Interpretation of special relativity as applied to earth-centered locally inertial coordinate systems in Global Positioning System satellite experiments

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Abstract   In the Global Positioning System (GPS) satellites, either the earth-centered, earth-fixed, reference frame (ECEF frame) or the earth-centered locally inertial (ECI) coordinate system is used for calculations. In the application of the theory of special relativity to GPS satellites, we have to use the ECEF frame, which is practically a reference frame which is at rest. We point out that another reference frame at rest (for example, one based on the cosmic microwave background or the solar system) can be applied to GPS satellites experiments, however, the calculation needs not only special relativity, but also general relativity.

1. Introduction

The theory of relativity is used in our daily lives. The Global Positioning System (GPS) is used in car navigation systems. The use of special relativity in GPS has been summarized by Ashby [1]. The GPS satellites orbit in a region of low gravity (~20,000 km from ground level) at 4 km/s. Therefore, the difference in gravitational potential between the ground and the GPS satellites, and the effect of special relativity on the motion of the GPS satellites, which is a transverse Doppler shift, are considered. The transverse Doppler shift, or second-order Doppler shift, can be calculated by the Lorentz transformation of time. In GPS, the earth-centered, earth-fixed, reference frame (ECEF frame) or the earth-centered locally inertial (ECI) coordinate system is used for the calculation. The ECEF frame is practically a reference frame at rest (a stationary reference frame).

Twin paradox experiments are carried out every day in the GPS satellites. The difference in gravitational potential between the GPS satellites and the ground causes a 45.7 μs time gain every day. The transverse Doppler shift that can be calculated by the Lorentz transformation of time results in a 7.1 μs time delay every day. We restrict this discussion to within the special relativity theory, that is, we do not consider the effect of the gravitational potential. Thus, the traveling twin is an atomic clock in a GPS satellite, and the stationary twin is an atomic clock on earth. The traveling twin becomes another 7.1 μs "younger" than the twin on earth every day.

As mentioned above, one of a pair of twins who returns from a space trip will be younger than the other one

who remained on earth. However, from the viewpoint of relativity, the twin who has returned from the space trip can claim that the other twin who stayed on earth moved away from the space ship: this is the twin paradox.

Twin paradox experiments have been carried out in GPS satellites; the experimental results show that we see "old" and "young" atomic clocks. The results predicted by the theory of special relativity are correct, however, the interpretation of relativity gives rise to the paradox. That is, we can claim that the clock in the GPS satellite underwent motion relative to the clock on earth. Of course, the situation is different from the viewpoints of the GPS satellite and earth because the GPS satellite undergoes accelerated motion. It is easy to resolve the paradox if we adopt a reference frame at rest, while it is rather difficult without a reference frame at rest.

Finally, using simple numerical calculations, we show that if an inertial frame at rest is introduced, we will be able to predict the periodic orbital reference time deviation in GPS satellites. However, in the GPS satellite experiments, such a large periodic orbital deviation that critically depends on the motion of the orbital plane of the GPS satellites in the cosmic microwave background has not been detected [1]. Therefore, only the ECI coordinate system is correct. However, this conclusion looks incompatible with the orthodox interpretation, that is, there is no reference frame at rest. In this paper, one possible solution is proposed.
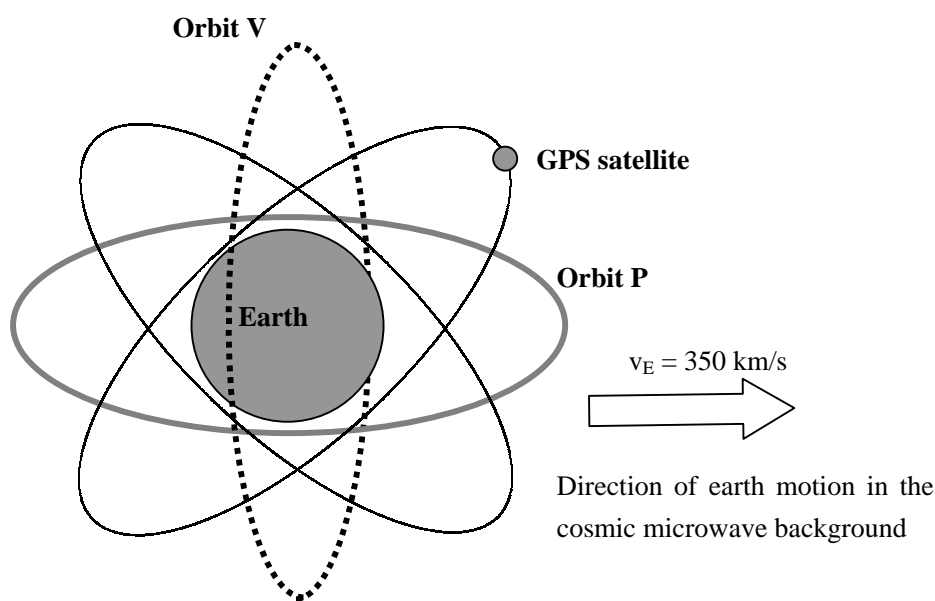

Fig. 1 Earth motion in the cosmic microwave background. Orbit P is parallel and orbit V is perpendicular to the direction of earth motion in the cosmic microwave background. In the cosmic microwave background, v_E= 350 km/s and in the solar system, v_E= 30 km/s.

2. Consideration of the ECEF frame

The time of the GPS satellite is calculated based on the ECEF frame and is operating well. Why does the inertial system of the ECEF frame operate well? Twenty four satellites are launched on six orbits and GPS is operating well by the ECEF frame. It is supposed that the earth is moving at 350 km/s in the cosmic microwave background. The relative velocity of the earth in the cosmic microwave background is set to v_E. The cosmic microwave background is assumed to be a reference frame at rest, and the time delay of the GPS
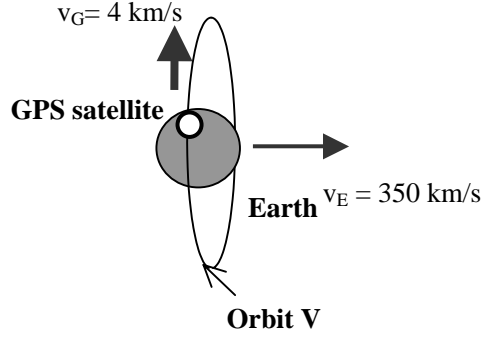
Fig. 2 Orbit V: the orbital plane of the GPS satellite is perpendicular to the direction of the earth motion in the cosmic microwave background.

satellite is examined on the basis of the reference time on earth. There are two types of GPS orbits, one is the orbit P that is parallel to v_E, and the other is the orbit V that is perpendicular to v_E. The orbit P and V are shown in **Fig.1**. The orbit V is shown in **Fig.2**, where a motion of the GPS satellite is perpendicular to v_E, thus causing the orbit V to become a spiral trajectory. The velocity of the GPS satellite relative to the earth is set to v_G. When the reference time of the reference frame at rest is set to t_0 and that of the earth is set to t_E and is expressed with t_0, the Lorentz transformation is as follows,

$$t_E = \frac{t_0}{\sqrt{1 - \left(\dfrac{350}{300000}\right)^2}} \quad . \tag{1}$$

Since the motion of a GPS satellite is perpendicular to that of the earth, that is $v_E \quad v_G$, the reference time t_G of the GPS is obtained by the following equation. This is because $v_E \quad v_G$ thus we can use Pythagorean proposition to obtain the summation of v_E and v_G.

$$t_G = \frac{t_0}{\sqrt{1 - \left(\dfrac{350^2 + 4^2}{300000^2}\right)}} \quad . \tag{2}$$

Therefore, the proportion of the time delay over the earth of the GPS satellite is as follows

$$\therefore \frac{t_E - t_G}{t_E} = 1 - \frac{\sqrt{1 - \left(\dfrac{350}{300000}\right)^2}}{\sqrt{1 - \left(\dfrac{350^2 + 4^2}{300000^2}\right)}} = -0.888890 \times 10^{-10} \quad . \tag{3}$$

Even if it is calculated in the ECEF frame, it is only after the 6th figure that a difference appears, as shown below

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$$\therefore \frac{t_0 - t_G}{t_0} = 1 - \frac{1}{\sqrt{1 - \left(\dfrac{4^2}{300000^2}\right)}} = -0.888889 \times 10^{-10} \quad . \tag{4}$$

Thus we obtain, $0.88889 \times 10^{-10} \times 60 \times 60 \times 24 = 7.6$ µs, and it becomes the delay of 7.6 µs per day (There is a difference with the experimental data of 7.1 µs, this difference critically depends on the value of the velocity v_G, in equations (3) and (4) we set $v_G = 4$ km/s).
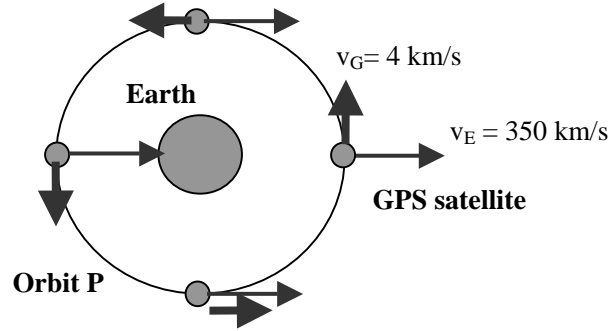

Fig. 3   The relative velocity of GPS satellite in the cosmic microwave background (orbit P): the relative velocity of the GPS satellite in the cosmic microwave background, where the orbital plane of the GPS satellite parallel to the direction of the earth motion in the cosmic microwave background has an orbital deviation of the reference time t_G^P. However, the calculated result predicted by this illustration is incompatible with the experimental data, that is, no periodic deviation was detected: we should adopt the illustration in **Fig. 4**.

Next, the orbit P is considered. In **Fig. 3**, the summation of the velocity of the earth v_E and the velocity of the GPS satellite in the cosmic microwave background is periodically changed. In this case, it becomes accelerated motion and the discussion using general relativity should be required. The orbit P of the GPS satellite and the orbit of the earth seen from arbitrary inertial systems are shown in **Fig. 4**. The orbit P is a cycloid thus acceleration and deceleration are repeated. The periodic derivation of the reference time t_G^P is expected; for example, the reference time t_G^P is calculated using equation (5) only from the theory of special relativity by setting $v_E = 350$ km/s and $v_G = 4$ km/s without including the effect of the theory of general relativity.

$$t_G^P = \frac{t_0}{\sqrt{1 - \left(\dfrac{v_E + v_G(t)}{c}\right)^2}} \tag{5}$$

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where t_G^P is the reference time of the GPS satellite when v_E and v_G are parallel, and the deviation is calculated as $\Delta t_G^P = \pm 1.56 \times 10^{-8}$. However, the deviation Δt_G^P becomes two orders larger comparing with the value -0.88889×10^{-10} obtained from equation (3). If the solar system is assumed to be an inertial system, that is v_E=30 km/s, there is a difference of one order larger comparing with the value obtained from equation (3). However, the ECEF frame operates well by the GPS satellites, that is, no periodic deviation is observed which depends on the orbits. The deviation of the reference time of the orbit P is similar to that of the orbit V.
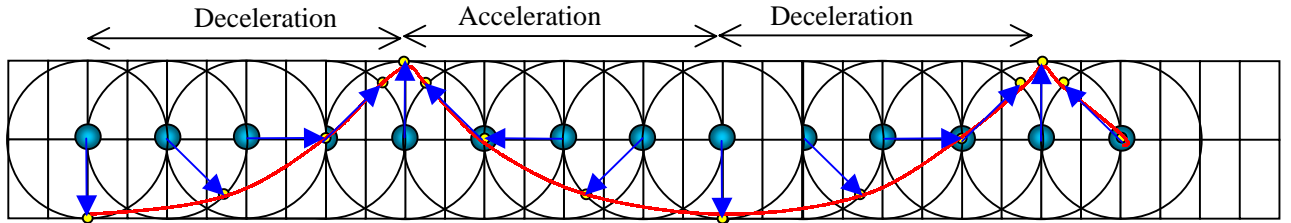

Fig. 4 Traveling path of the GPS satellite of orbit P in an arbitrary reference frame.

A clock in the satellite travels according to the red line. Therefore, a photon in an atomic clock in the satellite travels a further distance than an atomic clock on earth, which causes time delay in the satellite. From an arbitrary reference frame, satellite motion is periodic, orbital accelerated motion (that is, cycloid). I conclude that the calculation that includes acceleration will show the correct result: any inertial reference frame will give the same result.

Since the deviation of the reference time must be in agreement with the result of the ECEF frame, therefore the calculation using the ECEF frame is including not only the effects of the theory of special relativity, but also of general relativity. The orbit P and V are also considered so that the same equation of the reference time can be used as follows,

$$t_G = \frac{t_0}{\sqrt{1 - \dfrac{v_E^2 + v_G^2(t)}{c^2}}} \quad . \tag{6}$$

If the effect of the special relativity expressed by equation (6) is used, it seems that in any arbitrary orbit, the time delay of the GPS satellite will become the same. According to Ashby [1], there is no significant periodic deviation observed. This is considered that the discussion of equation (5) does not consider the effect of theory of general relativity. If the GPS satellite is seen from any inertial system, the effects of special relativity (that is, a periodic motion of velocity) and general relativity (the effect of acceleration and deceleration) offset each other, therefore equation (6) can be used. At present, the effect of the theory of general relativity has not been

calculated in this report.

At first, I did not know the reason why the ECEF frame or the ECI coordinate worked well. I considered that it may be caused by the Lense-Thirring effect, which is an effect that is produced by a gravitational field of the earth pulls an inertial system. This effect was predicted in the early stages of general relativity, and was checked in an experiment done in recent years [2]. I thought that the gravitational field of the earth had pulled the surrounding inertial system by the Lense-Thirring effect. At this stage, I do not consider the GPS experimental results to have any relation with the Lense-Thirring effect. I consider that either the ECEF frame or the ECI coordinate works correctly under the consideration of the theory of special and general relativity.

3. Conclusion

We discussed why either the ECEF frame or the ECI coordinate works well. We concluded that any inertial reference frame gives the same time delay result for the GPS satellite experiments. That is, the GPS experiments should be discussed from the viewpoint of general relativity (including the effect of acceleration) as well as special relativity (the effect of velocity).

References
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