CHAPTER I

NEWTON'S LAWS

The aim of classical mechanics is to describe and predict the motion of bodies and systems of bodies which are subject to various interactions. Newton's three laws of motion form the basis of this description, so let us begin by reviewing these.

Newton's laws[1]

Newton's first law deals with non-interacting bodies. It says that the **velocity** of an isolated body, one removed from the influence of other bodies, is constant. This law defines a set of preferred coordinate frames, **inertial frames**, as frames in which Newton's first law holds. Given an inertial frame, we can obtain others by translating the original in space and time, by rotating the original through some angle about some axis, or by giving the original frame a uniform velocity. Unless stated otherwise, we always refer motion to an inertial frame. To a first approximation a coordinate frame attached to the earth is an inertial frame. However, various physical phenomena such as the behavior of a Foucault pendulum, the flight of a ballistic missile, atmospheric and ocean currents, indicate its true non-inertial nature. Better approximations to inertial frames are frames attached to the sun or, even better, to the "fixed stars."

Newton's second and third laws deal with the effects of interaction between bodies on their motion. Interactions cause the velocities of the bodies to change; the bodies undergo **acceleration**. Let us consider two otherwise isolated interacting small bodies, "particles." We find that the accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ of the particles are oppositely directed, and that their magnitudes are related by

$$a_1/a_2 = k_{12},$$

where the ratio k_{12} is *independent* of the nature of the interaction between the particles. It does not depend on whether the interaction arises because the particles are in contact with one another, or because they are connected by a string or by a spring, or because they interact gravitationally, etc. The ratio k_{12} is thus a quantity which we can associate with the pair of particles themselves, as opposed to the particular interaction they happen to be undergoing. Further, if we consider three particles, a pair at a time, we find that the three acceleration ratios are not independent but are related by

$$k_{12}k_{23}k_{31} = 1.$$

This, together with $k_{12} = 1/k_{21}$, shows that the acceleration ratio can be written

$$k_{12} = m_2/m_1,$$

[1]Ernst Mach, *The Science of Mechanics*, (The Open Court Publishing Co., Chicago, 1893), trans. Thomas J. McCormack.

where m_1 (m_2) is a property of, something associated with, particle 1 (particle 2) alone. The quantity m_1 (m_2) is called the **inertial mass** of the particle. Putting these facts together, we see that the accelerations of two interacting particles are related by

$$m_1\mathbf{a}_1 = -m_2\mathbf{a}_2.$$

We describe the interaction by saying that particle 2 exerts a **force** $\mathbf{F}_{2\text{on}1}$ on particle 1, and particle 1 exerts a force $\mathbf{F}_{1\text{on}2}$ on particle 2, such that

$$m_1\mathbf{a}_1 = \mathbf{F}_{2\text{on}1} \quad \text{and} \quad m_2\mathbf{a}_2 = \mathbf{F}_{1\text{on}2}$$

with

$$\mathbf{F}_{2\text{on}1} = -\mathbf{F}_{1\text{on}2}.$$

This last equation is **Newton's third law**: the force which particle 2 exerts on particle 1 is equal and opposite to the force which particle 1 exerts on particle 2. It is another way of stating our conclusions about the acceleration ratio of two interacting particles.

If we now consider three interacting particles, we find that the acceleration of any one of them, say particle 1, is the vector sum of the acceleration of particle 1 due to particle 2 alone and the acceleration of particle 1 due to particle 3 alone (Fig. 1.01(a)), and thus

$$\begin{aligned} m_1\mathbf{a}_1 &= \mathbf{F}_{2\text{on}1} + \mathbf{F}_{3\text{on}1} \\ &= \mathbf{F}_{\text{total on}1}. \end{aligned}$$

This is **Newton's second law**: the acceleration of a particle is directly proportional to the total force acting on it (obtained by adding vectorially all the individual forces) and is inversely proportional to the mass of the particle. This law should be understood in the following way: we are meant to describe the interaction of our chosen particle with other particles by specifying the force acting on it in terms of the locations and velocities of all the particles. The form this takes depends, of course, on the nature of the interactions.

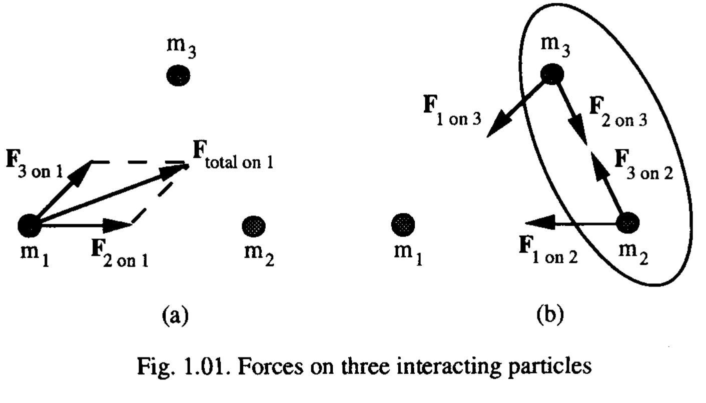


Fig. 1.01. Forces on three interacting particles

Similar equations apply to particles 2 and 3 (Fig. 1.01(b)),

$$m_2\mathbf{a}_2 = \mathbf{F}_{3\mathrm{on}2} + \mathbf{F}_{1\mathrm{on}2}$$
$$m_3\mathbf{a}_3 = \mathbf{F}_{1\mathrm{on}3} + \mathbf{F}_{2\mathrm{on}3}\,.$$

Now suppose that the forces $\mathbf{F}_{3\mathrm{on}2}$ and $\mathbf{F}_{2\mathrm{on}3}$ are such that particles 2 and 3 are bound together to form a single particle and move with a common acceleration

$$\mathbf{a}_2 = \mathbf{a}_3 = \mathbf{a}_{23}.$$

Then, adding the above two equations, we find

$$(m_2 + m_3)\mathbf{a}_{23} = \mathbf{F}_{1\mathrm{on}2} + \mathbf{F}_{1\mathrm{on}3},$$

the internal forces $\mathbf{F}_{3\mathrm{on}2}$ and $\mathbf{F}_{2\mathrm{on}3}$ canceling because of Newton's third law. The bound combination thus behaves as a single particle with mass

$$m_{23} = m_2 + m_3;$$

mass is additive. Further, the force acting on the bound combination can be taken to be the total *external* force; the internal forces which hold the combination together need not be taken into account.

To see how to apply these laws, we begin with some simple examples, most of which will already be familiar to you.

Free fall

For our first example, consider the motion of a body of mass m dropped near the surface of the earth. The only force acting on the body (ignoring air friction) is the downward **gravitational force** mg; here $g \approx 9.8\,\mathrm{m/s^2}$ is the approximately constant **gravitational field** due to the earth. Newton's second law gives

$$ma = mg,$$

so, canceling the m, we see that the body falls with constant acceleration g. Indeed, since g is body-independent, all bodies fall with the *same* constant acceleration.[2] The equation of motion for the falling body takes the form (measuring x downwards)

$$\frac{d^2x}{dt^2} = g.$$

[2]For the consequences, see A. Einstein in H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity* (Dover Publications, New York, NY, 1923), trans. W. Perrett and G. B. Jeffery, p. 99.

Integrating, we find that the velocity is given by

$$\frac{dx}{dt} = v_0 + gt = v(t)$$

where v_0 is the initial velocity. Integrating once again, we find that the position of the body is given by

$$x(t) = x_0 + v_0 t + \tfrac{1}{2}gt^2$$

where x_0 is the initial position. The position is thus determined as a function of the time; the expression also contains two constants of integration, adjustable parameters, which are set by the initial conditions, the initial position and velocity.

Simple harmonic oscillator

In the above example the integration could be done immediately since we knew the time dependence of the force: it was constant. Usually, however, the force is not known *a priori* as a function of time. Rather, it is known as a function of position. Take, for example, a mass m attached to a spring (Fig. 1.02).

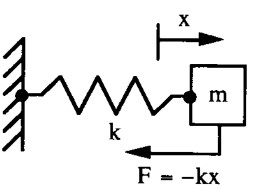


Fig. 1.02. Mass attached to a spring

It is found (**Hooke's law**) that the force F which the spring exerts on the mass is proportional to the amount x the spring is stretched and is directed opposite the stretch,

$$F = -kx.$$

The proportionality constant k, a measure of the strength of the spring, is called the **spring constant**. Newton's second law then gives

$$ma = -kx.$$

We cannot integrate this directly as in the free fall case. However, if we multiply the equation by the velocity v, the left-hand side becomes

$$mav = m\frac{dv}{dt}v = \frac{d}{dt}(\tfrac{1}{2}mv^2),$$

and the right-hand side becomes

$$-kxv = -kx\frac{dx}{dt} = -\frac{d}{dt}(\tfrac{1}{2}kx^2).$$

Thus the motion is such that

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

is constant in time. This quantity is called the **total energy** and is the sum of the **kinetic energy** $T = \frac{1}{2}mv^2$ and the **potential energy** $V = \frac{1}{2}kx^2$.

The energy equation can be rearranged to obtain

$$\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}} = \int_0^t dt = t.$$

To do the x-integration, we set

$$x = \sqrt{\frac{2E}{k}}\sin\phi \qquad dx = \sqrt{\frac{2E}{k}}\cos\phi\, d\phi$$

where ϕ is a new variable, the **phase**. The left-hand side then becomes

$$\sqrt{\frac{m}{k}}\int_{\phi_0}^{\phi} \frac{\cos\phi}{\sqrt{1-\sin^2\phi}}d\phi = \sqrt{\frac{m}{k}}\int_{\phi_0}^{\phi} d\phi = \sqrt{\frac{m}{k}}(\phi - \phi_0),$$

so that the phase is a linear function of time,

$$\phi = \phi_0 + \omega t.$$

The rate at which the phase increases with time, $\omega = \sqrt{k/m}$, is called the **angular frequency**. The position of the mass as a function of time is thus given by

$$x = A\sin(\omega t + \phi_0)$$

where $A = \sqrt{2E/k}$ is called the **amplitude** of the motion. The mass oscillates back and forth between $x = +A$ and $x = -A$, going through a complete cycle in a time $\tau = 2\pi/\omega$, the **period** of the motion. This very important motion is called **simple harmonic motion**. Its importance derives from the fact that, except in unusual circumstances, motion near any stable equilibrium point is simple harmonic motion.

The approach which we have used here to discuss the simple harmonic oscillator can be applied to any one-dimensional conservative system, for which $F = -dV/dx$; all we have to do is to replace $\frac{1}{2}kx^2$ by the appropriate potential energy $V(x)$. Such systems are thus in principle always integrable.

Central force

When we move from one-dimensional problems to three-dimensional problems, the degree of complexity increases enormously. Indeed, most three-dimensional problems cannot be integrated analytically. There is, however, a class of problems which can still be handled moderately easily, namely the motion of a particle acted on by a force $\mathbf{F} = F\hat{\mathbf{r}}$ which is always directed towards (or away from) a fixed point, the **force center**. This is the so-called **central force problem** (Fig. 1.03(a)). For such problems the **torque** $\mathbf{r} \times \mathbf{F}$ on the particle about the force center is zero, and the **angular momentum** $\mathbf{L} = \mathbf{r} \times (m\mathbf{v})$ is constant. The motion thus lies in a plane $\mathbf{L} \cdot \mathbf{r} = 0$ which is perpendicular to $\mathbf{L}$ and which passes through the force center, the **orbital plane**. Further, the motion is such that the magnitude L of the angular momentum about the force center is constant.

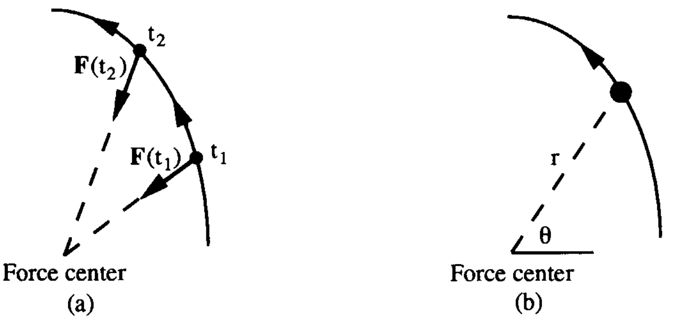


Fig. 1.03. (a) Central force, (b) Polar coordinates

It is convenient to introduce polar coordinates r and θ in the orbital plane, with the force center the pole (Fig. 1.03(b)). The position of the particle is then given by $\mathbf{r} = r\hat{\mathbf{r}}$, the velocity by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}$$
$$= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \ ,$$

and the acceleration by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \ .$$

In deriving these we have used the results

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}} \quad \text{and} \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\mathbf{r}}$$

which follow readily from Fig. 1.04.[3]

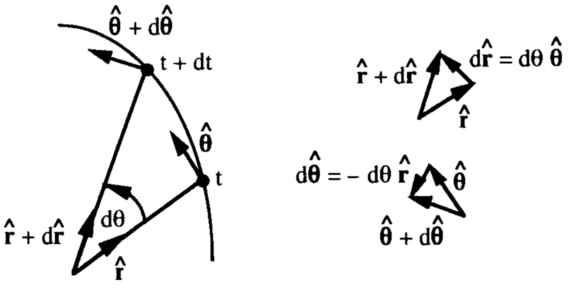


Fig. 1.04. Changes in the unit vectors

Newton's second law then gives the equations of motion

$$m(\ddot{r} - r\dot{\theta}^2) = F$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \ .$$

The second of these can be written

$$\frac{1}{r}\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

which shows that $L = mr^2\dot{\theta}$ is constant in time. The quantity L = (distance r from origin) × (component $mr\dot{\theta}$ of mv perpendicular to **r**) is the magnitude of the angular momentum, so this simply confirms what we already know. Its content can be expressed in a rather picturesque way.

[3]Alternate derivations can be found in Daniel Kleppner and Robert J. Kolenkow, *An Introduction to Mechanics*, (McGraw-Hill Book Company, New York, NY, 1973), pp. 27-38.

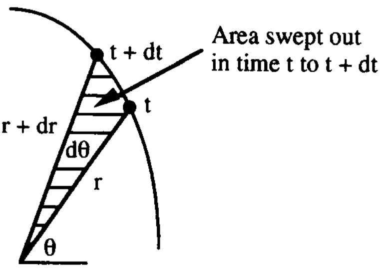


Fig. 1.05. Area swept out by radius vector

From Fig. 1.05, the radius vector sweeps out an area $dA = \frac{1}{2}(r)(r\dot{\theta}dt)$ in a time dt, so the rate at which it sweeps out area is $dA/dt = \frac{1}{2}r^2\dot{\theta} = L/2m$. This, as we have seen, is constant. Thus the particle moves along its orbit in such a way that the radius vector sweeps out equal areas in equal times. Applied to the solar system, this is known as **Kepler's second law** of planetary motion. We see, however, that it holds for any central force, not just for the gravitational force.

The fact that L is constant can be used to eliminate $\dot{\theta} = L/mr^2$ from the radial equation of motion. If, further, the central force is conservative, so that $F = -dV(r)/dr$ where V is the potential energy, we can write

$$m\ddot{r} = \frac{L^2}{mr^3} - \frac{dV}{dr} = -\frac{d}{dr}\left(\frac{L^2}{2mr^2} + V(r)\right).$$

The radial motion is thus the same as one-dimensional motion (but with $r > 0$) in an **effective potential**

$$V_{eff}(r) = \frac{L^2}{2mr^2} + V(r).$$

In particular, the total energy

$$E = \frac{1}{2}m|v|^2 + V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$
$$= \frac{1}{2}m\left(\dot{r}^2 + \frac{L^2}{m^2r^2}\right) + V(r) = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

is constant in time. This can be rearranged and integrated to obtain r as a function of t and the parameters E, L, and initial radius r_0. The result can then be substituted into $\dot{\theta} = L/mr^2$, and this equation integrated to obtain θ as a function of t and the parameters E, L, r_0, and initial angle θ_0. These four parameters, together with the two required to specify the orbital plane, specify the orbit (r_0 and θ_0 give the initial position in the orbital plane, and E and L then give the magnitude and direction of the initial velocity).

Before embarking on the integrations, either analytically or numerically, it is worthwhile to get an overview of the general behavior to be expected, to obtain qualitative pictures of the possible orbits and the ranges of the parameters over which they occur. These can be obtained by examining a sketch of $V_{eff}(r)$ versus r. We illustrate this idea by applying it to the important central force, gravitation.

Gravitational force: qualitative

The **gr ational force** between two bodies, which are small compared to the distance r betw.... them, is given by

$$F = -\frac{k}{r^2} = -\frac{dV}{dr}, \quad \text{where} \quad V = -\frac{k}{r}$$

is the **gravitational potential**. The constant k equals GmM where m and M are the masses of the bodies and G is the gravitational constant. The same expression may be used for the electrostatic force, the **Coulomb force**, between two electrically charged bodies, provided we set $k = -q_1q_2$ where q_1 and q_2 are the electric charges on the bodies. If one of the bodies is much lighter than the other, say $m << M$, as is the case for the planets compared with the sun, or artificial satellites with the earth, or an electron with a nucleus, we can regard the heavy body as providing an approximately fixed force center about which the lighter body orbits. The effective potential is then

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{k}{r}$$

and is shown in Fig. 1.06.

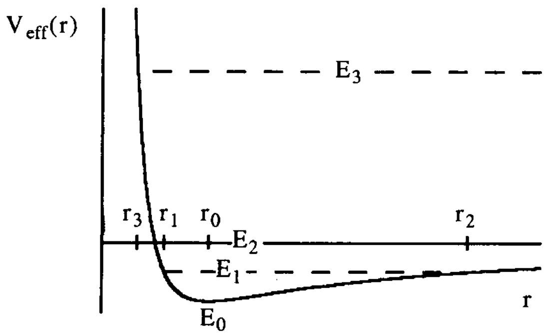


Fig. 1.06. Effective potential for the gravitational force

We have seen that the constant total energy E of an orbiting body is the sum of the kinetic energy $\frac{1}{2}m\dot{r}^2$ due to its radial motion and the effective potential V_{eff}. Since the kinetic energy is non-negative, the total energy must be greater than or equal to the energy E_0 at the bottom of the V_{eff} potential well (Fig. 1.06). If the energy is E_0, then the radius is fixed at r_0 and the orbit is a circle. If the energy is E_1 with $E_0 < E_1 < 0$, the radial motion is like that of a particle in a one-dimensional potential well, the orbit radius oscillating back and forth between an inner turning radius r_1 and an outer turning radius r_2, with $E = V_{eff}$ and $\dot{r} = 0$ at r_1 and r_2. All the while the angle θ is increasing. We see that the orbit then looks qualitatively like one of those in Fig. 1.07.

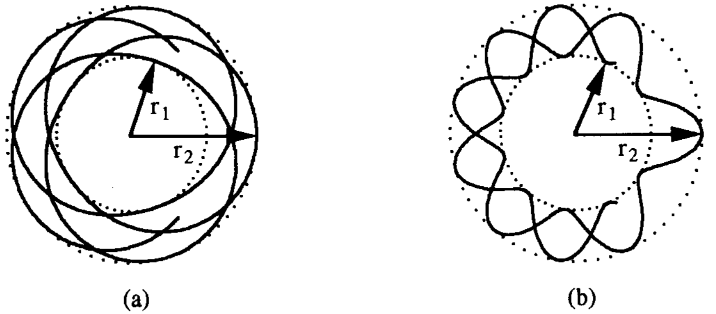


Fig. 1.07. Qualitative shape of bound orbit

Since for gravity the force is attractive rather than repulsive at the inner turning radius, the contact there must look like Fig. 1.07(a) rather than like (b). Detailed calculations to follow show that the orbit for this case is in fact an ellipse. If the energy is $E_2 = 0$ or $E_3 > 0$, there is an inner turning radius but no outer turning radius. The orbiting body comes in from infinity, reaches a minimum radius r_3, and moves out again to infinity. The orbit looks qualitatively like that in Fig. 1.08.

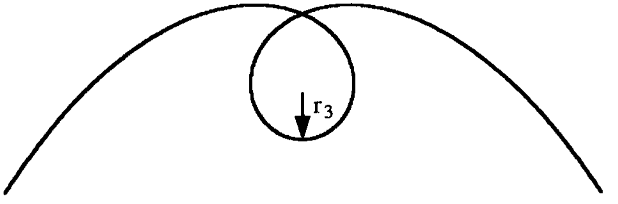
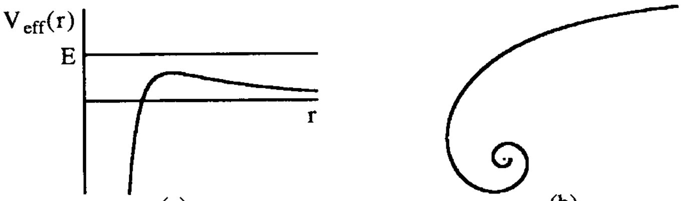


Fig. 1.08. Qualitative shape of scattering orbit

Detailed calculations to follow show that for the gravitational force the orbit is in fact a parabola for energy $E = 0$ and a hyperbola for energy $E > 0$.

The ideas used above to discuss the qualitative shape of the orbits in a gravitational potential can also be applied to an arbitrary central potential. In addition to the situations already considered for gravity, we may then encounter attractive potentials $V(r)$ which blow up faster than $-1/r^2$ as $r \to 0$. The effective potential V_{eff} then tends to $-\infty$ rather than $+\infty$ as $r \to 0$ and, depending on the energy, there may be no inner turning radius. For example, an orbit with energy E shown in Fig. 1.09(a) spirals in to the force center as in Fig. 1.09(b).



(a) (b)

Fig. 1.09. (a) A possible effective potential, (b) Qualitative shape of capture orbit

Gravitational force: quantitative

Let us now return to the gravitational force and consider the detailed integration. The energy and angular momentum equations

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{r} \quad \text{and} \quad L = mr^2\dot{\theta}$$

lead, on integration, to r and θ as functions of time. Rather than doing these integrations immediately, however, it is more useful first to obtain r as a function of θ; that is, to obtain the equation which describes the shape of the orbit. To do this, we set

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{L}{mr^2}$$

in the energy equation and hence, on rearranging, write

$$\int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{L^2} - \frac{1}{r^2} + \frac{2mk}{L^2 r}}} = \int_{\theta_0}^{\theta} d\theta = \theta - \theta_0.$$

The r-integration is performed by setting first $u = 1/r$, $du = -dr/r^2$ to give

$$-\int_{u_0}^{u}\frac{du}{\sqrt{\frac{2mE}{L^2}+\frac{2mk}{L^2}u-u^2}}=-\int_{u_0}^{u}\frac{du}{\sqrt{\frac{m^2k^2}{L^4}\left(1+\frac{2L^2E}{mk^2}\right)-\left(u-\frac{mk}{L^2}\right)^2}}.$$

Then set

$$u-\frac{mk}{L^2}=\frac{mk}{L^2}\sqrt{1+\frac{2L^2E}{mk^2}}\cos\alpha \quad \text{and} \quad du=-\frac{mk}{L^2}\sqrt{1+\frac{2L^2E}{mk^2}}\sin\alpha\, d\alpha.$$

Integration gives $\alpha=\theta-\theta_0$, which leads to the orbit equation

$$u=\frac{1}{r}=\frac{mk}{L^2}\left[1+\sqrt{1+\frac{2L^2E}{mk^2}}\cos(\theta-\theta_0)\right].$$

This has the form of a **conic section**

$$\frac{p}{r}=1+e\cos(\theta-\theta_0)$$

with **semi-latus-rectum** $p=\frac{L^2}{mk}$ and **eccentricity** $e=\sqrt{1+\frac{2L^2E}{mk^2}}$. We have chosen the constant of integration so that $\theta=\theta_0$ is the direction of **pericenter**, the point on the orbit closest to the force center. The angle $\alpha=\theta-\theta_0$ from pericenter is called the **true anomaly**. We can show that:

for $E=E_0=-\frac{mk^2}{2L^2}$, $e=0$ and the orbit is a **circle**,
for $E_0<E<0$, $0<e<1$ and the orbit is an **ellipse**,
for $E=0$, $e=1$ and the orbit is a **parabola**,
and for $0<E$, $1<e$ and the orbit is a **hyperbola**.

Let us first consider the bound orbits $E<0$, those for which the particle is confined to a finite region of space. We show that the above equation with $0<e<1$ represents an ellipse (clearly, the special case $e=0$ represents a circle). As the old geometry books put it, an ellipse is the locus of all points for which the sum of the distances to two fixed points is a constant. The two fixed points are called the **foci** of the ellipse. The sum of the distances is the **major axis** of the ellipse. We denote it by 2a, so a is the semi-major axis. The ratio of the distance between the foci to the major axis is the **eccentricity** e of the ellipse.

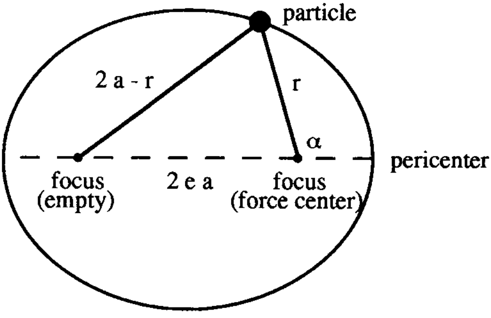


Fig. 1.10. Elliptic orbit

If we apply the trigonometric cosine law to the triangle in Fig. 1.10, we find

$$(2a - r)^2 = r^2 + 4e^2a^2 + 4ear\cos\alpha,$$

which, on rearranging, gives the polar form of the equation of an ellipse, as above, with semi-latus-rectum $p = a(1 - e^2)$. Applied to the solar system, this is **Kepler's first law** of planetary motion: the planets travel around the sun on an elliptical orbit with the sun at one focus.

The semi-major axis of the ellipse can be expressed in terms of the energy and angular momentum,

$$p = \frac{L^2}{mk} = a\frac{2L^2|E|}{mk^2} \quad \text{so} \quad a = \frac{k}{2|E|} \quad \text{or} \quad E = -\frac{k}{2a}.$$

The semi-major axis depends only on the energy, not on the angular momentum; alternatively, the energy depends only on the semi-major axis, not on the eccentricity.

Let us now turn our attention to the time dependence of the variables. According to Kepler's second law, the rate at which the radius vector sweeps out area is

$$\frac{dA}{dt} = \frac{L}{2m} = \frac{1}{2}\sqrt{\frac{ka}{m}(1 - e^2)}.$$

The time required to complete one orbit, the **period** τ, is the time to sweep out the complete area

$$A = \pi ab = \pi a^2\sqrt{1 - e^2}$$

enclosed by the ellipse (here $b = a\sqrt{1 - e^2}$ is the semi-minor axis of the ellipse). It is given by

$$\frac{\pi a^2\sqrt{1-e^2}}{\tau}=\frac{1}{2}\sqrt{\frac{ka}{m}(1-e^2)}\,,$$

which, on simplifying, yields

$$\tau=2\pi\sqrt{\frac{m}{k}}\,a^{3/2}.$$

For the family of planets orbiting the sun, $k/m = GM$ where M is the mass of the sun. Thus, the period of a planet is proportional to the $3/2$ power of the semi-major axis of the planet's orbit; it does not depend on the mass of the planet or the eccentricity of the orbit. This is **Kepler's third law** of planetary motion.[4] See Fig. 1.11 and note that the slope of the log-log plot is $3/2$.

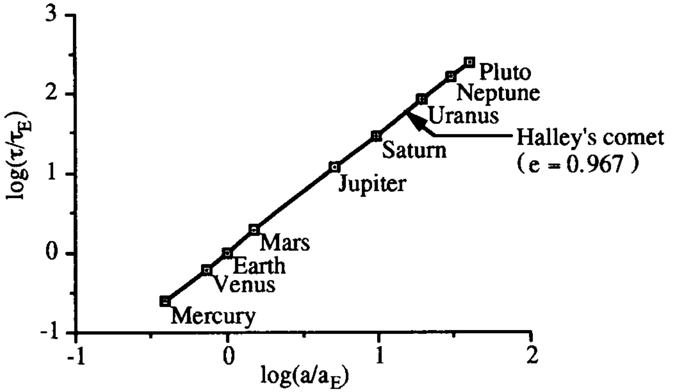


Fig. 1.11. Kepler's third law

Kepler's second law, plus some geometry, can also give the way the particle moves around the orbit as a function of time.[5] However, for modern readers more familiar with calculus than with geometry, it is probably easier to obtain this by integrating the equation for radial motion

$$\frac{1}{2}m\dot{r}^2+\frac{L^2}{2mr^2}-\frac{k}{r}=E.$$

[4]One sometimes sees a "corrected" version in which GM is replaced by G(M+m). This comes from treating the situation as a two-body problem involving the sun and the planet under consideration. But the solar system is a *many body* problem, and the solar motions which lead to "corrections" of this type are the result of the interaction of the sun not only with the planet under consideration but with all the planets. A proper treatment would have to take all these into account.

[5]See, for example, Forest Ray Moulton, *An Introduction to Celestial Mechanics*, (Macmillan, New York, NY, 1902, 1914, 1923), 2nd ed., p.159.

This can be rearranged in the form

$$\int_{r_0}^{r} \frac{dr}{\sqrt{E + \frac{k}{r} - \frac{L^2}{2mr^2}}} = \sqrt{\frac{2}{m}} \int_{t_0}^{t} dt = \sqrt{\frac{2}{m}}(t - t_0).$$

If the energy E and angular momentum L are expressed in terms of the semi-major axis a and eccentricity e, the left-hand side becomes

$$\sqrt{\frac{2a}{k}} \int_{a(1-e)}^{r} \frac{r dr}{\sqrt{a^2e^2 - (r-a)^2}}.$$

Here we have also chosen r_0 as the pericenter radius $a(1-e)$; t_0 is then the time of passage of pericenter. The integration is readily performed by setting

$$r - a = -ea\cos\psi \quad \text{and} \quad dr = ea\sin\psi\, d\psi$$

where ψ is a new variable called the **eccentric anomaly**, whose geometric significance can be seen from Fig. 1.12.

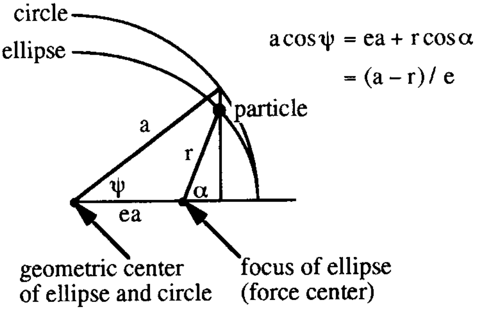


Fig. 1.12. Eccentric anomaly ψ

The integration then becomes

$$\sqrt{\frac{2a}{k}}\, a \int_0^{\psi} (1 - e\cos\psi)\, d\psi = \sqrt{\frac{2a}{k}}\, a(\psi - e\sin\psi),$$

and we find

$$\frac{r}{a} = 1 - e\cos\psi$$

with

$$\psi - e\sin\psi = \frac{2\pi}{\tau}(t - t_0)$$

where τ is the period of the motion. This last equation is known as **Kepler's equation**; it is a relation between the eccentric anomaly ψ and the time t, or the so-called **mean anomaly** $(2\pi/\tau)(t - t_0)$. Finally, the relation between the eccentric anomaly ψ and the true anomaly $(\theta - \theta_0)$ can be found by eliminating r/a between the above r-ψ equation and the orbit equation, to obtain

$$1 - e\cos\psi = \frac{1 - e^2}{1 + e\cos(\theta - \theta_0)}.$$

In principle, the determination of the motion is now complete. If, however, we want r and θ in terms of the time, as is often the case, we must first invert Kepler's equation to get ψ in terms of the time. This is in general difficult. In cases in which the eccentricity e of the orbit is small, however, expansions in powers of e are adequate. We use this approach in the next section to determine the parameters of earth's orbit.

Parameters of earth's orbit

The earth's orbit around the sun lies in the plane of the **ecliptic**, which is marked by the apparent path of the sun through the constellations of the zodiac over the course of a year. The plane of the earth's equator makes an angle of approximately 23° with the plane of the ecliptic, and the intersection of these two planes gives a direction in space. In September of each year the sun passes through the plane of the earth's equator going from north to south. This is known as the **autumnal equinox** (AE). The direction from sun to earth at this time provides a convenient reference (the **first point of Aries**) from which to measure the angle θ, so at autumnal equinox $\theta = 0$. As the year progresses, the midday sun (in the northern hemisphere) moves lower and lower in the sky, until at $\theta = \pi/2$, the **winter solstice** (WS) in December, it reaches its lowest point. The sun then moves higher in the sky and at $\theta = \pi$, the **vernal equinox** (VE) in March, the sun again passes through the plane of the earth's equator, this time going from south to north. The first point of Aries is thus also the direction from earth to sun at the vernal equinox. The sun continues to move higher in the sky, and at $\theta = 3\pi/2$, the **summer solstice** (SS) in June, the midday sun reaches its highest point. See Fig. 1.13.

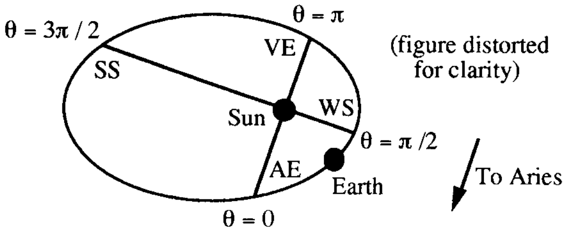


Fig. 1.13. Earth's orbit

Table I. Equinoxes, solstices, and seasons for 1994 - 1995

	Equinoxes and solstices		Seasons		
	Date (see 1)	Day (see 2)	(see 3)	(see 4)	
AE (94)	23 Sept. 01:19	266.0549			
			89.8361	0.245961	Autumn
WS (94)	21 Dec. 21:23	355.8910			
			88.9938	0.243654	Winter
VE (95)	20 Mar. 21:14	444.8847			
			92.7639	0.253977	Spring
SS (95)	21 June 15:34	537.6486			
			93.6521	0.256408	Summer
AE (95)	23 Sept. 07:13	631.3007			

1. Times in EST
2. 1 Jan. 1994 0:00 h = 1.0000
3. In days
4. In fractions of the year [AE (94) to AE (95)] = 365.2458 days

We wish to use the "observed" times of these seasonal events[6] to find the parameters of earth's orbit: the eccentricity e which determines the shape of the ellipse, the angle θ_0 of perihelion which determines the orientation of the ellipse in the plane of the ecliptic, and the time t_0 of passage of perihelion. The last equation of the previous section gives the relation between the eccentric anomaly ψ and the true anomaly $\theta - \theta_0$. Expanding the right-hand side of this as a power series in the eccentricity, we find

$$\cos\psi = \cos(\theta - \theta_0) + e\sin^2(\theta - \theta_0) - e^2\cos(\theta - \theta_0)\sin^2(\theta - \theta_0) + \cdots$$

which yields

[6]A convenient and inexpensive source of data is *The Old Farmer's Almanac*, (Yankee Publishing Inc., Dublin, NH). A more conventional source is *The Astronomical Almanac*, (US Government Printing Office, Washington, DC).

$$\psi = (\theta - \theta_0) - e\sin(\theta - \theta_0) + \tfrac{1}{4}e^2 \sin 2(\theta - \theta_0) + \cdots.$$

This, when substituted into Kepler's equation, then gives the relation between the observed quantities, the true anomaly and the time (in years),

$$(\theta - \theta_0) - 2e\sin(\theta - \theta_0) + \tfrac{3}{4}e^2 \sin 2(\theta - \theta_0) + \cdots = 2\pi(t - t_0).$$

Setting $\theta = 0$ at autumnal equinox, $\theta = \pi/2$ at winter solstice, $\theta = \pi$ at vernal equinox, and $\theta = 3\pi/2$ at summer solstice, we obtain the four equations

$$\begin{aligned}
-\theta_0 + 2e\sin\theta_0 - \tfrac{3}{4}e^2 \sin 2\theta_0 + \cdots &= 2\pi(t_{AE} - t_0) \\
\tfrac{\pi}{2} - \theta_0 - 2e\cos\theta_0 + \tfrac{3}{4}e^2 \sin 2\theta_0 + \cdots &= 2\pi(t_{WS} - t_0) \\
\pi - \theta_0 - 2e\sin\theta_0 - \tfrac{3}{4}e^2 \sin 2\theta_0 + \cdots &= 2\pi(t_{VE} - t_0) \\
\tfrac{3\pi}{2} - \theta_0 + 2e\cos\theta_0 + \tfrac{3}{4}e^2 \sin 2\theta_0 + \cdots &= 2\pi(t_{SS} - t_0) \ .
\end{aligned}$$

These can be combined to give

$$\begin{aligned}
\frac{2e}{\pi}\sin\theta_0 &= \frac{1}{2} - (t_{VE} - t_{AE}) = \frac{1}{2} - (\text{Autumn} + \text{Winter}) \\
-\frac{2e}{\pi}\cos\theta_0 &= \frac{1}{2} - (t_{SS} - t_{WS}) = \frac{1}{2} - (\text{Winter} + \text{Spring}) \\
4t_0 &= t_{AE} + t_{WS} + t_{VE} + t_{SS} + \frac{2\theta_0}{\pi} - \frac{3}{2} \\
\frac{1}{2} + \frac{3e^2}{2\pi}\sin 2\theta_0 &= t_{WS} - t_{AE} + t_{SS} - t_{VE} = \text{Autumn} + \text{Spring} \ .
\end{aligned}$$

The first two equations give the eccentricity and angle of perihelion; the third gives the time of perihelion; and the fourth is a check on the consistency of the data with the assumption of a Keplerian orbit. In particular, using the data in Table I we find

$$\begin{aligned}
\frac{2e}{\pi}\sin\theta_0 &= 0.010385 \\
-\frac{2e}{\pi}\cos\theta_0 &= 0.002369
\end{aligned}$$

which yield $e = 0.016732$ and $\theta_0 = 102.85°$. The third equation then gives

$$4t_0 = 1604.4792 + (2 \times 102.85/180 - 1.5) \times 365.2458$$

which yields $t_0 = 368.50 = 3$ January 1995, 12h. Finally, the fourth equation gives

$$0.499942 \approx 0.499937.$$

The equations we have been using are good to order e^2; corrections of order $e^3 = 4.7 \times 10^{-6}$ are expected to modify the results by a few parts in the last figure quoted. The uncertainty in the input times (assumed to be $\pm 1\,\text{minute} = \pm 1.9 \times 10^{-6}$ year) also affects the last figure.

The agreement with the almanac values $e = 0.01673 \pm 0.00002$ and $\theta_0 = 102.87° \pm 0.08°$, where the $\pm$ indicates the variation over the course of the year due to various perturbations, is excellent. However, our predicted t_0 is 18 hours early. The reason for this is interesting. We have actually been calculating the parameters of the earth-moon barycenter; in particular, our t_0 is the time of *barycenter* perihelion. What is listed in almanacs, however, is the time of *earth* perihelion. These differ by approximately $1.3 \sin\phi$ days, where ϕ is the (angular) phase of the moon near perihelion. In 1995 perihelion occurred about a third the way through the first quarter of the moon, so $\phi \approx 30°$ and the correction to our result is approximately +16 hours, as required.

Scattering

One of the primary ways for exploring the nuclear and sub-nuclear world is via scattering experiments. A beam of particles, such as electrons, protons, or α-particles, is directed at a thin target which contains the nucleus to be studied. The particles in the beam interact with the target nuclei and are scattered. A detector counts the number of particles per unit time scattered in various directions (Fig. 1.14(a)). This number is proportional to the (small) cross-sectional area ΔA of the detector and is inversely proportional to the square of the distance R from the target to the detector. That is, the number Δn of counts per unit time is proportional to the **solid angle** $\Delta\Omega = \Delta A / R^2$ subtended by the detector. Further, it is proportional to the **intensity** I of the incident beam (the number of incident particles per unit area per unit time) and to the number N of target nuclei in the path of the beam. To obtain a quantity from which these details of the experimental arrangement have been removed, we divide the number of counts per unit time by the solid angle subtended by the detector, by the intensity of the incident beam, and by the number of target nuclei. The resulting quantity,

$$\frac{d\sigma}{d\Omega} = \frac{1}{NI}\frac{dn}{d\Omega},$$

is called the **differential scattering cross section**. It has units of "area": $\Delta\sigma = (d\sigma/d\Omega)\Delta\Omega$ is the area an incident particle must strike, per target nucleus, in order to scatter into the solid angle $\Delta\Omega$.

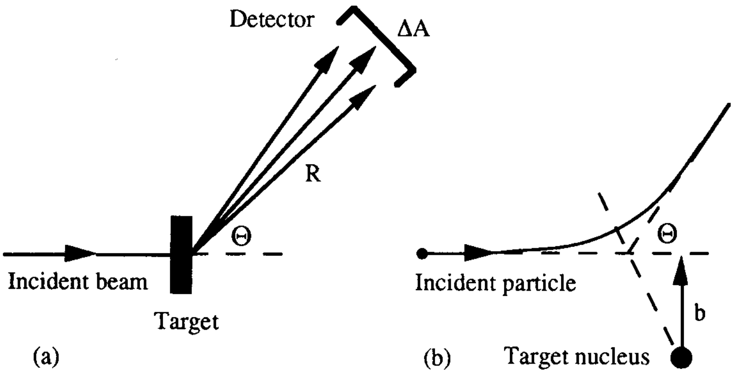


Fig. 1.14. Scattering (a) macroscopic view, (b) microscopic view

For thin targets the scattering of an incident particle is the result of a single collision between it and an individual target nucleus. We wish to relate the differential scattering cross section to this individual scattering process. The proper way to do this is to use quantum mechanics. It is nevertheless of interest to see how one approaches such a problem using classical mechanics, if only to introduce general ideas and to provide results which can then be compared with the quantum mechanical results. When the incident particle is far from the target nucleus, the force exerted on it by the target nucleus is small, and the particle moves along an incoming asymptotic straight line (Fig. 1.14(b)). If extended, this straight line would pass by the target nucleus with distance of closest approach b. This distance b is called the **impact parameter**. It is related to the (constant) angular momentum L and energy E of the incident particle by

$$L = mv_{\infty}b = b\sqrt{2mE}\,.$$

As a result of its interaction with the target nucleus, the incident particle is deflected from its original path, eventually emerging from the interaction region along an outgoing asymptotic straight line which makes an angle Θ $(0 \le \Theta \le \pi)$ with the incident direction. This angle Θ is called the **scattering angle**. It is a (single-valued) function $\Theta(b,E)$ of the impact parameter b and particle energy E. If we choose a spherical polar coordinate system with origin at the target and polar axis in the direction of the incident beam, the scattering angle Θ is the polar angle of the scattered particle. The azimuthal angle ϕ of the particle does not change (except possibly by π). The number of particles per unit time incident with impact parameter in the range b to $b + db$ and with azimuthal angle in the range ϕ to $\phi + d\phi$ is $dn = I\,b\,db\,d\phi$. These particles scatter into the specific polar angle range Θ to $\Theta + d\Theta$ where Θ is determined by b, and into the azimuthal angle range $\phi\,(+\pi)$ to $\phi + d\phi\,(+\pi)$. The solid angle which they subtend is thus $d\Omega = \sin\Theta\, d\Theta\, d\phi$. Substituting these results into the above definition of the differential scattering cross section then gives

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right|.$$

In writing this down we have assumed that the relationship between impact parameter and scattering angle (at fixed energy) is one to one. For some interactions this may not be the case, with more than one impact parameter yielding the same scattering angle. In such situations the above should be replaced by an appropriate sum.[7]

Coulomb scattering

The scattering of a low energy incident charged particle by a nucleus is largely the result of the electrostatic Coulomb force between it and the nucleus. We have seen that the orbit for a particle moving in a Coulomb potential $V = -k/r$ (with $k = -q_1q_2$ where q_1 and q_2 are the electric charges of the incident and target particles) is a conic section

$$\frac{p}{r} = 1 + e\cos\theta$$

where $p = L^2/mk$ is the semi-latus-rectum and $e = \sqrt{1 + 2L^2E/mk^2} = \sqrt{1 + (2bE/k)^2}$ is the eccentricity. Also, for convenience we have here taken pericenter in the direction $\theta = 0$ for an attractive force (unlike charges), or in the direction $\theta = \pi$ for a repulsive force (like charges). See Fig. 1.15.

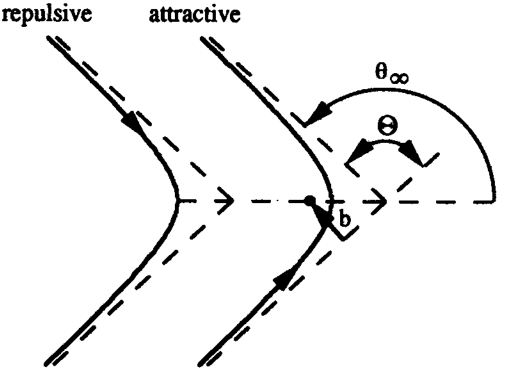


Fig. 1.15. Coulomb scattering

[7]For some of the things that can happen then see: Roger G. Newton, *Scattering Theory of Waves and Particles*, (McGraw-Hill Book Company, New York, NY, 1966), pp. 129-134; Herbert Goldstein, *Classical Mechanics*, (Addison-Wesley Publishing Company, Reading, MA, 1980), 2nd. ed., pp. 110-113.

For E positive, e is greater than one, and the orbit is a hyperbola. Far from the nucleus ($r \rightarrow \infty$) the incident particle travels along incoming or outgoing asymptotic straight lines, with directions $\mp\theta_\infty$ where

$$\theta_\infty = \cos^{-1}(-1/e) \qquad (\pi/2 \le \theta_\infty \le \pi).$$

Note that these directions are the same (apart from sign) for both attractive and repulsive Coulomb scattering. The scattering angle is given by

$$\Theta = 2\theta_\infty - \pi.$$

This, combined with the previous equations, yields the relation between the impact parameter and the scattering angle

$$b = \frac{|k|}{2E}\cot\frac{\Theta}{2}.$$

The scattering angle Θ is a monotonely decreasing function of the impact parameter b, decreasing from $\Theta = \pi$ at $b = 0$ to $\Theta = 0$ as $b \rightarrow \infty$. The differential scattering cross section follows from previous considerations and is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \csc^4\left(\frac{\Theta}{2}\right).$$

This is the famous **Rutherford scattering cross section**, first derived and used by Ernest Rutherford to interpret the experiments of Geiger and Marsden on α-particle scattering from thin metal foils. It led him to the discovery of the nuclear atom. As has already been pointed out, this is really a quantum mechanical problem. Fortunately for the development of atomic physics, however, in this instance classical mechanics, more by accident than anything, leads to the same result as quantum mechanics.[8]

Exercises

1. A particle of mass m moves in one dimension x in a potential well
$$V = V_0 \tan^2(\pi x/2a)$$
where V_0 and a are constants. Find, for given total energy E, the position x as a function of time and the period τ of the motion. In particular, examine and interpret the low energy ($E \ll V_0$) and high energy ($E \gg V_0$) limits of your expressions.

[8]See, for example: Kurt Gottfried, *Quantum Mechanics*: Volume I, (W. A. Benjamin, New York, NY, 1966), pp. 148-153; Gordon Baym, *Lectures on Quantum Mechanics*, (W. A. Benjamin, Reading, MA, 1969, 1973), pp. 213-224; J. J. Sakurai, *Modern Quantum Mechanics*, (Addison-Wesley, Redwood City, CA, 1985), ed. San Fu Tuan, pp. 434-444.

2. For each of the following central potentials $V(r)$ sketch the effective potential

$$V_{eff}(r) = \frac{L^2}{2mr^2} + V(r),$$

and use your sketch to classify and draw qualitative pictures of the possible orbits.

(a) $V(r) = \frac{1}{2}kr^2$ **3D isotropic harmonic oscillator**

(b) $V(r) = -V_1$ for $r < a$ **square well**
$V(r) = 0$ for $r > a$

(c) $V(r) = -\frac{k}{r^2}$

(d) $V(r) = -\frac{k}{r^4}$

(e) $V(r) = -k\frac{e^{-\alpha r}}{r}$ **Yukawa potential**

Note that the qualitative shape of $V_{eff}(r)$ versus r may depend on L and on the various parameters; consider all cases (but assume that the given parameters are positive).

3. The first U.S. satellite to go into orbit, Explorer I, which was launched on January 31, 1958, had a perigee of 360 km and an apogee of 2549 km above the earth's surface. Find:
(a) the semi-major axis,
(b) the eccentricity,
(c) the period,
of Explorer I 's orbit. The earth's equatorial radius is 6378 km and the acceleration due to gravity at the earth's surface is $g = 9.81 \mathrm{m/s^2}$.

4. Mars travels on an approximately elliptical orbit around the Sun. Its minimum distance from the Sun is about 1.38 AU and its maximum distance is about 1.67 AU (1 **AU** = mean distance from Earth to Sun). Find:
(a) the semi-major axis,
(b) the eccentricity,
(c) the period,
of Mars' orbit.

5. The most economical method of traveling from one planet to another, the **Hohmann transfer**, consists of moving along a (Sun-controlled) elliptical path which is tangent to the (approximately) circular orbits of the two planets. Consider a Hohmann transfer from Earth (orbit radius 1.00 AU) to Venus (orbit radius 0.72 AU). Find, in units of AU and year:
(a) the semi-major axis of the transfer orbit,
(b) the time required to go from Earth to Venus,
(c) the velocity "kick" needed to place a spacecraft in Earth orbit into the transfer orbit.
In this problem ignore the effects of the gravitational fields of Earth and Venus on the spacecraft.

6. Halley's comet travels around the Sun on an approximately elliptical orbit of eccentricity $e = 0.967$ and period 76 years. Find:
(a) the semi-major axis of the orbit (Ans. 17.9 AU),
(b) the distance of closest approach of Halley's comet to the Sun (Ans. 0.59 AU),
(c) the time per orbit that Halley's comet spends within 1 AU of the Sun (Ans. 78 days).

7. Define a "season" as a time interval over which the true anomaly increases by $\pi/2$. Find the duration of the shortest season for earth. Take the eccentricity of earth's orbit to be 0.0167.

8. A satellite of mass m moves in a circular orbit of radius a_0 around the earth.
(a) A rocket on the satellite fires a burst radially, and as a result the satellite acquires, essentially instantaneously, a radial velocity u in addition to its angular velocity. Find the semi-major axis, the eccentricity, and the orientation of the elliptical orbit into which the satellite is thrown.
(b) Repeat (a), if instead the rocket fires a burst tangentially.
(c) In both cases find the velocity kick required to throw the satellite into a parabolic orbit.

9. Show that the following ancient picture of planetary motion (in heliocentric terms) is in accord with Kepler's picture, if the eccentricity e is small and terms of order e^2 and higher are neglected:
(a) the earth moves around the sun in a circular orbit of radius a; however, the sun is not at the center of this circle, but is displaced from the center by a distance ea;
(b) the earth does not move uniformly around the circle; however, a radius vector from a point which is on a line from the sun to the center, the same distance from and on the opposite side of the center as the sun, to the earth does rotate uniformly.

10. (a) Show that

$$\frac{2r_0}{r} = 1 + \cos\theta$$

(the standard form for a conic section, on setting the eccentricity $e = 1$ and the semi-latus-rectum $p = 2r_0$) is the equation of a parabola, by translating it into cartesian coordinates with the origin at the focus and the x-axis through pericenter.
(b) A comet travels around the Sun on a parabolic orbit. Show that the distance r of the comet from the Sun is related to the time t from perihelion by

$$\frac{\sqrt{2}}{3}(r + 2r_0)\sqrt{r - r_0} = 2\pi t$$

where distances are measured in AU and time is measured in years.
(c) If one approximates the orbit of Halley's comet near the Sun by a parabola with $r_0 = 0.59$ AU, what does this give for the time Halley's comet spends within 1 AU of the Sun?
(d) What is the maximum time a comet on a parabolic orbit may spend within 1 AU of the Sun?

11. A particle of mass m moves in a central force field $\mathbf{F} = -(k/r^2)\hat{\mathbf{r}}$.
(a) By integrating Newton's second law $d\mathbf{p}/dt = \mathbf{F}$, show that the momentum of the particle is given by $\mathbf{p} = \mathbf{p}_0 + (mk/L)\hat{\boldsymbol{\theta}}$, where $\mathbf{p}_0$ is a constant vector and L is the magnitude of the angular momentum.
(b) Hence show that the orbit in momentum space (the so-called **hodograph**) is a circle. Where is the center and what is the radius of the circle?
(c) Show that the magnitude of $\mathbf{p}_0$ is $(mk/L)e$, where e is the eccentricity. Sketch the orbit in momentum space for the various cases, $e = 0$, $0 < e < 1$, $e = 1$, $e > 1$, indicating for the last two cases which part of the circle is relevant.
(See: Arnold Sommerfeld, *Mechanics*, (Academic Press, New York, NY, 1952), trans. Martin O. Stern, p. 33, 40, 242; Harold Abelson, Andrea diSessa, and Lee Rudolph, "Velocity space and the geometry of planetary orbits," Am. J. Phys. **43**, 579-589 (1975).)

12. Consider the motion of a particle in a central force field with potential $V = -k/r$. Since the force is central, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is constant and the orbit lies in a plane passing through the force center and perpendicular to **L**.
(a) Show that for the particular potential $V = -k/r$ there exists an additional vector quantity which is constant, the **Laplace-Runge-Lenz vector**

$$\mathbf{K} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}.$$

Further show that $\mathbf{K} \cdot \mathbf{L} = 0$, so that **K** and **L** are perpendicular and thus **K** lies in the orbital plane. (Hint: if you've done exercise 1.11, you need only show that $\mathbf{K} = \mathbf{p}_0 \times \mathbf{L}$).
(b) By taking the dot product of **K** with $\hat{\mathbf{r}}$ obtain the equation of the orbit

$$\frac{a(1-e^2)}{r} = 1 + e\cos\theta.$$

Hence find a and e in terms of K and L, and also find the direction that **K** points in the orbital plane.
(c) Express the energy $E = \frac{p^2}{2m} - \frac{k}{r}$ in terms of K and L.

13. Consider the motion of a particle of mass m in a central force field with potential

$$V = -\frac{k}{r} + \frac{h}{r^2}.$$

(a) Show that the equation for the orbit can be put in the form

$$\frac{a(1-e^2)}{r} = 1 + e\cos\alpha\theta,$$

and find a, e, and α in terms of the energy E and angular momentum L of the particle, and the parameters k and h of the potential.
(b) Show that this represents a precessing ellipse, and derive an expression for the average rate of precession in terms of the dimensionless quantity $\eta = h/ka$.
(c) The perihelion of Mercury precesses at the rate of $40''$ of arc per century, after all known planetary perturbations are taken into account. What value of η would

lead to this result? The eccentricity of Mercury's orbit is 0.206 and its period is 0.24 years.

14. A particle of mass m moves in a 3D isotropic harmonic oscillator potential well

$$V = \tfrac{1}{2}m\omega^2 r^2$$

where ω, the angular frequency, is a constant.

(a) Show that the equation for the orbit has the form

$$\frac{L^2}{mE}\frac{1}{r^2} = 1 + \sqrt{1 - \frac{\omega^2 L^2}{E^2}}\cos 2(\theta - \theta_0)$$

where E is the energy and L is the angular momentum.

(b) Show that this represents an ellipse with geometric center at the force center, and express the energy and angular momentum in terms of the semi-major axis a and eccentricity e of the ellipse. (Ans. $E = m\omega^2(a^2 + b^2)$ and $L = m\omega ab$ where $b = a\sqrt{1-e^2}$ is the semi-minor axis)

(c) Show that the period is $\tau = 2\pi/\omega$ independent of the energy and angular momentum, and that the radius is given as a function of time by

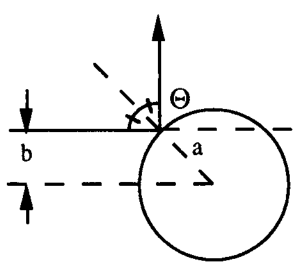
$$r^2 = \frac{E}{m\omega^2}\left[1 - \sqrt{1 - \frac{\omega^2 L^2}{E^2}}\cos 2\omega(t - t_0)\right].$$

15. A small meteor approaches the earth with impact parameter b and velocity v_∞ at infinity. Show that the meteor will strike the earth if

$$b < a\sqrt{1 + \left(v_0/v_\infty\right)^2}$$

where a is the radius and v_0 is the "escape velocity" for the earth.

16.



(a) Find the relation between the scattering angle Θ and the impact parameter b for scattering from a hard sphere of radius a (for which "angle of incidence = angle of reflection").

(b) Use your result to obtain the differential scattering cross section $d\sigma/d\Omega$. Integrate to find the **total scattering cross section** $\sigma = \int (d\sigma/d\Omega)d\Omega$, where the integration extends over the whole solid angle.

17. (a) Show that a particle of energy E is refracted in going from a region in which the potential is zero to a region in which the potential is $-V_1$, the angle of incidence θ_0 and the angle of refraction θ_1 being related by **Snell's law**

$$\frac{\sin\theta_0}{\sin\theta_1} = n$$

where angles are measured from the normal and $n = \sqrt{1 + V_1/E}$ is the **index of refraction**.

(b) Use Snell's law to show that a particle incident at impact parameter b on an attractive square well potential

$$V(x) = -V_1 \quad \text{for } r < a$$
$$V(x) = 0 \quad \text{for } r > a$$

is scattered through an angle Θ given by

$$\frac{b^2}{a^2} = \frac{n^2 \sin^2 \Theta/2}{n^2 + 1 - 2n\cos\Theta/2}.$$

In particular, show that for small impact parameters ($b << a$) the scattered particles are brought to a focus a distance $f \approx \left(\frac{n}{n-1}\right)\left(\frac{a}{2}\right)$ from the force center.

(c) Find the differential scattering cross section $d\sigma/d\Omega$.

18. (a) Show that

$$\frac{r_0}{r} = \cos\alpha\theta$$

is the equation of the orbit for a particle moving in a repulsive potential $V(r) = k/r^2$, determining α and r_0 in terms of the energy and angular momentum.

(Ans. $\alpha = \sqrt{1 + \frac{2mk}{L^2}}$, $r_0 = \frac{\alpha L}{\sqrt{2mE}}$)

(b) Show that the impact parameter b and scattering angle Θ are related by

$$b^2 = \frac{k}{E}\frac{(\pi - \Theta)^2}{\Theta(2\pi - \Theta)}.$$

(c) Show that the differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\pi^2 k}{E\sin\Theta}\frac{\pi - \Theta}{\Theta^2(2\pi - \Theta)^2}.$$