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Description of plate 3

Figure 1. IF emission spectrum (Raman glass spectrograph; Ilford long-range spectrum plate—exposure 10 min. top, $2\frac{1}{2}$ min. lower—iron arc comparison).

Figure 2. Rotational structure in IF bands. Note the overlap of the P and R branches in the 0, 4 (6031·2 Å) band (Glass Littrow spectrograph, H.P. 3 plate, exposure 6 hr.—iron arc comparison).

Figure 3. BrF emission spectrum (Raman glass spectrograph, Rapid Process Panchromatic plate. Exposure (*a*) 10 min. (*b*) 1 min.—iron arc comparison).

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The renormalization method in quantum electrodynamics

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(*Communicated by R. E. Peierls, F.R.S.—Received* 23 *January* 1951)

A new technique has been developed for carrying out the renormalization of mass and charge in quantum electrodynamics, which is completely general in that it results not merely in divergence-free solutions for particular problems but in divergence-free equations of motion which are applicable to any problem. Instead of using a power-series expansion in the whole radiation interaction, the new method uses expansions in powers of the high-frequency part of the interaction. The convergence of the perturbation theory is thereby much improved. The method promises to be especially useful in applications to meson theory.

The present paper contains a preliminary and non-technical account of a new method of handling problems in quantum electrodynamics. A full account of the method will be published in a series of papers of which the first only (Dyson 1951) is yet written.* That paper was occupied with a formal mathematical analysis of some expressions which arise in the matrix elements of Heisenberg operators. The analysis yielded a general rule by which any Heisenberg operator can be split into a sum of terms each of which has a simple structure. The results of that paper, although mathematical and not physical in character, are an indispensable tool in the successful development of the physical ideas which will now be introduced.

* That paper, with the title 'Heisenberg operators in quantum electrodynamics, I', is referred to hereafter as HOI.

These new ideas go considerably beyond the programme, outlined in the introduction to HOI, of proving the finiteness of Heisenberg operators after renormalization. The programme is now widened, so that the objective is a proof of the total disappearance of all divergences from quantum electrodynamics after the dynamical variables have been transformed by a suitable contact transformation. A unitary operator will be explicitly constructed which, when applied to the state-vector in the Tomonaga-Schwinger theory, gives a state-vector satisfying divergence-free equations of motion. The new equations of motion will describe exactly the behaviour of all electrodynamical systems, bound states being no longer excluded as they were from the S-matrix formalism (Dyson 1949). Furthermore, it is intended that the new divergence-free formulation of the theory shall be practically useful and adaptable to approximate numerical calculations. Within this widened programme, the proof of the finiteness of Heisenberg operators will appear as a special and important limiting case.

It is not surprising that quantum electrodynamics can be transformed by a single contact transformation into an explicitly divergence-free theory. For if this were not possible, the successes of the theory in giving finite answers to numerous problems would be hard to understand. It is also easy to foresee, from simple physical considerations, the general form of the required transformation. In the present introductory paper, the physical and heuristic ideas underlying the transformation method are briefly explained. Later papers will be concerned with the detailed mathematical verification that a transformation operator of the kind indicated by heuristic principles in fact does all that is expected of it.

One may imagine a physical picture, which includes an intuitive explanation for the success of the renormalization technique in quantum electrodynamics, roughly as follows. The two interacting fields, the electromagnetic and the matter field, are two characteristic properties of a single 'fluid' which fills the whole of space-time. The two fields are defined at every point, like the velocity and stress in a fluid in classical hydrodynamics. The fluid is in a state of violent quantum-mechanical fluctuation, the fluctuations becoming more and more noticeable as the region over which they are observed is made smaller. The fluctuations have the property that at sufficiently high frequencies and in sufficiently small regions they are essentially isotropic and uniform over the whole of space-time, like the fluctuations of a classical fluid in a state of isotropic turbulence. Statements concerning the behaviour of the fluid at a particular point are observationally meaningless; the description of the fluid in terms of operators defined at points is possible only in a formal sense and involves mathematical divergences. However, because of the isotropy and uniformity of the fluctuations, the macroscopic properties of the fluid are observable and well defined. A divergence-free description of the fluid will be obtained as soon as its behaviour is expressed entirely in terms of new dynamical variables which are averages of the original instantaneous variables over finite intervals of time.

In accordance with the foregoing crude picture, it is found mathematically that the successful removal of divergences in quantum electrodynamics by renormalization is always associated with an averaging-out of high-frequency fluctuations of the fields. The averaging-out is achieved by integrating the equations of motion of the fields

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explicitly with respect to the time. Thus in the Schwinger theory (Schwinger 1949*a*, *b*) the state-vector of the system is transformed so that the new state-vector refers to the behaviour of the system at a time which recedes in the limit into the infinite past. In the Feynman theory (Feynman 1949*a*, *b*, 1950) the description of events is directly in terms of an over-all space-time picture in which localization of processes in space and time is abandoned. In both theories, the time-averaging is performed not over a finite time-interval but over an infinite time. Averaging over an infinite time is also implicit in the definition of Heisenberg operators in HOI, because the multiple integrals in the series expansions of the operators extend into the infinite past.

Now it is precisely the averaging over an infinite time-interval which has hitherto introduced into every discussion of the renormalization method the two limitations mentioned in the introduction to HOI. Those limitations are: (i) because of the way in which the initial and final states are described, the customary S-matrix formalism is not applicable to problems involving bound states, (ii) the renormalization method has always been confined to quantities which are expanded as power series in the radiation interaction, while in many situations such expansions are demonstrably not convergent. The two limitations, of which the second is the more fundamental, are closely related. It is reasonable to hope that the theory can be freed from both limitations, if the averaging over infinite time-intervals is dropped and the removal of high-frequency fluctuations is accomplished by integrating the equations of motion over finite intervals.

That is to say, the transformation which leads to divergence-free equations of motion may be expected to be of the following type. The original state-vector Ψ of the interaction representation is replaced by a new one Φ according to

$$\Psi(t) = S(t)\,\Phi(t), \tag{1}$$

where t is the time and $S(t)$ is a unitary operator. The choice of S is guided by two principles. (i) Φ is to be a smoothed-out average of Ψ over a finite time-interval, or in other words S is to follow accurately the high-frequency fluctuations of Ψ but not the slow long-term variations. (ii) S is to be a power-series expansion not in the total radiation interaction but only in the high-frequency fluctuating part of the interaction. By 'high-frequency fluctuations' are here meant Fourier components with frequencies higher than a certain standard frequency which may be chosen arbitrarily. The standard will in general be chosen differently for different problems. It will probably be convenient to make it a little higher than the highest frequency that is physically important in a particular problem. Then the high-frequency part of the interaction is ineffective except in producing renormalization effects, and the expansion in powers of the high-frequency interaction may be expected to converge after the renormalizations have been carried out. In this series of papers no attempt will be made to prove the convergence. It is plausible, and in accordance with the original philosophy of the Schwinger theory, that the high-frequency interaction produces only small physical effects, and that an expansion in powers of the high-frequency interaction should be rapidly convergent and convenient for practical calculations.

A trial definition of S, satisfying the above requirements, can now be formulated. Let

$$H_1 = H_1(e_1, t) \tag{2}$$

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be the radiation interaction appearing in the Tomonaga-Schwinger equation, integrated over all space at a given time t. According to the results of the earlier S-matrix analysis (Dyson 1949), H_1 depends upon two divergent constants e and δm which are formal power-series (with divergent coefficients) in the finite e_1 which is the physically observed electronic charge. Thus H_1 is supposed to be expressed explicitly as a power-series in e_1, the higher terms representing the effects of mass and charge renormalization. Let a function $g(a)$ of the positive real variable a be chosen with the properties: $g(0) = 1$, $g(a) \to 0$ as $a \to \infty$, and $g(a)$ varies smoothly as a varies from 0 to ∞. Consider a fictitious world in which the radiation interaction at time t', instead of being given by (2), is for $t' \leqslant t$ given by

$$H_g(t, t') = H_1(e_1 g(t-t'), t'). \tag{3}$$

The fictitious world is one in which the charge e_1 of the electron is smoothly, but not adiabatically, varied, rising from zero in the remote past to its correct value at time t. In the fictitious world, there is a unitary transformation operator transforming the state-vector representing a system without interaction at $t' = -\infty$ into the state-vector representing the same system with interaction at $t' = t$; this transformation operator* is $S(t)$. In other words, $\Phi(t)$ is the state-vector at $t' = -\infty$ which, developing with time t' in the fictitious world, coincides with the actual state $\Psi(t)$ of the system in the real world at time t.

The definition which will finally be adopted for $S(t)$ is essentially that given above. Some modifications have to be made in the definition of H_g, in order to compensate a transient photon self-energy effect which appears while the charge e_1 is being varied.† Also, some more restrictive conditions will be imposed on the function g.

In the specification of g there will necessarily appear some constant T with the dimension of a time. Then T^{-1} is the standard frequency defining the division of frequencies into high and low. Consider first the meaning of the two limiting cases $T \to 0$ and $T \to \infty$. In the limit $T \to 0$, all frequencies are considered as low, $g(a) = 0$ and S is the identity operator. In this case Φ is the state-vector of the interaction representation, there is no smoothing-out of the fluctuations of Ψ, and there is no removal of divergences. In the limit $T \to \infty$, all frequencies are considered as high, $g(a) = 1$, and S satisfies the same equation of motion as Ψ. In this case Φ is the state-vector of the Heisenberg representation and is constant in time, all the fluctuations of Ψ have been smoothed out, the formal removal of divergences is complete, but S is an expansion in powers of the whole interaction which is generally not convergent. When T is given a finite value, the situation is intermediate between the two limiting cases. The representation in which Φ is the state-vector will be called the 'intermediate representation', meaning that it is intermediate between the interaction and Heisenberg representations. An expression of the form

$$Q_g(t) = S^{-1}(t)\, Q(t)\, S(t), \tag{4}$$

* The operator $S(t)$ has been previously studied in a series of papers by Ferretti (1950 *a*, *b*, *c*, *d*).

† The author is deeply indebted to Mr Abdus Salam for suggesting to him the correct definition to use for H_g. This suggestion was based upon an unpublished treatment of the renormalization technique by S. N. Gupta.

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where $Q(t)$ is an interaction representation operator referring to the time t, will be called an intermediate representation operator. In the intermediate representation, high-frequency fluctuations of a system are described by the field operators as in the Heisenberg representation, while low-frequency processes are described by the state-vector as in the interaction representation.

The programme of this series of papers is to prove that the intermediate representation provides a complete divergence-free formulation of quantum electrodynamics. The programme is divided into three parts, the first of which is an immediate generalization of the programme of the introduction to HOI. Let $Q(r,t)$ be a field-operator in the interaction representation defined at the point (r,t), for example, an electromagnetic field component. Let $Q_g(r,t)$ be the corresponding intermediate representation operator defined by (4). Let $R(r,t)$ be a scalar function of the point (r,t), vanishing outside a finite space-time region, and satisfying certain requirements of continuity. Then the operator

$$\int Q_g(r,t)\,R(r,t)\,d_3 r\,dt \tag{5}$$

is called an intermediate representation field-average; such operators represent a general class of locally defined physical quantities which are in principle precisely measurable. The first part of the programme is to prove that all matrix elements of intermediate representation field-averages are finite after the renormalizations of mass and charge are carried out.

The second part of the programme is to prove that the state-vector in the intermediate representation satisfies a divergence-free Schrödinger equation. The Schrödinger equation in the intermediate representation is

$$i\hbar(\partial\Phi/\partial t) = H'(t)\,\Phi, \tag{6}$$

where by (1) the Hamiltonian is

$$H'(t) = S^{-1}(t)\left[H_1(t) - i\hbar\frac{d}{dt}\right]S(t). \tag{7}$$

The objective is to prove that the operator H' is divergence-free.

The physical importance of the operator H' is made clearer by transforming back to the Schrödinger representation, in which all operators are independent of time. If H_0 is the Hamiltonian of the non-interacting fields, then

$$S(t) = \exp(iH_0 t/\hbar)\,S\exp(-iH_0 t/\hbar), \tag{8}$$

where S is a constant Schrödinger representation operator. The state-vectors Ψ_0 and Φ_0, defined by

$$\Psi_0(t) = \exp(-iH_0 t/\hbar)\,\Psi(t), \quad \Phi_0(t) = \exp(-iH_0 t/\hbar)\,\Phi(t), \tag{9}$$

are related by the time-independent unitary transformation

$$\Psi_0(t) = S\Phi_0(t). \tag{10}$$

Now Ψ_0 is the state-vector in the standard Schrödinger representation of quantum electrodynamics with Hamiltonian (H_0+H_1). In view of (10), Φ_0 may equally well

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be regarded as the state-vector of the system in the Schrödinger representation, satisfying a Schrödinger equation

$$i\hbar(\partial\Phi_0/\partial t) = (H_0+H')\,\Phi_0, \tag{11}$$

where H' is the time-independent operator corresponding to the time-dependent $H'(t)$. The identity

$$H_0+H' = S^{-1}(H_0+H_1)\,S, \tag{12}$$

from which (11) is derived, is equivalent to (7).

The transformed Schrödinger equation (11) gives a complete description of the behaviour of all systems in quantum electrodynamics, but is especially appropriate to the treatment of bound states and eigenvalue problems. The proof of finiteness of the operator H' will enable all such problems to be formulated in finite terms.

The third part of the programme is to devise methods for the approximate solution of the Schrödinger equation in practical situations, especially in circumstances where expansions in powers of the whole radiation interaction are forbidden. An important advantage of the intermediate representation is its flexibility, due to the fact that the function g may be left unspecified until the third part of the programme is reached. When a particular problem is attacked, it should be possible to minimize the labour involved in calculating the solution by making an appropriate choice of g. The change from one g to another is equivalent merely to a finite contact transformation which leaves all physically observable quantities invariant.

It is hardly necessary to repeat here the remarks made earlier (Dyson 1949) concerning the manipulations of infinite expressions which are involved in the renormalization method. The original equations of motion of quantum electrodynamics, and the S-operator itself, contain numerous divergent quantities; the equations of motion become divergence-free only after the infinities have been cancelled out as a result of formal analytical manipulations. The transformation of the theory by means of the S-operator is not a mathematically rigorous operation; the method must be justified *a posteriori* by the fact that it yields well-defined and physically reasonable results. It is regrettable that the S-transformation is not only divergent but also entirely non-covariant in form. The intermediate representation conceals both the Lorentz invariance and the gauge invariance of the theory. This lack of apparent Lorentz and gauge invariance does not, however, interfere with the consistent carrying out of renormalizations, and the practical inconvenience resulting from it is much less than might have been expected.

Though the methods discussed in this paper are here applied to quantum electrodynamics, they can be transferred without serious modifications to meson theory, whenever the meson theory is such as to give a divergence-free S-matrix. The removal of divergences from the S-matrix, using the renormalization technique, has recently been carried through by Salam (1951) for the meson theory of greatest practical importance, the theory of charged pseudo-scalar mesons interacting with a nucleon field and with the electromagnetic field.* The formal possibility now therefore certainly exists, of constructing a divergence-free formulation of pseudo-

* The removal of divergences in charged meson theory is, of course, very much more complicated than in electrodynamics.

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scalar meson theory, the results of which might be compared unambiguously with experiment. Such a divergence-free formulation will be a practical possibility, if the expansion of operators as power-series in the high-frequency part of the interaction is found to be convergent even when the coupling constant is large. On the basis of rough estimations, it seems probable that the expansion will be sufficiently convergent, if the function g is chosen suitably. The intermediate representation method thus already offers good prospects of overcoming the mathematical obstacles which have so long delayed a decisive verdict for or against the pseudo-scalar meson theory as a workable theory of nuclear phenomena.

The author is indebted to The Royal Society, the University of Michigan and the Institute for Advanced Study, Princeton, for financial support and hospitality. He wishes to thank Professor W. Pauli, Dr Res Jost, and above all Mr Abdus Salam, for much help received in discussions and correspondence.

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