Eur. Phys. J. Plus (2016) **131**: 296
DOI 10.1140/epjp/i2016-16296-x

THE EUROPEAN
PHYSICAL JOURNAL PLUS

Regular Article

Coordinate transformation between rotating and inertial systems under the constant two-way speed of light

Yang-Ho Choi[a]

Department of Electrical and Electronic Engineering, Kangwon National University, Chunchon, Kangwon-do, 200-701, South Korea

Received: 13 June 2016
Published online: 5 September 2016 –

Abstract. An observation system consists of the world lines of rest observers in the system. Recently a coordinate transformation between an isotropic and a rotating observation system has been presented which was derived through a relativistic circular approach based on the Lorentz transformation. It was formulated such that the relative speeds between the two systems are the same, but the two-way speed of light is not constant in the rotating observation system. The constancy of the two-way speed of light in inertial frames has been known to be experimentally verified. This paper presents the transformation that holds the constancy in the rotating system as well. Though the rotating system is in motion with acceleration, it can be regarded as locally inertial. Thus, in the limit, a transformation into a rotating system should be reduced to a transformation into an inertial systems. The transformation presented is consistent with the one between inertial systems so that the latter can be derived from the former in the limit. Moreover it allows us to theoretically analyze the generalized Sagnac effect, which involves rectilinear motion as well as circular motion. The theoretical analysis corresponds to the experimental results.

1 Introduction

For more than one hundred years since Sagnac's experiment in 1913 [1], circular motions could not have been relativistically dealt with through comprehensive analysis. Conventional methods based on special relativity or general relativity have usually employed Galilean-type transformations to relativistically explain the experimental results of circular motion such as the Sagnac effect [2–9]. However, no sufficient justifications for the use of the Galilean transformation, which is nonrelativistic, are found in the literature. No detailed information on motions in the rotating frame has been presented either. Even the speeds of the counter-propagating light beams on a rotating plate could not be clearly analyzed, though there is a variety of explanations and analyses on the Sagnac effect [2–4, 8–10]. Moreover the problem of time gap [3–5] that multiple times are defined at the same spatial point has remained unsolved.

The circular approach presented recently [11], which has been theoretically developed based on the Lorentz transformation reformulated in a complex Euclidean space (CES) with the time represented as an imaginary number, can deal with circular motions relativistically so that detailed information on motions including the speeds of objects in the rotating frame can be obtained. In the approach, four coordinate systems S, $\widetilde{S}$, $\widetilde{S}'$, and S' are employed to effectively accommodate circular motions in the primed and the unprimed frames. They represent the coordinate systems of single observers O, O', $\widetilde{O}$, and $\widetilde{O}'$, respectively. The coordinate systems $\widetilde{S}$ and $\widetilde{S}'$ with a tilde are rotating while the others without tildes are fixed. It has been shown that a circular motion in the unprimed system is represented as a circular motion also in the primed one, but with different angular velocity and radius, which in turn leads to the derivation of the coordinate transformation between the inertial and the rotating observation systems.

An observation system is different from the coordinate system for a single observer who sees the world via the Lorentz lens, and is identical to a collection of the world lines of observers so that it represents actual physical reality. The world line can be obtained by the circular approach which exploits the Lorentz transformation in the coordinate systems of observers. In the derivation of the circular approach, emphasis was placed on the relativity in the representation of motions that the transformation from the unprimed to the primed has the same forms as the reverse one, from the primed to the unprimed. Accordingly, the existing coordinate transformation between the inertial

[a] e-mail: yhochoi@kangwon.ac.kr

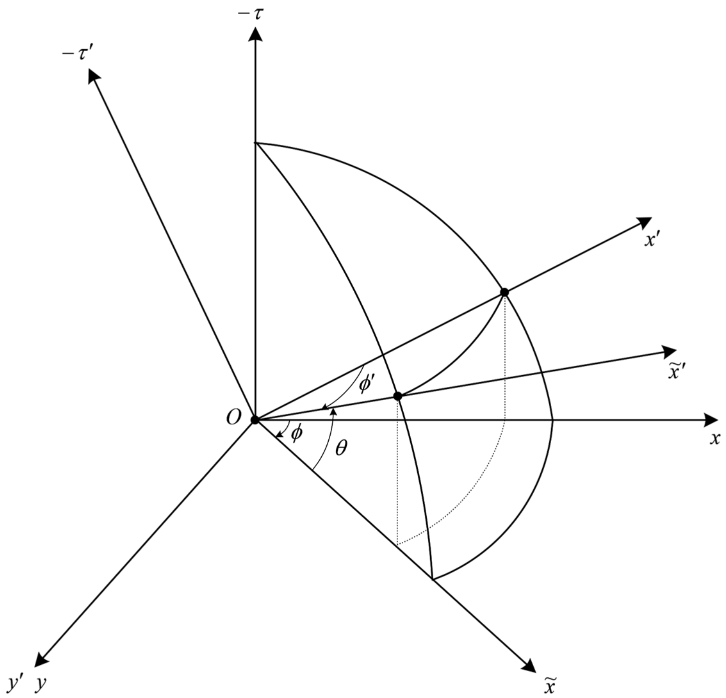

Fig. 1. Coordinate systems S, $\widetilde{S}$, $\widetilde{S}'$, and S' for single observers O, $\widetilde{O}$, $\widetilde{O}'$, and O', respectively.

and the rotating systems holds the constant relative speed such that the instantaneous speed of $\widetilde{O}'$ seen by O' is equal to that of $\widetilde{O}$ seen by O. We call it the transformation under the constant relative speed (TCR).

In inertial frames, the two-way speed of light has been known to be constant in every propagation direction since the Michelson-Morley experiment [12]. Circular motion in the limit can be considered as linear motion, and so the rotating system can be regarded as locally inertial though it is in motion with acceleration. However, TCR does not show the constancy of the two-way speed of light in the limit. In that sense, it is inconsistent with inertial motion. This paper presents the transformation that can allow the constancy of the two-way speed of light in the rotating observation system, which is referred to as the transformation under the constant light speed (TCL).

TCL is attained through the same circular approach used for the derivation of TCR. The transformations are made in cylindrical coordinates, and the coordinates from TCL except radius are the same as the ones from TCR. It is shown that TCL is consistent with the transformation for inertial observation systems so that the latter can be derived from the former in the limit to inertial motion. The Sagnac effect takes place by the difference between the travel times of two counter-propagating light signals on a rotating plate. The experiments of the generalized Sagnac effect [13,14] indicate that the difference of travel time is shown in inertial systems as well as in rotating ones. It implies that the one-way speed of light is anisotropic also in inertial frames. Theoretical analysis of the generalized Sagnac effect via TCL is presented, which corresponds to the experimental results.

2 Relativistic circular approach

This section summarizes the relativistic circular approach [11], using fig. 1 which illustrates the unprimed and the primed coordinate systems S, $\widetilde{S}$, $\widetilde{S}'$, and S' in CES. The speed of light is assumed to be isotropic in S. In fig. 1, the time coordinate of S is expressed as $\tau = ict$, where $i = (-1)^{1/2}$, t denotes time, and c is the speed of light in S. The coordinate time $\widetilde{\tau}$ of $\widetilde{S}$ is the same as τ. We denote the coordinate vectors of S and $\widetilde{S}$ by $\boldsymbol{p} = [\tau, x, y]^T$ and $\widetilde{\boldsymbol{p}} = [\widetilde{\tau}, \widetilde{x}, \widetilde{y}]^T$, respectively, where T stands for the transpose. We use a subscript "s" to represent the spatial vector from a space-time coordinate vector. For example, the spatial vector of $\widetilde{\boldsymbol{p}}$ is expressed as $\widetilde{\boldsymbol{p}}_s (= [\widetilde{x}, \widetilde{y}]^T)$. The observers O and $\widetilde{O}$ are located at radius r in the unprimed systems. The coordinate vector in S of O is written as $\boldsymbol{p}_O = [\tau, 0, -r]^T$ and the coordinate vector in $\widetilde{S}$ of $\widetilde{O}$ as $\widetilde{\boldsymbol{p}}_{\widetilde{O}} = [\tau, 0, -r]^T$.

The coordinate system $\widetilde{S}$ is rotating with an angular velocity ω relative to S, and its rotational angle at an instant $\boldsymbol{\tau}$ is

$$\phi = \omega_c \boldsymbol{\tau}, \tag{1}$$

where $\omega_c = \omega/ic$. Then the coordinate vectors $\boldsymbol{p}$ and $\widetilde{\boldsymbol{p}}$ are related by

$$\widetilde{\boldsymbol{p}} = \boldsymbol{A}(\phi)\boldsymbol{p}, \tag{2}$$

where $\boldsymbol{A}(\phi)$ is a rotation matrix

$$\boldsymbol{A}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}. \tag{3}$$

Clearly $\boldsymbol{A}^{-1}(\phi) = \boldsymbol{A}(-\phi)$. From (2), the coordinate vector in S of $\widetilde{O}$ is represented as $\boldsymbol{p} = \boldsymbol{A}(-\phi)\widetilde{\boldsymbol{p}}_{\widetilde{O}}$.

The coordinate systems S' and $\widetilde{S}'$ are the primed ones which correspond to S and $\widetilde{S}$, respectively. The coordinate vectors of S' and $\widetilde{S}'$ are denoted as $\boldsymbol{p}' = [\tau', x', y']^T$ and $\widetilde{\boldsymbol{p}}' = [\widetilde{\tau}', \widetilde{x}', \widetilde{y}']^T$, respectively. Though the spatial coordinates of S' and $\widetilde{S}'$ are different, their time coordinates are the same, *i.e.*, $\widetilde{\tau}' = \tau' = ict'$. The $\widetilde{\tau}$- and $\widetilde{x}$-axes of $\widetilde{S}$ and the $\widetilde{\tau}'$- and $\widetilde{x}'$-axes of $\widetilde{S}'$ lie on an identical plane, as can be seen in fig. 1 where the angle θ is a complex number and the trigonometric functions $\cos\theta$ and $\sin\theta$ are given by [11]

$$\cos\theta = \frac{1}{(1-\beta^2)^{1/2}}, \tag{4}$$

$$\sin\theta = \frac{\beta}{i(1-\beta^2)^{1/2}}, \tag{5}$$

with $\beta = r\omega/c = ir\omega_c$. The $\widetilde{y}'$-axis lies on the x-y plane and the locus of the $\widetilde{x}'$-axis forms a cone as ϕ increases from zero to 2π. In the primed, $\widetilde{S}'$ is rotated by ϕ' relative to S'. When the rotational angle of $\widetilde{S}$ is ϕ, as seen in fig. 1, ϕ' is given by

$$\phi' = -\phi\cos\theta_R, \tag{6}$$

where

$$\cos\theta_R = \frac{1}{(1+\beta^2)^{1/2}}. \tag{7}$$

In (6), ϕ' is defined in $\widetilde{S}'$. If defined in S', its sign is changed so that $\phi' = \phi\cos\theta_R$. The normalized instantaneous speed is represented from (4) and (5) as

$$\beta = i\tan\theta. \tag{8}$$

Motions when dealt with based on the Lorentz transformation can be relatively described. In the transformation from S to $\widetilde{S}'$, the latter can be viewed as fixed while the former as rotating with respect to it. Accordingly the spatial components of d$\boldsymbol{p}$, a differential vector of $\boldsymbol{p}$, are divided into the $\widetilde{x}$- and $\widetilde{y}$-components and then the Lorentz transformation is made in the $\widetilde{x}$-direction

$$\mathrm{d}\bar{\boldsymbol{p}} = \boldsymbol{A}(\phi)\mathrm{d}\boldsymbol{p}, \tag{9a}$$

$$\mathrm{d}\widetilde{\boldsymbol{p}}' = \boldsymbol{T}_L(\theta)\mathrm{d}\bar{\boldsymbol{p}}, \tag{9b}$$

where ϕ and θ are defined in S and $\boldsymbol{T}_L(\theta)$ is the Lorentz transformation matrix in CES, which is written as

$$\boldsymbol{T}_L(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

In case $\boldsymbol{\phi} = 0$ so that $\boldsymbol{A}(\phi) = \boldsymbol{I}$, where $\boldsymbol{I}$ is an identity matrix, (9) is reduced to the transformation between inertial frames.

Similarly, the transformation from the primed to the unprimed can be formulated. In the primed system, an observer $\widetilde{O}'$, whose spatial coordinate vector is written as $\widetilde{\boldsymbol{p}}_s' = [0, -r']^T$ in $\widetilde{S}'$, rotates with an angular velocity ω' relative to an observer O', who is located at $\boldsymbol{p}_s' = [0, -r']^T$ in S'. The instantaneous velocity of $\widetilde{O}'$ is given by $\beta' = ir'\omega_c'$, where $\omega_c' = \omega'/ic$. In the transformation from $\widetilde{S}'$ to S, the spatial components of the differential vector d$\widetilde{\boldsymbol{p}}'$ are divided into the x'- and y'-components and the resultant is Lorentz-transformed in the x'-direction

$$\mathrm{d}\bar{\boldsymbol{p}}' = \boldsymbol{A}(\phi')\mathrm{d}\widetilde{\boldsymbol{p}}', \tag{11a}$$

$$\mathrm{d}\boldsymbol{p} = \boldsymbol{T}_L(\theta')\mathrm{d}\bar{\boldsymbol{p}}', \tag{11b}$$

where θ' and ϕ' are defined in $\widetilde{S}'$, and the rotational angles ϕ' and ϕ are related by

$$\phi = -\phi' \cos\theta'_R. \tag{12}$$

The trigonometric functions $\cos\theta'$, $\sin\theta'$ and $\cos\theta'_R$ are the same as (4), (5), and (6) with β replaced by β'.

In case the primed and the unprimed notations in (9) are interchanged the resultant becomes virtually equal to (11), and vice versa. It implies that when motions are described based on the Lorentz transformation one motion can be represented relative to the other in circular motion as well as in linear motion, which indicates the relativity in the representation of motion. However, both the transformations (9) and (11) cannot be equally applicable. Only one of them can be valid. The valid transformation can be determined by the relationship of time between S and $\widetilde{S}'$. The effect of time dilation can be measured by exploiting the relativistic transverse Doppler effect (TDE). According to the experimental results of TDE by the Mössbauer spectroscopy [15,16], the dilation of time has been observed in $\widetilde{S}'$ and then (9) is valid.

3 Transformation between rotating and inertial observation systems

The coordinate systems S, $\widetilde{S}$, $\widetilde{S}'$, and S' are those for the single observers who see the world thorough the Lorentz lens. In contrast, an observation system consists of a collection of world lines, which can be obtained from (9). Consider an observation system S. An arbitrary observer O_k is located at a spatial point $p_s = (x_k, y_k)$ in S and its world line can be written, in terms of coordinates, as $W_k = \{(\tau, x_k, y_k)| -\infty < t < \infty\}$. The collection of these world lines constitutes S. That is to say, $\mathsf{S} = \{(\tau, x, y)|(t, x, y) \in \Re^3\}$ where $\Re$ is the set of real numbers. As S is composed of world lines, it represents actual physical reality. Coordinate systems are used to represent events. Hence a coordinate system can be viewed as a set of events and S can be expressed, in terms of a set of events, as

$$\mathsf{S} = \{e_{\mathsf{S}}(\tau, x, y)|(t, x, y) \in \Re^3\}, \tag{13}$$

where $e_{\mathsf{S}}(\tau, x, y)$ denotes an event at a point $p = (\tau, x, y)$ in S. The observation system S is assumed to be isotropic so that the speed of light is c in every direction. The coordinate system S is identical with S, which means $e_{\mathsf{S}}(\tau, x, y) = e_S(\tau, x, y)$ for every p where $e_S(\tau, x, y)$ is an event at p in S.

The primed rotating observation system $\widetilde{\mathsf{S}}'$ consists of the world lines of rotating observers and can be written as

$$\widetilde{\mathsf{S}}' = \left\{e_{\widetilde{\mathsf{S}}'}\left(\widetilde{\tau}', \widetilde{x}', \widetilde{y}'\right) \mid \left(\widetilde{t}', \widetilde{x}', \widetilde{y}'\right) \in \Re^3\right\}. \tag{14}$$

The world line of an arbitrary observer $\widetilde{O}'_k$ who is located at $\widetilde{p}'_s = (\widetilde{x}'_k, \widetilde{y}'_k)$ in $\widetilde{\mathsf{S}}'$ can be written as $\widetilde{W}'_k = \{(\widetilde{\tau}', \widetilde{x}'_k, \widetilde{y}'_k)| -\infty < \widetilde{t}' < \infty\}$, which corresponds to $\{e_{\widetilde{\mathsf{S}}'}(\widetilde{\tau}', \widetilde{x}'_k, \widetilde{y}'_k)|\widetilde{t}' \in \Re\}$, the world line at $\widetilde{p}'_s = (\widetilde{x}'_k, \widetilde{y}'_k)$ in $\widetilde{\mathsf{S}}'$. The observers O and $\widetilde{O}'$ are placed at $p_s = (0, r)$ in S and $\widetilde{p}'_s = (0, r')$ in $\widetilde{\mathsf{S}}'$, respectively.

3.1 Transformation under constant relative speed

With S given, the observation system S', which is the primed one corresponding to S, can be found by using (9). In TCR, the space-time coordinates of S are transformed into S' as follows [11,17]:

$$t' = \frac{t}{\cos\theta}, \qquad r' = \frac{r}{\cos\theta_R \cos\theta}, \qquad \phi' = \phi\cos\theta_R, \qquad z' = z, \tag{15}$$

where ϕ' is defined in S'. Note that the dilation of time takes place in the primed system. The observation system $\widetilde{\mathsf{S}}$ rotates at an angular frequency of ω_c relative to S and its rotation angle ϕ is given as (1). Then $\widetilde{\mathsf{S}}'$ rotates at an angular frequency of ω'_c with respect to τ' relative to S', which is calculated as $\omega'_c (= \mathrm{d}\phi'/\mathrm{d}\tau') = \omega_c \cos\theta_R \cos\theta$, where $\omega'_c = \omega'/ic$.

Consider the observers on the circumference of a circle of radius r in $\widetilde{\mathsf{S}}$. Their instantaneous speeds relative to S are the same. The radius r in the unprimed corresponds to r' in the primed which is given by the second equation of (15). A circle of radius r' in $\widetilde{\mathsf{S}}'$ is denoted by $\widetilde{\mathsf{S}}'_{r'}$. The normalized instantaneous speeds of $\widetilde{O}'$ in S' and $\widetilde{O}$ in S are $\beta' = ir'\omega'_c$ and $\beta = ir\omega_c$, respectively. Obviously $\beta' = \beta$. The instantaneous speeds are identical in TCR. The two-way speed of light has been known to be constant in inertial frames regardless of the propagation direction. A circle can be described by an infinite number of line segments. Hence $\widetilde{\mathsf{S}}'_{r'}$ can be considered to instantaneously consist of an infinite number of inertial frames with the same speed. Since the two-way speed of light is invariant in inertial frames, it would be reasonable to think that the two-way speed of light should be c also in $\widetilde{\mathsf{S}}'_{r'}$. However TCR is not consistent with the two-way speed constancy. Let us examine this matter.

In $\widetilde{\mathsf{S}}'$, a light beam traverses a circle of radius r' in the co- or counter-rotating direction. Using (15), we can find the one-way speed of light on the circumference [11,17]

$$c_{\mathrm{TCR}\pm} = c(1 \mp \beta) \tag{16}$$

where $c_{\mathrm{TCR}+}$ and $c_{\mathrm{TCR}-}$ are the one-way speeds of light in the co- and counter-rotating directions, respectively. The two-way speed is calculated as

$$c_{\mathrm{TCR}\updownarrow} = c(1 - \beta^2). \tag{17}$$

In TCR the two-way speed of light is other than c.

3.2 Transformation under constant two-way speed of light

The world lines that constitute an observation system can be found by using the Lorentz transformation of the coordinates of single observers. In the Lorentz transformation, the relative speeds between two observers are the same. Putting emphasis on the relativity in the representation, TCR has been formulated such that it holds the constant relative speed. The relative speeds are dependent on the clock synchronization and thus may not be accurately discovered because the problem of synchronization between two different places is raised. On the contrary, two-way speeds are independent of the synchronization. The rotating system can be regarded as locally inertial. The two-way speed of light is known to be c in inertial frames. However the two-way speed of light is not c in TCR, as shown above. Here we derive the transformation that has the constant two-way speed, rather than the constant relative speed.

The coordinate transformation between S and $\widetilde{\mathsf{S}}'$ which has the constancy of the two-way speed can be found by using (9). The spatial vector in S of $\widetilde{O}$ can be written as

$$\boldsymbol{p}_s = r[\sin\phi, -\cos\phi]^T. \tag{18}$$

One can see that the spatial coordinates of $\widetilde{O}$ are $\boldsymbol{p}_s = [0, -r]^T$ when $\tau = 0$. The differential coordinate vector is expressed as

$$\mathrm{d}\boldsymbol{p} = [\mathrm{d}\boldsymbol{\tau}, \mathrm{d}\boldsymbol{p}_s^T]^T, \tag{19}$$

where $\mathrm{d}\boldsymbol{p}_s = r\mathrm{d}\phi[\cos\phi, \sin\phi]^T$. Substituting (19) into (9), it follows that

$$\mathrm{d}\boldsymbol{\tau}' = \frac{\mathrm{d}\tau}{\cos\theta}\,, \tag{20a}$$

$$\mathrm{d}\widetilde{\boldsymbol{p}}'_s = \mathbf{0}. \tag{20b}$$

Equation (20b) indicates that $\widetilde{O}'$ is at rest in $\widetilde{S}'$ so that it rotates at an angular frequency relative to S'. As $\mathrm{d}\widetilde{\boldsymbol{p}}'_s = \mathbf{0}$, the radius of rotation of $\widetilde{O}'$ in S' cannot be obtained from (20b). However, fortunately it can be discovered through the transformation of $\widetilde{O}$ into the coordinate system S', which is made in such a way that [11]

$$\mathrm{d}\bar{\boldsymbol{p}}' = \boldsymbol{T}_L(\theta)\mathrm{d}\widetilde{\boldsymbol{p}}, \tag{21a}$$

$$\mathrm{d}\boldsymbol{p}' = \boldsymbol{A}(-\phi')\mathrm{d}\bar{\boldsymbol{p}}', \tag{21b}$$

where $\phi' = \phi\cos\theta_R$ when ϕ' is defined in S'. The observer $\widetilde{O}$ is at rest in $\widetilde{S}$ and its spatial coordinate vector in $\widetilde{S}$ can be written as $\widetilde{\boldsymbol{p}}_s = [0, -r]^T$. Then the $\mathrm{d}\widetilde{\boldsymbol{p}}$ is given by

$$\mathrm{d}\widetilde{\boldsymbol{p}} = [\mathrm{d}\tau, 0, 0]^T. \tag{22}$$

It is straightforward by using (21) and (22) to derive $\mathrm{d}\boldsymbol{p}'_s$ [11]

$$\mathrm{d}\boldsymbol{p}'_s = r' \begin{bmatrix} \mathrm{d}\phi'\cos\phi' \\ \mathrm{d}\phi'\sin\phi' \end{bmatrix}, \tag{23}$$

where the radius r' is given by

$$r' = \frac{r\cos\theta}{\cos\theta_R}\,. \tag{24}$$

Equation (23) results from the relativistic transformation of the coordinates in $\widetilde{S}$ of $\widetilde{O}$ to $\widetilde{S}'$, which is other than the transformation of the coordinates in S of $\widetilde{O}$. Nonetheless we can extract some information for the latter from the former. We have two options in determining r': one is such that the instantaneous speed of $\widetilde{O}'$ is equal to β as in (15) whereas the other is to employ the same radius as (24). TCR selects the former. However it seems reasonable to think that in the transformation of the unprimed coordinates into $\widetilde{S}'$, the radius would remain the same regardless of whether they are represented in $\widetilde{S}$ or S. In TCL, r' is given as (24) and the other coordinates of S' are equal to those in TCR

$$t' = \frac{t}{\cos\theta}, \qquad r' = \frac{r\cos\theta}{\cos\theta_R}, \qquad \phi' = \phi\cos\theta_R, \qquad z' = z, \tag{25}$$

where the initial condition that $t' = 0$ when $t = 0$ was used.

The angular frequency of $\widetilde{O}'$ is calculated as $\omega' = \mathrm{d}\phi'/\mathrm{d}t'$, which leads to

$$\omega' = \omega\cos\theta\cos\theta_R. \tag{26}$$

As $\widetilde{\mathsf{S}}'$ rotates at the angular frequency ω' with respect to S', the azimuthal angle $\widetilde{\phi}'$ is related to ϕ' by $\widetilde{\phi}' = \phi' - \omega' t'$ and the other coordinates of $\widetilde{\mathsf{S}}'$ are the same as the corresponding ones of S'. The coordinate transformation between $\widetilde{\mathsf{S}}'$ and S is written as

$$t' = \frac{t}{\cos\theta}, \qquad r' = \frac{r\cos\theta}{\cos\theta_R}, \qquad \widetilde{\phi}' = \phi\cos\theta_R - \omega' t', \qquad z' = z. \tag{27}$$

The coordinates of $\widetilde{\mathsf{S}}'$ are easily transformed into S by using (27). For example, let an arbitrary object $\widetilde{O}'_k$ be at rest on a circle of radius r' and its azimuthal angle be ϕ'_k in $\widetilde{\mathsf{S}}'$. As seen in S, the object $\widetilde{O}'_k$ is located at the circle of radius r and the azimuthal angle in S is $\phi = \omega t + \phi_k$, where $\phi_k = \phi'_k/\cos\theta_R$ and ω is related to ω' by (26).

3.2.1 Two-way speed of light

In the primed system, the length of an arc subtended by a differential angle $\mathrm{d}\phi'$ at a circle of radius r' is $r'\mathrm{d}\phi'$. Since the period of ϕ' is $2\pi\cos\theta_R$, the perimeter of the circle is given by

$$l'_p = 2\pi r'\cos\theta_R = l_p\cos\theta, \tag{28}$$

where $l_p = 2\pi r$ is the perimeter of the circle in the unprimed system. The frequency of rotation of $\widetilde{\mathsf{S}}'$ is $f' = f\cos\theta$, where $f = \omega/2\pi$. The instantaneous speed of $\widetilde{O}'$ is written as

$$\beta' = f' l'_p = \beta\cos^2\theta. \tag{29}$$

Note that β' is not equal to β.

As illustrated in fig. 2, a light signal takes a round trip between two spatial points $\widetilde{p}'_{s0}$ and $\widetilde{p}'_{s1}$ located at the same radius r' in the observation system $\widetilde{\mathsf{S}}'$ where $\widetilde{p}'_{s0} = (r', 0, 0)$ and $\widetilde{p}'_{s1} = (r', \mathrm{d}\widetilde{\phi}', \mathrm{d}z')$. The squared distance between the two spatial points is written as

$$\mathrm{d}\widetilde{l}'^2 = (r'\mathrm{d}\widetilde{\phi}')^2 + \mathrm{d}z'^2. \tag{30}$$

When a photon of the light signal moves in $\widetilde{\mathsf{S}}'$ by $r'\mathrm{d}\widetilde{\phi}'$ and $\mathrm{d}z'$ in the azimuthal and the z'-axis directions, respectively, it does in S by $r\mathrm{d}\phi$ and $\mathrm{d}z$ in the respective directions. Since the speed of light is c in S, it follows that

$$(c\mathrm{d}t)^2 - (r\mathrm{d}\phi)^2 - \mathrm{d}z^2 = 0. \tag{31}$$

Substituting $\mathrm{d}\phi = \mathrm{d}\widetilde{\phi} + \omega\mathrm{d}t$ into (31) and solving the quadratic equation of $c\mathrm{d}t$, we have

$$c\mathrm{d}t = \left[r\beta\mathrm{d}\widetilde{\phi} + ((r\mathrm{d}\widetilde{\phi})^2 + (1-\beta^2)\mathrm{d}z^2)^{1/2}\right]\cos^2\theta, \tag{32}$$

where $c\mathrm{d}t > 0$ irrespective of the sign of $\mathrm{d}\widetilde{\phi}$. It is easy to see from (25), (30), and (32) that $c\mathrm{d}t$ is expressed as

$$c\mathrm{d}t = r\beta\mathrm{d}\widetilde{\phi}\cos^2\theta + \mathrm{d}\widetilde{l}'\cos\theta. \tag{33}$$

The travel times $c\mathrm{d}t_+$ and $c\mathrm{d}t_-$ for $\widetilde{p}'_{s0} \to \widetilde{p}'_{s1}$ and $\widetilde{p}'_{s1} \to \widetilde{p}'_{s0}$, respectively, are given by

$$c\mathrm{d}t_\pm = \pm r\beta\mathrm{d}\widetilde{\phi}\cos^2\theta + \mathrm{d}\widetilde{l}'\cos\theta, \tag{34}$$

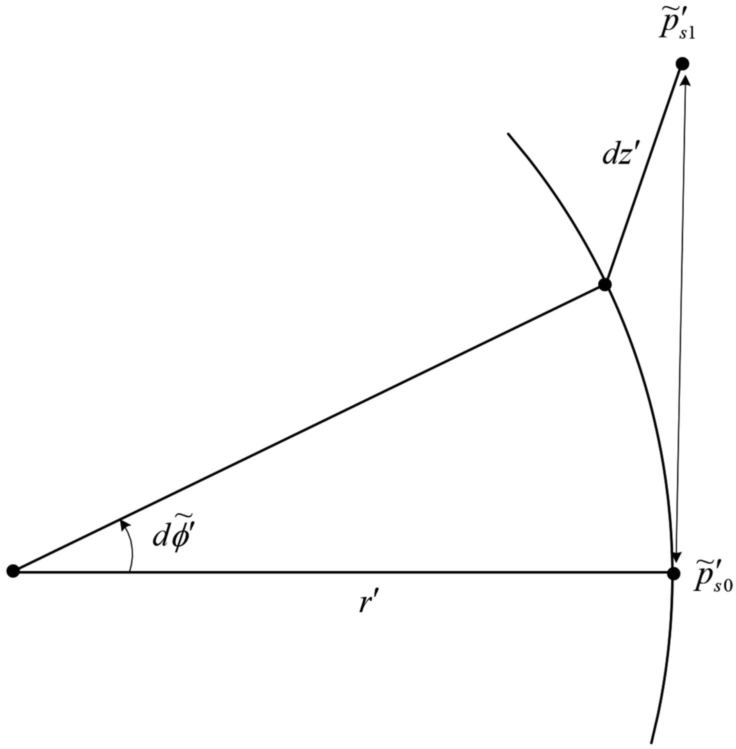

Fig. 2. Path of round trip at fixed r'.

where $\mathrm{d}\widetilde{\phi} > 0$. The time elapsed during the round trip is

$$c\mathrm{d}t_{\updownarrow} = c\mathrm{d}t_{+} + c\mathrm{d}t_{-} = 2\mathrm{d}\widetilde{l}' \cos\theta. \tag{35}$$

The round trip time in $\widetilde{\mathsf{S}}'$ is related to $\mathrm{d}t_{\updownarrow}$ by $\mathrm{d}\widetilde{t}'_{\updownarrow} = \mathrm{d}t_{\updownarrow}/\cos\theta$ according to the first equation of (27), which leads to

$$c_{\mathrm{TCL}\updownarrow} = \frac{2\mathrm{d}\widetilde{l}'}{\mathrm{d}\widetilde{t}'_{\updownarrow}} = c. \tag{36}$$

The round-trip speed of light is c in TCL.

3.2.2 Derivation of transformation for inertial observation systems

As r in (27) goes to infinity subject to the constraint that $\beta = r\omega/c$ is a constant, ω approaches zero so that $\widetilde{\mathsf{S}}'_{r'}$ becomes an inertial frame. In fig. 3, the circular plate rotates at an angular frequency of $\omega(\omega')$ relative to S in the unprimed (S' in the primed). If r' tends to infinity or $\mathrm{d}\widetilde{\phi}'$ approaches zero, the arc connecting points 1 and 2 in $\widetilde{\mathsf{S}}'_{r'}$ belongs to an inertial observation system S''. Hence it is expected that a transformation for inertial observation systems can be obtained from (27). As a consequence, the following transformation can be derived by taking the limit operation on (27):

$$\mathrm{d}\tau = \mathrm{d}\tau'' \cos\theta, \tag{37a}$$
$$\mathrm{d}x = \mathrm{d}\tau'' \sin\theta + \mathrm{d}x''/\cos\theta, \tag{37b}$$

where $\boldsymbol{p}'' = [\tau'', x'']^T$ represents the coordinate vector of S'' and the y''- and z''-coordinates, which are the same as the y- and z-coordinates of S, respectively, were suppressed. It is readily seen from (37) that the coordinate vector of S'' is related to that of S by

$$\boldsymbol{p}'' = \boldsymbol{T}_I(\theta)\boldsymbol{p}, \tag{38}$$

where

$$\boldsymbol{T}_I(\theta) = \begin{bmatrix} 1/\cos\theta & 0 \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{39}$$

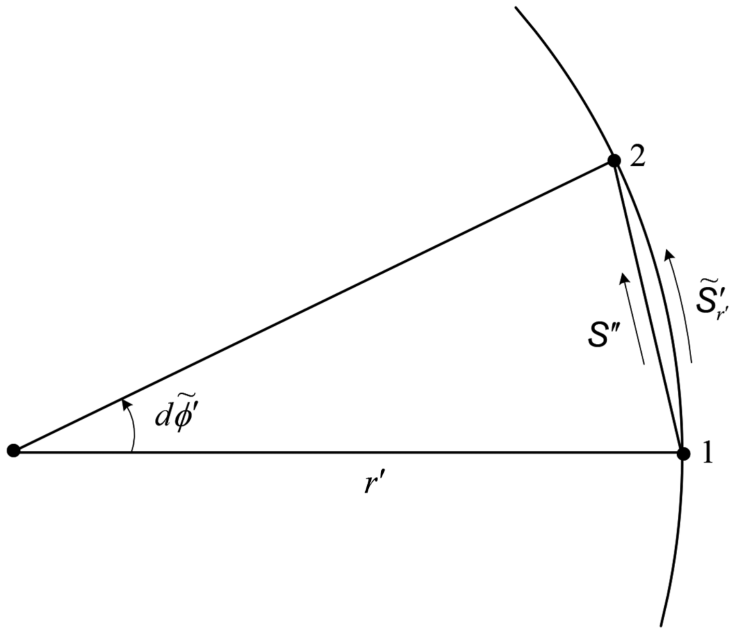

Fig. 3. Observation systems $\widetilde{\mathsf{S}}'_r$ and S''.

The transformation (38) has been investigated by several authors [18–22] and examined as the replacement of the Lorentz transformation in inertial motion. It has also been applied to explain the issues of relativistic rotation [3,4]. The one-way speed of light is anisotropic in S'', though its two-way speed is isotropic. Here (38) is referred to as the inertial transformation.

Suppose that in fig. 3, r' is so large or $\mathrm{d}\widetilde{\phi}'$ is so small that the arc subtended by $\mathrm{d}\widetilde{\phi}'$ can be approximated as a line segment which is in uniform rectilinear motion relative to S and lies in S''. From the third equation of (27), $\mathrm{d}\phi$ is expressed as

$$\mathrm{d}\phi = (\omega'\mathrm{d}t' + \mathrm{d}\widetilde{\phi}')/\cos\theta_R. \tag{40}$$

Multiplying by r both sides of (40), we have

$$r\mathrm{d}\phi = (r\omega'_c\mathrm{d}\tau' + r\mathrm{d}\widetilde{\phi}')/\cos\theta_R. \tag{41}$$

It is rewritten by using (26) and the second equation of (27) as

$$r\mathrm{d}\phi = r\omega_c\mathrm{d}\tau'\cos\theta + r'\mathrm{d}\widetilde{\phi}'/\cos\theta. \tag{42}$$

Substituting $r\omega_c(=\beta/i) = \tan\theta$ into (42), we have

$$r\mathrm{d}\phi = \mathrm{d}\tau'\sin\theta + r'\mathrm{d}\widetilde{\phi}'/\cos\theta. \tag{43}$$

As can be seen from fig. 3, the displacement $r'\mathrm{d}\widetilde{\phi}'$ in $\widetilde{\mathsf{S}}'$ corresponds to $\mathrm{d}x''$ in S'' and $r\mathrm{d}\phi$ corresponds to $\mathrm{d}x$. It is seen, by comparing the first equation of (27) and (43) with (37a) and (37b) that, when either r' tends to infinity or $\mathrm{d}\widetilde{\phi}'$ goes to zero, the transformation by (27) in a differential area is reduced to (37), which implies that the rotating observation system can be regarded as locally inertial. The inertial transformation can be derived from (27) in the limit, and the latter is consistent with the former.

The circular approach is based on the transformation of instantaneous inertial frames. It may be pointed out that if (38) is correct for the transformation between the isotropic and an inertial frame, the circular approach would not be accurate because it employs the Lorentz transformation for the instantaneous transformation as seen in (9b) and (11b). In (38) and the Lorentz transformation, their spatial components are the same and only the time components are different. In case the standard synchronization is introduced into the inertial transformation, the resulting one becomes equal to the Lorentz transformation. We use the circular approach to find the world lines of observers. The proper time is independent of clock synchronization. The circular approach is accurate as far as world lines are concerned.

4 Analysis of the Sagnac effect

A circular plate rotates around its center with an angular velocity ω_0 and a light source is located at a radius r_0 from the center as seen in a laboratory frame S''. Two counter-rotating light beams, which leave the light source at $t = t' = 0$, traverse the circular paths in opposite directions and then are received by a light detector $\widetilde{O}'$, which is placed at the same place as the light source. We denote the co-rotating and counter-rotating light beams by b_+ and b_-, respectively. It is assumed that S'' is isotropic so that $\mathsf{S}'' = \mathsf{S}$. Moreover we assume that a complete rotation in S corresponds to an angle of 2π. The travel times of $b_\pm$ when observed in S are calculated, respectively, as

$$t_{D\pm} = \frac{2\pi r_0}{c(1 \mp \beta)}\,, \tag{44}$$

where $\beta = r_0\omega_0/c$. Then the difference between the travel times is given by

$$\Delta t_d = \frac{4\pi r_0^2\omega_0}{c^2(1-\beta^2)}\,. \tag{45}$$

Examining the motions of $b_\pm$ in the primed system through the coordinate transformation (25) or (27), one can find the difference in travel time. In S, the speeds of $b_\pm$ are c and their rotational angles $\phi_{/\pm}$ are written, respectively, as

$$\phi_{/\pm} = \pm\frac{ct}{r_0}\,. \tag{46}$$

In S', from (46) and the first and third equations in (25), $\phi'_{/\pm}$ are given by

$$\phi'_{/\pm} = \phi_{/\pm}\cos\theta_R = \pm\frac{ct'\cos^2\theta}{r'_0}\,. \tag{47}$$

The angular frequencies of $b_\pm$ in S' are given by

$$\omega'_{/\pm} = \frac{\mathrm{d}\phi'_{/\pm}}{\mathrm{d}t'} = \pm\frac{c\cos^2\theta}{r'_0}\,. \tag{48}$$

When $b_\pm$ complete the travels, their rotational angles are $\phi_{/D\pm} = \pm 2\pi/(1\mp\beta)$ in S, which correspond to

$$\phi'_{/D\pm} = \pm\frac{2\pi\cos\theta_R}{1\mp\beta}\,, \tag{49}$$

as seen in S'. The length of the arc subtended by an angle $\mathrm{d}\phi'$ is $\mathrm{d}l' = r'_0\mathrm{d}\phi'$. The arrival times in the primed system are obtained as

$$t'_{D\pm} = \frac{r'_0\phi'_{/D\pm}}{r'_0\omega'_{/\pm}} = \frac{2\pi r'_0\cos\theta_R}{c(1\mp\beta)\cos^2\theta}\,. \tag{50}$$

It is rewritten by using the second equation in (25) as

$$t'_{D\pm} = \frac{2\pi r_0}{c(1\mp\beta)\cos\theta}\,. \tag{51}$$

The time difference $\Delta t'_d = t'_{D+} - t'_{D-}$ in S' is given by

$$\Delta t'_d = \frac{4\pi r_0^2\omega_0}{c^2(1-\beta^2)^{1/2}}\,. \tag{52}$$

It is seen by comparing (52) and (45) that $\Delta t'_d = \Delta t_d/\cos\theta$ and by comparing (51) and (44) that $t'_{D\pm} = t_{D\pm}/\cos\theta$. These time relationships are consistent with the first equation in (25).

Now we are in a position to calculate the speeds of $b_\pm$ in $\widetilde{\mathsf{S}}'$. The light beams as seen in $\widetilde{\mathsf{S}}'$ rotate the respective circular paths once, and so their travel distances are the same as the perimeter of the circle of radius r'_0. The times elapsed during the travels of $b_\pm$ are $t'_{D\pm}$. From (28) and (50), the speeds of $b_\pm$ are computed as

$$c_{\mathrm{TCL}\pm} = \frac{l'_p}{t'_{D\pm}} = c(1\mp\beta)\cos^2\theta. \tag{53}$$

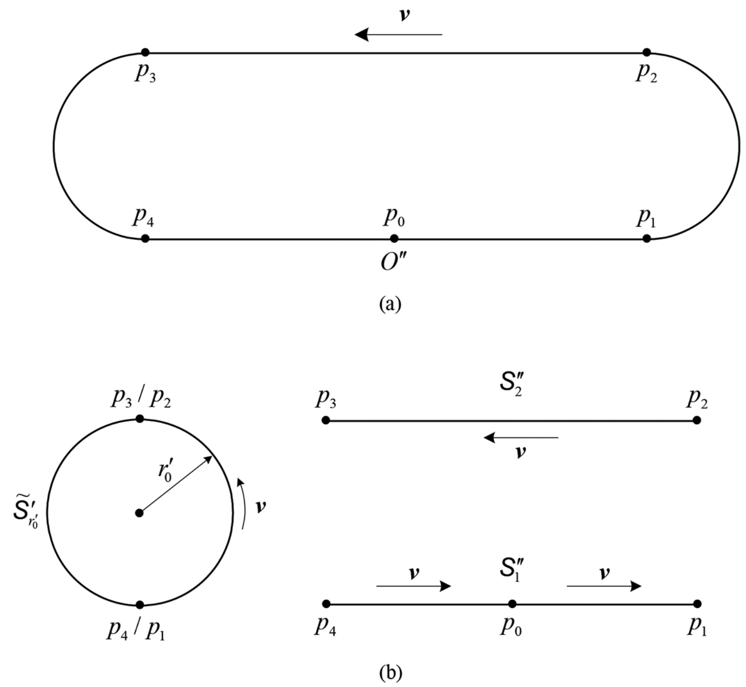

Fig. 4. (a) Optical fiber loop for an experiment of the generalized Sagnac effect. (b) Disintegration of the loop into circular and linear paths.

Once the direction of rotation of a light signal is given, its speed does not depend on the position on the circumference. The instantaneous speeds of $b_\pm$, which are defined as $\mathrm{d}l'/\mathrm{d}t'_\pm$ where $\mathrm{d}t'_\pm$ are the times elapsed when $b_\pm$ travel a differential distance $\mathrm{d}l'$, are the same as $c_{\mathrm{TCL}\pm}$. It is easy to see by using $\mathrm{d}l'/\mathrm{d}t'_\pm = c_{\mathrm{TCL}\pm}$ that the two-way speed of light on the circumference is c.

The Sagnac effect has been observed in linear motion as well as in circular motion [13,14]. An optical fiber loop for an experiment of the generalized Sagnac effect is illustrated in fig. 4(a) where it rotates at a speed of v, and a light source and a light detector O'' are placed at the same position p_0. Two counter-propagating light signals b_+ and b_- travel around the loop in the co- and counter-rotating directions, respectively. We can divide the loop into a circular path and two linear paths as in fig. 4(b) where the two half-circles are connected to each other. The circle belongs to a rotating observation system $\widetilde{\mathsf{S}}'$ and its radius is r'_0. Then the angular frequency of $\widetilde{\mathsf{S}}'_{r'_0}$ is given by $\omega'_0 = \omega_0 \cos\theta \cos\theta_R$, where $\omega_0 = v/r_0$, with r_0 denoting the radius of the circle in S. The line segments p_1p_4 and p_2p_3 in fig. 4(a) belong to different inertial observation systems S''_1 and S''_2 as their directions of motion are different, though the speeds are the same.

From (25) and (26) $r'_0\omega'_0 = r_0\omega_0 \cos^2\theta$. The time difference in $\widetilde{\mathsf{S}}'_{r'_0}$ is obtained as (52), which can be rewritten as

$$\Delta t'_c = \frac{2l'_c\beta}{c}\,, \tag{54}$$

where $\beta = v/c$ and l'_c is the perimeter of the circle, which is given as (28) with $r' = r'_0$. The time difference (54) results from the difference in the speeds of $b_\pm$ as seen in (53). Once the propagation direction of light is determined in $\widetilde{\mathsf{S}}'_{r'}$, its speed is constant anywhere on the perimeter of the circle of radius r'. Given a constant β, (53) is independent of radius r' and is valid even if r' goes to infinity. As shown above, S''_1 and S''_2 are equivalent to some parts of $\widetilde{\mathsf{S}}'_{r'_0}$ when r'_0 goes to infinity. Hence the time difference in the inertial observation systems S''_1 and S''_2 becomes equal to

$$\Delta t'_i = \frac{2l'_i\beta}{c}\,, \tag{55}$$

where l'_i is the sum of the lengths of the two linear paths in S''_1 and S''_2. In a different manner, the same $\Delta t'_i$ can be obtained by using the inertial transformations between S and S''_1 and between S and S''_2. The total time difference is given by

$$\Delta t'_d = \frac{2l'\beta}{c}\,, \tag{56}$$

where $l' = l'_c + l'_i$ is the rest length of the fiber loop. The analytic result of (56) corresponds with the experimental results of the generalized Sagnac effect [13,14].

5 Conclusions

The coordinate transformation between the rotating observation system $\widetilde{\mathsf{S}}'$ and the isotropic observation system S has been derived by using the differential equations of the circular approach. In the derivation we can choose one of the two options to determine the radius r' in $\widetilde{\mathsf{S}}'$. TCR was formulated such that the relative speeds of O' and $\widetilde{O}$ with respect to each other are the same, but does not hold the constancy of the two-way speed of light. In contrast, TCL employs the radius obtained by the transformation of $\widetilde{O}$ to $\widetilde{S}'$, and shows the constant two-way speed of light for fixed r. Moreover, $\widetilde{\mathsf{S}}'$ can be regarded as locally inertial, and accordingly the inertial transformation (38) can be obtained from TCL through the limit operation that r' tends to infinity or that $\mathrm{d}\widetilde{\phi}'$ approaches zero. These features of TCL lead to a comprehensive analysis of the generalized Sagnac effect which involves linear motion as well as circular motion. The theoretical analysis via TCL indicates that the one-way speed of light is anisotropic not only in the rotating system but also in the inertial one, though its two-way speed is isotropic. Also in the global positioning system (GPS), the one-way speed of light has been clearly shown to be anisotropic [17]. In the GPS, the Sagnac correction is required because the light speed is not a constant in the Earth centered Earth fixed (ECEF) frame. The anisotropy of the one-way speed in inertial frames is shown in the inertial transformation as well. The experimental results of the Sagnac effect including the ones of the generalized Sagnac effect, the empirical evidence in the GPS, and the consistency between TCL and the inertial transformation validate both the transformations.

References

1. M.G. Sagnac, C. R. Acad. Sci. Paris **157**, 708 (1913).
2. E.J. Post, Rev. Mod. Phys. **39**, 475 (1967).
3. G. Rizzi, M.L. Ruggiero (Editors), *Relativity in Rotating Frames* (Kluwer Academic, Dordrecht, The Netherlands, 2004).
4. R.D. Klauber, Found. Phys. **37**, 198 (2007).
5. D. Landau, E.M. Lifschitz, *The Classical Theory of Fields* (Pergamon Press, New York, 1971).
6. R.A. Mould, *Basic Relativity* (Springer, New York, 1994).
7. T.A. Weber, Am. J. Phys. **65**, 946 (1997).
8. P. Maraner, J.-P. Zendri, Gen. Relativ. Gravit. **44**, 1713 (2012).
9. A. Tartaglia, M.L. Ruggiero, Am. J. Phys. **83**, 427 (2015).
10. F. Hasselbach, M. Nicklaus, Phys. Rev. A **48**, 143 (1993).
11. Y.-H. Choi, Phys. Essays **29**, 167 (2016).
12. A.A. Michelson, E.W. Morley, Am. J. Sci. **34**, 333 (1887).
13. R. Wang, Y. Zheng, A. Yao, Phys. Rev. Lett. **93**, 143901 (2004).
14. R. Wang, Y. Zheng, A. Yao, D. Langley, Phys. Lett. A **312**, 7 (2003).
15. D.C. Champeney, G.R. Isaak, A.M. Khan, Proc. Phys. Soc. **85**, 583 (1965).
16. W. Kündig, Phys. Rev. **129**, 2371 (1963).
17. Y.-H. Choi, arXiv:1506.02688 (2015).
18. F.R. Tangherlini, Nuovo Cimento Suppl. **20**, 1 (1961).
19. F. Selleri, Found. Phys. **26**, 641 (1996).
20. F. Selleri, Found. Phys. Lett. **18**, 325 (2005).
21. G. Rizzi, M.L. Ruggiero, A. Serafini, Found. Phys. **34**, 1835 (2004).
22. R. de Abreu, V. Guerra, Eur. J. Phys. **29**, 33 (2008).