Compendium of vector analysis with applications to continuum mechanics

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1. Connection between integration and differentiation

Gauss-Ostrogradsky theorem

We transform the volume integral into a surface one:

$$\int_V \partial_i P dV = \int_V \partial_i P dx_i dx_j dx_k = \int_{S(V)} dx_j dx_k \Big|_{x_i^-(x_j,x_k)}^{x_i^+(x_j,x_k)} P =$$

$$= \int_{S(V)} dx_j dx_k \left[P\left(x_i^+(x_j,x_k), x_j, x_k\right) - P\left(x_i^-(x_j,x_k), x_j, x_k\right) \right] =$$

$$= \int_{S^+} \cos\theta_{ext}^+ dSP - \int_{S^-} \cos\theta_{\text{int}}^- dSP = \oint_S \cos\theta_{ext} dSP = \oint \mathbf{n}\cdot\mathbf{e}_i P dS$$

Here the following denotations and relations were used:
P is a multivariate function $P(x_i, x_j, x_k)$, $\partial_i = \partial/\partial x_i$, V volume,
S surface, $\mathbf{e}_i$ a basis vector, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, $\mathbf{n}$ the external normal to the element dS of closed surface with

$$dx_j dx_k = |\mathbf{n}\cdot\mathbf{e}_i| dS, \quad \mathbf{n}\cdot\mathbf{e}_i = \cos\theta \ .$$

Thus

$$\int_V \partial_i P dV = \oint_{S(V)} P \mathbf{n}\cdot\mathbf{e}_i dS \qquad (1.1)$$

Using formula (1.1), the definitions below can be transformed into coordinate representation.

Gradient

$$\oint\limits_{S(V)} P\mathbf{n}dS = \oint\limits_{S(V)} (\mathbf{n}\cdot\mathbf{e}_i)\mathbf{e}_i PdS = \int\limits_V \partial_i P\mathbf{e}_i dV$$

where summation over recurrent index is implied throughout. By definition

$$\mathrm{grad}P = \nabla P = \partial_i P\mathbf{e}_i$$

Divergence

$$\oint\limits_{S(V)} \mathbf{A}\cdot\mathbf{n}dS = \oint\limits_{S(V)} (\mathbf{n}\cdot\mathbf{e}_i)A_i dS = \int\limits_V \partial_i A_i dV \qquad (1.2)$$

By definition

$$\mathrm{di\upsilon}\mathbf{A} = \nabla\cdot\mathbf{A} = \partial_i A_i$$

Curl

$$\oint\limits_{S(V)} \mathbf{n}\times\mathbf{A}dS = \oint\limits_{S(V)} (\mathbf{n}\cdot\mathbf{e}_i)\mathbf{e}_i\times A_j\mathbf{e}_j dS = \int\limits_V \partial_i A_j \mathbf{e}_i\times\mathbf{e}_j dV \qquad (1.3)$$

By definition

$$\mathrm{curl}\mathbf{A} = \nabla\times\mathbf{A} = \partial_i A_j \mathbf{e}_i\times\mathbf{e}_j$$

Stokes theorem follows from (1.3) if we take for the volume a right cylinder with the height $h \to 0$. Then the surface integrals over the top and bottom areas mutually compensate each other. Next we consider the triad of orthogonal unit vectors

$$\mathbf{m},\ \mathbf{n},\ \boldsymbol{\tau}$$

where $\mathbf{m}$ is the normal to the top base and $\mathbf{n}$ the normal to the lateral face

$$\boldsymbol{\tau} = \mathbf{m}\times\mathbf{n}$$

Multiplying the left-hand side of (1.3) by $\mathbf{m}$ gives

$$\int\limits_{lateral} \mathbf{m}\cdot(\mathbf{n}\times\mathbf{A})dS = \int\limits_{lateral} (\mathbf{m}\times\mathbf{n})\cdot\mathbf{A}dS = \int\limits_{lateral} \boldsymbol{\tau}\cdot\mathbf{A}dS = h\oint\limits_l \boldsymbol{\tau}\cdot\mathbf{A}dl$$

where $\boldsymbol{\tau}$ is the tangent to the line. Multiplying the right-hand side of (1.3) by $\mathbf{m}$ gives

$$h\int\limits_S \mathbf{m}\cdot\mathrm{curl}\mathbf{A}dS$$

where $\mathbf{m}$ is the normal to the surface. Now, equating both sides, we come to the formula sought for

$$\oint\limits_l \boldsymbol{\tau}\cdot\mathbf{A}dl = \int\limits_S \mathbf{m}\cdot\mathrm{curl}\mathbf{A}dS$$

The Stokes theorem is easily generalized to a nonplanar surface (applying to it Ampere's theorem). In this event, the surface is approximated by a polytope. Then mutual compensation of the line integrals on common borders is used.

2. Elements of continuum mechanics

A medium is characterized by the volume density $\rho(\mathbf{x},t)$ and the flow velocity $\mathbf{u}(\mathbf{x},t)$.

Continuity equation

The mass balance in a closed volume is given by

$$\partial_t \int_V \rho dV + \oint_{S(V)} \rho \mathbf{u}\cdot\mathbf{n} dS = 0$$

where $\partial_t = \partial / \partial t$. We get from (1.2)

$$\oint \rho \mathbf{u}\cdot\mathbf{n} dS = \int \partial_i (\rho u_i) dV$$

Thereof the continuity equations follows

$$\partial_t \rho + \partial_i (\rho u_i) = 0$$

Stress tensor

We consider the force $d\mathbf{f}$ on the element dS of surface in the medium and are interested in its dependence on normal $\mathbf{n}$ to the surface

$$d\mathbf{f}(\mathbf{n})$$

where

$$d\mathbf{f}(-\mathbf{n}) = -d\mathbf{f}(\mathbf{n})$$

With this purpose the total force on a closed surface is calculated. We have for the force equilibrium at the coordinate tetrahedron

$$d\mathbf{f}(\mathbf{n}) + d\mathbf{f}(\mathbf{n}_1) + d\mathbf{f}(\mathbf{n}_2) + d\mathbf{f}(\mathbf{n}_3) = 0$$

where the normals are taken to be external to the surface

$$\mathbf{n}_1 = -sign(\mathbf{n}\cdot\mathbf{e}_1)\mathbf{e}_1 , \quad \mathbf{n}_2 = -sign(\mathbf{n}\cdot\mathbf{e}_2)\mathbf{e}_2 , \quad \mathbf{n}_3 = -sign(\mathbf{n}\cdot\mathbf{e}_3)\mathbf{e}_3$$

Thence

$$d\mathbf{f}(\mathbf{n}) = sign(\mathbf{n}\cdot\mathbf{e}_j) d\mathbf{f}(\mathbf{e}_j) \qquad (2.1)$$

The force density $\boldsymbol{\sigma}(\mathbf{n})$ is defined by

$$d\mathbf{f} = \boldsymbol{\sigma} dS$$

Insofar as

$$dS_j = |\mathbf{n}\cdot\mathbf{e}_j| dS$$

we have for (2.1)

$$d\mathbf{f}(\mathbf{n}) = sign(\mathbf{n}\cdot\mathbf{e}_j)\boldsymbol{\sigma}(\mathbf{e}_j)dS_j = sign(\mathbf{n}\cdot\mathbf{e}_j)|\mathbf{n}\cdot\mathbf{e}_j|\boldsymbol{\sigma}(\mathbf{e}_j)dS = \mathbf{n}\cdot\mathbf{e}_j\boldsymbol{\sigma}(\mathbf{e}_j)dS$$

i.e.

$$\boldsymbol{\sigma}(\mathbf{n}) = \mathbf{n}\cdot\mathbf{e}_j\boldsymbol{\sigma}(\mathbf{e}_j)$$
$$= \mathbf{n}\cdot\mathbf{e}_j\mathbf{e}_i\sigma_i(\mathbf{e}_j)$$

The latter means that $\boldsymbol{\sigma}(\mathbf{n})$ possesses the tensor property. The elements of the stress tensor are defined by

$$\sigma_{ij} = \sigma_i(\mathbf{e}_j)$$

Now, using (1.2), the force on a closed surface can be computed as a volume integral

$$\oint \boldsymbol{\sigma}(\mathbf{n})dS = \oint \boldsymbol{\sigma}(\mathbf{e}_j)\mathbf{e}_j\cdot\mathbf{n}dS = \int_V \partial_j\boldsymbol{\sigma}(\mathbf{e}_j)dV \qquad (2.2)$$

Euler equation

The momentum balance is given by the relation

$$\partial_t \int_V \rho\mathbf{u}dV + \oint_{S(V)} (\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}dS = \oint_{S(V)} \boldsymbol{\sigma}dS \qquad (2.3)$$

We have for the second term by (1.2)

$$\oint (\rho\mathbf{u})\mathbf{u}\cdot\mathbf{n}dS = \oint (\rho\mathbf{u})u_j\mathbf{e}_j\cdot\mathbf{n}dS = \int \partial_j(\rho u_j\mathbf{u})\, dV$$

Using also (2.2) gives for (2.3)

$$\partial_t(\rho\mathbf{u}) + \partial_j(\rho u_j\mathbf{u}) = \partial_j\boldsymbol{\sigma}(\mathbf{e}_j)$$

or

$$\rho\partial_t\mathbf{u} + \rho u_j\partial_j\mathbf{u} = \partial_j\boldsymbol{\sigma}(\mathbf{e}_j) \qquad (2.4)$$

Hydrodynamics

The stress tensor in a fluid is defined from the pressure as

$$\sigma_{ij} = -p\delta_{ij}$$

That gives for (2.4)

$$\rho\partial_t u_i + \rho u_j \partial_j u_i + \partial_j p = 0$$

Elasticity

The solid-like medium is characterized by the displacement $\mathbf{s}(\mathbf{x},t)$. For small displacements

$$\mathbf{u} = \partial_t \mathbf{s}$$

and the quadratic terms in the left-hand part of (2.4) can be dropped. For an isotropic homogeneous medium the stress tensor is determined from the Hooke's law as

$$\sigma_i(\mathbf{e}_j) = \lambda\delta_{ij}\partial_k s_k + \mu(\partial_i s_j + \partial_j s_i)$$

where λ and μ are the elastic constants. That gives

$$\partial_j\sigma_i(\mathbf{e}_j) = \lambda\partial_i\partial_k s_k + \mu(\partial_i\partial_j s_j + \partial_j^2 s_i) = (\lambda+\mu)\partial_i\partial_j s_j + \mu\partial_j^2 s_i$$

and

$$\partial_j \boldsymbol{\sigma}(\mathbf{e}_j) = (\lambda+\mu)\,\mathrm{graddi\upsilon}\,\mathbf{s} + \mu\nabla^2\mathbf{s}$$

$$= (\lambda+2\mu)\nabla^2\mathbf{s} + (\lambda+\mu)\,\mathrm{curlcurl}\,\mathbf{s}$$

$$= \lambda\,\mathrm{graddi\upsilon}\,\mathbf{s} - \mu\,\mathrm{curlcurl}\,\mathbf{s}$$

where $\mathrm{graddi\upsilon} = \nabla^2 + \mathrm{curlcurl}$ was used. Substituting it to (2.4) we get finally Lame equation

$$\rho\partial_t^2\mathbf{s} = (\lambda+\mu)\,\mathrm{graddi\upsilon}\,\mathbf{s} + \mu\nabla^2\mathbf{s}$$

where ρ is constant.