Mass–energy connection without special relativity

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September, 2020

Abstract. In 1905, Einstein carried out his first derivation of the mass–energy equivalence by studying in different reference frames the energy balance of a body emitting electromagnetic radiation and assuming special relativity as a prerequisite. In this paper, we prove that a general mass–energy relationship can be derived solely from very basic assumptions, which are the same made in Einstein's first derivation but completely neglecting special relativity. The general mass–energy relationship turns to a mass–energy equivalence when is applied to the case of a body emitting energy in the form of electromagnetic waves. Our main result is that if the core logic behind Einstein's approach is sound, then the essence of the mass–energy equivalence can be derived without special relativity. We believe that our heuristic approach, although not capable of giving the exact mathematical formula for the mass-energy equivalence, may represent a useful addition to the general discussion on the matter at the graduate level. Our finding suggests that the connection between mass and energy is at a deeper level and comes before any full-fledged physical theory.

Keywords: special relativity, mass–energy equivalence, non-relativistic classical electromagnetism, heuristic derivation, transformation of energy, history of physics

Submitted to: *Eur. J. Phys.*

1. Introduction

Mass–energy equivalence, known in the form of the celebrated equation $E = mc^2$, was derived by Einstein for the first time in a three-page paper published at the end of 1905 [1]. Einstein carried out his derivation by studying in different reference frames the energy balance of a body emitting electromagnetic radiation in two equal but oppositely directed amounts (thus, no change in the emitter velocity due to recoil). According to special relativity [2], the total energy of a plane light wave increases when is observed from a reference frame in uniform motion relative to the emitter's rest frame. Einstein

ascribed this increase to the fact that in the moving reference frame also the total energy of the emitter, where the radiation energy comes from, has increased: when the emitter is observed from the moving reference frame, its kinetic energy must also be added to its internal (proper) energy to get its total energy. Then, Einstein managed to derive that the increase of the emitted energy seen from the moving frame comes from a reduction in kinetic energy of the emitter after the emission. Since, for symmetry reasons, the velocity of the emitter does not change after the emission, Einstein concluded that the mass of the emitter must change by partially turning into radiation energy.

The correctness of this derivation was first criticized by Planck in 1907 [3]. He contended that it is valid "under the assumption permissible only as a first approximation that the total energy of a body is composed additively of its kinetic energy and its energy referred to a system in which it is at rest" [5]. Further criticism was later advanced by Ives in 1952 [4] and Jammer in 1961 [5]: they asserted that Einstein's derivation was but the result of a *petitio principii*. Several other authors (e.g. G. Holton, H. Arzeliés and A.I. Miller, to name a few) agreed with Ives and Jammer criticism. Recently, however, Stachel and Torretti [6] analyzed Ives's analysis and concluded that the logic behind Einstein's derivation is sound. In particular, they presented a proof from first principles of the assumption criticized by Planck. We shall return briefly to their analysis later on. In more recent times, Ohanian [7, 8] agreed with Stachel and Torretti's criticism of Ives, though he argued that Einstein's derivation was wrong mainly "because he assumed that the rest-mass change he found when using a non-relativistic, Newtonian approximation for the internal motions of an extended system would be equally valid for relativistic motions".

For the sake of completeness, let us review Einstein's first derivation in more detail. Einstein considered a body, at rest in an inertial frame S, that emits electromagnetic radiation of total energy L in two equal but oppositely directed amounts. He then considered the same emission process as seen from another inertial frame S', that of an observer moving in uniform parallel translation with respect to the system S and having its origin of coordinates in motion along the x-axis with velocity v (Fig. 1).

Let there be a stationary body in the system S, and let its energy referred to the system S be E_0. Let the energy of the body relative to the system S' moving as above with velocity v, be E'_0.

Let this body send out, in a direction making an angle θ with the x-axis, plane waves of light of energy $\frac{1}{2}\mathrm{L}$ measured relatively to S, and simultaneously an equal quantity of plane waves in the opposite direction, for a total emitted energy equal to L (see Fig. 1). Meanwhile, the body remains at rest in S.

Einstein showed that if the radiation is measured in S', then it possesses a total energy L' that is equal to

$$\mathrm{L}' = \frac{\mathrm{L}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{1}$$

where c is the velocity of light. This relation is the result established by using the law

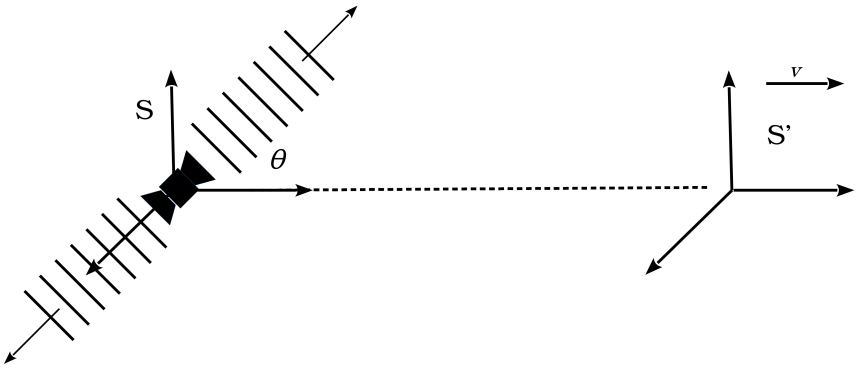

Figure 1. Sketch of the light emission process described in Einstein's paper [1].

for the transformation of the energy of a plane light wave from one inertial frame to the other, derived in the first paper on the special relativity [2].

If we call the energy of the body after the emission of the plane light waves E_1 or E'_1 respectively, measured relatively to the system S or S' respectively, then by making use of eq. (1) we have

$$\begin{aligned} \mathrm{E}_0 &= \mathrm{E}_1 + \mathrm{L}, \\ \mathrm{E}'_0 &= \mathrm{E}'_1 + \frac{\mathrm{L}}{\sqrt{1-\frac{v^2}{c^2}}}. \end{aligned} \tag{2}$$

By subtraction, Einstein obtained the following relation

$$(\mathrm{E}'_0 - \mathrm{E}_0) - (\mathrm{E}'_1 - \mathrm{E}_1) = L\left\{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right\}. \tag{3}$$

According to Einstein's reasoning, the two differences of the form $\mathrm{E}' - \mathrm{E}$ in eq. (3) have the following simple physical meaning. E' and E are the energy values of the same body referred to two reference frames that are in motion relatively to each other, the body being at rest in S. Thus, the difference $\mathrm{E}' - \mathrm{E}$ can differ from the kinetic energy K of the body, with respect to the system S', only by an additive constant C, which depends on the choice of the arbitrary additive constants of the energies E' and E and does not change during the emission of light. Without loss of generality, this constant can be taken equal to zero, and the difference can be written simply as $\mathrm{E}' - \mathrm{E} = \mathrm{K}$. This assumption drew the attention of most of the following literature on the first mass–energy equivalence derivation and generated some controversy on its validity. A careful discussion of this aspect is given in [6]. In the same paper, the authors give a formal derivation of Einstein's assumption from first principles, and their approach is presented as general. As already mentioned, according to these authors, Einstein's assumption turns out to be logically sound. In any case, the validity of what we shall present in Section 2 also relies on the acceptance of the validity of this assumption.

From eq. (3) we have

$$\mathrm{K}_0 - \mathrm{K}_1 = L \left\{ \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right\}. \tag{4}$$

What equation (4) tells us is that the kinetic energy of the body with respect to S' diminishes as a result of the emission of the plane light waves, and the amount of diminution is independent of the properties of the body. Moreover, like the kinetic energy, it depends on the relative velocity v. Neglecting quantities of the fourth and higher orders in v/c, eq. (4) becomes

$$\mathrm{K}_0 - \mathrm{K}_1 = \frac{1}{2} \left[\frac{\mathrm{L}}{c^2} \right] v^2. \tag{5}$$

From eq. (5), Einstein's mass–energy equivalence directly follows: if a body gives off the energy L (in the form of radiation), its mass diminishes by $\frac{\mathrm{L}}{c^2}$.

The rest of this paper is organized as follows. In Section 2, we prove that it is possible to derive a general mass–energy relationship by following the logic behind Einstein's original derivation and by applying the same fundamental assumptions but neglecting special relativity. Within the sphere of validity of these basic assumptions, the general mass–energy relationship would still be true even if special relativity would turn out to be false. We also notice that the general mass–energy relationship turns to a mass–energy equivalence when is applied to the case of a body emitting energy in the form of electromagnetic waves: this is the crucial step in Einstein's first derivation, and special relativity turns out to have no fundamental role in the realness of the equivalence. We shall show that mass–energy equivalence, although with a different mathematical equation, could have been derived even within Maxwell's theory of light (pre-Lorentz, classical ether theory).

In the concluding section, we summarize our findings and remark why they represent a useful addition to the general discussion on the matter.

2. The general mass–energy relationship

It is possible to heuristically derive a general mass–energy relationship by applying the core logic behind Einstein's original derivation but without special relativity. We only use few and very basic initial assumptions which are the same made in Einstein's derivation, exception made for the peculiar principles of special relativity.

Consider a body stationary in an inertial frame S that emits a total amount of energy equal to L. The energy can be emitted in any imaginable form but, like in Einstein's derivation, always in equal amounts in opposite directions to maintain a symmetry of emission that intuitively ensures the motionlessness of the body during the process. The equation of the energy balance in S is then $\mathrm{E}_0 = \mathrm{E}_1 + \mathrm{L}$, where E_0 and E_1 are the total energies of the body respectively before and after the emission referred to the system S.

If the same emission process is seen from an inertial reference frame S' moving in uniform parallel translation with respect to the system S and having its origin of coordinates in motion along the x-axis with velocity v, then it is reasonable to expect that the observed *total*‡ emitted energy L' is different from L and greater than that. This is what we heuristically expect in real life simply because the observer is moving relative to the emitter, and some energy is added to what he sees because of that motion. The equation of the energy balance in S' is then $\mathrm{E}'_0 = \mathrm{E}'_1 + \mathrm{L}'$, where E'_0 and E'_1 are the total energies of the body respectively before and after the emission referred to the system S'. So far, we have used only the principle of energy conservation in any inertial frame.

Without loss of generality, we can write the mathematical relation that connects L' and L as follows

$$\mathrm{L}' = \mathcal{F}(\mathrm{L}, v), \tag{6}$$

where $\mathcal{F}$ is a suitable mathematical function. Since the origin of reference frame S' moves along the x-axis, the functional dependence of eq. (6) on velocity is by construction on scalar velocity v. Moreover, let L' be directly proportional to L. If the body emits energy equal to 2L, the energy observed in S' must be equal to $2\mathrm{L}'$. Indeed, this seems a reasonable assumption: the body emitting energy 2L can, in theory, be composed of two distinct bodies emitting energy L each. Since in this second case the observer in S' sees a total energy of $2\mathrm{L}'$ (L' for each body), this must be also the case when we have a single body emitting energy equal to 2L. Thus, equation (6) becomes

$$\mathrm{L}' = \mathrm{L}f(v). \tag{7}$$

In order to determine the approximate mathematical form of the dimensionless function $f(v)$, consider the Maclaurin expansion of $f(v)$ up to $O(v^3)$

$$f(v) = \alpha + \beta v + \delta v^2 + O(v^3), \tag{8}$$

where α, β, and δ are numerical coefficients.

Since $f(0) = 1$, α must be equal to 1. Furthermore, we must have that $f(-v) = f(v)$ since, for symmetry reasons§, the *overall* energy L' observed by an observer in S' does not depend upon the arbitrary direction (towards the positive or the negative x-axis)

‡ We invite the reader to pay attention to the use of the word 'total' here. We know from experience (e.g. with sound waves, light waves, etc.) that the carried energy is 'perceived' as higher or lower according to the emission direction relative to the observer. However, here we consider the *overall* energy emitted by the source, namely the sum (integral) of the energy emitted in any direction. A corroboration of the fact that we expect greater overall energy is given further in the text when we calculate the energy of two light waves within Maxwell's theory of light, eqs. (17) to (22).

§ Whatever is the direction θ along which the energies $\mathrm{L}/2$ are emitted (see Fig. 1), the case in which we observe S and move in translational motion towards the positive x-axis ($+v$) is, as a whole, physically equivalent to the case in which we observe S and move in translational motion towards the negative x-axis ($-v$), provided that the whole setting is flipped over the x-axis. The amount of energy L' cannot change because of these symmetry (abstract) operations.

of the velocity of S' and thus $\beta = 0$. Since $f(-v) = f(v)$, function $f(v)$ must be even, and all other terms with odd powers must be absent. Therefore,

$$f(v) = 1 + \delta v^2 + O(v^4), \tag{9}$$

with constant δ having the physical units of an inverse square velocity. This velocity is the 'characteristic velocity' of the peculiar emission process.

Thus, we arrive at

$$\mathrm{L}' = \mathrm{L}(1 + \delta v^2 + O(v^4)). \tag{10}$$

Within the sphere of validity of the previous assumptions, equation (10) is very general and can be applied to all kinds of energy emission mechanisms. As a matter of fact, its derivation is completely independent of the energy emission process at play, exception made for the numerical value of the constant δ.

Now, the energy balance equations become

$$\begin{aligned} &\mathrm{E}_0 = \mathrm{E}_1 + \mathrm{L}, \\ &\mathrm{E}'_0 = \mathrm{E}'_1 + \mathrm{L}(1 + \delta v^2 + O(v^4)). \end{aligned} \tag{11}$$

Like Einstein in his 1905 paper, we subtract the first equation from the second

$$(\mathrm{E}'_0 - \mathrm{E}_0) - (\mathrm{E}'_1 - \mathrm{E}_1) = \mathrm{L}(\delta v^2 + O(v^4)), \tag{12}$$

and with Einstein's assumption $\mathrm{E}' - \mathrm{E} = \mathrm{K}$ we obtain

$$\mathrm{K}_0 - \mathrm{K}_1 = L(\delta v^2 + O(v^4)). \tag{13}$$

If, like in [6], we define the inertial mass for a body in translational motion (in keeping with the requirement that special relativistic dynamics have a Newtonian limit as $v \to 0$) by

$$m = \lim_{v \to 0} \frac{K}{v^2/2}, \tag{14}$$

then from eq. (13) it follows

$$-\Delta m = m_0 - m_1 = \lim_{v \to 0} \frac{(K_0 - K_1)}{v^2/2} = \lim_{v \to 0} \frac{L(\delta v^2 + O(v^4))}{v^2/2} = 2\delta \mathrm{L}. \tag{15}$$

In short,

$$-\Delta m = 2\delta \mathrm{L}, \tag{16}$$

and this is an exact, not an approximate result. If a body gives off the energy L, its mass diminishes by 2δL.

Notice that eq. (16) is not a mass–energy equivalence *per se*. If we apply eq. (16) to a body releasing two projectiles of mass m in opposite directions with non-relativistic velocity v_0 (relative to the parent body), then it is possible to prove that $\delta = 1/v_0^2$. Since $\mathrm{L} = 2 \cdot \frac{1}{2} m v_0^2$ (the emitted energy, in this case, is only kinetic), then $-\Delta m = 2m$. Namely, equation (16) gives simply the change of mass of the parent body due to the

loss of two projectiles of mass m each. Thus, in this case, eq. (16) does not give any mass–energy equivalence.

On the other hand, if we apply eq. (16) to the emission of energy in the form of electromagnetic waves, we obtain a mass–energy equivalence: *radiation energy comes from mass reduction, and thus mass transforms into radiation energy.* Special relativity is not essential for the derivation of this mass–energy equivalence: special relativity comes into play only in the numerical value of the constant δ. The constant δ has the physical units of an inverse square velocity, and in the case of electromagnetic phenomena, it must be heuristically proportional to $1/c^2$. In the case of Einstein's original derivation, we have that $\delta = 1/2c^2$.

In order to emphasize the implications of the derived general mass–energy relationship, consider that even within Maxwell's theory of light (and thus, no special relativity), one could have already come to mass–energy equivalence, albeit in the different form $E = \frac{1}{2}mc^2$.

Within Maxwell's theory of light (pre-Lorentz, classical ether theory), we have that $\delta = 1/c^2$. The total energy density associated with an electromagnetic wave is

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} = \epsilon_0 E^2, \tag{17}$$

where ϵ_0 and μ_0 are respectively the permittivity and the permeability of free space, and E and B denote the electric and magnetic fields of the wave. The last equality in eq. (17) holds because, for electromagnetic waves, we also have that $E = cB$ ($c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$).

Now, consider two plane waves of light, 1 and 2, emitted in opposite directions from the origin of the rest frame S along the x-axis, as shown in Fig. 2. Since we are working within Maxwell's theory of light, frame S shall also be considered as the reference frame of the ether. Consider further a reference frame S' moving away from the origin of S with velocity v in the direction of the positive x-axis (in the approximation $v \ll c$). In the present context, we cannot use the Lorentz transformations for the electromagnetic field to derive the electric field $\mathbf{E}'$ measured in the reference frame S'. Nonetheless, it is possible to obtain a suitable transformation law that applies to this specific case via the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ felt by a test charge q stationary in S', namely

$$\mathbf{E}' = \frac{\mathbf{F}}{q} = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \tag{18}$$

See also reference [9], where the same result is obtained by applying Faraday's law in the approximation of reference frames moving at speeds small compared to the speed of light.

According to the above transformation, the components $E'_{1\parallel}$ and $E'_{1\perp}$ of the electric field of wave 1 in the reference frame S' are ($\parallel$ and $\perp$ are referred to the plane $(c, \mathbf{B})$, see Fig. 2)

$$\begin{aligned} E'_{1\parallel} &= E_{1\parallel} = 0, \\ E'_{1\perp} &= E_{1\perp} + (\mathbf{v} \times \mathbf{B}_1)_\perp = E\left(1 - \tfrac{v}{c}\right), \end{aligned} \tag{19}$$

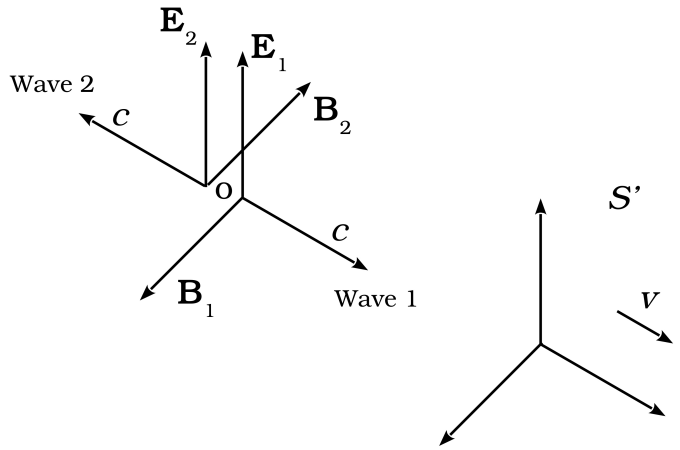

Figure 2. Emission of light waves 1 and 2.

since $E = E_{1\perp}$ and $B = E/c$.

The energy density u'_1 of wave 1 measured from S' is then

$$u'_1 = \epsilon_0 E^2 \left(1 - \frac{v}{c}\right)^2 = u\left(1 - \frac{v}{c}\right)^2. \tag{20}$$

By applying the same procedure to wave 2, the energy density u'_2 is

$$u'_2 = \epsilon_0 E^2 \left(1 + \frac{v}{c}\right)^2 = u\left(1 + \frac{v}{c}\right)^2. \tag{21}$$

In order to calculate the energy of the two plane waves of light, we now need to multiply the energy densities by the volumes of the plane waves measured in S'. After an interval of time T, wave 1 has traveled a distance cT from the origin of reference frame S, and its volume V is simply $V = AcT$, where A is the transverse area of the wave. For an observer in S', the volume is the same since, in Maxwell's theory of light, light propagates at speed equal to c only in the ether reference frame S, and no Lorentz contraction comes into play.

If the total energy of the two plane waves of light in S is $\mathrm{L} = 2uV$, then the total energy measured in S' is

$$\begin{aligned} \mathrm{L}' &= u'_1 V'_1 + u'_2 V'_2 = u\left(1 - \tfrac{v}{c}\right)^2 V + u\left(1 + \tfrac{v}{c}\right)^2 V = \\ &= 2uV\left(1 + \tfrac{v^2}{c^2}\right) = \mathrm{L}\left(1 + \tfrac{v^2}{c^2}\right), \end{aligned} \tag{22}$$

and thus $\delta = 1/c^2$.

3. Concluding remarks

We have shown that a general mass–energy connection can be heuristically derived by applying the core logic behind Einstein's original derivation with very basic assumptions but neglecting special relativity. Einstein's 1905 mass–energy equivalence is a special

case of this general relationship: the general mass–energy connection turns to a mass–energy equivalence when is applied to the case of a body emitting energy in the form of electromagnetic waves. Obviously, to obtain the exact mathematical equation for the mass–energy equivalence, we still need special relativity. Moreover, the validity of our result stands upon the acceptance of the validity and logical consistency of the basic assumptions in Einstein's original derivation. However, within these confines, our finding shows that the mass–energy equivalence seems to originate at a deeper, fundamental level and from first and general principles. In 1946, Einstein proposed an elementary derivation of the mass–energy equivalence that allegedly does not presume the formal machinery of special relativity but uses only three previously known laws of physics [10]: (i) the law of the conservation of momentum; (ii) the expression for the pressure of radiation; (iii) the well-known expression for the aberration of light. Our approach, instead, suggests that the mass–energy equivalence is almost inescapable, as happens with new laws of physics derived from dimensional analysis, and comes before any full-fledged physical theory.

We acknowledge that our approach is not capable of giving the exact mathematical formula for the mass-energy equivalence, and thus it is not a rigorous derivation of that equivalence. However, conceptually speaking, it is as rigorous as the proof attempts made by Einstein itself [11] since it derives from the same core logic behind most of them.

Acknowledgments

The author is indebted to Dr. Assunta Tataranni and Dr. Gianpietro Summa for key improvements to the manuscript. The author would also like to acknowledge three anonymous reviewers who made several suggestions for revision that have greatly improved this paper.

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