curves are only slightly less impressive, showing first the trading off of axial vs lateral color alluded to above and second a somewhat enlarged secondary spectrum in the oblique spherical. Whether or not this is real depends somewhat on the accuracy of the catalogue values for the refractive index given by the glass manufacturer. In any case, there appears no question that the requirement for 0.1-sec resolution is attained or that the image is actually or very nearly diffraction limited.

There appears no trace of the chromatic coma[6] referred to by Baker in the analogous Reflector-Corrector system. However, it may be pointed out that our system covers 1° at F:10, and this secondary aberration may be expected to show up at higher relative apertures and larger field angles.

[6] J. G. Baker in *Amateur Telescope Making* (Book 3) (Scientific American, Inc., New York, 1953), p. 8.

It is probable that at higher relative apertures, it may prove advantageous to get a closer approximation for the mirror by adding a sixth power term in which case the surface will depart more or less from the hyperboloidal form. In this regard one may designate the 4th power term as the "Seidel" or third-order approximation, and the 6th power and higher terms as belonging to the usual "higher order" category.

However, there is no doubt that the three element design discussed in this article may be extended over a wide range of apertures, both physical and relative.

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Curvature of Binocular Visual Space. An Experiment

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The sign of the curvature of any geometry is an intrinsic property independent of its coordinatization. Accordingly, it is possible in principle to determine the sign of the curvature of binocular visual space without employing knowledge of the particular relationship between the physical stimulus and the associated visual geometry. A simple experiment for making this determination is described and the outcome for a number of observers is presented. For most of the observers the indicated curvature is negative, in agreement with the preponderance of earlier findings.

1. INTRODUCTION

In the mathematical theory of binocular visual space developed by Luneburg and the author, the one feature which has evoked the greatest interest is the interpretation of certain experiments as evidence for the non-Euclidean character of visual space.[1,2] In a recent paper,[3] the author presented a theoretical analysis of known experimental results in this field. The indication of this analysis is that the experimental data can be given a mutually consistent interpretation only under the hypothesis that for most observers the curvature of visual space is negative as in the well-known hyperbolic geometry of Lobachevski and Bolyai.

No single experiment is absolutely decisive in reaching the conclusion of negative curvature. For the earlier experiments the conclusion in each case depends upon other experimentation either to determine the nature of the transformation connecting visual and physical coordinate frames or to assign otherwise a theoretical interpretation to the data. Since the sign of the curvature is an intrinsic property of the geometry, and is perfectly well defined independently of the coordinatization, it is feasible to make the determination of sign by a direct experiment independently of other considerations. Such an independent determination has obvious value if taken in relation to other experiments. Nonetheless, it remains an important consideration that no single experiment on any one observer can by itself be taken as a sufficient characterization of that observer's visual space. It is always possible that something entirely outside the realm of the experiment may affect the observer's performance and we shall see in Sec. 3 that this may conceivably occur on the part of at least one observer.

We shall characterize negative curvature by a property which not only is valid for Riemannian spaces but is so fundamental that it may actually be used to define negative curvature for a very broad class of metric spaces, the G spaces of Busemann.[4] It will be recalled

[1] R. K. Luneburg, *Mathematical Analysis of Binocular Vision* (Princeton University Press, Princeton, New Jersey, 1947).
[2] A. A. Blank, J. Opt. Soc. Am. 43, 717 (1953).
[3] A. A. Blank, J. Opt. Soc. Am. 48, 911 (1958).
[4] H. Busemann, *The Geometry of Geodesics* (Academic Press, Inc., New York, 1955).

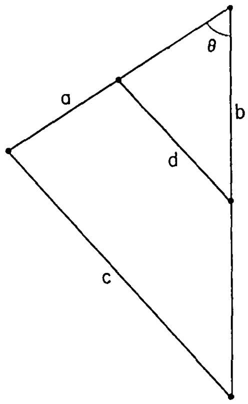


FIG. 1.

that in Euclidean geometry the line joining the midpoints of two sides of a triangle is equal in length to half the third side. A G space will have negative curvature if and only if every point is contained in a domain within which each triangle has the property that the line joining the midpoints of two of the sides is less in length than half the third side. The problem of assigning a definite numerical value to the curvature of a G space, rather than a sign alone, has yet to be solved completely; however, recently Kann has demonstrated how to obtain a numerical estimate of the absolute curvature in the form of an upper bound.[5]

We do not actually need all the generality of G spaces since it appears that binocular visual space may be described accurately enough as one of the Riemannian spaces, and, in fact, as one of the three homogeneous spaces, spherical, Euclidean, or hyperbolic.[6] It is desirable, however, to involve no further assumptions in determining the sign of the curvature of the space than are absolutely necessary. An experiment based on the definition of negative curvature given above will have additional value because of the fact that the conclusions obtained will be largely independent of other theoretical and experimental considerations.

It is not difficult to obtain the requisite inequalities analytically for the spherical and hyperbolic geometries. Let a, b, c denote simultaneously the sides of a triangle and their lengths (Fig. 1). Let d be the line segment joining the midpoints of a and b, and θ the angle included by a and b. From the law of cosines in the hyperbolic and spherical spaces we may calculate the relation between c and d. In hyperbolic space we have

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\theta \qquad \text{(1a)}$$

$$\cosh d = \cosh\tfrac{1}{2}a \cosh\tfrac{1}{2}b - \sinh\tfrac{1}{2}a \sinh\tfrac{1}{2}b \cos\theta; \qquad \text{(1b)}$$

in elliptic space,

$$\cos c = \cos a \cos b + \sin a \sin b \cos\theta \qquad \text{(2a)}$$

$$\cos d = \cos\tfrac{1}{2}a \cos\tfrac{1}{2}b + \sin\tfrac{1}{2}a \sin\tfrac{1}{2}b \cos\theta. \qquad \text{(2b)}$$

[5] E. D. Kann, dissertation, New York University, New York, 1960.

[6] A. A. Blank, J. Opt. Soc. Am. 48, 328 (1958).

From (1a) and (1b) it follows that

$$\cosh 2d = \cosh c - 2 \sinh^2\tfrac{1}{2}a \sinh^2\tfrac{1}{2}b \sin^2\theta. \qquad (3)$$

Since the hyperbolic cosine is a monotonically increasing function it follows that for all triangles in hyperbolic space

$$2d < c. \qquad (4)$$

Similarly, for spherical space we obtain

$$\cos 2d = \cos c - 2 \sin^2\tfrac{1}{2}a \sin^2\tfrac{1}{2}b \cos\theta. \qquad (5)$$

From the fact that the cosine is a monotonically decreasing function in the interval from 0 to π it can be shown that for all sufficiently small triangles (e.g., triangles such that a, b, c, $<\pi$) in elliptic space,

$$2d > c. \qquad (6)$$

2. EXPERIMENT

Three starlike lights A, B, C determining the experimental triangle are presented to the observer in the eye-level plane. In that plane a coordinate system is chosen so that the x axis is directed sagittally forward along the median line. The y axis is directed toward the observer's left along a line joining the apexes of his corneas. The coordinates of a point (x,y) will be given in inches and decimal fractions of an inch. For convenience in taking and presenting data the triangle is made symmetric to the median with vertexes $C=(108,0)$, $A=(28,12)$, $B=(28,-12)$. This does not preclude the use of asymmetric observers such as sufferers from aniseikonia. For a strongly asymmetric observer the data from the right and left sides cannot be pooled. No such observers were found in this series.

The observer is first presented the three lights A, B, C alone; these are kept fixed throughout the experiment. A fourth light is introduced somewhere to the left of the median and the observer is told to ask the experimenter to move the light in order to satisfy the instruction, "Place this light so that you see it as lying on the left side of the triangle exactly equidistant from the two endpoints. In executing this task be sure to look squarely at each light and fix its position carefully rather than superficially glide from light to light." The fourth light is then turned off and a fifth light is introduced at the right and the same task is performed on the right side of the triangle. After this initial setting, both lights are turned on and remain on simultaneously; the observer is asked to repeat the bisection of the sides of the triangle a variable number of times. Between the observer's settings the lights are displaced at random so that the observer makes a fresh beginning each time. The experiment is repeated until evidence of any continuing trend is not apparent in the last five or more settings. The medians of the x and y coordinates after the termination of the trend are used as the most convenient representative data. We denote these median data by (x_α,y_α) and (x_β,y_β) for the left and right sides,

TABLE I. Representative settings for seven observers in order of declining indication of negative curvature as given by $\bar{y}$. The entries n, k are, respectively, the total number of settings for the bisection points α and β, and the number of settings counted from the terminus of the series for the determination of the various averages.

Observer	Age	γ', γ'' $\bar{y}$	$\bar{x}$	α, β y^*	x^*	n	k
GAH	36	5.58 ± 0.34	28.40 ± 0.31	2.89 ± 0.26	91.00 ± 0.59	20	5
RGB	15	4.32 ± 0.31	28.43 ± 0.40	3.43 ± 0.32	90.69 ± 0.52	15	5
PE	16	4.07 ± 0.45	29.36 ± 0.47	3.93 ± 0.54	73.04 ± 1.19	14	5
RRC	35	3.40 ± 0.40	28.76 ± 0.17	2.14 ± 0.48	94.90 ± 0.98	11	5
MD	15	1.68 ± 0.64	28.15 ± 0.68	5.00 ± 0.57	71.46 ± 1.39	14	14
JG	17	0.80 ± 0.28	28.76 ± 1.00	4.67 ± 0.45	70.50 ± 0.68	11	5
WHF	14	-0.10 ± 0.74	28.34 ± 0.24	3.46 ± 0.39	93.25 ± 0.76	31	7

respectively. In the table we list the means on the right and left of the representative data, namely,

$$x^* = \tfrac{1}{2}(x_\alpha + x_\beta), \quad y^* = \tfrac{1}{2}(y_\alpha - y_\beta).$$

The tolerance indicated is the maximum of the root-mean-square deviations from the median on each side. In almost every case the distance of the medians on the two sides from their mean is definitely less than the maximum root-mean-square tolerance; in those few cases where it is larger, we give this larger figure instead. Also given are the number n of repetitions of the experiment and the number k of settings from which the median is taken.

In the second part of the experiment, two lights are fixed at $\alpha = (x^*, y^*)$ and $\beta = (x^*, -y^*)$ (Fig. 2). The observer is asked whether these lights satisfy the criterion of the instructions. In no case was the answer negative. Next he is instructed to set a light on the base AB of the triangle first so that (1) the distance from A to the new light equals that from α to β and then so that (2) the distance from B to the new light is equal to the distance from α to β. Five settings are made alternately under each of these instructions. The medians of the coordinates of the set points are used to determine points

$$\gamma' = (x', y'), \quad \gamma'' = (x'', y'')$$

which serve as representative data. The sensory relation assumed to hold for the interpretation of the experiment is

$$\alpha\beta = A\gamma' = \gamma'' B.$$

The means

$$\bar{x} = \tfrac{1}{2}(x' + x'') \quad \text{and} \quad \bar{y} = \tfrac{1}{2}(y' - y'')$$

and their tolerances (computed as for α and β) are given in Table I.

The interpretation of this experiment is that the curvature of visual space is positive if the segments $A\gamma'$ and $\gamma'' B$ lap over, and negative if they do not. In the table, positive curvature is indicated by a negative value of $\bar{y}$, and negative curvature by a positive $\bar{y}$. Thus it is seen that six observers set hyperbolically and that one observer does not exhibit significant curvature. In none of the hyperbolic observers did any of the five settings for $A\gamma'$ lap over any setting for $\gamma'' B$. We shall say that an observer is the more strongly hyperbolic the larger $\bar{y}$. The tabulation is in order of decreasing strength.

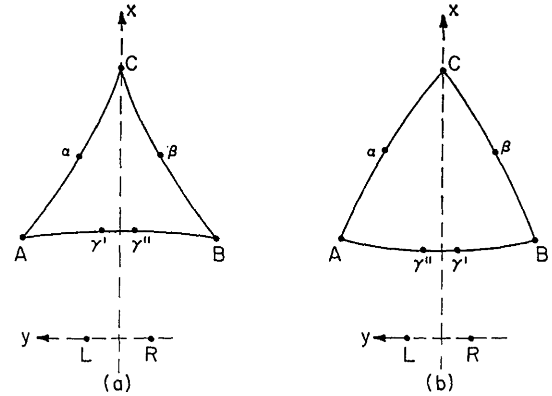


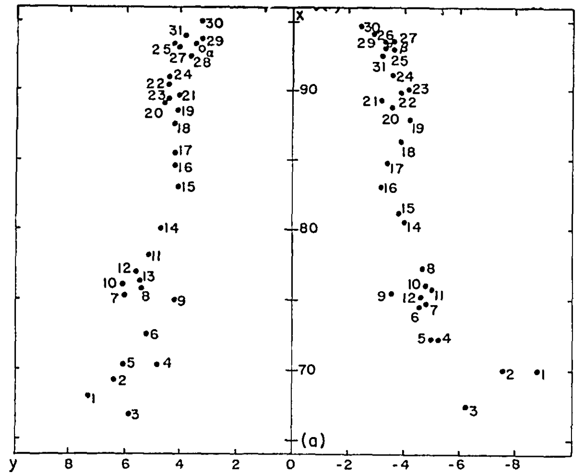
FIG. 2. Schematic representation of possible experimental settings. (a) Hyperbolic. (b) Spherical.

3. DISCUSSION OF THE EXPERIMENTAL RESULTS

The most curious fact about the data of the table is that there is no evidence that the manner of bisecting the sides is in any way directly related to the magnitude of y^*. This conflicts with intuitive feelings about the phenomenon of binocular "size constancy." The more distally α and β are set, the shorter the segment $\alpha\beta$ and hence, presumably, the shorter $A\gamma'$ and the greater the likelihood and the degree of hyperbolic settings. The evidence does not sustain any such presupposition in relating the performances of different individuals. In this connection, it should be noted that none of the observers had ever taken part in an experiment of this type before. As demonstrated by a marked initial trend in the data of GAH, RGB, RRC, and WHF, the observer's initial performance in the darkroom may not be the same as his eventual performance when he becomes accustomed to the conditions. This is typical of the novice observer and similar trends have been reported in other experiments.[7] For this reason, caution must be exerted in accepting the result of a single experiment on a single observer at face value. There is, moreover, a genuine possibility that some observers could not act independently of auditory clues from the experimenter's voice or the motion of the light standards. It would be desirable to use an apparatus in which no such secondary clues to the position of the lights are available.

The performance of the Euclidean observer WHF is distinctive enough to report in detail. As with a number of other observers, his settings exhibited a marked initial trend (Fig. 3). The only specially remarkable feature about this trend is its length. The first 18 settings were taken in one session and the conclusion of the experiment was put off to a later day when 13 more settings

[7] L. Hardy, G. Rand, and M. Rittler, A.M.A. Arch. Opthalmol. 45, 53 (1951).



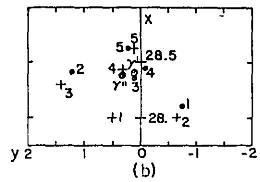


FIG. 3. Performance of Euclidean observer WHF exhibiting a marked initial trend in the settings for the visual midpoints of the sides. There is no clear distinction between the settings for γ' and γ''. (a) Settings for α and β. (b) Settings for γ and γ''.

were taken. Of these, the last seven were used to obtain the medians α and β.

Upon hearing instruction 2 WHF asked,

"Isn't there a theorem here?" (meaning the Euclidean theorem, of course) and was told

"Forget the theorem and go by what you feel directly."

"I can't forget the theorem."

In Fig. 4 we present another performance, that of PE, as typical of an observer without a well-marked trend.

From the kind of trend exhibited by observer WHF and others it can be seen why early data are discarded in Table I. General experience in the domain of pure binocular observation has demonstrated the need for practice on the part of the observer before he accustoms himself to a situation in which he operates on minimally sufficient clues. Sometimes the effect of practice on a given observer is exhibited by an increasing consistency in the settings, sometimes by an initial trend which leads eventually to stable settings. The effects of initially variable behavior can, of course, be reduced by taking averages of sufficiently many observations. Such a course proves to be prohibitively expensive, both in terms of time and the patience of the observer. For that reason, data are taken until it becomes definite that the observer's settings are random deviations from some statistical average. All earlier settings are discarded. Perhaps it is significant that the observer is generally not conscious of the change in character of his settings.

4. FURTHER MATHEMATICAL ANALYSIS

This experiment may also be used to determine the polar coordinates of the points of visual space in the manner indicated in the references.[2,3] In this way we are able to make a quantitative interpretation of the results of the experiment in the hyperbolic metric. So, for example, in evaluating the performances of our observers above, we utilized, rather loosely, the magnitude of $\bar{y}$ as a measure of the degree of curvature. The magnitude of the visual radial coordinate of point C in the units of the metric distance formula (1a) is a more precise measure. Finally a quantitative interpretation of the experiment permits the comparison of the results of this experiment with others for the same observer and enables us to check some of the predictions deriving from other aspects of the theory.

We shall assume that visual azimuth angle φ is a known function of physical coordinates. From the data of the experiment it is possible to calculate the values of the visual radial coordinate for points of the stimulus, using the known values of sensory visual angle. We ascribe visual polar coordinates to each point of the stimulus (Fig. 5) as follows: $C=(r_0,\varphi_0)$, $\alpha=(r_1,\varphi_1)$, $\beta=(r_1,-\varphi_1)$, $A=(r_2,\varphi_2)$, $B=(r_2,-\varphi_2)$, $\gamma'=(R,\psi)$, $\gamma''=(R,-\psi)$.

For the purposes of exposition we introduce auxiliary points, D the midpoint of $\alpha\beta$, and E the midpoint of

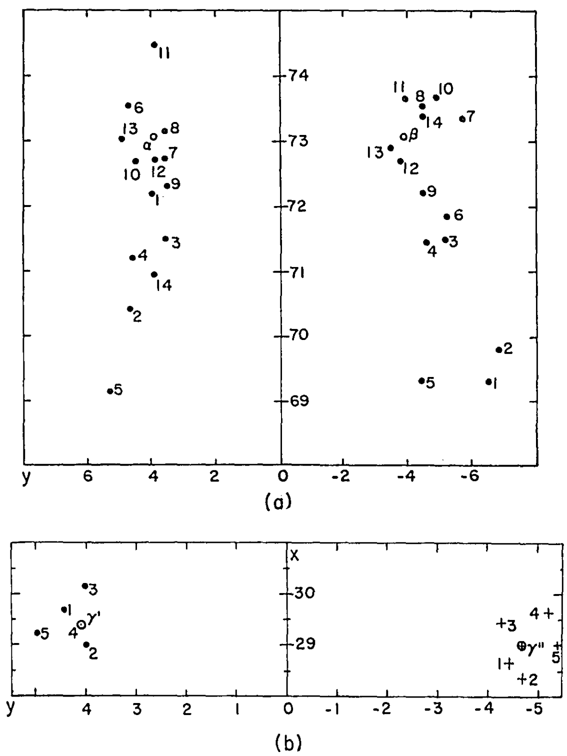


FIG. 4. Performance of hyperbolic observer PE exhibiting a slight, probably not significant, trend. The settings for γ' and γ'' are plainly distinguished. (a) Settings for α and β. (b) Settings for γ and γ''.

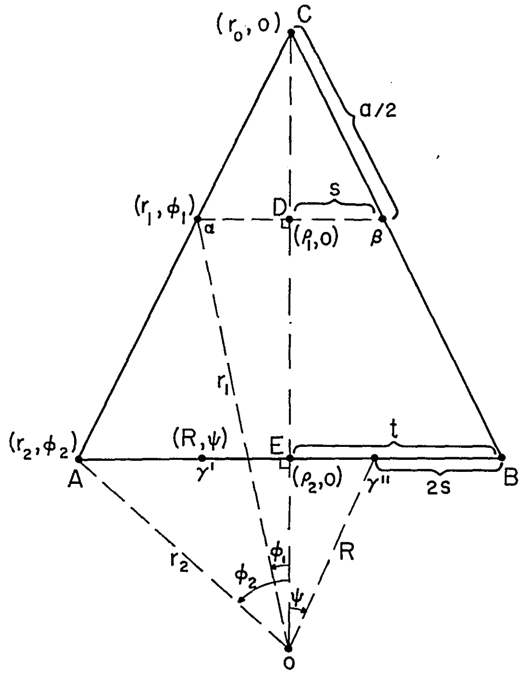


FIG. 5. Schematic representation of the configuration in visual space exhibiting the significant polar coordinates and visual distances.

AB. The visual lengths a, s, t, ρ_1, ρ_2 are defined by

$$a/2=\alpha A=\alpha C=\beta C=\beta B$$
$$s=\alpha D=\beta D$$
$$t=AE=BE$$
$$\rho_1=OD, \quad \rho_2=OE.$$

The analysis we give applies to the hyperbolic case; a very similar development may be obtained for the spherical case. The system of equations presented is not designed for elimination, but for the use of an iteration scheme in its solution. In any case, an attempt at a direct solution by elimination of variables seems to lead to a polynomial equation of high degree.

Given r_0 we may determine the value of r_2 by applying the law of sines to the angles $O\alpha A$ and $O\alpha C=\pi-O\alpha A$:

$$\sinh r_2=\sinh r_0 \sin\varphi_1/\sin(\varphi_2-\varphi_1). \tag{7}$$

From r_2 we may then calculate ρ_2 and t using

$$\tanh\rho_2=\tanh r_2 \cos\varphi_2 \tag{8}$$

and

$$\sinh t=\sinh r_2 \sin\varphi_2. \tag{9}$$

Having t and ρ_2, we determine s by

$$\tanh(t-2s)=\sinh\rho_2 \tan\psi, \tag{10}$$

and from s, we obtain r_1 from

$$\sinh r_1=\sinh s/\sin\varphi_1. \tag{11}$$

From r_1 we obtain ρ_1:

$$\tanh\rho_1=\tanh r_1 \cos\varphi_1. \tag{12}$$

We need one more equation to complete the system in the seven unknowns r_0, r_1, r_2, ρ_1, ρ_2, s, t. The last equation is obtained by applying the law of sines to $\sphericalangle \alpha AO=\sphericalangle CAO$:

$$\frac{\sinh r_0 \sin\varphi_2}{\sinh a}=\frac{\sinh r_1 \sin(\varphi_2-\varphi_1)}{\sinh\frac{1}{2}a} \tag{13a}$$

Together with Eq. (11) and

$$\cosh\tfrac{1}{2}a=\cosh s \cosh(r_0-\rho_1) \tag{13b}$$

we obtain from (13a) the relation

$$\cosh(r_0-\rho_1) =\sinh r_0 \sin\varphi_2 \sin\varphi_1/\sinh 2s \sin(\varphi_2-\varphi_1). \tag{13}$$

In principle we have given above a chain of eliminations which ends in an equation for r_0 alone. Rather than attempt to carry out the eliminations, we give an iteration scheme for the solution of the system (7)–(13).

Let ξ_1 be a first numerical approximation to the intended solution r_0. Go through the sequence of equations (7)–(12) determining the values of the other variables when r_0 is replaced by ξ_1. Use Eq. (13), however, to determine a second numerical approximation ξ_2 by

$$\cosh(\xi_2-\rho_1) =\sinh\xi_1 \sin\varphi_2 \sin\varphi_1/\sinh 2s \sin(\varphi_2-\varphi_1). \tag{14}$$

By iterating this scheme, putting ξ_2 in place of r_0, etc., we obtain a sequence of approximations $\xi_1, \xi_2, \xi_3, \ldots$ which converges to r_0. Note that the other conceivable iteration scheme,

$$\sinh\xi_2=\cosh(\xi_1-\rho_1) \sinh 2s\frac{\sin(\varphi_2-\varphi_1)}{\sin\varphi_2 \sin\varphi_1}$$

is unstable and its associated sequence $\{\xi_k\}$ tends either to the useless solution $r_2=0$ or diverges to infinity. The determination of a solution is greatly facilitated by the choice of a good first approximation to r_0. As a guide we have computed the values of r_0 for the strongest hyperbolic observer GAH and the weakest JG, based on the premise that visual angle φ may be equated to bipolar azimuth ϕ, where ϕ is the mean version of the two eyes.[8] For GAH we obtain $r_0=3.19$ and for JG, $r_0=0.93$.

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[8] A. A. Blank, Brit. J. Physiol. Optics 14, 154, 222 (1957).