Translation:On the Dynamics of Moving Systems

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By MAX PLANCK.

Introduction.

Recent researches in the field of thermal radiation lead to the conclusion (from the experimental as well as from the theoretical side), that a system devoid of ponderable matter and consisting only of electromagnetic radiation, obeys the basic laws of mechanics as well as the two laws of thermodynamics in such a complete way, that for all consequences drawn from it one has nothing left to wish for. Thus it has become necessary, that a number of ideas and laws which are usually considered as fixed and self evident conditions of all theoretical speculations in that field, are subject to a principle revision, and further consideration shows that some of the simplest and most important of them can in future only be characterized as approximations (though far reaching and practically very important), but in no way can be considered as exactly valid. Some examples will substantiate this in detail.

We are accustomed to regard the whole energy of a moving ponderable body as additively composed of a part, which varies only (regardless of the internal state of the body) with its speed: the energy of the kinetic motion, and a second part which (regardless of the speed) only depends on the internal state, namely on the density, temperature and chemical composition: the internal energy of the body. This decomposition is from now on, even principally, not allowed in any single case. For every ponderable body contains in its interior a specifiable finite amount of energy in the form of radiant heat, and if the body is imparted a certain velocity, then the heat radiation is also set into motion. As regards moving heat radiation, a separation of energy into internal and progressive energy is quite impossible, although its energy notably depends on the speed of motion; therefore such a separation of the total energy is not feasible. Although the internal radiation energy might have superiority in most cases over other energy forms, the latter will be nevertheless always present in detectable quantities and under well realizable circumstances even of the same order.

For gaseous bodies its amount is most notable. Take for example an ideal one-atomic gas at rest under the pressure p at temperature T, then the radiation energy within the gas is aVT^4, that is in absolute C.G.S. system, $a = 7{,}061 \cdot 10^{-15}$ and $V = \frac{RNT}{P}$ (N is the number of moles, $R = 8.31 \cdot 10^7$). In contrast, the internal energy of the gas, so far as it arises from the kinetic energy of molecular motions, is: Nc_vT+ const., where the molar heat c_v at constant volume in the same system of units is equal to $3 \cdot 4.19 \cdot 10^7 = 1.257 \cdot 10^8$. So if we supply heat to the gas from the outside at constant volume, then heat is distributed over the two mentioned energy forms in the ratio:

$$\frac{4aVT^3}{Nc_v} = \frac{4aRT^4}{c_v p}.$$

For 0.001mm of pressure at the temperature of melting platinum, in absolute measure $p = 1{,}33$ and $T = 1790 + 273 = 2063$, this proportion will be equal to 0.25 by using the given numbers; that is to say, in a heated one-atomic gas (by using the assumed values of pressure and temperature) the heat which causes an increase of radiation energy is already the fourth part of the heat due to molecular motions.

Another example concerns the *inertial mass* of a body. The concept of mass as an absolutely immutable quantum, neither modifiable by any physical nor by chemical influences, belongs to the foundations of mechanics since Newton. It appears that we can attribute constancy to it before all other quantities: it is what until very recently, even in Hertz's mechanics, was considered to be the fundamental property of matter and therefore is used in almost every physical world system as the first building stone. However, it can now be proven that the mass of each body depends on temperature. Because inertial mass is defined most directly by the kinetic energy. But since it is, as shown earlier, impossible to separate the energy of the kinetic motion of a body from its internal state, it follows immediately that a constant with the properties of inertial mass may not exist. The reason for this lies again in the internal energy of thermal radiation which contributes to the inertia of a body in a small but determinable way, i.e., by a term which depends on the radiation density or temperature. If, however, we want to define mass rather by momentum than by kinetic energy, namely as the ratio of momentum by velocity, we obtain no different result. For according to the investigations of H.A. Lorentz, H. Poincaré and M. Abraham, the internal heat radiation of a moving body, as well as in general any electromagnetic radiation, has a certain finite momentum included in the whole momentum of the body. However, it depends (as the radiation energy) on temperature, and consequently also on the mass defined by it.

The alternative, which is to distinguish between "real" and "apparent" mass and to attribute constancy only to the former, represents the same facts only in a modified formulation. While the "real" mass would now remain constant, it loses on the other side its previous significance for kinetic energy and momentum.

After this consideration a third example immediately follows, namely the question of the identity of *inertial* and *ponderable* mass. The thermal radiation in a fully evacuated space, bounded by reflecting walls, surely has inertial mass; but has it also ponderable mass? If this question is to be denied, which surely should be the obvious choice, then it seems that the identity of inertial and ponderable mass, which was confirmed by all previous experiences and was generally accepted, must be abolished. We must not object that the inertia of black cavity radiation is imperceptibly small compared to that of the limiting material walls. On the contrary: by a sufficiently large cavity volume, the inertia of radiation can be made arbitrarily great against that of the walls. Such a freely moving cavity radiation, separated form outer space by thin rigid reflective walls, provides a good example of a rigid body, whose laws of motion completely differ from those of ordinary mechanics. Although it differs, considered superficially, in no way from other rigid bodies and also possesses a certain inertial mass and obeys the law of inertia, its mass changes significantly with temperature and it also depends in a certain specifiable way from the magnitude of the velocity as well as from the direction of the moving force in respect to the velocity. However, there is absolutely nothing hypothetical about those properties, as they can be quantitatively derived from known laws in all details.

Given the situation described, by which some views and theorems are stripped of its general nature, hitherto considered to be the strongest support for the usual theoretical considerations of any kind, it must appear as a task of particular importance to single out and especially put into the foreground, those theorems that previously formed the bases of general dynamics, and which also proved absolutely accurate in the light of the results of recent research; for they alone will henceforth be entitled to find use as the foundations of dynamics. But it should of course not be said that the above theorems, noticeably marked as inexact, were to put out of use in future; for in the vast majority of cases, the enormous practical importance of the decomposition of energy in internal and kinetic, or the adoption of the absolute invariance of mass, or the condition of identity of inertial and ponderable mass, is indeed not affected at all by the considerations advanced, and we will never come in a position to dispense with those considerable simplifying assumptions. But from the standpoint of the general theory we must unconditionally and principally distinguish between such theorems, which can be regarded only as approximations, and those which claim exact validity, because today it is unknown to which consequences the further development of the exact theory will lead us: far reaching revolutions, also in practice, have often enough started with discoveries of almost imperceptibly small inaccuracies within a theory that was previously considered as generally exact.

If we therefore ask about the really exact basis of general dynamics, of all known theorems only the principle of least action remains at first, which includes, as it was proved by H. von Helmholtz[1], mechanics, electrodynamics and the two laws of thermodynamics in its application on reversible processes. The fact that the same principle is also contained in the laws of moving cavity radiation, I have especially shown in the following (see below (II (12)). But the principle of least action is not sufficient for a complete foundation of the dynamics of ponderable bodies; because by itself it provides no replacement - which has proven above to be untenable and which should not be introduced here - for splitting the energy of a body in kinetic and internal energy. On the other hand, such a replacement is in prospect in a complete way for the introduction of another theorem: the "*principle of relativity* as expressed by H. A. Lorentz[2] and in the most general form by A. Einstein[3]. Although only a single direct confirmation of the validity of this principle, yet very important, is to be mentioned: the result of the experiments of Michelson and Morley[4], yet on the other hand no fact is known so far, directly preventing us to ascribe general and absolute accuracy to that principle. On the other hand, the principle proves to be so pervasive and fertile that a possible in depth investigation appears desirable, and this can obviously only be done by examining the consequences which it contains.

Following this consideration, I felt it a worthwhile task to develop the conclusions, leading to a combination of the principle of relativity with the principle of least action for any ponderable body. On that occasion some other views have been obtained, as well as some conclusions that may be accessible for direct experimental confirmation.

First Section. Dynamics of a moving black cavity radiation.

§ 1.

The black-body radiation in pure vacuum is of all physical systems the only one, in which thermodynamic, electrodynamic and mechanical properties can be specified with absolute precision, independent of the opposition of special theories. Its treatment is therefore placed in front of the other systems. Imagine a radiation surrounded by vacuum and enclosed in absolutely reflecting moving walls, whose volume V may be chosen so great that the influence of the mass of the walls is too small to be considered. All changes happening in that system we consider as reversible, that is, they happen so slowly that there is in every moment a stationary state. Then the state of the system is completely determined by the speed q, which can be an arbitrarily large fraction of the speed of light c, the volume V and the temperature T. For an infinitesimal change of state, the change in energy E of the radiation is according to the first law of thermodynamics:

$$dE = A + Q.$$

where A is the mechanical work applied from the outside to the radiation, Q is the heat supplied from the outside: and after the second law, S is the change in entropy of the radiation:

$$dS = \frac{Q}{T} = \frac{dE - A}{T}.$$

By aid of the last equation we want to calculate the properties of the radiation in their dependence on the independent variables q, V and T. The energy of the radiation is:

$$E = \varepsilon \cdot V,$$

where ε is the spatial energy density, which depends only on q and T. Moreover, the external work A shall be additively composed of the translation work and the compression work. The former is equal to the product of velocity q and the increase of momentum G, the latter is equal to the product of pressure p and the decrease of volume V, thus:

$$A = qdG - pdV.$$

Now the pressure is[5]

$$p = \frac{c^2 - q^2}{3c^2 + q^2} \cdot \epsilon.$$

Furthermore, the momentum[6]

$$G = \frac{4q\epsilon V}{3c^2 + q^2}.$$

Substituting these values into the expression of A, and the values of A and E into the equation for dS, the latter is as follows:

$$dS = \frac{d(\epsilon V) - qd\left(\frac{4q\epsilon V}{3c^2+q^2}\right) + \frac{c^2-q^2}{3c^2+q^2}\epsilon dV}{T}.$$

The condition that this expression forms a complete differential of the three independent variables q, V and T (bearing in mind that ε only depends on q and T, not on V) gives as a necessary consequence the relations:

$$\epsilon = \frac{ac^4}{3} \cdot \frac{3c^2 + q^2}{(c^2 - q^2)^3} T^4 \tag{1}$$

and

$$S = \frac{4ac^4}{3} \cdot \frac{T^3 V}{(c^2 - q^2)^2} \tag{2}$$

where the constant a is determined by the fact that ε goes over to aT^4 for $q = 0$, which is in accordance with the STEFAN-BOLTZMANN radiation law.

With these values we obtain for the energy E, the pressure p and the momentum G of the moving cavity radiation as functions of the independent variables q, V and T, the following expressions:

$$E = \frac{ac^4}{3} \cdot \frac{3c^2 + q^2}{(c^2 - q^2)^3} T^4 V \tag{3}$$

$$p = \frac{ac^4}{3} \cdot \frac{T^4}{(c^2 - q^2)^2} \tag{4}$$

$$G = \frac{4ac^4 q}{3} \cdot \frac{T^4 V}{(c^2 - q^2)^3} \tag{5}$$

So, for example, if we impart some acceleration to the cavity radiation, while its volume V is kept constant and no heat is supplied from outside so that also the entropy S remains constant, the temperature T of the radiation is decreased by (2) in the ratio

$\left(1-\frac{q^2}{c^2}\right)^{\frac{2}{3}}$: 1. This result as well as various other related theorems are in line with the conclusions to which we are led by the study of K. Mosengeil,[7]. Below (in § 15), an even simpler and more direct derivation will be given.

Second Section. Principle of least action and principle of relativity.

§ 2.

In the following, we consider an arbitrary body in a steady state (consisting of a given number[8] of similar or different types of molecules), determined by the independent variables[9] *V, T* and the velocity components $\dot{x}, \dot{y}, \dot{z}$ of the body along the three axes *x, y, z* of a linear orthogonal reference frame at rest. The magnitude of the velocity *q* is then given by:

$$q^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2.$$

If the state of the body is changed in a reversible manner, then according to H. von Helmholtz[10] the differential equations derived from the principle of least action are given:

$$\frac{d}{dt}\frac{\partial H}{\partial \dot{x}} = \mathfrak{F}_x,\ \frac{d}{dt}\frac{\partial H}{\partial \dot{y}} = \mathfrak{F}_y,\ \frac{d}{dt}\frac{\partial H}{\partial \dot{z}} = \mathfrak{F}_z \qquad (6)$$

and

$$\frac{\partial H}{\partial V} = p,\ \frac{\partial H}{\partial T} = S. \qquad (7)$$

There, *H* is the kinetic potential of the body as a function of the above-mentioned five independent variables, where the velocity components $\dot{x}, \dot{y}, \dot{z}$ only occur in combination with *q*, and $\mathfrak{F}$ is the external moving force acting on the body.

We can use these five differential equations in the definition of the kinetic potential as well; but as we see, the function *H* is still not completely defined by them, because with certain $\mathfrak{F}$ and *p* and *S*, there remains in *H* an additive constant having no physical meaning and which can be arbitrarily determined. Further below (in § 9) we will give a proper disposal of these constant, and hence we will make the necessary complement as to the completion of *H*.

The momentum of the body is then given by the components:

$$\mathfrak{G}_x = \frac{\partial H}{\partial \dot{x}},\ \mathfrak{G}_y = \frac{\partial H}{\partial \dot{y}},\ \mathfrak{G}_z = \frac{\partial H}{\partial \dot{z}} \tag{8}$$

or by the resulting momentum:

$$G = \frac{\partial H}{\partial q} \tag{9}$$

and the total energy of the body by:

$$E = q\frac{\partial H}{\partial q} + T\frac{\partial H}{\partial T} - H = \dot{x}\mathfrak{G}_x + \dot{y}\mathfrak{G}_y + \dot{z}\mathfrak{G}_z + TS - H, \tag{10}$$

whence the equation for the energy principle is given:

$$dE = \mathfrak{F}_x dx + \mathfrak{F}_y dy + \mathfrak{F}_z dz - pdV + TdS, \tag{11}$$

which on its right side contains the translation work, the compression work, and the heat supplied from the outside.

All these relations are also valid, of course, for the special case of pure cavity radiation as discussed in the previous section, as one can easily convince himself if one substitutes in the above equations the value for the kinetic potential:

$$H = \frac{ac^4T^4V}{3(c^2 - q^2)^2} \tag{12}$$

So far, in the application to ponderable bodies it was always proceeded (also by H. *von Helmholtz*) in such a way that the kinetic potential H was split into two parts:

$$H = \frac{1}{2}Mq^2 - F,$$

and it was assumed that the mass of the body M is constant, while the free energy of the body F was assumed to be independent of q. Then equations (6) goes over into the equations of ordinary mechanics, and equations (7) into those of ordinary thermodynamics.

However, as shown by the example of cavity radiation, which has been elaborated above in the introduction, such a decomposition, strictly speaking, cannot be allowed in any single case: for every ponderable body contains in its interior radiant energy in specifiable amount. We therefore don't wish to make here such a decomposition, but instead we want to rely on the principle of relativity and develop its consequences for the considered case.

§ 3.

The principle of relativity says that instead of the previously used reference frame (*x, y, z, t*) we can use with exactly the same justification also the following reference frame:

$$x' = \frac{c(x-vt)}{\sqrt{c^2-v^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{c^2t - vx}{c\sqrt{c^2-v^2}}$$

for the basic equations of mechanics, electrodynamics and thermodynamics, and therefore describe them as "at rest". We want to denote in the following all quantities measured in the new reference frame by a prime, and denote accordingly the two reference systems as "primed" and "unprimed". Then the content of the principle of relativity can also be expressed in this way: All the equations between primed, unprimed or both quantities remain true, if we replace the primed quantities by the unprimed quantities of the same name, and simultaneously replace the unprimed by the primed quantities. And we have to set $c' = c$ and $v' = -v$.

This general theorem, which is of course valid for the defining equations (from above) of the primed coordinates, provides for any relation derived, a reciprocal relation that is often useful for verification.

§ 4.

Now, our next task is to establish the relation between each of the previously used quantities and the primed quantities of the same name. It will be shown that this may be done in a completely unambiguous way, so that we finally, for example, can calculate from the energy of a body at rest in one reference frame, the energy of the same body in the other reference frame, for which it possesses a certain finite speed.

First, for the primed velocity components ($\dot{x}' = \frac{dx'}{dt'}$, etc.) it is found in a purely mathematical way:

$$\dot{x} = \frac{c^2(\dot{x}-v)}{c^2 - v\dot{x}}, \quad \dot{y} = \frac{c\sqrt{c^2-v^2}\,\dot{y}}{c^2-v\dot{x}}, \quad \dot{z} = \frac{c\sqrt{c^2-v^2}\,\dot{z}}{c^2-v\dot{x}}. \qquad (13)$$

Furthermore,[11]

$$\sqrt{\frac{c^2-q'^2}{c^2-q^2}} = \frac{c\sqrt{c^2-v^2}}{c^2-v\dot{x}} = \frac{c^2+v\dot{x}'}{c\sqrt{c^2-v^2}} = \frac{V'}{V} = \frac{dt}{dt'}. \qquad (14)$$

We want to prove now that the *entropy* of the considered body has with respect to the primed system the same value as with respect to the unprimed system. We could found this prove, more generally, on the close connection of entropy with probability, whose quantity can impossibly depend on the choice of the reference frame; however, here we prefer a more direct way, completely independent of the introduction of the concept of probability.

We think of a body brought from a state at rest in the unprimed reference frame, into a second state by any reversible adiabatic process, so that it is at rest in the primed reference frame. If we denote the entropy of the body for the unprimed frame in the initial state by S_2, in the final state by S_2, then because of reversibility and adiabasy $S_1 = S_2$. But also for the primed reference frame the process is reversible and adiabatic, so we also have: $S'_1 = S'_2$.

If S'_1 would not be equal to S_1, but $S'_1 > S'_1$, then this would mean: the entropy of the body for that reference frame for which it is in motion, is greater than for that reference frame for which it is at rest. Then according to this proposition it should be $S_2 > S'_2$ as well; for in the second condition the body rests in the primed reference frame, while it is in motion for the unprimed reference frame. However, these two inequalities contradict the equations stated above. Nor can $S'_1 < S_1$; hence $S'_1 = S_1$, and in general:

$$S' = S, \tag{15}$$

that is, the entropy of the body does not depend on the choice of the reference frame.

§ 5.

Hence it follows the important conclusion: If a body (which in the initial state is at rest in the unprimed system) is brought in any way (reversible and adiabatic) at the speed of $\dot{x} = v,\ \dot{y} = 0,\ \dot{z} = 0$, so that the final volume V_2 is in relation to the initial volume V_1 by the relation:

$$V_2 = V_1 \cdot \sqrt{1 - \frac{v^2}{c^2}}, \tag{16}$$

then the final state 2 for the primed system is identical in all respects to the initial state 1 for the unprimed system.

The correctness of this proposition follows from the consideration that the condition of the body is defined by five independent variables, for which we can choose the volume and entropy in addition to the three velocity components. Now, under those conditions $\dot{x}'_2, \dot{y}'_2$ and $\dot{z}'_2 = 0$ are the 3 velocity components of the body in the final state for the primed system, furthermore by (15) the entropy $S'_2 = S_2 = S_1$, finally the volume by (14):

$$V'_2 = V_2 \frac{c^2 + v\dot{x}'_2}{c\sqrt{c^2 - v^2}} = V_2 \frac{c}{\sqrt{c^2 - v^2}} = V_1,$$

so every 5 condition-variables in the final state 2 for the primed system have the same value as in the initial state 1 for the unprimed system, thus the above theorem is proved.

§ 6.

Now we think of any number of different bodies separated from each other, which initially are at rest for the unprimed system and which all have the same temperature T_1 and are subjected to the same pressure p_1. Each of these bodies will somehow be brought to the speed v in a reversible and adiabatic way, and its final volume will be regulated according to relation (16). Then finally, all bodies have in turn a common temperature T_2 and a common pressure p_2. Because for the primed system every body is finally in the same condition as initially for the unprimed frame, thus for the primed system the final temperatures and the final pressure are all equal. However, the same is true for the unprimed frame; for two bodies, having the same temperature and same pressure for one reference system, i.e., they are in thermal and mechanical equilibrium with each other, have the same property also in every other frame of reference.

Thus we can state the following theorem: Different types of bodies of same temperature and same pressure, which are somehow brought from velocity 0 to velocity v (separately and in a reversible and adiabatic way) so that the volume is diminished by the ratio $\sqrt{1-\frac{v^2}{c^2}}:1$ for any body, will adopt the same temperature and pressure. Therefore, if we know for a single body the change of temperature set forth by such a process, then we know the change for any arbitrary body in nature.

Now, especially for a black cavity radiation we have for $q_1=0$, $q_2=v$ according to (2)

$$S_1=\frac{4aT_1^3V_1}{3},\quad S_2=\frac{4ac^4T_3^2V_2}{3(c^2-v^2)^2},$$

consequently, since under the condition $S_1 = S_2$ and $V_2=V_1\cdot\sqrt{1-\frac{v^2}{c^2}}$,

$$T_2=T_1\cdot\sqrt{1-\frac{v^2}{c^2}}$$

and by (4):

$$p_1 = p_2,$$

that is, the common final pressure is equal to the common initial pressure. The last two relations are thus generally valid for any arbitrary body subjected to that process.

It also follows that we can replace the volume condition (16) of § 5 by the simpler condition, in which the final pressure p_2 is equal to the initial pressure p_1. Then we can say: By a reversible adiabatic isobaric (i.e. p = const.) acceleration (in an arbitrary way) of any body from velocity 0 up to velocity v, both the volume and the temperature of the body is diminished in the ratio $\sqrt{1-\frac{v^2}{c^2}}:1$. In this theorem, of course, the direction of the velocity v is negligible. Therefore, the same theorem is valid even if we substitute the arbitrarily oriented velocity q instead of velocity v directed into the x-axis.

§ 7.

The last theorem makes it possible to express in a very general way the relation between the values of the temperature and the pressure of an arbitrarily moving body for the two reference frames used by us. We imagine that a moving body with arbitrarily directed velocity is given. The magnitude of velocity for the unprimed frame is q, and q' for the primed frame. If, from the given condition, the body is brought in a reversible, adiabatic and isobaric way to rest for the unprimed reference frame, then its volume has grown from V' to $\frac{V}{\sqrt{1-\frac{q^2}{c^2}}}$, its temperature from T' to $\frac{T}{\sqrt{1-\frac{q^2}{c^2}}}$. However, if from the given condition the body is brought in a reversible, adiabatic and isobaric way to rest for the primed reference frame, then its volume has grown from V to $\frac{V'}{\sqrt{1-\frac{q'^2}{c^2}}}$, its temperature from T to $\frac{T'}{\sqrt{1-\frac{q'^2}{c^2}}}$. However, the state of rest achieved in this way in the unprimed system is in all respects identical to the previously obtained state of rest in the primed system. For the conditions, under which the theorem of § 5 is valid, are all satisfied here when we think that the body (at rest for the unprimed system) is brought from the initially given state in a reversible, adiabatic and isobaric way to rest for the primed system. Consequently:

$$p=p',\ \frac{V}{\sqrt{1-\frac{q^2}{c^2}}}=\frac{V'}{\sqrt{1-\frac{q'^2}{c^2}}} \text{ and } \frac{T}{\sqrt{1-\frac{q^2}{c^2}}}=\frac{T'}{\sqrt{1-\frac{q'^2}{c^2}}}, \text{ or:}$$

$$\frac{V'}{V}=\frac{T'}{T}=\sqrt{\frac{c^2-q'^2}{c^2-q^2}},\ p'=p,\ S'=S \tag{17}$$

as a *generally valid relation between the primed and unprimed the variables*.

§ 8.

Now we are mainly concerned with the comparison of the values of the *kinetic potential* in both reference frames. For this purpose we first write the differential equations (7) in accordance with the principle of relativity for the primed system:

$$\frac{\partial H'}{\partial V'}=p',\ \frac{\partial H'}{\partial T'}=S' \tag{18}$$

These two equations give with respect to the equations (7) and the relations (17):

$$\frac{\partial}{\partial V}\left(H'\sqrt{\frac{c^2-q^2}{c^2-q'^2}}\right)=\frac{\partial H}{\partial V},\quad \frac{\partial}{\partial T}\left(H'\sqrt{\frac{c^2-q^2}{c^2-q'^2}}\right)=\frac{\partial H}{\partial T}. \tag{19}$$

Before we perform the integration, we derive the relevant equations for the velocity components $\dot{y}$ and $\dot{z}$. In addition to the differential equations (6) with respect to the primed system we have to use:

$$\frac{d}{dt'}\frac{\partial H'}{\partial \dot{x}'}=\mathfrak{F}'_{x'},\quad \frac{d}{dt'}\frac{\partial H'}{\partial \dot{y}'}=\mathfrak{F}'_{y'},\quad \frac{d}{dt}\frac{\partial H'}{\partial \dot{z}'}=\mathfrak{F}'_{z'} \tag{20}$$

the relations between the primed and unprimed components of the moving force $\mathfrak{F}$. To find them, we consider a special case, namely, an infinitely small diathermanous solid body charged with electricity e, in an arbitrary, evacuated electromagnetic field. Then, for the unprimed system:

$$\mathfrak{F}_x=e\mathfrak{E}_x+\frac{e}{c}(\dot{y}\mathfrak{H}_x-\dot{z}\mathfrak{H}_y)$$

$$\mathfrak{F}_y=e\mathfrak{E}_y+\frac{e}{c}(\dot{z}\mathfrak{H}_x-\dot{x}\mathfrak{H}_y)$$

$$\mathfrak{F}_z=e\mathfrak{E}_z+\frac{e}{c}(\dot{x}\mathfrak{H}_y-\dot{y}\mathfrak{H}_x),$$

where $\mathfrak{E}$ denotes the electric, $\mathfrak{H}$ the magnetic field intensity. The same equations apply according to the relativity principle, when all the variables, except e and c, were provided with primes. This leads with respect to the relations (13) and the relations:[12]

$$\begin{aligned}
\mathfrak{E}'_{x'} &= \mathfrak{E}_x & \mathfrak{H}'_{x'} &= \mathfrak{H}_x\\
\mathfrak{E}'_{y'} &= \tfrac{c}{\sqrt{c^2-v^2}}\left(\mathfrak{E}_y-\tfrac{v}{c}\mathfrak{H}_z\right) & \mathfrak{H}'_{y'} &= \tfrac{c}{\sqrt{c^2-v^2}}\left(\mathfrak{H}_y+\tfrac{v}{c}\mathfrak{E}_z\right)\\
\mathfrak{E}'_{z'} &= \tfrac{c}{\sqrt{c^2-v^2}}\left(\mathfrak{E}_z+\tfrac{v}{c}\mathfrak{H}_y\right) & \mathfrak{H}'_{z'} &= \tfrac{c}{\sqrt{c^2-v^2}}\left(\mathfrak{H}_z-\tfrac{v}{c}\mathfrak{E}_y\right)
\end{aligned}$$

the following equations between the primed and unprimed force components:

$$\mathfrak{F}'_{x'}=\mathfrak{F}_x-\frac{v\dot{y}}{c^2-v\dot{x}}\mathfrak{F}_y-\frac{v\dot{z}}{c^2-v\dot{x}}\mathfrak{F}_z, \tag{21}$$

$$\mathfrak{F}'_{y'}=\frac{c\sqrt{c^2-v^2}}{c^2-v\dot{x}}\mathfrak{F}_y,\ \mathfrak{F}'_z=\frac{c\sqrt{c^2-v^2}}{c^2-v\dot{x}}\mathfrak{F}_z. \tag{22}$$

The last two relations (22) we accept as generally valid; this give in combination with (6) and (20):

$$\frac{d}{dt'}\frac{\partial H'}{\partial y'} = \frac{c\sqrt{c^2-v^2}}{c^2-v\dot{x}}\frac{d}{dt}\frac{\partial H}{\partial \dot{y}}.$$

Now, by (13) and (14) we have:

$$\frac{\partial H'}{\partial \dot{y}'} = \frac{\partial H'}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \dot{y}'} = \frac{\partial H'}{\partial \dot{y}}\frac{c\sqrt{c^2-v^2}}{c^2+v\dot{x}'} = \frac{\partial}{\partial \dot{y}}\left(H'\sqrt{\frac{c^2-q^2}{c^2-q'^2}}\right) \tag{23}$$

and:

$$\frac{dt'}{dt} = \frac{c^2-v\dot{x}}{c\sqrt{c^2-v^2}}.$$

It follows:

$$d\frac{\partial}{\partial \dot{y}}\left(H'\sqrt{\frac{c^2-q^2}{c^2-q'^2}}\right) = d\frac{\partial H}{\partial \dot{y}}$$

and by integration:

$$\frac{\partial}{\partial \dot{y}}\left(H'\sqrt{\frac{c^2-q^2}{c^2-q'^2}}\right) = \frac{\partial H}{\partial \dot{y}}, \text{ ebenso: } \frac{\partial}{\partial \dot{z}}\left(H'\sqrt{\frac{c^2-q^2}{c^2-q'^2}}\right) = \frac{\partial H}{\partial \dot{z}}. \tag{24}$$

The constant of integration, an absolute constant, vanishes because only $q' = q$ H' goes over into H.

§ 9.

Now, the four equations (19) and (24) give by integration:

$$H'\sqrt{\frac{c^2-q^2}{c^2-q'^2}} = H + const.$$

The constant does not depend on *V, T,* $\dot{y}, \dot{z}$; but it can still depend on $\dot{x}$, or by (14), $\frac{c^2-q^2}{c^2-q'^2}$. We therefore write:

$$H'\sqrt{\frac{c^2-q^2}{c^2-q'^2}} = H + f\left(\frac{c^2-q^2}{c^2-q'^2}\right)$$

and determine the most general expression of the function *f*.

At first, we have:

$$\frac{H'}{\sqrt{c^2-q'^2}}-\frac{H}{\sqrt{c^2-q^2}}=\frac{1}{\sqrt{c^2-q^2}}\cdot f\left(\frac{c^2-q^2}{c^2-q'^2}\right). \qquad (25)$$

Since the function *H* only depends on *q, V* and *T*, and since *V'* and *T'* are only connected to *V* and *T* by the relations (17), then the right-hand side of the equation as well as the left-hand side, are of the form:[13]

$$\frac{1}{\sqrt{c^2-q^2}}\cdot f\left(\frac{c^2-q^2}{c^2-q'^2}\right)=Q'-Q,$$

where *Q* depends only on *q*. It necessarily follows:

$$\frac{1}{\sqrt{c^2-q^2}}\cdot f\left(\frac{c^2-q^2}{c^2-q'^2}\right)=\frac{C}{\sqrt{c^2-q'^2}}-\frac{C}{\sqrt{c^2-q^2}},$$

if *C* is an absolute constant.

This substituted into (25) gives the desired relation between *H'* and *H*:

$$\frac{H'-C}{\sqrt{c^2-q'^2}}=\frac{H-C}{\sqrt{c^2-q'^2}}.$$

Since the function *H-C* satisfies exactly the same differential equations (6) and (7) as the function *H*, we may easily imagine to set in all previous equations the function *H-C* instead of *H*, and we want from now on denote *H-C* simply by *H*. Then it is found:

$$\frac{H'}{\sqrt{c^2-q'^2}}=\frac{H}{\sqrt{c^2-q^2}}. \qquad (26)$$

In other words: If we set the constant *C* = 0, then this represents no physical limitation, but a useful supplement to the definition of the kinetic potential, which is not completely determined by the differential equations (6) and (7), as it was pointed out there already.

§ 10.

Having found the general relation between *H'* and *H* now, the relation of those values of any physical quantity of the two reference frames is directly given from the differential equations of the principle of least action. Consider first the *momentum*, whose components in the primed frame are:

$$\mathfrak{G}'_{x'}=\frac{\partial H'}{\partial\dot{x}'},\quad \mathfrak{G}'_{y'}=\frac{\partial H'}{\partial\dot{y}'},\quad \mathfrak{G}'_{z'}=\frac{\partial H'}{\partial\dot{z}'}. \qquad (27)$$

While the connection of the *y* and *z*-components of momentum is directly given from the comparison with (8) and (13):

$$\mathfrak{G}'_{y'} = \mathfrak{G}_y, \quad \mathfrak{G}'_{z'} = \mathfrak{G}_z, \tag{28}$$

while the connection between the x-components $\mathfrak{G}'_{x'}$ and $\mathfrak{G}_x$ is of a much more complicated nature.

From (27) we obtain for this connection in an easily understandable description:

$$\mathfrak{G}'_{x'} = \frac{\partial H'}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \dot{x}'} + \frac{\partial H'}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \dot{x}'} + \frac{\partial H'}{\partial \dot{z}} \frac{\partial \dot{z}}{\partial \dot{x}'} + \frac{\partial H'}{\partial V} \frac{\partial V}{\partial \dot{x}'} + \frac{\partial H'}{\partial T} \frac{\partial T}{\partial \dot{x}'},$$

This is by (26), (14) and (13):

$$\frac{\partial H'}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}} \left(H' \sqrt{\frac{c^2 - q'^2}{c^2 - q^2}} \right) = \frac{c\sqrt{c^2 - v^2}}{c^2 - v\dot{x}} \frac{\partial H}{\partial \dot{x}} + \frac{vc\sqrt{c^2 - v^2}}{(c^2 - v\dot{x})^2} H$$

$$\frac{\partial H'}{\partial \dot{y}} = \frac{c\sqrt{c^2 - v^2}}{c^2 - v\dot{x}} \frac{\partial H}{\partial \dot{y}}, \quad \frac{\partial H'}{\partial \dot{z}} = \frac{c\sqrt{c^2 - v^2}}{c^2 - v\dot{x}} \frac{\partial H}{\partial \dot{z}}$$

$$\frac{\partial H'}{\partial V} = \frac{c\sqrt{c^2 - v^2}}{c^2 - v\dot{x}} \frac{\partial H}{\partial V}, \quad \frac{\partial H'}{\partial T} = \frac{c\sqrt{c^2 - v^2}}{c^2 - v\dot{x}} \frac{\partial H}{\partial T}$$

$$\frac{\partial \dot{x}}{\partial \dot{x}'} = \frac{(c^2 - v\dot{x})^2}{c^2(c^2 - v^2)}, \quad \frac{\partial \dot{y}}{\partial \dot{x}'} = -\frac{v\dot{y}(c^2 - v\dot{x})}{c^2(c^2 - v^2)}, \quad \frac{\partial \dot{x}}{\partial \dot{x}'} = -\frac{v\dot{z}(c^2 - v\dot{x})}{c^2(c^2 - v^2)},$$

$$\frac{\partial V}{\partial \dot{x}'} = -\frac{v(c^2 - v\dot{x})}{c^2(c^2 - v^2)} V, \quad \frac{\partial T}{\partial \dot{x}'} = -\frac{v(c^2 - v\dot{x})}{c^2(c^2 - v^2)} T.$$

This is given by substitution with respect to (8) and (7):

$$\mathfrak{G}'_{x'} = \frac{1}{c\sqrt{c^2 - v^2}} \{(c^2 - v\dot{x})\mathfrak{G}_x + vH - v\dot{y}\mathfrak{G}_y - v\dot{z}\mathfrak{G}_z - vpV - vTS\}$$

or from the introduction of the energy E (10):

$$\mathfrak{G}'_{x'} = \frac{c}{\sqrt{c^2 - v^2}} \left(\mathfrak{G}_x - \frac{v(E + pV)}{c^2} \right). \tag{29}$$

If we introduce instead of the energy E the "thermal function at constant pressure" R by Gibbs:

$$R = E + pV, \tag{30}$$

whose variation in isobaric processes describes the supplied heat, then the last relation is simply given by:

$$\mathfrak{G}'_{x'} = \frac{c}{\sqrt{c^2 - v^2}} \left(\mathfrak{G}_x - \frac{v}{c^2} R \right). \tag{31}$$

§ 11.

By differentiating the equation (29) by time *t*:

$$\frac{d\mathfrak{G}'_{x'}}{dt} = \frac{d\mathfrak{G}'_{x'}}{dt'} \cdot \frac{dt}{dt} = \frac{c}{\sqrt{c^2 - v^2}} \left\{ \frac{d\mathfrak{G}_x}{dt} - \frac{v}{c^2} \left(\frac{dE}{dt} + p\frac{pV}{dt} + V\frac{dp}{dt} \right) \right\},$$

the relation between the x-components of the force $\mathfrak{F}$ follows with consideration of (27), (20), (14) and (11), namely:

$$\mathfrak{F}'_{x'} = \mathfrak{F}_x - \frac{v}{c^2 - v\dot{x}} (\mathfrak{F}_y \dot{y} + \mathfrak{F}_z \dot{z} + V\dot{p} + T\dot{S}). \tag{32}$$

Comparing this relation with the one found above (21), it follows that those have no general meaning, but only apply if $\dot{p} = 0$ and $\dot{S} = 0$, that is, when the process runs isobaric and adiabatic. In fact, this property is characteristically for the process under consideration at that time: the motion of an electrically charged, diathermanous solid body in an evacuated electromagnetic field.

Finally, there may be still room for general relations between the values for the energy of the body, as well as the performed external work, and the supplied heat for both reference systems.

For the energy E' we have, by (10):

$$E' = \dot{x}'\mathfrak{G}'_{x'} + \dot{y}'\mathfrak{G}'_{y'} + \dot{z}'\mathfrak{G}'_{z'} + T'S' - H',$$

consequently, by substituting the previously derived relations:

$$E' = \frac{c}{\sqrt{c^2 - v^2}} \left\{ E - v\mathfrak{G}_x - \frac{v(\dot{x} - v)}{c^2 - v\dot{x}} pV \right\}. \tag{33}$$

As to the *thermal function R* defined in (30), we have in the primed reference frame the simple relationship:

$$R' = \frac{c}{\sqrt{c^2 - v^2}} (R - v\mathfrak{G}_x).) \tag{34}$$

The performed translation work from outside (at an infinitesimal reversible change of state of the body) is for the primed frame of reference:

$$\mathfrak{F}'_{x'} dx' + \mathfrak{F}'_{y'} dy' + \mathfrak{F}'_{z'} dz' = \tag{35}$$

$$= \frac{c}{\sqrt{c^2-v^2}} \left\{ \mathfrak{F}_x dx + \mathfrak{F}_y dy + \mathfrak{F}_z dz - vdt \left(\mathfrak{F}_x + \frac{\dot{x}-v}{c^2-v\dot{x}} (V\dot{p} + T\dot{S}) \right) \right\}$$

Furthermore, the *compression work*:

$$-p'dV' = -\frac{c\sqrt{c^2-v^2}}{c^2-v\dot{x}} pdV - \frac{vc\sqrt{c^2-v^2}}{(c^2-v\dot{x})^2} pVd\dot{x} \tag{36}$$

finally, the *added heat*:

$$T'dS' = \frac{c\sqrt{c^2-v^2}}{c^2-v\dot{x}} TdS. \tag{37}$$

§ 12.

The relations derived above between the primed and unprimed quantities can be partly represented in a more simple way, if we examine those expressions that are invariant for the transformation from one reference frame to another. Such invariants are *y, z, p, s*, \ $\mathfrak{G}_y, \mathfrak{G}_z, \frac{H}{\sqrt{c^2-q^2}}, G\frac{\sqrt{c^2-q^2}}{q}$, furthermore the differential expressions $\sqrt{c^2-q^2}dt$, Hdt, Vdt, Tdt, $\mathfrak{F}_y dt,\ \mathfrak{F}_z dt, Edt - \mathfrak{G}_x dx, Rdt - \mathfrak{G}_x dx$, etc. All these quantities do not change their value, if they were replaced by the corresponding primed quantities.

It also follows that the value for the time integral, which is characteristic for the principle of least action, and which is taken from a certain initial state 1 to a certain final state 2, is:

$$W = \int_1^2 Hdt,$$

which may be described as the (corresponding to the process in question) "influence quantity", having the same value for the primed reference frame as for the unprimed. If we add the theorem that for the influence quantity there exists a certain elementary quantum[14]: h=6.55 · 10^{-27} erg. sec., we can also say: Every change in nature corresponds to a certain number of influence elements which are independent of the reference frame. It is understood that this theorem extends the importance of the principle of least action in a new direction. But at this point, these and related issues will not be further discussed.

Third Section. Applications.

§ 13.

The most important implication of the general relations established in the previous section concerns the dependence of the physical state of a body on its velocity. It can in fact be shown quite generally, that the kinetic potential *H* and thus all state variables, are directly specify as functions of velocity, volume and temperature, as soon as they are known for the velocity zero as functions of volume and temperature.

We want for this purpose denote by $H_0, p_0, S_0, E_0, \ldots$ those functions of two variables *V* and *T*, in which the functions *H, p, S, E, ...* of the three variables *q, V, T* go over, if we set *q* = 0 within them. We also want denote by $H'_0, p'_0, S'_0, E'_0, \ldots$ those functions of the three variables *q, V, T*, in which the functions $H_0, p_0, S_0, E_0, \ldots$ the two variables *V* and *T* go over, if we substitute $V' = \frac{c}{\sqrt{c^2-q^2}}V$ instead of *V* and $T' = \frac{c}{\sqrt{c^2-q^2}}T$ instead of T'.

Now we start from relation (26) and set therein *q'* = 0. Then it follows with respect to (17) in the recently introduced term:

$$H = \frac{\sqrt{c^2-q^2}}{c}H'_0, \tag{38}$$

and thus *H* is represented as a function of the three variables *q, V*, and *T*, if H_0 as a function of two variables *V* and *T* is known. By *H* all other physical state variables are determined according to (6) and (7). We at first obtain for the pressure:

$$p = p'_0\,. \tag{39}$$

If the pressure of the body at rest is known by the usual state equation as a function of volume and temperature, the state equation of the moving body follows immediately. Similarly, the entropy is given by:

$$S = S'_0. \tag{40}$$

Furthermore, the components of the momentum are given by:

$$\mathfrak{G}_x = G\frac{\dot{x}}{q},\quad \mathfrak{G}_y = G\frac{\dot{y}}{q},\quad \mathfrak{G}_z = G\frac{\dot{z}}{q},$$

where *G*, the resulting momentum, is according to (38):

$$G = \frac{\partial H}{\partial q} = -\frac{q}{c\sqrt{c^2-v^2}}H'_0 + \frac{\sqrt{c^2-q^2}}{c}\left\{\left(\frac{\partial H}{\partial V}\right)'_0\frac{cqV}{(c^2-q^2)^{\frac{3}{2}}} + \left(\frac{\partial H}{\partial T}\right)'_0\frac{cqT}{(c^2-q^2)^{\frac{3}{2}}}\right\}$$

$$G = \frac{q}{c^2-q^2}Vp'_0 + \frac{q}{c^2-q^2}TS'_0 - \frac{q}{c\sqrt{c^2-q^2}}H'_0. \tag{41}$$

Furthermore, the energy according to (10) is given by:

$$E = \frac{q^2}{c^2 - q^2} V p'_0 + \frac{c^2}{c^2 - q^2} T S'_0 - \frac{c}{\sqrt{c^2 - q^2}} H'_0. \tag{42}$$

Considering that $E_0 = T S_0 H_0$ and

$$E'_0 = \frac{cT}{\sqrt{c^2 - q^2}} S'_0 - H'_0,$$

thus we can write:

$$E = \frac{c}{\sqrt{c^2 - q^2}} E'_0 + \frac{q^2}{c^2 - q^2} V p'_0. \tag{43}$$

Finally, the thermal function *R* is by (30):

$$R = \frac{c^2}{c^2 - q^2} V p'_0 + \frac{c^2}{c^2 - q^2} T S'_0 - \frac{c}{\sqrt{c^2 - q^2}} H'_0 \tag{44}$$

or because:

$$R'_0 = \frac{cV}{\sqrt{c^2 - q^2}} p'_0 + \frac{cT}{\sqrt{c^2 - q^2}} S'_0 - H'_0$$

$$R = \frac{c}{\sqrt{c^2 - q^2}} R'_0 \tag{45}$$

By introduction of thermal function *R*, the momentum *G*simply writes by (41):

$$G = \frac{q}{c^2} R = \frac{q}{c\sqrt{c^2 - q^2}} R'_0. \tag{46}$$

§ 14.

The special relations which are contained in the above equations can all be summarized in a single differential equation, which is completely general for the function *H* of the three variables *q, V, T*. Namely, if we substitute in equation (46) for *G* the expression $\frac{\partial H}{\partial q}$, and for *R* the value $E + pV$, it follows with respect to (10) the equation:

$$T \frac{\partial H}{\partial T} + V \frac{\partial H}{\partial V} - \frac{c^2 - q^2}{q} \frac{\partial H}{\partial q} - H = 0. \tag{47}$$

This differential equation represents the general expression for the application of the principle of relativity on the kinetic potential. Its general integral is expressed by (38), of which one can easily convince himself. Thereafter, the kinetic potential *H* is a homogeneous function of first degree of the three variables *T, V,* and $\sqrt{c^2 - q^2}$.

§ 15.

Let us now at first give a special application to the black cavity radiation. Hereafter, all laws of motion of cavity radiation are resulting directly from the simple known thermodynamic formulas for static cavity radiation. Namely, for which the Stefan-Boltzmann law is given:

$$E_0 = aT^4 V.$$

Furthermore, Maxwell's radiation pressure is given by:

$$p_0 = \frac{1}{3} aT^4,$$

and the entropy of stationary radiation:

$$S_0 = \int \frac{dE_0 + p_0 dV}{T} = \frac{4}{3} aT^3 V.$$

For those values applying to $q = 0$, by definition (§ 13) the expressions follow:

$$E'_0 = \frac{ac^5 T^4 V}{(c^2 - q^2)^{\frac{5}{2}}}, \quad p'_0 = \frac{ac^4 T^4}{3(c^2 - q^2)^2}, \quad S'_0 = \frac{4ac^4 T^3 V}{3(c^2 - q^2)^2},$$

and with their aid by (39), (40), (43) and (46) the values valid for any velocity q follow:

$$P = \frac{ac^4 T^4}{3(c^2 - q^2)^2}, \; S = \frac{4ac^4 T^3 V}{3(c^2 - q^2)^2},$$

$$E = \frac{ac^4 (3c^2 + q^2)}{3(c^2 - q^2)^3} T^4 V, \; G = \frac{q}{c^2}(E + pV) = \frac{4ac^4 q}{3(c^2 - q^2)^3} T^4 V$$

in accordance with the equations of § 1

§ 16.

By momentum G, also its *inertial mass* is determined. This quantity, which plays in pure mechanics such a fundamental role, is degraded to a secondary expression within the general dynamics. For, once the momentum is no longer proportional to the velocity, the mass of a body is no longer constant; also we are led to completely different dependencies of mass on velocity, depending on whether we divide momentum G by velocity q, or if we differentiate velocity q, where in this case it is necessary to specify particularly the manner in which the differentiation took place: whether isothermal, adiabatic, etc. Again, a different value for the mass is found in general, if we start from the energy E and differentiate it to $\frac{q^2}{2}$. How to designate these different expressions, is of course a matter of definition.

Here, by "mass" M of a body we want to understand that quantity of a body independent of velocity, which is obtained if the momentum G is divided by velocity q and where we set the ratio $q = 0$, thus in our notation by (46):

$$M = \left(\frac{G}{q}\right)_0 = \frac{R_0}{c^2} = \frac{E_0 + pV_0}{c^2} \tag{48}$$

This quantity in general depends on the temperature T and volume V of the body.

If we set in the expression $\frac{G}{q}$ the velocity q not to zero, then we call the ratio, as usual,[15] the "transverse" mass of the body, while on the other hand, the derivative $\frac{dG}{dq}$ is the "longitudinal" mass. In the longitudinal mass is, however, the "isothermal-isochoric" mass to be distinguished from the "adiabatic-isobaric" mass, etc.; because the derivative has only one definite value when the path of differentiation is given. For the special speed $q = 0$, transverse and longitudinal masses of all forms become the same, i.e. they become (48).

The mass of a stationary cavity radiation is, therefore, given by (5):

$$\frac{4aT^4V}{3c^2}$$

the transverse mass of a moving cavity radiation:

$$\frac{G}{q} = \frac{4ac^4T^4V}{3(c^2 - q^2)^3},$$

the longitudinal isothermal isochoric mass of that[16]

$$\frac{\partial G}{\partial q}=\frac{4ac^4(c^2+5q^2)}{3(c^2-q^2)^4}T^4V,$$

the longitudinal adiabatic isochoric mass:[16]

$$\left(\frac{\partial G}{\partial q}\right)_{S,V}=\frac{4ac^4(3c^2-q^2)}{9(c^2-q^2)^4}T^4V,$$

the longitudinal adiabatic-isobaric mass, on the contrary:

$$\left(\frac{\partial G}{\partial q}\right)_{S,p}=\frac{4ac^6T^4V}{3(c^2-q^2)^4}.$$

$$\left(\frac{\partial G}{\partial q}\right)_{S,p}=\frac{4ac^6T^4V}{3(c^2-q^2)^4}.$$

§ 17.

Especially notable in relation (48) is the close connection of the mass of a body with the thermal function R_0. Since mass M can easily be measured in grams, then the quantity of R_0 immediately can be given by the absolute CGS system. But this value can not be tested directly by thermodynamic means; because an additive constant of the thermal function, as well as of energy, remains unspecified by pure thermodynamics. In this respect, relation (48) is essentially a complement to the thermodynamic definition of energy.

In contrast, there arises a possibility of an experimental test of the theory by taking into account the variability of the thermal function R_0 with temperature, volume and chemical composition. For according to equation (48), the inertial mass of a body changes due to thermal input and release, and the increase of mass is always equal to the amount of heat absorbed in an isobar change of the body from the outside, divided by the square of the velocity of light in vacuum.[17] It is particularly noteworthy, that the theorem not only applies to reversible processes, but in general also applies to any irreversible change of state; for the relation between the thermal function R and the heat supplied from the outside is based directly upon the first law of thermodynamics. Because the quantity is of the order of c^2, the mass variation caused by the simple heating or cooling of a body is so minimal, of course, that it is likely to escape forever any direct measurement. A stronger influence would be expected by consideration of chemical enthalpy changes, although even here the effect is unlikely to be measured.

Let us calculate, for example, the decrease in mass of 1½ moles oxygen-hydrogen (H_2 + ½O_2 = 18 gr), condensed at atmospheric pressure and room temperature to 1 mol of liquid water. This is equivalent to the heat in CGS units:

$$r = 68400 \cdot 419 \cdot 10^5 = 2.87 \cdot 10^{12}$$

Consequently, the decrease in mass: $\frac{r}{c^2} gr = 3.2 \cdot 10^{-6}$ mgr, is still a vanishingly small quantity.

§ 18.

According to the theory developed here, we therefore have to imagine an energy store in the interior of each body, whose amount is so enormous that the usually observed heating and cooling processes, and even quite deep invasive chemical transformations associated with considerable heat effects, changes it by only an imperceptible fraction. This is valid down to the lowest attainable temperatures: for both the specific heat of a body as well as the reaction enthalpy of chemical processes keep up their magnitude close to absolute zero. If the temperature of a body at rest is infinitely diminished (at constant external pressure), then its internal energy does not converge against zero, which is incidentally also excluded, because the reaction enthalpy of two bodies acting chemically on each other remains finite even at the lowest temperatures, but on the contrary it retains the same value at any finite temperature except for comparatively very minor terms. This energy store, which absolutely remains within the body at zero degrees, and against which all the usual physical and chemical processes within enthalpy changes are minimal, we want to denote as the "latent energy" of the body. The latent energy is quite independent of the temperature and the motion of chemical atoms,[18] its location is therefore to be found within the chemical atom; by its nature it could be of potential nature but just as well be of kinetic nature. For nothing hinders us to accept, which would even be considered very probably especially by the electrodynamic point of view, that within the chemical atoms certain stationary motion processes in the form of standing oscillations take place, associated with none or only small radiation. The energy of these oscillations, which can be very substantial, would (as long as the atoms remain unchanged) emerge in no other way than through inertia, by which it opposes to a translational acceleration of the oscillating system, and by the gravitational effect which is apparently in close connection to it. However, the views based on the kinetic theory of gases, which assume the inertial mass as something primarily given and the chemical atoms as rigid bodies or as simple material points, are insufficient for a further development of those ideas; namely, especially BOLTZMANN's law of even energy distribution in statistical equilibrium would also lose its meaning here. That in the field of intra-atomic processes the simple assumptions of the kinetic theory are in the need of profound additions, is indeed already suggested in view of the mercury spectrum, and is well recognized from all sides.

Although, after the foregoing, the existence and magnitude of the latent energy normally can only be inferred indirectly from theoretical considerations there is still a condition under which it comes into direct thermodynamic efficiency: that is the occurrence of a change or destruction of the chemical atoms; for in this case some latent energy has to be released in accordance with the energy principle. Although the prospect of the realization of such a radical operation appeared to be very low a decennium ago, now due to the discovery of radioactive elements and their transformations it is in close proximity, and in fact the observation of the strong persistence of heat production by radio-active substances gives almost a direct support for the assumption, that the source of that heat is just nothing else than the latent energy of the atoms. In accordance to (48), a large latent energy is connected with a large mass as well. This is well in line with the fact that the radio-active elements have a particularly high atomic weight, and that their bindings belong to those with the highest specific weights.

According to J. Precht[19] 1 gr of radium, when surrounded by a sufficiently thick layer of lead, gives 134.4 · 225 gr = 30240 gr cal per hour. This gives according to (48) a reduction of mass in an hour by

$$\frac{30240 \cdot 419 \cdot 10^5}{9 \cdot 10^{20}} gr = 1.41 \cdot 10^{-6} mgr$$

or in a year a reduction of mass around 0.012 mgr. This amount, however, particularly with regard to the high atomic weight of radium, is still so small that it is at first out of the realm of possible experience.

Incidentally, it might appear doubtful from the outset whether a weighing scale is the right instrument for this measurement. Because the relation (48) does not apply to ponderable, but to the inertial mass, and it has already been stressed in the introduction that these two factors are by no means identical, at least not if we attribute no gravitational effect to black cavity radiation in a vacuum, although it surely has inertia. However, inertia and gravitation are in every respect and by all our experience, for the most varied materials and from the lightest to the heaviest, so closely connected with each other that we may seek without concern the origin of these two effects at the same place, namely in the latent energy of the chemical atoms. Assuming that gravity is directly proportional to the latent energy, then the mass depending on temperature is only very slightly larger than the ponderable mass which is quite independent of temperature. In any case, however, a notable reduction in the latent energy would also result in a notable reduction of the ponderable mass. The future will teach us whether such an influence will ever be directly detected.

Fourth Section. Introduction of new independent variables.

§ 19.

The expression (38) for the kinetic potential H found in the previous section has the same form as that for the kinetic potential of a single material point moving with constant mass M, which was found by me in a previous study[20]:

$$-Mc^2\sqrt{1-\frac{q^2}{c^2}} + const \tag{49}$$

However, the agreement is not complete: for that it would be necessary that $M=-\frac{H'_0}{c^2}$, which according to equation (48) is not at all the case. The reason for this apparent contradiction is that the quantity H which was denoted as the kinetic potential, means something different than there, as can be seen most easily by considering the equations of motion (6). These equations can be found in my earlier paper in exactly the same form as here, but there the differential quotients $\frac{\partial H}{\partial\dot{x}}, \frac{\partial H}{\partial\dot{y}}, \frac{\partial H}{\partial\dot{z}}$ have a different meaning, because in that place the differentiation was not to be given in a isothermal but in a adiabatically-isobar way. As the material point moves without external heat supply under the constant external pressure zero, then according to § 6 it has variable volume and variable temperature. To make that difference clear, I will at this place refer by K to the former size H, so that we have the equations:

$$\left(\frac{\partial K}{\partial\dot{x}}\right)_{p,S}=\left(\frac{\partial H}{\partial\dot{x}}\right)_{V,T}\text{, etc.} \tag{50}$$

where, by (49):

$$K=-Mc^2\sqrt{1-\frac{q^2}{c^2}}+const \tag{51}$$

The full compatibility of these relations with the formulas of the previous section is most evident, when in equations (6) and (7) of the principle of least action the independent variables V and T are replaced by p and S. Thus they are:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{x}}\right)_{p,S}=\mathfrak{F}_x\text{, u.s.w.} \tag{52}$$

$$\left(\frac{\partial K}{\partial p}\right)_S=-V,\ \left(\frac{\partial K}{\partial S}\right)_p=-T, \tag{53}$$

where

$$K=H-pV-TS. \tag{54}$$

That these relationships, in fact, are quite equivalent with (6) and (7), can be directly and most easily seen by substituting the value (54) of K in equations (52) and (53), and the differentiation of H by the independent variables p and S is replaced by the differentiation of the independent variables V and T.

When we consider that by (10) and (30):

$$K=q\frac{\partial H}{\partial q}-pV-E=qG-R,$$

it follows by substitution in (46):

$$K = -\frac{\sqrt{c^2 - q^2}}{c} R'_0.$$

In order to compare this relation with that previously derived by me (51), we must restrict ourselves to adiabatic isobaric processes, because only for such (51) was derived. However, according to § 6 for an adiabatic-isobaric process $V' = \frac{V}{\sqrt{1 - \frac{q^2}{c^2}}}$ is constant, as well as $T' = \frac{T}{\sqrt{1 - \frac{q^2}{c^2}}}$ is constant, so R'_O is independent of q. Therefore we write R_O instead of R'_O, and then we obtain by (48):

$$K = -\frac{\sqrt{c^2 - q^2}}{c} R_0 = -Mc\sqrt{c^2 - q^2}$$

in full accordance with (51).

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1. H. Von Helmholtz, Wissenschaftl. Abhandl. III, p. 203, 1895.
2. H. A. Lorentz, Versl. Kon. Akad. v. Wet., Amsterdam S. 809, 1904.
3. A. Einstein, Ann. d. Phys. (4) 17, S. 891, 1905. (http://www.fourmilab.ch/etexts/einstein/specrel/www/)
4. A. A. Michelson and E. W. Morley, Amer. Journ. of Science (3) 34, p. 333, 1887.
5. Kurd Von Mosengeil, Ann. d. Phys. (4) 22, S. 867, 1907, gives (based on a formula for the pressure of a single ray on a moving mirror by M. Abraham (Elektromagnetische Theorie der Strahlung. Leipzig, B. G. Teubner 1905, p. 351) the equation (42):

$$p = \frac{4\pi}{3c} K(0) \left(1 - \frac{q^2}{c^2}\right)^{\frac{2}{3}}$$

   and as equation (44):

$$\epsilon = \frac{4\pi}{c} K(0) \frac{1 + \frac{q^2}{3c^2}}{\left(1 - \frac{q^2}{c^2}\right)^{\frac{1}{3}}}$$

   Combined, both equations give the above relation, which is by the way generally valid, not only for adiabatic processes.

6. According to K. Von Mosengeil, l.c. equation (24*) it is namely:

$$G = \frac{16\pi q}{3c^3} \frac{K\left(\frac{\pi}{2}, q\right)}{\left(1 - \frac{q^2}{c^2}\right)^3}$$

   where according to equation (25*):

$$\epsilon = \frac{4\pi}{c} K\left(\frac{\pi}{2}, q\right) \frac{1+\frac{1}{3}\frac{q^2}{c^2}}{\left(1-\frac{q^2}{c^2}\right)^3}.$$

7. K. Von Mosengeil, l. c. equation (47) etc.
8. This number can also be zero. Then the body is reduced to cavity radiation, as it was discussed in the previous section.
9. On the existence of a state equation. see Byk, Ann. d. Phys. (4) 19, p. 441, 1906.
10. H. Von Helmholtz, Ges. Abh. (Leipzig, J. A. Barth) III, S. 225, 1895. There, the kinetic potential is defined by the opposite sign.
11. All these relations are also valid for a non-uniformly moving medium in which the velocity continuously varies in magnitude and direction from point to point. In this case, *V* can be understood as any infinitesimal volume element.
12. A. Einstein, Ann. d. Phys. (4), 17, p. 909, 1905. (http://www.fourmilab.ch/etexts/einstein/specrel/www/)
13. This can be seen in the most simple way, when we take an arbitrary value *q″* and sum up the three expressions $\frac{H'}{\sqrt{c^2-q'^2}} - \frac{H}{\sqrt{c^2-q^2}}$, $\frac{H''}{\sqrt{c^2-q''^2}} - \frac{H'}{\sqrt{c^2-q'^2}}$ and $\frac{H}{\sqrt{c^2-q^2}} - \frac{H''}{\sqrt{c^2-q''^2}}$.
14. M. Planck, Vorlesungen über Wärmestrahlung (Leipzig. J. A. Barth), p. 162, 1906.
15. M. Abraham, Theorie der Elektricität, II, p. 186.
16. See K. von Mosengeil, l.c. § 9. There, the mass is not, as here, defined by the momentum, but by the energy.
17. Much the same conclusion was already drawn by A. Einstein (Ann. d. Phys. 18 p. 639, 1905) from the application of the relativity principle on a special radiation process, however, under the supposition only permitted in first approximation, that the entire energy of a moving body is additively composed of its kinetic energy, and its energy of its rest frame of reference. There, we find also a reference to a possible test of the theory by observation of radium salts.
18. For this, see for example the considerations of K. Bose, Physikalische Zeitschrift 5, p. 356, p. 731, 1904.
19. J. Precht, Ann. d. Phys. 21, p. 599, 1906.
20. Verhandlungen der Deutschen Physikalischen Gesellschaft 8, p. 140, 1906.
