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Relativistic theory for clock syntonization and the realization of geocentric coordinate times

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Abstract. General relativity predicts that two ideal clocks that move with a relative velocity, and are submitted to different gravitational fields will in general be observed to run at different rates. Similarly the rate of a clock with respect to the coordinate time of some space-time reference system is dependent on the velocity of the clock in that reference system and on the gravitational fields it is submitted to. For the syntonization of clocks and the realization of coordinate times (like TAI) this rate shift has to be taken into account with an uncertainty that should be below the frequency stability of the clocks in question, i.e. all terms that are larger than the instability of the clocks should be corrected for. Presently the best atomic clocks reach frequency stabilities of 1×10^{-15}, with new designs promising an improvement by several orders of magnitude. We present a theory for the calculation of the relativistic rate shift for clocks in the vicinity of the Earth, including all terms larger than one part in 10^{18}. This, together with previous work on clock synchronization (Petit & Wolf 1994), amounts to a complete relativistic theory for the realization of coordinate time scales at picosecond synchronization and 10^{-18} syntonization accuracy, which should be sufficient to accommodate future developments in time transfer and clock technology.

Key words: relativity – reference systems – time

1. Introduction

At its 1991 General Assembly, the International Astronomical Union (IAU) explicitly adopted the general theory of relativity as the theoretical framework for the definition and realization of space-time reference frames (IAU 1991). Barycentric and geocentric coordinate time scales and the relativistic transformations between them were defined, together with procedures for their realization.

Coordinate time scales in the vicinity of the Earth can be realized by a weighted average of atomic clocks operating on the Earth and on terrestrial satellites. For this purpose the clocks have to be synchronized and syntonized using a relativistic theory at the level of accuracy required.

A relativistic theory for the synchronization of clocks in the vicinity of the Earth has recently been developed to picosecond accuracy (Petit & Wolf 1994), which should be sufficient for present and near-future practical requirements which are at the level of $\approx$ 100 ps with an expected improvement by one order of magnitude.

A well known prediction of Einstein's general theory of relativity states that two ideal clocks that move with a relative velocity, and are submitted to different gravitational fields will in general be observed to run at different rates, a prediction that has been verified experimentally with a relative uncertainty of 7×10^{-5} using two hydrogen masers (Vessot 1980). This implies that the rate of a clock with respect to the coordinate time of some space-time reference system is dependent on the velocity of the clock in that reference system and on the gravitational fields to which it is submitted. If the clock is to be used for the realization of coordinate time this rate shift has to be taken into account at a level of accuracy which should be below the frequency stability of the clock in question, i.e. the clock has to be syntonized with respect to the coordinate time including all terms that are larger than the instability of the clock.

Presently hydrogen masers, the most stable atomic clocks, reach frequency stabilities of 1×10^{-15} (square root of the Allan variance) for averaging times of $\approx$ 10 000 s (Vessot et al. 1992; Busca 1993; Cutler 1993). For longer averaging times the best stabilities are displayed by caesium clocks and are below one part in 10^{14} (Cutler 1993). Several laboratories are currently developing caesium fountain standards which, in the near future, are expected to provide stabilities in the low 10^{-16} range (De Marchi 1993; Maleki 1993; Santarelli et al. 1994). In the longer term the advent of cooled hydrogen masers and trapped-ion standards may yield stabilities of a few parts in 10^{17} and 10^{18} respectively (Rolston & Phillips 1991; Itano 1991; Maleki 1993). Present and expected clock stabilities are summarized in Fig. 1.

In this paper we present a relativistic theory for the syntonization of clocks in the vicinity of the Earth (within a geo-

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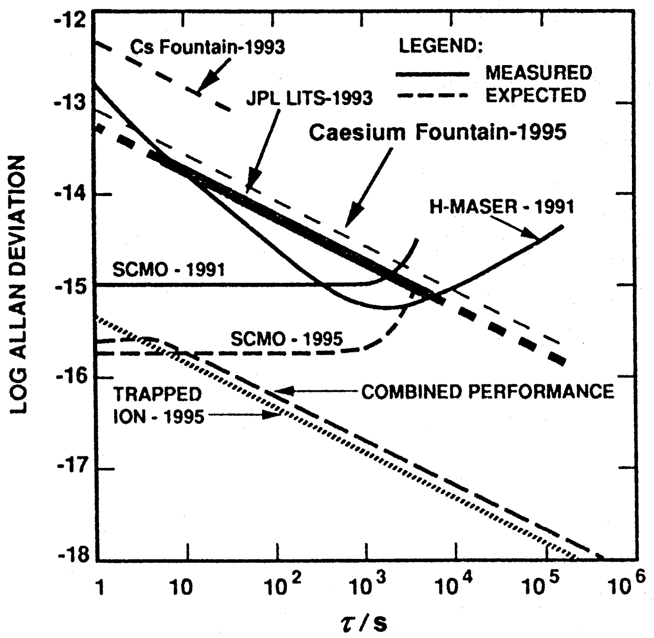

Fig. 1. Present and expected clock stabilities (from Maleki (1993)). SCMO stands for Superconducting Cavity stabilised Maser Oscilator and JPL LITS for the Jet Propulsion Laboratory Linear Ion Trap Standard. By combined performance is meant a combined time scale using an SCMO and a trapped ion standard

centric sphere of 300 000 km radius) including all terms larger than one part in 10^{18}. Outside this sphere the effect of the lunar quadrupole moment may exceed 1×10^{-18} and should be included.

On the surface of the Earth a relativistic frequency shift of 1×10^{-18} corresponds to a change in altitude of $\approx$ 1 cm. Therefore, effects on syntonization due to the size of the clocks cannot be neglected when such accuracies are required and should be accounted for when using the theory provided in this paper.

Section 2 clarifies the concept of syntonization in a relativistic context. In Sect. 3 we present the individual relativistic terms affecting the rate of a clock with respect to coordinate time based on the post-Newtonian approximation of general relativity. Evaluation of the terms of order c^{-4} shows that these amount to a few parts in 10^{19} in the vicinity of the Earth and are hence negligible. We provide explicit expressions for syntonization with respect to Geocentric Coordinate Time (TCG) to be applied to clocks on the surface of the Earth (Sect. 3.1) and on board terrestrial satellites (Sect. 3.2). The effects that need to be taken into account when using these expressions at the 10^{-17} and 10^{-18} accuracy level respectively are discussed. Terrestrial Time (TT), the ideal form of International Atomic Time (TAI), can be obtained from TCG via a transformation (IAU 1991) which is presented, together with its limitations, in Sect. 4. When determining the relative rate of two distant clocks one might be interested in time varying effects only (i.e. effects that influence the observed frequency stability). These are investigated to an accuracy of 10^{-18} in Sect. 5. We consider tidal variations of the gravitational field as well as those of non-tidal origin (atmospheric pressure changes, movements of the Earth's crust, polar motion etc.).

The space-time metric for the geocentric reference system, including terms of order c^{-4}, has been derived in Brumberg & Kopejkin (1988) and Kopejkin (1988). The relationship between TCG and the proper time of a clock in the vicinity of the Earth is given by Brumberg & Kopejkin (1990), Klioner (1992) and Brumberg et al. (1993). In these papers tidal terms of order 10^{-17}, including the response of an elastic Earth are given at an accuracy sufficient for our purpose. However, effects of oceanic tides and of non-tidal origin, which can contribute up to a few parts in 10^{17}, are not considered. Furthermore, the accuracy of the expressions given for the geopotential is limited at 10^{-14} with no information on possible improvements.

2. Syntonization in a relativistic context

When using the concept of syntonization in a relativistic context, certain ambiguities appear which can lead to confusion and misunderstanding. It is therefore essential to first clarify the different meanings of the expression as used in time metrology within a relativistic framework.

For the realization of coordinate time scales (like TAI) it is necessary to syntonize clocks with respect to the coordinate time in question, i.e. to determine the rate of a clock A with respect to an ideal coordinate time of some space-time reference frame. For example, using a geocentric non-rotating frame with TCG as coordinate time (as defined by the IAU (1991)), in the post-Newtonian approximation, the proper time τ_A of the clock A is related to TCG by

$$\frac{d\tau_A}{dTCG} = 1 - \frac{U(\boldsymbol{w}) + \frac{v^2}{2}}{c^2} + O(c^{-4}), \tag{1}$$

where $(cTCG, w^k)$ are coordinates in the geocentric frame with $\boldsymbol{w}$ representing the triplet w^k. The potential at the position of the clock $U(\boldsymbol{w})$ is the sum of the Earth's potential and the tidal potentials of external bodies, and v is the coordinate speed of the clock in the geocentric, non-rotating frame. Note that this rate depends entirely on the chosen reference frame. It is a coordinate quantity which cannot be obtained directly from measurement, but must be calculated theoretically using the definition of the reference frame in question with the appropriate metric equation.

When using repeated time transfers employing the convention of coordinate synchronization (Allan & Ashby 1986; Petit & Wolf 1994) for the determination of the relative rate of two clocks A and B the resulting rate predicted by theory is simply

$$\frac{d\tau_A}{d\tau_B} = \frac{d\tau_A}{dTCG}\frac{dTCG}{d\tau_B}, \tag{2}$$

with $d\tau/dTCG$ given in (1). This is a combination of coordinate dependent quantities and therefore entirely dependent on the chosen reference frame and the convention of synchronization.

The value of the quantity (2) is not to be confused with the result obtained by an observer measuring the relative frequency of two signals emitted by A and B, each signal being locked to the clocks in A and B. This quantity, $[f_A]_O/[f_B]_O$, is a measurable quantity which is specific to the measuring observer O. It is possible to relate the proper received frequency $[f_A]_O$ (the frequency of the signal measured by the observer O using an ideal clock) at the time of reception t_r to the proper emitted frequency $[f_A]_A$ (the frequency measured by an observer at A) at the time of emission t_e by the formula

$$\frac{[f_A]_O}{[f_A]_A} = \frac{d\tau_A}{d\tau_O}\left(\frac{1+\boldsymbol{k}_A\cdot\dot{\boldsymbol{w}}_O(t_r)/c}{1+\boldsymbol{k}_A\cdot\dot{\boldsymbol{w}}_A(t_e)/c}\right), \tag{3}$$

where, in the formulation of the Doppler effect, the Newtonian approximation is used with $\boldsymbol{k}_A$ being the unit vector along the propagation path, $\boldsymbol{k}_A = (\boldsymbol{w}_A(t_e) - \boldsymbol{w}_O(t_r))/|\boldsymbol{w}_A(t_e) - \boldsymbol{w}_O(t_r)|$. Applying this formula to A and B, it is clear that $[f_A]_O/[f_B]_O$ differs from the value given by (2) by the Doppler terms which are dependent on the relative motion of the clocks and the observer. In addition the two expressions differ in terms of order c^{-3} and higher, which are not represented in (3) but can reach 10^{-15}.

3. Syntonization with respect to TCG

In the general theory of relativity it is usually impossible to define space-time coordinates x^α that have a globally constant relation to measurable quantities (so called proper quantities). Instead this relationship is location dependent and defined by the space-time metric. For two events with an infinitesimal, time like separation such a relationship is given by:

$$ds^2 = -c^2 d\tau^2 = g_{\alpha\beta}(x^\lambda)dx^\alpha dx^\beta, \tag{4}$$

assuming a summation over repeated indices with ds representing the relativistic line element and τ the proper time elapsed between the two events as measured by a clock whose world-line passes through both. Here and throughout the paper greek indices run from 0 to 3 and latin ones from 1 to 3. Space-time coordinates are denoted by x^α with $x^0 = ct$ (t = coordinate time) and spatial coordinates x^i. The $g_{\alpha\beta}(x^\lambda)$ are coordinate dependent components of the space-time metric tensor.

In the post-Newtonian approximation, valid for weak gravitational fields and low velocities ($\varepsilon^2 \approx U/c^2 \approx v^2/c^2 \ll 1$), the components of the metric tensor can be expressed by a power series based on small corrections to the Minkowski metric of special relativity in terms of ε^n (see for example Brumberg 1991),

$$\begin{aligned} g_{00} &= -1 + \frac{h^{(2)}_{00}}{c^2} + \frac{h^{(4)}_{00}}{c^4} + O(c^{-6}) \\ g_{0i} &= \frac{h^{(3)}_{0i}}{c^3} + O(c^{-5}) \\ g_{ij} &= \delta_{ij} + \frac{h^{(2)}_{ij}}{c^2} + \frac{h^{(4)}_{ij}}{c^4} + O(c^{-6}), \end{aligned} \tag{5}$$

where the $(-+++)$ sign convention has been adopted. The $h^{(n)}_{\alpha\beta}/c^n$ are of order ε^n and δ_{ij} is the Kronecker symbol ($\delta_{ij} = 1$ for $i = j$; $\delta_{ij} = 0$ otherwise).

Substituting (5) into (4) and solving for $d\tau/dt$ gives the relationship between the proper time of a clock and coordinate time, i.e. an expression for the rate of the clock with respect to coordinate time,

$$\begin{aligned} \frac{d\tau}{dt} = 1 &- \frac{h^{(2)}_{00}}{2c^2} - \frac{v^2}{2c^2} - \frac{h^{(3)}_{0i}v^i}{c^4} - \frac{h^{(2)}_{ij}v^iv^j}{2c^4} - \frac{h^{(4)}_{00}}{2c^4} - \frac{{h^{(2)}_{00}}^2}{8c^4} \\ &- \frac{v^4}{8c^4} - \frac{h^{(2)}_{00}v^2}{4c^4} + O(c^{-6}), \end{aligned} \tag{6}$$

where $v^i = dx^i/dt$ is the coordinate velocity of the clock.

For a geocentric coordinate system with TCG as coordinate time and non-rotating spatial coordinates, the components of the space-time metric up to order $h^{(3)}_{\alpha\beta}$ are given, for example, by Brumberg et al. (1992). The fourth order term $h^{(4)}_{00}$ is derived in Kopejkin (1988). Substituting these results into (6) we find that in the vicinity of the Earth all terms in c^{-4} amount to a few parts in 10^{19} or less. In particular the $h^{(4)}_{00}/2c^4$ term and terms due to the geodesic precession (in $h^{(3)}_{0i}v^i/c^4$), which require the specification of coordinate conditions (harmonic, standard post-Newtonian etc...) and the state of rotation of the frame (kinematically or dynamically non-rotating), are below the 10^{-18} limit. The choice of coordinate conditions and of the state of rotation (in the above sense) of the frame is therefore not significant for syntonization at an accuracy of 10^{-18}.

For this reason only the $h^{(2)}_{00}$ component of the metric tensor is required for our purpose. It is reproduced from Brumberg et al. (1992) in Eq. (7) below.

$$h^{(2)}_{00} = 2[U_E(\boldsymbol{w}) + Q_k w^k + \overline{U}(\boldsymbol{x}_E + \boldsymbol{w}) - \overline{U}(\boldsymbol{x}_E) - \overline{U}_{,k}(\boldsymbol{x}_E)w^k], \tag{7}$$

where $(cTCB, x^k)$ (TCB = Barycentric Coordinate Time) are coordinates in the barycentric frame with $(x^k x^k)^{1/2} = r$ and $\boldsymbol{x}$ denoting the triplet x^k with the subscript E refering to the Earth's centre of mass. $U_E(\boldsymbol{w})$ and $\overline{U}(\boldsymbol{x})$ are the Newtonian gravitational potentials of the Earth and of external masses respectively, and Q_k is the correction for the non-geodesic barycentric motion of the Earth.

The second term in (7), arising from the interaction of the Earth's quadrupole moments and the external masses [given explicitly e.g. in Brumberg & Kopejkin (1990)] gives rise to a correction of less than a few parts in 10^{19} in the vicinity of the Earth and can be neglected for our purposes.

Hence the rate of a clock with respect to coordinate time (TCG) in the vicinity of the Earth, including all terms larger than one part in 10^{18}, is

$$\begin{aligned} \frac{d\tau}{dTCG} = 1 - \frac{1}{c^2}\Big[& U_E(\boldsymbol{w}) + \frac{v^2}{2} + \overline{U}(\boldsymbol{x}_E + \boldsymbol{w}) \\ & - \overline{U}(\boldsymbol{x}_E) - \overline{U}_{,k}(\boldsymbol{x}_E)w^k \Big]. \end{aligned} \tag{8}$$

Orders of magnitude of the individual terms in (8), and their calculation at the required accuracy, are considered in detail in the following sections.

Table 1. Effects on syntonization with respect to TCG of clocks on the Earth's surface; Orders of magnitude and uncertainties of the corrections

Effect	Order of magnitude	Uncertainty
Earth's gravitational potential	7×10^{-10}	10^{-17}
Centrifugal potential ($v^2/2/c^2$)	1×10^{-12}	$< 10^{-18}$
Volcanic and coseismic (highly localised)	$< 10^{-16}$	?
External masses (Moon, Sun)	10^{-17}	$< 10^{-18}$
Solid Earth tides	10^{-17}	$< 10^{-18}$
Ocean tides	10^{-17}	$< 10^{-18}$

3.1. Clocks on the Earth's surface

The limiting factor for syntonization with respect to coordinate time, of a clock on the surface of the Earth is the inaccuracy in the determination of the Earth's gravitational potential. Currently this uncertainty is $\approx 1\,\mathrm{m}^2\,\mathrm{s}^{-2}$ for the total (gravitational + centrifugal) potential on the geoid, W_0, (Bursa et al. 1992; Bursa 1993) which is equivalent to a 10 cm error in radial distance. This corresponds to an uncertainty of $\approx 1 \times 10^{-17}$ in (8). In this section we therefore only consider effects whose influence on the terms in (8) is larger than one part in 10^{17}. These are summarized in Table 1 together with orders of magnitude and uncertainties of the associated corrections.

The gravitational potential of the Earth, $U_E(w)$, can be expressed as a series expansion in spherical harmonics. However, owing to mass irregularities, such a series cannot be be considered convergent at the surface of the Earth (Moritz 1961). Nonetheless, due to the predominantly ellipsoidal shape of the Earth, one can use the first two terms of this series expansion as a first approximation (Allan & Ashby 1986; CCIR 1990; Klioner 1992). Thus,

$$U_E(w) = \frac{GM_E}{\rho} + \frac{GM_E a_1^2 J_2}{2\rho^3}(1 - 3\cos^2\theta) + \ldots, \quad (9)$$

where G is the Newtonian gravitational constant, M_E is the mass of the Earth, $\rho = (w^k w^k)^{1/2}$, a_1 and J_2 are, respectively, the equatorial radius and the quadrupole moment coefficient of the Earth ($a_1 = 6378136.3$ m, $J_2 = 1{,}0826 \times 10^{-3}$) and θ is the geocentric colatitude of the point of interest.

Substituting (9) into the second term of (8) gives terms which can amount to $\approx 7 \times 10^{-10}$ and $\approx 8 \times 10^{-13}$ for points on the surface of the Earth.

Considering the third term in (8), one can see that with

$$v = \omega\rho\sin\theta\,, \quad (10)$$

for a clock fixed on the surface of the Earth (where ω represents the angular velocity of rotation of the Earth) this term is equivalent to the centrifugal potential divided by c^2. Its magnitude can reach 1.2×10^{-12}.

The effect on this term of the movement of the pole ($\Delta\theta$) and variations in the length of day ($\Delta\omega$) are of order 10^{-18} and smaller and are treated in more detail in Sect. 5.

The surface obtained when setting $U_E(w) = W_0 - (\omega\rho\sin\theta)^2/2$ in (9) differs from the ellipsoid of the Earth model by less than 10 m. Therefore, an estimate of the accuracy of (9) can be obtained by considering the maximal difference between the geoid and the reference ellipsoid. This can amount to ≈ 100 m (Vanicek & Krakiwsky 1986), so expression (9) for the Earth's gravitational potential should not be used if accuracies better than one part in 10^{14} are required.

For improved accuracy the second and third term in (8) should not be computed separately using (9) and (10). Instead, their combined effect should be determined using

$$U_E(w) + \frac{(\omega\rho\sin\theta)^2}{2} = W_0 - \int_0^H g\mathrm{d}H = W_0 - \overline{g}H \quad (11)$$

where g is the Earth's gravitational + centrifugal acceleration, and H is the height above the geoid. A value of g averaged between 0 and H, $\overline{g}$, obtained from a gravimetric model, can be used instead of computing the integral when the required accuracy in (8) is of order 10^{-15}.

Using a geodetic GPS (Global Positioning System) receiver and a geoid model the height above the geoid, H, can be obtained with an accuracy of the order of 10 m. This allows the determination of the total (gravitational and centrifugal) effect on the clock with an accuracy of $\approx (1\text{–}2) \times 10^{-15}$. Similar accuracy can also be obtained by using a topographic map for the determination of H.

When higher accuracy is required, precise levelling should be used. Levelling measurements are referred to a zero-level reference point which can be compared to mean sea level using a tidal gauge. This level differs from the geoid by what is known as Sea Surface Topology (*SST*) which can amount to ± 0.7 m (Torge 1989). The *SST* can be determined with an accuracy of ≈ 0.1 m (Torge 1989) using oceanographic methods and satellite altimetry which induces an uncertainty of $\approx 1 \times 10^{-17}$ in (8). The uncertainty in the potential on the geoid, W_0, which is of order $1\,\mathrm{m}^2\,\mathrm{s}^{-2}$ (Bursa et al. 1992; Bursa 1993), contributes another part in 10^{17}. The sum of the gravitational and centrifugal potential differences between mean sea level and an arbitrary point far from the coast can be obtained by geometrical levelling with simultaneous gravimetric measurements. The accumulated uncertainty when using modern levelling techniques and gravimetry is below $(0.5\sqrt{D/\mathrm{km}})$ mm (Kasser 1989), where D is the distance between the reference point and the point of interest, and does therefore not exceed a few centimeters even over large distances. In many countries levelling networks have been established at accuracies of $\approx (2\sqrt{D/\mathrm{km}})$ mm for primary points, the use of which would again induce errors at the centimetric level. Alternatively levelling can be achieved with accuracies of order 10 cm (for distances of ≈ 100 km) using differential GPS (Milbert 1992).

Therefore the constant part of the total potential at any point on the Earth's surface can be determined with an ultimate uncertainty less than $2.5\,\mathrm{m}^2\,\mathrm{s}^{-2}$ using a tidal gauge and good geometrical levelling. The main contributions to this uncertainty are inaccuracies in the determination of W_0 and the SST. This limits the evaluation of (8) at the level of $(2\text{–}3) \times 10^{-17}$, which is the limit for syntonization of clocks with respect to coordinate time (TCG or TT) on the surface of the Earth.

Additionally, account has to be taken of the time varying part of the potential on the surface of the Earth caused by the gravitational attraction of external masses (tidal effects) and changes in the Earth's own gravitational field (non-tidal effects).

A number of effects give rise to relativistic rate shifts which are larger than one part in 10^{18} and are of a periodic nature. For clock comparisons using time transfers with a synchronization accuracy of one picosecond, such terms are negligible if their period is sufficiently short to prevent their amplitude in the time domain from exceeding this limit. They might, however, be of interest when using frequency transfers and are therefore included in this study.

At 10^{-17} accuracy non-tidal effects are highly localized and can be neglected in general. They relate mainly to movements of the Earth's crust caused by volcanic and coseismic processes. The resulting change in the second term of (8) can amount to $\approx 10^{-16}$ on time scales ranging from a few days to one year (Torge 1989). From gravimetric measurements Ervin & McGinnis (1986) have inferred local elevation changes in the Mississippi embayment of up to 15 cm, caused by the surface loading associated with changes in river stage. The resulting change in the second term of (8) is $\approx 1.5 \times 10^{-17}$. In Sect. 5. we present a more detailed treatment of non-tidal effects at the 10^{-18} accuracy level.

The third, fourth and fifth terms of (8) represent the effect of external masses, mainly the Moon and Sun. The Newtonian potential of external bodies can be expressed in the spherical approximation by,

$$\overline{U}(\boldsymbol{x}) = \sum_{\mathrm{A}\neq\mathrm{E}} \frac{GM_\mathrm{A}}{r_\mathrm{A}}, \tag{12}$$

where r_A is the coordinate distance between the point of interest and the centre of mass of the body A, M_A is the mass of body A, the summation is over all celestial bodies apart from the Earth, and multipole terms are neglected as their effect on the surface of the Earth does not exceed 10^{-18}. Note that it is not essential whether r_A is expressed in geocentric or barycentric coordinates, as the induced error in Eq. (8) is several orders of magnitude smaller than 10^{-18}.

Substituting (12) into the third, fourth and fifth terms of (8), expanding the third term in a Taylor series and using Love numbers to characterize the response of the Earth to the tidal potential (the solid Earth tide) gives,

$$U_T(\boldsymbol{w}) = (1 + k_2 - h_2) \sum_{\mathrm{A}\neq\mathrm{E}} \left[\frac{GM_\mathrm{A}}{2r^3_\mathrm{EA}} \left(\frac{3r^i_\mathrm{EA} r^j_\mathrm{EA}}{r^2_\mathrm{EA}} - \delta_{ij} \right) w^i w^j + O\left(\frac{GM_\mathrm{A} w^3}{r^4_\mathrm{EA}} \right) \right], \tag{13}$$

where $r^i_\mathrm{EA} = x^i_\mathrm{E} - x^i_\mathrm{A}$, $r_\mathrm{EA} = (r^i_\mathrm{EA} r^i_\mathrm{EA})^{1/2}$, and where k_2 and h_2 are the Love numbers. For most Earth models $(1+k_2-h_2) = 0.69$ (Farrell 1972).

Evaluation of expression (13) for the Moon and the Sun gives a correction in (8) which is smaller than 4×10^{-17}. Contributions from other planets and higher order terms in (8) add corrections which are smaller than 1×10^{-18}. The effect of oceanic tides can amount to 9×10^{-18} for the M2 lunar tide in a few regions, and roughly twice this value for the total tide (Scherneck 1994). It is discussed in more detail in Sect. 5.

3.2. Clocks on board terrestrial satellites

The accuracy of syntonization of satellite clocks with respect to TCG is limited by uncertainties in the geopotential model and the orbit determination. Solid Earth tides, ocean tides, polar motion and changes in atmospheric pressure may give rise to corrections of some parts in 10^{18} for low flying satellites, but can be neglected at altitudes exceeding 4000 km. The tidal potentials of external masses become more important with increasing altitude and so require an exact expression, rather than a series expansion as in (13), for their evaluation. Table 2 lists all relevant effects together with orders of magnitude and uncertainties of the associated corrections.

The rate of a clock with respect to TCG is given by Eq. (8). The geopotential in the second term can be expressed as a series expansion in spherical harmonics

$$U_\mathrm{E}(\boldsymbol{w}) = \frac{GM_\mathrm{E}}{\rho} \left[1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} P_{nm}(\cos\theta) \left(\frac{a_1}{\rho} \right)^n \times (C_{nm} \cos(m\lambda) + S_{nm} \sin(m\lambda)) \right], \tag{14}$$

where θ and λ are the geocentric colatitude and longitude of the satellite, $P_{nm}(\cos\theta)$ are associated Legendre polynomials, and the C_{nm} and S_{nm} are coefficients determined by fitting the data from satellite observations. For the latest model (GEM-T3) these coefficients are given with an accuracy of a few parts in 10^9 up to degree and order 50 (Lerch et al. 1992; IERS 1992). For low altitudes ($<$ 4000 km) this results in syntonization uncertainties which exceed one part in 10^{18}, but decrease with increasing altitude. The effects of solid Earth, oceanic, and pole tides can be included in the model as small variations of the coefficients. The effect of atmospheric pressure variations may amount to 1–2 parts in 10^{18} for altitudes $<$ 4000 km but corrections to an accuracy of 10^{-18} are possible (see Sect. 5).

The third, fourth and fifth terms in (8) characterize the effect of external masses on the rate of a clock relative to TCG. It is not practical to use the Taylor expansion of the third term as in the previous section, since at higher altitudes a large number of terms would be needed to achieve the required 10^{-18} accuracy. Instead, substituting (12) into (8) for the external potentials and differentiating the fifth term, we obtain

$$\overline{U}(\boldsymbol{x}_\mathrm{E} + \boldsymbol{w}) - \overline{U}(\boldsymbol{x}_\mathrm{E}) - \overline{U}_{,k}(\boldsymbol{x}_\mathrm{E}) w^k = \sum_{\mathrm{A}\neq\mathrm{E}} GM_\mathrm{A} \left[\frac{1}{r_\mathrm{PA}} - \frac{1}{r_\mathrm{EA}} + \frac{r^k_\mathrm{EA} r^k_\mathrm{PE}}{r^3_\mathrm{EA}} \right] \tag{15}$$

Table 2. Effects on syntonization with respect to TCG of clocks on board terrestrial satellites; Orders of magnitude and uncertainties of the corrections where h represents the altitude of the satellite

Effect		Order of magnitude	Uncertainty
Earth's gravitational potential		$< 6 \times 10^{-10}$	Few 10^{-18} (GEM-T3) $< 10^{-18}$ at $h > 4000$ km Few 10^{-18} (5 cm orbit uncertainty) $< 10^{-18}$ at $h > 10\,000$ km
2nd order Doppler ($v^2/2/c^2$)		$< 3 \times 10^{-10}$	$< 10^{-18}$ at $h > 10\,000$ km
External masses :	Moon	4×10^{-13}	$< 10^{-18}$
(at h = 300 000 km)	Sun	4×10^{-14}	
	Venus	6×10^{-18}	
Solid Earth tides Ocean tides Polar motion Atmospheric pressure		10^{-18} (at low altitudes)	$< 10^{-18}$

for the total tidal potential. Here $r^k_{PA} = r^k_P - r^k_A$ is the vector from the centre of mass of body A to the point of interest with magnitude $r_{PA} = (r^k_{PA} r^k_{PA})^{1/2}$, the subscript E stands for the Earth, and the summation is carried out over all celestial bodies except the Earth. At the required accuracy either barycentric or geocentric coordinates may be used.

The maximal magnitude of the tidal terms in (8) is $\approx 4 \times 10^{-13}$ for the moon, $\approx 4 \times 10^{-14}$ for the sun and $\approx 6 \times 10^{-18}$ for Venus. The effects of other planets and asteroids are negligible. The constraints on the knowledge of the planetary ephemerides are $\approx$ 117 m for the Earth-moon distance, $\approx$ 1200 km for the Earth-sun distance and $\approx 10^6$ km for the distance to Venus, which present no difficulties for modern astrometry.

Syntonization of satellite clocks with respect to TCG is limited mainly by orbitography errors. For syntonization to 10^{-18} the required accuracies are of order 1 cm on position and 1×10^{-5} m s^{-1} on velocity for a satellite at 1000 km altitude. For satellites in higher orbits the constraints are less severe, being about 40 cm and 3×10^{-5} m s^{-1} for a geostationary one.

The data necessary for orbit determination can be obtained by satellite ranging from a number of ground stations on the Earth using timing measurements of electromagnetic signals. For this purpose the clocks at the different stations have to be synchronized and syntonized, so the problems of satellite orbitography and time scale realization are not entirely independent. However, the accuracy required for station clock synchronization and syntonization for the realization of centimetric orbits is of order one microsecond and parts in 10^{11} respectively, values which can be achieved even with terrestrial methods. This is why the two problems can be separated.

At present satellite laser ranging produces measurements at a precision of a few millimetres, with an accuracy of roughly one centimeter (Degnan 1993), the limiting error source being uncertainty in the atmospheric propagation delay. The satellite orbit is determined from the ranging measurements using an orbital model, which introduces further inaccuracies. Differences between orbits obtained using different ranging techniques and models for the Topex/Poseidon mission (altitude 1200 km) are typically a few centimetres (Yunck 1994; Nouel 1994; Schutz 1994) (consistent with an uncertainty of $\approx 1 \times 10^{-5}$ m s^{-1} in velocity) which is an indication of the accuracy of the orbits obtained. Beutler et al. (1994) estimate the uncertainty of precise GPS ephemerides (altitude 20 000 km) to be $\approx$ 15 cm ($\approx 1 \times 10^{-5}$ m s^{-1} in velocity).

We conclude that syntonization of satellite clocks with respect to TCG is at present limited by uncertainties in the geopotential model (GEM- T3) and the satellite orbits to an accuracy of a few parts in 10^{18} at low altitudes, with a decrease of this limit below 1×10^{-18} with increasing altitude (> 10 000 km), which is an order of magnitude better than the uncertainty for clocks on the Earth's surface. Therefore it seems likely that future time will be provided from space.

4. Transformation to TT

TCG is related to TT by a relativistic transformation, so any clock syntonized with respect to TCG can also be syntonized with respect to TT. In this case the accuracy of syntonization may be limited by the uncertainty of the parameters participating in the transformation.

The IAU defined TT as a geocentric coordinate time scale differing from TCG by a constant rate, the scale unit of TT being chosen so that it agrees with the SI second on the geoid (IAU 1991). TT is an ideal form of the International Atomic Time

Table 3. Time varying effects on the Earth's surface for the determination of the relative rate of two clocks; orders of magnitude and uncertainties of the corrections

Effect	Order of magnitude	Uncertainty
Volcanic and coseismic (highly localised)	$< 10^{-16}$	?
Geodynamic and man-made (localised and long-term > 1 year)	$< 10^{-16}$	?
External masses (Moon, Sun)	10^{-17}	$< 10^{-18}$
Solid Earth tides	10^{-17}	$< 10^{-18}$
Ocean tides	10^{-17}	$< 10^{-18}$
Polar motion (long-term ≈ 430 days)	10^{-18}	$< 10^{-18}$
Atmospheric pressure	10^{-18}	$< 10^{-18}$

TAI, apart from a constant offset, and can be obtained from TCG by the transformation

$$\frac{dTT}{dTCG} = 1 - L_g \,, \tag{16}$$

where $L_g = W_0/c^2 = 6.9692903 \times 10^{-10} \pm 1 \times 10^{-17}$.

It follows that at present the accuracy of syntonization with respect to TT is limited to $\approx 1 \times 10^{-17}$ by uncertainties in the determination of the potential on the geoid W_0, even for clocks on board terrestrial satellites.

This limit is inherent in the definition of TT and can therefore only be improved by reducing the uncertainty of W_0. If highly stable clocks on terrestrial satellites are to be used for the realization of TT at accuracies exceeding this limit it may prove necessary to change this definition. One possibility would be to turn L_g into a defining constant with a fixed value. This would also provide a relativistic definition of the geoid (Bjerhammar 1985; Soffel et al. 1988).

5. Small time varying effects

Sections 3 and 4 treat the syntonization of clocks with respect to some ideal coordinate time (TCG or TT). We found that for clocks on the Earth's surface the uncertainty of this syntonization is limited at 10^{-17} by uncertainties in the geopotential. For this reason several time varying effects with amplitudes below this level (ocean tides, polar motion, atmospheric pressure etc...) were neglected. In this section we will consider the case where only the stability of the relative rate between two clocks is of interest. Then only time varying effects need be considered which, as shown below, can be calculated to 10^{-18} accuracy even for clocks on the Earth's surface. These effects are summarized in Table 3, together with orders of magnitude and uncertainties of the associated corrections.

The correction due to external masses (Moon and Sun) and the solid Earth tide can be calculated (see Eq. (13)) with an uncertainty of $\approx 4 \times 10^{-19}$ due to the uncertainty in the determination of the Love numbers (Farrell 1972).

The effect of the ocean tide, including the associated loading, can be obtained using global oceanic tide models (Schwiderski 1983) and the loading deformation coefficients of Farrell (1972). The results of one such calculation (Scherneck 1994) describe the total effect of the oceanic tides on the potential at the surface of the Earth in a 1° × 1° grid covering most of the globe. For the M2 lunar tide, the correction is 9×10^{-18} in a few regions with a maximum of twice this value for the total tide, the uncertainty of these values being $\approx 2 \times 10^{-19}$ (IERS 1992).

Atmospheric pressure variations may cause rate shifts of a few parts in 10^{18}. The direct gravitational potential of the atmosphere can be calculated adapting a method employed by Merriam (1992), in order to obtain the effect on the gravitational potential rather than the gravitational acceleration. The contribution to the potential at some point P, due to a thin column of air of infinitesimal area at azimuth α and a geocentric angular distance ϕ from the point of interest can be calculated as a function of the pressure and temperature at its base. Using the ideal gas law and assuming hydrostatic equilibrium and an isothermal atmosphere gives for the potential of one column

$$U(\phi,\alpha) = \frac{GP_0 a_1^2 \sin\phi \, d\phi \, d\alpha}{RT_0} \times \int_0^{z_{max}} \frac{e^{-z/H}}{\left[\rho^2 + (a_1+z)^2 - 2\rho(a_1+z)\cos\phi\right]^{1/2}} dz \tag{17}$$

where P_0 and T_0 are the pressure and the temperature at the base of the column, R is the specific gas constant for dry air (R = 287.05 J kg$^{-1}K^{-1}$), z_{max} is the height of the atmosphere ($z_{max} \approx$ 50 km), and H is the scale height of the atmosphere. Typically, H varies from about 8 km near the surface to about 7 km in the stratosphere (Merriam 1992). Summing these functions over the globe using surface pressure and temperature data and small increments $d\alpha$ and $d\phi$ gives the total gravitational potential of the atmosphere. The additional change caused by the associated atmospheric loading can be calculated using the above method and the surface load Love numbers k'_n and h'_n (Farrell 1972). Alternatively, a simple regression formula by Rabbel & Zschau (1985) can be used, giving the radial displacement of a point on the Earth's surface by

$$\Delta\rho/\text{mm} = (-0.35p - 0.55\overline{p})/\text{mbar} \tag{18}$$

where p is the pressure variation at the point of interest and $\overline{p}$ the average of the pressure variation in the surrounding area of

2000 km radius with the pressure values set equal to zero over ocean areas. Rabbel & Zschau (1985) estimate the uncertainty of this expression to be less than 1 mm. For pressure variations of 10 mbar on a global scale (corresponding to seasonal changes) the effect on the rate of a clock on the Earth's surface can reach 2 parts in 10^{18} due to the direct potential (Eq. (17)) but the magnitude of the displacement effect remains below 10^{-18}. Local pressure changes ((anti)cyclones) can cause displacements of up to 2.5 cm, corresponding to a correction of 2.7×10^{-18}, but have a negligible direct potential. Finally the secondary potential due to the deformation of the Earth can be neglected as the appropriate surface load Love numbers k'_n are at least a factor 4 smaller than the corresponding h'_n which are used to calculate the displacement effect (Farrell 1972).

For a clock on the Earth's surface the centrifugal potential is given by $(\omega\rho\sin\theta)^2/2$. Differentiating this expression with respect to ω and to θ, dividing by c^2 and including Love numbers allows the calculation of the total correction due to polar motion $\Delta\theta$ and variations in the angular velocity of the Earth $\Delta\omega$,

$$(1 + k_2 - h_2)\frac{\Delta(\omega\rho\sin\theta)^2}{2c^2} = (1 + k_2 - h_2) \times \frac{1}{c^2}\left(\frac{1}{2}\omega^2\rho^2\sin 2\theta\Delta\theta + \omega\rho^2\sin^2\theta\Delta\omega\right). \quad (19)$$

Because the spherical harmonic contribution of the centrifugal potential is restricted to degree two (Hinderer et al. 1982) the total effect can be obtained using the classical Love numbers k_2, h_2 with $(1 + k_2 - h_2) = 0.69$. Maximum values for $\Delta\theta$ and $\Delta\omega$ are 2.4×10^{-6} rad and 7×10^{-12} rad s^{-1} respectively (Torge 1989) which result in corrections of up to 2×10^{-18} and 1.6×10^{-19} for the first and second terms in (19).

Finally we mention long term effects of a geodynamic nature, and give some examples of highly localised effects of volcanic, coseismic and man made origin which might have to be taken into account at certain sites.

On tectonic plate boundaries, geodynamic effects may give rise to corrections of up to 10^{-16} over a period of several years. For example in northern Iceland elevation changes of 1 m (corresponding to a correction of the order 10^{-16}) were observed between 1975 and 1980 (Torge 1989). In other regions the magnitude of geodynamic effects may only marginally reach the 10^{-18} level on time scales exceeding 1 year.

Volcanic and coseismic activities observed in Hawaii and California caused elevation changes of the order of 1 m (Torge 1989) over periods up to several months.

Mass displacements by human interference (e.g. exploitation of oil, gas, geothermal fields) may lead to local corrections of order 10^{-17} per year (Torge 1989).

6. Conclusion

In Sects. 3 and 4 we present a theory for the syntonization of clocks with respect to geocentric coordinate times (TCG and TT) which includes all terms greater than 10^{-18} for clocks on board satellites at altitudes exceeding 10 000 km. For this purpose terms of order c^{-3} and c^{-4} in the metric can be neglected, which implies that the specification of coordinate conditions and the state of rotation of the reference system is not necessary.

Syntonization with respect to terrestrial time (TT), an ideal form of TAI, is limited to 1×10^{-17} by uncertainties in the determination of the potential on the geoid, W_0, inherent to its definition.

For clocks on the Earth's surface, syntonization with respect to TCG or TT is limited to $(2\text{–}3)\times 10^{-17}$ by uncertainties in the determination of the geopotential at the location of the clock.

These sections were concerned with the syntonization of clocks with respect to some ideal coordinate time scale, i.e. the determination of the rate between a physical time scale provided by a clock and the ideal coordinate time of some reference system using corrections which are provided by theory. In Sect. 5 we consider the syntonization of two distant clocks, i.e. the determination of the relative rate of two physical clocks. Corrections due to time varying effects that could affect the stability of this rate are provided to an accuracy of 10^{-18}.

At present the stability of atomic clocks is approaching 10^{-16} (Maleki 1993) with further improvements expected in the near future. For comparisons of these highly stable clocks over large distances, and their application in experimental relativity, geodesy, geophysics etc.., a sufficiently accurate relativistic theory for their syntonization, like the one presented in this paper, seems indispensable.

Combined with a previous paper (Petit & Wolf 1994) the results obtained here amount to a complete relativistic theory for the realization of a geocentric coordinate time scale with uncertainties in synchronization and syntonization of one picosecond and 10^{-18} respectively.

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