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# General Covariance, Gauge Theories and the Kretschmann Objection.

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How can we reconcile two claims that are now both widely accepted: Kretschmann's claim that a requirement of general covariance is physically vacuous and the standard view that the general covariance of general relativity expresses the physically important diffeomorphism gauge freedom of general relativity? I urge that both claims can be held without contradiction if we attend to the context in which each is made.

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## 1. Introduction

### Two views…

When Einstein formulated his general theory of relativity, he presented it as the culmination of his search for a generally covariant theory. That this was the signal achievement of the theory rapidly became the orthodox conception. A dissident view, however, tracing back at least to objections raised by Erich Kretschmann in 1917, holds that there is no physical content in Einstein's demand for general covariance. That dissident view has grown into the mainstream. Many accounts of general relativity no longer even mention a principle or requirement of general covariance.

What is unsettling for this shift in opinion is the newer characterization of general relativity as a gauge theory of gravitation, with general covariance expressing a gauge freedom. The recognition of this gauge freedom has proven central to the physical interpretation of the theory. That freedom precludes certain otherwise natural sorts of background spacetimes; it complicates identification of the theory's observables, since they must be gauge invariant; and it is now recognized as presenting special problems for the project of quantizing of gravitation.

### …That We Need not Choose Between

It would seem unavoidable that we can choose at most one of these two views: the vacuity of a requirement of general covariance or the central importance of general covariance as a gauge freedom of general relativity. I will urge here that this is not so; we may choose both, once we recognize the differing contexts in which they arise. Kretschmann's claim of vacuity arises when we have some body of physical fact to represent and we are given free reign in devising the formalism that will capture it. He urges, correctly I believe, that we will always succeed in finding a generally covariant formulation. Now take a different context. The theory—general relativity—is fixed both in its formalism and physical interpretation. Each formal property of the theory will have some meaning. That holds for its general covariance which turns out to express an important gauge freedom.

### To Come

In Section 4 I will lay out this reconciliation in greater detail. As preparation, in Sections 2 and 3, I will briefly review the two viewpoints. Finally in Section 5 I will relate the reconciliation to the fertile "gauge principle" used in recent particle physics. An Appendix discusses the difficulty of making good on Kretschmann's claim that generally covariant reformulations are possible for any spacetime theory.

## 2. Einstein and Kretschmann's Objection

### Einstein…

In November 1915 an exhausted and exhilarated Einstein presented the gravitational field equations of his general theory of relativity to the Prussian Academy of Science. These equations were generally covariant; they retained their form under arbitrary transformation of the spacetime coordinate system. This event marked the end of a seven year quest, with the final three years of greatest intensity, as Einstein struggled to see that a generally covariant theory was physically admissible. [2]

Einstein had several bases for general covariance. He believed that the general covariance of his theory embodied an extension of the principle of relativity to acceleration. This conclusion seemed automatic to Einstein, just as the Lorentz covariance of his 1905 formulation of special relativity expressed its satisfaction of the principle of relativity of

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[2] Over the last two decades there has been extensive historical work on this episode. Earlier works include Stachel (1980) and Norton (1984); the definitive work will be Renn et al. (in preparation).

inertial motion. [3] He also advanced what we now call the "point-coincidence" argument. The physical content of a theory is exhausted by a catalog of coincidences, such as the coincidence of a pointer with a scale, or, if the world consisted of nothing but particles in motion, the meetings of their worldlines. These coincidences are preserved under arbitrary coordinate transformations; all we do in the transformations is relabel the spacetime coordinates assigned to each coincidence. Therefore a physical theory should be generally covariant. Any less covariance restricts our freedom to relabel the spacetime coordinates of the coincidences and that restriction can be based in no physical fact.

## …and Kretschmann

Shortly after, Erich Kretschmann (1917) announced that Einstein had profoundly mistaken the character of his achievement. In demanding general covariance, Kretschmann asserted, Einstein had placed no constraint on the physical content of his theory. He had merely challenged his mathematical ingenuity. For, Kretschmann urged, any spacetime theory could be given a generally covariant formulation as long as we are prepared to put sufficient energy into the task of reformulating it. In arriving at general relativity, Einstein had used the "absolute differential calculus" of Ricci and Levi-Civita (now called "tensor calculus.") Kretschmann pointed to this calculus as a tool that made the task of finding generally covariant formulations of theories tractable. [4]

Kretschmann's argument was slightly more subtle than the above remarks. Kretschmann actually embraced Einstein's point-coincidence argument and turned it to his own ends. In his objection, he agreed that the physical content of spacetime theories is

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[3] The analogy proved difficult to sustain and has been the subject of extensive debate. See Norton (1993).

[4] For further discussion of Kretschmann's objection, Einstein's response and of the still active debate that follows, see Norton (1993) and Rynasiewicz (1999)

exhausted by the catalog of spacetime coincidences; this is no peculiarity of general relativity. For this very reason all spacetime theories can be given generally covariant formulations.[5]

Kretschmann's objection doe s seem sustainable. For example, using Ricci and Levi-Civita's methods it is quite easy to give special relativity a generally covariant formulation. In its standard Lorentz covariant formulation, using the standard spacetime coordinates (t, x, y, z), special relativity is the theory of a Minkowski spacetime whose geometry is given by the invariant line element

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \quad (1)$$

Free fall trajectories (and other "straights" of the geometry) are given by

$$d^2x/dt^2 = d^2y/dt^2 = d^2z/dt^2 = 0 \quad (2)$$

We introduce arbitrary spacetime coordinates $x^i$, for i = 0,…,3 and the invariant line element becomes

$$ds^2 = g_{ik}\, dx^i\, dx^k \quad (3a)$$

where the matrix of coefficients $g^{ik}$ is subject to a field equation

$$R_{iklm} = 0 \quad (3a)$$

with $R_{iklm}$ the Riemann-Christoffel curvature tensor. The free falls are now governed by

$$d^2x^i/ds^2 + \{{}_k{}^i{}_m\}\, dx^k/ds\, dx^m/ds = 0 \quad (4)$$

where $\{{}_k{}^i{}_m\}$ are the Christoffel symbols of the second kind.

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[5] Rhetorically, Kretschmann's argument was brilliant. To deny it, Einstein may need to deny his own point-coincidence argument. However a persistent ambiguity remains in Einstein's original argument. Just what is a point-coincidence? Einstein gives no general definition. He gives only a list of illustrations and many pitfalls await those who want to make the argument more precise. For example, see Howard (1999).

Examples such as this suggest that Kretschmann was right to urge that generally covariant reformulations are possible for all spacetime theories. While the suggestion is plausible it is certainly not proven by the examples and any final decision must await clarification of some ambiguities. See Appendix 1: Is a Generally Covariant Reformulation Always Possible? for further discussion.

## 3. The Gauge Freedom of General Relativity

### Active General Covariance

Einstein spoke of general covariance as the invariance of form of a theory's equations when the spacetime coordinates are transformed. It is usually coupled with a so-called "passive" reading of general covariance: if we have some system of fields, we can change our spacetime coordinate system as we please and the new descriptions of the fields in the new coordinate systems will still solve the theory's equations. Einstein's form invariance of the theory's equations also licenses a second version, the so-called "active" general covariance. It involves no transformation of the spacetime coordinate system. Rather active general covariance licenses the generation of many new solutions of the equations of the theory in the same coordinate system once one solution has been given.

For example, assume the equations of some generally covariant theory admit a scalar field $\varphi(x^i)$ as a solution. Then general covariance allows us to generate arbitrarily many more solutions by, metaphorically speaking, spreading the scalar field differently over the spacetime manifold of events. We need a smooth mapping on the events—a diffeomorphism—to effect the redistribution. For example, assume we have such a map that sends the event at coordinate $x^i$ to the event at coordinate $x'^i$ in the same coordinate system. Such a map might be a uniform doubling, so that $x^i$ is mapped to $x'^i = 2.\, x^i$. To define the redistributed field $\varphi'$, we assign to the event at $x'^i$ the value of the original field $\varphi$ at the event with coordinate $x^i$.[6] If the field is not a scalar field, the transformation rule is slightly more complicated. For further details of the scalar case, see Appendix 2: From Passive to Active Covariance.

## Why it is a Gauge Freedom

The fields $\varphi(x^i)$ and $\varphi'(x^i)$ are mathematically distinct. But do they represent physically distinct fields? The standard view is to assume that they do not, so that they are related by a gauge transformation, that is, one that relates mathematically distinct representations of the same physical reality. That this is so cannot be decided purely by the mathematics. It is a matter of physics and must be settled by physical argumentation.

A vivid way to lay out the physical arguments is through Einstein's "hole argument."[7] The transformation on the manifold of events can be set up so that it is the identity everywhere outside some nominated neighborhood of spacetime ("the hole") and comes smoothly to differ within. We now use the transformation to duplicate diffeomorphically all the fields of some generally covariant theory. Do the new fields represent the same physical reality as the old? It would be very odd if they did not. Both systems of fields agree completely in all invariants; they are just spread differently on the manifold. Since observables are given by invariants, they agree in everything observable. Moreover, the two systems of field will agree everywhere outside the hole, but they differ only within. This means that, in a generally covariant theory, fixing all fields outside this neighbor fails to fix the fields within. This is a violation of determinism. In short, if we assume the two systems of fields differ in some physical way we must insist upon a difference

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[6] To visualize this redistribution in the two dimensional case, imagine that the original field is represented by numbers written on a flat rubber membrane. If we now uniformly stretch the rubber membrane so it doubles in size, we have the new field.

[7] See Earman and Norton (1987), Norton (1999).

that transcends both observation and the determining power of the theory. The ready solution is that these differences are purely ones of mathematical representation and that the two systems of fields represent the same physical reality.

## Its Physical Consequences

Accepting that this gauge freedom has important consequences for the physical interpretation of a theory such as general relativity. [8] The theory is developed by positing a manifold of spacetime events which is then endowed with metric properties by means of a metric tensor field $g_{ik}$. The natural default is to take the manifold of events as supplying some kind of independent background spacetime in which physical processes can unfold. The gauge freedom makes it very difficult to retain this view. For, when we apply a diffeomorphism to the field and spread the metrical properties differently over events, the transformation is purely gauge and we end up changing nothing physical. So now the same events are endowed with different properties, yet nothing physical has changed. The simplest and perhaps only way to make sense of this is to give up the idea of an independent existence of the events of the manifold. In so far as we can associate an event of the manifold with real events in the world, that association must change in concert with our redistribution of the metrical field over the manifold.

Our notion of what is observable is affected by similar considerations. What is observable is a subset of the physically real and that in turn is expressed by the invariants of a theory. Might an observable result consist of the assertion that an invariant of some field has such and such a value at some event of the manifold? No. The invariance must also include invariance under the gauge transformation and the assertion would fail to be invariant under the gauge transformation. In redistributing the fields, the transformation

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[8] For further discussion of these and related issues and their import for the quantization of gravity see Rovelli (1997).

might relocate that invariant with that value at quite another event of the manifold. If some result is eradicated by a gauge transformation, it cannot have been a result expressing physical fact since the gauge transformation alters nothing physical. We must resort to more refined ways of representing observables. For example, they may be expressed by an assertion that two invariants are equal. The event at which the equality resided may vary under gauge transformation; but the transformation will preserve the equality asserted.

## 4. Reconciliation

### The Context in which Kretschmann's Objection Succeeds

Kretschmann's objection s ucceeds because he allows us every freedom in reformulating and reinterpreting terms within a theory. Thus we easily transformed special relativity from its Lorentz covariant formulation (1), (2) to a generally covariant formulation (3a), (3b), (4). In doing so, we introduced new variables not originally present. The are the coefficients of the metric tensor $g_{ik}$ and the Christoffel symbols $\{^{\,i}_{k\,m}\}$.

With this amount of freedom, it is plausible that we can arrive at formulations of any theory that have any designated formal property. [9] Imagine, for example, that we wanted a

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[9] I am distinguishing the formalism of the theory (and its formal properties) from its interpretation. The formalism of a theory would be the actual words used, if the theory consisted of an English language description, independently of their meanings. Formal properties would include such things as the choice of English and the number of words. More commonly physical theories use mathematical structures in place of words. These structures can be considered quite independently of what we take them to represent in the world. The properties we then consider are the purely formal properties. A real valued field on some manifold is just a mathematical structure until we specify what it may represent in the world. That specification is the job of the interpretation. See next footnote.

formulation of Newtonian particle mechanics in which the string of symbols " $E=mc^2$" appears. (This is a purely formal property since we place no conditions on what the string might mean.) Here is one way we can generate it. We take the usual expression for the kinetic energy K of a particle of mass m moving at velocity v, $K=(1/2) mv^2$. We introduce a new quantity E, defined by E = 2K, and also a new label "c" for velocity v, so that c=v. Once we substitute these new variables into the expression for kinetic energy, our reformulated theory contains the string "$E=mc^2$".

The physical vacuity arises because we are demanding the formal property of general covariance (or some other formal property) without placing further restrictions that would preclude it always being achievable. The vacuity would persist even if we demanded a fixed physical content; we must simply be careful not to alter our initial physical content as we adjust its formal clothing. In the case of the discovery of general relativity, Einstein did not keep the physical content fixed. It became fully fixed only after he found a generally covariant formulation that satisfied a number of restrictive physical limitation.

## The Context in Which the Diffeomorphism Gauge Freedom has Physical Content

Matters are quite different if we fix the formalism of the theory and its interpretation. So we might be given general relativity in its standard interpretation. [10] If a theory has any content at all, we must be able to ascribe some physical meaning to its assertions. A fortiori there must some physical meaning in the general covariance of general

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[10] By "interpretation" I just mean the rules that tell us how to connect the various terms or mathematical structures of the theory with things in the physical world. These rules can vary from formulation to formulations and theory to theory. So, in ordinary formulations of special relativity, "c" refers to the speed of light. In thermodynamics "c" would refer to specific heat.

relativity. It may be trivial or it may not. [11] Consulting the theory, as we did in Section 2 above reveals that the content is not trivial.

Things are just the same in our toy example of forcing the string " $E=mc^2$" into a formulation of Newtonian particle mechanics. Let us fix the formulation to be the doctored one above. We had forced the string " $E=mc^2$" into it. But now that we have done it, the string uses symbols that have a meaning and, when we decode what it says about them, we discover that the string expresses something physical, the original statement that kinetic energy is half mass x (velocity) $^2$. Mimicking Kretschmann, we would insist that, given Newtonian particle mechanics or any other theory, some reformulation with the string is assuredly possible; so the demand for it places no restriction on the physically possible. But, once we have the reformulation, that string will express something.

The analogous circumstance arises in the generally covariant reformulation of special relativity. The existence of the reformulation is assured. Once we have it, its general covariance does express something. In this case, it is a gauge freedom of the geometric structure just like that of general relativity. The Lorentz covariant formulation of (1) and (2) admits preferred coordinate systems. In effect, some of the physical content of the theory is encoded in them. They specify, for example, which are the inertial motions; a body moves inertially only if there is a coordinate system in which its spatial coordinates do not change with the time coordinate. In the transition to the generally covariant formulation, this

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[11] Indeed the assertion may prove to be a logical truth, that is, it would be true by the definition of the terms it invokes or it may amount to the definition of term.While their truth is assured, such assertions need not be trivial. For example in a formulation of special relativity we may assert that that coefficients of the metric tensor are linear functions of the coordinates. This turns out to place no physical restriction on the theory; it merely restricts us to particular coordinate systems. It is what is known as a coordinate condition that defines the restricted class of coordinate systems in which the formulation holds.

content is stripped out of the coordinate systems. We can no longer use constancy of spatial coordinates to discern which points move inertially. This content is relocated in the Christoffel symbols, which, via equation (4) determine whether a particular motion is inertial. The general covariance of (3a), (3b) and (4) leave a gauge freedom in how the metric $g_{ik}$ and the Christoffel symbols $\{ {}_{k}{}^{i}{}_{m} \}$ may be spread over some coordinate system. In one coordinate system, they may be spread in many mathematically distinct but physically equivalent ways.

### To summarize

There is no restriction on physical content in saying that there exists a formulation of the theory that has some formal property (general covariance, the presence of the string of symbols "$E=mc^2$", etc.) But once we fix a particular formulation and interpretation, that very same formal property will express something physical, although there is no assurance that it will be something interesting.

## 5. Gauge Theories in Particle Physics

This summary generates a new puzzle. One of the most fertile strategies in recent decades in particle physics has been to extend the gauge symmetries of non-interacting particles and thereby infer to new gauge fields that mediate the interaction between the particles. Most simply, the electromagnetic field can be generated as the gauge field that mediates interactions of electrons. This power has earned the strategy the label of the "gauge principle." How can this strategy succeed if Kretschmann is right and there is no physical content in our being able to arrive at a reformulation of expanded covariance? In the particle context, this corresponds to a reformulation of expanded gauge freedom. So why doesn't Kretschmann's objection also tell us that the strategy of the gauge principle is physically vacuous?

The solution lies in the essential antecedent condition of Kretschmann's objection. The physical vacuity arises since there are no restrictions placed how we might reformulate a theory in seeking generally covariance. It has long been recognized that the assured achievement of general covariance can be blocked by some sort of additional restriction on how the reformulation may be achieved. Many additional conditions have been suggested, including demands for simplicity and restrictions on which extra variables may introduced. (For a survey, see Norton, 1993, Section 5; Norton, 1995, Section 4.) The analogous solution is what gives the gauge principle its content. In generating gauge fields, we are most definitely not at liberty to expand the gauge freedom of some non-interacting particle field in any way we please. There is a quite precise recipe that must be followed: we must promote a global symmetry of the original particle field to a local symmetry, using the exemplar of the electron and the Maxwell field, and the new field arises from the connection introduced to preserve gauge equivalence. [12]

There is considerably more that should be said about the details of the recipe and the way in which new physical content arises. The recipe is standardly presented as merely expanding the gauge freedom of the non-interacting particles, which should mean that the realm of physical possibility is unaltered; we merely have more gauge equivalent representations of the same physical situations. So how can physically new particle fields

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[12] The transition from special relativity in (1) and (2) to the generally covariant formulation (3a), (3b) and (4) can be extended by one step. We replace the flatness condition (3a) by a weaker condition, a natural relaxation, $R_{ik} = g^{lm}R_{ilmk} = 0$. The result is general relativity in the source free case. Arbitrary, source free gravitational fields now appear in the generalized connection $\{ {}_{k}{}^{i}{}_{m} \}$. We have what amounts to the earliest example of the use of the gauge recipe to generate new fields. The analogy to more traditional examples in particle physics is obvious.

emerge? This question is currently under detailed and profitable scrutiny. See Martin (2000), (manuscript) and contributions to this volume.

## Appendix 1: Is a Generally Covariant Reformulation Always Possible?

As Earman (manuscript, Section 3) has pointed out, it is not entirely clear whether a generally covariant reformation is always possible for any spacetime theory. The problem lies in ambiguities in the question. Just what counts as "any" spacetime theory? Just what are we expecting from a generally covariant reformulation? Let me rehearse some of the difficulties and suggest that for most reasonable answers to these questions generally covariant reformulation will be possible though not necessarily pretty.

### The Substitution Trick…

Let us imagine that we are given a spacetime theory in a formulation of restricted covariance. It is given in just one spacetime coordinate systems $X^i$. Let us imagine that the laws of the theory happen to be given by n equations in the 2n quantities $A^k$, $B^k$

$$A^k(X^i) = B^k(X^i) \quad (5)$$

where k = 1, …, n and the $A^k$ and $B^k$ are functions of the coordinates as indicated. Consider an arbitrary coordinate system $x^i$ to which we transform by means of the transformation law

$$x^i = x^i(X^m) \quad (6)$$

We can replace the n equations (5) by equations that hold in the arbitrary coordinate system by the simple expedient of inverting the transformation of (6) to recover the expression for the $X^m$ as a function of the $x^i$, that is $X^m = X^m(x^i)$. Substituting these expressions for $X^m$ into (5), we recover a version of (5) that holds in the arbitrary coordinate system

$$A^k(X^i(x^m)) = B^k(X^i(x^m)) \quad (5a)$$

We seemed to have achieved a generally covariant reformulation of (5) by the most direct application of the intuition that coordinate systems are merely labels and we can relabel spacetime events as we please.

## …Yields Geometric Objects

While (5a) is generally covariant, we may not be happy with the form of the general covariance achieved—one of the ambiguities mentioned above. We might, as Earman (manuscript, Section 3) suggests, want to demand that (5a) be expressed in terms of geometric object fields. The standard definition of a geometric object field is that it is an n-tuple valued field of components on the manifold, with one field for each coordinate system, and that the transformation rule that associates the components of different coordinate systems have the usual group properties.

While this definition may appear demanding, it turns out to be sufficiently permissive to characterize each side of (5a) as a geometric object field. For example, in each coordinate system $x^m$, the geometric object field A has components $A^k(X^i(x^m))$, which I now write as $A^k(x^m)$. The transformation rule between the components is induced by the rule for coordinate transformations. That is, under the transformation $x^m$ to $y^r(x^m)$, $A^k(x^m)$ transforms to $A^k(x^m(y^r))$, where $x^m(y^r)$ is the inverse of the coordinate transformation. With this definition of the transformation law for $A^k$, the components will inherit as much group structure as the coordinate transformations themselves have; that is, it will be as much of a geometric object field as we can demand. [13] For example, assume the transformations of

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[13] Why the hedged "as much group structure as the coordinate transformations themselves have"? These general coordinate transformations may not have all the group properties if the domains and ranges of the transformations do not match up well. Assume $T_1$ maps coordinates $x^i$ on a neighborhood A to coordinates $y^i$ on a neighborhood B that is a proper subset of A and $T_2$ maps coordinates $y^i$ on B to coordinates $z^i$ on A. Then the composition

coordinate systems $z^p$ to $y^r$ and $y^r$ to $x^m$ conform to transitivity. Then this same transitivity will be inherited by A. We will have $A^k(x^m(z^p)) = A^k(x^m(y^r(z^p)))$ since the transitivity of the coordinate transformation yields $x^m(z^p) = x^m(y^r(z^p))$.

## But areThey the Geometric Objects We Expect?

While the components $A^k$ turn out to be geometric object fields, they are probably not the ones we expected. In brief, the reason is that the transformation rule induced by the substitution trick does not allow any mixing of the components. That precludes it yielding vectors or tensor or like structures; it turns everything into scalar fields. To see how odd this is, take a very simple case. Imagine that we have special relativity restricted to just one coordinate system $X^i$. Our law might be the law governing the motion of a body of unit mass, $F^i = A^i$, where $F^i$ is the four force and $A^i$ the four acceleration. Under a Lorentz transformation

$$Y^0 = \gamma(X^0 - vX^1) \quad Y^1 = \gamma(X^1 - vX^0) \quad Y^3 = X^3 \quad Y^4 = X^4$$

with velocity v in the $X^1$ direction, c=1 and $\gamma = (1-v^2)^{-1/2}$.The usual Lorentz transformation for the components $A^i$ of the four acceleration would be

$$A'^0 = \gamma(A^0 - vA^1) \quad A'^1 = \gamma(A^1 - vA^0) \quad A'^3 = A^3 \quad A'^4 = A^4 \tag{6}$$

Note that the transformed $A'^0$ and $A'^1$ are linear sums of terms in $A^0$ and $A^1$. For this same transformation, the substitution trick merely gives us

$$A'^i = A^i(X^m(Y^r)) \tag{6a}$$

That is, $A'^0$ is a function of $A^0$ only and $A'^1$ is a function of $A^1$ only.

This oddity becomes a disaste r if we apply the substitution rule in a natural way. Instead of starting with $A^i$ in one fixed coordinate system $X^i$, we might start will the full set

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$T_2T_1$ cannot coincide with the direct transformation of $x^i$ to $z^i$ since the composition has lost that part of the transformation outside B.

of all components of $A^i$ in all coordinate systems related by a Lorentz transformation to $X^i$. If we now try and make this bigger object generally covariant by the substitution trick, we will end up with two incompatible transformation laws for the transformation $X^i$ to $Y^i$ when we try to transform the components $A^i$—the law (6) and law (6a). We no longer have a geometric object field since we no longer have a unique transformation law for the components.

The escape from this last problem is to separate the two transformation groups. We consider $A^i$ in coordinate system $X^i$ and $A'^i$ in coordinate system $Y^i$ separately and convert them into distinct geometric object fields by the substitution trick. As geometric object fields they have become, in effect, scalar fields. The Lorentz transformation then reappears as a transformation between these geometric objects.

## The Coordinates as Scalars Trick

If this is our final goal, then another general trick for generating generally covariant reformulations could have gotten us there much faster. We return to $A^k(X^i)$ of (5). We can conceive the $X^i$ as scalar fields on the manifold—that is really all they are.[14] Scalar fields are geometric object fields already. The $A^i$ are functions of $X^i$, that is, functions of scalar fields. Therefore they are also geometric objects. So we can conceive of the entire structure $A^k(X^i)$ as a geometric object field. We have gotten general covariance on the cheap. We cannot avoid a cost elsewhere in the theory however. Our reformulation is overloaded with structure, one geometric object field for each of what was originally a component. There is clearly far more mathematical structure present than has physical significance. So the theory will need a careful system for discerning just which parts of all this structure has physical significance.

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[14] Ask, what is the $X^0$ coordinate in coordinate system $X^i$ of some event p? The answer will be the same number if we ask it from any other coordinate system $y^i$ as long as we are careful to ask it of the original coordinate system $X^i$. That is, each coordinate can be treated as a scalar field.

### Temptations Resisted

These devices for inducing general covariance are clumsy but they do fall within the few rules discussed. We might be tempted to demand that we only admit generally covariant formulations if their various parts fall together into nice compact geometric objects. But what basis do we have for demanding this? Are we to preclude the possibility that the theory we started with is just a complicated mess that can only admit an even more complicated mess when given generally covariant reformulations. (Newtonian theory has been accused of this!) And if we are to demand only nice and elegant reformulations, just how do we define "nice and elegant"?

My conclusion is that generally covariant reformulations are possible under the few rules discussed and that efforts to impose further rules to block the more clumsy ones will cause more trouble than they are worth elsewhere.

## Appendix 2: From Passive to Active Covariance

As above, assume the equations of some generally covariant theory admit a scalar field $\varphi(x^i)$ as a solution. We can transform to a new coordinate system by merely relabeling the events of spacetime; $x^i$ is relabeled $x'^i$, where the $x'^i$ are smooth functions of the $x^i$. The field $\varphi(x^i)$ transforms to field $\varphi'(x'^i)$ by the simple rule $\varphi'(x'^i) = \varphi(x^i)$. Since the equations of the theory hold in the new coordinate system, the new field $\varphi'(x'^i)$ will still be a solution. The two fields $\varphi(x^i)$ and $\varphi'(x'^i)$ are just representations of the same physical field in different spacetime coordinate systems.

This is the passive view of general covariance. It can be readily transmogrified into an active view, a transition that Einstein had already undertaken with his 1914 statements of the "hole argument". What makes $\varphi'(x'^i)$ a solution of the theory under discussion is nothing special about the coordinate system $x'^i$. It is merely the particular function that $\varphi'$ happens to be. It is a function that happens to satisfy the equations of the theory. We could take that very same function and use it in the original coordinate system, $x^i$. That is, we could form a new field $\varphi'(x^i)$. Since this new field uses the very same function, it retains every property except the mention of the primed coordinate system $x'^i$. Thus it is also a solution of the equations of the theory.

In short, the passive general covariance of the theory has delivered us two fields, $\varphi(x^i)$ and $\varphi'(x^i)$. They are not merely two representations of the same field in different coordinate systems. They are defined in the same coordinate system and are mathematically distinct fields, in so far as their values at given events will (in general) be different. Active general covariance allows the generation of the field $\varphi'(x^i)$ from $\varphi(x^i)$ by the transformation $x^i$ to $x'^i$.

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