$$\vec{B} = \frac{\vec{n}}{c} \times \vec{E} \tag{61b}$$

$$\vec{n} = \frac{c}{\omega}\vec{k} \tag{61c}$$

to calculate the index of refraction for a rotating system. Eliminating $\vec{E}$, $\vec{H}$, and $\vec{B}$ from these equations we obtain an equation for the refractive index $n = |\vec{n}|$ of the rotating optical medium

$$[n^2 - 2(\frac{\varepsilon'}{\varepsilon_o}) \ \mu_r \ \vec{n}\cdot \frac{(\vec{\Omega} \times \vec{r})}{c} - \varepsilon_r\mu_r] \ \vec{D} = 0 \tag{62}$$

The solution of this equation gives the refractive index

$$n = n_o - (\frac{\varepsilon'}{\varepsilon_o}) \ \mu_r \frac{\vec{n}_o}{n_o} \cdot \frac{(\vec{\Omega} \times \vec{r})}{c} \tag{63}$$

where $n_o = \sqrt{\varepsilon_r\mu_r}$. The index of refraction is a function of not only the magnitude, but also the direction of $\vec{\Omega}$ and the value of n is different according to whether the light circulates clockwise or counterclockwise, since it is a function of $\vec{n}_o/n_o$.

To properly study the dispersion phenomena, it is necessary to calculate the group and phase velocities of a wave in the rotating cavity. We have derived the wave equation for a rotating laser with optically transparent comoving medium to be

$$\frac{\partial^2\Psi}{\partial s^2} - 2\frac{\Omega R}{c^2}\frac{\partial^2\Psi}{\partial s\partial t} - \frac{n^2}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \tag{64}$$

where $\varepsilon_o\mu_o = 1/c^2$, $n_o = \sqrt{\varepsilon_r}$, $\mu_r = 1$

Assuming an Eikonal of the type $(k's - \omega't)$, one obtains for $|\vec{k}| = k$

$$k' = +k \mp \omega\frac{R\Omega}{c^2}, \tag{65}$$

and,

$$\omega' = \omega \tag{66}$$

where the prime refers to the accelerated frame. These relations tell us that during the time $\Delta\tau$, the frequency of the light beam does not change while its wave number is modified due to the extended or shortened optical path. In the rotating frame, the material is at rest, and ω' and k' are therefore related by $ck' = \omega' n(k')$. It is more appropriate in this case to choose the index of refraction as a function of the wave number k'. Substituting these expressions into $ck' = \omega' n(k')$, and expanding the function $n(k')$ about the central wave number k, one obtains

$$k_\pm = \frac{\omega n}{c} \pm \omega\frac{R\Omega}{c^2}\left[1 - \frac{\omega}{c}\frac{dn}{dk} \pm \frac{1}{2}\left(\frac{\omega}{c}\right)^2\frac{R\Omega}{c}\frac{d^2n}{dk^2}\right] \tag{67}$$

where + and - signs refer to the light beams circulating counterclockwise and clockwise respectively.

The group velocity v_g and the phase velocity v_p are given by

$$v_{g\pm} = \frac{\partial\omega}{\partial k\pm} \tag{68}$$

$$v_{p\pm} = \frac{\omega}{k\pm} \tag{69}$$

$$v_{g\pm} = \frac{c}{n} \frac{1 - \frac{\omega}{c}\frac{dn}{dk\pm} \pm (\frac{\omega}{c})^2 \frac{R\Omega}{c} \frac{d^2n}{dk^2\pm}}{1 \pm \frac{R\Omega}{nc}[1 - \frac{\omega}{c}\frac{dn}{dk\pm} \pm \frac{3}{2} (\frac{\omega}{c})^2 (\frac{R\Omega}{c}) \frac{d^2n}{dk^2\pm}]} \tag{70}$$

$$v_{p\pm} = \frac{c}{n} \frac{1}{1 \pm \frac{\Omega R}{nc}[1 - \frac{\omega}{c}\frac{dn}{dk\pm} \pm \frac{1}{2} (\frac{\omega}{c})^2 (\frac{R\Omega}{c}) \frac{d^2n}{dk^2\pm}]} \quad . \tag{71}$$

Note that in the vacuum ($n = 1$), the rotation of the observer causes him to record both the group and phase velocities to be greater or less than the speed of light. In the vacuum, group velocity equals phase velocity; however, in a medium $n > 1$ phase velocity is greater than the group velocity, and both are less than the speed of light if $\frac{dn}{dk}$ is a positive quantity, and if $\frac{R\Omega}{c} < n(n - 1)$. For Ω and $\frac{dn}{dk}$ small, the conventional relation between the group and phase velocities is recovered

$$v_g = v_p \; (1 - \frac{k}{n}\frac{dn}{dk}). \tag{72}$$

It is instructive to consider the contribution of dispersion in a wave guide first. The independence of δz from the mode number and index of refraction suggests that δz is in general independent of the optical properties of the path. To show that this is in fact true, let us examine the dispersion in the wave guide through the geometric optical analysis. Consider only the waves circulating in the counterclockwise direction, and let $k+ = k$. We start with the general expression previously derived

$$\delta z = \frac{1}{2\pi} \oint \delta\vec{k}\cdot d\vec{r} + \frac{1}{2\pi} \vec{k}\cdot\vec{q}\,\Big|_1^2$$

where the material properties have not yet been introduced. The quantities δk and $\vec{k}\cdot\vec{q}|_1^2$ now have to be calculated by taking into account the dispersion of the comoving (and coaccelerating) optical medium.

For $k = \frac{\omega}{v_p} = \omega \frac{n}{c}$, one finds for δk

$$\delta k = - \frac{\omega n^2}{c^2} \delta v_p$$

where, δv_p is the change in the velocity of propagation due to the drag. The coefficient of drag α may be calculated from consistency requirements. The expression for the fringe shift δz derived in the geometrical optical theory was

$$\delta z = \frac{1}{c\lambda_o} \oint n^2 (1 - \alpha) \vec{v}\cdot d\vec{r}.$$

If $\vec{v}$ is a uniform translational velocity field, then δz must vanish in order to satisfy the well established special theory of relativity. However, δz can not vanish for uniform motion if the integrand is a function of the index of refraction. That is, an integral $\oint f(n) \vec{v}\cdot d\vec{r}$, where f is some function of n, will not in general vanish for arbitrary n if $\vec{v}$ is a uniform translation. Hence, f(n) has to be a constant. f(n) = constant implies[14]

$$n^2 (1 - \alpha) = C$$

where C is a constant. Coefficient of drag, therefore equals

$$\alpha = 1 - \frac{C}{n^2} \qquad (73)$$

which is identical with the Fresnel-Fizeau coefficient of drag for translational motion. Upon setting, C = 1, the expression for δz then takes the form

$$z = \frac{1}{c\lambda_o} \oint \vec{v}\cdot d\vec{r}, \tag{74}$$

which has already been shown to provide the correct result for the fringe shift z in rotating wave guides. The coefficients of drag α when taking into account the dispersion is known to be[14]

$$\alpha = (1 - \frac{1}{n^2} + kn \frac{\partial n}{\partial k}) \tag{75}$$

Hence, $\delta v_p = \alpha\vec{v}$, if $\vec{v}$ is the velocity field describing the motion of the optical loop. The substitution of the drag α expression into the k formula thus gives

$$\delta\vec{k} = -\frac{\omega}{c^2}(n^2 - 1 + kn \frac{\partial n}{\partial k})\vec{v}.$$

For the evaluation of the term $\frac{1}{2\pi}\vec{k}\cdot\vec{q}\,|_1^2$, one may attempt to re-express the integrated term of wave-vector $\vec{k}$ dotted into the small displacement $\vec{q}$ as a loop integral. No generality is lost by assuming that $\vec{q}_1$, the displacement at the beginning of the light circulation, is zero. One may then say that the term $\vec{k}\cdot\vec{q}_2$ comes about gradually by the velocity field $\vec{v}$ so that one may write, if τ is the circulation time and v_g is the group velocity,

$$(\vec{k}\cdot\vec{q}_2 - \vec{k}\cdot\vec{q}_1) = \int_o^\tau \vec{k}\cdot\vec{v}\,dt = \oint \frac{k}{v_g}\vec{v}\cdot d\vec{r}.$$

The vectors $\vec{k}$ and $d\vec{r}$ have, of course, the same direction.

The group velocity is known to be related to the phase velocity v_p according to the formula

$$v_g = v_p\,(1 - \frac{k}{n}\,\frac{\partial n}{\partial k})$$

If the dispersion is small $\frac{k}{n}\frac{\partial n}{\partial k} << 1$ one may write

$$\frac{1}{v_g} = \frac{1}{v_p} (1 + \frac{k}{n} \frac{\partial n}{\partial k}).$$

And one obtains,

$$\vec{k}\cdot\vec{q}_2 - \vec{k}\cdot\vec{q}_1 = \frac{\omega}{c^2} (n^2 + kn \frac{\partial n}{\partial k}) \vec{v}\cdot d\vec{r}.$$

Substitution of the values of $\delta\vec{k}$ and $\vec{k}\cdot\vec{q}|_1^2$, in δz shows that the material contributions of refraction and dispersion both cancel. One ends up again with the expression

$$\delta z = \frac{1}{2\pi} \frac{\omega}{c^2} \oint \vec{v}\cdot d\vec{r}, \tag{76}$$

which does not depend on the physical properties of the refracting material comoving with the optical loop.

This result surprising at first sight can be understood if one considers that the optical path length change is proportional to the circulation time of the light around the loop. An increase of circulation time due to an index of refraction $n > 1$, exposes the circuit proportionally longer to the velocity field $\vec{v}$. However, the actual length change Δs, so increased, is now measured (locally) in terms of wavelength λ that has been correspondingly increased by the presence of an index of refraction $n > 1$.

The same arguments used for the fringe shifts in a wave guide may be used for the numerator of the frequency shift equation for moving ring lasers with comoving mediums. Hence, for $\alpha = 1 - \frac{1}{n^2}$, one

$$|\frac{\delta\omega}{\bar{\omega}}| = \frac{\oint \vec{v}\cdot d\vec{r}}{c^2 \oint \frac{k}{\omega} ds},$$

while for $|\frac{\Delta\omega}{\omega}|$ one writes,

$$\left|\frac{\Delta\omega}{\omega}\right| = \frac{\oint \vec{v}\cdot d\vec{r}}{c^2 \oint \frac{k + ds}{\omega}} + \frac{\oint \vec{v}\cdot d\vec{r}}{c^2 \oint \frac{k - ds}{\omega}}$$

where now considering second order dispersion $\frac{\partial^2 n}{\partial k^2}$

$$\oint \frac{k_+}{\omega} ds = \oint \left[\frac{n}{c} + \frac{R\Omega}{c^2}\left(1 - \frac{\omega}{c}\frac{dn}{dk} - \frac{1}{2}\frac{\omega^2}{c^2}\frac{R\Omega}{c}\frac{\partial^2 n}{\partial k^2}\right)\right] ds$$

and

$$\oint \frac{k_-}{\omega} ds = \oint \left[\frac{n}{c} - \frac{R\Omega}{c^2}\left(1 - \frac{\omega}{C}\frac{dn}{dk} - \frac{1}{2}\frac{\omega^2}{c^2}\frac{R\Omega}{c}\frac{\partial^2 n}{\partial k^2}\right)\right] ds.$$

Hence,

$$\left|\frac{\Delta\omega}{\omega}\right| = \frac{1}{c}\oint \vec{v}\cdot d\vec{r}\left[\frac{2}{\oint nds}\left(1 - \frac{1}{\oint nds}\oint \frac{\omega^2}{c^2}\frac{(R\Omega)^2}{c^2}\frac{\partial^2 n}{\partial k^2} ds\right)\right]. \quad (77)$$

This is a qualitative result, for the derivation of k± depended on a first order theory in $\frac{R\Omega}{c}$. Nevertheless, the independence of the frequency shift from a first order term $\frac{\partial n}{\partial k}$ is clearly established. In the case of the wave guide, where the operational frequencies are much below the anomalous region the term $\frac{\partial^2 n}{\partial k^2}$ need not be considered at all. However, contributions of this second order term near a point of anomalous dispersion may not be quite negligible in the case of the ring lasers.

4. Frequency Locking Phenomena in Ring Lasers

Laser gyroscopes are still in the development stage. The experimental and theoretical studies of the instrument have not yet been exhausted. However, the laser gyroscope already competes successfully with the better models of mechanical gyroscopes. Laser gyroscopes are not only useful as highly sensitive rotation indicators but they have found use as gyro direction finder, sextants, liquid-flow and wind-flow velocity meters.

One of the major problems encountered in the development of gyroscopes is the frequency-locking phenomena.[13,52,9,10] It is caused by the non-linear conductivity of the plasma in the gas discharge tube, that is, the laser element. The atomic transitions which sustain the laser process produce left and right handed circularly polarized laser modes. When the circularly polarized modes have equal frequencies, frequency locking occurs. Since rotation of the system causes a frequency shift, this problem occurs at low angular velocities. At higher angular velocities the frequency locking disappears. Near the frequency locking region there is also observed jump-like changes in the beat frequency.

It can also be shown that a magnetic field may be used to create frequency separation. The magnetic field placed parallel to the laser axis splits up the degenerate atomic levels of the active laser atoms (see Figs. 8 and 9). If the left and right handed circularly polarized laser modes arise from different cavity resonances, then a frequency shift appears which again eliminates locking.

To explain these experimentally observed phenomena, let us consider an idealized circular laser gyroscope symmetric about the z-axis. Let us assume a magnetic field in the z-direction, that is the magnetic field of a current carrying wire along the z-axis. For simplicity, let us assume $J = 1$ to $J = 0$ atomic transitions alone sustain the laser process.

In the presence of a magnetic field there are three transitions with different frequencies and polarizations. The $\Delta m = \pm 1$ transitions are right and left handed circularly polarized in the plane perpendicular to the magnetic field with frequencies ν_- and ν_+, respectively. The

$\Delta m = 0$ transition is linearly polarized parallel to the magnetic field and therefore may not be amplified by the laser process. The circularly polarized modes are polarized in the z - r plane and therefore may propagate along the s-axis and be amplified by the laser.

i. The Inhomogeneous Wave Equation

To take into account the nonlinearity in the conductivity, we shall now consider the inhomogeneous wave equation, that is, we will assume a non zero current to be present in the laser plasma. We will describe the electromagnetic field by the vector potential $\vec{A}$.

The inhomogeneous vector wave equation

$$\partial_\nu \chi^{\lambda\nu\sigma\kappa} \partial_\sigma A_{.\kappa} = j^\lambda \tag{78}$$

is form invariant for nonuniformly moving frames of reference provided is transformed properly. If we consider the case where $j^\lambda \neq 0$, then we must prescribe a consistent transformation procedure for j^λ. In the inertial frame of reference the constitutive equation that relates the current vector to a field variable reads

$$j^i = \sigma^{ij} E_j \tag{79}$$

where $i,j, = 1,2,3$, and σ^{ij} is the conductivity tensor. As shown previously, the conductivity tensor obeys the following transformation of coordinates

$$\sigma^{i'j'} = |\Delta|^{-1} A_i^{i'} A_j^{j'} \sigma^{ij}. \tag{80}$$

Therefore, in Cartesian system, assuming asymmetric diagonal terms for the conductivity tensor, we write $\sigma_x = \sigma''$, $\sigma_y = \sigma^{22}$, $\sigma_z = \sigma^{33}$, which in cylindrical coordinates takes the form $\sigma_r = r\sigma''$, $\sigma_\phi = r^{-1}\sigma^{22}$, $\sigma_z = r\sigma^{33}$. If we assume that σ does not change due to rotation, we may write $j' = \sigma E'$. But, since we measure all field variables on the non-inertial frame of reference, no further transformational procedure is necessary. Again dropping the superscripts, we have for the components of the nonhomogeneous wave equation using the relation $\vec{E} = -\partial_t \vec{A}$,

$\lambda = 1$

$$\varepsilon\, r_o\, \partial_t^2 A_r + 2\Omega\, \varepsilon'\, r_o \partial_t \partial_\phi A_r - \frac{1}{\mu r_o} \partial_\phi^2 A_r = - r_o \sigma\, \partial_t A_r$$

or,

$$\varepsilon\, \mu \partial_t^2 A_r + 2\mu\varepsilon'\Omega\, r_o\, \partial_t \partial_s A_r - \partial_s^2 A_r = -\mu\sigma_r\, \partial_t A_r$$

and for

$\lambda = 3$

$$\varepsilon\mu\, \partial_t^2 A_z + 2\mu\varepsilon'\Omega\, r_o\, \partial_t \partial_s A_z - \partial_s^2 A_z = -\mu\, \sigma^{33}\, \partial_t A_z = -\mu\, \sigma_z\, \partial_t A_z,$$

or in general we have

$$\varepsilon\, \mu\, \partial_t^2 A_i + 2\mu\, \varepsilon'\Omega\, r_o\, \partial_t \partial_s A_i - \partial_s^2 A_i = - \mu\, \sigma_i\, \partial_t A_i,$$

where for rotational convenience, we have redefined $\sigma_r \equiv \sigma^{11}$, $\sigma_z = \sigma^{33}$. The constitutive relations previously derived may be rearranged to give

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

or

$$\vec{P} = \varepsilon_o\, (\varepsilon_r - 1)\vec{E} + \mu\varepsilon'(\vec{\Omega} \times \vec{r}) \times \vec{H}$$

to first order in $\vec{\Omega}$ and further simplified to

$$\vec{P} = \varepsilon_o(\varepsilon_r - 1)\vec{E} + \varepsilon'(\vec{\Omega} \times \vec{r}) \times \vec{B}$$

$$\vec{P} = -\varepsilon_o(\varepsilon_r - 1)\dot{\vec{A}} + \varepsilon'(\vec{\Omega} \times \vec{r}) \times (\vec{\nabla} \times \vec{A}) \qquad (81)$$

And for the case of a system rotating about the z-axis with angular velocity $\vec{\Omega}$, one obtains

$$\vec{P} = -\varepsilon_o(\varepsilon_r - 1)\ \partial_t\vec{A} + \varepsilon' r\ \Omega[r(\frac{1}{r}\ \partial_r r\ A_\phi - \frac{1}{r}\ \partial_\phi A_r) - k(\frac{1}{r}\ \partial_\phi A_z - \partial_z A_\phi)].$$

For the idealized laser gyroscope again ∂_r, $\partial_z \to 0$, thus, we write

$$\vec{P} = -\varepsilon_o(\varepsilon_r - 1)\ \partial_t\vec{A} + \varepsilon' r_o\Omega\ (-\frac{r}{r_o}\ \partial_\phi A_r - k\frac{1}{r_o}\ \partial_\phi A_z). \tag{82}$$

Rearranging terms and taking the time derivative of both sides

$$\varepsilon\ \partial_t^2\ \vec{A} = -\ \partial_t\vec{P} + \varepsilon_o\ \partial_t^2\ \vec{A} - \varepsilon'\ \Omega\ r_o(r\ \partial_s\partial_t A_r + k\ \partial_s\partial_t A_z) \tag{83}$$

Hence, the wave equation may be written as

$$-\mu\ \partial_t P_i + \mu\varepsilon_o\ \partial_t^2\ A_i - \mu\varepsilon'\Omega r_o\ \partial_s\partial_t A_i + 2\mu\varepsilon'\Omega r_o\ \partial_t\partial_s A_i - \partial_s^2\ A_i = -\mu\sigma_i\ \partial_t A_i$$

or,

$$-\partial_s^2\ A_i + \mu\varepsilon_o\ \partial_t^2\ A_i + \mu\varepsilon'\Omega r_o\ \partial_t\partial_s A_i + \mu\sigma_i\ \partial_t A_i = \mu\ \partial_t P_i$$

For the case of the medium and observer corotating, we have shown that

$\varepsilon' = \varepsilon_o$, hence

$$-\partial_s^2\ A_i + \mu\varepsilon_o\ \partial_t^2\ A_i + \mu\varepsilon_o\Omega r_o\ \partial_t\partial_s A_i + \mu\sigma_i\ \partial_t A_i = \mu\ \partial_t P_i.$$

Now, we define an average conductivity $\sigma = \dfrac{\sigma_r + \sigma_z}{2}$ and $\Delta\sigma = \dfrac{\sigma_z - \sigma_r}{2}$.

We have therefore

$$-\partial_s^2\ A_r + \mu\varepsilon_o\ \partial_t^2\ A_r + \mu\varepsilon_o\Omega r_o\ \partial_t\partial_s A_r + \mu\sigma\ \partial_t A_r - \mu\Delta\sigma\ \partial_t A_r = \mu\ \partial_t P_r, \tag{84}$$

and,

$$-\partial_s^2\ A_z + \mu\varepsilon_o\ \partial_t^2\ A_z + \mu\varepsilon_o\Omega r_o\ \partial_t\partial_s A_z + \mu\sigma\ \partial_t A_z + \mu\Delta\sigma\ \partial_t A_z = \mu\ \partial_t P_z \tag{85}$$

From which we can write the equation for $u = A_r + i\ A_z$

$$-\partial_s^2\ u + \mu\varepsilon_o\ \partial_t^2\ u + \mu\varepsilon_o\Omega r_o\ \partial_t\partial_s u + \mu\sigma\ \partial_t\ + \mu\Delta\sigma\ \partial_t u^* = \mu\ \partial_t P \tag{86}$$

where $P = P_r + i\ P_z$.

Having derived the fundamental differential equation to be analyzed we will revert to Gaussian system of units. In the Gaussian system of units and defining $\kappa \equiv 2\pi\sigma$, we have

$$-\partial_s^2 u + \frac{1}{c^2} \partial_t^2 u + \frac{r_o \Omega}{c^2} \partial_t \partial_s u + \frac{2}{c^2} \kappa \partial_t u - \frac{2}{c^2} \Delta\kappa \partial_t u^* = \frac{4\pi}{c} \partial_t P \quad (87)$$

We may now write the vector potential in circularly polarized modes, since only the circularly polarized waves are amplified. For the sake of simplicity, we will assume one left handed and one right handed circularly polarized mode above the threshold, and write the vector potential in the form

$$u = \sqrt{\frac{8\pi \hbar c^2}{V}} \left\{ \frac{B_+(t)}{\sqrt{\Omega_+}} e^{i[\Omega_+ t + \phi_+(t)]} e^{ik_+ s} + \frac{B_-(t)}{\sqrt{\Omega_-}} e^{-i[\Omega_- t + \phi_-(t)]} e^{-ik_- s} \right\} \quad (88)$$

where $\omega\pm$ are the cavity resonances.

$B_+ e^{i(\Omega_+ t + \phi_+)}$, and $B_- e^{-i(\Omega_- t + \phi_-)}$ describe the left handed and the right handed circularly polarized modes with frequencies Ω_+ and Ω_-, respectively. $\Omega_\pm$ are time dependent. $B_\pm$ are real amplitudes and $\phi_\pm$ are the phases. The term $e^{\pm i \Omega_\pm t}$ describes the rapid variation with time and therefore $B_\pm$ and $\phi_\pm$ may be assumed to be slowly varying functions of time.

To treat the source term $\frac{4\pi}{c} \partial_t P$, we may employ the density matrix formalism. The average value of P may be calculated from the electric dipole operator $\mu^+ = e(r + iz)$ by

$$P = \frac{N}{V} \text{Trace } (\rho\mu^+), \quad (89)$$

where ρ is the statistical operator with matrix elements $\rho_{1,2m}$. The subscripts 1 and 2 correspond to the J = 0 and J = 1 atomic energy levels respectively, and m is the azimuthal quantum number. Since the selection rules for the electric dipole operator μ^+ are

$$\mu^+_{1,2m} = \mu^+_{12}\ \delta_{m,1} \tag{90a}$$

$$\mu^+_{2m,1} = \mu^+_{21}\ \delta_{m,-1} \tag{90b}$$

the polarization takes the form

$$P = \frac{N}{V}\,(\rho_{1,2-1}\ \mu^+_{21} + \rho_{21,1}\ \mu^+_{12}). \tag{91}$$

One can now write the differential equation for the density matrix ρ and solve it for the elements $\rho_{1,2-1}$ and $\rho_{21,1}$. We shall postpone this for the next section.

We substitute the expression for the vector potential into the inhomogeneous wave equation and multiply both sides of the equation by $\sin k_\pm s$ and integrate over a symmetric element of volume. Next, we neglect second order terms like $\dot{B}_\pm$, $\dot{\phi}_\pm$ and keep the largest coupling term to arrive at equations for the amplitudes B_+ and B_-.

$$(1 + \frac{r_o\Omega}{2c})\dot{B}_+ + [-i(\omega_+ - \Omega_+ - \dot{\phi}_+ - \frac{r_o\Omega}{2}\ \frac{\omega+}{c}) + \kappa]B_+ - \Delta\kappa\ B_-\ e^{i\Psi}\ \delta_{\omega+,\omega-} =$$

$$-\nu_+ \frac{N}{V}\, e^{-i(\Omega_+ t + \phi_+)}\, i \int g^+_{21\ \omega+}\ \rho_{1,2-1}\ dV \tag{92}$$

$$(1 - \frac{r_o\Omega}{2c})\dot{B}_- + [i(\omega_- - \dot{\phi}_- - \Omega_- + \frac{r_o\Omega\ \omega_-}{2c}) + \kappa]B_- - \Delta\kappa B_+\ e^{i\Psi}\ \delta_{\omega+,\omega-} =$$

$$-\nu_- \frac{N}{V}\, e^{i(\Omega_- t + \phi_-)}\, i \int g^+_{12\omega_-}\ \rho_{21,1}\ dV, \tag{93}$$

where

$$\Psi = (\Omega_- - \Omega_+)t + \phi_- - \phi_+$$

$$g^{+}_{12_\pm} = \sqrt{(2\pi)/(hV\Omega_\pm)}\ \mu^{+}_{12} \sin k_\pm s,\ g^{+}_{21_\pm} = \sqrt{(2\pi)/(hV\Omega_\pm)}\ \mu^{+}_{21} \sin k_\pm s$$

The coupling term $\Delta\kappa\ B_\pm\ e^{i\Psi}$ appear if $\omega_+ \sim \omega_-$, which can occur if the angular velocity is very small and if the left and right handed circularly polarized laser mode arise from the same cavity resonance. If a weak magnetic field is applied, the splitting of the atomic levels may be still too small compared to the frequency difference of adjacent cavity resonances (see Fig. 10). Thus, again, both laser modes may arise from the same cavity resonance, i.e. $\omega_+ \sim \omega_-$, causing coupled cavity oscillations. At higher magnetic fields, laser modes will arise from two different cavity resonances since the atomic frequencies are separated enough to agree approximately with different cavity resonances. Then, $\omega_+ \neq \omega_-$ and no mode coupling occurs. At higher angular velocities again $\omega_+ \neq \omega_-$ and the coupling terms vanish.

ii. Density Matrix Description of Active Atoms

The active atoms which contribute to the laser process are described by the density matrix formalism. The differential equation for the density matrix is

$$\partial_t \rho = \frac{1}{i\hbar} [H,\rho] - \frac{1}{2} \{\Gamma,\rho\} + \lambda \tag{94}$$

where, [] and {} stand for commutation and anti-commutation operations. We take into account the coherent motion $[H,\rho]$, the spontaneous emission $\{\Gamma,\rho\}$ and the pumping process λ, where

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 \\ 0 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & \gamma_2 \end{pmatrix} \qquad \lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_{2-1} & 0 & 0 \\ 0 & 0 & \lambda_{20} & 0 \\ 0 & 0 & 0 & \lambda_{21} \end{pmatrix}$$

$$H = \begin{pmatrix} E_1 & H^{int}_{1,2-1} & H^{int}_{1,20} & H^{int}_{1,21} \\ H^{int}_{2-1,1} & E_2 - n_L & 0 & 0 \\ H^{int}_{20,1} & 0 & E_2 & 0 \\ H^{int}_{21,1} & 0 & 0 & E_2 + n_L \end{pmatrix}$$

The $H^{int}_{2m,2n} = 0$ because of the selection rules for dipole transitions. The atomic damping constants γ for the $J = 1$ level are assumed to be the same. On the other hand, pumping constants of the $J = 1$ are assumed to be different.

We can now write the density matrix differential equation in component form

$$\partial_t \rho_{1,1} = -\gamma_1 \rho_{1,1} + \lambda_1 + \frac{1}{i\hbar} \sum_{m=-1}^{1} \left(H^{int}_{1,2m} \rho_{2m,1} - \rho_{1,2m} H^{int}_{2m,1}\right) \quad (95a)$$

$$\partial_t \rho_{1,2m} = -\gamma_{12} \rho_{1,2m} + \frac{1}{i\hbar} \sum_{\ell=-1}^{1} H^{int}_{1,2\ell} \rho_{2\ell,2m} - \frac{1}{i\hbar} \rho_{1,1} H^{int}_{1,2m} \quad (95b)$$

$$\partial_t \rho_{2\ell,2m} = -\gamma_2 \rho_{2\ell,2m} + \lambda_{2m} \delta_{\ell,m} + \frac{1}{i\hbar} H^{int}_{2\ell,1} \rho_{1,2m} - \frac{1}{i\hbar} \rho_{2\ell,1} H^{int}_{1,2m} \quad (95c)$$

where $\gamma_{12} = \frac{1}{2}(\gamma_1 + \gamma_2)$. To calculate the H^{int} terms which describe the interaction of an electron with the electromagnetic field, consider

$$H^{int} = -\frac{e}{mc}\vec{p}\cdot\vec{A}$$

$$H^{int} = -\frac{e}{mc}(p_r A_r + p_z A_z).$$

We now define,

$p^{\pm} = p_r \pm i\, p_z$, and recall that $u = A_r + i\, A_z$. Then,

$$p_r = \frac{1}{2}(p^+ + p^-),\ p_z = \frac{1}{2i}(p^+ - p^-),\ A_r = \frac{1}{2}(u + u^*),\ A_z = \frac{1}{2i}(u - u^*)$$

Substituting these into the H^{int} expression, we obtain

$$H^{int} = -\frac{e}{2mc}(p^- u + p^+ u^*),$$

which for matrix elements of H^{int} takes the form

$$H^{int}_{a,b} = -\frac{e}{2mc}(p^-_{ab}\, u + p^+_{ab}\, u^*).$$

If we write the elements of the momentum operators $p^{\pm}$ as

$$p^{\pm}_{ab} = m\frac{dr^{\pm}_{ab}}{dt},$$

where $r^{\pm}_{ab} = r^{\pm}_{o}\, e^{\nu}ab$ and $h\nu_{ab} = E_a - E_b$, momentum operators take the form

$$p^{\pm}_{ab} = im\nu_{ab}\, r^{\pm}_{ab}.$$

Defining the electric dipole operator $\mu^{\pm}_{ab} = r^{\pm}_{ab}\, e$, we obtain for the momentum operator,

$$p^{\pm}_{ab} = i\nu_{ab}\frac{m}{\ell}\mu^{\pm}_{ab},$$

hence,

$$H^{int}_{a,b} = -\frac{i\nu_{ab}}{2c}(\mu^-_{ab}\, u + \mu^+_{ab}\, u^*). \qquad (96)$$

The coupled differential equations for the density matrix elements may now be solved and the approximate values of $\rho_{1,2-1}$ and $\rho_{21,1}$

may be inserted to the source terms of the equations for the amplitudes $B_{\pm}$. The solutions for the density elements are given in the appendix.

iii. Laser Frequencies, Amplitudes and Phases

Splitting the amplitude equations into real and imaginary parts, we find equations for the amplitudes and frequencies.

$$B_{\pm} = \frac{1}{1 \pm \frac{r_o \Omega}{2c}} [\alpha_{\pm} B_{\pm} - \beta_{\pm} B_{\pm}^3 - \Theta_{\pm} B_{\pm} B_{\pm}^2 + \Delta\kappa B_{\pm} \cos \Psi \delta\omega_{+}\omega_{-}] \tag{97}$$

$$\Omega_{\pm} + \phi_{\pm} = \omega_{\pm} (1_{\pm} \frac{r_o \Omega}{2c}) \pm \Delta\kappa \frac{B_{-}}{B_{+}} \sin \Psi \delta\omega_{+}, \omega_{-} + \sigma_{\pm} - \rho_{\pm} B_{\pm}^2 - \tau_{\pm} B^2, \tag{98}$$

where values for the constants $\alpha_{\pm}$, $\beta_{\pm}$, $\Theta_{\pm}$, $\sigma_{\pm}$, $\rho_{\pm}$ &$\tau_{\pm}$ are derived in the appendix. We observe that the laser frequencies and the rate of change of amplitude is modulated due to macroscopic rotation of the laser and the atomic contributions from the multimode operation due to the presence of the weak magnetic field.

The frequency equation can further be split into time dependent and time independent parts. We assumed that the frequencies $_{\pm}$ of the circularly polarized modes were time dependent while the phases $_{\pm}$ were functions of time. If we ignore the small temporal modulation of $B_{\pm}^2$, we can write

$$\Omega_{\pm} = \omega_{\pm}(1 \pm \frac{r_o \Omega}{2c}) + \sigma_{\pm} - (\rho_{\pm} B_{\pm}^2 + \tau_{\pm} B_{\pm}^2) \tag{99}$$

$$\phi_{\pm} = \pm\Delta\kappa \frac{B_{-}}{B_{+}} \sin \Psi \delta\omega_{+},\omega_{-}. \qquad (100$$

The frequencies of the laser modes are given in general by $\Omega_{\pm} + \phi_{\pm}$; however, when $\Delta\kappa = 0$, and $\omega_{+} \neq \omega_{-}$ the frequencies are given by only the time independent parts $\Omega_{\pm}$. Frequencies become time dependent only if the laser is in a weak magnetic field and its cavity has symmetric conductivity and its angular velocity is very small. In higher magnetic fields, the modes may arise from different cavity resonances and $\phi_\pm$ becomes zero. At higher angular velocities again $\omega_+ \neq \omega_-$ and the $\phi_\pm$ vanish. In both cases, frequencies become time independent. $\Omega_\pm$ becomes approximately equal to $\omega_\pm$. The deviation from $\omega_\pm$ comes about due to the angular rotation factor $(1 + \frac{r_o\Omega}{2c})$, the pulling terms $\sigma_\pm$ and pushing terms $-(\rho_\pm B_\pm^2 + \tau_\pm B_\mp^2)$.

iv. Beat Frequencies and Frequency Locking

When the frequencies $\Omega\pm + \phi_\pm$ are functions of time, the electromagnetic waves are frequency modulated and the modulation frequency is given by the beat frequency of the right and left handed circularly polarized modes. The differential equation for the beat frequency may be expressed as

$$\Psi = \Omega_- - \Omega_+ + \phi_- - \phi_+$$

$$\Psi = \Omega_- - \Omega_+ - a \sin \Psi \delta_{\omega+,\omega-} \tag{101}$$

where $a = -\Delta\kappa(\frac{B_-}{B_+} + \frac{B_+}{B_-})$.

We observe that when $\omega_+ = \omega_-$, the terms $a \sin \Psi$ describes the time variation of the beat frequency aroung $\Omega_- - \Omega_+$.

We shall now consider the solutions of the differential equation for $\Psi(t)$ in a weak magnetic field when the laser is rotating at a low angular velocity. The temporal behaviour of Ψ depends on whether $|a| < |\Omega_- - \Omega_+|$ or $|a| > |\Omega_- - \Omega_+|$. Hence, we shall seek the solutions of

$$\Psi = \Omega_- - \Omega_+ - a \sin \Psi$$

for the two mentioned cases.

Case 1: a $\quad _- \, ^- \, _+$

$$\int \frac{d\Psi}{\Omega_- - \Omega_+ - a \sin \Psi} = \int dt$$

$$\Psi = 2 \tan^{-1} \{\frac{1}{\Omega_- - \Omega_+} [a + \sqrt{(\Omega_- - \Omega_+)^2 - a^2} \tan \sqrt{(\Omega_- - \Omega_+)^2} (\frac{t}{2})]\}$$

To find the beat frequency, we differentiate the expression for

$$\Psi = \frac{(\Omega_- - \Omega_+)^2 - a^2}{(\Omega_- - \Omega_+) + a \sin [\sqrt{(\Omega_- - \Omega_+)^2 - a^2}\, t + \sin^{-1} (\frac{a}{\Omega_- - \Omega_+})]} \qquad (102)$$

We observe that $\Psi_{min} = (\Omega_- - \Omega_+) - a$, and $\Psi_{max} = (\Omega_- - \Omega_+) + a$, that is the beat frequency varies between $|\Omega_- - \Omega_+| \pm |a|$ as a function of time (see Fig. 11). However, since $|\Omega_- - \Omega_+| > |a| > 0$, $\Psi > 0$ and no frequency locking occurs. The expression $|\Omega_- - \Omega_+| > |a|$ may be considered the frequency locking criterion. In zero magnetic field disregarding the atomic contributions but considering only the rotation $\Omega_- - \Omega_+ = \Delta\Omega$, where $\Delta\omega$ is the frequency shift expression already calculated in article 1 of this section. Thus for a single mode operation, the frequency locking criterion may be expressed

$$|\omega_o \frac{2r_o\Omega}{nc}| > |a| \qquad (103)$$

where $\omega_o = \omega_+ = \omega_-$. The equation determines the magnitude of the angular velocity Ω necessary to avoid frequency locking

Case 2: $|a| > |\Omega_- - \Omega_+|$

Integrating the differential equation for Ψ, we obtain

$$t = \frac{1}{\sqrt{a^2 - (\Omega_- - \Omega_+)^2}} \ln \frac{(\Omega_- - \Omega_+) \tan\frac{1}{2}\Psi - a - \sqrt{a^2 - (\Omega_- - \Omega_+)^2}}{(\Omega_- - \Omega_+) \tan\frac{1}{2}\Psi - a + \sqrt{a^2 - (\Omega_- - \Omega_+)^2}} \quad (104)$$

This expression can not be solved for explicitly. However, we note that t has two singularities, the first of which occurs when

$$\Psi = \Psi_o = 2 \tan^{-1} \left[\frac{a - \sqrt{a^2 - (\Omega_- - \Omega_+)^2}}{\Omega_- - \Omega_+}\right]. \quad (105)$$

In other words, as $t \to \infty$, Ψ takes the constant value Ψ_o. The time derivate of Ψ vanishes, resulting in a vanishing beat frequency, this is called frequency locking. In turn, no information about the rotation of the system can be detected. For a single mode operation when $|2\omega_o r_o \Omega/nc| < |a|$ frequency locking occurs. When $|2\omega_o r_o \Omega/nc| < |a|$ the frequency locking may be removed by turning on the magnetic field. As the magnetic field is increased, the atomic frequencies will excite different cavity frequencies thus causing a jump from the locking region. Similarly a jump from the locking region may occur as the angular velocity Ω is increased. The width of the locking region is proportional to the asymmetric damping $\Delta\kappa$.

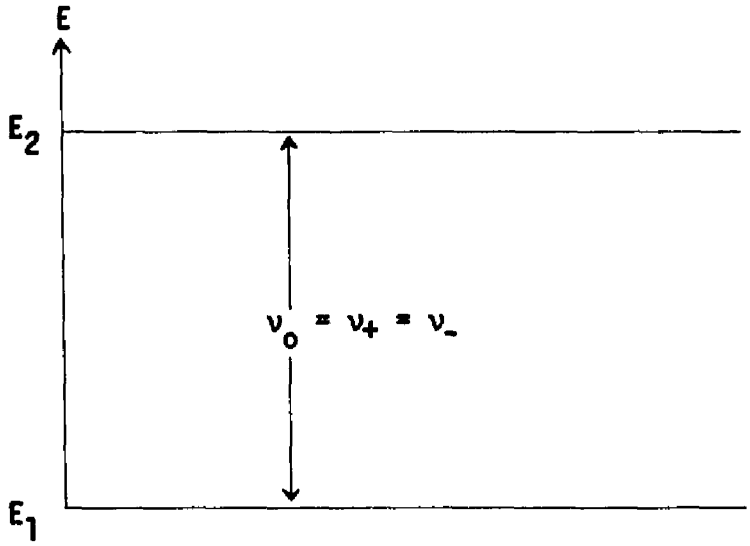

FIGURE 8

Fig. 8. Atomic transition from a degenerate energy level E_2.

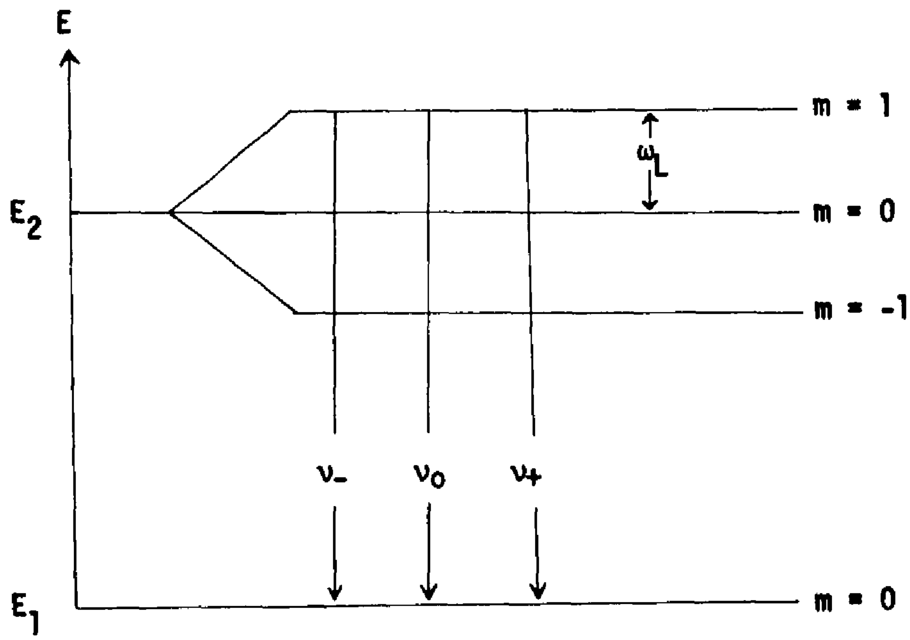

FIGURE 9

Fig. 9. Atomic transitions under a weak magnetic field.

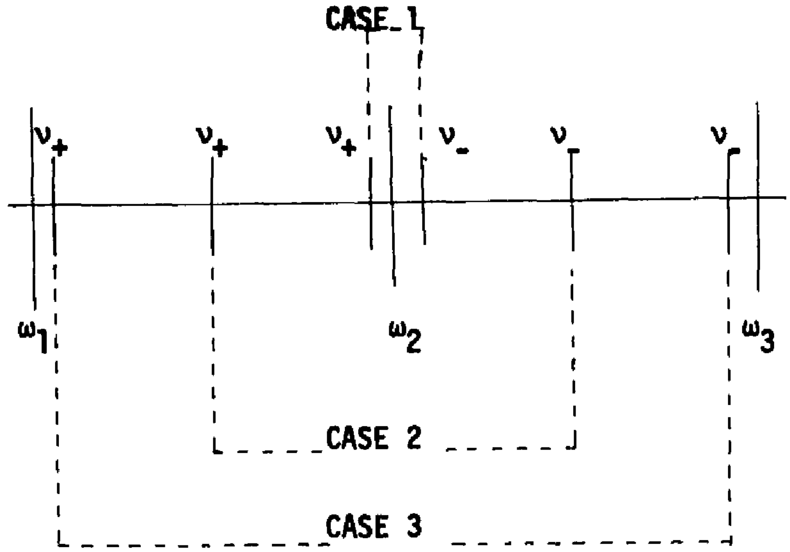

FIGURE 10

Fig. 10. Comparison of the atomic frequencies and resonator frequencies for different magnetic fields.

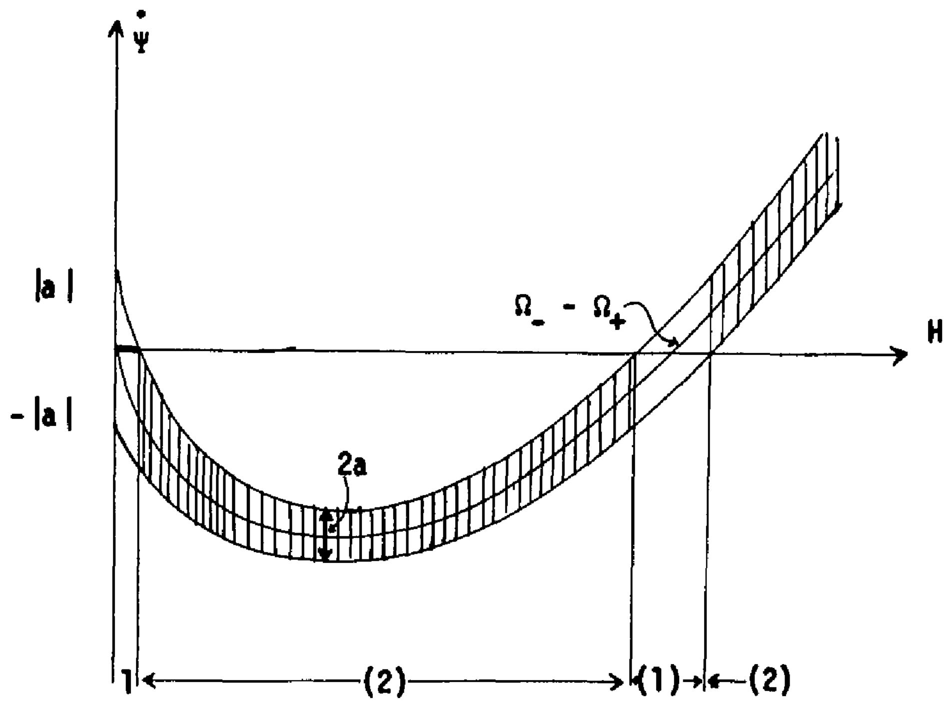

FIGURE 11

Fig. 11. The beat frequency as a function of the magnetic field (1)... locking regions, (2)... regions with frequency modulated modes.

VII. RESULTS AND DISCUSSION

We have reviewed the classical and modern contributions to the electromagnetic theory in non-inertial systems. This author has found that the method of natural invariance as formulated by Post, Yildiz, and Tang best suits the study of the electromagnetic cavity resonances in rotating systems. Using the constitutive equations developed, we have been able to explain Sommerfeld's paradox and the Pegram and Kennard effects. We have also been able to supply the method of natural invariance to rotating optical systems and rotating self oscillating devices.

The disagreement between different constitutive equations developed by different authors can only be resolved through Pegram and Kennard type of experiments. These experiments must include systems with corotating dielectrics with $\epsilon_r \neq 1$ and $\mu_r \neq 1$ to resolve the discrepancies between competing constitutive equations.

The results we have obtained for the frequency locking phenomenon qualitatively agree with the known experiments. Furthermore, we have established limits on the frequency locking which may be verified experimentally. Further studies to the nature of the laser itself is necessary for a complete understanding of this phenomenon.

We have shown that wave propagation in rotating optical fibers is independent of mode and the properties of the optical path. The use of optical fibers may open new areas of experimental investigation in the study of ring lasers.

The dispersion phenomenon in rotating systems has been shown to be independent of first order dispersion. This result disagrees with the results of Khromykh (1966). However, our results are consistent with the special theory of relativity.

A natural extension and further work that should be carried out in the application of the PYT formulation is the scattering and radiation from rotating objects. This work would be especially useful to further our understanding of the radiation from rotating stars.

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APPENDIX A

Transformation of the Constitutive Tensor and Derivation of the Wave Equation

The electromagnetic field in a rotating cavity in the vacuum will be studied by using covariant notation. For such a cavity, the generalized D'Alembertian reads

$$\partial_{\nu} \chi^{\lambda\nu\sigma\kappa} \partial_{\sigma} A_{\kappa} = 0.$$

This general expression is equivalent to writing Maxwell's equations for free or matter spaces, where $\chi^{\lambda\nu\sigma\kappa}$ and A_{κ} represent the constitutive tensor and the four-vector potential respectively.

This system of equations will be subjected to a rotational transformation which is the Galilean transformation in this case. The coordinate relation may be represented by the following table:

	t	r	ϕ	z
t'	1	0	0	0
r'	0	1	0	0
ϕ'	Ω	0	1	0
z'	0	0	0	1

TABLE 1

where $\phi' = \Omega + \Omega t$ is assumed. Now identifying t, r, ϕ, z by 0, 1, 2, 3 respectively, we can write the transformation Jacobian in the following manner:

$$\begin{bmatrix} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial r} & \frac{\partial t'}{\partial \phi} & \frac{\partial t'}{\partial z} \\ \frac{\partial r'}{\partial t} & \frac{\partial r'}{\partial r} & \frac{\partial r'}{\partial \phi} & \frac{\partial r'}{\partial z} \\ \frac{\partial \phi'}{\partial t} & \frac{\partial \phi'}{\partial r} & \frac{\partial \phi'}{\partial \phi} & \frac{\partial \phi'}{\partial z} \\ \frac{\partial z'}{\partial t} & \frac{\partial z'}{\partial r} & \frac{\partial z'}{\partial \phi} & \frac{\partial z'}{\partial z} \end{bmatrix} = \begin{bmatrix} A_o^{o'} & A_1^{o'} & A_2^{o'} & A_3^{o'} \\ A_o^{1'} & A_1^{1'} & A_2^{1'} & A_3^{1'} \\ A_o^{2'} & A_1^{2'} & A_2^{2'} & A_3^{2'} \\ A_o^{3'} & A_1^{3'} & A_2^{3'} & A_3^{3'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \Omega & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad A_\lambda^{\lambda'} = 1$$

And the inverse matrix as,

$$\begin{bmatrix} \frac{\partial t}{\partial t'} & \frac{\partial t}{\partial r'} & \frac{\partial t}{\partial \phi'} & \frac{\partial t}{\partial z'} \\ \frac{\partial r}{\partial t'} & \frac{\partial r}{\partial r'} & \frac{\partial r}{\partial \phi'} & \frac{\partial r}{\partial z'} \\ \frac{\partial \phi}{\partial t'} & \frac{\partial \phi}{\partial r'} & \frac{\partial \phi}{\partial \phi'} & \frac{\partial \phi}{\partial z'} \\ \frac{\partial z}{\partial t'} & \frac{\partial z}{\partial r'} & \frac{\partial z}{\partial \phi'} & \frac{\partial z}{\partial z'} \end{bmatrix} = \begin{bmatrix} A_{o'}^{o} & A_{1'}^{o} & A_{2'}^{o} & A_{3'}^{o} \\ A_{o'}^{1} & A_{1'}^{1} & A_{2'}^{1} & A_{3'}^{1} \\ A_{o'}^{2} & A_{1'}^{2} & A_{2'}^{2} & A_{3'}^{2} \\ A_{o'}^{3} & A_{1'}^{3} & A_{2'}^{3} & A_{3'}^{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\Omega & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad A_{\lambda'}^{\lambda} = 1.$$

To demonstrate the procedure of transforming the constitutive tensor, we shall consider only the vacuum part of $\chi^{\lambda\nu\sigma\kappa}$. For the rest system in the vacuum, the matrix form for the elements of the constitutive tensor is as shown in the following table:

The permittivity $\varepsilon_{\ell\kappa}$, and inverse permeability $\chi_{\ell\kappa}$ are diagonal and in general would contain free-space plus the medium properties. However, since $= \chi_{(o)} + \chi_{(m)}$, the procedure is the same for $\chi_{(m)}$, as for $\chi_{(o)}$.

The rotation about the z-axis will be considered for the case where free-space is observed from the rotating cavity. Now according to the transformation previously described, we write

$$\chi^{\lambda'\nu'\sigma'\kappa'} = \Delta^{-1} A^{\lambda'\nu'\sigma'\kappa'} \chi^{\lambda\nu\sigma\kappa}_{(o)} = \Delta^{-1} A^{\lambda'}_{\lambda} A^{\nu'}_{\nu} A^{\sigma'}_{\sigma} A^{\kappa'}_{\kappa} \chi_{(o)},$$

where $\Delta^{-1} = +1$.

$\chi^{\lambda\nu\sigma\kappa}$	01 $-E_1$	02 $-E_2$	03 $-E_3$	23 B_1	31 B_2	12 B_3
01 D_1	$-r\varepsilon_o$	0	0	0	0	0
02 D_2	0	$\frac{\varepsilon_o}{r}$	0	0	0	0
03 D_3	0	0	$-r\varepsilon_o$	0	0	0
23 H_1	0	0	0	$\frac{1}{r\mu_o}$	0	0
31 H_2	0	0	0	0	$\frac{r}{\mu_o}$	0
12 H_3	0	0	0	0	0	$\frac{1}{r\mu_o}$

TABLE 2

I. The first submatrix (clockwise from the top left)

1. $\chi^{o'1'o'1'} = A^{o'}_{o} A^{1'}_{1} A^{o'}_{o} A^{1'}_{1} \chi^{o1o1} = - r\varepsilon_o$

2. $\chi^{o'1'o'2'} = 0$

3. $\chi^{o'1'o'3'} = 0$

4. $\chi^{o'2'o'1'} = 0$

5. $\chi^{o'2'o'2'} = A^{o'2'o'2'}_{o\,2\,o\,2}\ \chi^{0202} = -\dfrac{\varepsilon_o}{r}$

6. $\chi^{o'2'o'3'} = 0$

7. $\chi^{o'3'o'1'} = 0$

8. $\chi^{o'3'o'2'} = 0$

9. $\chi^{o'3'o'3'} = A^{o'3'o'3'}_{o\,3\,o\,3}\ {}^{o3o3} = -\ \varepsilon_o r$

II. The second submatrix

1. $\chi^{o'1'2'3'} = 0$

2. $\chi^{o'1'3'1'} = 0$

3. $\chi^{o'1'1'2'} = -\ A^{o'1'1'2'}_{o\,1\,1\,o}\ \chi^{o1o1} = \Omega\ r\varepsilon_o$

4. $\chi^{o'2'2'3'} = 0$

5. $\chi^{o'2'3'1'} = 0$

6. $\chi^{o'2'1'2'} = 0$

7. $\chi^{o'3'2'3'} = A^{o'3'2'3'}_{o\,3\,o\,3}\ \chi^{o3o3} = -\ \Omega\ r\varepsilon_o$

8. $\chi^{o'3'3'1'} = 0$

9. $\chi^{o'3'1'2'} = 0$

III. The third submatrix

1. $\chi^{2'3'o'1'} = 0$

2. $\chi^{2'3'o'2'} = 0$

3. $\chi^{2'3'o'3'} = A^{2'3'o'3'}_{o\,3\,o\,3}\ {}^{o3o3} = -\ \Omega\ r\varepsilon_o$

4. $\chi^{3'1'o'1'} = 0$

5. $\chi^{3'1'o'2'} = 0$

6. $\chi^{3'1'o'3'} = 0$

7. $\chi^{1'2'o'1'} = - A^{1'2'o'1'}_{1\ o\ o\ 1} \ \chi^{o1o1} = \Omega \ r\varepsilon_o$

8. $\chi^{1'2'o'2'} = 0$

9. $\chi^{1'2'o'3'} = 0$

IV. The fourth submatrix

1. $\chi^{2'3'2'3'} = A^{2'3'2'3'}_{2323} \ \chi^{2323} + A^{2'3'2'3'}_{o3o3} \ \chi^{o3o3} = \frac{1}{r\mu_o} - \Omega^2 \ r\varepsilon_o$

2. $\chi^{2'3'3'1'} = 0$

3. $\chi^{2'3'1'2'} = 0$

4. $\chi^{3'1'2'3'} = 0$

5. $\chi^{3'1'3'1'} = A^{3'1'3'1'}_{3131} \ \chi^{3131} = \frac{r}{\mu_o}$

6. $\chi^{3'1'1'2'} = 0$

7. $\chi^{1'2'2'3'} = 0$

8. $\chi^{1'2'3'1'} = 0$

9. $\chi^{1'2'1'2'} = A^{1'2'1'2'}_{1212} \ \chi^{1212} + A^{1'2'1'2'}_{1o1o} \ \chi^{1o1o} = \frac{1}{r\mu_o} - \Omega^2 \ r\varepsilon_o$

The constitutive tensor in the rotational frame becomes

$\chi^{\lambda'\nu'\sigma'\kappa'}$	o'1' $-E_1$	o'2' $-E_2$	o'3' $-E_3$	2'3' B_1	3'1' B_2	1'2' B_3
0'1' D_1	$-\varepsilon_o r$	0	0	0	0	$\Omega\, r\varepsilon_o$
0'2' D_2	0	$-\frac{\varepsilon_o}{r}$	0	0	0	0
0'3' D_3	0	0	$-r\varepsilon_o$	$-\Omega\, r\varepsilon_o$	0	0
2'3' H_1	0	0	$-\Omega r\varepsilon_o$	$\frac{1}{\mu_o r} - \Omega^2 r\varepsilon_o$	0	0
3'1' H_2	0	0	0	0	$\frac{r}{\mu_o}$	0
1'2' H_3	r	0	0	0	0	$\frac{1}{\mu_o r} - \Omega^2 r\varepsilon_o$

TABLE 3

The constitutive equations for this case can be written in the vector form for first order in Ω as

$$\vec{D} = \varepsilon_o \vec{E} + \varepsilon_o(\vec{\Omega} \times \vec{r}) \times \vec{B}$$

and,

$$\vec{H} = \frac{1}{\mu_o} \vec{B} + \varepsilon_o(\vec{\Omega} \times \vec{r}) \times \vec{E}.$$

Based on the above table, $\partial_{\nu'}\, \chi^{\lambda'\nu'\sigma'\kappa'}\, \partial_{\sigma'}\, A_{\kappa'} = 0$ can be developed for $\lambda' = 0, 1, 2, 3$. Omitting primes for convenience, we obtain for $\lambda = 0, 1, 2, 3$ the following equations:

I. $\lambda = 0$

$$\partial_\nu\, \chi^{o\nu\sigma\kappa}\, \partial_\sigma\, A_\kappa = [\partial_1(\chi^{o1o1}\, \partial_o\, A_1) + \partial_1(\chi^{o11o}\, \partial_1\, A_o) + (\chi^{o112}\, \partial_1\, A_2) +$$

$$\partial_1(\chi^{o121}\, \partial_2\, A_1)] + [\partial_2(\chi^{o2o2}\, \partial_o\, A_2) + \partial_2(\chi^{o22o}\, \partial_2\, A_o)] +$$

$$[\partial_3(\chi^{o3o3}\, \partial_o\, A_3) + \partial_3(\chi^{o323}\, \partial_2\, A_3) + \partial_3(\chi^{o33o}\, \partial_3\, A_o) +$$

$$\partial_3(\chi^{o332}\ \partial_3\ A_2)]$$

II. $\lambda = 1$

$$\partial_\nu\ \chi^{1\nu\sigma\kappa}\ \partial_\sigma\ A_\kappa = [\partial_o(\chi^{1o1o}\ \partial_1\ A_o) + \partial_o(\chi^{1oo1}\ \partial_o\ A_1) + \partial_o(\chi^{1o12}\ \partial_1\ A_2) +$$

$$\partial_o(\chi^{1o21}\ \partial_2\ A_1)] + [\partial_2(\chi^{12o1}\ \partial_o\ A_1) + \partial_2(\chi^{121o}\ \partial_1\ A_o) +$$

$$\partial_2(\chi^{1212}\ \partial_1\ A_2) + \partial_2(\chi^{1221}\ \partial_2\ A_1)] + [\partial_3(\chi^{1313}\ \partial_1\ A_3) +$$

$$\partial_3(\chi^{1331}\ \partial_3\ A_1)]$$

III. $\lambda = 2$

$$\partial_\nu\ \chi^{2\nu\sigma\kappa}\ \partial_\sigma\ A_\kappa = [\partial_o(\chi^{2o2o}\ \partial_2\ A_o) + \partial_o(\chi^{2oo2}\ \partial_o\ A_2)] + [\partial_1(\chi^{21o1}\ \partial_o\ A_1) +$$

$$\partial_1(\chi^{211o}\ \partial_1\ A_o) + \partial_1(\chi^{2112}\ \partial_1\ A_2) + \partial_1(\chi^{2121}\ \partial_2\ A_1)] +$$

$$[\partial_3(\chi^{23o3}\ \partial_o\ A_3) + \partial_3(\chi^{233o}\ \partial_3\ A_o) + \partial_3(\chi^{2323}\ \partial_2\ A_3) +$$

$$\partial_3(\chi^{2332}\ \partial_3\ A_2)]$$

IV. $\lambda = 3$

$$\partial_\nu\ \chi^{3\nu\sigma\kappa}\ \partial_\sigma\ A_\kappa = [\partial_o(\chi^{3oo3}\ \partial_o\ A_3) + \partial_o(\chi^{3o3o}\ \partial_3\ A_o) + \partial_o(\chi^{3o23}\ \partial_2\ A_3) +$$

$$\partial_o(\chi^{3o32}\ \partial_3\ A_2)] + [\partial_1(\chi^{3131}\ \partial_3\ A_1) + \partial_1(\chi^{3113}\ \partial_1\ A_3)] +$$

$$[\partial_2(\chi^{32o3}\ \partial_o\ A_3) + \partial_2[\chi^{323o}\ \partial_3\ A_o) + \partial_2(\chi^{3223}\ \partial_2\ A_3) +$$

$$\partial_2(\chi^{3232}\ \partial_3\ A_2)].$$

Now using the symmetry properties of χ

I. $\lambda = 0$

$$\partial_\nu\ \chi^{o\nu\sigma\kappa}\ \partial_\sigma\ A_\kappa = \partial_1\chi^{o1o1}(\partial_o\ A_1 - \partial_1\ A_o) + \partial_1\chi^{o112}(\partial_1\ A_2 - \partial_2\ A_1) +$$

$$\partial_1\chi^{o2o2}(\partial_o\ A_2 - \partial_2\ A_o) + \partial_3\chi^{o3o3}(\partial_o\ A_3 - \partial_3\ A_o) +$$

$$\partial_3\chi^{o323}(\partial_2\ A_3 - \partial_3\ A_2)$$

II. $\lambda = 1$

$$\partial_\nu\ \chi^{1\nu\sigma\kappa}\ \partial_\sigma\ A_\kappa = \partial_o\chi^{1o1o}(\partial_1\ A_o - \partial_o\ A_1) + \partial_o\chi^{1o12}(\partial_1\ A_2 - \partial_2\ A_1) +$$

$$\partial_2\chi^{12o1}(\partial_o\ A_1 - \partial_1\ A_o) + \partial_2\chi^{1212}(\partial_1\ A_2 - \partial_2\ A_1) +$$

$$\partial_3\chi^{1313}(\partial_1\ A_3 - \partial_3\ A_1)$$

III. $\lambda = 2$

$$\partial_\nu\ \chi^{2\nu\sigma\kappa}\ \partial_\sigma\ A_\kappa = \partial_o\chi^{2o2o}(\partial_2\ A_o - \partial_o\ A_2) + \partial_1\chi^{21o1}(\partial_o\ A_1 - \partial_1\ A_o) +$$

$$\partial_1\chi^{2112}(\partial_1\ A_2 - \partial_2\ A_1) + \partial_3\chi^{23o3}(\partial_o\ A_3 - \partial_3\ A_o) +$$

$$\partial_3\chi^{2323}(\partial_2\ A_3 - \partial_3\ A_2).$$

IV. $\lambda = 3$

$$\partial_\nu \chi^{3\nu\sigma\kappa} \partial_\sigma A_\kappa = \partial_o\chi^{3oo3}(\partial_o A_3 - \partial_3 A_o) + \partial_o\chi^{3o23}(\partial_2 A_3 - \partial_3 A_2) +$$

$$\partial_1\chi^{3131}(\partial_3 A_1 - \partial_1 A_3) + \partial_2\chi^{32o3}(\partial_o A_3 - \partial_3 A_o) +$$

$$\partial_2\chi^{3223}(\partial_2 A_3 - \partial_3 A_2).$$

Now let $(o,1,2,3,) \to (t,r,\phi,z)$ and substitute the calculated values for the elements of χ

I. $\lambda = 0$

$$\partial_r(-r\varepsilon_o)(\partial_t A_r - \partial_r A_t) + \partial_r(\Omega r\varepsilon_o)(\partial_r A_\phi - \partial_\phi A_r) + \partial_\phi(-\frac{\varepsilon_o}{r})(\partial_t A_\phi - \partial_\phi A_t)$$

$$+ \partial_z(-r\varepsilon_o)(\partial_t A_z - \partial_z A_t) + \partial_z(-\Omega r\varepsilon_o)(\partial_\phi A_z - \partial_z A_\phi) = 0$$

II. $\lambda = 1$

$$\partial_t(-r\varepsilon_o)(\partial_r A_t - \partial_t A_r) + \partial_t(-\Omega r\varepsilon_o)(\partial_r A_\phi - \partial_\phi A_r) + \partial_\phi (\Omega r\varepsilon_o)(\partial_t A_r - \partial_r A_t)$$

$$+ \partial_\phi (\frac{1}{\mu r}) - \Omega^2 r\varepsilon_o)(\partial_r A_\phi - \partial_\phi A_r) + \partial_z(\frac{r}{\mu_o})(\partial_r A_z - \partial_z A_r) = 0$$

III. $\lambda = 2$

$$\partial_t(-\frac{\varepsilon_o}{r})(\partial_\phi A_t - \partial_t A_\phi) + \partial_r(-\Omega r\varepsilon_o)(\partial_t A_r - \partial_r A_t) + \partial_r(\frac{-1}{\mu_o r} + \Omega^2 r\varepsilon_o)$$

$$(\partial_r A_\phi - \partial_\phi A_r) + \partial_z(-\Omega r\varepsilon_o)(\partial_t A_z - \partial_z A_t) + \partial_z(\frac{1}{\mu_o r} - \Omega^2 r\varepsilon_o)(\partial_\phi A_z - \partial_z A_\phi) = 0$$

IV. $\lambda = 3$

$$\partial_t(r\varepsilon_o)(\partial_t A_z - \partial_z A_t) + \partial_t(\Omega r\varepsilon_o)(\partial_\phi A_z - \partial_z A_\phi) + \partial_r(\frac{r}{\mu_o})(\partial_z A_r - \partial_r A_z)$$

$$+ \partial_\phi(\Omega r\varepsilon_o)(\partial_t A_z - \partial_z A_t) + \partial_\phi (\frac{-1}{\mu_o r} + \Omega^2 r\varepsilon_o)(\partial_\phi A_z - \partial_z A_\phi) = 0.$$

Now in a circular propagation, where the light beam goes around with r = constant, z = constant, we assume zero variation in these directions. Hence, $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial z} \rightarrow 0$. Also neglecting second order terms in Ω, we write

I. $\lambda = 0$

$$\partial_\phi \partial_t A_\phi - \partial_\phi \partial_\phi A_t = 0$$

II. $\lambda = 1$

$$+ r\varepsilon_o \partial_t \partial_t A_r + \Omega r\varepsilon_o(\partial_t \partial_\phi + \partial_\phi \partial_t) A_r - \mu_o r \partial_\phi \partial_\phi A_r = 0$$

III. $\lambda = 2$

$$\partial_t \partial_\phi A_t - \partial_t \partial_t A_\phi = 0$$

IV. $\lambda = 3$

$$r\varepsilon_o \partial_t \partial_t A_z + \Omega r\varepsilon_o \partial_t \partial_\phi A_z + \Omega r\varepsilon_o \partial_\phi \partial_t A_z - \frac{1}{\mu_o r} \partial_\phi \partial_\phi A_z = 0.$$

APPENDIX B

Derivation of the Coupled Amplitude Equations for a Ring Laser

We assume a vector potential of the form:

$$u = u_o \{\frac{B_+}{\sqrt{\Omega_+}} e^{i\beta_+} A_+ + \frac{B_-}{\sqrt{\Omega_-}} e^{-i\beta_-} A_-\}$$

where $u_o = \sqrt{(8\ nc^2)/V}$, $\beta_\pm = \Omega_\pm + \phi_\pm$, $A_\pm = e^{\pm ik_\pm s}$. The expression for u is to be substituted into the wave equation

$$-\partial_s^2 u + \frac{1}{c^2}\ddot{u} + \frac{r_o\Omega}{c^2}\partial_s\dot{u} + \frac{2}{c^2}\kappa\dot{u} - \frac{2}{c^2}\Delta\kappa\ \dot{u}^* = \frac{4\pi}{c}\dot{P}.$$

Hence, we write down the appropriate derivatives of u

$$-\partial_s^2 u = u_o\{\frac{B_+k_+^2}{\sqrt{\Omega_+}} e^{i\beta_+} A_+ + \frac{B_-k_-^2}{\sqrt{\Omega_-}} e^{i\beta_-} A_-\}$$

$$\dot{u} = u_o \{\frac{A_+e^{i\beta_+}}{\sqrt{\Omega_+}} [\dot{B}_+ + i(\Omega_+ + \dot{\phi}_+)B_+] + \frac{A_-e^{-i\beta_-}}{\sqrt{\Omega_-}} [\dot{B}_- - i(\Omega_- + \dot{\phi}_-)B_-]\}$$

$$\partial_s\dot{u} = u_o \{\frac{ik_+A_+e^{i\beta_+}}{\sqrt{\Omega_+}} [\dot{B}_+ + i(\Omega_+ + \dot{\phi}_+)B_+] - \frac{ik_-A_-e^{-i\beta_-}}{\sqrt{\Omega_-}} [\dot{B}_- - i(\Omega_- + \dot{\phi}_-)B_-]$$

$$\ddot{u} = u_o(\frac{A_+e^{i\beta_+}}{\sqrt{\Omega_+}}\{\ddot{B}_+ + 2i(\Omega_+ + \dot{\phi}_+)\dot{B}_+ + [i\ddot{\phi}_+ - (\Omega_+ + \dot{\phi}_+)^2]B_+\} + \frac{A_-e^{-i\beta_-}}{\sqrt{\Omega_-}}$$

$$\{\ddot{B}_- - 2i(\Omega_- + \dot{\phi}_-)\dot{B}_- - [i\ddot{\phi}_- + (\Omega_- + \dot{\phi}_-)^2]B_-\}).$$

Now multiplying each derivative by $-i \sin k_+s\ e^{-i\beta_+}$ and integrating over a symmetric spatial volume element, dV, while defining

$d_{\pm} = \sqrt{(hc^2V)/(\Omega_{\pm})}$, and $\alpha = (\Omega_+ + \Omega_-)t + (\phi_+ + \phi_-)$, we obtain

$$-\int \partial_s^2\, u(-i \sin k_+s\, e^{-i\beta_+})dV = d_+ k_+^2 B_+ - d_- k_-^2 B_- \delta_{\omega+,\omega-}\, e^{-i\alpha}$$

$$\frac{1}{c^2} \int \ddot{u}(-i \sin k_+s\, e^{-i\beta_+})dV = \frac{d_+}{c^2} \{\ddot{B}_+ + 2i(\Omega_+ + \dot{\phi}_+)\dot{B}_+ + [i\ddot{\phi}_+ - (\Omega_+ + \dot{\phi}_+)^2]B_+\}$$

$$\frac{d_-}{c^2}\ \ddot{B}_- - 2i(\Omega_- + \dot{\phi}_-)B_- - [i\ddot{\phi} + (\Omega_- + \dot{\phi}_-)^2]B_-\ \ e^{-i\alpha}\ \delta_{\omega+,\omega-}$$

$$\frac{r_o\Omega}{c^2} \int \partial_s\, \dot{u}(-i \sin k_+s\, e^{-u\beta_+})dV = \frac{r_o\Omega}{c^2} i d_+ k_+ [\dot{B}_+ + (\Omega_+ + \dot{\phi}_+)B_+] + \frac{r_o\Omega}{c^2} i d_- k_-$$

$$[\dot{B}_- - i(\Omega_- + \dot{\phi}_-)B_-]\ e^{-i\alpha}\ \delta_{\omega+,\omega-}$$

$$\frac{2\kappa}{c^2} \int \dot{u}\ (-i \sin k_+s\, e^{-i\beta_+})dV = \frac{2\kappa}{c^2} d_+ [\dot{B}_+ + i(\Omega_+ + \dot{\phi}_+)B_+] - \frac{2\kappa}{c^2} d_-$$

$$[\dot{B}_- - i(\Omega_- + \dot{\phi}_-)B_-]\ e^{-i\alpha}\ \delta_{\omega+,\omega-}$$

$$-\frac{2\Delta\kappa}{c^2} \int \dot{u}^*(-i \sin k_+s\, e^{-i\beta_+})dV = +\frac{2\delta\kappa}{c^2} d_+ e^{-2\beta_+} [\dot{B}_+ - i(\Omega_+ + \dot{\phi}_+)B_+] -$$

$$\frac{2\Delta\kappa}{c^2} d_-\ e^{i(\beta_- - \beta_+)}\ [\dot{B}_- + i(\Omega_- + \dot{\phi}_-)B_-]\ \delta_{\omega+,\omega-}$$

Furthermore, if one assumes harmonic oscillations for the elements of the density matrix; that is, $\rho_{m=1} = \tilde{\rho}_1\, e^{-i\nu_- t}$, and $\rho_{m=-1} = \tilde{\rho}_{-1}\, e^{i\nu_+ t}$, then

$$\dot{P} = \frac{iN}{V} (\nu_+\ \rho_{1,2-1}\ \mu_{21}^+ - \nu_-\ \rho_{21,1}\ \mu_{12}^+,$$

and

$$\frac{4\pi}{c} \int \dot{P} \cos k_+s\ e^{-i\beta_+}\, d\tau = i\nu_+ \frac{4\pi}{c}\ \frac{N}{V}\, e^{-\beta_+} \int \rho_{1,2-1}\ \mu_{21}^+ \cos k_+s\ dV$$

$$- i\nu_- \frac{4\pi}{c}\frac{N}{V} e^{-\beta_+} \delta_{\omega+,\omega-} \int \rho_{21,1} \mu^+_{12} \cos k_+ s \; dV.$$

We observe that the largest coupling term contribution will come from the term containing the slowest oscillations. Hence, we neglect all coupling terms other than the expression containing $e^{i(\beta_- - \beta_+)}$, which represents the slowest variation in time. Hence, the B_+ equation containing the largest coupling term is

$$\frac{d_+}{c^2}\{[\omega_+^2 + i\ddot{\phi}_+ - (\Omega_+ + \dot{\phi}_+)^2 - \frac{r_o \Omega\omega_+}{c}(\Omega_+ + \dot{\phi}_+) + 2\kappa i \; (\Omega_+ + \dot{\phi}_+) - 2\Delta\kappa \; e^{-2i\beta_+}$$

$$(\Omega_+ + \dot{\phi}_+)]B_+ + [2i(\Omega_+ + \dot{\phi}_+) + i\frac{r_o\Omega\omega_+}{c} - 2\kappa + 2\Delta\kappa \; e^{-2i\beta_+}]\dot{B}_+ - i2\Delta\kappa\frac{d_-}{d_+}$$

$$(\Omega_- + \dot{\phi}_-)B_- \; e^{i(\beta_- - \beta_+)} \; \delta_{\omega+,\omega-}\} = \nu_+ \frac{4\pi}{c}\frac{N}{V} e^{-i\beta_+} \int \rho_{1,2-1} \mu^+_{21} \sin k_+ s \; dV.$$

Near resonance, the equation is simplified by neglecting terms like $\ddot{\phi}$, $\ddot{B}$ since they vary slowly in time and terms like $e^{-2i\beta_+}$ are neglected since they oscillate rapidly in time. Also, near resonance $\frac{d_-}{d_+} \simeq 1$. For a cavity with high Q, $\frac{\kappa}{\Omega_+} \to 0$. One finally obtains,

$$(1 + \frac{1}{2}\frac{r_o\Omega}{c})\dot{B}_+ + [-i(\omega_+ - \Omega_+ - \dot{\phi}_+ - \frac{1}{2}\frac{r_o\Omega}{c}\omega_+) + \kappa]B_+ - \Delta\kappa \; B_- \; e^{i(\beta_- - \beta_+)}$$

$$\delta_{\omega+,\omega-} = - i\nu_+ \frac{4\pi}{c}\;\frac{N}{V} e^{-i\beta_+} \int \rho_{1,2-1} \mu^+_{21} \sin k_+ s \; dV.$$

Similarly, now an equation for the amplitude B_- is obtained.

$$(1 - \frac{1}{2}\frac{r_o\Omega}{c})\dot{B}_- + [i(\omega_- - \Omega_- - \dot{\phi}_- + \frac{1}{2}\frac{r_o\Omega}{c}\omega_-) + \kappa]B_- - \Delta\kappa \; B_+ \; e^{i(\beta_- - \beta_+)}$$

$$\delta_{\omega+,\omega-} = - i\nu_- \frac{N}{V} e^{i\beta_-} \int g^+_{12\omega_-} \; \rho_{21,1} \; dV.$$

APPENDIX C

Solution of the Density Matrix Differential Equation

The components of the density matrix equation

$$\dot{\rho} = \frac{1}{\hbar i} [H,\rho] - \frac{1}{2} \{\Gamma,\rho\} + \lambda$$

may be written

$$\dot{\rho}_{1,1} = -\gamma_1 \rho_{1,1} + \lambda_1 + \frac{1}{i\hbar} \sum_{m=-1} (H^{int}_{1,2m} \rho_{2m,1} - \rho_{1,2m} H^{int}_{2m,1})$$

$$\dot{\rho}_{1,2m} = i(\nu_o + m\omega_L) \rho_{1,2m} - \gamma_{12} \rho_{1,2m} + \frac{1}{i\hbar} \sum_{\ell} H^{int}_{1,2\ell} \rho_{2\ell,2m} - \frac{1}{i\hbar} \rho_{1,1} H^{int}_{1,2m}$$

$$\dot{\rho}_{2,2m} = i(m - \ell) \omega_L \rho_{2\ ,2m} - \lambda_2 \rho_{2\ell,2m} + \lambda_{2m} \delta_{\ell,m} + \frac{1}{i\hbar} H^{int}_{2\ell,1} \rho_{1,2m} \frac{1}{i\hbar}$$

$$\rho_{2\ell,1} H^{int}_{1,2m},$$

where we have split the matrix elements in a term rapidly varying with time, and a slowly varying amplitude $\tilde{\rho}$, that is,

$$\rho_{1,1} = \tilde{\rho}_{1,1},\ \rho_{1,2m} = \tilde{\rho}_{1,2m}\, e^{i(\nu_0 + m\omega_L)t}\ ,\ \rho_{2\ell,2m} = \tilde{\rho}_{2\ell,2m}\, e^{i(m - \ell)\omega_L t},$$

and defined $\gamma_{12} = \frac{1}{2} (\gamma_1 + \gamma_2)$.

The components of the interaction Hamiltonian are given by

$$H^{int}_{a,b} = -\frac{i\nu_{ab}}{2c} (\mu^{-}_{ab} u + \mu^{+}_{ab} u^*).$$

Using the dipole selection rules

$$\mu^{+}_{1,2m} = \mu^{+}_{12}\, \delta_{m,1} \qquad\qquad \mu^{+}_{2m,1} = \mu^{+}_{21}\, \delta_{m,-1}$$

$$\mu^{-}_{1,2m} = \mu^{-}_{12}\ \delta_{m,-1} \qquad\qquad \mu^{-}_{2m,1} = \mu^{-}_{21}\ \delta_{m,1}$$

and defining

$$\mu^{+}_{12} = \mu^{-}_{12} = \mu^{+}_{21} = \mu^{-}_{21} = \mu$$

let us calculate a typical element of the Hamiltonian. Using the expression for $H^{int}_{a,b}$, one writes for the $H^{int}_{1,2-1}$ element

$$H^{int}_{1,2-1} = \frac{-i\ \nu_{1,2-1}}{2c}\ (\mu^{-}_{1,2-1}\ u + \mu^{+}_{1,2-1}\ u^{*}),$$

where $\mu^{+}_{1,2-1} = 0$, due to the dipole selection rules. Since the frequency

$$\nu_{1,2-1} = \frac{E_1 - E_{2,-1}}{\hbar} = \frac{\hbar\omega - \hbar(\omega + \nu_+)}{\hbar} = -\nu_+,$$ and $\mu^{-}_{1,2-1} = \mu\ \delta_{m,-1}$, one

obtains

$$H^{int}_{1,2-1} = \frac{i\nu_+\mu}{2c}\ u\ \delta_{m,-1}.$$

Assuming a vector potential of the form

$$u = \frac{8\pi\hbar\ c^2}{V}\ \{\frac{B_+\ e^{i\beta+}}{\sqrt{\Omega_+}}\ A_+ + \frac{B_-\ e^{-i\beta-}}{\sqrt{\Omega_-}}\ A_-\}$$

we note that

$$u\ \delta_{m,-1} = \sqrt{\frac{8\pi\hbar c^2}{V}}\ \frac{B_+\ e^{i\beta+}}{\sqrt{\Omega_+}}\ A_+$$

hence,

$$H^{int}_{1,2-1} = i\hbar\nu_+ b_{\omega+}$$

where

$$b_{\pm} = \sqrt{\frac{2\pi}{\hbar V\Omega_+}}\ \mu\ A_{\pm}\ B_{\pm}\ e^{\pm i\beta_{\pm}}$$

The components of the density matrix equation may be written as

$$\dot{\rho}_{1,1} + \gamma_1 \rho_{1,1} = \lambda_1 + \nu_+(b_+ \rho_{2-1,1} + b_+^* \rho_{1,2-1}) + \nu_-(b_-^* \rho_{21,1} + b_- \rho_{1,21})$$

$$\dot{\rho}_{1,2m} - i(\nu_o + m\omega_L)\rho_{1,2m} + \gamma_{12} \rho_{1,2m} = \nu_+ b_+(\rho_{2-1,2m} - \rho_{1,1} \delta_{m,-1}) + \nu_- b_-^*$$

$$(\rho_{21,2m} - \rho_{1,1} \delta_{m,1})$$

$$\dot{\rho}_{2\ell,2m} - i(m - \ell)\omega_L \rho_{2\ell,2m} + \gamma_2 \rho_{2\ell,2m} = \lambda_{2m} \delta_{\ell,m} - \nu_+(b_+^* \rho_{1,2m} \delta_{\ell,-1} +$$

$$b_+ \rho_{2\ell,1} \delta_{m,-1}) - \nu_-(b_- \rho_{1,2m} \delta_{\ell,1} + b_-^* \rho_{2\ell,1} \delta_{m,1}).$$

These coupled differential equations for the matrix elements of ρ may be solved approximately by treating the terms on the right hand side like source terms. If we assume that amplitudes $\tilde{\rho}$, $B_\pm$ and phases $\phi_\pm$ do not vary much within the time $\frac{1}{\gamma}$, we find the following linear equations for the elements of ρ.

$$\rho_{1,1} = \frac{\lambda_1}{\lambda_1} + \quad +[\frac{b_+ \rho_{2-1,1}}{\gamma_1 + i(\Omega_+ - \nu_+)} + \frac{b_+^* \rho_{1,2-1}}{\gamma_1 - i(\Omega_+ - \nu_+)}] + \nu_-[\frac{b_-^* \rho_{21,1}}{\gamma_1 + i(\Omega_- - \nu_-)} +$$

$$\frac{b_- \rho_{1,21}}{\gamma_1 - i(\Omega_- - \nu_-)}]$$

$$\rho_{1,2m} = \nu_+ b_+ \{\frac{\rho_{2-1,2m}}{\gamma_{12} + i(\Omega_+ - \nu_+)} - \frac{\rho_{1,1} \delta_{m,-1}}{\gamma_{12} + i[\Omega_+ - (\nu_o + m\omega_L)]}\} + \nu_- b_-^*$$

$$\{\frac{\rho_{21,2m}}{\gamma_{12} + i(\Omega - \nu_-)} - \frac{\rho_{1,1} \delta_{m,1}}{\gamma_{12} + i[\Omega_- - i(\nu_o + m\omega_L)]}\}$$

$$\rho_{2\ell,2m} = \frac{\lambda_{2m} \delta_{\ell,m}}{\gamma_2 - i*m - \ell)_L} - \nu_+ \{\frac{b_+^* \rho_{1,2m} \rho_{\ell,-1}}{\gamma_2 - i[\Omega_+ - (\nu_o + m\omega_L) + (m - \ell)\omega_L]} +$$

$$\frac{b_+ \rho_{2\ell,1} \delta_{m,-1}}{\gamma_2 + i[\Omega_+ - (\nu_o + \ell\omega_L) - (m - \ell)\omega_L]}\} - \nu_- \frac{b_- \rho_{1,2m} \delta_{\ell,1}}{\gamma_2 - i[\Omega_- - (\nu_o + m\omega_L) + (m - \ell)\omega_L]}$$

$$+ \frac{b_-^* \rho_{2\ell,1} \delta_{m,1}}{\gamma_2 + i[\Omega_- - (\nu_o + \ell\omega_L) - (m - \ell)\omega L]}$$

Now using the property that $\rho^*_{a,b} = \rho_{b,a}$ and neglecting terms of higher order than B^3 we can write expressions for $\rho_{1,2-1}$ and $\rho_{21,1}$

$$\rho_{1,2-1} = \frac{\nu_+ b_+}{\gamma_{12} + i(\Omega_+ - \nu_+)} d_{2-1} - \frac{\nu_+^3 |b_+|^2 b_+ d_{2-1}}{\gamma_{12} + i(\Omega_+ - \nu_+)} \{\frac{1}{\gamma_{12} + i(\Omega_+ - \nu_+)}$$

$$[\frac{1}{\gamma_2 - i(\Omega_+ - \nu_+)} + \frac{1}{\gamma_1 - i(\Omega_+ - \nu_+)}] + \frac{1}{\gamma_{12} - i(\Omega_+ - \nu_+)}[\frac{1}{\gamma_2 + i(\Omega_+ - \nu_+)} +$$

$$\frac{1}{\gamma_1 + i(\Omega_+ - \nu_+)}] - \nu_+ \nu_-^2 b_+ |b_-|^2 \{\frac{1}{\gamma_{12} + i(\Omega_+ - \nu_+)} \frac{1}{\gamma_{12} + i(\Omega_- - \nu_-)}$$

$$[\frac{d_{21}}{\gamma_1 - i(\Omega_- - \nu_-)} + \frac{d_{2-1}}{\gamma_2 - i(\Omega_- - \nu_-)}] + \frac{d_{21}}{\gamma_{12} - i(\Omega_- - \nu_-)} [\frac{1}{\gamma_{12} + i(\Omega_+ - \nu_+)}$$

$$\frac{1}{\gamma_1 + i(\Omega_- - \nu_-)} + \frac{1}{\gamma_{12} + i(\Omega_- - \nu_-)} \frac{1}{\gamma_2 + i(\Omega_+ - \nu_+)}]\}$$

$$\rho_{21,1} = \frac{\nu_- b_-}{\gamma_{12} - i(\Omega_- - \nu_-)} d_{21} - d_{21} \nu_-^3 \frac{b_- |b_-|^2}{\gamma_{12} - i(\Omega_- - \nu_-)} \{\frac{1}{\gamma_{12} - i(\Omega_- - \nu_-)}$$

$$[\frac{1}{\gamma_2 + i(\Omega_- - \nu_-)} + \frac{1}{\gamma_1 + i(\Omega_- - \nu_-)}] + \frac{1}{\gamma_{12} + i(\Omega_- - \nu_-)} [\frac{1}{\gamma_2 - i(\Omega_- - \nu_-)} +$$

$$\frac{1}{\gamma_1 - i(\Omega_- - \nu_-)}]\} - \nu_- \nu_+^2 b_- |b_+|^2 \{\frac{1}{\gamma_{12} - i(\Omega_+ - \nu_+)} \frac{1}{\gamma_{12} - i(\Omega_- - \nu_-)}$$

$$[\frac{d_{21}}{\gamma_2 + i(\Omega_+ - \nu_+)} + \frac{d_{2-1}}{\gamma_1 + i(\Omega_+ - \nu_+)}] + \frac{d_{2-1}}{\gamma_{12} + i(\Omega_+ - \nu_+)}[\frac{1}{\gamma_{12} - i(\Omega_+ - \nu_+)}$$

$$\frac{1}{\gamma_2 - i(\Omega_-^? - \nu_-)} + \frac{1}{\gamma_{12} - i(\Omega_- - \nu_-)} \frac{1}{\gamma_1 - i(\Omega_+ - \nu_+)}]\}$$

where $d_{2-1} = \frac{\lambda_{2-1}}{\gamma_2} - \frac{}{1}$ $\qquad d_{21} = \frac{\lambda_{21}}{\gamma_2} - \frac{\lambda_1}{\gamma_1}.$

To calculate the source terms in the amplitude equation, one separates the expression for $\rho_{1,2-1}$ and $\rho_{21,1}$ into real and imaginary parts and substitutes into the amplitude equation. The values of the constants mentioned in section V1-4 are

$$\alpha_\pm = -\kappa + \frac{1}{2}\frac{\nu_\pm^2}{\Omega_\pm} N g d_{2+1} \frac{\gamma_{12}}{\gamma_{12}^2 + (\nu_\pm - \Omega_\pm)^2},$$

$$\beta_\pm = \frac{3}{4}\frac{\nu_\pm^4}{\Omega_\pm^2} N g^2 d_{2+1} \frac{\gamma_{12}}{[\gamma_{12}^2 + (\nu_\pm - \Omega_\pm)^2]^2} (\frac{\gamma_{12}\gamma_2 + (\nu_\pm - \Omega_\pm)^2}{\gamma_2^2 + (\nu_\pm - \Omega_\pm)^2} +$$

$$\frac{\gamma_{12}\gamma_1 + (\nu_\pm - \Omega_\pm)^2}{\gamma_1^2 + (\nu_\pm - \Omega_\pm)^2}),$$

$$\theta_\pm = (\frac{1}{4} + \frac{1}{8}\delta_{\omega+,\omega-}) \frac{\nu_+^2 \nu_-^2}{\Omega_+ \Omega_-} N g^2 \frac{1}{\gamma_{12}^2 + (\nu_+ - \Omega_+)^2} \{2 \frac{\gamma_{12}[\gamma_{12}\gamma_1 + (\nu_\pm - \Omega_\pm)^2]}{[\gamma_{12}^2 + (\nu_\pm - \Omega_\pm)^2]} \times$$

$$\frac{d_{2\pm1}}{[\gamma_1^2 + (\nu_+ - \Omega_+)^2]} + \frac{\gamma_2 d_{2\pm1}}{\gamma_2^2 + (\nu_\pm - \Omega_\pm)^2} + \frac{\gamma_{12}^2\gamma_2 - (\nu_+ - \Omega_+)(\nu_- - \Omega_-)\gamma_2 +}{[\gamma_{12}^2 + (\nu_\pm - \Omega_\pm)^2] \times}$$

$$\frac{\gamma_{12}(\nu_+ - \Omega_+)(\nu_+ - \Omega_+ + \nu_- - \Omega_-)}{[\gamma_2^2 + (\nu_+ - \Omega_+)^2]} d_{2+1},$$

$$\sigma_\pm = \frac{1}{2}\frac{\nu_\pm^2}{\Omega_\pm} N g d_{2+1} \frac{\nu_\pm - \Omega_\pm}{\gamma_{12}^2 + (\nu_\pm - \Omega_\pm)^2},$$

$$\rho_{\pm} = \frac{3}{4}\frac{\nu_{\pm}^4}{\Omega_{\pm}^2} N g^2 d_{2} + 1 \quad \frac{\nu_{\pm} - \Omega_{\pm}}{[\gamma_{12}^2 + (\nu_{\pm} - \Omega_{\pm})^2]^2} \left(\frac{\gamma_{12}\ \gamma_2 + (\nu_{\pm} - \Omega_{\pm})^2}{\gamma_2^2 + (\nu_{\pm} - \Omega_{\pm})^2} + \right.$$

$$\left.\frac{\gamma_{12}\ \gamma_1 + (\nu_{\pm} - \Omega_{\pm})^2}{\gamma_1^2 + (\nu_{\pm} - \Omega_{\pm})^2}\right),$$

$$\tau_{\pm} = \left(\frac{1}{4} + \frac{1}{8}\,\delta_{\omega+,\omega-}\right) \frac{\nu_+^2\ \nu_-^2}{\Omega_+\ \Omega_-} \frac{Ng^2}{\gamma_{12}^2 + (\nu_{\pm} - \Omega_{\pm})^2} \left\{\frac{2(\nu_{\pm} - \Omega_{\pm})[\gamma_{12}\ \gamma_1 + (\nu_+ - \Omega_+)^2]\mathrm{x}}{[\gamma_{12}^2 + (\nu_{\pm} - \Omega_{\pm})^2]\ \mathrm{x}}\right.$$

$$\frac{d_{2} \pm 1}{[\gamma_1^2 + (\nu_+ - \Omega_+)^2]} + \frac{\nu_{\pm} - \Omega_{\pm}}{\gamma_2^2 + (\nu_{\pm} - \Omega_{\pm})^2}\, d_{2} \pm 1 + \frac{\gamma_{12}\ \gamma_2(\nu_+ - \Omega_+ + \nu_- - \Omega_-) -}{[\gamma_{12}^2 + (\nu_{\pm} - \Omega_{\pm})^2]}$$

$$\frac{(\nu_+ - \Omega_+)[\gamma_{12}^2 - (\nu_+ - \Omega_+)(\nu_- - \Omega_-)]}{[\gamma_2^2 + (\nu_+ - \Omega_+)^2]}\, d_{2} + 1$$

where,

$$g = \frac{2\pi}{nV}\,|\mu|^2.$$