Doc. 2

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Covariance Properties of the Field Equations of the Theory of Gravitation Based on the General Theory of Relativity

by Albert Einstein in Berlin and Marcel Grossmann in Zurich

In a paper[1] published in 1913 we based a generalized theory of relativity upon absolute differential calculus in a manner such that it also embraces the theory of
[3] gravitation. Two basically different kinds of systems of equations occur in this theory. For a gravitational field considered as given, we first established systems of equations for material (e.g., mechanical, electrical) processes. These equations are covariant under arbitrary substitutions of the space-time variables ("coordinates") and can be considered as generalizations of the corresponding equations of the original theory of relativity. Second, we established a system of equations that determines the gravitational field insofar as the quantities that determine the material processes are given; and this system can be considered a generalization of the Poisson equation of Newton's theory of gravitation. In the original theory of relativity, there is no corresponding system of equations for this. In contrast to the equations mentioned above, we could not demonstrate general covariance for those "gravitational equations." The reason is that their derivation was based (besides the conservation theorems) only upon the covariance with respect to *linear* transformations, and thus left it an open question as to whether or not there exist other substitutions that would transform the equations into themselves.

There are two reasons why the resolution of this question is of particular importance to the theory. The answer to this question gives, first, information on how far the basic idea of relativity theory can be developed; and this is of great import to
[p. 216] the philosophy of space and time. And second, the judgment about the value of the theory from the point of view of physics depends to a high degree upon the answer to this question, as is shown by the following consideration.

The entire theory evolved from the conviction that all physical processes in a gravitational field occur just in the same way as they would without it, if an appropriately accelerated (three-dimensional) coordinate system would be introduced ("hypothesis of equivalence"). This hypothesis, which is based upon the experimental fact of the equality between gravitational and inertial mass, gains additional convincing force if the "apparent" gravitational field—which exists relative to the

[1] [1]"Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation" (Leipzig: B. G. Teubner, 1913). [In the following it is abbreviated as "Outline." The paper is
[2] printed in this journal, *Zeitschrift für Mathematik und Physik,* vol. 62, pp. 225–261.]

accelerated three-dimensional coordinate system—can be viewed as a "real" gravitational field; in other words, if acceleration-transformations (i.e., nonlinear transformations) become permissible transformations in the theory.

At first glance it appears desirable to look for gravitational equations that are covariant toward arbitrary transformations. However, in §2 of the present paper[2] we will show by a simple consideration that the quantities $g_{\mu\nu}$ which characterize the gravitational field cannot completely be determined by generally-covariant equations.

In the following we shall demonstrate that the gravitational equations established by us are generally covariant just to the degree imaginable under the condition that the fundamental tensor $g_{\mu\nu}$ *must be completely determined. It follows in particular that the gravitational equations are covariant with respect to quite varied acceleration transformations (i.e., nonlinear transformations).* [5]

§1. The Basic Equations of the Theory

We characterized the energetic response of a physical process by means of a covariant tensor $T_{\mu\nu}$ or its reciprocal contravariant tensor $\Theta_{\mu\nu}$, respectively. This tensor satisfies equations (10) of the "Outline," viz., [6]

$$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\,\gamma_{\sigma\mu} T_{\mu\nu}\right) = \frac{1}{2} \sum_{\mu\nu} \sqrt{-g} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x_\sigma} T_{\mu\nu},$$

or respectively

$$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\,g_{\sigma\mu} \Theta_{\mu\nu}\right) = \frac{1}{2} \sum_{\mu\nu} \sqrt{-g} \cdot \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \Theta_{\mu\nu},$$

and they represent the energy-momentum equations of the material process. All equations of the theory take a particularly comprehensive form if one introduces the quantities [p. 217]

$$(1) \qquad \mathfrak{T}_{\sigma\nu} = \sum_{\mu} \sqrt{-g}\,\gamma_{\sigma\mu} T_{\mu\nu} = \sum_{\mu} \sqrt{-g}\,g_{\sigma\mu} \Theta_{\mu\nu},$$

which differ from the components of a mixed tensor[3] only by a factor of $\sqrt{-g}$. Conceptually we call them the *complex of energy-density* of the physical process. Our equations above can now be rewritten as

[2]Compare also the remark in the appendix of the reprint in *Zeitschr. f. Math. u. Phys.*, vol. 62. [4]

[3]Compare §1 of part II of the "Outline." [7]

[8] (I)
$$\sum_{\nu} \frac{\partial \mathfrak{T}_{\sigma\nu}}{\partial x_{\nu}} = \frac{1}{2} \sum_{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x_{\sigma}} \gamma_{\mu\varrho} \mathfrak{T}_{\varrho\nu}.$$

If one introduces in place of the energy tensor of the gravitational field the "complex of the energy-density of the gravitational field," that is, the quantities

(2)
$$\mathfrak{t}_{\sigma\nu} = \sum_{\mu} \sqrt{-g} \cdot \gamma_{\sigma\mu} t_{\mu\nu} = \sum_{\mu} \sqrt{-g} \cdot g_{\sigma\mu} \vartheta_{\mu\nu},$$

then the "Outline" equations (14) and (13) respectively yield

(2a)
$$-2\varkappa \mathfrak{t}_{\sigma\nu} = \sqrt{-g} \left(\sum_{\beta\varrho\tau} \gamma_{\beta\nu} \frac{\partial g_{\varrho\tau}}{\partial x_{\sigma}} \frac{\partial \gamma_{\varrho\tau}}{\partial x_{\beta}} - \frac{1}{2} \sum_{\alpha\beta\varrho\tau} \delta_{\sigma\nu} \gamma_{\alpha\beta} \frac{\partial g_{\varrho\tau}}{\partial x_{\alpha}} \frac{\partial \gamma_{\varrho\tau}}{\partial x_{\beta}} \right),$$

where $\delta_{\sigma\nu} = 0$ or 1 depending on $\sigma \neq \nu$ or $\sigma = \nu$.

In place of the gravitational equations (21) and (18), respectively, of the "Outline" we now get the equations

(II)
$$\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_{\alpha}} \left(\sqrt{-g} \gamma_{\alpha\beta} g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_{\beta}} \right) = \varkappa (\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}).$$

In a manner analogous to the one used in §5 of the "Outline" one can now get from (I) and (II) the general conservation theorems, which can take the form

(III)
$$\sum_{\nu} \frac{\partial}{\partial x_{\nu}} (\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}) = 0.$$

§2. Remarks on the Choice of the Coordinate System

We want to show now that, completely independent of the gravitational equations we established, a complete determination of the fundamental tensor $\gamma_{\mu\nu}$ of a gravitational field with given $\Theta_{\mu\nu}$ by a generally-covariant system of equations is impossible.

[p. 218] We can prove that if a solution for the $\gamma_{\mu\nu}$ for given $\Theta_{\mu\nu}$ is already known, then
[9] the general covariance of the equations allows for the existence of further solutions.

Assume a domain L within our four-dimensional manifold such that no "material process" shall exist within L, i.e., where the $\Theta_{\mu\nu}$ therefore vanish. By virtue of the given $\Theta_{\mu\nu}$ the $\gamma_{\mu\nu}$ are assumed determined everywhere outside of L and, therefore, also inside L (assumption *a*).

Instead of the original coordinates x_ν we now imagine new coordinates x'_ν introduced in the following manner. Everywhere outside of L we have $x'_\nu = x_\nu$, but inside L at least for part of it and at least for one index let there be $x'_\nu \neq x_\nu$.
[10] Obviously, at least for part of L, this substitution achieves $\gamma'_{\mu\nu} \neq \gamma_{\mu\nu}$. On the other hand we have $\Theta'_{\mu\nu} = \Theta_{\mu\nu}$ everywhere, that is, outside of *L*, because for this
[11] domain $x'_\nu = x_\nu$, and inside of *L* because for this domain $\Theta'_{\mu\nu} = 0 = \Theta_{\mu\nu}$.

Therefore, if all substitutions would be permitted, the same system of $\Theta_{\mu\nu}$ would have more than one system of the $\gamma_{\mu\nu}$ belonging to it, and this is a contradiction to assumption *a*).[4]

Once it is understood that an acceptable theory of gravitation implies necessarily a specialization of the coordinate system, it is also easily seen that the gravitational equations, given by us, are based upon a special coordinate system. An x_ν-differentiation of equations (II) and summation over ν, under simultaneous consideration of equations (III), yields the relations

$$\text{(IV)} \qquad \sum_{\alpha\beta\mu\nu} \frac{\partial^2}{\partial x_\nu \partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\, g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) = 0,$$

and these are four differential conditions for the quantities $g_{\mu\nu}$. We want to write (IV) in the abbreviated form

$$B_\sigma = 0\,.$$

These four quantities B_σ do not form a generally-covariant vector, as will be shown in §5. From this one can conclude that the equations $B_\sigma = 0$ represent a true condition for the choice of the coordinate system.[5] [p. 219]

§3. The Hamiltonian Form of the Gravitational Equations

In the following proof of covariance of the gravitational equations we will use the fact that these equations can be brought into the form of a variational principle.[6]

The gravitational equations (II) can be shown to be equivalent to the statement

$$\text{(V)} \qquad \int \left(\delta H - 2\varkappa \sum_{\mu\nu} \sqrt{-g}\, T_{\mu\nu}\, \delta\gamma_{\mu\nu}\right) d\tau = 0,$$ [15]

[4]This train of thought is already among the notes in the appendix of the reprint of the "Outline" in volume 62 of the *Zeitschr. f. Math. u. Phys.* The claim appended there about the restriction on the coordinate systems, however, does not apply; the restriction to linear substitutions follows from (III) only if the quantities $t_{\sigma\nu} / \sqrt{-g}$ have tensorial character, and this turned out to be not justified. [12] [13]

[5]The equations $B_\sigma = 0$ can also be obtained by imposing the divergence operator upon the gravitational equations in the manner of the absolute differential calculus, thereby using the conservation law of matter.

[6]We owe thanks to Mr. Paul Bernays in Zurich for suggesting the idea of simplifying the proof by such a procedure. [14]

where

(Va) $$H = \frac{1}{2}\sqrt{-g}\sum_{\alpha\beta\tau\varrho}\gamma_{\alpha\beta}\frac{\partial g_{\tau\varrho}}{\partial x_\alpha}\frac{\partial \gamma_{\tau\varrho}}{dx_\beta}$$

and the $\gamma_{\mu\nu}$ are varied independently of each other such that the variation on the boundary of the four-dimensional domain of integration vanishes.

Utilizing for the calculation of δH the easily understood formulas

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sum_{\mu\nu}\sqrt{-g}g_{\mu\nu}\delta\gamma_{\mu\nu},$$

$$\delta\left(\frac{\partial g_{\tau\varrho}}{\partial x_\alpha}\right) = \frac{\partial}{\partial x_\alpha}(\delta g_{\tau\varrho}) = -\sum_{\mu\nu}\frac{\partial}{\partial x_\alpha}(g_{\tau\mu}g_{\varrho\nu}\delta\gamma_{\mu\nu}),$$

$$\delta\left(\frac{\partial \gamma_{\tau\varrho}}{\partial x_\beta}\right) = \frac{\partial}{\partial x_\beta}(\delta\gamma_{\tau\varrho}),$$

and considering the fact that variations of surface integrals vanish, one finds

$$\int\delta H d\tau = \int\sum_{\mu\nu\alpha\beta\tau\varrho}\Big(-\frac{\partial}{\partial x_\alpha}\Big(\sqrt{-g}\gamma_{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x_\beta}\Big) + \sqrt{-g}\gamma_{\alpha\beta}\gamma_{\tau\varrho}\frac{\partial g_{\mu\tau}}{\partial x_\alpha}\frac{\partial g_{\nu\varrho}}{\partial x_\beta}$$

[16] $$+\frac{1}{2}\sqrt{-g}\cdot\frac{\partial g_{\tau\varrho}}{\partial x_\mu}\frac{\partial g_{\tau\varrho}}{\partial x_\nu} - \frac{1}{4}g_{\mu\nu}\gamma_{\alpha\beta}\frac{\partial g_{\tau\varrho}}{\partial x_\alpha}\frac{\partial g_{\tau\varrho}}{\partial x_\beta}\Big)\delta\gamma_{\mu\nu}\cdot d\tau.$$

Utilizing the definitions (14) and (16) of the "Outline," our condition (V) takes [p. 220] the form

$$\int\sum_{\mu\nu}(D_{\mu\nu}(g) + \varkappa(t_{\mu\nu} + T_{\mu\nu}))\delta\gamma_{\mu\nu}\cdot\sqrt{-g}d\tau = 0.$$

As the $\delta\gamma_{\mu\nu}$ are supposed to be mutually independent, the equations (21) of the "Outline," i.e., our gravitational equations in covariant form, now become a consequence of this condition.

§4. Proof of a Lemma. Adapted Coordinate Systems

Our next task is the investigation of the covariance properties of equation (V). For this purpose we look first for the transformational properties of the integrals

[17] $$J = \int H d\tau = \int\sqrt{-g}\sum_{\alpha\beta\tau\varrho}\gamma_{\alpha\beta}\frac{\partial g_{\tau\varrho}}{\partial x_\alpha}\frac{\partial \gamma_{\tau\varrho}}{\partial x_\beta}\cdot d\tau.$$

Let there be an arbitrary four-dimensional manifold *M*, referred to a coordinate system *K* of the x_ν. Furthermore, we refer the same manifold *M* to a second coordinate system *K'* of the x'_ν such that

$$dx_\nu = \sum_\mu \frac{\partial x_\nu}{\partial x'_\mu} dx'_\mu = \sum_\mu p_{\nu\mu} dx'_\mu$$

are the transformation formulas. J and J' shall be the values of the integral above relative to K and K', respectively. This gives

$$J' = \int \sqrt{-g'} \cdot \sum_{\alpha\beta\tau\varrho} \gamma'_{\alpha\beta} \frac{\partial g'_{\tau\varrho}}{\partial x'_\alpha} \frac{\partial \gamma'_{\tau\varrho}}{\partial x'_\beta} d\tau'.$$

Considering that $\sqrt{-g'} \cdot d\tau$ is a scalar, the transformation of J' in terms of the coordinate system K gives

$$J' = \int \sqrt{-g} \sum_{\substack{\mu\nu\alpha\beta \\ rsik \\ mn\tau\varrho}} \left(\pi_{r\alpha} \pi_{s\beta} \gamma_{rs} \cdot p_{i\alpha} \frac{\partial}{\partial x_i} (p_{m\tau} p_{n\varrho} g_{mn}) p_{k\beta} \frac{\partial}{\partial x_k} (\pi_{\mu\tau} \pi_{\nu\varrho} \gamma_{\mu\nu}) \right) d\tau,$$

hence

$$J' = \int \sqrt{-g} \sum_{\mu\nu mnik\varrho\tau} \left(\gamma_{ik} \frac{\partial}{\partial x_i} (p_{m\tau} p_{n\varrho} g_{mn}) \frac{\partial}{\partial x_k} (\pi_{\mu\tau} \pi_{\nu\varrho} \gamma_{\mu\nu}) \right) d\tau.$$

In further calculations we shall assume that the coordinate systems K and K' differ only by infinitesimals, i.e., the transformation is infinitesimal. We then have to set

$$x_\nu = x'_\nu - \Delta x_\nu,$$

therefore [p. 221]

$$p_{\nu\mu} = \frac{\partial x_\nu}{\partial x'_\mu} = \delta_{\nu\mu} - \frac{\partial(\Delta x_\nu)}{\partial x'_\mu} = \delta_{\nu\mu} - \frac{\partial(\Delta x_\nu)}{\partial x_\mu},$$

and

$$\pi_{\mu\nu} = \frac{\partial x'_\nu}{\partial x_\mu} = \delta_{\nu\mu} + \frac{\partial(\Delta x_\nu)}{\partial x_\mu},$$

where the Δx_ν are understood as infinitesimal quantities whose squares and products are negligible. This results in

$$J' - J = -4 \int \sqrt{-g} \sum_{mnik\tau} \gamma_{ik} g_{mn} \frac{\partial \gamma_{\tau n}}{\partial x_k} \frac{\partial^2 (\Delta x_m)}{\partial x_\tau \partial x_i} \cdot d\tau.$$

Partial integration turns this into

$$(3) \qquad J' - J = -4 \int \sum_{mnik\tau} \frac{\partial}{\partial x_\tau} \left(\sqrt{-g} \gamma_{ik} g_{mn} \frac{\partial \gamma_{\tau n}}{\partial x_k} \frac{\partial (\Delta x_m)}{\partial x_i} \right) d\tau$$

$$+ 4 \int \sum_{mnik\tau} \frac{\partial}{\partial x_i} \left(\sqrt{-g} \gamma_{ik} g_{mn} \frac{\partial \gamma_{\tau n}}{\partial x_k} \Delta x_m \right) d\tau \qquad [18]$$

$$- 4 \int \sum_{mnik\tau} \frac{\partial^2}{\partial x_\tau \partial x_i} \left(\sqrt{-g} \gamma_{ik} g_{mn} \frac{\partial \gamma_{\tau n}}{\partial x_k} \right) \Delta x_m \cdot d\tau.$$

We notice that the first two integrals can be written as surface integrals which we abbreviated as 0_1 and 0_2, respectively. The factor of Δx_μ in the third integral is readily recognized as B_m according to the notation introduced subsequent to equation (V). Equation (3) in abbreviated notation is now

$$J' - J = O_1 + O_2 - 4\int \sum_m B_m \Delta x_m \cdot d\tau. \tag{3a}$$

The reasons which led to a preference for coordinate systems in which the quantities $B_m = 0$ have been explicated in §2. We want to call these coordinate systems "adapted" to the manifold. It follows from equation (3a) that adapted coordinate systems are selected such that under fixed boundary values of the coordinates and their first derivatives (considered in an arbitrary coordinate system), the integral J becomes an extremum.

We now want to call a transformation between appropriate coordinate systems *admissible*.* When the transformation from K to K' is admissible, equation (3a) yields

$$J' - J = O_1 + O_2.$$

[p. 222]

§5. Proof of Covariance of the Gravitational Equations

In §4 we investigated a manifold M. We shall now consider a second manifold $\bar{M}$ which differs only infinitesimally from the former, and for which the quantities $g_{\mu\nu}$ and their first derivatives coincide on the boundary of the domain L with those of the corresponding manifold M. We impose the coordinate systems $\bar{K}$ and $\bar{K'}$ in the following manner:

a) Both coordinate systems be adapted ones for the manifold $\bar{M}$.
b) On the boundary of the domain L, let the coordinates $\bar{x_\nu}$ coincide with the x_ν and the $\bar{x'_\nu}$ with the x'_ν.
c) The coincidence of the coordinate systems shall not only apply on the boundary of the domain, but also for quantities of first order that are infinitesimally close to the boundary; this condition implies that the $\partial(\Delta x_\nu)/\partial x_\sigma$ coincide with the $\partial(\Delta \bar{x_\nu})/\partial x_\sigma$.

Conditions (b) and (c) do not contradict each other, as can be seen in the following manner. Since manifold M is referred to as an adapted coordinate system, §4 shows the choice of coordinate system K is such as to make the integral J an

*Translator's note. The word "berechtigt" in the German original is today mathematically understood as "admissible because of justification by previously stated conditions"; modern German texts also favor "zulässig" (= admissible) over the older "berechtigt" (= justified).

extremum under fixed boundary values of the coordinates and their first derivatives. It is then possible to put into the varied manifold $\overline{M}$ an adapted coordinate system $\overline{K}$ which coincides with the coordinate system K outside of L and deviates from K only inside of L; because an extremum of the integral J must also exist for the manifold $\overline{M}$ under unchanged boundary values—whereupon the satisfiability of the equations $B_m = 0$ follows also for the varied manifold.

Let us assume the coordinate systems K and K', which are used in the manifold M, are both adapted. According to (3b) the equations

$$J' - J = O_1 + O_2,$$
$$\bar{J}' - \bar{J} = \overline{O}_1 + \overline{O}_2,$$

or after subtraction

$$(\bar{J}' - J') - (\bar{J} - J) = (\overline{O}_1 - O_1) + (\overline{O}_2 - O_2).$$

are valid.

The specifications (b) and (c) and the relations between M and $\overline{M}$, together with (3), imply that both $\overline{0}_1 - 0_1$ and $\overline{0}_2 - 0_2$ vanish.

$\overline{M}$ can be called a manifold developed by variation of M. Therefore, we denote analogously [p. 223]

$$\bar{J} - J = \delta_a J,$$
$$\bar{J}' - J' = \delta_a J',$$

and consequently get

$$\delta_a J' = \delta_a J. \tag{4}$$

The index α is meant to express that, together with the manifold, the coordinate system is co-varied such that the varied coordinate system and the varied manifold are always adapted relative to each other, while on the boundary the coordinate system remains, of course, unvaried (so-called "adapted variation").

Our aim is to demonstrate that an equation

$$\delta J' = \delta J$$

is satisfied for *any* variation of the manifold, not just for an *adapted* variation as equation (4) says. However, we can let any variation of $g_{\mu\nu}$ evolve from an adapted one if we follow it with another variation of the coordinate system. It turns out that for a variation of $g_{\mu\nu}$, equivalent to just one variation of the coordinate system, the variation of J, denoted by $\delta_k J$, vanishes, provided we assume the variations δx_ν and their first derivations vanish on the boundary of the domain, and also provided furthermore that the coordinate system to be varied is an adapted system. The reason is that equation (3a) leads to the direct consequence

$$\delta_k J = O_1 + O_2 - 4\int \sum_m B_m \delta x_m \cdot d\tau = 0.$$

Therefore, we can associate to equation (4) the equation

$$\delta_k J' = \delta_k J = 0. \tag{5}$$

From these two equations—together with the fact that the superposition of an adapted variation and a mere coordinate variation is equivalent to any variation of the $\gamma_{\mu\nu}$—it follows that for such arbitrary variation one has

$$\delta J' = \delta J. \tag{6}$$

From this equation, however, one can prove in a simple manner the covariance of equation (V). The $\delta\gamma_{\mu\nu}$ are after all contravariant, the $T_{\mu\nu}$ covariant, and thus
[p. 224] $\sum_{\mu\nu} T_{\mu\nu}\delta\gamma_{\mu\nu}$ is a scalar; and the same is true of $\sqrt{-g} \cdot d\tau$. Consequently,

{1} $$\int \sqrt{-g'} \cdot \sum_{\mu\nu} T'_{\mu\nu} \delta\gamma'_{\mu\nu} \cdot d\tau' = \int \sqrt{-g} \cdot \sum_{\mu\nu} T_{\mu\nu} \delta\gamma_{\mu\nu} \cdot d\tau. \tag{7}$$

It follows from (6) and (7) that equation (V) is covariant toward all admissible transformations of the coordinate systems, provided the variations are chosen such that the $\delta\gamma_{\mu\nu}$ *and their first derivatives* vanish on the boundary of the domain. The variational theorem whose covariance has been proven in this manner is then a little less general than the one used in §3 for the derivation of the gravitational equations. However, a glance at the development of §3 shows that the derivation of those gravitational equations is not hampered by these restricting boundary conditions of the variation. With this it is proven that:

The gravitational equations (II) are covariant under all admissible transformations of the coordinate systems, i.e., under all transformations between coordinate systems which satisfy the conditions

$$B_\sigma = \sum_{\alpha\beta\mu\nu} \frac{\partial^2}{\partial x_\nu \partial x_\alpha}\left(\sqrt{-g} \cdot \gamma_{\alpha\beta} g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) = 0\ . \tag{IV}$$

We have claimed in §2 that the expressions B_σ do not form a covariant vector. We shall give the proof of it only now because it is especially simple when we utilize the results we have just obtained. All coordinate systems used in the foregoing (and called adapted systems) would be arbitrary coordinate systems if the B_σ were covariant. None of the steps in the proof would lose its convincing force due to this circumstance. The final result of the proof would be the completely general covariance of the gravitational equations. Then the following would be a general mixed tensor

{2} $$\mathfrak{A}_{\sigma\nu} = \frac{1}{\sqrt{-g}}\left(\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta} g_{\sigma\mu} \frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) - \varkappa \mathfrak{t}_{\sigma\nu}\right) = \frac{\varkappa}{\sqrt{-g}} \cdot \mathfrak{T}_{\sigma\nu}$$

and consequently

$$\sum_{\sigma}\mathfrak{A}_{\sigma\sigma} = -\sum_{\alpha\beta\tau\varrho}\left(\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\frac{\partial \lg g}{\partial x_\beta}\right) - \frac{1}{2}\gamma_{\alpha\beta}\frac{\partial g_{\varrho\tau}}{\partial x_\alpha}\frac{\partial \gamma_{\varrho\tau}}{\partial x_\beta}\right)$$

would be a scalar under arbitrary transformations. However, as is known from the theory of differential invariants,[7] this quantity does not coincide with the only differential invariant of second order, viz., [p. 225]

$$\sum_{imk}\gamma_{im}\,\{ik, km\}.$$ [19]

This new theory of gravitation gains convincing power by the far-reaching covariance of the gravitational equations, even if the foregoing deliberations may not provide complete transparency of adapted coordinate systems and admissible transformations. We believe to have shown that the covariance of the equations is the optimum imaginable, since the conditions $B_\sigma = 0$, by which we restricted the coordinate systems, are a direct consequence of the gravitational equations.

Additional notes by translator

{1} "$\delta\gamma'_{\mu}$" on the left-hand side has been corrected to "$\delta\gamma'_{\mu\nu}$."

{2} The parenthesis ")" was missing after $\partial\gamma_{\mu\nu}/\partial x_\beta$.

[7]See §4 of part II of the "Outline."