

Can we derive the Lorentz force from Maxwell's equations?

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The Lorentz force can be obtained from Maxwell's equations in the Coulomb gauge provided that we assume that the electric portion of the force acted on a charge is known, and the magnetic component is perpendicular to the velocity of motion of the charged particle.

Strictly speaking, the Lorentz force can not be derived merely from Maxwell's equations. To find it, additional postulates are needed. As you will see below, these postulates are too strong in order to view the procedure as a derivation. However, the job is not useless. For, it helps us to comprehend the structure of classical electrodynamics.

We proceed from the general form of Maxwell's equations

$$\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} + \nabla \varphi = 0 \quad (1)$$

$$\frac{\partial \mathbf{E}}{\partial t} - c \nabla \times (\nabla \times \mathbf{A}) + 4\pi \rho \mathbf{v} = 0 \quad (2)$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho \quad (3)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (4)$$

which will be taken in the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0 \quad (5)$$

The system (1)-(5) is not complete. For, it includes an uncoupled function $\mathbf{v}(\mathbf{x}, t)$. From the physical point of view the system of Maxwell's equations describes only kinematics of motion of an electric charge. In order to close it up, we must supplement (4) with a dynamic equation

$$\rho \frac{dv_i}{dt} = \frac{\partial \sigma_{ik}}{\partial x_k} + \rho f_i \quad (6)$$

where a stress tensor σ_{ik} and the term \mathbf{f} of an external force should be defined. The portions of \mathbf{f} are found from the same Maxwell's equations. Thus, the problem can be posed as follows: to define the minimal set of additional assumptions and, using them, to extract \mathbf{f} from (1)-(5). Before introducing new assertions we will do some preparatory work for the second step of the problem.

From (1) and (2) the well-known integral can be obtained:¹

$$\frac{1}{8\pi} \frac{\partial}{\partial t} \int [\mathbf{E}^2 + (\nabla \times \mathbf{A})^2] d^3x + \int \rho \mathbf{v} \cdot \mathbf{E} d^3x = 0 \quad (7)$$

Manipulating (1) and (2) in another manner we may construct the following relation:

$$\frac{1}{8\pi} \frac{\partial}{\partial t} \int [(\frac{\partial \mathbf{A}}{c \partial t})^2 + (\nabla \times \mathbf{A})^2] d^3x - \int \rho \mathbf{v} \cdot \frac{\partial \mathbf{A}}{c \partial t} d^3x = 0 \quad (8)$$

(see Appendix A). We have from (1)

$$\mathbf{E}^2 = -\mathbf{E} \cdot \frac{\partial \mathbf{A}}{c \partial t} - \mathbf{E} \cdot \nabla \varphi = (\frac{\partial \mathbf{A}}{c \partial t})^2 + \nabla \varphi \cdot \frac{\partial \mathbf{A}}{c \partial t} - \mathbf{E} \cdot \nabla \varphi \quad (9)$$

Substitute (9) into (7) taking integrals by parts and using (3) and (5):

$$\frac{1}{8\pi} \frac{\partial}{\partial t} \int [(\frac{\partial \mathbf{A}}{c \partial t})^2 + 4\pi \rho \varphi + (\nabla \times \mathbf{A})^2] d^3x + \int \rho \mathbf{v} \cdot \mathbf{E} d^3x = 0 \quad (10)$$

Subtract (8) from (10):

$$\frac{1}{2} \frac{\partial}{\partial t} \int \rho \varphi d^3x + \int \rho \mathbf{v} \cdot \frac{\partial \mathbf{A}}{c \partial t} d^3x + \int \rho \mathbf{v} \cdot \mathbf{E} d^3x = 0 \quad (11)$$

Next, we will consider the system of two point electric charges at $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$:

$$\rho(\mathbf{x}, t) = q_1 \delta(\mathbf{x} - \mathbf{x}_1) + q_2 \delta(\mathbf{x} - \mathbf{x}_2) \quad (12)$$

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_1(t) I(\mathbf{x} - \mathbf{x}_1) + \mathbf{v}_2(t) I(\mathbf{x} - \mathbf{x}_2) \quad (13)$$

$$\mathbf{v}_1 = \frac{d\mathbf{x}_1}{dt}, \quad \mathbf{v}_2 = \frac{d\mathbf{x}_2}{dt}$$

where $\delta(\mathbf{x})$ is the Dirac delta-function and $I(\mathbf{x})$ the indicator function. Substituting (12) and (13) into (11) gives

$$\frac{1}{2} (q_1 + q_2) \frac{\partial \varphi}{\partial t} + \int [(\delta(\mathbf{x} - \mathbf{x}_1) q_1 \mathbf{v}_1 + \delta(\mathbf{x} - \mathbf{x}_2) q_2 \mathbf{v}_2)] \cdot \frac{\partial \mathbf{A}}{c \partial t} d^3x + (q_1 \mathbf{v}_1 + q_2 \mathbf{v}_2) \cdot \mathbf{E} = 0 \quad (14)$$

Remark that in Maxwell's equations the fields are the functions of \mathbf{x} and t . After the above integration over the space coordinate they become functions of $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$. Let us extract from (14) cross terms. We have from Maxwell's equations for the electrostatic potential:

$$\varphi = \varphi_1 + \varphi_2 \quad (15)$$

$$\varphi_1 = q_1 \phi(|\mathbf{x}_2 - \mathbf{x}_1|) \quad (16)$$

$$\varphi_2 = q_2 \phi(|\mathbf{x}_1 - \mathbf{x}_2|) \quad (17)$$

where a function ϕ is written in variables after the integration. That gives for cross terms in the first term of (14):

$$\frac{1}{2} (q_1 \varphi_2 + q_2 \varphi_1) = q_1 \varphi_2 = q_2 \varphi_1 \quad (18)$$

We have from Maxwell's equations for the magnetic vector-potential:

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 \quad (19)$$

$$\mathbf{A}_1 = q_1 \mathbf{v}_1 \alpha(|\mathbf{x} - \mathbf{x}_1|) \quad (20)$$

$$\mathbf{A}_2 = q_2 \mathbf{v}_2 \alpha(|\mathbf{x} - \mathbf{x}_2|) \quad (21)$$

where a function α is written in variables before the integration. Then we have for cross terms in the second term of (14):

$$\frac{\partial \mathbf{A}_1}{\partial t} = q_1 \alpha \frac{\partial \mathbf{v}_1}{\partial t} + q_1 \mathbf{v}_1 \frac{\partial \alpha}{\partial t} \cdot \frac{\partial \alpha(|\mathbf{x} - \mathbf{x}_1|)}{\partial \mathbf{x}_1}$$

$$\frac{\partial \mathbf{A}_2}{\partial t} = q_2 \alpha \frac{\partial \mathbf{v}_2}{\partial t} + q_2 \mathbf{v}_2 \frac{\partial \alpha}{\partial t} \cdot \frac{\partial \alpha(|\mathbf{x} - \mathbf{x}_2|)}{\partial \mathbf{x}_2}$$

$$\int \delta(\mathbf{x} - \mathbf{x}_1) q_1 \mathbf{v}_1 \cdot \frac{\partial \mathbf{A}_2}{\partial t} d^3x = q_1 q_2 \left[\alpha \mathbf{v}_1 \cdot \frac{\partial \mathbf{v}_2}{\partial t} + (\mathbf{v}_1 \cdot \mathbf{v}_2) \frac{\partial \alpha}{\partial t} \cdot \frac{\partial \alpha(|\mathbf{x}_1 - \mathbf{x}_2|)}{\partial \mathbf{x}_2} \right] \quad (22)$$

$$\int \delta(\mathbf{x} - \mathbf{x}_2) q_2 \mathbf{v}_2 \cdot \frac{\partial \mathbf{A}_1}{\partial t} d^3x = q_1 q_2 \left[\alpha \mathbf{v}_2 \cdot \frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_2 \cdot \mathbf{v}_1) \frac{\partial \alpha}{\partial t} \cdot \frac{\partial \alpha(|\mathbf{x}_2 - \mathbf{x}_1|)}{\partial \mathbf{x}_1} \right] \quad (23)$$

Summing up (22) and (23) we get the cross terms of the second term in (14):

$$\begin{aligned} \frac{1}{c}q_1q_2\left[\alpha\frac{\partial(\mathbf{v}_1\cdot\mathbf{v}_2)}{\partial t}+(\mathbf{v}_1\cdot\mathbf{v}_2)\frac{\partial\alpha}{\partial t}\right]&=\frac{1}{c}q_1q_2\frac{\partial}{\partial t}(\alpha\mathbf{v}_1\cdot\mathbf{v}_2) \\ &=\frac{1}{c}q_1\frac{\partial}{\partial t}(\mathbf{v}_1\cdot\mathbf{A}_2)=\frac{1}{c}q_2\frac{\partial}{\partial t}(\mathbf{v}_2\cdot\mathbf{A}_1) \end{aligned} \quad (24)$$

We have for the electric field

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2 \quad (25)$$

Then cross terms in the third term of (14) will be

$$q_1\mathbf{v}_1\cdot\mathbf{E}_2+q_2\mathbf{v}_2\cdot\mathbf{E}_1 \quad (26)$$

Gathering (18), (24) and (26) gives for (14):

$$\frac{\partial}{\partial t}(q_1\varphi_2+\frac{1}{c}q_1\mathbf{v}_1\cdot\mathbf{A}_2+\varepsilon_0)+q_1\mathbf{v}_1\cdot\mathbf{E}_2+q_2\mathbf{v}_2\cdot\mathbf{E}_1+w_0=0 \quad (27)$$

where ε_0 and w_0 are self-interaction terms. Expression (27) was obtained from Maxwell's equations and it is a key relation for further calculations.

Now, some assumptions will be done concerning the form of the force term \mathbf{f} in (6). We postulate for the equation of motion of a point charge

$$m_1\frac{d\mathbf{v}_1}{dt}=q_1\mathbf{E}_2+\mathbf{v}_1\times(\dots) \quad (28)$$

The last term in (24) means simply that the magnetic force is perpendicular to the velocity of motion. Multiply (28) by \mathbf{v}_1 :

$$\frac{d}{dt}\left(\frac{1}{2}m_1\mathbf{v}_1^2\right)=q_1\mathbf{v}_1\cdot\mathbf{E}_2 \quad (29)$$

Substituting (29) into (27) we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}m_1\mathbf{v}_1^2+\frac{1}{2}m_2\mathbf{v}_2^2+q_1\varphi_2+\frac{1}{c}q_1\mathbf{v}_1\cdot\mathbf{A}_2+\varepsilon_0\right)+w_0=0 \quad (30)$$

Expression (30) enables us to construct the interaction Lagrangian

$$L=\frac{1}{2}m_1\mathbf{v}_1^2+\frac{1}{2}m_2\mathbf{v}_2^2-q_1\varphi_2+\frac{1}{c}q_1\mathbf{v}_1\cdot\mathbf{A}_2 \quad (31)$$

We get from (31) the exact form of (28) for a first charge moving in the field of a second charge

$$m_1\frac{d\mathbf{v}_1}{dt}=q_1\mathbf{E}_2+\frac{1}{c}q_1\mathbf{v}_1\times(\nabla\times\mathbf{A}_2) \quad (32)$$

APPENDIX A

In derivation of the integrals (7) and (8) we proceed from Maxwell's equations (1) and (2). To obtain (7) we take the curl of (1):

$$\frac{\partial}{\partial t}\nabla\times\mathbf{A}+c\nabla\times\mathbf{E}=0 \quad (A1)$$

Multiply (2) by \mathbf{E} and (A1) by $\nabla\times\mathbf{A}$. Summing up the results we get

$$\frac{1}{2}\frac{\partial}{\partial t}\mathbf{E}^2-c\mathbf{E}\cdot\nabla\times(\nabla\times\mathbf{A})+4\pi\rho\mathbf{v}\cdot\mathbf{E}+\frac{1}{2}\frac{\partial}{\partial t}(\nabla\times\mathbf{A})^2+c(\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{A})=0 \quad (A2)$$

Integrate (A2) over the whole space and take the second integral by parts supposing that the fields are vanishing at infinity. Then the respective integrals obtained from the second and fifth terms of (A2) cancel each other. Thus we come to (7) sought for.

In order to derive (8) we firstly operate (1) with ∂_t :

$$\frac{\partial^2 \mathbf{A}}{c\partial t^2} + \frac{\partial \mathbf{E}}{\partial t} + \nabla \frac{\partial \varphi}{\partial t} = 0 \quad (\text{A3})$$

Then exclude $\partial \mathbf{E}/\partial t$ from (A3) and (2):

$$\frac{\partial^2 \mathbf{A}}{c\partial t^2} + c \nabla \times (\nabla \times \mathbf{A}) + \nabla \frac{\partial \varphi}{\partial t} = 4\pi \rho \mathbf{v} \quad (\text{A4})$$

Multiply (A4) by $\partial \mathbf{A}/\partial t$:

$$\frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{A}}{\partial t} \right)^2 + c \frac{\partial \mathbf{A}}{\partial t} \cdot \nabla \times (\nabla \times \mathbf{A}) + \frac{\partial \mathbf{A}}{\partial t} \cdot \nabla \frac{\partial \varphi}{\partial t} = 4\pi \rho \mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial t} \quad (\text{A5})$$

Integrate (A5) over the whole space. Take the intergals of the second and third terms by parts. The third integral vanishes due to (5). Thus we arrive at the relation (8) sought for.

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¹ L.D.Landau and E.M.Lifshitz, *Mechanics. Electrodynamics*, (Nauka, Moscow, 1969).