Special relativity and the Lorentz equations. Errors in Einstein's 1905 paper

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Abstract: The explanation of Einstein's special theory of relativity in his original 1905 paper is examined. His analysis is confusing, as terms x, y, z, t, etc., have different meanings at various points, and he presents equations based on different and inconsistent assumptions. Adding subscripts clarifies these issues but exposes errors in his reasoning. To calculate his transformation equations, he selects a combination of equations which gives results matching the Lorentz transformation but he ignores other possible valid solutions. Also his calculations contain serious errors. Therefore, he fails to prove that his theory leads to the Lorentz equations as a unique solution. Einstein's analysis includes "moving" clocks which show "stationary" time t, so the idea that a moving clock should run slower than a stationary clock is incompatible with his theory. Also, his calculation of time dilation contains serious errors. As a result, he fails to provide a theoretical justification for his famous "clocks paradox." © *2024 Physics Essays Publication.* [http://dx.doi.org/10.4006/0836-1398-37.1.46]

Résumé: L'explication de la théorie de la relativité restreinte d'Einstein dans son article original de 1905 est examinée. Son analyse prête à confusion, car les termes x, y, z, t, etc. ont des significations différentes à différents moments et il présente des équations basées sur des hypothèses différentes et incohérentes. L'ajout d'indices clarifie ces problèmes mais induit des erreurs dans son raisonnement. Pour calculer ses équations de transformation, il sélectionne une combinaison d'équations qui donne des résultats correspondant à la transformation de Lorentz mais il ignore les autres solutions valides possibles. Ses calculs contiennent également de graves erreurs. Il ne parvient donc pas à prouver que sa théorie conduit aux équations de Lorentz comme solution unique. L'analyse d'Einstein inclut des horloges 'en onseque' qui affichent un temps 'stationnaire' t, donc l'idée selon laquelle une horloge en mouvement devrait fonctionner plus lentement qu'une horloge stationnaire est incompatible avec sa théorie. De plus, son calcul de la dilatation du temps contient de graves erreurs. En conséquence, il ne parvient pas à fournir une justification théorique à son fameux 'paradoxe des horloges'.

Key words: Einstein; Special Relativity; Lorentz Equations; Time Dilation; Transformation Equations; Clocks Paradox; Mathematical Errors.

I. INTRODUCTION

Einstein's special theory of relativity was first published in 1905.[1] In this paper, he presented calculations to demonstrate that his theory leads to transformation equations for time and distance matching those previously derived by Lorentz from ether theory[2] and thus the idea of an ether is "superfluous." Most scientists now accept Einstein's theory and his paper is regarded as one of the most important in modern physics. In 2007, Hawking stated:

> "The details of Einstein's reasoning, and the simple algebra behind it, are explained nowhere better than as found here, in Einstein's own words."[3]

However, there are still some who have raised theoretical objections to Einstein's theory and also questioned its reconciliation with results from experiments and the satellite global positioning system (GPS), e.g., Refs. 4–16. The present paper examines the logic and validity of Einstein's analysis and calculations in Part I (Kinematical Part) of his 1905 paper.

As terms such as x, x', y, and t have more than one possible meaning in his calculations, subscripts are added in the present paper to identify these and clarify the analysis. For ease of cross-reference, Einstein's symbols and general notation are adopted rather than modern notation.

II. DEFINITIONS

In his introduction, Einstein states:

> "… the view to be developed here will not require an 'absolutely stationary space' provided with special properties."

However in subsequent passages he refers to "the stationary system," "the stationary system of co-ordinates," "a stationary rigid rod," "the time of the stationary system," "stationary space," "stationary clocks," and "the time of the

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stationary system;" he also compares the timekeeping of "travelled clocks" which have "moved" with clocks which "remained at rest." To understand this it is necessary to refer to his definition of "stationary system" in Sec. I:

> "Let us take a system of co-ordinates in which the equations of Newtonian mechanics hold good. In order to render our presentation more precise and to distinguish this system of co-ordinates verbally from others which will be introduced hereafter, we call it the 'stationary system.'"

Thus Einstein's "stationary system" is not stationary as normally understood: To comply with his definition it need only be in uniform translatory motion. Also it is not unique: It could be any of the coordinate systems he introduces in Secs. II and III, as these are all in uniform translatory motion. This should be borne in mind whenever he describes an item as stationary.

In Sec. II, he defines the principle of relativity and the principle of the constancy of the velocity of light:

> "1. The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to one or the other of two systems of co-ordinates in uniform translatory motion" (the principle of relativity).

> "2. Any ray of light moves in the 'stationary' system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body" (the principle of the constancy of the velocity of light).

Based on Einstein's definition of the stationary system, principle "2" could be more clearly stated as:

> "In a system of coordinates in uniform translatory motion, any ray of light moves with the determined velocity c, whether the ray be emitted by a body which is stationary or moving relative to that system" (the principle of the constancy of the velocity of light).

III. SYNCHRONISING CLOCKS

At the heart of Einstein's analysis is a procedure for synchronising clocks by light flashes. In Sec. I, he states that if observers and clocks at A and B are at rest in the stationary coordinate system:

> "Let a ray of light start at the 'A time' t_A from A towards B, let it at the 'B time' t_B be reflected at B in the direction of A, and arrive again at A at the 'A time' t'_A. In accordance with definition the two clocks synchronise if

$$t_B - t_A = t'_A - t_B \qquad (1)$$

Einstein then states:

> "It is essential to have time defined by means of stationary clocks in the stationary system."

For this synchronisation method to work, the velocity of light must be constant relative to the observers carrying out the measurements, so these observers and their clocks must be at rest in the relevant coordinate system. However if clock B has a reflective mirror face, observer A can read time t_B directly in its reflected image and it is unnecessary to have an observer at B. Time t_B depends only on the location of clock B at the moment when the light flash reaches it, not its state of motion, so, although Einstein does not discuss this, his light flash synchronisation method will work equally well if clock B is moving.

Therefore, it is possible for a moving clock to be synchronised to show stationary system time. Einstein implicitly accepts this in Sec. II, where he describes a moving rod on which:

> "... clocks are placed which synchronise with the clocks of the stationary system, that is to say that their indications correspond at any instant to the 'time of the stationary system' at the places where they happen to be. These clocks are therefore 'synchronous in the stationary system.'"

IV. THOUGHT EXPERIMENT A

In Sec. III, Einstein describes thought experiments which compare time and space measurements in stationary system K with those in moving system k travelling at velocity v relative to K. Each system has an observer and clock at rest at its origin. In system K, space coordinates are x, y, and z and clocks show time t; in system k, space coordinates are ξ, η, and ζ and clocks show time τ.

In the first thought experiment (referred to here as "experiment A") a light flash emitted at time τ_0 by an observer at the origin of system k is reflected at τ_1 from a clock on the X-axis remote from the origin and it returns to the origin at τ_2. As the remote clock is synchronised with the observer's clock,

$$1/2(\tau_0 + \tau_2) = \tau_1. \qquad (2)$$

Einstein then analyses this scenario as seen by the system K observer. He defines

$$x' = x - vt, \qquad (3)$$

and states

> "... it is clear that a point at rest in the system k must have a system of values x', y, z, independent of time."

He then expresses τ as a function of (x', y, z, t):

$$1/2\left[\tau(0,0,0,t) + \tau\left(0,0,0,t + \frac{x'}{c-v} + \frac{x'}{c+v}\right)\right]$$
$$= \tau\left(x',0,0,t + \frac{x'}{c-v}\right). \qquad (4)$$

In Einstein's equations (1) and (2) different values of variables are identified by subscripts, without which the equations would be meaningless. However, in Eq. (4), there

are no subscripts, despite some of its terms appearing to have more than one meaning

First set: $(\tau = \tau_0) : x' = 0, y = 0, z = 0, t = t$;

second set: $(\tau = \tau_2) : x' = 0, y = 0, z = 0, t = t + \frac{x'}{c-v} + \frac{x'}{c+v}$;

third set: $(\tau = \tau_1) : x' = x', y = 0, z = 0, t = t + \frac{x'}{c-v}$.

Before proceeding further, the meanings of these terms need to be determined. In the first set, it is fairly obvious that the meaningless "$t = t$" should actually be "$t = t_0$," i.e., the system K time when the flash is emitted, corresponding to τ_0 in Eq. (2). Similarly, in the second set, having $x' = 0$ and also $t = t + (x'/(c-v)) + (x'/(c+v))$ makes no sense; logically, the first x' must be the coordinate of the system k origin but the second x' (in the expression for t) must represent some other value, e.g., the coordinate of the remote clock. Likewise, in the third set x' and t must represent particular values of these variables.

Values of (x', y, z, t) which would make sense of Eq. (4) are listed below. In the analysis which follows, these are assumed to have been Einstein's intended meanings.

first set: $(\tau = \tau_0): x' = 0, y = 0, z = 0, t = t_0$,

second set: $(\tau = \tau_2) : x' = 0, y = 0, \quad z = 0, t = t_2 = t_0 + \frac{x'_1}{c-v} + \frac{x'_1}{c+v}$,

third set: $(\tau = \tau_1) : x' = x'_1; y = 0, z = 0, t = t_1 = t_0 + \frac{x'_1}{c-v}$,

where x'_1 is the K coordinate of the remote clock, t_0 is the system K time when the light flash is emitted (corresponding to τ_0), t_1 is the system K time when the light flash reaches the remote clock (corresponding to τ_1), and t_2 is the system K time when the reflected flash reaches the origin of k (corresponding to τ_2) (Fig. 1).

With these subscripts added to identify the meanings of individual terms and "$_\text{A}$" added throughout to identify this as an Experiment A equation, (4) becomes

$$1/2 \left[\tau(0,0,0,t_{A0}) + \tau \left(0,0,0,t_{A0} + \frac{x'_{A1}}{c-v} + \frac{x'_{A1}}{c+v} \right) \right]$$
$$= \tau \left(x'_{A1}, 0, 0, t_{A0} + \frac{x'_{A1}}{c-v} \right). \tag{4a}$$
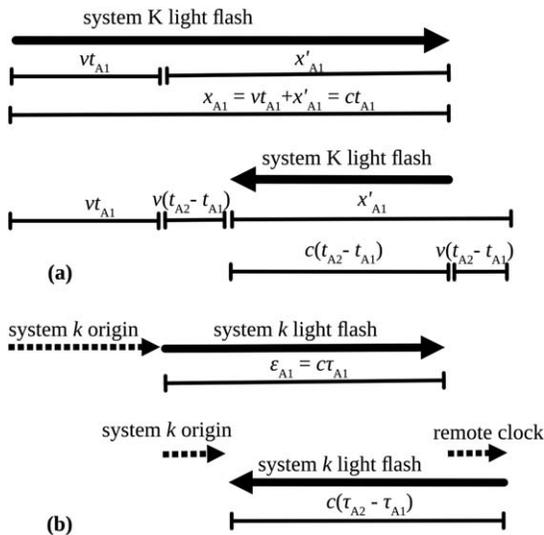

FIG. 1. Experiment A (a) light flash on X-axis as observed in system K. (b) Light flash on X-axis as observed in system k.

For the system k observer, the corresponding equation with coordinates (ξ, η, ζ, τ) is

$$1/2 \left[\tau(0,0,0,\tau_{A0}) + \tau \left(0,0,0, \tau_{A0} + \frac{2\xi_{A1}}{c} \right) \right]$$
$$= \tau \left(\xi_{A1}, 0, 0, \tau_{A0} + \frac{\xi_{A1}}{c} \right), \tag{5}$$

where ξ_{A1} is the k coordinate of the remote clock.

Note: In this experiment, the light flash velocity is c relative to the system K observer and it is also c relative to the system k observer, although these observers are moving relative to one another. To make the analysis easier to understand, in the following discussion the flash is described as if it is a pair of flashes, one travelling at velocity c relative to system k and the other at c relative to system K. In system k the flash travels between points which are all at rest in the system. However, in system K the flash is reflected at t_1 from a point which is moving at velocity v (it is "at rest in the system k" and has coordinate $x = x' + vt$) and it returns at t_2 to the origin of system k, which is also moving at velocity v. Times t_1 and t_2 could be determined by stationary system K observers positioned beside the locations where these occur or, alternatively, moving system K clocks could be positioned alongside the system k clocks and the times on their reflected images could then be read directly by the observer at the system K origin.

After Eq. (4), Einstein states

"Hence, if x' be chosen infinitesimally small:

$$1/2 \left(\frac{1}{c-v} + \frac{1}{c+v} \right) \frac{\partial \tau}{\partial t} = \frac{\partial \tau}{\partial x'} + \frac{1}{c-v} \frac{\partial \tau}{\partial t} \tag{6}$$

or

$$\frac{\partial \tau}{\partial x'} + \frac{v}{c^2 - v^2} \frac{\partial \tau}{\partial t} = 0. \tag{7}$$

…

Since τ is a *linear* function, it follows from these equations that

$$\tau = a \left(t - \frac{v}{c^2 - v^2} x' \right), \tag{8}$$

where a is a function $\varphi(v)$ at present unknown, and where for brevity it is assumed that at the origin of k, $\tau = 0$ when $t = 0$."

Note: At the origin of k, $\xi = x' = x - vt = 0$, so if $t = 0$ the origins of k and K coincide.

Again, Einstein fails to identify the meanings of terms clearly: In Eqs. (6)–(8) do x', t, and τ represent general variables, or particular values of these? If Eq. (4a) interprets Einstein's intended meanings correctly, then in Eqs. (6)–(8) x' should be x'_{A1}, the system K coordinate of the remote clock. However t is probably not the t which appears three times in Eq. (4), which is the experiment start time t_0: it is probably t_1 but is it $t_1 = t_0 + (x'_1/c - v)$, or is it $t_1 = 1/2(t_0 + t_2) = 1/2(t_0 + (x'_1/(c-v)) + (x'_1/(c+v)))$? This is unclear, as these are both combined in Einstein's subsequent calculations and (as discussed below) there are incompatibilities

between the "events" t_1 and t_2 are based on. To reflect this uncertainty, the subscript $_{1?}$ is applied here to t and τ:

$$\tau_{A1?} = a\left(t_{A1?} - \frac{vx'_{A1}}{c^2 - v^2}\right). \tag{8a}$$

Equation (4a) includes three pairs of events:

(i)   outward flashes emitted by the system K and k observers at $\tau = t = 0$;
(i)   reflection of these flashes from the remote clock at τ_{A1} and t_{A1}; and
(iii) the reflected flashes in both systems reaching the system k origin at τ_{A2} and t_{A2}.

For event (i), the origins of k and K coincide, so both flashes are emitted simultaneously from the same location.

For event (ii), in system K the flash path length and arrival time are

$$x_{A1} = \frac{x'_{A1}}{1 - v/c}, \tag{9}$$

$$t_{A1} = \frac{x_{A1}}{c} = \frac{x'_{A1}}{c - v}. \tag{10}$$

In system k, the flash has path length ξ_{A1} and arrival time

$$\tau_{A1} = \xi_{A1}/c. \tag{11}$$

For event (iii), in K the flash path length and arrival time are

$$(x_{A1} - x_{A2}) = \frac{x'_{A1}}{1 + v/c}, \tag{12}$$

$$t_{A2} = \left(t_{A1} + \frac{x'_{A1}}{c + v}\right) = \left(\frac{x'_{A1}}{c - v} + \frac{x'_{A1}}{c + v}\right). \tag{13}$$

In k the flash path length is ξ_{A1} and its arrival time is

$$\tau_{A2} = (\tau_{A1} + \xi_{A1}/c) = 2\xi_{A1}/c. \tag{14}$$

As noted earlier, when synchronising clocks the flash should have equal outward and return path lengths and travel times. Thus if it leaves and returns to the system k origin $\tau_2 = 2\tau_1$ and if it leaves and returns to the system K origin $t_2 = 2t_1$. However in Einstein's equation (4) both flashes return to the origin of system k. Therefore, the outward and return travel times of the flash are equal in system k but in system K they are unequal: $t_{A1} = (x'_{A1}/(c - v))$ and $(t_{A2} - t_{A1}) = (x'_{A1}/c + v)$. Therefore, $\tau_{A2} = 2\tau_{A1}$ but $t_{A2} \neq 2t_{A1}$; also $1/2(\tau_0 + \tau_2) = \tau_1$ but $1/2(t_0 + t_2) \neq t_1$.

Therefore, if the outward flashes are assumed to reach the remote clock simultaneously in both systems [event (ii)], the reflected flashes will not arrive simultaneously at the origin of k [event (iii)]—and *vice versa*. Therefore, events (ii) and (iii) are incompatible.

Einstein's time calculations in Eqs. (4)–(8) are based on a two-way light flash, including both its outward and return paths. However, he now considers a light flash which travels in only one direction:

"For a ray of light emitted at time $t = 0$ in the direction of increasing ξ:

$$\xi = c\tau \tag{15}$$

or

$$\xi = ac\left(t - \frac{v}{c^2 - v^2}x'\right). \tag{16}$$

But the ray moves relatively to the initial point of k, when measured in the stationary system, with the velocity $c{-}v$, so that

$$\frac{x'}{c - v} = t. \tag{17}$$

If we insert this value of t in the equation for ξ, we obtain

$$\xi = a\frac{c^2}{c^2 - v^2}x'. \tag{18}$$

[With subscripts added, these equations become

$$\xi_{A1} = c\tau_{A1}, \tag{15a}$$

$$\xi_{A1} = ac\left(t_{A1?} - \frac{v}{c^2 - v^2}x'_{A1}\right), \tag{16a}$$

$$\frac{x'_{A1}}{c - v} = t_{A1}, \tag{17a}$$

$$\xi_{A1} = a\frac{c^2}{c^2 - v^2}x'_{A1?}. \tag{18a}$$

Note: $X' = x'_{A1?}$ in (18a) because $t = t_{A1?}$ in (16a).]

Einstein does not explain his reasons for the change from a 2-way flash to a 1-way outward-only flash, or how the obvious differences between these scenarios are to be reconciled in his calculations. The change to a 1-way flash does not significantly affect equation (15), as the corresponding equation for a 2-way flash is: $\tau_{A2} = 2\tau_{A1} = 2\xi_{A1}$. However, Eq. (17) is affected by both the change to a 1-way flash and the choice of flash direction: [see Eq. (4)] the time for a 1-way return flash is $t = (x'/c + v)$ and the mean of the 2-way outward and return flash times is $t = 1/2((x'/c - v) + (x'/c + v))$. Therefore, Eq. (8), which is based on a combination of 2-way flash times, is not compatible with (17) and Eq. (18), which Einstein derives by combining Eq. (8) with (17), is invalid.

V. THOUGHT EXPERIMENT B

After Eq. (18), Einstein states:

"In an analogous manner we find, by considering rays moving along the other two axes …."

In this scenario, the origins of k and K coincide at $t = \tau = 0$ and the system k origin moves along the X-axis at velocity v but instead of the light flash travelling to a clock
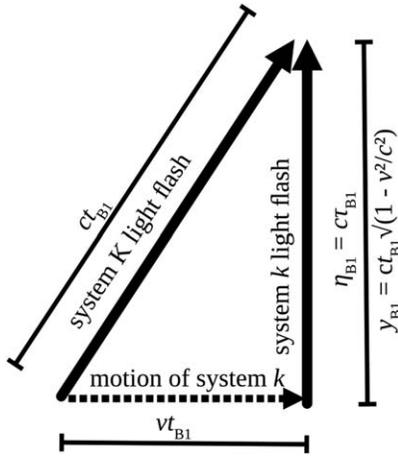
FIG. 2 Experiment B Light flash along Y-axis of system k.

at coordinate ξ on the X-axis, it travels to a clock at coordinate η on the system k Y-axis. Einstein compares how these events are seen by the system k and K observers but, unlike in experiment A Eq. (4), he considers only a 1-way outward flash. As this scenario is significantly different from experiment A (see discussion below), it is considered here separately as "experiment B" (Fig. 2).

Einstein's equations are

$$\eta = c\tau, \tag{19}$$

$$\frac{y}{\sqrt{c^2 - v^2}} = t, \tag{20}$$

$$x' = 0. \tag{21}$$

Unfortunately, once more he fails to identify the meanings of individual terms with subscripts or to identify these as experiment B equations. This is important because

- In experiment A, $x_1 = ct_1$ but in B (see below) $x_1 = vt_1$ (22);
- in experiment A, $x'_1 = t_1(c - v)$ (17) but in B, $x'_1 = 0$ (21);
- in experiment A: $y_1 = 0$ (4a), $\tau_1 = \xi_1/c$ (11) and $\eta_1 = 0$ (4a); but in B: $y_1 = t_1\sqrt{c^2 - v^2}$ (20a), $\xi_1 = 0$ (23), and $\eta_1 = c\tau_1$ (19a).

In each case, it is clear that the experiment A and experiment B equations are incompatible, so any results obtained by combining them will not be meaningful.

If suffix "$_B$" is added generally, together with numerical suffixes as in experiment A, Einstein's experiment B equations become

$$\eta_{B1} = c\tau_{B1}, \tag{19a}$$

$$\frac{y_{B1}}{\sqrt{c^2 - v^2}} = t_{B1}, \tag{20a}$$

$$x'_{B1} = 0, \tag{21a}$$

where τ_{B1} is the system k time when the light flash reaches the remote clock, η_{B1} is its system k coordinate, t_{B1} is the corresponding system K time and y_{B1} is the system K coordinate.

Also

$$x_{B1} = vt_{B1}, \tag{22}$$

$$\xi_{B1} = 0. \tag{23}$$

VI. COMBINING EQUATIONS FROM EXPERIMENTS A AND B

After experiment A equation (7), Einstein stated:

"It is to be noted that instead of the origin of the co-ordinates we might have chosen any other point for the point of origin of the ray, and the equation just obtained is therefore valid for all values of x', y, z …"

He provided no evidence or analysis to support this assertion but he relies on it in his subsequent analysis, where he assumes that equations from experiment A will be valid in experiment B and *vice versa*.

He combines experiment B equation (19) $\eta = c\tau$ with experiment A equation (8) to obtain

$$\eta = ac\left(t - \frac{v}{c^2 - v^2}x'\right). \tag{24}$$

He combines Eq. (24) with (21) to obtain

$$\eta = act. \tag{25}$$

He combines Eq. (25) with (20) to obtain

$$\eta = a\frac{c}{\sqrt{c^2 - v^2}}y. \tag{26}$$

He then substitutes

$$a = \varphi(v)/\beta, \tag{27}$$

$$\beta = 1/\sqrt{1 - v^2/c^2}, \tag{28}$$

to obtain

$$\eta = \varphi(v)y. \tag{29}$$

To obtain his transformation equations Einstein combines (27) and (28) with experiment A equations (8) and (18)

$$\tau = a\left(t - \frac{v}{c^2 - v^2}x'\right) \tag{8}$$

$$\Rightarrow \tau = \varphi(v)\beta\left(t - vx/c^2\right), \tag{30}$$

$$\xi = a\frac{c^2}{c^2 - v^2}x' \tag{18}$$

$$\Rightarrow \xi = \varphi(v)\beta(x - vt), \tag{31}$$

$$\eta = \varphi(v)y, \tag{32}$$

$$\zeta = \varphi(v)z. \tag{33}$$

He states

"If no assumption whatever be made as to the initial position of the moving system and as to the zero point of τ, an additive constant is to be placed on the right side of each of these equations" (see Refs. 17 and 18).

With subscripts added, Eqs. (30) and (31) are

$$\tau_{A1} = \varphi(v)\beta(t_{1?} - vx_1/c^2), \tag{30a}$$

$$\xi_1 = \varphi(v)\beta(x_1 - vt_{1?}). \tag{31a}$$

Einstein then introduces "a third system of co-ordinates K′″ which, according to the Perrett and Jeffery translation, "… relatively to the system k is in a state of parallel translatory motion parallel to the axis of X, such that the origin of coordinates of system k moves with velocity $-v$ …."[19] However this is incorrect: The phrase "of system k" is not in the original German text, so it should read: "… relatively to the system k is in a state of parallel translatory motion parallel to the axis of X, such that the origin of coordinates moves with velocity $-v$ …." Thus K′ is moving at $-v$ relative to k and it is simply system K viewed from system k.

By combining transformation equations for these systems, he obtains

$$t' = \varphi(-v)\beta(-v)(\tau + v\xi/c^2) = \varphi(v)\varphi(-v)t, \tag{34}$$

$$x' = \varphi(-v)\beta(-v)(\xi + v\tau) = \varphi(v)\varphi(-v)x \tag{35}$$

(NB: Here x' is the system K′ coordinate of the system K origin, not $x' = x - vt$ as elsewhere.)

Einstein concludes that K′ and K must be identical, so $\varphi(v)\varphi(-v) = 1$ and, because the length of a rod perpendicular to the X-axis must be the same for $+v$ and $-v$, $\varphi(v) = \varphi(-v)$, so:

$$\varphi(v) = 1. \tag{36}$$

Thus his transformation equations become

$$\tau = \beta(t - vx/c^2), \tag{37}$$

$$\xi = \beta x' = \beta(x - vt), \tag{38}$$

$$\eta = y, \tag{39}$$

$$\zeta = z, \tag{40}$$

where

$$\beta = 1/\sqrt{1 - v^2/c^2} \tag{28}$$

Although he does not mention it, Einstein's previous statement again applies: a constant must be added if the origins of k and K do not coincide at $t = 0$.

In these equations x and ξ are x_1 and ξ_1, the system K and k coordinates of the remote clock when the light flash is reflected back from it; t and τ also relate to this event but, as noted earlier, Einstein does not define their meanings clearly in Eqs. (6)–(8).

VII. ERRORS IN EINSTEIN'S ANALYSIS

There is a major error at the heart of Einstein's calculation of transformation equations:

(i)   He combines experiment A equation (8) $\tau = a(t - vx'/(c^2 - v^2))$ with Eq. (15) $\xi = c\tau$ to obtain Eq. (16) $\xi = ac(t - vx'/(c^2 - v^2))$; and

(ii)   he also combines (8) with experiment B equation (19) $\eta = c\tau$ to obtain Eq. (24) $\eta = ac(t - vx'/(c^2 - v^2))$. Therefore if Eqs. (16) and (24) are both true: $\eta = c\tau = \xi$.
However,

(iii)   in experiment A, $\xi_1 = c\tau_1$ (11) and $\eta_1 = 0$ (5), so $\eta \neq \xi$; and

(iv)   in experiment B, $\xi_1 = 0$ (23) and $\eta_{B1} = c\tau_{B1}$ (19a), so again $\eta \neq \xi$.

Therefore, Einstein's assumption $\eta = c\tau = \xi$ is false and his calculations based on this are invalid.

Also, as noted earlier, some of his equations are based on a 2-way light flash in experiment A, some are based on a 1-way flash in experiment A and some are based on a 1-way flash in experiment B. As these are not compatible with one another, when combined they will produce false results.

Furthermore, Einstein calculates his transformation equations by selecting one particular combination of experiment A and experiment B equations and he ignores other possible combinations. His analysis is summarised below, with relevant equations included for ease of reference.

(i)   As noted above, he combines experiment A equations (8) and (15) to obtain Eq. (16) $\xi = ac(t - vx'/(c^2 - v^2))$ and he also combines (8) with experiment B equation (19) $\eta = c\tau$ to obtain (24) $\eta = ac(t - vx'/(c^2 - v^2))$;

(ii)   he combines (24) with experiment B equation (21) $x' = 0$ to obtain (25) $\eta = act$ and combines this with (20) $y/\sqrt{(c^2 - v^2)} = t$ to obtain (26) $\eta = acy/\sqrt{(c^2 - v^2)}$; by combining this with (27) $a = \varphi(v)/\beta$, (28) $\beta = 1/\sqrt{1 - v^2/c^2}$ and (36) $\varphi(v) = 1$ he obtains his transformation equation (39) $\eta = y$;

(iii)   by combining experiment A equation (8) with (27), (28), and (36), he obtains his transformation equation (37) $\tau = \beta(t - vx/c^2)$.

However, Einstein has ignored other possible combinations of his equations, which produce different results. Examples are listed below:

a.   If experiment B equations (19) $\eta = c\tau$ and (25) $\eta = act$ are combined with Eqs. (27), (28), and (36) the result is

$$\tau = t\sqrt{1 - v^2/c^2}. \tag{41}$$

b.   If experiment A equations (8) $\tau = a(t - vx'/(c^2 - v^2))$ and (17) $x'/(c - v) = t$ are combined with Eqs. (27), (28) and (36) the result is

$$\tau = t\frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}}. \tag{42}$$

c.   If experiment A equations (8) and (17) are combined with experiment B equation (21) $x' = 0$, the result is

$$\tau = t = 0. \tag{43}$$

These alternative transformation equations are all calculated from combinations of Einstein's equations, yet they differ from his Eq. (37). [Note: In (37) τ appears to depend on x, t and v/c, whereas in Eqs. (41) and (42) it depends on only

t and v/c and in Eq. (43) it depends on neither. However this difference is illusory: x and t can be interchanged via his Eqs. (3) $x' = x - vt$ and (17) $x'/(c - v) = t$.] Einstein does not explain why he selected only the particular combination of equations which produces transformation equation (37) and ignored other possible combinations. Alternative transformation equations (41)–(43) are all equally valid solutions to his analysis.

Thus Einstein has failed to prove that calculations based on his theory produce the Lorentz equations as a unique solution.

VIII. LENGTH SHORTENING AND TIME DILATION

In Sec. IV, "Physical Meaning of the Equations Obtained in Respect to Moving Rigid Bodies and Moving Clocks" Einstein states that if a rigid body moving at velocity v is viewed from system K:

"the X-dimension will appear shortened in the ratio $1:\sqrt{(1 - v^2/c^2)}$ … It is clear that the same results hold good of bodies at rest in the 'stationary' system, viewed from a system in uniform motion."

Thus its length only *appears* shortened and reciprocity applies: a rigid body at rest in the stationary system will also appear shortened when viewed from the "moving system." This is logical as, by Einstein's definition, the stationary system could be either system K or system k.

Einstein now considers a clock:

"… located at the origin of the co-ordinates of k and so adjusted that it marks the time τ. What is the rate of this clock, when viewed from the stationary system?

Between the quantities x, t, and τ, which refer to the position of the clock, we have, evidently, $x = vt$ and

$$\tau = \frac{1}{\sqrt{1 - v^2/c^2}}\left(t - vx/c^2\right). \tag{37}$$

Therefore,

$$\tau = t\sqrt{1 - v^2/c^2} = t - \left(1 - \sqrt{1 - v^2/c^2}\right)t, \tag{44}$$

whence it follows that the time marked by the clock (viewed in the stationary system) is slow by $\left(1 - \sqrt{1 - v^2/c^2}\right)$ seconds per second, or—neglecting magnitudes of fourth and higher order—by $1/2v^2/c^2$."

If reciprocity applies to relativistic shortening, logically it should also apply to time dilation but Einstein does not mention this. Also there are errors in his calculations.

(i) As noted previously, it is possible for a moving clock to be synchronised to show "stationary time" t and this was implicitly accepted by Einstein in Sec. II.

(ii) His equation $x = vt$ applies to a clock at the origin of system k but in his transformation equation (37) t is the time on a clock which is remote from the origin—and in experiment A the relevant equation for this is (10) $x_1 = ct_1$.

(iii) As shown previously, Einstein's calculation of his transformation equation contains serious errors.

(iv) Furthermore his time dilation calculation is fundamentally flawed. If a clock is at the origin of k, its k coordinates are $\xi = \eta = \zeta = 0$ and its K coordinates are $x' = y = z = 0$. In experiment A, if $\xi_{A1} = x'_{A1} = 0$ then according to Einstein's equations (4), (15), and (17): $\tau_{A1} = t_{A1} = 0$. In experiment B, if $\eta_{B1} = y_{B1} = \zeta_{B1} = z_{B1} = 0$ then according to his Eqs. (19) and (20) $\tau_{B1} = t_{B1} = 0$. Therefore if the clock's timekeeping is analysed correctly on the basis of experiments A and B only one conclusion is possible: $\tau = t = 0$ (cf. Ref. 20).

Therefore, Einstein's calculation of time dilation is flawed and does not support his claim that a clock at the system k origin will run slow by $\left(1 - \sqrt{1 - v^2/c^2}\right)$ s/s relative to a clock at rest in system K.

IX. EINSTEIN'S "CLOCK PARADOX" THOUGHT EXPERIMENTS

After his time dilation calculation, Einstein states:

"From this there ensues the following peculiar consequence:"

a. a clock moving at velocity v between two synchronised stationary clocks will be found to run slow by $1/2tv^2/c^2$ where t is its journey time;

b. the same is true if the clock moves on a polygonal line, including if its start and finish points coincide;

c. if a clock moves at speed v around a closed curve, it will be slow by $1/2tv^2/c^2$ when it returns to its starting position; and

d. a clock on the Earth's Equator will run slow by $1/2tv^2/c^2$ relative to a clock on a Pole.

These are his famous "clocks/twins paradox" thought experiments, which have led to much argument over the years (e.g., Refs. 4 and 8–13). Kelly[21] lists 54 different explanations of the "twins paradox" proposed by various authors (including Einstein). The present analysis adds further objections.

(i) In his main Sec. III analysis, Einstein defines v as the relative velocity of coordinate systems, not the relative velocity of clocks, or the velocity of a clock relative to a coordinate system.

(ii) It is possible to synchronise a moving clock by the light flash method to show stationary time t and this is implicitly accepted by Einstein in Sec. II. However if a moving clock shows time t, his predicted "peculiar consequence" will not occur.

(iii) Because of the errors in Einstein's calculation of time dilation, he fails to justify his claim that his theory
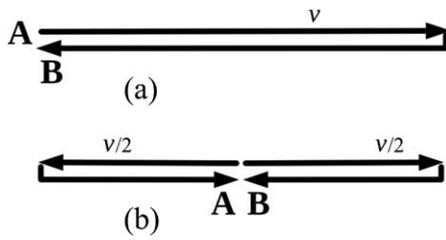
FIG. 3.　(a) Clock paradox: Clock A stationary, clock B travels away and returns at velocity v. (b) Clock paradox as amended by Ives: Clock A travels away and returns at velocity $v/2$; clock B travels away and returns at velocity $v/2$. Relative to observer A, clock B travels away and returns at speed v, as in the classic scenario, so, by Einstein's reasoning, at the finish it should be slow compared with A. However, relative to observer B, clock A travels away and returns at speed v, so at the finish it should be slow compared with B.

requires a clock at the system k origin to run slower than system K clocks by $1/2tv^2/c^2$.

(iv)　Einstein does not say whether reciprocity applies to time dilation. If it does, this would invalidate his "clock paradoxes."

(v)　The classic clocks paradox scenario [Fig. 3(a)] is asymmetrical, as accelerations are applied to the moving clock, but Einstein does not discuss the possible effects of these. In a revision proposed by Ives to eliminate this asymmetry,[22] clock B travels away eastwards and returns at speed $v/2$ and clock A travels away westwards and returns at $v/2$ [Fig. 3(b)].

(vi)　In Einstein's comparison of clocks on the Equator and Pole what is velocity v? As both clocks are at rest in a system rotating about the Earth's axis and the distance between them is constant, it could be argued that $v = 0$. Alternatively, if the equatorial clock is thought of as moving in a series of short straight lines at v relative to the polar clock, then the latter is also moving at v in short lines relative to the former. Einstein's assumption that the equatorial clock has velocity v and the polar clock has velocity 0 is only true if there is a preferred stationary system which travels with the Earth but does not rotate.

X. CONCLUSIONS

1.　By Einstein's definition, "the stationary system" can be any system in uniform translatory motion. In his analysis v is the relative velocity of the observers' coordinate systems, not the velocity of a clock relative to a coordinate system. Using light flashes, it is possible to synchronize a moving clock to show stationary time t and this is implicitly accepted by Einstein in Sec. II. Therefore, it cannot be true that his theory requires a moving clock to run slow relative to a stationary clock.

2.　Einstein identifies times t_A, t_B, t'_A, t'_B, τ_0, τ_1, and τ_2 by subscripts in Eqs. (1) and (2) but although terms such as t and x also have multiple possible meanings in his later equations, these are not identified by subscripts. Similarly, he does not identify equations derived from different assumptions to distinguish them from one another. This leads to confusion and errors in his analysis.

3.　Einstein's main analysis considers two thought experiments. In the first (referred to here as experiment A) all observers, clocks and light flashes are on the X-axis but his equations are based on a variety of different assumptions. In the second (experiment B) the remote clock and light flash are on the system k Y-axis.

Einstein then combines equations from experiments A and B, despite the fact that these are based on incompatible assumptions. He assumes $\eta = c\tau = \xi$ but this is false: in both experiment A and in experiment B, $\eta \neq \xi$. Furthermore, to calculate his transformation equations he selects a particular combination of equations which produces a result matching the Lorentz equations and ignores other possible combinations of equations which produce different results.

Therefore Einstein fails to justify his claim that calculations based on his theory lead to the Lorentz transformation equations as a unique solution.

4.　Einstein's calculation of time dilation $\tau = t\sqrt{1 - v^2/c^2}$ for a clock at the origin of system k contains further fundamental errors in addition to those in "3".

5.　As Einstein has failed to prove that his theory requires a moving clock to run slow by $1/2tv^2/c^2$ relative to a stationary clock, any detailed discussion of his clock paradox thought experiments is largely pointless. However these also contain further illogicalities and anomalies of their own.

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