

# Newton-Machian analysis of Neo-tychonian model of planetary motions

**Luka Popov**

University of Zagreb, Department of Physics, Bijenička cesta 32, Zagreb, Croatia

E-mail: lpopov@phy.hr

**Abstract.** The calculation of the trajectories in the Sun-Earth-Mars system will be performed in two different models, both in the framework of Newtonian mechanics. First model is well-known Copernican system, which assumes the Sun is at rest and all the planets orbit around it. Second one is less-known model developed by Tycho Brahe (1546-1601), according to which the Earth stands still, the Sun orbits around the Earth, and other planets orbit around the Sun. The term “Neo-tychonian system” refers to the assumption that orbits of distant masses around the Earth are synchronized with the Sun’s orbit. It is the aim of this paper to show the kinematical and dynamical equivalence of these systems, under the assumption of Mach’s principle.

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## 1. Introduction

The discussion of motion of celestial bodies is one of the most interesting episodes in the history of science. There are two diametrically opposite schools of thought: one that assumes that the Sun stands still, and Earth and other planets orbit around it; and another that assumes that the Earth stands still, and Sun and other planets in some manner orbit around the Earth. The first school of thought comes from Aristarchus (310-230 BC) and is generally addressed as *heliocentrism*, another from Ptolemy (90-168 BC) and is generally known as *geocentrism*. Since Aristotle, the ultimate authority in science for more than two millennia, accepted the geocentric assumption, it became dominant viewpoint among scientists of the time. The turnover came with Copernicus (so-called “Copernican revolution”) who in his work *De Revolutionibus* proposed a hypothesis that the Sun stands in the middle of the known Universe, and that Earth orbits around it, together with other planets. Copernicus’ system was merely better than Ptolemy’s, because Copernicus assumed the trajectories of the planets are perfect circles, and required the same number of epicycles (sometimes even more) as Ptolemy’s model [1]. The accuracy of Ptolemy’s model is still a subject of vivid debates among historians of science [2].

The next episode in this controversy is Kepler’s system with elliptical orbits of planets around the Sun. That system did not require epicycles, it was precise and elegant. It is therefore general view that Kepler’s work finally settled the question whether it is the Sun or the Earth that moves. But what is less known is that Tycho Brahe, Kepler’s tutor, developed a geostatic system that was just as accurate and elegant as Kepler’s: the Sun orbits around the Earth, and all the other planets orbit around the Sun. The trajectories are ellipses, and all the Kepler’s laws are satisfied. In that moment of history, the Kepler’s and Brahe’s models were completely equivalent and equally elegant, since neither of them could explain the mechanism and reason why the orbits are the way they are. It had to wait for Newton.

Sir Isaac Newton, as it is generally considered, gave ultimate explanation of planetary motions that was in accord with Kepler’s model, and excluded Brahe’s one. The laws of motions and the inverse square law of gravity could reproduce all the observed data only with the assumption that the Sun (i.e. the center of mass of the system, which can be very well approximated by the center of the Sun) stands still, and all planets move around it. According to Newton’s laws, it is impossible for small Earth to keep the big Sun in its orbit: the gravitational pull is just too weak. This argument is very strong, and it seemed to settle the question for good.

But in the end of 19th century, the famous physicist and philosopher Ernst Mach (1839-1916) came with the principle which states the equivalence of non-inertial frames. Using the famous “Newton’s bucket” argument, Mach argues that all so-called pseudo-forces (forces which result from accelerated motion of the reference frame) are in fact *real* forces originating from the accelerated motion of distant masses in the Universe, as observed by the observer in the non-inertial frame. Some go even further, stating that

“every single physical property and behavioral aspect of isolated systems is determined by the whole Universe” [3]. According to Mach’s principle, the Earth could be considered as the “pivot point” of the Universe: the fact that the Universe is orbiting around the Earth will create the exact same forces that we usually ascribe to the motion of the Earth.

Mach’s principle played a major role in the development of the Einstein’s General Theory of Relativity [4], as well as other developments in gravitation theory, and has inspired some interesting experiments [5]. This principle still serves as a guideline for some physicists who attempt to reformulate (“Machianize”) Newtonian dynamics [6, 7], or try to construct new theories of mechanics [8]. Some arguments and critiques against Mach’s principle have also been raised [9]. Since the time of its original appearance [10, 11, 12], Mach’s principle has been reformulated in numbers of different ways [13, 14]. For the purpose of this paper, we will only focus on the one of the consequences of Mach’s principle: that the inertial forces can be seen as resulting from real interactions with distant matter in the Universe, as was for example shown by A. Zylbersztajn [15].

The only question remains: *are these forces by themselves enough to explain all translational motions that we observe from Earth, and can they reproduce the Tycho Brahe’s model?* The discussion in this paper will show that the answer to this question is positive. In order to demonstrate it, we will consider the Sun-Earth-Mars system.

The paper is organized as follows. In section 2 an overview of two-body problem in the central potential and of Kepler’s problem is given. In section 3 the calculations of Earth’s and Mars’ trajectories are performed in the heliocentric system, both analytically (by applying the results from previous section) and numerically. In section 4 the calculations of Sun’s and Mars’ trajectories are performed in geocentric system, due to the presence of pseudo-potential originating from the fact of accelerated motion of the Universe. Finally, the conclusion of the analysis is given.

## 2. Two-body problem in the central potential

### 2.1. General overview

We start with the overview of two body problem in Newtonian mechanics. Although there are alternative and simpler ways to solve this problem [16, 17], we will follow the usual textbook approach [18, 19]. The Lagrangian of the system reads

$$L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(|\mathbf{r}_1 - \mathbf{r}_2|), \quad (2.1)$$

where  $U$  is potential energy that depends only on the magnitude of the difference of radii vectors (so-called *central potential*). We can easily rewrite this equation in terms of relative position vector  $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ , and let the origin be at the centre of mass, i.e.  $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \equiv 0$ . Solution of these equations are

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}, \quad \mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}. \quad (2.2)$$

The Lagrangian (2.1) so becomes

$$L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r), \quad (2.3)$$

where  $r \equiv |\mathbf{r}|$  and  $\mu$  is the reduced mass,

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad (2.4)$$

In that manner, the two-body problem is reduced to one-body problem of particle with coordinate  $\mathbf{r}$  and mass  $\mu$  in the potential  $U(r)$ .

Using polar coordinates, the Lagrangian (2.3) can be written as:

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r) \quad (2.5)$$

One can immediately notice that variable  $\phi$  is cyclic (it does not appear in the Lagrangian explicitly). Consequence of that fact is momentum conservation law, since  $(\partial/\partial t)(\partial L/\partial \dot{\phi}) = \partial L/\partial \phi = 0$ . Therefore,

$$\ell \equiv \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{const.} \quad (2.6)$$

is the integral of motion.

In order to find a solution for the trajectory of a particle, it is not necessary to explicitly write down the Euler-Lagrange equations. Instead, one can use the energy conservation law,

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \quad (2.7)$$

Straightforward integration of (2.7) gives equation for the trajectory,

$$\phi(r) = \int \frac{\ell \, dr/r^2}{\sqrt{2\mu [E - U(r)] - \ell^2/r^2}} \quad (2.8)$$

## 2.2. Kepler's problem

Let us now consider the particle in the potential

$$U(r) = -\frac{k}{r}, \quad (2.9)$$

generally known as *Kepler's problem*. Since our primary interest is in the planetary motions under the influence of gravity, we will take  $k > 0$ . Integration of eq. (2.8) for that potential gives:

$$\frac{p}{r} = 1 + e \cos \phi, \quad (2.10)$$

where  $2p$  is called *latus rectum* of the orbit, and  $e$  is *eccentricity*. These quantities are given by

$$p = \frac{\ell^2}{\mu k}, \quad e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} \quad (2.11)$$

Expression (2.10) is the equation of a conic section with one focus in the origin. For  $E < 0$  and  $e < 1$  the orbit is an ellipse.

One can also determine minimal and maximal distances from the source of the potential, called *perihelion* and *aphelion*, respectively:

$$r_{min} = \frac{p}{1+e}, \quad r_{max} = \frac{p}{1-e}. \quad (2.12)$$

These parameters can be directly observed, and often are used to test a model or a theory regarding planetary motions.

### 3. Earth and Mars in heliocentric perspective

According to Newton's law of gravity, the force between two massive objects reads:

$$\mathbf{F} = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2). \quad (3.1)$$

Which leads to a potential ( $\mathbf{F} = -\nabla U$ )

$$U(|\mathbf{r}_1 - \mathbf{r}_2|) = -\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}. \quad (3.2)$$

This is obviously Kepler's potential (2.9) with  $k = Gm_1m_2$ , where  $G$  is Newton's gravitational constant.

Since the Sun is more than 5 orders of magnitude more massive than Earth and Mars, we will in all future analysis use the approximation

$$\mu \approx m_i, \quad (3.3)$$

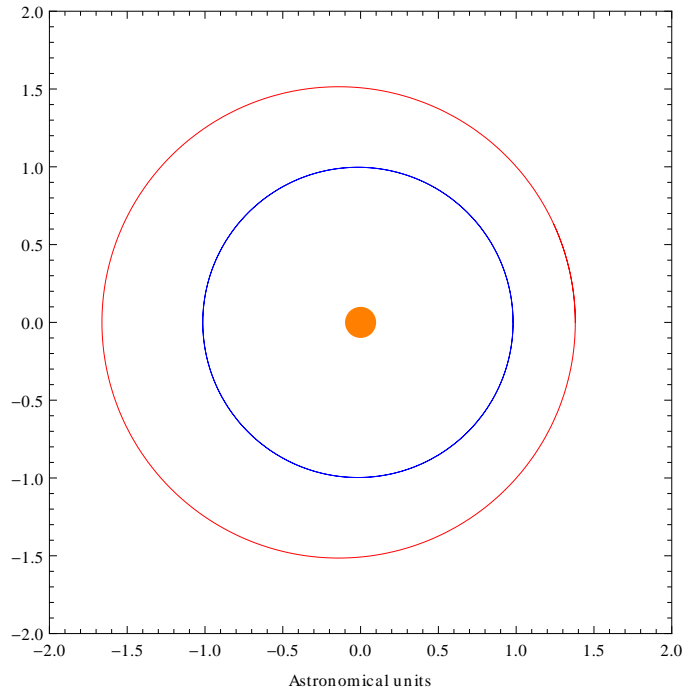
where  $m_i$  is mass of the observed planet. For the same reason, gravitational interaction between Earth and Mars can be neglected, since it is negligible compared with the interaction between Earth/Mars and the Sun.

Using these assumptions, we can write down corresponding Lagrangians,

$$\begin{aligned} L_{ES} &= \frac{1}{2}m_E\dot{\mathbf{r}}_{ES}^2 + \frac{Gm_EM_S}{r_{ES}}, \\ L_{MS} &= \frac{1}{2}m_M\dot{\mathbf{r}}_{MS}^2 + \frac{Gm_MM_S}{r_{MS}}, \end{aligned} \quad (3.4)$$

where  $m_E$  and  $m_M$  are masses of Earth and Mars, respectively. Subscripts  $ES$  ( $MS$ ) correspond to the motion of Earth (Mars) with respect to the Sun. These trajectories can be calculated using the exact solution (2.10) with appropriate strength constants  $k$  and initial conditions which determine  $E$  and  $\ell$ . Another way is to solve the Euler-Lagrange equations numerically, using astronomical parameters [20] (e.g. aphelion and perihelion of Earth/Mars) to choose the initial conditions that fit the observed data. The former has been done using *Wolfram Mathematica* package. The result is shown on Fig. 1.

For latter comparison, one could write out the expressions for the  $e$  and  $p$  parameters for the Earth. Putting the expressions for energy (2.7) and momentum (2.6) into



**Figure 1.** Trajectories of Earth and Mars in heliocentric system over the period of 2 years. Blue and red lines represent Earth's and Mars' orbits, respectively (color online).

Equations (2.11) it is straightforward to obtain

$$p = \frac{\dot{\phi}^2 r^4}{GM_S},$$

$$e = \sqrt{1 - \frac{2GM_S \dot{\phi}^2 r^3 - \dot{r}^2 \dot{\phi}^2 r^4 - \dot{\phi}^4 r^6}{G^2 M_S^2}}, \quad (3.5)$$

where  $\dot{\phi}$ ,  $\dot{r}$  and  $r$  are angular velocity, radial velocity and distance respectively, taken in the same moment of time (e.g. in  $t = 0$ ).

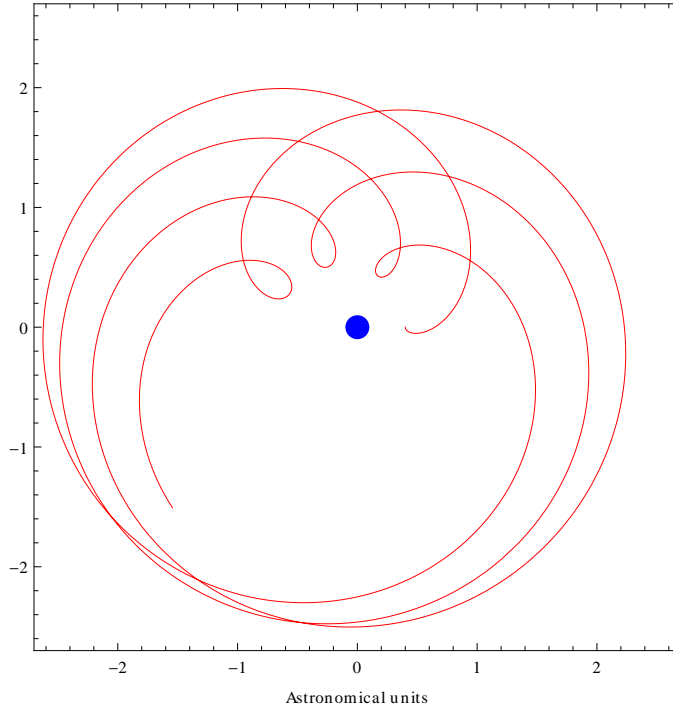
Fig. 2 displays motion of the Mars as viewed from the Earth, gained by trivial coordinate transformation

$$\mathbf{r}_{ME}(t) = -\mathbf{r}_{ES}(t) + \mathbf{r}_{MS}(t), \quad (3.6)$$

where  $\mathbf{r}_{ES}(t)$  and  $\mathbf{r}_{MS}(t)$  are solutions of Euler-Lagrange equations for the Lagrangians (3.4). Equation (3.6) is just the mathematical expression of the Tycho Brahe's claim. Retrograde motion of the Mars can be useful in the attempt to understand and determine orbital parameters, as was qualitative and quantitative shown by B. Thompson [21].

The acceleration that Earth experiences due to the gravitational force of the Sun is usually referred as *centripetal acceleration* and is given by

$$\mathbf{a}_{cp} = \frac{\mathbf{F}_{cp}}{m_E} = -\frac{GM_S}{r_{ES}^2} \hat{\mathbf{r}}_{ES}, \quad (3.7)$$



**Figure 2.** Trajectory of the Mars as seen from the Earth over the period of 7 years. Calculation of this trajectory is done numerically in the heliocentric system.

where  $\hat{\mathbf{r}}$  is the unit vector in the direction of vector  $\mathbf{r}$ ,  $\mathbf{r}_{ES}(t)$  is radius vector describing motion of Earth with respect to the Sun, and  $\mathbf{F}_{cp}$  is centripetal force, i.e. the force that causes the motion.

#### 4. Sun and Mars in geocentric perspective

##### 4.1. The pseudo-potential

From the heliocentric perspective, the fact that the Earth moves around the Sun results with centrifugal pseudo-force, observed only by the observer on the Earth. But if we apply Mach's principle to the geocentric viewpoint, one is obliged to speak about the *real* forces resulting from the fact that the Universe as a whole moves around the observer sitting on the stationary Earth. Although these forces will further be considered as the real forces, we will keep the usual terminology and call them pseudo-forces, for the sake of convenience. Our focus here will be on the annual orbits, not on diurnal rotation which requires some additional physical assumptions [8] [22] that are beyond the scope of this paper.

The Universe is regarded as an  $(N + 1)$ -particle system ( $N$  celestial bodies plus planet Earth). From the point of a stationary Earth, one can write down the Lagrangian that describes the motions of celestial bodies:

$$L = \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i^2 - \frac{1}{2} \sum_{i=1}^N \frac{Gm_i m_j}{r_{ij}} - \sum_{i=1}^N \frac{Gm_E m_i}{r_i} - U_{ps}, \quad (4.1)$$

where  $r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|$ ,  $U_{ps}$  stands for pseudo-potential, satisfying  $\mathbf{F}_{ps} = -\nabla U_{ps}$ .  $\mathbf{F}_{ps}$  is the pseudo-force given by

$$\mathbf{F}_{ps} = -m \sum_{i=1}^N \mathbf{a}_{cp,i}, \quad (4.2)$$

where  $\mathbf{a}_{cp,i}$  is centripetal acceleration for given celestial body (with respect to the Earth) and  $m$  is a mass of the object that is subjected to this force. It's easy to notice that the dominant contribution in these sums comes from the Sun. The close objects (planets, moons, etc) are much less massive than the Sun, and massive object are much further away. The same approximation is implicitly used in section 3.

In the Machian picture, the centripetal acceleration is a mere relative quantity, describing the rate of change of relative velocity. Therefore, centripetal acceleration of the Sun with respect to Earth is given by Equation (3.7), with  $\mathbf{r}_{ES} = -\mathbf{r}_{SE}$ . All that considered, Equation (4.2) becomes

$$\mathbf{F}_{ps} = -\frac{GmM_S}{r_{SE}^2} \hat{\mathbf{r}}_{SE}, \quad (4.3)$$

where  $\mathbf{r}_{SE}(t)$  describes the motion of the Sun around the Earth, and  $m$  is the mass of the body under consideration.

We can now finally write down the pseudo-potential which influences every body observed by still observer on Earth:

$$U_{ps}(\mathbf{r}) = \frac{GmM_S}{r_{SE}^2} \hat{\mathbf{r}}_{SE} \cdot \mathbf{r}, \quad (4.4)$$

where  $\mathbf{r}(t)$  describes motion of particle of mass  $m$  with respect to the Earth. Notice that this is not a central potential.

#### 4.2. Sun in Earth's pseudo-potential

In order to determine Sun's orbit in Earth's pseudo-potential, one needs to take dominant contributions of the Lagrangian (4.1), as was explained earlier. Taking into account the expression for pseudo-potential given in Equation (4.4), one ends up with

$$L_{SE} = \frac{1}{2} M_S \dot{\mathbf{r}}_{SE}^2 - \frac{GM_S^2}{r_{SE}}. \quad (4.5)$$

This Lagrangian has the exact same form as the reduced Lagrangian (2.3). That means that we can immediately determine the orbit by means of Equations (2.11) by substituting  $\mu = M_S$  and  $k = GM_S^2$ . This leads to the following result (subscript  $SE$  will be omitted):

$$p = \frac{\dot{\phi}^2 r^4}{GM_S},$$

$$e = \sqrt{1 - \frac{2GM_S \dot{\phi}^2 r^3 - \dot{r}^2 \dot{\phi}^2 r^4 - \dot{\phi}^4 r^6}{G^2 M_S^2}}, \quad (4.6)$$



which is the exact equivalent as the previous result given in Equations (3.5), since  $\dot{\phi}$ ,  $\dot{r}$  and  $r$  are relative quantities, by definition equivalent in both models. We can therefore conclude that the Sun's orbit in the Earth's pseudo-potential is equivalent to that observed from the Earth in the heliocentric system.

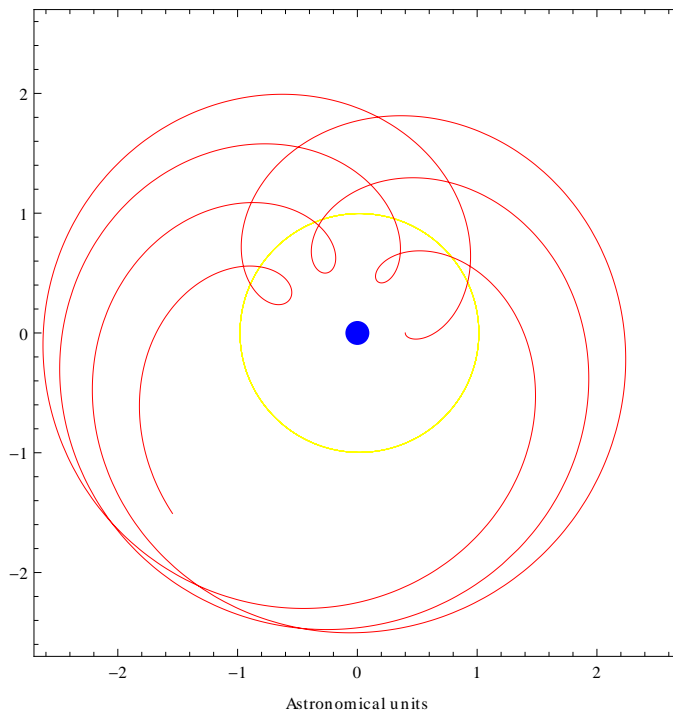
It remains to show the same thing for Mars' orbit.

#### 4.3. Mars in Earth's pseudo-potential

In the similar way as before, we take the dominant contributions of Lagrangian (4.1) together with Equation (4.4) and form the following Lagrangian

$$L_{ME} = \frac{1}{2}m_M\dot{\mathbf{r}}_{ME}^2 + \frac{Gm_MM_S}{|\mathbf{r}_{ME} - \mathbf{r}_{SE}|} - \frac{Gm_MM_S}{r_{SE}^2}\hat{\mathbf{r}}_{SE} \cdot \mathbf{r}_{ME}, \quad (4.7)$$

where subscript  $ME$  refers to the motion of Mars with respect to the Earth, and  $\mathbf{r}_{SE}(t)$  is the solution of Euler-Lagrange equations for the Lagrangian (4.5).



**Figure 3.** Trajectories of the Sun (yellow) and the Mars (red) moving in Earth's pseudo-potential over the period of 7 years (color online). Calculation of this trajectory is performed numerically in the geocentric system.

The Euler-Lagrange equations for  $\mathbf{r}_{ME}(t)$  using Lagrangian (4.7) are too complicated to be solved analytically, but they can easily be solved numerically. The numerical solutions for the equations of motion for both the Sun and Mars are displayed in Fig. 3. The equivalence of trajectories gained in two different ways is obvious, justifying the model proposed by Tycho Brahe.

## 5. Conclusion

The analysis of planetary motions has been performed in the Newtonian framework with the assumption of Mach's principle. The kinematical equivalence of the Copernican (heliocentric) and the Neo-tychonian (geocentric) systems is shown to be a consequence of the presence of pseudo-potential (4.4) in the geocentric system, which, according to Mach, must be regarded as the real potential originating from the fact of the simultaneous acceleration of the Universe. This analysis can be done on any other celestial body observed from the Earth. Since Sun and Mars are chosen arbitrarily, and there is nothing special about Mars, one can expect to come up with the same general conclusion.

There is another interesting remark that follows from this analysis. If one could put the whole Universe in accelerated motion around the Earth, the pseudo-potential corresponding to pseudo-force (4.2) will immediately be generated. That same pseudo-potential then causes the Universe to stay in that very state of motion, without any need of exterior forces acting on it.

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