INVESTIGATION OF VISUAL SPACE

The Blumenfeld Alleys

LE GRAND H. HARDY, M.D.

GERTRUDE RAND, Ph.D.

AND

M. CATHERINE RITTLER, A.B.

NEW YORK

THE PROBLEM of space perception is an old and important problem in visual physiology. It is assumed that there is some relationship between the world we live in and our visual sensations. But it is known that there is not a one-to-one correspondence between physical stimuli and visual sensation. One unit of physical measurement does not cause one unit of perception. Our sensations do not change directly with the physical change of the stimuli causing them. Many inconsistencies and paradoxes arise when we try to correlate our perceptions with the physical configuration of the stimuli. Until recently it seemed such a correlation was impossible. The essence of Luneburg's "Mathematical Analysis of Binocular Vision" [1] is that there exists such a basic correlation limited to the assignment of apparent size to line elements and that if this concept proves true much valuable information regarding our space perceptions may be derived mathematically and subjected to experimental testing.

To develop such a thesis a differentiation of "physical space" from "visual space" was necessary. "Physical space" is a measurable, engineering space of the outside world. It is the space in which houses, bridges, roadways and railways are mapped and built with great precision. It is the space with which the engineer is constantly concerned.

"Visual space" is the immediate instantaneous impression we have of our environment as a three dimensional manifold; the immediate impression we have of objects about us in relative terms: farther-nearer; larger-smaller-equal.

The necessity for such a differentiation arises from certain well established phenomena in visual optics. Important among these are the following:

Dr. Rudolf Luneburg and Dr. Paul Boeder acted as mathematical consultants in these studies.

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From the Knapp Memorial Laboratory for Physiological Optics, Institute of Ophthalmology, Columbia University College of Physicians and Surgeons.

1. Luneburg, R. K.: Mathematical Analysis of Binocular Vision, Princeton, N. J., Princeton University Press, 1947.

1. The sensed size of objects at different distances does not follow the visual angle law, but is in some way modified by the distance to which they are projected. However, the sensed size of objects is by no means proportional to their physical size and their physical distance, particularly at great distances. (The physical size of the moon cannot be sensed; a star distance cannot be sensed; the sky appears not as an infinite space but as a finite curved dome; a man does not appear smaller at 40 feet than at 10 feet.)

2. Physically straight lines in the frontal plane appear as curved lines, and certain physically curved lines appear straight (the horopter experiments of Helmholtz [2] and Hillebrand [3]).

3. Lines sensed as parallel, straight and perpendicular to the frontal plane are physically not parallel, straight or perpendicular to that plane (Hillebrand's alley experiment).

4. Alleys of lights set to appear parallel and perpendicular to the frontal plane do not coincide with alleys set to appear "equidistant" (Blumenfeld's alley experiments [4]). Hence, in visual space "equidistant" and "parallel" do not mean the same thing.

5. Markedly different physical configurations of stimuli arouse the same sensation. Ames's "distorted rooms" [5] is an elegant example of the difference between "physical" and "visual" spaces.

In dealing with this problem, one is forced to conclude that if physical space is Euclidean in its nature (at least in the range wherein our human stereopsis functions) visual space cannot be, and Luneburg was led to consider the possibilities other (non-Euclidean [6]) geometries held for finding a "psychometric," i. e., a coordination of the sensed points of a visual sensation to the points of a geometric manifold such that the apparent distance of any two sensed points is always proportional to the geometric distance of the correlated points. He was led to conclude that the geometry of the visual space is non-Euclidean in nature and corresponds closely to the hyperbolic geometry of Bolyai and Lobachevski. Utilizing this concept, Luneburg was able to clarify considerably the apparently inconsistent and paradoxical phenomena we have listed above (Helmholtz, Hillebrand, Blumenfeld and Ames). In addition, the Weber-Fechner law is very simply derived mathematically from this information.

The problem is of great importance in art, architecture, ophthalmology, industry and war. It was particularly in reference to the determination of the K constants

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2. Helmholtz, H. L. F.: Physiological Optics, edited by J. P. C. Southall, Rochester, N. Y., Optical Society of America, 1925, vol. 3, p. 318.

3. Hillebrand, F.: Theorie der scheinbaren Grösse bei binocularem Sehen, Denkschr. d. k. Akad. d. Wissensch. Math.-Naturw. Klasse **72**:255-307, 1902.

4. Blumenfeld, W.: Untersuchungen über die scheinbare Grösse im Sehraume, Ztschr. f. Psychol. u. Physiol. d. Sinnesorg. **65**(Abt. 1):241-404, 1913.

5. Ames, A., cited by Luneburg.[1]

6. "Non-Euclidean" is a term used by Gauss to describe a system of geometry different from Euclid's in that the Fifth Postulate regarding parallelism was not used. Bolyai, in Hungary, and Lobachevski, in Russia, had each developed a non-Euclidean system about 1820. Later Riemann, in Germany, and Cayley, in England, developed another system differing radically from Euclid's. In 1871 Klein classified the various systems, giving the names parabolic, hyperbolic and elliptic to the respective systems of Euclid, Bolyai-Lobachevski and Riemann-Cayley.

of a number of observers and a correlation of this factor with their mechanical abilities that the naval contract under which this work has been carried out was offered. Eight empirical methods of studying the visual space of an observer have been formulated. This report deals with the first. Since Blumenfeld [4] had acquired a great mass of data which most strongly indicates the non-Euclidean nature of visual space, it was decided first to set up and repeat his "parallel" and "equidistant" alley experiments in the hope of finding a simple method of determining the individual constants of different observers. (Blumenfeld had not used any non-Euclidean analysis of his results—they were simply paradoxical.)

At the time our experiments were begun (late in 1947) Luneburg's analysis had indicated that at least two personal constants characterized each observer. These were sigma (σ)[7] and K, which should be determinable experimentally and which should yield a description of the sensed space of the observer. Sigma is a constant determining the sensitivity of depth perception compared with the perception of lateral size and thus must be related to the interpupillary distance of the observer. Since the threshold of depth perception is considerably smaller than the threshold of size, sigma (σ) will be considerably greater than 1. The constant K determines the geometric character of the visual space and thus must be considered as a significant psychological constant which is a characteristic for an individual observer (analogous to his myopia, hypermetropia, astigmatism, aniseikonia, etc.). It is of importance for the relation of sensory judgment of size to physical size. Though the two will never be identical, one finds that the best approximation of physical size by sensory estimation is represented by the case $K = -1$. Indeed, in this limiting case, infinite distances are possible in the visual space. Thus, we conclude that the visual estimation of an observer agrees with physical size the nearer -1 the constant K is found. The possibility that a high correlation of mechanical abilities (in drivers, pilots, steersmen, gun sighters, etc.) to the values of K could be established was a principal motivation in undertaking this study.

THE BLUMENFELD ALLEYS

Hillebrand [3] in 1902 had demonstrated that alley walls, outlined by black threads against a white background, which appeared to an observer as parallel, actually converged toward the eyes and were not straight, but slightly curved. In these experiments no differentiation was made between the apparent parallelism and the apparent equidistance of the walls of the alley. Blumenfeld [4] in 1913 modified Hillebrand's procedure by giving in separate experiments the instruction to form the alley walls so that they appeared parallel and so that they appeared equidistant. He found with seven observers that the alleys so formed were not the same. The apparently equidistant walls have a wider lateral extent and a different curvature

7. Luneburg (The Metric of Binocular Visual Space, J. Optic. Soc. America **40**:627-642 [Oct.] 1950) defined the parameters of visual space as $f(\gamma)$ and K instead of σ and K. In doing this, he abandoned the necessity of always considering σ as a constant. He defines $f(\gamma)$ as a "non negative function of γ which increases monotonically if γ decreases from positive values to $\gamma = 0$." If future experiments indicate that σ is a constant, then $f(\gamma)$ may be specified as $2e^{-\sigma\gamma}$.

than the apparently parallel walls. Blumenfeld worked in a darkroom, using as stimuli tiny gas flames. Six distances were employed, ranging from 400 to 80 cm. from the eyes. The work was done both with the eyes at the level of the lights and with the eyes 7 cm. above the height of the flames.

According to Luneburg's mathematical analysis of such alleys formed with binocular vision, the condition in which the apparently equidistant alley walls lie outside the apparently parallel walls occurs when the personal constant K of the observer assumes a value less than zero. This, mathematically, means that the metric of the observer's visual space corresponds to hyperbolic geometry. If the two types of alley had been identical, K would have had a value of zero and the metric of his visual space would have been Euclidean. If, on the other hand, the apparently equidistant walls had occurred inside the apparently parallel walls, the personal constant K would have been greater than zero and the metric of his visual space would have corresponded to elliptic geometry. According to Blumenfeld's results as interpreted by Luneburg, therefore, the metric of the visual space of these observers is usually hyperbolic. The second personal constant, σ, can also be derived from the experimental data of these alleys.

EXPERIMENTAL INVESTIGATION

Apparatus.—An alley board, 20 feet (6 M.) long and 4 feet (122 cm.) wide, was constructed which permitted the experimental formation of alleys whose walls, one to the right and one to the left of the median plane, appear to the observer as parallel (Blumenfeld's Parallel Alley), and of alleys whose walls appear to the observer as equidistant (Blumenfeld's Distance Alley.) The walls of the alleys were delineated by minute points of light, resembling stars, spaced at 10 intervals through a distance of 500 to 50 cm. from the eyes. In order to eliminate as far as possible all external clues, the experiments were conducted in a darkroom.

The stimuli were 20 small General Electric lamps, 2.5 volts, 0.27 ampere. Ten lamps were used to form the alley wall on each side of the median plane. The lamps were connected to a transformer with rheostat control, so that they produced tiny points of light seen as stars, the intensity of which was so low that there was no perceptible surrounding illumination. They were mounted on small rectangular blocks of wood with the luminous filaments 3.5 cm. above the base of the mounting. Ten experimental distances were used: 500, 387, 300, 232, 180, 139, 108, 83, 65 and 50 cm. from the nodal points of the observer's eyes. These distances will be called stations 1 to 10, respectively. At station 1, 500 cm. from the eyes, two lamps were fixed in position, one 35 cm. to the right and the other 35 cm. to the left of the median line of the alley board. These fixed lamps served as the standard against which the pair of lights at each nearer station were to be adjusted (*a*) to form an alley with apparently parallel walls or (*b*) to form an alley such that the lateral space between the fixed pair of lights and that between each other pair appear equal. In fixing the position of the standard lights at the most distant station, we are following Blumenfeld's procedure.

To aid in the lateral adjustment of the lights in the darkness, nine metal strips were fastened across the width of the alley board. When a lamp mounting was in contact with the edge of any strip, the filament was located at the desired experimental distance. Two experimenters, one on each side of the alley board, were easily able to move the lamp mountings along the metal strips as directed by the observer.

Only binocular observations were made. The observer's head was fixed in a three-way adjustable head rest with his nodal points, taken as 7 mm. behind the cornea, the indicated distance from the lamp filaments. The height of the eyes was set at 5.5 cm. above the filaments in order to avoid interposition of the lights at different distances and to prevent as far as possible annoyance from double images. The median plane of the observer was alined by parallax to the median line of the alley board. For this adjustment two additional lights placed 2 M. apart on the median line served as stimuli.

Procedure.—To form alleys with apparently parallel walls, the pairs of lights of stations 1 and 2 were first connected, and those of station 2 were adjusted laterally under the direction of the observer until they appeared to approach him as parallel walls originating from station 1. The pair of lights of station 3 were next added to continue the formation of apparently parallel walls; then the lights of station 4, and so on. This procedure conforms to Blumenfeld's technic for setting the "simultaneous" parallel alley. During the formation of the alley and after the entire alley had been constructed, the observer was allowed to request the readjustment of any light. Measurement of the position of the lamp filaments was made only after the observer had retired to an adjoining room.

To form alleys whose walls appeared equidistant, the lights of stations 1 and 2 were connected and those of station 2 adjusted under the observer's direction until he was satisfied that the lateral spaces between the pairs of lights at the two stations appeared of equal size. The lights of station 2 were then disconnected; those of station 3 were connected and adjusted under direction until the distance between this pair of lights appeared equal to that between the fixed pair of station 1. The experiment continued in this way with the adjustment of each successive pair of lights in comparison with the fixed pair of station 1. This procedure conforms to Blumenfeld's technic for setting the "successive" distance alley. In this case no readjustment of the lights was permitted after each station was set and the observer did not see the entire equidistant alley he had formed.

The specific instructions given the observer were in general similar to those used by Blumenfeld. They are as follows:

"This is an experiment dealing with space perception. We know one does not always perceive objects in space where they actually are in physical space, or to be of their actual physical size. We want to measure some of these differences.

"In the first experiment we shall show you some small lights which we shall arrange under your direction so that when you look down between them they appear to you to form straight, parallel lines of light. We wish you to think not of where the lights actually are, but merely of how you sense them. When they are all arranged, we want you to be able to say that these straight lines of lights as you see them could never, if extended, meet at any distance in front of you or at any distance behind you, that is, that they form walls that appear to you as parallel walls that appear neither to converge nor to diverge."

To familiarize the observer with the observation, a trial run utilizing only stations 1, 3, 5 and 8 was made, no measurements being taken of this trial. The instructions continued:

"In the second experiment we shall give you two pairs of lights at a time. The position of one pair will be fixed. We want you to direct us to move the lights of the other pair so that the lateral distance between them appears to you the same as that between the first pair. We want you to make an immediate, instantaneous judgment of whether the distance between the lights of the second pair is greater or smaller than or equal to that between the first pair. Do not think in terms of physical units of distance between the lights, for example, inches or centimeters. Just direct us in adjusting them until you immediately sense the two pairs of lights as being the same distance apart."

Again a trial run utilizing only stations 1 and 3, 1 and 5, and 1 and 8 was made, again without measurements. After these preliminary observations, the experiment continued with the formation of the complete parallel and equidistant alleys in that sequence. The usual procedure was to give a second trial of each alley on the same day and to repeat the series on a second day.

Results.—In chart 1 are presented the results for two observers, both of whom were experienced in making visual judgments but neither of whom had prior experience with this type of experiment. In *A* are shown for one of these observers three trials of the apparently parallel alley (dash lines); in *B*, three trials of the apparently equidistant alley (solid lines), each made after the formation of a parallel alley, and in *C*, the average parallel and equidistant alleys formed by this observer. In *D* is shown for the other observer a single formation of the two types

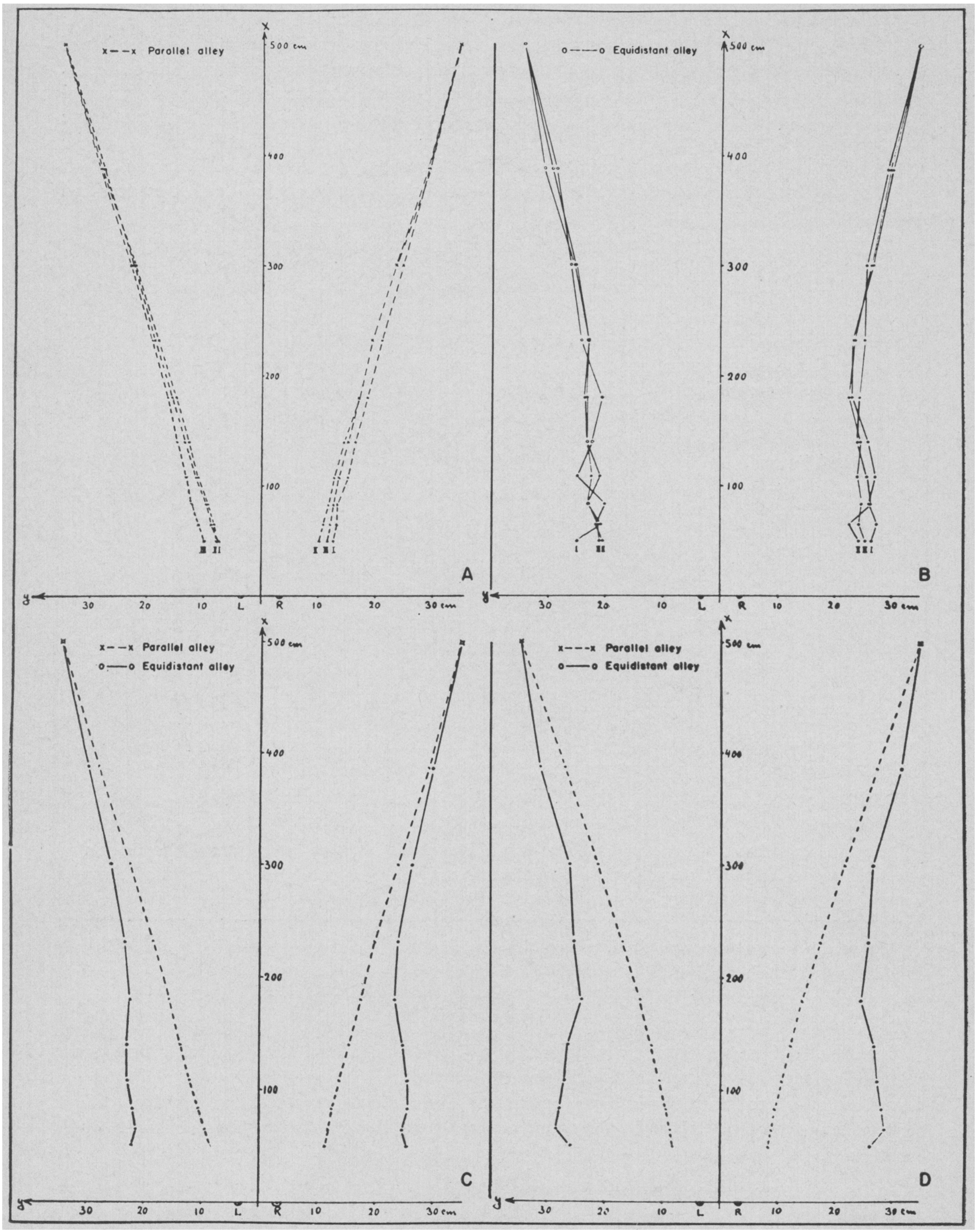

Chart 1.—Examples of parallel and equidistant alleys formed by observers whose curvature of visual space is shown by the alley experiment to be that of hyperbolic geometry.

A, three trials of parallel alley, *B,* three trials of equidistant alley, *C,* average parallel and equidistant alleys for one observer; *D,* a single trial of the alleys for another observer (right and left walls of alley averaged).

of alley. In this, and in the following cases presented, the alley walls formed on the right and left sides of the median line were averaged, since only average values are needed for purposes of computation of the personal constants. However, to simulate the appearance of an alley in the charts, the average wall is drawn on each side of the median line. Unimportant asymmetries on the right and left sides are thus not shown.

The results for these two observers are in general agreement with those of Blumenfeld's observers in that the equidistant alley lies outside the parallel alley. The curvature of the visual space is, then, according to Luneburg's analysis, that of hyperbolic geometry, K having a value less than zero. Among our observers

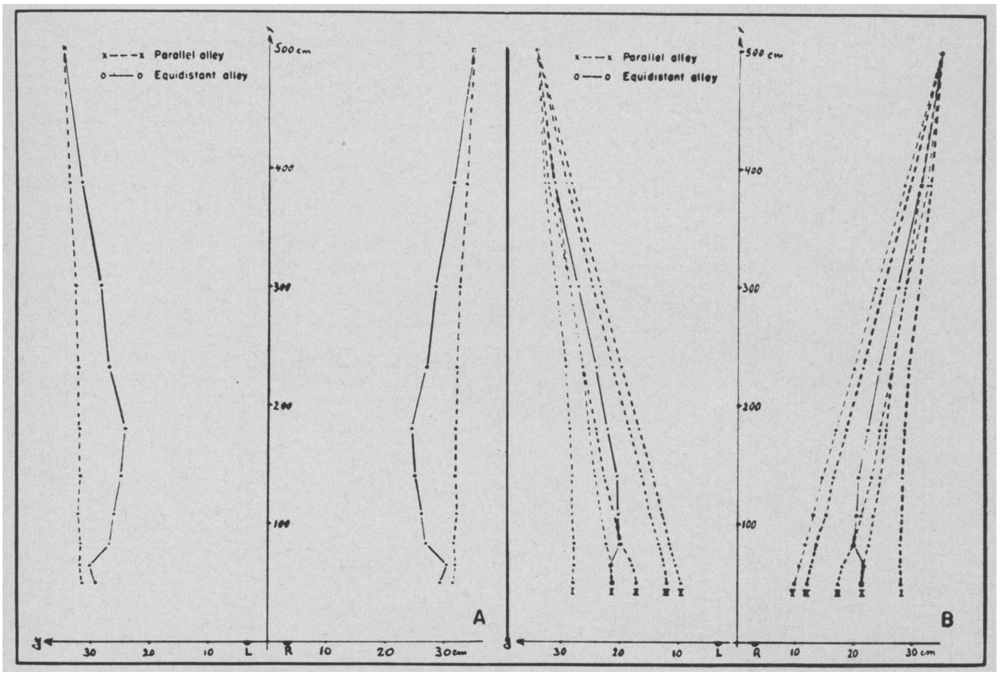

Chart 2.—Examples of parallel and equidistant alleys formed by observers who experienced difficulty in dissociating physical and sensory parallelism.

A, alleys formed by one observer with the initial instructions (average of two trials).

B, for another observer, average equidistant alley (five trials) and five successive trials of parallel alley. Trial 1 was set with the initial instructions; trials 2 to 5, were set after a discussion of the difference between physical and sensory parallelism. This illustrates how instruction and experience modify the perceptual form of the alley for some observers.

whose results were adequate for computation of the personal constants of Luneburg, eight were of this type.

The situation was not, however, so simple with all observers. For example, difficulty was experienced with some who tended to set the walls of the parallel alley not to appear parallel but, rather, as they would appear if they were actually parallel. In these cases the alley which was formed approximated fairly closely physical parallelism and was wider than the equidistant alley. Five observers were of this type. According to Luneburg's analysis, the curvature of the visual space of these observers is that of elliptic geometry, K being greater than zero.

Three of these observers, however, after the alleys had been measured, were asked how railroad tracks appear to them. All three admitted that parallel lines of tracks appear to converge in the distance, and that the parallel alleys they had formed appeared to converge in the same way. After this discussion, the instructions were repeated and further trials were made. Each successive trial resulted in alleys of increasingly greater convergence toward the eyes. Our belief is that, at least in these three cases, the curvature of the visual space was that of hyperbolic, not elliptic, geometry; that the observer in the initial trials had been unable to dissociate physical and sensory parallelism. An average of the results obtained before and after the discussion of the situation would not be representative unless the initial trials were discarded.

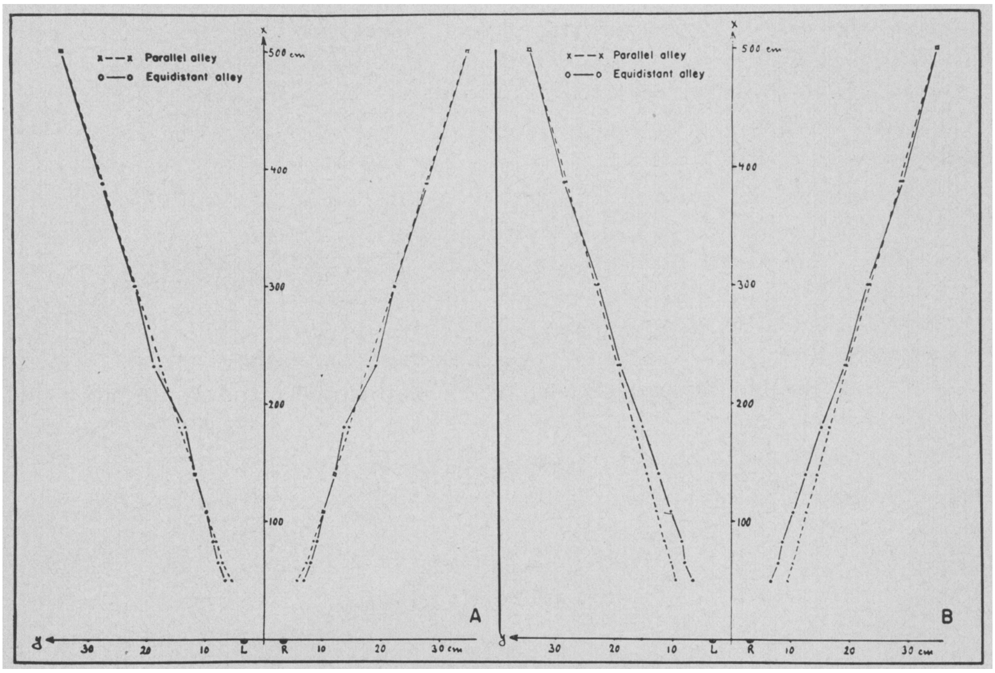

Chart 3.—Examples of parallel and equidistant alleys formed by two observers who had had training in art and the portrayal of vanishing point and central projection perspective.

In chart 2 *A* is presented for one of these three observers the parallel and the equidistant alley originally formed (average of two trials). In *B* of this figure is shown for another of these observers each of five trials of the parallel alley and the average equidistant alley (five trials). In this case only trial 1 was formed with the initial instructions. After the discussion of the appearance of railroad tracks, trials 2 to 5 were made without further comment on the part of the experimenters. On the completion of the fifth trial of the parallel alley the observer volunteered the statement that this was the best he had done, that this alley appeared most nearly parallel. Throughout the series the equidistant alley showed no consistent changes.

It is possible that this type of response might have been avoided, at least in part, had we given different initial instructions. For example, the observer might have been shown in advance an alley that was physically parallel and the apparent convergence in the distance discussed with him. The direction to set the alley so that the walls appeared parallel could then have been given. This type of instruction would have followed Hillebrand's, but not Blumenfeld's, procedure. Even with these instructions, however, we doubt if a complete dissociation of physical and sensory parallelism would have been obtained with all observers.

A third type of observer formed apparently parallel and equidistant alleys that were approximately identical. There were two observers of this type. Both had had training in art and the portrayal of vanishing point and central projection perspective, and by some learned formula set both alleys so that their walls converged almost directly to the eyes. The results for these observers are shown in chart 3, *A* and *B*. According to Luneburg's analysis, the curvature of the visual space in these cases may be considered to approximate that of Euclidean geometry, K being slightly less than zero in one case and slightly greater in the other. Whether the metric of visual space is Euclidean in these cases would have to be determined by other experiments which do not involve an acquired technic for the estimation of size and parallelism.

The experimental results of the equidistant and parallel alleys when plotted in cartesian coordinates (x, y), as in charts 1 to 3, give some evidence of the geometric character of the visual space. However, in order to facilitate the mathematical investigation of the relationship of visual space to physical space, Luneburg has suggested a special bipolar system of angular coordinates (γ,ϕ,θ) which is more closely related to the physiological aspects of binocular vision. In this system γ represents the bipolar parallax; ϕ, the bipolar latitude, and θ, the angle of elevation for any point.

An evaluation of the angular coordinates γ_0 and ϕ_0[8] for the fixed end point of the alleys (station 1) was obtained for the experimental values x and y by the relationships

$$\tan \gamma_0 = \frac{2\,x_0}{x_0^2 + y_0^2}$$

and

$$\tan \phi_0 = \frac{y_0}{x_0}$$

where x_0 and y_0 represent the experimental x and y of the fixed end point corrected for the interpupillary distance of the observer. That is, $x_0 = \frac{x}{\frac{1}{2}\text{ P.D.}}$ and $y_0 = \frac{y}{\frac{1}{2}\text{ P.D.}}$.

It was essential for the evaluation of the personal constants, sigma (σ) and K, that the relationship be obtained between $d\,\gamma$ and $d\,\phi$, as it is shown by the equidistant and parallel alleys of the observer. This relationship was found from the graph by drawing the best tangent to the more remote parts of each curve and obtaining the two intersection points on the y axis. These values, corrected for the interpupillary distance of the observer, were called a_D and a_P, respectively.

8. We have assumed that $\angle\,\theta = 0$, as a first approximation, although in our experiments the eyes were adjusted slightly above the level of the lights to avoid, as far as possible, double images.

By the use of the preceding relationships, sigma (σ) and K were evaluated by the equations [7]

$$\sigma = \frac{1}{2 \tan \phi_0} \sqrt{a_D\, a_P}$$

and

$$K = e^{2\sigma\gamma_0} \left[\frac{\sqrt{\frac{a_P}{a_D}} - 1}{\sqrt{\frac{a_P}{a_D}} + 1} \right]$$

In the table are given the values of sigma σ and K for 15 observers whose results were adequate for computation of these personal constants.

Luneburg's Personal Constants σ and K as Determined by the Alley Experiment *

Observer	σ	K
1	9.0	− 0.73
2	55.0	− 0.63
3	43.0	− 0.62
4	20.6	− 0.35
5	25.3	− 0.23
6	24.5	− 0.19
7	45.0	− 0.18
8	28.4	− 0.18
9	7.1	− 0.06
10	11.4	+ 0.08
11	46.5	+ 0.17
12	19.4	+ 0.21
13	69.1	+ 0.26
14	73.6	+ 0.37
15	48.5	+ 0.47

* Values for 15 observers whose results were adequate for computation.

For an evaluation of the constants shown in the table, it should be pointed out that they are based on the average of the results obtained with the original instructions and that in many cases reproducibility was poor. Some observers tended to change the criterion of the judgment of apparent parallelism and equidistance on successive trials of the experiment, often under the influence of the alley previously formed. In some of these the apparently parallel alley was set wider, and in others it was set narrower with repetition, while the equidistant alley was most frequently set wider on successive trials, particularly at the near stations.

It is interesting to note that Blumenfeld also encountered these difficulties. His results are based on a large number of trials, some of which are not listed in his tables and some of which are listed but not included in the average computations. Often he was obliged to discuss the instructions with the observer and to change them before the expected results were obtained. For example, with the type of observer who set apparently parallel walls as they would appear if they were physically parallel, he added, during a discussion of the appearance of lights in a subway, the instruction to set the walls to appear "perpendicular to the frontal plane." His observers were usually required to set a consecutive series of parallel alleys, then a series of distance alleys, then, again, the parallel alley. Frequently he found that the second group of parallel alleys was set wider than the first group because, he believed, of the influence of the distance judgment. Instances of this type are given in his tables but were not averaged in the results.

CONCLUSIONS

The results obtained have led us, in consultation with Luneburg and Boeder, to conclude that the alley experiment presents insuperable difficulties to practical application. Some of our reasons for this conclusion are as follows: 1. The inability of some observers to follow instructions. For example, the instruction to set the walls of the alley so that they appear parallel often resulted in a setting of the walls as they would appear if they were physically parallel. 2. The tendency of some observers to change their criterion for the judgment of "parallelism" and equidistance" on successive trials of the experiment. 3. The strong influence on these judgments of the prior associations, experience and judgment habits of the observer. This influence was particularly evident with observers who had either mechanical experience or artistic training. For some observers of this type it is obvious that frequent discussions of the criteria employed in making their judgments and many repetitions of the experiments would be necessary before the judgments could be stabilized so that reproducible values of the constants could be obtained.

Our conclusion is, therefore, that the alley experiment is inadequate as a test of the personal constants, sigma (σ) and K, of an observer's visual space because of the effect on the results of preconception, experience and judgment habits of the observer. We are forced to this conclusion in spite of the fact that the apparently parallel and apparently equidistant alleys of Blumenfeld demonstrate, perhaps as clearly as any other spatial phenomenon, that the metric of the visual space is that of non-Euclidean geometry. It is our hope and belief that other procedures can be devised which will present to the observer spatial judgments which are outside the realm of his previous experience and associations. Work is under way with several procedures of this type.

23 East Seventy-Ninth Street.