ON THE STABILITY OF TWO-DIMENSIONAL CONVECTION IN A LAYER HEATED FROM BELOW

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1. Introduction. The problem of convection in a layer heated from below has received considerable attention since it is a simple example of fluid motion generated by the instability of a basic state. This instability occurs when the Rayleigh number, which describes the balance between the available potential energy and the energy loss due to dissipation, exceeds a critical value R_c.

For Rayleigh numbers close to the critical value it has been shown by Schlüter, Lortz and Busse (1965)[1] that among all possible convective motions only the two-dimensional convection in the form of rolls is a stable stationary solution of the equation of motion in the Boussinesq approximation. Because the two-dimensional solution is relatively simple, it has been explored by various methods. Kuo (1961) used an expansion in powers of the convection amplitude. Veronis (1965) and Lortz, in an unpublished work, obtained results up to Rayleigh numbers of about 40 times the critical value in the case of free boundaries by expanding the depending variables in Fourier series. The equations have also been solved numerically by Deardorff (1964) and for a very extended range by Fromm (1965).

With the exception of an attempt by Lortz, the problem of the stability of the two-dimensional solution at higher Rayleigh number has not been attacked. Since instabilities have been observed experimentally by Malkus (1954) and others, the question of stability is of particular physical interest. The mathematical problem, however, is very difficult because in addition to the gravitational cause of instability, the convection flow may become hydrodynamically unstable. Yet in the case of large Prandtl number the hydrodynamic cause of instability can be disregarded, since the corresponding nonlinear terms are small. For this reason we will restrict ourselves in this paper to the limit of infinite Prandtl number in which only the gravitational cause for instability is present. The importance of this type of instability can be shown by the following simple consideration.

The Rayleigh number for a horizontal layer of depth d with temperature difference ΔT between the boundaries is defined by

$$R = \frac{\alpha g \Delta T d^3}{\nu \kappa}$$

where α is the expansion coefficient, g is the gravitational acceleration, ν is the kinematic viscosity and κ is the thermometric conductivity. At high Rayleigh numbers when the heat is transported mainly by convection, the horizontal average of the temperature is almost constant throughout the layer. Only near the boundaries does it change rapidly to satisfy the prescribed values of the tem-

[1] Hereafter referred to as "I".

perature at the boundaries. Neglecting the influence of all fluctuating quantities, one can define a local Rayleigh number R_b for these boundary layers with the thickness δ:

$$R_b = \frac{1}{2} R \left(\frac{\delta}{d}\right)^3$$

Since near the boundary the heat is transported by conduction only, the ratio d/δ is approximately equal to the Nusselt number Nu which is defined as the ratio between the heat transport with convection and the heat transport without convection. Thus the condition for the static stability of the boundary layer $R_b < R_c$ gives a lower bound for the heat transport of a stable convective motion:

$$Nu > \left(\frac{R}{2R_c}\right)^{\frac{1}{3}}$$

In the case of free boundaries this condition seems to be satisfied by the two-dimensional solution. In the more realistic case of rigid boundaries, which impose a larger constraint on the convection, we expect instability due to this rough estimate.

The problem of stability in dependence on the Rayleigh number has not been the only motive for the work described in this paper. Since no horizontal boundaries are prescribed the two-dimensional solution depends on a wave number α corresponding to its spatial periodicity. For supercritical Rayleigh numbers stationary solutions are possible for a certain range of α, depending on the Rayleigh number. It turns out that they are stable only in a much smaller range of α, and it is interesting to compare their stability property with other properties as for example the heat transport.

The following analysis is divided into two parts. In the first part we describe the method for the solution of the stationary convection equations. The method can be called a Galerkin procedure applied to a nonlinear problem since it reduces the nonlinear partial differential equation to a system of nonlinear algebraic equations for the coefficients of a complete set of functions. The algebraic equations are solved numerically. In the second part we analyze the stability of the stationary solution against infinitesimal disturbances which are nonoscillatory with respect to time. This problem leads to a linear homogeneous equation for the disturbances with the growth rate σ as eigenvalue. The solution of this problem can be obtained in close analogy to the stationary problem.

2. The Stationary Solution. For the description of convective motions in a fluid layer heated from below, generally the Navier-Stokes equations of motion are used in the Boussinesq approximation. This takes into account the temperature dependence of the density in the gravity term only. Thus the equations for the velocity vector u_i of the fluid and the heat equation for the deviation θ of the temperature from the static state have the following form in the usual dimensionless units based on the depth d as length scale, $d^2\varkappa^{-1}$ as time scale, and

$R^{-1}\Delta T$ as scale of the temperature:

$$\begin{aligned}\Delta u_i + \lambda_i\theta - \partial_i p &= Pr^{-1}\left(u_j \partial_j u_i + \frac{\partial}{\partial t} u_i\right) \\ \Delta\theta + R\lambda_j u_j &= u_j\partial_j\theta + \frac{\partial}{\partial t}\theta \\ \partial_j u_j &= 0\end{aligned} \tag{1}$$

We assume a system of Cartesian coordinates with x, y in the horizontal dimensions and z in the vertical direction opposite to the direction of the gravity force. In this system the unit vector λ_i has the components (0, 0, 1). The fluid layer is confined between two horizontal solid boundaries at $z = \pm\frac{1}{2}$, on which constant temperatures are prescribed. Hence the boundary conditions of the problem are given by

$$u_i = 0, \qquad \theta = 0, \qquad \text{at } z = \pm\tfrac{1}{2}.$$

The equations (1) contain two different nonlinear terms. The term with the inverse of the Prandtl number as factor describes the momentum advection; the nonlinear term in the second equation gives the divergence of the convective heat flux. The latter term is characteristic for the convection problem, and therefore the essential features of finite amplitude convection are still present in the case of infinite Prandtl number in which the nonlinear term of the first equation in (1) vanishes. In this limiting case it is easily shown that the vertical component of the curl of the velocity vanishes, since the corresponding part of the equation of motion

$$\Delta(\partial_x u_y - \partial_y u_x) = 0$$

together with the boundary condition admits only the vanishing solution. Using this fact, we will write the horizontal components of the velocity as the gradient of a potential $\partial_z v$. Because the vertical component is determined by the continuity equation, the general form of a velocity field with vanishing divergence and vanishing vertical component of the vorticity can be written

$$u_i = \delta_i v$$

with

$$\delta_i = \partial_i\partial_j\lambda_j - \lambda_i\Delta = (\partial_x\partial_z\,,\ \partial_y\partial_z\,,\ -\partial_{xx}^2 - \partial_{yy}^2)$$

Eliminating the pressure term we obtain the following equations for v and θ:

$$\begin{aligned}\Delta\Delta v - \theta &= 0 \\ \Delta\theta - R\,(\partial_{xx}^2 + \partial_{yy}^2)\,v &= \delta_j v\partial_j\theta + \frac{\partial}{\partial t}\theta\end{aligned} \tag{2}$$

with the boundary conditions

$$\theta = v = \partial_z v = 0 \qquad \text{at } z = \pm\tfrac{1}{2}. \tag{3}$$

Since we are interested in stationary two-dimensional solutions of the problem, we will neglect the derivatives with respect to the t- and y-coordinates for the rest of this section. Assuming further that the solution is periodic in the x-direction, we expand θ into a complete set of Fourier modes which satisfy the boundary condition for θ.

$$\theta = \sum_{\lambda,\nu} b_{\lambda\nu} e^{i\lambda\alpha x} \sin \nu\pi(z + \tfrac{1}{2}) \tag{4}$$

The summation runs through all integers $-\infty < \lambda < \infty$, $1 \leqq \nu < \infty$ with $b_{\nu\lambda} = \overline{b_{\nu-\lambda}}$. Using the general form (4) for θ we can solve the linear equation (2) for v exactly:

$$v = \sum_{\lambda,\nu} b_{\lambda\nu} e^{i\lambda\alpha x} v_\nu(\lambda\alpha,z)$$

where

$$v_\nu(\lambda\alpha,z) = \frac{\sin \nu\pi(z+\frac{1}{2})}{[(\lambda\alpha)^2 + (\nu\pi)^2]^2} + \frac{\nu\pi}{[(\lambda\alpha)^2 + (\nu\pi)^2]^2} \begin{cases} \dfrac{\frac{1}{2}\sinh \lambda\frac{\alpha}{2} \cosh \lambda\alpha z - z\cosh\lambda\frac{\alpha}{2}\sinh\lambda\alpha z}{-\sinh\lambda\frac{\alpha}{2}\cosh\lambda\frac{\alpha}{2} - \lambda\frac{\alpha}{2}} & \text{(for odd } \nu) \\[2ex] \dfrac{\frac{1}{2}\cosh \lambda\frac{\alpha}{2} \sinh \lambda\alpha z - z\sinh\lambda\frac{\alpha}{2}\cosh\lambda\alpha z}{\sinh\lambda\frac{\alpha}{2}\cosh\lambda\frac{\alpha}{2} - \lambda\frac{\alpha}{2}} & \text{(for even } \nu) \end{cases}$$

is the solution of

$$[\partial_{zz}^2 - (\lambda\alpha)^2]^2 v_\nu(\lambda\alpha,z) = \sin \nu\pi(z + \tfrac{1}{2})$$

with the boundary condition (3) for v_ν in place of v.

In order to determine the unknown coefficients $b_{\lambda\nu}$ we multiply the remaining second equation in (2) by $e^{-i\mathcal{K}\alpha x} \sin \mu\pi(z + \frac{1}{2})$ and take the average over the fluid layer. Using for $\varkappa$ and μ all integers in the range of λ and ν, we obtain an infinite set of algebraic equations for the coefficients $b_{\lambda\nu}$ as functions of the Rayleigh number R. The analysis of these equations can be simplified because of the symmetry of the equation. The symmetry of the equation with respect to x allows us to assume the same symmetry for the solution. Furthermore, the nonlinear part is quadratic and antisymmetric in z. Because of this fact the complete set of equations contains a subset of equations in which only coefficients with even $\lambda + \iota$ appear. We will restrict ourselves to this subset of equations since the corresponding subset of solutions contains all possible solutions at low Rayleigh number, i.e., below the critical Rayleigh number of solutions corresponding to a layer with half the depth of the given layer.

Since no method is known to solve the infinite system analytically, we have to approximate the solution by the solution of a finite system which can be solved numerically. It turns out to be appropriate to reduce the infinite system by omitting all modes with $|\lambda| + \nu > N$, where N is a positive even integer. We will regard a solution of this finite system as a satisfactory approximation if it differs by a sufficiently small amount from the solution obtained with $N + 2$ instead of N. As a measure for the quality of an approximation we introduce the convective heat transport. This quantity is given by the average value of $-\partial_z\theta$ at the boundary

$$H = -\sum_{\nu=1}^{\infty} \nu\pi b_{0\nu}$$

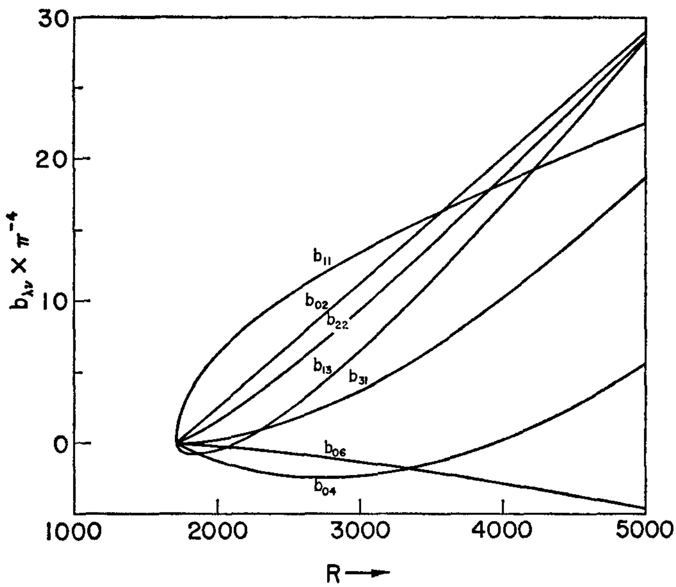

FIG. 1: The dependence of the Fourier components of θ on the Rayleigh number for $\alpha = 3.117$.

and is very sensitive to the influence of the higher modes. We will follow Veronis (1965) in regarding the approximation as satisfactory if the corresponding approximation of the heat transport

$$H_N = -\sum_{\nu=1}^{N} \nu\pi b_{0\nu}$$

differs from H_{N+2} by less than 1%. Since the number of equations increases rapidly with N and the results show a strong convergence of H_N, the deviation from the exact value H is of the same magnitude. The results of the calculations are shown in several figures. Figures 1 and 2 show the dependence of the lowest Fourier components $b_{\lambda\nu}$ on the Rayleigh number. It is interesting to note that they vary approximately as powers of $R - R_c$. The convective heat transport approaches the power law

$$\frac{H}{R - R_c} = \left(\frac{R - R_c}{645.3}\right)^{.250}$$

We have plotted the results only for $\alpha = 3.117$ which is the value of the solution at the critical Rayleigh number $R_c = 1707.8$. The dependence of the Fourier components for other values of α is qualitatively the same. As an example the dependence on α for the Rayleigh number 10,000 is shown in Figure 3. In slight violation of our criterion above, we used data obtained with $N = 6$ in the case of $R = 5{,}000$, for which H_8 differs from H_6 by 1.1%. This is indicated in Table I where we give values of the heat transport for different values of α and for dif-

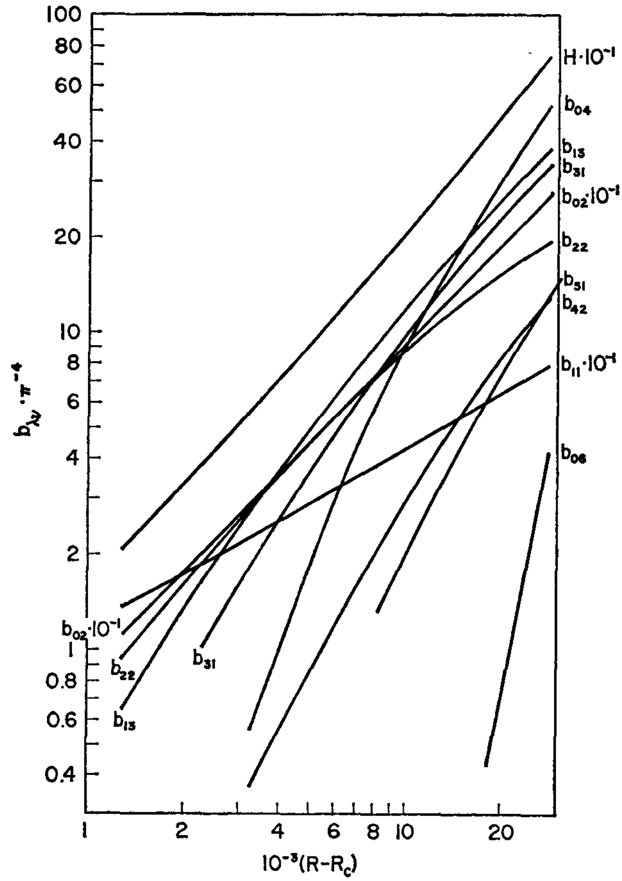

FIG. 2: The Fourier components of θ in dependence of $R - R_c$ for $\alpha = 3.117$

ferent approximations. They show that the value α of the solution with maximum heat transport increases with R.

3. Stability Analysis. The method of solution for the stationary problem which was described in the last section has the advantage that it can be easily combined with a stability analysis. For the determination of the stability of a stationary solution we superpose infinitesimal disturbances. If any disturbance exists with growing time dependence, the stationary solution is unstable; otherwise we

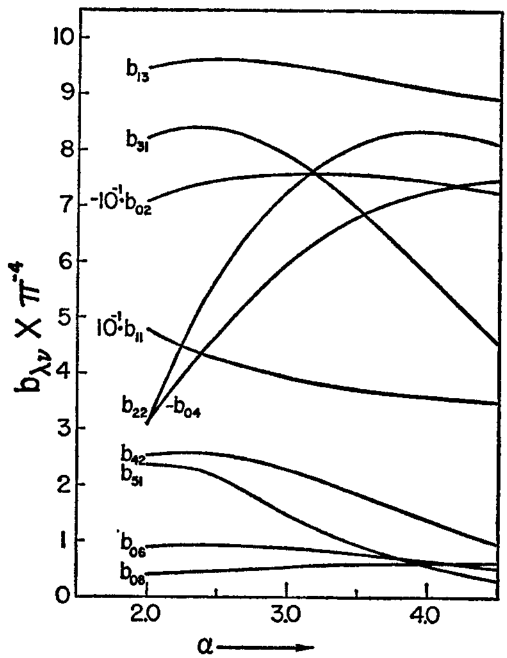

FIG. 3: The Fourier components of θ at $R = 10000$

TABLE I
CONVECTIVE HEAT TRANSPORT H_N

R	N	$\alpha = 2.4$	$\alpha = 2.7$	$\alpha = 3.117$	$\alpha = 3.6$	$\alpha = 4.0$	$\alpha = 4.5$
2000	6			430.5	352.5		
2500	6			1202.8	1120.5		
3000	6			2013.5	1953.6	1776.4	1403.3
5000	6		5420.3	5614.5	5627.1	5483.4	
	8			5551.1			
10000	6			16595			
	8	15234	15761	16233	16482	16483	16244
	10			16133			
20000	10	40552	41859	42272	43260	43669	
30000	10			72802	74969	76022	
	12				74431		

will regard it as stable. The equations for the infinitesimal disturbances $\tilde{v}$, $\tilde{\theta}$ follow from eq. (2):

$$\Delta\Delta\tilde{v} - \tilde{\theta} = 0$$
$$\Delta\tilde{\theta} - R(\partial_{xx}^2 + \partial_{yy}^2)\tilde{v} = \delta_j v \partial_j \tilde{\theta} + \delta_j \tilde{v} \partial_j \theta + \frac{\partial}{\partial t}\tilde{\theta} \qquad (6)$$

In order to make a complete stability analysis we have to admit disturbances

$\tilde{\theta}$ with arbitrary dependence on the three spatial coordinates restricted only by the boundary condition. However, since the equations (6) are linear differential equations with constant coefficients with respect to the time and the y-coordinate and with periodic coefficients with respect to the x-coordinate, the general solution can be written as a sum of solutions which depend exponentially on the three coordinates multiplied by a function of x with the same periodicity as the stationary solution. Hence it is sufficient to discuss disturbances of the general form

$$\tilde{\theta} = \left(\sum_{\lambda\nu} \tilde{b}_{\lambda\nu} e^{i\lambda\alpha x} \sin \nu\pi(z + \tfrac{1}{2}) \right) e^{i(dx+by)+\sigma t}$$

where the summation runs through all integers $-\infty < \lambda < \infty$, $1 \leqq \nu < \infty$. As in the stationary problem we satisfy the first equation in (6) and the boundary conditions for $\tilde{v}$ exactly by writing

$$\tilde{v} = \left\{ \sum_{\lambda\nu} \tilde{b}_{\lambda\nu} e^{i\lambda\alpha x} v_\nu \left(\sqrt{(\lambda\alpha + d)^2 + b^2}, z \right) \right\} e^{i(dx+by)+\sigma t}$$

By multiplying the second equation in (6) with $\sin \mu\pi(z + \frac{1}{2}) e^{-i(\varkappa\alpha x+dx+by)-\sigma t}$ and averaging it—the indices $\varkappa$, μ, running through the same range as the indices λ, ν—we transform the equation into a system of linear equations for the coefficients $\tilde{b}_{\lambda\nu}$. Because of the symmetry of the stationary solution the system separates into a system for the coefficients with even $\lambda + \nu$ and a system with odd $\lambda + \nu$. To approximate these infinite systems by a finite number of equations, we neglect—as in the case of the stationary solution—all modes with

$$|\lambda| + \nu > N.$$

Thus we have reduced the stability problem to the determination of the eigenvalue σ with the highest real part for two finite systems of equations.

Since the stability problem not only depends on the parameters R and α introduced by the stationary solution but also on the free parameters b and d, the amount of numerical calculations is still considerable. For this reason we calculate only the eigenvalue with the lowest absolute value using an iterative method [see for example Zurmühl (1964)]. Since it is known from the analysis in I that stable two-dimensional solutions exist in the neighborhood of the critical Rayleigh number, this method determines the growth rate with the highest real part as long as this growth rate is real.

It seems unlikely that complex growth rates with higher real parts exist because all eigenvalues are real in the neighborhood of the critical Rayleigh number according to the analysis in §I, and because the calculated eigenvalues turn out to be real in all cases. We can not exclude, however, this possibility, and hence the stability analysis is incomplete in this sense. For simplicity we will use the term "stability" in the further discussion as an abbreviation for "stability with respect to disturbances with nonoscillatory time dependence".

Since the calculations showed in all cases that the most critical disturbances with the highest value of the growth rate for given values of R, α, b, correspond to vanishing d, we can neglect the parameter d in the further discussion.

The range of stability of the stationary solution is mainly limited by the solutions of the system with odd $\lambda + \nu$. The highest growth rates in this case are plotted in Fig. 4 with horizontal Rayleigh numbers as labels. The value b of the most critical disturbance is approximately constant for a given Rayleigh number and varies from 3.117 at the critical Rayleigh number to 4.5 at the second critical Rayleigh number at which the stability range of the stationary solution closes. The amount of the higher modes in the disturbances increases more rapidly than in the stationary solution, indicating the destabilizing influence of the temperature boundary layer. The calculation of the growth rate nevertheless remains sufficiently accurate and differs by less than .05 when N is replaced by

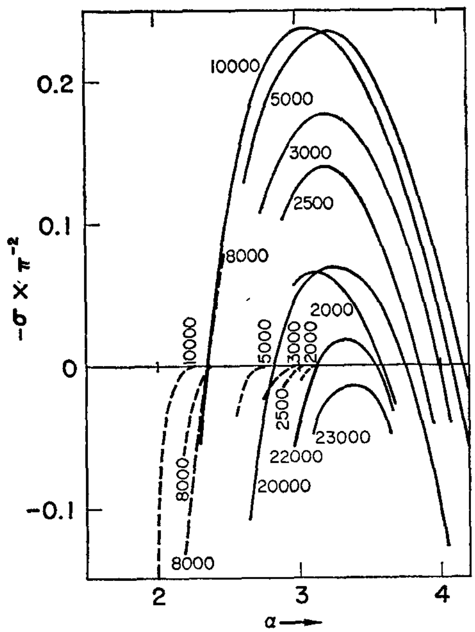

FIG. 4: The growth rates of disturbances with odd $\lambda + \nu$ (solid lines) and with even $\lambda + \nu$ (dashed lines).

$N + 2$. The corresponding difference for the second critical Rayleigh number can be used as an error estimate:

$$R_2 = 22600 \pm 100$$

Although the calculations have been carried out for selected values of R and α only, we have plotted the range of stability in Fig. 5 because the results seem to depend very smoothly on the parameters.

The stability boundary for $\alpha < \alpha_c$ and Rayleigh numbers less than 8000 is formed by solutions of the system with even $\lambda + \nu$. An exact solution of this system is given by $\sigma = 0$, $\tilde{v} = \partial_x v$, $\tilde{\theta} = \partial_x \theta$, which follows by differentiation of the stationary equations. The solutions with positive growth rate are adjacent to this exact solution. Since the growth rate and the value of b for the most critical disturbance increase quadratically from zero after crossing the stability boundary, the calculations in this case are less accurate. The growth rates for these disturbances are plotted in Fig. 4 with dashed lines. The connection with the exact solution shows that this type of instability tends to establish a roll solution of higher wave number. On the other side the relatively high wave number b of the critical disturbances with odd $\lambda + \nu$ at higher Rayleigh numbers indicates that this kind of instability leads to a new three-dimensional form of convection. In the neighborhood of the critical Rayleigh number, R_c, the form of the stability region is compatible with the prediction of the theory in I.[2]

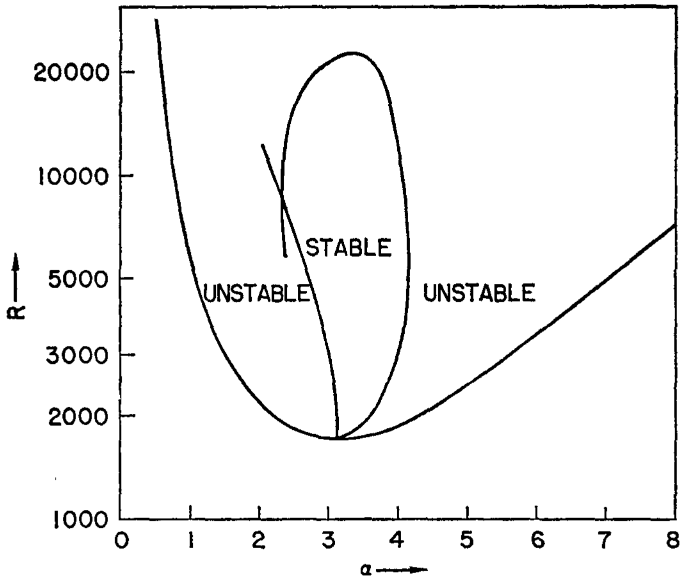

FIG. 5: The region of stability of two-dimensional convection with respect to disturbances with non-oscillatory time dependence.

4. Conclusion. The results of the stability analysis show that stationary two-dimensional solutions are stable in a small part of the wave-number range of possible solutions up to the Rayleigh number 22600 where all two-dimensional solutions become unstable. The value of the parameter α for the most stable solution remains essentially constant with increasing Rayleigh number. Like the stationary solution with maximum heat transport, however less pronounced, the most stable solution has a slightly increasing wave number α. A third physically distinguished solution, which shows a similar dependence on the wave number α, is the solution with minimum potential energy. Besides these analogies, however, no direct connections can be drawn between the property of stability and other simple physical properties.

Although the limit case of infinite Prandtl number can only be approximated in experiments, some evidence seems to exist for a transition corresponding to the

[2] Figure 1 in I has been drawn incorrectly. The right side of the stability boundary is given by a parabola instead by a straight line.

second critical Rayleigh number. Heat transport measurements by Malkus (1954) indicate transitions either at the Rayleigh numbers 11000 and 26000 or at 18000. The optical observations by T. Rossby (1966) show a change from two-dimensional flow to three-dimensional flow at Rayleigh numbers of about 20000. To confirm this evidence, however, more experiments are necessary.

5. Acknowledgments. The author wishes to thank Professors C. C. Lin and L. N. Howard for making possible his visit to Massachusetts Institute of Technology. The facilities of the M.I.T. Computation Center were used for the calculations.

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(Received July 13, 1966)