THE PHYSICAL PRINCIPLES OF THE QUANTUM THEORY

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One has first to determine the true fluidity of the material at rest, as explained above. If we know this we redraw the consistency curve in accordance with the equation

$$\frac{[V - P(\varphi_o' - \varphi_o)]\,\varphi_w}{\varphi_w + \varphi_o - \varphi_o'} = \sum_{n=o} \quad . \quad . \quad \Bigg| \quad \frac{[V - P(\varphi_o' - \varphi_o)]\,{}^2\varphi_w}{{}^2\varphi_w + \varphi_o - \varphi_o'} = \sum_{n=o} . \quad (31)$$

The first term of the series on the right side, i.e. for $n = 0$, is identical with the first term for a liquid showing *no* wall effect. In the case of a simple Newtonian liquid showing a wall effect, this curve is therefore the correct consistency curve of the bulk of the material and may accordingly be named the " bulk " consistency curve. The curve is identical with the consistency curve of the bulk of a generalized Newtonian liquid showing a wall effect, if either small *P's* are considered or $1/R$ is small. For larger *P's* and $1/R$*'s* there is no way of plotting consistency curves independently of the dimensions of the apparatus, if the law of flow of the material is not known.

Regions of very small shearing velocities require special consideration. Very thorough investigations were undertaken by Scott Blair and collaborators (compare Scott Blair *Introduction* [35], pp. 32–34). The consistency curve of clay and soil pastes was found to divide itself into four regions. In region I there is no flow, region II is a straight line, region III is markedly curved and region IV becomes asymptotically straight. In regions I and IV one recognizes the picture of Fig. 4. Region II is due to slippage. It was Scott Blair's contribution to find Region I, i.e. the fact that a definite stress is necessary to *start* slippage.

9. Lubricating and Roughening Wall Layers

When considering wall effects in the foregoing, we have spoken of " slippage " as occurring in a two-phase system, with the dispersion medium forming the wall layer. But this is taking too narrow a view of wall effects. As Bingham early (1922) pointed out, it is " quite possible that the fluidity of a liquid near the surface is not identical with that within the body of the liquid ". Bartsch [36] found, e.g. that when oleic acid is in contact with a metal surface a layer about 10,000 molecules thick adheres to the wall. Such molecules are at the same time oriented. Generally there may be two sorts of effects. The molecules may be either oriented *parallel* to the wall, which should reduce friction, or they may become oriented *normal*, which must increase friction. The first made a " lubricating ", the second a "roughening" layer. If we introduce in (29)

$$\left.\begin{aligned}\varphi_o &= \alpha\varphi_w\\ d &= R/n\end{aligned}\right\} \tag{32}$$

Fig. 4 shows φ_0 as a function of $1/n$ for different values of α.

For $\alpha < 1$ the wall layer acts as lubricant, for $\alpha > 1$ as a roughener. It must, however, be said that it is doubtful whether these effects are pronounced enough to affect viscosity measurements in the case of simple liquids. An attempt by R. Bulkley [37] in the case of different oils, using capillary viscometers and raising the precision of the measurements to unusual levels, had negative results.

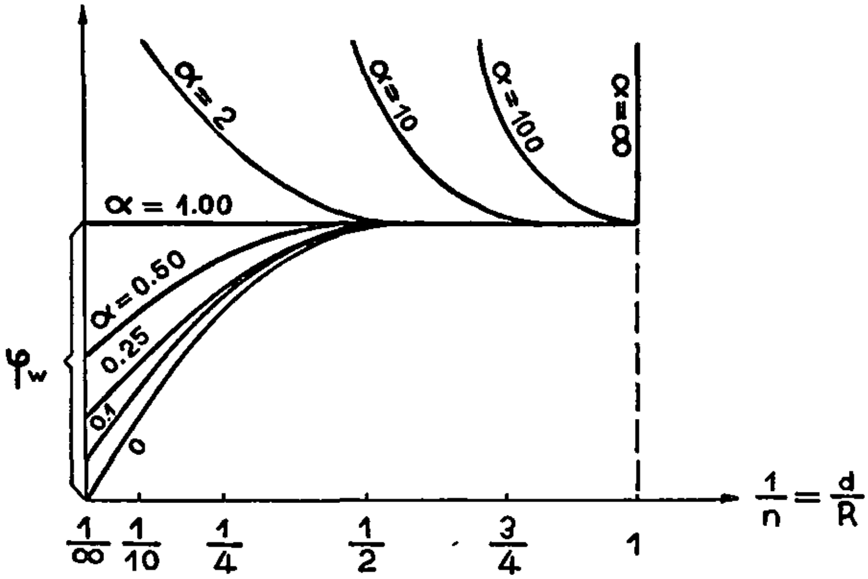

FIG. VIII, 4. Apparent zero fluidities of liquids in the presence of wall effects as determined in capillary instruments.
φ_w fluidity of the wall layer.
R radius of capillary.
d thickness of wall layer.

When a liquid does not wet the solid, as in the case of mercury, one cannot properly speak of slippage in the sense in which this word is ordinarily used. In this case there exists a layer of some other medium, either air or the vapour of the liquid, between the material and the solid wall. In the case of air at room temperature $\varphi_w > 5{,}000$. The thickness d must be $>10^{-8}$ cm., which is the order of magnitude of the diameter of simple molecules. Bingham and Thompson (1928) determined the viscosity of mercury in a capillary of $R = 0{\cdot}012$ cm. Therefore in this case $\varphi' - \varphi$ (compare (29))$> 0{\cdot}016$. The observed fluidity of mercury at room temperature <66. Therefore $\frac{\varphi' - \varphi}{\varphi'} > 0{\cdot}024$ per cent. This is less than the accuracy of the Bingham instrument, which is about 0·1 per cent. The presence of the air layer can therefore not make itself felt and the viscosity of mercury can be determined from the usual formulas assuming that the mercury adheres to the solid wall [1]. Erk [2] has pointed out that the viscosity of air or vapour, which is in a state of adsorption on the solid surface, must be assumed as much higher and may even have a " roughening " influence.

10. Summary

In the flow of a dispersed system, the dispersion medium may form a lubricating layer between the body of the liquid and a solid wall. This is especially so with suspensions which are soft plastic solids. In liquids in contact with solids there may be wall layers acting either as " lubricants " or " rougheners ".

In the presence of a wall layer the apparent fluidity at rest will be in the case of a plastic solid in the

capillary	rotating-cylinder
instrument	
$\varphi_o' = \varphi_w[1-(1-d/R)^4]$	$\varphi_o' = 2\varphi_w\left[1-\frac{1-2d/(R_e-R_i)}{(1+d/R_i)^2\,(1-d/R_e)^2}\right]$ (VIII, a)

and in the case of a liquid, the fluidity at rest of which is φ_o,

$\varphi_o' = \varphi_o + \varphi_w[1-(1-d/R)^4]$	$\varphi_o' = \varphi_o + 2\varphi_w\left[1-\frac{1-2d/(R_e-R_i)}{(1+d/R_i)^2(1-d/R_e)^2}\right]$ (VIII, b)

When plotting calculated φ_o' values against varying

$1/R$	$1/R_i$, keeping " a " constant

extrapolation of the resultant curve to zero gives $\varphi_o' = 0$ in the case of a plastic solid and φ_o in the case of a liquid.

CHAPTER IX

VOLUME ELASTICITY AND VISCOSITY

1. Epistomological Digression

We have said in Chapter I that under isotropic pressure all materials are elastic and only elastic, and we have called this an axiom of rheology. We did not give any proof for this statement. We relied upon the fact that, being an axiom, it is, as the *Oxford Dictionary* puts it, " a self-evident proposition, not requiring demonstration, but assented to as soon as stated ". This use of the term axiom, going back (again on the authority of the *Oxford Dictionary*) to the year 1600, is, however, not in accordance with modern ideas. We now, even in geometry, and still more so in rheology, do not accept the truth of a proposition without recourse to experience and if, as in the case of an axiom, experience is not drawn upon, the axiom cannot be more than the definition (an " implicit " definition) of a word appearing in the proposition. If we now look back to our statement that " under isotropic pressure all materials are elastic and only elastic " we find that there is no word in it which we have not already defined before, and, therefore, as it stands, the statement *cannot be true*. And so it is. We arrived at the statement by considering such materials as steel, plasticene and water. But if we had put a piece of *wood* under a high hydrostatic pressure we would have observed, after release of the pressure, a Permanent Set, i.e. an instantaneous deformation which is not recoverable. The reason is that there are *pores* in the wood and, under high pressure, stresses are produced which partly crush the material around the hollow spaces of the pores, forcing it into the pores. This local deformation which exceeds the strength of the material, is not recovered on the removal of the pressure, and the aggregate effect of many such microscopic deformations becomes manifest in the macroscopic set. To make our statement true we have to re-word it as follows : " Under isotropic pressure all *non-porous* materials are elastic and only elastic ", and in this form the axiom is an implicit (rheological) definition for " non-porous ". If we now read this statement together with the second rheological axiom that " every real material possesses *all* rheological properties ", which, as we now realize, is an implicit definition for " real material ", we must conclude that no real material is non-porous and we shall not be surprised to learn that Eyring [38] has propounded a theory of the viscosity of liquids which assumes that even a liquid possesses " holes ".

Some readers may think this digression to be out of place, but the situation as described means that the assertions which I have made and shall make about properties of materials are *never quite correct.* They are always only provisional, to be amplified or qualified later. If the reader will keep this in mind, I will be able to go ahead, speaking often in a loose way, without having to add to every quantitative statement the qualification " approximately ". Therefore, let it be granted that there are materials, which on the whole, are under isotropic pressure and tension elastic and only elastic, and let us examine what the laws of such elasticity may be.

2. The Bulk Modulus

If a hydrostatic pressure is applied to a body, its volume V is reduced and its density ρ, which is the mass per unit volume, or

$$\rho = m/V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

increased. Conversely, an isotropic tension (which it is much more difficult to apply) will increase the volume and reduce the density. Let V_0 be the volume for vanishing isotropic tension $p = 0$ and let ΔV_0 be the increase of volume produced by p, then an Elastic Bulk Modulus κ is usually defined by

$$p = \kappa \Delta V_0 / V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from which

$$\Delta V_0 = p/\kappa . V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and therefore

$$V = V_0 + \Delta V_0 = V_0 (1 + p/\kappa) \quad . \quad . \quad . \quad (4)$$

Equation (2) is regarded as a special case of Hooke's law, but this definition of the bulk modulus is objectionable for the reason that the latter cannot be constant. We defined the coefficient of viscosity in a certain way and said that in certain liquids it will be a constant, while in others it may vary with the stress. Rheological coefficients should generally be defined in such a way that for *certain simple materials they are constants.* The coefficient κ as defined above, can, however, under no circumstances be a constant because then the volume V_0 would vanish for $p = -\kappa$, with the density increasing to infinity. Actually, if κ is defined as in (2), it is generally found to increase with increasing compression ; i.e. in order to produce the same increase of compression the pressure has to be comparatively more increased the greater the compression from which we start. For instance, Bridgman (1923) found that κ^{-1} varies for metals in accordance with the following empirical formulas :

$$\left.\begin{array}{ll} \text{for iron } \kappa^{-1} & = 10^{-7}\,(5{\cdot}87 - 2{\cdot}10 \times 10^{-5}p) \quad . \quad . \quad . \quad . \\ \text{for copper} & = 10^{-7}\,(7{\cdot}32 - 2{\cdot}7 \times 10^{-5}p) \quad . \quad . \quad . \quad . \\ \text{for lead} & = 10^{-7}\,(23{\cdot}73 = 17{\cdot}25 \times 10^{-5}p) \quad . \quad . \quad . \end{array}\right\} \qquad (5)$$

where p is the hydrostatic *pressure* measured in kg. cm.$^{-2}$ up to 20,000 kg. cm.$^{-2}$.

Other approximate bulk moduli are listed in the Table.

TABLE IX, 1. *Bulk Moduli*

Material	Ether	Alcohol	Water	Tuff	Clay	Glass	Mercury	Iron	Steel
$\kappa \times 10^{-8}$ in megabars	0·08	0·1	0·2	0·5	1	4	5·4	15	18

3. Volumetric Strain

We can derive from (2) a more suitable definition by keeping in mind that this equation was put up to suit a comparatively small range of stresses.

Let that range be infinitely small, then (2) becomes

$$dp = \kappa dV_0/V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now let this equation be assumed to be valid not only for the volume V_0, but for any volume V, then

$$dp = \kappa dV/V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and this equation defines a bulk modulus which, in principle, could be a constant. Because let κ be a constant, then (6) can be integrated and gives

$$p = \kappa \ln(V/V_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or

$$V = V_0 e^{p/\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In this case, therefore, V vanishes for $p = -\infty$ or, practically never, the negative sign denoting a pressure.

We call the quantity

$$e_v = \ln(V/V_o) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

the Volumetric Strain or Cubical Dilatation and we can accordingly write (7) in the form

$$p = \kappa e_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which is the complete analogy to (I, d). Introducing again $V = V_o + \Delta V_o$, $e_v = ln\ (1 + \Delta V_o/V_o)$ and if $\Delta V_o/V_o$ is small, this expression can be developed into a series, the first two terms of which are

$$e_v = \Delta V_o/V_o - \tfrac{1}{2}(\Delta V_o/V_o)^2 \quad . \quad . \quad . \quad . \quad (12)$$

In a first approximation we find again (2), but in a second approximation we have

$$p = \kappa \Delta V_o/V_o - \kappa/2 . (\Delta V_o/V_o)^2 \quad . \quad . \quad . \quad (13)$$

Indicating by κ' an " apparent " bulk modulus in accordance with (2), we find by introducing

$$\Delta V_o/V_o = p/\kappa' \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$p = p\kappa/\kappa' - \kappa/2 . p^2/\kappa'^2 \quad . \quad . \quad . \quad . \quad (15)$$

from which

$$\kappa' = \kappa - p\kappa/2\kappa' \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$\kappa' = \kappa/2\,[1 + \sqrt{1 - 2p/\kappa}] \quad . \quad . \quad . \quad . \quad (17)$$

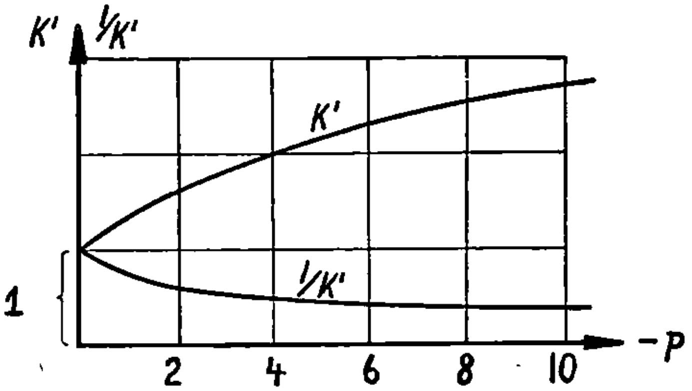

Fig. IX, 1. Apparent bulk modulus (κ') and apparent coefficient of compressibility ($1/\kappa'$) as a function of the hydrostatic pressure ($-p$).

If we plot κ' against p we get a curve as shown in Fig. 1. The Apparent Coefficient of Compressibility (κ'^{-1}) has also been plotted. It can be seen from these curves that the assumption that κ in (11) is a constant gives a better approximation to the compressibility of materials because the compressibility in accordance with this formula decreases with increasing pressure. However, comparison with the empirical equations (5) shows that quantitatively for large pressures the position is still far from satisfactory. Following Hencky [39] we therefore proceed to apply two corrections.

4. The Reduced Stress

For the first correction we must remember that the resistance of the material to the external forces comes from reactive forces between the molecules or atoms, as the case may be, caused by a change in their stable distances. The forces, accordingly, are bound up with the quantity of *matter*. The stresses, on the other hand, are defined in respect of the areas through which the forces act and a unit isotropic pressure will be a unit force applied upon the sides of a cube of unit length of edge and unit *volume*, say 1 cm.^{3} That cubic centimetre of volume, however, contains in the strained state, assuming tension and $V > V_o$, fewer molecules or atoms than in the unstrained state, in short, less matter. Conversely, the unity of matter in the unstrained state will in the strained state take up a volume increased in the proportion ρ_o/ρ where ρ_o is the density in the unstrained, ρ the density in the strained state. To calculate from the stresses the reactive force caused *by the same piece of matter*, we must increase the stress in the same proportion, i.e. we must replace the stress as defined in Chapter I by the Reduced Stress

$$p_{red} = \frac{\rho_o}{\rho} p = V/V_o \cdot p = V_{rel} p \quad . \quad . \quad . \quad (18)^*$$

where V_{rel} is the Relative Volume $= V/V_o$. This makes (11)

$$p = \kappa \ln V_{rel}/V_{rel} = \kappa e_v \mathrm{e}^{-e_v} \quad . \quad . \quad . \quad . \quad (19)$$

where e is the basis of " natural " logarithms.

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[*] Note that V_{rel} here is an entirely different quantity from what is denoted in the same way in Chapter V.

5. The Limiting Relative Volume

The second correction refers to the feature of (11) or (19) that with *infinite* pressure the volume of the body would be *compressed to zero.* There must, however, be a limiting volume containing all the molecules even if in a collapsed and crumpled-up state (the latter was realized by Bridgmann reaching pressures up to 20,000 kg. cm.$^{-2}$). Hencky, on applying (19) to Bridgman's observations, found that for a small number of solids this was a satisfactory formula. Most of the experiments, however, suggested that there was a Limiting Relative Volume Ψ which could not be reached by finite pressure. He therefore suggested the formula

$$p = (\kappa)\,\frac{1 - \Psi}{V_{rel} - \Psi}\,\frac{ln/V_{rel}}{V_{rel}} \quad . \quad . \quad . \quad . \qquad (20)$$

which he compared with Bridgman's observations and especially of those substances which Bridgman had marked out as very irregular and therefore particularly fit for a crucial test. Near atmospheric pressure (20) does not give as good results as at higher pressures and the simpler formula (11) may be used there. Hencky concluded that the individual structure of the molecule apparently has much influence at low pressures, but it seems that at pressure from 10,000 kg. cm.$^{-2}$ upwards even gases acquire a constant bulk modulus in accordance with (20). The following Table gives the results of his calculations. Equation (20) can be considered as expressing the modulus κ of (11) by $(\kappa)\ (1 - \Psi)/(V_{rel} - \Psi)$. While now κ is a variable coefficient and not a constant, (κ) and Ψ are supposed to be constants.

TABLE IX, 2. *Hencky's Bulk Modulus* (κ) *and Limiting Volume* (Ψ) *of certain Substances calculated with Hencky's Theory* (20) *from Bridgman's Observations.*

Substance	Temperature Centigrade	$\kappa \times 10^4$ in kg/cm^{-2}	Ψ
Iron	30	170	0·8
Rubidium	50	1·66	0·44
Mercury	20	24·9	0·75
Water	40	2·34	0·50
Alcohol	20	4·17	0·52
Hydrogen	65	0·82	nearly zero
Helium	65	0·47	,,

6. Pulsations

We have said in Section 4 that the internal forces with which the material reacts to the external forces are due to the displacements of the molecules or atoms from their stable relative positions. To effect such displacement, external work has to be expended which is converted into the potential elastic energy of the strained body. Let us look into this process in more detail.

If we apply a finite isotropic pressure $\bar{p}$ upon a body,* its particles will be brought into movement in accordance with (I, a), i.e. after some time through the operation of the acceleration a, they will attain certain velocities v and will by virtue of these possess kinetic energy of the magnitude

$$E_\kappa = mv^2/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

At the same time they will undergo by displacements u and this will produce in their surroundings elastic stresses in accordance with (11) or (20) or some other rheological equation. Let us assume the body to be a sphere of radius R_o, the centre of which is fixed, then, for reasons of symmetry, we may assume that the direction of the different u's and v's passes through the centre. To every $\bar{p}$ there corresponds a certain definite e_v and therefore a certain definite radius R at which the internal stresses p raised in the body through the strain will balance the external pressure $\bar{p}$. If $\bar{p}$ is applied from the start to the full extent, this means that before R_o is reduced to R there is no equilibrium; this causes the appearance of kinetic energy. When the sphere is compressed to V, p balances $\bar{p}$, but the particles will move on towards the centre because they have kinetic energy. This increases p beyond $\bar{p}$ and there are unbalanced stresses acting *away* from the centre. This, in turn, causes a negative acceleration a or retardation $-a$, a diminution of v and ultimately a state of rest when v vanishes. That state of rest, however, does not correspond to a state of equilibrium. The stress p exceeds the external pressure $\bar{p}$; rest lasts only through an element of time; the body starts to expand. We need not continue this story; the result will be voluminal oscillations or pulsations around the position of

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* The bar above $\bar{p}$ indicates the *external* force.

equilibrium $e_v = p/\kappa$; a position which, however, is not one of rest.

7. Volumetric Strainwork

If we want to determine the compressibility of a material in a suitable test, we shall therefore have to apply the external pressure very slowly, practically *infinitely slowly*, i.e. we shall have to increase the pressure from zero to $\bar{p}$ so slowly that it is always equal to p and the body passes continuously *through states of equilibrium*. Considering now the sphere of radius R which is at rest under the action of the isotropic tension p applied upon its surface A ; if p is slightly increased the radius will slightly be increased by ΔR and the forces acting upon the surface perform work. The force acting upon an element of the surface (ΔA) is $p.\Delta A$ and the work $p.\Delta A$, ΔR (compare Fig. 1). Adding up all elements of work over the whole surface of the sphere, we get $p.A.\Delta R$. If ΔR is very small, the increase of volume is $dV = A\ dR$ and, therefore,

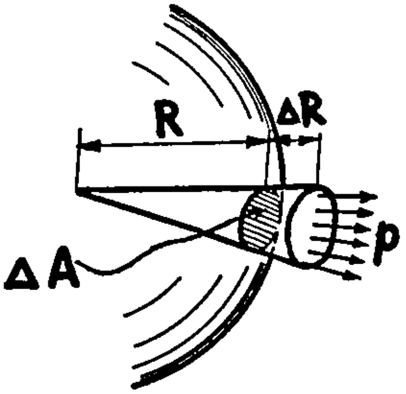

FIG. IX, 2. Volumetric strainwork.
p isotropic tension. ΔR increase of radius.

$$dW = pdV \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

or the elementary work per unit volume

$$dw = pdV/V = pe_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and the power per unit volume

$$\dot{w} = sw/dt = p\dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If $\tilde{p}$ is gradually increased from zero to p, the work performed per unit volume is

$$w = \int_{o}^{e_v} p\,de_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If the load p is decreased from p to zero,

$$w' = \int_{e_v}^{o} p\,de_v = -\int_{o}^{e_v} p\,de_v = -w \quad . \quad . \quad . \quad (26)$$

and the strainwork is completely regained. The only condition for this reversibility is that there should be a one-to-one relation between p and e_v, but the *form* of the relation, whether straight line or curved, is immaterial. Generally, *in a perfectly elastic body, all strainwork performed by the external forces through states of equilibrium is stored up in the body as potential energy and completely regained upon removal of the external forces, if this is likewise carried out through states of equilibrium.*

If p is expressed as a function of e_v by means of a rheological equation, the integration can be carried out. For instance, if we use (11),

$$w = \kappa e_v^2/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or

$$w = p^2/2\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

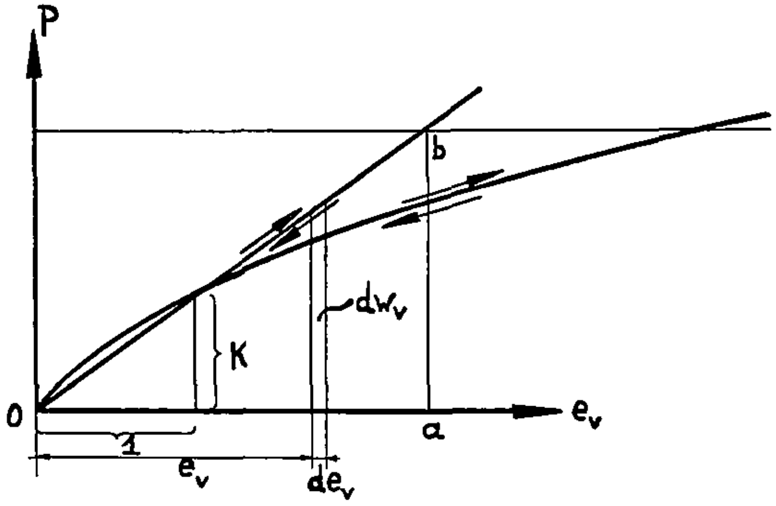

FIG. IX, 3. Fundamental rheological curve for infinitely slow compression. Area $o\ a\ b$ measures the strainwork.
e_v volumetric strain.
p isotropic stress.
k modulus of compressibility.

The reader will note that (22) is the analogue of (III, 41); (25) of (III, 42); (27) of (III, 43); and (28) of (III, 44).

If we plot correlated observed values of p against e_v, the resultant curve is a graphical representation of the rheological equation (compare Fig. IX, 3). If the curve is a straight line, κ in (11) is a constant; otherwise κ is variable. Comparing (23), we see that the elementary work per unit volume is represented in the Figure by an elementary strip parallel to the p-axis and of width de_v. The total work performed when increasing the pressure from zero to p is therefore represented by the area enclosed by the curve, the e_v axis and the ordinate p. If all stored-up energy is to be regained as the material expands with decreasing pressure, the points representing descending observations must lie on the same curve or, as we said before, p must be a one-valued function of e_v.

8. Damping of Free Pulsations

Let us now consider what will happen if, having compressed a sphere, by gradually applying an isotropic pressure p, from the radius R_0 to the radius R, the external pressure is suddenly released. Now, in expanding, the internal stresses meet no resistance against which to perform work. The particles will therefore acquire kinetic energy and oscillations will start. Because of the absence of external forces, the oscillations are called *free oscillations*. The elastic potential energy (E_p) is converted into kinetic energy (E_k), which gradually increases until it reaches a maximum for $R - R_0$ when the elastic energy vanishes. Then the sphere continues to expand with kinetic energy being converted into potential energy, etc., all the time the law of conservation of energy requiring that the sum of both is constant. This, however, cannot go on indefinitely. The process just described of recurring conversion of kinetic into potential energy requires that the velocities and accelerations or retardations of all particles are at all times directed towards the centre of the sphere. But if the oscillations started in this way, imperfections in the structure of the material making up the sphere, such as pores, local variations of density, or, generally, heterogeneities and local æolotropies, etc., will soon make themselves felt. While the sum of the kinetic energies of all particles will still conform to the law of conservation, the velocities will gradually become less oriented towards the centre and more and more acquire a random orientation. The oscillations of the sphere as a whole will gradually be replaced by individual oscillations of the particles which macroscopically manifest themselves in an increase of temperature. This is called Damping of Oscillations, with which we shall deal in more detail in Chapter XV. But as we have said in Section 9 of Chapter III, a rheological experiment must be isothermal. This can be realized by connecting the body with a large heat reservoir kept at constant temperature. The heat into which the kinetic energy of the oscillations is gradually converted is drawn off into the reservoir and dissipated (E_d). The law of conservation of energy now requires

$$E_p + E_k + E_d \text{ const.} \qquad (29)$$

No other form of energy appears in rheology. What we have derived here on the example of isotropic stress and voluminal oscillations is, as will be shown later, valid for every other kind of stress and deformation. Equation (29), together with the proviso that in every process E_d can only increase, form the energetic basis of rheology.

9. The Coefficient of Volume Viscosity

If there is dissipation of energy whenever the cubical dilatation is not produced infinitely slowly, or whenever there is a finite *rate* of dilatation, $\dot{e}_v$, this implies a sort of viscosity η_v which we may call Volume Viscosity. The complete isotropic rheological equation is accordingly

$$p = \kappa e_v + \eta \dot{e}_v \qquad (30)$$

It should be noted that in deriving (30) we did not say whether we were speaking of a liquid or a solid. This is in accordance with the first rheological axiom which (in other words) says that for simple changes of volume or density it is irrelevant whether the material is a solid or liquid. A liquid, therefore, *must* have *two* kinds of viscosity, viz. the ordinary Newtonian viscosity in shear η and the volume viscosity η_v. One could, of course, in principle assume that for a certain class of liquids η_v vanishes and we may call this class of liquids the Stokes Liquid, because this was what Stokes (1851) assumed when deriving the famous Stokes-Navier differential equation of viscous flow named after him and Navier (1823). Until recently this was generally assumed to conform to actual conditions, but Tisza [40] has pointed out that η_v must in real liquids be rather high and I have pointed to other consequences connected with a vanishing η_v which are not likely to conform to experience and about which we shall say more in Chapter X.

10. Elastic After-effect and Total Strainwork

Equation (30) yields an interesting consequence. Let a certain volumetric strain $[e_v]$ be produced by an isotropic external tension $\bar{p}$ which is then removed. If η_v is large so that no oscillations, but only an aperiodic movement results (or, conversely, if inertia-forces can be neglected) we find from (30) for $p = 0$

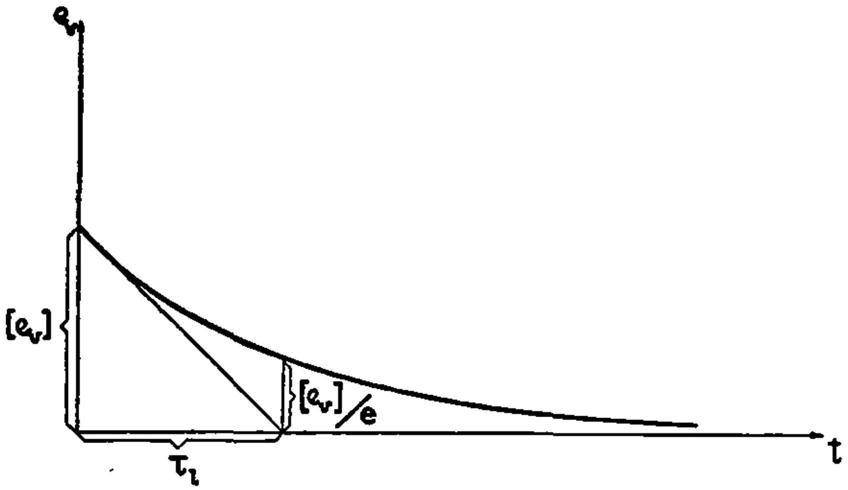

FIG. IX, 4. Elastic after effect in volumetric strain upon removal of stress. t time, τ time of lagging, e basis of natural logarithms.

$$\kappa e_v + \eta_v \dot{e}_v = 0 \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and introducing

$$\dot{e}_v = de_v/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$\eta_v \, de_v/e_v = -\kappa dt \quad . \quad . \quad . \quad . \quad . \quad (33)$$

which by integration yields

$$\eta_v \ln e_v = -\kappa t + C \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The integration constant C is determined from $e_v = [e_v]$ for $t = o$ and we ultimately have

$$e_v + [e_v]\mathrm{e}^{-\kappa/\eta_v{}'t} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The quotient η_v/κ is of the dimension of time (τ) which is a measure of the lagging of the elastic strain. τ may be called Time of Lagging. In accordance with (35), the original volume is recovered (i.e. $e_v = 0$) when $t/\tau = \infty$, which is $t = \infty$ for any finite τ. The curve corresponding to (35) is shown in Fig. IX, 4.

When the load is removed, the original volume for which $e_v = 0$ is therefore generally not instantaneously regained; this takes time and, theoretically, even infinite time. Conversely, when the load is applied, it also takes time (and again theoretically infinite time) until the cubical dilatation corresponding to the load is attained. This phenomenon is called Elastic After-Effect. It should be noted that the elastic after-effect does not constitute an *imperfection* of elasticity, according to both definitions of perfect elasticity, viz. (i) total disappearance of strain on removal of load, (ii) complete conservation of strainwork in infinitely slow deformation. If, however, the deformation is produced with a finite velocity, part of the strainwork is dissipated through volume viscosity.

Volume viscosity in liquids makes itself felt in the absorption and dispersion of ultrasonic waves, the rate of which is higher than if shear viscosity (η) only was present. In liquids $-\kappa e_v$ is called the *hydrostatic* pressure. It is, as follows from (30), not identical with the isotropic component of the stress.

In the general case (30), the strainpower per unit volume is

$$\dot{w} = p\dot{e}_v = \kappa e_v \dot{e}_v + \eta_v \dot{e}_v{}^2 \quad . \quad . \quad . \quad . \quad (36)$$

The first term on the right-hand side reverses its sign when e_v decreases and $\dot{e}_v$ is negative. On de-straining, this work is therefore regained and the first term accordingly constitutes the conserved part of the strainwork. The second term is always positive, whether e_v increases (positive $\dot{e}_v$) or decreases (negative $\dot{e}_v$). On de-straining as well as on straining, work is expended and the second term accordingly constitutes the dissipated part of the strainwork. The total strainwork is

$$w = \int^t w\,dt = \int_o^{lv} \kappa e_v\,de_v + \int_o^t \eta_v \dot{e}_v{}^2 dr \quad . \quad . \quad (37)$$

We shall come back to the rheological equation of form (30) in Chapter XIV showing that it embodies other properties besides those dealt with in the present chapter. Here let us only add the remark that we have introduced another qualification into our first axiom. Under isotropic stress non-porous materials are elastic only if the stress is applied *infinitely slowly*. Otherwise they also show a viscous resistance.

11. Summary

The rheological behaviour of every *homogeneous* material solid or liquid, conforms under isotropic stress p to the rheological equation

$$p = \kappa e_v + \eta_v \dot{e}_v \quad . \quad . \quad . \quad . \quad (\text{IX, a})$$

where e_v, the volumetric strain,

$$e_v = \ln(V/V_o) = \ln(V_{rel}) \quad . \quad . \quad . \quad (\text{IX, b})$$

In liquids κe_v is called " hydrostatic pressure."

In cases where the density ρ of the material is much changed by pressure, p must be replaced by the reduced stress

$$p_{red} = \frac{\rho_o}{\rho} p \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{IX, c})$$

The volumetric strain-power is

$$w_v = \kappa e_v \dot{e}_v + \eta_v \dot{e}_v^2 \quad . \quad . \quad . \quad . \quad (\text{IX, d})$$

of which the first part is conserved, the second dissipated. If κ is constant, the conserved (potential) energy is

$$E_p = \kappa e_v^2/2 = p^2/2\kappa \quad . \quad . \quad . \quad . \quad (\text{IX, e})$$

It follows from (IX, a) that there is an elastic after-effect with lagging time

$$\tau = \eta_v/\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{IX, f})$$

so that a strain $[e_v]$ disappears upon removal of the load in accordance with the equation

$$e_v = [e_v] e^{-t/\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{IX, g})$$

where it is assumed that τ is constant.

Where κ is a variable coefficient, Hencky has proposed

$$\kappa = (\kappa) \frac{1 - \Psi}{V_{rel} - \Psi} \quad . \quad . \quad . \quad . \quad . \quad (\text{IX, h})$$

Ψ being the relative limiting volume and (κ) a constant.

CHAPTER X

SIMPLE TENSION AND SIMPLE EXTENSION

1. Simple Stresses and Deformations

So far we have become acquainted with *two simple cases of stress and deformation.* They are simple shearing stress and simple isotropic stress on the one hand—simple shear and simple cubical dilatation on the other. These are *co-ordinated* pairs, simple shearing stress producing simple shear, and simple isotropic stress producing simple cubical dilatation. There exist, however, two other important cases of stress and deformation with which we have not yet dealt and which are of an entirely different character. They are *simple tension* on the one hand and *simple extension* on the other. Simple tension is a one-dimensional stress and simple extension is a one-dimensional deformation, *but the latter is not produced by the former.*

Simple tension and its opposite, simple pressure, can both be included under the term simple Normal Stress, designated by p_n, where the subscript n indicates " normal ". A simple normal stress is produced by either simple Pull ($+ P_n$) or simple Push ($- P_n$) upon a prismatic bar, acting in the direction normal to, and passing through the centre of the cross-section.* Under the action of such simple pull, the bar is elongated ; at the same time, however, it contracts laterally. *A one-dimensional traction is here accompanied by a three-dimensional deformation.*

Simple extension, or its opposite, simple compression, both included under the term simple Normal Deformation, (e_n), because the displacement is in the direction *normal* to the cross-section, is not so easily realised. A plastic material filling a cylinder, where the latter is so rigid that it does not extend in diameter, would, under pressure from a piston, be deformed in the manner of simple compression. Here the walls of the cylinder will also produce pressure upon the material and *the*

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* If the force does not pass through the centre of the cross-section, *bending* is caused, with which we shall deal in Chapter XII.

one-dimensional deformation is therefore accompanied by a three-dimensional stress. However, in order to produce in a plastic material a measurable compression, the pressure would have to be so high that the walls of the cylinder, too, would give way slightly, albeit elastically. Nevertheless, in porous materials, simple pressure may produce simple compression. One such material is cork, another porous rubber; still another, as we shall learn in Chapter XIII, is concrete. These, however, are exceptions.

2. Young's Modulus

The solution of this puzzle—viz. that simple normal stress does not correspond to simple normal deformation or strain (and *vice versa*)—is that " simple " normal stress and deformation are not as simple as one may think.

Let us first consider simple normal stress, for instance as produced by the simple pull of a cylinder or prismatic steel bar in the so-called " tensile test ". This is the predominant test for metals, but has also been in use for such materials as cement, pitch, bitumen, flour dough, etc. In this test a short bar, say of mild steel, of length l_o is fixed between two pairs of jaws (or some similar device), one stationary and the other movable, and is elongated by a gradually increasing load P_n. If A is the cross-sectional area of the bar, a traction is produced equal to

$$p_n = P_n/A. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

acting in the longitudinal direction, or normal to the cross-section, and which may be assumed as uniformly distributed over the cross-section. The assumption is not correct near the ends, where the bar is fixed between the jaws, but it will be valid at some distance from the ends, especially if the bar is slender.* Up to a certain point, the *elongation* Δl is proportional to the load and therefore follows Hooke's law. For this part we write in analogy to (I, 5)

$$\Delta l/l_o = (P_n/A)/\epsilon \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where the coefficient ϵ is called Young's Modulus.

The ratio $\Delta l/l_o$ is usually taken as the measure of the normal strain, here called Extension, positive or negative. For small

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* This is the Principle of St. Venant about which compare *Ten Lectures*, pp. 58–59, 65.

elongations Δl there is no objection to this. For large elongations, however, this definition breaks down; firstly, for the same reason as advanced in Section 3 of Chapter IX in respect of the volumetric strain; and, secondly, because there is no reason why $\Delta l/l_o$ should be a more correct measure than $\Delta l/l$, where $l = l_o + \Delta l$. For $\Delta l = l_o$, the first gives 100 per cent, the second a 50 per cent. increase.* A consistent definition would result from relating a differential of increase to the *instantaneous* length so that the element of strain is dl/l and the total strain

$$e_n = \int_{l_o}^{l} dl/l = (\ln l/l_o) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which formula both l and l_o are of equal standing.

This logarithmic measure of extension was first introduced by Roentgen, of X-ray fame. It was first systematically employed by Hencky (1929, 1) (compare also *Ten Lectures*, pp. 23–25).

For small elongations

$$e_n = \ln(l/l_o) = \ln\frac{l_o+\Delta l}{l_o} = \ln(1+\Delta l/l_o) = \Delta l/l_o - \tfrac{1}{2}(\Delta l/l_o)^2 + \quad . \quad (4)$$

With this definition in mind we write for the Hooke solid

$$e_n = p_n/\epsilon \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Values of Young's modulus for different materials are listed in the following Table :—

Material	Lead	Concrete	Tin	Glass	Zinc	Copper	Wrought iron	Steel
$\epsilon \times 10^{-5}$ in in megabars	1	2	4	7	8	12	18	25

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* Recently a local paper quoted the President of an American College as having said that " the price of automatic pilot devices had *dropped by several hundred per cent.* during the war " It must be assumed that he did not intend to say that every buyer of a pilot device is being paid a premium.

3. Poisson's Ratio

As we have already pointed out, the pull P_n produces in the bar not only an axial extension but, generally, at the same time also lateral contractions e_c of magnitude.

$$e_c = -\sigma e_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where σ is a material constant called " Poisson's ratio ". If e_n is negative, e_c will be positive or be a lateral expansion. Such lateral expansion cannot be observed in a cork-cylinder under pressure, in which case $\sigma = 0$.

Both e_c and e_n are to be measured logarithmically, and (6) therefore defines Poisson's ratio for *finite* strains as well.

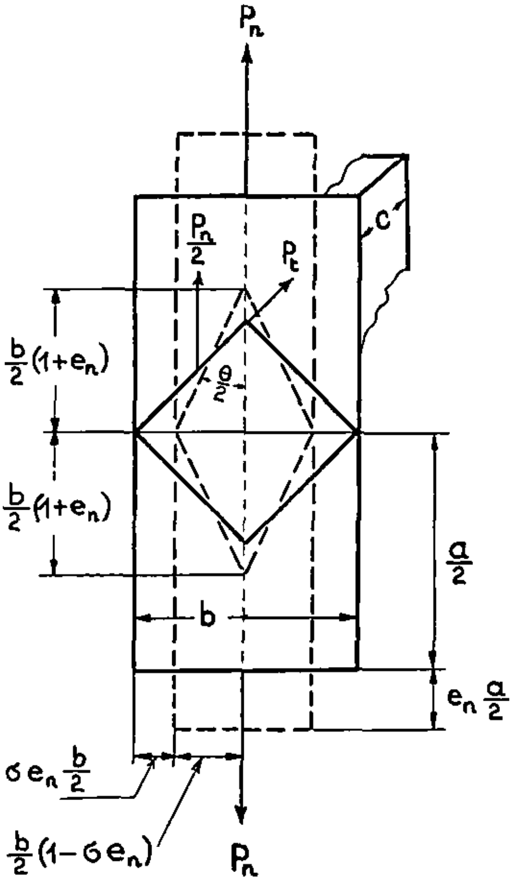

FIG. X, 1. Simple pull of a prismatic bar.

The strain has been assumed as very small, so that " a " can be put for " a_0 ".

We have thereby introduced for the Hooke solid two new constants, ϵ and σ, so that altogether we have four constants γ, κ, ϵ and σ. However, the values for γ and κ completely determine the rheological behaviour of a Hooke solid and it must therefore be possible to express ϵ and σ by the former. We regard γ and κ as " fundamental ", ϵ and σ as " derived " constants.

Poisson's ratio is a pure number. Approximate values of Poisson's ratio for different materials are listed in the following Table :—

Material	Cork	Aluminium	Zinc	Glass	Steel	Copper	Lead	Rubber
σ	0	0·13	0·20	0·24	0·30	0·32	0·40	0·49

Let us consider an axial section of the bar shown in Fig. X, 1. Generally, when being extended, the volume of the bar will be increased. At the same time, as can be seen from that figure, certain right angles will be changed and certain parallel planes accordingly sheared against each other. Therefore, the simple normal stress p_n will be accompanied by a volumetric strain e_v and a tangential strain e_t. Similarly, it is obvious that a simple extension e_n is accompanied by an isotropic stress p and a tangential stress p_t.

Our task will consist in finding the stresses p and p_t equivalent to p_n in the first case and the strains e_v and e_t equivalent to e_n in the second case.

4. Principal Strains and Deformations

If the bar is not itself a prism, it can be considered as made up of a large number of " elementary " prisms, the deformation of which we shall presently investigate.

Let the prism of edges a, b, c be extended or contracted in the directions of its edges. These directions are at right angles to each other before deformation and remain so after deformation. If we look at Fig. I, 1, we see that in the case of simple shear there also exist three such directions. They are called the Principal Axes of Deformation. In the case of simple shear, as again to be seen from Fig. I, 1, there is not only deformation but also rotation. One axis (the one normal to the plane of the figure) is fixed, but the other two are rotated by the same angle around the third. In our case, where the straining of the prism is in the direction of the edges, there is no such rotation. A deformation without rotation is called Pure Deformation.

We denote the directions of the principal axes by i, j and k. If these directions are the same throughout the body, the deformation is said to be *homogeneous*. This is the case in simple extension.

Now, let the edges of the prism before strain be a_o, b_o, c_o, so that $V_o = a_o, b_o, c_o$, and after strain a, b, c, so that $V = abc$.

The principal strains or deformations are

$$e_i = \ln(a/a_o),\ e_j = \ln(b/b_o),\ e_k = \ln(c/c_o) \quad . \quad . \quad (7)$$

Therefore

$$e_v = \ln(V/V_o) = \ln(abc/a_o b_o c_o) = \ln(a/a_o) + \ln(b/b_o) + \ln(c/c_o) = e_i + e_j + e_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If we define Mean Normal Strain or Deformation e_m by

$$e_m = (e_i + e_j + e_n)/3 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

we have

$$e_v = 3e_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If we deduct from each of the three principal strains the mean strain in the following manner

$$e_{oi} = e_i - e_m,\ e_{oj} = e_j - e_m,\ e_{ok} = e_k - e_m \quad . \quad . \quad (11)$$

three other principal strains result, the cubical dilatation of which vanishes in accordance with

$$e_{ov} = e_{oi} + e_{oj} + e_{ok} = e_i + e_j + e_k - 3e_m = 0 \quad . \quad (12)$$

Therefore $e_{oi} \,.\, e_{oj} \,.\, e_{ok}$ are the principal strains of the *distortion* resulting from the strains or, more generally, deformations e_i e_j, e_k. *Every combination of principal deformations* e_i, e_j, e_k, *can be resolved into a dilatation* $e_v = e_i + e_j + e_k$ and a *distortion* $e_i - e_v/3$, $e_j - e_v/3$, $e_k - e_v/3$.

We shall indicate the distortional deformation by the subscript " 0 ".

In the case of simple normal stress (or more correctly, traction) p_n, the principal strains are e_n in the longitudinal and $-\sigma e_n$ in the two transverse directions and, therefore,

$$\left.\begin{aligned}&e_v=e_n(1-2\sigma)\,;\ e_m=e_v/3=e_n(1-2\sigma)/3\\&e_{oi}=e_{oj}=-\sigma e_n-e_m=-(1+\sigma)/3\,.\,e_n\,;\ e_{ok}=e_n-e_m=\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2(1+\sigma)e_n/3\end{aligned}\right\} \qquad (13)$$

For an incompressible material, e_v must vanish or $\sigma = \frac{1}{2}$, in which case $e_{oi} = e_{oj} = - e_n/2$; $e_{ok} = e_n$.

In the case of simple normal strain in the k-direction we have

$$\left.\begin{aligned}e_i = e_j = 0\,;\ e_k = e_n\,;\ e_v = e_n\\ e_{oi} = e_{oj} = - e_n/3\,;\ e_{ok} = 2e_n/3\end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

5. Principal Stresses

The foregoing considerations in respect of strains can be repeated with suitable modifications in respect of stresses. Here *Principal Stress Axes* exist and *in isotropic materials these coincide with the principal axes of strain*, so that we can use for them the same subscripts, i, j, k. The Mean Normal Stress

$$p_m = (p_i + p_j + p_k)/3 \quad . \quad . \quad . \quad . \quad . \quad (15)$$

is that part of the stress which, when applied isotropically in the manner of a hydrostatic stress, causes the cubical dilatation (with positive or negative sign as the case may be) of the material, while the normal stresses

$$p_{oi} = p_i - p_m\,;\ p_{oj} = p_j - p_m\,;\ p_{ok} = p_k - p_m \quad . \quad (16)$$

cause the distortion. In the case of simple normal tension

$$\left.\begin{aligned}p_k = p_n\,;\ p_i = p_j = 0\,;\ p = p_m = p_n/3\\ p_{oi} = p_{oj} = - p_n/3\,;\ p_{ok} = 2p_n/3\end{aligned}\right\} \quad . \quad . \quad (17)$$

Now if the cubical dilatation e_v, as calculated in (13), is caused by the mean stresses p_m as expressed in (16), Equation (IX, a) requires in the case of rest ($\dot{e}_v = 0$), $p = \kappa e_v$, or $p_n/3 = \kappa e_n\,(1-2\sigma)$ and because of (5)

$$\kappa = \epsilon/3(1 - 2\sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

6. The Definition of Finite Shearing Strain

Let us now consider the shear produced by simple normal stress. In Fig. X, 1 a square has been inscribed on the longitudinal section of the prism, which by the extension e_n is converted into a rhomb. Let us first assume the extension to be very small ; in fact so small that we may use the first term only in the development (4) of e_n ; then the originally right angle of the square will be slightly changed to, say, θ. In Section 5 of Chapter I we have defined the change of a right angle produced by shear, if *small*, as the shearing strain. We find from the Figure

$$tg\theta/2 = (1 - \sigma e_n)/(1 + e_n) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The change of the right angle is $(90 - \theta)$ and its tangent is equal to the cotangent of θ, or $tg(90 - \theta) = ctg\theta = 1/tg\theta$.

From known formula

$$tg\theta = (2tg\theta/2)/(1 - tg^2\theta/2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and introducing $tg\,\theta/2$ from (19)

$$tg\theta = 2\frac{(1 - \sigma e_n)(1 + e_n)}{(1 + e_n)^2 - (1 - \sigma e_n)^2} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Neglecting powers of e_n higher than the first, this gives

$$1/tg\theta = e_n(1 + \sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Therefore, for a small change of the right angle into θ,

$$e_t = e_n(1 + \sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)^*$$

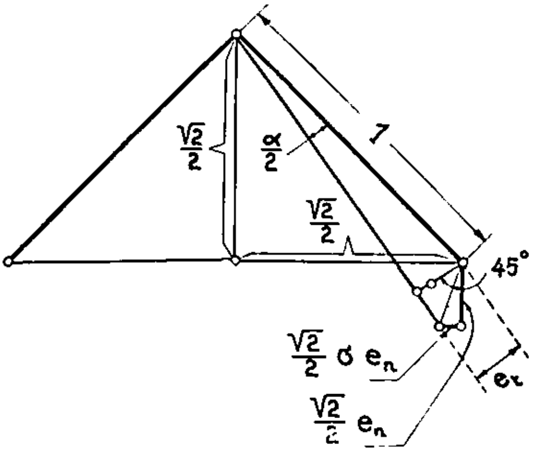

FIG. X, 2. The maximum shearing strain in pure extension.

$$\alpha/2 = \frac{\sqrt{2}}{2\sqrt{2}}e_n + \frac{\sqrt{2}}{2\sqrt{2}}\sigma e_n = \frac{e_n}{2}(1 + \sigma)$$

We now make a *new definition* for the shearing strain. We postulate that if e_n is large so that the *complete* (3) applies, e_t is defined by (23). This definition is consistent and for small strains conforms with the usual definition of the classical theory of elasticity.

7. The Law of Elastic Distortion

The shear of the square inscribed into the rectangle of Fig. X, 1 is produced by a shearing stress resulting from the action of

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* For those readers who prefer geometrical methods, Fig. X, 2 shows another way of deriving (23).

forces $P_n/2$ acting on all sides of the square. These forces which act in the longitudinal direction, have tangential components $P_t = \frac{P_n}{2}/\sqrt{2}$. The tangential forces act upon areas $= \frac{b\sqrt{2}}{2}.c$ and the tangential stresses are therefore

$$p_t = P_t/\frac{b\sqrt{2}}{2}c = P_n/2bc.$$

Introducing $P_n/bc = p_n$, this gives

$$p_t = p_n/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For a Hookean solid we can go back to (I, d) and get

$$p_n/2 = \gamma e_n(1 + \sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

which gives, because of (5),

$$2\gamma = \epsilon/(1 + \sigma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

From the two equations (18) and (26) we can now express the derived coefficients ϵ and σ through the fundamental coefficient κ and γ :—

$$\epsilon = 9\kappa\gamma/(3\kappa + \gamma) ; \; \sigma = (3\kappa - 2\gamma)/2(3\kappa + \gamma) \quad . \quad . \quad (27)$$

For an incompressible material $\kappa = \infty$ and therefore

$$\epsilon = 9\gamma/(3 + \gamma/\kappa) = 3\gamma ; \; \sigma = \tfrac{1}{2} \quad . \quad . \quad . \quad (27')$$

From (17) we find

$$p_{oi} = p_{oj} = -1/3.\epsilon_n ; \; p_{ok} = 2/3.\epsilon e_n \quad . \quad . \quad . \quad (28)$$

and considering (13)

$$p_{oi} = \epsilon/(1+\sigma).e_{oi} ; \; p_{oj} = \epsilon/(1+\sigma).e_{oj} ; \; p_{ok} = \epsilon/(1+\sigma).e_{ok} \quad . \quad (29)$$

Because of (26), these now become

$$p_{oi} = 2\gamma e_{oi} ; \; p_{oj} = 2\gamma e_{oj} ; \; p_{ok} = 2\gamma e_{ok} \quad . \quad . \quad . \quad (30)$$

and from the symmetry of the expressions it is clear that the *relations* (30) *express Hooke's law for elastic distortion*, independent of the conditions of simple normal stress from which they were derived. We can accordingly write more concisely

$$p_{on} = 2\gamma e_{on} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30')$$

where n now stands for *any* direction of the normal to *any* section however oriented.

If the distortion is accompanied by a cubical dilatation, very much changing the density of the body, the stresses p_o must be replaced by the reduced stresses $\frac{\rho_o}{\rho} p_o$ in accordance with (IX,c).

8. Strain-work in Simple Tension and Simple Extension

Let us go back to the tensile test of a steel bar extended elastically in simple tension with which we started in Section 2. In extending the bar, the pull P_n will perform work. In the testing machine, the speed of extension is moderate and the change will be through states of equilibrium so that kinetic energy is not generated. Therefore all work of the external forces will be converted into strain-work. Consider again an elementary prism with edges a, b, c (where a is in the direction of the axis of the test-piece) which is in equilibrium under the action of the simple normal stress p_n. If p_n is slightly increased, the edge a will be increased by Δa, while b and c will be decreased by Δb and Δc respectively. However, while in the direction of a, the force $p_n bc$ is acting, no forces act in the directions b and c. The increment of work will therefore be

$$\Delta W = p_n bc\Delta a = p_n abc . \Delta a/a \quad . \quad . \quad . \quad (31)$$

If Δa is very small, $\Delta a/a = e_n$ and the element of work per unit volume

$$dw = p_n de_n = \epsilon e_n de_n \quad . \quad . \quad . \quad . \quad . \quad (32)$$

This equation is perfectly analogous to (IX, 23) and analogous conclusions can be drawn which need not all be enunciated here.

For the simple Hooke body, where ϵ is constant, (32) gives by integration

$$w = \epsilon e_n^2/2 = p_n^2/2\epsilon \quad . \quad . \quad . \quad . \quad . \quad (33)$$

In simple extension, while there are stresses in all three directions a, b and c, displacement is only in the direction a. We therefore arrive at the simple expression (32), but here (5) is not applicable. To calculate the stress-strain relation, we start from (14) from which

$$p = \kappa e_v = \kappa e_n \; ; \; p_{ok} = 2\gamma e_{on} = 4/3 . \gamma e_n \quad . \quad . \quad (34)$$

and therefore

$$p_k = p_{ok} + p = e_n (\kappa + 4/3 . \gamma) \quad . \quad . \quad . \quad (35)$$

This gives for constant coefficients

$$w = (\kappa + 4/3 \,.\, \gamma) \,.\, e_n^2/2 \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

For an incompressible material ($\kappa = \infty$) both p_k and w become infinite, which means that simple extension is not possible with an incompressible material.

9. The Coefficient of Viscous Traction

We said in Section 2 that the tensile test is used not only for metals but also for such materials as pitch. Of course, in the latter case the rheologics of the test is entirely different. It is generally assumed that if a rod of mild steel is loaded below the yield point it extends elastically and then stays put to the end of times. This, now, is a statement where the epistomological considerations of Section 1 of Chapter IX come in. It has been maintained that even at room temperature mild steel will flow slowly or " creep ". About creep we shall speak in Chapter XIII. But even if mild steel should flow at the same temperature as pitch, there still remains a very definite *quantitative* difference between both. In the case of a rod of pitch it is not necessary to employ exceedingly sensitive instruments to discover the continuous elongation ; it is visible to the naked eye—so much so that Trouton (1906) regarded pitch as a very viscous liquid. This, too, is a gross simplification and we shall therefore consider here *the continuous elongation under constant load* of a rod made *of a Newtonian liquid of very high viscosity* instead of pitch, which, as we shall see in Chapter XIV, manifestly possesses other properties in addition to viscosity.

For such exceedingly viscous liquids, which to all appearances are solids, the two methods for the determination of viscosity, dealt with in Chapter I, are inapplicable. Whilst the viscosity of bitumen has been determined in rotation and tube viscometers, it may reach such values that it would either take tremendous force to produce a flow measurable in short time, or an impracticable long time under ordinary forces. In such cases (and for similar materials) Trouton applied such testing methods as are used for the determination of the elasticity of solids, e.g. the tensile test. He loaded a rod made of the material under test and measured the velocity of its extension. It is clear that the load per unit area of the section (p_n) divided by the velocity of extension ($\dot{e}_n$) must be a measure of the viscosity. Trouton called it the coefficient of viscous traction.

$$\lambda = p_n/\dot{e}_n \qquad \ldots\ldots\ldots (37)$$

10. The Mohr-Circle for Simple Tension

The question now arises : how is this quantity related to the coefficient of viscosity η as defined by (I, f) ? In the latter equation the coefficient of viscosity η is defined by considering a case of simple shear, i.e. a case where neither the strain nor the stress in respect of a given surface element has a normal component, i.e. $e_n = 0$ and $p_n = 0$. Moreover, simple shear is a laminar displacement and therefore not accompanied by any change of volume, which means that we also have $e_v = 0$.

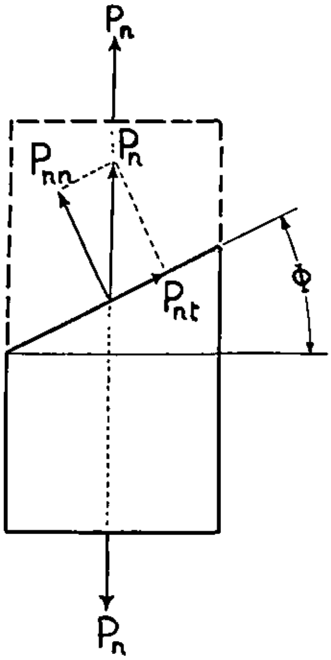

FIG. X, 3.
Stresses in simple tension. P_{nn} normal component of pull P_n. P_{nt} tangential component of pull P_n.

Now, is there any section in the loaded rod inclined against the cross-section at the angle ϕ, where the only stress is a shearing stress ? If we examine Fig. X, 3, it is clear that in every section through the rod, the pull P is equivalent to a tangential force

$$P_{nt} = P_n \sin\phi \quad . \; . \; . \; . \; . \; . \; . \quad (38)$$

together with a normal force

$$P_{nn} = P_n \cos\phi \quad . \; . \; . \; . \; . \; . \; . \quad (39)$$

Let the cross-section of the rod be unity, then the area of the inclined section is

$$A_\phi = 1/\cos\phi \quad . \; . \; . \; . \; . \; . \; . \quad (40)$$

and the tractions in the inclined section

$$\left.\begin{aligned} p_{nt} &= P_{nt}/A_\phi = P_n \sin\phi \cos\phi = \frac{p_n}{2}\sin 2\phi \\ p_{nn} &= P_{nn}/A_\phi = p_n \cos^2\phi = \frac{p_n}{2}(1 + \cos 2\phi) \end{aligned}\right\} \quad . \; . \; . \quad (41)$$

Therefore, when ϕ varies from zero to 90°, the normal component of the traction acting upon the inclined section increases from zero to p_n, while the tangential component goes from zero to zero, passing through a maximum for $\phi = 45°$

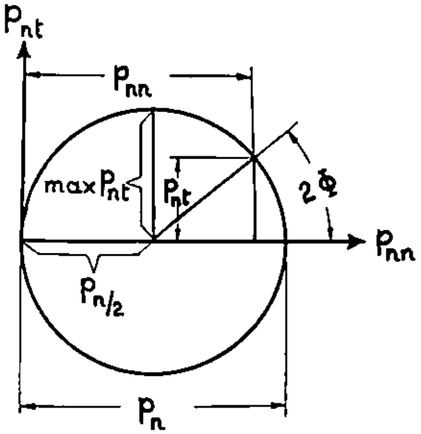

FIG. X, 4. Mohr's Circle for simple tension. ϕ slope of inclined section.

when *max* $p_{nt} = p_n/2$, while in that section p_{nn} also $= p_n/2$. These relations can be pictured graphically in a co-ordinate system where the p_{nn} are abscissas and p_{nt} ordinates. As can be seen from Fig. X, 4, comparing (41), all points representing connected values of p_{nn} and p_{nt} lie on a circle.

This is a special case of the so-called Mohr circle (compare *Ten Lectures*, pp. 123–128).* The reply to the question, whether in any section the only stress is a shearing stress, is therefore in the negative.

11. Trouton's Derivation of the Relation between λ and η

The reply being in the negative, we cannot directly apply (I, f). However, have we forgotten our first rheological axiom that, in order to examine rheological properties, we have to consider distortions ? Let us correct this oversight.

This is usually done as follows, the method apparently being due to Lord Kelvin. Let us resolve the longitudinal tensile stress p_n into three superposed tensile stresses, each equal to $p_n/3$. The strains produced will not be affected by this procedure. Also let each side face be subjected to two opposite normal stresses, each equal to $p_n/3$; these stresses will not produce any strain, and thus original strains remain unchanged. We can now group the stresses applied to the prism in any way we please, and the resultant strains must be identical with those produced by tensile stresses applied to the end faces. Each side face is subjected to a tensile stress $p_n/3$, indicated in Fig. X, 5 by a black arrow; grouping these stresses with the component tensile stresses $p_n/3$ applied to the end faces, we obtain a uniform dilatational stress $p_n/3$. We then group the stresses indicated by white arrows and those indicated by alternatively black and white arrows together.†

As can be seen from the Inset 1 of Fig. X, 5, these stresses—black and alternate arrows separately—are equivalent to shearing stresses equal to any of them (and therefore $= p_n/3$) and making angles of 45° with the length of the prism (compare Inset 2). Therefore, if a pyramid as shown in Inset 3 is cut out of the prism, it is in equilibrium under the action of the forces shown, with the traction p_n requiring the superposition of a hydrostatic tension p_n. In his calculation of the coefficient

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* This gives us the opportunity to explain, with an example, the difference between "traction" and "stress". We have used both indiscriminately, but the stress is represented by the *complete circle*, which contains *all tractions* (p_{nn}, p_{nt}), while the tractions are represented by *points* on the circle. This shows that stress is an entity of a higher order than traction.

† The reader will recognise that this is a geometrical derivation of what we found analytically in Section 5.

of viscous traction, Trouton followed this method as is evident from the quotation: "The tractional force applied to a rod may be resolved, as is usual in questions of elasticity, into two

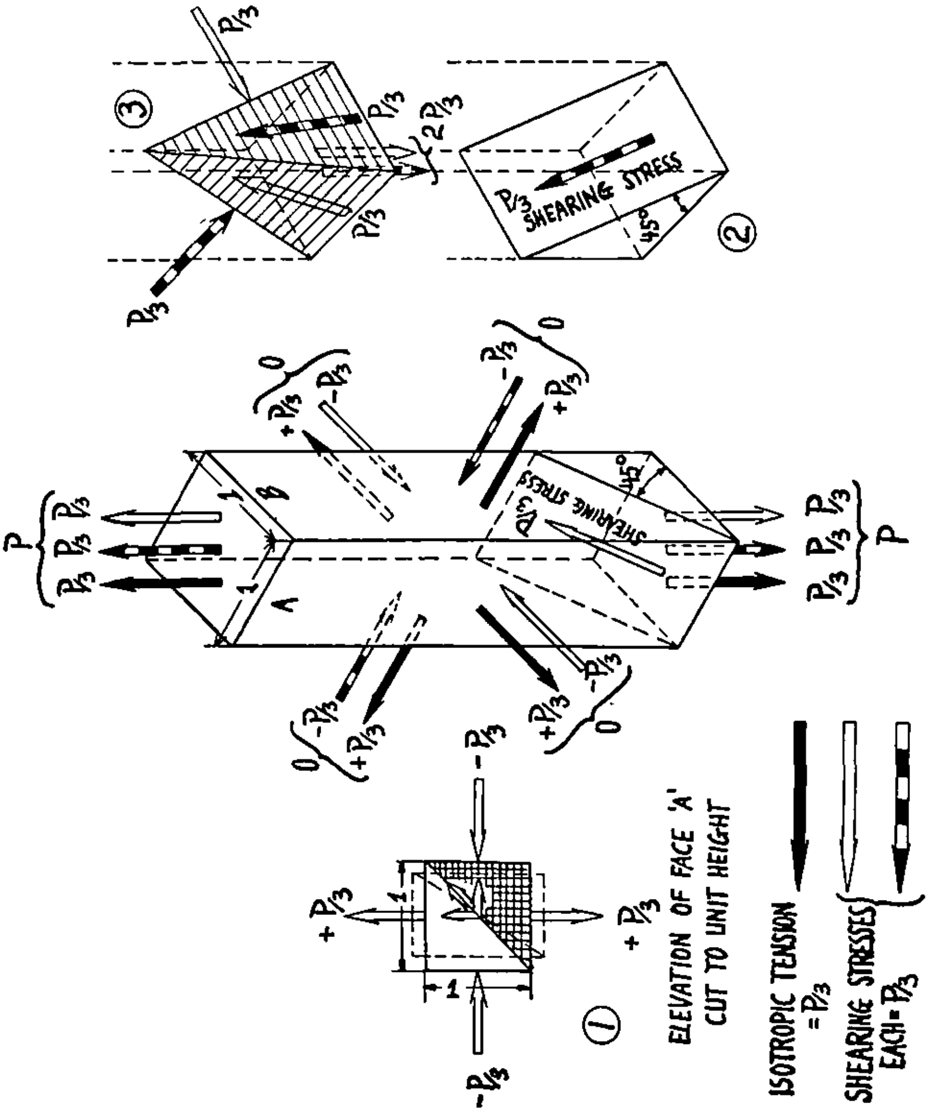

Fig. X, 5. Simple tension of a prism resolved into states of isotropic tension and shear.

equal shears (= shearing stresses or tractions, M.R.) which are situated at right angles to each other and at 45° to the direction of traction, along with a uniform force of dilatation.

The value of either shearing stress and also of the dilatation stress (hydrostatic tension) is in each case one-third of that of the tractive stress (p_n)."

It should be noted that Trouton is wrong in stating that the two shears are " at right angles to each other ". Their *horizontal projections* are at right angles to each other, but not they themselves ; because the planes in which the shears act make

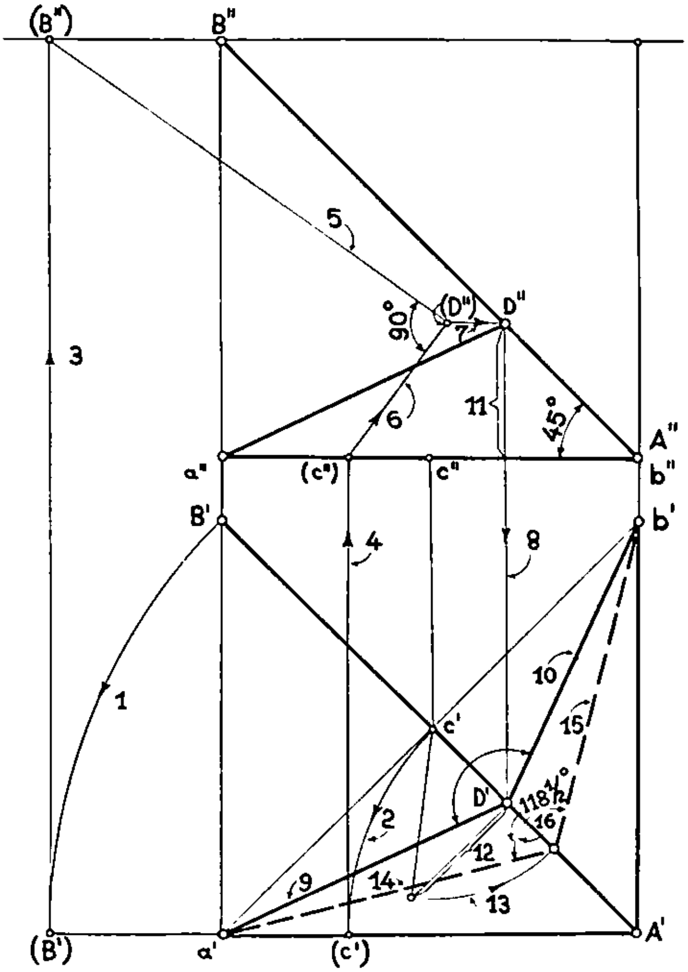

FIG. X, 6 Graphical determination of angle between planes of shear ABa and ABb.
Arrows and figures 1 to 16 indicate procedure of construction.

an angle which is necessarily greater than 90°.* Trouton continues: "In the first instance on the application of the tractive force, the resolved effects produced corresponding to these resolved stresses, will consist of a dilatation and of shearing strain. It can only be to the flow resulting from the latter that the *continued* elongation of the rod is due. Nothing similar can take place in the case of the stress of dilatation, which can only have an initial effect." I.e. should the material be compressible, and this it will generally be, then hydrostatic tension will only change its density immediately after the application of the traction and this will be all hydrostatic tension can produce: it will have no influence upon the flow. "The continued application of each shear will produce a corresponding flow given in each case by $p_t = \eta \dot{e}_t$, where p_t is the shearing stress, η the coefficient of viscosity, and $\dot{e}_t$ the rate of change of direction of any line in the material in the plane of the shear, as it passes through the direction normal to the shearing stress" (see Inset 1 in Fig. X, 5). This, however, involves two assumptions, which have not been expressly stated: Firstly the assumption that added hydrostatic pressure or tension does not effect the value of the coefficient of viscosity. This is only approximately correct. Secondly, it should be noted that (I, f) defines η for the case of *one simple* shear, while here we have two shears superposed. But what is worse, these two shears are *not* at right angles to each other, as was in Trouton's mind when advancing his argument. Trouton finishes as follows: "The resulting flow in the direction of the axis is obtained by adding the resolved components of the two flows in that direction; so that resolving the two effects, adding the components, and reducing the axial alongation to that per unit length, we find that $\dot{e}_n = \dot{e}_t$. Since $p_t = \eta \dot{e}_t$ and $p_t = p_n/3$, where p_n is the tractive force per square centimeter, we get $\eta = \lambda/3$, so that the coefficient of viscosity is equal to one-third of the coefficient of viscous traction."

This can be seen as follows:—

We consider a square with sides of unit length, subjected to simple shear. The square shown in Fig. X, 7, in dotted lines

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* It is approximately = 118⅓°. For those of my readers who studied *Descriptive Geometry* and liked it, I have shown in Fig. X, 6, how the angle can be determined on that method.

becomes a rhomboid, as shown in full lines. If the displacement $BB' = CC'$ is very small, the originally right angle between the diagonals AC and BD, which after strain are AC' and $B'D$, remains a right angle. AC, however, is extended to AC' and BD contracted to $B'D$. Simple shear is therefore equivalent to an extension and contraction without shear in two directions which are at right angles to each other and inclined by 45° against the direction of shear. As can be seen $BB'' = BB'/\sqrt{2}$. On the other hand DB'' is equal to DB' with a good enough approximation.

Furthermore, since $BD = \sqrt{2}$, we have $(DB - DB')/DB = (DB\text{-}DB'')/DB = BB''/DB = \dfrac{BB'}{\sqrt{2}}/\sqrt{2} = BB'/2$. Similarly it is found that $(AC' - AC)/AC$ is equal to $CC'/2$. As AB is unity, $BB' = CC' = e_t$ and a *simple shear* e_t *is equivalent to an extension* $e_t/2$ *combined with a contraction* $e_t/2$ *in directions which are mutually perpendicular, while either is inclined at an angle of 45° to the direction of the shearing displacement.*

Therefore, comparing Fig. X, 7, we find that from the action of

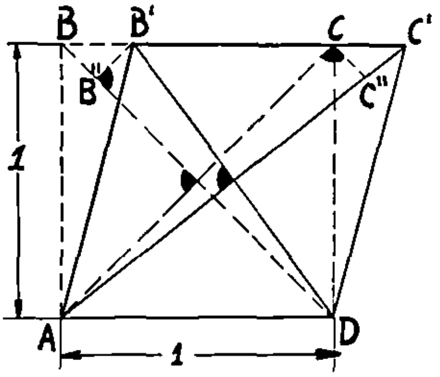

FIG. X, 7. Simple shear of square ABCD into rhomboid A, B′, C′, D.
◀ indicates right angle (exactly or approximately.)

the white arrows the prism will extend by $e_t/2$ and from the action of the alternately black and white arrows likewise by $e_t/2$. The total extension is therefore twice $e_t/2$ or $= e_t$. Now $\dot{e}_t$ can be calculated from (I, f) and is equal to p_t/η. However, as $p_t = p_n/3$, we have $\dot{e}_t = p/3\eta$ and therefore also $\dot{e}_n = p_n/3\eta$.

From the definition of the coefficient of viscous traction (1) on the other hand, $\dot{e}_n = p_n/\lambda$. Comparing the two last equations, we obtain

$$\lambda = 3\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

12. The General Relation between λ and η

We have reproduced Trouton's proof of the relation between the coefficient of viscous traction and the coefficient of viscosity because most rheologists have followed the same lines. It must, however, be said that at the same time many have felt some uneasiness. The subject has its difficulties, and I have therefore dealt with it by applying exact methods of tensor analysis, reducing assumptions to a minimum [39′]. The result is that Trouton's relation (42) is not generally correct, but perhaps correct in most practical cases. To indicate this, we shall write Trouton's special λ with an asterisk, so λ^*.

I shall here give the mathematical derivation in an elementary form. This will give me the opportunity to correct a mistake of mine in *Ten Lectures*, p. 49.

In Trouton's experiment we have to distinguish two stages. The stress due to the pull p_n has, in accordance with (17), an isotropic component $p_m = p_n/3$. When the pull is applied there will be an initial stage which starts with an accelerated and ends with a retarded movement of the particles with—in general—pulsations in between. During this initial stage the material expands, the measure of the cubical dilatation at every movement being e_v. This cubical dilatation produces an elastic isotropic reaction $= \kappa e_v$. It is accompanied by a viscous resistance $= \eta \dot{e}_v$ due to volume viscosity. When e_v has so much increased that $\kappa e_v = p_n/3$, the elastic isotropic reaction balances the isotropic component of the pull p_n. The cubical dilatation then ceases to increase and $\dot{e}_v$ therefore vanishes. Then the second stage sets in, in which the movement is steady.

Now let us assume that the rate of deformation of the prism or the velocities of flow of its particles are so small that their kinetic energy may be neglected; in which case there will be no pulsations but only an aperiodic movement leading to the second stage, when the movement becomes steady.

For the isotropic component of the traction we have accordingly (compare (IX, a))

$$p_n/3 = \kappa e_v + \eta_v \dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

For viscous distortion, because of the perfect analogy of (I, d) and (I, f), there are equations in force analogous to those of elastic distortion (30′), viz.

$$p_{on} = 2\eta\dot{e}_{on} \qquad \ldots\ldots\ldots \quad (44)$$

In our case (compare (17))

$$p_{ok} = 2p_n/3 \qquad \ldots\ldots\ldots \quad (45)$$

while (compare (10) and (11))

$$\dot{e}_{ok} = \dot{e}_k - \dot{e}_v/3 = \dot{e}_n - \dot{e}_v/3 \quad \ldots\ldots \quad (46)$$

Therefore

$$p_n/3 = \eta(\dot{e}_n - \dot{e}_v/3) \quad \ldots\ldots\ldots \quad (47)$$

and

$$\dot{e}_v = 3\dot{e}_n - p_n/\eta \quad \ldots\ldots\ldots \quad (48)$$

Introducing this expression for $\dot{e}_v$ into (43) we find

$$p_n/3 = \kappa e_v + \eta_v(3\dot{e}_n - p_n/\eta) \quad \ldots\ldots \quad (49)$$

or

$$\dot{e}_n = (p_n/3 - \kappa e_v)/3\eta_v + p_n/3\eta \quad \ldots \quad (50)$$

The coefficient of viscous traction λ is defined by (37) in accordance with which we find

$$\lambda^{-1} = \dot{e}_n/p_n = (1 - 3\kappa e_v/p_n)/9\eta_v + 1/3\eta \quad . \; . \quad (51)$$

or

$$\lambda = \left(9\eta\frac{\eta_v}{1 - 3\kappa e_v/p_n}\right) \Big/ \left(\eta + \frac{3\eta_v}{1 - 3\kappa e_v/p_n}\right) \quad . \; . \quad (52)$$

Equation (52) is the viscous analogy to the first of (27), viz. $\epsilon = \dfrac{9\gamma_k}{\gamma + 3\kappa}$. The analogy is, however, *not exact*. While $\epsilon = 3\gamma$ for $k = \infty$ and *only* in this case, the situation is very different in the case of λ.

In interpreting (52) one should keep in mind that $1 - 3\kappa e_v/p_n$ is either >0 in the first stage of Trouton's experiment, or $= 0$ in the second stage. For the volume viscosity we have $0 \gtreqless \eta_v \leqq \infty$. It is clear that the volume viscosity cannot be negative as "otherwise the more alternate expansion and compression, alike in all directions, of a fluid, instead of demanding the exertion of work upon it, would cause it to give work out" (Stokes). As can be seen, if η_v vanishes, λ

also vanishes in the first stage. On the other hand, $\lambda/\eta = 3$ in both stages if $\eta_v = \infty$ and also in the second stage, when $1 - 3\kappa e_v/p_n = 0$, whatever the magnitude of η_v. Because η_v cannot be negative, λ/η cannot exceed the value 3. Therefore

$$0 \overset{=}{<} \lambda \leqq 3\eta \qquad . \; . \; . \; . \; . \; . \quad (55)$$

with $\lambda^* = 3\eta$ in the second stage only.

13. The Stokes Liquid

These results are of interest in connection with the rheologics of the classical viscous liquid or what we called the Stokes Liquid. This was derived by Stokes by assuming $\eta_v = 0$, but in the first stage of the Trouton experiment a vanishing volume viscosity would mean $\lambda = 0$ or vanishing viscous resistance against extension of a liquid cylinder *no matter how high the ordinary viscosity* η *of the liquid*, a result at variance with our ideas of viscous flow. Tisza [40] has recently drawn attention to the following quotation from Stokes: " Of course we may at once put $\eta_v = 0$ if we assume that in the case of a uniform motion of dilatation the pressure at any instant depends only on the actual density and temperature at that instant and not at the rate at which the former changes with the time. In most cases in which it would be interesting to apply the theory of friction of fluids, the density of the fluid is either constant or may without sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same and in the third nearly the same whether η_v were equal to zero or not. Consequently, if theory and experiments should in such cases agree, the experiments must not be regarded as confirming that part of the theory which relates to supposing η_v to be equal to zero." Stokes does not seem to have realised the consequences which his assumption of vanishing volume viscosity carries in respect of the viscous resistance in simple tension, and he was definitely mistaken in equating the influence of either $\dot{e}_v = 0$ or $\eta_v = 0$ on experimental results. Examination of (52) shows that the same result follows from either $\dot{e}_v = 0$ or $\eta_v = \infty$ and *not* $\eta_v = 0$. I was misled by Stokes to make the same mis-statement in my *Ten Lectures*, p. 49, lines 17–20. If we assume with Stokes vanishing volume viscosity, the material would in the first stage expand purely elastically and instantaneously in no time as is natural in the absence of the damping influence of a volume viscosity. If, on the other hand, η_v were so great that it may be put $= \infty$, the expansion would be infinitely delayed and the second stage never reached. Tisza deduces from observations on supersonic adsorption in certain liquids a value of $\eta_v/\eta = 2{,}000$. This, when introduced in (54), is practically infinite. It is true, as Trouton found, that in the second stage $\lambda/\eta = 3$; but it cannot be assumed *a priori that in an actual experiment with some real material that second stage is reached during the experiment.* This depends entirely upon the magnitude of η_v.

Assuming that the second stage has in fact been reached in his experiments, Trouton calculated the values for λ^* shown in the following table :—

Material	Pitch	Mixture of pitch and		Shoemaker's Wax
		Tar	More tar	
$\lambda^* \times 10^{-10}$ in poises	3·6 – 4·3	1·29	0·67	0·0059

14. Summary

In every deformed body there exists at every point a triple of normal axes i, j, k which were at right angles, too, before deformation. They are called *principal axes of deformation* (or strain, as the case may be). As the right angles between them do not change, they are not sheared and the deformation consists of positive or negative extensions in the direction of the principal axes or normal to the planes containing them. They are accordingly *normal* deformations (e_n) called *principal deformations* and denoted by e_i, e_j, e_k. In an *isotropic* material these deformations are connected with *normal stresses* (p_n) called *principal stresses* and denoted by p_i, p_j, p_k.

If the directions of i, j, k in space are the same before and after deformation, the latter is called *pure* deformation.

If the directions i, j, k and the magnitudes of e_i, e_j, e_k are the same throughout the body, the deformation is said to be *homogeneous*.

In a prism with edges a, b, c oriented in the directions i, j, k respectively with lengths of edges a_o, b_o, c_o before deformation, the principal deformations are *extensions*

$$e_i = \ln(a/a_o),\ e_j = \ln(b/b_o),\ e_k = \ln(c/c_o) \quad . \quad . \quad (\text{X, a})$$

and the cubical dilatation as defined by (IX, 6)

$$e_v = e_i + e_j + e_k \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{X, b})$$

The principal deformations

$$e_{oi} = e_i - e_v/3,\ e_{oj} = e_j - e_v/3,\ e_{ok} = e_k - e_v/3 \quad . \quad (\text{X, c})$$

for which $e_{oi} + e_{oj} + e_{ok}$ vanishes, constitute a *distortion.*

In accordance with the first axiom of rheology (compare Section 3 of Chapter I) the characteristic rheological equation of a material connects e_{oi}, e_{oj}, e_{ok}, with stresses p_{oi}, p_{oj}, p_{ok} which are to be derived from p_i, p_j, p_k through

$$p_{oi} = p_i - p,\ p_{oj} = p_j - p,\ p_{ok} = p_k - p \quad . \quad (\text{X, d})$$

where the isotropic stress p is the mean of the principal stresses

$$p = (p_i + p_j + p_k)/3 \quad . \quad . \quad . \quad . \quad (\text{X, e})$$

In the case of simple tension, p_n,

$$p = p_n/3\ ;\ p_{oi} = p_{oj} = -\,p_n/3\ ;\ p_{ok} = 2p_n/3 \quad . \quad (\text{X, f})$$

The isotropic stress p is connected with the cubical dilatation e_v by means of (IX, a), which is the isotropic rheological equation valid for every kind of material.

For the distortional part in the case of the Hooke solid

$$p_{on} = 2\gamma e_{on} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{X, g})$$

In the case of the Newtonian liquid

$$p_{on} = 2\eta \dot{e}_{on} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{X, h})$$

When a prismatic bar of length l_o is acted upon by a pull $(+\,P_n)$ or a push $(-\,P_n)$ in the direction of its axis (direction k) they produce a simple one-dimensional stress p_n. This causes a three-dimensional pure deformation. In a Hooke-solid this consists of a longitudinal extension without shear

$$e_n = \ln\,(l/l_o) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{X, i})$$

or for small extensions

$$e_n = \Delta l/l_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{X, i}')$$

which is connected with p_n through

$$e_n = p_n/\epsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{X, j})$$

and lateral contractions of the magnitude

$$e_c = -\sigma e_n \qquad \ldots\ldots \qquad \text{(X, k)}$$

In sections inclined by 45° against the axis there is a maximum shearing strain

$$e_t = e_n(1 + \sigma) \qquad \ldots\ldots \qquad \text{(X, l)}$$

The " derived " elastic coefficients ϵ and σ can be calculated from the " fundamental " coefficients κ and γ in accordance with

$$\epsilon = 9\kappa\gamma/(3\kappa + \gamma) \qquad \ldots\ldots \qquad \text{(X, m)}$$

$$\sigma = (3\kappa - 2\gamma)/2(3\kappa + \gamma) \qquad \ldots \qquad \text{(X, n)}$$

When a viscous liquid is subjected to simple normal stress the analogy to ϵ is Trouton's coefficient of viscous traction

$$\lambda = p_n/\dot{e}_n \qquad \ldots\ldots \qquad \text{(X, o)}$$

and the analogy to (X, 1)

$$\lambda = \frac{9\eta \dfrac{\eta_v}{1 - 3\kappa e_v/p_n}}{\eta + \dfrac{3\eta_v}{1 - 3\kappa e_v/p_n}} \qquad \ldots\ldots \qquad \text{(X, p)}$$

When $\dot{e}_v$ vanishes and the movement becomes steady, $e_v = p_n/3\kappa$, and $\lambda^* = 3\eta$. This is also so whenever $e_v = 0$.

A Newtonian liquid in the special condition when $\dot{e}_v$ vanishes is called a Stokes Liquid.

Simple extension is a one-dimensional extension which generally is connected with a three-dimensional stress. In the case of simple shear e_t, the principal deformations are $e_i = e_j = e_t/2$ while e_k normal to the plane of shear vanishes. The axes i and j are inclined at angles of 45° to the direction of the shearing displacement.

The strain-work per unit volume in simple tension is

$$w_n = \int p_n \, de_n = \int \epsilon e_n \, de_n \qquad \ldots\ldots \qquad \text{(X, q)}$$

which for the simple Hooke body when ϵ is constant

$$w_n = \epsilon e_n^2/2 = p_n^2/2\epsilon \qquad \ldots\ldots \qquad \text{(X, r)}$$

CHAPTER XI

WORK-HARDENING

1. Technical and Rheological Test Curves

We have dealt in the preceding Chapter with the tensile test and have mentioned that this is the predominant test for metals. Among these the most important from the economical point of view is, without doubt, mild steel. Let us therefore look more closely into the tensile test of mild steel.

A record of the observations consists in plotting the load (P) against the elongation (Δl) or *vice versâ*. In Section 5 of Chapter V we have called such a curve, where directly measured quantities are plotted, a Technical Curve. It forms the basis of the theoretical analysis; the latter aims at *expressing the rheological properties of the material under test by* (i) *a rheological equation and* (ii) *the numerical values of its rheological coefficients.* For accomplishing this task there are certain difficulties to be overcome. First of all, there is a difference depending upon which one of the two variables P and Δl is taken as the independent and which one as the dependent variable. If we gradually increase the load, the rod of mild steel behaves first *elastically*, more or less as a Hooke solid; but with a certain load the yield point is reached, when the rod starts to flow *plastically* at a more or less *constant load.* If, disregarding this, we continue to increase the load, there will be no equilibrium, the material will flow at an accelerated rate and soon break. This difficulty does not arise if the *elongation* is taken as independent variable. Here the load first increases; then remains constant; then again increases; then, after what is called the " necking " of the piece, decreases, until the piece breaks. This is the arrangement in most tensile testing machines for metals, where one end of the rod is fixed, the other travels at constant rate; and where Δl is therefore the independent variable, and the co-related P is observed.

Another method of taking account of the difficulty consists in considering the traction p and not the load P as the independent variable. With increasing elongation the cross-sectional area gradually decreases. If we let the traction gradually increase, this will often mean that the load must be decreased. If the elongation continues under *constant* traction, as it is supposed in a St. Venant body or in viscous traction, the load must be gradually decreased in the same ratio as the cross-sectional area decreases. There exist a number of devices to effect this automatically. One, by Andrade for pull, is described in *Ten Lectures*, p. 83. For another one, also for pull, compare Andrade and Chalmers (1932). Two methods for push, keeping compression stresses constant, are described by Scott Blair in *Survey*, pp. 120–122. Compare also Caffyn [41].

A curve in which stress is plotted against deformation in such a manner that the curve is independent of the dimensions of the apparatus may be called, in contradistinction to the technical curve, a Rheological Test-Curve. A consistency curve, in which the stress and deformation are consistency variables, is such a rheological test curve (provided there is no wall-effect). A rheological test curve will not necessarily be independent of the *kind* of apparatus. If the variables are such that this is the case, the curve may be called the Fundamental Rheological Curve of the material. This is a *graphical representation of the rheological equation of the material.* The upper part of Fig. III, 3, a, shows the fundamental rheological curve of the Bingham body, while Fig. III, 4, shows two of its rheological test curves.

2. Strain-hardening

I have mentioned the necking of a mild steel bar after prolonged plastic extension. When necking starts, the rod ceases to be cylindrical or prismatic and obtains a shape which makes it less suitable as a test-piece ; stresses and deformations being no longer more or less evenly distributed over the length of the test-piece, but now concentrated at the section of striction where the cross-sectional area is a minimum.

If one comes to think of it, it is rather strange that the cylindrical shape can be maintained in *any* tensile test, either throughout the test or up to a certain elongation. The material tested (whatever it may be) cannot be of uniform quality over the length of the bar ; *its quality must at some section be lowest.*

At the same time it will not be absolutely cylindrical, but possess somewhere a minimum cross-sectional area. Therefore when loaded, it will first give way at one or the other of those sections. If so, the cross-sectional area will there be reduced and become markedly less than along the rest of the test piece. This will increase the stress at that section and worsen the position.* If this does not actually happen there must be some counteracting influence. The counteracting influence is *strain hardening.*† Strain Hardening *is the rise of the yield point of a material with increased deformation.*‡ When the material first gives way at the weakest section it is there strained. When the strain brings with it strain-hardening, the material at that section may become harder than throughout the rest of the length and, even with increased local stress, deformation will not continue at the section but start somewhere else, i.e. at the next worst place. By attacking one section after the other, deformation is equally distributed over the length of the rod, the shape of which is finally again cylindrical. Therefore, *if a bar of any material remains cylindrical (or prismatic) in a tensile test, this is an indication that the material possesses the property of strain-hardening.*

3. The " Stress-Strain " Curve of Mild Steel

Let us go back to the technical curve for the tensile test of mild steel. If we plot the load P against the elongation Δl, this gives from the start a different curve for every different diameter of bar of the same material. Therefore, denoting by A_0 the original cross-sectional area and by l_0 the original length of the test-piece, the " stress " P/A is plotted against the " strain " $\Delta l/l_0$ in what is called § the " stress-

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* The tensile test, accordingly, proceeds through states of *unstable* equilibrium; in contradistinction to the compression test where the equilibrium is stable, any local deformation resulting in a reduction of stress.

† This is the usual designation. It should, however, be noted that after the first yield point has been exceeded we should (in accordance with our terminology) not speak of " strain " but of " deformation ". We shall later propose to replace " strain hardening " by *work hardening*, a term which is more appropriate.

‡ This is the customary case. Schofield and Scott Blair [42] have observed strain hardening in flour dough where the *viscosity increases with strain.*

§ The terms " nominal stress ", " nominal strain " and " nominal stress-strain curve " are also used, to distinguish them from the " true " stress, strain and stress-strain curve, about which more will presently be said.

strain curve " of mild steel. While this curve is more or less independent of the dimensions of the test-piece, it can still not be regarded as a rheological test curve, the designation of " stress-strain curve " being, as we shall presently see, not fully justified.

As we have said, the elongation Δl is at first proportional to the load, until the yield point (a in Fig. XI, 1) is reached. This is the elastic or Hooke-range. After that the elongation grows quickly at approximately constant load. This is the plastic range. When the point c is reached the load must again be increased, but at a much lower rate than in the Hooke range. This is the strain-hardening range. When the point b is reached the load must be decreased ; the rod starts " necking " until it finally breaks at point e. Plotting P/A_0 against $\Delta l/l_0$.

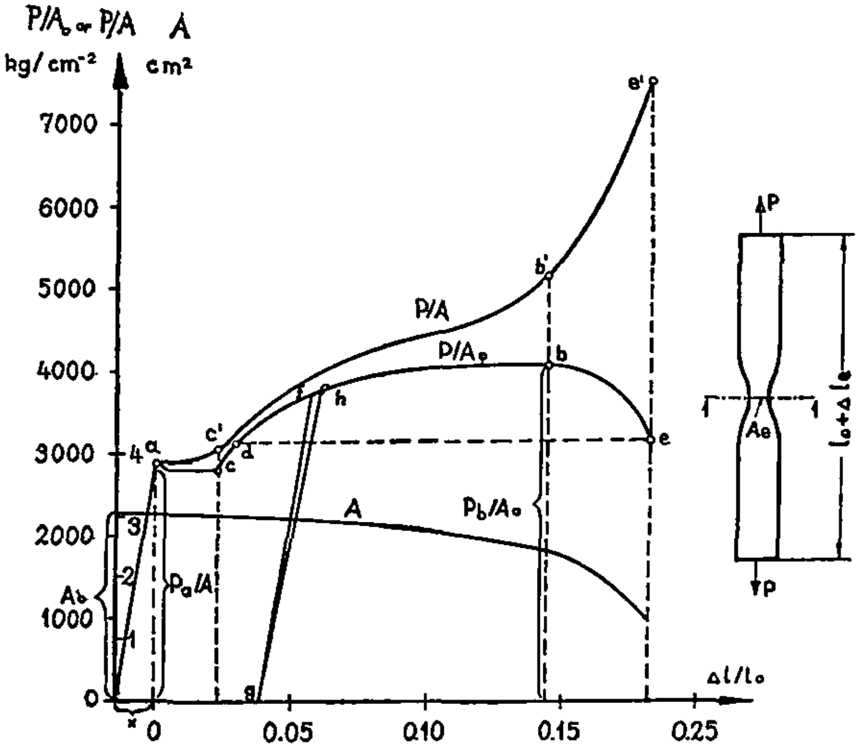

FIG. XI, 1. Technical " stress-strain curve " of mild steel. " a " first yield point. " h " yield point after strain-hardening. A cross-sectional area. P load. Δl elongation. (The elastic strain has been exaggerated.)

results in the curve *o–a–c–d–b–e* of Fig. 1. However, it is clear that this does not tell us much about the rheological properties of the material and even gives a misleading picture, because it is not obvious why, if the test-piece actually breaks at the load P_e, it can sustain the same load at an earlier point *d* without breaking ; and later on at point *b* an even higher load. The explanation is, of course, that with increasing elongation the cross-sectional area is gradually reduced and, therefore, the stress increases even for the same load. If we plot the stress P/A against $\Delta l/l_0$, the curve *o–a–c′–b′–e′* results and this curve looks more reasonable in that there is no fall of stress with increasing strain. However, the strain is here understood as the ratio of elongation to the *total* length of the test-piece, while after necking elongation takes place at the section of striction and should be related to the length of the latter. For this reason the denotion " stress-strain curve " is not applicable after the point " *b* ".

4. The St. Venant Range

Our description of what we called in the preceding section the plastic range, but what in accordance with our nomenclature we shall call the Simple St. Venant Range, was rather superficial. When the bar first yields, an immediate drop of the load to a lower level can be observed and yielding continues at that lower level. To describe this phenomenon, Bach [43] coined the terms " upper " and " lower " yield point. It has often been maintained that the phenomenon is not due to intrinsic rheological properties of the material, but to a " machine-effect " resulting from the combined mechanics of test-piece-cum-testing-machine. It has been stated that the rise of the yield point is caused by the inertia of the machine and is due to the sudden stretching of the specimen and the inability of the machine to react satisfactorily to this quick change. This question of the influence of the mechanical behaviour of the testing machine upon the observational results is certainly of importance. When the reaction of the apparatus is of the same order of magnitude as the phenomenon to be observed, these can hardly be separated. This has lead in the realm of the submicroscopic, where the observed is an electron and the means of observation a photon, to Heisenberg's uncertainty principle. If, in the realm of the macroscopic, the testing machine is a steel structure and the test-piece a steel rod introduced into that structure, conditions become similar. We cannot go into the problem here. It has been dealt with in great detail in a book by W. Spath [44]. In order to find out about the reality of the upper yield point, Edwards, Phillips and Liu [45] used a simple arrangement of direct loading by adding weights on a pan attached to the lower end of the test-piece, avoiding mechanical devices for applying loads so that the difficulties from inertia arising from machine bearings, levers and other features of design would be overcome. It was found that, once plastic deformation commenced, it continued at much lower loads than those initially required. The upper yield point could be raised by careful and *very slow loading*.

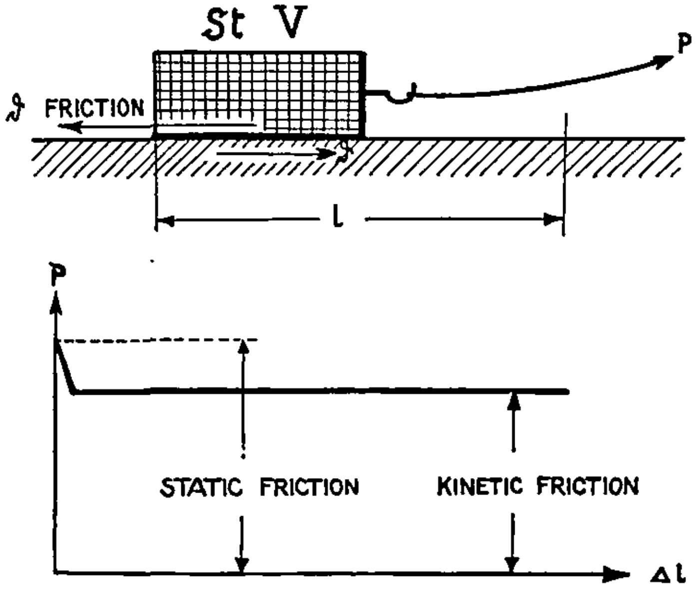

Fig. XI, 2. Model for a St. Venant body.

This suggests for the St. Venant body a model as shown in Fig. XI, 2. A weight rests upon a table top and there is solid friction between both, a friction which is higher at rest and lower when movement takes place. A string is connected to the weight and a gradually increasing pull P applied to its free end.* First the string is slack, then it is tightened.

As long as the pull is smaller than static friction, the weight remains at rest. When P reaches and slightly exceeds static friction, the weight starts to move, and in order to maintain equilibrium with constant velocity of movement, P must be reduced to kinetic friction. The resulting load-movement diagram is shown in the lower half of Fig. XI, 2. If we introduce the stress p_n for the load P and the strain e_n for the movement Δl, we see that the model qualitatively reproduces the behaviour of the test-piece during the St. Venant range. To include the Hooke range, we introduce an elastic spring as a model for the Hooke solid and replace the string by the spring or couple the StV element with the H element in series, an operation which we indicate symbolically by $StV - H$. In

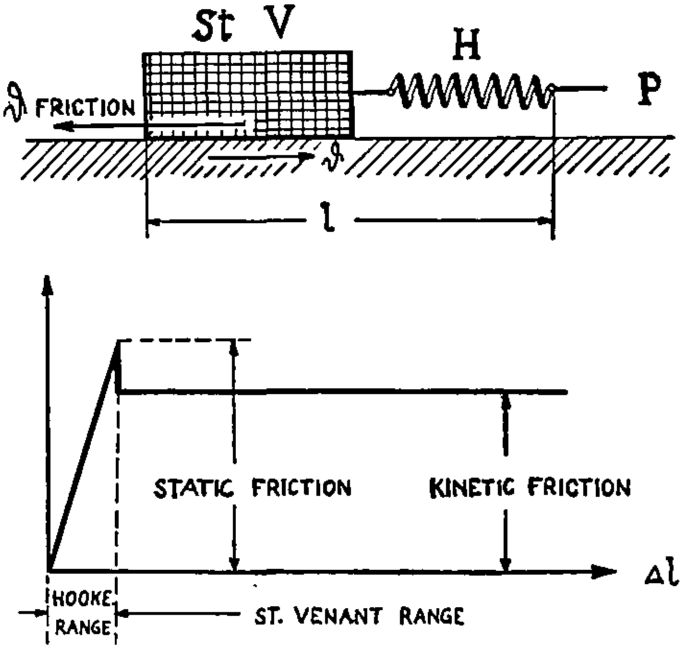

Fig. XI, 3. Model for a combined Hooke-St. Venant body.

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* If the reader has seen Fig. VIII, 1, (a) in *Ten Lectures* or the figure in my paper [46], he will notice that I have now changed the model by introducing the "string" as a structural element. This was suggested to me by Mr. W. Fuchs and has many advantages.

Fig. XI, 3, the model and the load-movement diagram for this arrangement are shown.

5. The Strain-hardening Range

It is not difficult to adapt this model so as to include the strain-hardening range. Fig. XI, 4 shows a model and the

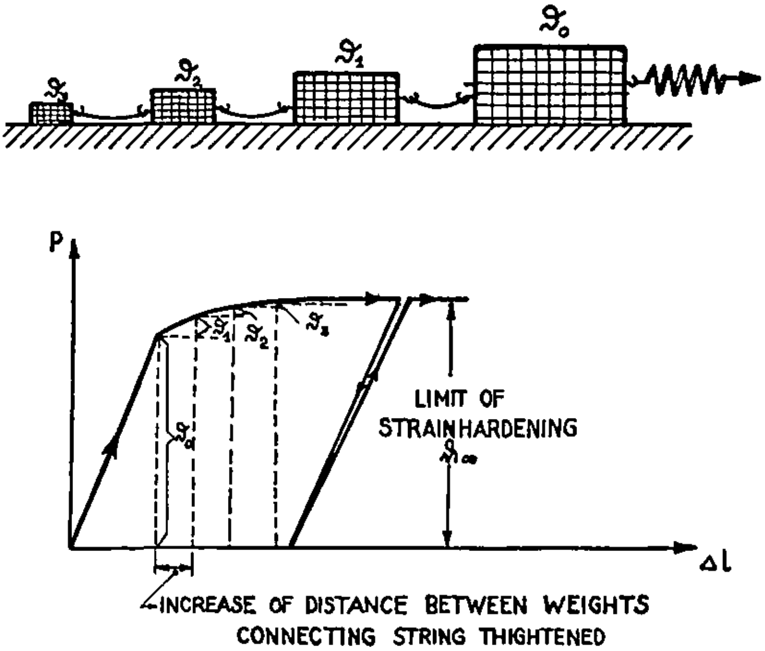

Fig. XI, 4. Model for a generalised St. Venant body.

respective load diagram. The model consists of a number of St. Venant elements connected in series and preceded by an H element. As deformation proceeds, or more strings are tightened, more and more weights are called into action, with the yield point rising. It has been found that in mild steel, as in other materials showing strain-hardening, this does not go on indefinitely as the deformation increases. In the tensile test the process is interrupted by the breakage of the test-piece at about 20 per cent. overall extension or 30 per cent. local extension, but in wire drawing that same small steel rod can be converted into a thin wire by drawing it cold through a number of decreasing holes. By this process its diameter is reduced to a fraction and its length increased by many thousands per cent. The process has a limit only when the wire thins to crystallite dimensions. While, therefore, the deformation is practically without limit, no such parallel unlimited increase in yield point can be observed. The yield point gradually increases with diminishing rate of increase until a Maximum Yield stress, ϑ_∞ is reached. This implies the existence of a Limit of strain-hardening. To this fact there corresponds in the model a limited number of decreasing weights.

6. The Tensile Strength

The models of Figs. XI, 2, 3 and 4 do not include considerations of " strength ". We shall deal with strength in Chapter XVIII. However, as the main object of the tensile test for mild steel is the determination of its " strength " we cannot avoid including a preliminary treatment of strength already in the present Chapter.

The tensile test is carried out in accordance with specifications for which we may take as typical the British Standard Specification (B.S.S.) No. 785—1938 for rolled steel bars and hard-drawn steel wire for concrete reinforcement. That specification asks (Clause 5) for " the tensile breaking strength, yield point (where specified) and elongation ". The terms are not defined, but there can be no question that the " yield point " is the stress P_a/A_o (Fig. XI, 1) and the " elongation " (extension) $\Delta l/l_o$. There is, however, a curious point about the " tensile breaking strength ". The specification says that it " shall conform to the requirements specified in Appendix A, page 18 ", but if we turn to that Appendix no tensile breaking strength is specified, but something which is called Ultimate Tensile Stress. The question naturally arises : are these two one and the same and how is it that they are named differently ?

British terminology is not as definite about the concepts involved as is American. In the United States the stress P_b/A_o is called Ultimate Tensile Strength and the stress P_e/A_o Tensile Breaking Strength. In the well-known English textbook, *The Strength of Materials*, by E. S. Andrews, P_b/A_o is called Commercial Maximum Stress. The Germans call P_b/A_o *Zugfestigkeit*. There can be no doubt that what the British Standard Specification requires is P_b/A_o. In view, however, of the fact that at the Load P_b no *breaking* occurs, the term "*tensile breaking strength*" is not very fortunate, especially if it is not defined.

The purpose of the present discussion is neither philological nor legalistic. There can be no question that in the use of a steel rod or bar as a structural member *in tension*, what governs its usefulness is the magnitude of the load P_b. The *load* which the member is to take is given. It does not help that the steel member can resist *higher stresses* when the point b has been passed (compare Fig. XI, 1), if the load has at the same time to be decreased—because this cannot be done. There are, however, other uses of mild steel and the question then arises: Is the stress P_b/A_0 or, more correctly, P_b/A of *physical* significance? Does it tell us something of the *intrinsic* properties of the steel under test? What is the necking due to? Until the point b is reached the cylindrical rod remains cylindrical, albeit with increased length and reduced cross-section. Why does this not continue down to actual breaking? And why does breakage occur at a certain extension, while in the wire-drawing process that extension can be infinitely enlarged?

We shall presently discuss these questions using the matter provided by experiments which I first undertook in collaboration with W. Fuchs and H. Ilberg at the Laboratory for Testing Materials of the Standards Institution of Palestine at Tel-Aviv [47] and continued in collaboration with A. Freudenthal [48] and which will be described in the following sections.

7. A Close-up of the Tensile Test Piece

A test-piece was shaped as shown in Fig. XI, 5. It was assumed that the yield stress of the steel was about 3,000 kg/cm.^{2} and the test-piece was first loaded with 4,500 kg., giving a stress of 1,460 kg./cm.^{2} After that the rod was unloaded and the length found to be the same within 1/100 part of a mm. This, there-

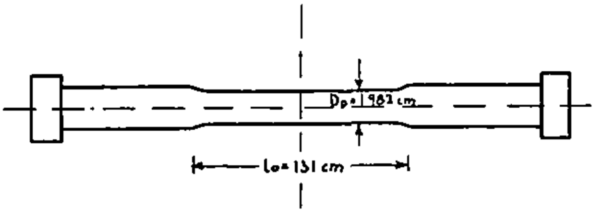

FIG. XI, 5. Original form and dimensions of test piece.

fore, was a purely elastic strain. After that, experiment 2 was undertaken.

In experiment 2, point *a* of Fig. XI, 1 was reached and exceeded at a load of between 8,900 and 9,000 kg., giving a stress of about 2,900 kg./cm.[2] The rod was again unloaded and measured.

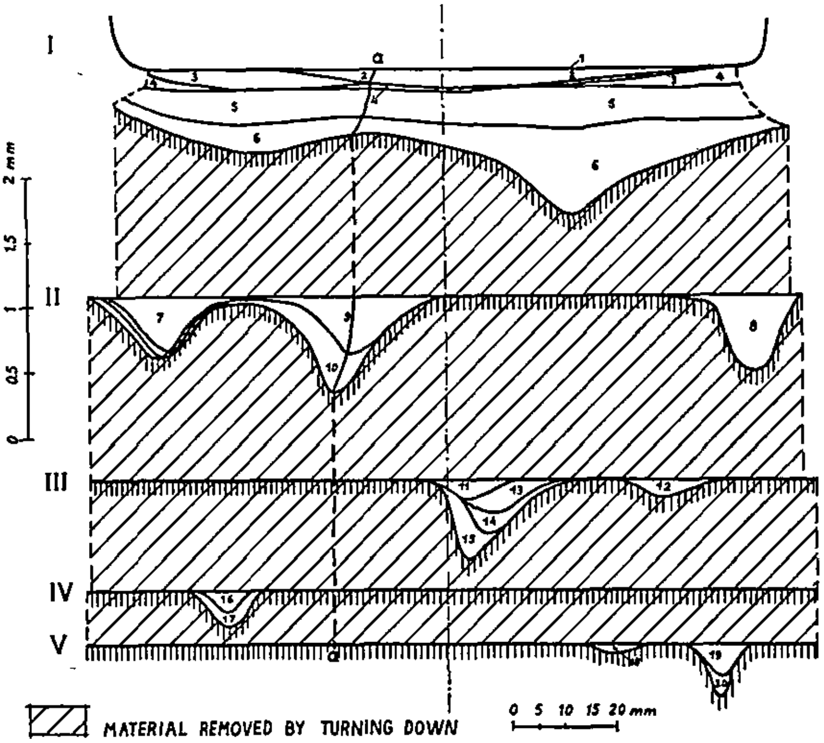

Fig. XI, 6. Lateral contractions of testpiece in linear scale.

The testpiece was turned down several times. I shows the contractions of the original testpiece; II, III, IV of testpieces turned down. The outline of the last shape before turning down has been hatched vertically. Numbers indicate the experiment (as listed in the Table) in which the deformation was effected. For the meaning of "*a*" compare section 11 below.

There was now a permanent elongation of the central piece of 1·65 mm. which gives an extension = elongation/original length of 1·26 per cent. within a hundredth of 1 per cent. Elongations were measured in two longitudinal sections at right angles and differed slightly. They were averaged.

This procedure was continued. The rod was loaded until the yield point was reached, then unloaded, then measured.

There were large intervals of time between experiments; not on purpose, but due to the conditions under which the investigation was undertaken. However, as will be explained later, this brought to better light the phenomenon of "ageing" of steel.

Table I gives the data of the experiments made. With the sixth experiment necking set in, the point *b* in Fig. XI, 1 being reached. The tensile strength, as usually defined, was found to be 4,120 kg./cm.^{2} In the same Table the reduced cross-sectional areas as determined by measurement have also been entered, and the "true" stress calculated accordingly. The "true" tensile strength was 5,200 kg./cm.^{2} but it should be kept in mind that actually no breaking occurred at this stress. There was no point in continuing the experiments in the same way. One knew what was to be expected: the rod would cease to be extended over the whole length, it would thin down at the section of striction where it would locally be extended; finally the rod would break at that place at a certain smaller load but higher stress.

TABLE I. *Data for Rheological Tensile Test Curve of Mild Steel*

	Experiment No.	Date 0=2 6 43 in days	Load P in kg	Stress P/A_0 in kg/cm^2	Elongation Δl in cm	Unit Extension $\Delta l/l_0$ in %	Initial dimensions for series	Smallest cross-sectional area A in cm^2	"True" stress P/A in kg/cm^2
1st Series	1	0	4500	1460	0	0	$D_0=1{\cdot}982$ cm $A_0=3{\cdot}085$ cm^2 $l_0=13{\cdot}1$ cm	3·085	1460
	2	0	8050	2900	0·165	1·26		3·040	2938
	3	46	9010	2920	0·263	2·01		2·986	3030
	4	113	9730	3150	0·379	2·90		2·970	3260
	5	103	12380	4020	1·202	9·18		2·814	4400
	6	193	12720	4730	1·861	14·20		2·436	5200
2nd Series	7	266	10060	4780	0·186	1·26	$D_0=1{\cdot}636$ cm $A_0=2{\cdot}102$cm^2 $l_0=14{\cdot}8$cm	NOTE: Elongation Δl and smallest cross-sectional area A are those attained at the end of the experiment.	
	8	278	10900	5180	0·302	2·65			
	9	323	11020	5240	0·558	3·77			
	10	341	11220	5340	0·761	5·14			
3rd Series	11	356	7760	5400	0·066	0·43	$D_0=1{\cdot}353$ cm $A_0=1{\cdot}438$ cm^2 $l_0=15{\cdot}5$ cm		
	12	371	7900	5490	0·118	0·76			
	13	443	8140	5660	0·220	1·42			
	14	449	7970	5550	0·287	1·85			
	15	471	8030	5590	0·308	1·99			
4th Series	16	502	5850	5710	0·077	0·18	$D_0=1{\cdot}141$ cm $A_0=1{\cdot}024$ cm^2 $l_0=15{\cdot}9$ cm		
	17	512	5850	5710	0·121	0·76			
5th Series	18	543	5150	5870			$D_0=1{\cdot}058$cm $A_0=0{\cdot}878$ cm^2 $l_0=16{\cdot}0$ cm		
	19	553	5100	5810	0·152	0·95			
	20	566	5240	5960	0·218	1·36			
	21	572	5100	5810					
	22	580	5220	5940					

When measuring the length after each experiment, the diameters along the length were also measured to 1/100 mm. In the uppermost part of Fig. XI, 6 the shapes of the rod up to the necking are shown in exaggerated scale. As can be seen, the statement in most textbooks that " until the point *b* is reached the cylindrical rod remains cylindrical " is correct only as far as observation with the naked eye goes. Actually the rod loses its perfect cylindrical shape with the first permanent deformation in experiment 2.

In experiment 3 the section where the diameter had been reduced before to a minimum, did *not* move ; deformation took place where the diameter had not been changed and the stress was therefore the smallest. In experiment 4 also the shallow microscopic neck formed in experiment 2 was not changed ; deformation took place at the ends which had not moved in either experiment 2 or 3. As we said in Section 2, the point where the diameter is most reduced in each experiment wanders along the length of the rod, to find the material which has so far been least strained. Just before necking, in experiment 5, the shape was again fairly cylindrical.

8. The Cause of Necking

We can now understand what causes necking. As can be seen from Fig. XI, 1, the rate of strain-hardening, i.e. the rise in yield point per unit increase of strain, falls off between points *c* and *b*. At the same time, with the reduction of the cross-sectional area, the stress increases and has its maximum at the smallest section. At that section the yield point has also been raised most. *There is accordingly a competition between the increase in yield-point and increase in stress at the smallest section.* As long as the former prevails the deformation wanders away to some other place where the yield point is lower. As soon, however, as the rate of strain-hardening has become so low that the latter prevails, deformation continues at the same place and necking sets in.

This answered the question as to the cause of necking. At the same time it showed in which way the experiments were to be continued. After necking has set in the stresses are not more or less uniformly distributed over the length of the rod, but concentrated at the point of striction. Therefore, before continuing the experiments, the cylindrical form was restored.

This was done by turning down the test-piece from a diameter of 1·982 cm. to 1·636 cm. The experiments were then continued as before.

9. Ageing

Now necking occurred at once, but at an entirely different place from that in experiment 6 (compare Fig. XI, 6). When the

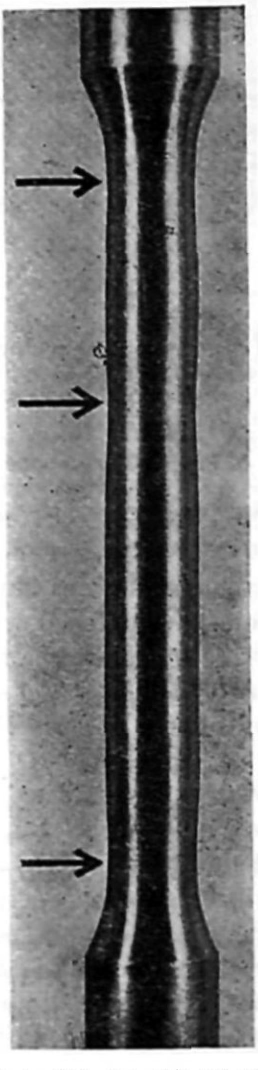

FIG. XI, 7. Test-piece after the 9th experiment, showing three strictures. (Compare Fig. XI, 6.)

test-piece was given a "rest", it further hardened at the place where it had been strained most or, as is said, it "aged". Therefore in the next experiment the deformation did not continue at the same place but occurred elsewhere, etc. In the end we had *three strictions in the one test-piece, a* rather unusual phenomenon (compare Fig. XI, 7). After that, the test-piece was again turned down and this was repeated several times. In Fig. XI, 8, all deformations have been plotted in one figure. As can be seen, the material well "remembers" its previous history. Here also deformation wanders from point to point: from a more to a less strained section, even if the test-piece, after turning down, is perfectly cylindrical.

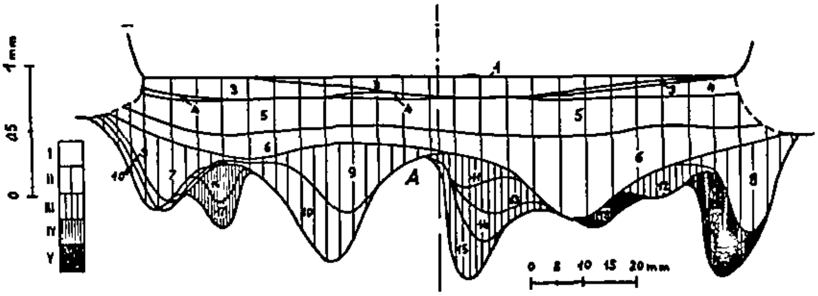

FIG. XI, 8. Superposed lateral contractions of all testpieces. Numbers correspond with number of experiment in Table. Roman numerals refer to the series, of which each represents a new turning-down. Note how the testpiece in the first experiment to which it was subjected in series II (No. 7) remembers, so to speak, where its least contraction had taken place before it was turned down after series I (and similarly in further experiments).

The fifth testpiece has again been turned down and is now perfectly cylindrical. Necking was expected at point A and so happened in experiment 21.

10. Local Deformations

It is clear that the value for Δl has no significance after necking has set in. Before that it gives a good average of the deformation in the different cross-sections. After necking, however, all deformation occurs at *one* place, while the rest of the rod is not deformed. Locally, as we pointed out in Section 3, the deformation will be quite considerable; while its contribution to the increase in length will be very small. The apparent high rate of strain hardening between points b' and e' in Fig. XI, 1 is therefore misleading. Actually the length on the abscissa related to these two points will be much longer in terms of deformation than shown here in terms of $\Delta l/l_o$.

To get a true picture one has to relate the stress at a section to the *local* deformation which manifests itself in the reduction of the diameter. Let us imagine the rod divided into disks of originally equal thickness d_o. The elongation Δl is the sum of the increases of thicknesses of these disks. Every such increase in thickness is accompanied by a decrease in diameter and we may assume (as a first approximation) that in this process the volume of the disk is not changed. Let d be the thickness of the disk after extension, then its contribution to the total axial extension is in, the correct logarithmic measure $e_n = \ln(d/d_o)$. Furthermore, let D_o be the original diameter and D the reduced diameter of the disk, then constancy of volume requires that

$$D^2d = D_o^2d_o \quad \ldots\ldots\ldots \quad (1)$$

or

$$\ln(D^2d/D_o^2d_o) = 2\ln(D/D_o) + \ln(d/d_o) = 0 \quad . \; . \quad (2)$$

from which with

$$\ln(D/D_o) = e_c \quad \ldots\ldots\ldots \quad (3)$$

$$e_n = -2e_c \quad \ldots\ldots\ldots \quad (4)^*$$

It is, therefore, possible to calculate the local axial extension from the local relative contraction. The former is twice as large as the latter.

11. The Rheological Tensile Test Curve

For the rheological test curve we have to plot local stresses $P/\frac{D^2\pi}{4}$ against local strains $e_n = \ln(d/d_o)$. This requires the tracing of the different disks in the consecutive experiments as they change their distance from the axis.

In Fig. XI, 6 I have shown how we can follow the course of one such disk, marked "*a*," and it obviously is a cumbersome process. In Fig. XI, 9 the successive extended lengths of the test-piece have therefore been reduced to the original length taking account of local extensions and in this way the disks

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[*] The same result is obtained from (X, 8) considering that the length and two diameters at right angles are principal axes of strain and therefore $e_r = e_n + 2e_c$. If e_r vanishes, $e_n = -2e_c$.

keep their places. At the same time the diameters have been plotted in logarithmic scale and the strains, or rather deformations, can therefore be read directly from the figure. If in the first series of experiments (I) any one disk suffered a reduction of its diameter from D_o to D_1 and was then turned down to a diameter D_2 which by deformation was reduced to D_3, etc., its final deformation is $e_c = \ln(D_1/D_o) + \ln(D_3/D_2) + \ldots$ which is therefore simply additive. This is not the case with the usual definition of strains and deformations. In Fig. XI, 9 all deformations have been superposed, in accordance with this principle, upon those of the first series as if no turning down had taken place.

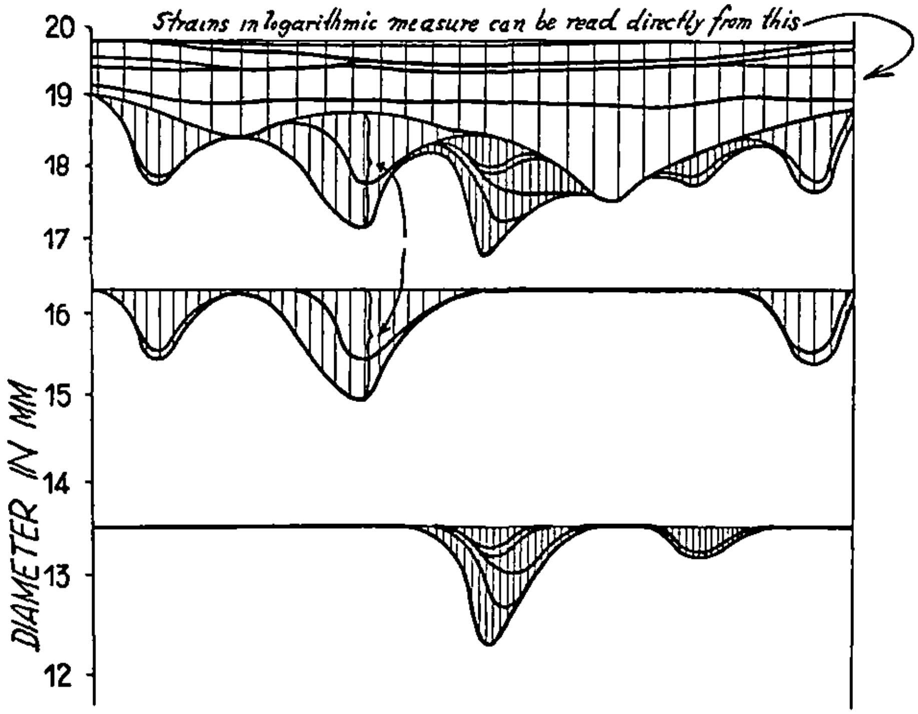

FIG. XI, 9. Lateral deformations of testpiece. The contractions shown in Fig. XI, 5 are here plotted in logarithmic scale and the lateral deformation can be read off directly.

It is now easy to plot local yield stresses against local strains, which has been done in Fig. XI, 10. No special precautions were taken either to exclude ageing or to ensure a definite and equal degree of ageing in the course of the experiments and there is therefore considerable scattering of points. Nevertheless the

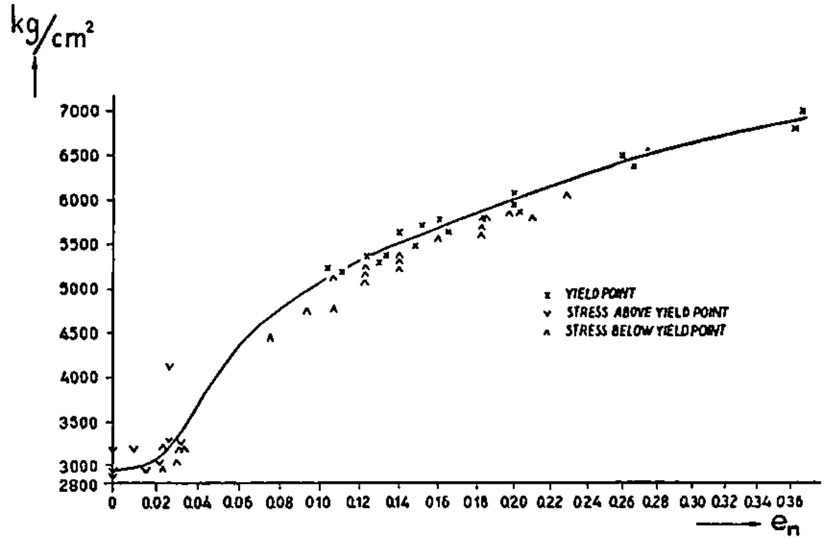

FIG. XI, 10. Rheological stress-deformation curve. The true yield stress has been plotted against double the local contraction.

curve can serve as the basis for the discussion of the problem of the determination of the rheological equation governing the strain-hardening of a material, i.e. a relation between ϑ and e.

12. The Mechanism of Strain-hardening of a Polycrystal

Before attacking this problem we must enquire to what cause the strain-hardening of mild steel may be due. First of all it should be mentioned that single crystals of metals are in all cases very soft. Metals, as we ordinarily know them, are, however, *assemblages* of very small crystals ; they are *polycrystals*. It is to this fact that they owe their hardness. At the same time it has been found that the hardness of a metal is greater the finer its crystalline structure (compare Andrade [49]). Therefore, whatever makes for smaller crystals will make for greater hardness and strain-hardening, on this view, is due to the diminution of the crystals through individual rupture on plastic deformation.

But then the question arises : why should a polycrystalline structure make for hardness ? To answer this question two theories were proposed, one by Beilby and Rosenhain and the other by Ludwik. In accordance with the first, between the crystals of a metal there exists a layer of metal in the amorphous state—" the intercrystalline cement "—to which the hardness is attributed. This theory was discarded because no difference in hardness in the interior of the crystals and in the intercrystalline cement could be found. The second theory attributes the greater hardness of the polycrystalline metal to the random arrangement of adjacent crystals, with their glide planes in different directions, which leads to a *jamming*, which in turn renders glide more difficult. This theory is now generally accepted. It has been proposed also by Jeffries and Archer under the name of the Slip-Interference Theory. We cannot here go more fully into these questions, but the reader may look up the very instructive and easily understandable article by Sir Lawrence Bragg [50] from which Fig. XI, 11 is reproduced.*

This shows a model for a single crystal and for a polycrystal made of small soap-bubbles, each one representing an " atom ". It can be seen at once why the first can be easily deformed by gliding between rows of bubbles, while in the second gliding is more difficult through the different orientations of the rows.

13. The Bauschinger Effect

While, therefore, rupture of individual crystals increases the resistance against gliding and raises the yield point, nevertheless a weakness is introduced in the surface along which rupture has taken place. This weakness possibly makes itself felt in the so-called Bauschinger effect. After plastic deformation and a reversal of direction of loading (from tension to compression or *vice versa* or from twisting in one sense to twisting in the opposite sense) comparatively large plastic deformations are produced by very small loads ; in other words, the yield point for the opposite kind of stress is lowered. But when a mild steel bar is twisted in one sense beyond the yield point there will be small internal rupture surfaces in the crystals along which there will be minute gliding. If the twisting is reversed, some of the ruptures will open up in the same way as the strands in a rope open up when the rope is twisted against its thread. This must cause large deformations or, in other

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* *Endeavour*, 1943, 2, 43–51 : by permission.

words, a Bauschinger effect in torsion. The explanation for the reversal from tension to compression is similar. Now, when we deformed the mild steel test-piece beyond the yield point and then allowed it to rest before reloading, we found that the yield point is raised and it is raised the more the longer the rest. This phenomenon is the ageing of which we spoke before. Ageing in this sense of the word is accelerated by heating to low temperatures up to, but not exceeding about 300° C. and the Bauschinger effect disappears. This is the tempering of the strain-hardened steel. It can be explained as brought about by a *healing* of internal rupture surfaces and the term " ageing " is therefore rather misleading. Rupture has taken place because the distance between the atoms on both sides of an interface is increased so much that they leave the range of atomic cohesion. Now, due to the heat energy of the body, every atom is in constant vibrations, the amplitude of which is determined by the temperature. If the amplitude of a vibration is sufficiently great, an atom on one side of the rupture surface may come within the range of attraction of an atom on the other side of the surface and a connection is made across the rupture. In this way the rupture heals. The process will also take place at ordinary temperature (albeit at a lower rate), because the vibrations are not all of the same amplitude but statistically distributed around some mean magnitude, and from time to time an extraordinarily great amplitude will effect a connection and local healing of the rupture.

When, however, the temperature is raised above 300° C. the vibrations become so strong that they not only heal the ruptures, but the atoms re-arrange themselves in their most stable grids. This is recrystallisation ; the crystals increase in size and the yield point is lowered until all strain-hardening may vanish. This is the annealing of the strain-hardened steel.

14. In Search of a Law of Strain-hardening

We are now in a better position to take up the problem which we put to ourselves in the closing sentence of Section 11. The tensile test of mild steel shows that the yield stress increases with increasing deformation. The question is, how is the former related to the latter ?

Let us go back to our test as represented graphically in Fig. XI, 1. If we, after reaching point "f" on the curve, unload the test-piece, a certain elastic strain is recovered, corresponding to the difference of abscissas of points "f" and "g", while the deformation og is plastic and permanent. Now, again increasing the load to the amount corresponding to point "f", we reach approximately the same point (denoted in the figure by "h") in an *elastic straining* with the *same elastic modulus* as in the first loading. This shows itself in the figure by the slope of gh being the same as that of oa. The curve a–c–b–e is therefore *the geometrical locus of all the yield points corresponding to the successive deformations.*[*] Nevertheless, as becomes clear for reasons which we similarly encountered before in two other cases,[†] the yield stress cannot depend directly upon the deformation. We mentioned in Section 13 the raising of the yield stress through a *twisting* of the bar. It is immediately obvious that this effect cannot depend upon whether we twist the bar clockwise or anti-clockwise. The yield stress ϑ_t must therefore be an *even* function of the tangential deformation e_t or a function of the e_t^2. Let us remember (compare Section 7, Chapter III) that ϑ_t itself is calculated by taking the root of another quantity, the plastic resilience E_{pl} which itself is an even function of the stress. It will also help us in our problem if we recollect that strain-hardening is also called *work-hardening*. What governs the rise of the yield point is clearly the *work* expended in the plastic deformation and not the deformation itself. Let us imagine a giant of such strength that he would be able to knead mild steel as we knead flour dough. Let us hand him a steel ball which he will knead into all sorts of shapes, at the end restoring the sphere. When he has handed back the ball to us, its deformation is *nil* : all distortions, positive and negative, having cancelled each other. The strain-work, however, has all the time increased to a distinct amount. If, in order to make our considerations more definite, we assume that the deformations are simple shears, alternatively in positive and negative directions, the deformational work in terms of strain is, in accordance with (III, 43), $= \gamma e_t^2/2$.

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[*] The existence of a yield point is therefore best revealed on *unloading* a testpiece.

[†] *Vide* Section 5 of Chapter III and Section 2 of Chapter VII.

The resilience on the other hand is, in accordance with (III, d) $= p_t^2/2\gamma$, and the plastic resilience in accordance with (III, b) $= \vartheta_t^2/2\gamma$. The modulus of elasticity being in that process constant (as we said before), we may lump it together with the numerical factor 2. In the case of tangential deformations the law of work-hardening would therefore relate ϑ_t^2 to e_t^2 and should be expressed as a functional relationship $\vartheta_t^2 = f(e_t^2)$. We may denote by θ the limiting strain corresponding to ϑ, so that θ_t is the ultimate shearing strain when the yield point is reached, or

$$\vartheta_t = \gamma\theta_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In other words, therefore, the law of work-hardening must relate *the square of the tangential strain to the square of the tangential deformation.**

15. The Mises-Hencky Flow Condition for Simple Tension

We cannot apply such a functional relationship directly to our case, which is one of simple tension and not of simple shear. We must first express the Mises-Hencky flow condition for the case of simple tension. As we remarked in Section 1 of Chapter X, simple tension is not as simple as one may think. We have shown in Section 5 of that Chapter and in Fig. V, 5, that a simple tension p_n can be considered as the result of the superposition of an isotropic tension $p = p_n/3$ and a distortional stress the components of which are a tension $2p_n/3$ in the axial direction and two compressions $p_n/3$ normal to the axis and at right angles to each other. In accordance with the first axiom of rheology, the distortional stress only produces the plastic deformation and the plastic resilience E_{pl} must be calculated not from the total strain-work of simple tension but only from its distortional component. The first is, in accordance with (X, r), $= p_n^2/2\epsilon$. The latter can be calculated as follows:—

In accordance with (X, g) we have

$$e_{on} = p_{on}/2\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5')$$

Therefore the distortional work

$$w_o = \int p_{on} de_{on} = \frac{1}{2\gamma}\int p_{on} dp_{on} = p_{on}^2/4\gamma \quad . \quad . \quad . \quad (6)$$

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* Keeping in mind the definition of strain as the *recoverable* part of the deformation.

Now, in the case of simple extension, in accordance with (X, f)

$$p_{oi} = p_{oj} = -p_n/3\,;\quad p_{ok} = 2p_n/3 \quad . \quad . \quad . \quad . \quad (7)$$

and, therefore,

$$w_o = 1/4\gamma . p_n^2/9 . (1 + 1 + 4) = p_n^2/6\gamma \quad . \quad . \quad . \quad (8)$$

If we increase the tension p_n until we reach the yield point, when $p_n = \vartheta_n$, the strain-work w_o reaches the plastic resilience E_{pl} and therefore

$$\vartheta_n = +\sqrt{6\gamma E_{pl}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

On the other hand from (III, b)

$$E_{pl} = \vartheta_t^2/2\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which makes

$$\vartheta_n = \vartheta_t\sqrt{3} = 1{\cdot}73\ \vartheta_n, \text{ or } \vartheta_t = 0{\cdot}578\ \vartheta_n \quad . \quad . \quad (11)$$

and if we determine ϑ_t from a torsion experiment we can predict the yield stress ϑ_n in a tensile experiment with the same material. We may compare (11) with St. Venant's flow condition,* i.e. the condition that the shearing stress reaches a certain maximum (compare Section 6 of Chapter III). We know from Mohr's circle, Fig. X, 4, that in the case of a simple tension p_n, the shearing stress reaches its maximum in a section inclined at 45° to the axis of the test-piece and is there equal to $p_n/2$. The yield point will therefore be reached in simple tension when $p_n = \vartheta_n = 2\vartheta_t$. The normal yield stress in accordance with St. Venant would accordingly be $2/1{\cdot}73 = 1{\cdot}15$ times the yield stress in accordance with Mises-Hencky. The difference is not great. Geiringer and Prager [51] are of the opinion that St. Venant's flow condition furnishes in certain cases, e.g. the first starting of flow of mild steel, the best description and must be abandoned only when *full* plastic flow has developed. I hope, however, to have made it clear that, theoretically, the Mises-Hencky flow condition is highly superior to the St. Venant flow condition. Speaking of flow conditions, I may mention that Beltrami in 1885 postulated that the yield point is reached when the strain-work reaches a certain limit. This hypothesis was soon refuted by experimental evidence and we know why: it contradicts our first rheological axiom; the

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* This condition was postulated independently by Couloumb (1801) and Tresca (1868)

dilatational strain-work should not cause any appreciable plastic flow.

16. A "Primitive" Law of Work-hardening

We said at the end of Section 14 that the law of strain-hardening would be one relating θ_t^2 to e_t^2, or, as we can now say, θ_n^2 to e_n^2. In physical language we express the plastic resilience, which is the resilience at the yield point, as a function of what we may call the Hardening Work; the first is proportional to θ_t^2 or θ_n^2, the second to e_t^2 or e_n^2. If we plot θ_n^2 against e_n^2 the resulting curve has a striking resemblance to the one (Fig. V, 8) for the variable fluidity of a liquid as a function of p_t^2, with θ_o^2 corresponding to φ_o, θ_∞^2 to φ_∞ and e_n^2 to p_t^2. We may therefore tentatively write down in analogy to (VII, i)

$$\theta^2 = \theta_\infty^2 - (\theta_\infty^2 - \theta_o^2)\mathrm{e}^{-e^2/\psi} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

or in another form

$$E_{pl} = E_{pl\infty} - (E_{pl\infty} - E_{plo})\mathrm{e}^{-w_h/\bar{\psi}} \quad . \quad . \quad . \quad (13)$$

where E_{pl} is the plastic resilience which gradually, but more and more slowly, increases from E_{plo}, the plastic resilience in the annealed state, to $E_{pl\infty}$, the maximum plastic resilience which can be stored up in the material after it undergoes the maximum work-hardening. The hardening work w_h is the work of the external forces spent in overcoming internal plastic friction and in changing the structure of the metal by breaking down its constituent crystal grains. The coefficient ψ is therefore quite analogous to the coefficient χ; it also is a Coefficient of Structural Stability. As the expressions e^2/ψ and $w_h/\bar{\psi}$ must be dimensionless, the quantity $\bar{\psi}$ has the dimension of work, and ψ (compare (III, d)) the dimension of a work per unit rigidity.

Equation (13) was proposed by me as a "primitive" work-hardening law at the Paris Congress for Applied Mechanics, 1946.

We found in Chapter VII that χ was not a constant. Neither can we expect ψ or $\bar{\psi}$ to be constants. From what has been said in Section 12 it is clear that the larger the crystal the less its resistance against slip. With the continuous breaking down of the crystals ψ and $\bar{\psi}$ must therefore increase.

In the second of the researches mentioned at the end of Section 6 above [48′], a strain-hardening curve was obtained first by discontinuous pulling in a tensile testing machine, and then by successive wire-drawing operations. A hot-rolled low-carbon steel rod of 8 mm. diameter was drawn down to a wire of 0·8 mm. diameter resulting in an extensional deformation of 4·6 in the logarithmic measure. It was found that the work-hardening curve consisted of *three* e-curves of the form of Equations (12) and (13) following each other with short transitional curves between them. The transition from the first to the second e-curve occurred at the maximum tensile load (point *b* in Fig. XI, 1); that from the second to the third at an extension at which the tensile test fracture (point *e* in Fig. XI, 1) takes place. In the paper, an " atomistic " interpretation is given, using ideas developed by Sir Lawrence Bragg [50], which, however, belong to what may be termed Metarheology.

17. Summary

In a static test, i.e. one proceeding through states of equilibrium, mild steel shows a Hooke range, a simple St. Venant range reached through an " upper " and " lower " yield point, and a " generalised " St. Venant range in which the yield point increases at a decreasing rate, reaching an upper limit. An elastic spring can serve as a model for the Hooke-body (indicated symbolically by H) and a weight resting on a table, with solid friction between them, as a model for the St. Venant body (StV). Both coupled in series (H–StV) represent the Hooke and simple St. Venant range, while $H\text{–}(StV)_1\text{–}(StV)_2\text{–}\ldots\text{–}(StV)_n$ includes the work-hardening or generalised St. Venant range, provided the series $(StV)_1, \ldots (StV)_n$ is arranged in monotonously decreasing order.

The distortional work in simple tension is

$$w_{on} = p_n^2/6\gamma. \quad \ldots\ldots \quad \text{(XI, a)}$$

and, therefore, the yield stress

$$\vartheta_n = \pm\sqrt{6\gamma E_{p2}} \quad \ldots\ldots \quad \text{(XI, b)}$$

which gives

$$\vartheta_n = 1{\cdot}73\ \vartheta_t. \quad \ldots\ldots \quad \text{(XI, c)}$$

in accordance with Mises-Hencky, while

$$\vartheta_n = 2\,\vartheta_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(XI, d)}$$

in accordance with St. Venant.

The rise of the yield point in the work-hardening range is due to the breaking up of large into small crystals. For a hypothetical polycrystal consisting of large crystals of *equal size* which in each slip are converted into small crystals of minimum size, the rheological equation

$$\vartheta^2 = \vartheta_o{}^2 - (\vartheta_\infty{}^2 - \vartheta_o{}^2)e^{-\epsilon^{s}/\psi} \quad . \quad . \qquad \text{(XI, e)}$$

is proposed, where ψ is a coefficient of structural stability. For a real polycrystal ψ gradually increases and the equation must be amended, taking into account a distribution of crystals of different sizes.

CHAPTER XII

BENDING AND TORSION*

1. Homogeneous Deformation and Stress

In the foregoing we have become acquainted with three cases of simple deformation, viz., simple shear, simple cubical dilatation and simple extension. These three are Homogeneous Deformations, i.e. they are the same throughout the body. If we assume that the prism of Fig. I, 2 has no weight and subdivide it into a number of smaller prisms, each one of these is deformed in exactly the same manner as the large prism. In other words, e_t or e_v or e_n in each one of the three cases does not depend upon the *ordinates* of the particles of the body. As the stress is related to the deformation by means of a rheological equation in which the co-ordinates do not appear, homogeneity of deformation carries with it homogeneity of stress.

On the other hand, if we subdivide a prism which has weight into horizontal layers, the weight of the material will cause a pressure increasing from zero at the top to a maximum at the bottom of the prism. It is therefore clear that homogeneous stress, and consequently homogeneous deformation, are possible only in the absence of weight, or, more generally, of Body Forces, such as inertia. Only if external forces acting upon the surface of the body, or Surface Forces, *prevail over body forces so that the latter can be neglected,* is homogeneous stress and consequently homogeneous deformation possible.

Generally, we must relate the rheological equation to an " infinitely " small volume element to which we apply the laws of mechanics (I, a) or (I, b) and (I, c), deriving, by integration, the rheological behaviour of the whole body. Sometimes, as

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* The subject-matter of this Chapter can be found in essence in any of the numerous textbooks on the Strength of Materials ; I have, however, included it for the benefit of those readers who have not had an engineering training in order to make the book self-contained, as in the following chapters reference is made to certain equations derived here.

in laminar distortion, we can treat a *portion* of the body, the lamina, finite in two dimensions but infinitely small in the third, as the volume element. This was the case when we dealt with the flow through the tube and in the rotation instrument where e_t was assumed to be constant throughout the length of the cylinder and the same in all radial sections, depending upon r only. This may be called ' quasi-homogeneous " deformation. In homogeneous deformation we do not need to relate the rheological equation to the volume element. If the deformation is homogeneous, the *whole body* can be considered as an "element"; there is no need for integration and all the rheologics of the body is contained in its rheological equation. This is the case in simple shear, simple dilatation and simple extension.

2. Simple Bending

There is an important case of " simple " deformation which is neither homogeneous nor quasi-homogeneous. It is Simple Bending.

Let a prismatic bar be subjected at both ends to couples which are in equilibrium, the plane in which the couples act passing through the axis (x) of the beam. Such couples are called " bending couples " (see Fig. XII, 1).

In order to find the deformation and the stresses by an elementary method, we reason as follows :—

For the sake of simplicity of argument let us first assume that the cross-section of the bar is symmetrical about an axis (y) and that the plane in which the bending couples act passes through this axis of symmetry. This plane is therefore the xy plane of the co-ordinate system. Now let the beam be divided into a number of shorter lengths.

As each length is in equilibrium, this requires that the internal stresses give rise to couples equal to the couples of the external forces. With exactly the same forces acting on each free body there is no reason why one should be strained differently from another. Yet, if all are strained in the same manner, the axes of all pieces are bent to the same curvature. As the length of the pieces can be taken as infinitely small, this implies that the whole beam is bent to a curve of constant curvature, i.e. to a *circle*, the plane of which is the x-y plane. Furthermore, it is

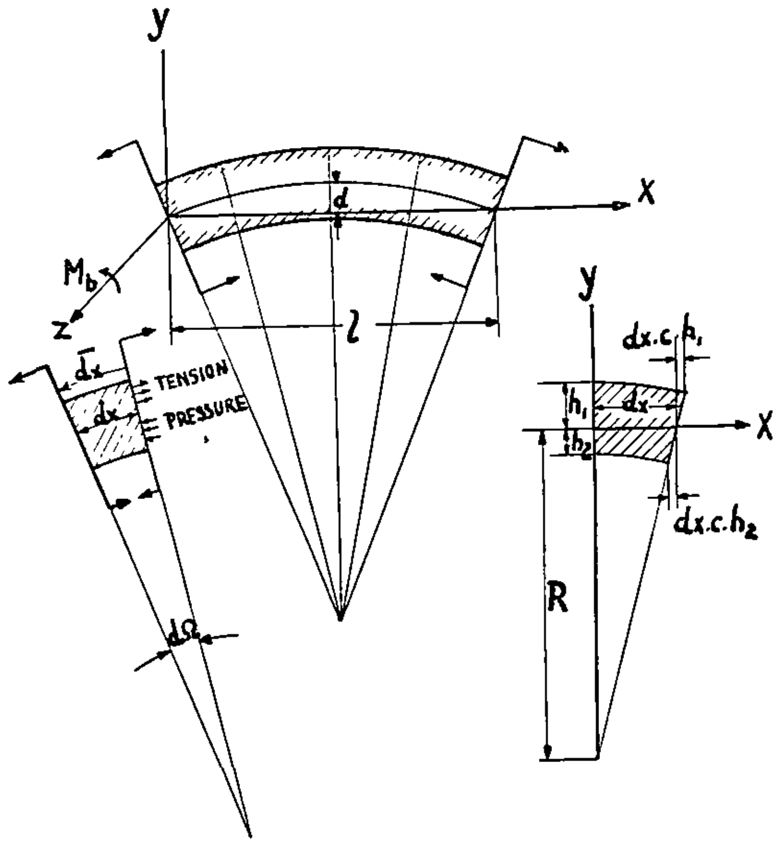

FIG. XII, 1. Simple bending of a prismatic beam.
The first length has been given a rotation without altering the deformation.

obvious that the plane end sections of the pieces must remain plane and normal to the bent axis otherwise they could not fit together to complete the beam. This is known as Bernoulli's assumption (1705).

We now consider such a piece of length dx. As has been said above, the internal stresses acting upon an imagined cross-section are equivalent to a couple and to a couple only. The resultant of the stresses therefore vanishes, and the short prismatic piece is, accordingly, *as a whole*, neither extended nor compressed. The length dx of the axis will therefore not be changed and as it is bent, as we have seen, to a circular curve, the "fibres" of the prism of length dx which are parallel to the axis are either extended or shortened in accordance with the equation

$$\overline{dx} - dx = (R + y)\, d\Omega - dx. \quad . \quad . \quad . \quad . \quad (1)$$

If the arc to which the beam is bent is flat, the strains will be small and the bending strain e_b will be

$$e_b = (\overline{dx} - dx)/dx - (R + y)\, d\Omega/dx - 1 \quad . \quad . \quad (2)$$

However, $d\Omega \doteq dx/R$, and, therefore,

$$e_b = y/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These strains are directed *normal* to the cross-section. They are therefore normal strains as dealt with in Chapter X, and if the material is a Hooke body, Hooke's law in the form (X, j) can be applied, or

$$p_b = \epsilon y/R \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The stresses are accordingly distributed linearily over the cross-section in triangles. Let the distance from the axis of the outermost fibre in tension be h_1 and of the innermost fibre in compression h_2, then the maximum tensile stress is $\epsilon h_1/R$, and the maximum compressive stress $\epsilon h_2/R$.

Now let us assume the cross-section to be a rectangle of width b, then the stresses give a resultant force in the x-direction

$$P_x = b\,\frac{\epsilon h_1}{R}\,\frac{h_1}{2} - b\,\frac{\epsilon h_2}{R}\,\frac{h_2}{2} = \frac{b\epsilon}{2R}(h_1^2 - h_2^2) \quad . \quad . \quad (5)$$

In our case, as there is no external force in the x-direction, P_x must vanish or $h_1 = h_2$ and the axis of the beam the length of which is not changed, or the so-called Neutral Axis, will pass through the centre of the section. This will be so in the case of every cross-section symmetrical about the z-axis, but the reader will easily find that in a triangular cross-section the neutral axis will be nearer to the base and that, generally, the neutral axis will pass through the centre of gravity of the section.

The stresses must, however, give a finite bending couple M_b, the couple of the external forces.

Consider a strip of the cross-section having a length b parallel to the z-direction and the width dy. The increment of force acting on this strip will be $\frac{y\epsilon}{R}bdy$ and its moment in respect of the axis $\left(\frac{\epsilon y}{R}bdy\right)y = \frac{\epsilon b}{R}y^2dy$. The bending moment is therefore $M_b = \frac{\epsilon b}{R}\int_{-h/2}^{+h/2} y^2dy = \frac{\epsilon b}{R}\frac{h^3}{12}$.

The reader will have no difficulty in finding that in the more general case

$$M_b = \epsilon I/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where I is the moment of inertia of the cross-section about the neutral axis.

From (6) we find the radius of curvature

$$R = \epsilon I/M_b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

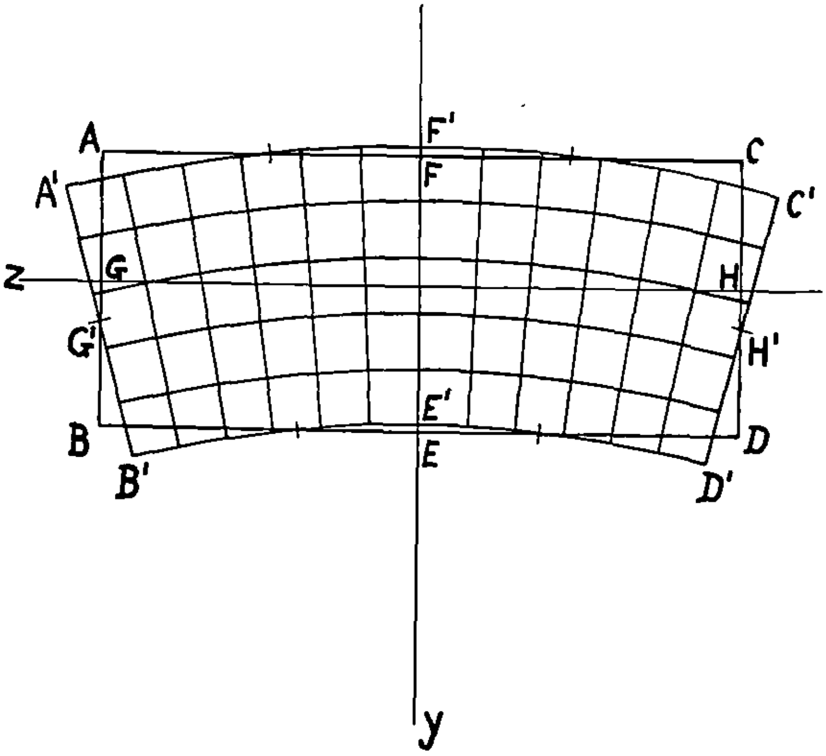

FIG. XII, 2. Deformation of the cross-section of a bent beam.

and, therefore, considering (3) and (4)

$$e_b = M_b y/\epsilon I \; ; \; p_b = M_b y/I \quad . \quad . \quad . \quad . \quad (8)$$

The maximum stress is found for the greatest value of $y = h_1$ in tension and $y = h_2$ in compression. If the neutral axis passes through the geometrical centre of the section, $h_1 = h_2 = h/2$. The quantity $SM = 2I/h$ is called the Section Modulus and

$$\max. p_b = M_b/SM \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For a rectangular section $I = bh^3/12$ and $SM = bh^2/6$ and, therefore,

$$\max. p_b = 6M_b/bh^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9')$$

While the p_b stresses are the only stresses acting on the cross-section, the e_b strains are not, of course, the only strains. With the extension of the fibres on the positive y-side, there goes hand-in-hand a contraction in the z-direction which is its $1/\sigma$ part. With the shortening of the fibres on the negative y-sides, there goes hand-in-hand an extension in z-direction. An originally rectangular cross-section therefore becomes of a shape as shown in Fig. XII, 2.

The " deflection " of the beam (d) can be calculated from well-known geometrical properties of a circle, according to which

$$(L/2)^2 = d(2R - d) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

or, if d is small,

$$L^2/4 = 2Rd \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$d = L^2/8R = L^2M/8\epsilon I \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If the bending couple M is known and L and I are determined and d is measured, Young's modulus ϵ can be calculated from

$$\epsilon = L^2M/8Id \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Simple bending is produced if a beam rests on two supports and loads are suspended outside the supports as shown in Fig. XII, 3. Only the piece of the beam *between the supports* is bent to a circle, because only in that part are the external forces equivalent to a constant couple $M = P.l$ *provided the weight of the beam can be neglected.*

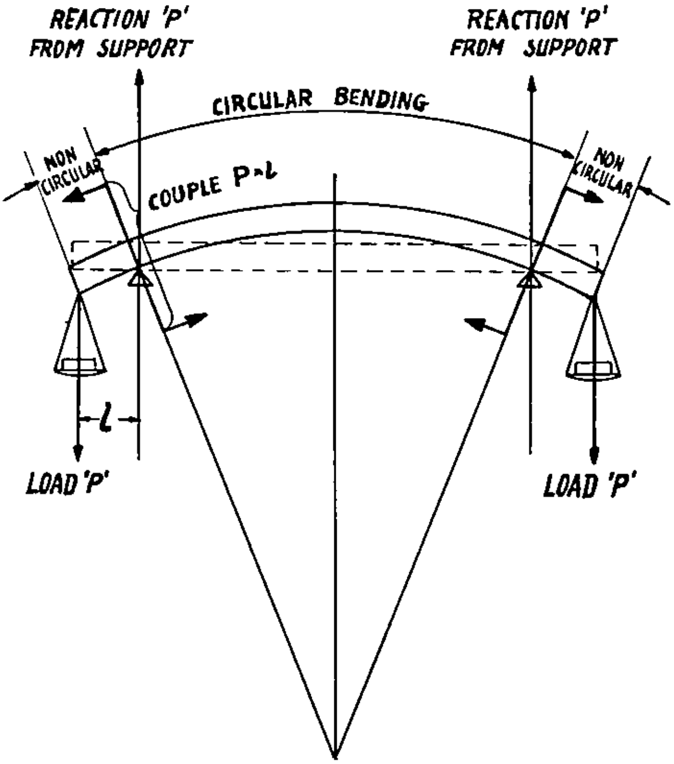

FIG. XII, 3. Scheme for simple bending of a beam.

Bernoulli's assumption is correct only in this single case of simple bending. It is *not correct in any other case.* E.g. in a beam resting upon two supports and bent by its own weight, plane cross-sections do *not remain plane* and the neutral-axis is *not bent to a circle.*

The general problem of bending has been treated by Reiner [52]. It transcends the possibilities of elementary methods. If, however, it is desired to find such an " overall " macro-quantity as the deflection d of the beam, when the exact micro-distribution of *local* strains and stresses is irrelevant, (8) gives a sufficient approximation.

If the curve is not a circle, its curvature at any point is, from known principles

$$1/R = \frac{y''}{[1 - (y')^2]^{3/2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $y' = dy/dx$ and $y'' = d^2y/dx^2$. If the curvature is moderate and the arc to which the beam is bent is accordingly flat, dy/dx can be neglected and $1/R$ approximately $= d^2y/dx^2$. This makes (8)

$$\frac{d^2y}{dx^2} = -M_b/\epsilon I \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)^*$$

from which the deflection can be calculated if M_b is known as a function of x.

3. Bending under its Own Weight

Let the weight of the beam itself be w per unit length, then the reactions from the supports are $wL/2$ and, therefore, the bending moment at the point x

$$M_x = \frac{wL}{2}x - wx\frac{x}{2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

(compare Fig. XII, 4).

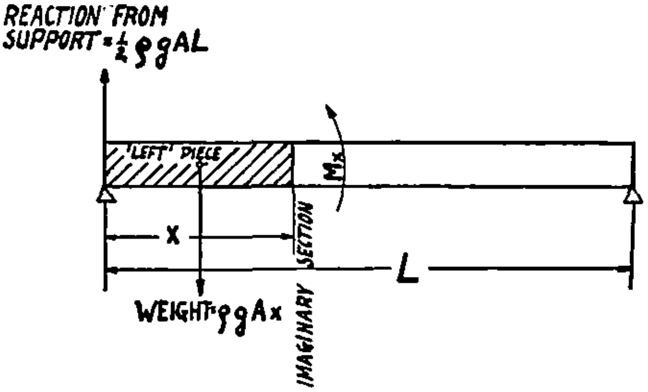

Fig. XII, 4. Bending of a beam by a distributed load.

Therefore

$$d^2y/dx^2 = -\frac{w}{2\epsilon I}x(L - x) \quad . \quad . \quad . \quad . \qquad (17)$$

which gives by integrating twice

$$y = -\frac{wx^3}{12\epsilon I}(L - x/2) + C_1x + C_2 \quad . \quad . \quad . \qquad (18)$$

where C_1 and C_2 are integration constants.

---

* The negative sign is due to d being positive for a negative curvature.

We determine these from the kinematic boundary conditions $y = 0$ for $x = 0$ and $x = L$, which give $C_2 = 0$, $C_1 = \frac{wL^3}{24\epsilon I}$, so that

$$y = wx/24\epsilon I \,.\, (L^3 + x^3 - 2Lx^2) \quad . \quad . \quad . \quad . \quad (19)$$

and

$$d = y/_{x=L/2} = 5wL^4/384\epsilon I \quad . \quad . \quad . \quad . \quad (20)$$

4. Bending by a Concentrated Load in the Centre

Here the reactions from the supports are $P/2$ and the moment

$$M_x = Px/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Compare Fig. XII, 5.

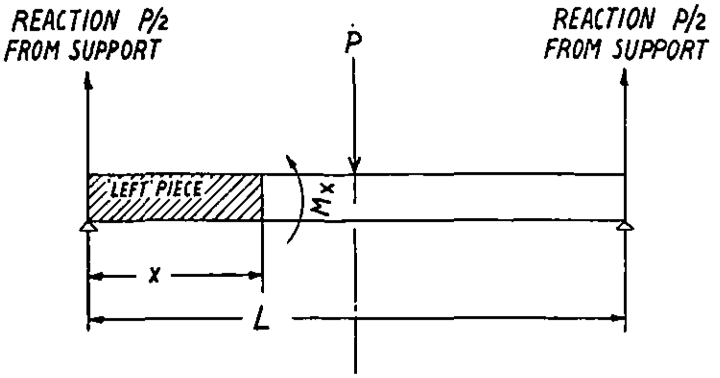

FIG. XII, 5. Bending of a beam by a concentrated load in the centre.

Therefore

$$d^2y/dx^2 = -Px/2\epsilon I \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

which gives by integrating twice

$$y = -Px^3/12\epsilon I + C_1 x + C_2 \quad . \quad . \quad . \quad . \quad (23)$$

The deflection line consists of two parts which meet at the centre without a break. The tangent to the deflection line at the centre is therefore horizontal. The integration constants can accordingly be determined from $y = 0$ for $x = 0$ and $dy/dx = 0$ for $x = L/2$. We find $C_2 = 0$, $C_1 = PL^2/16\epsilon I$, and therefore

$$y = PL^2/16\epsilon I \,.\, (L^2 - 4x^2/3) \quad . \quad . \quad . \quad . \quad (24)$$

This makes

$$d = y_{x=L/2} = PL^3/48\epsilon I \quad . \quad . \quad . \quad . \quad . \quad (25)$$

5. Bending by Two Concentrated Loads arranged Symmetrically about the Centre

Here the reactions from the supports are $M = P$, and the moment is (compare Fig. XII, 6)

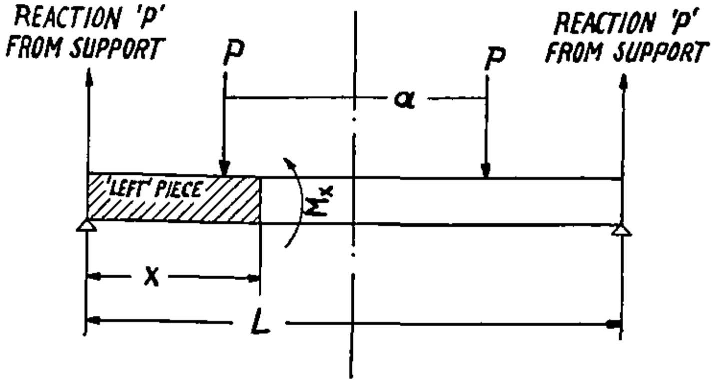

FIG. XII, 6. Bending of a beam by two concentrated loads arranged symmetrically about the centre.

$$M_x = Px \qquad \ldots\ldots\ldots \quad (26)$$

between $x = 0$ and $x = (L - a)/2$,
and

$$M_x = P(L - a)/2 \qquad \ldots\ldots \quad (27)$$

between both loads.

Therefore

$$d^2y/dx^2 = - Px/\epsilon I \qquad \ldots\ldots \quad (28)$$

between $x = 0$ and $x = (L - a)/2$, while

$$d^2y/dx^2 = - P(L - a)/2 \, . \quad \ldots\ldots \quad (29)$$

between the loads.

Integration of both differential equations gives

$$y = - Px^3/6\epsilon I + C_1 x + C_2 \, . \quad \ldots \quad (30)$$

$$y = - P(L - a)x^2/4\epsilon I + C_3 x + C_4 \quad \ldots \quad (31)$$

From the condition $y = 0$ for $x = 0$ we find $C_2 = 0$.

From the condition $dy/dx = 0$ for $x = L/2$, $C_3 = PL(L - a)/4$.

The other two integration constants can be calculated from the condition that at $x = (L - a)/2$ the deflection and the slope must be the same whether calculated from (30) or (31), i.e.

$$-\frac{P(L-a)^3}{48\epsilon I}+C_1\frac{L-a}{2}=-\frac{P(L-a)^3}{16\epsilon I}+\frac{PL(L-a)^2}{8\epsilon I}+C_4 \quad (32)$$

and

$$-\frac{P(L-a)^2}{8\epsilon I}+C=-\frac{P(L-a)^2}{4\epsilon I}+\frac{PL(L-a)}{4\epsilon I} \quad . \quad (33)$$

We find

$$C_1=\frac{P(L^2-a^2)}{8\epsilon I},\ C_4=-\frac{P(L-a)^3}{48\epsilon I} \quad . \ . \quad (34)$$

and therefore

$$\begin{aligned} y &= -Px^3/6\epsilon I+P(L^2-a^2)x/8\epsilon I \\ y &= -P(L-a)x^2/4\epsilon I+PL(L-a)x/4\epsilon I-P(L-a)^3/48\epsilon I \end{aligned} \quad (35)$$

The maximum deflection follows from the second equation for $x = L/2$ with

$$d = P(L-a)(2L^2+2La-a^2)/48\epsilon I \quad . \ . \ . \ . \ . \ . \ . \quad (36)$$

6. Simple Elastic Torsion

Another case of simple deformation is simple torsion. It is laminar, as shown in Fig. I, 1, and accordingly quasi-homo-

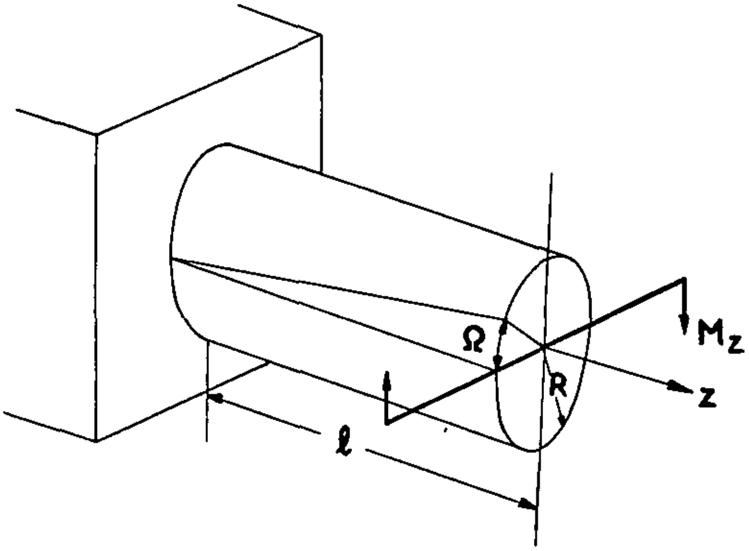

Fig. XII, 7. Torsion of a circular cylinder.

geneous. Let the circular cylinder of radius R and length l be fixed at one end and, at the other end, loaded by a torsional couple M_z. The cylinder will be twisted, i.e. the section $z = l$ will be rotated against the section $z = 0$ by an angle, say Ω. The problem is to express Ω in terms of M_z and the dimensions of the cylinder, i.e. the geometrical quantities R and l.

In the elementary method which we are following in the present book we cannot solve this problem without the introduction of an assumption similar to Bernoulli's assumption. This assumption can be expressed in different ways. The first to attempt a solution of the problem of torsion was Coulomb and he started from the assumption that shearing stresses only act in the cross-sections and that they are proportional to their distances from the axis and directed normal to the radius, i.e. he started from the equation

$$p_t = cr \qquad \ldots\ldots\ldots \quad (37)$$

where c is a constant.

From this he found by applying Hooke's law

$$e_t = cr/\gamma \qquad \ldots\ldots\ldots \quad (38)$$

Comparing Fig. XII, 7 with Fig. I, 2, we see that we can express e_t, introducing dz for H, as follows :—

$$e_t = du/dz \qquad \ldots\ldots\ldots \quad (39)$$

or

$$du = \frac{cr}{\gamma}dz \qquad \ldots\ldots\ldots \quad (40)$$

and by integration

$$u = cr/\gamma . z \qquad \ldots\ldots\ldots \quad (41)$$

considering that $u = 0$ for $z = 0$.

In order to find c we compose the stresses taken over the section to the torsional couple

$$M_z = \int (p_t 2r\pi dr)r = 2\pi c\int_0^R r^3\, dr = \pi c r R^4/2 \ . \quad . \quad (42)$$

and get $c = 2M_z/\pi R^4$ so that (compare with (37))

$$p_t = 2M_z r/\pi R^4 \qquad \ldots\ldots\ldots \quad (43)$$

Equation (41), on the other hand, gives

$$u = 2M_z/\pi R^4 . r/\gamma . z \ . \qquad \ldots\ldots \quad (44)$$

The free end-section is rotated through the angle

$$\Omega = u/r|_{z=l} = 2M_z l/\pi R^4\gamma \quad . \quad . \quad . \quad . \quad . \qquad (45)$$

and this equation connects the observable macromechanical quantities M_z and Ω. Ω may be called the rotational displacement.

Equation (45) can also be written

$$M_z = \Omega\frac{\pi R^4\gamma}{2}/l = \Omega D/l \quad . \quad . \quad . \quad . \qquad (45')$$

where D/l is the " restoring moment " of (2, 45) and D the " torsional resistance " of Table III, Chapter V.

From (41) it can be seen that the section $z = z$ will be rotated against the section $z = 0$ by an angle of the magnitude $u/r = c/\gamma . z = \Omega/_l . z$.

The quantity $\Omega/_l$ is called the *twist*. In our case, the twist is constant. Generally where it is not constant, the twist is measured by the relative rotation of one slice against the other or by $d\Omega/dz$. From (41) it also follows that if the cylinder is imagined to be composed of " slices " of thickness dz, the slices are displaced *as wholes* so that the displacements have neither radial nor longitudinal components, the only component being a rotational displacement. This confirms the picture of Fig. II, 1.

We could also have started with this, as a natural kinematical assumption, and would have found from

$$u_\theta = Crz \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46)$$

$$e_t = du_\theta/dz = Cr \quad . \quad . \quad . \quad . \quad . \quad . \qquad (47)$$

and, in accordance with Hooke's law, again

$$p_t = \gamma e_t = \gamma Cr \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

where

$$\gamma C = c \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49)$$

While it might seem natural to assume that in torsion the shearing stress is proportional to the distance from the axis; or that plane cross-sections are rotated against each other, but remain plane (both assumptions, as has just been shown, being identical), it can easily be shown that Coulomb's assumption cannot be valid for any shape of section other than a circular section. If the section is other than circular its edge makes an angle with the radius vector r; and the shearing stress p_t, which has the direction of θ, or is normal to r, must then have a component normal to the linear element of the edge (see Fig. XII, 8), i.e. in the direction of n. However, in accordance with the law of corresponding shearing stresses (compare Section 6, Chapter I), for every shearing stress *in the cross-section* there exists an equal shearing stress *in a longitudinal section* normal to the cross-section. In our case, if there existed a shearing stress in the cross-section having the direction n, there would have to exist a shearing stress acting upon the side in a direction normal to n. We have, however, assumed that no surface forces act upon the side of the cylinder and, therefore, such a shearing stress as mentioned last does not exist. Therefore, p_t cannot have a component in the direction n. Hence p_t cannot be directed normal to r and it has for non-circular sections generally the *two* components p_θ and p_r.

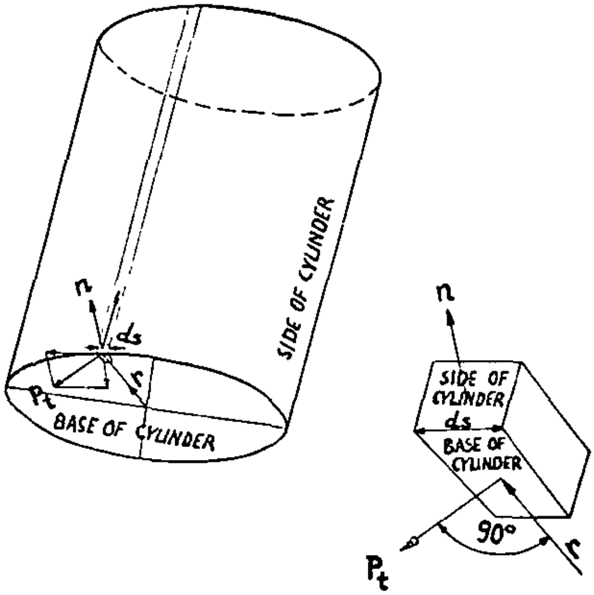

FIG. XII, 8. Torsion of a rod of non-circular cross-section.

The general solution for non-circular sections was found by St. Venant (1855). St. Venant assumed, as we have assumed above, that no forces act upon the sides of the prism. Actually, however, in order to twist a rod it must be fixed at the ends in such a way that surface forces *are* exerted upon short lengths of the sides. It is true that at some distance from the fixing-points, the sides will be free from forces and there the stresses will be those of St. Venant's theory. These stresses may, however, not be the maximum stresses and if the material fails it will do so at the fixed ends. Therefore, for calculations of strengths, a better approximation to actual conditions than provided by St. Venant's theory is required. In three papers I have made an attempt at such a generalisation of St. Venant's theory by considering the action of torsional surface forces applied upon the sides of a cylinder or prism [52′, 53, 54].

7. Plastic Torsion

If an elastic rod is subjected to torsion, we find (compare (45)) that in order to rotate its free end through an angle Ω a torque

$$M_z = \pi R^4 \gamma \Omega / 2l \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

has to be applied. From this equation it follows that there is a linear relationship between M_z and Ω. If, however, Ω is gradually increased through an increase of the external forces, we find that this linear relationship has a limit and that the rod either breaks, when it is brittle, or that it can from a certain stage on be further considerably increased by a very small increase of M_z, when the material is ductile or plastic. We may therefore try to apply St. Venant's law (I, e) to this case.

The application of St. Venant's law is easy enough if we have a case of homogeneous stress. Then the yield point is reached in all parts of the body at the same time and while, before reaching the yield point, the deformation of the whole body was elastic, after passing the yield point the deformation of the whole body is plastic. It is not so in cases of heterogeneous stress. In such cases there must be a surface or surfaces dividing the body in two parts, the rheological equation of which will be either (I, d) or (I, e). The form of the dividing surface is generally not known and this causes the greatest mathematical difficulties. In certain cases, however, the form of the surface suggests itself. This is especially so if we are dealing with a laminar displacement. The form of the surface can then be assumed, the assumption can be verified and the position of the surface found from the condition that for the dividing surface the tangential stresses of the elastic part are all equal in magnitude and equal to ϑ_t. We followed this procedure in deriving the Buckingham-Reiner and the Reiner and Riwlin equations. We shall take the same course in the present case.

As we have shown in the preceding Section, the tangential stress in the twisted elastic rod is

$$p_t = 2M_z r/\pi R^4 = \gamma\Omega r/I \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The tangential stress has therefore the same value in every point of a cylindrical surface with the radius r and its maximum value for $r = R$

$$\max.\, p_t = 2M_z/\pi R^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

The dividing surface will accordingly be a coaxial cylindrical surface. If Ω is gradually increased, maximum p_t also increases, until it reaches the value ϑ_t and the other stresses are increased correspondingly. If Ω is still further increased, maximum p_t cannot, in accordance with St. Venant's law, increase beyond ϑ_t and a plastic body develops as the outer shell of the cylinder, enclosing an elastic co-axial cylinder. As the tangential stress p_t of this elastic core is the same over one and the same concentric cylindrical surface, the dividing surface between the plastic and the elastic parts must, as has been said before, be a cylinder of, say, radius r_o. Within this cylinder the stresses will still follow Hooke's law as given in (51) and r_0 can be calculated from

$$\vartheta_t = \gamma r_o \Omega/l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

to be

$$r_o = \vartheta_t l/\gamma\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

In the outer shell the tangential stresses will all be equal to ϑ_t. Analogous to (42) we then have

$$M = 2\pi\int_o^R p_t r^2 dr = 2\pi\left[\int_o^{r_o}\gamma r^2/l\,.\,r^2 dr + \int_{r_o}^R \vartheta_t r^2 dr\right] = 2\pi\vartheta_t R^3/3 - \pi\vartheta_t^4 l^3/6\gamma^3\Omega^3 \quad . \quad . \quad (55)$$

Fig. XII, 9 shows M_z as a function of Ω.

It is interesting to note that while in the case represented in Fig. I, 5, b, when the shear has exceeded the value ϑ/γ, the material starts to flow under constant stress ; in our case, when plastic deformation starts there is no flow, but the external forces may even be increased and there will still be an arresting deformation up to a maximum moment (max. $M_z = 2\vartheta_t\pi R^3/3$). This is due to the fact that in the present case there always remains a solid elastic core of diminishing, but never vanishing, radius r_0. It should, however, be noted that we have treated our problem as a case of equilibrium. The torsional moment of the external forces can, of course, be greater than maximum M_z. In this case equilibrium is not possible and also not steady flow. If $M_z > 2\vartheta_t\pi R^3/3$, there will be accelerated flow under the action of a torsional moment which is $M_z - 2\vartheta_t\pi R^3/3$.

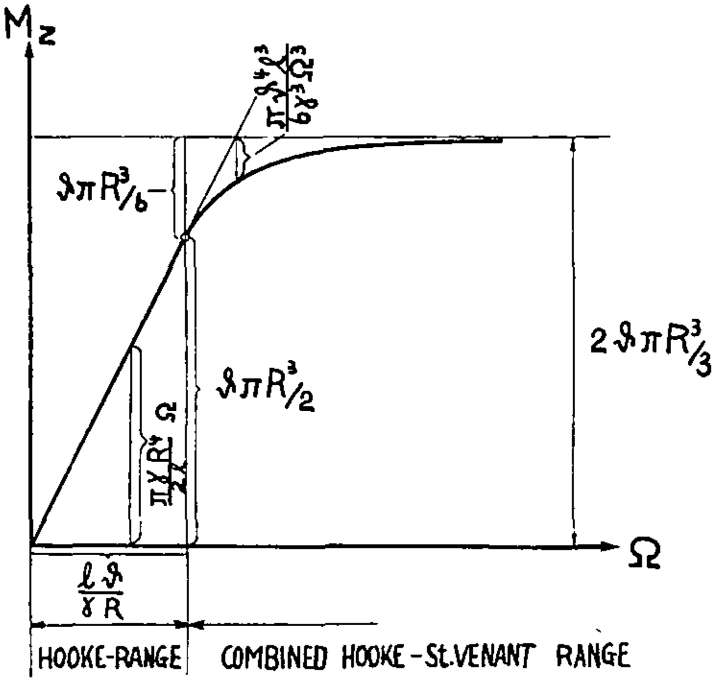

FIG. XII, 9. Plastic torque-twist diagram.

8. Viscosity-Elasticity Analogies

In Section 1 of Chapter III we have pointed out the analogy of (I, d) and (I, f) in accordance with which, if we know the solution of a problem of elasticity, we can write down at once the solution of the analogous problem of viscosity. These equations refer to cases of shear, but in Section 12 of Chapter X we have shown that a similar analogy exists also in simple tension. In shear, η corresponds to γ ; in simple tension in the second stage after the cubical dilatation has reached its maximum, Trouton's coefficient of viscous traction λ corresponds to Young's modulus ϵ in the case of an incompressible material. If, for instance, we place a beam made of, say, fairly hard bitumen upon two supports and load the beam in such a way that simple bending is produced, the beam will gradually and continuously sag and, as long as the deflection is not too great, *the rate of sagging*, $\dot{d}$ can be found from (12) to be

$$\dot{d} = L^2M/8\lambda I = L^2M/24\eta I \quad . \quad . \quad . \quad . \quad . \quad (56)$$

If the bending is by a weight w per unit length of the beam, the rate of deflection is in accordance with (20)

$$\dot{d} = 5wL^4/384\lambda I = 5wL^4/1152\eta I \quad . \quad . \quad . \quad (57)$$

A concentrated load P in the centre will produce a rate of deflection in accordance with (25) of

$$\dot{d} = PL^3/48\lambda I = PL^3/144\eta I \quad . \quad . \quad . \quad . \quad (58)$$

Bending by two concentrated loads P arranged symmetrically about the centre with a as the distance between them will produce in accordance with (36) a rate of deflection

$$\dot{d} = P(L - a)(2L^2 + 2La - a^2)/48\lambda I = P(L - a)$$
$$(2L^2 + 2La - a^2)/144\eta I \quad . \quad (59)$$

These are all examples where the main stress is " normal ". Torsion of a viscous liquid gives, in accordance with (45)

$$\dot{\Omega} = 2ML/\pi R^4\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

9. The Advantages of Bending and Torsion Tests and Hooke's Lucky Chance

The bending and torsion tests are often more suitable for the determination of rheological constants than simple tension. In a rheological test the kinematical quantity observed is seldom directly a strain (or a deformation) or rate of strain (or rate of deformation). It is rather a *displacement* or *rate of displacement*. In simple tension, where the deformation is *pure*, the total displacement is the sum of the elementary displacements. In the bending of a rod, where there is rotation of the elements, the displacements are increased along the length of the rod as in a rotating pointer arrangement. Take, for instance, a short rod of any elastic material in your hand and apply with the other hand some force. If the force is a pull in the direction of the axis of the rod, the displacement of the free end will hardly be noticeable. If the force is applied at the free end in the direction normal to the axis, there will be a noticeable displacement, provided the rod is not too stiff. To make this example more definite, let us assume the rod to be of mild steel of a cross-section 1 mm. square and 10 cm. long. Applying a pull of 100 g., the extension will be in accordance with (X, j), $e_n = 3 \times 10^{-6}$ and, therefore, in accordance with (X, i′) the displacement of the free end $\Delta l = 3 \times 10^{-5}$ cm. Applying the same force in the direction normal to the axis, the deflection will be found to be the same as if twice the load had been applied in the centre of a beam of double length supported at both ends. This, in accordance with (25) is $d = 2P(8L^3)/48\epsilon I$ and considering $I = bh^3/12 = 1/12$ *mm.*^{4}, we shall find d to be slightly over *1* cm. or a displacement one million times that of the first case.

Because of the viscosity-elasticity analogies mentioned in the preceding section, similar considerations apply in respect of the determination of coefficients of viscosity. Here also the " sagging-beam method " is more striking than simple traction.

In torsion also, due to the cumulative effect of rotations, the displacement is much greater than in simple shear.

The disadvantage of both the bending and torsion test is the heterogeneity of deformation and stress which means that the rheological test curve is not a fundamental curve in the meaning of Section 1, Chapter XI.

These considerations are relevant when considering the elastic deformation of a helical spring. Even when every element of the spring undergoes *infinitesimal* strain only, the combined and additive effect of rotations due to bending and torsion of the elements will produce a very pronounced *displacement* of the ends under an axial pull. Even should the material of the spring not follow Hooke's law of proportionality, the displacement *will* follow such a law because deviations from the law become noticeable at *finite* strain only. It was Hooke's lucky chance to experiment with *springs*, to measure *displacements*, and to relate force and displacement. Had he attempted a law between stress and strain, he would have failed because (i) he would not have been able to calculate either stress or strain in a spring, a problem which was solved by Kirchhoff centuries later only, (ii) had he therefore chosen the case of simple tension of a straight rod, the mechanics of which were accessible to him, his means of measurement would not have been accurate enough. Most rheological laws were discovered in such loose ways.

10. Summary

If a beam of elastic material is acted upon at both ends by bending couples M_b which are in equilibrium, it is bent to a circular arc of radius R which is

$$R = \epsilon I/M_b \qquad \text{(XII, a)}$$

The deflection of the beam, if small, is given by

$$d = L^2 M_b/8\epsilon I \qquad \text{(XII, b)}$$

The equation of the neutral axis can generally be calculated for small deflections from the differential equation

$$d^2y/dx^2 = -M_b/\epsilon I \qquad \text{(XII, c)}$$

By integration we find for the deflection of a beam through bending from its own weight w

$$d = 5wL^4/384\epsilon I \qquad \text{(XII, d)}$$

from a load P concentrated in the centre

$$d = PL^3/48\epsilon I \qquad \text{(XII, e)}$$

and from two concentrated equal loads P with a the distance between them

$$d = P(L-a)(2L^2 + 2La - a^2)/48\epsilon I \qquad \text{(XII, f)}$$

The twist of a circular rod of elastic material by the torsional moment M_t is

$$\Omega/l = 2M_t/\pi R^4 \gamma \qquad \text{(XII, g)}$$

and the stress

$$p_t = 2M_t r/\pi R^4 \qquad \text{(XII, h)}$$

If the material is plastic there is, in the case of equilibrium

$$M_t = 2\pi\vartheta_t R^3/3 - \pi\vartheta^4 l^3/6\gamma^3\Omega^3 \quad . \quad . \quad . \qquad \text{(XII, i)}$$

The viscosity-elasticity analogy permits the calculation of the rate of deflection from (XII, b to f) by replacing d by $\dot{d}$ and ϵ by λ and of the rate of twist from (XII, g) by replacing Ω by $\dot{\Omega}$ and γ by η.

CHAPTER XIII

CREEP

1. Cement Stone a Liquid, not a Solid—Glass a Solid, not a Liquid?

COMMERCIAL cement is a powder which on mixing with water hardens to an artificial stone. Many will be astonished to hear that this stone, which looks solid enough, flows, if given enough time, and is therefore actually a liquid. One may well ask: if cement stone is a liquid, what *is* a liquid? The answer is: a liquid *flows*, and flow is a *continuous* deformation under constant stress. A solid either does not flow at all, or it flows *plastically*. Plastic flow requires a stress larger than a definite stress, the yield stress, and a plastic solid does *not flow* under the action of a stress *below that yield stress*. The limiting stress below which the material does not flow may be quite low: this makes a Soft Solid. Such a soft solid may be mistaken for a liquid for quite a long time, as was the case with oil paint. In contradistinction to the plastic solid, *a liquid flows under any stress, however small.**

If this definition is accepted, cement stone is, as will presently be described, at least up to an age of five or six years, a liquid. (After that, it possibly has hardened to a solid.)

But if cement stone is a liquid, one may become doubtful, whether there are any solids at all. There is an inclination among physicists to admit a single crystal only to the status of "true solid" and to consider every amorphous material as a liquid. Glass, for instance, is spoken of as an "undercooled liquid"; but it is interesting to hear that "the 200-inch mirror for Mount Palomar is made of glass not for any optical property . . . but for its mechanical properties. In this case we are concerned with permanence of shape. . . . For in that mirror the silver or aluminium is on the front face, not the back one. . . . The glass is a purely mechanical support for the

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* Never forget that first axiom of rheology. If I speak of "any" stress, I exclude, of course, an isotropic stress.

almost infinitesimally thin mirror. We use glass in this case, not because it is transparent, but *because its general rigidity and permanence of shape are better than steel or concrete* " [55].

If this is so, glass would be—at least rheologically—a solid, not a liquid. However, let us see what Lord Rayleigh had to say on this subject: " I have tried the following experiment: A piece of optically flat crown glass 3·5 cm. long and 1·5 cm. broad and 0·3 cm. thick, was supported on wood at the extreme ends, and the middle was loaded with 6 kg. applied by means of wooden chisel edge. It remained in position from April 6th, 1938, to December 13th, 1939. At the end of that time the glass was taken out and tested on an optical flat by means of interference fringes. It was found to have been bent, the sagita of the arc amounting to 2·5 bands or 1·25 waves, that is about 6×10^{-5} cm." [56]. With the help of (III, e) and the viscosity elasticity analogy it is easy to calculate the viscosity of that particular glass at room temperature:

We have

$$\eta_c = PL^3/144\dot{d}I \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where, in our case, $I = bh^3/12 = 1{\cdot}5 \times 0{\cdot}3^3/12 = 0{\cdot}0034$ cm.^{4}; $P = 6{,}000 \times 981 = 5{,}886 \times 10^3$ dyne; $L = 3{\cdot}5$ cm.; $\dot{d} = 6 \times 10^{-5}/616 \times 24 \times 60 \times 60 = 1{\cdot}25 \times 10^{-12}$ cm/sec. We find $\eta_c = 6{\cdot}3 \times 10^{18}$ poises, using the subscript c in η_c to indicate Creep Viscosity. This is a viscosity not essentially different from the ordinary liquid shear viscosity, creep not being essentially different from slow viscous flow.

In the above-mentioned paper [55], Preston finds η_c by extrapolating to room temperature a graph showing viscosities of glass at high temperatures to be around 10^{60} or 10^{70} poises and then demonstrates that in our universe such a viscosity is without meaning. If Lord Rayleigh's experiment can be trusted, this shows the well-known dangers of extrapolation. We shall presently calculate the creep viscosity of concrete and find it to be about 3×10^{17} poises and this confirms that the " permanence of shape (of glass) is somewhat better than . . . concrete." I am not sure that it is better than steel.

Comparatively speaking, therefore, cement stone and concrete may be considered as liquids and glass as a solid—but actually this brings us back to what we said in the first Chapter, namely that *the strictly defined rheological divisions belong to ideal abstract bodies and not to real materials.* If we say that concrete *is* a liquid, every builder will laugh at us and the structural engineer dismiss the idea as fantastic ; if we say that glass is a solid, the theoretical physicists will consider us to be simple and crude. But neither the one nor the other can answer if we say : " a Hookean material is a solid, a Newtonian is a liquid ". And we shall presently have to postulate another liquid to take account of the fact that while both concrete and glass flow slowly or *creep*, both are also elastic, a property absent in a Newtonian liquid.

2. The Permanent Deformation of Concrete

The discovery of the creep of cement stone was made through the discovery of the creep of a still more " solid " material, of which cement is a constituent, viz. the creep of concrete and even reinforced concrete.

When a concrete prism of height h is loaded, it is compressed, i.e. shortened by, say Δh. When the load is removed, part of the reduction of height is recovered at once (Δh_e). After a day or two we find that still more is recovered, and this process goes on for several days up to a maximum (Δh_a). A residuum (Δh_p) is not recovered even if we wait for a long time.

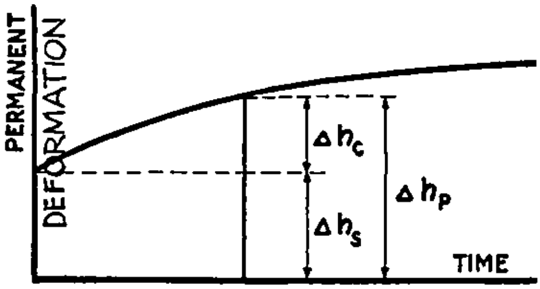

FIG. XIII, 1. Creep and permanent set.
Δh_p permanent deformation.
Δh_s permanent set.
Δh_c creep.

We accordingly have,

$$\Delta h = \Delta h_e + \Delta h_a + \Delta h_p \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Δh_e is the ordinary *elastic strain*, and the phenomenon to which Δh_a is due is called *delayed elasticity*, about which we shall speak in Chapter XV. Δh_p is called the Permanent Deformation, but needs a further analysis.

If we keep our load in position for, say, a year, we shall see that Δh_e and Δh_a are not affected, but Δh_p increases with time. If we plot the Δh_p against time and extrapolate to time $t = o$, we get a permanent Δh_s and a variable Δh_c so that

$$\Delta h_p = \Delta h_s + \Delta h_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Δh_s is called the Permanent Set and creep is the phenomenon to which Δh_c is due (compare Fig. XIII, 1). The same considerations can be applied to the deflections of a loaded beam, where h then stands for " deflection ".

The permanent deformation (Δh_p) was described by Bach as early as 1888, but as its increase in time was not noticed, it was thought to be entirely due to the permanent set (Δh_s).*

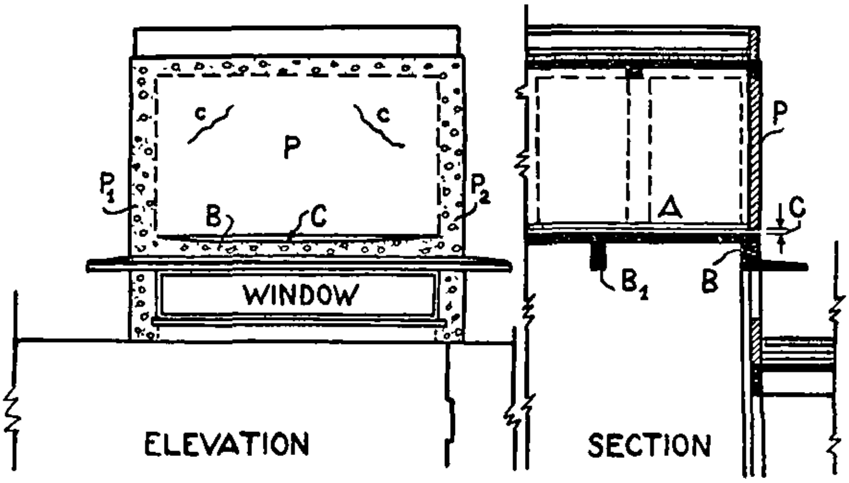

Fig. XIII, 2. Creep of a reinforced concrete beam in a building. Reinforced concrete in elevation shown dotted, in section shown full.

During the first world war, American investigators first observed and described creep. McMillan in 1921 reported on a concrete column in compression which at the end of a period of 600 days still showed a deformation proceeding at the average rate. In 1928 Faber first described creep in England. He called it " plastic yield " but it is important to keep in mind

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* We shall say more about the permanent set in Section 11 below.

that creep is *not* what we called plastic deformation. The plastic deformation of metals is easily produced by impact. In contradistinction, creep is *slow viscous* flow of a very viscous liquid and not the plastic deformation of a plastic solid. For instance, bitumen will flow slowly, i.e. will creep, but cannot be quickly deformed plastically. If an attempt is made to produce the permanent deformation quickly by impact, the bitumen breaks in a brittle manner. Also, a plastically deformable body can sustain loads up to the yield point without further deformation, while a creeping material has no yield point : it flows under the smallest load.

Faber discovered creep in a case very similar to what I had occasion to observe in a building in Jerusalem some years later. The building is a monolithic reinforced concrete frame with concrete panels later filled in (compare Fig. XIII, 2).

The floor A was heavily loaded. It was carried by beams of large spans (B and B_1). After a few years, a crack (C) was noticed. The calculation and design of beam B were checked and found in order. It was also found that the width of the crack was of such magnitude that the sagging of the beam could not have been elastic. An elastic deflection of such magnitude would have been accompanied by stresses exceeding the strength of the concrete. Actually the following had happened : the heavily loaded floor had caused high stresses in the beams and these in turn had caused creep, i.e. a deflection of the beams increasing with time. The deflection was too small to be noticed at one of the intermediate beams (B_1), but at the front wall the rigid panel (P) which was fixed between the pillars (P_1 and P_2) had not partaken in the movement of beam B and the joint between panel and beam had accordingly opened up to form a crack. Faber was led to the discovery of creep by the observation of cracks (c) in partitions which *did* partake in the movement of the beams on which they were standing.

The nature of the creep of concrete was at first not very well understood. In 1931, Straub [57], speaking of the plastic flow in concrete arches, proposed a " power law " or what would now be called a Nutting-Scott Blair equation for the creep of concrete, of the form

$$e_c = ap^m t^n \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where e_c is the deformation due to creep, p is the stress, t is time and a, m and n are constants. The equation did not allow for a yield strength and in the discussion the question was raised: Is concrete a viscous liquid or a plastic solid? This question could not be answered on the basis of the experimental material existing at the time. In 1930, Glanville [58] had stated that the creep of a 1 : 2 : 4 mix* is approximately twice that of 1 : 1 : 2 mix and it is generally assumed that the creep increases as the quantity of aggregate is increased. This would imply that the aggregates flow as loose sand may flow, and that it is the cement, which binds the aggregates together, that prevents flow.

3. The Creep of Cement and Cement Mortar

Without knowing of Glanville's paper, I co-operated in 1932 with Prof. Bingham in an investigation which aimed at finding out what actually *was* flowing in reinforced concrete [59]. Reinforced concrete consists of four macroscopic phases: cement, sand, stone, steel, and several microscopic phases in addition, among them water. It is heterogeneous and æolotropic and cannot be considered even as quasi-homogeneous or quasi-isotropic. The æolotropy is introduced through the steel which is inserted into the concrete in the shape of rods, i.e. of bodies one dimension of which exceeds the other two. Concrete, which is an aggregation of the first three material constituents, i.e. cement, sand and stone, can be considered as quasi-isotropic, but is heterogeneous because of the large size of the stones in the mix. Mortar, which consists of cement and sand only, is quasi-isotropic and can be treated to a first approximation as quasi-homogeneous. But cement itself, after setting, although it will contain water which is not chemically bound, is macroscopically isotropic and homogeneous. We therefore started with the simplest of these mixtures, i.e. hardened neat cement and cement mortar (1 : 3) using the "sagging beam" method. The beams, 2·27 cm. square in section, were placed on two supports, 76·1 cm. apart and allowed to sag under their own weight. The deflections were determined as functions of time. The curves were all of the same shape, viz. starting as parabolas and then proceeding as roughly straight lines for a limited time. The slope of the

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* *i.e.* 1 part cement, 2 parts sand, 4 parts crushed stone.

curve is a measure of the Rate of Creep, and the creep curves, therefore, show that the rate of creep first falls off rapidly and then for some time remains nearly constant. The decrease in rate of creep in the parabolic part is mainly due to chemical hardening, i.e. the setting of the cement up to an age of about 60 days when following a short curing. In beams allowed to set or kept wet for curing for a longer period before loading, the parabolic part nearly vanishes, being reduced to a small parabolic *start* extending over a few days only, which is probably due to an elastic after-effect, a phenomenon, as will be shown, entirely different from creep.

Our investigations brought out two main results :—

(i) The rate of creep in its approximately constant stage is the same whatever the previous history of the material as to setting and curing ; (ii) The rate of creep of neat cement stone is about twice or more than that of a 1 : 3 cement mortar stone.

However, our investigations were limited in two respects : firstly, there was one load only, i.e. the weight of the beams ; secondly, there were two different mixes only (1 : 0 and 1 : 3). To decide upon the relation between cement and sand in the mechanism of flow, I undertook, about ten years later, a further series of investigations in co-operation with A. Arnstein at the Laboratory for Testing Materials at Tel Aviv [60].

4. The Creep Viscosities of Different Mortar Mixes

In these investigations four neat cement beams and two beams each of the mortar mixes 1 : $\frac{1}{2}$, 1 : 1, 1 : 2 and 1 : 3 were observed under the action of their own weight and of superposed weights of 460 and 920 g. in the centre, producing bending moments approximately twice and three times the moments produced by their own weight. The dimensions of the beams were the same as of those of Bingham and Reiner.

First of all it was found that the rate of creep under the two superposed loads was roughly twice and three times that under its own weight. It had already been observed by Glanville [58] that the creep of concrete " can be considered for practical purposes as proportional to the stress." This is a characteristic property of viscous flow, in contradistinction to plastic flow. In the latter case, as there is no flow below the yield point and

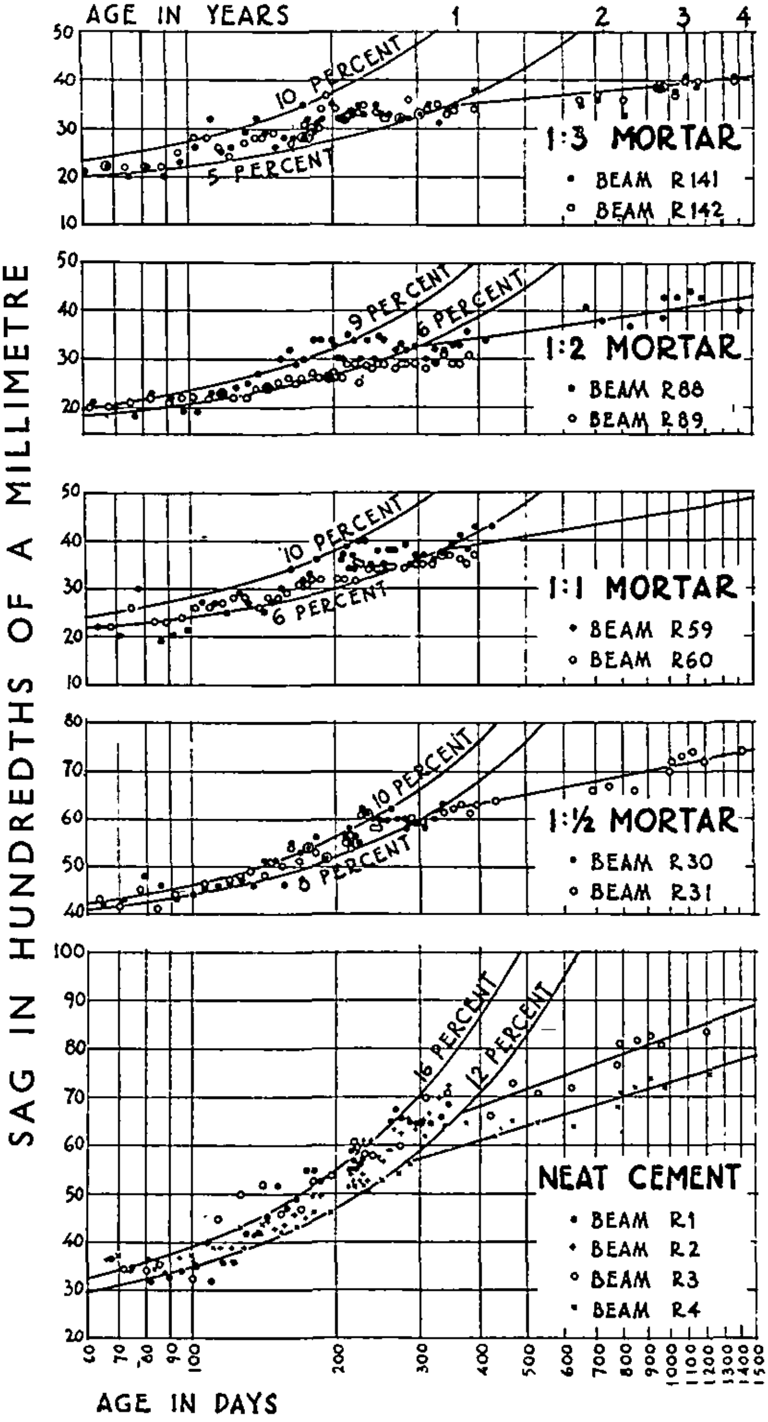

Fig. XIII, 3. Creep curves of cement and mortars. Deflections plotted against log of time.

therefore zero flow corresponds to a finite stress, there can be no proportionality. This was also Glanville's conclusion " the movement appears to be . . . in the nature of viscous flow " and Glanville and Thomas' [62] : " if there is a yield stress for concrete, its value is negligible ". This justified the calculation of creep viscosities from rates of creep, making use of the formulæ of Chapter XII.

The creep curves are shown in Fig. 3 on a semi-log scale. This scale has the advantage that prolonged times can be plotted on convenient lengths of abscissa ; it has the disadvantage that a straight line of constant slope appears as curved. The parabolas up to the age of 60 days have not been shown. The creep under superposed loads has been proportionately reduced to creep under the beam weight. The curves confirmed that there exists a roughly straight line portion as shown by Bingham and Reiner. It is followed by a curve of gradually diminishing slope as shown by Glanville.

In Table I (p. 242) the calculated creep viscosities have been entered in columns 13–15. Glanville observed compression in cylinders and deflection in beams of the type to which (XIII, f) refers ; Glanville and Thomas observed also extensions in cylinders. The materials were concretes of different mixes, different cements, and mortars. The viscosities change with age and the age of 60 days was selected as the greatest for which observations were available for all different materials, and as one which is past the first parabolic part of the creep curve during which the setting of the cement is proceeding at a high rate.*

In the present case also, as in the case of Bingham and Reiner's beams, the rate of creep of neat cement was over double that of the 1: 3 mortar. This was the more remarkable as the rate of creep of the beams prepared in Easton from the American " Atlas " cement was about four times the rate of creep of the cement beams prepared in Tel Aviv from " Nesher " cement. One conclusion only can be drawn from this as expressed by Thomas [62], viz. : " Concrete (and mortar, M.R.) is considered as *comprising two parts (or phases M.R.)* : (i) the cementitious

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* The rate of creep had to be estimated from the slope of the tangent to the creep curves. This procedure does not permit of great accuracy, and in columns 9–11 the whole range of estimated slopes has been entered, and in columns 13–15 corresponding ranges of viscosities.

TABLE I
CALCULATION OF CREEP VISCOSITIES

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	SOURCE.		FROM PAPERS						FROM CURVES				EQUS (2), (3) AND (4)			⑤/3·15	⑥/1	⑦/2·65	⑯+⑰+⑱	⑱/⑲	⑰/⑯		㉒/㉑+1	EQU (10)	⑭/㉔	㉕-1	㉖/⑳	EQUS. (11), (14)
2	AUTHORS	DESIGNATION OF TEST PIECE	BRAND OF CEMENT	MIX BY WEIGHT				KIND OF TEST	RATE OF CREEP ε AT AGE OF 60 DAYS			LOAD	CREEP VISCOSITY IN 10^{-16} POISES			MIX BY VOLUME				VOLUME CONCENTRATION OF AGGR C_v	VOLUME W/C RATIO	EXCESS WATER IN CEMENT BASE	VOLUME CONCENTRATION OF EXCESS WATER C_w	MEAN VISCOSITY OF CEMENT BASE 10^{-16} η_0	MEAN RELATIVE VISCOSITY η	MEAN SPECIFIC VISCOSITY η_{sp}	OC FROM OBSERVATIONS	10^{-16} CALCULATED
				NOMINAL	CEMENT	WATER	AGGREGATES		MIN	MAX	MEASURE		MIN	MEAN	MAX	CEMENT	WATER	AGGREGATE	TOTAL									
3	ARNSTEIN & REIMER	R1,2 3,4	PORTLAND CEMENT NEGHER	1 0	1000	250	—	BENDING	12	16	DEFLECTION IN 10^5 DAYS CENTIMETERS	OWN WEIGHT OF BEAM	37	43	49	317	250	—	567	0	0 79	0	0	43	1	0	(2 5)	43
4		R30,31		1 ½	800	200	400		8	10			59	67	74	254	200	151	605	0 25	0·79	0	0	43	1 56	0 56	2 24	70
5		R59,60		1 1	600	150	600		6	10			59	78	98	190	150	226	566	0 40	0 79	0	0	43	1 62	0 62	2 05	86
6		R88,89		1 2	400	120	800		6	9			65	82	98	127	120	302	549	0 55	0 95	0 16	0 08	40	2 05	1 05	1 90	95
7		R141 142		1 3	300	100	900		5	10			59	88	118	95	100	339	534	0·63	1 05	0·26	0·13	37	2 38	1·38	2 19	95
8	BINGHAM & REINER	3/84	ATLAS	1 0	1000	240	—							11·1		317	240	—	557	0	0 76	0	0	11·1	1	0	(2·5)	11 1
9		1/14 4/10 6/4		1 3	300	126	900							26 6		95	126	339	560	0·61	1·33	0 57	0 24	8 4	3 17	2·17	3 56	21·3
10	GLANVILLE		BRITISH	1 0	1000	220	—		48	55	INS	15 LBS	18 6	20	21 3	317	220	—	537	0	0 69	0	0	20	1	0	(2 5)	20
11		Nº 30		1 1 2	1000	400	3000	COMPRESSION	5 6	8 0	MOVEMENT PER UNIT LENGTH IN 10^5 MONTHS	1 LB PER SQ IN	75	90	106	317	400	1130	1847	0 61	1 26	0 57	0 25	15	6 00	5 00	8 2	65
12				1 2 4	1000	700	6000		16 6	21 0			28	32	36	317	700	2260	3277	0 69	2 21	1·52	0 47	10 6	3·02	2 02	2·9	36
13				1 3 6	1000	850	9000		30	45			13·2	16 6	19 9	317	850	3390	4557	0 74	2·69	2·00	0·54	9 2	1 80	0 80	1·08	14·5
14	GL. & THOMAS		RAPID HARDENING	1 1	1000	?	1000	BG	13	16	DEFLECT 10^4 INS	24 LBS	99	110	121	317	220?	377	914	0 41	0·69	0	0	54·4	2 03	1 03	2·5	—
15	GL	RH P28		1 2 4	1000	700	6000	CM	4 2	5·6	MOV. 10 L	1 LB/IN²	106	124	142	317	700	2260	3277	0·69	2 21	1·52	0·47	28 8	—	—	—	98
16				1 0			—							54 4				—		0		0	0	54 4	1	0		

material which behaves in a viscous manner when loaded, and (ii) inert aggregate, which does not flow (but moves) under load ". On this view (*a*) the rate of creep of a mortar depends *ceteris paribus*, upon the nature of the cement, but not the nature of the sand ; (*b*) the increase in creep viscosity of the mortar over the neat cement is due to the sand taking up space which it immobilises. Therefore *it is the cement which creeps*, and by embedding rigid particles into it to make mortar, the resistance to creep is naturally increased.

5. The Application of Einstein's Viscosity Equation to the Creep of Mortar

But if this is so, then the mortar is in its rheological behaviour essentially not different from any other suspension of solid particles in a liquid. We can therefore try to apply the generalized Einstein equation (IV, b)

$$\eta_{spec} = \eta_i c_v \qquad \ldots\ldots \qquad (2)$$

and determine how the intrinsic viscosity η_i depends upon the volume-concentration c_v in the case of cement mortar concretes.

In the present case, c_v is the volume concentration of the aggregate in the mix. This can be calculated from the given weight concentrations if the specific weights of the ingredients making up the mortar are known. We may assume for the latter the values shown in Table II.

TABLE II

Material	Water	Cement	Aggregates
Specific Weight in g/cm³	1	3.15	2.65

The recorded mixes by weight are shown in columns 5, 6, 7 of Table I and the calculated mixes by volume in columns 16, 17, 18. The volume concentration of the aggregates in the mix, c_v, was calculated from the volume of the aggregates as shown in column 18, and the total as added up in column 19, and it was entered in column 20. In Fig. XIII, 4, the creep viscosities η_c from columns 13–15 have been plotted against the volume concentrations c_v of the aggregates as shown in column 20. A remarkable fact results: all observed points for the mortars lie in a first approximation on straight lines radiating from a point (A) which lies at the distance of 0·4 from the origin. This distance is equal to 1/2·5 and, therefore, η_i is in a first approximation a constant and equal to 2·5. In a second approximation the points drop below the lines; the more so the leaner the mix. One reason for this becomes evident from an examination of column 21 of Table I. The water-cement ratio is not the same for all mixes. On the contrary, it increases for the leaner mixes. However, in the same way as the sand *increases* the viscosity by immobilising volume, water (with a viscosity negligible in comparison to the viscosity of the cement stone) *decreases* the viscosity. Denoting by η_o the viscosity of the neat cement stone, the viscosity of any other cement "base" in a mortar ($\bar{\eta}_o$) will be, in accordance with Guth and Mark [63],

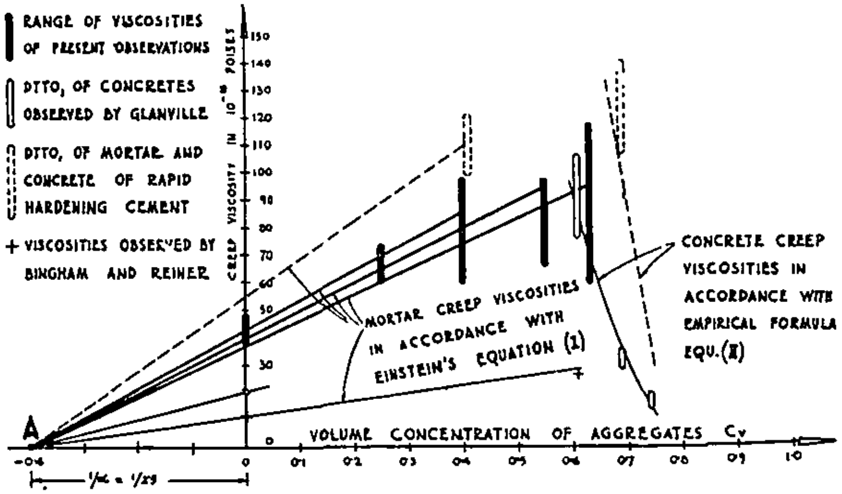

FIG. XIII, 4. Creep viscosities plotted against volume concentration of aggregates.

$$\bar{\eta}_o = \eta_o(1 - c_w) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where c_w is the volume concentration of water which the cement base contains *in addition to the water of the neat cement stone.* E.g. the neat cement stone in our case was composed of 317 parts cement and 250 parts water by volume, i.e. in a volume water-cement ratio of 0·79. In the 1 : 3 mortar, the volume water-cement ratio was 1·05 (compare column 21, row 7 of Table I). On 100 parts of cement there were, therefore, in the first case 79 parts of water and in the second case 105 parts of water, or 26 parts in excess. The total volume of cement and water in the mortar is 205 parts, and 26 parts, therefore, correspond to a volume concentration of excess-water $c_w = 26/105 = 0{\cdot}13$. The values of c_w have been calculated accordingly and entered in column 23. From these the viscosities η_o of the cement base were calculated and entered in column 24. When these viscosities are plotted on the vertical axis of Fig. XIII, 4, and connected with the point A, the observed mortar viscosities lie *well* on the straight lines radiating from *A*.

6. The Creep Viscosities of Mortar confirming Einstein's Equation

I must confess that when I had found that Einstein's equation (IV, a) is valid for cement mortars up to a concentration of over 60(!) per cent., I thought there was something wrong in the observations or their interpretation. In all text-books dealing with this equation, the experiments of Bancelin (1911) and Oden (1913) are quoted as providing a fair proof of the validity of the equation for spherical particles and concentrations of not more than 3 per cent. In the present case the particles (Leighton Buzzard British Standard sand) were fairly spherical, but the concentrations unbelievably high. Nevertheless, on second thoughts, the agreement with Einstein's equation can be understood. As we said in Section 2 of Chapter IV, there are two factors which may invalidate the equation. Firstly, if the suspended article is not exactly spherical, the change of orientation during flow will change the degree of interference with the flow of the dispersion medium. Secondly, if the concentration is not very small the modification in the flow of the dispersion medium caused by any one particle reaches into the field of neighbouring particles, increasing the resistance to flow still more. However, if the viscosity of the dispersion medium, in our case the cement base, is of the order of 10^{17} poises, the flow during one year will be too small to cause either an appreciable change in the orientation of the sand particles or an interference beyond the volume proper of the particle. One year in the flow of cement is as much as

10^{-19} years or about 10^{-12} seconds in the flow of water. If the observation were to last 10^{-12} seconds only we would likewise in a suspension in water not observe any interference other than the simple taking up of volume. As a matter of fact, in such a short period, a state of steady flow could not establish itself. To change the picture from a temporal into a spatial one: the distances between particles in a " liquid " of a viscosity of the order of 10^{17} poises are equivalent to distances in water 10^{19} times as great. We can accordingly see that a concentration at 60 per cent. in cement is infinitesimally small in terms of concentration in water.

7. A Formula for the Creep-viscosity of Cement Mortars

On the basis of the foregoing, the creep viscosity of a cement mortar η_c can be predicted if the creep viscosity of the neat cement η_o used in the mix is known.

We should have

$$\eta_c = \eta_o \frac{1 + WC}{1 + \overline{WC}} (1 + 2{\cdot}5c_v) \quad . \quad . \quad . \quad . \quad (4)$$

where WC is the volume water-cement ratio of the neat cement, $\overline{WC}$ is that of the cement base used in the mortar and c_v the volume concentration of the sand in the mortar. Equation (4) results from (3) through

$$c_w = \frac{WC - \overline{WC}}{1 + \overline{WC}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

8. An Empirical Formula for the Creep-viscosity of Concrete

Equation (4) does not hold good for concretes. First of all, the relative creep viscosity of the 1 : 1 : 2 concrete, which has a concentration of inert material equal to that of a 1 : 3 mortar ($c_v = 0{\cdot}61$) is much higher than that of the mortar. The reason may well be that besides cement, water and aggregates, there is always another material present in the mix, viz. *air*, filling voids. The viscosity of air is negligible and its volume should be added to the volume of the excess water. It may be assumed that the leaner the mix the more voids in the mortar or concrete. At the same time, however, it will be seen that a 1 : 1 : 2 concrete, because of the better grading of the aggregate, will have less pore space than a 1 : 3 mortar. Therefore, the relative viscosity of the former should be greater—and so it is.

The values for η_i were calculated for concretes from (1), where η_{spec} and c_v were introduced from columns 26 and 20 respectively. They can be approximated by a straight line

$$\eta_i = 40 - 53c_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $0{\cdot}60 < c_v < 0{\cdot}75$. This gives for concrete

$$\eta_c = \eta_o \frac{1 + WC}{1 + \overline{WC}} (1 + 40c_v - 53c_v^2) \quad . \quad . \quad . \quad (7)$$

In order to check (6) and (7), the creep viscosities of the mortars and concretes were calculated from these equations, using the values for c_v from column 20 and for c_w from column 23. The results were entered in column 28. As can be seen, the viscosities calculated from the equation fall within the ranges of the observed viscosities.

9. Dissimilarity of Creep-Behaviour of Mortars and Concretes

The dissimilarity of the behaviour of mortar and concretes—the first having their rate of creep decreased with increased quantity of aggregate, the second, on the contrary, having the rate of creep increased—can be understood on the analogy of the behaviour of a sand-water mixture. When the sand is dry, its cohesion is very small; its internal friction, as measured by the angle of repose, is also comparatively small. When we add a little water its cohesion is increased and there is an optimum water content, which we used when as boys we built castles on the beach. Still more water reduces the cohesion and 100 per cent. of water, of course, flows freely. The complete curve will, therefore, pass through the zero for $c_w = 1$ and $c_w = 0$ where c_w is the volume concentration of water. When c_w decreases from 1 to 0 the curve will go up at the nearer end and go down at the farther end. The rôle of water is played in our case by the cement. When the aggregates are on the increase and the cement on the decrease, the mortars lie on the rising part, the concretes on the descending part of the curve.

10. The Volume-Flow of Concrete

In our foregoing considerations we have over-simplified conditions. We assumed in accordance with our first rheological axiom

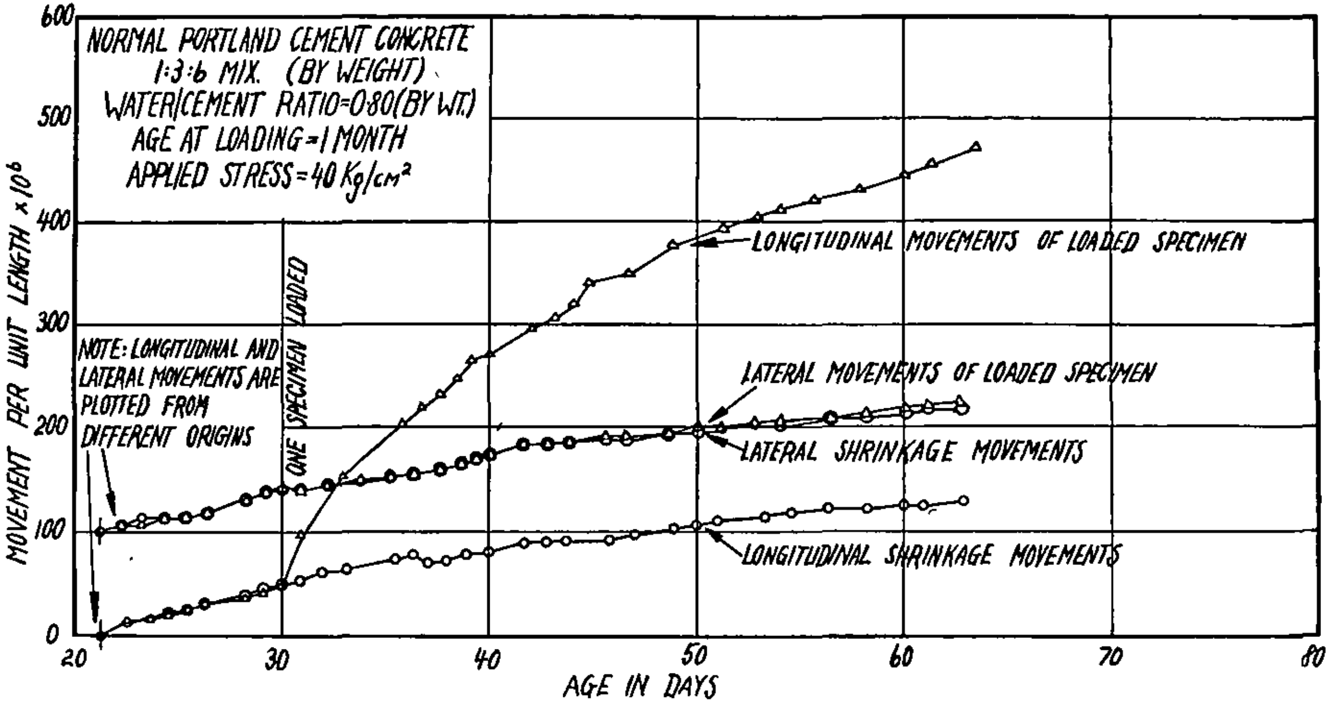

FIG. XIII, 5. Longitudinal and lateral creep.

that there is no voluminal flow, i.e. that all flow consists in a continuous distortion and that there is no such thing as a continuous condensation or rarefaction of the material. The assumption is so natural that Hencky based his theory of plastic flow upon it without questioning it and we also accepted it as an axiom. One reasons that, say, in compression, the condensation can, in Trouton's words (compare Section 11, Chapter X) be an " initial " effect only, or that we are, as we called it, in the *second* stage of viscous traction. But when we come to materials with a viscosity of the order of 10^{17} poises " initial " may be a period exceeding our lifetime. This is what Glanville and Thomas [62] found when investigating " lateral " creep. Concrete, on hardening, " shrinks ", i.e. all its dimensions are reduced proportionally. There is a negative cubical dilatation, e_v, going on for some time with a decreasing rate $\dot{e}_v$ without any corresponding stress. This is one of those time-effects with which we shall deal in Chapter XVIII.

In the sagging beam test, shrinkage will reduce the radius to which the beam is bent, but the influence upon the deflection is negligible. It is not so when a concrete cylinder is observed under compression and longitudinal and lateral movements are measured. Here shrinkage movements must be deducted. These were taken account of by measuring both longitudinal and lateral movements in a loaded and an unloaded specimen. It was found that the longitudinal movement increased very rapidly as a result of loading, but the lateral movement was practically unaffected, continuing to increase at about the same rate as the unloaded specimen (compare Fig. XIII, 5). From the results of the tests, which the authors call " surprising ", the conclusion must be drawn that there is practically *no lateral creep*.

Following (X, k) we can define a viscous analogue to Poisson's ratio σ by

$$\dot{e}_c = -\sigma_v \dot{e}_n \qquad \dots\dots \quad (8)$$

If σ_v is a constant, this gives by integration

$$e_c = -\sigma_v e_n \qquad \dots\dots \quad (9)^*$$

---

* Here the subscript " c " indicates " contraction " and not creep, while " v " indicates " viscous " and not volume.

In the case of concrete, as observed by Glanville and Thomas, σ_v vanished while σ was *0·14*.

This surprised the authors because, as we recognise, they had tacitly assumed that the first axiom was valid and that, therefore, the volumetric deformation could only be an elastic strain taking place in the " first stage ". Later, in the second stage there would be no change of volume, or (compare (X, 27′)) $\sigma_v = 0{\cdot}5$. A vanishing Poisson-ratio indicates a case of simple extension (in our case negative extension). This, we said in Section 1 of Chapter X, is " not . . . easily realized " and we gave a very far-fetched example. Now we meet such a case under quite ordinary conditions of an everyday material. But let us remember that we found it necessary in Section 1 of Chapter X to restate the first rheological axiom in the following form : " Under isotropic pressure all non-porous materials are elastic and only elastic " and that we said that this was an implicit (rheological) definition for " non-porous ". Glanville and Thomas' experiment accordingly revealed by rheological means the porosity of concrete which, of course, is known from other facts. In creep the concrete flows *into its pores and concrete shows voluminal flow.*

11. Trouton's Coefficient in the Case of Simple Extension

This gives us occasion to deal with the following problem :—

Let a material, the rheological equation of which is (IX, a) and which, therefore, has both volume elasticity (κ) and volume viscosity (η_v) undergo simple extension ; what will the magnitude of Trouton's coefficient be, and in what manner will flow take place ?

Our case is the one defined by (X, 14), in accordance with which $e_v = e_n$.

Introducing this expression into (X, 51), we get

$$\dot{e}_n + \frac{\kappa}{3\eta_v} e_n = p_n(\eta + 3\eta_v)/9\eta\eta_v \quad . \quad . \quad . \quad (10)$$

This equation is of the type

$$dy/dx + My = N \quad . \quad . \quad . \quad . \quad . \quad (11)^*$$

the solution of which is

$$y = e^{-\int M dx}\left(\int N e^{\int M dx} dx + C\right) \quad . \quad . \quad . \quad . \quad (12)^*$$

---

[*] We shall meet the same differential equation again in Chapter XIV.

as can easily be verified by differentiation and substitution into (11).

Accordingly, we find

$$e_n = p_n \frac{\eta + 3\eta_v}{3\eta\kappa}(1 - \mathrm{e}^{-\kappa/3\eta_v . t}) \quad . \quad . \quad . \quad . \quad (13)$$

and, therefore,

$$\dot{e}_n = p_n \frac{\eta + 3\eta_v}{3\eta\eta_v} \mathrm{e}^{-\kappa/3\eta_v . t} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This gives

$$\lambda = \frac{p_n}{\dot{e}_n} = \frac{9\eta\eta_v}{\eta + 3\eta_v} \mathrm{e}^{\kappa/3\eta_v . t} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

At the beginning of loading, i.e. for $t = 0$, we have

$$\lambda_o = \frac{9\eta\eta_v}{\eta + 3\eta_v} = 3\eta - \eta^2/\eta_v + \eta^3/3\eta^2{}_v - \dots \quad . \quad . \quad (16)$$

and only if $\eta_v = \infty$ is $\lambda_o = 3\eta$, in which case λ is constant and throughout $= 3\eta$. If $\eta_v < \infty$, $\lambda_o < 3\eta$. In this case λ is not constant, but increases with increasing time. There will be a limiting strain, which from (13) can be derived at

$$e_\infty = p_n \frac{\eta + 3\eta_v}{9\eta\kappa} = p_n \frac{\eta_v}{\lambda_o \kappa} \quad . \quad . \quad . \quad . \quad (17)$$

The increase of creep viscosity in concrete after some years, may have other causes besides the filling up of holes. It is not likely to be due to a chemical change, because if such a one has not taken place for quite a prolonged period during which the creep is a straight line, there is no reason why it should suddenly appear at a later date. It may, however, be that the modification of flow of the cement in the vicinity of a sand particle, caused by the *other sand particles* about which we spoke in Section 6 and which at first is absent, may make itself felt after years, thereby increasing the overall viscosity of the mortar, bringing its operative concentration up to the nominal concentration. Gradually a 1 : 3 mortar would tend to acquire the viscosity of a suspension of 63 per cent. concentration and only when this was the case would the state of flow become steady.

We have not yet explained the nature of the "permanent set" which we mentioned in Section 2. This is a deformation which appears immediately on loading—in contradistinction to the elastic after-effect and creep—and which is not recovered—in contradistinction to the instantaneous or acoustical elasticity. The appearance of a permanent set on deforming a body can be understood from the fact that no body is perfectly homogeneous and that all possess pores or holes and also inclusions of foreign material. To these imperfections should be added sharp local depressions on the surfaces, scratches and the like, forming notches. A stressed body will have stress-concentrations at these places. When considering the surface of a body these concentrations are spoken of as " notch-effects ", but the stress concentrations at holes and inclusions are not essentially different.

These flaws form weak spots and, while the body as a whole may be strong enough to resist the forces causing the deformation, at these irregularly interspersed imperfections the strength of the material may be exceeded.

In the case of a plastic material this will cause local plastic deformations ; in the case of brittle material local rupture. When the body is released from the external forces, *the aggregate of these minute plastic deformations or ruptures will show itself as a permanent set.* The stress concentration is the more marked the sharper the angle at the notch or the smaller the smallest radius of curvature at the hole. A local plastic deformation will tend to flatten the angle or to increase the radius of the curvature and a second loading of the same extent may therefore be more nearly perfectly elastic. This is well known in many metals.

A local rupture, however, may make the angle even sharper and produce sharp re-entrant angles starting from holes. In this case second and further loadings will again produce permanent sets and after many such repeated local breakages the rupture may spread across the whole body and the body breaks. Such rupture of a material after repeated loading and unloading is called Fatigue. The nature of fatigue which is the *failure under repeated cycles of stress or strain* is not yet fully understood. Fatigue occurs not only in brittle materials but also, and predominantly, in plastic metals. It should, of course, be kept in mind that in metals the local plastic deformation must raise the local yield point by causing local strain-hardening and therefore brittleness. It may also be assumed that apart from imperfections of structure of microscopic size there should also be sub-microscopic variations of cohesional bonds (atomic or molecular) of the material, and some may break under comparatively small stresses. This will cause a redistribution of stresses, increasing the latter beyond the average and bringing about in turn other breakages of cohesional bonds. Conditions of fatigue would accordingly be governed by principles of statistics. It appears that whatever laws of fatigue have been established until now, can simply be explained from such statistical considerations (Freudenthal [64]), but the *mechanism* of fatigue has still to be discovered.

It is, however, clear that no direct relation will exist between fatigue and damping of oscillations of the material in question. Whatever causes dissipation of energy will cause damping of oscillations. We shall see in Chapter XIV that there may be damping without any change in the structure of the material and that fatigue can only be one of the contributory causes of damping.

12. The Loading Test for Reinforced Concrete Structures

The considerations of the last Section are important in connection with the Load Test for reinforced concrete structures.

When there is doubt about the safe bearing capacity of a reinforced concrete structure, many regulations prescribe a "load test". For instance, the American Concrete Institute Standard prescribes that a beam or slab shall, under a certain load, show a deflection not greater than a certain maximum which is directly proportional to the square of its span and inversely to its thickness. However, "if the deflection exceeds (that) value . . . the construction shall be considered to have passed the test if within twenty-four hours after the removal of the load the portion of the structure shows a recovery of at least 25 per cent. of the observed deflection". The British Code of Practice has an analogous stipulation but uses the words "if after the removal of the load" without saying *when* after the removal. The experience of the Testing Laboratory of the Standards Institution of Palestine has almost invariably been that if the recovery is determined *immediately* after the removal of the load, most structures must be considered to have failed, but if the recovery is measured after twenty-four hours, there is hardly any structure so deficient that it would not pass.

The history of the load test and the theory underlying it are difficult to trace. It looks as if somebody somewhere introduced the test as a rule of thumb and its provisions were then copied by others for lack of something better. The theory would seem to be that a structural member should be elastic and the more elastic the better its performance. The non-recovered deformation indicates a deficiency in elasticity and the ratio of recovery to total deformation would accordingly be a measure of the fitness of the member.

This view, however, does not take into account the two important phenomena happening with time, viz. (i) creep, (ii) elastic after-effect.

In creep, the molecules of the material change places without any cracks or discontinuities appearing in the material and, therefore, without any loss in strength. This has been confirmed by experiment. The non-recovered part of the deflection being partly due to creep, the view that non-recovery of deflection is an indication of weakness cannot be entirely correct.

The nature of the elastic after-effect, which we mentioned in Chapter IX, will be fully discussed in Chapter XIV. It will be seen that an elastic after-effect does not indicate a lack in strength of the material—rather the opposite. It is the permanent set only which points to a weakness in the concrete and which on repeated loading may lead to destruction. On first loading a permanent set resulting from local failures such as crushing around holes or at notches is unavoidable. On second loading one of two things may happen. If the concrete has internally adapted itself to the load by *closing* holes, cracks and notches, there will be no *new* permanent set. If, however, the structure of the concrete is such that in the first loading notches are developing into cracks and cracks are lengthened, the second loading will result in an *additional* permanent set. In plotting loads increasing from zero and then decreasing to zero, against deflections, this reveals itself in the existence of a " power-diagram " enclosing between the loading and the unloading curve an area which is a measure of the work performed. In this case the structure behaves like a machine, in fact as a stone-crusher working for its own destruction.

Accordingly, it appears that a proper specification for a load test should not use the *total* recovery or non-recovery of the deflection as the criterion, but only the existence or the absence of permanent sets on second and further loadings. A specification on these lines is at present in hand at the Standards Institution of Palestine [65].

13. Summary

A liquid flows under any distortional stress, however small, while a solid flows plastically if the yield stress is exceeded. Seemingly solid materials such as pitch, bitumen, concrete, glass and metals may flow in the liquid manner, but exceedingly slowly: this slow flow is called creep. That permanent deformation of a body, which is neither plastic nor due to creep, is called permanent set, *i.e.* the aggregate of minute local microscopic failures of the material caused by random stress concentrations. Accumulating permanent sets may cause fatigue or rupture under repeated cycles of stress or strain. The latter may also be due to submicroscopic variations of cohesional bonds and local breakages.

The creep viscosity η_c of cement mortar can in a first approximation be calculated from the following modification of Einstein's equation (IV, a)

$$\eta_c = \eta_o \frac{1 + WC}{1 + \overline{WC}} (1 + 2{\cdot}5c_v) \quad . \quad . \quad \text{(XIII, a)}$$

where WC is the volume-water-cement ratio of the neat cement of viscosity η_o and $\overline{WC}$ that of the cement base used in the mortar.

For concrete the following empirical formula can be used

$$\eta_c = \eta_o \frac{1 + WC}{1 + \overline{WC}} (1 + 40c_v - 53c_v^2) \quad . \quad \text{(XIII, b)}$$

If the material is porous, as is the case with concrete, part of the flow is into the holes and the flow has accordingly a volume-component accompanied by a volume viscosity η_v.

A viscous analogue to Poisson's ratio can be defined by

$$\dot{e}_c = - \sigma_v \dot{e}_n \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII, c)}$$

where $\dot{e}_c$ is the rate of contraction. When there is no volume flow $\sigma_v = \frac{1}{2}$, otherwise $0 \leq \sigma_v < \frac{1}{2}$. When σ_v is constant, it can be calculated from

$$e_c = -\sigma_v e_n \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII, d)}$$

For $\sigma_v = 0$, in which case there is no lateral creep, i.e. no change of lateral dimensions, we have

$$\lambda = \frac{9\eta\eta_v}{\eta + 3\eta_v} \mathrm{e}^{-\kappa/3\eta_v \cdot t} \quad . \quad . \quad . \quad . \quad \text{(XIII, e)}$$

When all holes are filled, the limiting deformation

$$e_e = p_n \frac{\eta_v}{\lambda_o \kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII, f)}$$

is reached, where

$$\lambda_o = 9\eta/(3 + \eta/\eta_v) \quad . \quad . \quad . \quad . \quad \text{(XIII, g)}$$

CHAPTER XIV

ELASTICO-VISCOSITY AND FIRMO-VISCOSITY

1. Elastic Liquids

In Chapter XIII we have described the rheological behaviour of materials which, while they impress us as very solid, nevertheless, under constant (distortional) stress of however small a magnitude are continuously deformed, or, in other words, flow. Such flow is very slow and is therefore termed creep and it can be detected only through exceedingly accurate measurements over long periods. However, while they flow, these materials are not Newtonian liquids. In addition to viscosity, they exhibit *elasticity*. Their complex rheological behaviour has therefore been named by Jeffreys [66] Elastico-Viscosity. The first component of this property is not necessarily connected with a high value of the second, as is the case in cement, glass and asphalt. Hess [67] has described the elastic behaviour of a 1·5 per cent. starch solution which is elastic, while not exceedingly viscous. The liquid is brought into rotation by rotating its container, which is then suddenly stopped. " When a state of rest is apparently about to be reached with gradual diminution of the rotational velocity, we see that the liquid starts to move again in the opposite direction. A state of rest is again approached, first with increasing and then with diminishing velocity, but it is not final. Only after several oscillations does the liquid come to rest. The oscillations do not appear in the case of a glycerine-water mixture of corresponding viscosity: here the first rest is definite. These oscillations must be due to elastic forces which arise in the liquid from internal structures."

A 1·5 per cent. starch solution is obviously a liquid. Here the flow will not be spoken of as creep and this will underline that there is nothing in creep which distinguishes it from viscous flow except its low rate.

A few words must be added about the difference between an elastic liquid and a soft plastic solid. The latter also " flows ", but it flows *plastically*, i.e. only after the yield stress has been exceeded. Up to the yield point it is elastic, but *its elasticity is of a very different kind.* In a soft plastic solid the elastic component becomes predominant at *small* velocities. In contradistinction the elasticity of a true liquid makes itself felt at *high* velocities, when no time is given to its elastic stresses to " relax ". Twenty years ago, when rheology was in its infancy, that difference was not so well understood and I published a few papers on the theory of flow of soft plastic solids under titles referring to elastic liquids (compare Section 1 of Chapter IV). The theory was correct, but the terminology was misleading.

2. Relaxation of Stress

The elasticity of a solid, *even of a soft solid,* is of the Hooke type. The strainwork which has been put into a distorted Hooke body is stored there safely for as long as it is loaded; and it can be regained at any time by removing the load. So, on releasing the pressure after the second stage shown in Fig. I, 6, the segment disappears and the elastic potential energy stored in the paste, small as it may be, is regained. On increasing the pressure to the third stage, that potential energy is not affected and in the fourth stage it is regained in the same amount. In contradistinction, an elastico-viscous material, so to speak, *leaks:* the longer we wait, the less complete is the recovery on removal of the load. In other words, the strained material loses resilience, its internal stresses *relax.*

In his investigations on the dynamical theory of gases (1868), Maxwell was the first to treat the *relaxation of elastic stresses.* In his words, but using our notation and interspersing our comments :—

" *A distortion or strain of some kind, which we may call* e, *is produced in the body by displacement. A state of stress or elastic force which we may call* p *is thus excited. The relation between the stress and the strain may be written* $p = \epsilon e$, *when* ϵ *is a coefficient of elasticity for that particular kind of strain. . . .*" That coefficient is Young's modulus ϵ if p and e are *normal* stresses and strains, and it is the modulus of rigidity if they are *tangential* stresses and strains. " *In a solid body free from relaxation,** p *will remain* $= \epsilon e$, *and*

$$\dot{p} = \epsilon \dot{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

---

* Here I have slightly changed Maxwell's wording.

"*If, however, the body relaxes, p will not remain constant, but will tend to disappear, at a rate depending on the value of p and of the nature of the body. If we suppose this rate proportional to p, the equation may be written*

$$\dot{p} = \epsilon \dot{e} - p/\tau \quad . \quad . \quad . \quad . \quad . \quad . \text{''} \qquad (2)$$

τ is a material constant, the nature of which will be explained later.

"... *which will indicate the actual phenomena in an empirical manner.*"

For "empirical" we would say "phenomenological".

"*For if e be constant,*

$$p = \epsilon e e^{-t/\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

showing that p gradually disappears, so that if the body is left to itself it gradually loses any internal stress. . . ."

Equation (3) can easily be derived from (2) as follows: If e is constant $\dot{e}$ vanishes and (2) becomes

$$dp/dt = -p/\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

or

$$dp/p = -1/\tau \,.\, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

which by integration gives

$$\ln p - \ln p_0 = -t/\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

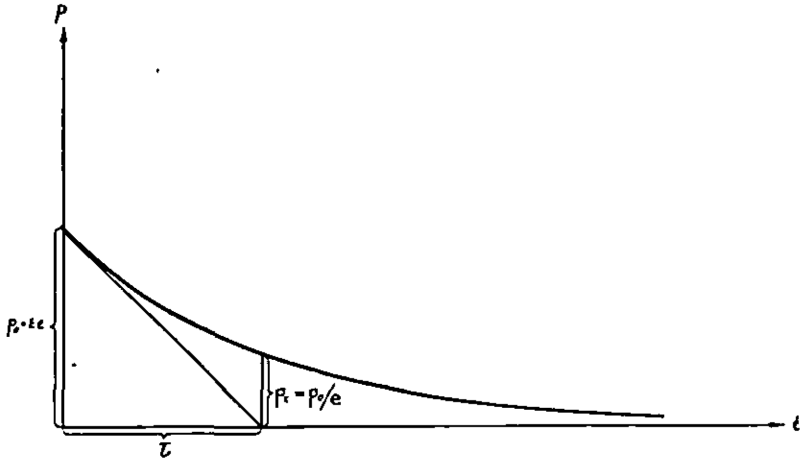

FIG. XIV, 1. Relaxation curve of a Maxwell liquid. p_0 stress at the time $t = 0$. τ relaxation time. e basis of natural logarithms.

where p_0 is the stress produced originally, i.e. at time $t = 0$, in the body. This stress, however, is equal to ϵe.

From (6) follows (3).

If we plot the stress against time the resulting curve is as shown in Fig. 1. The constant τ has the dimension of time, because the exponent t/τ of e must be a dimensionless number.

If we introduce in (3) τ for t, we get

$$p_\tau = p_0/\mathrm{e} \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

The time of relaxation is therefore that time in which the stress relaxes to the " eth " part of its original magnitude.

3. Viscosity Arising from Relaxation

Maxwell, actually, was not so much interested in the relaxation of the elastic stresses in solids. Following an earlier attempt by Navier to derive the equations of viscous flow of liquids from the elasticity equations by assuming instantaneous relaxation, he postulated his law (2) as a starting point for a theory of viscosity. He accordingly continues :—

" *If $\dot{e}$ is constant (= c), that is if there is a steady motion of the body . . .*"

This is meant to read " of the particles of the body ", because this is not a question of the motion of the body as a whole. Even so the statement is not quite correct because $\dot{e}$ is not the velocity of a motion, but the rate of a strain.

" . . . *which continuously increases the displacement,*

$$p = \epsilon\tau c + C\mathrm{e}^{-t/\tau} \quad . \; . \; . \; . \; . \; . \; . \quad (8)$$

showing that p tends to a constant value depending on the rate of displacement " (read " rate of strain."—M.R.) " *The quantity $\epsilon\tau$, by which the rate of displacement* " (as before) " *must be multiplied to get the force* " (read " traction "), " *may be called the ' coefficient of viscosity.'* "

Equation (8) can be verified by differentiation. We get $\dot{p} = -\frac{c}{\tau}\mathrm{e}^{-t/\tau}$ and as from (8) $c\mathrm{e}^{-t/\tau} = p - \epsilon\tau c$, this gives (2).

The identification of $\epsilon\tau$ with a coefficient of viscosity can be supported by the following consideration : If p is constant, $\dot{p}$ vanishes and (2) gives

$$\dot{e} = p/\epsilon\tau \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

Under constant stress the rate of strain is therefore also constant which means that there is continuous flow. In addition, assuming ϵ and τ to be constants, the relation between stress and rate of strain is linear, This, if stress and strain are tangential, is Newton's law of viscous flow ; and if they are normal (as we have assumed in (1)) it is Trouton's law.

Let us continue the quotation from Maxwell. " *It* ($\epsilon\tau$) *is the product of a coefficient of elasticity* (ϵ) *and a time* (τ) *which may be called the Time of Relaxation of the elastic force. In mobile fluids* τ *is a very small fraction of a second and* ϵ *is not easily determined experimentally. In viscous solids . . .* "

In our terminology we call such materials *liquids* and not solids, reserving the term " viscous solids " for very different materials as will be explained in Section 6. We would say here " *In very viscous elastic liquids . . .* " and, continuing the quotation, " . . . τ *may be several hours or days, and then* ϵ *is easily measured. It is possible that in some bodies* τ *may be a function of p. . . .* "

Introducing

$$\lambda = \epsilon\tau \quad . \; . \; . \; . \; . \; . \; . \quad (10)$$

Equation (2) can be written in the form

$$\dot{e} = \dot{p}/\epsilon + p/\lambda \quad . \; . \; . \; . \; . \; . \; . \quad (11)$$

In the quoted paper Maxwell arrived at the notion that the " rigidity " of air is equal to its pressure, i.e. in the free atmosphere equal to *1·014* × *10*^{6} dyne cm^{-2}. This sounds paradoxical, but one should bear in mind that the *elastic* deformation is the *smaller*, the *greater* γ. Therefore, for an inelastic Newtonian liquid $\gamma = \infty$.

Having determined the viscosity of air at atmospheric pressure by experiment (which is about 0·0002 poises), he easily found its time of relaxation to be 1·961 × 10^{-10} seconds. We can therefore say that the order of magnitude of the smallest time of relaxation is 10^{-10} seconds, and we may assume that even for water it does not vanish altogether. Experiments to find a rigidity of water must therefore have been unsuccessful because of imperfections of experimental technique only.

4. Remember the First Axiom

There is a flaw in the foregoing considerations. Maxwell must have been well aware of what we called the first axiom of rheology, but he did not pay attention to it. This was first emphasized by Reiger [68] who reports* that Zaremba [69] pointed out before him that, in principle, there must be two different times of relaxation just as the elastic state of a body is determined by two independent elastic constants. In hydrodynamical calculations a liquid is usually assumed as incompressible. This may be justified when one considers flow, because the deformations due to liquid flow are incomparably greater than those due to volumetric strain. However, when we consider an *elasticity* of the liquid, similar to the one of a solid, we must remember (compare the Table in Chapter IX) that actually liquids are much more compressible than solids.

We therefore have to assume, in principle, two different times of relaxation, viz.

$$\tau_t = \eta/\gamma, \ \tau_v = \eta_v/\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

one tangential, the other voluminal.

Reiger assumed that there is no relaxation of volume elasticity, or $\tau_v = \infty$ which implies that $\eta_v = \infty$. This was also our point of view when postulating the " first axiom of rheology " in Section 2 of Chapter I. Since then, what we learned about the voluminal flow of concrete (compare Section 10 of Chapter XIII) has made us more careful and our first axiom is more modestly reduced to the statement that *for every property there are two independent rheological coefficients*, one referring to changes of volume, the other to changes of shape. We therefore replace Equation (11) by the two independent equations

$$\dot{e}_t = \dot{p}_t/\gamma_t + p_t/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\dot{e}_v = \dot{p}/\kappa + p/\eta_v \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where γ and κ are the two elastic moduli, η and η_v the two coefficients of viscosity referring to change of shape and change of volume respectively.

To find the relaxation equation for simple tension we must first resolve the simple tension p_n into its two components in accordance with Equation (X, f)

$$p = p_n/3; \ p_{on} = 2p_n/3 \quad . \quad . \quad . \quad . \quad . \quad (15)$$

---

* I must warn the reader desirous of studying Reiger's paper of its numerous mistakes, which I cannot point out in detail.

while the corresponding strains are in accordance with Equations (X, b), (X, c) and (X, k)

$$e_v = e_n(1 - 2\sigma)\,; \; e_{on} = 2(1 + \sigma)e_n/3 \quad . \quad . \quad . \quad (16)$$

Taking first the voluminal part, Equation (14), we find

$$\dot{e}_n(1 - 2\sigma) = \dot{p}_n/3\kappa + p_n/3\eta_v \quad . \quad . \quad . \quad . \quad (17)$$

where, in differentiating the first of (16) we have assumed that σ is a constant.

For the distortional part we go back to Equations (X, g) and (X, h) and replace (13) by

$$\dot{e}_{on} = p_{on}/2\gamma + p_{on}/2\eta \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which, introducing (15) and (16), becomes

$$2(1 + \sigma)\dot{e}_n = \dot{p}_n/\gamma + p_n/\eta \quad . \quad . \quad . \quad . \quad (19)$$

Elimination of σ from (17) and (19) by adding both gives

$$\dot{e}_n = \frac{\dot{p}_n}{3}\left(\frac{1}{3\kappa} + \frac{1}{\gamma}\right) + \frac{p_n}{3}\left(\frac{1}{3\eta_v} + \frac{1}{\eta}\right) \quad . \quad . \quad (20)$$

However, in accordance with (X, m), $1/3\kappa + 1/\gamma = 3/\epsilon$, while from (X, p)

$$\frac{1}{3\eta_v} + \frac{1}{\eta} = \frac{3}{\lambda} + \frac{\kappa e_v/p_n}{\eta\eta_v} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This yields, considering the first of (16)

$$\dot{e}_n = \dot{p}_n/\epsilon + p_n/\lambda + \frac{\kappa e_n(1 - 2\sigma)}{3\eta\,\eta_v} \quad . \quad . \quad . \quad (22)$$

Equation (11), therefore, is correct only if either $\eta_v = \infty$ or $\sigma = \frac{1}{2}$.

If one assumes with Stokes—an assumption which possibly was also in Maxwell's mind—that the volume viscosity vanishes, the last term on the right-hand side, contrary to expectations, becomes infinitely great.

5. The Maxwell Liquid

We may call a material the rheological equation of which is (13) or (18) a Maxwell Liquid. To indicate its nature as a liquid, the subscript "*l*" has been added to the modulus of rigidity γ. If its volume viscosity η_v is not *infinite*, the voluminal rheological equation (14) must also be taken into consideration. These rheological equations can be derived through a combination of those of the Hooke solid (I, d) and the Newtonian liquid (I, f). The nature of this combination is found by considering in accordance with Equation (13) an infinitely small displacement *du* attained during the time *dt*. This is

$$du = dp_t/\gamma_l + p_t/\eta \, . \, dt = du_e + du_v \quad . \quad . \quad . \quad (23)$$

where du_e is the elastic and du_v the viscous displacement. Accordingly, in a Maxwell liquid the displacements are additive.

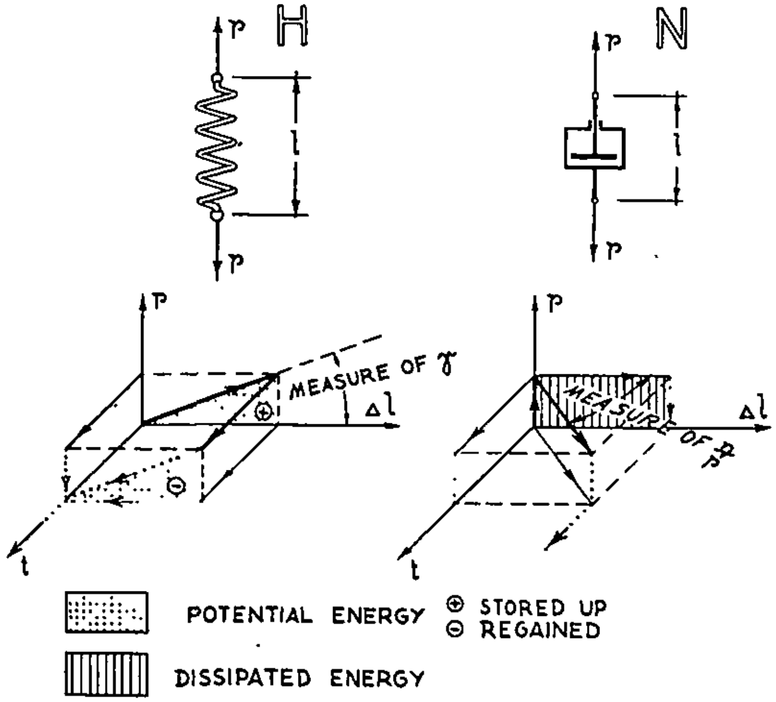

FIG. XIV, 2. Models for the Hooke solid and the Newtonian liquid. In working out the diagrams, the inertia of the mechanism has been neglected and the elongation assumed as starting instantaneously with applied force.

In Section 4 of Chapter XI we have proposed a spring as a model for an elastic (Hooke) material. We now introduce a dashpot as shown in Fig. XIV, 2, as a model for the Newtonian liquid.

If *displacements* of both should *add up*, we must combine them *in series* in order to get a model for the Maxwell liquid. Fig. XIV, 3, shows the model together with its stress-displacement-time curves under load and after removal of the load (the latter in dotted lines ......). Using, as before, the horizontal dash — for " coupling in series ", we write the structural formula of the Maxwell liquid in the form

$$M = H - N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

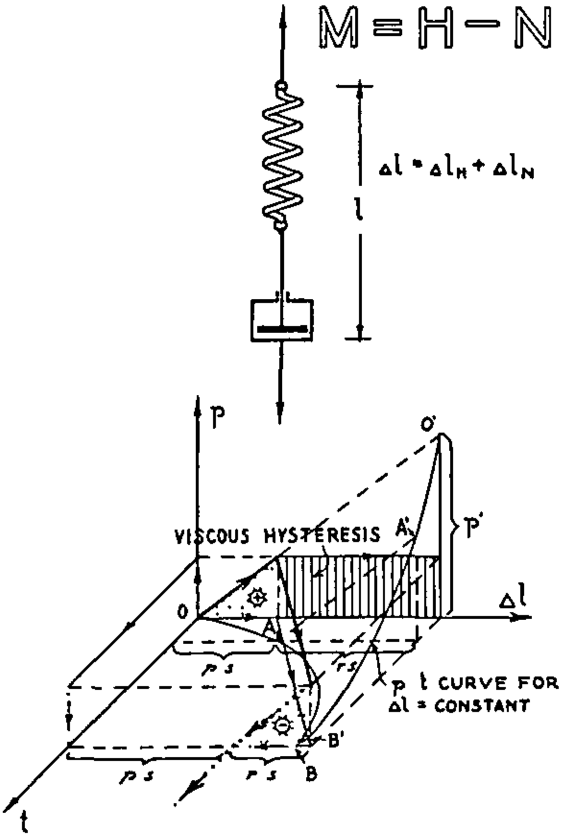

Fig. XIV, 3. Model for the Maxwell liquid. OAB = elastic recovery for elongation Δl. O'A'B' = relaxation curve for stress p'. p.s. = permanent strain; r.s. = recovered strain.

Equation (13) is of the form of Equation (XIII, 11). Provided γ and η are constants, its solution is, in accordance with Equation (XIII, 12).

$$p = e^{-\gamma/\eta\, t}\left(p_o + \gamma \int_0^t \dot{e} e^{\gamma/\eta\, t} dt\right) \quad . \quad . \quad . \quad . \quad (25)$$

where we have, for simplicity, omitted the subscripts " t " and " l ".

If the deformation is kept constant, or $\dot{e} = 0$, we find from Equation (25) the relaxation curve

$$p = p_o e^{-\gamma/\eta\, t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

as in Equation (3). On the other hand $\int^t e^{\gamma/\eta\, t}\, dt = \frac{\eta}{\gamma}(e^{\gamma/\eta . t} - 1)$

and, therefore, if $\dot{e}$ is constant c,

$$p = \eta c + (p_o - \eta c)e^{-\gamma/\eta\, t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

For $c_o = p_o/\eta$ this gives $p = \eta c_o = p_o$ and the stress remains constant. For $c > c_o$ the stress increases, for $c < c_o$ the stress decreases as shown in Fig. 2.

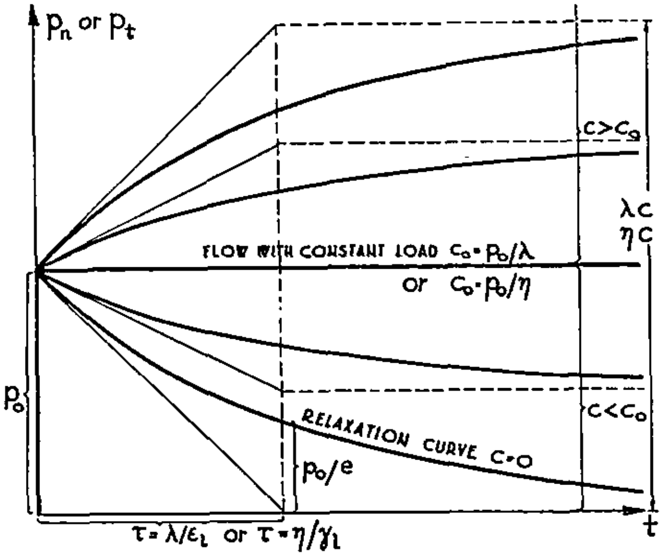

FIG. XIV, 4. The test piece is first ($t=o$) suddenly stressed to the value p_0 and then strained at the constant rate c. There exists for every initial stress p_0 a certain rate of strain c_0 so that the stress is not changed.

6. Viscous Solids

At about the same time when Maxwell was pondering over the relaxation of stresses, William Thomson, later Lord Kelvin, published a paper " On the Elasticity and Viscosity of Metals " [70], which was subsequently incorporated into his article on Elasticity in the *Encyclopædia Britannica.* He reasoned from thermodynamical considerations that " *no change of volume or of shape can be produced in any kind of matter without dissipation of energy* ". He also proceeded from theoretical reasoning to experiment and found " by vibrating a spring alternately in air of ordinary pressure and in the exhausted receiver of an air-pump, that there is an internal resistance to its motions immensely greater than the resistance of the air ". He concluded that " there is in elastic solids a molecular friction which may be properly called Viscosity of Solids, because, as being an internal resistance to change of shape depending on the rapidity of change, it must be classed with fluid molecular friction, which by general consent is called viscosity of fluids ". He also proposed a model for the better understanding of the phenomenon as follows : " *Consider a perfectly elastic vesicular solid, whether like a sponge with communications between the vesicles, or with each vesicle separately enclosed in elastic solid : imagine its pores and interstices filled up with a viscous fluid, such as oil. Static experiments on such a solid will show perfect elasticity of bulk and shape ; kinetic experiments will show losses of energy such as are really shown by vibrators of . . . elastic homogeneous solids. . . . According to Newton's** law of viscosity of fluids, our supposed vesicular vibrator would follow the law of subsidence of a simple vibrator experiencing a resistance simply proportional to the velocity of its motion.*" However, he adds as a result of his experiments on the damping of torsional oscillations of metal wires that while the resistance depended in some manner upon the velocity " *no such simple law is applicable* ".

The property named by Kelvin " viscosity of solids " was called by Jeffreys [66] Firmo-Viscosity. He remarks that this type of " imperfection of elasticity " implies not weakness but additional stiffness, the resistance to deformation being smaller when the viscosity is absent. However, in accordance with Kelvin " the elasticity is said to be perfect when the body always requires the same force *to keep it at rest,* in the same bulk

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* Kelvin says " Stokes's law ", but Stokes only generalised Newton's relation between shearing force and velocity gradient, Equation (I, f), into a relation between stress and rate of strain.

and shape . . . through whatever variations of bulk and shape it be brought." Therefore, firmo-viscosity should not be regarded as an imperfection of elasticity (compare also Section 10 of Chapter IX).

7. The Kelvin Solid

We may call a solid constituted on the model of the "vesicular solid", or, if truly homogeneous, *behaving in the same manner*, a

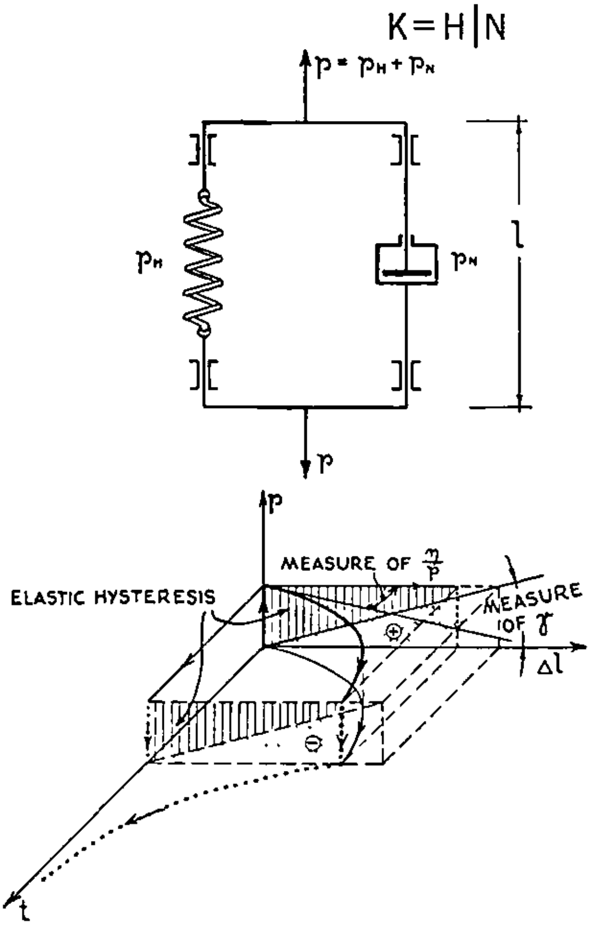

FIG. XIV, 5. The mechanisms in the figures show extensions and pulls, but one should not forget that these stand for shearing strains and shearing stresses, respectively. Also, the stress-strain relations have been pictured as linear; but the models can be applied in a general manner for nonlinear relations as well.

Kelvin Solid.* It is important to keep the italicised generalisation in mind. Mardles [76] has objected to what he calls my " assumption that the behaviour of a given material † can be *represented in terms of a number of elements each obeying a* simple law " as, e.g. in the sponge the solid substance between the pores and the liquid filling them. But, as Kelvin said when introducing his model : " *to form a rough* (or qualitative—M.R.) *idea of the results, irrespectively of the ultimate molecular* (or structural—M.R.) *theory* . . . " and I have shown in Section 8 of Chapter IX how even in a perfectly elastic homogeneous material, damping of free pulsations or dissipation of energy arises. The model does not serve any other purpose than the rheological equation. The elements of the model are symbols not essentially different from the mathematical symbols of the differential equation. To make this more obvious, we may replace the " vesicular solid " or sponge by a structural formula similar to the structural formula of the Maxwell liquid, Equation (24). When the sponge, which is full of liquid, is loaded, part of the load is taken up by the solid substance and part by the liquid. This causes the sponge to be deformed and in this process load is transferred from the liquid which yields through flow, to the solid which is strained. The solid and liquid stresses are additive, adding up to the total load ; but the deformation is the same for both phases. This is in contradistinction to the Maxwell liquid where the same stress is taken by both elements, while the deformations are additive. Combining the spring and the dashpot, we get a picture as shown in Fig. XIV, 5.

Indicating by a vertical dash, thus, |, *parallel coupling,* the structural formula of the Kelvin Solid is accordingly

$$K = H \mid N \qquad \ldots\ldots \qquad (28)$$

The rheological equation can easily be written down from (I, d) and (I, f). The stresses being additive, we have

$$p = \gamma e + \eta \dot{e} \qquad \ldots\ldots \qquad (29)$$

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* Jeffreys says that the firmo-viscous law was suggested to him by Sir J. Larmor. Weissenberg [71] quotes Jeffreys. Mises [72] ascribed it to Voigt [73] and I [74] followed him. Alfrey and Doty [75] speak of the Voigt model. I came across Lord Kelvin's article in the *British Encyclopaedia*, which has the priority, by chance and I can strongly recommend its study to the reader. It contains a wealth of forgotten knowledge, as we shall have to point out again in Chapter XVIII.

† Mardles uses the term " system."

which, in accordance with our modified first axiom, stands for two equations

$$p = \kappa e_v + \eta_v \dot{e}_v \quad . \; . \; . \; . \; . \; . \; . \quad (30)$$

$$p_t = \gamma e_t + \eta_s \dot{e}_t \quad . \; . \; . \; . \; . \; . \; . \quad (31)$$

In Equation (31) we have given the η the subscript " s " to indicate the " solid " viscosity.

It will be noted that Equation (30) is identical with (IX, a). However, while Equation (30) has here been derived with the help of a model representing the material as made up of two elements, Equation (IX, a) was derived for a perfectly homogeneous, non-porous material.

8. Delayed Elasticity

Equations (30) and (14), (31) and (13) are built up analogously. We can therefore write down the solution of either of them on the model of Equation (25). For instance, omitting for simplicity the subscripts " t " and " s ", the solution of Equation (31) is

$$e = \mathrm{e}^{-\gamma/\eta . t} \left(e_o + \frac{1}{\eta}\int_o^t p\mathrm{e}^{\gamma/\eta\, t} dt\right) \quad . \; . \; . \; . \quad (32)$$

If the stress is constant, we find as analogy to Equation (27)

$$e = p/\gamma + (e_o - p/\gamma)\mathrm{e}^{-\gamma/\eta . t} \quad . \; . \; . \; . \quad (33)$$

For $p = e_o\gamma$, this gives $e = e_o$ and the strain remains constant. For $p > e_o\gamma$ the strain increases, for $p < e_o\gamma$ the strain decreases, in both cases reaching p/γ asymptotically. If a stress p acts on an unstrained body, i.e. $e_o = 0$, we find

$$e = \frac{p}{\gamma}(1 - \mathrm{e}^{-\gamma/\eta\, t}) \quad . \; . \; . \; . \; . \quad (34)$$

The elastic strain p/γ, therefore, does not appear instantaneously or rather with acoustic speed, but is *delayed* and reaches this value only after infinite time. Likewise, if the load is removed from a strained body, it will not revert to the unstrained state with acoustic speed, but in accordance with ($p = 0$)

$$e = e_o\mathrm{e}^{-\gamma/\eta\, t} \quad . \; . \; . \; . \; . \; . \; . \quad (35)$$

and it will take infinite time until the strain completely vanishes.

The analogous equation for volume strain has already been derived in Equation (IX, g). The phenomenon was called elastic after-effect, but this is only a special case of Delayed Elasticity * and the effect described by Equation (34) is sometimes called elastic Fore-Effect, a term not altogether happy, suggesting an effect preceding the cause—which, of course, cannot be. As can be seen, every Kelvin body will show delayed elasticity.

As we pointed out in Section 10 of Chapter IX, η/γ is of the dimension of time. We called it there " time of lagging ", but Time of Retardation, as has also been proposed, may be a better term. The use of the term " time of relaxation ", as sometimes met, is misleading and should be reserved for the Maxwell liquid only.

9. Damping of Oscillations

As we saw in Section 6, Kelvin conceived the notion of solid viscosity in order to explain the damping of oscillations. This is " viscous damping ". However, viscous damping is not the only kind of damping. Whatever causes dissipation of either elastic potential or kinetic energy produces damping.

In analysing the causes of damping we must distinguish between the first two kinds, viz. *external* and *internal*. External causes are (i) friction at the fixing points of the vibrating body, and (ii) the resistance of the air. Neither tells us anything about the nature of the material : for the rheologist they are nuisances which must be eliminated. The internal resistances only interest him. He will make experiments on damping of oscillations for the purpose of learning something about those rheological properties which produce the internal resistances. The latter are of four kinds, which can be arranged in two pairs. We have first the viscous damping due to solid viscosity. Solid viscosity dissipates kinetic energy. The second is due to viscosity present in liquids, or liquid viscosity. It will operate in damping oscillations of a rod of asphalt or a slab of concrete or of any body consisting of a Maxwell liquid. Liquid viscosity dissipates elastic potential energy. The third kind is due to plastic resistance. A rod made of a plastic material and subjected to the torsion described in Section 7 of Chapter XII

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* Also termed Retarded Elasticity.

will show such damping. The fourth is due to local microscopic or submicroscopic destructions such as occur in " permanent sets " the nature of which was explained in Section 11 of Chapter XIII. In the first two instances the resistance depends upon the velocity; in the second two it is independent of the velocity. We shall deal here in detail with the mathematics of viscous damping only, simplifying our problem by considering a homogeneous *one-dimensional* case.

Let us imagine an arrangement as indicated in Fig. XIV, 6. A body of mass m oscillates on a frictionless horizontal plane in a straight line under the action of an elastic force exerted by springs which obey Hooke's law. We assume that the experiment is carried out *in vacuo* and that the springs are stressed below the elastic limit. The only resistance which we consider is internal viscous resistance of the springs in accordance with the equation

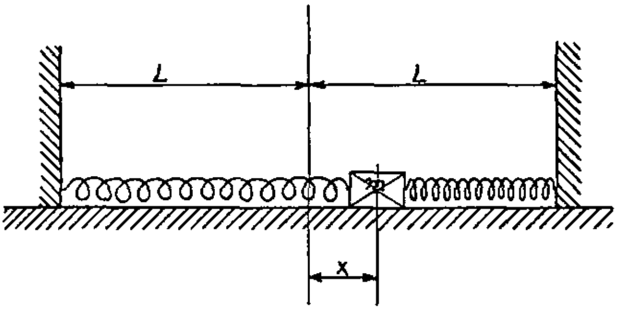


FIG. XIV, 6. Model for linear oscillator.

$$p = \epsilon e + \lambda_s \dot{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

In the Figure we have shown two springs, such as would be suitable for a working model, but for the calculation we replace the springs by a straight elastic rod which is capable of resisting push as well as pull, ϵ being its Young's modulus and λ_s its "solid" viscous Trouton coefficient.

Applying Newton's law (I, a) we write

$$m \cdot d^2x/dt^2 = P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The force is $P = A \cdot p$ where " A " is the cross-sectional area of the rod.

Introducing $e = x/l$ and $\dot{e} = \dot{x}/l$ we write

$$lp = -\epsilon\dot{x} - \lambda_s dx/dt \quad . \; . \; . \; . \; . \; . \quad (38)$$

where the negative sign for $\epsilon\dot{x}$ is introduced because the elastic force acts in the direction opposite to an increasing elongation and the negative sign for $\lambda\dot{x}$ because the viscous resistance is opposed to the direction of the velocity.

From Equations (37) and (38) we get

$$d^2x/dt^2 + \epsilon^*x + \lambda^*dx/dt = 0 \quad . \; . \; . \; . \quad (39)$$

where

$$\epsilon^* = A\epsilon/lm \; ; \; \lambda^* = A\lambda/lm \quad . \; . \; . \; . \quad (39^1)$$

The solution of the differential equation (39) can be expressed in two different forms, depending on whether $\lambda^*{}_s{}^2/4 > \epsilon^*$, or $\lambda^*{}_s{}^2/4 < \epsilon^*$.

(i) If we first take the case $\lambda^*{}_s{}^2/4 > \epsilon^*$, the solution can be expressed as follows :

$$x = x_o/2a . \left\{(\lambda^*/2 + a)e^{(-\lambda^*/2+a)t} + (-\lambda^*/2 + a)e^{(-\lambda^*/2-a)t}\right\} \quad (40)$$

where

$$a = \sqrt{\lambda^{*2}/4 - \epsilon^*} \quad . \; . \; . \; . \; . \; . \quad (41)$$

That (40) is a solution of (39) can be verified by differentiation and substitution.

Introducing $t = 0$ in (40), we get

$$x\big|_{t=o} = x_o \quad . \; . \; . \; . \; . \; . \quad (42)$$

and x_o is therefore the initial amplitude.

By differentiation of (40) we find

$$\dot{x} = x_o(a^2 - \lambda^{*2}/4)/2a\left\{e^{(-\lambda^*/2+a)t} - e^{(-\lambda^*/2-a)t}\right\} \quad . \; . \; . \; . \quad (43)$$

From (43) we find that $\dot{x}$ vanishes for $t = 0$ and $t = \infty$, but not between these extreme values, and from (40) it follows that x vanishes for $t = \infty$ but not before. This is therefore not a case of oscillations, but an aperiodic movement. In this case the viscosity is so great that the body asymptotically approaches the unstrained state, without any kinetic energy left to produce oscillation.

(ii) Secondly, we consider the case $\lambda^*{}_s{}^2/4 < \epsilon$.

The solution is

$$x = x_o e^{-\lambda^*/2\,t}(\cos bt + \lambda^*/2b \, . \sin bt) \quad . \; . \; . \quad (44)$$

where

$$b = \sqrt{\epsilon^* - \lambda^{*}_{s}{}^{2}/4} \quad . \quad . \quad . \quad . \quad (45)$$

This also can be easily verified by differentiation and substitution. Again we find

$$x\Big|_{t=o} = x_o \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and

$$\dot{x} = x_o\epsilon^*/b \,.\, e^{-\lambda^*/2\,t} \sin bt \quad . \quad . \quad . \quad . \quad . \quad (47)$$

From (47) we see that $\dot{x}$ vanishes for $t = 0, \pi/b, 2\pi/b \ldots n\pi/b$. For these values of t the amplitudes are x_o , $-\, x_oe^{-\lambda^*\pi/2b}$, $+\, x_oe^{-2\lambda^*\pi/2b}$, ... The movement therefore consists of oscillations with the period

$$T = 2\pi/b = 2\pi/\sqrt{\epsilon^* - \left(\frac{\lambda^*}{2}\right)^2} \quad . \quad . \quad . \quad . \quad (48)$$

This period is a constant. This is quite remarkable. Oscillations with viscous damping are accordingly *isochronous*, the same as oscillations without damping. When the amplitudes become smaller, the velocity decreases correspondingly, so that the time for making the smaller movement remains the same. The period itself is, however, greater in the damped than in the undamped oscillations, because " b " becomes larger if, in (45), the coefficient of the damping viscosity λ^*_s vanishes. The greater the damping, the slower, therefore, the oscillations. The amplitude of the oscillations gradually diminishes in accordance with our last series. In infinite time ($t = \infty$) $\dot{x}$ vanishes, i.e. the movement ceases, and at the same time x also vanishes, i.e. *the position of rest is the unstrained state.* The amplitudes form a geometrical series with the coefficient $e^{-\lambda^*\pi/2b}$ and the logarithms of the amplitudes form an arithmetical series with a negative increment or decrement, $\lambda^*\pi/2b$. This is called the Logarithmic Decrement of the oscillations. If we consult (48) we find the logarithmic decrement to be equal to $\lambda^*T/4$.

Plotting the elongation on a logarithmic scale we get a straight line.

The main difference between damped and undamped oscillations is in the factor $e^{-\lambda^*/2\,t}$ in (44). In the beginning of the oscillations as long as t is small, this factor is not very different from unity and the movement of the damped oscillations does not differ much from the movement of undamped oscillations. If λ^* is also small, several oscillations may pass before the diminution of amplitudes is noticeable. As soon, however, as *t has increased to such an extent that the exponential factor is* markedly different from unity, the amplitudes decrease very quickly. If after one second the exponential factor was, say, 0·9 and after two seconds = 0·8 which may be entirely within the error of observations, it becomes after 10 seconds $0{\cdot}9^{10} = 0{\cdot}35$, i.e. very noticeable; and after 20 seconds, when it is 0·1, the original oscillations have probably been entirely superseded by vibrations from accidental causes.

10. Summary

The property of elastico-viscosity is typified by the Maxwell liquid, the structural formula of which is

$$M = H - N \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV, a)}$$

with the rheological equation

$$\dot{e}_t = \dot{p}_t/\gamma_l + p_t/\eta \quad . \quad . \quad . \quad . \quad \text{(XIV, b)}$$

where γ_l is the " liquid " rigidity.

For constant stress ($\dot{p}_t = 0$), $\dot{e}_t = p_t/\eta$ and the Maxwell body behaves as a Newtonian liquid.

If γ_l and η are constant, Equation (XIV, a) gives on integration

$$p_t = \mathrm{e}^{-\gamma_l/\eta\, t}\left(p_{t,o} + \gamma_l \int_o^t \dot{e}_t \mathrm{e}^{\gamma_l/\eta\, t} dt\right) \quad . \quad . \quad \text{(XIV, c)}$$

For $\dot{e}_t = 0$ this is reduced to

$$p_t = p_{t,o}\, \mathrm{e}^{-\gamma_l/\eta\, t} . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV, d)}$$

which shows that for constant strain the elastic stress relaxes.

The quantity

$$\tau_M = \eta/\gamma_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV, d)}$$

is called the relaxation time.

The property of firmo-viscosity is typified by the Kelvin solid, the structural formula of which is

$$K = H|N \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV, f)}$$

with the rheological equation

$$p_t = \gamma e_t + \eta_s \dot{e}_t \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV, g)}$$

where η_s is the " solid " viscosity.

For constant strain ($\dot{e}_t = 0$), $p_t = \gamma e_t$ and the Kelvin body behaves as a Hookean solid.

If γ and η_s are constant, Equation (XIV, g) gives on integration

$$e_t = \mathrm{e}^{-\gamma/\eta_s \cdot t}\,(e_{t,o} + \frac{1}{\eta_s}\int_o^t p_t \mathrm{e}^{\gamma/\eta_s \cdot t}\,dt) \quad . \quad . \quad \text{(XIV, h)}$$

If a constant stress acts upon an unstrained body

$$e_t = \frac{p_t}{\gamma}(1 - \mathrm{e}^{-\gamma/\eta_s \cdot t}) \quad . \quad . \quad . \quad . \quad \text{(XIV, i)}$$

and the elastic strain p_t/γ does not appear with acoustic speed, but is delayed and reaches this value only after infinite time. Likewise, if the load is removed from a body the strain of which is $e_{t,o}$, this reverts to the unstrained state in accordance with

$$e_t = e_{t,o}\mathrm{e}^{-\gamma/\eta_s\, t} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV, j)}$$

The first phenomenon is called elastic fore-effect, the second elastic after-effect, both constituting delayed elasticity.

The quantity

$$\tau_K = \eta_s/\gamma \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV, k)}$$

is called the retardation time.

A Kelvin solid shows viscous damping of oscillations.

The oscillations are isochronous with the period

$$T = 2\pi/\sqrt{\epsilon^* - (\lambda^*/2)^2} \,. \quad . \quad . \quad . \quad \text{(XIV, l)}$$

The amplitudes form a geometrical series with the coefficient $e^{-\lambda^*\pi/2b}$ and

$$\Delta = \lambda^*\pi/2b = \lambda^* T/4 \quad . \quad . \quad . \quad \text{(XIV, m)}$$

is called the logarithmic decrement.

In these equations

$$\epsilon^* = A\epsilon/lm\,;\ \lambda^* = A\lambda_s/lm \quad . \quad . \quad . \quad \text{(XIV, n)}$$

where A is the cross-sectional area and l the length of a rod executing longitudinal vibrations, ϵ and λ_s being its Young's modulus and solid Trouton coefficient respectively, while m is the mass of the vibrator attached to the rod.

CHAPTER XV

COMPLEX BODIES

1. Delayed Elasticity in Concrete and the Burgers Body

In the preceding Chapters we have become acquainted with materials showing the simple properties of elasticity and viscosity; the more complex property of plasticity which can only be realised together with the property of elasticity; and finally the still more complex properties of elastico-viscosity and firmo-viscosity. These materials were idealised in the abstractions of the Hooke-, Newton-, St. Venant-, Maxwell- and Kelvin-Body. Of these, the first three only are elementary and it was shown how the other two could be related qualitatively to the first two by means of structural formulas. Quantitatively, rheological equations between p, $\dot{p}$, e and $\dot{e}$ were postulated in which the three parameters γ, η and ϑ appeared in different forms, constituting " rheological coefficients ". For every material two such independent rheological equations exist, one for change of volume, the other for change of shape. The first was indicated by the subscript " v ", the second by " t ". In the case of " simple " materials, the rheological coefficients are constants, while in the " generalised " materials they are functions involving deformational work or power and parameters, which may be constants. These developments provided us with ever better approximations for the description of the behaviour of real materials. In Chapter XIII we described in detail the rheological behaviour of concrete. In the first place concrete possesses elasticity: this makes it suitable for a structural material. One can accordingly regard concrete, in a first approximation, as a Hooke solid. As a matter of fact, structural engineers base their designs nearly exclusively upon that abstraction. Only when creep must be taken into consideration, will they proceed to the second approximation and consider concrete as a Maxwell liquid. However, we have mentioned that in evaluating a " load test ", one has to take cognizance of the fact that concrete also shows delayed elasticity. If we want to take this also into consideration, we must reach a third approximation by proceeding beyond the Maxwell liquid.

Delayed elasticity, as we have seen, is bound up with firmoviscosity, for which the Kelvin solid was devised. We therefore have to attach the K to the M complex and the question is only whether this is to be done in series or parallel. Concrete, because of its slow viscous flow, should not, even in a further approximation, lose its character of a *liquid*. The connection must therefore be in series. Such a model was proposed by Burgers [77] and we may name the ideal material corresponding to it the Burgers body (Bu). We accordingly write

$$Bu = M - K = (H - N) - (H|N) \quad . \quad . \quad . \quad (1)$$

and the reader will have no difficulty in making himself a sketch of the model representing the Bu-body in accordance with Equation (1).

The " second axiom " of rheology must lead us to expect that the Burgers body will be applicable to many other materials besides concrete. Lee and Markwick [29] found that there was " good qualitative agreement between the behaviour of actual (bituminous road-) surfacing materials and the behaviour of the model. The model also illustrates the behaviour of . . . binders such as bitumen and pitch."

In order to proceed from the structural formula (1) to a rheological equation, we note that there will be four rheological coefficients, two of viscosity and two of elasticity. If we consider shear (or, more generally, distortion), these will be the " ordinary " viscosity (η) in the M-complex and " solid " viscosity (η_s) in the K-complex ; " liquid " rigidity (γ_l) in the first and " ordinary " rigidity (γ) in the second case. Jeffreys [66] who was the first to propose a rheological equation for a M-K complex proceeded as follows :—

Being arranged in series, both the M- and K-complex must take the same stress. For the first we find from Equations (XIV, b) and (XIV, e)

$$\gamma_l e_t = p_t + \frac{1}{\tau_M}\int p_t dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For the second we find from Equations (XIV, g) and (XIV, k)

$$p_t = \gamma(e_t + \tau_K \dot{e}_t) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equating both, we get

$$\gamma(e_t + \tau_K \dot{e}_t) = p_t + \frac{1}{\tau_M}\int p_t dt \quad . \quad . \quad . \quad . \quad (4)$$

Jeffreys remarks that the substance would follow the firmo-viscous law if τ_M was infinite, and the elastico-viscous one if τ_K was zero. The substance will flow indefinitely with long-continued stresses, but the partial recovery on release will be gradual. If any problem of elastic strain had been solved for a simple elastic solid, the behaviour of that substance, so long as squares of the displacements can be neglected, could be inferred simply by writing $\gamma(1 + \tau_K \frac{d}{dt})/\left(1 + \frac{1}{\tau_M(d/dt)}\right)$ for γ. A body *nearly* simply elastic has τ_M large and τ_K small and has approximately $\gamma[1 + \tau_K(d/dt) - 1/\tau_M(d/dt]$ for γ. The damping of surface waves of earthquakes as they advance would suggest that either $\tau_M = 750$ sec. and $\tau_K = 0$ or $\tau_K = 0{\cdot}004$ sec. and $\tau_M = \infty$. In the rocky shell approximately $\tau_M > 3 \times 10^8$ sec., $\tau_K = 0{\cdot}004$ sec., $\gamma = 1{\cdot}7 \times 10^{12}$ dynes cm^{-2}, $\eta = 5 \times 10^{20}$ poises.

In the above derivation one may object to the equating of $\gamma_t e_t$ with p_t. It is true that the stress p_t must be the same in both the *H*- and the *N*-element of the *M*-complex, but to calculate p_t one would have to take into account in the first case the strain of the spring only and in the second the rate of flow in the dashpot. Actually, however, e_t in Equation (2) is derived from the *sum* of both. Nevertheless, disregarding the derivation, one may *postulate* Equation (4) as a rheological equation, permitting even the coefficients γ and τ to be variable and test it against the rheological behaviour of some real material.

A more consistent method for the derivation of the rheological equation of the Bu-body seems to me to be the one used by Reiner [46] by taking into account that the *displacements* of the *M*- and K-complex are *additive*; the same method as used above in the derivation of the equation of the *M*-body.

From

$$\dot{e}_{Bu} = \dot{e}_M + \dot{e}_K \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we find by means of Equations (XIV, b) and (XIV, h)

$$\dot{e}_t = p/\eta + \dot{p}/\gamma_t + \frac{d}{dt}\left[e^{-\gamma/\eta_s \cdot t}\left(e_o + \frac{1}{\eta_s}\int_o^t p e^{\gamma/\eta_s \cdot t}\, dt\right)\right] \quad . \quad (6)$$

After carrying out the differentiation, this yields

$$\dot{e} = p\frac{\eta + \eta_s}{\eta\eta_s} + \dot{p}/\gamma_l - \frac{\gamma}{\eta_s}\,\mathrm{e}^{-\gamma/\eta_s t}\,(e_o + \frac{1}{\eta_s}\int p\mathrm{e}^{\gamma/\eta_s \cdot t}\,dt)\ . \quad (7)$$

Here also, while in the integration the coefficients were assumed to be constants, the derivation of Equation (7) may be disregarded and the equation postulated as the rheological equation of the Burgers body, even should the coefficients turn out to be variable.

If a stress is applied to a *Bu*-body when it is in an unstressed and unstrained state, we find from Equation (7), putting $e_o = 0$ and $p = 0$, $\dot{e} = p/\gamma_l$. This is an instantaneous strain due to the H_M— element. If, on the other hand, the stress is kept constant or $p = 0$, we find

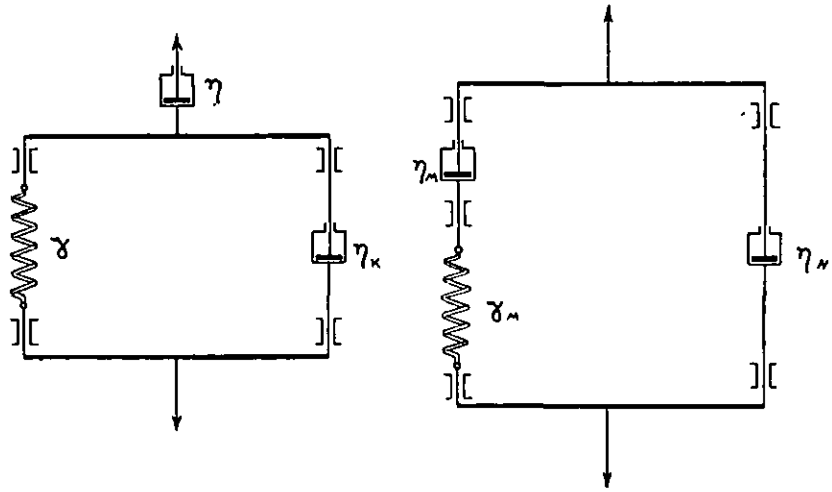
$$\dot{e} = p/\eta + p/\eta_s \,.\, \mathrm{e}^{-\gamma/\eta_s \cdot t} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and two kinds of continuous deformation take place simultaneously, the second vanishing in infinite time. We have called the first creep and the second elastic fore-effect, but if, as has recently been observed by Lethersich [79], in some materials that fore-effect lasts for *years* it is difficult to separate them and there may be some justification in naming the phenomenon to which that $\dot{e}$ is due *in toto*, " creep ", in which case there will be a partial " creep-recovery ". That term " creep-recovery " was used by Glanville and Thomas [62] and while I voiced my objection to it at one time, I am now rather inclined to think that there might be an advantage in having a term for slow continuous deformation *prior to an analysis* of which part of it is recoverable and which not, an analysis which might require further years of observation.

2. The Rheological Behaviour of Bitumen and the Truncated Burgers Body

If we let γ_l increase $\rightarrow \infty$, the second term on the right side of Equation (7) disappears and with it the H_M element in the structural formula which becomes *N–K*. Such material which, until a proper name is put forward, may be termed Truncated Burgers Body, has been proposed by Lethersich [79] to represent the behaviour of bitumen. This would mean that with the apparatus or the kind of bitumen used by him, Lethersich could not observe the acoustic elasticity, which Lee and Markwick [29] had observed. We accordingly see that bitumen will be represented by successive approximations by N, N–K, M–K ; its most prominent rheological property being flow (N), the next prominent delayed elasticity (K) and the least easily observable acoustic elasticity (H).

Actually Lethersich proposed two alternative models as shown in Fig. XV, 1. Their structural formulas are respectively K–N and $M|N$. By making use of the transformation, $K\text{–}N = (N|H) - N = N|(H - N) = N|M$, one could come to the conclusion that they are equivalent. This, however, would be too rash, as we shall presently show.



(a) $N - K = N - (H|N)$ (b) $M|N = (N - H)|N$

FIG. XV, 1. Models proposed by Lethersich to represent the rheological behaviour of bitumen.

3. Rheological Models, Structural Formulas and Rheological Equations

In order to treat rheological problems by mathematical methods, we found it necessary to create concepts of ideal bodies having rigorously defined (rheological) properties. This process is facilitated by building up in imagination models consisting of various combinations of mechanical elements which under the action of appropriate forces register displacements of particular kinds, similar to those exhibited by the materials the behaviour of which we desire to describe.

We have introduced in our exposition such models. The first, in Section 4 of Chapter X, was a model for the St. Venant body in the form of a weight resting upon a table top with solid friction between them. To describe more complex behaviour, we added a spring as a model for the Hooke body. Then, in Section 5 of Chapter XIV, we brought in the dashpot for the Newtonian liquid. At the same place we showed how coupling a spring and dashpot in series reproduces the properties of the Maxwell liquid, while, when coupled in parallel, the same elements reproduce the so different behaviour of the Kelvin solid. We then saw in the present chapter how other rheological properties can be understood from the contemplation of some combination of these three elements. To give another example we shall presently show how a model for the Bingham body is to be built up. We met this body in Chapter III. We found that when stressed below a certain stress it is strained elastically. We therefore take as our first element a spring. However, when the yield stress is exceeded, the body is deformed plastically. We therefore have to attach the spring to a weight. When the rate of plastic deformation is increased, the stress also increases. This required the attaching of a dashpot to the weight at the side of it which is opposite of the spring. The model is shown in Fig. XV, 2. When putting the elements together, one should keep in mind that, in contradistinction to the spring and the dashpot, the StV element is not symmetrical. When coupling a StV element it is therefore necessary to indicate whether the connection is to the " weight " or to both " weight

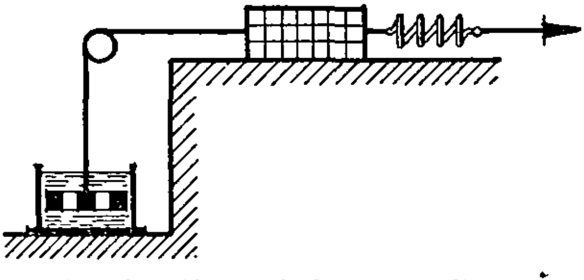


FIG. XV, 2. Model for a Bingham body corresponding to the structural formula $B = N - StV - H$.

and table ". In our case we had to connect the spring in the first, the dashpot in the second manner. When using the symbol *StV* we shall assume that *St* stands for the table and *V* for the weight. The friction ϑ acts between the *St* and the *V* part of the *StV* element.

It should be pointed out that the displacements which the models can register are increases in length or elongations produced by pulls. These, however, may be translated into every other kind of deformation and corresponding stress. One may consider the elongation as representing a shear, when the pull will represent a shearing stress; or the first may be a cubical dilatation and the second a hydrostatic tension, etc. It should also be noted that while, if we should build real models (which can conveniently be done, using Meccano parts), the materials and arrangements for our mechanical elements would correspond to *constant* rheological coefficients, this does not preclude their application in cases where the coefficients are variable.

From the model we proceeded to the structural formula, which is a sort of shorthand representation of the model's blueprint. For instance, the structural formula for the Bingham body is

$$B = N - StV - H \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This gives us all the information which we may draw from the picture of the model as shown in Fig. XV, 2, but is, of course, much more economical.

The models and structural formulas help us to understand rheological behaviour *qualitatively*. But they also help us to establish the rheological equations of the different ideal bodies, which we need for a *quantitative description*. These equations connect certain *types* * of stress and deformation. For instance the *H*-spring with its elastic extensions leads us to a rheological equation of any one of the three following types

$$p_t = \gamma e_t\,;\; p = \kappa e_v\,;\; p_n = \epsilon e_n \quad . \quad . \quad . \quad . \quad (10)$$

where it does not matter that ϵ can be expressed by γ and κ in accordance with Equation (X, m). If, therefore, in writing the rheological equation, we omit the subscripts "t", "v" or

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* In advanced rheology they connect the *tensors* of stress and deformation. Compare *Ten Lectures*, p. 38.

" n ", the reader may put them in as required, taking care to use at the same time the appropriate coefficient, which e.g. in the case of viscous flow will be one of the following, viz. η, η_v . or λ.

We shall now examine systematically in what ways complex bodies can be built up from the elements and how the corresponding rheological equations are to be derived.

We first note that elements can be connected either (i) in series (—), or (ii) in parallel (|).

As will be seen, all elements, when coupled in series, act as if they were links in one chain and they must therefore all take the same stress, while the elongation of the combination will be the sum of the elongations of each element.

In contradistinction, when coupled in parallel, all elements are forced to undergo the same displacement, while the stress which the combination will take is the sum of the stresses taken by each single element. These two principles will guide us when composing the rheological equations.

Let us start with the H-body, the rheological equation of which we write in the form

$$p = \gamma e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If we have two different springs, viz. H_1 for which $p_1 = \gamma_1 e_1$ and H_2 for which $p_2 = \gamma_2 e_2$, and couple them in series, H_1—H_2, we have in accordance with the first of the above principles

$$e = e_1 + e_2 = p/\gamma_1 + p/\gamma_2 = p(1/\gamma_1 + 1/\gamma_2) = p/\gamma \quad . \quad (12)$$

and we see that *n H-bodies connected in series are equivalent to one single H-body the coefficient of elasticity* (reverse of modulus of elasticity) *of which is the sum of the individual coefficients of elasticity.*

If we couple the springs in parallel, $H_1|H_2$, we have, in accordance with the second principle,

$$p = p_1 + p_2 = \gamma_1 e + \gamma_2 e = (\gamma_1 + \gamma_2)e = \gamma e \quad . \quad . \quad (13)$$

from which we see that *n H-bodies connected in parallel are equivalent with one single H-body the modulus of elasticity of which is the sum of the n individual moduli.*

Conditions with regard to the N-bodies are entirely analogous.

4. M- Complexes

We have already dealt with $N - H = M$ and $N \mid H = K$ in detail.

We now examine the combination $M_1 - M_2$. This will give

$$\dot{e} = p/\eta_1 + p/\eta_2 + \dot{p}/\gamma_1 + \dot{p}/\gamma_2 = p(1/\eta_1 + 1/\eta_2) + \dot{p}(1/\gamma_1 + 1/\gamma_2) = p/\eta + \dot{p}/\gamma \quad . \quad . \quad (14)$$

where

$$1/\eta = 1/\eta_1 + 1/\eta_2\,;\ 1/\gamma = 1/\gamma_1 + 1/\gamma_2 \quad . \quad . \quad . \quad (15)$$

Several *M-complexes coupled in series are* accordingly *equivalent* to *one M-complex* in which all springs and all dashpots are taken together to form one spring and one dashpot each.

It is very different with $M_1 \mid M_2$. Here we have to add stresses in accordance with Equation (XIV, c) and find, putting $p_o = \gamma e_o$,

$$p = e_o(\gamma_1 e^{-\gamma_1/\eta_1 t} + \gamma_2 e^{-\gamma_2/\eta_2 t}) + (\gamma_1 e^{-\gamma_1/\eta_1 \cdot t}\int_o^t \dot{e} e^{\gamma_1/\eta_1 \cdot t}\,dt + \gamma_2 e^{-\gamma_2/\eta_2 \cdot t}\int_o^t \dot{e} e^{\gamma_2/\eta_2 \cdot t}\,dt) \quad . \quad . \quad (16)$$

Neither sums within brackets can be replaced by a single term of similar form. As a matter of fact, the $M \mid M$ body behaves very differently from a single M-body, as we shall presently see.

If we put $\dot{e}$ = const. $= c$, we get from Equation (15)

$$p = c(\eta_1 + \eta_2) + e^{-\gamma_1/\eta_1 \cdot t}(e_o\gamma_1 - c\eta_1) + e^{-\gamma_2/\eta_2 \cdot t}(e_o\gamma_2 - c\eta_2) \qquad (17)$$

Now, we can select such a rate of deformation, c, that either the second or the third term vanishes, but *both* these terms will vanish only if both constituent M-complexes have *the same relaxation time*. Generally, therefore, if the material flows at a constant rate, the stress will relax. Conversely, under a constant stress, the material will not flow at a constant rate of deformation. In other words there will be no state of simple viscous flow, as is possible in a Maxwell liquid.

The problem of the rheological behaviour of a system built up by a combination of Maxwell bodies connected partly in series and partly in parallel has lately become important in the investigation of the "relaxation spectrum" of rubbery materials. From our considerations it can be seen that only *parallel* connections will become manifest, while all different

M-bodies *when connected in series* will give *one simple relaxation curve* and not be detectable (compare W. Kuhn [83]).

5. Limitations of the Structural Formula

We can now come back to the comparison of the truncated *Bu*-body (N–K) and the body proposed by Lethersich ($M \mid N$).

The rheological equation of the first can easily be written down from Equation (7) by omitting the second term on the right side. The rheological equation of the second is found by adding the stresses taken by M and N, or

$$p = p_M + p_N = e^{-\gamma_M/\eta_M t}(\eta_M \dot{e}_o + \gamma_M \int^t \dot{e} e^{\gamma_M/\eta_M t} dt) + \eta_N \dot{e} \quad . \quad (18)$$

where I have introduced $\eta_M \dot{e}_o$ for p_{oM}.

We now compare the behaviour of both bodies under constant stress, $p = p_o$, applied upon the unstrained and unstressed body. For the first we find a rate of deformation as given by Equation (8) in accordance with which it gradually decreases from

$$\dot{e}_o = p_o(1/\eta + 1/\eta_s) \quad . \quad . \quad . \quad . \quad . \quad (19)$$

to

$$\dot{e}_\infty = p_o/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In contradistinction, it can easily be found by substitution into Equation (17) that the second body will, under a constant stress p_o flow at a constant rate

$$\dot{e} = p_o/(\eta_M + \eta_N) \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and will therefore behave no differently from a single M-complex.

We draw the conclusion that we must write in structural formulas each complex within brackets and that the equivalence of different models built up from the same elements must be investigated separately in every single case.

6. The Plastic Strength of a Gelatine Solution and the Schwedoff Body

The examples of the rheological behaviour of cement and concrete, bitumen and asphalt (the relation of asphalt to bitumen being the same as that of concrete to cement) showed us that there are kinds of materials where it becomes necessary to consider a combination of the Maxwell liquid with the Kelvin solid. Investigations carried out by Schwedoff [84] with a gelatine solution showed that in this case the Maxwell liquid must be combined with the St. Venant plastic. Schwedoff tested a 0·5 per cent. gelatine solution at an age of 24 hours in an apparatus consisting of the same elements as the Couette-Hatschek viscometer (compare Section 7 of Chapter II). If the external cylinder does not rotate, and the top of the wire on which the internal cylinder is suspended is given a twist through the angle Ω, the internal cylinder, if the material between both cylinders is an elastic solid, will be rotated through another angle, say θ. Making use of the elasticity-viscosity analogy, the angle θ can be calculated from Equation (II, k) with the result that

$$\theta = M_z/4\gamma\pi h \,.\, (1/R_i^2 - 1/R_e^2) \quad . \quad . \quad . \quad . \quad (22)$$

where M_z is the torque exerted upon the wire.

If, however, the material between the cylinders is a Newtonian liquid, the inner cylinder will follow the wire immediately with decreasing velocity until it has moved through the angle Ω, i.e. until no torsion is left in the wire. Thirdly, if the material is a Maxwell liquid, the cylinder will not follow immediately, but will in the first instance rotate through the angle θ ($< \Omega$), the elastic resistance of the liquid balancing the torque M_z of the wire, which is (compare Equation (XII, g))

$$M_z = \pi R_w^4 \gamma_w/2l_w(\Omega - \theta) = (\Omega - \theta)D_w/l_w \quad . \quad . \quad (23)$$

where the subscript " w " indicates " of the wire ".

Introducing $\Omega - \theta = \phi$, Equation (22) becomes

$$\theta = \frac{D_w}{l_w}\phi \;/4\gamma\pi h \;.\; (1/R_i^2 - 1/R_e^2) \quad . \quad . \quad (24)$$

From Equation (24) the rigidity of the elastic liquid can be determined and is found to be

$$\gamma_l = \frac{D_w}{l_w}\phi(1/R_i^2 - 1/R_e^2)/4\pi h\theta \quad . \quad . \quad . \quad (25)$$

The internal cylinder does not remain at the deflection θ but relaxation sets in and the cylinder gradually follows the wire. The relaxation of the stresses required to maintain the elastic strain θ can be studied by reducing the torsion of the wire (Ω)

from time to time, so as to maintain the cylinder at the deflection θ and by plotting the torques $\phi_1, \phi_2, \phi_3 \ldots$ against the times $t_1, t_2, t_3. \ldots$ If the material is a Maxwell liquid, the ϕ–t curve is of the exponential type and the stress vanishes for $t = \infty$. Schwedoff, however, found that in his gelatine solution the relaxation was not of this type, but that the material apparently maintained a small residual deformation permanently; i.e. if Ω was maintained at a constant value, the inner cylinder did not follow until $\theta = \Omega$ or $\phi = 0$, but a finite ϕ remained. Instead of Equation (XIV, d) he found the equation

$$p = \vartheta + (p_o - \vartheta)e^{-t/\tau} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

to be applicable, where ϑ is the residual stress which does not relax after practically infinite time. Hatschek [85] says that " the magnitude of this permanent deformation is very uncertain, as the whole apparatus and the condition of the liquid are highly susceptible to even small vibrations ", but if Schwedoff's observations were only qualitatively correct, and similar observations were made by Hatschek and Jane [86] on a number of other solutions, this would mean that the materials in question were in fact not liquids, as was thought from their appearance, but solids, or, in another terminology, not sols but gels.

To represent such materials, we postulate the Schwedoff body, built up of three elements in accordance with the structural formula

$$Schw = M - StV - H \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The model is shown in Fig. XV, 3. The rheological equation is

$$\dot{e} = (p - \vartheta)/\eta_{pl} + \dot{p}/\gamma_1 \quad . \quad . \quad . \quad . \quad . \quad (28)$$

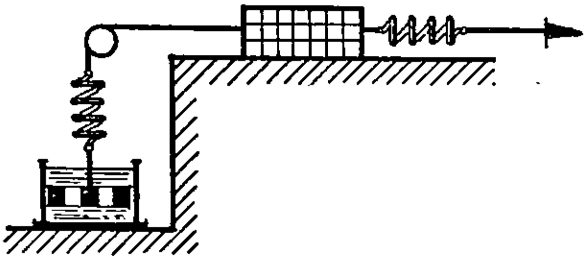

FIG. XV, 3. Model for a Schwedoff body corresponding to the structural formula Schw = M − StV − H.

7. The Schofield-Scott Blair Body

So far the most complex body which it was necessary to postulate in order adequately to describe the observations on a material, appears to be the Schofield-Scott Blair body, the material being flour dough.

In a series of investigations [42], the authors first stretched out long cylindrical pieces of dough for a measured time, at the end of which the cylinders were cut loose. It was found that part of the extension was recovered, but part was permanent (Fig. 1 in [42], part I). This showed that while flour dough (FD), because of the permanent extension of the cylinders, is not a Hooke body, it has a Hooke component; and in a first approximation we write $FD = H - X$, where X represents another element or elements to be deduced from further experiments. In the next series of experiments it was found that the elastic recovery decreased with time under stress in accordance with an exponential law (Fig. 4, in the same paper). Comparison with our Fig. XIV, 3, shows that we must write N for X, so that in a second approximation $FD = H - N = M$. This was confirmed by further experiments, in which the decay of internal stresses in pieces of dough which had been stretched and held stretched was followed (Fig. 6 in [42], part I). The shape of the curve conformed to our Fig. for Δl = const. The second report of the authors describes observations in which the rate of extension of cylinders of dough, hung vertically and allowed to elongate under the action of gravity, is related to the stress. It was found that, while the rate of extension generally decreased with decreasing stress, there was a finite stress at which the rate of extension vanished; in fact, a yield point. This shows that a St. Venant element must further be added; and in a third approximation $FD = N - H - StV = M - StV = Schw$. It had however been noticed that " a considerable time often elapses between the release of stress and the cessation of contraction ". This indicated an elastic after-effect, to the investigation of which a third report is devoted. *For an elastic after-effect a K body must be coupled on.* As the structural formula for FD contains a StV element, the question arises as to which end of the StV element the K body is to be connected (compare Section 3 above). Experiments (Fig. 2 in report III) showed that the elastic after-effect makes its appearance in strains *below the yield point.* This means that the K-body must be connected at the spring end. It could be formed there by parallel coupling of the spring with an N-element. The same figure, however, illustrates that, beside the "lagging" elastic recovery, there is also "instantaneous" recovery, i.e. the spring of the St. Venant element is not impaired in its working and the K-body is therefore connected to it "in series". We accordingly obtain in a fourth approximation the structural formula

$$SchScB = N - H - StV - K = M - StV - K = Schw - K \quad . \quad (28)$$

Fig. 4 in report III confirms the formula for deformations above the yield point.

Guided by Equation (28) it is not difficult to design a model for the Schofield-Scott Blair body. Equation (28), in addition, helps us to establish the rheological equation. The authors expressed it, translated into our notation, as follows :—

$$\dot{e} = p/\eta + \dot{p}/\gamma - \alpha \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where α represents the elastic after-effect. We note that they do not take into account that flour dough, as they themselves had found, possesses a yield point. Their rheological equation, therefore, represents only part of their experimental findings ; as a matter of fact it corresponds to the structural formula $Bu = M - K$ which results from the second form of Equation (28) when omitting the StV element. If we compare the rheological equation (7) of the Burgers body with Equation (29) we find that Schofield and Scott Blair's α is actually

$$\gamma/\eta_s e^{-\gamma/\eta_s t}(e_o + \frac{1}{\eta_s}\int_o^t p e^{\gamma/\eta_s t} dt) - p/\eta_s$$

and only if p is equal to zero, as in an elastic after-effect, is this reduced to $e_o \frac{\gamma}{\eta_s} e^{-\gamma/\eta_s \cdot t} = \frac{e_o}{\tau\kappa} e^{-/t.\tau_\kappa}$. Equation (29), therefore, cannot adequately express an elastic fore-effect.

If we take into account the StV element, the complete rheological equation of the Schofield-Scott Blair body is

$$\dot{e} = (p - \vartheta)\frac{\eta_{pl} + \eta_s}{\eta_{pl} \quad \eta_s} + \dot{p}/\gamma_l - \frac{\gamma}{\eta_s} e^{-\gamma/\eta_s \cdot t}$$
$$\left[e_o + \frac{1}{\eta_s}\int_o^t (p - \vartheta) e^{\gamma/\eta_s \cdot t} dt\right] \qquad (30)$$

Under a constant stress, exceeding the yield value, the body is plastically deformed at a rate which, in analogy to Equation (8) is

$$\dot{e} = (p - \vartheta)\,(1/\eta_{pl} + \frac{1}{\eta_s} e^{-\gamma/\eta_s\, t}) \quad . \quad . \quad . \quad . \quad (31)$$

Therefore, as in the case of Equations (19) and (20), the rate gradually decreases from

$$\dot{e}_o = (p - \vartheta)\,(1/\eta_{pl} + 1/\eta_s) \quad . \quad . \quad . \quad . \quad (32)$$

to

$$\dot{e}_\infty = (p - \vartheta)/\eta_{pl} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

It all depends upon the magnitude of the retardation time $\tau_\kappa = \eta_s/\gamma$ how long it will take for $\dot{e}$ to become practically constant and indistinguishable from $\dot{e}_\infty$. However, Equation (33) is the same as Equation (III, a), the rheological equation of the Bingham body. It is therefore not astonishing that Wolarowitsch and Samarina [87], when testing flour dough in a plastometer of the co-axial rotating cylinder type, found that the Reiner and Riwlin equation (III, g), which is based upon the *B*-body, was applicable. They probably sheared the material long enough to reach a steady state, and this was one corresponding to a *B*-body.

8. Regarding the Second and Third Axioms

This furnishes an illustration for the working of what I called the second and third axioms of rheology. The second (compare Section 12, Chapter I) says that every real material possesses all rheological properties, while the third states that every " simple " behaviour is a degeneracy of a more complex one. If flour dough is put into an apparatus where its properties under *steady* laminar shear are observed, *nothing more complicated than corresponds to the B-body can be observed.* In order to find out about the other properties of flour dough, i.e. those which Schofield and Scott Blair discovered, Wolarowitsch and Samarina would have had to use their apparatus in the manner of Schwedoff; they would then have found its *M*-component, etc., in accordance with the second axiom. On the other hand, we see now that Scott Blair was not entirely correct when he stated on page 27 of his book [35] that to apply " Bingham's principles to . . . flour dough . . . is contrary to the author's [Scott Blair's] experience." It is true that flour dough *is not* a Bingham body, but from Schofield and Scott Blair's observations the conclusion can be drawn with the help of the third axiom that flour dough *must* under certain conditions show the properties of a Bingham body.

Let us go back to Equation (30). If $\vartheta \rightarrow 0$, the *SchScB*-body degenerates, as we have already shown, into the *Bu*-body. If $\eta_s \rightarrow 0$, it degenerates into the *Schw*-body. The *Bu*-body itself can degenerate into the *M*- and *K*-bodies respectively and both these into the *N*- and *H*-bodies. The *Schw*-body degenerates into the *B*-body for $\gamma_l \rightarrow \infty$ and the latter into the *StV*-body for $\eta_{pl} \rightarrow 0$. Ultimately the *SchScB*-body is thereby reduced to either the *N*-, *H*- or *StV*-body and in accordance with the conditions of testing one or the other will predominate. One investigator may lay stress upon the elasticity of flour dough, the other upon its viscosity, while a third may emphasise its plasticity. Each one will be right, but not entirely so. Similarly, we have seen that Lee and Markwick [29] approximated bitumen by the *Bu*-body, while Lethersich assumed a degeneracy of the latter, which we called the truncated *Bu*-body. As far as I know, nobody has yet found a yield value in bitumen, but this may only mean that it was not observed under suitable conditions. When these will be established and a yield value found, for which there is some likelihood in blown bitumen, one will have to use the *SchScB*-body to describe the rheological properties of bitumen.

9. Summary

Complex ideal bodies can be built up in the form of models, structural formulas and rheological equations, from the elemental bodies of (i) the Hooke elastic (*H*) represented by a spring; (ii) the St. Venant plastic (*StV*) represented by a weight sliding over a table; and (iii) the Newtonian liquid (*N*) represented by a dashpot. The mechanical elements may be connected either " in series " (—) or " in parallel " (|). The rheological equation of a complex is derived from the rheological equations of the components by (a) adding deformations, when the coupling is in series; (b) adding stresses, when the coupling is in parallel.

When *elements* of the same kind are connected the resultant body possesses the same properties, e.g. $H_1 - H_2 = H$, etc. For complexes, this is generally not true, e.g. $M_1 \mid M_2$ is *not* an M-body. But $M_1 - M_2 = M$; $K_1 \mid K_2 = K$; $M - N = M_1$; $M_1 - H = M_2$; $K \mid N = K_1$; $K_1 \mid H = K_2$.

The following ideal bodies have been applied for the representation of the behaviour of certain materials, but are applicable for many others :—

(I) For bitumen, asphalt and concrete, the Burgers body

$$Bu = K - M \quad . \quad . \quad . \quad . \quad . \quad \text{(XV, a)}$$

with the rheological equation

$$\dot{e} = p\,\frac{\eta + \eta_s}{\eta\eta_s} + \dot{p}/\gamma_l - \frac{\gamma}{\eta_s}\,\mathrm{e}^{-\gamma/\eta_s \cdot t}\,(e_o + \frac{1}{\eta_s}\int_o^t p\mathrm{e}^{\gamma/\eta_s\, t}\,dt) \quad . \text{(XV, b)}$$

When the stress is constant

$$\dot{e} = p/\eta + p/\eta_s\,.\,\mathrm{e}^{-\gamma/\eta_s\, t} \; . \quad . \quad . \quad . \quad \text{(XV, c)}$$

and there is both viscous flow and an elastic after-effect starting with

$$\dot{e}_o = p(1/\eta + 1/\eta_s) \quad . \quad . \quad . \quad . \quad \text{(XV, d)}$$

and finishing up with

$$\dot{e}_\infty = p/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XV, e)}$$

(II) Also for bitumen, Lethersich proposed the two bodies $N - K$ and $M \mid N$. The first can be considered as a truncated Burgers body.

(III) For a concentrated gelatine solution, the Schwedoff body

$$Schw = M - StV - H \quad . \quad . \quad . \quad \text{(XV, f)}$$

with the rheological equation

$$\dot{e} = (p - \vartheta)/\eta_{pl} + \dot{p}/\gamma \; . \quad . \quad . \quad . \quad \text{(XV, g)}$$

If the deformation is kept constant, the stress relaxes in accordance with

$$p = \vartheta + (p_o - \vartheta)e^{-\gamma/\eta\, t} \quad . \quad . \quad . \quad \text{(XV, h)}$$

(IV) For flour dough, the Schofield-Scott Blair body

$$SchScB = N - H - StV - K = M - StV - K = Schw - K \quad . \text{(XV, i)}$$

with the rheological equation

$$\dot{e} = (p - \vartheta)\frac{\eta_{pl} + \eta_s}{\eta_{pl}\eta_s} + \dot{p}/\gamma_l - \frac{\gamma}{\eta_s}\mathrm{e}^{-\gamma/\eta_s\, t}\left[e_o + \frac{1}{\eta_s}\int_o^t (p - \vartheta)\mathrm{e}^{\gamma/\eta_s\, t}\,dt\right] \text{(XV,j)}$$

Of these bodies the most complex is the *SchScB*-body and all others can be derived from it as degeneracies, assuming that certain rheological cofficients vanish.

In the rheological equations the rheological coefficients appropriate for shear (e_t, p_t) have been used. Similar equations can be written down for isotropic (e_v,p) and normal (e_n, p_n) deformation, when, for elasticity the moduli κ and ϵ and for viscosity the coefficients η_v (ζ) and λ must be used.

CHAPTER XVI

STRENGTH

1. Theories of Strength

THE *strength* of a material is that property by which it resists either rupture or excessive deformation, the latter in many cases ultimately leading to rupture. *Rupture* is the visible separation of parts of a material body. When the body is strained beyond its strength, it *fails*. Failure can therefore be either through (i) excessive deformation, or (ii) rupture. Excessive deformation may lead to rupture either (*a*) in tension

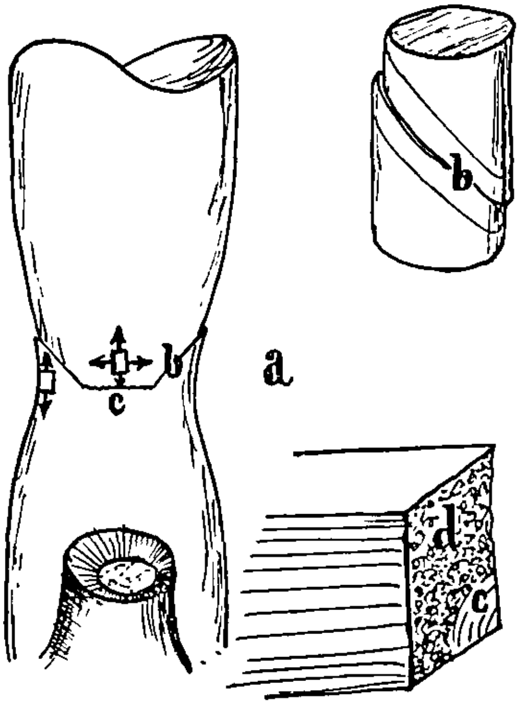

FIG. XVI, 1. Types of rupture. '*a*' tension-rupture of a non-porous material. '*b*' rupture by glide. '*c*' brittle rupture. '*d*' rupture by flow of porous material.

under constant load when the cross-sectional area has substantially decreased and the traction accordingly much increased ; or (*b*) one part of the body may be sheared against the other until both are separated : this is rupture by *glide*. In rupture by glide the separation is *tangential* to the surface of the separation, in (*c*) brittle rupture it is *normal* to that surface. There may, however, also be (*d*) a separation normal to the surface which is not brittle, but occurs *in flow*. There does not exist an established terminology and it will help the reader to compare Fig. XVI, 1.

To prevent failure is a very important task in engineering which makes it imperative to have a *quantitative measure* of strength. This is accomplished by means of a *theory of strength*. The following are the main theories of strength proposed to date :—

The material fails when a certain limit, which is of a different character in each case, is exceeded by

(i) the greatest of the principal stresses (Rankine),
(ii) the greatest of the principal strains (St. Venant),
(iii) the maximum strain-work (Beltrami, Haigh),
(iv) the maximum shearing stress (Coulomb, Mohr, Guest),
(v) the maximum distortional strain-work (Huber, Hencky),
(vi) the maximum *conserved* isotropic or distortional strain-work (Reiner and Weissenberg [89]).

Nadai [8] has critically discussed the theories (i) to (v), of which (i) to (iii) are of historical interest only. He finally accepts (v) under the heading " New Theories ". It should be noted that (vi) was not known at the date of the publication of Nadai's book. Regarding (v), the mathematical expressions for both Huber's and Hencky's theories are not very different, but while Huber is concerned with *rupture*, Hencky is concerned with *plastic flow*.

2. Importance of Rate of Deformation

All the above-mentioned theories, except the last one, have in common that *the rate of deformation* does not enter. They are purely *statical*. Recent improvements in the methods of testing and the frequent use of high-speed tests as well as of creep tests, and specially the testing of synthetic plastics, have made it, however, increasingly evident that the rate of deformation or, what comes to the same thing, the rate of application of the external forces, considerably influences the results of the tests. It is therefore obvious that purely statical concepts cannot lead to a satisfactory theory of strength. A theory taking account of *velocity* is required, both from theoretical and practical considerations. For instance, standard specifications for mild steel until recently asked for a certain minimum breaking stress to be determined in a tensile test without, however, specifying the rate at which the test is to be carried out. It is known that less conscientious suppliers of rolled steel, if their product was not up to standard at ordinary speeds, sometimes used to resort to the device of increasing the speed, with the result that a higher breaking strength was recorded. This loophole has been closed in the latest British Standard Specification by prescribing a definite rate of application of the load. On the extreme end of a series of materials where the influence of the testing speed makes itself felt, stands such a material as rayon, where the concept of strength has a meaning only in relation to a maximum rate of deformation.

3. The Huber-Hencky Theory

Because of the satisfactory evidence for the Huber-Hencky theory at low speeds, a good dynamical theory must include Huber-Hencky's statical theory as a special case for vanishing rate. This requirement is satisfied by Reiner and Weissenberg's theory, which, as far as I know, is the most general rheological theory of strength.

Huber's theory arose from Beltrami's theory of maximum strainwork. Beltrami overlooked the important fact of the radically different behaviour of all materials in isotropic strain and distortion, which we have emphasised at the very beginning of the present book. Many experiments have shown that under high *hydrostatic pressure* increasing amounts of elastic energy may be stored in a body without any limit from fracture or permanent deformation. Huber, therefore, considered isotropic strain and distortion separately. He assumed that there are two different measures for strength in the case of simple pull and push respectively. Let w_v be the strain-work per unit volume in isotropic (voluminal) strain and w_o in distortion, then Huber assumed that in push the measure of breaking strength is given by a maximum value of w_o and in pull by a maximum value of $w_v + w_o$. Hencky was concerned with finding a measure of plastic yield strength. He argued that there cannot be isotropic flow—and therefore also not isotropic *plastic* flow—either in compression or in tension. Therefore the plastic Flow-Condition must be expressible in terms of *distortion only*. As we have already mentioned in Chapter III, he accordingly pictured every plastic material as a vessel capable of absorbing in the unit volume a limited amount of distortional energy. When more is " poured in " the vessel overflows, or the material yields.

Huber's and Hencky's theories can be combined by assuming that there are three independent measures of strength, viz. :—

(A) Against rupture by isotropic tension ; the resilience E_v.

(B) Against plastic yielding in distortion : the resilience for $E_{o,\,pl}$.

(C) Against rupture in distortion : the resilience $E_{o,r}$.

Here E is " resilience " (energy), while the subscripts indicate " voluminal " (v), " distortional " (o), " plastic " (pl) and " rupture " (r).

In accordance with this, and in contradistinction to Huber, in pull a material may break in *two* ways, depending upon whether in the particular case E_v or $E_{o,pl}$ is exceeded.

4. Cohesional Strength

In Chapter IX we considered a body under the action of isotropic stress. When the stress is a pressure, the volume is decreased and we assumed that there is a limiting relative volume which cannot be reduced, even with the pressure becoming infinite. This would not exclude that the body may *break* under such pressure with the total volume of all parts remaining constant, which could well be the case in an æolotropic material. (This gives us the opportunity to remark again that we are dealing in the present book with isotropic or quasi-isotropic materials only.) Secondly, one has to keep in mind the possibility of *local* or *internal* ruptures in porous materials, manifesting itself in a permanent set. With these qualifications it may be said that for isotropic *pressure* the strength of every material is infinite. It is not so if the stress is a tension. The volume is then increased, the distances between the molecules or atoms which make up the material are also increased and by this, correspondingly, the attractive reactions between them, which balance the external forces. This cannot go on indefinitely because there is a maximum attractive force which cannot be substantially exceeded. When the external isotropic tension reaches that maximum, the strength of the material is reached. When the maximum is exceeded, the body breaks through rupture, its parts are torn asunder, it loses cohesion. We can therefore say that *for isotropic tension the strength of every material is determined by its cohesion.* The cohesion itself is the resultant of the attractive and repulsive atomic forces. In the unstrained state the resultant vanishes, each atom being at the bottom of a potential trough, about which its heat movements make it oscillate; in this state the material does not possess any elastic potential energy. At smaller distances there is practically unlimited repulsion, at larger distances an attraction prevails, which is limited. The elastic energy which is produced by isotropic tension is therefore also limited by the value E_v. This is a measure of its cohesional strength.

Rupture in isotropic tension may be of two kinds. Let the material have no pores or holes, then, when cohesion fails, there cannot be any flow in which the particles of the body move away radially with preservation of the continuity of matter. In this case the rupture is brittle. But if the material has holes, such as was found by Reiner, Rigden and Thrower [90] in asphalt, the rupture will be preceded by flow, if given enough time (as at " *d* " of Inset of Fig. XVI, 1). If the stresses are very great, rupture will in this case also be brittle, there being not enough time for flow (as at " *c* " of Inset of Fig. XVI, 1). The statement made by me on page 119 of *Ten Lectures* that rupture under isotropic tension is always brittle must accordingly be qualified.

5. Breaking Energy

When the body breaks into pieces the process is accompanied by the appearance of certain forms of energy. There is an audible report which indicates the presence of vibrations. These vibrations will gradually die down and ultimately be converted into heat. Of greater importance is the increase of surface energy due to the increase of the surfaces of the broken parts over the surface of the entire body. These energies and others appearing in rupture are provided for by the conversion of the potential energy E. There have been attempts to calculate on this basis the strength of a material from the value of its surface tension. The results were many orders of magnitude out. We shall not go into this side of the problem : it belongs to metarheology. But one thing is clear : the expenditure of energy connected with rupture cannot come from the *total* strain-work performed by the external forces. Only that part of the strain-work which is *conserved* in the body is available for conversion into other forms of energy. Generally, part of the strain-work is *dissipated*, and this, of course, is not available. For instance, when isotropic tension is not applied infinitely slowly through states of equilibrium but with finite speed, this will cause, as we learned in Chapter IX, viscous resistance and part of the strain-work will be dissipated. In order to reach the limiting value E_v of potential energy, the external forces will have to perform a *greater amount of work.* This can be done only by increasing the isotropic stress over that stress at which the material breaks when the forces are applied infinitely slowly. This will, accordingly, manifest itself in a *rise of strength.*

6. Isotropic Tensile Strength

Let us consider this case in greater detail. In the infinitely slow application of the tension, E_v can be calculated in accordance with Equation (IX, d) and expressed in terms of stress or strain or relative volume or density—there being a one-valued relation between all these quantities. In this case there is therefore a *definite isotropic tension* (or cubical dilatation, relative volume, density) at which the material will break. As we said before, that isotropic tension is equal to the molecular or atomic cohesion. For non-porous materials the isotropic tensile strength must accordingly be very high. In classical hydrodynamics it is assumed that liquids have no such strength, but from thermodynamic considerations Poynting and Thomson [91] suggested that the isotropic tensile strength of water is about 25,000 atmospheres, while van der Waals calculated from his equation a value of about 10,000 atmospheres. O. Reynolds found by actual experiment that water could sustain a tension of about 5 atmospheres without rupture. I suggested in a letter [92] that the well-known phenomenon of cavitation-erosion of metals may be due to pieces of metal being torn off by the water before its own tensile strength is reached. This would imply that the isotropic tensile strength of a metal was lower than that of water. In reply to my letter, R. S. Silver [93] pointed out that " the failure of a liquid to sustain the calculated tensile strength in an enclosed and filled space is due . . . to the formation of vapour bubbles. It is the formation of vapour cavities around nuclei which prevents the full tensile strength from being attained." This means that the liquid actually is not *non-porous*, it contains microscopic holes around which there exist *non-isotropic stress-concentrations*. Now, if even a liquid the molecules of which easily flow into internal holes to reduce and close them actually has pores, still more must this be assumed of solids, where pores developed in the process of formation will be stable. Therefore, while theoretical cohesion might be very high, actually, because of holes, pores, flaws and cracks, the isotropic tensile strength will be comparatively low.

7. The Isotropic Rupture Condition

Let us now consider the action of isotropic tension when the rate of dilatation *does not vanish*. As we said in Section 5 above, this will cause a rise in strength. We shall presently translate this qualitative statement into quantitative language.

The general volumetric rheological equation of a homogeneous material is in accordance with Equation (IX, a).

$$p = \kappa e_v + \zeta_\kappa \dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ζ_κ is the Kelvin coefficient of volume viscosity. In Equation (IX, a) we wrote η_v, but, in accordance with the principles laid down in the Summary to Section XV, we must use the subscript K in order to make clear that the function of *this* volume viscosity is to delay the appearance of the elastic strain and/or to cause viscous damping, but that, actually, there is *no flow*. In short, it is a *solid* viscosity.

The strain-work per unit volume is in accordance with the Equation (IX, 37)

$$w_v = \int_o^{e_v} \kappa e_v \, de_v + \int_o^{t} \zeta \dot{e}_{\kappa v}{}^2 dt \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As explained in Section 10 of Chapter IX, the first part on the right side is conserved as elastic potential energy, while the second part is converted into heat and dissipated. Rupture, therefore, takes place when the first part reaches and exceeds the *isotropic resilience* of the material measured by E_v. The Rupture Condition will accordingly be

$$w_v - \int_o^{t} \zeta_\kappa \dot{e}_v{}^2 dt = \int_o^{e_c} \kappa e_v de_v \geq E_v \quad . \quad . \quad . \quad . \quad (3)$$

If κ is constant, the integral on the right-hand side can be calculated and is $\kappa e_v{}^2/2$ and, therefore, the cubical dilatation at which rupture takes place

$$e_{vr} = \pm \sqrt{2E_v/\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, Equation (1) gives for the tensile stress at which rupture takes place

$$p_r = \kappa e_{vr} + \zeta_\kappa \dot{e}_v = \sqrt{2E_v\kappa} + \zeta_\kappa \dot{e}_v \quad . \quad . \quad . \quad (5)$$

Let the breaking stress in the static state for which $\dot{e}_v = 0$ be p_{rs}

$$p_{rs} = \pm \sqrt{2E_v\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

then

$$p_r = p_{rs} + \zeta_\kappa \dot{e}_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Therefore: *the stress at which a material fails in isotropic tension increases with the rate of dilatation.* If ζ_κ is constant that increase is linear; otherwise it follows some other law determined by $\zeta_\kappa = f(\dot{e}_v)$.

It should be noted that this result does not depend on κ being constant. From Equation (2) it follows directly that the strain-work to bring about rupture (w_{vr}) is for any given rate of dilatation ($\dot{e}_v$) always greater than w_{vrs}, the strain-work expended in infinitely slow dilatation (for which $\dot{e}_v = 0$).

While, therefore, in the absence of viscous resistance to the cubical volume expansion of a material there is a definite stress at which rupture occurs; when there exists volume viscosity, the dilatation e_{vr} is definite (compare (4)), but the stress p_r is not.

8. The Distortional Failure Condition for a Hooke Solid in Simple Shear

We now pass to simple shear. Here we have before us not mainly *one* general rheological equation as in the case of cubical dilatation. For shear we have to consider a great variety of rheological equations corresponding to the infinite variety of real materials.

Let us first consider the Hooke solid.

Its rheological equation is (I, d) and the strain-work is given by (III, c).

The rupture condition is accordingly

$$w_{tr} = \int_o^{lt} \gamma e_t de_t \geq E_{tr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If γ is constant this gives, considering (III, d)

$$\left.\begin{aligned} e_{tr} &= \pm \sqrt{2E_{tr}/\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \\ p_{tr} &= \pm \sqrt{2\gamma E_{tr}} \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}\right\}(9)$$

Both e_{tr} and p_{tr} are definite constants depending upon the same definite quantities γ and E_{tr} and are independent of the rate of strain or the rate of application of load.

Should the body be plastic, similar reasoning, starting from Equation (I, e) results in the Equation (III, b) and

$$e_{t,pl} = \pm \sqrt{2E_{t,pl}/\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

9. Failure of a Newtonian Liquid

At the other extreme of the rheological bodies is the Newtonian liquid with the rheological equation (I, f), the strainwork being in accordance with (VII, a).

$$w_t = \int_o^t \eta \dot{e}_t^2 dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In accordance with our criterion, rupture in a Newtonian liquid would therefore only be possible if E_{tr} was equal to zero and then rupture would always and continuously take place. This is not what we observe. Rupture has been defined by us as a visible separation of the parts of the body, and we do not observe such visible separation in a flowing liquid in laminar flow.* If the velocity increases so that turbulence sets in, a visible separation of the parts of the liquid can be observed. Turbulence is of great interest to the hydraulic engineer, but of little interest to the rheologist, except perhaps in the case of what Wo. Ostwald termed Structural Turbulence and about which we shall presently have to say a few words. We may, at this occasion point out the analogy which exists between turbulence and rupture as conceived by us. The inception of turbulence is connected with a dimensionless figure called Reynolds' Number. Weissenberg (compare Rabinowitsch [94]) has expressed this figure as the ratio of dissipated to kinetic energy. In this interpretation the kinetic energy E_k plays in turbulence the same part as the potential energy E_p in rupture. Turbulence is accordingly determined by a certain point on the side E_k–E_d of the triangle, Fig., page 117 of *Ten Lectures*, while rupture is determined by a point on the side E_p–E_d of the triangle.

10. Structural Turbulence

If one were to assume the Coulomb theory of maximum shearing stress (iii of Section 1 above) to be valid for every kind of material, one would say that in a liquid p_t could also not exceed a certain maximum value. Otherwise, as there is no inherent limit to $\dot{e}_t$, the shearing stress could also increase without limit and water would be stronger than steel. In a very early paper [95], I assumed that there was a shearing strength for liquids, the same as for solids, and when this is exceeded a discontinuity in the flow appears so that, e.g. in the case of Fig. I, 4, there would be a break in the straight line representing the increase in velocity from zero to V. There does not appear to be any evidence to support this view in regard to simple Newtonian liquids, but Wo. Ostwald and Auerbach [96] maintained that in liquids showing structural viscosity, turbulence sets in long before Reynolds' critical velocity is reached. They suggested as the reason the internal rupture of the structure of the system,

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* The breaking up of a jet of liquid into drops due to *surface tension* belongs to metarheology.

which causes the appearance of eddies such as occur in turbulence.

If this is so, there would, e.g. in laminar flow of a liquid through a tube, be a maximum shearing stress

$$\text{max.}\ p_t = R\Delta p/2l \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

compare (II, c). If we assume that in a first approximation Poiseuille's law is valid, the mean velocity is

$$\text{mean}\ v = Q/R^2\pi = R^2\Delta p/8\eta l \quad . \quad . \quad . \quad . \quad (13)$$

and, therefore,

$$\text{crit.}\ v = R/4\eta \,.\, p_{t,max} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the critical velocity for the appearance of structural turbulence.

As is known, Reynolds' critical velocity is

$$\text{crit.}\ u = \eta/R \,.\, C/\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where C is a constant of the apparatus, but not of the liquid. These two critical velocities are accordingly of opposite character. Crit. v increases, while crit. u decreases with R.

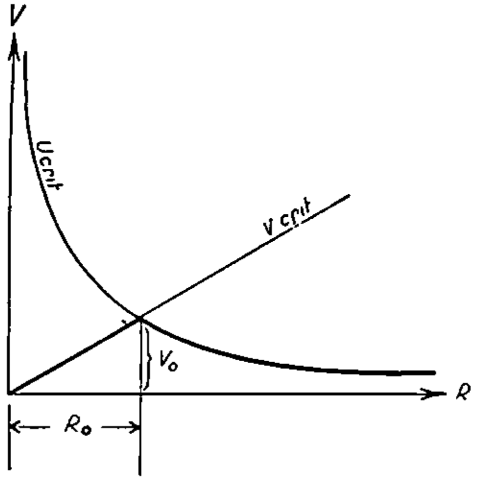

FIG. XVI, 2. Structural and Reynolds' Turbulence.

In respect of η the opposite takes place. For one and the same liquid, crit. v and crit. u vary as shown in Fig. XVI, 2. Both are equal when the radius of the tube is

$$R_o = 2\eta\sqrt{C/\rho p_{t,max}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

A solution exhibiting structural viscosity would therefore be

" weaker " than its solvent. While the latter could sustain shearing stresses up to the inception of Reynolds' turbulence, the former would break up before, namely when the structure of the dispersed phase is destroyed (Reiner [97]).

11. A Model of Liquid Strength

For a better understanding of the strength of a liquid a modification of Hencky's model (compare Section 6 of Chapter III) will serve. Let the bottom of Hencky's vessel be

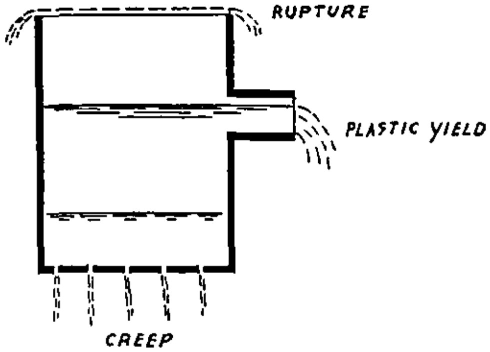

FIG. XVI, 3. Model of strength of a liquid.

perforated as shown in Fig. XVI, 3. Let there be an overflow at some level above the bottom. The liquid contained in the vessel represents the elastic distortional energy which can be stored in unit volume of the material. If energy is expended upon the material, or liquid poured into the vessel, so slowly that it leaks out entirely through the holes in the bottom, in other words ; if the velocity of deformation is less than or equal to the velocity of relaxation, there will be no failure by distortion. The material, even if to all appearances it is taken as a solid, will flow at some rate depending upon the area of the openings. If that area is small the liquid will leak out slowly, i.e. there will be creep. However, by pouring in energy at a higher rate, i.e. by increasing the *power* of deformation (remember : power = work/time) the vessel is steadily filled until the overflow opening is reached (assuming that such one exists) and further energy is lost by plastic yield. This is the case for instance with a metal which creeps under stresses below the yield point, and flows plastically when the yield point is exceeded. If the combined leakage by relaxation and plastic yield is not sufficient to dissipate the energy of work of the external forces, the level in the vessel rises above the overflow and reaches its brim. The moment the energy flows over the brim, the material breaks.

One or the other of the just-mentioned features of the vessel may be absent or especially prominent. If there are no holes in the bottom and no overflow, the material can only fail through brittle rupture. If there is an overflow so large that it will dispose of any supply of energy at whatever rate, the material will yield plastically, but will not break. If there are holes in the bottom but no overflow, the material will show creep and brittle rupture. This is the case with asphalt and concrete which, in spite of their ability to creep, cannot be deformed plastically and fail only through brittle rupture. If the holes are not at the bottom but in the sides at some level above the bottom, there will be viscous dissipation of energy down to that level : this is the case of the Schwedoff body. A vessel without a bottom (if there was such an absurdity) through which infinite quantities of energy can be poured at any rate is the model of a Newtonian liquid, the velocity of relaxation of which is infinite and which, through its viscosity, can dissipate energy at any rate desired. From considering the model we therefore come again (as in Section 10 above) to the result that a Newtonian liquid, say water, should be able to withstand very high, practically unlimited shearing stresses, a result which we *cannot very well accept. This result, therefore, points to some* defect or limitation in our theory. The limitation arises from our having considered steady states of equilibrium only, when there are *no accelerations*. When the particles are accelerated, rupture may occur through an entirely different mechanism. Let two contiguous parts of the body have at some time the same or slightly different velocities : if the state is steady and the movements are not accelerated, the body will either move as a whole and/or be deformed. The same happens if both parts are accelerated in equal amounts. But let the accelerations differ, however little, then after some time there will be reached a state when both parts must separate, i.e. there will be rupture. It can be shown that the stresses at which rupture then occurs may be very small or even vanish : this result can be deduced from the complete stress equations (Equation (1), page 30, *Ten Lectures*) by considering d'Alembert forces, which balance or nearly balance the external forces with the tractions absent or small (compare Reiner [98]).

12. Failure of a Maxwell Liquid

Passing from qualitative to quantitative considerations, let us now consider the failure of a Maxwell liquid with the rheological equation (XIV, b). Its elastic stresses relax and the " vessel " will accordingly have holes at the bottom.

We have

$$\dot{w}_t = p_t\, \dot{e}_t = p_t\dot{p}_t/\gamma_M + p_t^2/\eta \quad . \quad . \quad . \quad . \quad (17)$$

of which the conserved part is $p_t\dot{p}_t/\gamma_M$. Conditions for failure by either plastic yielding or rupture are entirely analogous and we shall therefore write $E_{o,t}$ for both $E_{o,pl}$ and $E_{o,r}$ of Section 3, (B) and (C). We shall also, for simplicity, omit the subscript " t ", but the reader should mentally add it.

The failure condition is accordingly

$$\int_o^t p\dot{p}/\gamma_M = \int_o^t (\dot{w} - p^2/\eta)dt \geq E_{o,f} \quad . \quad . \quad . \quad (18)$$

If γ is constant, this results in

$$p_f^2 = 2\gamma_M E_{o,f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and Equation (XIV, b) now gives

$$\dot{e}_f = \dot{p}_f/\gamma_M + \frac{1}{\eta}\sqrt{2\gamma_M E_{o,f}} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Now, let $\dot{e}_{f,s}$ be the rate of deformation at which the material fails under a constant stationary load, when $\dot{p}$ vanishes so that

$$\dot{e}_{f,s} = \tfrac{1}{2}\sqrt{2\gamma_M E_{o,f}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

then

$$\dot{e}_f = \dot{e}_{f,s} + \dot{p}_f/\gamma_M \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The Maxwell liquid, therefore, fails in distortion *when the tangential stress reaches a definite limit,* $p_{t,f} = \pm\sqrt{2\gamma E_{o,f}}$. *There is, however, no limit to distortional deformation, while the rate of deformation at which the material fails increases with the rate at which the load is applied.*

If γ_M is not constant, this increase is not linear.

13. Failure of a Kelvin Solid

We proceed in the same manner when investigating the failure of a Kelvin solid. The results are similar to those found in Section 7 above for isotropic strain and stress, because we assumed there the presence of solid viscosity.

Starting from the rheological Equation (XIV, g) the failure condition is found to be

$$\int_o^t \gamma e\dot{e} + \int_o^t (\dot{w} - \eta_K \dot{e}^2) \geq E_{o,f} \quad . \quad . \quad . \quad . \quad (23)$$

Let $p_{f,s}$ be the stress at which the material fails in a static test when $\dot{e}$ vanishes, then

$$p_f = p_{f,s} + \eta_K \dot{e} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The Kelvin solid, therefore, fails in distortion *when the tangential strain reaches a definite limit* $e_{t,f} = \pm \sqrt{2E_{o,f}/\gamma}$. *There is, however, no limit to the rate at which the stress is applied, while the tangential stress at which the material fails increases with the rate of strain.* If η_K is not constant, this increase is not linear.

14. The Maxwell Liquid in Simple Pull

The examples treated so far show the way by which to arrive at the failure condition for other more complicated materials, e.g. the Burgers body, the Bingham body and others. This we shall not do here, but shall apply the theory in greater detail to the problem of failure of a Maxwell liquid in simple pull, following Reiner and Freudenthal [99].

Let the material under investigation be formed into a cylindrical rod, as Schofield and Scott Blair did with flour dough [42] and Lethersich [82] with bitumen. Let l be the length of the cylinder and R the radius of the circular section of area A; l_o, R_o and A_o denoting their values at the beginning of the experiment, when $t = 0$. Let a pull P be applied in the longitudinal direction. For reasons of symmetry, that direction will be one of the principal axes (compare Chapter X) of stress and deformation, say " k ", while tractions and deformations in the directions " i " and " j " normal to the axis will be equal. The normal traction in the direction of the axis will be

$$p_n = P/A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

causing a longitudinal extension e_l and lateral contractions $-e_c$. The latter are not accompanied by stresses and the power of deformation is therefore

$$\dot{w} = p_n \dot{e}_l \quad . \; . \; . \; . \; . \; . \; . \; . \quad (26)$$

In accordance with Equation (X, f) the stress can be resolved into its isotropic and deviatoric components, or, in other terminology, hydrostatic tension and a system of shears (stress differences),

$$p = \frac{1}{3}\frac{P}{A}\,;\; p_{ol} = \frac{2}{3}\frac{P}{A}\,;\; p_{oc} = -\frac{1}{3}\frac{P}{A} \quad . \; . \; . \quad (27)$$

where the subscript "*o*" indicates the deviator.

The deformation can be resolved in the same way in accordance with Equations (X, b) and (X, c)

$$e_v = e_l + 2e_c\,;\; e_{ol} = \frac{2}{3}(e_l - e_c)\,;\; e_{oc} = -\frac{1}{3}(e_l - e_c) \quad . \quad (28)$$

The power of deformation consists of two parts, viz. one of cubical dilatation

$$\dot{w}_v = p\dot{e}_v = \frac{1}{3}\frac{P}{A}(\dot{e}_l + 2e_c) \quad . \; . \; . \; . \quad (29)$$

the other of distortion

$$\dot{w}_o = p_{ol}\dot{e}_{ol} + 2p_{oc}\dot{e}_{oc} = \frac{2}{3}\frac{P}{A}(\dot{e}_l - \dot{e}_c) \quad . \; . \quad (30)$$

both adding up to $\dot{w}$ of (26).

We now arrange the test so that one end of the rod is fixed, while the other end at the distance l moves with the velocity $v(t) = dl/dt$. The extension is, in accordance with (X, a),

$$e_n = \ln(l/l_o) \quad . \; . \; . \; . \; . \; . \; . \quad (31)$$

Therefore the rate of extension

$$\dot{e}_n = d(\ln l - \ln l_o)/dt = \frac{1}{l}\frac{dl}{dt} = \frac{v}{l} \quad . \; . \; . \quad (32)$$

Now, in accordance with (X, b)

$$\dot{e}_v = \dot{e}_n + 2\dot{e}_c \quad . \; . \; . \; . \; . \; . \; . \quad (33)$$

But

$$V = lA \quad . \; . \; . \; . \; . \; . \; . \; . \quad (34)$$

and therefore $\ln V = \ln l + \ln A$, or

$$\frac{1}{V}\frac{dV}{dt} = \frac{1}{l}\frac{dl}{dt} + \frac{1}{A}\frac{dA}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Comparison of (33) and (35) yields

$$\dot{e}_c = \frac{1}{2A}\frac{dA}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Expressions (32) and (36) make Equations (29) and (30).

$$\dot{w}_v = \frac{P}{3A}\left(\frac{v}{l} + \frac{1}{A}\frac{dA}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$\dot{w}_o = \frac{2P}{3A}\left(\frac{v}{l} - \frac{1}{2A}\frac{dA}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

So far the expressions of this Section are valid for any material and any magnitude of deformation. We now consider a special material which, we assume, behaves as a Maxwell liquid in distortion, while its cubical dilatation is purely elastic. In other words, we assume that its distortional stresses relax, while its isotropic stresses do *not* relax. The latter assumption is safe for all dense materials with no appreciable voids. Let us furthermore assume, for simplicity, that the material has no " solid " volume viscosity, i.e. that its volume-elasticity acts instantaneously without delay. Then its " isotropic " rheological equation will be

$$p = \kappa e_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Even should the material have delayed volume elasticity, our results will be valid when the volume does not change or when it changes slowly.

For the " distortional " equation we have to take into account that we are dealing here with simple tension and not with simple shear for which case we postulated Equation (XIV, b). An appropriate modification gives for our present case

$$\dot{e}_n = \dot{p}_n/\epsilon_M + p_n/\lambda_M \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

or introducing p_n from (25)

$$\dot{e}_n = \frac{1}{\epsilon_M}\frac{d(P/A)}{dt} + \frac{1}{\lambda_M}(P/A) \quad . \quad . \quad . \quad . \quad (41)$$

When the load is applied, the volume expands to its maximum which it assumes under the stress p and which is p/κ.

After this, the volume either remains constant or changes so slowly that we may assume $\dot{e}_v = 0$.

When this is the case lA is constant and Equations (33) and (36) yield

$$\dot{e}_n = -\frac{1}{A}\frac{dA}{dt} \quad . \; . \; . \; . \; . \; . \; . \quad (42)$$

and Equation (32)

$$v = -\frac{l}{A}\frac{dA}{dt} \quad . \; . \; . \; . \; . \; . \; . \quad (43)$$

With this Equation (41) becomes

$$\frac{1}{A}\frac{dA}{dt} = -\frac{1}{\epsilon_M}\frac{d(P/A)}{dt} - \frac{1}{\lambda_M}(P/A) \quad . \; . \; . \quad (44)$$

while Equation (38) is reduced to

$$\dot{w}_o = \dot{w} = -\frac{P}{A^2}\frac{dA}{dt} = \frac{1}{2\epsilon_M}\frac{d(P/A)^2}{dt} + \frac{1}{\lambda_M}(P/A)^2 \; . \quad (45)$$

The strain-work expended during the time t is accordingly

$$w = \int_o^t \dot{w}dt = \frac{1}{2\epsilon_M}(P/A)^2 + \frac{1}{\lambda_M}\int_o^t (P/A)^2 dt \quad . \; . \quad (46)$$

where it should be kept in mind that in order to calculate the potential elastic energy we have to start from the unstressed state, or $P = 0$ for $t = 0$.

Rupture occurs either when the isotropic stress $P/3A$ reaches a certain value $= \sqrt{2\kappa E_{v,r}}$ in accordance with Equation (6) or when (as in Equation (18))

$$\frac{1}{2\epsilon_M}\int \frac{d}{dt}(P/A)^2\, dt = E_{o,r} \quad . \; . \; . \; . \quad (47)$$

which, as in Equation (19), gives

$$(P/A)_r^2 = 2\epsilon_M E_{o,r} \quad . \; . \; . \; . \; . \; . \quad (48)$$

Therefore, there will be brittle or deformational rupture in accordance with whether $E_{v,r}$ is smaller or larger than $\frac{\epsilon}{9\kappa}E_{o,r}$.

The material can be tested under the following conditions:

(i) constant stress p_n

(ii) constant load P

(iii) constant velocity of elongation v

(iv) constant rate of strain $\dot{e}_n$

(v) constant rate of stress $\dot{p}_n$.

(i) With one of the apparatuses mentioned in Section 1 of Chapter XI we keep the stress P/A constant $= P_o/A_o$, or

$$P = \frac{P_o}{A_o}A \quad . \; . \; . \; . \; . \; . \; . \quad (49)$$

Introducing expression (48) into Equation (44) yields a differential equation in dA/dt as follows :

$$dA/dt = -P_oA/A_o\lambda_M \quad . \; . \; . \; . \quad (50)$$

the solution of which is

$$A = A_o e^{-P_o/A_o\lambda_m t} \quad . \; . \; . \; . \; . \; . \; . \quad (51)$$

and therefore

$$dA/dt = -\frac{P_o}{\lambda_M}e^{-P_o/A_o\lambda_M t} \quad . \; . \; . \; . \; . \; . \quad (52)$$

We now easily find the following relations :

(*a*) Velocity-time function (compare (43))

$$v = \frac{l^2}{l_oA_o}\frac{P_o}{\lambda_M}e^{-P_o/A_o\lambda_M t} \quad . \; . \; . \; . \; . \; . \; . \quad (53)$$

(*b*) Length-time function

$$l = l_o e^{-P_o/A_o\lambda_M t} \quad . \; . \; . \; . \; . \; . \; . \quad (54)$$

(*c*) Length-load function

$$l = l_oP_o/P \quad . \; . \; . \; . \; . \; . \; . \quad (55)$$

(*d*) Strainwork-time function (compare (46))

$$w = (1/2\epsilon_M + 1/\lambda_M)(P_o/A_o)^2 \quad . \; . \; . \; . \quad (56)$$

Because $P/A = P_o/A_o$, rupture takes place either immediately when the load is applied or not at all. The rupture condition is accordingly independent of time and identical with the Huber-Hencky condition.

(ii) If we apply to the rod a constant load $P =$ const. $= P_o$, this constitutes the *creep* test :

Equation (44) now becomes

$$\frac{1}{A}\frac{dA}{dt} = -\frac{P}{\epsilon_M}\frac{d(1/A)}{dt} - \frac{P}{\lambda_M}\frac{1}{A} \quad . \; . \; . \; . \quad (57)$$

the solution of which is

$$A/A_o - \frac{P_o}{A_o\epsilon_M}\ln(A/A_o) = 1 - \frac{P_o}{A_o\lambda_M}\,t \qquad . \quad . \quad (58)$$

When ϵ_M is very large, as it mostly will be, or the rate of stress small, the last equation can be approximated between the limits $A_o \geq A \geq 0{\cdot}5\,A_o$ by the linear equation

$$A = A_o\left(1 - \frac{P_o}{A_o\lambda_M}\,t\right) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

From this

$$dA/dt = -\,P_o/\lambda_M \qquad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Hence

(*a*) Velocity-time function

$$v = l\,\frac{P_o/\lambda_M}{A_o(1 - P/A_o\lambda_M\,.\,t)} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

(*b*) Length-time function

$$l = \frac{l_o}{1 - P_o/A_o\lambda_M\,.\,t} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

(*c*) Stress-deformation function

$$\dot{e}_n = p_n/\lambda_M \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

(*d*) Strainwork-time function

$$w = \frac{1}{2\epsilon_M}(P_o/A_o)^2\frac{1}{(1 - P_o/A_o\lambda_M\,.\,t)^2} + (P_o/A_o)\frac{1}{1 - P_o/A_o\lambda_M\,.\,t} \quad . \quad (64)$$

Rupture takes place when either

$$P_o/A_o = 3(1 - P_o/A_o\lambda_M\,.\,t)\sqrt{2\kappa E_{v,r}} \qquad . \quad . \quad (65)$$

or

$$P_o/A_o = (1 - P_o/A_o\lambda_M\,.\,t)\sqrt{2\epsilon_M E_{o,r}}\,. \quad . \quad . \quad (66)$$

i.e. at the time

$$t = \lambda_M(A_o/P_o - 1/3\sqrt{2\kappa E_{v,r}}) \quad . \quad . \quad . \quad . \quad (67)$$

or

$$t = \lambda_M(A_o/P_o - 1/\sqrt{2\epsilon_M E_{o,r}}.) \quad . \quad . \quad . \quad . \quad (68)$$

Comparing Equation (63) with (X, o) we see that by assuming ϵ_M to be very large we have in some respects reduced the Maxwell to the Newtonian liquid. For the latter, Equation (62) is of special interest when written in the form

$$1/l \doteq 1/l_o - \frac{P}{V\lambda_M} t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

In accordance with this equation, if we plot $1/l$ against time, a Newtonian liquid, and in a first approximation a Maxwell liquid also, should give a straight line the slope of which is $P/V\lambda_M$ and a measure of the fluidity of the material. Mr. T. J. Qadura of the Arab Training College, Jerusalem, independently found this relation and tried it with success on numerous observations with bitumen, one of which is shown in Fig. XVI, 4. If it does not break before, the rod would become of infinite length in finite time $= \frac{A_o\lambda_M}{P}$.

For behaviour under the other testing conditions, the reader is referred to the paper by Reiner and Freudenthal [99]. The apparently so simple ordinary tensile test with constant velocity of elongation leads to the most complicated equations.

15. Summary.

Strength is that property of a material by which it resists failure through *either (i) excessive deformation, or (ii) rupture.* There are two independent modes of failure in each one of the cases mentioned, viz. (*a*) in isotropic, (*b*) in distortional strain. Excluding the exceptional case of volume-flow, failure in isotropic strain can only be brittle rupture.

In *steady states* of deformation, the strength is reached when a definite amount of elastic potential energy, the *resilience* E of the material, is produced in unit volume of it. There are ordinarily three independent measures of strength, viz., E_v, $E_{o,pl}$, $E_{o,r}$; in exceptional cases E_v may be either $E_{v,pl}$ or $E_{v,r}$.

The resilience E is the time-integral of that part of the rate of strain-work which changes sign on change of sense of strain from positive to negative, while that part of the power which is always positive is dissipated.

The following criteria are found for some ideal materials :

A. Hooke-solid.

$$e_f = \sqrt{2E_f/\gamma}\,;\; p_f = \sqrt{2\gamma E_f} \quad . \quad . \quad . \quad (\text{XVI, a})$$

There exists a definite stress at which the material fails and likewise a definite limit to deformation.

B. Newtonian liquid.

A Newtonian liquid will not fail in steady states of flow. In non-steady states, a simple Newtonian liquid may fail through Reynolds' turbulence, while a generalized Newtonian liquid possessing structural viscosity may fail before that through structural turbulence.

C. Maxwell liquid.

$$p_f = \sqrt{2\gamma_M E_f} \quad . \quad . \quad . \quad . \quad . \qquad \text{(XVI, b)}$$

$$\dot{e}_f = \dot{p}_f/\gamma_M + \frac{1}{\eta}\sqrt{2\gamma_M E_f} \quad . \quad . \quad . \qquad \text{(XVI, c)}$$

There exists a definite failure-stress, but no limit to deformation. The rate of deformation at which the material fails increases as the rate at which the load is applied.

D. Kelvin-solid.

$$e_f = \sqrt{2E_f/\gamma} \quad . \quad . \quad . \quad . \quad . \qquad \text{(XVI, d)}$$

$$p_f = \eta_\kappa \dot{e}_f + \sqrt{2\gamma E_f} \quad . \quad . \quad . \quad . \qquad \text{(XVI, e)}$$

There is a definite limit to deformation, The stress, however, at which the material fails increases with the rate of deformation.

In all the above equations the subscript "f" can be replaced by one of the following, "o, pl", "o, r". For the isotropic stress-strain case when E_f is either $E_{v,pl}$ or $E_{v,r}$, the modulus of elasticity γ must be replaced by the modulus κ and the coefficient of viscosity η by the coefficient ζ. Similarly in simple pull these coefficients will be ϵ and λ.

When a Newtonian liquid is formed into a rod and pulled with constant load, plotting $1/l$ against time gives the straight line

$$1/l = 1/l_o - \frac{P}{V}t \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(XVI, f)}$$

where V is the constant volume of the rod. This relation can in a first approximation also be used for a Maxwell liquid.

CHAPTER XVII

DILATANCY AND THE WEISSENBERG EFFECT

1. Independence of dilatation and distortion.

In the foregoing, when postulating rheological equations, we resolved both deformation and stress into an isotropic component and another one by which it " deviates " from isotropy. When a deformation, the first was called volumetric deformation or strain, cubical dilatation, or, simply, dilatation; the second we called distortion. When a stress, the first is often spoken of as " hydrostatic " pressure or tension and the second as stress-*difference*; the latter can also be regarded as a combination of shearing tractions. We then assumed that every material was characterised by two independent rheological equations: one between the isotropic, the other between the deviatoric components of stress and strain; the latter being equivalent to a relation between shear and shearing stress. It was assumed that an isotropic stress produced a cubical dilatation only or a change of volume, but no change of shape. Similarly, a shearing traction was supposed to produce shear or a change of shape only, but not to affect the volume or density of the material. It will presently be shown that while this view —the view of classical elasticity and hydrodynamics—is generally correct, there are two phenomena which contradict it. These are *dilatancy* discovered by O. Reynolds [100] and "cross-elasticity", or what may be named the Weissenberg effect.

2. Reynolds' Experiments.

In 1885 O. Reynolds described the following experiment: " If . . . we have an extremely flexible bag of indiarubber, this envelope, when filled with heavy spheres, imposes no sensible restraint on their distortion; standing on the table it takes nearly the form of a heap of shot. . . . Filling up the interstices between the shot with water so that the bag is quite full of water and shot, no bubble of air in it, and carefully closing the mouth, I now find that the bag has become absolutely rigid in whatever form it happened to be when closed. It is clear that the envelope . . . imposes no distortional constraint on the shot within it, nor does the water. What, then, converts the heap of loose shot into an absolutely rigid body ? Clearly the limit which is imposed on the volume by the pressure of the atmosphere. So long as the arrangement of the shot is such that there is enough water to fill the interstices the shot are free, but *any arrangement which requires more room** is absolutely prevented by the pressure of the atmosphere. If there is an (small, M.R.) excess of water in the bag when the shot are in their maximum density, the bag will change its shape quite freely for a limited extent, but then becomes instantly rigid . . . without further change. By connecting the bag with a graduated vessel of water so that the quantity which flows in and out can be measured, the bag again becomes susceptible of any amount of distortion. Getting the bag into a spherical form and its contents at *maximum density*, and then squeezing it between two planes, the moment the squeezing begins the water begins to *flow in*, and flows in at a diminishing rate until it ceases to draw more water. . . . The material in the bag is (then, M.R.) in a condition of *minimum* density. . . . If we continue to squeeze, water begins to flow out. . . . The very finest quartz sand, or glass balls $\frac{3}{4}$-inch in diameter, all give the same results. Sand is, on the whole, the most convenient material, and its extreme fineness reduces any effect of the squeezing of the indiarubber between the interstices of the balls at the boundaries."

3. Dilatancy of Granular Masses.

The explanation for this strange behaviour was given by Reynolds as follows :

" If . . . a group of spheres . . . (is) arranged as a pile of shot (Fig. XVII, 1, left) . . . the density of the media is $\pi/3\sqrt{2}$, taking the density of the sphere as unity. If arranged in a cubical formation (Fig. XVII, 1, right) the density is $\pi/6$, or $\sqrt{2}$ times less than in the former case. These arrangements are both controlled by the bounding spheres ; and in either case *the distortion necessitates a change of volume.* I have called this

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* The italics are mine.—M.R.

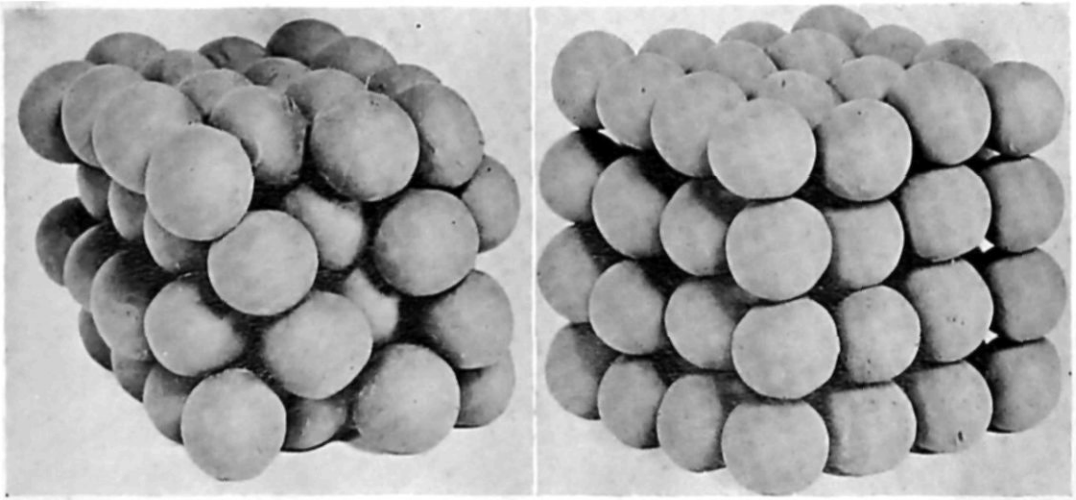

FIG. XVII, 1.* The figure (left) shows 64 spheres arranged approximately in closed packing. When this system is subjected to shear the spheres must first move into open packing as shown on right. It will be seen that this change in packing has resulted in an expansion.

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[*To face page* 318.

unique property of granular masses 'dilatancy' because the property *consists in a definite change of bulk, consequent on a definite change of shape* or distortional strain, any disturbance whatever causing a change of volume."

He then cites the following example: "A well-marked phenomenon receives its explanation at once from the existence of dilatancy in sand. When the falling tide leaves the sand firm, as the foot falls on it the sand whitens, or appears momentarily to dry round the foot, When this happens the sand is full of water, the surface of which is kept up to that of the sand by capillary attraction ; the pressure of the foot causing dilatation of the sand, more water is required, which has to be obtained either by depressing the level of the surface against the capillary attraction, or by drawing water through the interstices of the surrounding sand. The latter requires time to accomplish, so that for the moment the capillary forces are overcome ; the surface of the water is lowered below that of the sand, leaving the latter white or dryer until a sufficient supply has been obtained from below, when the surface rises and wets the sand again. On raising the foot it is generally seen that the sand under the foot and around becomes momentarily wet ; this is because, on the distortion forces being removed, the sand again contracts, and the excess of water finds momentary relief at the surface."

4. Dilatancy as a General Property.

It is historically interesting to note that Reynolds undertook his investigation in order to arrive at a mechanical theory of the ether. Accordingly he said that " as regards any results which may be expected to follow from the recognition of this property of dilatancy,—

" In a practical point of view it will place the theory of earth pressures on a true foundation. . . .

" The greatest results are likely to follow in philosophy. . . ."

We do not apply nowadays rheology to " explain the fundamental arrangement of the universe," but we have found that dilatancy is not confined to granular masses.

After positive dilatancy in sand, *negative* dilatancy was observed in clay. While sand particles are little spheres, clay particles are minute discs. A sandy soil will therefore settle in dense packing, while a clay will in its undisturbed state be in loose packing, many of the discs standing on edge. In shear these will collapse and the density of the clay will be *increased.* These cases may be regarded as such of " plastic " dilatancy. But about the same time when Reynolds discovered his remarkable phenomenon in settled sands, a famous contemporary predicted on purely theoretical grounds that an analogous phenomenon should be present in elastic solids. In 1875 Sir William Thomson, later Lord Kelvin, contributed the article on Elasticity in the ninth edition of the *Encyclopœdia Britannica,* which I have already mentioned before (Section 6 of Chapter XIV), and wrote there : " It is possible that a shearing stress may produce in a truly isotropic solid condensation or dilatation in proportion to the square of its value ; and it is possible that such effect may be sensible in indiarubber or cork, or other bodies susceptible of great deformations or compressions with persistent elasticity " [70]. Reynolds must certainly have read that article and it is extraordinary that he did not connect this remark with his observation. If he had attempted to relate the change of volume observed by him to the shear or rather the shearing stress causing it, he would without doubt have realised that a shear " to the right " must produce the same effect as a shear " to the left." It is inconceivable that a shear to the " right " should produce a " dilatation " and one to the " left " a " condensation." Therefore, the shear or shearing stress cannot enter into a rheological equation except in an even power, or in the first instance *in the square* of its value. This is what Kelvin maintained in the case of elasticity.

I do not know of any observations on elastic dilatancy in solid indiarubber or cork, but it is very easy to observe it in " expanded " or porous rubber, which has the property of negative elastic dilatancy to a very marked degree. This can easily be demonstrated with a commercial rubber sponge. Sticking a piece of board on each of its two flat surfaces, it can be observed that if one is shifted tangentially against the other, the surfaces also approach each other.

We can say that there will generally be elastic, plastic and viscous dilatancy. In an elastic solid, side by side with the volume component of the Hooke solid resulting from the isotropic stress p,

$$e_{v1} = p/\kappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

there will be one resulting from the shearing stress p_t,

$$e_{v2} = p_t^2/\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and we may define the Reynolds elastic by the rheological equation

$$e_v = p/\kappa + p_t^2/\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The parameter δ may be called the Modulus of Dilatancy.

There may also be an analogous Reynolds liquid, and we know that there are Reynolds plastics, positive and negative. In the Reynolds elastic an isotropic tension will cause a cubical dilatation and *vice versâ*, as in the Hooke solid; but cubical dilatation may also be caused in the absence of a hydrostatic tension by simple shearing stress. Likewise, a hydrostatic pressure may be required to maintain simple shear. Similar conditions may be present in liquids [101].

5. Second Order Phenomena.

Dilatancy, being dependent upon p^2_t, is a second order phenomenon. The stress necessary to produce a small strain may be a large figure in the usual measure, but then the elastic modulus is likewise a large figure and the quotient of both small. As long as the strain is small, the stress can therefore also be considered as small. We see from Equ. (2) that the modulus of dilatancy is of the dimension of the square of a stress. Therefore if the shear is small, dilatancy, depending upon the square of the shear, may be neglected. This is the reason why in classical elasticity dilatancy was not noticed. It has changed now when we have to consider in many fields large deformations, as was prophetically realised by Kelvin.

But then there may be other second order phenomena, which likewise cannot be neglected. If we consider shear, there is no reason why finite shear should affect not only the volume, but why it should not also cause *linear* extensions or contractions in some specific directions, likewise depending upon the square of the magnitude of the shear. Actually, Weissenberg [102] has observed phenomena which point to the existence of such " cross-elasticity," as we may call it for reasons to be seen later. These phenomena we shall presently describe.

6. The Weissenberg Effect.

In Fig. I, 3 we showed simple shear converting a prism into a parallelepiped. Now imagine the parallelepiped being the undeformed body and the force P_t acting from right to left : clearly the parallelepiped will be deformed into a prism. We show in Fig. XVII, 2, a section through the body and ask ourselves : what is the relative position of planes which are parallel to the boundary surfaces of the body ?

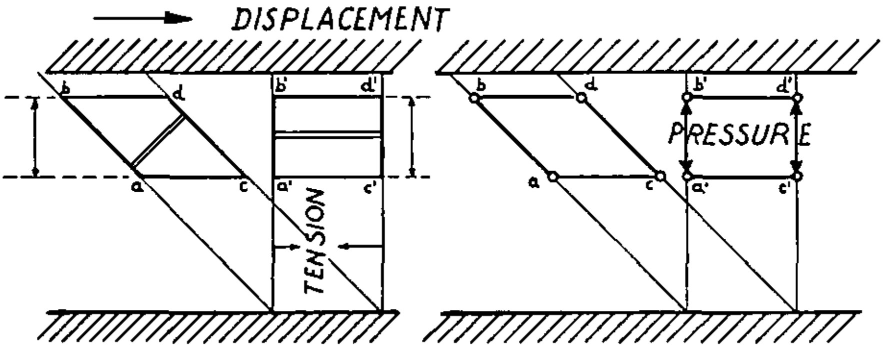

Fig. XVII, 2. Parallelepiped deformed into a prism through simple shear.

Let us consider planes *ab* and *cd* which after deformation become *ab'* and *cd'*. It can be seen from the figure at the left that their distance *increases* after deformation.* In contradistinction the distance of the two planes *cd* and *ac* remains the same. We do not know the mechanism which produces stresses in a strained solid, but Weissenberg assumes that it is the change of distances of such planes which governs the reactive forces or stresses. From this he predicted that in simple shear there would be not only a shearing stress, but also a pull or tension *in the direction of the displacement.* That tension would be a second order phenomenon because in a first approximation, if the displacement is very small, the distance between planes *ab* and *cd* is not changed.

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* Remember that the distance of two parallel planes is measured in the direction normal to both.

Now take an elastic liquid where the deformation naturally is very large and experiment with it in an arrangement such that the liquid is sheared in a gap between an outer vessel rotated with an angular velocity which is kept constant at various levels and an inner member which is held against rotation and either rigidly fixed in position or free to move up and down the axis of rotation. The conditions are so chosen that under the combined actions of the shear imposed at the boundaries, and the forces of gravity and inertia (centrifugal forces) the liquid executes a steady laminar shearing movement. As has been argued above, if the liquid is elastic, there should be in addition to the shear stress components, a pull along the lines of flow. If, as in our experiments, the lines of flow are closed circles, the pull along these lines " strangulates " the liquid and forces it inwards against the centrifugal forces and upwards against the forces of gravity. The effect, as predicted by Weissenberg, may be called " Weissenberg effect."

7. Experimental Evidence of the Weissenberg Effect.

Experiments carried out by C. H. Lander with a variety of materials, including saponified oils, solutions of rubber, starch, cellulose acetate, etc., showed that the predicted phenomena actually occur in a great variety of conditions of flow such as are reproduced by Fig. XVII, 3. It should be noted that Weissenberg calls a Newtonian liquid " special liquid," a designation natural enough considering that the Newtonian liquid is a special case of the more " general " Maxwell liquid for $\gamma_l = \infty$.

8. Cross Elasticity.

As I have shown by tensor analysis [102], the phenomenon can also be described without assuming anything about the mechanism of stress-strain connection. If we were to assume that the reactive forces or stresses are governed by changed distances *of material points* in the body and not of planes, Fig. XVII, 3, will show that there would be no pull in the direction of displacement, but a push normal to it. While after deformation the distances of points *ac* and *bd* have remained unchanged, the distances of points *ab* and *cd* are *decreased.* It can be shown that in the usual expressions for elastic stress-strain relations it all depends upon how you define the strain, whether you get a pull or a push. However, it is unacceptable that our definition of strain should prejudice an experimental result. Therefore, there is something wrong with the usual functional stress-strain relationship. I have shown how this position is to be amended by taking into account the second powers of the tensors [101].† It all comes to this : Let n be the normal to an element of interface in the interior or of surface on the boundary of an elastic body. Let the traction on this interface be resolved into three orthogonal components, one normal p_n and two tangential p_t and p_c. Now select the

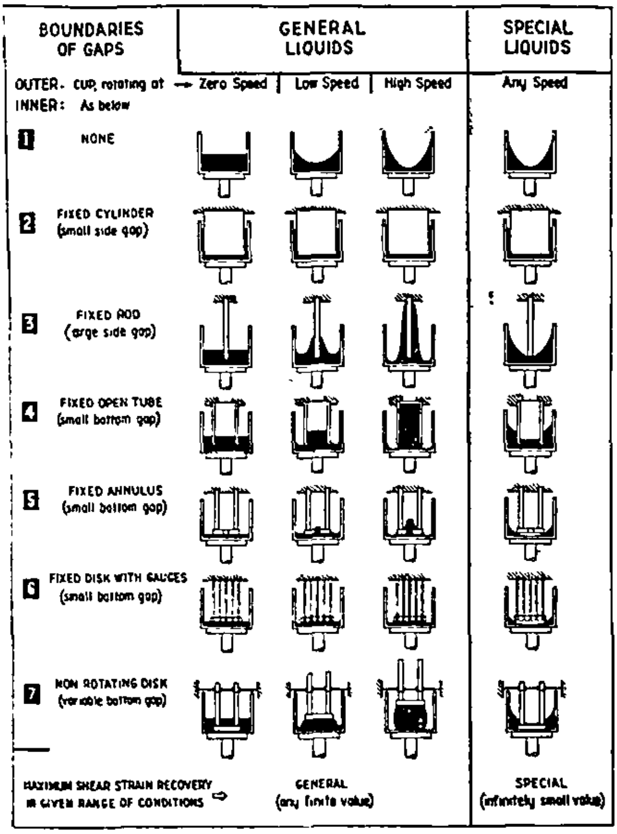

FIG. XVII, 3.* Flow of liquids under actions of steady shear in gaps.

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* Reproduced with the permission of Macmillian & Co. Ltd., from *Nature*, 159, 310 (1947).

† The same method was applied independently by Prager [103] in a more restricted manner.

directions t and c so that p_c vanishes. Let the strain be resolved in the same directions. Then the new theory requires that e_c does not vanish; or, in other words, that there is a strain sideways or in the crosswise direction. Conversely there may be a stress in a crosswise direction where there is no strain. This justifies the designation "cross-elasticity." If cross-elasticity is assumed, experimental results are not prejudiced. In this case simple shear may require for its maintenance not only a corresponding shearing stress but also a traction in the direction of the displacement (pull), or normal to it (push), or both.

CHAPTER XVIII

POSTSCRIPT

1. Rheology also Flows

Several years have elapsed between the time when I started writing the manuscript of this book and the time of its publication. But rheology, as was to be expected from a young science of that name, does not stand still, but also flows. What is more, I myself have made some progress during that period and have changed my views on several points. It was impossible to re-write the manuscript, because at the time I would have finished I would have had to start anew. The reader will have noticed, however, how I have modified or qualified in later chapters views expressed in earlier ones.

2. Volume Flow

When I started writing this book it was a generally accepted axiom of rheology that, as said in Section 3 of Chapter I, " under isotropic stress all materials are purely and simply elastic." This was at the base of Hencky's theory of plasticity.* Hencky started with the statement : " It is clear that a hydrostatic pressure or tension can have no influence on plastic flow." However, as somebody has said, " if you find a sentence starting with the words ' it is clear that ' you may be sure that they will be followed by something which is not clear at all." And so Hencky had to continue : " If tension experiments simulate such an effect it must be due to the disturbance produced by invisible rupture phenomena." This is nothing else but an allusion to the *permanent set*, by which I qualified the axiom when I came to writing Chapter IX. But this was not enough. If there was volume elasticity, this was necessarily connected with *solid* viscosity. This I introduced as *volume* viscosity, denoting it by η_v. I overlooked that this viscosity was *not* analogous to the shear viscosity η. It was not connected with *flow* as the latter is, but with the *viscous resistance to elastic strain*. It did not occur to me that there might be real *isotropic*

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* As already mentioned in Section 10 of Chapter XIII.

flow. If there was isotropic or volume flow, say under hydrostatic pressure, where would the material go to ? The reply, of course, was : into its holes. I knew Glanville and Thomas' [62] observations which showed that the volume of a body (of concrete) may gradually decrease over a considerable period of time. But I still did not recognise its true nature. As an excuse I can only mention the fact that Dr. Glanville himself, as he told me, had not realised that this meant volume flow. I reported on this phenomenon in Section 10 in Chapter XIII, but I was still mixing up the viscosity of that flow with the " volume-viscosity " of Chapter IX. Only last year did Dr. Glanville put his original observations at my disposal, and in an as yet unpublished paper I used them to clear up this problem of volume flow. In this way it has now become clear to me that in principle the only difference in the isotropic and distortional behaviour of a material is that in the former case *every* material can be considered *in a first approximation* as a Hooke body and in a second approximation as a Kelvin body, both belonging to the group of *solids*, so that there are, strictly speaking, no first-approximation-liquids under isotropic stress. Both water and steel are, under hydrostatic pressure, elastic solids : so much remains of my first axiom. Nevertheless, in a third approximation some materials may be regarded in their isotropic behaviour as Burgers bodies and these, as we know, will under certain circumstances flow in the manner of a Newtonian liquid.

There is, therefore, not *one* coefficient of volume-viscosity, but *at least two*. For this reason it is better to adopt a special symbol for volume viscosity, for which I introduced in Section 5 of Chapter XV the Greek letter ζ, which has the mnemotechnic advantage of following η. We may use ζ without subscript as the analogy to η, and ζ_K for what I denoted by η_v in Chapter IX, Sections 12 and 13 of Chapter X, and Section 11 of Chapter XIII, the subscript " K " indicating " of a Kelvin solid." I accordingly request the reader to go back over the matter wherever I mentioned a " volume-viscosity " and to make it clear to himself of what sort of viscosity I am speaking, whether " solid " or " liquid," and what sort of body I am dealing with, whether "Kelvin " or "Maxwell " or " Newton " or " Burgers."

I also cannot see how we can differentiate in a phenomenological way between permanent set and plastic deformation, and we must consider the former as indicating the existence of just what Hencky wanted to exclude, namely *volume-plasticity.* Therefore to sum up: all properties which make themselves manifest in distortion will also occur under isotropic stress, albeit in very different degrees and in very different materials. In Chapter XIII we have found it necessary to consider concrete under hydrostatic pressure as a liquid, but while I cannot imagine any circumstances under which it would be necessary to consider water under the same stress type of other than as an elastic solid, such circumstances may make themselves manifest some day.

3. Deformation and Strain

In my terminology I should now be more careful in using the word strain for the recoverable part of the deformation *only*, deformation remaining the more general term. There would then be no strain without stress, while a body may have been deformed and remain deformed when all stress has been removed. This should show itself also in the notation, and I propose to use accordingly " D " for " deformation " and " e " for " strain." It should be kept in mind that when a body is deformed plastically by some stress, there will at the same time also be some strain, but then " D " is so much larger than " e " that the former may be regarded as including the latter. Solid or Kelvin viscosity is accordingly bound up with rate of strain $\dot{e}$, but ordinary or flow—or Maxwell—viscosity with rate of deformation $\dot{D}$.

4. The Tangential Component of Strain.

In Section 5 of Chapter I, I defined $u/H = e_t$ as the tangential component of strain. This is in accordance with the usage in the classical theory of elasticity, including Love's standard work, and in technical literature. I followed this usage in my *Ten Lectures* and was censured for it by Prof. Prager because, as defined here, e_t is *not* a component of the tensor $||e||$, even in infinitesimal strain, the correct component being $1/2 \,.\, u/H = 1/2 \,.\, e_t$. For tensor notation it is, of course, much simpler to define e_t by $1/2 \,.\, u/H$, and this is followed in more modern works. But in the present book, which is meant to be elementary, I was not using tensor notation. An escape from this dilemma can be found by denoting u/H by G, for gradient, in this case " displacement gradient." One would then use $\dot{G}$ for v/H, i.e. the " velocity gradient." It can be shown that while $\dot{G}$ is double the tangential component of the " velocity-deformation tensor," G is double the component of the " displacement-deformation-tensor " only when the deformation is very small. This is a complicated subject and I cannot say more about it without using very advanced mathematical language.

5. Strain-work versus Second Invariant

There is another point requiring advanced mathematics, and with which I cannot deal here in full, but which never-the-less I want to mention. It starts with the difference between Mises' and Hencky's flow conditions in the theory of plasticity. Many authors deal with them as if they were actually identical, i.e., one single Mises-Hencky flow condition. In Chapters III and XI, I have used the same expression, but actually there is an essential difference between them, which disappears only in the determination of the first yield point of a simple Hooke solid. Briefly, every tensor is determined by three scalar quantities of first, second and third degree which are called its *invariants*. The first invariant of the deformation is the cubical dilatation, the first invariant of the stress its isotropic component. The second invariant of the tensor of strain, if it is a *shear* e_t, is e_t^2 ; of the tensor of stress, if it is a shearing stress p_t, p_t^2. When the material is a simple Hooke solid the strain-work is proportional to either e_t^2 or p_t^2 or proportional to the second invariant of either strain or stress. Mises' flow condition prescribes a limit of the second invariant, Hencky's flow condition of the strain-work ; when γ is *not constant, both are different.* I knew of this difference and had pointed it out on several occasions, but I thought the Mises condition to be more general (not supposing simple Hookean behaviour) and therefore superior. The strain-work, of course, is also an invariant and therefore in the general case also expressible in terms of the second invariant. I had come across these considerations in two investigations, namely when trying to establish (i) a law for *generalized* Newtonian behaviour (Chapter VII), (ii) a law of work-hardening (Chapter XI). In both cases I had taken as the argument (in the mathematical sense) the second invariant, following Mises. In the first case I tried to express the variable fluidity φ as a function of p_t^2; in the second I tried to express the square of the yield stress ϑ as a function of e_t^2. In the first case I did not succeed in establishing a law, in the second case I thought I did [48] and this may be so. I have, however, lately conceived the notion that possibly the Hencky condition, being abstract mathematical concept but having a concrete physical content, might be the superior one. As I said before, the strain-work can be expressed as a function of the second invariant and *vice versa*, but, except in the uninteresting case when the fluidity φ or the yield stress ϑ is constant, we do not know in what manner this reduction is to be carried out. The laws which we try to find will, however, look very differently in accordance with the argument of the function; they may be simple and obvious in one case, but very complicated and hidden in the other.

6. Envoi

I am sending off these last pages to the Publisher in the midst of a war which has for over half a year prevented me from giving the book its final touch. I must apologize to the reader for it and can only hope that even in this imperfect form it will be found useful.

TEL AVIV, ISRAEL,
July, 1948.

CHAPTER XIX

NOTATION

One of the difficulties in studying scientific books and papers is the diversity of notation. This is sometimes accidental and often unsystematic. In the present book a systematic notation has been attempted, based on mnemotechnic principles. These are :—

I. A Roman letter is, as far as possible, the first letter of the term in the Latin language for which it stands as a symbol, e.g., P for force (*pondus*) and not F (force).

II. Capital and small letters are used for the same term, the former in a wider sense and the latter in a more restricted meaning, e.g., P for total force, p for a force per unit area (traction).

III. Small Greek letters are used for physical (material) constants (parameters), e.g. σ for Poisson's ratio (and not for stress as in American technical literature). Similarly γ for modulus of rigidity and not G. This principle will only make consistent an often-followed practice, e.g. φ for fluidity, ρ for density. The π for 3.1415 . . . is an exception.

IV. Capital Greek letters will be used for angles, e.g. θ for angular distance.

V. Newton's methōd of notation for the differential quotient in respect of time is used, e.g. $\dot{\theta} = d\theta/dt$ for angular velocity and not ω. This saves a letter for some other use.

VI. The same symbol is used for quantities which differ only algebraically, e.g. p for pressure, where p is tension and not p for pressure and σ for tension.

It is hoped that this systematic notation will facilitate study, and the reader is recommended, when studying a rheological paper, to translate the notation used by the author into a notation familiar to him, e.g. that of the present book.

In order to economize in the cost of printing, division will often be indicated by the sloped bar / and not by the horizontal bar —. The influence of the sloped bar / is meant to reach to the nearest operational sign, where operational signs are +, —, and . but not $\sqrt{}$, $\int$, Σ. Therefore $a/bc = \frac{a}{bc}$, but $a/b.c = \frac{a}{b}c$.

Equations and figures are numbered separately in every chapter.

A	area, also a constant.
A_o	original area.
a	acceleration, also $\doteq$ (Ri/Re)^2.
α	an angle ; also a material constant or coefficient ; also spherical equivalent volume coefficient.
B	Bingham body.
Bu	Burgers body.
b	breadth, also "of bending" (qualificatory index in e_b, p_b, M_b).
β	a material constant or coefficient.
C	an integration constant.
c	concentration, also "of creep" (qualificatory index in e_c).
c_v	volume concentration.
χ	coefficient of structural stability.
D	deformation, also diameter.
D/l	restoring moment.
d	differential, also distance, also thickness, also deflection.
$\dot{d}$	rate of deflection.
Δ	increment.
Δl	elongation.
Δp	pressure difference.
Δs	deflection.
E	energy.
E_d	dissipated energy.
E_f	ultimate resilience (energy at failure).
E_k	kinetic energy.
E_o	distortional energy.
E_p	potential energy.
E_{pl}	plastic resilience.
E_r	rupture resilience.
E_v	volume resilience.

e	strain, also " external " (qualificatory index in R_e).
e_b	bending strain.
e_c	contraction.
e_i, e_j, e_k	principal strains.
e_m	mean normal strain.
e_n	normal strain, extension.
e_o	distortion.
e_t	tangential strain, shearing strain.
e_v	volume strain, cubical dilatation.
$\dot{e}$	rate of strain.
$\dot{e}_t$	rate of shear.
$\dot{e}_v$	rate of cubical dilatation.
e	basis of natural logarithms.
ϵ	Young's modulus.
η	coefficient of viscosity.
η_c	creep viscosity.
η_i	intrinsic (reduced) viscosity.
$[\eta]$	limiting intrinsic viscosity.
η_o	viscosity of liquid at rest.
η_{pl}	plastic viscosity, stiffness.
η_{rel}	relative viscosity.
η_s	solid viscosity.
η_{solu}	viscosity of solution.
η_{solv}	viscosity of solvent.
η_{spec}	specific viscosity.
η_v	volume viscosity (better ζ).
η_∞	viscosity of liquid at maximum shear.
η'	apparent viscosity.
f	function, " of failure " (qualificatory index in E_f).
f'	first derivative of function f.
$f^{(n)}$	nth derivative of function f.
φ	coefficient of fluidity.
φ_o	zero fluidity, fluidity of the liquid at rest.
φ_∞	maximum fluidity.
φ'	apparent fluidity.
φ_w	fluidity of wall layer.
$\phi(x)$	Gauss' error integral.
ϕ	angle.
G	gradient, displacement gradient.
$\dot{G}$	velocity gradient.

g	acceleration of gravity.
γ	modulus of rigidity.
γ_l	liquid shear elasticity.
H	height, also Hooke body.
h	height.
I	moment of inertia.
i, j, k	principal axes.
i	qualificatory index for " internal " in R_i.
K	Kelvin-body, also constant.
k	of kinetic (qualificatory index in E_k), also constant.
κ	bulk modulus.
κ'	apparent bulk modulus.
L	length.
l	length, also of liquid (qualificatory index in γ_l), also " lagging " (qualificatory index in e_l), also " longitudinal " (qualificatory index in e_l).
lim	limes, limiting.
ln	natural logarithm.
l_o	original length.
λ	Trouton's coefficient of viscous traction.
M	moment, couple, also Maxwell liquid.
M_b	bending moment.
M_x	moment in respect of axis x.
M_z	moment in respect of axis z, torsional moment.
m	mass, also constant (power).
N	pull ($-$N push), also Newtonian liquid.
n	number of revolutions, also constant (power), also qualificatory index for normal in e_n.
$n\,!$	$= 1, 2, 3 \ldots$ n.
o	" of deviator " (qualificatory index in E_o), also qualificatory index for " original " in l_o, A_o.
Ω	angle of rotation.
$\dot{\Omega}$	angular velocity.
P	force, also dynamic consistency variable.
P_{nn}	normal component of force P_n.
P_{nt}	tangential component of force P_n.
P_t	tangential force.
p	traction (stress), hydrostatic tension = isotropic tension, also qualificatory index for " permanent " in h_p.

$\dot{p}$	rate of stress $= dp/dt$.
p_b	bending stress.
p_i, p_j, p_k	principal stresses.
p_m	mean normal stress.
p_n	normal stress.
p_{nn}	normal component of traction p_n.
p_{nt}	tangential component of traction p_n.
p_o	deviator of stress.
p_r	rupture stress.
p_{rs}	static rupture stress.
p_{red}	reduced stress.
p_t	tangential stress, shearing stress.
Ψ	limiting relative volume, also coefficient of structural stability.
π	3·1415 . . .
Q	quantity of flow.
R	rheological function, also radius.
R_e	radius of external cylinder.
R_i	radius of internal cylinder.
r	radius.
ρ	density.
Sch Sc B	Schofield-Scott Blair body.
Schw	Schwedoff body.
St V	St. Venant body.
Σ	the sum of . . .
σ	Poisson's ratio.
σ_d	deformational Poisson ratio.
σ_v	viscous Poisson ratio.
T	period of oscillation.
t	time, also tangential (qualificatory index in p_t, e_t).
ϑ	yield stress.
ϑ_n	normal yield stress.
ϑ_t	tangential yield stress.
θ	angle, also limiting strain.
$\dot{\theta}$	angular velocity.
τ	time-constant.
τ_r	relaxation time $= \tau_M$
τ_l	lagging time, retardation time $= \tau_\kappa$.
u	displacement.

V	velocity, also volume, also kinematic consistency variable.
V_o	original volume.
v	velocity, also qualificatory index for " viscous," also qualificatory index for " voluminal."
v_o	velocity at centre.
v_θ	velocity in the direction of increasing θ.
W	work.
WC	water-cement ratio.
w	strain work per unit volume, also weight per unit length, also qualificatory index for " of wire."
$\dot{w}$	power of deformation $= dw/dt$.
w_v	volumetric strainwork.
x, y, z	co-ordinates and co-ordinate axes.

CHAPTER XX

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INDEX

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