The indicated limit of error in this latter value is the standard deviation derived from seven measurements.

The fact that the difference $_{83}Bi^{209}-{}_{82}Pb^{208}$ is 5 mMU larger than unity indicates a sharp increase in the slope of the packing fraction curve. This agrees with the expectation since Bi^{209} has one proton more than the magic number 82. The addition of this single proton adds, in this case, only 3 Mev to the binding energy of the nucleus. This result is in reasonable agreement with the difference of 1.004 mass units derived from the disintegration data of Harvey.[5]

[5] J. A. Harvey, Phys. Rev. **81**, 353 (1951).

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Divergence of Perturbation Theory in Quantum Electrodynamics

F. J. Dyson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

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An argument is presented which leads tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge. The divergence in no way restricts the accuracy of practical calculations that can be made with the theory, but raises important questions of principle concerning the nature of the physical concepts upon which the theory is built.

ALL existing methods of handling problems in quantum electrodynamics give results in the form of power-series in e^2. The individual coefficients in these series are finite after mass and charge renormalization. The technique of renormalization can at present be applied only to the separate coefficients, and not to the series as a whole. If the series converges, its sum is a calculable physical quantity. But if the series diverges, we have no method of calculating or even of defining the quantity which is supposed to be represented by the series.

Several authors have remarked[1] that the series after renormalization will be divergent in a trivial way, if the series represents a scattering amplitude of a free particle, in circumstances where the particle has a possibility of being captured into a permanently bound system. In this situation a perturbation expansion of the scattering amplitude will diverge, even in nonrelativistic quantum mechanics,[2] and in the relativistic theory the series will diverge for the same reason. It is to be expected that such trivial divergences will not impose any fundamental limitations on the use of the renormalization method. In fact, a new method of carrying through the renormalization program has been developed,[3] a method which is applicable to problems involving bound systems and in which divergences of this elementary nature cannot occur. In the new method the series expansion arises from a formal integration of the equations of motion over a finite interval of time, and in an elementary nonrelativistic theory such a perturbation expansion would necessarily be convergent. For this reason it was claimed as probable[4] that the power series arising from the application of the new method in quantum electrodynamics would always converge. If the claim had been accompanied by a proof of convergence, then the theoretical framework of quantum electrodynamics could have been considered closed, being within its limits a complete and consistent theory.

The purpose of this note is to present a simple argument which indicates that the power-series expansions obtained by integrating the equations of motion in quantum electrodynamics will be divergent after renormalization. The divergence is of a basic character, different from the trivial divergences mentioned above, and is present equally in the results obtained from the new and the older methods of calculation. The argument here presented is lacking in mathematical rigor and in physical precision. It is intended only to be suggestive, to serve as a basis for further discussions. To me it seems convincing enough to merit publication in its present incomplete form; also I am glad to have this opportunity to withdraw the erroneous argument previously put forward[5] to support the claim that the power series should converge.

The argument for divergence is as follows. According to Feynman,[6] quantum electrodynamics is equivalent to a theory of the motion of charges acting on each other by a direct action at a distance, the interaction between two like charges being given by the formula

$$e^2\delta_+(s_{12}^2), \tag{1}$$

where e is the electron charge. The action-at-a-distance formulation is precisely equivalent to the usual formulation of the theory, in circumstances where all emitted radiation is ultimately absorbed. We shall suppose that

[1] B. Ferretti, Nuovo cimento **8**, 108 (1951); K. Nishijima, Prog. Theor. Phys. **6**, 37 (1951).

[2] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

[3] F. J. Dyson, Proc. Roy. Soc. (London) **A207**, 395 (1951). Phys. Rev. **83**, 608, 1207 (1951).

[4] Phys. Rev. **83**, 608 (1951), Section XII.

[5] See reference 4. The error in the argument lay in using the concept "the number of times that an interaction operates" in an intuitive and imprecise way.

[6] R. P. Feynman, Phys. Rev. **76**, 769 (1949), Eq. (4); Phys. Rev. **80**, 440 (1950), Appendix B.

conditions are such as to justify the use of the Feynman formulation of the theory. Then let

$$F(e^2)=a_0+a_2e^2+a_4e^4+\cdots \quad (2)$$

be a physical quantity which is calculated as a formal power series in e^2 by integrating the equations of motion of the theory over a finite or an infinite time. Suppose, if possible, that the series (2) converges for some positive value of e^2; this implies that $F(e^2)$ is an analytic function of e at $e=0$. Then for sufficiently small values of e, $F(-e^2)$ will also be a well-behaved analytic function with a convergent power-series expansion.

But for $F(-e^2)$ we can also make a physical interpretation. Namely, $F(-e^2)$ is the value that would be obtained for F in a fictitious world where the interaction between like charges is $[-e^2\delta_+(s_{12}^2)]$ instead of (1). In the fictitious world, like charges attract each other. The potential between static charges, in the classical limit of large distances and large numbers of elementary charges, will be just the classical Coulomb potential with the sign reversed. But it is clear that in the fictitious world the vacuum state as ordinarily defined is not the state of lowest energy. By creating a large number N of electron-positron pairs, bringing the electrons together in one region of space and the positrons in another separate region, it is easy to construct a "pathological" state in which the negative potential energy of the Coulomb forces is much greater than the total rest energy and kinetic energy of the particles. This can be done without using particularly small regions or high charge densities, so that the validity of the classical Coulomb potential is not in doubt. Suppose that in the fictitious world the state of a system is known at a certain time to be an ordinary physical state with only a few particles present. There is a high potential barrier separating the physical state from the pathological states of equal energy; to overcome the barrier it is necessary to supply the rest-energy for the creation of many particles. Nevertheless, because of the quantum-mechanical tunnel effect, there will always be a finite probability that in any finite time-interval the system will find itself in a pathological state. Thus every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization. In these circumstances it is impossible that the integration of the equations of motion of the theory over any finite or infinite time interval, starting from a given state of the fictitious world, should lead to well-defined analytic functions. Therefore $F(-e^2)$ cannot be analytic and the the series (2) cannot be convergent.

The divergence of the series in the real world is associated with virtual processes in which large numbers of particles are involved. Therefore the divergence will only become noticeable when terms of very high order in the expansion (2) are considered. A crude quantitative estimate indicates that the terms of (2) will decrease to a minimum and then increase again without limit, the index of the minimum term being roughly of the order of magnitude 137. This estimate assumes the system to be such that the trivial kind of divergence discussed earlier does not occur. The non-trivial and unavoidable divergence will not prevent practical calculations being made with the series (2), to an accuracy far beyond anything at present required or contemplated. Only if similar arguments should be found to be applicable to meson theory, the divergence might impose a severe limitation on the possible accuracy of practical calculations in that field.[7]

If the conclusion of the foregoing argument is accepted, then there are two alternative possibilities for the future development of quantum electrodynamics. Alternative A: There may be discovered a new method of carrying through the renormalization program, not making use of power series expansions. In this case every physical quantity $F(e^2)$ will be well-defined and calculable, and the series (2) will be an asymptotic expansion for it in the limit of small e. Since $F(e^2)$ is not analytic at $e=0$, the asymptotic expansion will not be sufficient to determine the function uniquely. The additional information necessary to determine $F(e^2)$ will be obtained from the existing formalism, using no new physical hypotheses but only some improved mathematical methods. Alternative B: All the information that can in principle be obtained from the formalism of quantum electrodynamics is contained in the coefficients a_0, a_2, a_4, $\cdots$ of series such as (2). In this case the quantity $F(e^2)$ is neither physically well-defined nor mathematically calculable, except in so far as the asymptotic expansion (2) gives some workable approximation to it. In order to define $F(e^2)$ precisely, not merely new mathematical methods but a new physical theory is needed.

I wish to call attention to the attractive features of alternative B in the present state of physics. If B were true, it would imply that quantum electrodynamics is in its mathematical nature not a closed theory, but only a half-theory giving insufficient information for the exact prediction of events. Experimentally we know that the world contains one group of phenomena which is accurately in agreement with the results of quantum electrodynamics, and another group of phenomena which is not understood at all. We need to develop new physical ideas to understand the second group, and still we cannot abandon the theory which successfully accounts for the first. If quantum electrodynamics were a closed theory, this would be a difficult dilemma. But if the theory itself leaves room for new ideas, no such dilemma arises. In conclusion, I wish to thank Professors Pauli, Bethe, Pais, and Oppenheimer for valuable discussions of these problems.

[7] C. A. Hurst in a private communication informs me that he has discovered by direct calculation the fact that the S-matrix diverges in the way here described, in the case of a simple scalar meson theory, assuming that certain terms which are not yet calculated do not decisively change the behavior of the series.