The Luneburg Theory of Binocular Visual Space*

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A theoretical treatment of binocular space perception based on the methods of Rudolph K. Luneburg. A simplified axiomatics is employed. New experimental results are brought within the framework of the theory. The experimental evidence is seen to support Luneburg's hypothesis as to the hyperbolic character of visual space. The relation between visual and physical coordinates proposed by Luneburg is found to hold only as a special case of a more general transformation.

1. AIMS AND LIMITATIONS

THE ultimate objective of a theory of three-dimensional space perception is to state in some precise way what an observer really "sees" when he looks out upon the physical world. Such an ambitious undertaking, stated in all generality, goes far beyond the scope of the theory presented here. So many and various are the factors contributing to the perception of space that it becomes impossible to comprehend their effects when all are considered together. It is necessary to isolate a limited number of factors, at best one, and to consider the effect of this restricted set of factors in an environment as free as possible from the contamination of other influences.

The Luneburg theory is concerned with the sole factor of binocularity. No attempt will be made here to account for the known three-dimensional effects attributable to perspective, color, brightness, etc. To insure that only the one factor of two-eyedness is present requires a certain amount of elaborate precaution in experiment. The optimum results are obtained when the subject is allowed to view only simple configurations consisting of lights so tiny as to approach the mathematical ideal of a point. The lights are adjusted to appear of equal intensity to the subject and so low that there is no perceptible surrounding illumination. The only factor remaining under these conditions besides binocularity would be accommodation and it has been shown that the effect of accommodation in this situation is not important.[1–3] It cannot be emphasized too strongly that experiments conducted without these precautions cannot be expected to give results comparable to those cited here.

The problems engendered by including the factor of motion are again too complicated to be brought at once within the scope of the theory. Motion is avoided by fixing the subject's head in a headrest and he is exposed only to static stimuli. However, no artificial restriction is placed on the subject's binocular function. The subject makes his observations by allowing his eyes to rove at will over the entire range of the stimulus configuration until a stable perception of the geometry of the situation is achieved.

The objective of the Luneburg theory is to establish the relationship between the physical space and the binocular visual space. In other words, it offers an answer to the question, what connection is there between the physical stimuli of form and localization and the qualities of form and localization that we perceive in binocular vision. To those who feel that we see the physical space just as it is, the question seems pointless. So intimately and reliably do our visual and proprioceptive senses bring us into contact with the objects distributed in our physical surroundings, that we are ordinarily completely oblivious of the distinction that can be made between the world as measured by our yardsticks and the world as measured by our senses. We must be prepared to abandon primitive intuition in these regards. No one would insist that a color is perceived as a particular blend of electromagnetic vibrations. On the contrary, we think of a color as a perception, a concept quite apart from the stimulus, and ask what stimuli give a perception of that kind. Much the same approach will be taken to the problems of binocular vision.

Once the dichotomy between physical and sensory measurement is recognized there is nothing remarkable in the demonstrable fact that the binocular visual space is non-Euclidean. What is remarkable is that so many visual observations are encompassed in one of the simplest of geometries, the hyperbolic geometry of Bolyai and Lobachewski, the Riemannian geometry of constant negative curvature.

* This work was supported in part by a contract between the U. S. Office of Naval Research and Columbia University.

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[1] C. J. Campbell, "An experimental investigation of the size constancy phenomenon" (Columbia University thesis, 1952). Campbell explored carefully the role of accomodation under precisely these conditions.

[2] A. v. Tschermak-Seysenegg and P. Boeder (trans.), *Introduction to Physiological Optics* (Charles C. Thomas, Springfield, Illinois, 1952), p. 189 ff.

[3] A. Linksz, *Physiology of the Eye*, (Grune and Stratton, New York, 1952), Vol. 2 (Vision). This work contains a very readable discussion of the factors involved in visual space perception.

2. THE GEOMETRY OF BINOCULAR VISUAL SPACE

2a. Existence of the Metric

To characterize the binocular visual geometry, a visual "yardstick" is utilized in a manner not very different from the way in which a measuring stick is used to determine the geometrical characteristics of the physical world. Consider, for example, a binocular stimulus consisting of three points of light Q_1, Q_2, Q_3. A subject presented with such a stimulus will perceive three points P_1, P_2, P_3 in a three-dimensional continuum and will be able to distinguish inequalities among the distances between pairs of points. He might say, for example, that the distance between P_1 and P_2 is larger than that from P_2 to P_3. The subject may even take the first distance as a unit and characterize the second numerically as being so many times larger or smaller than the first. Such visual relationships of size need not have any directly obvious connection to the physically measurable relations in the stimulus configuration.

By taking some fixed visual distance as a standard unit, it would be possible at least in conception to determine any other visual length in terms of that unit. In this way a positive numerical value $D(P_1, P_2)$ could be assigned to the distance between any pair of points P_1, P_2 in the visual space. Once such a function of two points was determined, it would be expected that any other function constructed in the same way would differ from $D(P_1, P_2)$ only in a positive constant factor depending on the choice of unit.

It is not difficult to show experimentally that there is a subjective sense of straightness in the visual space. Given three points P_1, P_2, P_3 a subject judges readily whether or not the point P_3 lies on a line with P_1 and P_2. (Configurations which are sensorially straight need not, of course, be physically straight.[4,5]) In fact, the sense of alignment is one of the strongest characteristics of binocular visual perception. This sense of straightness may be described in terms of the visual distance function by the inequality

$$D(P_1, P_2)+D(P_2, P_3)\geqslant D(P_1, P_3).$$

This inequality expresses the idea that it is not shorter to go from P_1 to P_3 by way of a third point P_2 than it is to go "straight" from P_1 to P_3. Thus, three points are seen on a straight line if and only if the relation above is an equality. In order to distinguish between physical straightness and the visual sense of straightness, a configuration which is visually straight will be called a *visual geodesic.*

The foregoing considerations suggest that the binocular visual space may be described mathematically as a metric space. This means that there exists a visual distance function or metric $D(P_1, P_2)$, satisfying the following conditions:

(A1) $$D(P, P)=0.$$

A point has zero distance from itself.

(A2) $$D(P_1, P_2)=D(P_2, P_1)>0, \text{ if } P_1\neq P_2.$$

If P_1 and P_2 are distinct points then the distance between them is positive and independent of the order in which the points are taken.

(A3) $$D(P_1, P_2)+D(P_2, P_3)\geqslant D(P_1, P_3).$$

For all points P_1, P_2, P_3, the distance from P_1 to P_3 by way of any third point P_2 is not shorter than that from P_1 to P_3.

Such a function $D(P_1, P_2)$ is called a metric. For such a metric to have significance as the yardstick of visual measurement it must satisfy in addition the following conditions:

(B1) *If the distance between P_1 and P_2 is sensed as greater than that between P_3 and P_4, then*

$$D(P_1, P_2)>D(P_3, P_4).$$

(B2) *Points P_1, P_2, P_3 are sensed as being on a straight line if and only if*

$$D(P_1, P_2)+D(P_2, P_3)=D(P_1, P_3).$$

A function $D(P_1, P_2)$ satisfying the conditions A and B is called a *metric* or *psychometric distance function* for binocular visual space.

2b. The Indeterminacy of the Metric

The function $D(P_1, P_2)$ is not completely determined by the conditions of Sec. 2a. In fact, it is clear that the function will satisfy these conditions if multiplied by any positive constant whatever. That such an indeterminacy exists is not surprising for it amounts simply to freedom in the choice of a unit. Under the following general assumptions it can be shown that this kind of indeterminacy is the only one possible:

(C) *The visual space is convex.*

Between every pair of distinct points P_1, P_2 of the visual space, there is another point P_3 on the straight segment between P_1 and P_2.

In terms of the metric, this condition implies the existence of a point P_3 satisfying the equation

$$D(P_1, P_3)+D(P_3, P_2)=D(P_1, P_2).$$

(D) *The visual space is compact.*

Every infinite set of points of the visual space has at least one cluster point.

For an infinite sequence of points, $P_1, P_2, P_3, \ldots, P_i, \ldots$ this condition states the existence of a point

[4] Helmholtz, v. Kries, and Southall (trans.), *Physiological Optics* (Optical Society of America, Rochester, New York, 1925), Vol. 3, p. 318.

[5] Hardy, Rand, Rittler, and Blank, "The geometry of binocular space perception," (Final Report to U. S. Office of Naval Research from the Knapp Memorial Laboratory of Physiological Optics, Columbia University College of Physicians and Surgeons, New York, 1953), Fig. 19.

P of the visual space such that for some subsequence P_1', P_2', P_3', . . ., P_j', . . . the points P_j' approach P in the limit,

$$\lim_{j\to\infty} D(P, P_j') = 0.$$

Under this pair of conditions, the metric must be completely determined to within a constant positive factor.[6] Furthermore, there is nothing in these conditions to conflict even with naive intuition, except perhaps for the necessity of considering infinite aggregates of points.

The class of compact convex metric spaces is so large that it becomes necessary to postulate further properties of the visual space in order to reduce the latitude of choice. This can be done with great simplicity.

2c. The Hyperbolic Character of Visual Space[7]

It is in reasonable accord with experience to postulate the existence of visual perceptions of *planeness*. Such a perception would be described by the postulates:

(E1) *Every triple of points in the visual space is contained in a visual plane.*

(E2) *Together with any pair of points in a visual plane, the plane contains the segment of visual geodesic joining them.*

A geometry possessing surfaces which satisfy these properties is called *Desarguesian.*

Experience again permits the supposition that the visual space is Euclidean in the small, for we are not generally aware of any distortion in viewing small configurations such as geometrical diagrams on a printed page. This could only be the case if the geometry were Euclidean in the small, no matter what the relation between physical and visual coordinates. This state of affairs is described by the postulate:

(F) *The visual geometry is locally Euclidean.*

The property (F) is perhaps the simplest way of characterizing the visual geometry as Riemannian.

The surprising fact is that a geometry which is both Desarguesian and Riemannian can be only one of three, the Riemannian spaces of constant positive, zero, and negative Gaussian curvature.

The Riemannian geometry of zero curvature is precisely ordinary Euclidean geometry. The elliptic geometry of constant positive curvature is familiar to us in the two-dimensional case as the geometry on the surface of the sphere. The geometry of constant negative curvature is the hyperbolic geometry of Lobachewski and Bolyai.

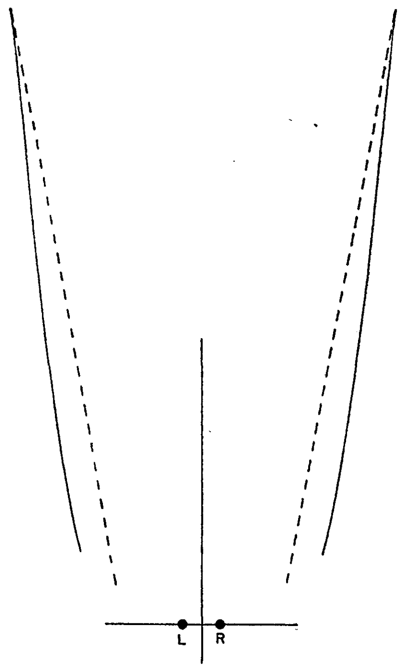

FIG. 1. Blumenfeld alleys. Schematic representation of experimental settings. – – – – Parallel alley. ——— Distance alley.

It is not difficult to discover which of the three is the correct model for the binocular visual space. The answer is contained in the now classic alley experiment of Blumenfeld.[8,9] In Blumenfeld's experiment two rows of lights on either side of the median plane are exhibited to the subject in the horizontal plane of the eyes. The farthest pair of lights is fixed symmetrically with respect to the median plane and the subject is asked first to construct a "parallel alley" and then a "distance alley" extending toward him from the fixed lights. To construct a parallel alley the subject arranges the lights in two rows symmetric to the median which give him the impression of being straight and parallel to each other. The distance alley, on the other hand, is a physical arrangement of rows symmetric to the median in which the visual separation between the two points of a symmetric pair is kept constant and equal to the visual distance between the two fixed points. Luneburg assumed that the parallel alleys are normals to the sensed frontal plane and subsequent experiment has shown this interpretation to be correct.[5]

The result of the Blumenfeld alley experiment is typically of the form depicted in Fig. 1. The parallel alley is set nearer to the median than the distance alley. Since the two criteria, equidistance and parallelism, do not give the same result, it is clear that the geometry is not Euclidean. If the geometry were elliptic, then as noted above, in the two-dimensional case, it has

[6] R. K. Luneburg, "Metric methods in binocular visual perception," in *Studies and Essays, Courant Anniversary Volume,* (Interscience Publishers, Inc., New York, 1948).

[7] This kind of axiomatics has been given a thoroughgoing treatment elsewhere (see H. Busemann, *Metric Methods in Finsler Spaces* (Princeton University Press, Princeton, 1942)).

[8] W. Blumenfeld, Z. Psychol. u. Physiol. d. Sinnesorgane **65,** 241 (1913).

[9] Hardy, Rand, and Rittler, Arch. Ophthalmol. (Chicago) **45,** 53 (1951).

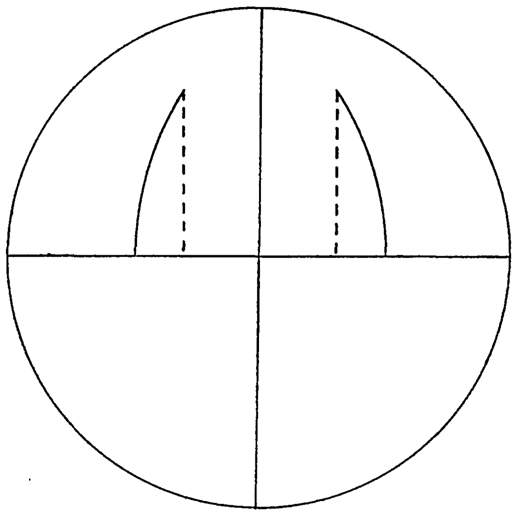

FIG. 2. Blumenfeld alleys. Representation of a spherical surface exhibiting the impossibility of making the visual space conform to an elliptic model. ---- Parallel alley. ——— Distance alley.

an isometric representation on the surface of a sphere. In this representation, let the axis along which the sensed directions of left and right are represented be the equator. The parallel alleys will then consist of segments of two great circles normal to the equator and passing through the poles. The distance alleys will be segments of circles of latitude perpendicular to the equator (Fig. 2). Hence, in the elliptic geometry, the distance alley falls nearer to the median than the parallel alley. This observation leaves only one possibility. The geometry of the binocular visual space must be hyperbolic.

2d. The Hyperbolic Metric

Luneburg chose as coordinate axes the sensed lateral, frontal, and vertical directions. To correspond to these axes he chose coordinates (ξ, η, ζ). The origin (0, 0, 0) represents the subjective center of observation. The subjective frontal, medial, and horizontal planes are given by the equations $\xi=0$, $\eta=0$, $\zeta=0$, respectively. In the hyperbolic geometry it is possible to choose such coordinates (ξ, η, ζ) so that the distance D between two points (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) of the visual space is given by

$$\sinh\left(\frac{K^{\frac{1}{2}}}{2}\frac{D}{C}\right) = \left[\frac{\frac{K}{4}\{(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2\}}{\left(1-\frac{K}{4}\rho_1^2\right)\left(1-\frac{K}{4}\rho_2^2\right)}\right]^{\frac{1}{2}}, \quad (1)$$

where $\rho_i^2=\xi_i^2+\eta_i^2+\zeta_i^2$ ($i=1, 2$). The constant C is the factor of indeterminacy in the metric and K is the absolute value of the Gaussian curvature.[10]

Though this metric is useful in giving a very simple mapping of the hyperbolic space into Euclidean space, it is more convenient for the purpose of this paper to use polar coordinates (r, φ, ϑ). These coordinates are connected with the coordinates (ξ, η, ζ) through the transformation equations

$$\begin{aligned}\xi&=\rho\cos\varphi\cos\vartheta,\\ \eta&=\rho\sin\varphi,\\ \zeta&=\rho\cos\varphi\sin\vartheta,\end{aligned} \quad (2)$$

and the radial coordinate r is given by

$$\rho=(\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}=\frac{2}{K^{\frac{1}{2}}}\tanh\frac{r}{2}. \quad (3)$$

If the standard form of the metric D is taken by setting $C=1$, then the metric is given in terms of the polar coordinates by

$$\cosh D=\cosh r_1\cosh r_2-\sinh r_1\sinh r_2\{\cos(\varphi_2-\varphi_1) -\cos\varphi_1\cos\varphi_2[1-\cos(\vartheta_2-\vartheta_1)]\}. \quad (4)$$

The radial coordinate r is to be interpreted as a quantity measuring sensed distance from the subject in relation to other sensed distances. It cannot be equated to any conception of the subject as to the absolute distance of points. The equation $r=$ constant represents a locus of points having the appearance of equidistance from the subject. It describes the subject's sensation of a sphere with himself at the center. The coordinate ϑ represents the perceived angle of elevation above the subjective horizontal. The perceived angle of deviation from the subjective median plane is given by the coordinate φ. On the sphere $r=$ constant, the curves $\vartheta=$ constant represent meridians of longitude passing through poles on the left and right of the subjective center. The curves $\varphi=$ constant on the visual sphere represent latitude circles parallel to the subjective median plane.

It is a simple matter to derive the laws of hyperbolic plane trigonometry from the metric (4). The right-triangle laws of ordinary trigonometry have similar counterparts in the hyperbolic space. Let A, B, C denote the angles of a right triangle with $C=90°$, and let a, b, c, respectively, denote the sidelengths of the opposite sides, c being the hypotenuse. The counterpart of the Pythagorean theorem is

$$\cosh c=\cosh a\cosh b. \quad (5)$$

The angle functions are given by

$$\sin A=\sinh a/\sinh c, \quad (6)$$

$$\cos A=\tanh b/\tanh c, \quad (7)$$

$$\tan A=\tanh a/\sinh c. \quad (8)$$

[10] R. K. Luneburg, J. Opt. Soc. Am. 40, 627 (1950).

For small triangles it may be seen that the hyperbolic laws approach the Euclidean ones. This is the meaning of the postulate (*F*) that the geometry is locally Euclidean.

Although the character of the visual space has been determined and the metric written down explicitly, the problem is only half solved. It is necessary also to determine the relation between the visual coordinates (r, φ, ϑ) and the coordinates of physical space.

3. RELATION OF VISUAL TO PHYSICAL SPACE

In binocular space perception, as in the perception of color, the same perception may arise in many different ways. There is an extensive category of stimuli all of which produce the same effect. The first and most appealing evidence to appear in support of this point were the equivalent rooms of A. Ames constructed at the Dartmouth Eye Institute. In these constructions,

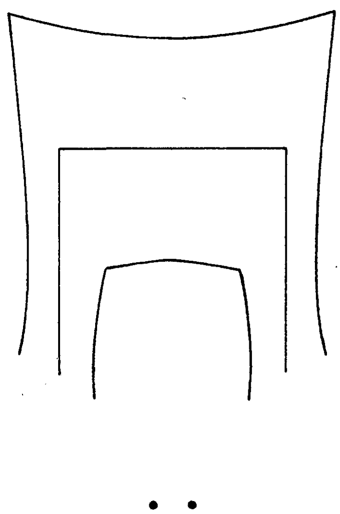

FIG. 3. Schematic representation of the Ames rooms.

Ames succeeded by empirical methods in building a set of distorted rooms which were indistinguishable from a given rectangular room with respect to binocular vision. The walls of these rooms could, in fact, have marked curvature as indicated in Fig. 3.

Luneburg found two possible ways to account for a result of this kind.[11] The first possibility he considered was that the distorted rooms are equivalent in that they supply the same binocular clues; the sequence of retinal images in gazing from point to point of one room is the same as that for any of the others. The second hypothesis was that the distorted rooms are related to the rectangular original by translatory displacements of the visual hyperbolic space. Although Luneburg favored the latter point of view and explored its consequences in his papers, he could have proceeded just as easily from the other hypothesis and arrived by the same methods at results just as plausible and internally self-consistent. Without experimental evidence to guide him he took the second and somewhat simpler choice. Only recently, after several years of laboratory work has it become clear that the other course is the correct one.

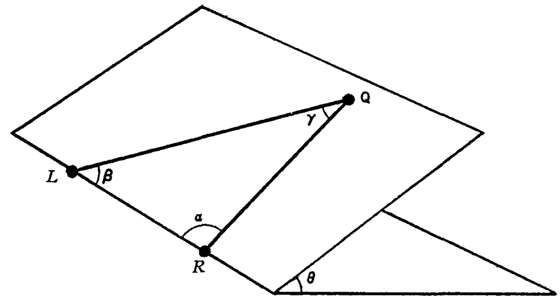

FIG. 4. Bipolar coordinates of a physical point.

3a. The Bipolar Coordinates

The position of a point in physical space will be fixed by means of special angular coordinates, and these coordinates will be seen to have a profound intrinsic significance for binocular visual perception.

The physical situation is idealized somewhat by localizing the eyes at points in a horizontal plane. In practice, these points may be taken to be the rotation centers of the eyes. Let Q denote any binocularly effective point and let R and L denote the right and left eyes, respectively (Fig. 4). The lines RQ and LQ are called the lines of sight and the angles made by the lines of sight with the base segment RL are denoted, respectively, by α and β. The position of the point Q is uniquely defined by three bipolar coordinates:

the *bipolar parallax* γ, the angle of convergence between the lines of sight,

$$\gamma=\pi-\alpha-\beta, \tag{9a}$$

the *bipolar latitude* ϕ, the average direction of the lines of sight measured counterclockwise with respect to the median plane,

$$\phi=\tfrac{1}{2}(\beta-\alpha), \tag{9b}$$

the *elevation* θ, the angle made by the plane RQL with the horizontal plane.

By means of the bipolar coordinates the phenomenon of the equivalent rooms can be described mathematically in a particularly elegant way.

3b. Equivalent Configurations. Iseikonic Coordinates

Luneburg first approached the phenomenon of the equivalent rooms with the hypothesis that such structures provide identical binocular clues to the subject. If it is assumed that two sets of binocular stimuli are equivalent when the sequence of retinal images in looking from point to point is the same for

[11] R. K. Luneburg, *Mathematical Analysis of Binocular Vision* (Princeton University Press, Princeton, 1947), pp. 17*ff* and pp. 89*ff*.

both, then it is possible to describe the relation between the rooms very simply. The bipolar coordinates γ', ϕ', θ' of the points of one are related to the associated points γ, ϕ, θ of another through an *iseikonic transformation*,

$$\gamma'=\gamma+\lambda, \quad \phi'=\phi+\mu, \quad \theta'=\theta+\nu, \tag{10}$$

where λ, μ, ν are constants. In particular, the Ames rooms could be constructed by employing the special transformation

$$\gamma'=\gamma+\lambda, \quad \phi'=\phi, \quad \theta'=\theta.$$

It is not necessary for the immediate purposes of this paper to assume anything concerning the hypothesis which led to these transformations. It is sufficient to note here that the validity of the transformations has been established experimentally, at least in the two-dimensional case, $\theta=0$.

Since the binocular metric relations between the points of a configuration are not changed by iseikonic transformation it is convenient to express the visual coordinates in terms of coordinates invariant under iseikonic transformation rather than directly in terms of the bipolar coordinates. For any particular stimulus configuration this is accomplished by singling out three values γ_0, ϕ_0, θ_0 associated with the configuration and setting

$$\Gamma=\gamma-\gamma_0, \quad \Phi=\phi-\phi_0, \quad \Theta=\theta-\theta_0. \tag{11}$$

The three values Γ, Φ, Θ are called *iseikonic coordinates* and it is clear that they are invariants under iseikonic transformation.

The values γ_0, ϕ_0, θ_0 may be chosen arbitrarily. However, for reasons which will appear subsequently, the value of γ_0 is always chosen as the smallest value of γ in the configuration under consideration. Thus, for points of the configuration for which the convergence is a minimum, $\Gamma=0$. For all points of stronger convergence, Γ is positive.

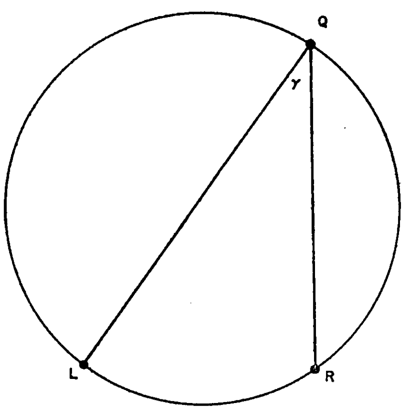

FIG. 5. The Vieth-Müller circle, $\gamma=$ constant.

3c. Visual Orientation and Perceived Direction

A subject will usually orient himself so that the subjective frontal, median, and horizontal planes described in Sec. 2d are correctly coordinated with the corresponding principal planes of objective physical space. This coordination can easily be upset, in a moving airplane or seagoing vessel, for example. Indeed the possibility of such a dislocation of the senses is suggested by the iseikonic transformations. However, for the moment, the important fact is that a correct coordination of the principal visual planes with the corresponding physical planes is possible.

Two physical points Q_1 and Q_2 will lead to the perception of points P_1 and P_2 lying in the same direction from the subject, if and only if they both have the same angular coordinates ϕ and θ. In other words, when the ϕ and θ coordinates of two points are the same, the subject will perceive the two points as being in line with his subjective center of observation. It follows that the hyperbolas $\theta=$ constant, $\phi=$ constant of physical space, are interpreted as radial lines $\vartheta=$ constant, $\varphi=$ constant in the visual space.

Moreover, equal changes in the physical coordinates ϕ and θ are perceived as equal changes in the visual coordinates φ and ϑ. When the subjective orientation of the principal planes agrees with the physical orientation, it follows that we may set $\varphi=\phi$, $\vartheta=\theta$. In general, with allowance for the iseikonic transformations, it may be stated that perceived differences in φ and ϑ in looking from one point to another are equal to the respective differences in ϕ and θ.

3d. Perceived Radial Distance. The Vieth-Müller Torus

Once the relations between visual direction and physical coordinates are understood, the determination of the relation between the visual radial coordinate r and the physical coordinates can be determined empirically. It is found on experiment that the loci in the horizontal plane $\theta=0$ which give the impression of circles about the subjective center are very nearly the Vieth-Müller circles $\gamma=$ constant.[5] These are the one-parameter family of circles in the horizontal plane passing through the ocular centers R and L. Thus, for a given point Q, the physical locus of points which gives the impression of being at the same distance from the subjective center as Q is the Vieth-Müller circle through Q (Fig. 5).

The surface $\gamma=$ constant which is generated by rotating the Vieth-Müller circle about the axis LR is known as a Vieth-Müller torus. On the basis of the above experiment it will be assumed that a Vieth-Müller torus is perceived as a sphere about the subjective center. The toruses $\gamma=$ constant are mapped onto spheres $r=$ constant in the visual space.

3e. Equations for the Visual Coordinates

The result of Sec. 3c suggests setting

$$\varphi=\Phi,\quad \vartheta=\Theta, \tag{12a}$$

where φ, ϑ are the sensory angular coordinates and Φ, Θ are the iseikonic coordinates of (11). It will be recalled, however, that Φ and Θ depend on the arbitrary parameters ϕ_0 and θ_0. It does not matter, nevertheless, what particular values of ϕ_0 and θ_0 are chosen. The choice of different values of ϕ_0 and θ_0 would amount to changing the values of φ and ϑ by constants. Such a transformation would be simply a special rotation of the hyperbolic visual space. Since the metric relations in the visual space are not changed by rotation, one arbitrary choice of ϕ_0 and θ_0 will do as well as any other. In practice these values are chosen in any convenient way.

The situation is somewhat different when the result of Sec. 3d is considered. The experiment suggests that the visual radial coordinate r is some unspecified function of Γ,

$$r=r(\Gamma). \tag{12b}$$

It is not true that freedom in the value of γ_0 may be allowed, for a change in the value of the radial coordinate alone cannot possibly be a rigid transformation of the hyperbolic space. In other words, if a relation of the form (12b) is valid, then a change in the value of γ_0 would result in a change in the distance relationships described by the metric (4). For the relation (12b) to hold, then, the value of γ_0 must be determined by the stimulus configuration.

There remains the unpleasant possibility that the value of γ_0 might depend in some very complicated way upon the stimulus configuration presented to the subject. The experimental determinations of r show, however, that the points which have the minimum value of γ for any particular configuration under consideration, must always be assigned the same value ω of the radial coordinate, independently of the configuration. In other words, the points of greatest sensed distance from the subject, whatever the total stimulus configuration, are always located at the same fixed hyperbolic distance ω from the subjective center. The visual space is bounded and in any stimulus situation the points of greatest visual distance are located exactly on the boundary. This result is in accord with the fact that for all observation and experiment there is nothing in our perceptions corresponding to the ideas of infinitely far away or infinitely large. The sky over our heads is a finite dome. The sun is a disk which sometimes has been compared in size to a dime.

In any stimulus situation, the value of γ_0 is then taken to be the least among all points available to the binocular vision of the subject. Thus, under this convention, Γ will be zero for the points of greatest sensed distance and positive for those sensed as being nearer.

The relations between the visual and physical coordinates are, in summary,

$$r=r(\Gamma),\quad \varphi=\Phi,\quad \vartheta=\Theta, \tag{13a}$$

where Γ, Φ, Θ are related to the bipolar coordinates through the equations,

$$\Gamma=\gamma-\gamma_0,\quad \Phi=\phi-\phi_0,\quad \Theta=\theta-\theta_0, \tag{13b}$$

the values of ϕ_0 and θ_0 being arbitrary, and the value of γ_0 being chosen as above.

The significant fact about Eqs. (13a) is that it is only necessary to determine the single function $r(\Gamma)$ for a given subject in order to have a complete characterization of his visual space. In practice, it may even be possible to express $r(\Gamma)$ in terms of a limited number of parameters and so be able to describe an individual's visual space in terms of a few characteristic constants. It is to be stressed that the function $r(\Gamma)$ will not necessarily be the same for different subjects. On the other hand, nothing is known about the mean behavior of man in this regard, and an investigation of that point would have considerable significance.

In Luneburg's papers[6,10,11] it was assumed that the radial coordinate is a function of γ. It is clear from the present point of view, that this would be valid only in the special case when all the configurations under consideration are restricted to have the same value for γ_0. Since the value of γ_0 in daily life is extremely frequently close to zero the difference between the two ideas would not ordinarily lead to great apparent discrepancies.

On the basis of the theory it is possible to determine a function $r(\Gamma)$ which will completely specify the visual space for a given subject. On the other hand, the usefulness of the theory rests upon the possibility of determining such a function. It remains therefore to show how the function is determined in practice and to give the results obtained.

4. THE EXPERIMENTAL DETERMINATION OF $r(\Gamma)$

By using the hyperbolic right-triangle laws (5), (6), (7), (8) it is not difficult to devise trigonometric methods for determining $r(\Gamma)$. For experimental convenience, all determinations of the function were made in the horizontal plane $\theta=0$. There would be no difficulty, in theory, in doing experiments involving the third coordinate, but the laboratory arrangements would have to be more elaborate than those which were available. In the following, three methods employed for the determination of $r(\Gamma)$ are described.

The difficult and tedious experimental work was performed at the Knapp Laboratory of Physiological Optics of the Institute of Ophthalmology in New York under the direction of L. H. Hardy by G. Rand and M. C. Rittler.[5] This paper could not have been completed without their ableness and persistence.

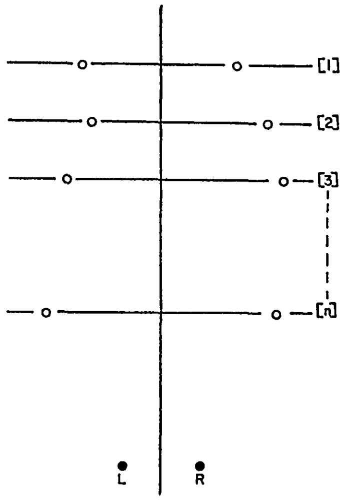

FIG. 6. Experimental scheme for the Blumenfeld alleys experiment.

4a. The Blumenfeld Alleys

Blumenfeld's experiment, Sec. 2c, is not merely striking evidence of the hyperbolic character of visual space. It can be used also to measure the function $r(\Gamma)$.

In this experiment, two rows of n lights each are set down in the horizontal plane, one row to the left and one to the right of the median. The lights in the left and right rows will be denoted by Q_i^L and Q_i^R, respectively, ($i=1, 2, 3, \ldots, n$). The two lights Q_i^L and Q_i^R of a pair are located on a fronto-parallel line, known as the station [i], but no other restriction is placed upon them. However, the two lights at the most distant station are completely fixed in positions symmetric to the median. The stations [1], [2], [3], . . ., [n] are labeled in order of increasing nearness to the subject. Thus, at the beginning of any experiment the lights will be set in more or less irregular rows as in Fig. 6.

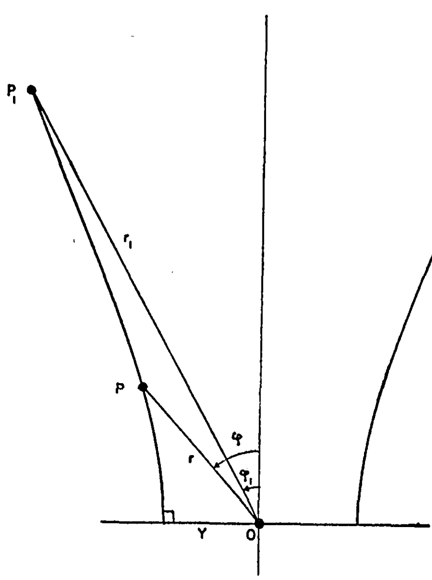

FIG. 7. Schematic representation of parallel alleys in visual space.

The subject is asked to construct a *parallel alley* according to the instruction:

(a) Keeping the most distant pair of lights fixed arrange the two rows of lights so that they appear to be straight, parallel to each other, and parallel to the median.

The *distance alley* is constructed by shutting off all the lights except the fixed pair at station [1] which is taken as a standard and a test pair at any of the stations [i]. The subject is asked to set the test pair according to the instruction:

(b) Set the nearer pair of lights symmetrically so that the distance between the two lights appears to be the same as that for the fixed lights.

The results obtained with the instructions (a) and (b) show the characteristic differences of Fig. 1. If all the distance settings in (b) are exhibited simultaneously

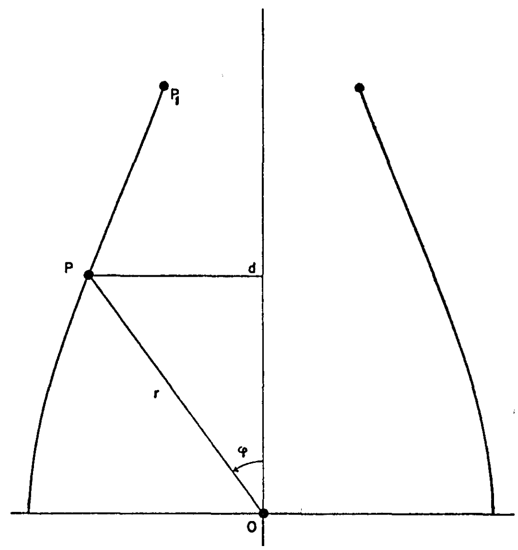

FIG. 8. Schematic representation of distance alleys in visual space.

to the subject upon completion of the experiment, the two rows of lights appear to be neither parallel nor straight.

The iseikonic coordinates in this experiment are taken as

$$\Gamma=\gamma-\gamma_1, \quad \Phi=\phi, \tag{14}$$

where γ_1 is the bipolar parallax for the fixed points at station [1].

The parallel alleys are characterized in the visual space as visual geodesics, located symmetrically with respect to the subjective median, which intersect the subjective frontal plane perpendicularly (Fig. 7). Let Y denote the hyperbolic distance of the point of intersection from the subjective center and let $P=(r, \varphi)$ denote a variable point on the left-hand alley. The equation of the curve is given by the right-triangle formula (7) as

$$\sin\varphi=\cos\left(\frac{\pi}{2}-\varphi\right)=\tanh Y/\tanh r. \tag{15a}$$

The value of Y is determined by the coordinates of the fixed point P_1 through the equation

$$\tanh Y = \tanh\omega \sin\varphi_1, \qquad (15b)$$

where $\omega = r(0)$. The equation of the parallel alley in visual coordinates is therefore

$$\tanh r \sin\varphi = \pm\tanh\omega \sin\varphi_1. \qquad (16)$$

The distance alleys, on the other hand, are characterized as the loci of constant perceived distance d from the median (Fig. 8). For a variable point $P=(r, \varphi)$ on the left-hand locus, Eq. (6) gives

$$\sin\varphi = \sinh d/\sinh r. \qquad (17a)$$

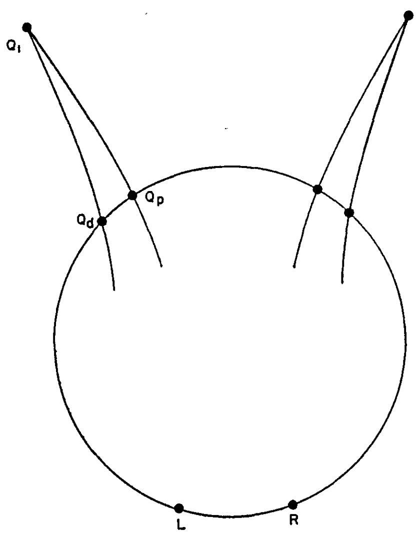

FIG. 9. Intersections of a Vieth-Müller circle with a pair of Blumenfeld alleys.

The value of d is determined by the position of the point P_1 according to the relation

$$\sinh d = \sinh\omega \sin\varphi_1. \qquad (17b)$$

The equation for the distance alley in visual space must then be

$$\sinh r \sin\varphi = \pm\sinh\omega \sin\varphi_1. \qquad (18)$$

Let $Q_d = (\Gamma_0, \phi_d)$ and $Q_p = (\Gamma_0, \phi_p)$ be the intersection of a Vieth-Müller circle with the left-hand distance and parallel alleys, respectively (Fig. 9). Since both points have the same coordinate Γ_0, the visual radial coordinate $r_0 = r(\Gamma_0)$ is the same for the two points. These values are set in Eqs. (16) and (18) to yield

$$\tanh r_0 \sin\phi_p = \tanh\omega \sin\phi_1,$$
$$\sinh r_0 \sin\phi_d = \sinh\omega \sin\phi_1.$$

The value of r_0 may readily be eliminated from these equations to give an equation for ω,

$$\cosh^2\omega = \frac{\sin^2\phi_d - \sin^2\phi_1}{\sin^2\phi_p - \sin^2\phi_1}. \qquad (19)$$

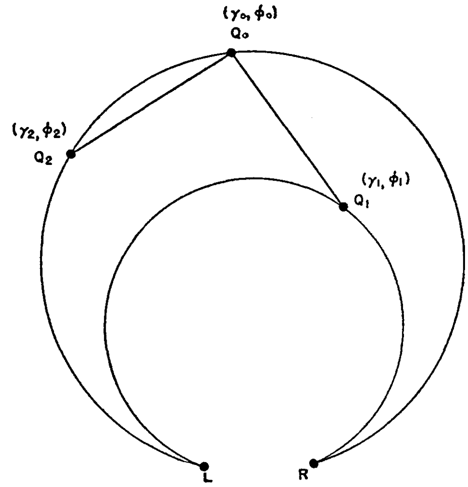

FIG. 10. The double Vieth-Müller circles. Three-point experiment.

Once the value of $\omega = r(0)$ is obtained, the values of r for other values of Γ can be determined from the settings of the alleys through Eqs. (16) and (18).

4b. The Double Vieth-Müller Circles

These experiments were designed by Luneburg.[10] They are of special interest in that the subject is required only to make visual matches of size. They have uniformly indicated the hyperbolic character of visual space.

(*i*) *The Three-Point Experiment*

Consider the two Vieth-Müller circles defined by the equations $\gamma=\gamma_0$, $\gamma=\gamma_1$, with $\gamma_0<\gamma_1$. A point $Q_0=(\gamma_0, \phi_0)$ is fixed on the outer circle and a point $Q_1=(\gamma_1, \phi_1)$ is fixed on the inner circle. A variable point $Q_2=(\gamma_0, \phi_2)$ is allowed to take any position on the outer circle (Fig. 10). The iseikonic coordinates for this configuration are given by taking γ_0 and ϕ_0 as the parameters in (13b). The corresponding points of visual space will be $P_0=(\omega, 0)$, $P_1=(r_1, \varphi_1)$, and $P_2=(\omega, \varphi_2)$, respectively, where

$$\varphi_1=\phi_1-\phi_0, \quad \varphi_2=\phi_2-\phi_0$$

and

$$\omega=r(0), \quad r_1=r(\gamma_1-\gamma_0).$$

The subject is instructed to set the point Q_2 in the position where the sensory distance P_2P_0 is matched to the distance P_0P_1. For a setting of this kind, the metric (4) immediately gives the equation

$$\cosh^2\omega - \sinh^2\omega \cos\varphi_2 = \cosh r_1 \cosh\omega - \sinh r_1 \sinh\omega \cos\varphi_1,$$

whence,

$$\cos\varphi_2 = m\cos\varphi_1 + b \qquad (20a)$$

and the coefficients

$$m=\frac{\sinh r_1}{\sinh\omega}, \quad b=\frac{\cosh^2\omega - \cosh\omega\cosh r_1}{\sinh^2\omega} \qquad (20b)$$

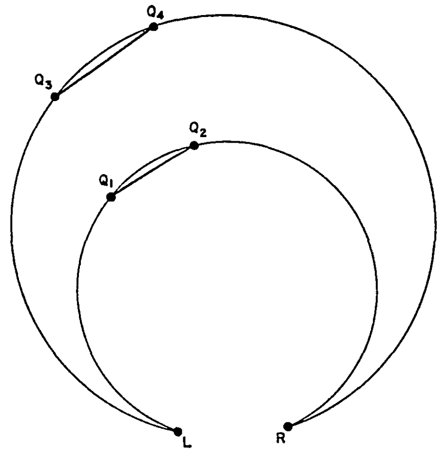

FIG. 11. The double Vieth-Müller circles. Four-point experiment.

are constants depending only on the value $\gamma_1-\gamma_0$ and not the particular values φ_1 and φ_2. Consequently, if different values of $\varphi_1=\phi_1-\phi_0$ are taken and the corresponding values of $\varphi_2=\phi_2-\phi_0$ found experimentally, the plot of $\cos\varphi_2$ as ordinate against $\cos\varphi_1$ as abscissa should be a straight line. The experimental findings do conform to this expectancy. Luneburg showed that such a result implied constant Gaussian curvature for the visual space.

The values of m and b are easily determined from the graph of $\cos\varphi_2$ against $\cos\varphi_1$. The value of ω, obtained by eliminating r_1 from Eqs. (20b), is given by

$$\cosh^2\omega=\frac{b^2}{(1-b)^2-m^2}. \tag{21}$$

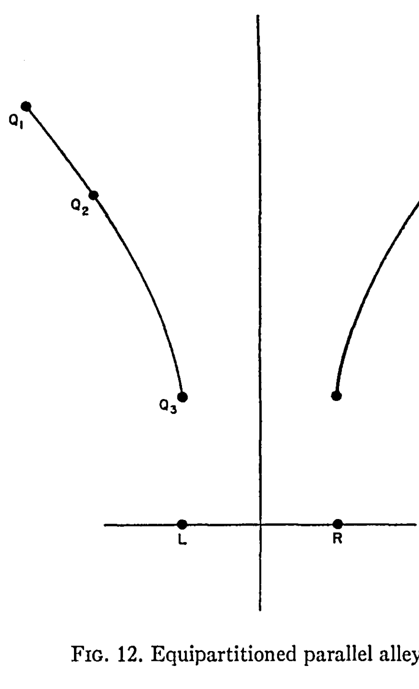

FIG. 12. Equipartitioned parallel alleys.

Given ω, the value of r_1 is found from the equation (20b) for m.

If ω is to be a real quantity the left side of (21) must be greater than unity. The fact that this is so, as an experimental result, provides further evidence for the hyperbolic character of the geometry. It is not difficult to show that the geometry is hyperbolic, Euclidean, or elliptic according to whether m^2 is greater than, equal to, or less than $1-2b$.

In practice, it is found that the experimental limitations are such that the constants m and b cannot be determined with sufficient accuracy. What is done instead is to determine by extrapolation the value a of $\cos\varphi_2$ at $\varphi_1=0$. This can be done with considerable accuracy. A more precise determination of m is provided by the following experiment and b is given by $b=a-m$.

(ii) The Four-Point Experiment

This experiment is executed on the same pair of Vieth-Müller circles as the three-point experiment. Let $Q_1=(\gamma_1,\ \phi_1)$ and $Q_2=(\gamma_1,\ \phi_2)$ be two points fixed on the inner Vieth-Müller circle, and let $Q_3=(\gamma_0,\ \phi_3)$ and $Q_4=(\gamma_0,\ \phi_4)$ be variable points on the outer circle (Fig. 11). The corresponding sensed points are denoted by P_1, P_2, P_3, P_4, respectively. The subject is instructed to set the points Q_3 and Q_4 so that the sensory distance between the points P_3 and P_4 of the visual space is equal to that between P_1 and P_2. Set $r_1=r(\gamma_1-\gamma_0)$ as in the three-point experiment, and put

$$\Delta_1=\phi_2-\phi_1, \quad \Delta_2=\phi_4-\phi_3.$$

From the metric (4) it follows immediately that

$$\cosh^2\omega-\sinh^2\omega\cos\Delta_2=\cosh^2r_1-\sinh^2r_1\cos\Delta_1,$$

whence,

$$m^2=\frac{\sinh^2r_1}{\sinh^2\omega}=\frac{1-\cos\Delta_2}{1-\cos\Delta_1}.$$

The trigonometric identity

$$1-\cos\Delta=2\sin^2\tfrac{1}{2}\Delta$$

gives the simple relation

$$m=\frac{\sinh r_1}{\sinh\omega}=\frac{\sin\frac{1}{2}\Delta_2}{\sin\frac{1}{2}\Delta_1}. \tag{22}$$

This value of m may then be used for a better determination of b as indicated above.

From the values of m and b we may then calculate the value of ω from (21). Once the value of ω is obtained experimentally, the three-point experiment need no longer be used; by repeated use of the four-point experiment with differing values of $\Gamma=\gamma_1-\gamma_0$, the values of $r(\Gamma)$ can be calculated from (22).

4c. Equipartitioned Parallel Alleys

This experiment is significant in that the calculation of the radial coordinate r is altogether independent of any hypothesis concerning the dependence of r on the physical coordinates. This is not the case in the preceding two experiments. The results of this experiment can therefore be considered as an independent check on the assumption $r=r(\Gamma)$.

The subject is shown two rows of three lights on either side of the median and instructed to arrange them in a parallel alley as in 4a. In addition, he is instructed to locate the intermediate light in each row at a position exactly midway in distance between the two end lights. The configuration so constructed is called an equipartitioned parallel alley.

Let Q_1, Q_2, Q_3 denote the points in the left-hand row in order of decreasing distance from the subject (Fig. 12). The light Q_1 is fixed and the light Q_3 is permitted to move only along a fronto-parallel line as in Sec. 4a. The light Q_2 may be moved freely in the two dimensions of the horizontal plane. Let the coordinates of the points be denoted by

$$Q_i=(\gamma_i, \phi_i) \qquad (i=1, 2, 3).$$

Iseikonic coordinates for this configuration are given by

$$\Gamma=\gamma-\gamma_1, \quad \Phi=\phi.$$

The points P_i in visual space corresponding to the points Q_i will have coordinates (r_i, φ_i) satisfying Eq. (15a)

$$\tanh r_i \sin\varphi_i=\tanh Y. \tag{23}$$

The value Y, determined from the condition that the sensory distances P_1P_2 and P_2P_3 are equal, is found to satisfy the equation

$$\sinh^2 Y=\tan^2\varphi_2\left[\frac{2-(S+T)}{(S+T)-2ST}\right], \tag{24}$$

where

$$S=\frac{\tan\varphi_2}{\tan\varphi_1}, \quad T=\frac{\tan\varphi_2}{\tan\varphi_3}. \tag{24a}$$

The values r_i may be found from the value of Y through Eq. (23). From $r_1=r(0)=\omega$, it follows that

$$\tanh\omega=\tanh Y/\sin\varphi_1. \tag{25}$$

The equipartitioned alleys also give evidence that the geometry is hyperbolic. This is a consequence of the fact that the expression on the right in (24) is found experimentally to be positive. If it were zero or negative the result would mean that the geometry is Euclidean or elliptic in the respective cases.

4d. The Function $r(\Gamma)$

In Fig. 13 the experimental values of $r(\Gamma)$ are given for two different subjects and for all three of the experiments described. It is noteworthy that individual differences are clearly brought out. The values of the function obtained for MCR are significantly lower than those for GR.

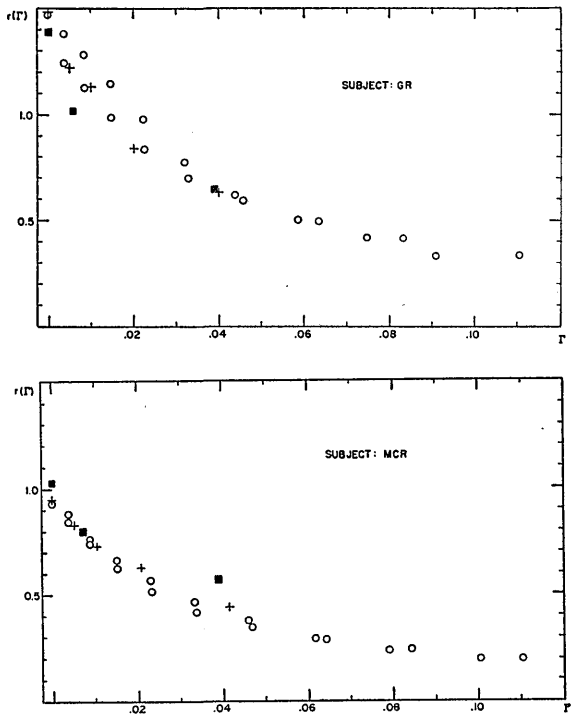

FIG. 13. Experimentally determined values of $r(\Gamma)$ for two subjects. ○ Blumenfeld alleys. + Double Vieth-Müller circles. ■ Equipartitioned parallel alleys.

5. THE TIME-DEPENDENT METRIC

All considerations, up to this point, have been restricted to stimulus patterns which are fixed with respect to the observer. The problem of including effects due to motion is especially interesting since it appears to be necessary to employ a space-time metric analogous to that of relativity theory. Obviously, there is a limiting value to the sensation of velocity, since a moving light will produce only the sensation of a streak if it moves rapidly enough. If the homogeneity of visual space is preserved when motion is allowed, it is to be expected that the space-time metric would take the form

$$c^2dt^2-ds^2, \tag{26}$$

where c is the limiting velocity in the visual space and ds^2, if motion is limited to the horizontal plane, is

$$ds^2=dr^2-\sinh^2 r d\varphi^2. \tag{27}$$

The constant c, when determined, would probably be found to be connected to the maximum angular velocity of eye movements.

These considerations suggest the possibility that something like the Lorentz contraction of physics ought to be a part of our visual sensations. In fact, both Dr. Luneburg and the author, on separate occasions, had the good fortune to observe such an effect; that is, an apparent foreshortening in the direction of motion of rapidly moving objects.