QUANTUM FIELD THEORY

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PREFACE

This book has pleasant associations with the group of physicists who were working at Manchester from 1953 to 1955. When Dr. UMEZAWA arrived in Manchester he brought with him a small Japanese book which he had written; the many formulae and names of European physicists printed in Roman type were sufficient clues to the contents: it was clearly a textbook of quantum electrodynamics. UMEZAWA himself was full of that subject, but for some time his conversation was very much like his book to us, with the added obstacle that the names of the European physicists had to be recognised by ear rather than by sight! This tantalising situation was much relieved in the first place by UMEZAWA's rapid improvement in the mastery of the English language, but also by his more specific proposal to prepare an English translation of his book. Everybody was at once willing to assist in the carrying out of this project. UMEZAWA took it in his stride as a minor part of his impressive activity. He mustered collaborators around him, kept up a scientific correspondence with his Japanese colleagues, and almost at the same rate as the pages of the translation were produced other pages were coming forth containing the results of the investigations he was concurrently pursuing.

With great eagerness the successive chapters of the translation were read and discussed among the group as they came, rather untidy I must say, from the stencil roller. The stencils were also produced by collective effort in the Common Room, to which the bustle of people in shirt sleeves gave a suspiciously experimental air. In this process he gave us most, but he also received from us some constructive criticism. At any rate his book in its present form is rather different from the original Japanese, and one may say that it has passed a rather searching test of its suitability as an introduction into the various aspects of the subject, starting from the first begin

nings and leading up to the latest developments. Among these I am pleased to single out the elegant exposition of the theory of propagators which was developed in our Department by UMEZAWA and VISCONTI and could be incorporated in the last revision of the translation. I have no doubt the book will prove as useful to all readers as it has been to us, although they will miss the charm which the author's personality added to its teachings as it gradually took shape under our eyes.

L. ROSENFELD

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A^*	hermitian conjugate of the q-number A or complex conjugate of the c-number A.
A_μ^*	hermitian conjugate of the four dimentional vector, i.e., $A_\mu^* = (A_1^*, A_2^*, A_3^*, A_4^* = -iA_0^*)$.
$A_\mu^\dagger$	four dimensional vector whose components are A_1^*, A_2^*, A_3^* and iA_0^*.
A^T	transposed matrix of A.
$\mathbf{A}$	three dimensional vector whose components are A_k.
$\mathbf{A} \wedge \mathbf{B}$	vector product of $\mathbf{A}$ and $\mathbf{B}$.
$[A, B]_-$ or $[A, B]$	commutator $AB - BA$.
$[A, B]_+$	anticommutator $AB + BA$.
$A \propto B$	means A is proportional to B.
C	light velocity.
delta functions	$\delta(\mathbf{x}) = \delta(x_1)\ \delta(x_2)\ \delta(x_3)$ $\delta^4(x) = \delta(x_1)\ \delta(x_2)\ \delta(x_3)\ \delta(x_0)$.
derivation operators	$\partial_\mu = \frac{\partial}{\partial x_\mu}$, $\Box = \partial_\mu \partial_\mu$, $\Delta = \partial_k \partial_k$.
$\varepsilon(Z)$	$= \begin{cases} +1 & Z_0 > 0 \\ -1 & Z_0 < 0. \end{cases}$
field quantities	$Q(x) = [Q_\alpha(x)]$ for unspecified fields, $\psi(x)$ when the field equations are given as a set of first order differential equations, $U_\mu \ldots(x)$ for quantities of Bose fields in the tensor representation.
ϕ	c-number potential.
H'	interaction Hamiltonian.
h	Planck constant.
$\hbar$	$h/2\pi$.

integration symbols	$\int d\mathbf{k} = \int d^3k = \int dk_1 dk_2 dk_3$, $\int d^4k = \int dk_1 dk_2 dk_3 dk_0$, $\int d^3x = \int dx_1 dx_2 dx_3$. $\int d^4x = \int dx_1 dx_2 dx_3 dt$, $(t = x_0 = -ix_4)$.
k, l, m cycl. (1, 2, 3)	implies that (k, l, m) is any one of (1, 2, 3), (2, 3, 1) and (3, 1, 2).
K_μ	energy-momentum of a *free* particle, i.e., $K_\mu = (k_1, k_2, k_3, iK_0)$ with $K_0 = \sqrt{\mathbf{k}\cdot\mathbf{k} + \varkappa^2}$.
$\varkappa$	mass of an unspecified particle.
L	total Lagrangian.
L^0	Lagrangian of free fields.
L'	interaction Lagrangian.
masses	m for the electron, $\varkappa$ for an unspecified particle.
$\mathbf{m}$	operator of the magnetic moment
$n_\mu(x)$	normal vector of a space-like surface σ at the point x
$P_{\mu\nu}$	operator of the total angular momentum of fields.
state vector	$\Psi[\sigma]$ for states on the surface σ, Φ_0 for the vacuum state.
suffices	μ runs from 1 to 4, e.g. $a_\mu = (a_1, a_2, a_3, a_4 = ia_0)$, k runs from 1 to 3, e.g. $a_k = (a_1, a_2, a_3)$.
$\mathrm{Sp}(A)$	trace of the matrix A.
σ	space-like surface.
$\sigma(x)$	space-like surface through the point x.
σ_k $(k = 1, 2, 3)$	spin matrices.
T_μ	operator of the energy-momentum of fields.

VALUES OF CONSTANTS

$$\alpha \equiv e^2/4\pi = \left(\frac{1}{137.3}\right)\hbar C$$

electron mass

$$m \begin{cases} = 0.78 \times 10^{21} \text{ sec.}^{-1} \times \left(\frac{1}{\hbar}\right) \\ = 0.26 \times 10^{11} \text{ cm.}^{-1} \times \left(\frac{C}{\hbar}\right) \end{cases}$$

classical electron radius

$$r_0 \equiv \frac{\alpha}{mC^2} = 2.80 \times 10^{-13} \text{ cm.}$$

Compton wave length of electron

$$\lambda_0 \equiv \frac{\hbar}{mC} = 3.85 \times 10^{-11} \text{ cm.}$$

radius of hydrogen atom

$$a \equiv \frac{k^2}{m\alpha} = 0.528 \times 10^{-8} \text{ cm.}$$

CHAPTER I

HISTORICAL INTRODUCTION

§ 1. The Search for a Theory of Elementary Particles

It is convenient to classify the various forms of matter according to the physical laws which are applicable to them. Thus it is that we distinguish between the solid, liquid and gaseous states of matter in bulk. In our present subject a classification that depends on the scale of the phenomena considered is appropriate and familiar; thus we distinguish between nebula, heavenly bodies, matter on the laboratory scale, atoms, nuclei and elementary particles. These levels, or strata, are not, however, independent – atoms are constituted of nuclei, nuclei of elementary particles.

The laws that govern the motion of matter are not the same at all levels; on the laboratory scale the laws of Newtonian mechanics are valid—on the atomic scale quantum mechanics is valid. And we can say that the object of the theory of elementary particles is the codification of the laws of this hitherto unknown stratum.

Our only approach to the theory of elementary particles is to attempt to base it on a suitable modification of the laws of the preceding level. Although serious contradictions with the first theory are certain to arise in this process, usually it is just these contradictions that provide the key to the new theory. A classical example is the contradiction between the Bohr quantum condition and Newtonian mechanics, which eventually led to the new quantum mechanics.

The limits of applicability of a theory define the boundaries of the corresponding level. However there are some smooth transitions from one level to the next in the sense that certain results of one theory correspond with certain results of the other – this is the correspondence principle. For example, quantum mechanics and relativity are respectively characterised by the constants h and C and, when h and $(1/C)$ are taken as vanishingly small, the results of these theories are identical with those of Newtonian mechanics.

We shall consider the history of the theory of elementary particles from this point of view. Although the essential features of the

behaviour of elementary particles are not yet contained in the present theory, many important advances in this direction have been made. For example, the recent discovery of several new elementary particles has made it possible to discuss fundamental problems in a more general way. Of great importance is the progress made in the development of meson theory.

Many of the fundamental difficulties concerning the structure of elementary particles were present even in the classical Lorentz theory of the electron. However, they were disregarded, unsolved, in the discovery of quantum mechanics and the provisional formulation of the theory of elementary particles is an extension of quantum mechanics (cf. Ch. VI). Certain approximate results of the latter theory are in excellent agreement with experimental observations of electrons and the electromagnetic field; nonetheless it has long been known that the theory can make no exact prediction because of the effects of the so-called *proper fields*, which appear more closely bound up with the difficulties of the classical theory of electrons. These difficulties, it might be said, are the fundamental problem which an accurate theory of elementary particles must solve.

It is natural to ask how the present theory can be applied to electrons and electromagnetic fields and, to a certain approximation, be successful in spite of the fundamental difficulties involved. This question is strictly that of the limits of applicability of the present theory of elementary particles.

At first the problems presented by the proper fields were regarded as academic ones, although a satisfactory solution of the general problem depends on an investigation of the associated physical effects. The first important step towards this solution was the Tomonaga–Schwinger renormalisation of quantum electrodynamics, which succeeded in explaining certain real physical properties of the proper fields of electrodynamics.

In the following paragraph we shall outline the history of these problems—the discovery of the various elementary particles and the demonstration that the physical effects of the proper fields are real. We shall also discuss the limits of applicability of the present theory.

§ 2. Theory of Elementary Particles

We define elementary particles as being those that, as far as we know, have no internal structure. The most familiar are the electron

and the photon, which played an important part in the discovery of quantum mechanics. For this reason the present theory of elementary particles has been developed as an extension of quantum mechanics (cf. Ch. VI).

The first hints of the existence of electrons were provided by the experimental data of electro-chemistry, which indicated that the electric charges on ions are always integral multiples of some elementary charge e. By means of his experiments on cathode rays Thomson proved the existence of a particle with a negative and elementary electric charge – the electron. The elementary nature of the electric charge is a characteristic property of elementary particles.

The electron has an intrinsic angular momentum or *spin*. The spin angular momentum is not zero even when the electron is at rest. The value $\hbar/2$ of the electron spin that it was necessary to assume in order to explain the fine-structure of atomic spectra and the anomalous Zeeman effects of alkali atoms was accounted for by Dirac's relativistic quantum theory of the electron.

Since every atomic mass is approximately equal to the mass of the hydrogen atom multiplied by some integer, we may surmise that the proton is a unit of which nuclei are made (cf. PROUT's Hypothesis, [1815]). However, we cannot conclude that nuclei are made of protons alone because, for example, the deuteron, which has a mass twice that of the hydrogen atom, has only a charge e (and not $2e$).

Furthermore, even though the electron mass is negligibly small in comparison with the proton mass, the deuteron cannot be made of two protons and one electron (i.e. three particles with spin $\hbar/2$), because this assumption would imply a half-integer spin for the deuteron, in conflict with the experimental value $\hbar$.

This difficulty was resolved by the discovery of the neutron in 1932. JOLIOT CURIE [1931, 1932] found that a neutral radiation was produced in the artificial disintegration of Be nuclei, and that this could transfer considerable energies to hydrogen nuclei on collision with them. CHADWICK [1932] demonstrated that the radiation consisted of neutral particles with roughly the same mass as that of the proton.

As early as 1920 RUTHERFORD had postulated the existence of a neutral particle of mass roughly the same as that of the proton; he supposed it to be constituted of one proton and one electron and to penetrate matter easily because of its absence of charge. Nevertheless

GLASSON [1921] and ROBERTS [1922] failed to find such a particle in electrical discharges in hydrogen and, in this way, to demonstrate the truth of Rutherford's hypothesis. But the hypothesis was not rejected. And it was not surprising that Chadwick, who has worked with Rutherford at the Cavendish Laboratory, should discover the neutron – indeed. CHADWICK's second paper [1932] refers to Rutherford's predictions.

On the basis of this discovery IWANENKO [1932] suggested that all nuclei are constituted of protons and neutrons. Independently HEISENBERG [1932] made a detailed investigation of the various properties of nuclei with the object of accounting for the interrelationship. Ever since the neutron and proton have been regarded as different states of the same particle – the *nucleon* (cf. Example 10, Ch. VII). This doctrine became the foundation of the theory of nuclei and, on the other hand, led to difficulties which stimulated the formulation of meson theory.

Experimental information on nuclear structure was enormously increased with the development of a high energy accelerator by COCKCROFT and WALTON at the Cavendish Laboratory [1932]. Many nuclei could be disintegrated at will by means of this machine. Progress in the field of high energy accelerators has been marked by the development of the cyclotron by LAWRENCE [1930], the high voltage static electric generator by Van de Graaf, the synclotron by MCMILLAN [1945] and VEKSLER [1945], and the betatron by KERST [1941]. Recently machines have been built with energies sufficiently high to produce some elementary particles artificially.

Because the proton-neutron theory of nuclear structure does not allow for the existence of electrons in nuclei, it is necessary to interpret the β-disintegration of nuclei as the production of electrons by nuclei. The fact that the energies of the electrons produced should be continuously distributed in spite of the exact differences of energy between the parent and daughter nuclei seems, at first sight, incompatible with the energy conservation law. In a lecture at Pasadena in 1931 PAULI showed that the requirement of the conservation of energy would be fulfilled if it were postulated that a neutral particle of very small mass (called the neutrino) is emitted at the same time as the electron. This idea, which was discussed at the 7th SOLVAY CONGRESS [1933], is substantiated by the fact that the greatest energy with which an electron can be emitted in β-decay is roughly equal to

the difference between the energies of the parent and daughter nuclei (ELLIS and MOTT, [1933]). In 1934 a theory of β-disintegration based on the neutrino hypothesis was developed (FERMI [1934]) and this, as will be shown later, prepared the way for the meson theory.

The positron now enters the picture. The relativistic quantum theory (DIRAC [1928a, b]) involved serious difficulties in the interpretation of negative energy states (KLEIN [1929]). For this reason Dirac put forward the so-called hole theory, which introduced a particle of mass equal to that of the electron and of positive charge (DIRAC, [1931a, b]). This particle was called the positron and could be identified with that discovered independently by Anderson in Wilson's cloud chamber experiments on cosmic rays. (ANDERSON [1932]; also BLACKETT, CHADWICK and OCCHIALINI [1934]). Dirac's theory treats electrons and positrons symmetrically [1]), and is thus the prototype of later quantum field theories of charged particles, which are always assumed to exist in positive and negative form (cf. Ch. IX). On these grounds we would anticipate the existence of a particle of mass equal to that of the proton and of opposite and equal charge. Although there are some experimental data which suggest that such a particle does exist, this cannot be affirmed or denied. The resolution of this matter would indicate whether or not the proton was amenable to the present theory of elementary particles.

§ 3. Mesons

The existence of the electromagnetic and gravitational forces between material objects was established a long time ago. The former can be derived from quantum electrodynamics. In fact, the present quantum field theory always gives a force between elementary particles (e.g. electrons) which is mediated by other elementary particles (e.g. the photons) when both types of particle (e.g. the electrons and the photons) interact with each other. We shall leave out of consideration the gravitational force [2]), because it is not yet clear if its effects can be properly described by the elementary particles formalism; in any

[1]) The 1947 Congress of the International Union of Physics agreed to name electrons and positrons *negatons* and *positons* respectively, and to reserve the name *electron* for the generic reference to particles of both signs. In this book we shall adhere to this convention.

[2]) Therefore, throughout this book the special relativity is simply called the relativity.

case the coupling constant of the gravitational to other fields is known to be very small in comparison with other known coupling constants. The forces between nucleons cannot be purely electromagnetic because neutrons have no electric charge, and because nuclear forces are experimentally known to be about 100 times stronger than electromagnetic forces (cf. Example 11 of Ch. VII).

Since Heisenberg published his theory of nuclear structure, people have tried to develop nuclear theory in two ways: by formulating phenomenological nuclear forces, and by applying the theory of elementary particles to nuclei. Many authors have developed phenomenological theories of nuclear forces in which the theoretical results derived from the various assumed potentials were compared with the experimental data. They studied the stationary states of light nuclei, especially the deuteron, and nucleon-nucleon scattering, and then applied the nuclear forces obtained from these experimental data to derive properties of heavier nuclei.

Heavy nuclei exhibit a characteristic feature, the so-called saturation property. If all nucleons in a nucleus (of mass number n) interacted with each other by ordinary two-body forces, the binding energies would be proportional to n^2, not to n as is found experimentally. This suggests a similarity with the valences of chemical bonds. Each particle seems to select a small number of particles with which to interact. Taking into account this analogy, Heisenberg introduced exchange forces in his phenomenological investigations of nuclear forces. The range of the force between two nucleons given by the phenomenological investigations is $\approx 2 \times 10^{-13}$ cm, much smaller than that of the electromagnetic interaction.

As explained above, nuclear forces cannot be reduced to an electromagnetic interaction. The other elementary particles (known two decades ago) which could interact with nucleons were electrons and neutrinos. IWANENKO [1934] and TAMM [1934] estimated the nuclear force due to the electron-neutrino interaction with nucleons (i.e. β-interaction). We shall show later that in the present quantum field theory the magnetic moment of each elementary particle must include a contribution from the interaction with other elementary particles. Therefore, if nuclear forces could be interpreted as due to the β-interaction, this would also be responsible for the nucleon magnetic moments. WICK [1935] and WEIZSACKER [1936] investigated if this assumption could give the experimental values (2.8 $e\hbar/2Mc$,

$-1.9\, e\hbar/2\, MC$; M is the nucleon mass) of the magnetic moments of the proton and the neutron. However, the nuclear force and anomalous parts of the magnetic moments given by the β-interaction were too small compared with the experimental data. At that time no other elementary particles had been observed experimentally, and the question arose of whether nuclear forces could be derived from quantum field theory.

However, this difficulty was resolved by a very simple idea, which can be confirmed only by reliable experimental data. In the present quantum field theory we can show (WICK [1938]) by means of the uncertainty principle alone (cf. Ch. XII) that the mass $\varkappa$ of the elementary particles responsible for a force of range r must be $\varkappa = (\hbar/rC)$. Substituting in this formula the value of r given by the phenomenological investigations of the nuclear forces, we see that the nuclear force is due to elementary particles of rest mass $\varkappa \approx 200\, m$ (m electron mass). This particle, the meson, was introduced theoretically by YUKAWA [1935]. Furthermore, from meson theory one can derive the exchange forces in a sensible way. However, exchange forces have not yet given a complete solution of the problem of saturation of nuclear forces. As pointed out above, the problems of nuclear forces and anomalous magnetic moments of nucleons are intimately connected. We shall discuss later the calculations of the anomalous magnetic moments of nucleons from meson theory. In 1936, ANDERSON and NEDDERMEYER discovered a particle of rest mass ≈ 200 m in his cloud chamber observations of cosmic rays [1]) and OPPENHEIMER. STUECKELBERG [1937] and YUKAWA [1937] pointed out that this new particle could be Yukawa's meson. This stimulated a renewed interest in elementary particle theory, and provided confidence in the quantum field formalism which was an extension of quantum electrodynamics.

§ 4. Short Review of the known Elementary Particles

After the discovery of the meson, the existence of many sorts of elementary particle was reported.

Cosmic radiation consists of two components—the hard component (which can penetrate 10 cm in lead) and the soft component (which is absorbed by 10 cm of lead). The first of these consists mainly of mesons. Experimental observations show that the cosmic ray meson

[1]) KUNZE had reported the existence of a particle like the meson in 1932.

decays into an electron and neutral particles in a very short time ($\approx 10^{-6}$ sec.). This decay process was observed by WILLIAMS and ROBERTS [1940], in Wilson cloud chambers. The above value for the lifetime had been suggested by various phenomena observed in cosmic radiation. RASETTI [1940] and MAZE [1941] succeeded in measuring the lifetime for each decay process directly. On account of the energy conservation law we must expect that neutral particles are produced together with decay electrons.

The decay process of the meson was also taken into account in Yukawa's theory, which assumed that β-disintegration is not a direct transmutation of nucleons with the emission of electrons and neutrinos, but that mesons decaying into electrons and neutrinos occur as an intermediate stage. Since the magnitude of the nucleon-meson interaction and of the nucleon-electron indirect interaction could be estimated from the experimental data on nuclear forces and β-disintegrations respectively, the magnitude of the direct meson-electron interaction, and therefore of the lifetime of the mesons, could be estimated. However, these estimates give a lifetime 10^{-8} sec., which is smaller than the observed lifetime by a factor 10^{-2} (BETHE and NORDHEIM [1940], YUKAWA, SAKATA, KOBAYASHI and TAKETANI [1938]).

Since mesons decay naturally into electrons, we cannot avoid the assumption that all observed cosmic ray mesons are produced near the earth. Moreover, the fact that the main part of the primary cosmic radiation approaching the earth consists of protons (SCHEIN, JESSE and WOLLAN [1940]) shows that the observed mesons must be produced by collisions of these protons with atoms of the atmospheric gases. The meson-nucleon interaction must therefore be strong enough to explain the production rate of observed mesons. In fact, in Yukawa's theory, this interaction is strong enough to give the right nuclear force. However, this fact seemed to be incompatible with the fact that cosmic ray mesons have a small cross-section for interaction with matter (WILSON [1939]).

These difficulties were made obvious by the cosmic ray experiments of the Rome group. (CONVERSI, PANCINI and PICCIONI [1947]). Previously, TOMONAGA and ARAKI [1940] had pointed out that in Yukawa's theory almost all negatively charged mesons must be captured by nuclei on account of the attractive Coulomb force and the meson-nucleon interaction. However, the experiments of the

Rome group showed that an appreciable proportion of the negatively charged mesons are not absorbed in carbon but decay into electrons. The detailed calculations of FERMI, TELLER and WEISSKOPF [1947] showed that the theoretical and experimental capture cross-sections differ by a factor $\approx 10^{12}$

Since this difficulty seemed too serious to be settled by any improvement of the calculations, it led people to a "two meson theory", in which it was assumed that the cosmic ray mesons observed near sea level are not Yukawa's nuclear mesons, but are produced as secondary particles in the decay of the latter, and interact only weakly with matter. Various attempts to resolve this difficulty were made. For example, WEISSKOPF [1947] considered an interesting possibility based on the "meson-pregnant states of nucleons". These two attempts had in common the separation of the meson absorption process from the meson production process. In 1942, the two mesons theory was presented by SAKATA, TANIKAWA and INOUE. MARSHAK and BETHE [1947] independently presented the two mesons theory.

The two mesons theory was experimentally confirmed by the Bristol Group (LATTES, OCCHIALINI and POWELL, [1947]). They showed that the heavier mesons produced in nuclear interactions decay into lighter mesons in the emulsions. The emulsion technique has, since then, made a brilliant contribution to our knowledge of the properties of the new elementary particles.

In the two mesons theory we have the following interaction scheme: the nuclear meson (which is called a π-meson since the Bristol Group experiments) interacts strongly with nucleons to give the nuclear force. Since it is just equivalent to the particle introduced by Yukawa, we may say that the conclusive identification of the Yukawa meson was given by the Bristol group experiments. As shown previously, the cosmic ray mesons observed near sea level interact weakly with nucleons and decay naturally into electrons with a lifetime $\approx 10^{-6}$ sec. They are now called μ-mesons. Many authors (e.g. LEIGHTON, ANDERSON and SERIFF [1949]) have shown experimentally that the number of neutral particles produced in the decay process of a μ-meson is more than two. The π-mesons decay naturally into μ-mesons with a lifetime $\approx 10^{-8}$ sec. A precise measurement of the π-meson lifetime was made by RICHARDSON [1948] using artificially produced mesons. The energy conservation law requires that a neutral particle of very small mass accompanies the μ-meson in the above decay

process. Thus we see that the protons of the primary cosmic radiation produce π-mesons by collisions with atoms of the atmospheric gases, and these π-mesons decay into μ-mesons which in their turn decay into electrons. There still remain various possibilities for the detailed structure of the interaction scheme for the nucleons, π- and μ-mesons and electrons; e.g. both of the direct interactions between the nucleon and the electron and the coexistence of (π-nucleon) – and (π-electron) – interactions can lead to the β-disintegration of nuclei. Much work has been done (e.g. TIOMNO and WHEELER [1949], TAKETANI, NAKAMURA and SASAKI [1949], YUKAWA [1949]) to examine which scheme gives results nearest to the experimental values (cf. Example 11, Ch. VII). From this the conclusion is that the β-interaction is not completely mediated by π-mesons.

In 1948 π-mesons were produced artificially for the first time by bombardment of a target with α-particles accelerated in the Berkeley cyclotron. Great progress has been made in meson physics by means of artificially produced mesons. In particular, it has been shown (MARSHAK [1951], CHESTON [1951], DURBIN, LOAR and STEINBERGER [1951], CLARK, ROBERTS and WILSON [1951]) that the spin of the π-meson is zero and moreover, the π-meson seems to be of the pseudoscalar type (cf. Example 6 of Ch. X).

We cannot observe directly tracks of neutral mesons in Wilson chambers or emulsions. However, experimental data on nuclear forces have suggested the existence of neutral nuclear mesons (which are now called π^0-mesons). The fact (BREIT *et al.* [1939]) that cross-sections for low energy nucleon-nucleon scattering do not depend on the charge state of nucleons (except for the Coulomb force acting in pp scattering) shows that there must be a contribution of the neutral nuclear mesons to the nuclear forces. (FRÖHLICH, HEITLER and KEMMER [1938], YUKAWA, SAKATA, KOBAYASHI and TAKETANI [1938]). In 1940 Sakata and Tanikawa pointed out that according to the present quantum field theory the neutral nuclear meson must decay quickly into photons (cf. Example 3, Ch. XIII); it decays into two or three photons, if its spin is zero or one respectively. This led observers to expect that photons produced in the decay of neutral mesons should contribute considerably to the soft component of cosmic radiation. The so-called air showers observed in cosmic radiation also suggested the existence of photons produced in the decay of the neutral mesons. Furthermore, emulsion experiments at altitudes of

100,000 feet (KAPLON, PETER and BRADT, [1949]) had given photographs which seemed to show the decay of neutral mesons into photons. In 1950, the Berkeley group succeeded in producing artificially the neutral meson, which decayed into two photons, by bombarding the nuclei in a Be target with accelerated protons. (BJORKLUND *et al.*, [1950]). According to the present quantum field theory, we may say that the spin of the neutral meson is zero (cf. Example 2, Ch. XIII).

In 1947, ROCHESTER and BUTLER found new positively charged and neutral particles of masses ≈ 1000 m. Many observations of new elementary particles have been reported since then. They are called Λ, θ, τ, χ and $\varkappa$-particles etc. (cf. § 5, Ch. IV). In the present quantum field theory they are assumed to have definite spins, masses and charges.

The assumption that all these particles are quanta of really distinct fields is, probably, a provisional one, valid only under some restrictions (e.g. in the region of not very high energies, not very strong couplings etc.) such that the structure of particles can be neglected. But still one can hope, in this way, to aim at understanding their mutual relations and finally constructing a synthetic theory for them.

The problem of the structure of particles has been often discussed in connection with the fundamental difficulties inherent to the present quantum field theory. We shall describe this problem in the following paragraphs.

§ 5. The Quantum Field Theory and its Difficulties

As pointed out above, the transmutations are a characteristic feature of elementary particles. The present quantum field theory was formulated (HEISENBERG and PAULI [1929]) by extending quantum mechanics so as to satisfy the relativity requirements and to treat the various transmutations.

Quantum mechanics, as originally formulated, was restricted to problems in which the number of particles remained constant and their velocities were low enough for a non-relativistic approximation to be valid. Its first extension, the relativistic quantum mechanics of particles, was given by DIRAC [1928] and this led us to the theory of positions in order to settle the difficulties of the negative energy states introduced by the theory (DIRAC [1931]). In the latter theory we have transmutations between electrons and photons, and we know now

that this is equivalent to the quantum theory of the electron field (KRAMERS [1937], IWANENKO and SOKOLOW [1937]).

The electromagnetic field requires a more direct introduction of the quantum field theory. In this case we have the Maxwell theory which is covariant in form, and the radiation phenomena of electromagnetic waves can be regarded as the classical form of the transmutations of elementary particles (electrons and photons): — in the quantum field theory, the radiation and absorption of electromagnetic waves are regarded as the creation and annihilation of photons, respectively.

We can roughly express the feature of the quantum field theory as follows: The state of fields at a certain time can be determined by observations at all points at this time. In order that such a state can be uniquely determined, it is necessary that the observations of two points at the same time do not disturb each other. This requirement can be automatically satisfied by the relativity requirement, because the propagation velocity of the disturbances must be smaller than the light velocity. A photon of large energy ($> 2\,m$: m the electron mass) induces a change of the state of the electron field--the pair production of a negaton and a positon—in the matter. The state vector can be treated according to the same argument (e.g. the superposition law, the time change according to the Schrödinger equation etc.) as in quantum mechanics.

The quantum electrodynamics has been based on some experimental features of the quantum properties of the system of the electron and electromagnetic fields (cf. Ch. VI) and has achieved some brilliant successes, e.g. the cascade showers. By 1932 it was clear that the penetrability of cosmic rays in lead was stronger than that calculated by taking into account the radiation and ionization losses. In 1933, BLACKETT and OCCHIALINI discovered phenomena in which many charged particles were produced simultaneously, i.e. the showers. These phenomena were explained by the cascade theory (CARLSON and OPPENHEIMER [1936], BHABHA and HEITLER [1936]) as the repetition of electron pair production by photons and the radiation of photons by the high energy electrons. This shows that quantum electrodynamics can be successful even in the region of very high energies. However there remained unexplained a part of the highly penetrable cosmic rays, i.e. the hard component discussed in § 3 and § 4.

In 1936 DIRAC extended his relativistic theory of the electron to

the case of general spin. This theory was investigated in detail by FIERZ and PAULI [1939], (cf. Ch. IV). The theory of the case of zero spin was developed already in 1934 by PAULI and WEISSKOPF. PAULI [1940] clarified the general relation between the spins and the statistics; the elementary particles of the half-integer spins (e.g. the electrons, nucleons, etc.) must obey Fermi statistics and those of the integer spins (e.g. the photons) Bose statistics (cf. Ch. VIII). General kinds of relativistic wave equations will be discussed in Ch. V.

These theories of elementary particles of general types found their application in meson theory. YAKAWA, SAKATA, TAKETANI and KOBAYASHI [1938], KEMMER [1938] and BHABHA [1938] investigated the meson theory assuming various types of mesons. Speaking theoretically, it is to be expected at present that all kinds of elementary particles given by the general theory of relativistic wave equations may exist in nature. Therefore, it is an interesting question to ask how the particles realised in nature can be selected from the general framework.

To answer this question seems to require a clarification of the structure of the elementary particles. The latter problem has been investigated by many authors in connection with the serious difficulties of the present theory of elementary particles. However, since this problem has been very academic, more fruitful results have arisen from the more concrete problem, the determination of the mutual interactions of the elementary particles realized in nature. We may mention that the latter problem is expected to change the more academic problem into a practical one.

An example of the difficulties which arose in connection with the structure of elementary particles is to be found in the Lorentz classical theory of the electron. In the classical electromagnetic theory the concentration into a point of a finite charge with a definite sign requires an infinite energy on account of the Coulomb repulsive force. According to the well known Einstein relation connecting energy and mass, such an electron has an infinite mass and will explode quickly. In the Lorentz theory of the electron, any electron is under the following force (besides a possible external force)

$$\mathbf{F} = -\delta m \frac{d^2}{dt^2}\mathbf{x} + \frac{2}{3}\frac{e^2}{C^3}\frac{d^3}{dt^3}\mathbf{x} + \dots .$$

The first term corresponds to the above-mentioned increase in energy due to the proper field, i.e. the electromagnetic field induced

by the electron itself. In fact, since it is proportional to the acceleration, it gives the change δm in the mass. The second term comes from the damping effect, i.e. the decrease in energy due to radiation. The remaining terms (....) depend on the charge distribution inside the electron [1]).

The effects given by (F) are called the reactions of the proper field. Since $\delta m \propto 1/a$ (a : the electron radius) the mass becomes infinite as the electron contracts to a point.

In order to avoid this difficulty, many attempts have been made to introduce suitable extended models of the electron. However, the problem has been too purely theoretical to give a conclusive answer. It seems to be better to search for the experimental effects of the proper field which gives rise to the difficulty.

Attempts to settle the difficulty of the infinite mass of the electron were made by many authors (BORN and INFELD [1934], BOPP [1940], DIRAC [1938], etc.). Bopp succeeded in obtaining a stable electron by introducing an energy, the sign of which is opposite to that of the infinite electromagnetic energy of the electron. This method may be regarded as an example of the method of the cohesive field, in which a field giving an attractive force between charges of the same sign is introduced to cancel the Coulomb repulsive force to give a stable electron. The concept of the cohesive field was introduced by POINCARÉ [1905]. A detailed investigation of the cohesive field in the classical theory was given by STUECKELBERG [1939], who showed that a consistent theory of the electron can be obtained by introducing a cohesive field of the neutral scalar type.

This idea was examined in the quantum field theory of electrons by SAKATA [1947] and PAIS [1947] independently. They showed that a cohesive field of the neutral scalar type is able to eliminate the difficulties of the infinite mass. This cohesive field is called the C-meson or f-field, and their theory is called the mixture theory or the compensation theory.

Since the cohesive field can be treated in a similar way to that used in quantum electrodynamics, it can be easily applied to various problems and can yield consequences which can be claimed to be true independently of any assumption involved in the introduction of a

[1]) The Dirac classical theory of the electron [1938] is formulated to be covariant and its higher terms ($\geqslant 3$) of (F) are zero. His "new classical theory of the electron" was presented in 1952.

cohesive field (SAKATA [1947]). Therefore we may construct a consistent theory without using any cohesive field by abstracting the latter features from a cohesive field theory. Sakata called this method that of "the concrete and the abstract".

In any theory of the structure of elementary particles, we have to take into account their internal degrees of freedom. In the mixture theory we may say that the various spins and masses of the cohesive fields correspond to these degrees. There have been many attempts to introduce internal degrees of freedom in connection with internal motions (UHLENBECK and GOUDSMIDT model [1925], WESSEL [1938], HÖNL [1938], HÖNL and PAPAPETROU [1939]). In these theories, we have an energy of internal motion instead of an energy due to the interaction with the cohesive field. If we assume the angular momentum mav of a rotating electron with an angular velocity v/a (v: velocity of points on the circumference of the electron, a: the electron radius) to be equal to the spin $(1/2)\hbar$, we obtain $a=(1/2)(\hbar/mv)$. Since $v \leqslant C$ (C: light velocity), we see that the electron radius is not much smaller than the Compton wave length $(\hbar/mC)$. On the other hand, taking the rest mass mC^2 to be equal to the electromagnetic energy, we have $a \approx (e^2/mC^2)=(1/137)(\hbar/mC)$, which is incompatible with the above result when we assume $v \leqslant C$. This argument seems to show the necessity of introducing a different energy from the electromagnetic one (BORN and SCHRÖDINGER [1935]).

Just as the electron carries its electromagnetic field, any electromagnetic field is accompanied by an infinite number of electrons, whose effects are called those of vacuum polarisation. According to the quantum field theory the latter effects appear to be infinite in various problems. To these problems the idea of the mixture theory has been applied (RAYSKI [1948], UMEZAWA, YUKAWA and YAMADA [1948]). As an abstract theory corresponding to the mixture theory, we have the regularization theory [1], presented by PAULI and VILLARS [1949], where no cohesive fields are assumed to exist.

When an extended model of the electron is used, the electron and the electromagnetic field interact not at a point but over an extended region. Such an interaction is called non-local. Various characteristics

[1] The regularization conditons assumed in this theory can be derived from a formalistic mixture theory. (PAIS and UHLENBECK [1950]). In their discussion Pais and Uhlenbeck pointed out the important featur of the "propagation character".

of non-local interactions have been clarified by many authors (e.g. WATAGIN [1934], BOPP [1946], BLOCH [1950], KRISTENSEN and MØLLER [1952], PAULI [1953], CHRÉTIEN and PEIERLS [1953], UMEZAWA and TAKAHASHI [1953], KATAYAMA [1953], HAYASHI [1954] etc.). YUKAWA [1950] developed a non-local theory, where the fields have a non-local character even when there are no interactions. This theory has been also investigated by many authors (e.g. YENNIE [1950], RAYSKI [1951], HARA and SHIMAZU [1951], TOKUOKA and KATAYAMA [1951] etc.). These theories automatically give various states of mass and spin (FIERZ [1950], RAYSKI [1953]). Yukawa discussed the mass spectrum by connecting the theory of the non-local field with that of the non-local interaction.

An extended model of the electron has frequently involved the principle of causality; the usual formulation of the principle of causality may not hold inside the electron. Indeed, STUECKELBERG [1950] showed that the requirement of causality leads to the Dyson S-matrix given by the present quantum field theory of the structureless elementary particle (cf. Ch. XIII). HEISENBERG [1950] presented a theory [1]) in which all particles are constructed of a fundamental field of spin 1/2. HEISENBERG and FIERZ [1950] discussed this theory in connection with the problem of causality to show that in this theory, though it eliminated some difficulties of infinities, the usual causality cannot hold in a region near the elementary particles. For example, using the usual interpretations we have curious processes (in a "very short time"), in which some particles annihilate (from a state where they do not exist) and after that they are created (not the creation and then annihilation!). Therefore we must first find a consistent interpretation of these mathematical results.

In almost all of the theories discussed in this section we find some relations between the masses of elementary particles which is one of the problems of mutual relations of elementary particles.

§ 6. Real Effects of Proper Fields

It is impossible to obtain definite conclusions from the above theories on account of their academic character. Since the difficulties come from the effects of the proper fields, we may expect that the observed effects of the proper fields in the behaviour of many known

[1]) It is not possible to construct any particle of half-integer spin by means of particles of integer spin.

elementary particles give a real footing for the study of the problems of the structure of elementary particles.

The observed effects of the proper fields may also give important contributions to the problem of the applicability of the present quantum field theory. In other words, we expect that they may throw some light on the question of why quantum electrodynamics based on some approximations, (in which the effects of the proper fields are not taken into account) has shown excellent agreement with the experimental results.

In the present quantum field theory, we have a probability of observing all particles which are distributed like clouds in a region immediately around their source particle, and this source particle and the cloud (i.e. the proper field) are observed as a particle (cf. Ch. XII). For example, every electron forms its cloud of the electromagnetic field. Furthermore, the proper fields interact with their source particle to give the so-called reaction of proper fields. The densities of high energy particles of the proper fields are larger at points near the origin and so give various difficulties of infinities, e.g. the infinite mass.

The cloud of charged particles carried by an electromagnetic field give the effects of the so-called vacuum polarization, in which, for example, the observed value of the charge e' includes the contribution δe of the charged particles of the cloud. However, the actual calculations give an infinite value for δe (cf. Example 6, Ch. XIII).

The contact of the cloud of an elementary particle with another particle gives rise to interactions between them. This is the quantum theoretical expression of the contact interactions (cf. Ch. XII). The cloud of a proper field (the mass $\varkappa$) extends mainly in a region of a radius $r=(\hbar/\varkappa C)$ around its source particle (WICK [1938]). The main part of the nuclear force is given by the cloud of the meson field around the nucleon.

The particles together with their proper fields move differently from bare particles and have, consequently, different angular momenta. Moreover, the proper fields themselves have angular momenta. Therefore, the particles with their proper fields show different magnetic moments from those which are without. For example, we can expect on theoretical grounds, a small anomalous magnetic moment of the electron (i.e. a deviation from the proper magnetic moment, $-\hbar e/2mC$) due to the electromagnetic proper field, and since the

interaction between the nucleons and the mesons are stronger than the electromagnetic interaction, we may expect easily observable effects of the mesonic proper field: e.g. the main part of the observed anomalous magnetic moments of the nucleons may be given by their mesonic proper fields. However, the theoretical answers are again infinite.

Since 1936 Heisenberg has analysed the applicability of the quantum field theory in connection with the actual effects of the proper fields and the strange successes of the quantum electrodynamics (HEISENBERG [1936, 1938, 1939]).

Since the effects of the proper fields are intimately connected with the structure of elementary particles, it may be natural to expect that the applicability of quantum field theory depends on whether the proper fields have significant effects or not, and that when these effects are too great for this theory to be applicable the phenomena should depend on the structure of the elementary particles. In quantum electrodynamics the effects of proper fields may be regarded as small perturbation effects on account of the weak interaction of the proper fields. HEISENBERG [1939] began his analysis by a classification of interactions into two classes, viz. the interactions of the 1st and 2nd kinds. In the first case, quantum field theory shows that the density of particles of the proper field does not depend essentially on their energies, in contrast with the second case, in which the density strongly increases with the energies of particles of the proper field.

The mathematical expression of this classification is given in terms of the dimensions of the interaction constants. In general, the interaction constants have dimensions $[L^\eta]$ with various η ($[L]$: the dimension of length). Then, interactions of the 1st and 2nd kinds have $\eta \leqslant 0$ and $\eta > 0$ respectively (cf. Ch. XV). Heisenberg expected that the interactions of the 1st and 2nd kinds are within and without the range of applicability of the quantum field theory, respectively. Since this applicability condition was examined in detail by OPPENHEIMER *et al.* [1940] it has been called the Heisenberg–Oppenheimer condition.

In addition Heisenberg gave a suggestion for the theory beyond the region of applicability based on a fundamental constant r_0 of the dimension of length, the existence of which was expected on account of the following reasons: The proper fields characterise the properties of regions near the elementary particles in the present quantum field

theory. Since the infinity difficulties come from the contributions of the high energy particles of the proper fields, their properties given by the present quantum field theory should be modified in a future theory. Thus it appears necessary to introduce a constant $r_0 = (\hbar C/E_0)$ discriminating the two regions of low and high energies in such a way that the above-mentioned high energy region corresponds to energies larger than E_0. In the same way that the high energy radiations, which give rise to a difficulty pointed out by Rayleigh, are cut off by means of the Planck constant h in the Planck theory, r_0 may play the role of cutting off the high energy region of the proper fields. The infinities would appear when the present quantum field theory was applied unrestrictedly without taking into account the effects depending on r_0. In the case of particles with wave lengths much greater than r_0, the effects of the proper fields may be expected to be contained only in constants, i.e. the masses and charges. On the other hand, in phenomena concerning particles with short wave lengths ($\ll r_0$), the proper fields show dynamical effects.

That the high energy region of the proper field should be cut off seems to be especially necessary for interactions of the second kind because the probability distributions of high energy particles correspond to a much greater density than those for interactions of the first kind.

The other reason for introducing r_0 is that in a theory, which explains the mass spectra, we must have such a constant, because it is impossible to make any constant of the dimension of mass by using only h and C. However, the characteristic features depending on r_0 have not yet been clarified, since no definite value of r_0 is known, and we do not know how r_0 can be measured. The interactions of the 2nd kind may give phenomena depending strongly on the properties of the proper fields. As an example of the latter phenomena Heisenberg discussed the multiple production of particles. If, in the mesonic proper field of a nucleon, there are many high energy mesons, these mesons might escape from the nucleon under the influence of an external field. Such a theory of the multiple production of mesons was discussed by LEWIS, OPPENHEIMER and WOUTHUYSEN [1948] (cf. Ch. XII). HEISENBERG [1949, 1952] discussed multiple production in analogy with the theory of turbulence. There has been another theory of multiple production based on an analogy with black body radiation. This theory was suggested by HEISENBERG [1939] and was developed

by FERMI [1950]. In this theory the multiple production phenomena do not show the properties of the meson proper field, because the collision energy is distributed quickly over the meson cloud to give new states of equilibrium, from which the mesons are produced. Recently, multiple production phenomena have been observed in photographic-emulsion experiments (Bristol group). Although it has not yet been decided whether the meson-nucleon interaction belongs to the 2nd kind or not, multiple production phenomena are expected to give some new features of the theory of elementary particles.

According to the above analysis of Heisenberg's, the experimental success of quantum electrodynamics are explained by the fact that the interaction between electrons and electromagnetic fields belongs to the 1st kind. However, this explanation seemed to be incomplete because the infinities appear in quantum electrodynamics despite its supposed interaction of the first kind. On the other hand, WATAGIN [1934] presented the so-called Heisenberg-Watagin applicability condition which says that all the high energy ($> E_0$) phenomena are out of the range of applicability of the quantum field theory independently of the types of interactions. However, this condition cannot explain the successes of the quantum electrodynamics. As discussed later, the reason for this contradiction has been partly clarified by the "renormalisation theory" in connection with the real effects of the electromagnetic proper fields.

We have previously had some indication of the reality of the effects of the electromagnetic proper field. For example a calculation which does not take into account the effects of the proper fields, gives an infinite probability for the scattering of an electron by an external force (the infra red catastrophe). In 1937, BLOCH and NORDSIECK showed that this infinity disappears if we take into account the contributions of the low energy photons of the proper field (cf. Example 4, Ch. XIII) [1]. However, it is not consistent to include the low energy photons and then to neglect the effects of the high energy photons. The high energy photons give rise to another infinity (the ultra violet catastrophy). This difficulty, which was discussed in detail by PAULI and FIERZ [1938], prevents our obtaining a definite answer regarding the effects of the proper field in the scattering problem.

[1] The Lewis, Oppenheimer and Wouthuysen theory of the multiple production of mesons discussed above is also based on the Bloch-Nordsieck method.

A more definite demonstration of the effects of the proper field was given by the analysis of the atomic spectra. As is well known, one of the successes of Dirac's relativistic theory of electrons was to explain the fine-structure of energy levels of atomic electrons. However, in these calculations the effects of the proper fields were not taken into account. More detailed investigations to examine the results of the Dirac theory had been prevented by Doppler effects and the collisions of atoms which broaden the spectrum lines (cf. Ch. XVI). However, the fact that there is a few $^0/_0$ difference between the theoretical and experimental results had been noticed by many investigators. In particular, PASTERNACK [1938] pointed out that the $2S$ level of the hydrogen atom is larger than the theoretical result by about 0.03 cm^{-1}.

Using the 3 cm microwave techniques, LAMB and RETHERFORD [1947] succeeded in measuring the precise value of the $2^2S_{1/2}$-level of the hydrogen atom by means of the following method: The hydrogen is excited from the ground-state to the $2^2S_{1/2}$-state by electron bombardment. The $2^2S_{1/2}$-state, which is metastable (lifetime 0.15 sec.), can be quickly changed into the $2^2P_{1/2}$-state by means of electromagnetic microwave radiation (wave length about 3 cm), and the $2^2P_{1/2}$-state decays quickly (lifetime 1.6×10^{-9} sec.) into the $1S$-state. The precise value of the wave length corresponding to the energy difference between the $2^2P_{1/2}$- and $2^2S_{1/2}$-states can be measured on account of the resonance effect.

The results indicated that, contrary to the Dirac theory but in essential agreement with Pasternack's expectation, the $2^2S_{1/2}$-state is higher than the $2^2P_{1/2}$ by about 1000 Mc/sec. (i.e. about 0.033 cm^{-1}). This level shift is called the Lamb shift.

Thus, a calculation disregarding the effects of the proper field fails even in quantum electrodynamics. Previously, many authors attempted to explain the deviation of the energy level of the atomic electron from the result of the Dirac theory by modifying the Coulomb potential near the proton. (KEMBLE and PRESENT [1933], PASTERNACK [1938], FRÖHLICH, HEITLER and KAHN [1939]). On the other hand, UEHLING [1935] calculated the contribution of the electron proper field alone (i.e. the vacuum polarization effect) induced by the electromagnetic field due to the proton, to the energy level of the electron in the hydrogen atom. This idea could not succeed on account of the incorrect sign of its result.

§ 7. Development of the Renormalisation Theory

In 1947, BETHE gave a clear discussion of the Lamb shift in terms of the effects of the electromagnetic proper field. The contribution of the proper field to the electron mass must be included in the observed value of the electron mass, although this contribution is infinite. Therefore, we can expect that the Lamb shift can be calculated as the difference between the contributions of the proper field to the electron energy in the free state and in the Coulomb field of the proton. It may be possible that, although the contributions of the proper field are infinite, the above difference is finite. According to this idea, Bethe succeeded in obtaining a level shift which is about the same as the experimental result. However, since there was no consistent formulation suitable for this idea, he used a non-relativistic calculation and a technical management of the infinity.

Thus, we have the question of whether the infinity appearing in the contribution of the electromagnetic proper field of the electron can be included in the electron mass independently of the state of the electron.

The electromagnetic radiative reactions involved in various electron phenomena have been calculated by many authors. In 1939, DANCOFF stimulated by the Bloch–Nordsieck and Pauli–Fierz results (discussed in the last paragraph) calculated the cross-section of the electron scattering by an external field by taking into account the effects of the proper field. However, his calculations contained some mistakes; these were corrected by ITO, KOBA and TOMONAGA [1947], [1948]. After the Bethe theory, many authors made a calculation of the scattering problem. (ITO, KOBA and TOMONAGA [1948], LEWIS [1948], EPSTEIN [1948], UMEZAWA, YUKAWA and YAMADA [1948]). These investigations show that the infinity due to the electromagnetic proper field can be included completely in the observed value of the electron mass independently of the state of the electron and, moreover, the infinity due to the vacuum polarization effects can be included in the observed value of the charge at least in the 4th order approximation of the perturbation calculation. Thus we obtain a quantum electrodynamics, in which the proper fields give finite effects, by "renormalising" the observed values of the mass and charge to include their effects. Such a renormalisation theory was fast given a covariant formulation by the super-many-time theory, which was presented by TOMONAGA [1943]. This covariant renormalisation theory gave the

Lamb shift, showing excellent agreement with the experimental result. (SCHWINGER [1948], FUKUDA, MIYAMOTO and TOMONAGA [1949]) (cf. Example 2 of Ch. XIV). Tomonaga, Koba and Ito attacked this problem in connection with the discussion of the effectiveness of the C-meson theory in this problem.

In the last paragraph we discussed the anomalous magnetic moment of the electron given by the electromagnetic proper field. In 1947 the Columbia group (NAFE, NELSON and RABI [1947]) investigated the fine structure of the ground states of electrons in hydrogen, deuterium, sodium and potassium atoms, to find the anomalous magnetic moment of the electron, which is about 0.1 % of the proper magnetic moment. This experimental result agrees with the magnetic moment given by the renormalisation theory (cf. Example 2 of Ch. XIV).

These successes of the renormalisation theory seem to show that the electromagnetic proper fields have real effects which cannot be disregarded. It is easily seen from the Bloch–Nordsieck theory, discussed in the last paragraph, that there is no infra-red catastrophe in the renormalisation theory of quantum electrodynamics on account of the effects of low energy photons in the proper field. Tomonaga pointed out that the renormalisation theory can be regarded as the continuation of the Bloch–Nordsieck and Pauli–Fierz investigations.

The super-many-time theory is the completely covariant reformulation of the quantum field theory, and was obtained by extending the Jordan and Pauli covariant theory [1928] of free fields and the Dirac many-time theory [1932a, b]. Its physical content is equivalent to the Heisenberg and Pauli quantum field theory. However, the covariance has not been shown explicitly in spite of its covariant content. It is obviously important to make theories consistent, from the beginning, with such fundamental principles as those of special relativity. In fact, we can remember the important part played by the canonical formulation of classical mechanics in the discovery of the quantum mechanics.

A similar formulation was developed also by SCHWINGER [1948], and by FEYNMAN [1949] starting from a different standpoint. This covariant quantum field theory has been applied not only to the problems of quantum electrodynamics, but also to most problems of elementary particles in connection with the renormalisation method. This has led to remarkable progress in the theory of elementary particles.

In 1949, DYSON proved that the perturbation calculations of the renormalisation theory in quantum electrodynamics have no infinities in any order of approximation.

In 1948, WELTON showed that the Lamb shift calculated by the renormalisation theory may be obtained by using an intuitive image of an electron moving in its proper field (cf. Example 1 of Ch. XIV), and a calculation of the anomalous magnetic moment of the electron was given by KOBA [1949] by improving Welton's method. Thus the problem of the applicability of the quantum field theory has changed into that of the renormalisation theory. Furthermore, we can show theoretically that the present renormalisation procedure can give a closed theory for interactions of the 1st kind, but not for those of the 2nd kind (cf. Ch. XV). Thus, we see that the Heisenberg–Oppenheimer applicability condition is just the one for the renormalisation theory (SAKATA, UMEZAWA and KAMEFUCHI [1952]).

Then, we have an important question: do all interactions realized in nature belong to the 1st kind or not? If there exist interactions of the 2nd kind in nature, it would be difficult to understand the successes of quantum electrodynamics because the electron and electromagnetic fields interact also with other fields having interactions of the 2nd kind in the higher order terms of the perturbation approximation according to the present quantum field theory. For example, the β-interactions seem to belong to the interactions of the 2nd kind [1]).

Many attempts have been made to investigate the properties of the mesonic proper field. They have shown that the meson-nucleon interaction is much stronger than the electromagnetic one and have sometimes suggested the existence of an excited state of nucleons. Thus, since the perturbation procedure cannot be applied, many attempts have been made in meson theory since 1951 without using any approximation. One of the most important results of these investigations is the determination of the spin of the π-meson (cf. Example 6, Ch. X). Observations on the high energy nucleon-nucleon scattering has also suggested various possible features of the meson field. In particular, they have suggested the existence of a strong repulsive force with a short range (1/3 of that of the nuclear force). Although it is not yet clear whether this repulsive force comes from the heavy particles discovered recently, or from the higher

[1]) See TANIKAWA [1953], UMEZAWA [1952], TANAKA and ITO [1953] on the attempts to explain the β-interactions only by using interactions of the 1st kind.

order radiative corrections of the meson field, it may be intimately connected with the structure of nucleons.

The renormalisation theory discussed above was formulated in terms of perturbation expansions. Thus even in cases of interactions of the first kind there remain many problems, e.g. we must examine the convergence of the perturbation series and find a formulation of renormalisation theory without being based on perturbation expansions (cf. Ch. XIV). A simple model of interacting fields which is renormalisable and which may be solved completely was presented by LEE [1954]. The application of the renormalisation method to this model leads to a new difficulty which will be discussed in Ch. XVIII.

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Note added in proof

Since this chapter was written, great progresses have been made in problems of the renormalisation theory and of the meson-nucleon phenomena. They will be discussed in Note added in proof in Ch.XVIII, which can be read as the continuation of the historical discussions in the present chapter.

CHAPTER II

RELATIVISTIC WAVE EQUATION

§ 1. Relativistic Wave Equation

In the preceding Chapter we have enumerated the various particles whose existence has been demonstrated experimentally and which, provisionally, we shall assume to be elementary. Because interactions give rise to transmutation processes, the number of these particles will not remain constant in time. For this reason it is necessary that the state of a system of particles at a particular time is defined by the specification of the observables—e.g. energy, momentum, number of particles—whose values are determined by observations made throughout space. Such a definition may be compared with that of an electromagnetic field at a particular time by the specification of the values of the electromagnetic field strength throughout space.

These considerations lead to the concept of a *field state*—one whose definition at a particular time depends on the values of certain quantities at all points of space. We shall assume that the state of elementary particles is a field state and shall denote the various field components by $Q_\alpha(x)$ with $\alpha = 1, 2,$ etc. .... For example, the electromagnetic field will be described by the four components $A_\mu(x)$ of the vector potential ($\mu = 1, 2, 3, 4$).

A quantum field theory is obtained from a classical field theory by means of the customary quantisation procedure. Thus the c-numbers which represent the classical field components in the classical field equations (e.g. the Maxwell equation of the electrodynamics) are replaced by q-numbers. This substitution would seem to be justified by the correspondence principle. Nevertheless, it cannot yield quantum field equations associated with particles—electrons, for example,—for which no classical field theory exists.

In such a case we adopt following procedure. When the number of particles is constant, the quantum state of a particle is described by a wave function which satisfies a wave equation. In the quantum field theory, which admits of the creation and annihilation of particles,

it is postulated that the wave functions are certain q-numbers $Q_\alpha(x)$ and that the wave equations are those which the $Q_\alpha(x)$ satisfy.

Briefly, this postulate is necessary in order that the results of the quantum field theory should agree with those of the quantum mechanics of particles when the effects of creation and annihilation may be disregarded.

In addition, the field equations must fulfill the requirements of the principle of special relativity, which we must needs regard as a general principle of nature. Wave equations of such kind are called *relativistic wave equations.*

§ 2. The Free Field

Because the time and space derivatives do not appear in a symmetrical way, the wave equation of the customary non-relativistic quantum mechanics are not of Lorentz invariant form. Indeed such equations which describe free particles are derived from the non-relativistic equation $K_0=(1/2\varkappa)k_ik_i$ which refers to the energy K_0 and the momentum k_i $(i=1, 2, 3)$. These quantities are interpreted as the operators

$$\left.\begin{aligned} K_4 \equiv iK_0 &= -i\partial_4 \\ k_i &= -i\partial_i . \end{aligned}\right\} \tag{2.1}$$

It is well known that the fundamental relation (2.1) embodies the connection between the wave and particle aspect of a field in that, by its means, the energy and momentum of a particle are related to the frequency and wave number of a wave. It would seem that the use of (2.1) in the derivation of relativistic wave equations is justified. Now the relativistic relation between energy K_0 and momentum k_i is

$$K_0^2 = k_ik_i + \varkappa^2 \tag{2.2}$$

Accordingly, making use of (2.1), the wave equation

$$(\Box - \varkappa^2)\, Q_\alpha(x) = 0 \tag{2.3a}$$

is obtained. This is known as the Klein–Gordon equation.

For this reason we assume that field quantities of any quantum field theory must satisfy the following condition—K.G. *Condition. The field quantities $Q_\alpha(x)$ corresponding to free elementary particles must satisfy equation* (2.3a). A familiar illustration is provided by the components $A_\mu(x)$ of the electromagnetic field. It must be remarked that if the field components $Q_\alpha(x)$ correspond to a set of particles with

various masses $(\varkappa_1, \ldots, \varkappa_n)$, their equation (2.3$a$) must be replaced by

$$\prod_{s=1}^{n} (\Box - \varkappa_s^2)\, Q_\alpha(x) = 0. \tag{2.3b}$$

However, we shall not often be concerned with this generalisation.

A difference between (2.3a) and the non-relativistic wave equation is that (2.3a) contains the second power of the time-derivative operator. It is well known that any differential equation can be transformed into a system of first order equations by increasing the number of variables. The latter set of variables has an important physical meaning. Indeed, the fact that the non-relativistic wave equation is linear in the time derivation operator, is essential for quantum mechanics in the canonical formalism, in which the canonical invariance of the theory ensures the conservation of probability. In order to preserve the latter conservation law, we adopt the canonical formalism in the quantum field theory. Then, the wave equations must lead to a set of canonical equations (see (7.27)), which are of first order differential equations. Thus, the canonical theory of quantum fields are based on field quantities, satisfying first order differential equations. As shown in Ch. IX, the canonical theory can always be applied to the quantisation of fields, when they satisfy the K.G. condition.

A well known example of field equations being first order in the derivatives, is found in electromagnetism. The Maxwell equations are a system of first order differential equations for the four components of the vector potential A_μ and the six components of the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which satisfy (2.3a). We give another example in a case of a scalar field U satisfying (2.3a). We can convert equation

$$(\Box - \varkappa^2) U(x) = 0 \tag{2.4}$$

into the system of first order differential equations

$$\left.\begin{aligned} U_\mu(x) &= \partial_\mu\, U(x) \\ \partial_\mu\, U_\mu(x) &= \varkappa^2\, U(x) \end{aligned}\right\} \tag{2.5}$$

by introducing five components (U, U_μ).

In general, the relativistic wave equation for wave functions Q_α $(\alpha = 1, \ldots, n)$ has the form:

$$A_{\alpha\beta}(\partial)\, Q_\beta(x) = 0. \tag{2.6}$$

K.G. condition limits this wave equation because it requires the existence of a derivation operator $d_{\alpha\beta}(\partial)$ which satisfies the relation

$$d_{\alpha\beta}(\partial)\, \Lambda_{\beta\varrho}(\partial) = (\Box - \varkappa^2)\delta_{\alpha\varrho} \tag{2.7}$$

and is a function of the ∂_μ's:

$$d(\partial) \equiv [d_{\alpha\beta}(\partial)] = \alpha + \alpha_\mu \partial_\mu + \ldots + \alpha_{\mu_1 \ldots \mu_l} \partial_{\mu_1} \ldots \partial_{\mu_l} + \ldots . \tag{2.8}$$

The coefficients α, α_μ, ... are matrices of dimension n. We denote the highest order of the derivation operators in (2.8) by b.

$$\alpha_{\mu_1 \ldots \mu_l} = 0 \qquad \text{for} \qquad l > b. \tag{2.9}$$

The relativistic wave equation, as a set of first order differential equations, can be written in the form

$$(\varrho_\mu \partial_\mu + \varkappa\beta)\, Q(x) = 0, \tag{2.10}$$

where ϱ_μ and β are certain matrices of n dimension and Q is the one column matrix $[Q_\alpha(x)]$. By substituting (2.8) with

$$\Lambda(\partial) \equiv [\Lambda_{\alpha\beta}(\partial)] = \varrho_\mu \partial_\mu + \varkappa\beta$$

into (2.7), we have

$$\varkappa\alpha\beta = -\varkappa^2 I. \tag{2.11}$$

Here, I is the identity matrix of dimension n. The relation (2.11) shows that β cannot be singular and therefore its inverse matrix β^{-1} must exist. By multiplying (2.10) by β^{-1} we have

$$(\beta_\mu \partial_\mu + \varkappa)\, Q(x) = 0, \tag{2.12}$$

where

$$\beta_\mu \equiv \beta^{-1} \varrho_\mu. \tag{2.13}$$

Since, inversely, we can derive (2.10) from (2.12), we see that (2.10) and (2.12) give an equivalent theory. Therefore, we can always write the relativistic wave equation in the form of (2.12) without any loss of generality under K.G. condition.

When $d(\partial)$ is a first order differential operator:

$$d(\partial) = \alpha + \alpha_\mu \partial_\mu \tag{2.14}$$

(i.e. $b = 1$ in (2.9)), (2.7) with $\Lambda = -(\beta_\mu \partial_\mu + \varkappa)$ leads to

$$-(\alpha + \alpha_\mu \partial_\mu)\, (\beta_\mu \partial_\mu + \varkappa) = (\Box - \varkappa^2) I.$$

From this we obtain

$$\alpha\varkappa = \varkappa^2 I$$

$$\varkappa\alpha_\mu + \alpha\beta_\mu = 0 \tag{2.15}$$

$$\alpha_\mu\beta_\nu + \alpha_\nu\beta_\mu = -2\delta_{\mu\nu}$$

which leads to

$$d(\partial) = -(\beta_\mu\partial_\mu - \varkappa) \tag{2.16a}$$

$$\beta_\mu\beta_\nu + \beta_\nu\beta_\mu = 2\delta_{\mu\nu}. \tag{2.16b}$$

As is shown in Ch. V, the number b appearing in (2.9) is equal to $2f$ where f is the maximum value of the spin of fields described by Q_α. In the case of (2.14), $b=1$ and therefore the spin of the field Q_α is $\frac{1}{2}$. Thus, (2.12) is the equation of the field with spin $\frac{1}{2}$, when β_μ satisfies the relation (2.16b). By means of a similar consideration, we shall derive, in Ch. V, the field equations of the general spin.

CHAPTER III

THE DIRAC EQUATION

§ 1. The Equation

In this Chapter we shall explore the consequences of the wave equation appropriate to particles of spin $\frac{1}{2}$. This equation may be written in the form [1])

$$(\gamma_\mu \partial_\mu + \varkappa)\psi = 0, \tag{3.1}$$

where $\varkappa$ is the mass of the particle and the quantities γ_μ satisfy th commutation relationships

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \tag{3.2}$$

on account of (2.16a, b).

It follows from (3.1) and (3.2) that

$$(\Box - \varkappa^2)\psi = 0. \tag{3.3}$$

Accordingly, it is clear that the K.G. condition of the previous Chapter is fulfilled by the theory based on the Dirac equation.

It is a consequence of (3.2) that any algebraic function X of the quantities γ_μ may be rewritten as a linear form

$$X = c^A \gamma^A. \tag{3.4}$$

The quantities γ^A are defined as

$$\gamma^A = \left\{ \begin{array}{l} I, \\ \gamma_1, \gamma_2, \gamma_3, \gamma_4, \\ i\gamma_2\gamma_3,\ i\gamma_3\gamma_1,\ i\gamma_1\gamma_2,\ i\gamma_1\gamma_4,\ i\gamma_2\gamma_4,\ i\gamma_3\gamma_4 \\ i\gamma_1\gamma_2\gamma_3,\ i\gamma_1\gamma_2\gamma_4,\ i\gamma_3\gamma_1\gamma_4,\ i\gamma_2\gamma_3\gamma_4, \\ \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4. \end{array} \right\} \tag{3.5}$$

The sequence of 16 quantities in this array is taken to be that of the γ^A as A runs from 1 to 16. The occurrence of the imaginary factor i ensures that all the γ^A satisfy the relation

$$(\gamma^A)^2 = 1. \tag{3.6}$$

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[1]) See DIRAC [1928]. The original form of the equation was that of (3.45a).

The quantities γ^A, which are so far unspecified in kind, can be represented by matrices. The traces, or spurs, of the representative matrices, are important, and may be calculated by means of the general matrix relation

$$\mathrm{Sp}(\alpha \cdot \beta) = \mathrm{Sp}(\beta \cdot \alpha). \tag{3.7}$$

For example, the spin of the matrix representing γ_1 is zero, because

$$\begin{aligned} \mathrm{Sp}(\gamma_1) &= \mathrm{Sp}(\gamma_1\gamma_2\gamma_2) = -\mathrm{Sp}(\gamma_2\gamma_1\gamma_2) \\ &= -\mathrm{Sp}(\gamma_1) = 0. \end{aligned}$$

In general,

$$\left.\begin{aligned} &\mathrm{Sp}\,(\gamma^A) = 0 \quad \text{if} \quad \gamma^A \neq I \\ \text{and} \quad &\mathrm{Sp}\,(\gamma^A \gamma^B) = 0 \quad \text{if} \quad A \neq B. \end{aligned}\right\} \tag{3.8}$$

It can also be shown that the spur of the product of any odd number of matrices representing γ_μ is zero — a useful property in practice.

We shall now show that the matrices representing the quantities γ^A are linearly independent of each other in the sense that there cannot be a non-trivial relation of the form

$$c^A \gamma^A = 0. \tag{3.9}$$

For such a relation implies that

$$\sum_{A \neq B} c^A \gamma^A \gamma^B + c^B = 0 \qquad \text{for all } B.$$

On taking the spurs of these matrices and using (3.8), it follows that

$$c^B = 0 \qquad (B = 1, \ldots, 16). \tag{3.10}$$

This means that the γ^A are linearly independent of each other. They are also a complete set in that any algebraic function of γ^A can be expressed in the form (3.4).

All the representations of the γ^A by matrices that are irreducible are also equivalent to each other. For if a matrix X commutes with the representation of all the γ^A, then (3.4) can be used to show that X is a multiple of the unit matrix I with a c-number, or of the form

$$X = cI. \tag{3.11}$$

Since the number of independent quantities γ^A is sixteen, this representation must be in terms of four-dimensional matrices. In these terms, ψ is represented by a four-dimensional column matrix.

The complex conjugate of a c-number and the hermitian conjugate of a q-number will both be indicated by an asterisk. For example, the hermitian conjugate of a matrix $A = [a_{\alpha\beta}]$ is denoted by $A^* = [a^*_{\beta\alpha}]$, where $a^*_{\beta\alpha}$ is the complex conjugate of $a_{\beta\alpha}$.

Different representations of the γ_μ, say γ'_μ and γ''_μ are related to each other by transformation by non-singular matrices. Thus

$$\gamma'_\mu = S^{-1}\gamma''_\mu S, \tag{3.12}$$

where S is some non-singular matrix. At least one of these representations consists of hermitian matrices. For the quantities γ_μ and $-\gamma_\mu$ form a finite group of order 32, which can be represented by unitary matrices. But such a representation, say γ'_μ, is also hermitian. For

$$\gamma'^2_\mu = 1 \text{ and } \gamma'_\mu \gamma'^*_\mu = 1 \qquad \text{(no summation)}$$

and therefore

$$\gamma'_\mu = \gamma'^*_\mu.$$

Different hermitian representations are obtained by transformation with unitary matrices.

If γ_μ is a set of matrices satisfying (3.2), the sets γ^*_μ and γ^T_μ (where γ^T_μ denotes the transposed matrix of γ_μ) also satisfy (3.2). Therefore there must be matrices A and B such that

$$\gamma^*_\mu = A\gamma_\mu A^{-1}, \tag{3.13}$$

$$\gamma^T_\mu = B\gamma_\mu B^{-1}. \tag{3.14}$$

In the following discussion we shall assume, unless otherwise stated, that the γ_μ are represented by a definite set of hermitian matrices, so that, in (3.13), $A = I$. Moreover, B can be chosen to be unitary or

$$B^* = B^{-1}. \tag{3.15}$$

We shall now prove the important property (PAULI [1936]),

$$B^T = -B. \tag{3.16}$$

Equation (3.14) is transposed to read

$$\gamma_\mu = (B^T)^{-1}\gamma^T_\mu (B^T).$$

Substituting (3.14) and using (3.11), it follows that

$$B^T = cB. \tag{i}$$

Substituting the value of B given by the transposed equation of (i) we conclude that $c^2=1$ and, therefore, that

$$c=\pm 1. \tag{ii}$$

Now we can exclude the value $c=1$. For this would imply the existence of no less than ten independent matrices that change sign on being transposed (namely, the products of B with the elements of the third and fourth lines of (3.5) — $iB\gamma_2\gamma_3$, ..., $iB\gamma_3\gamma_4$; $iB\gamma_1\gamma_2\gamma_3$, ..., $iB\gamma_2\gamma_3\gamma_4$). In other words, there would be ten linearly independent elements that are anti-symmetric about the principal diagonals. This cannot be, for it is known that the number of anti-symmetric linearly independent quantities in four-dimensional space is six — an anti-symmetric tensor in four-dimensional space has six components.

On the other hand, the value of $c=-1$ is acceptable in that it implies that there are but six anti-symmetrical matrices — B, $B\gamma_1$, $B\gamma_2$, $B\gamma_3$, $B\gamma_4$ and $B\gamma_5$. Because there is nothing in the preceding argument that depends on the special assumption that the γ_μ are hermitian, we can conclude that the result (3.16) is of general validity.

§ 2. The γ-Matrices Under Lorentz Transformation

We turn to a consideration of the Lorentz invariance of (3.1).

We shall assume that two sets of coordinate x_μ and $'x_\mu$ are connected by a Lorentz transformation

$$'x_\mu = a_{\mu\nu}x_\nu, \tag{3.17a}$$

with

$$a_{\mu\nu}a_{\lambda\nu}=\delta_{\mu\lambda}. \tag{3.17b}$$

Further, we shall assume that ψ is transformed linearly according to

$$'\psi = \Lambda\psi, \tag{3.18}$$

where Λ is a four-dimensional matrix. Substituting (3.18) into (3.1), the necessary and sufficient condition for the Lorentz invariance of (3.1) is found to be

$$\Lambda^{-1}\gamma_\mu\Lambda = a_{\mu\nu}\gamma_\nu. \tag{3.19}$$

(A more detailed discussion of such points is given in Ch. IV). Nothing prevents the assumption that Λ is normalised so that

$$|\det \Lambda| = 1. \tag{3.20}$$

An infinitesimal Lorentz transformation is expressed by

$$a_{\mu\nu} = \delta_{\mu\nu} + \delta w_{\mu\nu}, \tag{3.21a}$$

with

$$\delta w_{\mu\nu} = -\delta w_{\nu\mu}. \tag{3.21b}$$

(The second equation is a consequence of the orthogonality relations (3.17*b*) and the infinitesimal character of the $\delta w_{\mu\nu}$). Now, with the neglect of powers of $\delta w_{\mu\nu}$, Λ can be expanded in the form

$$\Lambda = 1 + \tfrac{1}{2} S_{\mu\nu}\, \delta w_{\mu\nu}, \tag{3.22a}$$

where

$$S_{\mu\nu} = -S_{\nu\mu}. \tag{3.22b}$$

Further properties of the matrix $S_{\mu\nu}$ can be discovered by substituting (3.22*a*) and (3.21*a*) into (3.19) to obtain

$$[\gamma_\mu, S_{\lambda\nu}] = \delta_{\lambda\mu}\gamma_\nu - \delta_{\nu\mu}\gamma_\lambda. \tag{3.23}$$

Taking account of (3.2), $S_{\mu\nu}$ can be written as

$$S_{\mu\nu} = \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{3.24}$$

From the facts that δw_{i4} are imaginary and δw_{ik} real ($k, i \neq 4$), it follows that

$$\Lambda^* = \gamma_4 \Lambda^{-1} \gamma_4. \tag{3.25}$$

These results indicate that any quantity that commutes with all the γ_μ can be added to $S_{\mu\nu}$ without affecting the validity of (3.23). According to (3.11), such a quantity must have the form $\Delta_{\mu\nu} I$, and must lead to the addition of a term $\Delta \equiv (1/2)\, \Delta_{\mu\nu} \delta w_{\mu\nu} I$ in the expression (3.22*a*) for Λ. Because the (a_{4i}, a_{i4}) are imaginary and the (a_{ik}, a_{ki}) are real, (3.19) and its hermitian conjugate relations

$$\Lambda^* \gamma_k \Lambda^{*-1} = a_{ki}\gamma_i - a_{k4}\gamma_4,$$
$$\Lambda^* \gamma_4 \Lambda^{*-1} = -a_{4i}\gamma_i + a_{44}\gamma_4,$$

require that $\Lambda^* = \pm \gamma_4 \Lambda^{-1} \gamma_4$. But the negative sign would contradict (3.25), and must therefore be excluded. It follows that Δ is imaginary and, from (3.18), that the imaginary infinitesimal quantity in Λ has no other effect than a change of the phase of ψ. But since this phase can be arbitrarily assigned, we can always make $\Delta = 0$ (and so $\Delta_{\mu\nu} = 0$). Therefore $S_{\mu\nu}$ can be given the form (3.24).

The class of infinitesimal transformations described above does not include those that have the properties of reflections in the time

and space coordinates. A discussion of such transformations is given in § 9 — it can be verified, however, that (3.19) is satisfied by the following forms of

$$\Lambda = i\gamma_4 \quad \text{for } {}'x_i = -x_i,\ {}'x_4 = x_4, \tag{3.26}$$

$$\Lambda = \gamma_1\gamma_2\gamma_3 \text{ for } {}'x_i = x_i,\ {}'x_4 = -x_4, \tag{3.27}$$

$$\Lambda = i\gamma_5 \quad \text{for } {}'x_\mu = -x_\mu. \tag{3.28}$$

The replacement of $i\gamma_4$ by γ_4 in (3.26), and of $\gamma_1\gamma_2\gamma_3$ by $i\gamma_1\gamma_2\gamma_3$ in (3.27) is also consistent with (3.19).

It is convenient to collect together certain special properties of Λ in the statement:—

$$\Lambda^{-1}\gamma_5\Lambda = \begin{cases} +\gamma_5 \text{ for infinitesimal Lorentz transformation} \\ -\gamma_5 \text{ for space or time reflection ((3.26), (3.27)).} \end{cases} \tag{3.29}$$

It will be shown in Ch. IV that any Lorentz transformation can be regarded as a continued product of infinitesimal transformations of the form (3.21a) together with the reflections (3.26), (3.27) and (3.28). If there are two Lorentz transformations $x \to {}'x$ and ${}'x \to {}''x$ defined by

$${}'x_\mu = a_{\mu\nu}\, x_\nu,$$

$${}''x_\mu = b_{\mu\nu}\, {}'x_\nu,$$

and if Λ^a and Λ^b correspond to $a_{\mu\nu}$ and $b_{\mu\nu}$, then $(\Lambda^b\Lambda^a)$ corresponds to the repeated transformation $b_{\mu\nu}a_{\nu\lambda}$. For

$$b_{\mu\nu}\, a_{\nu\lambda}\, \gamma_\lambda = (\Lambda^a)^{-1}\, b_{\mu\nu}\, \gamma_\nu\, (\Lambda^a) = (\Lambda^b\Lambda^a)^{-1}\, \gamma_\mu\, (\Lambda^b\Lambda^a),$$

which shows that (3.19) is satisfied by these quantities. Further, (3.20) is satisfied by virtue of the rules for the multiplication of determinants. Now the repeated transformation entails that

$$\psi \to {}''\psi = \Lambda\psi \tag{3.30}$$

with

$$\Lambda = \Lambda^b\Lambda^a. \tag{3.31}$$

§ 3. The Wave Function under Lorentz Transformation. Charge Conjugation

Introducing a row matrix $\bar{\psi}$ by the definition

$$\bar{\psi} = \psi^*\gamma_4, \tag{3.32}$$

it is found that $\bar{\psi}$ satisfies the equation

$$\partial_\mu\bar{\psi}\gamma_\mu - \varkappa\bar{\psi} = 0. \tag{3.33}$$

It is not a difficult matter to discover the properties of $\bar{\psi}$ under Lorentz transformations of the form (3.17a). In the first place, there is the transformation

$$'\bar{\psi} = \bar{\psi}\gamma_4 \Lambda^* \gamma_4. \tag{3.34}$$

Moreover, from (3.25), (3.26) and (3.27) it follows that

$$\left.\begin{aligned} '\bar{\psi} &= \bar{\psi}\Lambda^{-1} \text{ for an infinitesimal transformation (3.22}a\text{)} \\ &\quad \text{or the space reflection,} \\ '\bar{\psi} &= -\bar{\psi}\Lambda^{-1} \text{ for the time reflection.} \end{aligned}\right\} \tag{3.35}$$

Therefore $\bar{\psi}\psi$ changes sign under the time reflection. However, it will be shown in Ch. VIII that as far as quantum field theories are concerned, it is permissible to regard $\bar{\psi}\psi$ as a scalar under time reflection by making a simultaneous transformation of the field state vector.

We shall apply similar considerations to the quantities $\bar{\psi}A\psi$ and $\bar{\psi}\gamma_5 A\psi$ where A is some matrix. By (3.29), these quantities become $-\bar{\psi}\Lambda^{-1}A\Lambda\psi$ and $\bar{\psi}\gamma_5\Lambda^{-1}A\Lambda\psi$ respectively under the time reflection, and $\bar{\psi}\Lambda^{-1}A\Lambda\psi$ and $-\bar{\psi}\gamma_5\Lambda^{-1}A\Lambda\psi$ respectively under the space-reflection. In particular, under the time reflection $\bar{\psi}\psi$ changes sign, while $\bar{\psi}\gamma_5\psi$ remains unchanged; under space reflection $\bar{\psi}\psi$ and $\bar{\psi}\gamma_5\psi$ are scalar and pseudoscalar respectively [1]). On the other hand, in quantum field theory $\bar{\psi}\gamma_5\psi$ changes sign under time-reflection while $\bar{\psi}\psi$ does not.

We now introduce a matrix C in terms of the matrix B of (3.14) by

$$C = \gamma_5 B^{-1}. \tag{3.36}$$

It follows that

$$\gamma_\mu^T = -C^{-1}\gamma_\mu C \tag{3.37}$$

and that

$$C^T = -C. \tag{3.38}$$

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[1]) A quantity, say a, which is invariant under any infinitesimal Lorentz transformation is either a scalar or a pseudoscalar; the two categories are distinguished from each other by their properties under space-reflection. Thus

$$\begin{aligned} &\text{if } a \to \quad a, \ a \text{ is a scalar,} \\ &\quad a \to -a, \ a \text{ is a pseudoscalar.} \end{aligned}$$

Similarly, if quantities a_μ transform vectorially under infinitesimal Lorentz transformations, they constitute either a vector or a pseudovector; indeed, for the space-reflection,

$$\begin{aligned} &\text{if } a_k \to -a_k, \ a_4 \to a_4, \ a_\mu \text{ is a vector,} \\ &\quad a_k \to \quad a_k, \ a_4 \to -a_4, \ a_\mu \text{ is a pseudovector.} \end{aligned}$$

Since B is unitary (when the γ_μ are hermitian), then

$$C^*C = 1. \tag{3.39}$$

The matrix C has an important physical significance which can be demonstrated by using (3.37). Let $\psi'(x)$ and $\bar{\psi}'(x)$ be defined by the transformations

$$\psi'(x) = C\bar{\psi}^T(x), \qquad \bar{\psi}'(x) = (C^{-1}\psi(x))^T. \tag{3.40}$$

It is apparent that ψ' satisfies the wave equation (3.1) and that $\bar{\psi}'$ satisfies the wave equation (3.33). However, it will be shown later that in the presence of an electromagnetic field, ψ and ψ' satisfy the wave-equation of particles of charge e and $-e$ respectively. For example, when ψ represents the negaton of mass $\varkappa$ and charge $-e$, ψ' represents the positon of mass $\varkappa$ and charge $+e$. For this reason the equation (3.40) is called the *charge conjugation transformation.*

If ψ and ψ' are assumed to have the same properties under Lorentz transformation then, by means of (3.34), the condition for the invariance of (3.40) is found to be

$$C\Lambda^{T*}C^{-1} = \gamma_4\Lambda\gamma_4. \tag{3.41}$$

But we have already remarked that it would have been possible to take $\Lambda = \gamma_4$ and $i\gamma_1\gamma_2\gamma_3$ in (3.26) and (3.27) respectively; it is now evident that such a choice would be inconsistent with (3.41). In other words, if ψ and ψ' are to behave similarly under Lorentz transformation, Λ must be chosen in the way indicated in (3.26) and (3.27) (see RACAH [1937]).

We can now derive an important property of the quantities γ^A (PAULI [1936]). If X is an arbitrary four-dimensional matrix, then (cf. (3.4))

$$X = c^A\gamma^A \tag{3.42}$$

where c^A $(A = 1, \ldots, 16)$ are ordinary c-numbers. From (3.6) and (3.8) it follows that

$$\mathrm{Sp}\,(X\gamma^B) = 4c^B \quad (B = 1, \ldots, 16). \tag{3.43}$$

This leads to

$$X_{\alpha\beta} = \tfrac{1}{4}\, X_{\varrho\lambda}\, \gamma^A_{\lambda\varrho}\, \gamma^A_{\alpha\beta}.$$

If the matrix elements of X are such that $X_{\sigma\varrho} \neq 0$ and all others vanish, then

$$\delta_{\alpha\sigma}\, \delta_{\beta\varrho} = \tfrac{1}{4}\, \gamma^A_{\varrho\sigma}\, \gamma^A_{\alpha\beta}. \tag{3.44a}$$

Then, if F and G are two arbitrary four-dimensional matrices,

$$\left.\begin{aligned} F_{\alpha\sigma}\, G_{\varrho\beta} &= F_{\alpha\alpha'}\, \delta_{\alpha'\sigma}\, \delta_{\varrho\beta'}\, G_{\beta'\beta} \\ &= \tfrac{1}{4}\, \gamma^A_{\varrho\sigma}\, (F\, \gamma^A\, G)_{\alpha\beta}. \end{aligned}\right\} \tag{3.44b}$$

An example of the application of (3.44*b*) is provided by Example 8, Ch. VII.

§ 4. The Wave-Function of a Free Electron

It is now convenient to write the wave-equation (3.1) in the form

$$(i\partial_4 + \alpha_k \partial_k + i\varkappa\beta)\psi = 0, \tag{3.45a}$$

with

$$\left.\begin{aligned} \alpha_k &= i\, \gamma_4\, \gamma_k \\ \beta &= \gamma_4. \end{aligned}\right\} \tag{3.45b}$$

If the γ_μ are hermitian, so are the matrices α_k and β.

Let ψ have the form of a plane wave, or

$$\psi = u\, e^{ik_\mu x_\mu}, \tag{3.46}$$

where k_μ are the components of a four vector. Substituting in (3.45*a*), it follows that

$$k_0 u = (\alpha_i k_i + \varkappa\beta)u. \tag{3.45c}$$

This indicates that the energy k_0 is the eigenvalue of the hermitian operator $H = (1/i)\alpha_i \partial_i + \varkappa\beta$, where H is the Hamiltonian of a free particle. It follows from (3.3) that $k_0^2 = k_i k_i + \varkappa^2$, so that

$$k_0 = \begin{cases} K_0 \\ -K_0 \end{cases}, \tag{3.47}$$

where

$$K_0 = \sqrt{k_i\, k_i + \varkappa^2}. \tag{3.48}$$

We now define *spin matrices* σ_k by

$$\left.\begin{aligned} \sigma_1 &= -i\gamma_2\gamma_3 = -i\alpha_2\alpha_3, \\ \sigma_2 &= -i\gamma_3\gamma_1 = -i\alpha_3\alpha_1, \\ \sigma_3 &= -i\gamma_1\gamma_2 = -i\alpha_1\alpha_2. \end{aligned}\right\} \tag{3.49}$$

If $\mathbf{e}$ is a unit vector parallel to the vector with components k_1, k_2 and k_3, it is easily shown that $(\boldsymbol{\sigma}\cdot\mathbf{e})$ commutes with H, or that

$$[H,\ (\boldsymbol{\sigma}\cdot\mathbf{e})] = 0. \tag{3.50}$$

Moreover

$$(\boldsymbol{\sigma}\cdot\mathbf{e})^2=(\mathbf{e}\cdot\mathbf{e})=\mathbf{1}. \tag{3.51}$$

Therefore $(\boldsymbol{\sigma}\cdot\mathbf{e})$ has eigenvalues $\pm\,\mathbf{1}$.

The eigenvectors of H (each component of an eigenvector is a function) can be classified in terms of the values of k_0 and $(\boldsymbol{\sigma}\cdot\mathbf{e})$. Thus we have four eigenvectors u^1, etc., such that

$$\left.\begin{aligned} u^1&: \quad k_0=K_0, \quad (\boldsymbol{\sigma}\cdot\mathbf{e})\,u^1=u^1\\ u^2&: \quad k_0=K_0, \quad (\boldsymbol{\sigma}\cdot\mathbf{e})\,u^2=-u^2\\ u^3&: \quad k_0=-K_0, \quad (\boldsymbol{\sigma}\cdot\mathbf{e})\,u^3=u^3\\ u^4&: \quad k_0=-K_0, \quad (\boldsymbol{\sigma}\cdot\mathbf{e})u^4=-u^4 \end{aligned}\right\} \tag{3.52}$$

The components of u^1, etc. will be indicated by Greek suffixes, e.g., u^1_α. Since the eigenvectors u^ϱ are normalised and are orthogonal to each other, it follows that

$$\left.\begin{aligned} u^{\varrho *}_\alpha\, u^\sigma_\alpha &= \delta_{\varrho\sigma},\\ u^{\varrho *}_\alpha\, u^\varrho_\beta &= \delta_{\alpha\beta}. \end{aligned}\right\} \tag{3.53}$$

The second of these conditions is a consequence of the fact that the matrix $[u^\varrho_\alpha]$ (α and ϱ designate row and column respectively) is unitary. It will be remarked that negative values of the energy are possible, and that the absolute difference between associated positive and negative values of the energy is at least $2\varkappa$.

The physical significance of σ_k can be understood by deriving the relation

$$\left[\frac{1}{i}\,\mathbf{r}\wedge\boldsymbol{\partial}+\frac{1}{2}\,\boldsymbol{\sigma}, H\right]=0, \tag{3.54}$$

where $\boldsymbol{\partial}$ is the three-dimensional vector $\partial_1, \partial_2, \partial_3$. Since H is the Hamiltonian of the particle, it follows that the total angular momentum, $(1/i)\mathbf{r}\wedge\boldsymbol{\partial}+(1/2)\boldsymbol{\sigma}$ is a constant of the motion. Spin angular momentum $(1/2)\boldsymbol{\sigma}$ is an intrinsic property in the sense that it is not zero even when the particle is at rest. Moreover, equation (3.54) remains valid for a particle that is not free, but subject to central forces — (a more general discussion of angular momentum in quantum field theory appears in Example 1 in Ch. VII and Example 5 in Ch. IX).

It is apparent from (3.52) that for a particle described by the Dirac equation (3.1) the energy can be $\pm K_0$ and the component of the spin angular momentum in the direction of the vector $\mathbf{e}$ can be $\pm\,\frac{1}{2}$.

That the quantity $(1/2)\boldsymbol{\sigma}$ has the properties of angular momentum can, indeed, be shown by other means. From (3.49) and (3.2) it follows that

$$[\tfrac{1}{2}\sigma_k,\ \tfrac{1}{2}\sigma] = i\tfrac{1}{2}\sigma_m \quad (k, l, m = \text{cycl. } (1, 2, 3)). \tag{3.55}$$

This is just the relation that is typical of angular momentum (see DIRAC, Principles of Quantum Mechanics). From (3.55) it follows that $(1/2)\sigma_k$ has eigenvalues of $\pm \frac{1}{2}$ corresponding to the two orientations of spin angular momentum. The result (3.51) is a particular instance of this in which the component of $\boldsymbol{\sigma}$ in the direction of the vector $\mathbf{e}$ is taken.

It is clear from (3.52) that the function $u_{\mathbf{x}}^{\varrho}$ refers to electrons of positive energy when ϱ is 1 or 2 and to electrons of negative energy when ϱ is 3 or 4. For many practical purposes it is necessary to select the projection vector on the vector-manifold referring to electrons whose energy has a particular sign. This may be accomplished by means of the operators

$$\Lambda_{\pm}(\mathbf{k}) = \frac{1}{2K_0}\{K_0 \pm (\boldsymbol{\alpha}\cdot\mathbf{k} + \varkappa\beta)\}. \tag{3.56}$$

For it follows from (3.45c) and (3.47) that

$$\Lambda_{+}(\mathbf{k})\, u^{\varrho} = \begin{cases} u^{\varrho} & \text{for } \varrho = 1, 2 \\ 0 & \text{for } \varrho = 3, 4 \end{cases} \tag{3.57}$$

$$\Lambda_{-}(\mathbf{k})\, u^{\varrho} = \begin{cases} 0 & \text{for } \varrho = 1, 2 \\ u^{\varrho} & \text{for } \varrho = 3, 4. \end{cases} \tag{3.58}$$

In other words, Λ_+ and Λ_- are projection operators which act on state vectors u to yield the components of positive and negative energy respectively.

These projection operators have the further useful property that they may be used to calculate the expectation values of a matrix, say A, in states of positive or negative energy. This can be accomplished without using the actual form of the function u^1 and u^2 (or u^3 and u^4). For

$$\sum_{\varrho=1,2} u^{\varrho *} A\, u^{\varrho} = u^{\varrho *} A\, \Lambda_+\, u^{\varrho} = \mathrm{Sp}\,(A\, \Lambda_+). \tag{3.59}$$

The last quantity may be calculated easily by using the fact (3.4) that $A\Lambda_+$ can be written as a linear combination of γ^4.

The velocity of a free particle may be calculated as

$$\frac{d}{dt}\mathbf{x} = i\,[\mathbf{x}, H] = \boldsymbol{\alpha}. \tag{3.60}$$

Since α_k has eigenvalues ± 1, it would seem that the speed of the particle is always equal to that of light. This result, however, has no direct interpretation in that we are, in the real world, concerned with the average over a short time of the velocity of a particle whose energy is positive. This value is different from the velocity of light, for

$$\sum_{\varrho=1,2} u^{\varrho *} \frac{d}{dt}\mathbf{x}\, u^{\varrho} = \mathrm{Sp}(\boldsymbol{\alpha}, \Lambda_{+}(\mathbf{k})) = \frac{\mathbf{k}}{K_0}. \tag{3.61}$$

Indeed, it is apparent that the classical relation between velocity and momentum is still valid.

§ 5. Foldy-Wouthuysen-Tani Transformation

It may be convenient to illustrate the preceding paragraph by an example of the methods by which practical calculations are made. For this purpose we shall describe the Foldy–Wouthuysen–Tani transformation, (FOLDY and WOUTHUYSEN [1950]; TANI [1951]) which provides a convenient method of calculating the effect of Dirac operators on states of definite energy.

The transformation is defined by

$$\psi \to \varphi = S\psi$$

with

$$S = \beta\,\frac{(\boldsymbol{\alpha}\cdot\mathbf{k}) + \beta(K_0+\varkappa)}{\{2K_0(K_0+\varkappa)\}^{1/2}} \qquad S^{-1} = \frac{(\boldsymbol{\alpha}\cdot\mathbf{k}) + \beta(K_0+\varkappa)}{\{2K_0(K_0+\varkappa)\}^{1/2}}\,\beta.$$

The Hamiltonian of a free particle may be transformed according to

$$H' \equiv SHS^{-1} = S((\boldsymbol{\alpha}\cdot\mathbf{k}) + \varkappa\beta)S^{-1} = K_0\beta.$$

It is convenient to choose the representation in which β is diagonal and of the form

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

It is then clear that one pair of functions (φ_1, φ_2) refers to positive values of the energy, and another (φ_3, φ_4) to negative values.

The transformation S transforms a matrix A according to

$$A' = SAS^{-1}.$$

If the matrix A' has elements which connect only states with energies of the same (different) sign it is called even (odd). It is easy to show that β and σ_k are even and that α_k is odd.

Table I collects together the expression for certain operators after transformation to the φ representation.

TABLE I

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x} - i\frac{1}{2K_0}\beta\boldsymbol{\alpha} + \frac{i\beta(\boldsymbol{\alpha}\cdot\mathbf{k})\mathbf{k} - [\boldsymbol{\sigma}\wedge\mathbf{k}]k}{2K_0(K_0+\varkappa)k}$$

$$\mathbf{k} \to \mathbf{k}' = \mathbf{k}$$

$$H \to H' = K_0\beta$$

$$\boldsymbol{\alpha} \to \boldsymbol{\alpha}' = \boldsymbol{\alpha} + \frac{\mathbf{k}}{K_0}\beta - \frac{(\boldsymbol{\alpha}\cdot\mathbf{k})\mathbf{k}}{K_0(K_0+\varkappa)}$$

$$\beta \to \beta' = \frac{1}{K_0}(\varkappa\beta - (\boldsymbol{\alpha}\cdot\mathbf{k}))$$

$$\boldsymbol{\sigma} \to \boldsymbol{\sigma}' = \boldsymbol{\sigma} + \frac{i}{K_0}\beta[\boldsymbol{\alpha}\wedge\mathbf{k}] - \frac{[\mathbf{k}\wedge[\boldsymbol{\sigma}\wedge\mathbf{k}]]}{K_0(\varkappa+K_0)}.$$

We shall use the transformation to calculate a value for the operator $\beta\boldsymbol{\alpha}$ referring to nucleon states of positive energy only. This operator is sometimes assumed to be a factor in that part of the Hamiltonian, which represents the interaction for the β-decay, and to express the effect of transition between nucleon states. Transformation S leads to

$$S\beta\boldsymbol{\alpha}S^{-1} = \frac{1}{2K_0(K_0+\varkappa)}[-\{(K_0+\varkappa)^2+k^2\}\boldsymbol{\alpha}\beta + 2k^2\boldsymbol{\alpha}\beta + 2\mathbf{k}\beta(\boldsymbol{\alpha}\cdot\mathbf{k}) + 2i(K_0+\varkappa)[\mathbf{k}\wedge\boldsymbol{\sigma}]].$$

Since states of negative energy are irrelevant, odd matrices may be omitted from this expression. If terms of order two or more in $(\mathbf{k}/\varkappa)$ are neglected ($\varkappa$ is the nucleon mass), there results the expression

$$\beta\boldsymbol{\alpha} \to S\beta\boldsymbol{\alpha}S^{-1} \approx \frac{i}{\varkappa}[\mathbf{k}\wedge\boldsymbol{\sigma}].$$

In this matter of β-decay, $\mathbf{k}$ in this equation is interpreted as the average momentum of a nucleon (AHRONS and FEINBERG [1952]).

It is of interest that if ψ is a function with properties akin to those of a delta-function, φ is written as an integral over a region whose

dimension is of the order of the Compton wavelength $(1/\varkappa)$. This is a consequence of the appearance of the space derivation operator $\mathbf{k}$ in S.

This method has been extended to deal successfully with particles in external fields (FOLDY and WOUTHUYSEN [1950]).

§ 6. Charged Particles in an Electromagnetic Field. Hole Theory

The form of the wave equation of a charged particle in an electromagnetic field (of potential A_μ) is determined by the requirements of gauge invariance. A detailed discussion is given in Ch. VII, and leads to the equation

$$\{(\gamma_\mu, \partial_\mu - ieA_\mu(x)) + \varkappa\}\psi(x) = 0. \tag{3.62a}$$

The charge conjugation transformation of (3.40) transforms this to

$$\{(\gamma_\mu, \partial_\mu + ieA_\mu(x)) + \varkappa\}\psi'(x) = 0. \tag{3.62b}$$

It is now apparent that ψ' and ψ, related to each other by the charge conjugation transformation, refer to particles of opposite charge. Further, when ψ is a state of positive energy, ψ' is a state of negative energy. Indeed, if ψ is a state of positive energy, $\psi = \Lambda_+(\mathbf{k})\psi$; transforming this by charge conjugation and using (3.36) and (3.14) there is obtained the result

$$\Lambda_+(-\mathbf{k})\psi' = \Lambda_+(-\mathbf{k})\gamma_5\gamma_4\Lambda_+(\mathbf{k})B^{-1}\psi^* = \Lambda_+(-\mathbf{k})\Lambda_-(-\mathbf{k})\gamma_5\gamma_4 B^{-1}\psi^* = 0.$$

In other words, the charge conjugation provides a correspondence between states of negative energy of particles with charge $-e$ and states of positive energy of particles with charge $+e$ (and the same mass).

The existence of states of negative energy results in a considerable physical difficulty, which may be illustrated by the so-called Klein Paradox (KLEIN [1929]).

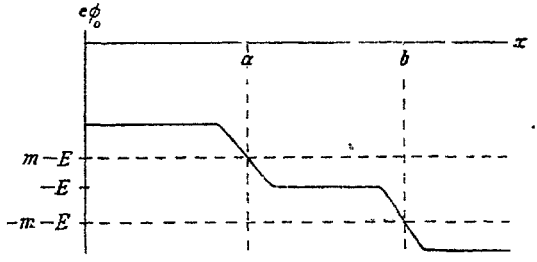

Fig. 3.1

In the real world electrons always have positive energy, and such states must be stable. However, a simple argument can lead to a contradictory result. The electrostatic potential $\phi_0(x)$ of Fig. 3.1 is such that this potential energy $e\phi_0(x)$ has the properties

$$\left.\begin{aligned} E+e\phi_0(x) &> m && \text{for } x<a \\ m>E+e\phi_0(x) &> -m && \text{for } a<x<b \\ -m &> E+e\phi_0(x) && \text{for } x>b \end{aligned}\right\}. \tag{3.63}$$

It is supposed that an electron of total energy E moves in the positive direction of x. According to classical mechanics the electron can never be found in the region $a<x<b$ because its momentum, $p(x)=[(E+e\phi_0(x))^2-m^2]^{1/2}$ is imaginary there. Thus, according to classical theory, an electron initially in the region $x<a$ can never find itself in the region $x>b$. However, quantum theory does not forbid this transition — it may occur by a process which is analogous to the tunnel effect. Thus it would appear that an electron in a state of positive energy need not necessarily remain in such a state indefinitely.

An escape from this difficulty is provided by the hole theory of electrons (DIRAC [1931]). It is postulated that a vacuum is not empty of electrons — that in a vacuum all states of positive energy are unoccupied and all states of negative energy, on the other hand, are fully occupied. (According to Pauli's exclusion principle, a state of given momentum is fully occupied by two electrons of opposite spin). Dirac's theory supposes that the sea of electrons of negative energy cannot be observed; further, it is supposed that this sea of electrons becomes apparent only when an external source provides an amount of energy greater than $2m$ and causes an electron of negative energy to make a transition to a state of positive energy. When this happens the sea of electrons is characterised by the absence of an electron from some states of negative energy, or by a "hole". By such means the Klein Paradox is resolved — it is not possible for an electron of positive energy to appear in a state of negative energy because all such states are fully occupied under normal conditions. Only when a hole is present can a transition to a state of negative energy take place.

A hole in this basic sea of negatively charged particles will have the properties of a particle of positive charge and the mass of an electron. In general this particle is called an "antiparticle". The transition of an electron (of negative charge) from a state of negative

energy to one of positive energy can then be interpreted as the creation of a pair of particles — a negaton and a positon. In fact the prediction of the creation of such pairs was the most striking success of the hole theory.

That feature of Dirac's theory that predicts the appearance of electrons in, and the disappearance from, an infinity of states is also required of a quantum field theory. In such a theory, however, the creation of a negaton–positon pair is interpreted not as the transition of an electron from a state of negative energy to one of positive energy, but as the creation of both a negatively charged and a positively charged particle. Antiparticles can still be described by wave-functions ψ' that are related to ψ by charge conjugation. The stability of electrons is ensured, given a suitable definition of the vacuum, by the fact that they are made subject to Fermi statistics — a consequence of the commutation relationships that are applied (see Ch. IX).

Although the hole theory resolves the Klein Paradox, it is not itself free from similar difficulties. Thus an external electromagnetic field can polarise the infinite number of electrons that are present in the vacuum in such a way that some electrons appear in states of positive energy. The current induced by such means is infinite in intensity (cf. Example 6, Ch. XIII).

§ 7. Majorana's Theory of Particles of Spin 1/2

The relationship between particles and antiparticles appears in an especially simple way in Majorana's theory (MAJORANA [1937]). This is a theory of particles of spin 1/2, but is not the most general theory of such particles.

The four-vectors $\psi^{(1)}$ and $\psi^{(2)}$ are introduced, in terms of the charge-conjugate pair ψ and ψ', by the equation

$$\left.\begin{aligned} \psi^{(1)} &= \frac{1}{\sqrt{2}}(\psi + \psi') \\ \psi^{(2)} &= \frac{1}{i\sqrt{2}}(\psi - \psi'). \end{aligned}\right\} \tag{3.64}$$

This implies that

$$\left.\begin{aligned} \psi &= \frac{1}{\sqrt{2}}(\psi^{(1)} + i\psi^{(2)}) \\ \psi' &= \frac{1}{\sqrt{2}}(\psi^{(1)} - i\psi^{(2)}). \end{aligned}\right\} \tag{3.65}$$

Equation (3.32) is used to introduce the vectors $\bar{\psi}^{(1)}$ and $\bar{\psi}^{(2)}$ defined as

$$\left.\begin{aligned} \bar{\psi}^{(1)} &\equiv \psi^{(1)*}\gamma_4 = \frac{1}{\sqrt{2}}(\bar{\psi}+\bar{\psi}') \\ \bar{\psi}^{2} &\equiv \psi^{(2)*}\gamma_4 = -\frac{1}{i\sqrt{2}}(\bar{\psi}-\bar{\psi}'). \end{aligned}\right\} \tag{3.66}$$

Then (3.40) gives

$$\left.\begin{aligned} \bar{\psi}^{(1)'} &\equiv (C^{-1}\psi^{(1)})^T = \bar{\psi}^{(1)}, \quad \psi^{(1)'} \equiv C\,\bar{\psi}^{(1)T} = \psi^{(1)}, \\ \bar{\psi}^{(2)'} &\equiv (C^{-1}\psi^{(2)})^T = \psi^{(2)}, \quad \psi^{(2)'} \equiv C\,\bar{\psi}^{(2)T} = \psi^{(2)}. \end{aligned}\right\} \tag{3.67}$$

which shows that $\psi^{(1)}$ and $\psi^{(2)}$ are invariant under charge conjugation. It will be remarked that if $\psi^{(2)}=0$, then $\psi=\psi'$ by (3.64). This must be interpreted to mean that the particles described by such wave-functions are neutral particles. Majorana's theory is concerned with particles of this kind — $\psi^{(2)}$ is assumed to be zero.

Thus, in Majorana's scheme, not only are the particles neutral but there is no logical distinction between a particle and the antiparticle. This has important consequences that may be illustrated by the process of spontaneous neutron decay. If the neutrino, ν, and anti-neutrino can be distinguished and the neutron, N, can create not the antineutrino but neutrino;

$$N \to P+e+\nu$$
$$\nu'+N \to P+e \quad (\nu';\ \text{antineutrino})$$

then it follows that in the double β-decay two neutrinos must be produced

$$N+N \to N+P+e+\nu \to P+P+e+e+\nu+\nu.$$

On the other hand, Majorana's theory, in which there is no distinction between neutrino and antineutrino, can lead to the scheme (FURRY [1939])

$$N+N \to N+P+e+\nu \to P+P+e+e.$$

The neutrino emitted by one neutron may be assumed to be absorbed, in its role of antineutrino, by the second decaying neutron. Study of double β-decay therefore provides a means of testing the validity of the application of Majorana's theory to neutrinos [1]). Unfortunately

[1]) SAKATA [1955] and MADDOX (private discussion) independently suggested the existence of β-interactions, in cases of which the study of double β-decay cannot decide whether or not the neutrino is of the Majorana type.

the available experimental results (see McCARTHY [1953], FIREMAN and SCHWARZER [1952]) are not sufficiently accurate to be valuable in this context.

§ 8. Spinors

Under Lorentz transformation ψ behaves in a rather special way, and to take advantage of this we shall now introduce certain quantities called spinors.

If σ_k are the spin matrices (3.49), a hermitian representation of the γ_μ is provided by

$$\gamma_k = -\Sigma_k \varrho_2, \quad \gamma_4 = \varrho_1, \quad \varrho_3 = -\gamma_1\gamma_2\gamma_3\gamma_4 \tag{3.68}$$

$$\varrho_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \ \varrho_2 = \begin{bmatrix} 0 & -iI \\ iI & \end{bmatrix}, \ \varrho_3 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \ \Sigma_k = \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix} \tag{3.69}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.70}$$

It is easily shown that the matrices γ_μ of (3.68) satisfy the commutation relationship (3.2). (For practical purposes it is sometimes more convenient to choose the representation $\gamma_k = -\Sigma_k\varrho_2$ and $\gamma_4 = \varrho_3$). If (3.68) is substituted into (3.24)

$$\left.\begin{aligned} S_{kl} &= \tfrac{1}{2}\,\Sigma_k\,\Sigma_l = \frac{i}{2}\begin{bmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{bmatrix} \quad (j, k, l = \text{cycl. } (1, 2, 3)) \\ S_{k4} &= \tfrac{1}{2}\,\Sigma_k\,\varrho_2\,\varrho_1 = \frac{i}{2}\begin{bmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{bmatrix}. \end{aligned}\right\} \tag{3.71}$$

The last equations show that under infinitesimal Lorentz transformation of the form (3.18) and (3.22a) the pairs of components ψ_1 and ψ_2, ψ_3 and ψ_4 transform according to

$$'\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \Big(I + \frac{i}{2}\sum_{\text{cycl.}} \delta w_{23}\,\sigma_1 + \frac{i}{2}\,\delta w_{k4}\,\sigma_k\Big)\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{3.72}$$

and

$$'\begin{bmatrix} \psi_3 \\ \psi_4 \end{bmatrix} = \Big(I + \frac{i}{2}\sum_{\text{cycl.}} \delta w_{23}\,\sigma_1 - \frac{i}{2}\,\delta w_{k4}\,\sigma_k\Big)\begin{bmatrix} \psi_3 \\ \psi_4 \end{bmatrix}. \tag{3.73}$$

In these formulae

$$\sum_{\text{cycl.}} \delta w_{23}\sigma_1 \equiv \delta w_{23}\sigma_1 + \delta w_{12}\sigma_3 + \delta w_{31}\sigma_2.$$

It is apparent that the transformations (3.72) and (3.73) are unimodular (the determinant of the coefficients of the transformation

is unity); this is consequently true of all transformations of the continuous Lorentz group (i.e. all transformations that are products of infinitesimal transformations).

Binary row or column matrices whose infinitesimal Lorentz transformations are unimodular are called *spinors*. If the transformation is of the form (3.73), the spinor is said to be *contravariant* and we shall denote such spinors by superscripts. Thus the elements ψ_3 and ψ_4 in (3.73) will be written ψ^1 and ψ^2 and, in general, the elements of a contravariant spinor, b^r. A covariant spinor is a binary row of elements a_r such that $a_r b^r$ is invariant under Lorentz transformation. It follows that the spinor transforms according to

$$'(a_1, a_2) = (a_1, a_2)\left(I - \frac{i}{2}\sum_{\text{cycl.}} \delta w_{23}\,\sigma_1 + \frac{i}{2}\,\delta w_{k4}\,\sigma_k\right) \tag{3.74}$$

under infinitesimal transformation.

The column formed of the complex conjugates of the elements of a covariant spinor (a row) is also a spinor, called the *complex conjugate spinor*. The transformation properties of such a column are those of (3.72), because δw_{k4} and δw_{ki} in that equation are imaginary and real respectively. In general the elements of a spinor having such properties will be written as $b_{\dot{r}}$; for this reason the components ψ_1 and ψ_2 of ψ will be written as $\psi_{\dot{1}}$ and $\psi_{\dot{2}}$. Evidently spinors with elements $a^{\dot{r}}$ exist such that $a^{\dot{r}} b_{\dot{r}}$ is invariant.

It is easy to see that from a covariant spinor a_r can be deduced a contravariant spinor a^r by putting

$$a^1 = a_2 \quad , \quad a^2 = -a_1. \tag{3.75a}$$

Similarly the spinor (a row) with elements $a^{\dot{r}}$ given by

$$a^{\dot{1}} = a_{\dot{2}} \quad , \quad a^{\dot{2}} = -a_{\dot{1}} \tag{3.75b}$$

is such that $a^{\dot{r}} a_{\dot{r}}$ is invariant.

Under spatial rotations, ψ^r and $\psi_{\dot{r}}$ transform in the same way — this is seen by putting $\delta w_{k4} = 0$ in (3.72) and (3.73). The same equations also show that ψ^r and $\psi_{\dot{r}}$ transform independently under continuous Lorentz transformations.

The last property does not obtain under spatial reflections. Indeed, in the sense of (3.26) a spatial reflection is characterised by the matrix

$$\mathcal{A} = i\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \tag{3.76}$$

This has the effect of interchanging ψ^r and ψ_t (apart from the factor i). Time reflection, on the other hand, is characterised by the matrix

$$\Lambda = i \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \tag{3.77}$$

leading to the transformation

$$\psi_t \to -i\psi_t \quad , \quad \psi^r \to i\psi^r. \tag{3.78}$$

In general, when dealing with transformations that contain spatial reflections, it is not possible to regard ψ as divided into two spinors; the complete vector (ψ^r, ψ_t) is called an *undor* (BELINFANTE [1939]).

In terms of the spinors ψ_t and ψ^r the wave equation takes the form

$$\left.\begin{aligned} -(I\,\partial_4 + i\,\sigma_k\,\partial_k) \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} &= \varkappa \begin{bmatrix} \psi_{\dot{1}} \\ \psi_{\dot{2}} \end{bmatrix} \\ -(I\,\partial_4 - i\,\sigma_k\,\partial_k) \begin{bmatrix} \psi_{\dot{1}} \\ \psi_{\dot{2}} \end{bmatrix} &= \varkappa \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix}. \end{aligned}\right\} \tag{3.79}$$

This may be further simplified by introducing the quantities

$$\left.\begin{aligned} \sigma_{k:ts} &\equiv [\sigma_k]_{ts} \\ \sigma_k{}^{rs} &\equiv -[\sigma_k]_{rs} \\ \sigma_{4:ts} &\equiv -i\,[I]_{ts} \\ \sigma_4{}^{rs} &\equiv -i\,[I]_{rs}. \end{aligned}\right\} \tag{3.80}$$

Here $[\sigma_k]_{rs}$ and $[I]_{rs}$ denote the (rs) components of the matrix σ_k and I. Putting

$$\left.\begin{aligned} \partial_{ts} &\equiv \sigma_{\mu:ts}\,\partial_\mu \\ \partial^{st} &\equiv \sigma_\mu{}^{st}\,\partial_\mu, \end{aligned}\right\} \tag{3.81}$$

the wave equation becomes

$$\left.\begin{aligned} \partial^{st}\psi_t &= i\varkappa\psi^s \\ \partial_{ts}\psi^s &= i\varkappa\psi_t. \end{aligned}\right\} \tag{3.82}$$

Since the wave equation is invariant under Lorentz transformation and ψ^s and ψ_t are spinors, it is clear that ∂_{ts} and ∂^{ts} have the transformation properties of spinors of the appropriate type in virtue of each index — they are said to be spinors of the second degree. Similarly, (3.81) is evidence that the quantities $\sigma_{\mu:ts}$ and $\sigma_\mu{}^{st}$ transform as four-vectors in virtue of the index μ and as spinors in virtue of the indices

$\dot{r}$ and s. Inspection of $\sigma_{\mu:\dot{r}s}$ and the other element related to this shows that

$$\left.\begin{aligned}\sigma_{\mu:\dot{r}s} &= \sigma_{\mu:s\dot{r}}\\ \sigma_{\mu}^{\dot{r}s} &= \sigma_{\mu}^{s\dot{r}}.\end{aligned}\right\} \tag{3.83}$$

Here $\sigma_{\mu:s\dot{r}}$ and $\sigma_{\mu}^{\dot{r}s}$ are the quantities derived from $\sigma_{\mu}^{s\dot{r}}$ and $\sigma_{\mu:\dot{r}s}$ according to (3.75a, b). It can be seen from (3.81) that $\sigma_{\mu:s\dot{r}}$ and $\sigma_{\mu}^{s\dot{r}}$ transform four vectors into spinors of the second degree by contraction of the suffix μ.

§ 9. Pseudo Spinors

As shown in § 1, the Dirac functions transform in four different ways under space reflections, namely

$$'\psi = \left\{\begin{array}{ll} i\gamma_4\psi & (A)\\ -i\gamma_4\psi & (B)\\ \gamma_4\psi & (C)\\ -\gamma_4\psi. & (D)\end{array}\right\} \tag{3.84}$$

Transformation (A) is described in detail by (3.26). We have seen in § 3 that ψ and ψ' have different transformation properties for the space reflections that give (C) or (D). In other words, ψ' obeys (C) or (D) when ψ obeys (D) or (C) respectively and ψ' obeys (A) or (B) when ψ obeys (A) or (B) respectively. Since in the Majorana theory a linear combination of ψ and ψ' is used, the Lorentz invariance of the theory requires that ψ and ψ' should have the same transformation properties under a Lorentz transformation. Therefore ψ must obey (A) or (B) in the Majorana theory. As we shall show later, the physical quantities contain ψ and $\bar{\psi}$ in pairs of the form $(\bar{\psi}^a O\psi^b)$ where O is a product of γ_μ's. Equation (3.84) shows that when ψ^a and ψ^b describe the same kind of particle $(a=b)$, there is no difference between the types (A), (B), (C), and (D). On the other hand, when ψ^a and ψ^b describe different kinds of particle $(a \neq b)$, the four types in (3.84) give different results, because ψ^a and ψ^b may have different transformation properties under space reflection. We shall denote the wave function of fields of the types (A), (B), (C), and (D) by ψ_A, ψ_B, ψ_C and ψ_D. Wave functions ψ_B, ψ_C, ψ_D are called pseudospinors, while ψ_A is a spinor.[1]) (YANG and TIOMNO [1950]). Then $\bar{\psi}_A O\psi_B$ and $\bar{\psi}_C O\psi_D$ transform differently from $\bar{\psi}_A O\psi_A$ under space

[1]) Some authors call ψ_C spinor and ψ_A, ψ_B, ψ_D pseudospinors.

reflection. It is easily seen that time reflections give similar results.

We shall now illustrate the physical importance of pseudospinors by the example of β-disintegration.

The interactions relevant to β-disintegration are described in detail in Example 8, Ch. VII. In the present example we consider only the scalar coupling

$$(\bar{\psi}^P \psi^N)\,(\bar{\psi}^e \psi^\nu) \tag{3.85a}$$

where ψ^P, ψ^N, ψ^e and ψ^ν are the wave functions of a proton, neutron, electron and neutrino respectively.

If ψ^P, ψ^N, ψ^e and ψ^ν are of the same type in the sense of (3.84), then (3.85a) is invariant under space reflection. On the other hand, if ψ^P, ψ^N, and ψ^e belong to type A and ψ^ν belongs to type B, we cannot take (3.85a) as a Lorentz invariant interaction. Then we can take the following Lorentz invariant interaction instead of (3.85a):—

$$(\bar{\psi}^P \psi^N)\,(\bar{\psi}^e \gamma_5 \psi^\nu). \tag{3.85b}$$

This shows that interactions of pseudospinor fields are different from the usual ones.

If the β-disintegration scheme were

$$N \to P + e + \nu \tag{3.86}$$

then the inverse process would be

$$P \to N + e' + \nu' \tag{3.87}$$

Here e' and ν' denote the positon and antineutrino respectively. Since the mass of the neutron is larger than that of the proton, (3.87) is forbidden by the energy conservation law. However, the process (3.87) could be induced by an external field. The positon emission of nuclei is an example of such a process.

We now consider the process

$$P \to N' + e' + \nu' \quad (N' \text{: antineutron}). \tag{3.88}$$

This induces the process $P + N \to e' + \nu'$ and would imply the instability of nuclei. On the other hand (3.87) induces $P + N' \to e' + \nu'$, which is not incompatible with the stability of nuclei because in nuclei we have N and not N'. Therefore we must reject (3.88).

If we want to have (3.87) and not (3.88), we can proceed as follows. Choose ψ^N of type (C) or type (D), then ψ^N and $\psi^{N'}$ obey different

transformation laws under space reflection, and therefore we can have an interaction that allows (3.87) and forbids (3.88) [1]). We see that the transformation properties of spinors under space and time inversion provide a wide framework in which can be explained simultaneously various transmutation processes.

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[1]) However, this is not the only possibility for which (3.88) is forbidden.

CHAPTER IV

THE GENERAL RELATIVISTIC WAVE EQUATION (I)

§ 1. Spinors

In the last Chapter we introduced the spinors a_r, a^r, $b_{\dot r}$, $b^{\dot r}$. We shall now discuss their general characteristics (VAN DER WAERDEN [1929], LAPORTE and UHLENBECK [1931]). The rules (3.75a) and (3.75b) may be written in the form

$$\left.\begin{aligned} a^r &= \varepsilon^{rs}\, a_s\,, \quad a_s = \varepsilon_{sr}\, a^r\,, \\ b^{\dot r} &= \varepsilon^{\dot r\dot s}\, a_{\dot s}\,, \quad b_{\dot s} = \varepsilon_{\dot s\dot r}\, a^{\dot r}\,, \end{aligned}\right\} \tag{4.1}$$

with

$$[\varepsilon^{rs}] = [\varepsilon^{\dot r\dot s}] = -[\varepsilon_{rs}] = -[\varepsilon_{\dot r\dot s}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.2}$$

It follows that

$$\varepsilon^{rs} = -\varepsilon^{sr}\,, \quad \varepsilon_{rs} = -\varepsilon_{sr}\,, \quad \varepsilon^{\dot r\dot s} = -\varepsilon^{\dot s\dot r}\,, \quad \varepsilon_{\dot r\dot s} = -\varepsilon_{\dot s\dot r}\,, \tag{4.3a}$$

$$\varepsilon^{rs}\,\varepsilon_{ts} = -\,\delta_{tr}\,, \quad \varepsilon^{\dot r\dot s}\,\varepsilon_{\dot u\dot s} = -\,\delta_{\dot u\dot r}\,, \tag{4.3b}$$

$$\varepsilon^{rs}\,\varepsilon^{ts} = \varepsilon_{rs}\,\varepsilon_{ts} = \delta_{tr}\,, \quad \varepsilon^{\dot r\dot s}\,\varepsilon^{\dot u\dot s} = \varepsilon_{\dot r\dot s}\,\varepsilon_{\dot u\dot s} = \delta_{\dot u\dot r} \tag{4.3c}$$

$$\varepsilon^{rs}\,\varepsilon^{ut} = \varepsilon_{rs}\,\varepsilon_{ut} = \delta_{ru}\,\delta_{st} - \delta_{rt}\,\delta_{su}\,. \tag{4.3d}$$

As shown in Ch. III, $a_r a'^r$ and $b_{\dot r} b'^{\dot r}$ are invariant under continuous Lorentz transformations.

From (4.1) it follows that

$$a^s\, b_s = -\,a_s\, b^s\,, \quad a^{\dot s}\, b_{\dot s} = -\,a_{\dot s}\, b^{\dot s} \tag{4.4}$$

and therefore that

$$a^s\, a_s = 0. \tag{4.5}$$

From (4.3d) it follows that

$$a^s\, b_s\, c_t + a_s\, b_t\, c^s + a_t\, b^s\, c_s = a^v\, b^r\, c^u\,(\varepsilon_{vr}\,\varepsilon_{tu} + \varepsilon_{uv}\,\varepsilon_{tr} + \varepsilon_{tv}\,\varepsilon_{ru}) = 0. \tag{4.6}$$

The relations (4.5) and (4.6) remain valid when the undotted suffices and superfices are replaced by dotted suffices and superfices respectively. As is shown in Ch. III the complex conjugate of a spinor can be obtained by dotting undotted suffices and superfices and removing the dots from the others.

Quantities $a^{rs\ldots}_{\dot{u}\dot{v}\ldots}$ that have the same properties under continuous Lorentz transformation as the products $(a^r a^s \ldots b_{\dot{u}} b_{\dot{v}} \ldots)$ are called spinors of degree n, where n is the sum of the numbers of suffices and superfices. Suffices can be raised and lowered according to the rule (4.1), e.g.

$$a^{12} = a_2{}^2 = -\, a^1{}_1.$$

Since $a^{rs\ldots}_{\dot{u}\dot{v}\ldots}$ has the same transformation properties as $(a^r a^s \ldots b_{\dot{u}} b_{\dot{v}} \ldots)$, and $a_s a^s$ is an invariant, the degree of a spinor can be reduced by contraction. For example the product $a_{st} b^{ur}$ can be reduced to a second degree spinor by forming the sum

$$a_{st} b^{tr} = c_s{}^r. \tag{4.6}$$

Since, as shown in Ch. III, $\sigma_\mu^{r\dot{s}}$ has the spinor property for superfices $(r\dot{s})$, they can be lowered according to the rules (4.1) to give $\sigma_{\mu,s}{}^{\dot{t}}$, $\sigma_{\mu,\dot{t}}{}^{s}$ and $\sigma_{\mu,\dot{r}\dot{s}}$.

A vector a_μ can be derived from a 2nd-degree spinor $a_{\dot{t}}{}^s$ by means of the $\sigma_{\mu,s}{}^{\dot{t}}$ as follows:

$$a_\mu = \tfrac{1}{2}\, \sigma_{\mu,s}{}^{\dot{t}}\, a_{\dot{t}}{}^s \tag{4.7a}$$

because $\sigma_{\mu,s}{}^{\dot{t}}$ has the vector property for suffix μ. In detail, (4.7a) is

$$\left.\begin{aligned} a_1 &= \frac{1}{2}\,(a_{\dot{2}1} + a_{\dot{1}2}), \\ a_2 &= \frac{1}{2i}\,(a_{\dot{2}1} - a_{\dot{1}2}), \\ a_3 &= \frac{1}{2}\,(a_{\dot{1}1} - a_{\dot{2}2}), \\ a_4 &= \frac{i}{2}\,(a_{\dot{1}1} + a_{\dot{2}2}). \end{aligned}\right\} \tag{4.7b}$$

In general, a tensor of degree n can be derived from a spinor of degree $2n$ in the following way:

$$a_{\mu_1\ldots\mu_n} = \prod_{i=1}^{n} (\tfrac{1}{2}\, \sigma_{\mu_i,s_i}{}^{\dot{t}_i})\, a^{s_1\ldots s_n}_{\dot{t}_1\ldots\dot{t}_n} \tag{4.8a}$$

$$= (-1)^n \prod_{i=1}^{n} (\tfrac{1}{2}\, \sigma_{\mu_i}{}^{s_i\dot{t}_i})\, a_{s_1\ldots s_n \dot{t}_1\ldots\dot{t}_n}. \tag{4.8b}$$

From (3.80) we can derive the relations

$$\sigma_{\mu,s}{}^{\dot{t}}\, \sigma_{\nu,\dot{t}}{}^{s} = -\, \sigma_{\mu,\dot{r}s}\, \sigma_\nu^{s\dot{r}} = \mathrm{Sp}(\sigma_\mu\, \sigma_\nu) = 2\, \delta_{\mu\nu}, \tag{4.9a}$$

$$\sigma_\mu{}^{r\dot{s}}\,\sigma_{\nu,\dot{s}u} + \sigma_\nu{}^{r\dot{s}}\,\sigma_{\mu,\dot{s}u} = -\,(\sigma_\mu\,\sigma_\nu + \sigma_\nu\,\sigma_\mu)_{ru} = -\,2\,\delta_{\mu\nu}\,\delta_{ru}\,, \tag{4.9b}$$

$$\sigma_{\mu,u\dot{r}}\,\sigma_\nu{}^{r\dot{s}} + \sigma_{\nu,u\dot{r}}\,\sigma_\mu{}^{r\dot{s}} = -\,2\,\delta_{\mu\nu}\,\delta_{u\dot{s}}\,, \tag{4.9c}$$

and

$$\sigma_{\mu,r\dot{s}}\,\sigma_{\mu,u\dot{w}} = \sigma_\mu{}^{rs}\,\sigma_\mu{}^{u\dot{w}} = 2\,(\delta_{r\dot{w}}\,\delta_{u\dot{s}} - \delta_{r\dot{s}}\,\delta_{u\dot{w}}), \tag{4.9d}$$

where

$$\delta_{r\dot{s}} \equiv \begin{cases} 1 & \text{for } r = s, \\ 0 & \text{for } r \neq s. \end{cases}$$

The relation (4.9*d*) leads to

$$\sigma_\mu{}^{rs}\,\sigma_\mu{}^{uw} + \sigma_\mu{}^{us}\,\sigma_\mu{}^{r\dot{w}} = 0, \tag{4.10a}$$

$$\sigma_{\mu,rs}\,\sigma_{\mu,uw} + \sigma_{\mu,u\dot{s}}\,\sigma_{\mu,r\dot{w}} = 0, \tag{4.10b}$$

$$\sigma_{\mu,r}{}^{\dot{s}}\,\sigma_{\mu,u}{}^{\dot{w}} + \sigma_{\mu,u}{}^{\dot{s}}\,\sigma_{\mu,r}{}^{\dot{w}} = 0 \tag{4.10c}$$

and

$$\left.\begin{aligned} \sigma_\mu{}^r{}_{\dot{s}}\,\sigma_{\mu,u}{}^{\dot{p}} &= 2\,\varepsilon^{rv}\,\varepsilon^{\dot{p}\dot{w}}\,(\sigma_{\mu,v\dot{s}}\,\sigma_{\mu,u\dot{w}}) \\ &= 2\,\varepsilon^{rv}\,\varepsilon^{\dot{p}\dot{w}}\,(\delta_{v\dot{w}}\,\delta_{u\dot{s}} - \delta_{v\dot{s}}\,\delta_{u\dot{w}}) \\ &= 2\,\varepsilon^{rw}\,\varepsilon^{pw}\,\delta_{u\dot{s}} - 2\,\varepsilon^{rs}\,\varepsilon^{pu} \\ &= 2\,\delta_{r\dot{p}}\,\delta_{u\dot{s}} - 2\,(\delta_{r\dot{p}}\,\delta_{u\dot{s}} - \delta_{ru}\,\delta_{\dot{s}\dot{u}}) \\ &= 2\,\delta_{ru}\,\delta_{\dot{s}\dot{p}} \end{aligned}\right\} \tag{4.10d}$$

because of (4.3*c*) and (4.3*d*).

Using (4.8) and (4.10*d*) we can derive a spinor of degree $2n$ from a tensor of degree n as follows

$$a^{\dot{s}_1\ldots\dot{s}_n}_{\dot{r}_1\ldots\dot{r}_n} = \prod_{i=1}^{n}\,(\sigma_{\mu_i}{}^{s_i}_{\dot{r}_i})\,a_{\mu_1\ldots\mu_n}\,. \tag{4.11}$$

From (4.9*c*) and (4.11) we have

$$\left.\begin{aligned} a_{u\dot{r}}\,a^{\dot{w}r} &= \sigma_{\mu,u\dot{r}}\,\sigma_\nu{}^{\dot{w}r}\,a_\mu\,a_\nu \\ &= \tfrac{1}{2}\,(\sigma_{\mu,u\dot{r}}\,\sigma_\nu{}^{\dot{w}r} + \sigma_{\nu,u\dot{r}}\,\sigma_\mu{}^{\dot{w}r})\,a_\mu\,a_\nu \\ &= -\,a_\mu\,a_\mu\,\delta_{u\dot{w}} \end{aligned}\right\} \tag{4.12a},$$

and

$$a^{s\dot{t}}\,a_{\dot{t}t} = -\,a_\mu\,a_\mu\,\delta_{st}\,. \tag{4.12b}$$

In particular

$$\partial_{u\dot{r}}\,\partial^{\dot{w}r} = -\,\Box\,\delta_{u\dot{w}} \tag{4.12c}$$

$$\partial^{s\dot{t}}\,\partial_{\dot{t}t} = -\,\Box\,\delta_{st}\,. \tag{4.12d}$$

The skewsymmetric spinor a^{rs} ($a^{rs} = -\,a^{sr}$) has only the one independent component

$$a^{12} = -\,a^{21} = \tfrac{1}{2}\,a_r^r\,, \tag{4.13a}$$

which can be written on account of (4.1) as

$$a^{rs} = \tfrac{1}{2}\,\varepsilon^{rs}\,a_t{}^t\,. \tag{4.13b}$$

Equation (4.13*b*) shows that a^{rs} is invariant under continuous Lorentz transformation.

From two spinors a_{su} and a^{rt} that are symmetric and of degree 2 there can be derived a skew-symmetric tensor by the expression

$$a_{\mu\nu} = -\tfrac{1}{4}\,[\sigma_\mu{}^{st}\,\sigma_{\nu,ls}{}^{u}\,a_{tu} - \sigma_{\mu,ts}\,\sigma_\nu{}^{s}{}_{t}\,a^{vt}]. \tag{4.14a}$$

The tensor $a_{\mu\nu}$ is seen to be skew-symmetric if account is taken of (4.4). Equation (4.14*a*) implies that

$$\left.\begin{aligned} a_{tu} &= \tfrac{1}{2}\,\sigma_{\mu,tl}\,\sigma_{\nu,u}{}^{l}\,a_{\mu\nu} \\ a^{vt} &= \tfrac{1}{2}\,\sigma_{\mu,s}{}^{v}\,\sigma_\nu{}^{st}\,a_{\mu\nu} \end{aligned}\right\} \tag{4.14b}$$

— because using (4.9*a*, *b*) it follows: —

$$\begin{aligned} &\tfrac{1}{2}\,\sigma_{\mu,tl}\,\sigma_{\nu,u}{}^{l}\,a_{\mu\nu} = \\ &\quad -\tfrac{1}{8}\,[\sigma_{\mu,tl}\,\sigma_{\nu,u}{}^{l}\,\sigma_\mu{}^{sv}\,\sigma_{\nu,s}{}^{q}\,a_{pq} - \sigma_{\mu,tl}\,\sigma_{\nu,u}{}^{l}\,\sigma_{\mu,sv}\,\sigma_\nu{}^{w}{}_{l}\,a^{vl}] \\ &\quad = a_{tu} + \tfrac{1}{4}\,\sigma_{\mu,tl}\,\sigma_{\mu,uv}\,a^{vl} \\ &\quad = a_{tu} + \tfrac{1}{8}\,(\sigma_{\mu,tl}\,\sigma_{\mu,uv} + \sigma_{\mu,tv}\,\sigma_{\mu,ul})\,a^{vl} \\ &\quad = a_{tu}\,. \end{aligned}$$

Transformation (4.14*b*) may be illustrated by the effect on the skew-symmetric tensor

$$a_{\mu\nu} = \partial_\mu\,a_\nu - \partial_\nu\,a_\mu\,. \tag{4.15a}$$

Equation (4.14*b*) gives

$$\left.\begin{aligned} a_{tu} &= \tfrac{1}{2}\,(\partial_{tl}\,a_u{}^{l} + \partial_{ul}\,a_t{}^{l}) \\ a^{vt} &= \tfrac{1}{2}\,(\partial^{vt}\,a_t{}^{t} + \partial^{tt}\,a_t{}^{v})\,, \end{aligned}\right\} \tag{4.15b}$$

where $a_u{}^t$ is the spinor derived from the vector a_μ by means of (4.11). When $a_u{}^t$ satisfies the relations

$$\left.\begin{aligned} \partial_{tl}\,a_u{}^{l} &= \partial_{ul}\,a_t{}^{l} \\ \partial^{vt}\,a_t{}^{t} &= \partial^{tt}\,a_t{}^{v}\,, \end{aligned}\right\} \tag{4.16}$$

(4.15*b*) leads to

$$\left.\begin{aligned} a_{tu} &= \partial_{tt}\,a_u{}^{t} \\ a^{vt} &= \partial^{vt}\,a_t{}^{t}\,. \end{aligned}\right\} \tag{4.17}$$

From (4.1) and (4.16) the following relation can be derived

$$\begin{aligned}\partial_{\dot{\tau}t}\, a^{\dot{\tau}t} &= \partial_{\dot{\tau}t}\, \varepsilon^{\dot{\tau}\dot{u}}\, a_{\dot{u}}{}^{t} = \partial_{\dot{u}t}\, \varepsilon^{\dot{\tau}\dot{u}}\, a_{\dot{\tau}}{}^{t} \\ &= -\partial_{\dot{u}t}\, \varepsilon^{\dot{u}\dot{\tau}}\, a_{\dot{\tau}}{}^{t} = -\partial_{\dot{u}t}\, a^{\dot{u}t}.\end{aligned}$$

This leads to

$$\partial_{\dot{\tau}t}\, a^{\dot{\tau}t} = 0, \tag{4.18a}$$

which is equivalent to

$$\partial_{\mu}\, a_{\mu} = 0 \tag{4.18b}$$

by (4.9a) and (4.11). From (4.12c, d) and (4.17) we obtain

$$\left.\begin{aligned}\partial^{v\dot{\tau}}\, a_{\dot{\tau}u} &= -\Box\, a_{u}{}^{v} \\ \partial_{\dot{\tau}t}\, a^{v\dot{t}} &= -\Box\, a_{\dot{\tau}}{}^{v}.\end{aligned}\right\} \tag{4.19}$$

§ 2. Lorentz Transformation

A Lorentz transformation is a linear transformation under which the expression

$$x_{\mu}\, x_{\mu} = x_{k}\, x_{k} - t^{2}$$

is invariant. The Lorentz transformations form a group (WIGNER [1939]). It follows that a linear transformation

$${}'x_{\mu} = a_{\mu\nu}\, x_{\nu} \tag{4.19}$$

must have the properties

$$|a_{\mu\nu}| = \pm 1, \tag{4.20a}$$

$$a_{\mu\nu}\, a_{\mu\sigma} = \delta_{\nu\sigma}, \tag{4.20b}$$

if it is to be a Lorentz transformation. In particular, for $\nu = \sigma = 4$,

$$a_{\mu 4}\, a_{\mu 4} = 1. \tag{4.20c}$$

Since a_{k4} and a_{44} are imaginary and real respectively, (4.20c) shows that

$$a_{44}^{2} \geqslant 1$$

so that either

$$a_{44} \geqslant 1 \tag{4.21a}$$

or

$$a_{44} \leqslant -1 \tag{4.21b}$$

We see from (4.20a) that the transformations are of two different kinds. It is impossible to find transformations characterised by the positive sign that are arbitrarily close to a given transformation

characterised by a negative sign. The same is true of the two cases in (4.21*a*, *b*). We therefore say that the Lorentz group of transformations consists of four disconnected parts, namely

$$\text{Lorentz transformation} \begin{cases} L^+ \begin{cases} L^\uparrow_\uparrow & a_{44} \geqslant 1, \quad |a_{\mu\nu}| = 1, \\ L^\downarrow_\downarrow & a_{44} \leqslant -1, \quad |a_{\mu\nu}| = 1, \end{cases} \\ L^- \begin{cases} L^\uparrow_\downarrow & a_{44} \geqslant 1, \quad |a_{\mu\nu}| = -1, \\ L^\downarrow_\uparrow & a_{44} \leqslant -1, \quad |a_{\mu\nu}| = -1. \end{cases} \end{cases}$$

The first part consists of elements which are continuously connected with the identity element, and form a group; it is usually called the continuous Lorentz group, or the $L^\uparrow_\uparrow$ group. The product of every element of $L^\uparrow_\uparrow$ and any particular element of $L^\downarrow_\downarrow$, $L^\uparrow_\downarrow$ or $L^\downarrow_\uparrow$ generates $L^\downarrow_\downarrow$, $L^\uparrow_\downarrow$ and $L^\downarrow_\uparrow$ respectively. Space reflections and time reflections belong to $L^\uparrow_\downarrow$ and $L^\downarrow_\uparrow$ respectively. Total inversion, the product of space and time reflection, belongs to $L^\downarrow_\downarrow$. It follows, for example, that an element of $L^\uparrow_\downarrow$ is equivalent to the product of a space reflection and some element of $L^\uparrow_\uparrow$. The combination of $L^\uparrow_\downarrow$ with any of the other parts is also a group

A Lorentz invariant theory of elementary particles is obtained if each kind of elementary particle is described by an irreducible quantity in the Lorentz group. A reducible quantity is regarded as an assembly of several kinds of elementary particle. For example, vectors and scalars are both irreducible quantities under Lorentz transformations. On the other hand the transverse wave and the longitudinal wave of a vector field are mixed by a Lorentz transformation and therefore cannot describe two elementary particles.

We now introduce the following products of $(k+l-2)$ degree spinors:–

$$P^{(k,l)}_{mn} = \frac{a_1^{k-m-1}\, a_2^m}{\sqrt{m!\,(k-m)!}} \frac{a_1^{l-n-1}\, a_2^n}{\sqrt{n!\,(l-n)!}},$$

$$0 \leqslant m \leqslant k-1, \; 0 \leqslant n \leqslant l-1.$$

These have (kl) components. Group theory (WIGNER [1939]) shows that linear combinations of these components give the irreducible representations of the continuous Lorentz Group. By introducing the rule that the dotted and undotted suffices in $P^{(k,l)}_{mn}$ are to be replaced by undotted and dotted suffices (according to (3.76)) respectively in the case of space reflections, we can include the $L^\uparrow_\downarrow$ Lorentz group.

Similarly, $L^{\pm}$ and $L^{\mp}$ Lorentz transformations can be included by using the transformation property (3.78) for each spinor suffix in the case of time reflection, because total reflection is equivalent to the product of space and time reflections. Thus we can assume that the states of elementary particles are described by linear combinations of P_{mn} that are denoted by $U(k, l)$ $(\equiv a_{mn} P_{mn}^{(k,l)})$. For example, the wave functions are $U(1, 1)$ for a scalar field, $U(2, 1)$ and $U(1, 2)$ for a first degree spinor field, and $U(2, 2)$ for a vector field.

If q is the number of linearly independent components of $U(k, l)$ for an elementary particle at rest with non-vanishing mass $\varkappa$, we can define the spin S of this particle as follows:

$$q = 2S + 1. \tag{4.22}$$

Equation (4.22) shows that there are $(2S+1)$ states for the elementary particle at rest. We shall show in Ch. VII that these states correspond to the different eigenvalues $(S, S-1, \ldots, -S)$ of a specified component of an angular momentum operator, the spin angular momentum, and that a conservation law can be established for the sum of orbital and spin angular momenta. A special example of this was given in § 4 of Ch. III for an elementary particle with spin $\frac{1}{2}$.

EXAMPLE. THE WAVE EQUATION FOR A PARTICLE WITH SPIN 1

We shall consider the wave equation

$$\left.\begin{aligned} F_{\mu\nu} &= \partial_\mu U_\nu - \partial_\nu U_\mu, \\ \partial_\mu F_{\mu\nu} - \varkappa^2 U_\nu &= 0, \end{aligned}\right\} \tag{4.23a}$$

where U_μ is a vector and $\varkappa$ is a constant. This is the Proca equation (PROCA [1936]). The equation (4.23a) is equivalent to

$$\left.\begin{aligned} (\Box - \varkappa^2)\, U_\mu &= 0 \\ \partial_\mu U_\mu &= 0. \end{aligned}\right\} \tag{4.23b}$$

Thus, we see that $\varkappa$ denotes the mass of the particle. The second equation in (4.23b) shows that, in the rest system, $U_4 = 0$. Therefore, the number of independent components, q, is 3, and the spin, $S = 1$. Comparing (4.23a) and (4.23b) with (4.15a), (4.18b) and (4.19) we obtain:

$$\partial_{ts}\, \varphi_{\dot{u}}^{\ s} = i\varkappa\, \chi_{tu},$$
$$\partial^{rs}\, \chi_{su} = i\varkappa\, \varphi_{\dot{u}}^{\ r},$$

where U_μ, $F_{\mu\nu}$, $\varphi_{\dot{u}}^{\ s}$ and $\chi_{t\dot{u}}$ correspond to a_μ, $a_{\mu\nu}$, $a_{\dot{u}}^{\ s}$ and $(1/i\varkappa)\, a_{t\dot{u}}$ respectively. These are the equations for the particle with spin 1 in the spinor representation. With $\varkappa = 0$, (4.23) reads:

$$\left.\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ \partial_\mu F_{\mu\nu} &= 0. \end{aligned}\right\} \tag{4.24a}$$

These equations with the Lorentz condition

$$\partial_\mu A_\mu = 0 \tag{4.24b}$$

are the Maxwell equations. One essential difference between the theories with $\varkappa \neq 0$ and with $\varkappa = 0$ is that, in the latter case (4.24*b*) cannot be derived from (4.24*a*). Equations (4.24*a*) and (4.24*b*) are equivalent to

$$\left.\begin{aligned} \Box A_\mu &= 0 \\ \partial_\mu A_\mu &= 0. \end{aligned}\right\} \tag{4.24c}$$

The Maxwell equations are invariant under the transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda, \tag{4.25}$$

where Λ is a scalar function satisfying the relation

$$\Box \Lambda = 0.$$

This is called the gauge transformation.

We have seen that the particle of spin 1 can be described by a spinor of degree 2. In the next paragraph we shall show that the particle of spin S may be described by a spinor of degree $2S$.

§ 3. General Relativistic Wave Equations

We shall derive the equations for elementary particles of general kind by extending the spinor equation (3.82), referring to spin $\frac{1}{2}$, to refer to any spin.

Equations (3.82) can be extended to the general case of a spinor of degree n, $\varphi_{\dot{u}\ldots}^{st\ldots}$, $\chi_{t\dot{u}\ldots}^{t\ldots}$:—

$$\left.\begin{aligned} \partial_{ts}\, \varphi_{\dot{u}\ldots}^{st\ldots} &= i\varkappa\, \chi_{t\dot{u}\ldots}^{t\ldots} \\ \partial^{vs}\, \chi_{s\dot{u}\ldots}^{t\ldots} &= i\varkappa\, \varphi_{\dot{u}\ldots}^{vt\ldots} \end{aligned}\right\} \tag{4.26}$$

(see DIRAC [1936], FIERZ [1939], FIERZ and PAULI [1939]). From (4.26) and (4.12*c*, *d*) we have

$$\left.\begin{aligned} (\Box - \varkappa^2)\, \varphi_{\dot{u}\ldots}^{st\ldots} &= 0 \\ (\Box - \varkappa^2)\, \chi_{t\dot{u}\ldots}^{t\ldots} &= 0. \end{aligned}\right\} \tag{4.27}$$

We see that the K. G. Condition of Ch. II for relativistic wave equations is satisfied.

We now show if $\varphi^{st}_{\dot{u}\dots}$ and $\chi^{\dot{t}}_{r\dot{u}\dots}$ are symmetric spinors of degree n. they then describe particles of spin $(n/2)$. For the particle with non-vanishing mass ($\varkappa \neq 0$) we can take the Lorentz frame in which the particle is at rest and rewrite (3.81) in the form

$$\partial_{\dot{r}\dot{s}} = -\varkappa \delta_{\dot{r}\dot{s}}$$

by using $\partial_k \to 0$, $\partial_4 \to (1/i)\varkappa$. Thus in the rest system

$$\varphi^{rt}_{\dot{u}\dots} = -i\,\chi^{\dot{t}\dots}_{r\dot{u}\dots},$$

which leads to

$$\varphi^{st}_{\dot{u}\dots} = -i\,\chi^{\dot{t}\dots}_{s\dot{u}\dots} = -i\,\chi^{\dot{t}\dots}_{u\dot{s}\dots} = \varphi^{ut}_{\dot{s}\dots}.$$

In other words, $\varphi^{st}_{\dot{u}\dots}$ is symmetric with respect to dotted and undotted indices.

In the rest system the continuous Lorentz transformations are reduced to the space rotations. Thus in this case it is unnecessary to discriminate between dotted and undotted indices; we can regard $\varphi^{st}_{\dot{u}\dots}$ as $\varphi^{st\dots}_{u\dots}$.

Since each suffix and superfix can take 2 values (1 and 2), the number of independent components is

$$\binom{n+2-1}{2-1} = 2(n/2) + 1. \tag{4.28}$$

Equations (4.22) and (4.28) show that (4.23) corresponds to particles with spin $S = n/2$.

If the numbers of dotted and undotted suffices in $\varphi^{st}_{\dot{u}\dots}$ are $k-1$ and $l-1$ respectively, (where $k+l-2=n$), those in $\chi^{\dot{t}\dots}_{r\dot{u}\dots}$ are k and $l-2$ respectively. By taking $k = 1, 2, \dots, n$, we obtain $n(=2S)$ possible theories for elementary particles with spin S. Let us denote $\chi_{t_1\dots t_n}$ and $\varphi^{t}_{t_1\dots t_{n-1}}$ $(k=n,\ l=2)$ by $\overset{(1)}{\chi}$ and $\overset{(1)}{\varphi}$ respectively and introduce $(\overset{(q)}{\varphi}, \overset{(q)}{\chi})$ $(q = 1, \dots, n)$ as follows:

$$\left.\begin{aligned} \overset{(q)}{\varphi}{}^{t_{n-q+1}\dots t_n}_{t_1\dots t_{n-q}} &= -\frac{i}{\varkappa}\,\partial^{t_{n-q+1}\,r_{n-q+1}}\,\overset{(q-1)}{\varphi}{}^{t_{n-q+2}\dots t_n}_{r_1\dots r_{n-q+1}} \\ \overset{(q)}{\chi}{}^{t_{n-q+2}\dots t_n}_{t_1\dots t_{n-q+1}} &= \overset{(q-1)}{\varphi}{}^{t_{n-q+2}\dots t_n}_{t_1\dots t_{n-q+1}}. \end{aligned}\right\} \tag{4.29}$$

Then we can see that all $(\overset{(q)}{\varphi}, \overset{(q)}{\chi})$ satisfy (4.23) if $(\overset{(1)}{\varphi}, \overset{(1)}{\chi})$ does so. In

other words, n theories ($q=1, \ldots, n$) are equivalent to each other. However, as is shown in § 2, under space reflection $(\overset{(q)}{\varphi}, \overset{(q)}{\chi})$ are replaced by $(\overset{(n-q+1)}{\chi}, \overset{(n-q+1)}{\varphi})$. Thus we must take into account $(\overset{(q)}{\varphi}, \overset{(n-q+1)}{\chi}, \overset{(q)}{\chi}, \overset{(n-q+1)}{\varphi})$ in a general Lorentz transformation. The quantities $(\overset{(q)}{\varphi}, \overset{(q)}{\chi})$ and $(\overset{(n-q+1)}{\chi}, \overset{(n-q+1)}{\varphi})$ are equivalent to each other for half integer spin S, if $q=(n+1)/2$. We see that there are S (for $S=$integer) or $(S+\frac{1}{2})$ (for $S=$half-integer) theories of elementary particles with spin S.

If S is an integer we can use $(\overset{(S)}{\varphi}, \overset{(S)}{\chi}, \overset{(S+1)}{\varphi}, \overset{(S+1)}{\chi})$:

$$\left.\begin{aligned} &\overset{(S)}{\varphi} \equiv \varphi^{i_1 \ldots i_S}_{\dot{u}_1 \ldots \dot{u}_S}, \qquad \overset{(S+1)}{\varphi} \equiv \varphi^{i_1 \ldots i_{S-1} vt}_{\dot{u}_1 \ldots \dot{u}_{S-1}}, \\ &\overset{(S)}{\chi} \equiv \chi^{i_1 \ldots i_S}_{t\dot{u}\dot{u}_2 \ldots \dot{u}_S}, \qquad \overset{(S+1)}{\chi} \equiv \overset{(S)}{\varphi}. \end{aligned}\right\} \tag{4.30}$$

Since $\overset{(S)}{\chi}_{t\dot{u}\ldots}$ and $\overset{(S+1)}{\varphi}_{\ldots}{}^{vt}$ are symmetrical with respect to the suffices $(\dot{r}\dot{u})$ and superfices (vt) respectively, it can be transformed, according to (4.14a) and (4.8a), into a tensor $F_{[\mu,\nu]\mu_3 \ldots \mu_S}$ which is antisymmetrical with respect to the suffices (μ, ν). Taking into account the fact that (4.17) has the same forms as (4.26) with respect to the suffices $(\dot{r}\dot{u})$ and the superfices (vt) we can see that (4.26) can be rewritten as tensor equations similar to (4.15a) and (4.18b), namely

$$\partial_\nu U_{\mu\mu_2 \ldots \mu_S} - \partial_\mu U_{\nu\mu_2 \ldots \mu_S} = F_{[\nu,\mu]\mu_2 \ldots \mu_S}, \tag{4.31a}$$

$$\partial_\mu U_{\mu\mu_2 \ldots \mu_S} = 0. \tag{4.32}$$

Since (4.27) leads to

$$(\Box - \varkappa^2)\, U_{\mu_1 \ldots \mu_S} = 0. \tag{4.33}$$

we have

$$\partial_\nu F_{[\nu,\mu]\mu_2 \ldots \mu_S} = \varkappa^2 U_{\mu\mu_2 \ldots \mu_S}. \tag{4.31b}$$

The quantity $U_{\mu_1 \ldots \mu_S}$ is a symmetric tensor, because $\varphi^{i_1 \ldots i_S}_{\dot{u}_1 \ldots \dot{u}_S}$ is a symmetric spinor. Moreover, using (4.10c) we have

$$U_{\mu\mu\mu_3 \ldots \mu_S} = 0. \tag{4.31c}$$

On the other hand, it is easily proved that (4.26) can be derived from (4.31a, b, c) by using (4.9). Thus we see that (4.31a, b, c) give a theory equivalent to that of (4.26) and therefore one of elementary particles with spin S. This fact may also be proved by counting the number of independent components of $U_{\mu_1 \ldots \mu_S}$ in the rest system.

Equations (4.31a, b, c) are equivalent to

$$\left.\begin{aligned}&(\Box - \varkappa^2)\, U_{\mu_1 \ldots \mu_S} = 0\\ &\partial_\mu U_{\mu\mu_1 \ldots \mu_S} = 0\\ &U_{\mu\mu\mu_1 \ldots \mu_S} = 0.\end{aligned}\right\} \tag{4.34}$$

In particular, for spin $S=1$, (4.31c) becomes meaningless and (4.31a, b) become

$$\left.\begin{aligned}&\partial_\mu U_\nu - \partial_\nu U_\mu = F_{\mu\nu}\\ &\partial_\mu F_{\mu\nu} - \varkappa^2 U_\nu = 0,\end{aligned}\right\} \tag{4.35a}$$

where $F_{[\mu,\nu]}$ has been denoted by $F_{\mu\nu}$. With $S=1$, (4.34) also becomes

$$\left.\begin{aligned}&(\Box - \varkappa^2)\, U_\mu = 0\\ &\partial_\mu U_\mu = 0.\end{aligned}\right\} \tag{4.35b}$$

We have seen that the wave equations (4.26) for $(\overset{(S)}{\varphi}, \overset{(S)}{\chi})$ can be written in terms of the tensors $(U_{\mu_1 \ldots \mu_S}, F_{[\nu,\mu]\mu_1 \ldots \mu_S})$ and take the form (4.31a, b, c). On the other hand, (4.15), (4.17) and (4.29) show that the spinors $(\overset{(S-1)}{\varphi}, \overset{(S-1)}{\chi})$ correspond to tensors

$$(F_{[\nu,\mu]\mu_1 \ldots \mu_S},\ F_{[\nu_1,\mu_1],[\nu_2,\mu_2]\mu_3 \ldots \mu_S}).$$

The last tensor is defined by

$$\left.\begin{aligned}&F_{[\nu_1,\mu_1][\nu_2,\mu_2]\mu_3 \ldots \mu_S}\\ &\quad = \partial_{\nu_2} F_{[\nu_1,\mu_1]\mu_2\mu_3 \ldots \mu_S} - \partial_{\mu_2} F_{[\nu_1,\mu_1]\nu_2\mu_3 \ldots \mu_S}\\ &\quad = \partial_{\mu_2\nu_2:\mu'} F_{[\nu_1,\mu_1]\mu'\mu_3 \ldots \mu_S},\end{aligned}\right\} \tag{4.37}$$

where

$$\partial_{\mu\nu:\mu'} \equiv \partial_\mu \delta_{\mu'\nu} - \partial_\nu \delta_{\mu'\mu}. \tag{4.38}$$

In this way, we can show that, corresponding to S possible formulations based on $(\overset{(q)}{\varphi}, \overset{(q)}{\chi}, \overset{(n-q+1)}{\varphi}, \overset{(n-q+1)}{\chi})$ for the theory of an integer spin S, we have the same number of possible tensor theories, which are derived from each other by means of the operator $\partial_{\mu\nu:\mu'}$.

We shall use $\varphi^{s i_1 \ldots i_k}_{\dot{u}_1 \ldots \dot{u}_k}$ and $\chi_{\dagger \dot{u}_1 \ldots \dot{u}_k}^{i_1 \ldots i_k}$ for the case of half integer spin $S = k + \frac{1}{2}$ and non-vanishing mass ($\varkappa \neq 0$). We introduce the quantities

$$\left.\begin{aligned}&\psi^s_{\mu_1 \ldots \mu_k} = \prod_{i=1}^{k} \left(\tfrac{1}{2}\, \sigma_{\mu_i,\, i_i \dot{u}_i}\right) \varphi^{s i_1 \ldots i_k}_{\dot{u}_1 \ldots \dot{u}_k}\\ &\psi_{\dagger:\mu_1 \ldots \mu_k} = \prod_{i=1}^{k} \left(\tfrac{1}{2}\, \sigma_{\mu_i,\, i_i \dot{u}_i}\right) \chi_{\dagger \dot{u}_1 \ldots \dot{u}_k}^{i_1 \ldots i_k}\end{aligned}\right\} \tag{4.39}$$

where $\psi^1_{\mu_1 \ldots \mu_k}$ and $\psi^2_{\mu_1 \ldots \mu_k}$ are tensors of degree k with respect to suffices μ_i, and spinors of degree 1 with respect to suffices $(s, \dot{r})$. We shall denote them by $\psi_{\mu_1 \ldots \mu_k}$ of the form

$$\psi_{\mu_1 \ldots \mu_k} = \begin{bmatrix} \psi^1_{\mu_1 \ldots \mu_k} \\ \psi^2_{\mu_1 \ldots \mu_k} \\ \psi_{1:\mu_1 \ldots \mu_k} \\ \psi_{2:\mu_1 \ldots \mu_k} \end{bmatrix}. \tag{4.40}$$

If $k = 0$, this is equivalent to a ψ of spin $\frac{1}{2}$. In general, representation by $\psi_{\mu_1 \ldots \mu_k}$ is the so-called Rarita–Schwinger formalism (RARITA and SCHWINGER [1941]). The tensor $\psi_{\mu_1 \ldots \mu_k}$ is symmetric with respect to suffices $\mu_1 \ldots \mu_k$.

The wave equation (4.26) may be written in the form

$$(\gamma_\mu \partial_\mu + \varkappa)\, \psi_{\mu_1 \ldots \mu_k} = 0. \tag{4.41}$$

The γ_μ are the matrices with four rows and columns that satisfy the relation (3.2). Since $\varphi^{s\dot{t}_1 \ldots \dot{t}_k}_{\dot{u}_1 \ldots \dot{u}_k}$ is a symmetric spinor, (4.5) gives

$$\varphi^{s\dot{t}_1 \ldots \dot{t}_k}_{\dot{u}_1 \ldots \dot{u}_k} = 0. \tag{4.42}$$

From this, (3.68) and (3.80) we can derive:

$$\gamma_\mu \psi_{\mu\mu_1 \ldots \mu_k} = 0. \tag{4.43}$$

On the other hand we can show that (4.26) can be derived from (4.41) and (4.43). Therefore (4.41) and (4.43) provide a theory for spin $S = k + \frac{1}{2}$.

From (4.41) and (4.43) can be derived

$$(\Box - \varkappa^2)\, \psi_{\mu_1 \ldots \mu_k} = 0, \tag{4.44}$$

$$\partial_\mu \psi_{\mu\mu_1 \ldots \mu_k} = 0, \tag{4.45}$$

$$\psi_{\mu\mu\mu_1 \ldots \mu_k} = 0. \tag{4.46}$$

We now consider the elementary particle with zero mass ($\varkappa = 0$). The wave equation (4.34) gives for $\varkappa = 0$:

$$\left.\begin{aligned} \Box A_{\mu_1 \ldots \mu_S} &= 0 \\ \partial_\mu A_{\mu\mu_2 \ldots \mu_S} &= 0 \\ A_{\mu\mu\mu_3 \ldots \mu_S} &= 0, \end{aligned}\right\} \tag{4.47}$$

where $A_{\mu_1 \ldots \mu_S}$ is a symmetric tensor of degree S. Equation (4.47) is equivalent to:

$$\left.\begin{aligned} &F_{[\nu,\mu]\mu_2\ldots\mu_S} = \partial_\nu A_{\mu\mu_2\ldots\mu_S} - \partial_\mu A_{\nu\mu_2\ldots\mu_S} \\ &\partial_\nu F_{[\nu,\mu]\mu_2\ldots\mu_S} = 0 \\ &\partial_\mu A_{\mu\mu_2\ldots\mu_S} = 0 \\ &A_{\mu\mu\mu_3\ldots\mu_S} = 0. \end{aligned}\right\} \tag{4.48}$$

One essential difference between the theories with $\varkappa \neq 0$ and with $\varkappa = 0$ is that, in the latter case, the third equation of (4.48) cannot be derived from the others.

Putting $S = 1$ in (4.48) we obtain Maxwell equations. The third equation of (4.48) gives the Lorentz condition:

$$\left.\begin{aligned} &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ &\partial_\mu F_{\mu\nu} = 0 \\ &\partial_\mu A_\mu = 0 \qquad \text{(Lorentz condition).} \end{aligned}\right\} \tag{4.49}$$

Since there is no rest system when $\varkappa = 0$, we cannot use the definition of spin given in § 2. Therefore, for $\varkappa = 0$, we shall define the spin as the magnitude of highest eigenvalue of the spin angular momentum, which will be introduced in Example 1 of Ch. VII. It can be shown that the spin thus defined is equal to the degree S of the tensor.

We now introduce the symmetric tensor $N_{\mu_1 \ldots \mu_S}$ of the degree S — namely

$$N_{\mu_1\ldots\mu_S} = \partial_{\mu_1} C_{\mu_2\ldots\mu_S} + \partial_{\mu_2} C_{\mu_1\mu_3\ldots\mu_S} + \ldots + \partial_{\mu_S} C_{\mu_1\ldots\mu_{S-1}}. \tag{4.50}$$

Here $C_{\mu_1 \ldots \mu_{S-1}}$ forms a symmetric tensor of degree $(S-1)$ satisfying the equation (4.47). We then see that the wave equations (4.47) are invariant under transformation

$$A_{\mu_1\ldots\mu_S} \to A'_{\mu_1\ldots\mu_S} = A_{\mu_1\ldots\mu_S} + N_{\mu_1\ldots\mu_S}. \tag{4.51}$$

This is called a gauge transformation. Thus the theory of an elementary particle with vanishing mass ($\varkappa = 0$) is invariant under a gauge transformation. This is not so for particles with $\varkappa \neq 0$. Since the representations connected by a gauge transformation are physically equivalent to each other, the number of independent components of the wave functions is decreased by the number of independent components of $N_{\mu_1\ldots\mu_S}$.

We shall consider the plane wave with propagation vector $k_\mu = (0, 0, k, ik)$ and progressing in the x_3-direction. Then the second equation in (4.47) gives

$$i A_{4\mu_2 \ldots \mu_S} = - A_{3\mu_2 \ldots \mu_S}. \tag{4.52}$$

Since $A_{\mu_1 \ldots \mu_S}$ is a symmetric tensor, the number of independent relations given by (4.52) is

$$\binom{S-1+4-1}{S-1}.$$

On the other hand, it is easily seen that in the third equation of (4.47), the number of relations that are independent of (4.52) is

$$\binom{S-2+3-1}{S-1}.$$

Thus, taking into account the fact that $A_{\mu_1 \ldots \mu_S}$ is a symmetric tensor, we obtain the following formula for the number of independent components of

$$\binom{S+4-1}{S} - \binom{S+2}{S-1} - \binom{S}{S-2} = 2S+1. \tag{4.53}$$

Then, the number of independent components of $C_{\mu_1 \ldots \mu_{S-1}}$ is $2(S-1)+1$, for $C_{\mu_1 \ldots \mu_{S-1}}$ also satisfies the equation (4.47). Since the representations connected by a gauge transformation (4.51) are physically equivalent to each other, the number of independent states is given as 2 by subtracting the number of independent components of $C_{\mu_1 \ldots \mu_{S-1}}$ from (4.53):

$$(2S+1) - (2S-1) = 2. \tag{4.54}$$

This is true also for the cases of half integer spin.

An example of a theory with $\varkappa = 0$ is the electromagnetic field. In this case the Maxwell equations are invariant under the gauge transformation

$$\left.\begin{aligned} A_\mu \to A'_\mu &= A_\mu + \partial_\mu \Lambda \\ \Box \Lambda &= 0. \end{aligned}\right\} \tag{4.55}$$

It is easily seen that (4.55) is a particular case of (4.51). Moreover, as is well known, electromagnetic waves are transverse waves with two components. This fact agrees with (4.54).

Thus we have seen that the equations (4.26) are reasonable general wave equations, which include the Dirac and Maxwell equations as special cases. However, since they are derived by a formal extension of the Dirac equation, it is necessary to discuss the theoretical basis of the latter. We shall give a more detailed discussion of relativistic wave equations in Ch. V.

EXAMPLE. RARITA–SCHWINGER FORMALISM FOR SPIN 3/2

The wave equations for ψ_μ given by (4.41) and (4.43), are

$$\left.\begin{aligned}(\gamma_\mu \partial_\mu + \varkappa)\,\psi_\nu &= 0\\ \gamma_\mu \psi_\mu &= 0.\end{aligned}\right\} \tag{4.56a}$$

It follows that

$$(\Box - \varkappa^2)\,\psi_\mu = 0,$$
$$\partial_\mu \psi_\mu = 0.$$

It can be shown that the equations (4.56a) are equivalent to

$$-[(\gamma_\varrho \partial_\varrho + \varkappa)\,\delta_{\mu\nu} - \tfrac{1}{3}(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu) + \tfrac{1}{3}\gamma_\mu(\gamma_\varrho \partial_\varrho - \varkappa)\,\gamma_\nu]\,\psi_\nu(x) = 0. \tag{4.56b}$$

In fact, multiplying by γ_μ or ∂_μ on the left side of (4.56b) we have

$$\tfrac{2}{3}\partial_\nu \psi_\nu - \tfrac{1}{3}\varkappa \gamma_\nu \psi_\nu = 0, \tag{4.56c}$$

and

$$\varkappa \partial_\mu \psi_\mu + \partial_\mu \gamma_\mu (\tfrac{2}{3}\partial_\nu \psi_\nu - \tfrac{1}{3}\varkappa \gamma_\nu \psi_\nu) = 0, \tag{4.56d}$$

respectively. (4.56a) can be derived from (4.56b), (4.56c) and (4.56d). Substituting

$$\Lambda_{\mu\nu}(\partial) = -[(\gamma_\varrho \partial_\varrho + \varkappa)\,\delta_{\mu\nu} - \tfrac{1}{3}(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu) + \tfrac{1}{3}\gamma_\mu(\gamma_\varrho \partial_\varrho - \varkappa)\,\gamma_\nu]$$

and (2.8) into (2.7), we obtain $d_{\mu\nu}(\partial)$ namely,

$$\left.\begin{aligned}d_{\mu\nu}(\partial) = &-(\gamma_\varrho \partial_\varrho - \varkappa)\Big[\delta_{\mu\nu} - \frac{1}{3}\gamma_\mu \gamma_\nu\\ &+ \frac{1}{3\varkappa}(\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu) + \frac{2}{3\varkappa^2}\partial_\mu \partial_\nu\Big]\\ &+ \frac{1}{3\varkappa^2}(\Box - \varkappa^2)\,[(\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu) + (\gamma_\varrho \partial_\varrho - \varkappa)\,\gamma_\mu \gamma_\nu]\end{aligned}\right\} \tag{4.56e}$$

(TAKAHASHI and UMEZAWA [1953]).

§ 4. Half-Integer Spin and Negative Energy

As shown in § 2 and § 3, the wave functions for the cases of integer and half-integer spins are $U(k, l)$ with $(k+l)$ even and odd respectively,

We first consider the case with $(k+l)$ even, and introduce the following symbols:

$$U(k,l) = \left\{ \begin{matrix} U^- & \text{for even } k,l \\ U^+ & \text{for odd } k,l. \end{matrix} \right\} \tag{4.57}$$

It is easily seen by use of the Clebsch–Gordon theorem that $U(k,l) \times U(k',l')$ can be represented by a linear combination of $U(k'',l'')$ with

$$\left. \begin{aligned} k'' &= k+k'-1,\ k+k'-3,\ \ldots,\ |k-k'|+1 \\ l'' &= l+l'-1,\ l+l'-3,\ \ldots,\ |l-l'|+1. \end{aligned} \right\} \tag{4.58}$$

It follows that

$$\left. \begin{aligned} &U^+\, U^+ \doteqdot U^-\, U^- \doteqdot U^+ \\ &U^+\, U^- \doteqdot U^-, \end{aligned} \right\} \tag{4.59}$$

where $\doteqdot$ means the same type with respect to $(\pm)$. The relation (4.59) shows that the even degree tensors are of the U^+-type, since they transform like a product of an even number of vectors $U(2,2) \doteqdot U^-$. Similarly, the odd degree tensors belong to the U^--type. In particular, a vector ∂_μ is of the U^--type.

When we write the relativistic wave equations as differential equations of the first degree (cf. Ch. II), they take the form

$$\partial\, U^+ = U^-, \quad \partial\, U^- = U^+, \tag{4.60}$$

where the relations (4.59) have been taken into account. In (4.60), ∂ means the derivation operator ∂_μ. It is easily seen that (4.60) is invariant under the transformation

$$\left. \begin{aligned} &\partial_\mu \to -\partial_\mu \\ &U^+ \to U^+ \\ &U^- \to -U^- \end{aligned} \right\} \tag{4.61}$$

It must be noted that the first transformation in (4.61) is equivalent to a total inversion of the axes. Under this transformation, odd degree tensors change sign, even degree tensors do not. Thus we see that odd degree tensors composed of wave functions of integer spin cannot have a definite sign. Current density which will be introduced in Ch. VII is an example of an odd degree tensor.

In the case of half integer spin we introduce the symbols

$$U(k,l) = \left\{ \begin{matrix} U^{+\prime} & k = \text{odd},\ l = \text{even} \\ U^{-\prime} & k = \text{even},\ l = \text{odd}. \end{matrix} \right\} \tag{4.62}$$

By using (4.58) we have

$$\left.\begin{aligned} &U^{+\prime}\,U^{+\prime} \doteqdot U^{-\prime}\,U^{-\prime} \doteqdot U^{+} \\ &U^{+\prime}\,U^{-\prime} \doteqdot U^{-} \end{aligned}\right\} \tag{4.63}$$

where $U^{\pm}$ is defined in (4.57). Equation (4.63) gives the wave equations

$$\left.\begin{aligned} &\partial\, U^{+\prime} = U^{-\prime} \\ &\partial\, U^{-\prime} = U^{+\prime} \end{aligned}\right\} \tag{4.64}$$

It is easily seen that (4.64) is invariant under the transformation

$$\left.\begin{aligned} &\partial_\mu \to -\partial_\mu \\ &U^{+\prime} \to i\, U^{+\prime} \\ &U^{-\prime} \to -i\, U^{-\prime}. \end{aligned}\right\} \tag{4.65}$$

The imaginary factor i appears in (4.65) because (4.62) shows that the complex conjugate of $U^{+\prime}$ is $U^{-\prime}$, so that we cannot take $U^{+\prime} \to U^{+\prime}$, $U^{-\prime} \to -U^{-\prime}$.

We can see from (4.63) that the even degree tensor U^{+} and the odd degree tensor U^{-} have the structures $(U^{+\prime}U^{+\prime},\ U^{-\prime}U^{-\prime})$ and $(U^{+\prime}U^{-\prime})$ respectively. Since the former changes sign under (4.65) and the latter does not, even degree tensors made of wave functions of half integer spin have no definite sign. Since the energy which we shall introduce in Ch. VII is the space integral of T_{44}, we obtain the following important theorem:—

The energy of elementary particles with half integer spin has no definite sign (PAULI [1939]). *The negative energy state in the case of spin* $\frac{1}{2}$ *is an example of this theorem.*

We must therefore adopt the hole theory in any case of half-integer spin in order to avoid the Klein paradox. This fact requires that elementary particles with half-integer spin should obey Fermi–Dirac statistics. This requirement is further discussed in the quantum field theory in Ch. VI.

§ 5. The Properties of Known Elementary Particles

The particles whose properties are well known are tabulated below [1]).

The determination of the spins of the π- and π^0-mesons is discussed in Ch. X and Ch. XIII. It is not certain that the proton obeys the Dirac

[1]) Since the experimental information on elementary particles is increasing rapidly, the mass values in this table must be regarded as provisional.

TABLE I

	Photon	Electron e	Proton P	Neutron N	Neutrino	π-meson	μ-meson	π^0-meson
Charge	0	$\pm e$	$+e$	0	0	$\pm e$	$\pm e$	0
Spin	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	half-integer	0	$\frac{1}{2}$	0
Mass	0	m	1836 m	1838 m	$\ll$ m	276 m	215 m	266 m

equation (3.1). One difficulty is that a negative proton has not been discovered. The π^0-meson decay supports, but not conclusively, the use of the Dirac equation for protons (cf. Example 2 in Ch. XIII). The mutual interactions between the above elementary particles are discussed in Ch. VII.

Neutrons, π-mesons, μ-mesons and π^0-mesons decay naturally according to the following schemes, and with the lifetimes indicated: –

$$N \to P + e + \nu, \qquad \sim 12 \text{ minutes}$$
$$\pi \to \mu + (\text{light neutral particle}), \qquad \sim 10^{-8} \text{ sec.}$$
$$\mu \to e + (\text{light neutral particles, the number of which is} \geqq 2), \qquad \sim 10^{-6} \text{ sec.}$$
$$\pi^0 \to \gamma + \gamma, \qquad < 10^{-11} \text{ sec.}$$

There is a high probability for the absorption of negative π-mesons in heavy nuclei (cf. Example 1 in Ch. VII). This provides a convenient method for determining the sign of the charge of π-meson. The interesting fact that there is a slight difference, i.e. 2.47 m, between the masses of the proton and the neutron is shown in the above Table. The proton and the neutron are treated as different states of the same particle, i.e. the nucleon (cf. Example 10 in Ch. VII). On this viewpoint it is an interesting problem to explain the $P-N$ mass difference. Since, as is shown in Ch. XIII, the electromagnetic field around the proton makes a contribution to the proton mass, it seems to be natural that the electromagnetic effect is the origin of $P-N$ mass difference. However, since the present quantum field theory gives an infinite result for this mass difference, it cannot give any conclusive answer to this problem [1].

[1] C-meson theory gives a mass difference, but its conclusion is still not definite. (SAKATA and UMEZAWA [1950], KAWABE and UMEZAWA [1949], ENATSU [1951], PAIS [1947].

Recently a new approach to this problem has been given by WEISKOPF [1954], FEYNMAN and SPEISMAN [1954] and PETERMANN [1954].

Recently many new particles have been discovered and they are called Λ, θ, k, τ particles, etc. Their mass and decay schemes are given in the following table (see, for example, POWELL [1952a], [1952b] Report of the Fifth Rochester Conference [1955]).

TABLE II

	Mass	Decay scheme	Natural lifetime
θ^0	$\sim$ 960 m	$\pi^+ + \pi^-$	$\sim 1.5 \times 10^{-10}$ sec.
$\tau^\pm$	$\sim$ 960 m	$\pi^\pm + \pi^+ + \pi^-$	$\sim 5 \times 10^{-9}$ sec.
$k_{\mu 2}$	$\sim$ 960 m	$\mu + ?$	$\sim 10^{-8}$ sec.
$k_{\pi 2}$	$\sim$ 960 m	$\pi^\pm + \pi^0$	$\sim 10^{-8}$ sec.
$k_{\mu 3}$	$\sim$ 960 m	$\mu + ? + ?$	$\sim 10^{-8}$ sec.
k_{e3}	$\sim$ 960 m	$e + ? + ?$	$\sim 10^{-8}$ sec.
Λ^0	2190 ± 10 m	$P + \pi^-$	$\sim 3 \times 10^{-10}$ sec.
$\Sigma^\pm$	$\sim$ 2325 m	$\begin{cases} n + \pi^\pm \\ P + \pi^0 \end{cases}$	$\sim 5 \times 10^{-10}$ sec.
Ξ^-	2570 m	$\Lambda^0 + \pi^-$	$\sim 10^{-10}$ sec.

The symbols 0, + and − mean that the particle is neutral, positively charged and negatively charged respectively; "?" means that the properties of the neutral particles have not yet been clarified. Moreover, it is possible that many as yet undiscovered particles exist in nature. For example, some particles have not yet been observed because of their short lifetimes. At any rate we can see that the relation between the elementary particles in nature is not simple. Thus, we are faced with many questions: How many sorts of elementary particles are there in nature? Why are they selected from the infinitely wide framework of elementary particles given by the present quantum field theory? How can we find the relations between and the synthetic aspects of the elementary particles? These are questions which must be answered by the theory of elementary particles; in fact the investigation of these questions may lead us to the theory. It was the investigation of the inherent relations between the atoms which led to the discovery of their structure and of quantum mechanics.

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Note added in proof

Discussions in § 5 will be supplemented by Note added in proof in Ch. XVIII. Here it is noted only that the Berkeley group (1956) succeeded to establish the existence of the negative proton.

CHAPTER V

THE GENERAL RELATIVISTIC WAVE EQUATION (II)

§ 1. The General Relativistic Wave Equation

In the previous Chapter we obtained relativistic wave equations (4.26) by extending the Dirac equations of the spinor form (3.82). We shall now discuss in detail the general relativistic equation by means of discussions given in Ch. II. This has provided a theoretical basis for the Dirac equation. The Duffin–Kemmer–Petiau theory for particles of spin 0 or 1 will also be seen to be a particular example of the general equations; this theory is discussed in § 2.

As shown in Ch. II the relativistic field equation can be written in the form:

$$(\beta_\mu \partial_\mu + \varkappa)\psi = 0. \tag{5.1}$$

We assume that under the Lorentz transformation

$$'x_\mu = a_{\mu\nu} x_\nu \tag{5.2}$$

ψ is linearly transformed according to

$$'\psi = \Lambda\psi. \tag{5.3}$$

We can see by a discussion analogous to that in Ch. III that the Lorentz invariance of equation (5.1) requires the following relations for β_μ:

$$\Lambda^{-1}\beta_\mu \Lambda = a_{\mu\nu}\beta_\nu. \tag{5.4}$$

In particular, for the infinitesimal Lorentz transformations

$$\left.\begin{aligned} a_{\mu\nu} &= \delta_{\mu\nu} + \delta w_{\mu\nu} \\ \delta w_{\mu\nu} &= -\delta w_{\nu\mu} \end{aligned}\right\} \tag{5.5a}$$

the matrix Λ has the form

$$\Lambda = 1 + \tfrac{1}{2} S_{\mu\nu}\delta w_{\mu\nu}, \tag{5.5b}$$

with

$$S_{\nu\mu} = -S_{\mu\nu}. \tag{5.5c}$$

By substituting (5.5*b*) and (5.5*a*) into (5.4), we obtain

$$[\beta_\mu, S_{\lambda\nu}] = \delta_{\lambda\mu}\beta_\nu - \delta_{\nu\mu}\beta_\lambda. \tag{5.6}$$

In general, every tensor in β-algebra can be written as a product of the β_μ and $\delta_{\mu\nu}$ (HARISH-CHANDRA [1947]). Since (5.6) shows that $S_{\mu\nu}$ is a 2nd degree tensor, we obtain from (5.4)

$$A^{-1} S_{\mu\nu} A = a_{\mu\varrho}\, a_{\nu\sigma}\, S_{\varrho\sigma}. \tag{5.7}$$

Substituting (5.5*a*) and (5.5*b*) into (5.7) we have:

$$[S_{\mu\nu}, S_{\varrho\sigma}] = -\delta_{\mu\varrho} S_{\nu\sigma} + \delta_{\nu\varrho} S_{\mu\sigma} + \delta_{\mu\sigma} S_{\nu\varrho} - \delta_{\nu\sigma} S_{\mu\varrho}. \tag{5.8}$$

It can be proved (HEPNER [1951]) that there is only one $S_{\mu\nu}$ which satisfies (5.6) and (5.8). In fact, (5.6) shows that the quantity $t_{\mu\nu} = S^{(1)}_{\mu\nu} - S^{(2)}_{\mu\nu}$, in which $S^{(1)}_{\mu\nu}$ and $S^{(2)}_{\mu\nu}$ satisfy (5.6) and (5.8), commutes with every β_μ and so with all quantities in β_μ-algebra. Thus we have

$$[S^{(1)}_{\mu\nu}, S^{(1)}_{\varrho\sigma}] - [S^{(2)}_{\mu\nu}, S^{(2)}_{\varrho\sigma}] = [S^{(1)}_{\mu\nu}, t_{\varrho\sigma}] + [t_{\mu\nu}, S^{(2)}_{\varrho\sigma}] = 0.$$

On the other hand, (5.8) gives

$$[S^{(1)}_{\mu\nu}, S^{(1)}_{\mu\sigma}] - [S^{(2)}_{\mu\nu}, S^{(2)}_{\mu\sigma}] = -2\, t_{\nu\sigma}$$

which shows that $t_{\nu\sigma} = 0$ so that $S^{(1)}_{\nu\sigma} = S^{(2)}_{\nu\sigma}$. It is easily seen by calculation that the $S_{\mu\nu}$ for spin $\frac{1}{2}$, (equation (3.24)) satisfy (5.8).

Since spur $(\beta_{\mu_1} \beta_{\mu_s})$ has the same transformation properties as the tensor with suffices $(\mu_1 \ldots \mu_s)$, it must be constituted of $\delta_{\mu\nu}$. It is impossible to obtain odd degree tensors constituted by combining $\delta_{\mu\nu}$ and therefore the spur of the product of an odd number of β_μ's is zero, that is,

$$\mathrm{Sp}\,(\beta_{\mu_1} \ldots \beta_{\mu_{2n+1}}) = 0 \tag{5.9a}$$

(HARISH-CHANDRA [1947]). In particular,

$$\mathrm{Sp}(\beta_\mu) = 0. \tag{5.9b}$$

We have found an example of this general theorem for spin 1/2 in Ch. III.

We now introduce spin matrices σ_k $(k = 1, 2, 3)$ by means of

$$\sigma_k = -iS_{lm} \quad (k, l, m = \text{cycl. } (1, 2, 3)). \tag{5.10}$$

Then, (5.8) gives

$$[\sigma_k, \sigma_l] = i\sigma_m \quad (k, l, m = \text{cycl. } (1, 2, 3)). \tag{5.11}$$

An example [1]) of (5.11) is provided by (3.55) which is valid for spin 1/2.

Equation (5,11) is just the commutation relation of the angular momentum matrices. Therefore, σ_k have the properties of the angular momentum matrix, and have eigenvalues $(f, f-1, \ldots, -f)$, where f is an integer or half-integer (DIRAC [1947]). Thus, the characteristic equation of σ_k is:

$$(\sigma_k - f)(\sigma_k - f + 1) \ldots (\sigma_k + f) = 0. \tag{5.12}$$

In general, representations of the angular momentum matrices σ_k satisfying (5.12) can be decomposed into irreducible representations $D_{|f|}, D_{|f-1|}, \ldots$ of the three-dimensional rotation group; (VAN DER WAERDEN [1932]); these irreducible representations correspond to the values $|f|, |f-1|, \ldots$ of the angular momentum. Since, in an irreducible representation D_l, the wave function ψ has $(2l+1)$ independent components corresponding to various directions of the angular momentum, (4.22) shows that ψ describes a state of a particle with the spin $S=l$. Thus, we see that the wave function ψ in (5.1) describes states of particles with spin $|f|, |f-1|, \ldots$. This angular momentum is an intrinsic property of particles, because it does not vanish even when particles are at rest. However, (5.6) and (5.8) are not sufficient to determine completely the β-algebra. We now give two important cases of the relativistic wave equations.

CASE I

The wave function ψ satisfies the Klein–Gordon equation (2.3a)

$$(\Box - \varkappa^2)\psi = 0 \tag{5.13}$$

and therefore describes states of a particle with unique mass $\varkappa$. Then, as shown in Ch. II, there must exist a derivation operator matrix $d(\partial) = [d_{\alpha\beta}(\partial)]$ which satisfies the relation

$$A(\partial)\, d(\partial) = (\Box - \varkappa^2) I, \tag{5.14}$$

$$A(\partial) = -(\beta_\mu \partial_\mu + \varkappa). \tag{5.15}$$

Here I is the unit matrix. The quantity $d(\partial)$ has the form given in (2.8) and (2.9). By substituting (2.8) into (5.14) we obtain the recurrence relation

[1]) It must be noted that the definitions of σ_k in (3.49) and (5.11) differ by a factor $(\frac{1}{2})$.

$$\left.\begin{aligned} &\alpha = \varkappa I \\ &\alpha\beta_\mu + \varkappa\alpha_\mu = 0 \\ &(\beta_\mu\alpha_\nu + \beta_\nu\alpha_\mu) + 2\varkappa\alpha_{\mu\nu} = -2\delta_{\mu\nu} \\ &\Sigma^{(P)}(\beta_{\mu_1}\alpha_{\mu_2\,.\,\mu_l} - \varkappa\alpha_{\mu_1\,..\,\mu_l}) = 0 \qquad \text{for } l>2 \end{aligned}\right\} \tag{5.16}$$

where $\Sigma^{(P)}$ indicates a summation over terms given by taking all possible permutations of the suffices.

Equation (5.16) may be solved to give

$$\left.\begin{aligned} &\alpha = \varkappa I \\ &\alpha_\mu = -\beta_\mu \\ &\alpha_{\mu\nu} = -\frac{1}{\varkappa}[\delta_{\mu\nu} - \tfrac{1}{2}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu)] \\ &\alpha_{\mu_1\ \mu_l} = \left(\frac{-1}{\varkappa}\right)^{l-1}\frac{1}{l!}\Sigma^{(P)}\beta_{\mu_1}\dots\beta_{\mu_{l-2}}[\delta_{\mu_{l-1}\mu_l} - \beta_{\mu_{l-1}}\beta_{\mu_l}] \quad \text{for } l>2. \end{aligned}\right\} \tag{5.17}$$

Thus, we have

$$\left.\begin{aligned} d(\partial) &= \varkappa I - \beta_\mu\partial_\mu - \frac{1}{\varkappa}[\Box - \beta_\mu\beta_\nu\partial_\mu\partial_\nu] + \dots \\ &\quad + \left(\frac{-1}{\varkappa}\right)^{l-1}[\beta_{\mu_2}\dots\beta_{\mu_l}\Box - \beta_{\mu_1}\dots\beta_{\mu_l}\partial_{\mu_1}\partial_{\mu_2}]\partial_{\mu_3}\dots\partial_{\mu_l} - \dots \\ &= \varkappa I - \beta_\mu\partial_\mu - \frac{1}{\varkappa}[\Box - (\beta_\varrho\partial_\varrho)^2] + \dots \\ &\quad + \left(\frac{-1}{\varkappa}\right)^{l-1}[\Box - (\beta_\varrho\partial_\varrho)^2](\beta_\varrho\partial_\varrho)^{l-2} + \dots \end{aligned}\right\} \tag{5.18a}$$

(UMEZAWA and VISCONTI [1955]).

Equation (2.9) leads to [1])

$$\Sigma^{(P)}\beta_{\mu_1}\dots\beta_{\mu_{b-1}}[\delta_{\mu_b\mu_{b+1}} - \beta_{\mu_b}\beta_{\mu_{b+1}}] = 0 \tag{5.18b}$$

on account of (5.17) (UMEZAWA and VISCONTI [1955]).

This is the equation that the matrices β_μ must satisfy for a finite integer b in order that (2.3a) can be derived from the wave equation (5.1).

We shall now prove an important theorem concerning the properties of $d(\partial)$ (UMEZAWA [1952]).

The order of the differential operator $d(\partial)$, called b is given by

$$b = 2f, \qquad \text{if } \varkappa \neq 0, \tag{5.19}$$

[1]) This equation was obtained by HARISH-CHANDRA [1947] in a different way. See also BHABHA [1949].

where f is the maximum value of the spin of the various fields described by the field quantities ψ (see (5.12)).

Under continuous Lorentz transformations the operator $d_{\alpha\beta}(\partial)$ transforms in the same way as $\psi_\alpha \times \psi_\beta$. Since the quantities ψ_α describe fields with the maximum spin f, they form an invariant space for a representation of the 3-dimensional rotation group, which breaks up into irreducible representations $D_f, D_{f-1}, \ldots$ (cf. (5.12)). Therefore that representation whose basis is given by the direct product $\psi_\alpha \psi_\beta$ can be decomposed into $D_{2f}, D_{2f-1}, \ldots$ (Clebsch–Gordon theorem). Analogously, $\partial_{\mu_1} \ldots \partial_{\mu_l}$ corresponds to a set of representations $D_l, D_{l-1}, \ldots$ unless some of the ∂_μ build up the scalar operator $\Box$. Thus, $\alpha_{\mu_1 \ldots \mu_l} \partial_{\mu_1} \ldots \partial_{\mu_l}$ in (2.9) can appear only in the form

$$\alpha_{\mu_1 \ldots \mu_l} \partial_{\mu_1} \ldots \partial_{\mu_l} = \tilde{\alpha}_{\mu_1 \ldots \mu_{2f}} (\Box)^{\frac{l-2f}{2}} \partial_{\mu_1} \ldots \partial_{\mu_{2f}}$$

when l is larger than $2f$. This relation shows that $\alpha_{\mu_1 \ldots \mu_l}$ is zero if $l-2f\,(>0)$ is odd. Then the recurrence formula (5.16) shows that $\alpha_{\mu_1 \ldots \mu_l}$ is zero for even positive $l-2f$:

$$\alpha_{\mu_1 \ldots \mu_l} = 0, \qquad \text{for } l > 2f,$$

which is equivalent to (5.19).

From (5.18b) with $b=2f$ we have the characteristic equation for β_μ, namely

$$\beta_\mu^{2f-1}(\beta_\mu^2 - 1) = 0. \tag{5.20}$$

where summation over the suffix μ is not carried out.

It can be shown that β_μ cannot be hermitian if $2f > 2$ (i.e. the highest spin $f > 1$). In fact, if β_μ was hermitian, there would be a representation in which β_μ was diagonal, and therefore had eigenvalues $\pm 1, 0$ on account of (5.20). Such a matrix β_μ could satisfy a relation:

$$\beta_\mu(\beta_\mu^2 - 1) = 0.$$

This shows that (5.20) cannot be the characteristic equation when $b > 2$.

HARISH-CHANDRA [1947] pointed out that (5.18b) does not generate a finite algebra for $b > 1$ (i.e. $f \geqslant 1$). Some other stronger condition (on the quantities β_μ) compatible with (5.18b) is necessary to make the algebra finite.

For spin $S=1/2$ (i.e. $f=1/2$) (5.18b) leads to

$$\beta_\mu\beta_\nu+\beta_\nu\beta_\mu=2\delta_{\mu\nu} \tag{5.21}$$

which is just the relation (3.2) in the Dirac theory. Here $d(\partial)$ is

$$d(\partial)=-(\beta_\mu\partial_\mu-\varkappa). \tag{5.22}$$

For spin $S=1$ or 0 (i.e. $f=1$), (5.18) gives

$$\Sigma^{(P)}\,\beta_\mu(\beta_\nu\,\beta_\sigma-\delta_{\nu\sigma})=0. \tag{5.23a}$$

This can be rewritten as

$$\Sigma^{(P)}(\beta_\mu\,\beta_\nu\,\beta_\sigma+\beta_\sigma\,\beta_\nu\,\beta_\mu-\beta_\mu\,\delta_{\nu\sigma}-\beta_\sigma\,\delta_{\mu\nu})=0 \tag{5.23b}$$

and $d(\partial)$ is given by (5.18a) as follows:—

$$d(\partial)=-\left[\frac{1}{\varkappa}(\Box-\varkappa^2)+\beta_\mu\,\partial_\mu-\frac{1}{2\varkappa}(\beta_\mu\,\beta_\nu+\beta_\nu\,\beta_\mu)\,\partial_\mu\,\partial_\nu\right] \tag{5.24}$$

(Takahashi and Umezawa [1953]).

Case II

When we require that (2.3b) should be derivable from the wave equation (5.1), and therefore, that ψ should describe states of particles with various masses ($\varkappa_1, \varkappa_2 \ldots$) the matrices β_μ satisfy relations other than (5.18b). An example of this is provided by assuming that $S_{\mu\nu}$ contain no term that is a product of more than two matrices. Then, since $S_{\mu\nu}$ is an antisymmetric tensor made up of the β_μ's, it must have the form

$$S_{\mu\nu}=g\,(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu) \tag{5.25}$$

where g is a constant c-number (see Bhabha [1945, 1949]).

By substituting (5.25) into (5.6),

$$\beta_\mu\,\beta_\lambda\,\beta_\nu-\beta_\mu\,\beta_\nu\,\beta_\lambda-\beta_\lambda\,\beta_\nu\,\beta_\mu+\beta_\nu\,\beta_\lambda\,\beta_\mu=\frac{1}{g}\,\delta_{\lambda\mu}\,\beta_\nu-\frac{1}{g}\,\delta_{\nu\mu}\,\beta_\lambda. \tag{5.26a}$$

In particular,

$$\beta_\mu^2\,\beta_\nu+\beta_\nu\,\beta_\mu^2-2\,\beta_\mu\,\beta_\nu\,\beta_\mu=\frac{1}{g}\,\beta_\nu \qquad \text{for } \mu\neq\nu \tag{5.26b}$$

where summation over the suffix μ is not taken.

On the other hand, from (5.6) and (5.25) we have

$$\left.\begin{aligned}&a^2[\beta_k,\beta_l]=i\,\sigma_m,\ [\sigma_k,\beta_l]=i\,\beta_m,\ [\beta_k,\sigma_l]=i\,\beta_m\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{for } k,l,m=\text{cycl. }(1,2,3),\end{aligned}\right\} \tag{5.27}$$

$$a^2[\beta_4\,\beta_k]=i\,\delta_k,\ [\delta_k,\beta_4]=i\,\beta_k,\ [\beta_k,\delta_k]=i\,\beta_4 \qquad \text{for } k=1,2,3, \tag{5.28}$$

where

$$a^2 \equiv g, \tag{5.29}$$

$$\delta_k \equiv -iS_{4k} \quad (k=1, 2, 3), \tag{5.30}$$

These commutation relations show that $a\beta_\mu$ and δ_k have also eigenvalues $(f, f-1, \ldots, -f)$, as σ_k does (HEPNER [1951]). Consequently [1])

$$(a\beta_\mu - f)\,(a\beta_\mu - f+1) \ldots (a\beta_\mu + f) = 0. \tag{5.31}$$

In this way we can derive the fundamental relations for β_μ for various spins from (5.31).

We now show that the wave function ψ satisfies the equation

$$\left(\Box - \frac{a^2\varkappa^2}{f^2}\right)\left(\Box - \frac{a^2\varkappa^2}{(f-1)^2}\right) \ldots (\Box - a^2\varkappa^2)\,\psi = 0 \quad \text{for integer } f, \tag{5.32a}$$

$$\left.\begin{aligned}\left(\Box - \frac{a^2\varkappa^2}{f^2}\right)\left(\Box - \frac{a^2\varkappa^2}{(f-1)^2}\right) \ldots \left(\Box - \frac{a^2\varkappa^2}{(1/2)^2}\right)\psi = 0 \\ \text{for half integer } f.\end{aligned}\right\} \tag{5.32b}$$

In fact, from (5.31)

$$((a\beta_4\partial_4)^2 - f^2\partial_4^2)\,((a\beta_4\partial_4)^2 - (f-1)^2\partial_4^2) \ldots ((a\beta_4\partial_4)^2 - \partial_4^2)\,(a\beta_4\partial_4)\,\psi = 0$$

for an integer f. In the reference system of a particle at rest, we have

$$\Box = \partial_4^2,$$

$$\beta_\mu\partial_\mu = \beta_4\partial_4,$$

and therefore

$$(f^2\Box - (a\beta_\mu\partial_\mu)^2) \ldots (\Box - (a\beta_\mu\partial_\mu)^2)\,(a\beta_\mu\partial_\mu)\,\psi = 0.$$

This relation must hold in any reference system on account of the Lorentz invariance of the theory. Equation (5.32a) can be obtained by putting $(-\varkappa)$ into $\beta_\mu\partial_\mu$ on account of the wave equation (5.1). A similar proof can also be given for half-integer f.

Equation (5.32a, b) shows that the wave function ψ, for spin $S>1$ (i.e. $f>1$) satisfies not the Klein–Gordon equation but the equation of a type of (2.3b). In other words, in the theory based on (5.1) and (5.25), the rest mass of particles with higher spin $f>1$ can

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[1]) This relation can also be derived by a slight extension of (5.14). In general, the relation for the matrices β_μ can be derived for a given mass spectrum, by deriving $d(\partial)$ from

$$d(\partial)\,A(\partial) = \prod_s (\Box - \varkappa_s^2)\,I \tag{2.3c}$$

and by using (5.19) in a similar way to Case I. (UMEZAWA and VISCONTI [1955]).

take various values, namely $(a\varkappa/f), \ldots, a\varkappa$ for an integer spin f and $(a\varkappa/f), \ldots 2a\varkappa$ for a half-integer spin f.

However, in this book we shall limit our discussion to Case I. It must be remarked that, for lower spins $S=1, \frac{1}{2}, 0$, the theory based on (5.1) and (5.25) is included in Case I, because then equations (5.32a, b) are just the Klein–Gordon ones. In fact, by putting $f=\frac{1}{2}$ (i.e. $S=\frac{1}{2}$) in (5.31),

$$(a\beta_\mu - \tfrac{1}{2})\,(a\beta_\mu + \tfrac{1}{2}) = 0.$$

From this we find:

$$a^2\beta_\mu^2 = \tfrac{1}{4}$$

where summation over the suffix μ is not taken.

We can normalise β_μ^2 without loss of generality in such a way that

$$\beta_\mu^2 = 1, \qquad a = \tfrac{1}{2}, \qquad g = \tfrac{1}{4}. \tag{5.33}$$

Then (5.26b) and (5.33) give

$$\beta_\mu\beta_\nu + \beta_\nu\beta_\mu = 2\delta_{\mu\nu}.$$

Thus we have recovered the Dirac theory.

It can be seen that the right-hand side of (5.25) is the $S_{\mu\nu}$ (3.24) of the Dirac theory.

We next consider spins 1 and 0. Substituting $f=1$ into (5.31) we obtain

$$a^2\beta_\mu^3 = \beta_\mu. \tag{5.34}$$

Multiplying (5.26b) by β_μ on both sides, we obtain

$$2\,\beta_\mu^2\,\beta_\nu\,\beta_\mu^2 = \left(\frac{1}{g}\right)\beta_\mu\,\beta_\nu\,\beta_\mu \qquad \text{for } \nu \neq \mu.$$

Multiplying again by β_μ on both sides, we derive

$$\frac{2}{g}\,\beta_\mu\,\beta_\nu\,\beta_\mu = \beta_\mu^2\,\beta_\nu\,\beta_\mu^2 \qquad \text{for } \nu \neq \mu$$

and therefore

$$\beta_\mu\beta_\nu\beta_\mu = 0 \qquad \text{for } \nu \neq \mu. \tag{5.35}$$

On the other hand, (5.26a) gives

$$\beta_\mu\,\beta_\lambda\,\beta_\nu + \beta_\nu\,\beta_\lambda\,\beta_\mu = \beta_\mu\,\beta_\nu\,\beta_\lambda + \beta_\lambda\,\beta_\nu\,\beta_\mu \qquad \text{for } \mu \neq \nu,\ \nu \neq \lambda,\ \mu \neq \lambda.$$

Therefore, multiplying by β_λ^2 on the right, we have

$$\beta_\mu\,\beta_\nu\,\beta_\lambda + \beta_\lambda\,\beta_\nu\,\beta_\mu = 0 \qquad \text{for } \mu \neq \nu,\ \nu \neq \lambda,\ \lambda \neq \mu$$

because of (5.26*b*). Thus $(\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu)$ which is symmetric with respect to suffices μ, λ must have the form

$$\beta_\mu\,\beta_\nu\,\beta_\lambda+\beta_\lambda\,\beta_\nu\,\beta_\mu=\alpha_\mu\,\delta_{\nu\lambda}+\alpha_\lambda\,\delta_{\mu\nu}+\alpha'_\nu\,\delta_{\lambda\mu}.$$

Equations (5.35) and (5.34) show that

$$\alpha'_\nu=0$$

$$\alpha_\mu=\frac{1}{a^2}\,\beta_\mu.$$

Thus, by taking $a=1$ we have

$$\beta_\mu\,\beta_\nu\,\beta_\lambda+\beta_\lambda\,\beta_\nu\,\beta_\mu=\beta_\mu\,\delta_{\nu\lambda}+\beta_\lambda\,\delta_{\mu\nu}. \tag{5.36}$$

Equation (5.36) is compatible with (5.23*b*) in Case I. The matrix $d(\partial)$ is given by (5.24). Equations (5.1) and (5.36) give the theory proposed by Duffin, Kemmer and Petiau, which will be discussed in detail in § 2.

§ 2. Duffin-Kemmer-Petiau Theory

The fundamental equations in the Duffin–Kemmer–Petiau theory for spins 1 and 0 are

$$(\beta_\mu\partial_\mu+\varkappa)\,\psi=0 \tag{5.37}$$

$$\beta_\mu\,\beta_\nu\,\beta_\lambda+\beta_\lambda\,\beta_\nu\,\beta_\mu=\beta_\mu\,\delta_{\lambda\nu}+\beta_\lambda\,\delta_{\mu\nu} \tag{5.38}$$

(Duffin [1938], Kemmer [1939], Petiau [1936]).

From (5.25) and (5.29) we have

$$S_{\mu\nu}=\beta_\mu\,\beta_\nu-\beta_\nu\,\beta_\mu. \tag{5.39}$$

Equation (5.38) gives

$$\left.\begin{aligned}&\beta_\mu\,\beta_\nu^2=\begin{cases}(1-\beta_\nu^2)\,\beta_\mu, & \mu\neq\nu\\ \beta_\mu\quad, & \mu=\nu\end{cases}\\ &\beta_\mu^2\,\beta_\nu^2=\beta_\nu^2\,\beta_\mu^2\\ &\beta_\mu\,\beta_\nu\,\beta_\mu=\beta_\mu\,\delta_{\mu\nu}\end{aligned}\right\} \tag{5.40}$$

where summation over the suffix μ is omitted.

Then, for the matrices η_μ defined by

$$\eta_\mu\equiv 2\beta_\mu^2-1, \tag{5.41}$$

we can derive the relations

$$\left.\begin{aligned}&\eta_\mu^2=I\\ &\eta_\mu\,\eta_\nu=\eta_\nu\,\eta_\mu\\ &\beta_\mu\,\eta_\nu=-\eta_\nu\,\beta_\mu\quad(\text{for }\mu\neq\nu)\\ &\beta_\mu=\beta_\mu\,\eta_\mu=\eta_\mu\,\beta_\mu.\end{aligned}\right\} \tag{5.42}$$

Again summation over the suffix μ is omitted.

In the end of this Chapter we shall find a representation in which every β_μ is hermitian. We shall assume this property in the following discussion. It follows that η_μ is also hermitian.

We define $\bar{\psi}$ by

$$\bar{\psi} = \psi^* \eta_4. \tag{5.43}$$

It is easily seen that $\bar{\psi}$ satisfies the wave equation:

$$\partial_\mu \bar{\psi} \beta_\mu - \varkappa \bar{\psi} = 0. \tag{5.44}$$

Equations (5.5*b*) and (5.39) show that, under an infinitesimal Lorentz transformation, $\bar{\psi}$ becomes

$$\bar{\psi} \to {}'\bar{\psi} = {}'\psi^* \eta_4 = \psi^* \Lambda^* \eta_4 = \bar{\psi} \Lambda^{-1}, \tag{5.45}$$

because δw_{ik} and δw_{i4} in (5.5*a*) are real and imaginary respectively. For space reflection (5.4) is satisfied by

$$\Lambda = \pm \eta_4. \tag{5.46}$$

A detailed discussion of the two signs ($\pm$) will be given later.

Multiplying (5.37) by the $d(\partial)$ of (5.24) there is obtained the Klein–Gordon equation:

$$(\Box - \varkappa^2)\, \psi = 0. \tag{5.47}$$

As we have seen in § 1, this theory corresponds to the cases of spin 1 and 0. We can therefore separate wave functions of spin 1 and 0 as follows (FUJIWARA [1953]).

To select the wave function for spin 0, we introduce an operator

$$P \equiv \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^2, \tag{5.48a}$$

$$P_\mu \equiv P \beta_\mu. \tag{5.48b}$$

Then

$$P_\mu \beta_\nu = P \delta_{\mu\nu} \tag{5.49}$$

because of (5.40). This leads to

$$P S_{\mu\nu} = P(\beta_\mu \beta_\nu - \beta_\nu \beta_\mu) = 0$$

and so, under an infinitesimal Lorentz transformation, we have

$$P \Lambda \psi = P \psi. \tag{5.50}$$

Similarly, for the space reflection (5.46), we have, using

$$P \eta_4 = \beta_1^2 \beta_2^2 \beta_3^2 (2\beta_4^4 - \beta_4^2) = \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^2 = P, \tag{5.51}$$

the result

$$P\Lambda\psi = \begin{cases} -P\psi & \text{for } (-) \text{ type of (5.46)} \\ +P\psi & \text{for } (+) \text{ type of (5.46).} \end{cases} \tag{5.52}$$

Equation (5.52) shows that $P\psi$ is scalar or pseudoscalar according to whether the $(+)$ or $(-)$ sign in (5.46) is taken. On the other hand, under the Lorentz transformations (5.5*a*) and (5.46) we have, by using (5.42),

$$P_\mu\psi = \begin{cases} \text{pseudovector} & \text{for } (-) \text{ type of (5.46)} \\ \text{vector} & \text{for } (+) \text{ type of (5.46).} \end{cases} \tag{5.53}$$

If we introduce the notations $P\psi = U$ and $P_\mu\psi = U_\mu$, the wave equations

$$\left.\begin{aligned} \partial_\mu U_\mu &= -\varkappa U \\ \partial_\mu U &= -\varkappa U_\mu \end{aligned}\right\} \tag{5.54}$$

can be deduced from (5.37) on account of (5.48*b*) and (5.49). We can see that (5.54) are the equations for spin 0, and that (U, U_μ) describes the scalar or pseudoscalar field corresponding to the $+$ or $-$ signs in (5.46). Similarly we can pick out the wave function for spin 1 by introducing the operators

$$R_\mu \equiv \begin{cases} -\beta_1^2\beta_2^2\beta_3^2\beta_\mu\beta_4 & \mu = 1, 2, 3, \\ \beta_1^2\beta_2^2\beta_3^2(1-\beta_4^2) & \mu = 4, \end{cases} \tag{5.55}$$

$$R_{\mu\nu} \equiv R_\mu\beta_\nu. \tag{5.56}$$

These equations show that [1])

$$R_{\mu\nu} = -R_{\nu\mu} \tag{5.57}$$

$$R_\mu\beta_\nu\beta_\sigma = \delta_{\nu\sigma}R_\mu - \delta_{\mu\sigma}R_\nu. \tag{5.58}$$

By using (5.58) it can be shown that $S_{\mu\nu}$ satisfies the relations

$$R_\mu S_{\varrho\nu} = R_\mu(\beta_\varrho\beta_\nu - \beta_\nu\beta_\varrho) = \delta_{\mu\varrho}R_\nu - \delta_{\mu\nu}R_\varrho \tag{5.59}$$

which lead to

$$R_\mu\Lambda = a_{\mu\nu}R_\nu$$

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[1]) We have, for example

$$R_{12} + R_{21} = -\beta_1^2\beta_2^2\beta_3^2(\beta_1\beta_4\beta_2 + \beta_2\beta_4\beta_1) = 0$$

because of (5.38).

for the infinitesimal Lorentz transformation (5.5*a*, *b*). On the other hand we have

$$R_\mu \eta_4 = \left\{ \begin{array}{ll} R_\mu & \text{for } \mu = 1, 2, 3 \\ -R_\mu & \text{for } \mu = 4 \end{array} \right\} \tag{5.60}$$

because of (5.42).

It follows that

$$R_\mu \psi = \left\{ \begin{array}{l} \text{vector} \quad \text{for } (-) \text{ type of (5.46)} \\ \text{pseudovector} \quad \text{for } (+) \text{ type of (5.46).} \end{array} \right\} \tag{5.61}$$

If we introduce U_μ and $F_{\mu\nu}$ by means of

$$R_\mu \psi = \varkappa U_\mu, \qquad R_{\mu\nu} \psi = F_{\mu\nu},$$

from (5.37), (5.57) and (5.58) we can derive the wave equations

$$\left. \begin{array}{l} \partial_\mu F_{\mu\nu} - \varkappa^2 U_\nu = 0 \\ F_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu \end{array} \right\}. \tag{5.62}$$

The last is the wave equation (4.23) for spin 1.

Thus we have proved that the Duffin–Kemmer–Petiau theory is equivalent to the theories of spin 1 and 0. The general wave functions in this theory are the following superpositions:

$$\left. \begin{array}{l} \text{(pseudoscalar particle + vector particle) for } (-) \text{ type of (5.46)} \\ \text{(scalar particle + pseudovector particle) for } (+) \text{ type of (5.46)} \end{array} \right\}. \tag{5.63}$$

We now introduce a special representation for β_μ, in which two sets of suffices α and α' ($\alpha, \alpha' = 1, 2, 3, 4$) are used. We shall denote quantities belonging to these two spaces by undashed and dashed symbols respectively. Using γ_μ and γ'_μ which separately satisfy the relation (3.2), and unit matrices I and I', we can show that the fundamental relation (5.38) is satisfied by the matrices

$$\beta_\mu = \tfrac{1}{2} (\gamma_\mu I' + \gamma'_\mu I) \tag{5.64}$$

(KEMMER [1939]). Since (5.64) has four rows and four columns for both primed and unprimed quantities, ψ has 16 components. In particular, in the spinor representation (3.68) of γ_μ, ψ can be decomposed into the second degree spinors $\psi_{\dot{r}\dot{s}'}$, $\psi_{\dot{r}}{}^{\dot{s}'}$, $\psi^{\dot{r}}{}_{\dot{s}'}$, $\psi^{\dot{r}\dot{s}'}$ ($r = 1, 2$; $s = 1, 2$) where we adopt the notation ($\dot{1}$, $\dot{2}$) for $\alpha = (1, 2)$ and $(1, 2)$

for $\alpha=(3, 4)$. In this representation the wave equation (5.37) can be written as second degree spinor equations.

As was shown in Ch. IV, (5.54) and (5.62) give the irreducible representations for spins 0 and 1. Since they have 5 and 10 components respectively, we must find one more in order to obtain the 16 components of ψ. This one component constitutes an irreducible representation, so that it must be a scalar quantity satisfying a first order differential equation. However, it is impossible to find such an equation for one scalar component which must also satisfy the second order Klein–Gordon equation, unless this scalar component is equal to zero. Usually it is called the trivial component. If P_t is the operator picking out the trivial component from ψ we have

$$P_t\psi=0. \tag{5.65}$$

We can give a more concrete representation of the above discussion by means of the spinor representation.

Since there are three irreducible representations of which one is of the first, one of the fifth, and one of the tenth degree, the number of linearly independent matrices is $(1)^2+(5)^2+(10)^2=126$. We can prove, by using (5.38) that all products of β_μ's must be expressed as linear combinations of the following 126 matrices:

	Number of independent matrices
I	1,
β_μ	4,
$\beta_\mu\beta_\nu$	12,
$\beta_\mu\beta_\nu\beta_\varrho$	12,
$\beta_\mu\beta_\nu\beta_\varrho\beta_\sigma$	6,
η_μ	4,
$\eta_\mu\beta_\nu$	12,
$\eta_\mu\beta_\nu\beta_\varrho$	24,
$\eta_\mu\beta_\nu\beta_\varrho\beta_\sigma$	12,
$\eta_\mu\eta_\nu$	6,
$\eta_\mu\eta_\nu\beta_\varrho$	12,
$\eta_\mu\eta_\nu\beta_\varrho\beta_\sigma$	12,
$\eta_\mu\eta_\nu\eta_\varrho$	4,
$\eta_\mu\eta_\nu\eta_\varrho\beta_\sigma$	4.
$\eta_\mu\eta_\nu\eta_\varrho\eta_\sigma$	1,

(5.66)

The non-zero spurs of the matrices of (5.66) are:

	Spin 1	Spin 0	trivial
Sp I	10	5	1
Sp η_μ	2	−1	−1
Sp $\eta_\mu \eta_\nu$	−2	1	1
Sp $\eta_\mu \eta_\nu \eta_\varrho$	−2	3	−1
Sp $\eta_\mu \eta_\nu \eta_\varrho \eta_\sigma$	2	−3	1

(5.67)

The spurs of matrices which occur in (5.66) but not in (5.67) are zero.

We now give a matrix representation for the β_μ.

Trivial case: $\beta_1=\beta_2=\beta_3=\beta_4=0$.

Spin 0:

$$\beta_1=\left(\begin{array}{cccc:c} & & & & 0\\ & & 0 & & 1\\ & & & & 0\\ & & & & 0\\ \hdashline 0 & 1 & 0 & 0 & 0 \end{array}\right), \qquad \beta_2=\left(\begin{array}{cccc:c} & & & & 0\\ & & 0 & & 0\\ & & & & 1\\ & & & & 0\\ \hdashline 0 & 0 & 1 & 0 & 0 \end{array}\right),$$

$$\beta_3=\left(\begin{array}{cccc:c} & & & & 0\\ & & 0 & & 0\\ & & & & 0\\ & & & & 1\\ \hdashline 0 & 0 & 0 & 1 & 0 \end{array}\right), \qquad \beta_4=\left(\begin{array}{cccc:c} & & & & -i\\ & & 0 & & 0\\ & & & & 0\\ & & & & 0\\ \hdashline i & 0 & 0 & 0 & 0 \end{array}\right).$$

Spin 1:

$$\beta_1=\left[\begin{array}{ccc:ccc:ccc:c} & & & & & & & & & -1\\ & 0 & & & 0 & & & 0 & & 0\\ & & & & & & & & & 0\\ \hdashline & & & & & & 0 & 0 & 0 & \\ & 0 & & & 0 & & 0 & 0 & -1 & 0\\ & & & & & & 0 & 1 & 0 & \\ \hdashline & & & 0 & 0 & 0 & & & & \\ & 0 & & 0 & 0 & 1 & & 0 & & 0\\ & & & 0 & -1 & 0 & & & & \\ \hdashline -1 & 0 & 0 & & 0 & & & 0 & & 0 \end{array}\right]$$

$$\beta_2 = \left[\begin{array}{ccc|c} & & & 0 \\ 0 & 0 & 0 & -1 \\ & & & 0 \\ \hline & & 0\ 0\ 1 & \\ 0 & 0 & 0\ 0\ 0 & 0 \\ & & -1\ 0\ 0 & \\ \hline & 0\ 0\ -1 & & \\ 0 & 0\ 0\ 0 & 0 & 0 \\ & 1\ 0\ 0 & & \\ \hline 0\ -1\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$\beta_3 = \left[\begin{array}{ccc|c} & & & 0 \\ 0 & 0 & 0 & 0 \\ & & & -1 \\ \hline & & 0\ -1\ 0 & \\ 0 & 0 & 1\ 0\ 0 & 0 \\ & & 0\ 0\ 0 & \\ \hline & 0\ 1\ 0 & & \\ 0 & -1\ 0\ 0 & 0 & 0 \\ & 0\ 0\ 0 & & \\ \hline 0\ 0\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$\beta_4 = \left[\begin{array}{ccc|c} & & -i\ 0\ 0 & \\ 0 & 0 & 0\ -i\ 0 & 0 \\ & & 0\ 0\ -i & \\ \hline & & & \\ 0 & 0 & 0 & 0 \\ & & & \\ \hline i\ 0\ 0 & & & \\ 0\ i\ 0 & 0 & 0 & 0 \\ 0\ 0\ i & & & \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

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CHAPTER VI

PRELIMINARIES TO QUANTISATION

§ 1. Quantum Electrodynamics

The usual quantum field theory has been constructed as an extension of the quantum theory of electron and electromagnetic fields or quantum electrodynamics. Therefore it is an important problem to see if present theory is successful when applied to systems other than systems of electron and electromagnetic field. For many other elementary particles are now known (cf. Ch. I).

On the other hand, although the present quantum field theory is a very consistent formalism for free fields, there are still many fundamental difficulties for interacting fields. Such difficulties appear even in quantum electrodynamics. Therefore it would seem curious that quantum electrodynamics should have been so successful in explaining experimental facts. In Ch. XIV, XV and XVIII we shall discuss how the present theory has attacked these difficulties and clarified them.

In the present Chapter, we shall provide a basis for the general quantum field theory by seeking the fundamental characterisitics of quantum electrodynamics.

Although the quantum properties of photons emphasised in Planck's theory of black body radiation and Einstein's theory of light quanta played an important role in the discovery of quantum mechanics, the non-relativistic quantum mechanics of "particles" was developed before the establishment of the relativistic quantum theory of the electromagnetic "field". However, as shown in Ch. III, Dirac's relativistic wave equation derived as an extension of non-relativistic quantum mechanics leads us to the hole theory. This implies the existence of an infinite number of particles. We shall see in Ch. VIII and IX that this latter theory is equivalent to the quantum theory of fields with spin 1/2 (IWANENKO and SOKOLOW [1937], KRAMERS [1937]).

The electromagnetic field can be described by the vector potential A_μ which depends on continuous parameters (for example, space-time coordinate x_μ or Fourier amplitudes A_p). Thus an electromagnetic

field is a system of infinitely many degrees of freedom. Einstein's theory of light quanta shows that the state of the free electromagnetic field can be described by the numbers of light quanta in the various states. The emission and absorption of electromagnetic waves can be treated as a change in the numbers of quanta.

Planck's theory of black body radiation implies that a free electromagnetic field can be described as an assembly of simple harmonic oscillators. The energy of a simple harmonic oscillator of frequency ν has the eigenvalues $2\pi(n+\frac{1}{2})\hbar\nu$. Here n is a positive integer (0, 1, 2, ...) and can be interpreted as the number of quanta with energy $(2\pi\hbar\nu)$. Then the emission of one quantum corresponds to the transition from a (n)-state to a $(n+1)$-state. As is well known, the quantum theory of simple harmonic oscillations can be developed by using a canonical formalism. (See for example, DIRAC [1947].) In the latter the energy of a simple harmonic oscillator in its lowest energy state $(n=0)$ is not zero but $\frac{1}{2}(2\pi\hbar\nu)$ (zero-point energy). Therefore the free electromagnetic field, in its lowest energy state (i.e. the state in which there is no quanta), has energy [1])

$$E_0 = 2\int_0^\infty d\nu \, \frac{8\pi\nu^2}{C^3}\left(\frac{2\pi\hbar\nu}{2}\right).$$

This shows that the vacuum must not be defined as a state of zero energy; we shall define it as the state of lowest energy. Such a definition, of course, means that the vacuum has infinite energy. This strange consequence can be avoided by the so called "subtraction of the vacuum" device, which implies that observable effects are merely the differences between the total effects predicted by the theory and the unobservable vacuum effects. This subtraction is customary not only in calculation of the energy, but also in every other calculation. Nevertheless, it sometimes happens that the results obtained in this way are still infinite.

§ 2. Quantum Theory of General Fields

Maxwell's equations and Dirac's equation can be derived from appropriate Lagrangians by a variational method. We shall assume

[1]) It is well known from the theory of thermal radiation that the number of oscillators with interval $(\nu, \nu+d\nu)$ is $(8\pi\nu^2/C^3)d\nu$. The factor 2 appears on account of the two possible independent states of polarization of a transverse wave.

that this is true for general fields $Q_\alpha(x)$ $(\alpha = 1, 2, \ldots)$; that is, we shall assume that the relativistic field equation for $Q_\alpha(x)$ can be derived from a suitable Lagrangian. This Lagrangian must be a scalar and Q_α must be an irreducible representation of the Lorentz group on account of the Lorentz invariance of the theory. For these reasons Q_α obeys the relativistic wave equation given in preceding Chapters.

As shown in Ch. IV for fields with half integer spin, the Klein paradox arises from the existence of negative energy states, so it is necessary that the particles should obey Fermi statistics if the paradox is not to have absurd consequences.

The exact relationship between spin and statistics for particles of general kind will be discussed in Ch. VIII.

To extend the discussion of quantum electrodynamics given in § 1, we shall quantise free fields by regarding them as a system of simple harmonic oscillators. If the $Q_\alpha(x)$ satisfy the Klein–Gordon equation (2.3), then quantisation may be carried out. Indeed, each Fourier component defined by

$$Q_\alpha(x) = \int d^3k \, Q_\alpha(\mathbf{k}, t) \, e^{-i(\mathbf{k}\cdot\mathbf{x})}$$

satisfies the equation appropriate to a simple harmonic oscillator, namely

$$-\partial_t^2 Q_\alpha(\mathbf{k}, t) = (k_l k_l + \varkappa^2) \, Q_\alpha(\mathbf{k}, t). \tag{6.1}$$

It will be shown in Ch. VIII that the energy of a free field is equivalent to a sum of energies of particles of mass $\varkappa$. This exhibits the relationship between fields and their quanta.

The state $\Psi[\sigma_t]$ of a field at a given time is determined by observations on the three dimensional world, i.e. a space-like flat surface σ_t [1]). However, this description is not relativistic because the concept of "same time" is not Lorentz invariant. Since the concept "space-like" is relativistic, we shall consider a state of a field on a space-like surface σ. This state can be determined by independent observations at all the points of σ, because disturbances due to observation cannot propagate faster than light. Then $\Psi[\sigma]$ is a functional of the surface σ. We shall assume that the usual state vector properties of quantum mechanics (e.g. principle of superposition, etc.) apply also to $\Psi[\sigma]$.

In the usual quantum mechanics the Schrödinger equation describes

[1]) A surface is called space-like when any pair of points on it define a space-like interval.

the behaviour of the state vector in time. Extending this fact to quantum field theory, we shall assume that the change of state from surface to surface in a time-like direction is described by a Schrödinger equation for $\Psi[\sigma]$.

Now we shall introduce (TOMONAGA [1946], SCHWINGER [1949]) a vector $d\sigma_\mu$ normal to the surface σ and with a length equal to the area of surface element of σ; given by

$$d\sigma_\mu \equiv (dx_2dx_3dt,\ dx_1dx_3dt,\ dx_1dx_2dt,\ \tfrac{1}{i}\,dx_1dx_2dx_3). \tag{6.2}$$

It follows that when σ_t is replaced by σ, the volume integration $\int d^3x$ is replaced by $i\int d\sigma_\mu$ [1]). Occasionally we shall use a unit vector n_μ ($n_\mu n_\mu = -1$) normal to σ.

The functional derivative of a functional $F[\sigma]$ is defined by

$$\frac{\delta F[\sigma]}{\delta\sigma(x)} = \lim_{\sigma(x)\to\sigma} \frac{F[\sigma(x)] - F[\sigma]}{dw} \tag{6.3}$$

where $dw = \int d^4x$ is the "volume" of the four-dimensional domain between σ and a surface $\sigma(x)$ which differs infinitesimally from σ in a small region about a point x (cf. Fig. 6.1). Since (6.3) is a Lorentz invariant definition, we can use it in the covariant Schrödinger equation.

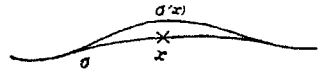

Fig. 6. 1

We shall give now some important theorems on functional derivatives.

When the functional $F[\sigma]$ can be written as a surface integral

$$F_\mu[\sigma] \equiv \int_\sigma d\sigma'_\mu F(x') \tag{6.4}$$

of the differentiable point function $F(x)$ on σ, we can derive from Green's theorem the relation

$$\frac{\delta}{\delta\sigma(x)} F_\mu[\sigma] = \lim_{\sigma(x)\to\sigma} [\int_{\sigma(x)} - \int_\sigma]\, d\sigma'_\mu\, F(x')/dw = \partial_\mu F(x). \tag{6.5a}$$

---

[1]) d^3x and d^4x mean
$d^3x \equiv dx_1dx_2dx_3$
$d^4x \equiv dx_1dx_2dx_3dt$.

Therefore, for the functional

$$F[\sigma] \equiv \int_\sigma d\sigma'_\mu \, F_\mu(x')$$

with the differentiable point function $F_\mu(x')$, we obtain the relation

$$\frac{\delta}{\delta\sigma(x)} F[\sigma] = \partial_\mu F_\mu(x). \tag{6.5b}$$

Thus we have the following theorem:—

$F[\sigma]$ is independent of the surface σ when $F_\mu(x)$ satisfies the continuity equation $\partial_\mu F_\mu(x) = 0$ [1]).

If $F[x, \sigma]$ depends explicitly on σ, (6.5a) can be extended to give

$$F_\mu[\sigma] \equiv \int_\sigma d\sigma'_\mu \, F[x', \sigma], \tag{6.6a}$$

$$\frac{\delta}{\delta\sigma(x)} F_\mu[\sigma] = \partial_\mu F[x, \sigma] + \int_\sigma d\sigma'_\mu \frac{\partial}{\partial\sigma(x)} F[x', \sigma], \tag{6.6b}$$

where $\partial/\partial\sigma$ signifies derivation with respect to arguments explicitly dependent on σ and $\delta/\delta\sigma(x)$ signifies the total variation resulting from the deformation of the surface σ.

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[1]) When σ is a flat surface σ_t, (6.4) is $F[\sigma_t] = -i \int d^3x F_4(x)$. As is well known, $F[\sigma_t]$ is time-independent when $F_\mu(x)$ satisfies a continuity equation: $\partial_\mu F_\mu(x) = 0$. The above theorem is a covariant expression of this fact.

CHAPTER VII

RELATIVISTIC QUANTUM FIELD THEORY

§ 1. Relativistic Quantum Field Theory

We shall consider fields described by the quantities $Q_\alpha(x)$ $(\alpha = 1, 2, \ldots)$ We assume that the Lagrangian, which is hermitic and invarian under the Lorentz transformation, neither contains x_μ explicitly no contains derivatives higher than those of first order. The forme assumption ensures that the field represents a conservative system the latter is made in this Chapter alone. Then the Lagrangian densit can be written as $L(Q_\alpha, Q_{\alpha;\mu})$, where $Q_{\alpha;\mu}(x) \equiv \partial_\mu Q_\alpha(x)$.

The wave equation for $Q_\alpha(x)$ is derived from the variation princip

$$\delta \int d^4x L(Q_\alpha, Q_{\alpha;\mu}) = 0. \tag{7.1}$$

For the variation $Q_\alpha \to Q_\alpha + \delta Q_\alpha$ subject to $\delta Q_\alpha = 0$ on the three dimen sional boundary of the region of integration, (7.1) gives the wav equation

$$\frac{\partial L}{\partial Q_\alpha} - \partial_\mu \left(\frac{\partial L}{\partial Q_{\alpha;\mu}} \right) = 0. \tag{7.2}$$

The (K.G.)-condition in Ch. II for the equation of the free field restricts the form of the free field term L^0 in Lagrangian density.

From (7.2) we have the following continuity equation:

$$\partial_\nu T_{\mu\nu} = 0 \tag{7.3a}$$

with

$$T_{\mu\nu} \equiv -\frac{\partial L}{\partial Q_{\alpha;\nu}} Q_{\alpha;\mu} + L\delta_{\mu\nu}. \tag{7.3b}$$

The tensor $T_{\mu\nu}$ is known as the canonical energy-momentum tensor, in terms of which the energy-momentum vector T_μ is defined by

$$T_\mu \equiv -i \int_\sigma d\sigma'_\nu T_{\mu\nu}(x'). \tag{7.4}$$

Here $W \equiv T_4$ and $G_k \equiv iT_k$ $(k = 1, 2, 3)$ are respectively the total energy and the total momentum of the field on the surface σ. From (6.5b) and (7.3a) it follows that

$$\frac{\delta}{\delta\sigma(x)} T_\mu = 0. \tag{7.5}$$

This is the covariant form for the conservation law of energy-momentum.

The various field quantities, for example the angular momentum and the charge, arise naturally by considering the invariance of the theory (PAULI [1939], [1941]).

We shall first consider the Lorentz invariance of the theory. In an infinitesimal transformation

$$x_\mu \to {}'x_\mu = a_{\mu\nu}x_\nu = x_\mu + \delta x_\mu$$

where

$$\delta x_\mu = \delta w_{\mu\nu}x_\nu, \quad \delta w_{\mu\nu} = -\delta w_{\nu\mu}, \tag{7.6a}$$

$$a_{\mu\nu} = \delta_{\mu\nu} + \delta w_{\mu\nu}, \tag{7.6b}$$

$Q_\alpha(x)$ is transformed (cf. Chapter V) according to

$$Q_\alpha(x) \to {}'Q_\alpha('x) = Q_\alpha(x) + \delta Q_\alpha(x) = \Lambda_{\alpha\beta}\, Q_\beta(x), \tag{7.7a}$$

$$\delta Q_\alpha(x) = \tfrac{1}{2}S_{\mu\nu;\alpha\beta}\, Q_\beta(x)\delta w_{\mu\nu}, \tag{7.7b}$$

$$\Lambda \equiv [\Lambda_{\alpha\beta}] = 1 + \tfrac{1}{2}S_{\mu\nu}\delta w_{\mu\nu}. \tag{7.7c}$$

Here $S_{\mu\nu}$ is a matrix with $(\alpha\beta)$-element $S_{\mu\nu\cdot\alpha\beta}$ and satisfies

$$S_{\mu\nu} = -S_{\nu\mu}. \tag{7.8}$$

From (5.8) we have [1])

$$[S_{\mu\nu}, S_{\varrho\sigma}] = -\delta_{\mu\varrho}S_{\nu\sigma} + \delta_{\nu\varrho}S_{\mu\sigma} + \delta_{\mu\sigma}S_{\nu\varrho} - \delta_{\nu\sigma}S_{\mu\varrho}. \tag{7.9}$$

When $\bar{\delta}Q_\alpha(x)$ denotes a change of $Q_\alpha(x)$ in which δx_μ is neglected, that is, when

$$\bar{\delta}Q_\alpha(x) \equiv {}'Q_\alpha(x) - Q_\alpha(x), \tag{7.10a}$$

by using the Taylor expansion of $'Q('x)$ we obtain

$$\begin{aligned}\bar{\delta}Q_\alpha(x) &= {}'Q_\alpha('x) - \partial_\mu Q_\alpha(x)\,\delta x_\mu - Q_\alpha(x)\\ &= \delta Q_\alpha(x) - \partial_\mu Q_\alpha(x)\,\delta x_\mu.\end{aligned} \tag{7.10b}$$

Equations (7.6a), (7.7b) and (7.10b) then give

$$\bar{\delta}Q_\alpha(x) = \sum_{\mu<\nu}\delta w_{\mu\nu}\,(x_\mu Q_{\alpha;\nu}(x) - x_\nu Q_{\alpha;\mu}(x) + S_{\mu\nu\cdot\alpha\beta}\,Q_\beta(x)). \tag{7.11}$$

From (7.11) and (7.3b) we can derive the relations

$$\frac{\partial L}{\partial Q_{\alpha,\sigma}}\bar{\delta}Q_\alpha + L\delta x_\sigma = \sum_{\mu<\nu}M_{\mu\nu,\sigma}\,\delta w_{\mu\nu}, \tag{7.12}$$

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[1]) It must be noted that in (5.8) we have used only the assumption of the Lorentz invariance of theory.

where

$$M_{\mu\nu,\sigma} \equiv x_\nu T_{\mu\sigma} - x_\mu T_{\nu\sigma} + \left(\frac{\partial L}{\partial Q_{\alpha;\sigma}}\right) S_{\mu\nu;\alpha\beta} Q_\beta. \tag{7.13}$$

It must be remarked that $M_{\mu\nu,\sigma}$ is antisymmetric with respect to μ and ν.

We shall denote the transformed Lagrangian density by $L'('x)$. Since the Lagrangian is invariant under the transformation (7.6*a*, *b*),

$$\delta L \equiv L'('x) - L(x) = 0.$$

Taking the Taylor expansion of $L'('x)$, namely

$$L'('x) = L'(x) + \partial_\mu L(x) \delta x_\mu,$$

we have

$$\delta L + \partial_\mu (L \delta x_\mu) = 0, \tag{7.14}$$

where δL is defined by

$$\begin{aligned} \delta L &\equiv L'(x) - L(x) \\ &= \frac{\partial L}{\partial Q_{\alpha;\mu}} \delta Q_{\alpha;\mu} + \frac{\partial L}{\partial Q_\alpha} \delta Q_\alpha = \partial_\mu \left(\frac{\partial L}{\partial Q_{\alpha;\mu}} \delta Q_\alpha\right). \end{aligned} \tag{7.15}$$

This follows from (7.2). From (7.14) and (7.15),

$$\partial_\mu \left(\frac{\partial L}{\partial Q_{\alpha;\mu}} \delta Q_\alpha + L \delta x_\mu\right) = 0. \tag{7.16}$$

Then we find, by using (7.12)

$$\partial_\sigma M_{\mu\nu,\sigma} = 0. \tag{7.17}$$

The quantity $M_{\mu\nu,\sigma}$ is called the angular momentum tensor in terms of which the total angular momentum $P_{\mu\nu}$ of the field on σ is defined by

$$P_{\mu\nu} = -P_{\nu\mu} \equiv -i \int d\sigma_\varrho \, M_{\mu\nu,\varrho}. \tag{7.18}$$

Equations (7.17) and (6.5*b*) give

$$\frac{\delta}{\delta\sigma(x)} P_{\mu\nu} = 0. \tag{7.19}$$

This is the covariant form of the conservation law for angular momentum.

We can also write $M_{\mu\nu,\sigma}$ in the form

$$M_{\mu\nu,\sigma} = x_\nu T_{\mu\sigma} - x_\mu T_{\nu\sigma} - f_{\mu,\nu\sigma} + f_{\nu,\mu\sigma}, \tag{7.20}$$

where $f_{\mu,\nu\sigma}$ is

$$f_{\mu,\nu\sigma} \equiv \tfrac{1}{2}\left[-\frac{\partial L}{\partial Q_{\alpha;\sigma}} S_{\mu\nu;\alpha\beta} Q_\beta - \frac{\partial L}{\partial Q_{\alpha;\nu}} S_{\sigma\mu;\alpha\beta} Q_\beta + \frac{\partial L}{\partial Q_{\alpha;\mu}} S_{\nu\sigma;\alpha\beta} Q_\beta\right]. \tag{7.21}$$

From (7.21) and (7.8)

$$\left.\begin{aligned} f_{\mu,\nu\sigma} &= -f_{\mu,\sigma\nu} \\ f_{\mu,\nu\sigma} - f_{\nu,\mu\sigma} &= -\left(\frac{\partial L}{\partial Q_{\alpha;\sigma}}\right) S_{\mu\nu;\alpha\beta} Q_\beta \end{aligned}\right\}. \tag{7.22}$$

We shall now introduce a symmetric energy-momentum tensor $\theta_{\mu\nu}$ by means of $f_{\mu,\nu\sigma}$. The symmetric energy-momentum tensor is defined as a symmetric tensor that satisfies the equation

$$-i\int_\sigma d\sigma_\nu\, \theta_{\mu\nu}(x) = T_\mu . \tag{7.23}$$

Then (7.23), (7.5) and (6.5*b*) give

$$\partial_\nu \theta_{\mu\nu} = 0. \tag{7.24}$$

We shall show that the symmetric tensor satisfying (7.23) is given by

$$\theta_{\mu\nu} \equiv T_{\mu\nu} - \partial_\sigma f_{\mu,\nu\sigma} . \tag{7.25}$$

(BELINFANTE [1939], ROSENFELD [1940]).

The right-hand side of this equation is symmetric with respect to the suffices μ, ν; for the substitution of (7.20) into (7.17) gives

$$T_{\mu\nu} - T_{\nu\mu} - \partial_\sigma f_{\mu,\nu\sigma} + \partial_\sigma f_{\nu,\mu\sigma} = 0$$

on account of (7.3*a*). Since $f_{\mu,\nu\sigma}$ is antisymmetric with respect to ν and σ,

$$\partial_\nu \partial_\sigma f_{\mu,\nu\sigma} = 0.$$

Then, (6.5*b*) shows that $\int_\sigma d\sigma_\nu(\theta_{\mu\nu} - T_{\mu\nu})$ is independent of the surface σ chosen. Moreover, since $f_{\mu,44} = 0$, we have, by using the partial integration,

$$i\int_\sigma d\sigma_\nu\, \partial_\sigma f_{\mu,\nu\sigma} = \int d^3x\, \partial_\sigma f_{\mu,4\sigma} = \int d^3x\, \partial_k f_{\mu,4k} = 0$$

when the space-like surface σ is flat. Thus, we see that $\int_\sigma d\sigma_\nu(\theta_{\mu\nu} - T_{\mu\nu})$ is zero for any space-like surface σ, and therefore that the condition (7.23) is satisfied.

In the same manner we can prove that

$$\int_\sigma d\sigma_\varrho\, (x_\mu \theta_{\nu\varrho} - x_\nu \theta_{\mu\varrho} + M_{\mu\nu,\varrho}) = 0.$$

It follows that

$$P_{\mu\nu} = i\int_\sigma d\sigma_\varrho\, (x_\mu \theta_{\nu\varrho} - x_\nu \theta_{\mu\varrho}) \tag{7.26}$$

on account of (7.18). This is the familiar relation between the angular momentum and the energy-momentum tensors.

The main problems of the quantisation of fields are those of deriving commutation relations for the q-numbers $Q_\alpha(x)$ and of rewriting the wave equations (7.2) as the canonical equations:

$$-\partial_\mu F(x) = [F(x), T_\mu]. \tag{7.27}$$

Here $F(x)$ is arbitrary functional of $Q_\alpha(x)$.

This canonical formulation will be discussed in detail in following Chapters. Equation (7.27) shows that T_μ is the displacement operator in the x_μ-direction.

The quantisation due to HEISENBERG and PAULI [1929a] and [1929b] is an example of such a procedure. The quantities $P_\alpha(x)$ that are conjugate to $Q_\alpha(x)$ are introduced as $P_\alpha(x) \equiv (-i)\partial L/\partial Q_{\alpha;4}$ and the canonical commutation relationships [1])

$$[Q_\alpha(\mathbf{x}, t),\ P_\beta(\mathbf{x}', t)] = i\delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{x}')$$

are assumed to hold. The $Q_{\alpha;4}$ can be eliminated in favour of P_α, and then the Hamiltonian $\bar{H} = \int d^3x (iP_\alpha \dot{Q}_{\alpha;4} - L)$ is identical with the T_4 of equation (7.27). In the following pages we shall adopt a procedure that is different from this.

In the present Chapter we shall assume that (7.27) has been derived by a suitable quantisation process. Then (7.3a), (7.24) and (7.17) give

$$[\theta_{\mu\nu}(x), T_\nu] = [T_{\mu\nu}(x), T_\nu] = 0, \tag{7.28}$$

and

$$[M_{\mu\nu,\sigma}(x), T_\sigma] = 0. \tag{7.29}$$

These equations are the quantum theoretical expressions of the conservation laws of energy-momentum and angular momentum.

We shall now consider the Schrödinger equation describing a change of state. Each point on σ has a different time coordinate $t = t(x_k)$ $(k = 1, 2, 3)$, which is the equation determining the three dimensional curved surface σ. Then the surface $\sigma(x)$, which deviates infinitesimally from σ only in the immediate neighbourhood of a point x, is determined by equation $t = t(x_k) + \delta t(x_k)$ where $\delta t(x_k)$ is an infinitesimal quantity in the neighbourhood of the point x. Taking into account the fact that observations at different points on σ do not influence each other, we can rewrite the functional derivative (6.3) for $\Psi[\sigma]$ as:

$$\frac{\delta}{\delta\sigma(x)}\,\Psi[\sigma] = \lim_{\sigma\to\sigma(x)} \frac{\Psi[t(x_k)+\delta t(x_k)] - \Psi[t(x_k)]}{\iiint d^3x\,\delta t(x_k)}. \tag{7.30}$$

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[1]) $\delta(\mathbf{x}-\mathbf{x}') \equiv \delta(x_1-x_1')\,\delta(x_2-x_2')\,\delta(x_3-x_3')$.

Since, in the Heisenberg representation, the state vector is time independent, we take the covariant Schrödinger equation suggested by (7.30), namely

$$\frac{\delta \Psi[\sigma]}{\delta \sigma(x)} = 0. \tag{7.31}$$

The wave equation (7.2) or (7.27) for Q_α, the commutation relations for Q_α and the Schrödinger equation are the foundations of quantum field theory.

§ 2. Complex Fields and Charged Fields

In general, the field quantities $Q_\alpha(x)$ that appear in the Lagrangian are components of various fields, some of which are complex. We call a field a real field or a complex field according to whether its field quantities are real or complex.

We shall suppose that there is a group of n complex fields described by n sets of field quantities $(Q_\alpha^{(j)}, Q_\alpha^{(j)*}; j=1, \ldots, n)$ and that the Lagrangian is invariant under the phase transformation of all field quantities belonging to this group; that is, that

$$\left.\begin{aligned} Q_\alpha^{(j)} &\to Q_\alpha^{(j)} e^{i\gamma} \\ Q_\alpha^{(j)*} &\to Q_\alpha^{(j)*} e^{-i\gamma} \end{aligned}\right\} \tag{7.32a}$$

Here $j=1, 2$ to n and γ is a constant. We call such a group the phase invariant set. If γ is infinitesimal, the invariance of the Lagrangian leads to [1])

$$\delta L \equiv i\gamma \left[\frac{\partial L}{\partial Q_\alpha^{(j)}} Q_\alpha^{(j)} + \frac{\partial L}{\partial Q_{\alpha;\mu}^{(j)}} Q_{\alpha;\mu}^{(j)} - Q_\alpha^{(j)*} \frac{\partial L}{\partial Q_\alpha^{(j)*}} - Q_{\alpha;\mu}^{(j)*} \frac{\partial L}{\partial Q_{\alpha;\mu}^{(j)*}}\right] = 0 \tag{7.33}$$

where we sum over j. This equation yields the continuity equation

$$\sum_{j=1}^{n} \partial_\mu N_\mu^{(j)} = 0 \tag{7.34a}$$

where the "j-field current operator" $N_\mu^{(j)}$ is defined by

$$N_\mu^{(j)} \equiv -i\left(\frac{\partial L}{\partial Q_{\alpha;\mu}^{(j)}} Q_\alpha^{(j)} - Q_\alpha^{(j)*} \frac{\partial L}{\partial Q_{\alpha;\mu}^{(j)*}}\right). \tag{7.35a}$$

Here we do not sum over j.

Taking into account (6.5b), we have, from (7.34a),

$$\sum_{j=1}^{n} \frac{\delta}{\delta \sigma(x)} N^{(j)} = 0, \tag{7.36a}$$

[1]) $Q_{\alpha;\mu}^{(j)*}$ means not $(\partial_\mu Q_\alpha^{(j)})^*$ but $\partial_\mu (Q_\alpha^{(j)*})$.

where

$$N^{(j)} \equiv \int_\sigma d\sigma_\mu N_\mu^{(j)}. \tag{7.37}$$

Equation (7.36) shows that $\Sigma N^{(j)}$ is constant in time. We shall show in Ch. VIII that the eigenvalues of $N^{(j)}$ are the differences of numbers of particles and antiparticles belonging to the j-field.

As an example, we shall consider the Lagrangian which represents an interaction

$$\bar{\psi}_P \psi_N U_\pi \tag{7.38}$$

Here ψ_P and ψ_N are complex quantities describing the proton and neutron fields and U_π the charged π-meson field. Since this interaction is invariant under the phase transformation of nucleon fields (ψ_P, ψ_N), (7.36) leads to the conservation of difference of numbers of nucleons and their antiparticles in any transmutation process induced by this interaction. This is called the nucleon conservation.

One of the most important phase invariant sets is that which consists of all charged fields $Q_\alpha^{(c)}$ ($c = 1, 2, \ldots$). We postulate that the Lagrangian is invariant under phase transformation for all charged fields – that is, that

$$\left.\begin{aligned} Q_\alpha^{(c)} &\to Q_\alpha^{(c)} e^{ie\gamma} \\ Q_\alpha^{(c)*} &\to Q_\alpha^{(c)*} e^{-ie\gamma} \end{aligned}\right\} \tag{7.32b}$$

Here e is the elementary electric charge. We define the electric current $J_\mu(x)$ of $Q_\alpha^{(c)}$-field

$$\left.\begin{aligned} J_\mu &\equiv -ieN_\mu^{(c)} \\ &= -ie\left(\frac{\partial L}{\partial Q_{\alpha;\mu}^{(c)}} Q_\alpha^{(c)} - Q_\alpha^{(c)*} \frac{\partial L}{\partial Q_{\alpha;\mu}^{(c)*}}\right), \end{aligned}\right\} \tag{7.35b}$$

where we do not sum over (c). Equations (7.34a) and (7.36a) give, in the present case,

$$\sum_c \partial_\mu J_\mu^{(c)} = 0, \tag{7.34b}$$

$$\frac{\delta}{\delta\sigma(x)} e_t = 0, \tag{7.36b}$$

where the "total charge operator" e_t is

$$e_t \equiv \sum_c \int_\sigma d\sigma_\mu J_\mu^c. \tag{7.36c}$$

Eigenvalues of e_t are interpreted in the value of the total electric charge on the surface σ. Equation (7.36b) expresses the law of charge conservation.

In the interaction (7.38), there are two charged fields $\bar{\psi}_P$, U_π, and (7.38) is invariant under their phase transformation. Thus, we see that total electric charge of the fields of protons and π-mesons is conserved in transmutation processes induced by the interaction (7.38).

We saw in the theory of electron (Ch. III) that the charges of a particle and its antiparticle are of opposite sign. We shall show in Ch. VIII that this is generally true. Therefore, we see that the conservation of difference of numbers of charged particles and their antiparticles is equivalent to the conservation of total charge.

When Q_α is real, J_μ, (7.34*b*), is zero. In other words real wave functions describe neutral fields. The Majorana theory is an example of a theory of a real neutral field. Well known neutral particles described by real quantities are the photon and the neutral π-meson. However, it must be noted that the wave functions of neutral fields are not always real; neutrons provide an example of this.

§ 3. Examples

We shall close this Chapter by applying the preceding arguments to some special examples.

EXAMPLE 1. $S_{\mu\nu}$ AND SPIN ANGULAR MOMENTUM

As shown in Ch. III and V, the $S_{\mu\nu}$ of (7.7*c*) are

$$_{\nu} = [S_{\mu\nu:\sigma\varrho}] = \left\{ \begin{array}{ll} \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) & \text{Spin } \frac{1}{2} \\ \beta_\mu\beta_\nu - \beta_\nu\beta_\mu & \text{Duffin–Kemmer Theory (spin 1.0)} \\ [\delta_{\mu\sigma}\delta_{\nu\varrho} - \delta_{\mu\varrho}\delta_{\nu\sigma}] & \text{vector or pseudovector [1])} \\ 0 & \text{scalar or pseudoscalar} \end{array} \right\} \quad (7.39$$

From (7.9) we have

$$[-iS_{12}, -iS_{23}] = i(-iS_{31}) \text{ etc.}$$

These commutation relations show that the $(-iS_{ik})$ ($i, k,$ = cycl. (1, 2, 3)) satisfy the commutation relations of angular momentum and therefore have eigenvalues $(S, S-1, \ldots, -S)$, where S assumes integer or half-integer values. Since each particle has $(2S+1)$ intrinsic degrees of freedom, we see from (4.22) that S is the spin and the $(2S+1)$ intrinsic states correspond to the various directions of spin angular momentum.

[1]) For the vector or pseudovector field U_μ, (7.7*a*) reads

$$'U_\sigma = a_{\sigma\varrho}U_\varrho = U_\sigma + \delta w_{\sigma\varrho}U_\varrho = U_\sigma + \tfrac{1}{2}(\delta_{\mu\sigma}\delta_{\nu\varrho} - \delta_{\mu\varrho}\delta_{\nu\sigma})\,\delta w_{\mu\nu}U_\varrho.$$

Thus we have the third line of (7.39).

The total angular momentum $P_{\mu\nu}$ is composed of two parts, viz.

$$P_{\mu\nu} = P^0_{\mu\nu} + P^S_{\mu\nu} \tag{7.40}$$

$$P^0_{\mu\nu} = i \int_\sigma d\sigma_\varrho \, (x_\mu T_{\nu\varrho} - x_\nu T_{\mu\varrho}), \tag{7.41}$$

$$P^S_{\mu\nu} = - i \int_\sigma d\sigma_\varrho \left(\frac{\partial L}{\partial Q_{\alpha\,\varrho}}\right) S_{\mu\nu:\alpha\beta} Q_\beta . \tag{7.42}$$

The quantity $P^S_{\mu\nu}$ has the familiar form of an orbital angular momentum, while $P^S_{\mu\nu}$ is interpreted as the spin angular momentum. We shall show in Example 5, Ch. IX, that $P^0_{\mu\nu}$ and $P^S_{\mu\nu}$ represent the sum of the orbital angular momentum and the spin angular momentum of the particles respectively.

EXAMPLE 2. FREE SCALAR OR PSEUDOSCALAR FIELDS $U(x)$

The wave equation of $U(x)$ is given in Ch. IV as

$$(\Box - \varkappa^2)\, U(x) = 0. \tag{7.43}$$

This equation can be derived from the Lagrangian

$$L = - \partial_\mu U^* \cdot \partial_\mu U - \varkappa^2 U^* U. \tag{7.44}$$

It follows that the canonical energy-momentum tensor and electric current are given by

$$T_{\mu\nu} = \partial_\mu U^* \cdot \partial_\nu U + \partial_\nu U^* \cdot \partial_\mu U + L\delta_{\mu\nu}, \tag{7,45}$$

$$J_\mu = ie(\partial_\mu U^* \cdot U - U^* \cdot \partial_\mu U). \tag{7.46}$$

In this case $T_{\mu\nu}$ itself can be taken as the symmetrical energy-momentum tensor. We can formulate this theory in terms of real quantities $(U^{(1)}, U^{(2)})$ defined as

$$\left.\begin{aligned} U(x) &= \frac{1}{\sqrt{2}} (U^{(1)}(x) - i\, U^{(2)}(x)), \\ U^*(x) &= \frac{1}{\sqrt{2}} (U^{(1)}(x) + i\, U^{(2)}(x)). \end{aligned}\right\} \tag{7.47}$$

Then, (7.44), (7.45) and (7.46) are

$$L = - \tfrac{1}{2} \{(\partial_\mu U^{(1)})^2 + (\partial_\mu U^{(2)})^2\} - \tfrac{1}{2} \varkappa^2 (U^{(1)2} + U^{(2)2}), \tag{7.48}$$

$$T_{\mu\nu} = \partial_\mu U^{(1)} \cdot \partial_\nu U^{(1)} + \partial_\mu U^{(2)} \cdot \partial_\nu U^{(2)} + L\delta_{\mu\nu} \tag{7.49a}$$

$$J_\mu = e\,(\partial_\mu U^{(1)} \cdot U^{(2)} - \partial_\mu U^{(2)} \cdot U^{(1)}). \tag{7.49b}$$

The theory of the real (and therefore neutral) field with spin 0 can be derived from the above system of equations by putting $U^{(2)} = 0$.

EXAMPLE 3. FREE ELECTROMAGNETIC FIELD $A_\mu(x)$

The wave equation of the free electromagnetic field is (cf. Ch. IV)

$$\Box A_\mu(x) = 0. \tag{7.50}$$

This can be derived from the Lagrangian density

$$L = -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} - \tfrac{1}{2}(\partial_\mu A_\mu)^2, \tag{7.51}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.52}$$

Here $F_{\mu\nu}$ is called the strength of the electromagnetic field. As shown in Ch. IV, this theory is invariant under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x), \tag{7.53}$$

$$\Box \Lambda(x) = 0 \tag{7.54}$$

where $\Lambda(x)$ is c-number. Consequently the electromagnetic wave is a transverse wave with two independent components (i.e. two states of polarization).

Equation (7.50) can be written in the form

$$\partial_\mu F_{\mu\nu} + \partial_\nu(\partial_\mu A_\mu) = 0. \tag{7.55}$$

As shown in Ch. IV, the vector potential A_μ must also satisfy the Lorentz condition $\partial_\mu A_\mu = 0$ which ensures that the theory is based on an irreducible representation of spin 1 [1]). However, it will be shown in Ch. VIII that this condition is incompatible with commutation relation of A_μ. Therefore, instead of $\partial_\mu A_\mu = 0$, we shall impose as the Lorentz condition, the auxiliary condition on the state vector given by

$$(\partial_\mu A_\mu)\, \Psi = 0 \qquad \text{Lorentz condition.} \tag{7.56}$$

(See FERMI [1932]).

The last equation implies that only states in which the eigenvalue of $\partial_\mu A_\mu$ is 0 occur in nature. We shall now prove that (7.56) is compatible with the wave equation When on a flat surface σ_t with time t, the conditions

$$\left.\begin{aligned} (\partial_\mu A_\mu(\mathbf{x}, t))\, \Psi &= 0 \\ (\partial_4 \partial_\mu A_\mu(\mathbf{x}, t))\, \Psi &= 0 \end{aligned}\right\} \tag{7.57}$$

are satisfied, then we can prove, by the repeated use of (7.50), that

$$(\partial_4)^n (\partial_\mu A_\mu(\mathbf{x}, t))\Psi = 0. \tag{7.58}$$

[1]) When this Equation is abandoned, a negative energy field with spin 0 is mixed with positive energy field with spin 1 in the electromagnetic field.

Indeed, (7.57) leads to

$$(\partial_4)^{2m}\,\partial_\mu A_\mu(\mathbf{x}, t)\,\Psi = (-1)^m(\partial_k\partial_k)^m\,\partial_\mu A_\mu(\mathbf{x}, t)\,\Psi = 0$$

and

$$(\partial_4)^{2m+1}\,\partial_\mu A_\mu(\mathbf{x}, t)\,\Psi = (-1)^m(\partial_k\partial_k)^m\,\partial_4\partial_\mu A_\mu(\mathbf{x}, t)\,\Psi = 0.$$

On the other hand, $\partial_\mu A_\mu(\mathbf{x}, t')$ on the surface σ_t' of time t', can be written as a series expansion

$$\partial_\mu A_\mu(\mathbf{x}, t') = \sum_{n=0}^{\infty}\frac{(i)^n}{n!}(t'-t)^n\,\partial_4^n(\partial_\mu A_\mu(\mathbf{x}, t)).$$

Thus we have

$$\partial_\mu A_\mu(\mathbf{x}, t')\Psi = 0. \tag{7.59}$$

This shows that Lorentz condition is satisfied throughout the universe as long as the initial condition (7.57) is satisfied. Therefore we can impose the Lorentz condition by ensuring that the state function satisfies the initial condition (7.57).

From (7.56) we have

$$\partial_\nu\partial_\mu A_\mu\Psi = 0, \tag{7.60}$$

which gives

$$\partial_\mu F_{\mu\nu}\Psi = 0. \tag{7.61}$$

This equation has a form similar to the classical Maxwell equations and shows that Maxwell theory is valid in the states realized in nature.

The canonical energy-momentum tensor $T_{\mu\nu}$ derived from Lagrangian (7.51) is

$$\left.\begin{aligned} T_{\mu\nu} = {} & \tfrac{1}{2}(F_{\mu\sigma}F_{\nu\sigma} + F_{\nu\sigma}F_{\mu\sigma}) - \tfrac{1}{4}F_{\varrho\sigma}F_{\varrho\sigma}\delta_{\mu\nu} + \partial_\varrho A_\varrho\cdot\partial_\mu A_\nu \\ & - \tfrac{1}{2}(\partial_\varrho A_\varrho)^2\delta_{\mu\nu} + \tfrac{1}{2}(F_{\nu\sigma}\cdot\partial_\sigma A_\mu + \partial_\sigma A_\mu\cdot F_{\nu\sigma}). \end{aligned}\right\} \tag{7.62}$$

The symmetric energy-momentum tensor can be obtained by using (7.25) and (7.62). Using (7.62), (7.56) and (7.4) we find the total energy of the free electromagnetic field on a flat surface σ_t to be

$$\Psi^* T_4\Psi = \tfrac{1}{2}\Psi^*\int d^3x[\mathbf{E}^2 + \mathbf{H}^2]\Psi \tag{7.63}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic strengths given by

$$\mathbf{E} = i(F_{14}, F_{24}, F_{34}),\quad \mathbf{H} = (F_{23}, F_{31}, F_{12}). \tag{7.64}$$

Then (7.63) has the familiar form of the energy of the classical Maxwell field.

EXAMPLE 4. INTERACTION BETWEEN ELECTROMAGNETIC FIELD AND CHARGED FIELDS

The Lagrangian for an electromagnetic field and a charged field in interaction is made up of three parts—the Lagrangian L^e (see (7.51)) of the free electromagnetic field, the Lagrangian L^c of the free charged field, and an interaction term L'. Thus

$$L = L^e + L^c + L'. \tag{7.65}$$

The electromagnetic interaction term L' must be constructed so that the total Lagrangian L is invariant under the gauge transformation

$$\left.\begin{aligned} &Q_\alpha(x) \to Q_\alpha(x)\, e^{ie\Lambda(x)},\ Q^*_\alpha(x) \to Q^*_\alpha(x)\, e^{-ie\Lambda(x)}, && \text{Gauge transformation of the first kind} \\ &A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x). && \text{Gauge transformation of the second kind} \end{aligned}\right\} \tag{7.66}$$

Here $\Lambda(x)$ is a real c-number satisfying the equation

$$\Box \Lambda(x) = 0. \tag{7.67}$$

A gauge transformation of the first kind is an extension of the phase transformation (7.32b) to the case of a $\Lambda(x)$ which is not constant. A gauge transformation of the second kind is identical with the transformation (7.53). Therefore L^e is invariant under the transformation (7.66). The following changes are induced by (7.66) in L^c:—

$$\left.\begin{aligned} \partial_\mu Q_\alpha &\to \{\partial_\mu Q_\alpha + ie\,(\partial_\mu \Lambda)\, Q_\alpha\}\, e^{ie\Lambda} \\ \partial_\mu Q^*_\alpha &\to \{\partial_\mu Q^*_\alpha - ie\,(\partial_\mu \Lambda)\, Q^*_\alpha\}\, e^{-ie\Lambda}. \end{aligned}\right\} \tag{7.68}$$

The changes induced by (7.68) in $L^c + L'$ must be cancelled by the contribution from the second kind gauge transformation in L'. Since L is invariant under phase transformation (7.32b), $L^c + L'$ must be derivable from L^e by the following procedures:

(i) $\partial_\mu Q_\alpha$ and $\partial_\mu Q^*_\alpha$ in L^c are replaced according to

$$\left.\begin{aligned} \partial_\mu Q_\alpha &\to (\partial_\mu - ie\, A_\mu)\, Q_\alpha \\ \partial_\mu Q^*_\alpha &\to (\partial_\mu + ie\, A_\mu)\, Q^*_\alpha \end{aligned}\right\} \tag{7.69}$$

(ii) Any terms which contain A_μ in the form $F_{\mu\nu}$ can be added. We shall call the interaction terms derived from (i) and (ii) the A_μ type and $F_{\mu\nu}$-type interactions respectively [1]).

[1]) Moreover we can add the terms containing derivatives in the form of the right hand side of (7.69).

It is easily seen from (7.69) that the electric current (7.35*b*) can be written, for the A_μ-type interactions L' as

$$J_\mu = \frac{\partial L'}{\partial A_\mu}. \tag{7.70}$$

When L' contains the $F_{\mu\nu}$-type interaction, we extend the definition of the electric current by

$$J_\mu = J_\mu^{(1)} + J_\mu^{(2)}, \tag{7.71}$$

$$\left.\begin{aligned} J_\mu^{(1)} &= \frac{\partial L'}{\partial A_\mu} \\ J_\mu^{(2)} &= 2\,\partial_\nu\left(\frac{\partial L'}{\partial F_{\mu\nu}}\right). \end{aligned}\right\} \tag{7.72}$$

Since $F_{\mu\nu}$ is an anti-symmetric tensor, $J_\mu^{(2)}$ satisfies the continuity equation

$$\partial_\mu J_\mu^{(2)} = 0. \tag{7.73}$$

From (7.34*b*) and (7.73) we obtain the continuity equation of the total current,

$$\partial_\mu J_\mu = 0. \tag{7.74}$$

We can rewrite (7.71) into

$$J_\mu = \frac{\delta L'}{\delta A_\mu} \tag{7.75}$$

where $\delta L'/\delta A_\mu$ is defined by

$$\frac{\delta L'}{\delta A_\mu} \equiv \frac{\partial L'}{\partial A_\mu} - \partial_\nu\left\{\frac{\partial L'}{\partial(\partial_\nu A_\mu)}\right\}.$$

Taking into account the wave equation (7.2) and the fact that L' and J_μ are defined by (7.51) and (7.75) respectively, we can easily prove that

$$\Box A_\mu = -J_\mu. \tag{7.76}$$

This shows that (7.75) is a reasonable definition of the electric current.

Since the wave equation (7.76) differs from that of the free field, we must reestablish the compatibility between the Lorentz condition (7.56) and wave equation. This proof is obtained in a similar way to the free field proof (cf. (7.57), (7.58), (7.59)), because we can derive again (7.58) from (7.57) by using

$$\Box\,\partial_\mu A_\mu = -\partial_\mu J_\mu = 0. \tag{7.77}$$

From (7.56) and (7.76) we have

$$(\partial_\mu F_{\mu\nu})\Psi = -J_\nu\Psi. \tag{7.78}$$

The last equation has the familiar form of the classical Maxwell equation.

We shall now consider a charged field interacting with a magnetic field **H** constant in space and time. To the approximation in which second and higher powers of **H** are neglected, the magnetic moment is defined by the space-vector **m** given by

$$(\mathbf{H}\cdot\mathbf{m}) = \int_\sigma d^3x L'. \tag{7.79}$$

Here L' is the interaction Lagrangian [1]). Then from (7.70) when the $F_{\mu\nu}$-type interaction is zero [2]),

$$\int_\sigma d^3x\, J_\mu\, A_\mu \approx (\mathbf{H}\cdot\mathbf{m}). \tag{7.80}$$

On the other hand, as is well known, we can take the gauge in the case of constant magnetic field such that

$$\mathbf{A} = \tfrac{1}{2}[\mathbf{H}\wedge\mathbf{x}],\ A_4 = 0. \tag{7.81}$$

Substituting (7.81) into (7.80),

$$\mathbf{m} = \tfrac{1}{2}\int_\sigma d^3x\, [\mathbf{x}\wedge\mathbf{J}]. \tag{7.82}$$

In the case of a charged field of spin 0 (i.e. a scalar or pseudoscalar field), the Lagrangian given by method (i) is

$$L = -(\partial_\mu + ieA_\mu)U^*\cdot(\partial_\mu - ieA_\mu)U - \varkappa^2 U^*U + L^e \tag{7.83}$$

which gives the electric current

$$J_\mu = -ie(\partial_\mu U^*\cdot U - U^*\cdot\partial_\mu U) - 2e^2 A_\mu U^*U \tag{7.84}$$

on account of (7.35*b*). This can also be derived directly from (7.70).

EXAMPLE 5. VECTOR OR PSEUDOVECTOR FIELDS $U_\mu(x)$

As shown in Ch. IV, the wave equations for $U_\mu(x)$ are

$$\partial_\nu U_{\mu\nu} + \varkappa^2 U_\mu = 0, \tag{7.85a}$$

$$U_{\mu\nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu. \tag{7.85b}$$

Equations (7.85*a*) and (7.85*b*) lead to

$$\partial_\mu U_\mu = 0. \tag{7.86}$$

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[1]) As shown in Ch. X, to the present approximation, L' is equal to $(-H')$: $L' \approx -H'$, where H' is the interaction energy.

[2]) Where the symbol $\approx$ means the neglect of second order and higher powers of $e\mathbf{H}$.

Now (7.85a) can be derived from the Lagrangian

$$L^c = -\tfrac{1}{2} U^\dagger_{\mu\nu} U_{\mu\nu} - \varkappa^2 U^\dagger_\mu U_\mu . \tag{7.87}$$

Here $U^\dagger_\mu$ and $U^\dagger_{\mu\nu}$ are defined by

$U^\dagger_\mu \equiv (U^*_1, U^*_2, U^*_3, iU^*_0) = (U^*_1, U^*_2, U^*_3, -U^*_4)$ and $U^\dagger_{\mu\nu} = \partial_\mu U^\dagger_\nu - \partial_\nu U^\dagger_\mu$.

Discussions in § 2 remain to be valid even when we replace * by †.

The Lagrangian (7.87) leads to the canonical energy-momentum tensor given as

$$T_{\mu\nu} = U^\dagger_{\nu\varrho} \cdot \partial_\mu U_\varrho + \partial_\mu U^\dagger_\varrho \cdot U_{\nu\varrho} + L^c \delta_{\mu\nu} . \tag{7.88}$$

The tensor (7.88) is not symmetric; symmetric energy-momentum tensor is obtained by using (7.21), (7.25) and (7.39) to give

$$\left.\begin{aligned} \theta_{\mu\nu} &= T_{\mu\nu} - \partial_\varrho (U^\dagger_{\nu\varrho} U_\mu + U^\dagger_\mu U_{\nu\varrho}) \\ &= U^\dagger_{\mu\varrho} U_{\nu\varrho} + U^\dagger_{\nu\varrho} U_{\mu\varrho} + \varkappa^2 (U^\dagger_\mu U_\nu + U^\dagger_\nu U_\mu) \\ &\quad - \delta_{\mu\nu} (\tfrac{1}{2} U^\dagger_{\varrho\sigma} U_{\varrho\sigma} + \varkappa^2 U^\dagger_\varrho U_\varrho). \end{aligned}\right\} \tag{7.89}$$

The A_μ-type current derived by method (i) in Example 4 is

$$\left.\begin{aligned} J^{(1)}_\mu &= ie (U^\dagger_{\mu\nu} U_\nu - U^\dagger_\nu U_{\mu\nu}) \\ &\quad - e^2 (2A_\mu U^\dagger_\nu U_\nu - A_\nu U^\dagger_\mu U_\nu - A_\nu U^\dagger_\nu U_\mu). \end{aligned}\right\} \tag{7.90}$$

According to method (ii) in Example 4 we can add the $F_{\mu\nu}$-type interaction Lagrangian

$$\frac{e}{2} \gamma (U^\dagger_\mu U_\nu - U^\dagger_\nu U_\mu) F_{\mu\nu}, \tag{7.91}$$

where γ is a real constant. This equation gives the electric current J_μ as

$$J^{(2)}_\mu = i\gamma \partial_\nu (U^\dagger_\mu U_\nu - U^\dagger_\nu U_\mu). \tag{7.92}$$

Since (7.91) has the form of (7.79) for constant magnetic field, we see that the contribution of (7.91) to the magnetic moment is

$$m^{(2)}_k = ie\gamma \int d^3x (U^\dagger_l U_m - U^\dagger_m U_l) \qquad (k, l, m = \text{cycl. } (1, 2, 3)).$$

On the other hand, the magnetic moment $\mathbf{m}^{(1)}$ due to the current $J^{(1)}_\mu$ is given by (7.90) as

$$\begin{aligned} m^{(1)}_k &= \tfrac{1}{2} \int d^3x [\mathbf{x} \wedge \mathbf{J}^{(1)}]_k d^3x \\ &\approx \frac{ie}{2} \int d^3x [x_l (U^\dagger_{m\nu} U_\nu - U^\dagger_\nu U_{m\nu}) - x_m (U^\dagger_{l\nu} U_\nu - U^\dagger_\nu U_{l\nu})], \\ &\qquad\qquad (k, l, m = \text{cycl. } (1, 2, 3)) \end{aligned}$$

where the symbol $\approx$ means the neglect of second order and higher

powers of e. This $m_k^{(1)}$ can be rewritten (on account of (7.85b) and (7.86)) as

$$m_k^{(1)} \approx -ie\int d^3x\,(U_l^\dagger U_m - U_m^\dagger U_l) - ie\int d^3x\; U_\nu^\dagger \mathsf{L}_{lm} U_\nu - \frac{ie}{2}\int d^3x\,[x_l \partial_4 (U_m^\dagger U_4 - U_4^\dagger U_m) - x_m \partial_4 (U_l^\dagger U_4 - U_4^\dagger U_l)], \quad (k, l, m = \text{cycl. } (1, 2, 3)).$$

Here L_{lm} is the orbital angular momentum operator of the particle, namely

$$\mathsf{L}_{lm} \equiv x_l \partial_m - x_m \partial_l.$$

For particles of low energy the second term is negligibly small ($\partial U_\mu \approx 0$) because of the orbital angular momentum operator L_{lm}.

The third term can also be neglected, because (7.85a) leads to

$$U_4 \approx \frac{1}{\varkappa^2}\partial_4\partial_4 U_4 = -\frac{1}{\varkappa^2}\partial_4\partial_k U_k \approx 0.$$

Thus, we have

$$m_k^{(1)} \approx -ie\int d^3x\,(U_l^\dagger U_m - U_m^\dagger U_l), \qquad (k, l, m = \text{cycl. } (1, 2, 3))$$

and therefore, the total magnetic moment

$$m_k^{(1)} + m_k^{(2)} \approx -ie(1-\gamma)\int d^3x\,(U_l^\dagger U_m - U_m^\dagger U_l)$$

for particles of low energy. In other words, the current $J_\mu^{(2)}$ affects the magnetic moment of particles of low energy by the factor $(1-\gamma)$.

For a real wave function $U_\mu(x)$ (i.e. a neutral field) we have [1])

$$L = -\tfrac{1}{4} U_{\mu\nu} U_{\mu\nu} - \tfrac{1}{2}\varkappa^2 U_\mu U_\mu \tag{7.93}$$

$$T_{\mu\nu} = U_{\nu\varrho}\partial_\mu U_\varrho + L\,\delta_{\mu\nu} \tag{7.94}$$

$$\theta_{\mu\nu} = U_{\mu\varrho} U_{\nu\varrho} + \varkappa^2 U_\mu U_\nu - \delta_{\mu\nu}(\tfrac{1}{4} U_{\varrho\sigma} U_{\varrho\sigma} + \tfrac{1}{2}\varkappa^2 U_\varrho U_\varrho). \tag{7.95}$$

EXAMPLE 6. STUECKELBERG FORMALISM FOR THE VECTOR FIELD

In this formalism (STUECKELBERG [1938]) the theory of the vector field takes on a form very similar to that of the electromagnetic field.

In this theory the vector field is described by the 5 components of a vector $A_\mu(x)$ and a scalar $B(x)$. Therefore, we must introduce a new auxiliary condition in order to eliminate the redundant component. This condition can be written in a form similar to the Lorentz condition.

[1]) It must be noted that, for the real vector field, the real field quantities are (U_1, U_2, U_3, U_0) and therefore $U_4 \equiv iU_0$ is imaginary.

The Lagrangian in this theory is [1])

$$L = -\partial_\mu A_\nu^\dagger \cdot \partial_\mu A_\nu - \varkappa^2 A_\mu^\dagger A_\mu - \partial_\mu B^\dagger \cdot \partial_\mu B - \varkappa^2 B^\dagger B, \quad (7.96)$$

which gives the wave equations

$$\left.\begin{aligned}(\Box - \varkappa^2) A_\mu(x) &= 0\\ (\Box - \varkappa^2) B(x) &= 0\end{aligned}\right\} \quad (7.97)$$

The initial conditions on a flat surface σ_t at time t are

$$\left.\begin{aligned}(\partial_\mu A_\mu + \varkappa B)\Psi &= 0\\ (\partial_\mu A_\mu^\dagger + \varkappa B^\dagger)\Psi &= 0\\ \partial_4(\partial_\mu A_\mu + \varkappa B)\Psi &= 0\\ \partial_4(\partial_\mu A_\mu^\dagger + \varkappa B^\dagger)\Psi &= 0.\end{aligned}\right\} \quad (7.98)$$

When these are satisfied, we can easily prove, by using (7.97), that at the same time t,

$$\left.\begin{aligned}(\partial_4)^n(\partial_\mu A_\mu + \varkappa B)\Psi &= 0\\ (\partial_4)^n(\partial_\mu A_\mu^\dagger + \varkappa B^\dagger)\Psi &= 0\end{aligned}\right\}$$

(see (7.58)). By the same discussion as that used in quantum electrodynamics, (see (7.59)), it can be proved that the wave equations (7.97) are compatible with the auxiliary conditions

$$\left.\begin{aligned}(\partial_\mu A_\mu + \varkappa B)\Psi &= 0\\ (\partial_\mu A_\mu^\dagger + \varkappa B^\dagger)\Psi &= 0.\end{aligned}\right\} \quad (7.99)$$

Under the initial condition (7.98) the equations (7.99) can be written as

$$(\partial_\mu U_\mu)\Psi = 0, \quad (\partial_\mu U_\mu^\dagger)\Psi = 0, \quad (7.99')$$

$$U_\mu \equiv A_\mu + \frac{1}{\varkappa}\partial_\mu B, \quad U_\mu^\dagger \equiv A_\mu^\dagger + \frac{1}{\varkappa}\partial_\mu B^\dagger. \quad (7.100)$$

This shows that the relation $\partial_\mu U_\mu = 0$ in the usual theory of vector fields (cf. (4.23b)) is valid for the states realized by nature.

It is easily seen that this theory is invariant under the transformation

$$\left.\begin{aligned}&A_\mu \to A_\mu + \partial_\mu \Lambda\ , \quad B \to B - \varkappa\Lambda\\ &(\Box - \varkappa^2)\Lambda = 0\end{aligned}\right\} \quad (7.101)$$

This ensures that the redundant component has no physical effect. Since the transformation for A_μ in (7.101) has the same form as the

[1]) $A_\mu^\dagger \equiv (A_1^*, A_2^*, A_3^*, iA_0^*)$, $B^\dagger \equiv B^*$.

gauge transformation for the electromagnetic field, we shall call it by that name.

For a real field (A_μ, B) (i.e. a neutral field) the Lagrangian and the auxiliary condition are

$$L = -\tfrac{1}{2}(\partial_\mu A_\nu \cdot \partial_\mu A_\nu + \varkappa^2 A_\mu A_\mu) - \tfrac{1}{2}(\partial_\mu B \cdot \partial_\mu B + \varkappa^2 B^2) \quad (7.102)$$

and

$$(\partial_\mu A_\mu + B)\,\Psi = 0 \quad (7.103)$$

respectively.

We shall discuss the detailed relationship between this neutral vector field and the electromagnetic field in Ch. XI.

EXAMPLE 7. DUFFIN–KEMMER–PETIAU THEORY $\psi(x)$

In the Duffin–Kemmer theory for spin (0, 1) the wave equation is

$$(\beta_\mu \partial_\mu + \varkappa)\psi = 0 \quad (7.104)$$

(cf. Ch. V).

This equation can be derived from the Lagrangian

$$L = -\bar{\psi}(\beta_\mu \partial_\mu + \varkappa)\psi. \quad (7.105)$$

Whence the canonical energy-momentum tensor and the electric current are

$$T_{\mu\nu} = \bar{\psi}\beta_\nu \partial_\mu \psi, \quad (7.106)$$

$$J_\mu = ie\bar{\psi}\beta_\mu \psi. \quad (7.107)$$

From (7.21), (7.39) and (5.38)

$$\left.\begin{aligned} f_{\mu,\nu\sigma} &= \tfrac{1}{2}\bar{\psi}\{\beta_\sigma[\beta_\mu, \beta_\nu] + \beta_\nu[\beta_\sigma, \beta_\mu] - \beta_\mu[\beta_\nu, \beta_\sigma]\}\psi \\ &= \bar{\psi}(\beta_\sigma \beta_\mu \beta_\nu - \beta_\sigma \delta_{\nu\mu})\psi \end{aligned}\right\} \quad (7.108a)$$

$$\left.\begin{aligned} \partial_\sigma f_{\mu,\nu\sigma} &= \partial_\sigma \bar{\psi}\beta_\sigma \beta_\mu \beta_\nu \psi + \bar{\psi}\beta_\sigma \beta_\mu \beta_\nu \partial_\sigma \psi - \bar{\psi}\beta_\sigma \partial_\sigma \psi \delta_{\nu\mu} - \partial_\sigma \bar{\psi}\beta_\sigma \psi \delta_{\mu\nu} \\ &= \varkappa \bar{\psi}\beta_\mu \beta_\nu \psi - \bar{\psi}\beta_\nu \beta_\mu \beta_\sigma \partial_\sigma \psi + \bar{\psi}\beta_\nu \partial_\mu \psi - \partial_\sigma \bar{\psi}\beta_\sigma \psi \delta_{\mu\nu} \\ &= \varkappa\{\bar{\psi}(\beta_\mu \beta_\nu + \beta_\nu \beta_\mu)\psi - \delta_{\mu\nu}\bar{\psi}\psi\} + \bar{\psi}\beta_\nu \partial_\mu \psi. \end{aligned}\right\} \quad (7.108b)$$

Thus the symmetric energy-momentum tensor is

$$\theta_{\mu\nu} = -\varkappa\{\bar{\psi}(\beta_\mu \beta_\nu + \beta_\nu \beta_\mu)\psi - \delta_{\mu\nu}\bar{\psi}\psi\}. \quad (7.109a)$$

In particular, the energy of the field is

$$W = -\theta_{44} = \varkappa \psi^* \psi \quad (7.109b)$$

(cf. (5.43)).

The electromagnetic interaction can be derived by using (i) and (ii) of Example 4.

EXAMPLE 8. SPINOR FIELD $\psi(x)$

The wave equations for ψ are

$$\left.\begin{aligned}(\gamma_\mu \partial_\mu + \varkappa)\,\psi &= 0\\ \partial_\mu \bar{\psi}\,\gamma_\mu - \varkappa\,\bar{\psi} &= 0,\end{aligned}\right\} \tag{7.110}$$

which can be derived from the Lagrangian

$$L = -\tfrac{1}{2}(\bar{\psi}\,\gamma_\mu\,\partial_\mu\,\psi - \partial_\mu\,\bar{\psi}\cdot\gamma_\mu\,\psi) - \varkappa\,\bar{\psi}\,\psi. \tag{7.111}$$

This gives

$$T_{\mu\nu} = \tfrac{1}{2}(\bar{\psi}\,\gamma_\nu\,\partial_\mu\,\psi - \partial_\mu\,\bar{\psi}\cdot\gamma_\nu\,\psi), \tag{7.112}$$

$$\begin{aligned} f_{\mu,\nu\sigma} &= \tfrac{1}{16}\,\bar{\psi}\{\gamma_\sigma[\gamma_\mu,\gamma_\nu] + \gamma_\nu[\gamma_\sigma,\gamma_\mu] - \gamma_\mu[\gamma_\nu,\gamma_\sigma]\\ &\quad + [\gamma_\mu,\gamma_\nu]\gamma_\sigma + [\gamma_\sigma,\gamma_\mu]\gamma_\nu - [\gamma_\nu,\gamma_\sigma]\gamma_\mu\}\psi\\ &= \tfrac{1}{4}\,\bar{\psi}\,(\gamma_\sigma\,\gamma_\mu\,\gamma_\nu - \gamma_\sigma\,\delta_{\mu\nu} - \gamma_\nu\,\delta_{\sigma\mu} + \gamma_\mu\,\delta_{\nu\sigma})\,\psi, \end{aligned}$$

$$\partial_\sigma f_{\mu,\nu\sigma} = \tfrac{1}{4}(\bar{\psi}\,\gamma_\nu\,\partial_\mu\,\psi - \bar{\psi}\,\gamma_\mu\,\partial_\nu\,\psi) - \tfrac{1}{4}(\partial_\mu\,\bar{\psi}\cdot\gamma_\nu\,\psi - \partial_\nu\,\bar{\psi}\cdot\gamma_\mu\,\psi),$$

$$\theta_{\mu\nu} = \tfrac{1}{4}(\bar{\psi}\,\gamma_\nu\,\partial_\mu\,\psi + \bar{\psi}\,\gamma_\mu\,\partial_\nu\,\psi) - \tfrac{1}{4}(\partial_\mu\,\bar{\psi}\cdot\gamma_\nu\,\psi + \partial_\nu\,\bar{\psi}\cdot\gamma_\mu\,\psi). \tag{7.113}$$

The electromagnetic interaction may be obtained by methods (i) and (ii) of Example 4, and is

$$L' = i\,e\,\bar{\psi}\,\gamma_\mu\,\psi\,A_\mu + \frac{e}{8}\frac{i}{\varkappa}\,\varrho\,\bar{\psi}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\,\psi\,F_{\mu\nu}, \tag{7.114}$$

with a constant ϱ. The first and second terms are A_μ-type and $F_{\mu\nu}$-type interactions respectively. Since the latter term has the form of (7.79) in a constant magnetic field $\mathbf{H}$, it gives rise to an intrinsic magnetic moment. The electric current derived from (7.114) is

$$J_\mu = J_\mu^{(1)} + J_\mu^{(2)}, \tag{7.115a}$$

$$J_\mu^{(1)} = i\,e\,\bar{\psi}\,\gamma_\mu\,\psi, \tag{7.115b}$$

$$J_\mu^{(2)} = \frac{i}{4\varkappa}\,\varrho\,e\,\partial_\nu\{\bar{\psi}\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\,\psi\}. \tag{7.115c}$$

Using (3.2) and the field equation (7.110), we can rewrite the current $J_\mu^{(1)}$ due to the A_μ-type interaction as

$$\begin{aligned} J_\mu^{(1)} &= -\frac{ie}{4\varkappa}\,\partial_\nu\{\bar{\psi}\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\,\psi\}\\ &\quad + \frac{ie}{2\varkappa}\,(\partial_\mu\,\bar{\psi}\cdot\psi - \bar{\psi}\cdot\partial_\mu\,\psi). \end{aligned}$$

The last term is usually called the convection current. The magnetic moment due to $J_\mu^{(1)}$ is:

$$\left.\begin{aligned} m_k^{(1)} &= \tfrac{1}{2}\int_\sigma d^3x\,[\mathbf{x}\wedge\mathbf{J}^{(1)}]_k \\ &= -\frac{ie}{4\varkappa}\int_\sigma d^3x\,\bar\psi\,(\gamma_l\gamma_m-\gamma_m\gamma_l)\,\psi \\ &\quad+\frac{ie}{4\varkappa}\int d^3x\,(\mathsf{L}_{lm}\bar\psi\cdot\psi-\bar\psi\cdot\mathsf{L}_{lm}\psi) \\ &\quad-\frac{ie}{4\varkappa}\int d^3x\,\{x_l\partial_4(\bar\psi\,\gamma_m\gamma_4\psi)-x_m\partial_4(\bar\psi\,\gamma_l\gamma_4\psi)\} \\ &\qquad\qquad (k,l,m=\text{cycl. }(1,2,3)) \end{aligned}\right\}\quad(7.115d)$$

where

$$\mathsf{L}_{lm}=x_l\partial_m-x_m\partial_l.$$

On the other hand, the magnetic moment due to $J_\mu^{(2)}$ is

$$m_k^{(2)}=\frac{ie}{4\varkappa}\,\varrho\int d^3x\,\bar\psi\,(\gamma_l\gamma_m-\gamma_m\gamma_l)\,\psi\qquad(k,l,m=\text{cycl. }(1,2,3).)$$

(cf. (7.79)). The second term in (7.115*d*) is negligibly small on account of the orbital angular momentum operator L_{lm} for the particles of low energy ($\partial\psi\approx 0$). Furthermore, the third term in (7.115*d*) contributes a negligibly small quantity to the expectation value of the magnetic moment of a low energy state ($\mathbf{k}\approx 0$) of the particle. For we have

$$\begin{aligned}(\mathbf{k}\approx 0\,|\,\partial_4(\bar\psi\,\gamma_m\gamma_4\psi)\,|\,\mathbf{k}\approx 0) &= (\mathbf{k}\approx 0\,|-\bar\psi\,\gamma_m(\partial_k\gamma_k+\varkappa)\,\psi+\\ &\quad+(\partial_k\gamma_k^T-\varkappa)\,\bar\psi\cdot\gamma_m\psi\,|\,\mathbf{k}\approx 0)\\ &= 2\varkappa i\,(\mathbf{k}\approx 0\,|\,\psi^*\alpha_m\psi\,|\,\mathbf{k}\approx 0)=2\varkappa i\,k_m/K_0\approx 0\end{aligned}$$

on account of (3.61). Thus the total magnetic moment of the low energy particle is

$$\begin{aligned}(\mathbf{k}\approx 0\,|\,m_k\,|\,\mathbf{k}\approx 0) &= (\mathbf{k}\approx 0\,|\,m_k^{(1)}+m_k^{(2)}\,|\,\mathbf{k}\approx 0)\\ &\approx -\frac{ie}{4\varkappa}\,(1-\varrho)\,(\mathbf{k}\approx 0\,|\int d^3x\,\bar\psi\,(\gamma_l\gamma_m-\gamma_m\gamma_l)\,\psi\,|\,\mathbf{k}\approx 0)\\ &\qquad\qquad (k,l,m=\text{cycl. }(1,2.3)).\end{aligned}$$

In other words, the current $J_\mu^{(2)}$ affects the magnetic moment of the low energy particle by a factor $(1-\varrho)$.

The scalar interaction (coupling constant f) and the vector interaction (coupling constant g) between a real (and therefore neutral) scalar field U and a spinor field ψ is

$$L'=f\bar\psi\psi U+ig\bar\psi\gamma_\mu\psi\partial_\mu U.\qquad(7.116)$$

The pseudoscalar interaction (coupling constant f) and the pseudovector interaction (coupling constant g) between a real (and so neutral) pseudoscalar field U and a spinor field ψ is

$$L' = i f \bar{\psi} \gamma_5 \psi U + i g \bar{\psi} \gamma_5 \gamma_\mu \psi \partial_\mu U. \tag{7.117}$$

The vector interaction between a real (and so neutral) vector field U_μ and a spinor field ψ is

$$L' = i f \bar{\psi} \gamma_\mu \psi U_\mu. \tag{7.118}$$

The pseudovector interaction between a real (and so neutral) pseudovector field U_μ and a spinor field ψ is

$$L' = i f \bar{\psi} \gamma_5 \gamma_\mu \psi U_\mu. \tag{7.119}$$

We shall explain how these interactions are derived by considering the particular case (7.118).

(i) The form $\bar{\psi} \gamma_\mu \psi U_\mu$ is adopted for reasons of Lorentz invariance.

(ii) Since the hermitean conjugate operator of $\bar{\psi} \gamma_\mu \psi U_\mu$ is $-\bar{\psi} \gamma_\mu \psi U_\mu$, an imaginary factor i is introduced in (7.118) because of the hermitean property of the Lagrangian. Thus we obtain (7.118). Other interactions can be derived by the same procedure.

We shall now consider the interactions between four spinor fields ψ^a, ψ^b, ψ^c, ψ^d. The linearly independent interactions that contain no derivatives are given by (3.5) as

$$\left.\begin{aligned}
L^{(1)}(a,b,c,d) &= g(\bar{\psi}^a \gamma^1 \psi^b)(\bar{\psi}^c \gamma^1 \psi^d) + h.c. \\
&= g(\bar{\psi}^a \psi^b)(\bar{\psi}^c \psi^d) + h.c. \qquad \text{(scalar type, } S\text{)} \\
L^{(2)}(a,b,c,d) &= \sum_{A=2}^{5} g(\bar{\psi}^a \gamma^A \psi^b)(\bar{\psi}^c \gamma^A \psi^d) + h.c. \\
&= g(\bar{\psi}^a \gamma_\mu \psi^b)(\bar{\psi}^c \gamma_\mu \psi^d) + h.c. \quad \text{(vector type, } V\text{)} \\
L^{(3)}(a,b,c,d) &= \sum_{A=6}^{11} g(\bar{\psi}^a \gamma^A \psi^b)(\bar{\psi}^c \gamma^A \psi^d) + h.c. \\
&= -\tfrac{1}{8} g(\bar{\psi}^a [\gamma_\mu, \gamma_\nu] \psi^b)(\bar{\psi}^c [\gamma_\mu, \gamma_\nu] \psi^d) + h.c. \\
&\qquad \text{(tensor type, } T\text{)} \\
L^{(4)}(a,b,c,d) &= \sum_{A=12}^{15} g(\bar{\psi}^a \gamma^A \psi^b)(\bar{\psi}^c \gamma^A \psi^d) + h.c. \\
&= -g(\bar{\psi}^a \gamma_5 \gamma_\mu \psi^b)(\bar{\psi}^c \gamma_5 \gamma_\mu \psi^d) + h.c. \\
&\qquad \text{(axialvector type, } A\text{)} \\
L^{(5)}(a,b,c,d) &= g(\bar{\psi}^a \gamma^{16} \psi^b)(\bar{\psi}^c \gamma^{16} \psi^d) + h.c. \\
&= g(\bar{\psi}^a \gamma_5 \psi^b)(\bar{\psi}^c \gamma_5 \psi^d) + h.c. \\
&\qquad \text{(pseudoscalar type, } P\text{)}
\end{aligned}\right\} \tag{7.121}$$

where "*h.c.*" means "hermitean conjugate". These interactions are obviously invariant under Lorentz transformations.

Any Lorentz invariant interaction containing no derivatives can be written as a linear combination of the five interactions in (7.121). In particular, if O is some product of the γ_μ, the interaction

$$(\bar{\psi}^a O\, \psi^d)\,(\bar{\psi}^c O'_1 \psi^b)$$

in which the order of (a, b, c, d) is interchanged can be written as such a linear combination. For example, we can show that the scalar interaction $L^{(1)}\,(a, d, c, b)$ can be written as

$$L^{(1)}(a, d, c, b) = g\,(\bar{\psi}^a\,\psi^d)\,(\bar{\psi}^c\,\psi^b) = \tfrac{1}{4}\sum_{k=1}^{5} L^{(k)}(a, b, c, d). \quad (7.122a)$$

In fact, using (3.44*a*) we have

$$(\bar{\psi}^a\,\psi^d)\,(\bar{\psi}^c\,\psi^b) = \tfrac{1}{4}\,\bar{\psi}^a_\alpha\,\psi^d_\beta\,\bar{\psi}^c_\varrho\,\psi^b_\delta\,\gamma^A_{\alpha\delta}\,\gamma^A_{\varrho\beta}$$

which agrees with (7.122*a*). Quite generally, we have (FIERZ [1936])

$$L^{(k)}(a, d, c, b) = a^{kl}\,L^{(l)}(a, b, c, d) \qquad (7.122b)$$

with

$$[a^{kl}] = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & -\frac{1}{2} & 0 & \frac{1}{2} & -1 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{2} \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad (7.123)$$

As an example we shall calculate a^{2l}. From (3.44*b*) we obtain

$$g\,(\bar{\psi}^a\,\gamma_\mu\,\psi^d)\,(\bar{\psi}^c\,\gamma_\mu\,\psi^b) = \tfrac{1}{4}\,g\,(\bar{\psi}^a\,\gamma^A\,\psi^b)\,(\bar{\psi}^c\,\gamma_\mu\,\gamma^A\,\gamma_\mu\,\psi^b) = L^{(1)}(a, b, c, d)$$
$$-\tfrac{1}{2}\,L^{(2)}(a, b, c, d) + \tfrac{1}{2}\,L^{(4)}(a, b, c, d) - L^{(5)}(a, b, c, d).$$

on account of the relation

$$\gamma_\mu\gamma\ \ \gamma_\mu = \begin{cases} 4 & \text{for } A = 1 \\ -2 & \text{for } A = 2, \ldots, 5 \\ 0 & \text{for } A = 6, \ldots, 11 \\ 2 & \text{for } A = 12, \ldots, 15 \\ -4 & \text{for } A = 16 \end{cases}$$

When one of the wave functions ψ^a, ψ^b, ψ^c, ψ^d is a pseudospinor, Lorentz invariance requires the interactions different from (7.121*a*) (cf. § 9 of Ch. III).

As shown by (7.122*b*), the classification into the types S, V, T, A, P depends on the order of the suffixes a, b, c, d. In the theory of β-disintegration we usually adopt the order a, b, c, d in which ψ^a denotes a proton, ψ^b a neutron, ψ^c an electron and ψ^d a neutrino.

Experimental information on these β-interactions is provided by the energy spectrum of electrons, life time of β-decay, spins and energy levels of parent and daughter nuclei, etc. In recent years, the experimental investigation of the energy distribution of the β-spectra from either allowed or forbidden transitions and the classification of the comparative half-lives of the various β-active substances yield substantial agreement with interactions in (7.121). (See FEINGOLD [1951]). Moreover, they support the existence of T and select a linear combination of (S, T, P) as being the most promising interaction. (See WU [1952].) However, it is not known whether β-disintegrations are induced by the direct interactions between four spinor fields or by intermediate fields of integer spin (cf. Ch. XV).

EXAMPLE 9. RARITA–SCHWINGER THEORY FOR SPIN 3/2 FIELD ψ_μ

The wave equation of ψ_μ, namely

$$(\gamma_\varrho \partial_\varrho + \varkappa)\,\psi_\mu - \tfrac{1}{3}(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu)\,\psi_\nu + \tfrac{1}{3}\gamma_\mu(\gamma_\varrho \partial_\varrho - \varkappa)\,\gamma_\nu \psi_\nu = 0$$

can be derived from the Lagrangian

$$\left.\begin{aligned} L = -\{&\bar{\psi}_\mu(\gamma_\varrho \partial_\varrho + \varkappa)\,\psi_\mu - \tfrac{1}{3}\bar{\psi}_\mu(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu)\,\psi_\nu + \\ &+ \tfrac{1}{3}\bar{\psi}_\mu \gamma_\mu(\gamma_\varrho \partial_\varrho - \varkappa)\,\gamma_\nu \psi_\nu\}. \end{aligned}\right\} \tag{7.124}$$

EXAMPLE 10. ISOTOPIC SPIN

Since the transmutation from a proton to a neutron is accompanied by the creation of a π^+-meson or the annihilation of a π^--meson, it must be induced by the interaction

$$\bar{\psi}^{(2)} O \psi^{(1)}\, U^* = \bar{\psi} O \tau_-\, \psi\, U^* \tag{7.125}$$

with

$$\psi \equiv \begin{bmatrix} \psi^{(1)} \\ \psi^{(2)} \end{bmatrix}, \quad \tau_- \equiv \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{7.126a}$$

Here $\psi^{(1)}$ and $\psi^{(2)}$ are the wave functions of a proton and a neutron respectively, and O is a certain product of the γ_μ. In (7.125) the spin of the π-meson field is assumed to be zero (cf. Example 6 of Ch. X), τ_- to be a matrix with 2 rows and 2 columns, in which all matrix elements except the (21) element (i.e. $P \to N$ transmutation), which

is 1, are zero. Therefore, the hermitian conjugate matrix τ_+ of τ_- induces the transmutation $N \to P$. The total interaction is obtained from the sum of (7.125) and its hermitian conjugate term [1])

$$L_c' = f\bar{\psi} O \tau_- \psi U^* + f\bar{\psi} O \tau_+ \psi U, \tag{7.127}$$

where τ_+ is

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{7.126b}$$

Using the matrices τ_k $(k=1, 2, 3)$ given by

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{7.128}$$

τ_+ and τ_- may be written as

$$\tau_\pm = \tfrac{1}{2}(\tau_1 \pm i\tau_2). \tag{7.129}$$

The matrices τ_k satisfy the commutation relations

$$[\tfrac{1}{2}\tau_k, \tfrac{1}{2}\tau_l] = i\tfrac{1}{2}\tau_m, \qquad k, l, m = \text{cycl. } (1, 2, 3). \tag{7.130}$$

This shows that $(1/2)\tau_k$ has the same properties as the angular momentum $(1/2)\sigma_k$ (see (3.55)). It is seen from (7.128) that the two eigenvalues $+(1/2)$, $-(1/2)$ of the matrix $(1/2)\tau_3$ correspond to proton and neutron states respectively. In this formulation proton and neutron are regarded as different states of the same particle, i.e. a nucleon. The matrix τ_k are usually called the isotopic spin of the nucleon.

From (7.47) we have

$$(\tau_+ U + \tau_- U^*) = \sqrt{2} \sum_{k=1}^{2} \tau_k U^{(k)}. \tag{7.131}$$

Thus, by (7.127), the interaction between a charged field $U^{(k)}$ $(k=1, 2)$ and a nucleon is

$$L_c' = \sqrt{2}\, f \sum_{k=1}^{2} \bar{\psi}\, O\, \tau_k\, \psi\, U^{(k)}. \tag{7.132}$$

On the other hand the interaction between real (and therefore neutral) scalar fields $U^{(3)}$, $U^{(4)}$ and a nucleon has in general the form

$$L_n' = \sqrt{2}\, f' \bar{\psi} O \tau_3 \psi\, U^{(3)} + g\bar{\psi} O \psi\, U^{(4)}. \tag{7.133}$$

We shall now introduce the meson field operators

$$\mathbf{L}_\mu \equiv (-i)\,(\partial_\mu \mathbf{U} \wedge \mathbf{U}) \tag{7.134}$$

$$\mathbf{L}[\sigma] = i \int_\sigma d\sigma_\mu \mathbf{L}_\mu, \tag{7.135}$$

[1]) We can always construct O to satisfy the relation $fO = f^* \gamma_4 O^* \gamma_4$ by adjusting the phase factor of the constant f.

where U means the three dimensional vector with components $U^{(k)}(k=1, 2, 3)$ and symbol $\wedge$ denotes a vector product. From (7.49b) we see that $e\mathbf{L}_3[\sigma]$ is equivalent to the total charge operator $e_t^{(m)}$ of the meson field, namely

$$e_t^{(m)} = e\mathbf{L}_3[\sigma]. \tag{7.136a}$$

The total charge operator $e_t^{(n)}$ for the nucleon field is given by (7.37) and (7.115b) as

$$\left.\begin{aligned} e_t^{(n)} &= i\,e\int_\sigma d\sigma_\mu\,\bar{\psi}\,\gamma_\mu\,\tfrac{1}{2}\,(1+\tau_3)\,\psi \\ &= \frac{e}{2}\,(\tau_3[\sigma]+I[\sigma]). \end{aligned}\right\} \tag{7.136b}$$

Here the quantity $\tau_3[\sigma]$ is the third component of the three dimensional vector $\boldsymbol{\tau}[\sigma]$ defined by

$$\boldsymbol{\tau}[\sigma] \equiv i\int_\sigma d\,\sigma_\mu\bar{\psi}\,\gamma_\mu\boldsymbol{\tau}\,\psi$$

and $I[\sigma]$ is

$$I[\sigma] \equiv i\int_\sigma d\,\sigma_\mu\,\bar{\psi}\,\gamma_\mu\psi.$$

Comparing this with (7.35a), we see that eigenvalues of $I[\sigma]$ are just numbers of nucleons. Introducing the operator J given by

$$\mathbf{J}[\sigma]=\tfrac{1}{2}\boldsymbol{\tau}[\sigma]+\mathbf{L}[\sigma], \tag{7.137a}$$

we find that

$$e_t = e_t^{(n)} + e_t^{(m)} = e\{J_3[\sigma]+\tfrac{1}{2}I[\sigma]\} \tag{7.137b}$$

is the total charge operator of the meson-nucleon system. This operator is time independent because of the charge conservation law. The operators $\mathbf{J}[\sigma]$, $(1/2)\boldsymbol{\tau}[\sigma]$ and $\mathbf{L}[\sigma]$ are called the total isotopic spin, nucleon isotopic spin and meson isotopic spin operators respectively.

We introduce a rotation by an angle α around the unit vector $\mathbf{e}$ in a three dimensional space, i.e. isotopic space. Under this rotation $U^{(k)}$ and $\bar{\psi}O\tau_k\psi$ are required to be transformed as two three-dimensional vectors according to

$$'U^{(k)} = a_{kl}\,U^{(l)} \tag{7.138a}$$

$$'\bar{\psi}\,O\,\tau_k\,'\psi = a_{kl}\,\bar{\psi}\,O\,\tau_l\,\psi \tag{7.138b}$$

where $[a_{kl}]$ is the well-known transformation matrix for a three dimensional vector under this rotation. The operator ψ is also linearly transformed by

$$'\psi = R\psi. \tag{7.138c}$$

From (7.138*b*) and (7.138*c*) we have

$$R^{-1}\,\tau_k\,R = a_{kl}\,\tau_l. \tag{7.139}$$

This equation is solved by the R given by

$$\left.\begin{aligned} R &= \exp\left\{-i\,\frac{\alpha}{2}\,(\boldsymbol{\tau}\cdot\mathbf{e})\right\} \\ &= \cos\frac{\alpha}{2} + i\,(\boldsymbol{\tau}\cdot\mathbf{e})\sin\frac{\alpha}{2}. \end{aligned}\right\} \tag{7.140a}$$

In particular, for an infinitesimal rotation, namely

$$\begin{aligned} a_{kl} &= \delta_{kl} + \delta w_{kl} \\ \delta w_{kl} &= -\,\delta w_{lk}, \end{aligned}$$

the equation (7.139) is solved by the R given by

$$R = I + \tfrac{1}{2}\tau_{kl}\,\delta w_{kl} \tag{7.140b}$$

with

$$\tau_{kl} \equiv \tfrac{1}{4}\,(\tau_k\,\tau_l - \tau_l\,\tau_k) = \frac{i}{2}\,\tau_m\,, \qquad (k, l, m = \text{cycl. } (1, 2, 3)).$$

On the other hand, by using the commutation relations given in Example 1 of Ch. VIII, we can derive the relations [1])

$$[\psi(x),\ \tau_{kl}[\sigma(x)]] = \tau_{kl}\psi(x) \tag{7.141a}$$

$$[U^{(m)}(x),\ \mathsf{L}_{kl}[\sigma(x)]] = \begin{cases} U^{(l)} & \text{for } m = k \\ -\,U^{(k)} & \text{for } m = l \\ 0 & \text{for } m \neq k,\ m \neq l. \end{cases} \tag{7.141b}$$

Here, the quantities $\tau_{kl}[\sigma]$ and $\mathsf{L}_{kl}[\sigma]$ are defined by

$$\left.\begin{aligned} \tau_{kl}[\sigma] &\equiv \frac{i}{2}\,\tau_m[\sigma] \\ \mathsf{L}_{kl}[\sigma] &\equiv i\,\mathsf{L}_m[\sigma] \end{aligned}\right\} \quad (k, l, m = \text{cycl. } (1, 2, 3))$$

and $\sigma(x)$ is a space-like surface passing through the point x. Since the field operators $\psi(x)$ and $U^{(k)}(x')$ with x and x' on a space-like surface, commute with each other, we have

$$[\tau_{k,l}[\sigma(x)],\ U^{(m)}(x)] = [\mathsf{L}_{kl}[\sigma(x)],\ \psi(x)] = 0.$$

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[1]) For example,

$$\begin{aligned} [U^{(1)}(x),\ \mathsf{L}_{12}[\sigma(x)]] &= i\int_{\sigma(x)} d\sigma'_\mu\,[U^{(1)}(x),\ \partial_\mu U^{(1)}(x')\cdot U^{(2)}(x') - \partial_\mu U^{(2)}(x')\cdot U^{(1)}(x')] \\ &= -\int_{\sigma(x)} d\sigma'_\mu\,\partial'_\mu\,\Delta(x-x')\cdot U^{(2)}(x') = \int d^3x'\,\delta(\mathbf{x}'-\mathbf{x})\,U^{(2)}(x') = U^{(2)}(x). \end{aligned}$$

Thus. (7.138a) and (7.138c) can be rewritten

$$'\psi(x) = \mathsf{R}^{-1}\psi(x)\mathsf{R} \tag{7.142a}$$

$$'U^{(k)}(x) = \mathsf{R}^{-1}U^{(k)}(x)\mathsf{R}. \tag{7.142b}$$

Here the operator R is defined by

$$\mathsf{R} \equiv I + \tfrac{1}{2} J_{kl}[\sigma]\delta w_{kl} \tag{7.143a}$$

with

$$\left.\begin{aligned} J_{kl}[\sigma] &\equiv i J_m[\sigma] \\ &= i\{\tfrac{1}{2}\tau_m[\sigma] + \mathsf{L}_m[\sigma]\}, \quad (k, l, m = \text{cycl. } (1, 2, 3)). \end{aligned}\right\} \tag{7.143b}$$

The relation (7.142a) can be proved as follows:

$$\begin{aligned} \mathsf{R}^{-1}\psi(x)\mathsf{R} &= \psi(x) + \tfrac{1}{2}[J_{kl}[\sigma], \psi(x)]\,\delta w_{kl} \\ &= \psi(x) + \tfrac{1}{2}\tau_{kl}\,\delta w_{kl}\,\psi(x) \\ &= R\psi(x) = '\psi\,(x). \end{aligned}$$

The relation (7.143b) can be proved in the same way. We see, from (7.143a, b), that a given field quantity $F(x)$ is transformed into $'F(x$ under the infinitesimal rotation, according to

$$'F(x) = \mathsf{R}^{-1}F(x)\mathsf{R}. \tag{7.144}$$

This with (7.143a) shows that the total isotopic spin $\mathbf{J}[\sigma]$ is the generating operator of the rotation, and commutes with any quantity that is invariant under the rotation in the three-dimensional τ-space. The theory is said to be charge independent when its interaction Lagrangian is invariant under the rotations in the isotopic space. In the charge independent theory, $\mathbf{J}[\sigma]$ commutes with the interaction Lagrangian and is therefore time independent; – eigenvalues for magnitude and the third components of $\mathbf{J}[\sigma]$ are good quantum numbers.

Since the vector $\mathbf{J}[\sigma]$ is the generating operator of the rotation, its components satisfy the commutation relations for angular momenta, that is:

$$[J_k[\sigma], J_l[\sigma]] = iJ_m[\sigma] \quad (k, l, m = \text{cycl. } (1, 2, 3)). \tag{7.145a}$$

This leads to

$$[\tfrac{1}{2}\tau_k[\sigma], \tfrac{1}{2}\tau_l[\sigma]] = \tfrac{1}{2}i\tau_m[\sigma] \quad (k, l, m = \text{cycl. } (1, 2, 3)) \tag{7.145b}$$

$$[\mathsf{L}_k[\sigma], \mathsf{L}_l[\sigma]] = i\mathsf{L}_m[\sigma] \quad (k, l, m = \text{cycl. } (1, 2, 3)). \tag{7.145c}$$

These relations can also be derived from (7.130), (7.141a) and (7.141b).

Therefore, $(1/2)\tau_m[\sigma]$ and $\mathsf{L}_m[\sigma]$ have the eigenvalues $(S, S-1, \ldots -S)$, where S assumes integer or half-integer values.

Since the one nucleon or one π-meson states have two or three charge states respectively, $S=1/2$ for $(1/2)\boldsymbol{\tau}[\sigma]$ and $S=1$ for $\mathbf{L}[\sigma]$ respectively. In other words, the magnitude of isotopic spin is $1/2$ for the nucleon, and 1 for the π-meson. Since $e\mathsf{L}_3[\sigma]$ and $(e/2)\,(I[\sigma]+\tau_3[\sigma])$ are charge operators for the π-meson and nucleon respectively, we obtain

$$\mathsf{L}_3[\sigma]\omega(\alpha)=\eta\omega(\alpha) \tag{7.146a}$$

$$\eta = \begin{cases} +1 & \text{for } \alpha=+ \\ 0 & \text{for } \alpha=0 \\ -1 & \text{for } \alpha=- \end{cases}$$

$$\tfrac{1}{2}(I[\sigma]+\tau_3[\sigma])\;\omega(P)=\omega(P) \tag{7.146b}$$

$$\tfrac{1}{2}(I[\sigma]+\tau_3[\sigma])\;\omega(N)=0. \tag{7.146c}$$

Here $\omega(+)$, $\omega(-)$ and $\omega(0)$ are the state vectors for the π^+, π^- and π^0-mesons, respectively, and $\omega(P)$ and $\omega(N)$ for the proton and neutron, respectively. Furthermore, since $I[\sigma]$ is equal to the number operator N for the nucleon (see (7.37)) its eigenvalues are the differences of numbers of nucleons and antinucleons. Thus we have:

$$I[\sigma]\;\omega(P)=\omega(P) \tag{7.147a}$$

$$I[\sigma]\;\omega(N)=\omega(N). \tag{7.147b}$$

The following relations can be obtained from (7.146*b*, *c*) and (7.147*a*, *b*):

$$\tfrac{1}{2}(I[\sigma]-\tau_3[\sigma])\;\omega(P)=0 \tag{7.148a}$$

$$\tfrac{1}{2}(I[\sigma]-\tau_3[\sigma])\;\omega(N)=\omega(N). \tag{7.148b}$$

As the projection operators of the three charge states of the one π-meson, we can adopt the operators n^+, n^-, n^0, given by [1])

$$n^+=\tfrac{1}{2}\{\mathsf{L}_3[\sigma]+(\mathsf{L}_3[\sigma])^2\} \tag{7.149a}$$

$$n^-=\tfrac{1}{2}\{-\mathsf{L}_3[\sigma]+(\mathsf{L}_3[\sigma])^2\} \tag{7.149b}$$

$$n^0=\tfrac{1}{2}(\mathbf{L}[\sigma]\cdot\mathbf{L}[\sigma])-(\mathsf{L}_3[\sigma])^2 \quad \text{or}$$

$$n^0=1-(\mathsf{L}_3[\sigma])^2 \tag{7.149c}$$

$$n^\alpha\omega(\beta)=\delta_{\alpha\beta} \qquad (\alpha,\beta = +, -, 0) \tag{7.149d}$$

(see WATSON [1950]). On the other hand, the projection operators

[1]) In (7.148*c*) the eigenvalue of $(\mathbf{L}[\sigma]\cdot\mathbf{L}[\sigma])$ is $S(S+1)$ for isotopic angular momentum S.

for the one proton state and the one neutron state, respectively, are written as

$$n^P = \tfrac{1}{2}(I[\sigma] + \tau_3[\sigma]) \tag{7.150a}$$

$$n^N = \tfrac{1}{2}(I[\sigma] - \tau_3[\sigma]) \tag{7.150b}$$

by using (7.146*b*, *c*) and (7.148*a*, *b*).

The interchange of proton and neutron (the so-called T-transformation) is represented by

$$\psi' = T\psi \tag{7.151a}$$

with

$$T \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \tau_1.$$

Since

$$\bar{\psi}' \tau_k \psi' = \varepsilon \bar{\psi} \tau_k \psi, \qquad (\tau_4 \equiv 1),$$

$$\varepsilon = \begin{cases} 1 & \text{for } k=1 \\ -1 & \text{for } k=2 \\ -1 & \text{for } k=3 \\ 1 & \text{for } k=4, \end{cases}$$

the interactions (7.132) and (7.133) are invariant under the T-transformation as long as $U^{(k)}$ is transformed as

$$U^{(k)\prime} = t_{kl}\, U^{(l)} \tag{7.151b}$$

$$[t_{kl}] \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We call this the T-transformation of the U-field. A theory which is invariant under the T-transformation is called charge symmetric.

In particular, if $f = f'$ and $g = 0$ in (7.132) and (7.133), the theory is charge symmetric and charge independent because of (7.138*a*, *b*). However, since the electromagnetic interaction is not invariant under the T-transformation or rotation in τ-space, we can have an approximately charge symmetrical or independent theories by neglecting the powers of (e/f), but only when $f \geqslant e$. The mass difference between the proton and the neutron also destroys the charge symmetry and independence (cf. § 5 of Ch. IV). However, it is possible that this mass difference arises just from the electromagnetic interactions.

As an example of the application of the isotopic spin conservation law, we now consider the production of π-mesons by collisions of two nucleons. The processes concerned are

$$P + P \to \pi^+ + D \tag{7.152a}$$

$$N + P \to \pi^0 + D. \tag{7.152b}$$

The isotopic spin of a $(P+P)$-system is 1. Since the state of the deuteron is ${}^3S$ (orbital angular momentum $l=0$, spin $S=1$), its isotopic spin must be zero (because of Pauli's exlusion principle). Thus the isotopic spin of the $(\pi+D)$ system is 1. On the other hand, the isotopic spin of the $(N+P)$ system has two eigenvalues, 1 and 0, which correspond to the symmetric part $(1/\sqrt{2})(\omega_1(P)\omega_2(N)+\omega_1(N)\omega_2(P))$ and the antisymmetric part $(1/\sqrt{2})(\omega_1(P)\omega_2(N)-\omega_1(N)\omega_2(P))$ of the wave function respectively. Since isotopic spin is conserved in the charge independent theory, the $(N+P)$ state with isotopic spin 0 makes no contribution to process (7.152*b*) above. Since the transition matrix elements of the two processes (7.152*a*, *b*) are equivalent because of the charge independence, the cross sections of these two processes are related by

$$\sigma(P+P \to \pi^+ + D) = 2\sigma(N+P \to \pi^0 + D) \tag{7.153}$$

(MESSIAH [1952]). Experimental results [1]) on the angular distributions of the two processes $N+P \to \pi^0+D$ and $\pi^+ + D \to P+P$ (the inverse process of (7.152*a*)) in the centre of gravity system show good agreement with (7.153). This fact lends support to the charge independence of the theory of the π-meson–nucleon system.

Quite generally the isotopic angular momentum vector of any field is defined as the operator generating the transformation of the field quantity $Q^{(i)}$ under rotations in the three dimensional isotopic space.

In the charge independent theory the third component of the total isotopic spin gives a good quantum number. Consider a state with numbers $n_i (i=1, \ldots)$ of particles [2]). The suffices signify the kinds of particles. It follows from (7.137*b*) that the total charge e_t is given by

$$e_t = eJ_3[\sigma] + \tfrac{1}{2}eN \tag{7.137c}$$

with

$$N = \Sigma\, n_i\, C_i. \tag{7.137d}$$

[1]) Proceeding of the Third Annual Rochester Conference on High Energy Physics. Dec. (1952).

[2]) The number of the antiparticle is counted with a negative sign.

In (7.137c) $J_3[\sigma]$ is the third component of the total isotopic spin due to all particles in the state. The C_i is defined such that $(eC_i/2)$ is the average value (or the charge centre) of charges in all possible charge-states of the (i)-particle. The conservation of the total electric charge together with the conservation of the third component of the total isotopic spin leads to the law of conservation of N appearing in (7.137c). For the nucleon-meson system, N is equal to the number of nucleons, and therefore is conserved in all known transmutation processes in this system.

As an example of the selection rule given by the conservation of N, we shall consider the process

$$\pi^- + P \to A^+ + B^-.$$

Here A and B have charge states e, o, $-e$ (isotopic spin 1) and $-e$, o (isotopic spin 1/2), respectively. This process cannot be induced by charge independent interactions, because $N = 1$ and -1 for the left and right hand sides of the above process respectively. By assuming that θ^0 and K^+ are two states of the θ-particle and that K^- is the antiparticle of K^+, we obtain the charge centre of θ to be $e/2$. This leads to $C_i = 1$ for the θ-particle. It must be noted that the conservation of N can be slightly affected by the electromagnetic and other weak interactions, which are not charge independent [1]).

EXAMPLE 11. INTERACTIONS BETWEEN KNOWN PARTICLES

The coupling constants of the interactions between known particles can be estimated from the experimental data on their mutual transformations, (see § 5 of Ch. IV).

Since, for the purposes of the present example, we require only a rough estimate of the coupling constants, we shall write the interactions L' as the products of wave functions without taking into account the detailed structure of the interactions (FERMI [1950]). As shown in Ch. XIII the probability per unit time of a process is, to the lowest order of the power series in the coupling constants, (cf. (13.6)).

$$\frac{dw}{dt} = 2\pi |S|^2 \varrho \tag{7.154a}$$

with

$$S \equiv \int d^3x\, L'(x). \tag{7.154b}$$

[1]) In effect the conservation law of N was used in the GELL-MANN, NAKANO and NISHIJIMA theory [1953] for the nucleon, k, θ, Λ particle system (cf. § 5 in Ch. IV).

Here ϱ is the density of final states per unit energy. For the present we can interpret (7.154a) as the natural extension of the perturbation calculation in quantum mechanics. In the interaction L', the matrix element of the wave function Q for a creation or annihilation of a particle (mass $\varkappa$) with small energy-momentum (k, k_0) can be *roughly* approximated by

$$Q(x) \approx \begin{cases} V^{-1/2}\, e^{\pm i(\mathbf{k}\cdot\mathbf{x}-k_0 t)} & \text{for spin } 1/2 \\ (2\varkappa V)^{-1/2}\, e^{\pm i(\mathbf{k}\cdot\mathbf{x}-k_0 t)} & \text{for spin } 0. \end{cases} \qquad (7.155a)$$

Here + or − signs in the exponential factors correspond to the annihilation or creation processes, respectively. The symbol V denotes the volume of the integration domain in (7.154b). A derivation of (7.155a) will be given in Ch. IX (cf. (9.43) and (9.78)). While postponing proofs of (7.154a) and (7.155a) to the following Chapters, we shall use them for the present rough estimation of the coupling constants. Since we are only interested in processes, through which the total energy and momentum conserve, we can replace $Q(x)$ in L' by

$$Q(x) \approx \begin{cases} V^{-1/2} & \text{for spin } 1/2 \\ (2\varkappa V)^{-1/2} & \text{for spin } 0 \end{cases} \qquad (7.155b)$$

without taking care about exponential factors in (7.155a).

In the present Chapter we shall denote the wave function of the fields by the same symbols as are used for the particles themselves. The mass of a particle "A" will be denoted by $\varkappa(A)$, unless A is not the electron whose mass will be denoted by m.

The electromagnetic coupling constant, i.e. the elementary electric charge e, has been accurately determined as $e/\sqrt{4\pi} = 1/\sqrt{137}$. As is well known, all charged particles so far observed have the same charge e. This fact is expressed in the theory of all charged fields in the presence of an electromagnetic field, by its gauge-invariance [1]). In fact, it is shown in Ch. IX (e.g. see (9.81)) that the charge conservation derived by using gauge-invariance is equivalent to the conservation of the difference between the numbers of positively and negatively charged particles. This means that each charged particle (antiparticle) carries the same charge $e(-e)$.

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[1]) On the other hand, if there were a field $\psi(n)$ with charge ne (n not necessarily an integer), charge conservation requires that the gauge transformation would have to be changed as follows:

$$\psi(n) \to \psi(n) \exp\{in\,\Lambda(x)\}.$$

The coupling constant g_π^n of the π-meson–nucleon interaction can be estimated from a knowledge of nuclear forces. But its accurate value is not yet known because of various difficulties in the theoretical analysis of the π-meson. The values obtained by various analyses fall between the limits $0.1 \sim 20$.

The coupling constant g_μ^π of the interaction between the π-meson and the μ-meson fields can be estimated from the life time $\tau_\pi = 2 \times 10^{-8}$ sec. of π–μ decay. For the π–μ interaction S has the form $S = g_\mu^\pi \int d^3x \pi \mu \mu^0$, where μ^0 denotes a light neutral particle created in the π–μ decay (cf. § 5 of Ch. IV). Since the spin of the π-meson is 0, we can replace π in S by $(2\varkappa(\pi)V)^{-1/2}$. Similarly, when we assume that both spins of the μ and μ^0 fields are $\frac{1}{2}$, μ and μ^0 in S can be replaced by $(V)^{-1/2}$ because of (7.155b). Thus S is approximately equal to $g_\mu^\pi / \sqrt{2\varkappa(\pi)V}$. In the centre of gravity system, for which $\mathbf{k} + \mathbf{k}' = 0$ ($\mathbf{k}$ and $\mathbf{k}'$ are the momenta of μ and μ^0), the number of states in the energy region $(E, E+dE)$ is [1])

$$\varrho dE = 4\pi k^2 \frac{V dk}{(2\pi)^3} = \frac{k^2 V}{2\pi^2} \frac{dk}{dE} dE = \frac{V k^2}{2\pi^2 (v_\mu + v_{\mu^0})} dE.$$

Here $k \equiv |\mathbf{k}|$, and v_μ and v_{μ_0} are the velocities of μ and μ^0 respectively. Since experimental analysis shows that the μ^0 mass is very small, we can take v_μ^0 to be the velocity of light $C\,(=1)$. When we adopt $\varkappa(\pi) = 276$ m and $\varkappa(\mu) = 210$ m, k can be calculated by means of the energy conservation law:

$$276 \text{ m} = \sqrt{k^2 + (210 \text{ m})^2} + k.$$

Thus, we have $k = 58$ m [2]), and so

$$v_\mu = \frac{k}{\sqrt{k^2 + \varkappa(\mu)^2}} = 0.22.$$

By using this value, we obtain the result

$$\frac{dw}{dt} = 1.2 \times 10^{21} (g_\mu^\pi)^2.$$

Since the life time τ_π is equal to $(dw/dt)^{-1}$, we obtain

$$g_\mu^\pi \approx 0.2 \times 10^{-6} = 2.4 \times 10^{-6}\,(e/\sqrt{4\pi}). \qquad (7.157)$$

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[1]) $\frac{dE}{dk} = \frac{d}{dk}\left(\sqrt{k^2 + \varkappa(\mu)^2} + \sqrt{k^2 + \varkappa(\mu^0)^2}\right) = v_\mu + v_{\mu^0}$.

[2]) In the unit system of $\hbar = C = 1$, m is $m = 0.26 \times 10^{11} \text{cm}^{-1} = 0.78 \times 10^{21}$ sec^{-1}.

The coupling constant g_β of the interaction between nucleon and electron fields can be estimated from the life-time $\tau_\beta = 12 \times 60$ sec. of the natural decay of a neutron. When we simply write the interaction as $g_\beta \int d^3x\, NPev$, by (7.155*b*) S can be replaced by $g_\beta V^{-1}$. The relation (7.154*b*) shows that the dimension of S is that of energy, that is [1])

$$[S] = [L^{-1}].$$

Thus g_β has the dimension

$$[g_\beta] = [S][V] = [L]^2.$$

Estimating g_β in the same way as g_μ^π we obtain

$$g_\beta = 2 \times 10^{-12} \{(4\pi/3) r_0^3\} \text{ erg. cm}^3 \qquad (7.158a)$$

where $r_0 = e^2/m$ the classical electron radius. Therefore we may call $(4\pi/3) r_0^3$ the classical electron volume. The units in which the classical electron volume is 1 are called relativistic units (or simply r.u.). Thus g_β is

$$g_\beta = 2 \times 10^{-12} \text{ erg. r.u.} = 1.3\, eV \text{ r.u.} \qquad (7.158b)$$

Equation (7.158*b*) shows that the g_β-interaction is roughly equivalent to a square well potential of height 1.3 eV and width r_0. Thus we can see that the g_β-interaction is much weaker than the nuclear force which is roughly represented by a square well potential of height $20 \times 10^6\, eV$ and width $\approx r_0$.

In a similar way we can estimate the coupling constant g_μ^e of the interaction between the electron field and the μ-meson field from the lifetime $\tau_\mu = 2.15 \times 10^{-6}$ sec. of the natural decay of a μ-meson. Using the interaction $g_\mu^e \int d^3x \mu e \nu\nu$, where it is assumed that the two light neutral particles in the final state are neutrinos (ν), we have:

$$g_\mu^e \approx 3 \times 10^{-12} \text{ erg. r.u.} \approx g_\beta. \qquad (7.159)$$

Although $g_\mu^e \approx g_\beta$, a neutron has a longer life time than a μ-meson because of the smallness of the mass difference between the proton and neutron.

The coupling constant between the proton and μ-meson fields is estimated from the capture process $P + \mu^- \to N + \nu$ of a negatively charged μ-meson by nuclei. The experimental results show that the probability of this capture process in nuclei of atomic number $Z = 11$ is about equivalent to that of the μ–e decay process (VALLEY [1947]),

[1]) $[L^n]$ means dimensions of the nth power of the length.

i.e. $1/\tau_\mu (= 0.5 \times 10^6\ \text{sec}^{-1})$. Since the total time before a μ^--meson of several Mev is slowed down and captured into the lowest Bohr orbit (i.e. K-orbit) is much smaller than the decay life-time (FERMI and TELLER [1947]), we can assume that all μ-mesons are first captured in K-orbits and then captured by the nucleus. The radius of the K-orbit, a', is

$$a' = \frac{m}{\varkappa(\mu)} \frac{1}{Z} a = 2.5 \times 10^{-11}/Z \text{ cm}, \tag{7.160}$$

where a is the Bohr radius of an electron in a Hydrogen atom. This shows that the larger Z is, the smaller the radius must be, and so the wave functions of the μ^--mesons and the nuclei overlap to a great extent. Thus, the capture probability increases with Z and is equal to the decay probability at $Z = 11$.

We can estimate the probability of the capture of a μ^--meson by a proton in a nucleus by means of the simple interaction $L' = g_\mu^P P N \mu \nu$. Then we have

$$\frac{dw}{dt} = 11400\ (g_\mu^P)^2\ m^2/V, \tag{7.161}$$

In this calculation, the wave function P of a proton must be taken as $(V)^{-1/2}$ inside the K-orbit and zero outside. Thus we have, for the volume of the domain of the interaction, $V \approx 4\pi a'^3/3$. The capture probability of a μ^--meson by the nucleus is (7.161) multiplied by Z, because all protons in the nuclei can capture the μ^--meson. Thus we can see that the capture probability by a nucleus is roughly proportional to Z^4. This fact is consistent with the experimental results. A proton cannot make a transition to an occupied neutron state because of the Pauli exclusion principle. This effect makes the capture probability by the nuclei even smaller, because the main part of the energy of the μ^--meson is taken up by the neutral particles ν. Estimating roughly this effect (FERMI [1950]), we obtain the following capture probability of a μ^--meson by a nucleus:

$$\frac{dw}{dt} \approx 5000\ (g_\mu^P)^2\ m^2\ Z^4 \left(\frac{\varkappa(\mu)}{m} \frac{1}{a}\right)^3. \tag{7.162}$$

Comparing (7.162) for $Z = 11$ with $1/\tau_\mu$ we have

$$g_\mu^P \approx 10^{-12} \text{ erg. r.u.} \approx g_\mu^e \approx g_\beta. \tag{7.163}$$

The above interaction scheme is represented graphically in Fig. 7.1. If it were assumed that the electron and the nucleon fields interact

via the π-meson field, we could estimate the coupling constant of the interaction between the electron and the π-meson fields from g_π^n and g_β. Such an estimate leads to the conclusion that most π-mesons would decay not into μ-mesons, but into electrons. This conclusion is not compatible with the experimental results, which show that not more than one π^+-meson in 1400 decays into a positon. (FRIEDMAN and RAINWATER [1951]). Thus we conclude that the β-interaction is not mediated by the π-meson field. On the other hand the assumption that g_μ^p-interaction is mediated by the π-meson field has some advantage. For, if the π-meson and the μ-meson fields did not interact directly but through the agency of the nucleon field, then, because the mass of the electron is smaller than that of the μ-meson, π–e decay would take place. Such a situation can be avoided by assuming suitable forms for the g_μ^p-interactions and the g_π^n-interactions. This is represented by Fig. 7.1, in which (P, N) and μ are connected by a dotted line.

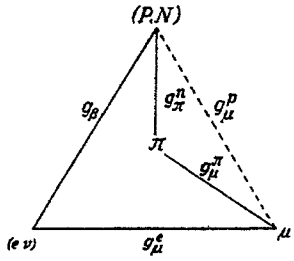

Fig. 7. 1

An interesting symmetry is apparent in Fig. 7.1. The particles (P, N), (e, ν) and μ on the vertices of the triangle have the same spin 1/2 and the coupling constants g_β, g_μ^e, $g_{\mu|}^p$ of their interactions are strikingly similar. This symmetry was first noticed by O. KLEIN [1948] (see also TIOMNO and WHEELER [1949]). Although the significance of this symmetry is not yet understood, it may well be more than coincidental. Extending this symmetry, many attempts have been made to examine the possibility that the interactions between all Dirac particles are of the same type. This interaction is termed the universal Fermi interaction. The first work on the symmetrical interactions between P, N, e and ν was made by CHRITCHFIELD and WIGNER, [1941]. They showed [1]) that there is no interaction which is completely symmetric between Dirac fields, and only one completely antisymmetric interaction [2]). The latter interaction is equivalent to the linear combination (scalar) + (axialvector) − (pseudoscalar). However, this interaction is not compatible with the experimental results which show the existence of a β-interaction of the

[1]) See also CHRITCHFIELD [1943].

[2]) More generally, PEYCE [1952] investigated the irreducible representations of the permutation group for four spinors.

Tensor-type (cf. Example 8). The assumption of the symmetry between charged states of the nucleon field and between charged states of the electron field leads us to two linear combinations (DE GROOT and TOLHOEK [1950]), namely (SAP) or (TV), which are also incompatible with the experimental results that exclude the existence of V. Many attempts have been made to improve the theory of the universal Fermi interaction by taking into account (P, N, e, ν, μ) and pseudospinors (MICHEL [1950], CAIANIELLO [1951]). However, there is the further uncertainty arising from the necessity of deciding which particles play corresponding roles in their processes. The electron spectrum of the μ–e decay provides important information on the types of the linear combination in the g_μ^e-interaction (MICHEL [1949, 1950]).

The coupling constants of known interactions may be divided into four classes. The first consists of strong interactions having coupling constants of about the same order as g_π^π. The experimental results for the Λ-production phenomena have suggested that the Λ-nucleon interaction also belongs to this class [1]). Many experiments on π-nucleon phenomena have shown their charge independence. If the strong Λ-nucleon interaction were not charge independent, it would disturb the charge independence of π-nucleon phenomena. Thus the strong Λ-nucleon interaction seems to be charge independent. The interaction of the second class is the electromagnetic one. In the third class we have weak interactions between Bose- and Fermi-particles, the coupling constants of which are about the same as g_μ^π (OGAWA, OKONOGI and ONEDA [1954]). The fourth class is made up of weak interactions between Fermi particles with coupling constants $\approx g_\beta$. It is hoped that experimental information on the interactions realized in nature will suggest the natural law which selects them from the wide range of possible interaction types in the present quantum field theory.

---

[1]) The long life-time of the Λ-decay process seems incompatible with the existence of a strong Λ-nucleon interaction. There have been some attempts to find selection rules based on the charge independence of strong interactions in order to make the probability of the Λ-decay process small. (PAIS, [1953], NAKANO and NISHIJIMA [1953], GELL-MANN [1953].)

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Note added in proof

Rough estimations being based on (7.154) and (7.155*b*) show that coupling constants of interactions for decay processes of Ξ, Σ, Λ-particles and those for decay processes of $K_{\mu 3}$, K_{e3}-particles belong to the third and fourth classes, respectively. Furthermore, interactions between Bose particles, i.e., interactions for decay processes of $\theta, K_{\pi 2}, \tau$-particles, also seem to belong to the third class; their coupling constants are about the same as g_μ^τ.

CHAPTER VIII

QUANTUM THEORY OF FREE FIELDS (I)

COMMUTATION RELATIONS

§ 1. Quantum Theory of Free Fields

The commutation relations between the wave functions at different times are not simple for interacting fields because of the non-linearity of their wave equations. On the other hand, when the wave functions of the free fields obey the Klein–Gordon equation we can construct their covariant commutation relations by means of the quantum mechanics of the simple harmonic oscillator.

We first consider a field described by complex quantities $Q_\alpha(x)$ and their Hermitian conjugate quantities $Q_\alpha^*(x)$. The wave equation and the Schrödinger equation for a free field in the Heisenberg representation are

$$\frac{\partial L^0}{\partial Q_\alpha} - \partial_\mu \left(\frac{\partial L^0}{\partial Q_{\alpha;\mu}} \right) = 0, \tag{8.1a}$$

$$\frac{\delta}{\delta \sigma(x)} \Psi[\sigma] = 0. \tag{8.1b}$$

Here L^0 is the Lagrangian of the free field.

We can write the linear differential equation (8.1a) as

$$\Lambda_{\alpha\beta}(\partial)\, Q_\beta(x) = 0 \tag{8.2a}$$

where $\Lambda_{\alpha\beta}(\partial)$ is an operator. This can be derived from the Lagrangian

$$\bar{L}^0 \doteqdot \int d^4x\, \bar{Q}_\alpha(x) \Lambda_{\alpha\beta}(\partial)\, Q_\beta(x). \tag{8.3a}$$

Here $\doteqdot$ means equivalence except for a surface integral over the boundary of the volume of integration arising from the partial integration and $\bar{Q}_\alpha(x)$ is connected with $Q_\alpha^*(x)$ by means of a matrix η according to

$$\bar{Q}_\alpha(x) \equiv Q_\gamma^*(x) \eta_{\gamma\alpha}. \tag{8.3b}$$

The matrix η must be non-singular. Indeed, the variation $Q_\alpha^* \to Q_\alpha^* + \delta Q_\alpha^*$ in the Lagrangian leads to the equation

$$\bar{\Lambda}_{\alpha\beta}(\partial) Q_\beta(x) = 0 \tag{8.2b}$$

where

$$\bar{A}(\partial) \equiv \eta A(\partial) \tag{8.4}$$

with

$$\bar{A}(\partial) \equiv [\bar{A}_{\alpha\beta}(\partial)], \quad A(\partial) \equiv [A_{\alpha\beta}(\partial)].$$

This equation could not give (8.2), if the matrix η were singular. An example of (8.3*b*) has been given by (3.32) for the Dirac theory, where $\eta = \gamma_4$.

According to K.G. condition of Ch. II, field quantities must satisfy the Klein Gordon equation

$$(\Box - \varkappa^2)\, Q_\alpha(x) = 0.$$

Since the commutation relation between $Q_\alpha(x)$ and $\bar{Q}_\beta(x')$ must be independent of any Lorentz frame, it depends on (x, x') through $(x-x')$ alone, and is of the form [1])

$$[Q_\alpha(x), \bar{Q}_\beta(x')]_\pm = \Delta_{\alpha\beta}(x-x'). \tag{8.5}$$

Indeed, any function of x, x' can be written as a function of $(x-x', x+x')$ and only $x-x'$ is invariant under the transformation $x \to x+a$, $x' \to x'+a$.

Under the $L_+^\uparrow$, $L_+^\downarrow$ Lorentz transformation, (cf. § 2 of Ch. IV), $\Delta_{\alpha\beta}(x)$ transforms in the same way as $Q_\alpha \times \bar{Q}_\beta$. It satisfies the equation

$$(\Box - \varkappa^2)\, \Delta_{\alpha\beta}(x) = 0. \tag{8.6}$$

The property of commutation relations under the time reflection requires special consideration. We shall postpone it to Example 5.

As shown later, the $(-)$ and $(+)$ types correspond to fields of the integer and the half-integer spins.

On the other hand, there is another requirement for $\Delta_{\alpha\beta}$. Since the velocity of the signal cannot be greater than that of light, the field quantities at two points that are space-like with respect to each other can be independently determined by the observations. This means that $\Delta_{\alpha\beta}(x-x')$ in the commutation relation (8.5) is zero where x and x' are space-like with respect to each other, i.e.

$$\Delta_{\alpha\beta}(x) = 0 \qquad \text{for a space-like vector } x_\mu. \tag{8.7}$$

---

[1]) The subscripts $(\pm)$ mean:

$$[A, B]_+ \equiv AB + BA, \quad [A, B]_- \equiv [A, B] = AB - BA.$$

From (8.6) it follows that the Fourier expansion of $\Delta_{\alpha\beta}(x)$ has the form [1])

$$\Delta_{\alpha\beta}(x) = \int d^4k\, e^{ik_\mu x_\mu}\, \delta(k_\mu k_\mu + \varkappa^2)\, \Delta_{\alpha\beta}(k). \tag{8.8}$$

Invariance under L_+^+, L_+^- Lorentz transformations requires that $\Delta_{\alpha\beta}(k)$ is given by [2])

$$\Delta_{\alpha\beta}(k) = F_{\alpha\beta}(ik)\, (a + b\varepsilon(k)) \tag{8.9}$$

where $F_{\alpha\beta}(ik)$ is a quantity which transforms in the same way as $Q_\alpha \times \bar{Q}_\beta$ under the Lorentz transformation and a and b are two constants. The symbol $\varepsilon(z)$ (where z is a vector z_μ) is defined as

$$\varepsilon(z) \equiv \begin{cases} +1 & z_0 > 0 \\ -1 & z_0 < 0. \end{cases} \tag{8.10}$$

Thus $\Delta_{\alpha\beta}(x)$ can be written as a linear combination of the two "invariant delta functions" Δ and $\Delta^{(1)}$ differentiated by an operator $F_{\alpha\beta}(\partial)$, where Δ and $\Delta^{(1)}$ are

$$\Delta(x) = -\frac{i}{(2\pi)^3} \int d^4k\, e^{ik_\mu x_\mu}\, \delta(k_\mu k_\mu + \varkappa^2)\, \varepsilon(k) \tag{8.11a}$$

$$= \frac{i}{(2\pi)^3} \int d^3k\, e^{ik_l x_l}\, (1/2\, K_0)\, (e^{iK_0 t} - e^{-iK_0 t}), \tag{8.11b}$$

$$\Delta^{(1)}(x) = \frac{1}{(2\pi)^3} \int d^4k\, e^{ik_\mu x_\mu}\, \delta(k_\mu k_\mu + \varkappa^2). \tag{8.12}$$

Here

$$K_0 \equiv \sqrt{(k_l k_l + \varkappa^2)}.$$

However, since, as we shall show later, $\Delta^{(1)}$ is not zero for a spacelike vector x, we must take $a = 0$ in (8.9) and therefore (8.5) has a form

$$[Q_\alpha(x), \bar{Q}_\beta(x')]_\pm = iF_{\alpha\beta}(\partial)\Delta(x - x'), \tag{8.13a}$$

where $F_{\alpha\beta}(\partial)$ is a differential operator with the same Lorentz transformation properties as $Q_\alpha \times \bar{Q}_\beta$.

As shown in Ch. II, there exists a derivation operator $d(\partial) \equiv [d_{\alpha\beta}(\partial)]$ which satisfies the relation (2.7), namely

$$d(\partial)\Delta(\partial) = (\Box - \varkappa^2)I. \tag{8.13b}$$

---

[1]) $d^4k \equiv dk_1\, dk_2\, dk_3\, dk_0$, $d^3k \equiv d\mathbf{k} \equiv dk_1\, dk_2\, dk_3$

[2]) It is easily seen that $\delta(k_\mu k_\mu + \varkappa^2)\, \varepsilon(k)$ is invariant under L_+^+, L_+^- transformations, because its value in the upper and lower parts of the light cone are $\delta(k_\mu k_\mu + \varkappa^2)$ and $-\delta(k_\mu k_\mu + \varkappa^2)$ respectively.

We shall prove in the next Chapter that $F_{\alpha\beta}(\partial)$ must be identical with $d_{\alpha\beta}(\partial)$ in (8.13*b*), and also that $Q_\alpha(x)$ ($\bar{Q}_\alpha(x)$) and $Q_\beta(x')$ ($\bar{Q}_\beta(x')$) commute with each other (TAKAHASHI and UMEZAWA [1953], RIVIER [1953]). Thus we have [1])

$$\left.\begin{aligned}&[Q_\alpha(x), \bar{Q}_\beta(x')]_\pm = id_{\alpha\beta}(\partial)\, \Delta(x-x')\\ &[Q_\alpha(x), Q_\beta(x')]_\pm = [\bar{Q}_\alpha(x), \bar{Q}_\beta(x')]_\pm = 0.\end{aligned}\right\} \tag{8.14a}$$

When $Q_\alpha(x)$ are real field operators, the commutation relations are

$$[Q_\alpha(x), Q_\beta(x')]_\pm = id_{\alpha\beta}(\partial)\, \Delta(x-x'). \tag{8.14b}$$

It must be remarked that we can derive

$$A(\partial)\, d(\partial) = (\Box - \varkappa^2) I \tag{8.13c}$$

from (8.13*b*). Indeed, (8.13*b*) leads to

$$\bar{d}(\partial)\, \bar{A}(\partial) = (\Box - \varkappa^2) I \tag{8.13d}$$

where

$$\bar{d}(\partial) \equiv d(\partial)\, \eta^{-1}. \tag{8.13e}$$

Since the Lagrangian is a real quantity, $\bar{A}(\partial)$ is hermitian on account of (8.3*a*). Thus, we see that $\bar{d}(\partial)$ is hermitian too, and therefore that

$$\bar{A}(\partial)\bar{d}(\partial) = (\Box - \varkappa^2) I.$$

This relation leads to (8.13*c*).

In (8.13*b*) it must be noted that $A(\partial)$ is completely defined, even to the extent of sign and a constant factor derived from the Lagrangian L^0. Thus, we can obtain $A(\partial)$ and $d(\partial)$ from the given Lagrangian L^0 by using (8.1*a*) and (8.13*b*). The operator $d(\partial)$ for various fields has been given in Ch. III, IV and V.

The wave equations (8.2*a*), the Schrödinger equation (8.1*b*) and the commutation relations [2]) (8.14*a*, *b*) are the fundamental equations

---

[1]) The commutation relations for fields that have many mass states and therefore satisfy (2.3*b*), can also be written in the form of (8.14*a*), (8.14*b*). In such a case $d(\partial)$ and Δ can be obtained from relations

$$d(\partial)A(\partial) = \prod_i (\Box - \varkappa_i^2), \quad \prod_i (\Box - \varkappa_i^2)\, \Delta(x) = 0.$$

For example, for the commutation relations of the Bhabha field, (cf. § 1, Ch. V) $d(\partial)$ was given by UMEZAWA and VISCONTI [1955]. General features of fields with various mass states have been discussed in detail by PAIS and UHLENBECK [1950].

[2]) Recently, an interesting theory of the derivation of the covariant commutation relations has been given by PEIERLS, [1952]. See also M. CINI [1952]. Their results agree with (8. 14*a*, *b*) in the case of free fields.

for the covariant quantum theory of free fields in the Heisenberg representation. As shown in the next Chapter, (8.2a) can be written in terms of the energy-momentum vector T_μ^0 of the free field as

$$-\partial_\mu F(x) = [F(x), T_\mu^0], \tag{8.15}$$

where $F(x)$ is any functional of $Q_\alpha(x)$.

§ 2. The Invariant Delta Functions, the Green's Functions and the Causality Condition

We shall now collect together the important properties of the invariant delta functions.

From (8.11b) we have

$$\left.\begin{aligned} &\Delta(x) = -\Delta(-x) \\ &\Delta(\mathbf{x}, x_4) = -\Delta(\mathbf{x}, -x_4) \\ &\Delta(\mathbf{x}, x_4) = \Delta(-\mathbf{x}, x_4) \end{aligned}\right\} \tag{8.16}$$

That is $\Delta(x)$ does not change in sign under a space reflection but does so under the time reflection. Moreover, (8.11a) and (8.11b) give [1])

$$(\Box - \varkappa^2)\Delta(x) = 0, \tag{8.17}$$

$$\left(\frac{\partial}{\partial t}\Delta(x)\right)_{t=0} = -\delta(\mathbf{x}). \tag{8.18}$$

From (8.16) we have

$$\Delta(\mathbf{x}, 0) = 0, \tag{8.19a}$$

$$(\partial_k \Delta(\mathbf{x}, x_4))_{t=0} = 0 \quad \text{for } k = 1, 2, 3, \tag{8.19b}$$

$$(\partial_k \Delta(\mathbf{x}, x_4))_{\mathbf{x}=0} = 0 \quad \text{for } k = 1, 2, 3. \tag{8.19c}$$

From equations (8.18) and (8.19a, b), we obtain the relations [2])

$$(\partial_\mu \varepsilon(x))\Delta(x) = 0 \tag{8.20a}$$

$$(\partial_\mu \varepsilon(x))(\partial_\nu \Delta(x)) = 2\delta_{\mu 4}\delta_{\nu 4}\delta^4(x) \tag{8.20b}$$

by using the relation [3])

$$\partial_\mu \varepsilon(x) = -2i\delta(t)\delta_{\mu 4}. \tag{8.20c}$$

---

[1]) $\delta(\mathbf{x}) \equiv \delta(x_1)\,\delta(x_2)\,\delta(x_3)$.

[2]) $\delta^4(x) \equiv \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,\delta(x_0)$.

[3]) From (8.10) we obtain

$$\partial_k \varepsilon(x) = 0 \quad (k = 1, 2, 3), \qquad \int_{-a}^{a} dt \frac{\partial}{\partial t}\varepsilon(x) = 2,$$

for any positive value of a. These relations lead to (8.20c).

From (8.20a, b) we have:

$$(\partial_\mu \partial_\nu \varepsilon(x)) \cdot \Delta(x) = -(\partial_\nu \varepsilon(x))(\partial_\mu \Delta(x)), \tag{8.20d}$$

$$(\Box - \varkappa^2)\,(\varepsilon(x)\,\Delta(x)) = 2\delta^4(x). \tag{8.20e}$$

The function $\Delta(x)$ can be written in terms of the Bessel function J_0 as

$$\Delta(x) = \frac{1}{4\pi}\frac{1}{r}\frac{\partial}{\partial r} F(r,t), \quad r \equiv (\mathbf{x}\cdot\mathbf{x})^{1/2} \tag{8.21a}$$

$$F(r,t) \equiv \left\{ \begin{array}{ll} J_0(\varkappa(t^2-r^2)^{1/2}) & \text{for } t > r \\ 0 & \text{for } -r < t < r \\ -J_0(\varkappa(t^2-r^2)^{1/2}) & \text{for } t < -r. \end{array} \right\} \tag{8.21b}$$

(See DIRAC, [1934]). In the particular case of mass $\varkappa = 0$, we have

$$\Delta(x)_{\varkappa=0} = \frac{1}{4\pi r}\{\delta(r-t) - \delta(r+t)\}. \tag{8.22}$$

(JORDAN and PAULI, [1928]).

Equation (8.21b) shows that $\Delta(x)$ vanishes outside the light cone with vertex at $x=0$ (i.e. in a space-like region). Since $F(r,t)$ varies discontinuously over the light cone, $\Delta(x)$ has a δ-function-like singularity on the light cone.

The function $\Delta^{(1)}(x)$ is a scalar satisfying the relation

$$(\Box - \varkappa^2)\,\Delta^{(1)}(x) = 0, \tag{8.23}$$

$$\Delta^{(1)}(x) = \Delta^{(1)}(-x), \tag{8.24}$$

$$\Delta^{(1)}(x) = \frac{1}{4\pi}\frac{1}{r}\frac{\partial}{\partial r} F_1(r,t), \tag{8.25a}$$

$$F_1(r,t) = \left\{ \begin{array}{ll} N_0(\varkappa(t^2-r^2)^{1/2}) & \text{for } t > r \text{ or } t < -r \\ -iH_0^{(1)}(i\varkappa(r^2-t^2)^{1/2}) & \text{for } r > t > -r. \end{array} \right. \tag{8.25b}$$

Here N_0, $H_0^{(1)}$ are the Neumann function and the Hankel function of the first kind, respectively. In the particular case of $\varkappa = 0$

$$\Delta^{(1)}(x)_{\varkappa=0} = \frac{1}{(2\pi)^2}\frac{1}{r^2-t^2}. \tag{8.26}$$

Equation (8.25b) shows that $\Delta^{(1)}(x)$ is not zero even outside the light cone.

We shall now introduce several functions that will appear frequently in the following discussion (SCHWINGER [1949]). These are

$$\bar{\Delta}(x) \equiv -\tfrac{1}{2}\varepsilon(x)\Delta(x), \tag{8.27a}$$

$$\Delta_F(x) \equiv \Delta^{(1)}(x) - 2i\bar{\Delta}(x), \tag{8.27b}$$

$$\begin{aligned}\Delta^{\mathrm{ret}}(x) &\equiv -\tfrac{1}{2}(1+\varepsilon(x))\Delta(x)\\ &= \bar{\Delta}(x) - \tfrac{1}{2}\Delta(x),\end{aligned} \tag{8.27c}$$

$$\begin{aligned}\Delta^{\mathrm{adv}}(x) &\equiv \tfrac{1}{2}(1-\varepsilon(x))\Delta(x)\\ &= \bar{\Delta}(x) + \tfrac{1}{2}\Delta(x).\end{aligned} \tag{8.27d}$$

The funtion $\bar{\Delta}(x)$ has the Fourier representation

$$\bar{\Delta}(x) = \frac{1}{(2\pi)^4} P \int d^4k\, e^{ik_\mu x_\mu} \frac{1}{k_\mu k_\mu + \varkappa^2}, \tag{8.28}$$

where P means that the value of the intergrand of (8.28), at its pole, is defined by the principal value (SCHWINGER [1949]).

From (8.27a) and (8.28),

$$\Delta(x) = -2\,\varepsilon(x)\frac{1}{(2\pi)^4} P \int d^4k\, e^{ik_\mu x_\mu} \frac{1}{k_\mu k_\mu + \varkappa^2}. \tag{8.29}$$

On the other hand we have [1])

$$\left.\begin{aligned}\delta_+(a) &\equiv \frac{1}{2\pi}\int_0^\infty d\beta\, e^{ia\beta} = \lim_{\epsilon\to+0}\frac{-1}{2\pi i(a+i\epsilon)}\\ &= -\frac{1}{2\pi i}\left(P\frac{1}{a} - i\pi\delta(a)\right)\end{aligned}\right\} \tag{8.30a}$$

where ϵ is an infinitesimal positive number.

The quantity $\delta_-(a)$ that is the complex conjugate of $\delta_+(a)$ is

$$\left.\begin{aligned}\delta_-(a) &\equiv \frac{1}{2\pi}\int_0^\infty d\beta\, e^{-ia\beta} = \frac{1}{2\pi}\int_{-\infty}^0 d\beta\, e^{ia\beta}\\ &= \lim_{\epsilon\to+0}\frac{1}{2\pi i(a-i\epsilon)} = \frac{1}{2\pi i}\left(P\frac{1}{a} + i\pi\delta(a)\right).\end{aligned}\right\} \tag{8.30b}$$

Therefore

$$\delta_+(a) + \delta_-(a) = \delta(a) \tag{8.31a}$$

$$\delta_+(a) - \delta_-(a) = \frac{i}{\pi} P\frac{1}{a}. \tag{8.31b}$$

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[1]) $\delta_+(a) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^\infty d\beta\, e^{ia\beta-\epsilon\beta} = \lim_{\epsilon\to+0}\frac{-1}{2\pi i(a+i\epsilon)}$.

Equations (8.28) and (8.12) then can be rewritten as

$$\left.\begin{aligned}\bar{\Delta}(x) &= -\frac{i\pi}{(2\pi)^4}\int d^4k\, e^{ik_\mu x_\mu}\{\delta_+(k_\mu k_\mu+\varkappa^2)-\delta_-(k_\mu k_\mu+\varkappa^2)\}\\ &= \frac{1}{2i(2\pi)^4}\int_{-\infty}^{\infty} d\beta\int d^4k\,\varepsilon(\beta)\, e^{i\beta(k_\mu k_\mu+\varkappa^2)+i(k_\mu x_\mu)},\end{aligned}\right\} \tag{8.32}$$

$$\Delta^{(1)}(x) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\beta\int d^4k\, e^{i\beta(k_\mu k_\mu+\varkappa^2)+i(k_\mu x_\mu)}. \tag{8.33}$$

By using (8.12), (8.27*b*) and (8.28), Δ_F can be written as

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^4k\, e^{ik_\mu x_\mu}\left\{2\pi\delta(k_\mu k_\mu+\varkappa^2)-2iP\frac{1}{k_\mu k_\mu+\varkappa^2}\right\} \tag{8.34}$$

$$= -\frac{2i}{(2\pi)^4}\lim_{\epsilon\to 0}\int d^4k\, e^{ik_\mu x_\mu}\frac{1}{k_\mu k_\mu+\varkappa^2-i\epsilon}. \tag{8.35}$$

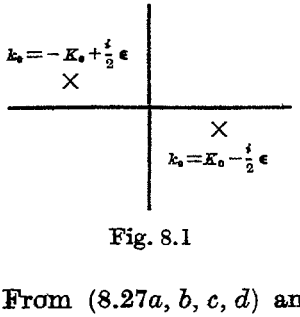

Fig. 8.1

As shown by (8.28) and (8.35), the Fourier amplitudes of $\bar{\Delta}$ and $\Delta_F(x)$ have poles at $k_0 = \pm\sqrt{(k_l k_l+\varkappa^2)}$ (cf. Fig. 8.1). The two poles of (8.35) at $k_0 = +K_0$ and $-K_0$ deviate from the real axis by $-(i\epsilon/2)$ and $+(i\epsilon/2)$ respectively and the limiting process $\epsilon\to 0$ is carried out after the integration d^4k.

From (8.27*a*, *b*, *c*, *d*) and (8.20*e*)

$$(\Box-\varkappa^2)\bar{\Delta}(x) = -\delta^4(x), \tag{8.36}$$

$$(\Box-\varkappa^2)\Delta_F(x) = 2i\delta^4(x), \tag{8.37}$$

$$(\Box-\varkappa^2)\Delta^{\mathrm{ret}}(x) = -\delta^4(x), \tag{8.38}$$

$$(\Box-\varkappa^2)\Delta^{\mathrm{adv}}(x) = -\delta^4(x). \tag{8.39}$$

It now follows from (8.13*b*) that the Green's functions of (8.2) involve $d(\partial)$, and are given by

$$\bar{G}(x) \equiv -d(\partial)\bar{\Delta}(x), \tag{8.40a}$$

$$G_F(x) \equiv -\tfrac{1}{2}id(\partial)\Delta_F(x), \tag{8.40b}$$

$$G^{\mathrm{ret}}(x) \equiv -d(\partial)\Delta^{\mathrm{ret}}(x), \tag{8.40c}$$

$$G^{\mathrm{adv}}(x) \equiv -d(\partial)\Delta^{\mathrm{adv}}(x), \tag{8.40d}$$

$$A(\partial)\,G(x) = \delta^4(x)I. \tag{8.41}$$

Here $G(x)$ denotes any one of the Green's functions defined in (8.40*a*, *b*, *c*, *d*). In general, when $\Delta_G(x)$ satisfies the relation

$$(\Box-\varkappa^2)\Delta_G(x) = \delta^4(x),$$

$d(\partial)\Delta_G(x)$ is the Green's function satisfying (8.41).

The physical significance of the Green's functions in (8.40*c*, *d*) may be understood by a consideration of equations (8.27*c*) and (8.27*d*). From these it follows that

$$G^{\text{ret}}(x-x')=0 \quad \text{for } \sigma(x)<\sigma(x'), \tag{8.42}$$

$$G^{\text{adv}}(x-x')=0 \quad \text{for } \sigma(x)>\sigma(x'), \tag{8.43}$$

where $\sigma(x)$ denotes a space-like surface through a point x and $\sigma(x)>\sigma(x')$ means that $\sigma(x)$ is a surface posterior to $\sigma(x')$. The last equations show that, at the point x, G^{ret} and G^{adv} contain contributions from anterior points x' and posterior points x' (i.e. the retarded effects or the advanced effects) respectively. Thus, G^{ret} and G^{adv} are, respectively, Green's functions when initial states and final states are specified.

The physical meaning of G_F may be understood by separating $\Delta(x)$ into parts of "positive" and "negative" frequency $\Delta^{\pm}(x)$ (SCHWINGER [1949]). Thus

$$\Delta(x)=\Delta^{+}(x)+\Delta^{-}(x), \tag{8.44}$$

where

$$\Delta^{\pm}(x)=\frac{1}{2\pi i}\int_{C_{\pm}}\Delta(x-\epsilon\tau)\frac{d\tau}{\tau}.$$

The contour of integration C_+ (C_-) goes from $-\infty$ to $+\infty$ (from $+\infty$ to $-\infty$) and below (above) the singular point $\tau=0$ in the complex τ-plane (cf. Fig. 8.2). The argument $x-\epsilon\tau$ is an abbreviation of the vector symbol $x_\mu-\epsilon_\mu\tau$ where ϵ_μ is a time-like unit vector satisfying the relation

$$\epsilon_0\,(=-i\epsilon_4)>0. \tag{8.45}$$

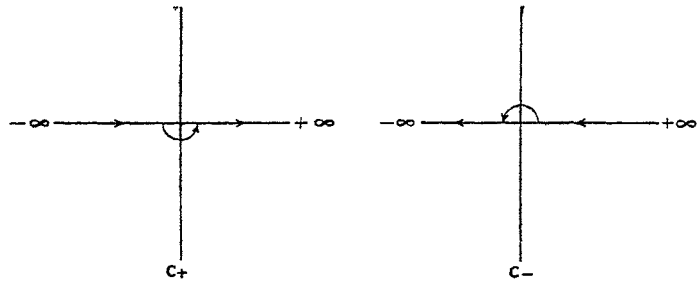

Fig. 8.2

Using the formulae:

$$\frac{1}{2\pi i}\int_{C_+} e^{-ik_\mu \epsilon_\mu \tau}\frac{d\tau}{\tau} = \begin{cases} 1 \text{ for } -k_\mu \epsilon_\mu > 0 \\ 0 \text{ for } -k_\mu \epsilon_\mu < 0 \end{cases} \tag{8.46a}$$

$$\frac{1}{2\pi i}\int_{C_-} e^{-ik_\mu \epsilon_\mu \tau}\frac{d\tau}{\tau} = \begin{cases} 0 \text{ for } -k_\mu \epsilon_\mu > 0 \\ 1 \text{ for } -k_\mu \epsilon_\mu < 0, \end{cases} \tag{8.46b}$$

we obtain from (8.44) and (8.11a) the equation

$$\left.\begin{aligned} \Delta^+(x) &= \frac{-i}{(2\pi)^3}\int_{-k_\mu \epsilon_\mu > 0} d^4k\ \delta(k_\mu k_\mu + \varkappa^2)\, e^{ik_\mu x_\mu} \\ \Delta^-(x) &= +\frac{i}{(2\pi)^3}\int_{-k_\mu \epsilon_\mu > 0} d^4k\ \delta(k_\mu k_\mu + \varkappa^2)\, e^{-ik_\mu x_\mu}. \end{aligned}\right\} \tag{8.47}$$

Here integrations are taken over the domain $(-k_\mu \epsilon_\mu > 0)$. It must be remarked that the condition $-k_\mu \epsilon_\mu > 0$ does not depend on the special vector ϵ_μ and is invariant under L_+^+, L_+^- Lorentz transformations. Indeed, we have

$$-k_\mu \epsilon_\mu = k_0 \epsilon_0 \left(1 - \frac{k_i \epsilon_i}{k_0 \epsilon_0}\right),$$

which leads to

$$k_0 (= -ik_4) > 0 \quad \text{for} \quad -k_\mu \epsilon_\mu > 0, \tag{8.48}$$

because k_μ and ϵ_μ are time-like vectors and, consequently, satisfy the relation

$$\left|\frac{(\mathbf{k}\cdot\boldsymbol{\epsilon})}{k_0 \epsilon_0}\right| \leqslant \frac{|\mathbf{k}|\,|\boldsymbol{\epsilon}|}{|k_0|\,|\epsilon_0|} < 1.$$

The inequality (8.48) does not depend on the vector ϵ_μ and is invariant under L_+^+, L_+^--transformations. Equation (8.47) shows explicitly that Δ^+ and Δ^- are the positive and negative frequency parts. Moreover, $\Delta^\pm$ satisfy the relations

$$(\Box - \varkappa^2)\Delta^\pm(x) = 0, \tag{8.49a}$$

$$\Delta^-(x) = -\Delta^+(-x). \tag{8.49b}$$

The relations (8.47) show that $\Delta^+(x)$ is the complex conjugate of $\Delta^-(x)$. We have, from (8.47) and (8.12),

$$\Delta^{(1)}(x) = i\{\Delta^+(x) - \Delta^-(x)\}. \tag{8.50}$$

Equations (8.27b), (8.44) and (8.50) lead to

$$\Delta_F(x - x') = \begin{cases} 2i\Delta^+(x - x') \text{ for } \sigma(x) > \sigma(x') \\ -2i\Delta^-(x - x') \text{ for } \sigma(x) < \sigma(x'). \end{cases} \tag{8.51}$$

We now introduce the function

$$M(x, x') = \begin{cases} a\,\Delta^{+}(x-x') & \text{for } \sigma(x) > \sigma(x') \\ b\,\Delta^{-}(x-x') & \text{for } \sigma(x) < \sigma(x'), \end{cases}$$

where a and b do not contain x and x'. Since $\Delta^{\pm}(x-x')$ are superpositions of plane waves of the form

$$\exp iE\{(\mathbf{v}, \mathbf{x}-\mathbf{x}') \mp (t-t')\}, \quad (\mathbf{v} \equiv \mathbf{k}/K_0)$$

with positive and negative frequencies, the effect of a point x' (in the present) on a point x in the future or from the past is represented, in $M(x, x')$, by a diverging wave $f((t-t')-(\mathbf{v}, \mathbf{x}-\mathbf{x}'))$ or a converging wave $f((t-t')+(\mathbf{v}, \mathbf{x}-\mathbf{x}'))$. That this should be so is in accord with the principle of causality, which requires that the present state should depend only on events in the past, and can only influence events in the future. When the theory is formulated in such a way that the initial and final states are involved symmetrically, we must have

$$M(x, x') = M(x', x).$$

This leads to

$$a = -b$$

on account of (8.49b). This is just the condition satisfied by $\Delta_F(x-x')$. Thus, we see that the propagation of influences between any two points are described by $G_F(x-x')$ in the formulation of the causal theory, in which the initial and final states are treated symmetrically (STUECKELBERG and GREEN [1951]). This will be demonstrated explicitly in Ch. XIII.

§ 3. Commutation Relationship and Spin

The commutation relationships of the field quantities are intimately connected with the spin of the particles which the field represents. This section is devoted to an exploration of this connection.

We first consider a field of integer spin. As shown in Ch. IV, the wave functions may now be chosen as a set of tensors. We shall suppose that the commutation relations are of (+)-type. Then, for Q_α and Q_α^* (with the same suffix α), we have

$$[Q_\alpha(x), Q_\alpha^*(x')]_+ = i\bar{d}_{\alpha\alpha}(\partial)\Delta(x-x')$$

with no summation over α. Since Q_α and Q_α^* are tensors of the same

degree, $d_{\alpha\alpha}(\partial)$ is a tensor of even degree made up of $\delta_{\mu\nu}$ and ∂_μ. Thus, we have

$$\bar{d}_{\alpha\alpha}(-\partial)=\bar{d}_{\alpha\alpha}(\partial).$$

By exchanging x and x' in this commutation relationship,

$$\begin{aligned}[Q_\alpha(x'), Q_\alpha^*(x)]_+ &= i\bar{d}_{\alpha\alpha}(-\partial)\Delta(x'-x)\\ &= -i\bar{d}_{\alpha\alpha}(\partial)\Delta(x-x')\end{aligned}$$

on account of (8.16). Adding both commutation relat onships we obtain

$$[Q_\alpha(x), Q_\alpha^*(x')]_+ + [Q_\alpha(x'), Q_\alpha^*(x)]_+ = 0,$$

which leads to [1])

$$[Q_\alpha(x), Q_\alpha^*(x)]_+ = 0.$$

This shows that

$$Q_\alpha = Q_\alpha^* = 0.$$

To avoid this triviality we are forced to the conclusion that, for a field of integral spin, the commutation relationship in (8.14*a*) and (8.14*b*) must be of the $(-)$-type (PAULI [1940]).

On the other hand, we have already shown that if the Klein Paradox is to be avoided, particles of half integer spin must obey Fermi statistics. We shall show in the next chapter that this is tantamount to the requirement that particles of half integer spin should be associated with commutation relationships of the $(+)$-type.

We have derived, in Ch. V, an important theorem concerning the properties of $d(\partial)$ (UMEZAWA [1952]).

The order of the differential operator $d(\partial)$, *called* $b^{(s)}$, *is given by*

$$b^{(s)} = 2S \text{ if } \varkappa \neq 0 \qquad (8.52a)$$

where S *is the maximum value of the spin of the various fields described by the field quantities* Q_α. (It will be recalled that, in general, the quantities Q_α belong to a reducible representation and represent a set of fields).

When the mass of the field is zero, this argument cannot be carried through. In general, dimensional considerations imply that high derivatives in $d(\partial)$ can only be introduced as products of $(\partial_\mu/\varkappa)$. If $\varkappa$ is zero, it follows that derivatives of higher order cannot be introduced in $d(\partial)$. The consequence of this may be that the commutation relationship may not be compatible with all the field equations. This

[1]) In general, the matrix equation $[A, A^*]_+ = 0$ leads to $A = A^* = 0$.

is the case, in fact, of the electromagnetic field (7.50). If $A(\partial) = \Box$, $d(\partial) = I$, the field equation

$$A(\partial) A_\mu(x) = 0$$

does not include the Lorentz condition

$$\partial_\mu A_\mu(x) = 0.$$

This situation is in complete contrast to that of the vector field U_μ with non-zero mass $\varkappa$. There the equation $\partial_\mu U_\mu = 0$ is not independent of the field equation (7.85*a*). It is for reasons of such a kind that in Example 3, Ch. VII, we found it expedient to replace the Lorentz condition, a restriction on the field quantities $A_\mu(x)$, by the condition (cf. (7.56))

$$\partial_\mu A_\mu(x) \Psi = 0,$$

a restriction on the state vector Ψ.

In general, for zero-mass fields ($\varkappa = 0$) with higher spins $S > 1/2$, the field equations consist of the two sets

$$A_{\alpha\beta}(\partial)\ Q_\beta(x) = 0, \tag{8.53}$$

$$A^s_{\alpha\beta}(\partial)\ Q_\beta(x) = 0. \tag{8.54}$$

As shown in § 3 of Ch. IV, although the field has spin S the number of its independent components is reduced to 2 on account of the gauge invariance of the theory. Like the Lorentz condition, the subsidiary condition (8.54) is required for this reduction of the number of independent states. The third equation in (4.48) is an example of such a subsidiary condition. We shall regard (8.53) as the complete field equations and shall derive the commutation relations from (8.53), (8.14*a*), (8.14*b*) and (8.13*b*) (without taking (8.54) into account). These commutation relations will not be compatible with the subsidiary condition (8.54). Therefore, we shall replace (8.54) by

$$A^s_{\alpha\beta}(\partial)\ Q_\beta(x)\ \Psi = 0. \tag{8.55}$$

This condition is a restriction not of the field quantities Q_α but of the state vector Ψ, and ensures that the redundant components do not appear in states realised in nature. Thus, for $\varkappa = 0$, only two components are susceptible to observation. For example, in quantum electrodynamics, the Lorentz condition (7.56) leads to two observable

states (i.e. two polarization states) of the electromagnetic plane wave. A more detailed discussion of this point will be given in Example 2 of Ch. IX.

In the case of the integer spin field $A_{\mu_1 \ldots \mu_S}$, (8.53) is given by the first line in (4.47). Thus we have

$$\Lambda(\partial) = \Box I,$$

which leads to

$$d(\partial) = I.$$

The equation (8.53) for the half-integer spin field $\psi_{\mu_1 \ldots \mu_k}$ $(S = k + \frac{1}{2})$ is given by (4.41) with $\varkappa = 0$. Thus

$$\Lambda(\partial) = -\gamma_\mu \partial_\mu$$

and

$$d(\partial) = -\gamma_\mu \partial_\mu.$$

Then, we obtain

$$b^{(S)} = \begin{cases} 0 & \text{for the integer spin field } A_{\mu_1 \ldots \mu_S},\ \varkappa = 0 \\ 1 & \text{for the half-integer spin field } \psi_{\mu_1 \ldots \mu_k},\ \varkappa = 0. \end{cases} \quad (8.52b)$$

§ 4. Examples

EXAMPLE 1. COMMUTATION RELATIONS OF VARIOUS FIELDS

We shall now list the commutation relations of the various fields (cf. (8.13a) and (8.14a, b)). The operators $\Lambda(\partial)$ are derived by comparing (8.3a) with the Lagrangians L^0 given in Ch. VII.

We obtain [1])

$$\Lambda(\partial) = \begin{cases} (\Box - \varkappa^2) I & \text{for the charged scalar or pseudoscalar fields } (U,\ U^*) \\ [(\Box - \varkappa^2)\delta_{a,b}]\ (a, b = 1, 2) & \text{for the charged scalar or pseudoscalar fields } (U^{(1)},\ U^{(2)}) \\ \Box I & \text{for the electromagnetic field } A_\mu \\ [\partial_\mu \partial_\nu - (\Box - \varkappa^2)\delta_{\mu\nu}] & \text{for the charged vector or pseudovector fields } (U_\mu,\ U_\mu^*) \\ -(\gamma_\mu \partial_\mu + \varkappa) & \text{for the spinor fields } (\psi,\ \bar{\psi}). \end{cases}$$

[1]) $\{\partial_\mu \partial_\nu - (\Box - \varkappa^2)\delta_{\mu\nu}\} U_\nu = \partial_\nu F_{\mu\nu} - \varkappa^2 U_\mu.$

Now $d(\partial)$ may be found by using (8.13*b*), and leads to the commutation relations

$$\left.\begin{aligned} &[U(x),\, U^*(x')]_- = i\Delta(x-x') \\ &[U^{(1)}(x),\, U^{(1)}(x')]_- = [U^{(2)}(x),\, U^{(2)}(x')]_- = i\Delta(x-x') \\ &[U^{(1)}(x),\, U^{(2)}(x')]_- = 0 \end{aligned}\right\} \tag{8.56}$$

for the scalar or pseudoscalar fields,

$$[A_\mu(x),\, A_\nu(x')]_- = i\delta_{\mu\nu}\,\Delta(x-x') \quad (\varkappa = 0) \tag{8.57}$$

for the electromagnetic field,

$$[U_\mu(x),\, U_\nu^*(x')]_- = i\left(\delta_{\mu\nu} - \frac{1}{\varkappa^2}\partial_\mu\partial_\nu\right)\Delta(x-x') \tag{8.58}$$

for the vector or pseudovector fields,

$$[\psi(x),\, \bar\psi(x')]_+ = \frac{1}{i}\,S(x-x') \tag{8.59a}$$

$$S(x-x') \equiv (\gamma_\mu\partial_\mu - \varkappa)\,\Delta(x-x') \tag{8.59b}$$

for the spinor fields.

Commutation relations for real (and therefore neutral) fields are obtained by omitting the stars in (8.56) and (8.58) or by putting $U^{(2)} = 0$ in (8.56). It is easily seen that (8.57) is not compatible with the equation $\partial_\mu A_\mu = 0$. It is for this reason that the Lorentz condition (7.56) is adopted in Example 3 of Ch. VII.

The commutation relations of a vector field in the Stueckelberg formalism are

$$\left.\begin{aligned} &[A_\mu(x),\, A_\nu^\dagger(x')]_- = i\delta_{\mu\nu}\,\Delta(x-x') \\ &[B(x),\, B^\dagger(x')]_- = i\,\Delta(x-x') \\ &[A_\mu(x),\, B^\dagger(x')]_- = [A_\mu^\dagger(x),\, B(x')]_- = 0 \end{aligned}\right\} \tag{8.60}$$

The commutation relations of a real (and therefore neutral) field are given by omitting the daggers in (8.60).

EXAMPLE 2. DUFFIN–KEMMER–PETIAU THEORY

From the Lagrangian (7.105) we obtain

$$\Lambda(\partial) = -(\partial_\mu\beta_\mu + \varkappa).$$

Since a field described by this theory has spin 1 or 0, we can see from (8.52*a*) that $b^{(S)} = 2$. The operator $d(\partial)$ has been given already by (5.24), and is

$$d(\partial) = -\left\{\frac{1}{\varkappa}\,(\Box - \varkappa^2) + \beta_\mu\partial_\mu - \frac{1}{2\varkappa}\,(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu)\,\partial_\mu\partial_\nu\right\}. \tag{8.61}$$

Whence

$$\left.\begin{aligned}[\psi(x), \bar{\psi}(x')]_- &= \frac{1}{i}\left\{\frac{1}{\varkappa}(\Box - \varkappa^2) + \beta_\mu \partial_\mu - \frac{1}{2\varkappa}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu)\,\partial_\mu\partial_\nu\right\} \Delta(x-x') \\ &= \frac{1}{i}\left\{\beta_\mu\partial_\mu - \frac{1}{2\varkappa}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu)\,\partial_\mu\partial_\nu\right\}\Delta(x-x').\end{aligned}\right\} \tag{8.62}$$

(TAKAHASHI and UMEZAWA [1953]). It must be noted that term $(\Box - \varkappa^2)$ does not contribute to the commutation relation (8.62) but rather to the Green's functions $G(x)$.

EXAMPLE 3. RARITA–SCHWINGER THEORY (SPIN 3/2)

From the Lagrangian (7.124) we obtain

$$A_{\mu\nu}(\partial) = -\{(\gamma_\varrho\partial_\varrho + \varkappa)\,\delta_{\mu\nu} - \tfrac{1}{3}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu) + \tfrac{1}{3}\gamma_\mu(\gamma_\varrho\partial_\varrho - \varkappa)\,\gamma_\nu\}. \tag{8.63}$$

Equation (8.52a) gives $b^{(S)} = 3$. The operator $d_{\mu\nu}(\partial)$ has already been given by (4.56e) and is

$$\left.\begin{aligned}d_{\mu\nu}(\partial) = &-(\gamma_\varrho\partial_\varrho - \varkappa)\left\{\delta_{\mu\nu} - \tfrac{1}{3}\gamma_\mu\gamma_\nu + \frac{1}{3\varkappa}(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu) - \frac{2}{3\varkappa^2}\partial_\mu\partial_\nu\right\} \\ &+ \frac{1}{3\varkappa^2}(\Box - \varkappa^2)\{(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu) + (\gamma_\varrho\partial_\varrho - \varkappa)\,\gamma_\mu\gamma_\nu\}\end{aligned}\right\} \tag{8.64}$$

(TAKAHASHI and UMEZAWA [1953]). The commutation relation is

$$[\psi_\mu(x), \bar{\psi}_\nu(x')] = i d_{\mu\nu}(\partial)\,\Delta(x-x'). \tag{8.65}$$

EXAMPLE 4. GENERAL CASES

We now consider the general field equation (5.15):—

$$A(\partial) = -(\beta_\mu\partial_\mu + \varkappa). \tag{8.66}$$

By (5.18a), $d(\partial)$ is

$$\left.\begin{aligned}d(\partial) = &\varkappa I - \beta_\mu\partial_\mu - \frac{1}{\varkappa}[\Box - (\beta_\varrho\partial_\varrho)^2] + \dots \\ &+ \left(\frac{-1}{\varkappa}\right)^{2S-1}[\Box - (\beta_\varrho\partial_\varrho)]^2\,(\beta_\varrho\partial_\varrho)^{2S-2}.\end{aligned}\right\} \tag{8.67}$$

Here S denotes the spin of the field. The commutation relation is then [1])

$$[\psi(x), \bar{\psi}(x')]_\pm = i d(\partial)\,\Delta(x-x') \tag{8.68}$$

(UMEZAWA and VISCONTI [1955]).

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[1]) The commutation relation of the Dirac-Fierz field (cf. § 3 of Ch. IV) has been given by FIERZ [1939] and UMEZAWA [1952].

EXAMPLE 5. TIME REFLECTION AND DETAILED BALANCE [1])

As shown in § 3 of Ch. III, the Lagrangian of the Dirac field (7.111), which may be regarded as a c-number, changes under time reflection [2]). On the other hand, the Lagrangian of the scalar field (7.44) is invariant. For time reflection in general, we can prove that

$$\int_{-\infty}^{\infty} d^4x\, L^0(x) = \begin{cases} \text{odd scalar for fields of half-integer spin} \\ \text{scalar for fields of integer spin.} \end{cases}$$

It now follows from (7.3) and (7.4) that

$$T^0_\mu = \begin{cases} \text{even vector for the half-integer spin fields} \\ \text{odd vector for the integer spin fields} \end{cases} \tag{8.69}$$

for the time reflection, because $d\sigma_\mu$ is transformed as an odd vector (cf. (6.2)). In other words, for the time reflection

$$\left.\begin{aligned} x &\to {}'x \\ Q(x) &\to {}'Q({}'x) = \Lambda Q(x) \end{aligned}\right\} \tag{8.70}$$

the energy-momentum vector transforms according to

$$\left.\begin{aligned} T^0_k[Q(x)] &= -\sigma T^0_k[\Lambda Q(x)] \quad (k = 1, 2, 3) \\ T^0_4[Q(x)] &= \sigma T^0_4[\Lambda Q(x)] \end{aligned}\right\} \tag{8.71a}$$

where the constant σ is given by

$$\sigma = \begin{cases} -1 \text{ for fields of half-integer spin} \\ +1 \text{ for fields of integer spin.} \end{cases} \tag{8.71b}$$

Since ∂_μ is always transformed as an even vector under time reflection, the transformation of (8.71a) seems to be incompatible with the canonical equation (cf. (8.15))

$$-\partial_\mu Q_\alpha(x) = [Q_\alpha(x), T^0_\mu]. \tag{8.72}$$

However, this conclusion is premature, because we have not taken account of the transformation of the state vector,

$$\Psi \to {}'\Psi = \Psi^*. \tag{8.73}$$

Indeed, because of the change of sign of the time, there is necessarily a change of phase of the state-vector Ψ. Equation (8.73) may be

---

[1]) See PAULI and BELINFANTE [1940], WATANABE [1951] [1955], SCHWINGER [1951], LÜDERS [1952], UMEZAWA, KAMEFUCHI and TANAKA [1954].

[2]) We have seen, for example, that $\varkappa\bar{\psi}\psi$ changes its sign under time reflection.

interpreted as the transformation connecting two different worlds. If there is a transition from a state of n particles to one of m particles in one world, this will be removed by a transition from a state of m particles to one of n particles in the other world. In addition, there may be, for example, associated changes of charge and spin of the particles. It follows that the field operators Q_α and Q_β^* must be transformed into linear combination of their transposed operators Q_α^T and Q_β^{*T}. Thus, denoting the field quantities after time reflection by ${}'Q_\alpha('x)$ and ${}'Q_\alpha^*('x)$ we must have

$$Q_\alpha(x) = a_{\alpha\beta}\, {}'Q_\beta^T('x) + b_{\alpha\beta}\, {}'Q_\beta^{*T}('x), \tag{8.74}$$

where $[a_{\alpha\beta}]$ and $[b_{\alpha\beta}]$ are, at present, unknown matrices. The last equation may be rewritten as

$$Q_\alpha(x) = \Lambda_{\alpha\beta}^{-1}\, r_{\beta\varrho}\, {}'Q_\varrho^T('x) + \Lambda_{\alpha\beta}^{-1}\, t_{\beta\varrho}\, {}'Q_\varrho^{*T}('x). \tag{8.75}$$

Here $[r_{\alpha\beta}]$ and $[t_{\alpha\beta}]$ are unitary matrices which must be determined by the invariance of the theory under time reflection.

From (8.72) and (8.71) we have [1])

$$-{}'\partial_\mu\, {}'Q_\alpha('x) = \sigma\,[{}'Q_\alpha('x), T_\mu^0\,[r\, {}'Q^T('x) + t\, {}'Q^{*T}('x)]^T]. \tag{8.76}$$

The invariance of the theory under time reflection requires that (8.76) must be identical with the relation

$$-{}'\partial_\mu {}'Q_\alpha('x) = [{}'Q_\alpha('x),\ T_\mu^0\ [{}'Q('x)]]. \tag{8.77}$$

That is, that

$$T_\mu^0[r'Q^T('x) + t'Q^{*T}('x)]^T = \sigma T_\mu^0\,[{}'Q('x)] + c\text{-number}. \tag{8.78}$$

As shown in Ch. VII, $T_\mu^0\,[Q(x)]$ has the form

$$T_\mu^0\,[Q(x)] = \int d^3x\, Q_\alpha^*(x)\, \Omega_{\mu,\alpha\beta}(\partial)\, Q_\beta(x), \tag{8.79}$$

where $\Omega_{\mu,\alpha\beta}(\partial)$ is a differential operator.

The condition (8.78) can be written, by using (8.79),

$$\left.\begin{aligned} &{}'Q('x)\,(r^{-1}\,\Omega('\partial)r)^T\, {}'Q^*('x) + {}'Q^*('x)\,(t^{-1}\,\Omega_\mu('\partial)t)^T\, {}'Q('x) \\ &+ {}'Q('x)\,(t^{-1}\Omega('\partial)r)^T\, {}'Q('x) + {}'Q^*('x)\,(r^{-1}\,\Omega_\mu('\partial)t)^T\, {}'Q^*('x) \\ &= \sigma\, {}'Q^*('x)\,\Omega_\mu('\partial)\, {}'Q('x) + c\text{-number}. \end{aligned}\right\} \tag{8.80}$$

There are two solutions, (i) and (ii), of this equation.

---

[1]) For the derivation of (8.76) we make use of the relation

$$[A, B]^T = -[A^T, B^T].$$

They are:—

(i) If r satisfies

$$r^{-1}\,\Omega_\mu(\partial)r=\Omega_\mu(\partial) \tag{8.81a}$$

and

$$t=0, \tag{8.81b}$$

and the commutation relations are of

$$\left.\begin{array}{l}(+)\text{-type for the half-integer spin fields}\\ (-)\text{-type for the integer spin fields,}\end{array}\right\} \tag{8.82}$$

(8.80) is solved (see (8.71b)).

In this case the intimate relationship between the spins and the commutation relations subsists. Equation (8.81a) is solved by $r=1$ [1]). Moreover, in the case of a field of spin $\frac{1}{2}$ and zero mass ($\varkappa=0$) it is easily seen from (7.111) that $r=\gamma_5$ is another solution of (8.81a).

(ii) Equation (8.80) is also solved if

$$r=0, \tag{8.83}$$

$$t^{-1}\,\Omega_\mu(\partial)t=\sigma\Omega_\mu^T(\partial). \tag{8.84}$$

In this case we have no restriction on the type of the commutation relationships.

We can prove that the matrix t of (8.84) is equal to the product of the r of (8.81a) and the charge conjugation matrix. As an example, we shall consider the field ψ of spin 1/2 with non-zero mass ($\varkappa\neq 0$), for which (8.81a) gave $r=1$, and $\Omega_4(\partial)$ is $\Omega_4(\partial)=\gamma_4(\gamma_k\partial_k+\varkappa)$.

From (8.84) and (8.71b) [2]) we have

$$t^{-1}\,\gamma_4\,\gamma_k\,t=(\gamma_4\,\gamma_k)^T,$$
$$t^{-1}\,\gamma_4\,t=-\gamma_4^T.$$

These relations lead to

$$\gamma_4^T=-t^{-1}\,\gamma_4\,t$$
$$\gamma_k^T=t^{-1}\,\gamma_k\,t,$$

which lead to

$$\gamma_\mu^T=-C^{-1}\,\gamma_\mu C$$

---

[1]) We can take r as complex c-number of the absolute value 1 and ± 1 for the complex and real fields, respectively. However, this arbitrariness of the phase factors can be absorbed into those of A.

[2]) On account of space integration in (8.79), we can take

$$\partial_k^T=-\partial_k.$$

This corresponds to taking the partial integration.

for the matrix $C=t\gamma_4^T$. Comparing this relation with (3.37) we see that t is the matrix connecting ψ with ψ'^* under the charge conjugation transformation (cf. (3.40))

$$\psi=C\bar{\psi}'^T=C\gamma_4^T\psi'^{*T}=t\psi'^{*T}.$$

Thus we can see that the transformation in the standpoint (i) is equivalent to a product of the transformation of the standpoint (ii) and the charge conjugation.

We shall now compare a physical quantity $F[Q(x)]$ with the transformed quantity $F['Q('x)]$. Since $F[Q(x)]$ must be hermitian, every eigenvalue of this operator must be real, so that

$$F[Q(x)]\Psi=f\Psi. \tag{8.85}$$

Here Ψ is the eigenvector with the eigenvalue f. By making the transformation (8.73), we have

$$F[Q(x)]^T\,'\Psi=f'\Psi.$$

Thus, by introducing a sign constant $\varepsilon=\pm 1$ defined by

$$F['Q('x)]=\varepsilon F[Q(x)]^T, \tag{8.86}$$

we see that $F['Q('x)]$ has eigenvalue εf, because

$$F['Q('x)]'\Psi=\varepsilon f'\Psi. \tag{8.87}$$

Equation (8.86) may also be written, using (8.75) as

$$F[r^{-1}\Lambda Q^T(x)]=\varepsilon F[Q(x)]^T \qquad \text{for the standpoint (i)}$$
$$F[Q^{*T}(x)\Lambda^{-1}t]=\varepsilon F[Q(x)]^T \qquad \text{for the standpoint (ii).}$$

Equation (8.87) shows that ε defines parities of the physical quantities with respect to the time reflection. Table 1 shows these parities for the various quantities of the Dirac fields.

We shall consider the component $\bar{\psi}\gamma_1\psi$ of the current as an example. The condition (8.87) requires that

$$\varepsilon(\bar{\psi}\gamma_1\psi)^T=\psi^{*T}\Lambda^{-1}\,r\,\gamma_4\,\gamma_1 r^{-1}\,\Lambda\psi^T$$

on the standpoint (i). By using the commutation relationship for ψ^* and ψ, and by taking $r=1$ and $\Lambda=\gamma_1\gamma_2\gamma_3$ (cf. (3.27)) we have

$$\psi^{*T}\Lambda^{-1}r\gamma_4\gamma_1 r^{-1}\Lambda\psi^T=\psi^T\gamma_1^T\,\gamma_4^T\,\psi^{*T}+\text{constant } c\text{-number}$$
$$=(\bar{\psi}\gamma_1\psi)^T\ +\ \text{constant } c\text{-number.}$$

This leads to $\varepsilon = 1$, because the constant c-number may be amalgamated into the zero-point current. On the other hand, on the standpoint (ii), (8.87) requires that

$$\varepsilon\,(\bar{\psi}\gamma_1\psi)^T = (t^{-1}\,\Lambda\psi^T)\,\gamma_4\gamma_1(\psi^{*T}\,\Lambda^{-1}\,t).$$

By taking $t = C\gamma_4^T$ and $\Lambda = \gamma_1\gamma_2\gamma_3$ we obtain

$$(t^{-1}\,\Lambda\psi^T)\,\gamma_4\gamma_1\,(\psi^{*T}\,\Lambda^{-1}\,t) = \psi^T\,\Lambda^T\,t^{T,-1}\,\gamma_4\gamma_1\,t^T\,\Lambda^{T,-1}\,\psi^{*T} =$$
$$= -\psi^T\,\gamma_1^T\,\gamma_4^T\,\psi^{*T} = -(\bar{\psi}\,\gamma_1\,\psi)^T$$

by using (3.37). This leads to $\varepsilon = -1$.

TABLE I

Type		ε in (i)	ε in (ii)
$\bar{\psi}\psi$		+ 1	+ 1
$i\bar{\psi}\gamma_\mu\psi$	k	+ 1	− 1
	4	− 1	+ 1
$i\bar{\psi}\gamma_\mu\gamma_\nu\psi$	ik	+ 1	− 1
	$4k$	− 1	+ 1
$i\bar{\psi}\gamma_5\psi$		− 1	− 1
$i\bar{\psi}\gamma_5\gamma_\mu\psi$	k	− 1	− 1
	4	+ 1	+ 1
$\bar{\psi}\gamma_5\gamma_\mu\gamma_\nu\psi$	ik	− 1	+ 1
	$4i$	+ 1	− 1

Table I shows that many physical quantities change their sign under time reflections. Table II, which can be easily derived from Table I, gives the relationship between the states of particles in worlds which are reciprocal to each other. The symbols (−) and (+) indicate that a physical quantity does, or does not, change its sign [1]).

For example, a negaton of momentum $\mathbf{k}$ and spin $\boldsymbol{\sigma}$ is represented in the reciprocal world by a positon or a negaton with momentum $-\mathbf{k}$ and spin $-\boldsymbol{\sigma}$ from the standpoints (i) and (ii) respectively.

The symbol ε has an important significance in the discussion of detailed balance; – the principle that the probability of a transition

[1]) The parities of the electromagnetic field A_μ can be determined by requiring the invariance of the interaction Lagrangian $\bar{\psi}\gamma_\mu\psi A_\mu$ under time reflection.

TABLE II

	(i)	(ii)
charge J_4	−	+
current J_k	+	−
momentum T_k	−	−
spin / orbital angular momentum	−	−
energy T_4	+	+
vector potential A_k of el. mag. field	+	−
scalar potential A_4 of el. mag. field	−	+

$A \to B$ is equal to that of the transition $B \to A$. The discussion above shows that the principle of detailed balance must be modified in such a way that the probability of a transition $A \to B$ is equal to that of a transition $B' \to A'$, where A' and B' are the states of the reciprocal world corresponding to A and B and determined by Table II.

We shall show in Ch. X that requirements of time reflection sometimes restrict forms of interaction, and these restrictions are different according to the standpoints (i) and (ii) which is chosen.

EXAMPLE 6. TRUE ORDER OF DIFFERENTIAL OPERATORS

The theorem (8.52) restricts $b^{(S)}$, the order of the differential operators $d(\partial)$. We shall now discuss the greatest order of the differential operators that appear in the commutation relations for $D(\partial)Q_\alpha(x)$; namely:

$$[D(\partial)Q_\alpha(x),\ D(\partial')Q_\beta(x')] = iD(\partial)D(\partial')d_{\alpha\beta}(\partial)\Delta(x-x'). \tag{8.88}$$

Here $D(\partial)$ is a differential operator and the Q_α's describe a field of non-zero mass $\varkappa$ [1]. When the highest order of the differential operators in $D(\partial)D(\partial)d(\partial)$ is $(b^{(S)}+2t)$ we call t the true order of $D(\partial)$ (SAKATA, UMEZAWA and KAMEFUCHI [1952]). There are some cases in which t is not the same as the highest order of the derivation operators in $D(\partial)$. The following discussion will show that these exceptions arise from the requirement that an elementary particle must be described by wave functions constituting an irreducible representation of the Lorentz group.

Since, as shown in Ch. IV, $(U_{\mu_1\ldots\mu_S},\ F_{[\mu_1,\nu_1]\mu_2\ldots\mu_S})$ and $(F_{[\mu_1,\nu_1]\mu_2\ldots\mu_S},$

[1] For simplicity the spinor or tensor suffixes of $D(\partial)$ are omitted.

$F_{[\mu_1,\nu_1][\mu_2,\nu_2]\mu_3\cdots\mu_S}$) (given by (4.37)) can describe a state of the same integer spin field, (8.52) implies that the $b^{(S)}$ appearing in the commutation relations are, in each case, the same. In other words, the true order of $\partial_{\mu\nu;\mu'}$ (cf. (4.38)) operating on $U_{\mu'\mu_2\cdots\mu_S}$ is zero. A similar situation obtains for fields of half-integer spin. Since every $(\varphi, x)^{(a)}$ (a) $(a=1, \ldots)$ (cf. (4.29)) can describe the same field, the true order of ∂^{rs} operating on $\varphi_{s\ldots}$ is zero. Thus $a^{(S)}$ is the sum of the orders of all the differential operators in $D(\partial)$ except for those of $\partial_{\mu\nu;\mu'}$ and ∂^{rs}.

We give an example of the fact that the true order of $\partial_{\mu\nu;\mu'}$ is zero. As shown in Example 1, the commutation relations of vector fields include the factors $R(\mu, \nu)$ given by

$$R(\mu,\nu) \equiv \delta_{\mu\nu} - \frac{1}{\varkappa^2}\partial_\mu\partial_\nu. \tag{8.89}$$

It follows that

$$\partial_{\mu\nu\cdot\mu'}\partial_{\varrho\sigma;\mu''}R(\mu',\mu'') = \partial_{\mu\nu;\mu'}\partial_{\varrho\sigma;\mu''}. \tag{8.90}$$

Equation (8.90) shows that the true order of $\partial_{\mu\nu;\mu'}$ is zero.

The concept of the true order will be used in Ch. XV.

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CHAPTER IX

QUANTUM THEORY OF FREE FIELDS

FIELDS AND PARTICLES

§ 1. Quantisation

We shall now prove that the commutation relations (8.14*a*) and (8.14*b*) lead to a covariant quantum field theory in which the states of fields represent assemblies of Klein–Gordon fields quantized in accordance with a *canonical theory* (TAKAHASHI and UMEZAWA [1953], UMEZAWA and VISCONTI [1955]).

We start from a charged field described by a set of field quantities $Q_\alpha(x)$ $(\alpha = 1, \ldots, n)$ and their hermitian conjugates $Q^*_\alpha(x)$. Since the differential operators [1])

$$\bar{\Lambda}(\partial) = \eta \Lambda(\partial) \text{ and } \bar{d}(\partial) = d(\partial)\eta^{-1} \text{ (cf. (8.4), (8.13}e\text{))}$$

are hermitian, there exists a unitary matrix $s = [s_{\alpha r}(\partial)]$ which transforms $\bar{\Lambda}(\partial)$ and $\bar{d}(\partial)$ into diagonal matrices given by

$$\mathbf{\Lambda}(\partial) = s^{-1}\bar{\Lambda}(\partial)\, s = [\lambda(r : \partial)\delta_{rs}] \tag{9.1a}$$

$$\mathbf{d}(\partial) = s^{-1}\, \bar{d}(\partial)\; s = [d(r : \partial)\delta_{rs}] \tag{9.1b}$$

$$s^*_{\alpha r}(\partial)\; s_{\alpha s}(\partial) = \delta_{rs} \tag{9.1c}$$

$$s_{\alpha s}(\partial)\; s^*_{\beta s}(\partial) = \delta_{\alpha\beta} \tag{9.1d}$$

$$\lambda(r : \partial)\; d(r : \partial) = \Box - \varkappa^2 \tag{9.2}$$

In (9.2) the suffix r runs from 1 to n and is not summed. The operators $\lambda(r : \partial)$ and $d(r : \partial)$ are real so that

$$d^*(r : \partial) = d(r : -\partial, \tag{9.3}$$

on account of the fact that ∂_μ corresponds to ik_μ in the momentum space.

---

[1]) Differential operator $f(\partial)$ may be defined in terms of functions $f(k)$ by means of the Fourier representation

$$f(\partial)F(x) = \int d^4k\; f(ik)\; F(k)e^{ik_\mu x_\mu}.$$

Here $F(x)$ is an arbitrary function whose Fourier expansion is

$$F(x) = \int d^4k\; F(k)e^{ik_\mu x_\mu}$$

The matrix s transforms the field equations (8.2b) into n independent equations,

$$\lambda(r : \partial)\, q_r(x) = 0. \tag{9.4}$$

Here summation over r is omitted and $q_r(x)$ is

$$q_r(x) \equiv s^{-1}_{r\alpha}(\partial) Q_\alpha(x). \tag{9.5}$$

The Lagrangian (8.3a) can be rewritten as

$$\bar{L}^0 \doteqdot \int d^4x\; q^*_r(x)\, \lambda(r : \partial)\, q^{\mathrm{v}}_r(x) \tag{9.6}$$

on account of (9.5) and (9.1a). Indeed, (9.6) is just the Lagrangian which gives the equations (9.4).

By rewriting $q_r(x)$ as

$$q_r(x) = (d(r : \partial))^{1/2}\, u_r(x), \tag{9.7}$$

we obtain, from (9.2) and (9.6),

$$\bar{L}^0 \doteqdot \int d^4x\, \bar{u}_r(x)\, (\Box - \varkappa^2)\, u_r(x) \tag{9.8a}$$

$$\doteqdot -\int d^4x [\partial_r \bar{u}_r(x) \cdot \partial_\mu u_r(x) + \varkappa^2 \bar{u}_r(x)\, u_r(x)]. \tag{9.8b}$$

The $\bar{u}_r$ is defined by $q^*_r(x) = (d(r : -\partial))^{1/2}\, \bar{u}_r(x)$.

On the other hand, because of (9.3), the commutation relations (8.14a) are transformed by (9.1b) into

$$\left.\begin{aligned} [q_r(x), q^*_s(x')]_\pm &= i d(r : \partial)\, \Delta(x - x')\, \delta_{rs} \\ [q_r(x), q_s(x')]_\pm &= [q^*_r(x), q^*_s(x')]_\pm = 0. \end{aligned}\right\} \tag{9.9}$$

Equation (9.9) is satisfied if

$$\left.\begin{aligned} [u_r(x), \bar{u}_s(x')]_\pm &= i \Delta(x - x')\, \delta_{rs} \\ [u_r(x), u_s(x')]_\pm &= [\bar{u}_r(x), \bar{u}_s(x')]_\pm = 0. \end{aligned}\right\} \tag{9.10}$$

These equations lead, for a time t, to the commutation relations

$$\left.\begin{aligned} [P_r(\mathbf{x}, t), u_s(\mathbf{x}', t)]_\pm &= i\delta(\mathbf{x} - \mathbf{x}')\, \delta_{rs} \\ [u_r(\mathbf{x}, t), \bar{u}_s(\mathbf{x}', t)]_\pm &= 0 \\ [u_r(\mathbf{x}, t), u_s(\mathbf{x}', t)]_\pm &= [\bar{u}_r(\mathbf{x}, t), \bar{u}_s(\mathbf{x}', t)]_\pm = 0 \end{aligned}\right\} \tag{9.11}$$

where

$$P_r(x) \equiv -i \partial_4 \bar{u}_r(x). \tag{9.12}$$

Since the Klein-Gordon equation is the second order differential

equation, (9.11) required at a time t can determine uniquely the commutation relations between $u_r(x)$ and $u_r^*(x')$ for any time difference $t-t'$; namely (9.10).

Now $P_r(x)$ is just the operator canonically conjugate $u_r(x)$ and can be derived from the Lagrangian (9.8b) (cf. § 1 of Ch. VII). Thus we see that according to the canonical theory of Klein–Gordon fields, the set of commutation relations (9.11) is necessary.

Since, for spin S, each plane wave solution of the field equations is a linear combination of $(2S+1)$ linearly independent components (cf. (4.22)), only $(2S+1)$ of the $q_r(k)$ (The Fourier amplitudes of $q_r(x)$ with k given by $k_\mu k_\mu + \varkappa^2 = 0$) are not zero. Then, (9.7) shows that only $(2S+1)$ of the $d(r:ik)$ $(r=1, \ldots, n)$ are not zero, when k satisfies the relation $k_\mu k_\mu + \varkappa^2 = 0$. In other words, the rank of the matrix $\mathbf{d}(ik)$ is $2S+1$ for $k_\mu k_\mu + \varkappa^2 = 0$.

Since $u_r(x)$ satisfies the Klein–Gordon equation

$$(\Box - \varkappa^2)\, u_r(x) = 0, \tag{9.13}$$

u_r, $\bar{u}_r$ can be expanded as [1])

$$\left.\begin{aligned} u_r(x) = \frac{V}{(2\pi)^3} \int d^3k (2K_0 V)^{-1/2} \{ & u_r^+(K)\, e^{i(\mathbf{k}\cdot\mathbf{x} - K_0 t)} \\ & + u_r^-(K)\, e^{-i(\mathbf{k}\cdot\mathbf{x} - K_0 t)} \}, \end{aligned}\right\} \tag{9.14a}$$

$$\left.\begin{aligned} \bar{u}_r(x) = \frac{V}{(2\pi)^3} \int d^3k (2K_0 V)^{-1/2} \{ & \varepsilon\, u_r^{*+}(K)\, e^{i(\mathbf{k}\cdot\mathbf{x} - K_0 t)} \\ & + u_r^{*-}(K)\, e^{-i(\mathbf{k}\cdot\mathbf{x} - K_0 t)} \}. \end{aligned}\right\} \tag{9.14b}$$

Here ε is $1(-1)$ for the $-(+)$ type of (9.10). The V is given by

$$V \equiv \lim_{\mathbf{p}\to 0} \int d^3x\, e^{i(\mathbf{p}\cdot\mathbf{x})} \tag{9.15}$$

and may be interpreted as the volume of the accessible world. If this volume V is taken to be a cube, with volume $V = L^3$ the wave numbers of the standing waves have discrete values $(2\pi n/L)$, where n is an integer. If it is assumed that the continuous energy spectrum of particles may be obtained by the limting process $L \to \infty$ (i.e. $(2\pi/L)^3 \to d^3k$) the summation symbols must be replaced according to

$$\lim_{V\to\infty} \frac{1}{V} \sum \to \frac{1}{(2\pi)^3} \int d^3k. \tag{9.16}$$

---

[1]) $K_\mu \equiv (k_1, k_2, k_3, iK_0)$, $K_0 = \sqrt{(\mathbf{k}\cdot\mathbf{k} + \varkappa^2)}$.

By substituting (8.11b) and (9.14) into the commutation relations (9.10), we have [1])

$$\left.\begin{aligned} [u_r^{*+}(K), u_s^-(K')]_\pm &= \delta_{rs}\,\delta_{KK'} \\ [u_r^+(K), u_s^{*-}(K')]_\pm &= \delta_{rs}\,\delta_{KK'} \\ [u_r^\pm(K), u_s^\pm(K)]_\pm &= 0 \\ [u_r^{*\pm}(K), u_s^{*\pm}(K)]_\pm &= 0. \end{aligned}\right\} \tag{9.17}$$

As will be shown in Examples in § 3 u_r^{*+} (u_r^{*-}) is not u_r^{+*} (u_r^{-*}), the hermitian conjugate of $u_r^+(u_r^-)$.

Equations (9.17) are the well-known commutation relations of the quantum mechanics (see DIRAC [1947]). In the representation in which the matrices $u_r^{*-}(K)u_r^+(K)$ and $u_r^-(K)u^{*+}(K)$ are diagonal, it follows that

$$\left.\begin{aligned} (n_r^+ + 1\,|\,u_r^{*-}(K)\,|\,n_r^+) &= (n_r^+\,|\,u_r^+(K)\,|\,n_r^+ + 1) = (n_r^+ + 1)^{1/2} \\ (n_r^- + 1\,|\,u_r^-(K)\,|\,n_r^-) &= (n_r^-\,|\,u_r^{*+}(K)\,|\,n_r^- + 1) = (n_r^- + 1)^{1/2} \end{aligned}\right\} \tag{9.18}$$

and all the other matrix elements are zero, for commutation relations of $(-)$-type, and that all matrix elements except

$$\left.\begin{aligned} &(n_r^+ = 0\,|\,u_r^+(K)\,|\,n_r^+ = 1),\ (n_r^+ = 1\,|\,u_r^{*-}(K)\,|\,n_r^+ = 0), \\ &(n_r^- = 0\,|\,u_r^{*+}(K)\,|\,n_r^- = 1),\ (n_r^- = 1\,|\,u_r^-(K)\,|\,n_r^- = 0), \end{aligned}\right\} \tag{9.19}$$

are zero for commutation relations of $(+)$-type. In (9.18) and (9.19), n_r^+ and n_r^- are the eigenvalues of the "number" operators

$$\left.\begin{aligned} N_r^+(K) &= u_r^{*-}(K)\,u_r^+(K) \\ N_r^-(K) &= u_r^-(K)\,u_r^{*+}(K). \end{aligned}\right\} \tag{9.20}$$

They can be the positive integers (0, 1, 2, ...) and the two integers (0, 1) for $(-)$ and $(+)$ type commutation relations respectively.

We shall interpret n_r^+ and n_r^- as the number of particles in a (particle, r, K_μ)- and (antiparticle, r, K_μ)-state. Then we see that (9.18) and (9.19) correspond to Bose statistics and Fermi statistics respectively. Taking into account the relation between commutation relations and spins (cf. § 3 of Ch. VIII), we can conclude that particles of integer and of half-integer spin obey Bose and Fermi statistics respectively. This conclusion is compatible with the fact that electrons and nucleons obey Fermi statistics and photons Bose statistics.

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[1]) $\delta_{KK'} = \begin{cases} 1 \text{ for } K_\mu = K'_\mu \\ 0 \text{ for } K_\mu \neq K'_\mu \end{cases}$.

According to (9.5), (9.7) and (9.14), $Q_\alpha(x)$ and $\bar{Q}_\alpha(x)$ can be written as

$$\left.\begin{aligned} Q_\alpha(x) = \frac{V}{(2\pi)^3}\int d^3k (2K_0V)^{-1/2}\, s_{\alpha\tau}(\partial)\,(d(r:\partial))^{1/2} \\ \{u_\tau^+(K)\, e^{i(\mathbf{k}\cdot\mathbf{x}-K_0t)} + u_\tau^-(K)\, e^{-i(\mathbf{k}\cdot\mathbf{x}-K_0t)}\} \end{aligned}\right\} \tag{9.21}$$

$$\left.\begin{aligned} \bar{Q}_\alpha(x) = \frac{V}{(2\pi)^3}\int d^3k (2K_0V)^{-1/2}\, s^*_{\tau\beta}(\partial)\,(d(r:-\partial))^{1/2}\,\eta_{\beta\alpha} \\ \{u_\tau^{*-}(K)\, e^{-i(\mathbf{k}\cdot\mathbf{x}-K_0t)} + \varepsilon\, u_\tau^{*+}(K)\, e^{i(\mathbf{k}\cdot\mathbf{x}-K_0t)}\}. \end{aligned}\right\} \tag{9.22}$$

These equations show that $u_\tau^+(K)$ and $u_\tau^-(K)$ ($u_\tau^{*+}(K)$ and $u_\tau^{*-}(K)$) correspond to the Fourier amplitudes of the positive and negative parts of $Q_\alpha(x)$ ($\bar{Q}_\alpha(x)$) respectively.

We shall denote positive and negative frequency parts of $Q_\alpha(x)$ ($\bar{Q}_\alpha(x)$) by $Q_\alpha^+(x)$ and $Q_\alpha^-(x)$ ($\bar{Q}_\alpha^+(x)$ and $\bar{Q}_\alpha^-(x)$), so that

$$Q_\alpha(x) = Q_\alpha^+(x) + Q_\alpha^-(x) \tag{9.23a}$$

$$\bar{Q}_\alpha(x) = \bar{Q}_\alpha^+(x) + \bar{Q}_\alpha^-(x) \tag{9.23b}$$

$$(\Box - \varkappa^2)\, Q_\alpha^\pm(x) = (\Box - \varkappa^2)\, \bar{Q}_\alpha^\pm(x) = 0. \tag{9.24}$$

These separations may be carried out directly by means of the method used to obtain (8.44), when

$$\left.\begin{aligned} Q_\alpha^+(x) &= \frac{1}{2\pi i}\int_{C_+} Q_\alpha(x - \epsilon\tau)\,\frac{d\tau}{\tau} \\ Q_\alpha^-(x) &= \frac{1}{2\pi i}\int_{C_-} Q_\alpha(x - \epsilon\tau)\,\frac{d\tau}{\tau} \\ \bar{Q}_\alpha^+(x) &= \frac{1}{2\pi i}\int_{C_+} \bar{Q}_\alpha(x - \epsilon\varkappa)\,\frac{d\tau}{\tau} \\ \bar{Q}_\alpha^-(x) &= \frac{1}{2\pi i}\int_{C_-} \bar{Q}_\alpha(x - \epsilon\tau)\,\frac{d\tau}{\tau} \end{aligned}\right\} \tag{9.25}$$

From the commutation relations (8.14a) we have

$$[Q_\alpha^\pm(x), \bar{Q}_\beta(x')]_\pm = id_{\alpha\beta}(\partial)\,\Delta^\pm(x - x').$$

By using (8.46a, b) we can prove that

$$\int_{C_+} e^{i\alpha\tau}\frac{d\tau}{\tau}\int_{C_+} e^{-i\alpha\tau'}\frac{d\tau'}{\tau'} = 0$$

whence we can obtain

$$[Q_\alpha^-(x), \bar{Q}_\beta^+(x')]_\pm = id_{\alpha\beta}(\partial)\,\Delta^-(x - x')$$
$$[Q_\alpha^+(x), \bar{Q}_\beta^-(x')]_\pm = id_{\alpha\beta}(\partial)\,\Delta^+(x - x').$$

If the field quantities Q_α are real, (9.21) can still be derived, but (9.17) must be replaced by

$$\left.\begin{aligned}[u_r^+(K), u_s^-(K')]_\pm &= \delta_{rs}\,\delta_{KK'} \\ [u_r^+(K), u_s^+(K')]_\pm &= [u_r^-(K), u_s^-(K)']_\pm = 0.\end{aligned}\right\} \tag{9.26}$$

The commutation relation (8.14*b*) follows from this.

§ 2. Energy-Momentum Vector and Definition of Vacuum

Since the canonical energy-momentum vector T_μ, (7.4), is a bilinear functional of Q_α^* and Q_α and is independent of time [1]), it can be written as a superposition of contributions from each plane wave solution $Q_\alpha(k, x) \equiv Q_\alpha(k) \exp(ik_\mu x_\mu)$ of the field equation (8.2*a*). Thus, using (7.3*b*), we can write [2])

$$T_\mu = i \sum_{\mathbf{k}} \int d\sigma_\nu \left\{ \frac{\partial L^0(k,x)}{\partial Q_{\alpha:\nu}(k,x)}\, Q_{\alpha:\mu}(k,x) + Q^*_{\alpha:\mu}(k,x)\, \frac{\partial L^0(k,x)}{\partial Q^*_{\alpha:\nu}(k.x)} \right\}_{k=k^0}$$
$$- i \int d\sigma_\mu\, L^0(k,x)_{k=k^0}, \qquad (k^0_\mu\, k^0_\mu + \varkappa^2 = 0)$$

where $L^0(k, x)$ is the Lagrangian density obtained by the substitution $Q_\alpha(x) \to Q_\alpha(k, x)$ in (8.3*a*).

By using the transformation (9.5) and the equation (9.4) we have

$$T_\mu = i \sum_r \sum_{\mathbf{k}} \int d\sigma_\nu \left\{ \frac{\partial L^0(k,x)}{\partial q_{r:\nu}(k.x)} \frac{\partial q_{r:\nu}(k,x)}{\partial Q_{\alpha:\nu}(k,x)}\, Q_{\alpha\cdot\mu}(k,x) \right.$$
$$\left. + Q^*_{\alpha:\mu}(k,x) \frac{\partial q^*_{r:\nu}(k,x)}{\partial Q_{\alpha:\nu}(k,x)} \frac{\partial L^0(k,x)}{\partial q^*_{r:\nu}(k,x)} \right\}_{k=k^0} - i \int d\sigma_\mu\, L_0(k,x)_{k=k^0}$$
$$= i \sum_r \sum_{\mathbf{k}} \int d\sigma_\nu \left\{ \frac{\partial L^0(k,x)}{\partial q_{r:\nu}(k,x)}\, s^{-1}_{r\alpha}(k)\, s_{\alpha s}(k)\, q_{s:\mu}(k,x) \right.$$
$$\left. + q^*_{s:\mu}(k,x)\, s^*_{s\alpha}(k)\, s^{*-1}_{\alpha r} \frac{\partial L^0(k,x)}{\partial q_{r:\nu}(k,x)} - q^*_r(k,x)\, \lambda(r:\partial)\, q_r(k,x) \right\}_{k=k^0}$$
$$= i \sum_r{}' \sum_{\mathbf{k}} \int d\sigma_\nu \left\{ \frac{\partial L^0(k,x)}{\partial q_{r:\nu}(k,x)}\, q_{r:\mu}(k,x) + q^*_{r:\mu}(k,x) \frac{\partial L^0(k,x)}{\partial q^*_{r:\nu}(k.x)} \right.$$
$$\left. - q^*_r(k,x)\, \lambda(r:\partial)\, q_r(k,x) \right\}_{k=k^0}$$

---

[1]) Such a functional has the form

$$T_\mu = \int d^3k\, Q^*_\alpha(k)\, \Omega_{\mu\alpha\beta}(k)\, Q_\beta(k)$$

where $Q_\alpha(K)$ is the Fourier amplitude of $Q_\alpha(x)$.

[2]) In calculations of Lagrangian we must not restrict the energy-momentum k_μ into $k^0_\mu(k^0_\mu k^0_\mu + \varkappa^2 = 0)$.

where Σ' means that the summation with respect to the suffix r is taken over only the $(2S+1)$ non-zero component of $q_r(k^0, x)$. By making the transformation (9.7) and by using (9.86),

$$T_\mu = i \sum_r{}' \sum_k \int d\sigma_\nu \frac{\partial L^0(k,x)}{\partial u_{r:\mu}(k,x)} u_{r:\mu}(k,x) + \bar{u}_{r:\mu}(k,x) \frac{\partial L^0(k,x)}{\partial \bar{u}_{r:\nu}(k,x)} + \bar{u}_r(k,x)\,(k_\mu k_\mu + \varkappa^2)\, u_r(k,x)\Big\}_{k=k^0}$$
$$= -i \sum_r{}' \sum_k \int d\sigma_\nu \{\bar{u}_{r:\nu}(k,x)\, u_{r:\mu}(k,x) + \bar{u}_{r:\mu}(k,x)\, u_{r:\nu}(k,x)\}_{k=k^0}$$

Taking the Fourier expansion of $u_r^\pm(x)$ as $u_r(k,x)$

$$T_\mu = -i \sum_r{}' \int d\sigma_\nu \{u^{*-}_{r:\nu}(x)\, u^+_{r:\mu}(x) + u^{*-}_{r:\mu}(x)\, u^+_{r:\nu}(x) + u^{*+}_{r:\nu}(x)\, u^-_{r:\mu}(x) + \varepsilon\, u^{*+}_{r:\mu}(x)\, u^-_{r:\nu}(x)\}, \tag{9.27}$$

which leads to

$$T_\mu = -i \sum_r{}' \sum_k K_\mu (N_r^+(K) + N_r^-(K)) + \text{constant } c\text{-number} \tag{9.28}$$

on account of (9.20).

Equation (9.28) implies that the total energy and momentum of the field are the sums of the energies and the momenta of all particles. This exhibits the intimate relation between particles and fields.

The vacuum is defined as a state in which there are no particles ($n_r^\pm = 0$). Then, (9.28) shows that the vacuum is the state of lowest energy, which is equal to the c-number (i.e. zero point energy) in (9.28) (cf. § 1 of Ch. VI).

From (9.28), (9.17) and (9.20), we have

$$[u_r^\pm(K), T_\mu] = \pm(-iK_\mu) u_r^\pm(K) \tag{9.29}$$
$$[u_r^{*\pm}(K), T_\mu] = \pm(-iK_\mu) u_r^{*\pm}(K), \tag{9.30}$$

which lead to

$$[Q_\alpha(x), T_\mu] = -\partial_\mu Q_\alpha(x) \tag{9.31}$$
$$[\bar{Q}_\alpha(x), T_\mu] = -\partial_\mu \bar{Q}_\alpha(x) \tag{9.32}$$

on account of (9.21) and (9.22). The canonical equations (8.15) may be derived from (9.31) and (9.32).

From (9.29) we obtain

$$W u_r^+(K)\Psi = (W' - w) u_r^+(K)\Psi \tag{9.33}$$
$$W u_r^{*+}(K)\Psi = (W' - w) u_r^{*+}(K)\Psi, \tag{9.34}$$

where

$$W = T_4, \quad w = K_0 \tag{9.35}$$

and Ψ is an eigenfunction of the energy operator W with eigenvalue W'. Equations (9.33) and (9.34) show that $u_r^{*+}(K)\Psi$ and $u_r^{+}(K)\Psi$ are also eigenfunctions of W with the eigenvalue $(W'-w)$. In other words $u_r^{+}(K)$ and $u_r^{*+}(K)$ are operators that decrease the energy by w. In a similar way we can prove that $u_r^{-}(K)$ and $u_r^{*-}(K)$ are operators that increase the field energy by w. These properties of $u_r^{\pm}(K)$ and $u_r^{*\pm}(K)$ are compatibele with the interpretation of § 1, i.e. $u_r^{+}(K)$ $(u_r^{*+}(K))$ and $u_r^{-}(K)$ $(u_r^{*-}(K))$ are the annihilation and creation operators of the particles (antiparticles) of the field.

Since the vacuum Φ_0 is the state of the lowest energy, we have (SCHWINGER [1949])

$$u_r^{+}(K)\Phi_0 = u_r^{*+}(K)\Phi_0 = 0 \tag{9.36}$$

$$\Phi_0^* u_r^{-}(K) = \Phi_0^* u_r^{*-}(K) = 0, \tag{9.37}$$

which gives

$$\left.\begin{aligned} Q_\alpha^{+}(x)\,\Phi_0 = \Phi_0^*\,Q_\alpha^{-}(x) = 0 \\ \bar{Q}_\alpha^{+}(x)\,\Phi_0 = \Phi_0^*\,\bar{Q}_\alpha^{-}(x) = 0. \end{aligned}\right\} \tag{9.38}$$

This is the mathematical expression of the definition of the vacuum.

In these terms, the zero point energy of the electron field corresponds to the energy of the vacuum electrons of the old "hole" theory. Since the vacuum in quantum field theory affects many real processes, it is necessary to subtract its effects from the calculations (the subtraction of vacuum effects). To this extent, theoretical predictions depend on the definition of the vacuum [1]).

We now give some formulae dependent on the definition of the vacuum. Introducing $Q_\alpha^{(1)}(x)$ as:

$$Q_\alpha^{(1)}(x) \equiv i\,\{Q_\alpha^{+}(x) - Q_\alpha^{-}(x)\} = \frac{1}{\pi}\,P\int_{-\infty}^{\infty} Q_\alpha(x-\epsilon\tau)\,\frac{d\tau}{\tau}, \tag{9.39}$$

we obtain

$$\left.\begin{aligned} [Q_\alpha(x),\,\bar{Q}_\beta(x')]_\mp = -\,i\,[Q_\alpha^{(1)}(x),\,\bar{Q}_\beta(x')]_\pm \\ +2\,(Q_\alpha^{-}(x)\,\bar{Q}_\beta(x') \mp \bar{Q}_\beta(x')\,Q_\alpha^{+}(x)). \end{aligned}\right\} \tag{9.40}$$

It follows that

$$([Q_\alpha(x),\,\bar{Q}_\beta(x')]_\mp)_0 = -\,i\,([Q_\alpha^{(1)}(x),\,\bar{Q}_\beta(x')]_\pm)_0 \tag{9.41}$$

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[1]) However, the definition (9.38) of the vacuum has the defect that Φ_0, an eigenstate of the energy operator of the free field, is not stationary in the case of interacting fields. So far there is no complete theory of the vacuum of interacting fields. We shall touch this problem in Ch. XVIII.

where $(\ )_0$ denotes an expectation value in the vacuum state. From (8.50) and (9.39) we have

$$([Q_\alpha(x), \bar{Q}_\beta(x')]_\mp)_0 = d_{\alpha\beta}(\partial)\, \Delta^{(1)}(x-x') \tag{9.42}$$

where we must take $(-)$ or $(+)$ on the left hand side, according to whether commutation relations (8.14a, b) are of $(+)$ or $(-)$ type.

§ 3. Examples

EXAMPLE 1. SCALAR OR PSEUDOSCALAR FIELDS $U(x)$

For such fields (9.21) and (9.22) can be written as

$$\left.\begin{aligned} U(x) &= \sum_{\mathbf{K}} (2K_0 V)^{-1/2} (u^+(K)\, e^{iK_\mu x_\mu} + u^-(K)\, e^{-iK_\mu x_\mu}), \\ U^*(x) &= \sum_{\mathbf{K}} (2K_0 V)^{-1/2} (u^{*+}(K)\, e^{iK_\mu x_\mu} + u^{*-}(K)\, e^{-iK_\mu x_\mu}), \end{aligned}\right\} \tag{9.43}$$

because (see Example 1 of Ch. VIII)

$$d(\partial) = 1.$$

The commutation relations (9.17) lead to

$$\left.\begin{aligned} [u^+(K),\ u^{*-}(K')] &= \delta_{KK'} \\ [u^{*+}(K),\ u^-(K')] &= \delta_{KK'}. \end{aligned}\right\} \tag{9.44}$$

From (9.28) we have

$$T_\mu = -i \sum_{\mathbf{K}} K_\mu \{u^{*-}(K)\, u^+(K) + u^{*+}(K)\, u^-(K)\}, \tag{9.45}$$

which we can also derive from the Lagrangian (7.44) by using (7.4)

The total electric charge is given by (7.46) as

$$\left.\begin{aligned} e^t &= e \sum_{\mathbf{K}} (u^{*-}(K)\, u^+(K) - u^-(K)\, u^{*+}(K)) + c\text{-number constant} \\ &= e \sum_{\mathbf{K}} (N^+(K) - N^-(K)) + c\text{-number constant} \end{aligned}\right\} \tag{9.46}$$

$$N^+(K) = u^{*-}(K)\, u^+(K), \quad N^-(K) = u^-(K)\, u^{*+}(K). \tag{9.47}$$

Here the c-number constant is the vacuum expectation value of e^t i.e., the zero-point charge. The total charge e^t can be written, by using $U^{(1)}$ and $U^{(2)}$ of Example 2, in Ch. VII, as

$$\left.\begin{aligned} e^t = i\, e \sum_{\mathbf{K}} \{u^{(1)+}(K)\, u_{\mathbf{i}}^{(2)-}(K) - u^{(2)+}(K)\, u^{(1)-}(K) + \\ + \text{constant } c\text{-number} \end{aligned}\right\} \tag{9.48}$$

From (9.42) we have

$$([U(x),\ U^*(x')]_+)_0 = \Delta^{(1)}(x-x'). \tag{9.49}$$

EXAMPLE 2. THE ELECTROMAGNETIC FIELD A_μ

From (9.21) we have

$$A_\mu(x) = \sum_{\mathbf{K}} (2K_0 V)^{1/2} (a_\mu^+(K)\, e^{iK_\mu x_\mu} + a_\mu^-(K)\, e^{-iK_\mu x_\mu}), \tag{9.50}$$

because $d(\delta) = 1$.

The commutation relations (9.17) lead to

$$[a_\mu^+(K),\ a_\nu^-(K')] = \delta_{KK'}\, \delta_{\mu\nu}. \tag{9.51}$$

However the Lorentz condition implies that $a_l^\pm \Psi$ $(l = 1, 2, 3)$ and $a_4^\pm \Psi$ are not independent but connected with each other by the relation

$$\left.\begin{aligned} \{(\mathbf{K}\cdot\mathbf{a}^+(K)) - K_0\, a_0^+(K)\}\, \Psi = 0 \\ \{(\mathbf{K}\cdot\mathbf{a}^-(K)) - K_0\, a_0^-(K)\}\, \Psi = 0. \end{aligned}\right\} \tag{9.52}$$

Here $\mathbf{a}^+(K)$ and $\mathbf{a}^-(K)$ denote the three dimensional vectors $a_l^+(K)$ and $a_l^-(K)$ $(l = 1, 2, 3)$ respectively.

It is usual to take A_0 as hermitean, and so $A_4 = iA_0$ as non-hermitian. Then a_4^+ and a_4^- are not the same as the matrices (9.18), because they are not hermitian conjugate to each other. On the other hand, from (9.51)

$$[a_0^-(K),\ a_0^+(K')] = \delta_{KK'}. \tag{9.53}$$

Since a_0^- and a_0^+ are hermitian conjugate to each other, (9.53) shows that a_0^- and a_0^+ (not a_0^+ and a_0^-) are the annihilation and creation operators. Therefore $N_0(K) = a_0^+(K) a_0^-(K)$ has the integral eigenvalues $(0, 1, 2, \ldots)$. Since the number operator $N_4(K) = a_4^-(K) a_4^+(K)$ of the $(K, 4)$-photons is

$$N_4(K) = -(N_0(K) + 1),$$

$N_4(K)$ has negative integral eigenvalues $(-1, -2, \ldots)$ and leads to the energy $(-iK_4)N_4(K) = -K_0(N_0(K) + 1)$. In other words, $(K, 0)$-photons have negative energy $-K_0$. The relation (9.38) implies that from the vacuum it is impossible to create (not annihilate !) any $(K, 0)$-photon. In other words, this vacuum is the state of maximum number of $(K, 0)$-photons. The $(K, 0)$-photons are usually called scalar photons. Since the $(K, 0)$-photons have negative energies, this

vacuum is the lowest energy state. This definition of the vacuum has other difficulties, in that it is not compatible with the second equation of the Lorentz condition (9.52). In fact, as will be shown later, $a_0^+\Psi$, in which Ψ satisfies the second equation of (9.52), cannot be zero. This contradiction can also be seen in the relation, derived from (9.42),

$$([\partial_\mu A_\mu(x),\ A_\nu(x')]_+)_0 = \partial_\nu \Delta^{(1)}(x-x'). \tag{9.54}$$

The right hand side must be zero on account of the Lorentz condition, in contrast with the non-zero left-hand side.

We shall now consider the physical content of this difficulty. Using the coordinate system in which $K_\mu = (0, 0, K_3, iK_0)$, the state vector Ψ can be expanded as

$$\Psi = \sum_{n,m}^{\infty} C(n, m)\, \Phi(n, m), \tag{9.55}$$

where $\Phi(n, m)$ are state vectors in which the numbers of the $(K, 3)$-photons (longitudinal photons) and $(K, 0)$-photons are n and m respectively. Substituting (9.55) into (9.52) we obtain the recurrence formulae (see (9.18))

$$\sqrt{n+1}\, C(n, m) = \sqrt{m+1}\, C(n+1, m+1),$$
$$\sqrt{n}\, C(n, m) = \sqrt{m}\, C(n-1, m-1),$$

which give

$$C(n, m) = c\delta_{nm}. \tag{9.56}$$

The constant c must be determined by the normalization [1]) of the state vector. From (9.55) and (9.56) we have

$$\Psi = c \sum_{n=0}^{\infty} \Phi(n, n) \tag{9.57}$$

which implies that the vacuum state has infinitely many longitudinal and scalar photons; but their energies cancel out and they do not produce (for free fields) any observable effects.

Thus we can see that $a_0^+\Psi$ cannot be zero for a state satisfying the Lorentz condition. For fields in interaction, the effects of the longitudinal and scalar photons do not cancel each other but give the coulomb potential.

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[1]) Equations (9.55) and (9.56) show that, if c is finite $\Psi^*\Psi$ is infinite and Ψ cannot be a vector in Hilbert space

Thus we must modify the definition of the vacuum for the electromagnetic field, and use

$$\left.\begin{aligned}&(\mathbf{e}^{(r)}\cdot\mathbf{a}^{+}(K))\,\Phi_0=0\\&[(\mathbf{K}\cdot\mathbf{a}^{\pm}(K))-K_0 a_0^{\pm}(K)]\,\Phi_0=0\end{aligned}\right\}\tag{9.58}$$

where $\mathbf{e}^{(r)}(r=1, 2)$ are three dimensional unit vectors satisfying

$$(\mathbf{e}^{(r)}, \mathbf{K})=0, \quad (\mathbf{e}^{(r)}, \mathbf{e}^{(s)})=\delta_{rs}.$$

The second equation in (9.58) is the Lorentz condition. These equations are not covariant in form, because the special unit vector $\mathbf{e}^{(r)}$ appears in it.

We shall now show that we can use (9.38) instead of (9.58) for the particular calculation of the transition probability between the infinitely past ($\sigma=-\infty$) and infinitely future ($\sigma=+\infty$) states, (i.e. the S-matrix, cf. Ch. XIII). From (9.38) we have

$$A_\mu^{+}(x)\,\Phi_0=0 \tag{9.59}$$

$$([A_\mu(x),\ A_\nu(x')]_{+})_0=\delta_{\mu\nu}\,\Delta^{(1)}(x-x'). \tag{9.60}$$

On the other hand, from (9.58),

$$([(\mathbf{e}^{(r)}\cdot\mathbf{a}^{-}(K)),\ (\mathbf{e}^{(s)}\cdot\mathbf{a}^{+}(K))]_{+})_0=\delta_{rs}.$$

This can be extended into the four dimensional expression

$$([a_\mu^{-}(K),\ a_\nu^{+}(K)]_{+})_0=\delta_{\mu\nu}+K_\mu K_\nu\,\phi(K) \tag{9.61}$$

on account of Lorentz invariance. Here $\phi(K)$ is a Fourier amplitude of a scalar function. The relation (9.61) leads to

$$([A_\mu(x),\ A_\nu(x')]_{+})_0=\delta_{\mu\nu}\,\Delta^{(1)}(x-x')+\partial_\mu\partial_\nu\,\phi(x-x'). \tag{9.62}$$

On the other hand, in the calculation of the transition matrix, $[A_\mu(x), A_\nu(x')]_{+}$ appear in the form

$$\int_{-\infty}^{\infty}d^4x\int_{-\infty}^{\infty}d^4x'\,K_{\mu\nu}(x, x')\,[A_\mu(x), A_\nu(x')]_{+} \tag{9.63}$$

in virtue of Lorentz invariance. Taking into account gauge-invariance, we have

$$\partial_\mu K_{\mu\nu}(x, x')=\partial'_\nu K_{\mu\nu}(x, x')=0. \tag{9.64}$$

Indeed, the gauge-invariance (see (7.66)) requires that

$$\begin{aligned}
&\int_{-\infty}^{\infty} d^4x \int_{-\infty}^{\infty} d^4x'\, K_{\mu\nu}(x, x')\, [A_\mu(x), A_\nu(x')]_+ \\
&= \int_{-\infty}^{\infty} d^4x \int_{-\infty}^{\infty} d^4x'\, K_{\mu\nu}(x, x')\, [A_\mu(x) + \partial_\mu \Lambda(x), A_\nu(x') + \partial_\nu \Lambda(x')]_+ \\
&= \int_{-\infty}^{\infty} d^4x \int d^4x'\, K_{\mu\nu}(x, x')\, [A_\mu(x), A_\nu(x')]_+ \\
&- 2 \int_{-\infty}^{\infty} d^4x \int_{-\infty}^{\infty} d^4x'\, \partial_\mu K_{\mu\nu}(x, x')\, \Lambda(x)\, A_\nu(x') \\
&- 2 \int_{-\infty}^{\infty} d^4x \int_{-\infty}^{\infty} d^4x'\, \partial'_\nu K_{\mu\nu}(x, x')\, A_\mu(x)\, \Lambda(x') \\
&+ 2 \int_{-\infty}^{\infty} d^4x \int_{-\infty}^{\infty} d^4x'\, \partial_\mu \partial'_\nu K_{\mu\nu}(x, x')\, \Lambda(x)\, \Lambda(x').
\end{aligned}$$

This leads to (9.64). The equation (9.64) shows that the second term of (9.62) does not contribute to the matrix elements, in which $[A_\mu, A_\nu]_+$ in (9.63) appears to be $([A_\mu, A_\nu]_+)_0$. Thus, we can use (9.60) instead of (9.62) (DYSON [1950]).

However, in order to carry out covariant calculations in more general problems than that of the S-matrix, it is necessary to formulate a covariant definition of the vacuum. For this purpose we shall go back to (9.59). Then the A_4^+ must be annihilation operators, in order that there should be no particle in the vacuum defined by (9.59). This requires that A_4 should be hermitian and A_0 anti-hermitian. Then (9.51) shows that $N_4(K)$ has integral eigenvalues (0, 1, 2, ...) and that there is no particle in the vacuum defined by (9.59), which is the lowest energy state. Since we have seen that (9.59) is not consistent with the Lorentz condition (9.52), we shall now modify the Lorentz condition, writing it as

$$\partial_\mu A_\mu^+(x) \Psi = 0. \tag{9.65a}$$

Since (9.65a) gives only the first equation of (9.52), it is compatible with the definition of the vacuum (9.59). This condition is also compatible with (9.54). Indeed, by (9.59) the right hand side of the relation (9.54) can be rewritten as

$$(\partial_\mu A_\mu^+(x) \cdot A_\nu^-(x') + A_\nu^+(x') \cdot \partial_\mu A_\mu^-(x'))_0.$$

This is not zero, but equal to $\partial_\nu \Delta^{(1)}(x - x')$ if we use (9.65a). When we deal with interacting fields, we cannot divide A_μ into two parts, corresponding to positive and negative frequency, because the A_μ do not satisfy the Klein–Gordon equation. However, we can still prove that $\partial_\mu A_\mu$ satisfies the Klein–Gordon equation

$$\Box \partial_\mu A_\mu(x) = -\partial_\mu J_\mu(x) = 0. \tag{9.66}$$

Since it is thus always possible to divide $\partial_\mu A_\mu$ into parts of positive and negative frequency $\partial_\mu A_\mu^{\pm}$, we can adopt (9.65a) as the modified Lorentz condition (GUPTA [1950], BLEULER [1950]). By a method similar to that of Examples 3 and 4, Ch. VII, it can be proved that (9.65a) is compatible with wave equations (7.50). We now show that, if there are no longitudinal or scalar photons at the time t, there are no such photons at any time.

In fact, if there are no longitudinal and scalar photons at a time t,

$$\partial_k A_k^+(\mathbf{x}, t)\Psi = 0$$
$$A_4^+(\mathbf{x}, t)\Psi = 0.$$

Then (9.65a) gives

$$\partial_4 A_4^+(\mathbf{x}, t)\Psi = -\partial_k A_k^+(\mathbf{x}, t)\Psi = 0$$

whence, by using the wave equations (7.50), it follows that

$$\partial_4^n A_4^+(\mathbf{x}, t)\Psi = 0.$$

Therefore, for an arbitrary time t',

$$A_4^+(x, t')\Psi = \sum_{n=0}^{\infty} \frac{1}{n!}(t'-t)^n \left\{\left(\frac{\partial}{\partial t}\right)^n A_4(\mathbf{x}, t)\right\}\Psi = 0.$$

which gives

$$\partial_4 A_4^+(\mathbf{x}, t')\Psi = 0.$$

Then (9.65a) gives

$$\partial_k A_k^+(\mathbf{x}, t')\Psi = 0.$$

Thus we see that, because of the Lorentz condition (9.65a), if there are no longitudinal and scalar photons at one given time, there cannot be any other time. We may therefore assume that there are no longitudinal and scalar photons in the states occurring in nature.

In this theory (9.60) can be established.

We shall now introduce the operator η and the state vector $\Psi^\dagger$ by

$$\eta^* = \eta^{-1},\ \eta A_k = A_k \eta,\ \eta A_4 = -A_4 \eta,\ \eta^2 = 1, \tag{9.67a}$$
$$\Psi^\dagger = \Psi^* \eta. \tag{9.67b}$$

Taking into account the fact that A_4 is hermitian and x_4 is imaginary, we obtain from (9.66a)

$$\Psi^\dagger \partial_\mu A_\mu^-(x) = 0. \tag{9.65b}$$

Equations (9.65a) and (9.65b) yield

$$\Psi^\dagger \partial_\mu A_\mu(x)\Psi = 0. \tag{9.65c}$$

Equation (9.65c) shows that the usual Lorentz condition can be established for the expectation value of $\partial_\mu A_\mu$ if we define the expectation value of an operator F not by $\Psi^* F\Psi$ but by $\Psi^\dagger F\Psi$. This fact exhibits the correspondence between the classical Maxwell theory and the present theory, the expectation values of q-numbers replacing the quantities of the classical theory.

In the representation in which N_4 is diagonal [1]), η is given by $(-1)^{N_4}$. Then $\Psi^\dagger\Psi$ is not always positive. For this reason the theory is called that of the indefinite metric. However, for the states realised in nature, $\eta = 1$ and therefore $\Psi^\dagger\Psi = \Psi^*\Psi$ for $N_3 = N_4 = 0$. Thus it is that calculations may be carried out in a covariant manner by using conditions (9.59), (9.60) and (9.65) and the initial conditions $N_3 = N_4 = 0$.

EXAMPLE 3. VECTOR OR PSEUDOVECTOR FIELDS

From (8.58) we have

$$d(ik) = [d_{\mu\nu}(ik)] = [\delta_{\mu\nu} + a_\mu a_\nu]$$

with

$$a_\mu = k_\mu/\varkappa.$$

It is easily seen that d has two eigen values

$$\lambda = 1 \text{ and } \lambda = 1 + a_\mu a_\mu.$$

For

$$\det|\lambda I - d(ik)| = (\lambda - 1)^3(\lambda - 1 - a_\mu a_\mu). \tag{9.68}$$

Since [2]) $1 + a_\mu a_\mu = 0$ for $k_\mu = K_\mu$, the rank of the matrix d (for $k_\mu = K_\mu$) is 3, which agrees with the number $2S+1 = 3$ of the independent components of U_μ.

Using the coordinate system in which $k_\mu = (0, 0, k, ik_0)$, we can transform the hermitian matrix $d(ik)$ into the diagonal form by means of the matrix given by

$$s = -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -a_4 & a_3 \\ 0 & 0 & a_3 & a_4 \end{bmatrix} \frac{1}{(a_3^2 + a_4^2)}. \tag{9.69}$$

---

[1]) Since A_4 is an operator which changes the number of scalar photons by one, we have: $(-1)^{N_4} A_4 = -A_4(-1)^{N_4}$.

[2]) $K_\mu K_\mu + \varkappa^2 = 0$.

Then, for $U'_\mu = s_{\mu\nu} U_\nu$,

$$U'_4(K) = \frac{1}{\varkappa} K_\mu U_\mu = 0 \tag{9.70}$$

because of (7.86). This shows that the three independent components are $U'_l(K)$ $(l = 1, 2, 3)$. Using (9.21) we obtain

$$\left.\begin{aligned} \mathbf{U}(x) = (2V)^{-1/2} \sum_{\mathbf{K}} \Big[\Big\{ (K_0)^{-1/2} \sum_{r=1,2} \mathbf{e}^{(r)} u_r^+(K) + \frac{K_0^{1/2}}{\varkappa} \mathbf{e}^3 u_3^+(K) \Big\} e^{iK_\mu x_\mu} \\ + \Big\{ (K_0)^{-1/2} \sum_{r=1,2} \mathbf{e}^{(r)} u_r^-(K) + \frac{K_0^{1/2}}{\varkappa} \mathbf{e}^3 u_3^-(K) \Big\} e^{-iK_\mu x_\mu} \end{aligned}\right\} \tag{8.71a}$$

$$U_4(x) = \frac{i}{\varkappa V^{1/2}} \sum_{\mathbf{K}} (2K_0)^{-1/2} |\mathbf{K}| \{ u_3^+(K) e^{iK_\mu x_\mu} + u_3^-(K) e^{-iK_\mu x_\mu} \}, \tag{9.71b}$$

where $\mathbf{e}^{(r)}$ $(r = 1, 2)$ are the unit vectors that are mutually orthogonal and orthogonal to $\mathbf{K}$, and $\mathbf{e}^3$ is the unit vector in the direction of $\mathbf{K}$. In other words, $\mathbf{e}^{(r)}$ are transverse directions and $\mathbf{e}^3$ is a longitudinal direction.

From (9.17),

$$[u_r^+(K),\ u_s^{*-}(K')] = \delta_{rs}\, \delta_{\mathbf{K}\mathbf{K}'} \quad (r, s = 1, 2, 3). \tag{9.72}$$

The number operators $N_r^\pm(K)$ of the positively and negatively charged particles are respectively

$$N_r^+(K) = u_r^{*-}(K)\, u_r^+(K),\ N_r^-(K) = u_r^-(K)\, u_r^{*+}(K).$$

The total energy-momentum T_μ and the total charge e^t of the field are

$$\left.\begin{aligned} T_\mu &= -i \sum_{\mathbf{K}} \sum_{r=1,2,3} K_\mu \{ N_r^+(K) + N_r^-(K) + 1 \} \\ e^t &= e \sum_{\mathbf{K}} \sum_{r=1,2,3} \{ N_r^+(K) - N_r^-(K) \}. \end{aligned}\right\} \tag{9.73}$$

EXAMPLE 4. SPINOR FIELD $\psi(x)$

In this case the transformation (9.5) and (9.7) can be written as [1])

$$\left.\begin{aligned} \psi_\alpha^+(K) &= 2K_0\, V^{1/2}\, \varphi_r^+(K)\, a_{r\alpha}(K) \\ \psi_\alpha^-(K) &= 2K_0\, V^{1/2}\, \varphi_r^-(K)\, b_{r\alpha}(K), \end{aligned}\right\} \tag{9.74}$$

---

[1]) α corresponds to four components of ψ and r takes two values (1, 2) corresponding to the two directions of the spin angular momentum.

where $a_{r\alpha}$ and $b_{r\alpha}$ satisfy the relations

$$a_{r\alpha}\, a^*_{r\beta} = \frac{1}{2K_0}\,(K_0 + \alpha_i\, k_i + \varkappa\beta)_{\alpha\beta} \tag{9.75}$$

$$b_{r\alpha}\, b^*_{r\beta} = \frac{1}{2K_0}\,(K_0 - \alpha_i\, k_i - \varkappa\beta)_{\alpha\beta}. \tag{9.76}$$

It must be noted that (9.75) and (9.76) agree with the projection operators $\Lambda_\pm$ (cf. (3.56)) that select the positive and negative energy states.

From (9.17)

$$\left.\begin{aligned} [\varphi^+_r(K),\ \varphi^{*-}_r(K')]_+ &= \delta_{rs}\,\delta_{\mathbf{KK}'} \\ [\varphi^{*+}_r(K),\ \varphi^-_r(K')]_+ &= \delta_{rs}\,\delta_{\mathbf{KK}'}. \end{aligned}\right\} \tag{9.77}$$

Equations (9.21) and (9.22) give

$$\left.\begin{aligned} \psi_\alpha(x) &= V^{-1/2} \sum_{\mathbf{K}} \{\varphi^+_r(K)\, a_{r\alpha}(K)\, e^{iK_\mu x_\mu} + \varphi^-_r(K)\, b_{r\alpha}(K)\, e^{-iK_\mu x_\mu}\} \\ \psi^*_\alpha(x) &= V^{-1/2} \sum_{\mathbf{K}} \{\varphi^{*+}_r(K)\, b^*_{r\alpha}(K)\, e^{iK_\mu x_\mu} + \varphi^{*-}_r(K)\, a^*_{r\alpha}\, e^{-iK_\mu x_\mu}\}. \end{aligned}\right\} \tag{9.78}$$

From (9.42) we have

$$\left.\begin{aligned} ([\psi_\alpha(x),\ \bar\psi_\beta(x')]_-)_0 &= -S^{(1)}_{\alpha\beta}(x-x') \\ S^{(1)}_{\alpha\beta}(x-x') &\equiv (\gamma_\mu \partial_\mu - \varkappa)_{\alpha\beta}\, \Delta^{(1)}(x-x'). \end{aligned}\right\} \tag{9.79}$$

From (7.112) the total energy-momentum vector is obtained as

$$T_\mu = -i \sum_{\mathbf{K}} \sum_{r=1,2} K_\mu \{\varphi^{*-}_r(K)\, \varphi^+_r(K) + \varphi^-_r(K)\, \varphi^{*+}_r(K) - 1\} \tag{9.80}$$

(see (9.28)). The zero point energy in (9.80) has a negative value which is the same as the total energy of the vacuum electrons in the hole theory.

The last equation shows that the difficulties due to the negative energy states in the c-number theory of the Dirac particles can be avoided by use of commutation relations of the (+) type, i.e. the use of the Pauli exclusion principle in the quantum theory since $\{T_\mu - (\text{zero point energy})\}$ given by (9.80) is positive.

The total charge of the Dirac field is

$$e^t = e \sum_{\mathbf{K}} \sum_{r=1,2} \{N^+_r(K) - N^-_r(K)\}, \tag{9.81}$$

where

$$N^+_r(K) = \varphi^{*-}_r(K)\, \varphi^+_r(K),\quad N^-_r(K) = \varphi^-_r(K)\, \varphi^{*+}_r(K). \tag{9.82}$$

EXAMPLE 5. ANGULAR MOMENTUM AND MAGNETIC MOMENT

First, we shall show that the expectation value of the orbital angular momentum $P^0_{\mu\nu}$, (7.41), and the spin angular momentum $P^S_{\mu\nu}$ (7.42), can be written as the sum of those of the particles.

The space component of the orbital angular momentum of a vector field is

$$P^0_{jk}=\int d^3x\,\{U^{\dagger *}_{4\varrho}(x_j\,\partial_k-x_k\,\partial_j)\,U_\varrho+(x_j\,\partial_k-x_k\,\partial_j)\,U^\dagger_\varrho\cdot U_{4\varrho}\} \tag{9.83}$$

becuase of (7.41) and (7.88). Substituting (9.71*a*, *b*) into (9.83), we obtain the expectation value of P^0_{jk} with respect to the state in which there are particles of momentum $\mathbf{K}$,

$$(P^0_{jk})_{\mathbf{K}}=i\,(x_j\,K_k-x_k\,K_j)\sum_{r=1,2,3}(N^+_r(K)+N^-_r(K)-1). \tag{9.84}$$

This is the sum of the orbital angular momenta of the particles.

The space component of the spin angular momentum is

$$P^S_{jk}=\int d^3x\,\{U^\dagger_{4j}\,U_k+U^\dagger_k\,U_{4j}-U^\dagger_{4k}\,U_j-U^\dagger_j\,U_{4k}\} \tag{9.85}$$

because of (7.39) and (7.42). The expectation value of P^S_{jk} for a particle at rest ($\mathbf{K}=0$) is obtained, by substituting (9.71*a*, *b*) into (9.85)

$$\left.\begin{aligned}(P^S_{jk})_{\mathbf{K}=0}=&-\{(u^{*-}_j(K)\,u^+_k(K)-u^{*-}_k(K)\,u^+_j(K))\\&+(u^-_j(K)\,u^{*+}_k(K)-u^-_k(K)\,u^{*+}_j(K)\}_{\mathbf{K}=0},\end{aligned}\right\} \tag{9.86}$$

where we use the coordinate system with $\mathbf{e}^{(r)}$ ($r=1, 2, 3$) as basis vectors.

The last equation can be written as

$$i(P^S_{jk})_{\mathbf{K}=0}=(N^+_+ + N^-_+ - N^+_- - N^-_-) \tag{9.87}$$

where

$$N^-_s=u^-_s\,u^{*+}_s,\quad N^+_s=u^{*-}_s\,u^+_s\quad(s=+,-) \tag{9.88}$$

and $u^\pm_\pm$ is defined by

$$\left.\begin{aligned}u^\pm_+&=\frac{1}{\sqrt{2}}(u^\pm_j-i\,u^\pm_k)\\u^\pm_-&=\frac{1}{\sqrt{2}}(u^\pm_j+i\,u^\pm_k)\end{aligned}\right\} \tag{9.89}$$

(WENTZEL [1943]). Using (9.72) we obtain

$$[u^{*+}_s,u^-_{s'}]=\delta_{ss'},\quad [u^+_s,u^{*-}_{s'}]=\delta_{ss'}. \tag{9.90}$$

Since (9.90) shows that the possible eigenvalues of N_s^r are (0, 1, 2, ...), we can regard these as the numbers of particles in the (r, s)-state. Equation (9.89) shows that the superscript r signifies the charge state similar to the $(\pm)$ in $N^{\pm}$, (9.20). Regarding the values of the suffices as labelling the spin states, we see that (9.87) gives the sum of the spin angular momenta $(\pm 1,0)$ of the particles.

From (9.85) we have

$$(i)^2 (P_{jk}^s P_{jk}^s)_{K=0}^{(+)} = 2 \tag{9.91}$$

where the superscript $(+)$ signifies the positive charge state, for which the expectation value is calculated. This shows that the magnitude of the spin angular momentum is $S(S+1)=2$.

We shall now consider the magnetic moment of low energy ($K_0 \approx \varkappa$) particles. Since $U_4 \approx 0$ for the vector particles of low energy (cf. Example in § 2 of Ch. IV), θ_{k4} given by (7.89) is

$$\theta_{k4} \approx U_{k\varrho}^{\dagger} \cdot \partial_4 U_{\varrho} + \partial_4 U_{\varrho}^{\dagger} \cdot U_{k\varrho}.$$

This leads to

$$(\theta_{k4})^{\pm} = \pm \varkappa (U_{k\varrho}^{\dagger} U_{\varrho} - U_{\varrho}^{\dagger} U_{k\varrho})^{\pm} = \mp \frac{i}{e} \varkappa (J_k)^{\pm}$$

on account of (7.90). Thus, we have

$$(J_k)^{\pm} \approx \pm \frac{ie}{\varkappa} (\theta_{k4})^{\pm}. \tag{9.92}$$

Here, following Example 4, Ch. VII, we have omitted the e^2-term in the calculation of the magnetic moment. The space component of the magnetic moment can be obtained from (7.82), and is

$$(m_{jk})^{\pm} \approx \pm \frac{ie}{2\varkappa} (P_{jk})^{\pm}. \tag{9.93}$$

because of (7.26). This reflects the intimate relationship between the magnetic moment and the angular momentum of the low energy particles. Since we can neglect the orbital magnetic moment of the low energy particles, we have from (9.87) and (9.93),

$$(m_{jk})^{\pm} \approx \pm \frac{e}{2\varkappa} (N_{\mp}^{\pm} - N^{\pm}). \tag{9.94}$$

This shows that a low energy particle of positive (negative) charge has a magnetic moment $e/2\varkappa$, the direction of which is the same as (opposite to) that of the spin angular momentum. The quantity $e/2\varkappa$ is called a magneton. As shown in Example 5 of Ch. VII, the $F_{\mu\nu}$-type

interaction (7.91) changes this magnetic moment into $(e/2\varkappa)(1-\gamma)$.

In the case of the electromagnetic field we cannot use the rest system of a photon. Introducing $a_{\pm}^{\pm}$ in terms of $a_r^{\pm}$ $(r=1, 2)$ by means of the transformation similar to (9.89),

$$(P_{12}^S)_{\mathbf{K}} = -i(N_+ - N_-). \tag{9.95}$$ [1]

Here the coordinate system, whose third axis is in the direction of **K**, is used. The last equation shows that the components of the spin angular momentum of a photon in the direction of **K** are ± 1. The numbers N_+ or N_- may be interpreted as the number of photons whose spin angular momenta are parallel or antiparallel to **K** respectively.

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[1] Since $A_4^+ \Psi = A_3^+ \Psi$ for a state occurring in nature in the case of the free electromagnetic field, we have $P_{33} = P_{13} = 0$.

CHAPTER X

QUANTUM THEORY OF INTERACTING FIELDS

§ 1. Interaction Representation

In this Chapter we shall present the canonical formulation of the quantum theory of interacting fields. We shall use the representation in which the field quantities satisfy the wave equation for free fields and the commutation relations (8.14*a*) and (8.14*b*) (i.e. the interaction representation). (TOMONAGA [1946], SCHWINGER [1948]). In such terms the states of the interacting fields can be interpreted as those of an assembly of simple harmonic oscillators which change from time to time.

We shall represent the unitary transformation connecting the states $\Psi[\sigma]$ and $\mathbf{\Phi}$ of the interaction and Heisenberg representations respectively by

$$\Psi[\sigma]=S[\sigma]\mathbf{\Phi}. \tag{10.1}$$

In this Chapter we use bold and ordinary type to denote quantities in the Heisenberg and interaction representations respectively.

Since, as shown in Ch. IV, the $Q_\alpha(x)$ generally involve not only the independent components but also the dependent components of the field, we cannot determine $S[\sigma]$ as the transformation connecting $Q_\alpha(x)$ and $\mathbf{Q}_\alpha(x)$.

The quantities $q_r(x)$ derived by the transformation (9.5) are a set of independent components. Therefore they must be connected with the independent components $\mathbf{q}_r(x)$ in the Heisenberg representation by the unitary transformation $S[\sigma]$:

$$\mathbf{q}_r(x)=(S^{-1}[\sigma]q_r(x)\,S[\sigma])_{x/\sigma}. \tag{10.2}$$

Here x/σ indicates that the point x lies on the surface σ. However, since a Lorentz covariant formulation is desired, (10.2) must be rewritten so as to contain covariant quantities.

For this purpose, use is made of the auxiliary quantities

$$Q_\alpha[x,\,\sigma]\equiv S^{-1}[\sigma]\,s_{r\alpha}(\partial)\,q_r(x)\,S[\sigma], \tag{10.3}$$

where $s_{r\alpha}(\partial)$ is the matrix of derivation operator in (9.5) (TAKAHASHI and UMEZAWA [1953]). It must be noted that $Q_\alpha[x, \sigma]$ is a function of the point x and a functional of the surface σ, and that x and σ are independent of each other. From (10.3) and (9.5) we have

$$Q_\alpha[x, \sigma] \equiv S^{-1}[\sigma]Q_\alpha(x)S[\sigma]. \tag{10.4a}$$

Now $\mathbf{q}_r(x)$ can be derived from $Q_\alpha[x, \sigma]$ by

$$\mathbf{q}_r(x) = (s_{r\alpha}^{-1}(\partial)Q_\alpha[x, \sigma])_{x/\sigma}. \tag{10.4b}$$

Here we must take x to be a point on σ after operating on $Q_\alpha[x, \sigma]$ with $s_{r\alpha}^{-1}(\partial)$. It is apparent that complete information about $\mathbf{q}_r(x)$ can be obtained from $Q_\alpha[x, \sigma]$ and, therefore, that (10.4a) can be regarded as the covariant relation corresponding to (10.2). In the next section we shall use (10.4a) to determine $S[\sigma]$. From (10.4a), (8.2a) and (8.14a) we can show that differential equations and commutation relations for $Q_\alpha[x, \sigma]$ are the same as those for $Q_\alpha(x)$, viz.

$$\Lambda_{\alpha\beta}(\partial)Q_\beta[x, \sigma] = 0 \tag{10.5}$$

$$[Q_\alpha[x, \sigma],\ \bar{Q}_\beta[x', \sigma]]_\pm = id_{\alpha\beta}\Delta(x-x'). \tag{10.6}$$

Here $\bar{Q}_\alpha[x, \sigma]$ (cf. (8.3b)) is

$$\bar{Q}_\alpha[x, \sigma] \equiv Q_\varrho^*[x, \sigma]\eta_{\varrho\alpha} = S^{-1}[\sigma]\bar{Q}_\alpha(x)S[\sigma]. \tag{10.7}$$

It must be noted that, in (10.6), the arguments σ of Q_α and $\bar{Q}_\beta$ are the same.

Since we wish the field quantities $\mathbf{Q}_\alpha(x)$ to be functions of the independent components $\mathbf{q}_r(x)$ (at the same point x), the quantities $Q_\alpha[x, \sigma]$ can be determined by requiring that [1]: (i) they satisfy (10.5) and (10.6), and (ii) they give $\mathbf{Q}_\alpha(x)$ as function of the $\mathbf{q}_r(x)$, which are related with $Q_\alpha[x, \bar{\sigma}]$ by (10.4b).

The unitary transformation $S[\sigma, \sigma']$ connecting $Q_\alpha[x, \sigma]$ and $Q_\alpha[x, \sigma']$ according to

$$Q_\alpha[x, \sigma] = S^{-1}[\sigma, \sigma']Q_\alpha[x, \sigma']S[\sigma, \sigma'] \tag{10.8}$$

is given by

$$S[\sigma, \sigma'] = S[\sigma]S^{-1}[\sigma']. \tag{10.9}$$

---

[1]) If we have two $Q_\alpha[x,\sigma]$ satisfying these requirements, they lead to two canonical theories. However, since these two theories have the same number of independent components q_r, they are also connected by a unitary transformation; i.e. they are different representations of the same theory.

The interaction Hamiltonian $H'[x : n]$ is defined (cf. (6.3)) by the functional derivative equation

$$i\frac{\delta S[\sigma]}{\delta\sigma(x)} = H'[x:n]S[\sigma]. \tag{10.10}$$

It will be shown below that in general $H'[x : n]$ depends on the vector $n_\mu(x)$, the unit normal to σ at the point x (cf. § 2, Ch. VI). The interaction Hamiltonian must also satisfy the integrability condition which implies that $S[\sigma]$ must be determined uniquely in term of (10.10) and specified initial conditions. This means that

$$\frac{\delta}{\delta\sigma(x)}\frac{\delta}{\delta\sigma(x')}S[\sigma] = \frac{\delta}{\delta\sigma(x')}\frac{\delta}{\delta\sigma(x)}S[\sigma],$$

where x, x' are two points on σ (TOMONAGA [1946]).

Equation (10.10) can be rewritten as

$$-i\,[H'[x:n],\ H'[x':n]] = \frac{\partial}{\partial\sigma(x)}H'[x':n] - \frac{\partial}{\partial\sigma(x')}H'[x:n], \tag{10.11}$$

where $\partial/\partial\sigma$ signifies derivation with respect to those arguments depending explicitly on σ (cf. § 2 of Ch. VI). The quantities introduced here are fundamental to the following discussion.

The integrability condition (10.11) contains the term

$$(\partial/\partial\sigma(x'))\,H[x : n].$$

For practical purposes the calculation of this term is facilitated by certain important formula for the normal vector $n_\mu(x)$. These are derived below.

First, we shall prove that

$$n_\mu(x)n_\nu(x) = -\int_\sigma d\sigma'_\mu\partial_\nu\Delta(x-x'), \tag{10.12}$$

$$\frac{\partial}{\partial\sigma(x')}\{n_\mu(x)\,n_\nu(x)\} = -\frac{\delta}{\delta\sigma(x')}\int_\sigma d\sigma'_\mu\,\partial_\nu\Delta(x-x') = \partial_\mu\partial_\nu\Delta(x-x'). \tag{10.13}$$

If the surface σ is flat, the left hand side of (10.12) is equal to $-\delta_{\nu 4}\delta_{\mu 4}$ on account of (8.18). Since $n_\mu n_\nu$ is the covariant form of $-\delta_{\nu 4}\delta_{\mu 4}$, (10.12) follows. Then (10.13) can be derived [1]) from (10.12) by using (6.5*b*).

---

[1]) We can show that the right hand side of (10.12) is symmetric with respect to the suffices μ and ν. In fact, it is symmetric when the surface σ is flat, or

$$\int_\sigma d\sigma'_\mu\,\partial_\nu\Delta(x-x') - \int_\sigma d\sigma'_\nu\,\partial_\mu\Delta(x-x') = 0.$$

This relation must hold for any space-like surface, because the left hand side does not depend on the form of surface σ. Indeed,

$$\frac{\delta}{\delta\sigma(x'')}\left\{\int_\sigma d\sigma'_\mu\,\partial_\nu\Delta(x-x') - \int_\sigma d\sigma'_\nu\Delta(x-x')\right\}$$
$$= -\partial_\mu\partial_\nu\Delta(x-x'') + \partial_\nu\partial_\mu\Delta(x-x'') = 0,$$

From (10.12) we can derive:

$$\partial_\varrho(n_\mu(x)n_\nu(x)) = -\int_\sigma d\sigma'_\mu \partial_\varrho \partial_\nu \Delta(x-x').$$

On the other hand, we have

$$\int_\sigma d\sigma'_\mu \partial_\varrho \partial_\nu \Delta(x-x') - \int_\sigma d\sigma'_\varrho \partial_\mu \partial_\nu \Delta(x-x') = 0. \tag{10.14}$$

This is true when σ is the flat surface because, then, each term is zero on account of (8.17) and (8.18). We can show that (10.14) is valid for any space-like surface by showing that the left hand side of (10.14) is independent of the surface:

$$\begin{aligned}\frac{\delta}{\delta\sigma(x'')}\{\int_\sigma d\sigma'_\mu \partial_\varrho \partial_\nu \Delta(x-x') - \int_\sigma d\sigma'_\varrho \partial_\mu \partial_\nu \Delta(x-x')\} \\ = -\partial_\mu \partial_\varrho \partial_\nu \Delta(x-x'') + \partial_\varrho \partial_\mu \partial_\nu \Delta(x-x') = 0.\end{aligned}$$

By using (10.14), (10.12) and (10.13) we obtain

$$\left.\begin{aligned}\partial_\varrho(n_\mu(x)\, n_\nu(x)) &= -\int_\sigma \partial_\mu \partial_\nu \Delta(x-x')\, d\sigma'_\varrho \\ &= -\int d\sigma'_\varrho \frac{\partial}{\delta\sigma(x')}(n_\mu(x)\, n_\nu(x)).\end{aligned}\right\} \tag{10.15}$$

The equation (10.15) can be generalised to give

$$\partial_\varrho(n_{\mu_1}(x) \ldots n_{\mu_{2n}}(x)) = -\int d\sigma'_\varrho \frac{\partial}{\delta\sigma(x')}(n_{\mu_1}(x) \ldots n_{\mu_{2n}}(x)). \tag{10.16}$$

This depends on the well-known distributive law for derivation operators.

We shall now give a useful lemma, namely, that

$$\left.\begin{aligned}&\frac{\delta}{\delta\sigma(x)}\int_\sigma F_{\mu_1 \ldots \mu_{2n}}(x')\, n_{\mu_1}(x') \ldots n_{\mu_{2n}}(x')\, d\sigma'_\varrho \\ &= (\partial_\varrho F_{\mu_1 \ldots \mu_{2n}}(\cdot))\, n_{\mu_1}(x) \ldots n_{2n}(x) \\ &- F_{\mu_1 \ldots \mu_{2n}}(x) \int_\sigma d\sigma'_\varrho \frac{\partial}{\delta\sigma(x')}\{n_{\mu_1}(x) \ldots n_{\mu_{2n}}(x)\} \\ &+ \int_\sigma d\sigma'_\varrho F_{\mu_1 \ldots \mu_{2n}}(x') \frac{\partial}{\delta\sigma(x)}\{n_{\mu_1}(x') \ldots n_{\mu_{2n}}(x')\}.\end{aligned}\right\} \tag{10.17}$$

This can be proved easily by means of (6.6*b*) and (10.16). As a special case ($n=1$), (10.17) leads to

$$\left.\begin{aligned}&\frac{\delta}{\delta\sigma(x)}\int_\sigma d\sigma'_\varrho F_{\mu\nu}(x')\, n_\mu(x')\, n_\nu(x') \\ &= (\partial_\varrho F_{\mu\nu}(x)) \cdot n_\mu(x)\, n_\nu(x) - F_{\mu\nu}(x) \int_\sigma d\sigma'_\varrho \frac{\partial}{\delta\sigma(x')}\{n_\mu(x)\, n_\nu(x)\} \\ &+ \int_\sigma d\sigma'_\varrho F_{\mu\nu}(x') \frac{\partial}{\delta\sigma(x)}\{n_\mu(x')\, n_\nu(x')\}.\end{aligned}\right\} \tag{10.18}$$

Another important relation is:

$$F_{\mu\nu}(x)n_\mu(x)n_\nu(x) = -\tfrac{1}{2}\int d^4x'\ F(x')(\partial_\mu\varepsilon(x-x'))(\partial_\nu\Delta(x-x')). \tag{10.19}$$

This can be proved by rewriting (10.19), for a flat surface ($n_\mu=(0, 0, 0, i)$) in a covariant form (cf. (8.20*b*)).

§ 2. Derivation of the Interaction Hamiltonian

The wave equations of the Heisenberg and interaction representations are [1])

$$A_{\alpha\beta}(\partial)\mathbf{Q}_\beta(x) = \mathbf{J}_\alpha(x) \tag{10.20a}$$

$$A_{\alpha\beta}(\partial)Q_\beta(x) = 0 \tag{10.20b}$$

where

$$\mathbf{J}_\alpha(x) \equiv -\frac{\partial \mathbf{L}'(x)}{\partial \mathbf{Q}_\alpha(x)} + \partial_\mu \frac{\partial \mathbf{L}'(x)}{\partial \mathbf{Q}_{\alpha:\mu}(x)}. \tag{10.21}$$

Here L' is the interaction Lagrangian.

From (10.20*a*) and (10.20*b*) we have

$$\mathbf{Q}_\alpha(x) = Q_\alpha(x) + \int d^4x' \{D'_a G_{\alpha\beta}(x-x')\} \cdot \mathbf{j}_{\beta:a}(x').$$

Here D_a and $\mathbf{j}_{\alpha:a}$ are quantities whose components are

$$D_a \equiv (1, \partial_\mu) \tag{10.22a}$$

$$j_{\alpha\cdot a}(x) \equiv \left(-\frac{\partial \mathbf{L}'(x)}{\partial \mathbf{Q}_\alpha(x)}, -\frac{\partial \mathbf{L}'(x)}{\partial \mathbf{Q}_{\alpha:\mu}(x)}\right). \tag{10.22b}$$

The symbol Da' denotes Da constituted of derivation operators ∂'_μ:

$$D'_a = (1, \partial'_\mu).$$

The function $G(x-x')$ is a Green's function satisfying (8.41). Different Green's functions lead to the various interaction representations. However, since these interaction representations must be connected by unitary transformations, we shall adopt the particular one $G^{\text{ret}}(x-x')$ (cf. (8.40*c*)). Thus we have

$$\mathbf{Q}_\alpha(x) = Q_\alpha(x) + \int d^4x' \{D'_a G^{\text{ret}}_{\alpha\beta}(x-x')\} \cdot \mathbf{j}_{\beta\cdot a}(x'). \tag{10.23}$$

---

[1]) We shall restrict the present discussions to the case in which L' has derivation operators of lower degree than those of the free Lagrangian L^0. For the investigation of quantization theory in the more general case, see references PAULI [1953], UMEZAWA and TAKAHASHI [1953], KATAYAMA [1953], HAYASHI [1953], RAYSKI [1952].

Furthermore, we can show that the quantities

$$Q_\alpha[x, \sigma] = Q_\alpha(x) + \int_{-\infty}^{\sigma} d^4x' \{D'_a d_{\alpha\beta}(\partial) \Delta(x-x')\} \cdot \mathbf{j}_{\beta,a}(x') \quad (10.24)$$

satisfy the relations (10.5). In (10.24) the point x is not necessarily on the surface σ. By using (10.4b) to derive $\mathbf{q}_r(x)$, it will be shown later that the field quantities $\mathbf{Q}_\alpha(x)$ can be written as functions of $\mathbf{q}_r(x)$. Therefore, we may assume that $Q_\alpha[x, \sigma]$ is given by (10.24). This assumption can be justified if there exists the unitary transformation $S[\sigma]$ connecting $Q_\alpha(x)$ with (10.24). For, in such a case, (10.24) satisfies (10.6). In the following discussion we shall deduce the condition [1]) for the existence of the unitary transformation which transforms $Q_\alpha(x)$ into (10.24). It is apparent from (10 10) that this condition is equivalent to that for the existence of the interaction Hamiltonian (i.e. (10.29)).

Now (10.23) and (10.24) yield (cf. (8.10), (8.27c))

$$\mathbf{Q}_\alpha(x) = Q_\alpha[x/\sigma] + \tfrac{1}{2} \int d^4x' \, \mathbf{j}_{\beta\ a}(x') \, [D'_a d_{\alpha\beta}(\partial), \varepsilon(x-x')] \, \Delta(x-x'). \quad (10.25)$$

Here $Q_\alpha[x/\sigma]$ means $Q_\alpha[x, \sigma]$ at a point x lying on σ.

The equation (10.24) also shows that

$$Q_\alpha[x, -\infty] = Q_\alpha(x)$$

and therefore, on account of (10.8), that

$$S[\sigma] = S[\sigma, -\infty].$$

From (10.10) we have

$$i \frac{\delta Q_\alpha[x, \sigma]}{\delta\sigma(x')} = S^{-1}[\sigma] \, [Q_\alpha(x), \ H'[x' : n]] \, S[\sigma]. \quad (10.27)$$

On the other hand, (10.24) gives

$$i \frac{\delta Q_\alpha[x, \sigma]}{\delta\sigma(x')} = i \, \mathbf{j}_{\beta\ a}(x') \{D'_a d_{\alpha\beta}(\partial) \Delta(x-x')\}. \quad (10.28)$$

Comparing this with (10.27), we obtain

$$[Q_\alpha(x), H[x' : n]] = i \, S[\sigma] \, \mathbf{j}_{\beta\ a}(x') \{D'_a d_{\alpha\beta}(\partial) \Delta(x-x')\} S^{-1}[\sigma]. \quad (10.29)$$

---

[1]) When the interaction Lagrangian contains fields of higher spin ($\geqslant 3/2$) or differential operators of higher order ($\geqslant 2$), it may sometimes happen that this condition (and therefore (10.29)) cannot be satisfied. For an extension of the formulation given in this chapter to such a case, see UMEZAWA and TAKAHASHI [1953], KATAYAMA [1953], HAYA[illegible] [1953]

This is the fundamental equation by which the interaction Hamiltonian $H'[x \cdot n]$ *must be determined* (TAKAHASHI and UMEZAWA [1953]). The unitary transformation $S[\sigma]$ connecting the Heisenberg and interaction representations can be obtained by (10.10).

From (10.29), $H'[x \cdot n]$ can be expressed as a power series in the coupling constants by rewriting $\mathbf{j}_{\beta\ a}(x')$ in terms of $Q_\alpha[x, \sigma]$. This can be done by means of the formula

$$\left.\begin{aligned} M(\partial)\,\mathbf{Q}_\alpha(x) &= (M(\partial)\,Q_\alpha[x,\sigma])_{x/\sigma} \\ &+ \tfrac{1}{2}\int d^4x'\,\mathbf{j}_{\beta\ a}(x')\cdot[M(\partial)\,D'_a\,d_{\alpha\beta}(\partial),\,\varepsilon(x-x')]\,\Delta(x-x')\end{aligned}\right\} \quad (10.30)$$

which is derived from (10.23) and (10.24). Here $M(\partial)$ is any given differential operator.

From (10.4*b*), we deduce

$$s^{-1}_{\tau\alpha}(\partial)\,\mathbf{Q}_\alpha(x) = Q_\tau(x) + \tfrac{1}{2}\int d^4x'\,\mathbf{j}_{\beta\ a}(x')\,[s^{-1}_{\tau\alpha}(\partial)\,D'_a\,d_{\alpha\beta}(\partial),\,\varepsilon(x-x')]\,\Delta(x-x').$$

This leads to.

$$\begin{aligned} M(\partial)\,\mathbf{Q}_\alpha(x) &= M(\partial)\,s_{\alpha\tau}(\partial)\,\mathbf{q}_\tau(x) \\ &+ \tfrac{1}{2}\,M(\partial)\,s_{\alpha\tau}(\partial)\int d^4x'\,\mathbf{j}_{\beta\ a}(x')\,[s^{-1}_{\tau\alpha}(\partial)\,D'_a\,d_{\alpha\beta}(\partial).\,\varepsilon(x-x')]\,\Delta(x-x').\end{aligned}$$

The second term depends only on x, on account of the relation (8.20*a*, *b*). Therefore the second term is a function of the field quantities $\mathbf{Q}_\alpha(x)$ (at the point x). By iteration we can therefore express $\mathbf{Q}_\alpha(x)$ as a series expansion, each term being a function of $\mathbf{q}_\tau(x)$. Thus, we see that the requirement (ii) for $Q_\alpha[x, \sigma]$ (cf. § 1), is satisfied.

We shall now prove that the interaction Hamiltonian $H'[x : n]$ derived from (10.29) satisfies the integrability condition (10.11).

Since the right hand side of (10.28) is not a functional of the surface, we have

$$\frac{\delta}{\delta\sigma(x'')}\,\frac{\delta}{\delta\sigma(x')}\,Q_\alpha[x,\sigma] = 0.$$

This with (10.27) leads to

$$\Big[Q_\alpha(x),\,[H\,[x':n],\,H'[x'':n]] + i\,[Q_\alpha(x),\,\frac{\partial}{\partial\sigma(x'')}\,H'[x':n]\Big] = 0,$$

which can be rewritten as

$$\left.\begin{aligned} &Q_\alpha(x)\,H'[x':n]\,H'[x'':n] + H'[x'':n]\,H'[x':n]\,Q_\alpha(x) \\ &+ i\Big[Q_\alpha(x),\,\frac{\partial}{\partial\sigma(x'')}\,H'[x':n]\Big] \\ &- \{H'[x':n]\,Q_\alpha(x)\,H'[x'':n] + H'[x'':n]\,Q_\alpha(x)\,H'[x':n]\} = 0.\end{aligned}\right\} \quad (10.31)$$

By subtracting from (10.31) the relation obtained by exchanging x' and x'' in it we deduce

$$\left[Q_a(x), [H'[x':n], H'[x'', n]] - i\frac{\partial}{\delta\sigma(x')}H'[x'':n] + i\frac{\partial}{\delta\sigma(x'')}H'[x':n]\right] = 0.$$

The right hand term in the brackets must be a c-number. This c-number must be zero because $H'[x : n]$ is a sum of products of at least two field quantities. The integrability condition (10.11) can now be derived.

The canonical energy-momentum vector $\mathbf{T}_\mu$ is defined by

$$\left.\begin{aligned}\mathbf{T}_\mu &= -iS^{-1}[\sigma]\int_\sigma d\sigma'_\nu\,(T^0_{\mu\nu}(x') - H'[x':n]\,\delta_{\mu\nu})\cdot S[\sigma]\\ &= S^{-1}[\sigma]\,(T^0_\mu + i\int_\sigma d\sigma'_\mu\,H'[x':n])\,S[\sigma],\end{aligned}\right\} \tag{10.32}$$

where $T^0_{\mu\nu}$ and T^0_μ are the energy-momentum tensor and vector for free fields. The last equation shows that the total energy is the sum of the interaction energy and the energy of the free fields. We shall define the total energy-momentum vector T_μ for the interaction representation as

$$T_\mu = S[\sigma]\mathbf{T}_\mu S^{-1}[\sigma] = T^0_\mu + i\int_\sigma d\sigma'_\mu H'[x : n]. \tag{10.33}$$

Although, as shown later, $\mathbf{T}_\mu$ is constant in time, T_μ does not have this property. However, since T_μ and $\mathbf{T}_\mu$ are connected by a unitary transformation, they have the same eigenvalues. The eigenvalues E_μ of an eigenstate $\Psi(E)$ of $\mathbf{T}_\mu$:

$$\mathbf{T}_\mu\Psi(E) = E_\mu\Psi(E) \tag{10.34}$$

give the total energy and momentum of the interacting fields. Moreover, $\Psi(E)$ is independent of σ as $\mathbf{T}_\mu$ is. Then $S[\sigma]\Psi(E)$ is an eigenstate of T_μ for the same eigenvalue E_μ:

$$T_\mu S[\sigma]\Psi(E) = E_\mu S[\sigma]\Psi(E). \tag{10.35}$$

We now prove that $\mathbf{T}_\mu$ satisfies the equations (7.5) and (7.27), namely

$$\frac{\delta\mathbf{T}_\mu}{\delta\sigma(x)} = 0 \tag{10.36}$$

$$-\partial_\mu\mathbf{F}(x) = [\mathbf{F}(x), \mathbf{T}_\mu]. \tag{10.37}$$

Since $H'[x : n]$ is a scalar quantity, it has the form

$$H'[x:n] = \sum_m h_{\mu_1\ldots\mu_{2m}}(x)\,n_{\mu_1}(x)\ldots n_{\mu_{2m}}(x)$$

where the $h_{\mu_1 \ldots \mu_{2m}}$ are functionals of the field quantities $Q_\alpha(x)$. From (8.15) we have

$$[T^0_\mu, H'[x:n]] = \sum_n \partial_\mu h_{\mu_1 \ldots \mu_{2m}}(x)\, n_{\mu_1}(x) \ldots n_{2m}(x).$$

Then, (10.32), (10.17) and (10.10) give

$$\frac{\delta \mathbf{T}_\mu}{\delta\sigma(x)} = -S^{-1}[\sigma] \int_\sigma d\sigma'_\mu \Big\{ [H'[x:n], H'[x':n]] - i\frac{\partial H'[x:n]}{\partial\sigma(x')} - i\frac{\partial H'[x':n]}{\partial\sigma(x)} \Big\} S[\sigma].$$

This is equivalent to (10.36) because of the integrability condition (10.11). It must be noted that this proof of the conservation of $\mathbf{T}_\mu$ is based on the integrability condition alone. Since we can construct various Hamiltonians $H'[x:n]$ satisfying the integrability condition, the conservation law for the energy-momentum vector is not sufficient to determine $\mathbf{T}_\mu$ uniquely. However, (10.32) with (10.29) determines $\mathbf{T}_\mu$ uniquely.

We shall now show that (10.37) is valid. Since, as shown by (10.36), $\mathbf{T}_\mu$ does not depend on the form of the surface σ, we choose a flat surface $\sigma(x) = \sigma(t)$ at a specified time t (i.e. $n_\alpha = 0, 0, 0, i$). Then, (10.32) leads to

$$\mathbf{T}_\mu = S^{-1}[\sigma(t)]\Big(T^0_\mu - \int_{\sigma(t)} d^3x H'[x' : n]\delta_{\mu 4}\Big) S[\sigma(t)]. \tag{10.38}$$

On the other hand, since $\varepsilon(x-x')$ changes its value only when x' passes through the surface $\sigma(x)$, contributions to the second term in (10.25) arise when $x = x'$ in the integrand (cf. 8.20*b*)). Thus, we can write $\mathbf{Q}_\alpha$ in the form

$$\mathbf{Q}_\alpha(x) = S^{-1}[\sigma(x)]\{Q_\alpha(x) + g_\alpha[x : n]\}\, S[\sigma(x)]. \tag{10.39}$$

This gives

$$\begin{aligned} \partial_\mu \mathbf{Q}_\alpha(x) &= S^{-1}[\sigma(t)]\{\partial_\mu Q_\alpha(x) + \\ &\quad + \sum_m \partial_\mu g_{\alpha\, \mu_1 \ldots \mu_m}(x)\, n_{\mu_1}(x) \ldots n_{\mu_m}(x)\} S[\sigma(t)] \\ &\quad - S^{-1}[\sigma(t)] \int_{\sigma(t)} d^3x\, [Q_\alpha(x) + g_\alpha[x:n], H'[x':n]]\, S[\sigma(t)]\, \delta_{\mu 4}. \end{aligned} \tag{10.40}$$

Here $g_\alpha[x : n]$ has the form

$$g_\alpha[x:n] = \sum g_{\alpha\, \mu_1 \ldots \mu_m}(x)\, n_{\mu_1}(x) \ldots n_{\mu_m}(x)$$

and the relation

$$i\frac{\partial}{\partial t}S[\sigma(t)] = \int_{\sigma(t)} d^3x\, H'[x\ \ n] \tag{10.41}$$

has been used. The relation (10.41) can be derived from (10.10). Moreover, (10.40) can be rewritten as

$$\partial_\mu \mathbf{Q}_\alpha(x) = -S^{-1}[\sigma(t)]\,[Q_\alpha(x) + g_\alpha[x:n], T^0_\mu + \int_{\sigma(t)} d^3x\, H'[x'.n]\,\delta_{\mu 4}]\,S[\sigma(t)]$$

on account of (8.15). This with (10.38) leads to (10.37).

Equations (10.22*b*) and (10.29) show that the interaction Hamiltonian $H'[x:n]$ has the form

$$H'[x\ .\ n] = -L'(x) + W'_{\mu\nu}(x)n_\mu(x)n_\nu(x) + \ \ldots, \tag{10.42}$$

where $L'(x)$ is obtained from the interaction Lagrangian $\mathbf{L}'(x)$ by replacing $\mathbf{Q}_\alpha$ by Q_α.

Since the $g_\alpha[x:n]$ contains only terms of degree $\geqslant 1$ in the coupling constant g, then all terms except $-L'(x)$ in (10.40) have degree ($\geqslant 2$) in g. It must be noted that, when $W'_{\mu\nu}(x)$ contains a term of the form $w(x)\delta_{\mu\nu}$, the second term in (10.42) gives $-w(x)$ which is independent of the surface σ.

§ 3. Relation with the Usual Canonical Theory

In the last paragraph, we developed the canonical theory of the quantum field theory by using the canonical variables $\mathbf{q}_r$ implicitly. Now, we shall show the relation between this formalism and that of Heisenberg and Pauli in which the canonical properties are exhibited explicitly.

In general, the Q_α's are made up of two sets of field quantities, $Q_{\alpha'}$ ($\alpha' = 1, \ldots, f$) whose canonically conjugate quantities

$$P_{\alpha'} \equiv (-i)\partial L^0/\partial Q_{\alpha'\,4}$$

are not zero; for other components, i.e. $Q_{\alpha''}$ ($\alpha'' = f+1, \ldots$),

$$\frac{\partial \mathbf{L}_4}{\partial \mathbf{Q}_{\alpha''}} = 0, \quad \frac{\partial L^0}{\partial Q_{\alpha''}} = 0. \tag{10.43}$$

In other words, $Q_{\alpha'}$, $P_{\alpha'}$ gives the independent canonical components and $Q_{\alpha''}$ are the dependent components. Therefore, by the unitary transformation $S[\sigma]$, $Q_{\alpha'}$ and $P_{\alpha'}$ are connected with the independent

canonical components $\mathbf{Q}_{\alpha'}$ and $\mathbf{P}_{\alpha'}$ ($P_{\alpha'} \equiv (-i)\partial \mathbf{L}/\partial \dot{\mathbf{Q}}_{\alpha' 4}$) in the Heisenberg representation, namely

$$\left.\begin{aligned} \mathbf{Q}_{\alpha'}(x) &= (S^{-1}[\sigma]\, Q_{\alpha'}(x)\, S[\sigma])_{x/\sigma} \\ \mathbf{P}_{\alpha'}(x) &= (S^{-1}[\sigma]\, P_{\alpha'}(x)\, S[\sigma])_{x/\sigma}. \end{aligned}\right\} \tag{10.44}$$

On the other hand, the differences between $\mathbf{Q}_{\alpha''}$ and $(S^{-1}[\sigma]Q_{\alpha''}S[\sigma])_{x/\sigma}$ are not zero but give the term $S^{-1}[\sigma]g_{\alpha''}[x : n]S[\sigma]$ in (10.39). We shall illustrate this in Example 3.

§ 4. Fundamental Equations

From (10.1) and (10.10) we can show that the Schrodinger equation in the interaction representation is

$$i \frac{\delta}{\delta\sigma(x)} \Psi[\sigma] = H'[x:n]\, \Psi[\sigma]. \tag{10.45}$$

This equation, the commutation relations (8.14*a*, *b*) and the field equation (10.20*b*) are the fundamental equations for the quantum field theory in the interaction representation.

The solutions of (10.45) can be written as

$$\Psi[\sigma] = S[\sigma, \sigma_1]\, \Psi[\sigma_1] = S[\sigma]\, \Psi[-\infty], \tag{10.46}$$

where $S[\sigma, \sigma_1]$ satisfies

$$\left.\begin{aligned} i \frac{\delta}{\delta\sigma(x)} S[\sigma, \sigma_1] &= H'[x.n]\, S[\sigma, \sigma_1] \quad \text{for } \sigma > \sigma_1 \\ S[\sigma_1, \sigma_1] &= 1 \qquad \text{(initial condition).} \end{aligned}\right\} \tag{10.47}$$

When an initial state $\Psi[-\infty]$ is given, we can obtain the expectation value of any quantity $F(x)$ by means of

$$\Psi^*[\sigma]\, F(x)\, \Psi[\sigma] = \Psi^*[-\infty]\, S^{-1}[\sigma]\, F(x)\, S[\sigma]\, \Psi[-\infty].$$

We write this as

$$(F(x))_\sigma = (S^{-1}[\sigma] F(x) S[\sigma])_{-\infty}, \tag{10.48}$$

where $(\ )_\sigma$ denotes the expectation value with respect to the state $\Psi[\sigma]$.

The probability amplitude of finding a state a on σ when $\Psi[\sigma_1]$ is a state b, is

$$\Psi_a^* \Psi[\sigma] = \Psi_a^* S[\sigma, \sigma_1] \Psi_b = (a|S[\sigma, \sigma_1]|b). \tag{10.49}$$

Now (10.47) can be written as the integral equation

$$S[\sigma, \sigma_1] = 1 - i \int_{\sigma_1}^{\sigma} d^4x'\, H'[x' : n]\, S[\sigma(x'), \sigma_1] \tag{10.50}$$

where the integration is taken over the region between two surfaces σ and σ_1. The integrability condition (10.11) ensures that $S[\sigma, \sigma_1]$ does not depend on the order of the integration in (10.50).

The transition matrix $S[\infty]$ between the states corresponding to $t = -\infty$ and $t = +\infty$ is called the scattering matrix or S-matrix.

§ 5. The Schrödinger Representation

We shall now derive the relations between the interaction and Schrödinger representations for flat surfaces $\sigma = \sigma(t)$ at times t. The Schrödinger equation (10.45) can be written as

$$i \frac{\partial}{\partial t} \Psi[\sigma(t)] = H'[\sigma(t)] \Psi[\sigma(t)] \tag{10.51}$$

with

$$H'[\sigma(t)] \equiv \int_{\sigma(t)} d^3x \, H'[x : n] \tag{10.52}$$

on account of (10.41).

From (8.15) we have

$$\frac{\partial}{\partial t} T^0_4 = 0. \tag{10.53}$$

Introducing $\overline{F}(x)$ given by

$$\overline{F}(x) \equiv \exp(-iT^0_4 t) \, F(x) \exp(iT^0_4 t), \tag{10.54}$$

where $F(x)$ is an operator in the interaction representation, (8.15) and (10.53) lead to

$$\frac{\partial}{\partial t} \overline{F}(x) = 0.$$

This shows that $\overline{F}(x)$ is an operator in the Schrödinger representation and that $\exp(iT^0_4 t)$ is the unitary transformation connecting the interaction and Schrödinger representations. From (10.54) we see that the energy operators in both representations agree with each other, or that

$$\overline{T}^0_4 = T^0_4. \tag{10.55}$$

The wave functions $\varphi(t)$ in the Schrödinger representation, i.e.

$$\varphi(t) = exp(-iT^0_4 t) \, \Psi[\sigma(t)] \tag{10.56}$$

satisfy the Schrödinger equation

$$i \frac{\partial}{\partial t} \varphi(t) = (H^0 + H') \varphi(t). \tag{10.57}$$

Here H' and H^0 are the interaction Hamiltonian and the free energy operator T_4^0 respectively in the Schrödinger representation, that is

$$H' \equiv \exp(-iT_4^0 t)\, H'[\sigma(t)] \exp(iT_4^0 t) \tag{10.58}$$

$$H^0 \equiv T_4^0. \tag{10.59}$$

Although we have used the flat surface $\sigma(t)$ in the present paragraph, this theory of the Schrödinger representation is Lorentz invariant because it has been derived from the covariant formalism of the interaction representation.

§ 6. Examples

EXAMPLE 1. VARIOUS INTERACTION HAMILTONIANS

We shall first consider the vector interaction between a real (neutral) scalar field and a Dirac field. The interaction Lagrangian was given in (7.116) as

$$\mathbf{L}' = ig\bar{\Psi}\gamma_\mu\Psi \cdot \partial_\mu \mathbf{U}. \tag{10.60}$$

Equation (10.29) shows that the interaction Hamiltonian can be determined by means of the relation

$$\left.\begin{aligned} [U(x), H'[x':n]] &= gS[\sigma]\,\partial'_\mu\,(\bar{\Psi}(x')\,\gamma_\mu\,\Psi(x'))\,S^{-1}[\sigma]\,\Delta(x-x') \\ [\psi(x), H'[x':n]] &= -\,ig\,\gamma_\mu S[\sigma]\,\psi(x')\cdot\partial'_\mu\,\mathbf{U}(x')\,S^{-1}[\sigma]\,S(x-x'). \end{aligned}\right\} \tag{10.61}$$

From (10.30) we have [1])

$$\left.\begin{aligned} \mathbf{U}(x) &= U[x/\sigma] - \frac{i}{2}g\int d^4x'\,\bar{\Psi}(x')\,\gamma_\nu\,\Psi(x')\,[\partial_\nu,\, \varepsilon(x-x')]\,\Delta(x-x') \\ &= U[x/\sigma] \end{aligned}\right\} \tag{10.62a}$$

$$\Psi(x) = \psi[x/\sigma] \tag{10.62b}$$

$$\left.\begin{aligned} \partial_\mu \mathbf{U}(x) &= (\partial_\mu U[x,\sigma])_{x/\sigma} \\ &\quad - \frac{i}{2}g\int d^4x'\,\bar{\Psi}(x')\,\gamma_\nu\,\Psi(x')\,[\partial_\mu\,\partial_\nu,\, \varepsilon(x-x')]\,\Delta(x-x') \\ &= (\partial_\mu U[x,\sigma])_{x/\sigma} + ig\bar{\psi}[x/\sigma]\,\gamma_\nu\,\psi[x/\sigma]\,n_\mu(x)\,n_\nu(x), \end{aligned}\right\} \tag{10.63}$$

where (10.19), (8.20a), (8.20b) and (8.20c) are used.

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[1]) $[\partial_\nu, \varepsilon(x-x')]\,\Delta(x-x') = \partial_\nu(\varepsilon(x-x')\,\Delta(x-x')) - \varepsilon(x-x')\,\partial_\nu\Delta(x-x') =$
$= \Delta(x-x')\,\partial_\nu\varepsilon(x-x') = 0$ (cf. (8.20a)).

Substituting (10.62a, b) and (10.63) into (10.61) we have

$$[U(x), H'[x':n]] = g\,\partial'_\mu\,(\bar\psi(x')\,\gamma_\mu\,\psi(x'))\,\Delta(x-x') \qquad (10.64a)$$

$$\left.\begin{aligned}[\psi(x), H'[x':n]] = &-i\,g\,\gamma_\mu\,\psi(x')\cdot\partial'_\mu\,U(x')\cdot S(x-x')\\ &+ g^2\gamma_\mu\,\psi(x')\cdot(\bar\psi(x')\,\gamma_\nu\,\psi(x'))\,S(x-x')\,n_\mu(x')\,n_\nu(x')\end{aligned}\right\} \qquad (10.64b)$$

which gives

$$H'[x:n] = -ig\bar\psi(x)\gamma_\mu\psi(x)\partial_\mu U(x) - \tfrac{1}{2}g^2(\bar\psi(x)\gamma_\mu\psi(x)n_\mu(x))^2. \qquad (10.65)$$

We now give the interaction Hamiltonians that are derived from the interaction Lagrangians given in Example 8 of Ch. VII by the same method. The scalar and vector interaction between a real (neutral) scalar field and a Dirac field is

$$H'[x:n] = -f\bar\psi\psi U - ig\bar\psi\gamma_\mu\psi\partial_\mu U - \tfrac{1}{2}g^2(\bar\psi\gamma_\mu\psi n_\mu)^2. \qquad (10.66)$$

The pseudoscalar and pseudovector interaction between a real (neutral) pseudoscalar field and a Dirac field is

$$H'[x:n] = -if\bar\psi\gamma_5\psi U - ig\bar\psi\gamma_5\gamma_\mu\psi\partial_\mu U - \tfrac{1}{2}g^2(\bar\psi\gamma_5\gamma_\mu\psi n_\mu)^2. \qquad (10.67)$$

The vector interaction between a real (neutral) vector field and a Dirac field is

$$H'[x:n] = -if\bar\psi\gamma_\mu\psi\,U_\mu - \frac{1}{2\varkappa^2}f^2(\bar\psi\gamma_\mu\psi n_\mu)^2. \qquad (10.68)$$

The pseudovector interaction between a real (neutral) pseudovector field and a Dirac field is

$$H'[x:n] = -if\bar\psi\gamma_5\gamma_\mu\psi\,U_\mu - \frac{1}{2\varkappa^2}f^2(\bar\psi\gamma_5\gamma_\mu\psi n_\mu)^2. \qquad (10.69)$$

EXAMPLE 2. ELECTROMAGNETIC INTERACTIONS IN CHARGED FIELDS OF GENERAL KIND

We shall consider the A_μ-type of interactions of general charged fields $Q_\alpha(x)$.

The interaction Lagrangian does not contain any differential operators that acts on A_μ (cf. Example 4 of Ch. VII), and $d_{\mu\nu}(\partial)$ for the electromagnetic field A_μ is $\delta_{\mu\nu}$ (cf. (8.57)). Therefore, (10.30) gives [1])

$$\left.\begin{aligned}\mathbf{A}_\mu(x) &= (S^{-1}[\sigma]\,A_\mu(x)\,S[\sigma])_{x/\sigma}\\ \partial_\mu\mathbf{A}_\mu(x) &= (S^{-1}[\sigma]\,\partial_\mu A_\mu(x)\,S[\sigma])_{x/\sigma}.\end{aligned}\right\} \qquad (10.70)$$

[1] $[\partial_\mu,\ \varepsilon(x-x')]\,\Delta(x-x') = 0$.

The fundamental equation (10.29) gives

$$\begin{aligned}[A_\mu(x), H'[x':n]] &= -i\Delta_p(x-x')S[\sigma]\frac{\partial \mathbf{L}'}{\partial \mathbf{A}_\mu}S^{-1}[\sigma] \\ &= -i\Delta_p(x-x')S[\sigma]\mathbf{J}_\mu(x')S^{-1}[\sigma].\end{aligned} \tag{10.71}$$

The symbol Δ_p means the Δ-functions for the photon field. In (10.71) $\mathbf{J}_\mu$ is the electric current vector (7.70). The electric current $J_\mu[x:n]$ in the interaction representation is defined in terms of $\mathbf{J}_\mu$ by the unitary transformation

$$J_\mu[x:n] = (S[\sigma]\mathbf{J}_\mu(x)S^{-1}[\sigma])_{x/\sigma}. \tag{10.72}$$

Thus we have, from (10.71),

$$[A_\mu(x), H'[x':n]] = -i\Delta_p(x-x')J_\mu[x':n]. \tag{10.73}$$

Taking into account the commutation relations (8.57) for A_μ, we obtain, from (10.73), the important relation

$$J_\mu[x':n] = -\frac{\partial}{\partial A_\mu(x)}H'[x:n]. \tag{10.74}$$

For simplicity, we assume that $H'[x:n]$ has the form

$$H'[x:n] = -e\,j_\mu(x)A_\mu(x) - \tfrac{1}{2}e^2 j_{\mu\nu}[x:n]A_\mu(x)A_\nu(x). \tag{10.75}$$

Here j_μ and $j_{\mu\nu}$ do not contain A_μ. This assumption can be established by practical calculations in cases of charged fields of low spin ($S \leqslant 1$). The last equation shows that

$$j_{\mu\nu}[x:n] = j_{\nu\mu}[x:n]. \tag{10.76}$$

Then (10.74) gives the current

$$J_\mu[x:n] = ej_\mu(x) + e^2 j_{\mu\nu}[x:n]A_\nu(x). \tag{10.77}$$

Moreover (10.42) shows that j_μ has the form of the electric current for free charged fields, and therefore satisfies:

$$\partial_\mu j_\mu(x) = 0.$$

We now prove the theorem (UMEZAWA [1952]).

$$j_{\mu\nu}[x:n]n_\nu(x) = 0. \tag{10.78a}$$

Equation (7.35*b*) gives

$$\mathbf{J}_\mu n_\mu = -ie\left(\frac{\partial \mathbf{L}}{\partial \mathbf{Q}_{\alpha:\mu}}\mathbf{Q}_\alpha - \mathbf{Q}_\alpha^*\frac{\partial \mathbf{L}}{\partial \mathbf{Q}_{\alpha:\mu}}\right)n_\mu,$$

which leads to

$$J_\mu[x:n]n_\mu = -ie\left(\frac{\partial L}{\partial Q_{\alpha:\mu}}Q_\alpha - Q_\alpha^*\frac{\partial L}{\partial Q_{\alpha:\mu}}\right)n_\mu$$

on account of (10.43) and (10.44). Since the Lagrangian L in the interaction representation (i.e. the Lagrangian for free charged fields) contains no field quantities $A_\mu(x)$, we have

$$\frac{\partial}{\partial A_\mu}\{J_\mu[x:n]\,n_\mu\} = 0.$$

Comparing this with (10.77), and taking into account (10.76), we obtain (10.78a). From (10.78a) we obtain the relation

$$S[\sigma(x)]\mathbf{J}_\nu(x)S^{-1}[\sigma(x)]n_\nu(x) = j_\nu(x)n_\nu(x). \tag{10.78b}$$

We shall now obtain the analogue of the continuity equation (7.34b),

$$\partial_\mu \mathbf{J}_\mu(x) = 0,$$

in the interaction representation.

From (10.72) we have

$$\left.\begin{aligned}\frac{\delta}{\delta\sigma(x)}\int_\sigma d\sigma'_\mu\,\mathbf{J}_\mu(x') = S^{-1}[\sigma]\Big\{&\frac{\delta}{\delta\sigma(x)}\int_\sigma d\sigma'_\mu J_\mu[x':n]\Big\}\\ &- i\int_\sigma d\sigma'_\mu\,[J_\mu[x':n],\,H'[x:n]]\Big\}S[\sigma].\end{aligned}\right\} \tag{10.79}$$

By applying the formulae (6.6b) and (10.17) to (10.79), we obtain

$$\left.\begin{aligned}\partial_\mu\,\mathbf{J}_\mu(x) = S^{-1}[\sigma]\Big\{&\partial_\mu J_\mu[x:n] - \int_\sigma d\sigma'_\mu\frac{\partial}{\partial\sigma(x')}J_\mu[x:n]\\ &+\int_\sigma d\sigma'_\mu\frac{\partial}{\partial\sigma(x)}J_\mu[x':n]\\ &- i\int_\sigma d\sigma'_\mu\,[J_\mu[x':n],\,H'[x:n]]\Big\}S[\sigma].\end{aligned}\right\} \tag{10.80}$$

On the other hand, the integrability condition (10.11) leads to [1])

$$i\left[\frac{\partial}{\partial A_\mu(x)}H'[x:n],\,H'[x':n]\right] = \frac{\partial}{\partial\sigma(x')}\frac{\partial}{\partial A_\mu(x)}H'[x:n],$$

$$i\left[H'[x:n],\,\frac{\partial}{\partial A_\mu(x')}H'[x':n]\right] = -\frac{\partial}{\partial\sigma(x)}\frac{\partial}{\partial A_\mu(x')}H'[x':n],$$

---

[1]) By operating $\partial/\partial A_\mu(x)$ on (10.11) we obtain

$$-i\left[\frac{\partial}{\partial A_\mu(x)}H'[x:n],\,H'[x':n]\right] - i\left[H'[x:n],\,\frac{\partial}{\partial A_\mu(x)}H'[x':n]\right]$$
$$= \frac{\partial}{\partial\sigma(x)}\frac{\partial}{\partial A_\mu(x)}H'[x':n] - \frac{\partial}{\partial\sigma(x')}\frac{\partial}{\partial A_\mu(x)}H'[x:n].$$

On the other hand,

$$\frac{\partial}{\partial A_\mu(x)}H'[x':n] = 0$$

because x and x' are two different points. Thus we have

$$i\left[\frac{\partial}{\partial A_\mu(x)}H'[x:n],\,H'[x':n]\right] = \frac{\partial}{\partial\sigma(x')}\frac{\partial}{\partial A_\mu}H'[x:n].$$

where x and x' are two different points on the space-like surface σ. By adding both relations and taking (10.76) into account,

$$i\,[J_\mu[x:n],\, H'[x':n]] = \frac{\partial}{\partial\sigma(x')}\,J_\mu[x:n] - \frac{\partial}{\partial\sigma(x)}\,J_\mu[x':n] - i\,[H'[x:n],\, J_\mu[x':n]].$$

This is obviously valid even when x agrees with x'. Substituting this into (10.80) we have

$$\partial_\mu \mathbf{J}_\mu(x) = S^{-1}[\sigma]\{\partial_\mu J_\mu[x:n] - i\int_\sigma d\sigma'_\mu[J_\mu[x:n],\, H'[x':n]]\}S[\sigma]$$

which leads to

$$\partial_\mu J_\mu[x:n] = i\int_\sigma d\sigma'_\mu[J_\mu[x:n],\, H'[x':n]]. \tag{10.81}$$

Here the derivation operator ∂_μ operates on the field quantities in $J_\mu[x:n]$ but not on the normal vector n_μ.

Substituting (10.75) and (10.76) into (10.81), and comparing the terms of the same power of A_μ in both sides of (10.81), we obtain

$$\partial_\mu j_\mu = 0 \tag{10.82a}$$

$$\int_\sigma d\sigma'_\mu[j_\mu(x),\, j_\nu(x')]A_\nu(x') = i\partial_\mu(j_{\mu\nu}[x:n]A_\nu(x)) \tag{10.82b}$$

$$\int_\sigma d\sigma'_\mu[j_\mu(x),\, j_{\varrho\nu}[x':n]]A_\varrho(x')A_\nu(x') = 0 \tag{10.82c}$$

(UMEZAWA [1952]).

Here x and x' are on the same surface and the derivation operator ∂_μ does not operate on the normal vector n_μ in $j_{\mu\nu}[x:n]$. Equations (10.82) and (10.78a) lead to the general properties of the electric current expressed by [1])

$$n_\mu(x)\,[j_\mu(x),\, j_\nu(x')] = i\,n_\mu(x)\,\partial_\varrho(j_{\varrho\nu}[x:n]\cdot\partial_\mu\Delta(x-x')) \tag{10.83a}$$

$$n_\mu\, n_\nu[j_\mu(x),\, j_\nu(x')] = 0 \tag{10.83b}$$

$$n_\mu[j_\mu(x),\, j_{\varrho\nu}[x':n]] = 0. \tag{10.83c}$$

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[1]) Since no specific property of A_μ was used for deriving the continuity equation, any function, $f(x)$ can replace $A_\mu(x)$ in (10.82b, c): —

$$\int_\sigma d\sigma'_\mu[j_\mu(x),\, j_\nu(x')]\,f(x') = i\,\partial_\mu(j_{\mu\nu}[x:n]\,f(x)].$$

By taking $\partial_4\Delta(x-x'')\,(=i\,\delta(\mathbf{x}-\mathbf{x}''))$ at a given point x'', on σ as $f(x)$, we obtain

$$-i\int[j_4(x),\, j_\nu(x')]\,\partial'_4\Delta(x'-x'') = i\,\partial_\mu(j_{\mu\nu}[x:n]\,\partial_4\Delta(x'-x'')).$$

This leads to

$$[j_4(x),\, j_\nu(x'')] = i\,\partial_\mu(j_{\mu\nu}[x:n]\cdot\partial_4\Delta(x'-x'')).$$

From this (10.83a) can be obtained. By taking $\nu = 4$ in (10.83a), we can deduce (10.83b).

Here x and x' are in space-like positions with respect to each other, and the derivation operator ∂_μ does not operate on the normal vector n_μ in $j_{\mu\nu}$. These relations will be used in the next chapter.

We shall now rewrite the Lorentz condition as a relation in the interaction representation. From (10.30), (10.72), (10.78a), (10.19) and (8.20d), we obtain

$$\left.\begin{aligned}\partial_\mu \partial_\nu A_\mu(x) &= (S[\sigma]\, \partial_\mu\, \partial_\nu\, \mathbf{A}_\mu(x)\, S^{-1}[\sigma])_{x/\sigma} \\ &\quad + \tfrac{1}{2}\int d^4x'\, S[\sigma]\, \mathbf{J}_\mu(x')\, S^{-1}[\sigma]\cdot[\partial_\mu\, \partial_\nu,\, \varepsilon(x-x')]\, \Delta_p(x-x') \\ &= (S[\sigma]\, \partial_\mu\, \partial_\nu\, \mathbf{A}_\mu(x)\, S^{-1}[\sigma])_{x/\sigma} - e\, j_\mu(x)\, n_\mu(x)\, n_\nu(x) \\ &= (S[\sigma]\, \partial_\mu\, \partial_\nu\, \mathbf{A}_\mu(x)\, S^{-1}[\sigma])_{x/\sigma} + e\int_\sigma d\sigma'_\mu\, j_\mu(x')\, \partial'_\nu\, \Delta_p(x'-x).\end{aligned}\right\} \quad (10.84)$$

Here x is on σ. On the other hand we have the formula

$$\partial_\mu\, A_\mu(x) = \int_\sigma d\sigma'_\nu \{\Delta_p(x-x')\, \partial'_\nu\, \partial'_\mu\, A_\mu(x') - \partial'_\mu\, A_\mu(x')\cdot \partial'_\nu\, \Delta_p(x-x')\} \quad (10.85)$$

where x is not necessarily on σ. We prove (10.85) by showing that the functional derivation $\delta/\delta\sigma$ of the right hand side is zero, so that (10.85) is independent of σ. On the other hand, (10.85) is easily established for the flat surface through the point x.

The conditions (7.59) and (7.60);—

$$\left.\begin{aligned}\partial_\mu\, \mathbf{A}_\mu(x)\, \mathbf{\Phi} &= 0 \\ \partial_\nu\, \partial_\mu\, \mathbf{A}_\mu(x)\, \mathbf{\Phi} &= 0,\end{aligned}\right\} \quad (10.86a)$$

lead to

$$\partial_\mu\, A_\mu(x)\, \Psi[\sigma] = \int_\sigma d\sigma'_\nu\, \Delta_p(x-x')\, \partial'_\mu\, \partial'_\nu\, A_\mu(x')\, \Psi[\sigma] \quad (10.86b)$$

on account of the relations (10.85) and (10.70). In (10.86b), x is not necessarily on σ. Substituting (10.84) into (10.86b) and using (10.86a) we have

$$\partial_\mu\, A_\mu(x)\, \Psi[\sigma] = e\int_\sigma d\sigma'_\nu \int_\sigma d\sigma''_\mu\, j_\mu(x'')\, \Delta_p(x-x')\, \partial''_\nu\, \Delta_p(x''-x')\, \Psi[\sigma]$$

which gives

$$[\partial_\mu A_\mu(x) - e\int_\sigma d\sigma'_\mu j_\mu(x')\Delta_p(x-x')]\Psi[\sigma] = 0. \quad (10.87)$$

Here x is not necessarily on σ. In the derivation of (10.87), we have used the relation

$$\int_{\sigma(x'')} d\sigma'_\nu \Delta(x-x')\partial''_\nu \Delta(x''-x') = \Delta(x-x'')$$

(cf. (8.18), (8.19b)).

Equation (10.87) is the Lorentz condition in the interaction representation.

Regarding (9.65a) as the Lorentz condition in the Heisenberg representation, the Lorentz condition in the interaction representation becomes

$$[\partial_\mu A_\mu^+(x) - e\int_\sigma d\sigma'_\mu j_\mu(x') \Delta_y^+(x-x')]\Psi[\sigma] = 0. \tag{10.88}$$

EXAMPLE 3. ELECTROMAGNETIC INTERACTION IN DUFFIN–KEMMER–PETIAU FORMALISM

We shall consider the electromagnetic interaction of a field of spin (1 or 0) in the Duffin–Kemmer–Petiau theory.

The interaction Lagrangian is, by (7.105) and (7.69),

$$\mathbf{L}' = ie\bar{\Psi}\beta_\mu\Psi\mathbf{A}_\mu. \tag{10.89}$$

From (8.61) and (10.25) we have

$$\mathbf{A}_\mu(x) = A_\mu[x/\sigma] \tag{10.90}$$

$$\left.\begin{aligned}\Psi(x) &= \psi[x/\sigma] + \frac{i}{2} e \int d^4x' \, \beta_\mu \, \Psi(x') \, \mathbf{A}_\mu(x') \, [d(\partial), \varepsilon(x-x')] \\ &= \psi[x/\sigma] + \frac{i}{\varkappa} e \, (1 + (\beta_\nu n_\nu)^2) \, \beta_\mu \, A_\mu[x/\sigma] \, \psi[x/\sigma],\end{aligned}\right\} \tag{10.91}$$

if the formula

$$(1+(\beta_\nu n_\nu)^2)\beta_\mu(1+(\beta_\varrho n_\varrho)^2) = 0 \tag{10.92}$$

is used. Then, (10.29) can be written as

$$\begin{aligned}[\psi(x), H'[x':n]] &= -\, e \, d(\partial) \, \Delta(x-x') \, S[\sigma] \, \beta_\mu \, \mathbf{A}_\mu(x') \, \Psi(x') \, S^{-1}[\sigma] \\ &= -\, e \, d(\partial) \, \Delta(x-x') \, \beta_\mu \, A_\mu(x') \left\{1 + \frac{i}{\varkappa} e \, (1 + (\beta_\nu n_\nu)^2) \, \beta_\varrho \, A_\varrho(x')\right\} \psi(x').\end{aligned}$$

This gives the interaction Hamiltonian

$$H'[x:n] = -\, i e \bar{\psi} \beta_\mu \psi A_\mu + \frac{e^2}{\varkappa} \bar{\psi} \beta_\mu (1 + (\beta_\nu n_\nu)^2) \, \beta_\varrho \psi A_\mu A_\varrho \tag{10.93}$$

(TAKAHASHI and UMEZAWA [1953]).

It must be noted that (10.93) has a e^2-term which is independent of n_μ (cf. § 2).

EXAMPLE 4. RESTRICTIONS ON COMMUTATION RELATIONS ARISING FROM INTERACTIONS

We shall now show that condition (10.29), which is satisfied by interaction Hamiltonians, restricts the possible type of the commutation relations.

It is easily seen that (10.29) can be written as

$$\left.\begin{aligned}[Q_\alpha(x),\, H'[x':n]] &= i\,\frac{\partial H'[x':n]}{\partial(D'_a\, Q^*_\alpha(x'))}\, d_{\alpha\beta}(\partial)\, D'_a\, \Delta(x-x') \\ &= [Q_\alpha(x),\, D'_a\, \bar{Q}_\alpha(x')]\,\frac{\partial H'[x':n]}{\partial(D'_a\, \bar{Q}_\alpha(x'))}.\end{aligned}\right\} \tag{10.94}$$

From (10.94) we have: (ONEDA and UMEZAWA [1953], UMEZAWA, PODOLANSKI and ONEDA [1955], KINOSHITA [1954]):

Theorem (i): Every term of the interaction $H'[x \cdot n]$ must contain an even number of field operators $Q^{(1)}, \ldots, Q^{(2n)}$ for which

$$[Q^{(i)}(\mathbf{x}, t),\, X(\mathbf{x}'.\, t)]_+ = 0.$$

Here X is an arbitrarily chosen field and $\mathbf{x}$ and $\mathbf{x}'$ are two different points.

To prove this theorem, we separate the field quantities appearing in the given term of $H'[x:n]$ into two categories $(Q^{(1)}, \ldots, Q^{(m)})$, $(Q^{(m+1)}, \ldots)$ which satisfy

$$\begin{aligned}[Q^{(i)}(\mathbf{x}, t),\, X(\mathbf{x}', t)]_+ &= 0 \qquad \text{for} \quad i \leqslant m \\ [Q^{(i)}(\mathbf{x}, t),\, X(\mathbf{x}', t)]_- &= 0 \qquad \text{for} \quad i > m\end{aligned}$$

($\mathbf{x}$ and $\mathbf{x}'$ are two different points). If m is an odd number $m = 2n+1$, then

$$\left.\begin{aligned}&[X,\, Q^{(1)} \ldots Q^{(2n+1)}\, Q^{(2n+2)} \ldots]_- \\ &\quad = \{[X,\, Q^{(1)}]_+\, Q^{(2)} \ldots\} - \{Q^{(1)}\, [X,\, Q^{(2)}]_+\, Q^{(3)} \ldots\} \\ &\quad + \ldots - \{Q^{(1)} \ldots Q^{(2n-1)}\, [X,\, Q^{(2n)}]_+\, Q^{(2n+1)} \ldots\} \\ &\quad + \{Q^{(1)} \ldots Q^{(2n)}\, [X,\, Q^{(2n+1)}]_-\, Q^{(2n+2)} \ldots\} \\ &\quad + Q^{(1)} \ldots Q^{(2n+1)}\, [X,\, Q^{(2n+2)} \ldots]_-.\end{aligned}\right\} \tag{10.95}$$

The appearance of $[X, Q^{(2n+1)}]_-$ in (10.95) implies that an interaction of this kind cannot satisfy (10.94).

Since all transmutation processes are induced by interactions satisfying Theorem (i), we have

Theorem (ii): In every transmutation process there must be an even number of fields $(Q^{(1)}, \ldots, Q^{(2n)})$, for which $[Q^{(i)}(\mathbf{x}, t), X(\mathbf{x}', t)]_+ = 0$. Here X is an arbitrarily chosen field and $\mathbf{x}$ and $\mathbf{x}'$ are two different points.

Since there exists the bremsstrahlung process:

$$P + e \to P + e + \gamma,$$

it follows from the theorem (ii) that A_μ must commute with any other field.

We shall now assume that the proton field ψ_P and the neutron field ψ_N are both Fermi fields, and that the charged π-meson field φ_π is a Bose field. In other words, we assume that

$$[\psi_P(\mathbf{x}, t), \bar{\psi}_P(\mathbf{x}', t)]_+ = [\psi_N(\mathbf{x}, t), \bar{\psi}_N(\mathbf{x}', t)]_+ = [\varphi_\pi(\mathbf{x}, t), \varphi_\pi^*(\mathbf{x}', t)]_- = 0$$

(for two different points $\mathbf{x}$, $\mathbf{x}'$).

Then, by theorem (ii), because of the existence of the transmutation process $P + P \to P + N + \pi^+$, it follows that either

$$\left.\begin{aligned} &[\psi_P, \psi_N]_+ = 0 \\ &[\psi_P, \varphi_\pi]_- = [\psi_N, \varphi_\pi]_- = 0 \end{aligned}\right\} \tag{10.96}$$

or

$$\left.\begin{aligned} &[\psi_P, \psi_N]_- = 0 \\ &[\psi_P, \varphi_\pi]_+ = [\psi_N, \varphi_\pi]_+ = 0. \end{aligned}\right\} \tag{10.97}$$

It must be noted that not all of the commutators

$$[\psi_P, \psi_N]_-, \quad [\psi_P, \varphi_\pi]_-, \quad [\psi_N, \varphi_\pi]_-$$

can simultaneously vanish. The process $P + P \to P + P + \pi^0$ implies that the neutral π-meson (π^0) must commute with all other fields (say X);

$$[\varphi_{\pi^0}, X] = 0. \tag{10.98a}$$

For example,

$$\left.\begin{aligned} &[\varphi_{\pi^0}, \varphi_{\pi^0}]_- = 0 \\ &[\psi_P, \varphi_{\pi^0}]_- = [\psi_N, \varphi_{\pi^0}]_- = 0. \end{aligned}\right\} \tag{10.98b}$$

Therefore, if we assume that the charged π-meson and the neutral π-meson have the same type of commutation relation, (10.97) is forbidden and (10.96) must obtain. This result seems to support the view that the proton and neutron are two different states of one particle, (i.e. the nucleon).

Thus, we see that the existence of various processes restricts the possible types of commutation relations.

EXAMPLE 5. THE INTERACTION AND TIME REFLECTION

A detailed discussion of time reflection in quantum field theory has been given in Example 5 of Ch. VIII. We shall now give some examples

of some interactions that are excluded by the requirement that the theory should be invariant under time reflection.

From (10.45) we obtain, by using (8.73),

$$i\frac{\delta\,'\Psi[\sigma]}{\delta\sigma('x)} = (H'[x:n])^T\,'\Psi[\sigma]. \tag{10.99}$$

Therefore the invariance of the Schrödinger equation (10.45) implies that the interaction Hamiltonian must satisfy the condition

$$H'['Q_\alpha('x)] = (H'[Q_\alpha(x)])^T. \tag{10.100}$$

Here $H'[Q_\alpha]$ means the $H'[x:n]$ made up of the field operators Q_α. The condition (10.100) requires that the interaction Hamiltonian must have positive parity under time reflection (cf. (8.86)).

We can now prove, if we accept standpoint (ii) of Example 5, Ch. VIII, that linear combinations of scalar interactions and a vector interaction cannot represent the true interaction between a Dirac field ψ and a *real* (and therefore neutral) scalar field U. In fact, as shown by the Table I in Example 5 of Ch. VIII, the invariances of $\bar\psi\psi U$ and $\bar\psi\gamma_\mu\psi\partial_\mu U$ lead to the impossible result that, under this transformation, U and $\partial_\mu U$ must be even scalar and odd vector respectively.

With the same premises we can prove that a linear combination of a pseudotensor interaction and a pseudovector interaction between a Dirac field and a *real* pseudovector field is impossible (LÜDERS [1952]).

In the case of interactions between complex Bose fields and Dirac fields, or of β-interactions, the coupling constants are complex and we find, accepting standpoint (ii) that their phases are restricted by the requirement of invariance under time reflection (UMEZAWA, KAMEFUCHI and TANAKA [1954]). We shall denote the determinant of the matrix Λ for α-field under the time reflection by ϱ_α; its absolute value $|\varrho_\alpha|$ must be 1 (cf. (3.20)). Further ϱ_α must be real for a real field. It may, however, be complex for complex fields.

For the interactions between complex Boson fields and a neutral spinor field ψ^α or a charged spinor field ψ^β we have

$$f\bar\psi^\alpha\psi^\beta U + f^*\bar\psi^\beta\psi^\alpha U^* \quad \text{(scalar interaction)} \tag{10.101a}$$

$$f\bar\psi^\alpha\gamma_5\psi^\beta U - f^*\bar\psi^\beta\gamma_5\psi^\alpha U^* \quad \text{(pseudoscalar interaction)} \tag{10.101b}$$

$$f\bar\psi^\alpha\gamma_\mu\psi^\beta U_\mu - f^*\bar\psi^\beta\gamma_\mu\psi^\alpha U_\mu^* \quad \text{(vector interaction)} \tag{10.101c}$$

$$if\bar\psi^\alpha\gamma_5\gamma_\mu\psi^\beta U_\mu - if^*\bar\psi^\beta\gamma_5\gamma_\mu\psi^\alpha U_\mu^* \quad \text{(pseudovector interaction)} \tag{10.101d}$$

Because of the invariance under the time reflection (from standpoint (ii)), we obtain the conditions

$$f\varrho_\alpha^*\varrho_\beta\varrho = f^*. \tag{10.102}$$

From the same standpoint, the five β-interactions (7.121) under time reflection provide the simple condition

$$g\varrho_a^*\varrho_b\varrho_c^*\varrho_d = g^*. \tag{10.103}$$

When we assume that all phase factors $\varrho_a, \ldots, \varrho_d$ are 1, this condition requires that the coupling constant g of the β-interaction is real.

EXAMPLE 6 SPIN OF CHARGED π-MESON

When the interaction is invariant under time reflection, the discussions of Example 5, Ch. VIII and the last example show that the matrix element of the S-matrix of a process (0-state, f-state) is equivalent to that of the inverse process (f-state, 0-state) after taking the average value with respect to states of quantities of negative parity, under the time reflection. Therefore, the ratio of the transition probability $dw_{f\leftarrow 0}$ and $dw_{0\leftarrow f}$ of these processes per unit time and unit energy of the final states is

$$\frac{dw_{f\leftarrow 0}}{dw_{0\leftarrow f}} = \frac{\varrho_{Ef}}{\varrho_{E_0}}, \tag{10.104}$$

where ϱ_{E_0} and ϱ_{E_f} are the number of final states per unit energy of the respective processes.

We shall consider a case in which 0- and f-states are those of a system of a π^+-meson and a deuteron D and a system of two protons $(P+P)$ respectively. Then, $dw_{f\leftarrow 0}$ and $dw_{0\leftarrow f}$ correspond to the absorption of a π^+-meson by a deuteron and the production of a π-meson and a deuteron by the collision of two protons, respectively. Denoting the momentum and energy of the proton and the π-meson by $(\mathbf{p}, E_p)$ and $(\mathbf{k}, \varepsilon_k)$ respectively we have, in the centre of mass system,

$$\left.\begin{aligned} \varrho_{E_f} &= \frac{2}{(2\pi)^3}\, p\, E_p\, d\,\Omega_f\,, \\ \varrho_{E_0} &= \frac{3(2S+1)}{(2\pi)^3}\, k\, \varepsilon_k\, d\,\Omega_0\,. \end{aligned}\right\} \tag{10.105}$$

Here S is the spin of a π^+-meson and $(2S+1)$ is the number of possible spin states of π^+-mesons. The factor 3 appears because the deuteron

is in a ^3S-state, and $d\Omega_f$ and $d\Omega_0$ are the small solid angles in which protons and π^+-mesons of the final states are scattered. Therefore, the ratio of the transition probability/unit time/unit solid angle/unit energy is (MARSHAK [1951], CHESTON [1951])

$$\frac{dw_{f\leftarrow 0}}{d\Omega_f} \Big/ \frac{dw_{0\leftarrow f}}{d\Omega_0} = \frac{2}{3}\frac{1}{(2S+1)}\frac{pE_p}{k\varepsilon_k}. \tag{10.106}$$

In fact, the spin of the π^+-meson has been determined as zero by comparing (10.106) with experimental data (DURBIN, LOAR and STEINBERGER [1951], CLARK, ROBERTS and WILSON [1951]). Consequently if the π^- and π^+-meson are different charge states of a field, the spin of the π^--meson must be also zero. Further, experimental data on the absorption of π^--mesons by deuterium seem to suggest that the charged π^--meson is a pseudoscalar field.

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CHAPTER XI

MISCELLANEOUS THEOREMS

§ 1. Gauge invariance

Quantum field theory in the Heisenberg representation has a form invariant under the gauge transformation (7.66). However, when this transformation is applied to the quantities of the interaction representation, it induces a factor $\exp(ie\Lambda(x)-ie\Lambda(x'))$ in the commutation relation (8.14a), for charged fields. Therefore, we rewrite the gauge-transformation in the interaction representation in the form (KOBA, TATI and TOMONAGA [1947], SCHWINGER [1948]):

$$\left.\begin{aligned} &A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda, \quad \Box\Lambda = 0 \\ &\Psi[\sigma] \to \Psi'[\sigma] = U^{-1}[\sigma]\Psi[\sigma] \\ &U[\sigma] = \exp\left[-ie\textstyle\int_\sigma d\sigma'_\mu j_\mu(x')\Lambda(x')\right]. \end{aligned}\right\} \tag{11.1}$$

We can now prove that quantum field theory is invariant under the transformation (11.1) (UMEZAWA [1952]). Using (10.78a), (10.82a), (10.83a) and (10.83b) we have

$$\left.\begin{aligned} &i\,U^{-1}[\sigma]\frac{\delta}{\delta\sigma(x)}U[\sigma] = i\frac{\delta}{\delta\sigma(x)} + e\,\partial_\mu\,(j_\mu(x)\,\Lambda(x)) \\ &+\frac{i}{2}e^2\int_\sigma d\sigma'_\mu\,[j_\mu(x'),\,j_\nu(x)]\,\Lambda(x')\,\partial_\nu\,\Lambda(x) \\ &+\frac{1}{3!}e^3\int_\sigma d\sigma'_\mu\int_\sigma d\sigma''_\varrho\,[j_\varrho(x''),\,[j_\mu(x'),\,j_\nu(x)]]\,\Lambda(x'')\,\Lambda(x')\,\partial_\nu\,\Lambda(x) \\ &+\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\ &= i\frac{\delta}{\delta\sigma(x)} + e\,j_\mu(x)\partial_\mu\Lambda(x) - \frac{e^2}{2}\int_\sigma d\sigma'_\mu\Lambda(x')\cdot\partial_\nu\Lambda(x)\cdot\partial'_\varrho\{j_{\varrho\nu}[x':n]\cdot\partial'_\mu\Delta(x'-x)\} \\ &+\frac{i}{3!}e^3\int_\sigma d\sigma''_\delta\int_\sigma d\sigma'_\mu\,\Lambda(x'')\,\Lambda(x')\cdot\partial_\nu\,\Lambda(x)\cdot\partial'_\delta\,\{[j_\varrho(x''),\,j_{\varrho\nu}[x':n]]\cdot\partial'_\mu\Delta(x'-x)\} \\ &+\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\ &= i\frac{\delta}{\delta\sigma(x)} + e\,j_\mu(x)\,\partial_\mu\,\Lambda(x) + \tfrac{1}{2}e^2\,j_{\nu\varrho}[x:n]\,\partial_\varrho\,\Lambda(x)\cdot\partial_\nu\,\Lambda(x). \end{aligned}\right\} \tag{11.2}$$

on account of the general formula

$$\begin{aligned} &\exp[-F]\frac{\delta}{\delta\sigma}\exp[F] \\ &\quad = \left\{1 - F + \frac{1}{2}F - \frac{1}{3!}F + \ldots\right\}\frac{\delta}{\delta\sigma}\left\{1 + F + \frac{1}{2}F + \frac{1}{3!}F + \ldots\right\} \\ &\quad = \frac{\delta}{\delta\sigma} + G - \frac{1}{2}[A, G] + \frac{1}{3!}[A, [A, G]] - \ldots \end{aligned}$$

with

$$G \equiv \delta F/\delta\sigma.$$

On the other hand, by using (10.75), (10.78a), (10.83a) and (10.83b) we obtain

$$\left.\begin{aligned}
&U^{-1}[\sigma]\, H'[x:n]\, U[\sigma] = -e\, j_\mu(x)\, A_\mu(x) - \tfrac{1}{2} e^2 j_{\mu\nu}[x:n]\, A_\mu(x)\, A_\nu(x) \\
&\quad - i e^2 \textstyle\int_\sigma d\sigma'_\mu\, [j_\mu(x'), j_\nu(x)]\, \Lambda(x')\, A_\nu(x) \\
&\quad - \frac{i}{2} e^3 \textstyle\int_\sigma d\sigma'_\nu\, [j_\nu(x'), j_{\mu\sigma}[x:n]]\, \Lambda(x')\, A_\mu(x)\, A_\sigma(x) + \dots \\
&= -e\, j_\mu(x)\, A_\mu(x) - \tfrac{1}{2} e^2 j_{\mu\nu}[x:n]\, A_\mu(x)\, A_\nu(x) - e^2 j_{\nu\mu}[x:n]\, \partial_\nu \Lambda(x) \cdot A_\mu(x) \\
&= -e\, j_\mu(x)\, A_\mu(x) \\
&\quad - \tfrac{1}{2} e^2 j_{\mu\nu}[x:n] \{A_\mu(x) + \partial_\mu \Lambda(x)\} \{A_\nu(x) + \partial_\nu \Lambda(x)\} \\
&\quad + \tfrac{1}{2} e^2 j_{\mu\nu}[x:n]\, \partial_\mu \Lambda(x) \cdot \partial_\nu \Lambda(x).
\end{aligned}\right\} \quad (11.3)$$

Since the Schrödinger equation can be written as

$$U^{-1}[\sigma] \left\{ i \frac{\delta}{\delta\sigma(x)} - H'[x:n] \right\} U[\sigma]\, \Psi'[\sigma] = 0,$$

we obtain

$$\left\{ i \frac{\delta}{\delta\sigma(x)} - e\, j_\mu(x)\, A'_\mu(x) - \tfrac{1}{2} e^2 j_{\mu\nu}[x:n]\, A'_\mu(x)\, A'_\nu(x) \right\} \Psi'[\sigma] = 0. \quad (11.4)$$

This equation and (10.75) exhibit the invariance of the Schrödinger equation under the gauge transformation (11.1).

Further, using (10.83b) we have

$$\begin{aligned}
&U^{-1}[\sigma][\partial_\mu A_\mu(x) - e \textstyle\int_\sigma d\sigma'_\mu j_\mu(x')\, \Delta_\nu(x-x')] U[\sigma] \\
&\quad = \partial_\mu A_\mu(x) - e \textstyle\int_\sigma d\sigma'_\mu j_\mu(x') \Delta_\nu(x-x') \\
&\quad = \partial_\mu A'_\mu(x) - e \textstyle\int_\sigma d\sigma'_\mu j_\mu(x') \Delta_\nu(x-x').
\end{aligned}$$

We see from these relations that the Lorentz condition (10.87) is invariant under the transformation (11.1).

§ 2. The Relation between a Real Vector Field U_μ and the Electromagnetic Field

We shall now prove that the quantum field theory of the electromagnetic interaction can be derived as the limiting case ($\varkappa \to 0$) of the theory of the vector interaction of a real (neutral) vector field U_μ (mass $\varkappa$) (GLAUBER [1953], UMEZAWA [1952]).

We shall call the interaction derived from the free Lagrangian of

a charged field Q_α a vector interaction, if it can be obtained by the substitution [1])

$$\partial_\mu \mathbf{Q}_\alpha \to (\partial_\mu - if\mathbf{U}_\mu)\mathbf{Q}_\alpha, \quad \partial_\mu \mathbf{Q}^*_\alpha \to (\partial_\mu + if\mathbf{U}_\mu)\mathbf{Q}^*_\alpha. \tag{11.5}$$

Here f is a coupling constant.

In contrast with the result $\mathbf{A}_\mu(x) = A_\mu[x/\sigma]$ of Example 2, Ch. X (cf. (10.70)), we now have

$$\mathbf{U}_\mu(x) = S^{-1}[\sigma]\left\{U_\mu(x) + \frac{1}{\varkappa^2} f\, j_\varrho(x)\, n_\varrho(x)\, n_\mu(x)\right\} S[\sigma], \tag{11.6a}$$

where x is on σ. In fact, (10.25) gives

$$\mathbf{U}_\mu(x) = U_\mu[x/\sigma] - \frac{1}{2}\frac{1}{\varkappa^2} f \int d^4x'\, \mathbf{J}_\nu(x')\, [\partial_\mu\, \partial_\nu,\, \varepsilon(x-x')]\, \Delta(x-x'), \tag{11.6b}$$

where the current $\mathbf{J}_\mu$ is defined by

$$f\mathbf{J}_\mu \equiv \partial \mathbf{L}'/\partial \mathbf{U}_\mu$$

and therefore, can be obtained by (7.35a) (cf. (11.5)). In other words, $\mathbf{J}_\mu$ is just the electric current (7.35b) without the charge e. Thus, $\mathbf{J}_\mu$ satisfies the continuity equation

$$\partial_\mu \mathbf{J}_\mu(x) = 0. \tag{11.7}$$

The second term in (11.6b) can be calculated by means of the relation

$$\begin{aligned}
&[\partial_\mu\partial_\nu,\, \varepsilon(x-x')]\Delta(x-x') \\
&\quad = \partial_\mu\partial_\nu(\varepsilon(x-x')\cdot\Delta(x-x')) - \varepsilon(x-x')\partial_\mu\partial_\nu\Delta(x-x') \\
&\quad = \partial_\mu\partial_\nu\varepsilon(x-x')\cdot\Delta(x-x') + \partial_\nu\varepsilon(x-x')\cdot\partial_\mu\Delta(x-x') \\
&\qquad + \partial_\mu\varepsilon(x-x')\cdot\partial_\nu\Delta(x-x'),
\end{aligned}$$

which, on account of (8.20d), is equal to

$$\partial_\mu\varepsilon(x-x')\cdot\partial_\nu\Delta(x-x').$$

By using (10.19), we have

$$\mathbf{U}_\mu(x) = U[x/\sigma] + \frac{1}{\varkappa^2} f\, \mathbf{J}_\nu(x)\, n_\mu(x)\, n_\nu(x).$$

On the other hand, we can show that

$$S[\sigma(x)]\mathbf{J}_\nu(x)S^{-1}[\sigma(x)]n_\nu(x) = j_\nu(x)n_\nu(x)$$

by the same method as that used to obtain (10.78b). Thus (11.6a) is established.

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[1]) In this Chapter we use bold and ordinary type to denote quantities in the Heisenberg and interaction representation respectively.

Corresponding to (10.75), we assume the interaction Hamiltonian to be

$$\left.\begin{aligned} H'[x:n] = &- f\, j_\mu(x)\, U_\mu(x) - \tfrac{1}{2} f^2\, j_{\mu\nu}[x.n]\, U_\mu(x)\, U_\nu(x) \\ &+ \frac{1}{2\varkappa^2} f^2\, (j_\varrho(x)\, n_\varrho(x))^2. \end{aligned}\right\} \tag{11.8a}$$

In fact, (11.8a) can be justified for charged fields of low spin ($S \leqslant 1$). The last term of (11.8) comes from the last term in (11.6a).

In the Stueckelberg formalism (cf. Example 6, Ch. VII) in which

$$\mathbf{U}_\mu(x) \equiv \mathbf{A}_\mu(x) + \frac{1}{\varkappa}\, \partial_\mu\, \mathbf{B}(x),$$

(11.8a) can be written as

$$\left.\begin{aligned} H'[x:n] = &- f\, j_\mu \left(A_\mu + \frac{1}{\varkappa}\, \partial_\mu\, B\right) \\ &- \frac{1}{2} f^2\, j_{\mu\nu} \left(A_\mu + \frac{1}{\varkappa}\, \partial_\mu\, B\right) \left(A_\nu + \frac{1}{\varkappa}\, \partial_\nu\, B\right) + \frac{1}{2\varkappa^2} f^2\, (j_\mu\, n_\mu)^2. \end{aligned}\right\} \tag{11.8b}$$

Moreover, we have the conditions

$$\{\partial_\mu \mathbf{A}_\mu(x) + \varkappa \mathbf{B}(x)\}\Psi = 0 \tag{11.9}$$

(cf. (7.103)).

Field equations for $\mathbf{A}_\mu$ and $\mathbf{B}$ can be obtained from (7.102) and (11.5) as

$$\begin{aligned} (\Box - \varkappa^2)\mathbf{A}_\mu(x) &= -f\mathbf{J}_\mu(x) \\ (\Box - \varkappa^2)\mathbf{B}_\mu(x) &= -f\partial_\mu \mathbf{J}_\mu(x) = 0. \end{aligned}$$

By using (10.30), we have

$$M(\partial)B(x) = (S[\sigma] M(\partial)\mathbf{B}(x) S^{-1}[\sigma])_{x/\sigma}.$$

In particular,

$$B(x) = (S[\sigma]\mathbf{B}(x)S^{-1}[\sigma])_{x/\sigma} \tag{11.6c}$$

$$\partial_\mu B(x) = (S[\sigma]\partial_\mu \mathbf{B}(x) S^{-1}[\sigma])_{x/\sigma}. \tag{11.6d}$$

From (10.30), and commutation relations (8.60), we obtain (cf. (10.84))

$$\left.\begin{aligned} A_\mu(x) &= (S[\sigma]\ \mathbf{A}_\mu(x)\ S^{-1}[\sigma])_{x/\sigma}, \\ \partial_\mu\, A_\mu(x) &= (S[\sigma]\ \partial_\mu\, A_\mu(x)\ S^{-1}[\sigma])_{x/\sigma} \\ &\quad + \frac{1}{2\varkappa^2} f \int d^4x'\, S[\sigma]\, \mathbf{J}_\nu(x')\, S^{-1}[\sigma]\, [\partial_\nu,\, \varepsilon(x-x')]\, \Delta(x-x') \\ &= (S[\sigma]\ \partial_\mu\, A_\mu(x)\ S^{-1}[\sigma])_{x/\sigma}, \end{aligned}\right\} \tag{11.6e}$$

$$\left.\begin{aligned} \partial_\nu\, \partial_\mu\, A_\mu(x) &= (S[\sigma]\ \partial_\nu\, \partial_\mu\, \mathbf{A}_\mu(x)\ S^{-1}[\sigma])_{x/\sigma} \\ &\quad + \frac{1}{2\varkappa^2} f \int d^4x'\, S[\sigma]\, \mathbf{J}_\nu(x')\, S^{-1}[\sigma]\, [\partial_\nu\, \partial_\mu,\, \varepsilon(x-x')]\, \Delta(x-x') \\ &= (S[\sigma]\ \partial_\nu\, \partial_\mu A_\mu(x)\ S^{-1}[\sigma])_{x/\sigma} + f \int_\sigma d\sigma'_\mu\, j_\mu(x')\, \partial'_\nu\, \Delta(x'-x). \end{aligned}\right\} \tag{11.6f}$$

These relations will be used to obtain the Lorentz condition.

Introducing the unitary transformation

$$\left.\begin{aligned} &\Psi[\sigma] \to \Psi'[\sigma] = U^{-1}[\sigma]\,\Psi[\sigma], \\ &U[\sigma] = \exp\left\{\frac{i}{\varkappa} f \int_\sigma d\sigma'_\mu\, j_\mu(x')\, B(x')\right\}, \end{aligned}\right\} \tag{11.10}$$

we can rewrite the Schrödinger equation as

$$U^{-1}[\sigma]\left\{i\frac{\delta}{\delta\sigma(x)} - H'[x:n]\right\} U[\sigma]\,\Psi'[\sigma] = 0,$$

which leads to

$$\left\{i\frac{\delta}{\delta\sigma(x)} - f\, j_\mu(x)\, A_\mu(x) - \tfrac{1}{2} f^2\, j_{\mu\nu}[x:n]\, A_\mu(x)\, A_\nu(x)\right\}\Psi'[\sigma] = 0. \tag{11.11}$$

This can be obtained by using the relations

$$\left.\begin{aligned} i\,U^{-1}[\sigma]\frac{\delta}{\delta\sigma(x)}U[\sigma] &= \left\{i\frac{\delta}{\delta\sigma(x)} - \frac{1}{\varkappa} f\, j_\mu(x)\,\partial_\mu B(x)\right. \\ &\quad \left. + \frac{1}{2\varkappa^2} f^2\, j_{\nu\varrho}[x:n]\,\partial_\varrho B(x)\,\partial_\nu B(x)\right\} \\ &\quad + \frac{i}{2\varkappa^2} f^2 \int_\sigma d\sigma'_\mu\, j_\mu(x')\, j_\nu(x)\,[B(x'),\,\partial_\nu B(x)] \\ &= \left\{i\frac{\delta}{\delta\sigma(x)} - \frac{1}{\varkappa} f\, j_\mu(x)\,\partial_\mu B(x) + \frac{1}{2\varkappa^2} f^2\, j_{\nu\varrho}[x:n]\,\partial_\varrho B(x)\,\partial_\nu B(x)\right. \\ &\quad \left. - \frac{f^2}{\varkappa^2}\,(j_\mu(x)\, n_\mu(x))^2\right. \end{aligned}\right\} \tag{11.10a}$$

and

$$\left.\begin{aligned} U^{-1}[\sigma]\,H'[x:n]\,U[\sigma] &= \left\{-f\, j_\mu(x)\left(A_\mu(x) + \frac{1}{\varkappa}\,\partial_\mu B(x)\right)\right. \\ &\quad - f^2\, j_{\mu\nu}[x:n]\left(A_\mu(x) + \frac{1}{\varkappa}\,\partial_\mu B(x)\right)\left(A_\nu(x) + \frac{1}{\varkappa}\,\partial_\nu B(x)\right) \\ &\quad + \frac{1}{2\varkappa} f^2\, j_{\mu\nu}[x:n]\,\partial_\nu B(x)\cdot\left(A_\mu(x) + \frac{1}{\varkappa}\,\partial_\mu B(x)\right) \\ &\quad + \frac{1}{2\varkappa} f^2\, j_{\mu\nu}[x:n]\left(A_\nu(x) + \frac{1}{\varkappa}\,\partial_\mu B(x)\right)\partial_\mu B(x)\Big\} \\ &\quad + \frac{1}{2\varkappa^2} f^2\,(j_\varrho(x)\, n_\varrho(x))^2 + \frac{i}{\varkappa^2} f^2 \int_\sigma d\sigma'_\mu\, j_\mu(x')\, j_\nu(x)\,[B(x'),\,\partial_\nu B(x)] \\ &= \left\{-f\, j_\mu(x)\left(A_\mu(x) + \frac{1}{\varkappa}\,\partial_\mu B(x)\right)\right. \\ &\quad - \frac{1}{2} f\, j_{\mu\nu}[x:n]\left(A_\mu(x) + \frac{1}{\varkappa}\,\partial_\mu B(x)\right)\left(A_\nu(x) + \frac{1}{\varkappa}\,\partial_\nu B(x)\right) \\ &\quad + \frac{1}{2\varkappa} f^2\, j_{\nu\mu}[x:n]\,\partial_\nu B(x)\cdot\left(A_\mu(x) + \frac{1}{\varkappa}\,\partial_\mu B(x)\right) \\ &\quad + \frac{1}{2\varkappa} f^2\, j_{\nu\mu}[x:n]\left(A_\nu(x) + \frac{1}{\varkappa}\,\partial_\nu B(x)\cdot\partial_\mu B(x)\right\} \\ &\quad - \frac{1}{2\varkappa^2} f^2\,(j_\mu(x)\, n_\mu(x))^2 \end{aligned}\right\} \tag{11.10b}$$

instead of (11.2), (11.3) (cf. (8.60)).

The terms in the first brackets in (11.10a) and (11.10b) may be obtained by replacing $eA(x)$ and $A_\mu(x)$, in (11.2) and (11.3), by

$$(-f/\varkappa)B(x)$$

and

$$A_\mu(x)+(1/\varkappa)B(x),$$

respectively. The last terms in (11.10a) and (11.10b) are the contributions due to the q-number properties of $B(x)$ and the last term in (11.8b).

On the other hand we can show that

$$\partial_\mu A_\mu(x)=\int_\sigma d\sigma'_\nu\{\Delta(x-x')\partial'_\nu\partial'_\mu A_\mu(x')-\partial'_\mu A_\mu(x')\cdot\partial'_\nu\Delta(x-x')\}$$
$$B(x)=\int_\sigma d\sigma'_\nu\{\Delta(x-x')\partial'_\nu B(x')-B(x')\partial'_\nu\Delta(x-x')\}$$

by the same method as that used to obtain (10.85). Then, conditions (11.9) lead to (cf. (11.6c, d, e, f))

$$[\partial_\mu A_\mu(x)+\varkappa B(x)-f\int_\sigma d\sigma'_\nu\int d\sigma''_\mu\,\Delta(x-x')\,j_\mu(x')\,\partial''_\nu\,\Delta(x''-x')]\,\Psi[\sigma]=0$$

on account of (11.6c, d, e, f). Thus we obtain

$$[\partial_\mu A_\mu(x)+\varkappa B(x)-f\int_\sigma d\sigma'_\mu j_\mu(x')\Delta(x-x')]\Psi[\sigma]=0. \qquad (11.12)$$

Here x is not necessarily on σ.

We see that, in the limit $\varkappa\to 0$, (11.12) and (11.11) are equivalent to the Lorentz condition (10.87) and the Schrödinger equation in quantum electrodynamics. Moreover, (11.11) shows that the B component in the vector interaction gives no contribution to the change of the state vector.

§ 3. The Vector Interaction of a Real Scalar Field

Equation (11.11) shows that the state vector $\Psi[\sigma]$ is a constant when $A_\mu=0$. In this case, the vector interaction in the last paragraph agrees with that between a real scalar field $U(=B)$ and the charged fields Q_α, and the substitution (11.5) can be written as

$$\partial_\mu\,\mathbf{Q}_\alpha\to\left(\partial_\mu-i\frac{f}{\varkappa}\,\partial_\mu\,\mathbf{U}\right)\mathbf{Q}_\alpha,\quad \partial_\mu\,\mathbf{Q}^*_\alpha\to\left(\partial_\mu+i\frac{f}{\varkappa}\,\partial_\mu\,\mathbf{U}\right)\mathbf{Q}^*_\alpha. \qquad (11.13)$$

Thus we see that the vector interaction of a real scalar field has no physical effect.

§ 4. Equivalence Theorem

A theorem analogous to that of § 3 can be established approximately [1]) for the charge independent interactions between fields $U^{(i)}$ $(i=1, 2, 3)$, whose spin and isotopic spin are 0 and 1 respectively, and the Dirac fields ψ of the isotopic spin 1/2 (cf. Example 10 of Ch. VII).

We denote the masses of the $U^{(i)}$- and ψ fields by $\varkappa$ and μ respectively.

For the vector interaction of the scalar field $U^{(i)}$, we have (cf. (10.66)),

$$\left.\begin{aligned} H'[x:n] = &-i\left(\frac{g}{\varkappa}\right)\bar{\psi}\,\gamma_\mu\,\tau_i\,\psi\,\partial_\mu\,U^{(i)} \\ &-\frac{g^2}{2\varkappa^2}\,(\bar{\psi}\,\gamma_\mu\,\tau_i\,\psi\,n_\mu)\,(\bar{\psi}\,\gamma_\varrho\,\tau_i\,\psi\,n_\varrho). \end{aligned}\right\} \tag{11.14}$$

Now, although we omit the proof, it is possible to establish the theorem:

In the approximation in which the higher order terms g^n $(n \geqslant 3)$ *of the power of* g *can be neglected, the vector interaction* (11.14) *has the same physical effect as that of the interaction*

$$-\sum_{(1,2,3)}\frac{i}{(2\mu)^2}\,g^2\,\bar{\psi}\,\gamma_\nu\,\{\tau_i\,(U^{(k)}\,\partial_\nu\,U^{(l)}-U^{(l)}\,\partial_\nu\,U^{(k)})\}\,\psi; \tag{11.15}$$

here $\sum_{(1,2,3)}$ *means that the summation is taken over suffices* (i, k, l) *which run cyclically over the numbers* (1, 2, 3).

For the pseudoscalar field $U^{(i)}$, we consider the pseudoscalar (PS) and pseudovector (PV) interactions (cf. (10.69))

$$H'[x:n] = -if\bar{\psi}\gamma_5\tau_i\psi U^{(i)} \qquad \text{for (PS)} \tag{11.16}$$

$$\left.\begin{aligned} H'[x:n] = &-\frac{g}{\varkappa}\,\bar{\psi}\,\gamma_5\,\gamma_\mu\,\tau_i\,\psi\,\partial_\mu\,U^{(i)} \\ &-\frac{g^2}{2\varkappa^2}\,(\bar{\psi}\gamma_5\gamma_\mu\tau_i\psi n_\mu)\,(\bar{\psi}\gamma_5\gamma_\varrho\tau_i\psi n_\varrho) \quad \text{for (PV).} \end{aligned}\right\} \tag{11.17}$$

Then we can prove the theorem:

In the approximation in which the higher order terms f^n $(n \geqslant 3)$ *can*

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[1]) The approximate property of this theorem is due to the fact that some charges of the Bose field can be converted into those of the Dirac field, so that the electric current of the Dirac field cannot satisfy the continuity equation which was used in § 1, 2 and 3.

be neglected, the pseudoscalar interaction has the same physical effect as that of the interactions

$$\left.\begin{aligned} &-\frac{i}{2\mu} f(\bar{\psi}\gamma_5\gamma_\mu\tau_i\psi\,\partial_\mu U^{(i)}) - \frac{1}{2}\frac{1}{(2\mu)^2} f^2(\bar{\psi}\gamma_5\gamma_\mu\tau_i\psi\, n_\mu)(\bar{\psi}\gamma_5\gamma_\varrho\tau_i\psi\, n_\varrho) \\ &+\frac{f^2}{2\mu}\bar{\psi}\psi\, U^{(i)}\, U^{(i)} \\ &+\sum_{(1,2,3)} i\frac{1}{(2\mu)^2} f^2\bar{\psi}\gamma_\nu\{\tau_i(U^{(k)}\partial_\nu U^{(l)} - U^{(l)}\cdot\partial_\nu U^{(k)})\}\psi. \end{aligned}\right\} \quad (11.18)$$

The proofs of these equivalence theorems are made in a way similar to the discussions of § 1, § 2 and § 3, i.e. by using a unitary transformation depending only on a surface σ (DRELL and HENLEY [1952]).

Theorem (11.15) shows that if the π-meson were a charged scalar field, its vector interaction would have no effect on the nuclear force in the g^2-approximation. On the other hand, as shown by (11.17) and (11.18), if the π-meson is the charged pseudoscalar field, its pseudovector interaction has the same effect on the nuclear force as that of the pseudoscalar interaction, the coupling constant of which is $f = -2\mu g/\varkappa$ (NELSON [1941], DYSON [1948], CASE [1949]). In these theorems we have assumed the equivalence of the masses of two spinor particles (e.g. the proton and the neutron). Therefore, the accuracy of the equivalence theorems may be slightly impaired in the case of interaction between the π-meson and the nucleon fields on account of the mass difference of the proton and the neutron. The third and fourth terms in (11.17) make contributions to the scattering probability of the π-meson and the nucleon.

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CHAPTER XII

QUANTUM FIELD THEORY IN HEISENBERG REPRESENTATION

§ 1. Proper Field around a Source

In the interaction representation states of fields are represented by assembly of free particles, and these states change from time to time on account of the interaction Hamiltonian in the Schrödinger equation. In the Heisenberg representation, the interaction effects are included in the field quantities; this property also obtains in classical field theory. Taking account of this fact, HEISENBERG [1931] presented a theory of radiation in the Heisenberg representation, in which the radiation intensity was calculated as the expectation value of the number operator of photons. The electromagnetic field around an electron changes from time to time due to its movement. The high energy photons can escape and lead to the Bremsstrahlung phenomenon, and the low energy photons, which cannot escape, give rise to the proper field (i.e. bound photons) of the electron.

In this Chapter we shall present this theory in a covariant formulation for general fields $Q_\alpha(x)$. All operators, in this Chapter, belong to the Heisenberg representation.

The wave equations in the Heisenberg representation are (cf. (10.20a))

$$\Lambda_{\alpha\beta}(\partial)Q_\beta(x)=J_\alpha(x). \tag{12.1}$$

We shall now suppose that an incoming wave $Q_\alpha^{(\mathrm{in})}(x)$ comes into interaction at a time that is in the infinite past, so that

$$Q_\alpha(x)=Q_\alpha^{(\mathrm{in})}(x) \qquad \text{for } \sigma(x)=-\infty. \tag{12.2}$$

Therefore, the $Q_\alpha^{(\mathrm{in})}(x)$ satisfy the wave equations of free fields

$$\Lambda_{\alpha\beta}(\partial)Q_\beta^{(\mathrm{in})}(x)=0. \tag{12.3}$$

Then it is easily seen that, in the discussion of Ch. X, we can replace the field operators in the interaction representation by the $Q_\alpha^{(\mathrm{in})}(x)$ in the Heisenberg representation.

We shall use the same notation with suffixes (in) for the quantities

in which the field operators in the interaction representation are replaced by $Q_\alpha^{(\mathrm{in})}(x)$.

The commutation relations of $Q_\alpha^{(\mathrm{in})}(x)$ are

$$[Q_\alpha^{(\mathrm{in})}(x),\ Q_\beta^{(\mathrm{in})}(x')] = i\, d_{\alpha\beta}(\partial)\, \Delta(x-x') \tag{12.4}$$

Moreover, $Q_\alpha(x)$ can be written as (cf. (10.23)):

$$Q_\alpha(x) = Q_\alpha^{(\mathrm{in})}(x) - \int d^4x'\, d_{\alpha\beta}(\partial)\, D'_a\, \Delta^{\mathrm{ret}}(x-x')\, j_{\beta\, a}(x'). \tag{12.5}$$

Using a unitary transformation $S[\sigma]$ satisfying the equation

$$i\,\frac{\delta}{\delta\sigma(x)}\, S[\sigma] = H'^{(\mathrm{in})}[x:n]\, S[\sigma], \tag{12.6}$$

we introduce $Q_\alpha[x, \sigma]$ as (cf. (10.4a)):

$$Q_\alpha[x, \sigma] = S^{-1}[\sigma]\, Q_\alpha^{(\mathrm{in})}(x) S[\sigma] \tag{12.7}$$

(YANG and FELDMAN [1950]). We therefore have

$$\Lambda_{\alpha\beta}(\partial) Q_\beta[x, \sigma] = 0. \tag{12.8}$$

The commutation relations of $Q_\alpha[x, \sigma]$ are given by (10.6). Then we can write $Q_\alpha(x)$ as (cf. (10.25)):

$$Q_\alpha(x) = Q_\alpha[x/\sigma] + \tfrac{1}{2} \int d^4x' \{[D'_a\, d_{\alpha\beta}(\partial),\ \varepsilon(x-x')]\, \Delta(x-x')\}\, j_{\beta\, \alpha}(x'). \tag{12.9}$$

The second term of (12.9), which depends only on the field operators on the surface σ, is zero when the $Q_\alpha(x)$ are independent canonical components (cf. (10.44)). Therefore, we can interpret (12.9) in the following way: if we could make the interaction vanish on σ, the particles, which were bound to the source by the interaction, would escape as free particles and their states would be described by the free field $Q_\alpha[x, \sigma]$.

Since the $Q_\alpha[x, \sigma]$ satisfy the wave equations of a free field, they can be separated into parts of positive and negative frequency; that is,

$$Q_\alpha[x, \sigma] = Q_\alpha^+[x, \sigma] + Q_\alpha^-[x, \sigma], \tag{12.10a}$$

$$Q_\alpha^{(\mathrm{in})}(x) = Q_\alpha^{(\mathrm{in})+}(x) + Q_\alpha^{(\mathrm{in})-}(x), \tag{12.10b}$$

$$Q_\alpha^\pm[x, \sigma] = S^{-1}[\sigma]\, Q_\alpha^{(\mathrm{in})\pm}(x)\, S[\sigma]. \tag{12.11}$$

Using the Fourier expansion of $Q_\alpha^\pm[x, \sigma]$ and $Q_\alpha^{(\mathrm{in})\pm}(x)$ given by (9.21), we have

$$u_\tau^\pm[k, \sigma] = S^{-1}[\sigma] u_\tau^{(\mathrm{in})\pm}(k) S[\sigma]. \tag{12.12}$$

The quantities $u_r^{(\mathrm{in})\pm}(k_\mu)$ can be regarded as annihilation and creation operators of a free particles with energy-momentum k_μ.

Since the vacuum Φ_0 is the lowest energy state, we can derive (cf. (9.36))

$$Q_\alpha^+[x,\sigma]\,\Phi_0[\sigma]=0, \tag{12.13}$$

where

$$\Phi_0[\sigma] \equiv S[\sigma]\Phi_0.$$

Using (9.20), we introduce the number operators $N_r[k,\sigma]$ of the free fields $Q_\alpha^\pm[x,\sigma]$. Thus $N_r[k,\sigma]$ gives the number of (k,r)-particles on the surface σ (UMEZAWA, TAKAHASHI and KAMEFUCHI [1952]). If $N_r^{(\mathrm{in})}(k)$ are the number operators of the free fields $Q_\alpha^{(\mathrm{in})}(x)$, we have

$$N_r[k,\sigma]=S^{-1}[\sigma]\,N_r^{(\mathrm{in})}(k)S[\sigma], \tag{12.14a}$$

$$N_r^{(\mathrm{in})}(k) = u_r^{(\mathrm{in})*-}(k)\,u_r^{(\mathrm{in})+}(k). \tag{12.14b}$$

The expectation value of the number of the (k,r)-particles on σ is

$$(N_r[k,\sigma])_\sigma=\Psi^*N_r[k,\sigma]\Psi=\Psi'^*[\sigma]\,N_r^{(\mathrm{in})}(k)\Psi'[\sigma], \tag{12.15}$$

where $\Psi'[\sigma]$ is (cf. (10.1)):

$$\Psi'[\sigma] \equiv S[\sigma]\Psi. \tag{12.16}$$

§ 2. Examples

First, we shall consider the scalar interaction $L'=f\bar\psi\psi U$ between a scalar field $U(x)$ and a Dirac field $\psi(x)$. We have

$$U[x,\sigma]=U^{(\mathrm{in})}(x)-f\textstyle\int_{-\infty}^{\sigma}d^4x'\Delta(x-x')\bar\psi(x')\psi(x'), \tag{12.17a}$$

which leads to

$$U^+[x,\sigma]=U^{(\mathrm{in})+}(x)-f\textstyle\int_{-\infty}^{\sigma}d^4x'\Delta^+(x-x')\cdot\bar\psi(x')\psi(x'). \tag{12.17b}$$

The number operator of the scalar particles on σ is

$$N[\sigma]=\frac{2}{i}\textstyle\int_\sigma (U^+[x,\sigma])^*\,\partial_\mu\,U^+[x,\sigma]\,d\sigma_\mu. \tag{12.18}$$

It is easily seen by using (9.43), that the Fourier amplitude of (12.18) agrees with (12.14a).

Now (12.17a) can be written as

$$U^+[x,\sigma]=U^{(\mathrm{in})+}(x)+\frac{f}{2}\int d^3k\,\frac{O(k)}{K_0(K_0-k_0)}\,e^{i\{K_\mu x_\mu+(K_0-k_0)t_\sigma\}}, \tag{12.19}$$

where $O(k)$ is the Fourier amplitude in

$$\bar\psi(x)\,\psi(x) = \int d^4k\,O(k)\,e^{ik_\mu x_\mu}, \tag{12.20a}$$

and t_σ is the time of σ, which we take to be a flat surface. In (12.19), K_0 has the value

$$K_0 \equiv \sqrt{(k_i k_i + \varkappa^2)}$$

and the four dimensional vector K_μ is defined by $K_\mu \equiv (\mathbf{k}, K_0)$. It is apparent from (12.19) and (12.20a) that k_0 and $\mathbf{k}$ are the change of the energy and momentum of the spinor particle due to the reaction of the scalar field.

When we can assume that the velocity v of the source spinor particle does not change appreciably, and that $O(k)$ can be treated as a c-number;

$$O(k) \approx \frac{1}{(2\pi)^3}\,\delta(k_0) \tag{12.20b}$$

we have

$$\left.\begin{aligned} U^+[x/\sigma] - U^{(\mathrm{in})+}(x) &= \frac{f}{2}\frac{1}{(2\pi)^3}\int d^3k\,\frac{1}{K_0^2}\,e^{ik_i x_i} \\ &= \frac{1}{8\pi}\,f\,\frac{1}{r}\,e^{-\varkappa r}. \end{aligned}\right\} \tag{12.21}$$

where r is $\sqrt{(x_i x_i)}$. Here we have made use of the formula

$$\frac{1}{(2\pi)^3}\int d^3k\,\frac{e^{ik_i x_i}}{k^2+\varkappa^2} = \frac{1}{4\pi}\,\frac{1}{r}\,e^{-\varkappa r}. \tag{12.22}$$

Since it follows from (12.20a) that $\bar{\psi}(x)\psi(x) \propto \delta(\mathbf{x})$, the origin of the coordinate system is the position of the source particle and r is the distance from this origin.

Equation (12.21) is the proper field of the source particle, i.e. the scalar field, induced by its interaction with the source particle. Therefore the $N[\sigma]$ of this scalar field gives the number of scalar particles of the proper field. This proper field has a space distribution $e^{-\varkappa r}/r$ and therefore its range a is $\approx 1/\varkappa$.

This fact can be understood physically as follows (WICK [1938]):

A free particle cannot radiate scalar particles on account of energy-momentum conservation law. However, when the energy is ill-defined on account of the uncertainty principle, some low energy particles can be radiated. We shall consider an experiment in which scalar particles radiated by the source particle are caught at the point r. Since this observation is made in the time interval $\approx r/v$ (v is the velocity of a scalar particle), the energy is uncertain by the amount [1])

$$\Delta E \approx (hv/r) < hC/r. \tag{12.23}$$

[1]) C is the light velocity.

Therefore, the radiation of n scalar particles is possible if

$$n\varkappa hC < \Delta E < hC/r,$$

that is, if

$$r < \frac{1}{n\varkappa}. \tag{12.24}$$

Thus we see that the probability of observing n scalar particles is very small in the region $r > 1/n\varkappa$. The region in which we can find at least one scalar particle, is defined by $r < 1/\varkappa$. In other words, the range of the proper field is $a = (1/\varkappa)$.

We see from the preceding discussion that if we do not use the approximation (12.20*b*), n scalar particles give terms of the form $e^{-n\varkappa r}$. Moreover, the pair creation of Dirac particles gives the proper Dirac field of the source particle, in which m pairs will give terms of the form $e^{-2m\mu r}$ (μ is the mass of the Dirac particle). Equation (12.24) shows that there are large numbers of high energy particles infinitesimally close to the source particle. These high energy particles give rise to infinite values of various physical quantities (e.g. total energy, total charge of the fields, etc.). The infinite energy of the field gives rise to the so-called self-energy divergence. We shall discuss this problem in the following chapters.

When two Dirac particles $\psi(x)$, $\psi(x')$ approach each other, one particle $\psi(x')$ interacts with the scalar field $U[x'/\sigma]$ around the other particle $\psi(x)$, the interaction being $-f\bar{\psi}(x')\psi(x')\,U[x'/\sigma]$. This, as shown by (12.21), implies that the interaction potential between the two particles is

$$V(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} f^2 \frac{1}{r} e^{-\varkappa r}. \tag{12.25}$$

Here r is the distance between x and x'. If we assume that these two particles and the scalar field are nucleons and the π-meson field respectively, $V(x, x')$ is the nuclear potential.

Applying a similar discussion to the system of two electrons and the electromagnetic field, (12.25) gives the electromagnetic potential between two electrons. In the latter case, $V(\mathbf{x}, \mathbf{x}')$ depends on r like $1/r$ and has an infinite range because $\varkappa = 0$. This is a well-known property of the Coulomb potential. Thus, we see that the proper fields act as intermediate for the interaction and, at the same time, give rise to the difficulties of divergence.

This picture of the proper fields has been confirmed by the recent

success of the predictions of quantum electrodynamics. Thus, we can expect that a search for further observable effects of the proper fields would provide valuable help in the formulation of a correct theory of elementary particles.

Substituting (12.19) into (12.18), we obtain the number of particles in the proper field as

$$\left.\begin{aligned} \langle N[\sigma]\rangle &= \int d^3k\, N(k), \\ N(k) &= \frac{f^2}{2}\frac{1}{K_0(K_0-k_0)^2}\,(O^*(k)\,O(k)). \end{aligned}\right\} \tag{12.26}$$

If we can assume that the velocity of the source particle remains approximately constant, we can take $k_0 \approx (\mathbf{k}\cdot\mathbf{v})$ for the energy transfer of the source particle [1]).

We can apply the above results to quantum electrodynamics by replacing $O(k)$ by $O_\mu(k)$ defined by

$$\bar{\psi}(x)\,\gamma_\mu\,\psi(x) = \int d^4k\, O_\mu(k)\, e^{ik_\mu x_\mu}. \tag{12.27}$$

We shall now appiy these results to the problem of the Bremsstrahlung of low energy photons by low energy electrons. We shall use a simplified argument, but the same results can be confirmed by a more correct method (BLOCH and NORDSIECK [1937], UMEZAWA, TAKAHASHI and KAMEFUCHI [1952], THIRRING and TOUSCHEK [1951]). As shown by (12.27), O_μ may be regarded, in the non-relativistic approximation, as

$$O_\mu \approx v_\mu \qquad (v_4 \equiv i). \tag{12.28}$$

On the other hand (10.24) gives

$$A_\mu^+[x,\sigma] = A_\mu^{(\mathrm{in})+}(x) + \frac{e}{2}\int d^3k\,\frac{O_\mu(k)}{K_0(K_0-k_0)}\, e^{i\{K_\mu x_\mu + (K_\eta - k_\eta)k_\sigma\}}. \tag{12.29}$$

The electromagnetic field $\Delta A_\mu^+[x,\sigma]$ radiated by the electron is given by the difference between the electromagnetic fields before and after the scattering of the electron, and is

$$\Delta A_\mu^+[x,\sigma] = \frac{e}{2}\int d^3k\,\frac{\Delta O_\mu(k)}{K_0\{K_0-(\mathbf{k}\cdot\mathbf{v})\}}\, e^{i\{K_\mu x_\mu - (K_\eta - k_\eta)k_\sigma\}}.$$

Here ΔO_μ is the change of the quantity O_μ due to the scattering of the electron.

---

[1]) If $\mathbf{p}$ is the momentum of the incoming particle, we obtain

$$k_0 = \sqrt{\{|\mathbf{p}-\mathbf{k}|^2+\varkappa^2\}} - \sqrt{\{|\mathbf{p}|^2+\varkappa^2\}} \approx \frac{(\mathbf{p}\cdot\mathbf{k})}{\sqrt{\{|\mathbf{p}|^2+\varkappa^2\}}} = (\mathbf{v}\cdot\mathbf{k}).$$

The number of photons of the electromagnetic field is given by (12.26), and is

$$\Delta N(k_\mu) = \frac{e^2}{2} \frac{|\Delta O_\alpha(k)|^2}{K_0\{K_0 - (\mathbf{k}\cdot\mathbf{v})\}^2}. \tag{12.30}$$

We can derive the expression of the cross-section for the creation of n photons expressed in the form in which photons are created by iteration of single productions for a very short time theoretically by successive changes of the electron states. Since the created photons have low energies, the reaction of each creation on the electron state is small and each creation can be regarded as a process independent of all others. In other words, the probability for creation of n photons has approximately the form of the Poisson distribution. Then, (12.30) can be regarded as the probability of the creation of a photon (k_μ), and the probability of the creation of n_a photons of energy-momentum $(k^a, k^a + dk^a)$ $(a = 1, 2, \ldots)$ is

$$dw = e^{-\bar{n}} |V|^2 \prod_a \left[\frac{1}{n_a!} (\Delta N(k^a))^{n_a} d^3k^a\right]. \tag{12.31}$$

Here $|V|^2$ is the probability of scattering of the electron in the external potential and $\bar{n}$ is

$$\bar{n} = \int d^3k \, \Delta N(k). \tag{12.32}$$

The domain of integration in (12.32) is determined by the energy-momentum conservation law.

By performing the integration in (12.31) we obtain the sensible result that the probability w for scattering of the electron accompanying the production of any photon agrees with the scattering probability of the electron, namely

$$w = |V|^2 \left\{\sum_{n=0}^{\infty} \frac{1}{n!} e^{-\bar{n}} \bar{n}^n\right\} = |V|^2. \tag{12.33}$$

These results will be applied to the problem of the infrared catastrophy in Example 4 of Ch. XIII. The same method has been applied to the multiple production of mesons (LEWIS, OPPENHEIMER and WOUTHUYSEN [1948]).

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CHAPTER XIII

PERTURBATION THEORY

§ 1. Perturbation Theory

Perturbation theory is the name given to those approximation methods in which the changes of state of a system are expressed as a combination of those induced by a series of small perturbation effects, only the terms of low order being taken into account for comparison with experiment. Such methods are widely used in physics.

In the quantum field theory various approximation methods have been used to solve the Schrödinger equations, because the q-number character of the interaction Hamiltonian makes it difficult to solve them completely. The most commonly used approximation is the perturbation theory of "weak coupling", in which the interaction Hamiltonian $H'[x : n]$ perturbs the system of free fields, (i.e. unperturbed system). The present Chapter is devoted to the perturbation theory of weak coupling in the interaction representation. This theory can be applied only when the contribution of lower order terms in the power series of the interaction Hamiltonian quickly approaches a definite result. However, interactions may exist for which this condition cannot be valid. For example, as shown in Example 11 of Ch. VII, the coupling constant of the interaction between the π-meson and the nucleon does not seem to be sufficiently small for this condition to be satisfied. Moreover, even in quantum electrodynamics, it is very strange that the results given by the lower approximations of perturbation theory have been experimentally confirmed because, as pointed out in the example of the last Chapter, the proper fields induce terms of infinite value in the higher perturbation power series. We shall see the reason for this strange fact in following Chapters.

The state vector $\Psi[\sigma]$ and the expectation value of the operator $F(x)$ on the surface σ are determined by $S[\sigma]$ in the way shown by (10.46) and (10.48). Calculated by the method of successive approximations applied to (10.50), $S[\sigma, \sigma']$ is

$$S[\sigma, \sigma'] = \sum_{m=0}^{\infty} S^{(m)}[\sigma, \sigma'], \tag{13.1}$$

$$\left.\begin{aligned} &S^{(0)}[\sigma, \sigma'] = 1 \\ &S^{(m)}[\sigma, \sigma'] = (-i)^m \int_{\sigma'}^{\sigma} d^4x_1 \int_{\sigma'}^{\sigma_1} d^4x_2 \ldots \int_{\sigma'}^{\sigma_{m-1}} d^4x_m \, H'[x_1 : n] \ldots H'[x_m : n], \end{aligned}\right\} \quad (13.2)$$

where σ_m denotes the space-like surface $\sigma(x_m)$ (through a point x_m). Relations (13.1) and (13.2) lead to

$$\left.\begin{aligned} &S^{-1}[\sigma(x), \sigma'] \, F(x) \, S[\sigma(x), \sigma'] = \sum_{m=0}^{\infty} (i)^m \int_{\sigma'}^{\sigma(x)} d^4x_1 \int_{\sigma'}^{\sigma_1} d^4x_2 \ldots \int_{\sigma'}^{\sigma_{m-1}} d^4x_m \\ &[H'[x_m : n], \, [H'[x_{m-1} : n], \, [\ldots [H'[x_1 : n], \, F(x)]] \ldots]. \end{aligned}\right\} \quad (13.3)$$

By taking $\sigma' = -\infty$ in (13.1) and (13.3) we obtain $S[\sigma]$ and

$$S^{-1}[\sigma] \, F(x) \, S[\sigma]$$

respectively.

The transition probability $w(f, i)$ between two states Ψ_f and Ψ_i is given by (10.49), and is

$$w(f, i) = (f|S^*[\sigma]|i) \, (f|S[\sigma]|i).$$

When σ is a flat surface with time t, we can calculate the transition probability per unit time by writing

$$\frac{d}{dt} w(f, i) = \left(f \left| \frac{d}{dt} S^*[\sigma] \right| i\right) (f|S[\sigma]|i) + (f|S^*[\sigma]|i) \left(f \left| \frac{d}{dt} S[\sigma] \right| i\right). \quad (13.4)$$

By assuming that, in (13.2), σ is a flat surface at the time t, we see that the matrix element of $S[\sigma]$ has the form

$$(f|S[\sigma(t)]|i) = S(E_f, E_i) \int_{-\infty}^{t} e^{i(E_f - E_i)t'} \, dt'.$$

Here f and i denote eigenstates of the free energy operator T_4^0 with eigenvalues E_f and E_i. Using (8.30b) we obtain

$$(f|S[\sigma(t)]|i) = 2\pi \, S(E_f, E_i) \, e^{i(E_f - E_i)t} \, \delta_-(E_f - E_i). \quad (13.5a)$$

Taking the limit $t \to \infty$ we obtain the S-matrix

$$(f|S[\infty]|i) = 2\pi S(E_f, E_i) \delta(E_f - E_i). \quad (13.5b)$$

Equations (13.5a) and (13.4) provide us with the probability per unit time of the transition from the state i to the state f (with energy E_f in the range $(E_f, E_f + \Delta E)$. Indeed

$$\left.\begin{aligned} \frac{d}{dt} w(E_f, E_i) &= 2\pi \int_{\Delta E} |S(E, E_i)|^2 \{\delta_-(E - E_f) + \delta_+(E - E_i)\} \, \varrho_E \, dE \\ &= 2\pi \int_{\Delta E} |S(E, E_i)|^2 \, \delta(E - E_i) \, \varrho_E \, dE \end{aligned}\right\} \quad (13.6)$$

where $\varrho_E dE$ is the number of states in the energy interval $(E_f, E_f + dE)$.

The perturbed energy δW of the system can be calculated from the relation

$$\delta W = (i|S^{-1}[\sigma] \, T_4 \, S[\sigma] - T_4^0|i), \quad (13.7)$$

where the state i is the unperturbed state Φ_i, an eigenstate of T_4^0:

$$T_4^0\Phi_i = E_i\Phi_i.$$

In the next section we shall rewrite the relation (13.7) in the form

$$\delta W = \sum_{m,l=0}^{\infty} \frac{1}{m+l+1}\,(i\,|S^{(l)*}[\sigma]\int_\sigma d^3x\, H'[x:n]\, S^{(m)}[\sigma]\,|\,i). \qquad (13.8)$$

The perturbed energy can now be calculated from (13.8) by using $S[\sigma]$.

§ 2. Non-Covariant Formulae

In this paragraph we shall assume [1]) that each σ_m is a flat surface $\sigma(t_m)$ at certain times t_m, and that the interaction is introduced adiabatically from the infinite past. In accordance with the last assumption, we shall replace the interaction $H'[x:n]$ in (13.2) by $H'[x:n]\exp(-\epsilon|t|)$ and proceed to the limit $\epsilon \to 0$ at the end of calculation.

Carrying out the time integrations in (13.2) the mth order term of $S(E_f, E_i)$ is given by (cf. (10.52))

$$S^{(m)}(E_f, E_i) = \sum_{m-1}\ldots\sum_{1}(f\,|H'[\sigma(t)]\,|\,m-1)\left[\frac{(m-1\,|H'[\sigma(t)]\,|\,m-2)}{(E_{m-1}-E_i)}\ldots\frac{(1\,|H'[\sigma(t)]\,|\,i)}{(E_1-E_i)}\right] \qquad (13.9)$$

(see HEITLER [1935]).

We shall derive equation (13.8). From (13.3), (8.15) and (10.50), we have

$$\begin{aligned}
&S^{-1}[\sigma]\,T_4^0\,S[\sigma] - T_4^0\\
&= \sum_{m=1}^{\infty} i^m \int_{-\infty}^{t} d^4x_1\, g(t_1)\int_{-\infty}^{t_1} d^4x_2\, g(t_2)\ldots\int_{-\infty}^{t_{m-1}} d^4x_m\, g(t_m)\,[H'[x_m:n],\,[\ldots[H'[x_1:n],\,T_4^0]\ldots]\\
&= \sum_{m=2}^{\infty} i^{m+1}\int_{-\infty}^{t} d^4x_1\, g(t_1)\ldots\int_{-\infty}^{t_{m-1}} d^4x_m\, g(t_m)\,[H'[x_m:n],\,[\ldots[H'[x_2:n],\,\partial_{t_1}H'[x_1:n]]\ldots]\\
&= -\int_{-\infty}^{t} d^4x_1\, g(t_1)\, S^{-1}[\sigma(t_1)]\,(\partial_{t_1}H'[x_1:n])\,S[\sigma(t_1)]\\
&= -\,g(t)\,S^{-1}[\sigma(t)]\,H'[x:n]\,S[\sigma(t)]\\
&\quad + \int_{-\infty}^{t} d^4x_1\,(\partial_{t_1}g(t_1))\,S^{-1}[\sigma(t_1)]\,H'[x_1:n]\,S[\sigma(t_1)]\\
&\quad - i\int_{-\infty}^{t} d^4x_1\,(g(t_1))^2\int_{\sigma(t_1)} d^3x\, S^{-1}[\sigma(t_1)]\,[H'[x_1:n],\,H'[x:n]]\,S[\sigma(t_1)].
\end{aligned}$$

[1]) Thus, n_μ is the constant vector $(0, 0, 0, i)$.

Here $g(t)$ is

$$g(t) = e^{-\epsilon|t|} \qquad \epsilon > 0. \tag{13.10}$$

The last term is zero [1]) on account of the integrability condition (10.11).

Thus we have

$$S^{-1}[\sigma]T_4 S[\sigma] - T_4^0 = \int_{-\infty}^{t} d^4x'(\partial_{t'}g(t'))S^{-1}[\sigma(t')]H'[x' : n]S[\sigma(t')]$$

on account of (10.32). Since $S^{(m)}[\sigma]$ is proportional to $(g(t))^m$:

$$S^{(m)}[\sigma(t)] = (g(t))^m \lim_{\epsilon\to 0} S^{(m)}[\sigma(t)],$$

we have

$$\begin{aligned}
&S^{-1}[\sigma]\, T_4\, S[\sigma] - T_4^0 \\
&= \sum_{m,l=0}^{\infty} \int_{-\infty}^{t} d^4x'\, (\partial_{t'}\, g(t'))\, (g(t'))^{m+l} \lim_{\epsilon\to 0} (S^{(l)*}[\sigma(x')]\, H[x' : n]\, S^{(m)}[\sigma(x')]) \\
&= \sum_{m,l=0}^{\infty} \frac{1}{m+l+1} \int_{-\infty}^{t} d^4x'\, \partial_{t'}(g(t'))^{m+l+1} \cdot \lim_{\epsilon\to 0} (S^{(l)*}[\sigma(x')]\, H'[x' : n]\, S^{(m)}[\sigma(x')]) \\
&= \sum_{m,l=0}^{\infty} \frac{1}{m+l+1} (g(t))^{m+l+1} \cdot \lim_{\epsilon\to 0} (S^{(l)*}[\sigma(x)] \int_{\sigma(x)} d^3x\, H'[x : n]\, S^{(m)}[\sigma(x')]) \\
&- \sum_{m,l=0}^{\infty} \frac{1}{m+l+1} \int_{-\infty}^{t} d^4x'\, (g(t'))^{m+l+1}\, \partial_{t'} [\lim_{\epsilon\to 0} (S^{(l)*}[\sigma(x')]\, H'[x' : n]\, S^{(m)}[\sigma(x')])]
\end{aligned}$$

However, the last term does not contribute to δW, because

$$(i|\partial_t F|i) = (E_i - E_i)\ (i|F|i) = 0$$

for any regular operator F. Thus we have, by taking the limit $\epsilon \to 0$,

$$\begin{aligned}
&(i|S^{-1}[\sigma]\, T_4\, S[\sigma] - T_4^0|i) \\
&\quad = \sum_{m,l=0}^{\infty} \frac{1}{m+l+1}\, (i\,|S^{(l)*}[\sigma] \int_\sigma d^3x\, H'[x : n]\, S^{(m)}[\sigma]|\, i).
\end{aligned}$$

This leads to (13.8).

By substituting (13.9) into (13.8), we have

$$\left.\begin{aligned}
\delta W = (i\,|H'[\sigma]|\,i) &- \sum_1 \frac{(i|H'[\sigma]|\,1)\ (1|H'[\sigma]|\,i)}{E_1 - E_i} \\
&+ \sum_1 \sum_2 \frac{(i|H'[\sigma]|\,2)\ (2|H'[\sigma]|\,1)\ (1|H'[\sigma]|\,i)}{(E_2 - E_i)\ (E_1 - E_i)} - \cdots
\end{aligned}\right\} \tag{13.11}$$

---

[1]) $\int d^3x_1 \int d^3x \left\{ \dfrac{\partial}{\partial\sigma(x_1)} H'[x : n] - \dfrac{\partial}{\partial\sigma(x)} H'[x_1 : n] \right\} = 0.$

The only matrix elements of $H'[\sigma]$ that do not vanish are those connecting states of the same total momentum; the space integration ensures this. In other words the conservation law of momentum obtains for virtual states. However there is no conservation law for the energies E_l of the virtual states. This is a consequence of the uncertainty principle applied to energy and time. In fact, in (13.2), $S[\sigma(t)]$ is constructed as a sum of the contributions of the states of various times t' ($t'<t$) and the energy of the states $\Psi[\sigma(t')]$ are uncertain by $(2\pi\hbar/\Delta T)$, where ΔT is the difference of the time t' and the time at which the interaction is introduced. It is for the same reason that proper fields had to be accounted for in § 2, Ch. XII. Indeed, the effects of proper fields are represented by the contributions of virtual states to the terms of higher order in the perturbation expansion of the S-matrix. The difficulties of infinities induced by the proper fields appear as contributions of virtual states of very high energy. However, the uncertainty in the energy disappears as $t \to \infty$, as shown by (13.5*b*).

§ 3. Covariant Perturbation Theory

In the last paragraph the non-covariant form of the perturbation theory was discussed in order to clarify some details of the physical picture on which this theory is based. However, in practical calculation it is more advantageous to write each stage of the calculations in an explicitly covariant form.

The matrix element of $S[\sigma, -\infty]$ connecting the state $\Psi[\sigma]$ on σ (any space-like surface) and $\Psi[-\infty]$ must be written as

$$\left.\begin{aligned}(f|S[\sigma,-\infty]|i) = \Phi_0^* \{Q^-_{\alpha_1}(k'^{(1)}) \dots Q^-_{\alpha_{m'}}(k'^{(m')})\} \\ f(k'^{(1)} \dots k'^{(m')}; k^{(1)} \dots k^{(m)}) \{Q^+_{\beta_1}(k_{\beta_1}) \dots Q^+_{\beta_m}(k_{\beta_m})\} \Phi_0 .\end{aligned}\right\} \quad (13.12)$$

Here $(\alpha_1 \dots \alpha_{m'})$ and $(\beta_1 \dots \beta_m)$ represent the states (spin, charge etc.) of particles with momenta $(k'^{(1)} \dots k'^{(m')})$ and $(k^{(1)} \dots k^{(m)})$ in the states $\Psi[\sigma]$ and $\Psi[-\infty]$ respectively, and Φ_0 denotes the vacuum state.

By separating Q_α into parts of positive and negative frequency, and by using the commutation relations (8.14*a*, *b*) and the vacuum expectation values (9.42), we can write $f(\dots)$ as covariant integrals of products of $\Delta(x)$, $\Delta^{(1)}(x)$ and their derivatives.

§ 4. Method of P-Symbol

Equation (13.2) is such that σ and σ' appear in an unsymmetrical

way. It can be rewritten in a symmetrical form (with respect to σ and σ') by using the P-symbol introduced by DYSON [1949].

We first introduce a set $\{\sigma_i\}$ of space-like surfaces between σ and σ' in such a way that for any point x_i between σ and σ' there is only one surface σ_i that contains x_i; then we construct a functional of the quantities $F_i[x_i : n]$ written as

$$P[F_1[x_1 : n], \ldots, F_m[x_m : n]], \tag{13.13}$$

in which the P-symbol means that the quantities $F_i[x_i : n]$ in the bracket occur from right to left in order of time (from past to future). Since $P[H'[x_1 : n] \ldots H'[x_m : n]]$ is symmetrical with respect to points $(x_1, \ldots, x_m)$,

$$\int_{\sigma'}^{\sigma} dx_1 \ldots \int_{\sigma'}^{\sigma} dx_m \, P[H'[x_1 : n], \ldots, H'[x_m : n]]$$

is equal, apart from a factor $m!$ to the integral of $[H'[x_1 : n] \ldots H'[x_m : n]]$ over the region between σ and σ' under the condition $\sigma_1 > \sigma_2 > \ldots > \sigma_m$.

Thus (13.1) can be written as

$$S[\sigma, \sigma'] = \sum_{m=0}^{\infty} (-i)^m \frac{1}{m!} \int_{\sigma'}^{\sigma} dx_1 \ldots \int_{\sigma'}^{\sigma} dx_m \, P[H'[x_1 : n], \ldots, H'[x_m : n]], \tag{13.14}$$

which has a symmetrical form with respect to σ and σ'. Then $S[\sigma]$ is obtained by putting $\sigma' = -\infty$.

In scattering problems the eigenstates of the free fields are usually represented as $\Psi[\infty]$ and $\Psi[-\infty]$, by assuming that the interaction is adiabatically switched on and off in past and future. It was pointed out in § 2 that this assumption can be expressed mathematically by replacing the coupling constant g by a time dependent quantity $g(t)$. There, we adopted $g \exp(-\epsilon|t|)$ as $g(t)$. Here, in order to facilitate the calculation, we shall assume $g(t)$ to be given by

$$g(t) = \lim_{\epsilon \to 0} \frac{g}{2\epsilon} \int_{-\epsilon}^{\epsilon} da \, e^{-iat} \tag{13.15a}$$

$$= \lim_{T \to \infty} \frac{T}{2it} \left(e^{i\frac{t}{T}} - e^{-i\frac{t}{T}}\right) \qquad (T \equiv 1/\epsilon) \tag{13.15b}$$

and, as the last stage of the calculation, we shall carry out the limiting process $\epsilon \to 0$.

Since this procedure normally gives the same result as constant g does, we shall usually take g to be a constant. However, in Example 5 we shall show that g cannot always be taken as a constant, but must

be of the form (13.15a). However, (13.15a) is not the most general from of $g(t)$. More generally, we can write

$$g(t) = \lim_{\epsilon \to 0} \frac{2m+1}{2\epsilon^{2m+1}} g \int_{-\epsilon}^{\epsilon} da\, e^{-iat} a^{2m} \qquad (m \geqslant 0) \tag{13.15c}$$

$$= \begin{cases} 0 & \text{for } |t| \gg T \\ g & \text{for } |t| \ll T. \end{cases} \tag{13.15d}$$

It will appear that, in some circumstance, we must use (13.15c) (with $m>0$) instead of (13.15a).

When the interaction Hamiltonian consists of two terms, i.e. $H'[x:n]+V[x:n]$, (13.14) can be written as

$$\left.\begin{aligned} S[\sigma, q'] = \sum_{l,m=0}^{\infty} (-i)^{l+m} \frac{1}{l!\, m!} \int_{\sigma'}^{\sigma} dx_1 \dots \int_{\sigma'}^{\sigma} dx_{m+} \\ P[V[x_1:n], \dots, V[x_m:n], H'[x_{m+1}:n], \dots, H'[x_{m+l}:n]. \end{aligned}\right\} \tag{13.16}$$

We shall now introduce a quantity $F^M[x:n]$ for every $F[x:n]$ which has the form

$$\begin{aligned} F^M[x:n] &\equiv S[\infty]S^{-1}[\sigma(x)]F[x:n]S[\sigma(x)] \\ &= S[\infty, \sigma(x)]F[x:n]S[\sigma(x), -\infty]. \end{aligned} \tag{13.17}$$

Using (13.16) we have

$$\left.\begin{aligned} F^M[x:n] = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{m!\, l!} (-i)^{n+m} \int_{\sigma(x)}^{\infty} dx_1 \dots \int_{\sigma(x)}^{\infty} dx_m \int_{-\infty}^{\sigma(x)} dx_{m+1} \dots \int_{-\infty}^{\sigma(x)} dx_{m+l} \\ P[F[x:n], H'[x_1:n], \dots, H'[x_{m+l}:n]] \\ = \sum_{n=0}^{\infty} (-i)^m \frac{1}{m!} \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_m\, P[F[x:n], H'[x_1:n], \dots, H'[x_m:n]]. \end{aligned}\right\} \tag{13.18}$$

The relation

$$\Psi^*[\sigma]F[x:n]\Psi[\sigma] = \Psi^*[\infty]F^M[x:n]\Psi[-\infty] \tag{13.19}$$

shows that the expectation values of F for given initial and final states are best calculated from the expectation values of F^M. Since the state vectors of both sides of $F^M[x:n]$ in (13.19) belong to different representations, $F^M[x:n]$ is called a quantity in the "mixed representation" (DYSON [1949]). A remarkable feature of the quantities in the mixed representation is that they have a symmetric form with respect to the direction of time.

We proceed to the calculation of (13.14). The quantity (13.18) can

be calculated by the same method. For simplicity we shall assume that the interaction between an integer spin field $U_\alpha(x)$ and a half-integer spin field $\psi_\alpha(x)$ is

$$H'[x : n] = C(\alpha\beta\gamma)\, \bar{\psi}_\alpha(x)\psi_\beta(x)\, U_\gamma(x)$$

where the $C(\alpha\beta\gamma)$ are constants. The matrix $S[\sigma]$ contains a term

$$H'[x_1 : n] \dots H'[x_m : n], \quad \sigma_1 > \sigma_2 > \dots > \sigma_m, \tag{13.20}$$

in which some operators ψ_α, U_α (i.e. external operators) play the role of creation operators for the particles in the final state or annihilation operators for the incoming particles, and other operators (i.e. internal operators) contribute to $f(\dots)$ in (13.12). Since the internal operators play the role of creation or annihilation operators of particles in virtual states, they may be grouped in pairs

$$(U_\alpha^+(k),\ U_\beta^-(k)),\ (\psi_\alpha^+(k),\ \bar{\psi}_\beta^-(k)),\ (\bar{\psi}_\alpha^+(k),\ \psi_\beta^-(k))$$

of annihilation and creation operators for the same particles. The two operators of each pair can be written in succession only by interchanges among the operators of particles in different states. Since these exchanges give rise merely to the factor $\varepsilon(= \pm 1)$, (13.20) (without the factors $C(\alpha\beta\gamma)$) can be written as

$$\left.\begin{array}{l} \varepsilon\,(P[U_{\alpha_1}(x_{r_1}),\ U_{\alpha'_1}(x_{r'_1})])_0\,(P[U_{\alpha_2}(x_{r_2}),\ U_{\alpha'_2}(x_{r'_2})])_0 \dots \\ (\varepsilon\,(s_1, s'_1)\ P[\psi_{\beta_1}(x_{s_1}),\ \bar{\psi}_{\beta'_1}(x_{s'_1})])_0 \dots \qquad \times \text{[external operators]} \end{array}\right\} \tag{13.21}$$

where $\varepsilon(a, b)$ is

$$\varepsilon(a, b) \equiv \varepsilon(x_a - x_b) = \left\{\begin{array}{ll} +1 & \text{for} \quad \sigma(x_a) > \sigma(x_b) \\ -1 & \text{for} \quad \sigma(x_a) < \sigma(x_b). \end{array}\right\} \tag{13.22}$$

The operators in each P-bracket in (13.21) are those of each pair.

As shown by (13.20), for any external operator $\psi(x_\alpha)$, there exists an operator $\bar{\psi}(x_\alpha)$. When $\bar{\psi}(x_\alpha)$ is an internal operator, there is an operator $\psi(x_{\alpha'})$ which is in the same P-bracket in (13.21). Then, we can always find $\bar{\psi}(x_{\alpha'})$, and so on. By repeating this process along the path $x_\alpha \to x_{\alpha'} \to \dots \to x_\beta$, we find an external operator $\bar{\psi}(x_\beta)$. We then shall arrange the external operators $\bar{\psi}(x_\beta)$ and $\psi(x_{\alpha'})$ in the order $\bar{\psi}(x_\beta)\,\psi(x_\alpha)$ in (13.21), by taking account of the sign constant ε. On the other hand, since the external operators of U_α satisfy commutation relations of the $(-)$-type, their order is immaterial.

The sign constant ε can be expressed as [1])

$$\varepsilon = (-1)^a, \tag{13.23}$$

where a is the number of interchanges of field operators $\psi, \bar{\psi}$ when we change them from the order in (13.20) into the order (cf. (13.21), (13.22)) [2])

$$U_{\alpha_1}(x_{\tau_1})\, U_{\alpha'_1}(x_{\tau'_1}) \ldots \psi_{\beta_1}(x_{s_1})\, \bar{\psi}_{\beta'_1}(x_{s'_1}) \ldots \text{ [external operators]}.$$

Here the orders of external operators are the same as those in (13.21).

The above calculation can be performed in a more intuitive way by means of the Feynman diagrams (FEYNMAN [1949]). As shown in Fig. 13.1, we fix m points $x_1, \ldots, x_m$ between two lines corresponding to the initial and final surfaces σ' and σ, and associate with each line connecting two points a pair of internal operators $(U(x_i), U(x_j))$ or $(\bar{\psi}(x_i), \psi(x_j))$; these lines are called internal lines. Then we associate, with each line connecting a point with σ' or σ, an external annihilation or creation operator; these lines are called external lines. The lines of different types (e.g. solid lines, dotted lines, wavy lines, etc.) are used for different fields. The arrows corresponding to the lines of the half-integer spin point from $\bar{\psi}$ to ψ. By means of the Feynman diagrams the calculation of (13.14) can be carried out according to the following prescription:

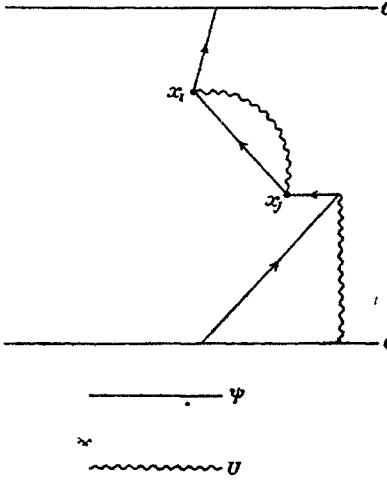

Fig. 13.1

(I) write down all possible Feynman diagrams corresponding to the given process, except the diagrams containing isolated graphs without external lines. These must be omitted because we must calculate the transition probability relative to that

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[1]) Constant ε is connected with the sign constant ε' defined by DYSON [1949] by $\varepsilon = (-1)^b \varepsilon'$, where b is the number of internal lines.

[2]) Take into account the relation

$$\varepsilon(x_s, x_{s'})\, P[\psi_\beta(x_s), \bar{\psi}_{\beta'}(x_{s'})] = \begin{cases} \psi_\beta(x_s)\, \bar{\psi}_{\beta'}(x_{s'}) & \text{for } \sigma(x_s) > \sigma(x_{s'}) \\ -\bar{\psi}_{\beta'}(x_{s'})\, \psi_\beta(x_s) & \text{for } \sigma(x_s) < \sigma(x_{s'}). \end{cases}$$

of the transition between vacuum states, which is given by all diagrams without external lines.

(II) construct the products, corresponding to internal lines $(x_i\ x_j)$, of the terms

$$\left.\begin{array}{ll}(P[U_\alpha(x_i),\ U_{\alpha'}(x_j)])_0 & \text{for the lines of the field } U\\ (\varepsilon(i,j)\ P[\psi_\alpha(x_i),\ \bar\psi_{\alpha'}(x_j)])_0 & \text{for the arrows of the field } \psi.\end{array}\right\} \quad (13.24a)$$

When the interaction Hamiltonian contains derivation operators $D_\alpha^{(i)}$ operating on the field operators $U_\alpha(x_i)$, $\psi_\alpha(x_i)$ or $\bar\psi_\alpha(x_i)$, the rule (13.24a) must be changed as follows:

$$\left.\begin{array}{ll}(P[D_\alpha^{(i)}U_\alpha(x_i),\ D_{\alpha'}^{(j)}U_{\alpha'}(x_j)])_0 & \text{for the internal lines of the field } U\\ (\varepsilon(i,j)\ P[D_\alpha^{(i)}\psi_\alpha(x_i),\ D_{\alpha'}^{(j)}\bar\psi_{\alpha'}(x_j)])_0 & \text{for the arrow of the field } \psi\end{array}\right\} \quad (13.24b)$$

(III) construct the products of external operators U_α, $\bar\psi_\alpha$, ψ_α (more generally, $D_\alpha U_\alpha$, $D_\alpha\psi_\alpha$, $D_\alpha\bar\psi_\alpha$), corresponding to the external lines.

(IV) arrange these factors in the order of the points on the lines.

(V) multiply by the sign constant ε and the factor $(-i)^m/m!$ (cf. (13.14)).

(VI) sum the contributions of all different diagrams obtained by all possible permutations of points $(x_1, \ldots, x_m)$ within a diagram. The diagrams which are topologically identical are called equivalent (WICK [1950]). For example, in Fig. 13.2, the diagrams (a) and (b) are equivalent but the diagrams (c) and (d) are not equivalent.

This rule (VI) can be expressed as follows: When there are

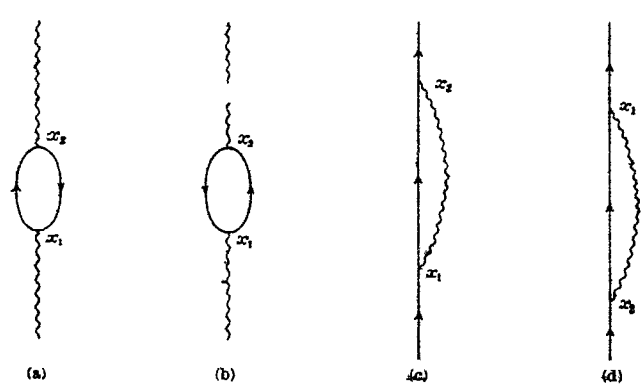

Fig. 13.2

g diagrams equivalent to a given one (e.g. $g=2$ for (2) in Fig. 13.2), multiply the contribution of this diagram [1]) by $m!/g$.

(VII) sum the contributions of all possible Feynman diagrams obtained without taking into account the permutation of points $(x_1, \ldots, x_m)$ and perform the integrations with respect to $(dx_1, \ldots, dx_m)$. A set of connected ψ-lines (with two external lines) is called an open polygon, and a diagram with no external ψ-lines is called a closed loop. The points $x_1, \ldots, x_m$ are called the vertices.

It can be proved that

$$\varepsilon = (-1)^a \tag{13.25}$$

where a is the numbers of the closed loops of the fields (cf. (13.23)).

According to the definition of the P-symbol, we have

$$\left.\begin{aligned}(P[U_\alpha(x_1),\, U_\beta(x_2)])_0 &= \tfrac{1}{2}\{1+\varepsilon(1,2)\}\,(U_\alpha(x_1)\,U_\beta(x_2))_0 \\ &\quad + \tfrac{1}{2}\{1-\varepsilon(1,2)\}\,(U_\beta(x_2)\,U_\alpha(x_1))_0 .\end{aligned}\right\} \tag{13.26a}$$

Then (8.14) and (9.42) give

$$\left.\begin{aligned}&(P[U_\alpha(x_1),\, U_\beta(x_2)])_0 \\ &= \tfrac{1}{2}\{d_{\alpha\beta}(\partial^{(1)})\,\Delta^{(1)}(x_1-x_2) + i\,\varepsilon(1,2)\,d_{\alpha\beta}(\partial^{(1)})\,\Delta(x_1-x_2)\} \\ &= \tfrac{1}{2}\,\bar{d}_{\alpha\beta}(\partial^{(1)})\,\Delta_F(x_1-x_2)\end{aligned}\right\} \tag{13.26b}$$

with

$$\partial^{(1)} \equiv \partial/\partial x_1 .$$

In a similar way we have

$$(\varepsilon(1,2)\,P[\psi_\alpha(x_1),\, \bar{\psi}_\beta(x_2)])_0 = \tfrac{1}{2}\,\bar{d}_{\alpha\beta}(\partial^{(1)})\,\Delta_F(x_1-x_2). \tag{13.27}$$

In (13.26b) and (13.27), $\bar{d}_{\alpha\beta}(\partial)$ mean the derivation operators $d_{\alpha\beta}(\partial)$ acting on $\Delta^{(1)}$ and Δ but not on $\varepsilon(1,2)$ in Δ_F. The Δ_F's in (13.26b) and (13.27) are Δ_F-functions of U and ψ-fields respectively.

§ 5. Method of P^*-Symbol

The calculation of $S[\sigma]$ can be simplified using the fundamental condition (10.29) on the interaction Hamiltonian.

For simplicity we shall consider the second order approximation in the coupling constant g. Equivalence to this approximation will be denoted by $\approx$. Equations (10.42), (13.14) and (13.24b) give

$$\left.\begin{aligned}S[\sigma,\sigma'] \approx &-\tfrac{1}{2}\int_{\sigma'}^{\sigma} d^4x_1 \int_{\sigma'}^{\sigma} d^4x_2\, P[L'(x_1),\, L'(x_2)] \\ &- i\int_{\sigma'}^{\sigma} d^4x\, W_{\mu\nu}(x)\, n_\mu\, n_\nu\end{aligned}\right\} \tag{13.28}$$

[1]) For the determination of g, see COESTER [1951].

$$\left.\begin{aligned} &= -\tfrac{1}{2}\int_{\sigma'}^{\sigma} d^4x_1 \int_{\sigma'}^{\sigma} d^4x_2\, P[D_a^{(1)} Q_\alpha(x_1),\, D_b^{(2)} Q_\beta(x_2)]\, j_{\alpha:a}(x_1)\, j_{\beta:b}(x_2) \\ &\quad - i\int_{\sigma'}^{\sigma} d^4x\, W_{\mu\nu}(x)\, n_\mu\, n_\nu \end{aligned}\right\} \tag{13.29}$$

where $D_a^{(1)}$ and $D_a^{(2)}$ are the derivation operators, defined by (10.22a), with respect to x_1 and x_2 respectively. In this equation, a matrix element of $S[\sigma]$ corresponding to a process in which $j_{\alpha:a}$ and $j_{\beta:b}$ play the role of external operators is considered. Then, the vacuum expectation value, must be taken for the P-bracket in the first term. From (8.40b) and (13.26),

$$\left.\begin{aligned} &(P[D_a^{(1)} Q_\alpha(x_1),\, D_b^{(2)} Q_\beta(x_2)])_0 = \\ &i\, D_a^{(1)}\, D_b^{(2)}\, G_{F\alpha\beta}(x_1-x_2) + \frac{i}{2}[\varepsilon(1,2),\, d_{\alpha\beta}(\partial^{(1)})\, D_a^{(1)}\, D_b^{(2)}]\, \Delta(x_1-x_2). \end{aligned}\right\} \tag{13.30}$$

On the other hand, selecting the g^2-term in both sides of (10.29) and taking into account (10.30), we have

$$\left.\begin{aligned} &[Q_\alpha(x), W_{\mu\nu}(x')]\, n_\mu(x')\, n_\nu(x') \\ &\qquad = \frac{i}{2} d_{\alpha\beta}(\partial)\, D'_a\, \Delta(x-x')\, [S^{-1}[\sigma]\, \mathbf{j}_{\beta:a}(x')\, S[\sigma] - j_{\beta\cdot a}(x')] \\ &\approx \frac{i}{2} d_{\alpha\beta}(\partial)\, D'_a\, \Delta(x-x')\, \frac{\partial j_{\beta.a}(x')}{\partial (D'_b Q_\varrho(x'))} \int d^4x''\, [D'_b\, D''_c\, d_{\varrho\gamma}(\partial'), \\ &\qquad\qquad \varepsilon(x'-x'')]\, \Delta(x'-x'')\, j_{\gamma:c}(x''). \end{aligned}\right\} \tag{13.31}$$

This is derived from the consideration that the g^2-term comes from the difference between $D_b \mathbf{Q}_\varrho$ (in $\mathbf{j}_{\beta:a}$) and $S^{-1}[\sigma] D_b Q_\varrho(x) S[\sigma]$.

From (13.31) we can derive the relation

$$\left.\begin{aligned} &\int d^4x'\, W_{\mu\nu}(x')\, n_\mu(x')\, n_\nu(x') \\ &= -\tfrac{1}{4}\iint d^4x'\, d^4x''\, j_{\alpha:a}(x')\, j_{\beta:b}(x'')\, [\varepsilon(x'-x''),\, D'_a D''_b\, d_{\alpha\beta}(\partial')]\, \Delta(x'-x'') \end{aligned}\right\} \tag{13.32}$$

by means of the relation (cf. (10.22b))

$$\frac{\partial j_{\beta:a}}{\partial (D_b Q_\varrho)} = -\frac{\partial^2 L'}{\partial (D_a Q_\beta)\cdot \partial (D_b Q_\varrho)} = \frac{\partial j_{\varrho:b}}{\partial (D_a Q_\beta)}. \tag{13.33}$$

Substituting (13.30) and (13.32) into (13.29) we obtain (UMEZAWA and TAKAHASHI [1953])

$$S[\sigma, \sigma'] \approx -\frac{i}{2}\int_{\sigma'}^{\sigma} d^4x_1 \int_{\sigma'}^{\sigma} d^4x_2\, j_{\alpha:a}(x_1)\, j_{\beta:b}(x_2)\, D_a^{(1)}\, D_b^{(2)}\, G_{F\alpha\beta}(x_1-x_2). \tag{13.34}$$

From this $S[\infty]$ can be obtained by substituting $\sigma = \infty$ and $\sigma' = -\infty$ into (13.34). Comparing (13.34) with (13.29) we have

$$S[\infty] = \sum_{m=0}^{\infty} i^m \frac{1}{m!} \int_{-\infty}^{\infty} d^4x_1 \ldots \int_{-\infty}^{\infty} d^4x_m\, P^*[L'(x_1), \ldots, L'(x_m)], \tag{13.35}$$

where the P^*-symbol corresponds to the same rules of calculation, (I), ..., (VII), except for the following modification of (II):

(II)′ construct the products of the Green functions

$$i G_{F\alpha\alpha'}(x_i - x_j)$$

corresponding to the internal lines (x_i, x_j).

More generally, as shown by (13.34), when the interaction contains differential operators (cf. (13.24*b*)), the quantities $P[(D_a^{(i)}Q_\alpha(x_i), D_b^{(j)}Q_{\alpha'}(x_j)]$ appearing in the P-symbol method must be replaced in the P^*-symbol method by $iD_a^{(i)}D_b^{(j)}G_{F\alpha\alpha'}(x_i - x_j)$.

An essential advantage of (13.35) is that $S[\sigma]$ can be calculated directly by using the interaction Lagrangian. This suggests the possibility of extending (13.35) to cases where there is no canonical theory and therefore no interaction Hamiltonian. It must be noted that (13.34) does not depend explicitly on the form of any surface σ_i between σ and σ' $(\sigma > \sigma_i > \sigma')$. This is reasonable (KOBA [1950]) because the interaction Hamiltonian satisfies the integrability condition.

The rule II′ of the P^*-symbol method is just what we anticipated in § 2 of Ch. VIII; the propagation of influences between two points are described by $G_F(x - x')$ in the formulation of the causal theory in which the initial and final states are treated symmetrically. Indeed, Stueckelberg (STUECKELBERG and GREEN [1951]) derived a theory of the S-matrix, equivalent to the Dyson's theory by using the causality requirement. There remains some arbitrariness in the S-matrix determined by the requirement of the causality, because this requirement determines only the propagation of effects between two points and does not determine point functions. Therefore, we can expect that it may be possible to introduce arbitrary local interactions so as to eliminate the infinite effects of the proper fields. Stueckelberg has shown that this method corresponds to the renormalisation theory which will be discussed in the next Chapter.

§ 6. Examples

EXAMPLE 1. DISPLACED POLE AND INTERFERENCE EFFECT

Relation (8.34) shows that the k_μ satisfying $k_\mu k_\mu + \varkappa^2 = 0$ contributes a term $\delta(k_\mu k_\mu + \varkappa^2)$ to the Fourier amplitude of $\Delta_F(x)$. The relation $k_\mu k_\mu + \varkappa^2 = 0$ implies that k_μ is the energy-momentum vector of a free

particle. For example, in the scattering process of a Neutron (N) and a Photon (γ) through intermediate virtual states with a proton (P), an electron (e) and a neutrino (ν) (a Feynman diagram of this process is given in Fig. 13.3), a *free* proton, electron and neutrino can appear in the virtual states on account of the natural decay (cf. § 5 of Ch. IV) of a neutron. The contribution of these free particles to the S-matrix is given by the $\delta(k_\mu k_\mu + \varkappa^2)$-terms in the Δ_F-functions corresponding to the lines P, ν and e. However, these poles of the Δ_F-function give rise to no difficulties because they deviate from the real axis by $i\epsilon$, as shown by (8.35) (cf. Fig. 8.1).

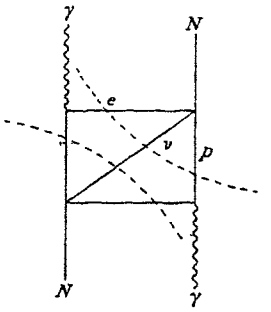

Fig. 13.3

For clarity's sake, we shall consider the simple example of the scattering process (I) of two particles (1) and (2) (with the same mass $\varkappa$, spin 0 and energy-momentum $P_0^{(1)} = P_0^{(2)}$, $\mathbf{P}^{(1)} = -\mathbf{P}^{(2)}$ in the centre of mass system), in which there is a virtual state in which two other particles (3) and (4) with masses $\varkappa'$ and spin 0 exist. If

$$P_0^{(1)} + P_0^{(2)} = 2\sqrt{(\mathbf{k}^2 + \varkappa'^2)} \geqslant 2\varkappa' \tag{13.36}$$

the process (II): (1) + (2) → (3) + (4) (final state) becomes also possible, and the process (I) is disturbed by the process (II). In (13.36) $\mathbf{k}$ is the momentum (in the centre of mass system) of the particle (3) in the final state. The value of $(P_0^{(1)} + P_0^{(2)})$ for which the *equality* holds in the right hand side of (13.36) is called the *interference threshold* of the two competing processes (EDEN [1952]).

In the transition matrix for the process (I) the Δ_F-functions of the particles (3) and (4) in the virtual state provides the integral

$$\left.\begin{aligned} &\lim_{\epsilon \to 0} F(P, \epsilon) \\ &\quad \equiv \lim_{\epsilon \to 0} \int d^4k \, \frac{1}{(k_\mu k_\mu + \varkappa'^2 - i\epsilon)\,\{(P_\varrho^{(1)} + P_\varrho^{(2)} - k_\varrho, P_\varrho^{(1)} + P_\varrho^{(2)} - k_\varrho) + \varkappa'^2 - i\epsilon\}} \end{aligned}\right\} \tag{13.37}$$

because the energy-momentum vectors of the particles (3) and (4) are k_μ and $P_\mu^{(1)} + P_\mu^{(2)} - k_\mu$ respectively.

The integrand of (13.37) has the four poles

$$k_0 = \left\{\begin{aligned} &\pm(\sqrt{(\mathbf{k}^2 + \varkappa'^2)} - i\epsilon) \equiv k_0^a(\pm) \\ &P_0^{(1)} + P_0^{(2)} \pm (\sqrt{(\mathbf{k}^2 + \varkappa'^2)} - i\epsilon) \equiv k_0^b(\pm). \end{aligned}\right\} \tag{13.38}$$

While $k_0^a(\pm)$ is symmetric with respect to the origin, $k_0^b(\pm)$ appears only on the same side of the imaginary axis when the condition

$$P_0^{(1)} + P_0^{(2)} \geqslant \sqrt{(\mathbf{k}^2 + \varkappa'^2)} \tag{13.39}$$

is satisfied. In this case we shall call $k_0^b(\pm)$ displaced poles (cf. Fig. 13.4).

Moreover, when $\mathbf{k}$ satisfies (13.36), $k_0^b(-)$ and $k_0^a(+)$ appear symmetrically with respect to the real axis. Therefore, it is not possible to deform the contour without cutting these singularities. This is a feature of the interference threshold.

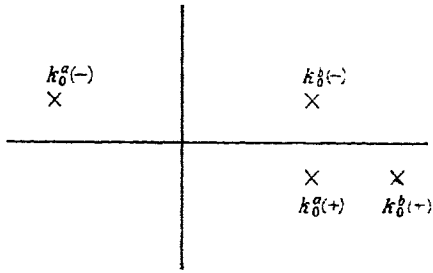

Fig. 13.4

We shall define the value of $F(P, \in)$ at $\in = 0$ by using the analytic continuation of $F(P, \in)$ for $\ni > 0$. Then we can expect that the interference effect due to the process (II) increases continously from zero, since it is zero at energies lower than the interference threshold.

In fact, the d^4k-integration of (13.37) gives (EDEN [1952])

$$A(P) + B(P)\{(P_0^{(1)} + P_0^{(2)})^2 - 4\varkappa'^2 - 4i\in\}^{1/2} \times \log\left\{\frac{(P_0^{(1)} + P_0^{(2)}) + \sqrt{(P_0^{(1)} + P_0^{(2)})^2 - 4\varkappa'^2 - 4i\in}}{(P_0^{(1)} + P_0^{(2)}) - \sqrt{(P_0^{(1)} + P_0^{(2)})^2 - 4\varkappa'^2 - 4i\in}}\right\},$$

where $A(P)$ and $B(P)$ are analytic functions of P. The second term corresponds to the interference effect due to the process (II), since it comes from the effect of the poles $k_0^b(\pm)$. We see that this term is zero at the interference threshold. The second term also shows that at the interference threshold the S-matrix integrals have branching points.

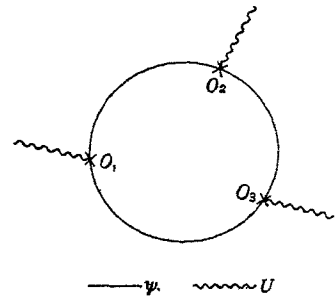

Fig. 13.5

EXAMPLE 2. FURRY'S THEOREM

We shall consider a transition matrix element corresponding to a Feynman diagram consisting of a closed loop of a Dirac field ψ interacting with a field U of integer spin (cf. Fig. 13.5). The interaction Hamiltonians at the

points $x_i (i=1, \ldots, n)$ are assumed to be $\bar{\psi} O_i \psi U$ respectively [1]), where O_i is a product of the matrices γ_μ. Then, the transition matrix element contains the factor [2])

$$M = S_P(O_1 S_F(x_1-x_2)\, O_2 S_F(x_2-x_3)\, O_3 \ldots O_n S_F(x_n-x_1)) \tag{13.40}$$

$$\left.\begin{aligned} S_F(x-x') &\equiv 2\,(\varepsilon(x-x')\, P[\psi(x), \bar{\psi}(x')])_0 \\ &= -(\gamma_\mu \partial_\mu - \varkappa)\, \Delta_F(x-x'). \end{aligned}\right\} \tag{13.41}$$

Since the spur of any product of an odd number of the matrices γ_μ is zero, M is invariant under the substitution $\gamma_\mu \to -\gamma_\mu$. Under this substitution the matrix O_i transforms into

$$\bar{O}_i = (-1)^{a_i} O_i \tag{13.42}$$

where a_i is the number of the matrices γ_μ in O_i. Since (13.41) shows that $S_F(x_i-x_j) \to S_F(x_j-x_i)$ under the substitution $\gamma_\mu \to -\gamma_\mu$, we have

$$M = (-1)^A S_P(O_1 S_F(x_2-x_1)\, O_2 S_F(x_3-x_2)\, O_3 \ldots O_n S_F(x_1-x_n)), \tag{13.43a}$$

where

$$A \equiv \sum_{i=1}^{n} a_i. \tag{13.43b}$$

Using the formula $S_P(\gamma_i \gamma \ldots \gamma_k) = S_P(\gamma_k \ldots \gamma_i \gamma_i)$ we have

$$M = (-1)^B S_P(S_F(x_1-x_n)\, O_n \ldots O_3 S_F(x_3-x_2)\, O_2 S_F(x_2-x_1)\, O_1). \tag{13.44}$$

Here B is defined by

$$B \equiv A + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + b_i), \tag{13.45a}$$

with b_i defined by

$$O_i' = (-1)^{b_i} O_i, \tag{13.45b}$$

where O_i' is a product of the matrices γ_μ in the opposite order to that in which they appear in O_i.

Equation (13.44) shows that $(-1)^B M$ is equivalent to the transition matrix M' corresponding to the Feynman diagram obtained from

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[1]) The following results are also valid even when the interaction Hamiltonians contain derivatives operating on U.

[2]) The present definition of $S_F(x)$ is equivalent to that $-S_F(-x)$ used by DYSON [1952].

Fig. 13.5 by rearranging the points $x_1, \ldots, x_n$ in the opposite order. Thus, we have

$$M + M' = M'((-1)^B + 1) = 0 \text{ for odd } B. \tag{13.46}$$

Since a positon line is obtained from a negaton line simply by changing the direction of the arrow, the last equation shows that the contribution of the negatons and the positons cancel each other when B is odd.

For example, any closed loop made up of an odd number of vertices of the electromagnetic interaction $\bar{\psi}\gamma_\mu A_\mu \psi$ ($a_i = 1$, $b_i = 0$) makes no contribution to the S-matrix. This fact was first established by FURRY [1937].

Using (13.46), we can generalise Furry's theorem as follows: *Any closed loop for which B is odd makes no contribution to the S-matrix.*

When O_i are members of $(1, \gamma_\mu, \gamma_5, \gamma'_\mu\gamma'_\nu - \gamma'_\nu\gamma'_\mu, \ldots)$, we have, for every O_i,

$$a_i + b_i = \left\{ \begin{array}{l} \text{even for } 1, \gamma_5, \gamma_\mu\gamma_5 \\ \text{odd for } \gamma_\mu, \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu, \gamma'_5(\gamma'_\mu\gamma'_\nu - \gamma'_\nu\gamma'_\mu) \end{array} \right\} \tag{13.47}$$

(cf. (3.5)). Therefore B is equal to the number of vertices of $O_i = \gamma_\mu$, $(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ and $\gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$.

As an example of the application of this theorem we shall consider the determination of the spin of the π^0-meson (SAKATA and TANIKAWA [1940]). If the π^0-meson were a vector field interacting with the proton by the vector coupling, the above theorem shows that it would decay not into two photons but three, because the Feynman diagram of the process $\pi^0 \to 2\gamma$ is given by Fig. 13.5 with $O_1 = \gamma_\mu$, $O_2 = \gamma_\mu$, $O_3 = \gamma_\mu$. However, experiments show that the π^0-meson decays into two photons with a very short life time. Moreover it can be proved by a more general discussion (YANG [1950]) that if the π^0-meson had spin 1, it could not decay into two photons. On the other hand there is no such selection rule for the π^0-meson of spin 0 decaying into two photons.

EXAMPLE 3. RELATIVE SELECTION RULES

We have theorems analogous to Furry's theorem for interactions between nucleon ψ and integer spin fields U (FUKUDA and MIYAMOTO [1950], NISHIJIMA [1951], PAIS and JOST [1952]). This theorem

depends on the charge symmetry [1]) of the theory (cf. Example 10 of Ch. VII). However, charge symmetry cannot be valid when the electromagnetic interaction is taken into account, because the electromagnetic field interacts with the proton but not with the neutron. Therefore, the selection rules given in the present example can be valid only when the electromagnetic interaction and the mass difference between proton and neutron can be neglected. They are called relative selection rules.

We assume, here, that the Lagrangian is invariant under the T-transformation (cf. Example 10 of Ch. VII), i.e., is charge symmetric. On the other hand, under the charge conjugation transformation of the Dirac field $\psi \to \psi' = C\bar{\psi}^T$ (cf. (3.40), (3.38)) we have

$$\left.\begin{aligned} \bar{\psi}' \Omega \psi' &= (C^{-1}\psi)^T \Omega C \bar{\psi}^T = -\psi^T C^{-1} \Omega C \bar{\psi}^T \\ &= \varepsilon \bar{\psi} \Omega \psi \end{aligned}\right\} \tag{13.48a}$$

with

$$\varepsilon = \begin{cases} -1 \text{ for } \Omega = \gamma_\mu,\ (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu),\ \gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu),\ \tau_2 \\ +1 \text{ for } \Omega = 1,\ \gamma_5,\ \gamma_\mu\gamma_5,\ \tau_1,\ \tau_3 . \end{cases}$$

The charge conjugation transformation $U \to U'$ of the Bose field is defined by [2])

$$U' = U^*,\ U'^* = U \tag{13.48b}$$

or

$$U'^{(1)} = U^{(1)},\ U'^{(2)} = -U^{(2)}, \tag{13.48c}$$

because, under this transformation, the electric current changes its sign (cf. (7.46)).

Then, the interaction Lagrangian $g\bar{\psi}\Omega\tau_{(i)}\psi D(\partial)U_{(i)}$ ($i = 1, 2, 3, 4$; $D(\partial)$ denotes differential operators) is invariant under the charge conjugation transformation, when the signs of the coupling constants referring to the interactions with $\Omega = \gamma_\mu,\ \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu,\ \gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ are changed. We call latter coupling constants "odd coupling constants", and the charge conjugation transformation with change of the sign of odd coupling constants the C-transformation.

Then, under the CT-transformation (the product of the C and the

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[1]) Here, we omit the tensor suffices for Boson fields. Charge symmetry should not be confused with charge independence.

[2]) $U^{(1)}$ and $U^{(2)}$ are defined by the same relations as (7.47).

T transformations) $U^{(k)}$ must transform according to

$$U^{(k)} \to \varepsilon_k U^{(k)}, \tag{13.49a}$$

$$\varepsilon_k \equiv \left\{ \begin{array}{l} -1 \text{ for } k=3 \\ 1 \text{ otherwise,} \end{array} \right\} \tag{13.49b}$$

(see (7.151b)).

We consider a term S of the transition matrix corresponding to a Feynman diagram for a given process in which only particles of the integer spin fields $U^{(k)}$ exist in the initial and final states. Then S is a product of coupling constants, external operators of the field quantities $U^{(k)}$ and G_F-functions [1]) corresponding to internal operators. It follows that, under the CT-transformation, $S \to S'$ given by

$$S' = (-1)^{n(\tau_3)+n(v)+n(t)+n(pt)} S.$$

Here $n(\tau_3)$, $n(v)$, $n(t)$ and $n(pt)$ are the total numbers of vertices with τ_3, γ_μ, $(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ and $\gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ (in the Feynman diagram) respectively. The numbers $n(v)$, $n(t)$ and $n(pt)$ appear because of the change of sign of odd coupling constants, while $n(\tau_3)$ is introduced by the transformation (13.49a).

S and S' correspond to the same transition process because, as shown by (13.49a), $U^{(1)}$ and $U^{(2)}$ are invariant and therefore, the charge states of the Bose fields do not change under the CT-transformation.

Since the Lagrangian is invariant under the CT-transformation in charge symmetric theory, the S-matrix must be also invariant. Thus we obtain the selection rule

$$S=0 \text{ for } n(\tau_3)+n(v)+n(t)+n(pt)=\text{odd}. \tag{13.50}$$

As an example we shall consider the process $B^+ \to \pi^+ + \pi^0$ where B^+ is a positively charged scalar particle with scalar interaction and π^+ and π^0 are pseudoscalar π-mesons with pseudoscalar interaction; when π^0 is described by $U^{(3)}$ (not $U^{(4)}$ in (7.151b)) according to (13.50), this process is forbidden.

If all the particles in the initial and final states are described by real wave functions $U^{(3)}$, $U^{(4)}$ of integer spin, the S-matrix element

[1]) Particles and their antiparticles have the same G_F-functions because they satisfy the same field equations in the interaction representation. Proton and neutron also have the same G_F-function, i.e. S_F in the approximation neglecting their mass difference.

S^0 corresponds to the same process after the T-transformation. When the Lagrangian is invariant under the T-transformation, the S-matrix element must be also invariant, and we conclude (cf. (7.151b)) that

$$S^0 = 0 \quad \text{for} \quad n(\tau_3) \text{ odd.} \tag{13.51}$$

In this case, using (13.50), we also find

$$S^0 = 0 \text{ for } \left\{ \begin{array}{l} n(\tau_3) = \text{even and} \\ n(v) + n(t) + n(pt) = \text{odd.} \end{array} \right\} \tag{13.52}$$

EXAMPLE 4. BREMSSTRAHLUNG AND THE INFRA-RED CATASTROPHE

We shall consider the bremsstrahlung of a photon by an electron scattered by an interaction Hamiltonian, i.e. $-ie\bar{\psi}\gamma_\mu\psi A^e_\mu$, where A^e_μ is the vector potential of an external electromagnetic field. The energy-momentum vectors of the electron in the initial and final states are denoted by p_μ, q_μ respectively; the photon is denoted by k_μ (cf. Fig. 13.6).

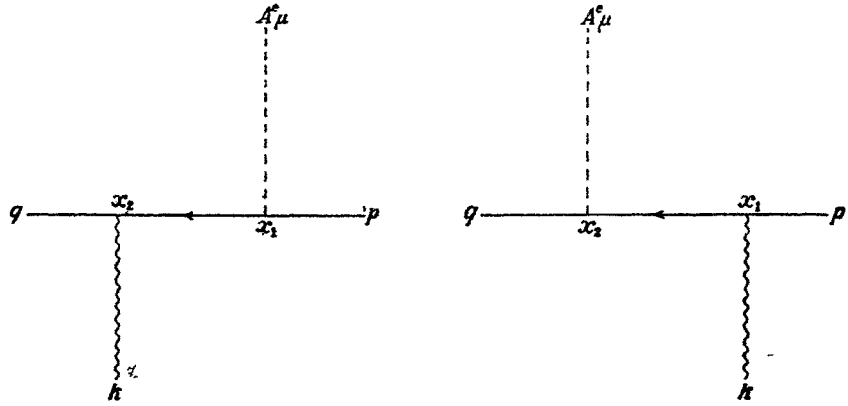

Fig. 13.6

The term $S^{(2)}$ of the S-matrix which corresponds to this process in e^2-approximation, is

$$\left.\begin{aligned} S^{(2)} = {} & \frac{e^2}{2}\int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2\, \bar{\psi}(x_2)\,\gamma_\mu S_F(x_2-x_1)\,\gamma_\nu A^e_\nu(x_1)\,\psi(x_1)\,A_\mu(x_2) \\ & + \frac{e^2}{2}\int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2\, \bar{\psi}(x_2)\,\gamma_\nu A^e_\nu(x_2)\,S_F(x_2-x_1)\,\gamma_\mu\psi(x_1)\,A_\mu(x_1), \end{aligned}\right\} \tag{13.53}$$

because $\varepsilon = 1$ and $g = 1$.

In this and the following examples, S_F and Δ_F correspond to the electron and the photon respectively. If, as in (13.5*b*), we put

$$(q_\mu, k_\mu|S^{(2)}|p_\mu) = 2\pi M \delta(l_0 + p_0 - q_0 - k_0),$$

we have from (9.50) and (9.78)

$$\left.\begin{aligned} M = \frac{ie^2}{\sqrt{2|\mathbf{k}|V^3}} a_r^*(q)\, \gamma_4 \Big\{ (\boldsymbol{\gamma}\cdot\mathbf{e}) \frac{[i(k_\mu + q_\mu, \gamma_\mu) - \varkappa]}{(k_\mu + q_\mu \cdot k_\mu + q_\mu) + \varkappa^2 - i\epsilon} \gamma_\nu A_\nu^e(q + k - p) \\ + A_\nu^e(q + k - p)\, \gamma_\nu \frac{[-i(k_\mu - p_\mu, \gamma_\mu) - \varkappa]}{(p_\mu - k_\mu, p_\mu - k_\mu) + \varkappa^2 - i\epsilon} (\boldsymbol{\gamma}\cdot\mathbf{e}) \Big\} a_s(p). \end{aligned}\right\} \quad (13.54)$$

Here $A_\mu^e(l)$ is a Fourier amplitude of $A_\mu^e(x)$ defined by

$$A_\mu^e(x) = \frac{1}{(2\pi)^3} \sum_{l_0} \int d^3l\, A_\mu^e(l) \exp(i\, l_\mu\, x_\mu),$$

and $\mathbf{e}$ is a unit vector in the direction of the polarisation of the photon. The symbol $\boldsymbol{\gamma}$ denotes the three dimensional vector $(\gamma_1, \gamma_2, \gamma_3)$. The transition probability dw/dt per unit time is given by (13.6) and is

$$\frac{dw}{dt} = 2\pi |M|^2 d\varrho_f, \quad (13.55)$$

where the final state density $d\varrho_f$ is

$$d\varrho_f = \frac{1}{(2\pi)^6} |\mathbf{q}|\, |\mathbf{k}|^2 E_q\, d\Omega_q\, d\Omega_k\, V^2. \quad (13.56)$$

Here $E_q\ (= (\mathbf{q}^2 + \varkappa^2)^{1/2})$ is the energy of the electron in the final state, and $d\Omega_q$ and $d\Omega_k$ are the solid angles in which the electron and the photon are ejected in the final state. We shall define the "cross-section" $d\phi$ to be

$$d\phi \equiv \frac{dw}{dt} \frac{V}{v}, \quad (13.57)$$

where v is the velocity of the incident electron, namely

$$v = |\mathbf{p}|/E_p, \quad E_p \equiv (\mathbf{p}^2 + \varkappa^2)^{1/2}.$$

The last equation shows that $d\phi$ does not depend on the volume V of the world and its dimensions are of the second power of length. Speaking intuitively, $d\phi$ is a transition probability when one electron enters in unit time through a unit surface, because the factor $d\phi/(dw/dt)$ can be interpreted as a change of the normalisation of the wave function of the incoming electron with this result that the electron is not contained in a volume V but in a volume v, so that one electron crosses a unit surface in unit time.

Integrating (13.57) with respect to $d\Omega_k$, we obtain the cross-section $\phi_k dk$ for the radiation of a photon of the energy k. The energy loss $(-dE_p/dx)$ of the electron per unit range by bremsstrahlung in the coulomb potentials of the atoms in the matter traversed can be calculated from

$$N \int_0^{E_p - \varkappa} k \, \phi_k \, dk,$$

where we consider an electron moving in the x-direction. The upper limit $E_p - \varkappa$ of the integration is the maximum energy of a radiated photon because of the energy conservation law. N is the number of atoms in unit volume. For electrons of extremely high energy moving through matter constituted of atoms of atomic number Z, we have (HEITLER [1935])

$$\phi_k = \frac{2Z^2}{137}\left(\frac{e^2}{\varkappa}\right)^2 \frac{E_q}{k}\left\{\frac{E_p^2+E_q^2}{E_p E_q} - \frac{2}{3}\right\}\left\{2\log\frac{2E_p E_q}{\varkappa k} - 1\right\}$$

and

$$-\frac{dE_p}{dx} = 4NE_p\left(\log\frac{2E_p}{\varkappa} - \frac{1}{3}\right)\frac{Z^2}{137}\left(\frac{e^2}{\varkappa}\right)^2.$$

This theoretical calculation of bremsstrahlung has successfully predicted experimental results for high energy electrons. In particular the cascade shower phenomena in cosmic rays, which at first seemed to indicate a failure of quantum electrodynamics in the high energy region, was theoretically explained as a repetition of the production of photons by bremsstrahlung and the transmutations of photons into electrons (i.e. the cascade theory) (CARLSON and OPPENHEIMER [1936], BHABHA and HEITLER [1936]). Nevertheless, we have seen in the preceding chapters that quantum electrodynamics has the serious defects concerning the appearance of infinities. We shall set out on an analysis of these difficulties by considering the infra-red catastrophy.

When the energy k of a photon is infinitesimally small, (13.54) can be written as

$$M \approx -\frac{e^2}{\sqrt{2|\mathbf{k}|V^3}}\,(a_\tau^*(q)\,\gamma_4\gamma_\nu A_\nu^\varepsilon(q+k-p)\,a_\sigma(p))\left(\frac{\mathbf{q}}{q_\mu k_\mu} - \frac{\mathbf{p}}{p_\mu k_\mu}, \mathbf{e}\right). \quad (13.58a)$$

This equation is obtained by using the formula (cf. (3.33), (3.1))

$$a_\tau^*(q)\gamma_4(iq_\mu\gamma_\mu + \varkappa) = 0$$
$$(ip_\mu\gamma_\mu + \varkappa)\,a_\sigma(p) = 0$$
$$q_\mu q_\mu = p_\mu p_\mu = -\varkappa^2,$$

and the commutation relations (3.2) for the γ-matrices. Substituting (13.58a) into (13.55) and integrating with respect to dk, we obtain a factor $\int dk/k$ in the transition probability. This has the strange consequence that the transition probability has a logarithmic infinity for infinitesimally small k. Such is called the infra-red catastrophe.

However, it would be premature to conclude that the infra-red catastrophe was a difficulty inherent in quantum field theory, because elastic scattering of the electron can compete with the bremsstrahlung of a photon of zero energy.

The Feynman diagrams of the elastic scattering in the e^2-approximation are (a), (b), (c), (d) of Fig. 13.7. The virtual state denoted by the intersected line in (b) has the same energy as the initial state when $k=0$. Therefore, this elastic scattering competes with the bremsstrahlung of a zero energy photon, which is also observed as the elastic scattering. Denoting the matrix-element of the S-matrix, which corresponds to the Feynman diagrams (a), (b), (c) and (d) respectively, by M_a, M_b, M_c and M_d respectively, we can write the transition probability dw/dt of the elastic scattering as

$$\frac{dw}{dt} = 2\pi\{|M_a+M_b+M_c+M_d|^2+|M_1+M_2|^2\}. \qquad (13.59)$$

Here M_1 and M_2 are the contributions due to the first and second diagrams in Fig. 13.6, respectively:—

$$\lim_{k\to 0} M = M_1+M_2.$$

In other words,

$$M_1 = -\lim_{k\to 0} \frac{e^2}{\sqrt{2|\mathbf{k}|V^3}} (a_r^*(q)\,\gamma_4\,\gamma_\nu\,A_\nu^s(q+k-p)\,a_s(p))\,\frac{(\mathbf{q}\cdot\mathbf{e})}{q_\mu k_\mu}, \quad (13.58b)$$

$$M_2 = \lim_{k\to 0} \frac{e^2}{\sqrt{2|\mathbf{k}|V^3}} (a_r^*(q)\,\gamma_4\,\gamma_\nu\,A_\nu^s(q+k-p)\,a_s(p))\,\frac{(\mathbf{p}\cdot\mathbf{e})}{p_\mu k_\mu}. \quad (13.58c)$$

As shown by Fig. 13.9, the terms $M_a^* M_b$, $M_a^* M_c$ etc. in (13.59) can be written graphically by connecting the graphs (b) and (a^*), (c) and (a^*) etc. respectively. Symbols (a^*), (b^*) ... mean that the relevant diagrams are mirror images with respect to right and left of the diagrams (a), (b) ..., respectively. As an example (a^*) is shown in Fig. 13.8. Each intersected line in Fig. 13.9 indicates the connecting point of two diagrams in the way referred to under each diagram. It can be shown that all the diagrams of Fig. 13.9 (with the photon line of zero energy) give finite contributions to dw/dt, so that the

infra-red catastrophe disappears. This fact stems from a simple relation (ITO [1951], KINOSHITA [1950], BETHE and OPPENHEIMER [1946]) – we shall consider the $(M_a^* M_b)$ term corresponding to the first diagram of Fig. 13.9 as an example. The chain $(M_1^* M_2)$ in the M_b-part shows that M_b has virtual states of the same energy as the

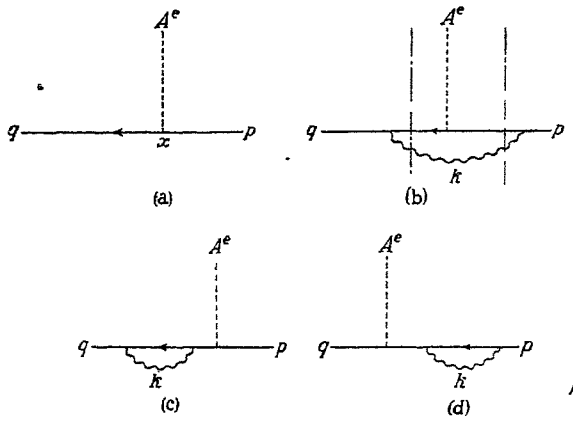

Fig. 13.7

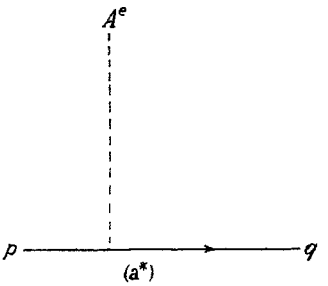

Fig. 13.8

initial state. We shall denote the singular term of $M_a^* M_b$ (at $k=0$) due to the above virtual states by $w_{M_a^* M_b}(M_1^* M_2)$. In a similar way, in the same diagram, the $M_1^* M_2$-term is singular (at $k=0$), on account of the virtual state denoted by the chain $M_a^* M_b$. We shall denote this singular term by $w_{M_1^* M_2}(M_a^* M_b)$. Then we can prove by actual calculation that

$$w_{M_a^* M_b}(M_1^* M_2) + w_{M_1^* M_2}(M_a^* M_b) = 0\,. \tag{13.60}$$

In fact, the S-matrix element, corresponding to the diagram (b) in Fig. 13.7, is

$$-\frac{e^3}{4}\int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \int_{-\infty}^{\infty} d^4x_3\, \bar{\psi}(x_3)\,\gamma_\varrho\, S_F(x_3-x_2)\,\gamma_\nu\, A_\nu^e(x_2)\, S_F(x_2-x_1)\,\gamma_\mu\,\psi(x_1)$$
$$(P[A_\varrho(x_3), A_\mu(x_1)])_0\,.$$

The zero energy photons contribute to $(P[A_\varrho(x_3), A_\mu(x_1)])_0$ through

$$\lim_{k\to 0}\frac{\delta_{\varrho\mu}}{2kV}\left\{\frac{1+\varepsilon(x_3-x_1)}{2}\,e^{i(k,x_3-x_1)} + \frac{1-\varepsilon(x_3-x_1)}{2}\,e^{i(k,x_1-x_3)}\right\} = \lim_{k\to 0}\frac{\delta_{\varrho\mu}}{2kV}$$

(cf. (13.26a)). Then, the singular term of M_b (due to the zero energy photon) can be calculated as

$$\lim_{k\to 0}\frac{e^3}{2kV}\,a_r^*(q)\,\gamma_4\,\gamma_\varrho\,\frac{\{i(q_\mu-k_\mu,\gamma_\mu)-\varkappa\}}{(q_\mu-k_\mu,q_\mu-k_\mu)+\varkappa^2-i\epsilon}\,\gamma_\nu\,A_\nu^e(q-p)\,\frac{\{-i(k_\mu-p_\mu,\gamma_\mu)-\varkappa\}}{(p_\mu-k_\mu,p_\mu-k_\mu)+\varkappa^2-i\epsilon}\,\gamma_\mu\,a_s(p)$$

$$=-\lim_{k\to 0}\frac{e^3}{2|\mathbf{k}|V}\,(a_r^*(q)\,\gamma_4\,\gamma_\nu\,A_\nu^e(q-p)\,a_s(p))\,\frac{(\mathbf{q}\cdot\mathbf{e})}{(q_\mu k_\mu)}\cdot\frac{(\mathbf{p}\cdot\mathbf{e})}{(p_\mu k_\mu)}$$

$$=-\lim_{k\to 0}\frac{e}{\sqrt{2|\mathbf{k}|V}}\,\frac{(\mathbf{q}\cdot\mathbf{e})}{(q_\mu k_\mu)}\,M_2.$$

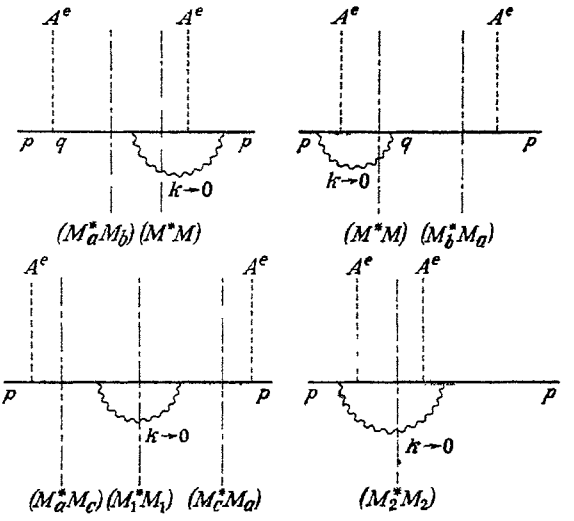

Fig. 13.9

On the other hand, it is apparent from Fig. 13.7 that

$$M_a=-\,e\,a_r^*(q)\,\gamma_4\,\gamma_\nu\,A_\nu^e(q-p)\,a_s(p).$$

Thus, zero energy photons contribute to $M_a^*\,M_b$ through

$$w_{M_a^*M_b}(M_1^*\,M_2)=\lim_{k\to 0}\frac{(\mathbf{q}\cdot\mathbf{e})}{(q_\mu k_\mu)}\,\frac{e^2}{\sqrt{2|\mathbf{k}|V}}\,(a_r^*(q)\,\gamma_4\,\gamma_\nu\,A_\nu^e(q-p)\,a_s(p))^*\,M_2.$$

This is equal to $-M_1^*\,M_2$, and leads to (13.60). We have the similar situation for each diagram of Fig. 13.8.

Thus we see that the infra-red catastrophe is eliminated by taking into account the contributions of the diagram (a), (b), (c) and (d) in Fig. 13.7.

This situation can be understood intuitively by means of (12.31) in § 2 of Ch. XII. Since, as shown by (12.30) and (12.32), at $k \to 0$, $\bar{n}$ (i.e. the number of the photons of the infinitesimally small energies) diverges logarithmically, we see from (12.31) that the probability of a finite number of photons of zero energy is zero, and therefore that the infra-red catastrophe disappears. Since, as shown by (12.33) the probability of the scattering of a radiating electron is not zero, the scattered electron must radiate an infinite number of photons of infinitesimally small energy. This proof of the disappearance of the infra-red catastrophe was given by BLOCH and NORDSIECK [1937].

Thus we see that the infra-red catastrophe disappears when the effects of the proper fields are taken into account. However, the proper field also has many high energy photons, which frequently give rise to difficulties connected with infinities of the kind discussed in the last Chapter. These difficulties are called the ultra-violet catastrophies. If we eliminated them by disregarding the effects of proper fields, we would be faced with an infra-red catastrophe. Clearly we must seek another escape.

EXAMPLE 5. ULTRA-VIOLET CATASTROPHE

In the last example only the effects of the low energy photons in the proper field were considered. We shall now consider the contribution of all the photons of the proper field to the elastic scattering of an electron by an external electromagnetic field $A^e_\mu(x)$ in the e^3-approximation. The Feynman diagrams of this process are shown in Fig. 13.10.

First we shall consider the Feynman diagram in Fig. 13.11, in which we shall assume that the operators $\bar{\psi}(x_2)$ and $\psi(x_1)$ corresponding to the external lines do not always satisfy the wave equations ((3.1), (3.33)) of the free electron.

In the S-matrix Fig. 13.11 leads to the term

$$\left.\begin{aligned}&\int d^4x\, \bar{\psi}(x)\, \Sigma^0\, \psi(x)\\ &\quad = \frac{e^2}{4} \int d^4x_1 \int d^4x_2\, \bar{\psi}(x_2)\, \gamma_\mu\, S_F(x_2 - x_1)\, \gamma_\mu\, \psi(x_1)\, \Delta_F(x_2 - x_1).\end{aligned}\right\} \quad (13.61a)$$

By actual calculation, using two infinite constants A^0 and B^0, we can write Σ^0 as

$$\Sigma^0 = -i\{A^0 + B^0\,(\gamma_\mu \partial_\mu + \varkappa) + \ldots\}, \quad (13.62)$$

where ... denotes the finite terms containing the factor $(\gamma_\mu \partial + \varkappa)^2$. This can be regarded as a power expansion in $(i\gamma_\mu p_\mu - \varkappa)$ (p_μ is the energy-momentum vector of the electron) in the momentum representation. We shall now show that the infinities introduced by each diagram can only appear as the coefficients of terms of low order in $(i\gamma_\mu p_\mu + \varkappa)$. For an increase of the power of $(i\gamma_\mu p_\mu + \varkappa)$ by unity means that the power n of the dimension $[L]^n$ of the coefficient increases by one, and therefore that the degree of the infinity of the coefficient decreases by one. This situation can be seen clearly as follows. By substituting (13.41) and (8.35) into (13.61a) we have

$$\left.\begin{aligned}&\int d^4x\, \bar\psi(x)\, \Sigma^0\, \psi(x) = \\ &\frac{e^2}{(2\pi)^8}\int d^4p \int d^4k\, \bar\psi(-p)\, \gamma_\varrho \frac{\{i(\gamma_\mu, p_\mu - k_\mu) - \varkappa\}}{\{(p_\mu - k_\mu, p_\mu - k_\mu) + \varkappa^2 - i\epsilon\}} \frac{1}{(k_\mu k_\mu - i\epsilon)}\, \gamma_\varrho\, \psi(p).\end{aligned}\right\} \quad (13.61b)$$

Here $\bar\psi(p)$ and $\psi(p)$ are Fourier amplitudes of $\bar\psi(x)$ and $\psi(x)$ given by

$$\psi(x) = \frac{1}{(2\pi)^4} \int d^4p\, \psi(p)\, e^{ip_\mu x_\mu}$$

$$\bar\psi(x) = \frac{1}{(2\pi)^4} \int d^4p\, \bar\psi(p)\, e^{-ip_\mu x_\mu}.$$

Integration, $\int d^4k$, in (13.61b), leads to infinite coefficients in (13.62). The integrand in (13.61b) is a function of $p_\mu - k_\mu$ and k_μ. It is easily seen that each differentiation d/dp operating on (13.61b) reduces the maximum order of the infinity by one. Since the highest powers of k_μ in the numerator and denominator, in the integrand in (13.61b), are one and four, respectively, the order of infinity in A^0 cannot be larger than 1. Thus, the order of the infinity in B^0 is at most logarithmic, and the remaining terms (...) in (13.62) are finite.

We shall now rewrite (13.61b) in the form of (13.62). By using the formula

$$a^{-1}b^{-1} = \int_0^1 du\, (au + b\,(1-u))^{-2},$$

we obtain

$$\begin{aligned}&\int d^4k\, \frac{f(k)}{\{(p_\mu - k_\mu, p_\mu - k_\mu) + \varkappa^2 - i\epsilon\}} \cdot \frac{1}{(k_\mu k_\mu - i\epsilon)} \\ &= \int d^4k \int_0^1 du\, \frac{f(k)}{\{k_\mu k_\mu - 2u p_\mu k_\mu + u p_\mu p_\mu + u\varkappa^2 - i\epsilon\}^2} \\ &= \int d^4l \int_0^1 du\, \frac{f(l + up)}{\{l_\mu l_\mu + u(1-u) p_\mu p_\mu + u\varkappa^2 - i\epsilon\}^2},\end{aligned}$$

where

$$l_\mu = k_\mu - u p_\mu.$$

Thus, we have

$$\left.\begin{aligned}&\int d^4x\, \bar{\psi}(x)\, \Sigma^0\, \psi(x)\\ &= \frac{e^2}{(2\pi)^8}\int_0^1 du \int d^4p \int d^4l\, \bar{\psi}(-p)\,\gamma_\varrho\, \frac{\{-i(1-u)\,(\gamma_\mu p_\mu)+i(\gamma_\mu l_\mu)-\varkappa\}}{\{l_\mu l_\mu+u(1-u)\,p_\mu p_\mu+u\varkappa^2-i\epsilon\}^2}\,\gamma_\varrho\,\psi(p)\\ &= \frac{e^2}{(2\pi)^8}\int_0^1 du \int d^4p \int d^4l\, \bar{\psi}(-p)\,\gamma_\varrho\, \frac{\{-i(1-u)\,(\gamma_\mu p_\mu)-\varkappa\}}{\{l_\mu l_\mu-u(1-u)\,p_\mu p_\mu+u\varkappa^2-i\epsilon\}^2}\,\gamma_\varrho\,\psi(p).\end{aligned}\right\} \quad (13.61c)$$

We now write

$$\begin{aligned}&\frac{1}{\{l_\mu l_\mu+u(1-u)\,p_\mu p_\mu+u\varkappa^2-i\epsilon\}^2}\\ &= \frac{1}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^2}-2\,\frac{u(1-u)}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^3}\,(p_\mu\, p_\mu+\varkappa^2)+\dots\\ &= \frac{1}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^2}-4\varkappa\,\frac{u(1-u)}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^3}\,(i\,\gamma_\mu\, p_\mu+\varkappa)+\dots.\end{aligned}$$

Since

$$\gamma_\varrho\{i(1-u)\,(\gamma_\mu p_\mu)-u\}\gamma_\varrho = -2(1-u)\,(i\gamma_\mu p_\mu+\varkappa)-2(1+u)\varkappa,$$

we can derive

$$\begin{aligned}&\gamma_\varrho\,\frac{\{-i(1-u)\,(\gamma_\mu p_\mu)-\varkappa\}}{\{l_\mu l_\mu+u(1-u)\,p_\mu p_\mu+u\varkappa^2-i\epsilon\}^2}\,\gamma_\varrho = -\frac{2(1+u)\varkappa}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^2}\\ &+\left[\frac{-2(1-u)}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^2}+8\varkappa^2\,\frac{u(1-u^2)}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^3}\right](i\,\gamma_\mu\, p_\mu+\varkappa)+\dots.\end{aligned}$$

By substituting this into (13.61c), we obtain (13.62), where

$$A^0 = \frac{-ie^2}{(2\pi)^4}\int_0^1 du \int d^4l\, \frac{2(1+u)u}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^2}. \qquad (13.61d)$$

In order to calculate this integral, we use the fact that

$$\begin{aligned}\int d^4l\,\frac{1}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^2} &= \int d^3l \int dl_0\,\frac{1}{(l_0^2-\mathbf{l}^2-u^2\varkappa^2+i\epsilon)^2}\\ &= 2\pi^2 i\int_0^\infty dl\, l^2\,\frac{1}{(l^2+u^2\varkappa^2)^{3/2}}, \qquad (l\equiv|\mathbf{l}|),\\ \int d^4l\,\frac{1}{\{l_\mu l_\mu+u^2\varkappa^2-i\epsilon\}^3} &= -\frac{3\pi^2 i}{2}\int_0^\infty dl\, l^2\,\frac{1}{(l^2+u^2\varkappa^2)^{5/2}}.\end{aligned}$$

These formulae can be obtained by calculating the residuals of the

integrands at their poles $l_0 = \pm \{(l^2+u^2\varkappa^2)-i\epsilon\}$. After the integration over du, we now have

$$\int_0^1 du \int d^4l \frac{2(1+u)\varkappa}{\{l_\mu l_\mu+\varkappa^2u^2-i\epsilon\}} = 4\,i\,\pi^2 \int l^2\,dl \left\{\frac{\varkappa}{l^2(l^2+\varkappa^2)^{1/2}} + \frac{1}{\varkappa l} - \frac{1}{\varkappa(l^2+\varkappa^2)^{1/2}}\right\}$$
$$= 2\,i\,\pi^2\,\varkappa \int_0^\infty dw \left\{\frac{1}{1+w} + \frac{w}{2(1+w)} - \frac{w^2}{2(1+w)^2}\right\}$$
$$= 6\,i\,\pi^2\,\varkappa \left\{\frac{1}{2}\log(w+1) - \frac{1}{6}\right\}_{w\to\infty},$$

where the new integration variable w is

$$w = \frac{2}{\varkappa^2}\{l^2 + l(l^2+\varkappa^2)^{1/2}\}.$$

It can be shown that this integration variable w has the physical significance of the four dimensional scalar product of the energy-momenta $(-\mathbf{l}, E_l; \mathbf{l}, l)$ of electrons and photons in virtual states (in the centre of mass system) and, therefore, that it is a Lorentz invariant variable:

$$w = \frac{2}{\varkappa^2}\{(\mathbf{l}\cdot\mathbf{l}) + l\,E_l\}.$$

It sometimes happens that taking integration variables, which are Lorentz invariant, simplifies calculations (PAULI and ROSE [1936], UMEZAWA and KAWABE [1949a]). From (13.61d) it follows that

$$A^0 = \frac{3\varkappa}{2\pi}\left(\frac{e^2}{4\pi}\right)\left\{\frac{1}{2}\log(w+1) - \frac{1}{6}\right\}_{w\to\infty} \tag{13.63}$$

(WEISSKOPF [1939]). When the external operators $\bar\psi, \psi$ satisfy the wave equation of the free electron, we obtain

$$\int d^4x \bar\psi(x)\Sigma^0\psi(x) = -iA^0\int d^4x\;\bar\psi(x)\,\psi(x). \tag{13.64}$$

This is just the contribution of interaction Lagrangian $-A^0\bar\psi(x)\psi(x)$ to the S-matrix in the e^2-approximation (cf. (7.111)). Thus, we see that the diagram of Fig. 13.10 gives the change of electron mass induced by the proper field as A^0 (i.e. $\varkappa \to \varkappa + A^0$). It is usual to call A^0 the self-mass. The interaction Lagrangian $-A^0\bar\psi\psi$ corresponds to the interaction Hamiltonian $A^0\bar\psi\psi$. By using this interaction Hamiltonian, we can calculate the perturbed energy (13.10), due to the self-mass term. For a free electron of energy-momentum $(\mathbf{p}, E_p)$ (cf. (9.75), (9.78)), we have, in the e^2-approximation [1]),

$$\Delta E = \tfrac{1}{2}A^0(a_r^*(p)\gamma_4 a_r(p)) = A^0\varkappa/E_p = A^0(\partial/\partial\varkappa)E_p. \tag{13.65}$$

[1]) The factor 1/2 in (13.65) is introduced to give the average value with respect to the two directions of the spin angular momentum.

This change ΔE of the energy due to the effects of the proper field is called the self-energy. As shown by (13.65), ΔE corresponds to the change (in the e^2-approximation) of the electron mass obtained by the substitution $\varkappa \to \varkappa + A^0$.

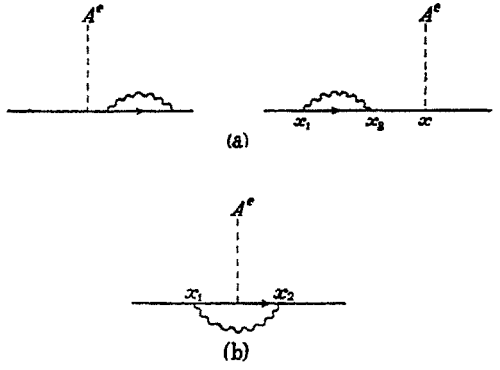

Fig. 13.10

Fig. 13.11

However, the difficulty arises that the self-energy is not a small correction but infinite, because the electron must have finite mass. Equations (10.1) and (10.10) show that the interaction (13.64) leads to a phase factor $\exp(i\Delta E t)$ $(t \to \infty)$ in $(\Psi[\sigma] = S[\sigma]\,\Psi[-\infty])_{\sigma\to\infty}$. Since ΔE is the change of the energy, this result is easily understood in terms of the usual quantum mechanics. Fig. 13.11 is called the self-energy graph of an electron.

In the case of the elementary particles which decay naturally into other elementary particles (e.g. π-mesons) with finite life times, we find the complex self-energies $\Delta E = \Delta E_1 + i(\Gamma/2)$. Then the real and complex parts of ΔE give the factors $\exp(i\Delta E_1 t)$ and $\exp(-\Gamma/2)t$ respectively. The former factor shows that ΔE_1 is the change of energy and the latter factor shows that the probability amplitude of the state in which a particle exists decreases exponentially with the time. This behaviour is called the damping effect. The decrease of the probability amplitude corresponds to the natural decay of an elementary particle with the life time $(1/\Gamma)$.

We shall now consider the Feynman diagrams of Fig. 13.10. The matrix element of the S-matrix given by the diagram on the right hand side of Fig. 13.10(a) can be written, by using (13.61a), as

$$\frac{e}{2}\int d^4x_1 \int d^4x\, \bar{\psi}(x_1)\, \Sigma^0\, S_F(x_1 - x)\, \gamma_\mu\, \psi(x)\, A^e_\mu(x). \qquad (13.66)$$

Substituting (13.62) into (13.66), we have

$$\left.\begin{aligned} &= \frac{e}{2} A^0 \int d^4x_1 \int d^4x\, \bar{\psi}(x_1)\, S_F(x_1 - x)\, \gamma_\mu\, \psi(x)\, A^e_\mu(x) \\ &\quad - \frac{ie}{2} B^0 \int d^4x\, \bar{\psi}(x)\, \gamma_\mu\, \psi(x)\, A^e_\mu(x). \end{aligned}\right\} \tag{13.67}$$

The second term of (13.67) comes from the term

$$M \equiv \frac{e}{2} B^0 \int d^4x_1 \int d^4x\, \bar{\psi}(x_1)\, (\gamma_\mu \partial^1_\mu + \varkappa)\, S_F(x_1 - x)\, \gamma_\mu\, \psi(x)\, A^e_\mu(x). \tag{13.68}$$

It must be noted that (13.68) would yield

$$M = -ieB^0 \int d^4x\, \bar{\psi}(x)\, \gamma_\mu \psi(x)\, A^e_\mu(x)$$

if we used the formula

$$(\gamma_\mu \partial_\mu + \varkappa)\, S_F(x) = -2i\delta(x).$$

However, we would find $M = 0$ if we used integration by parts for ∂^1_μ and the wave equation (3.33). The second term of (13.67) is equal to the mean value of these two results for M. In fact, we can derive this result for M consistently as follows (LÜDERS [1952]): Substituting $e(t)$ (defined by (13.15a)) into e (appearing in B^0) we have

$$\begin{aligned} M = -\lim_{\epsilon, \epsilon' \to 0} \frac{i B^0}{(2\pi)^3} \frac{e}{4\epsilon\epsilon'} \int_{-\epsilon}^{\epsilon} da \int_{-\epsilon'}^{\epsilon'} da' \int d^4x_1 \int d^4x \int d^4p \\ e^{-ia t_1}\, \bar{\psi}(x_1)\, (i\gamma_\mu p_\mu - \gamma_4 a' + \varkappa)\, e^{-ia' t_1}\, \frac{1}{(i\gamma_\mu p_\mu - \gamma_4 a' - \gamma_4 a + \varkappa)}\, \gamma_\mu\, A^e_\mu(x) \\ e^{i(p, x_1 - x) + i(a + a')(t_1 - t)}. \end{aligned}$$

Since the higher order ($\geqslant 2$) terms of a and a' can be neglected, we have

$$\begin{aligned} &\bar{\psi}(p)\, (i\gamma_\mu p_\mu - \gamma_4 a' + \varkappa)\, \frac{1}{(i\gamma p - \gamma_4 a' - \gamma_4 a + \varkappa)} \\ &\quad = \bar{\psi}(x)\, \gamma_4 a'\, \frac{(i\gamma p - \gamma_4 a' - \gamma_4 a - \varkappa)}{2 i p_4 (a' + a)} = \bar{\psi}(p)\, \frac{a'}{a + a'} \end{aligned}$$

where $\bar{\psi}$ satisfies the equations of the free electron, namely

$$\bar{\psi}(p)\, (i\gamma p + \varkappa) = 0.$$

Now, we can obtain $M =$

$$\begin{aligned} &- ie \lim_{\epsilon, \epsilon' \to 0} B^0 \left\{ \frac{1}{4\epsilon\epsilon'} \int_{-\epsilon}^{\epsilon} da \int_{-\epsilon'}^{\epsilon'} da'\, \frac{a'}{a' + a}\, e^{-i(a + a')t} \right\} \int d^4x\, \bar{\psi}(x)\, \gamma_\mu\, \psi(x)\, A^e_\mu(x) \\ &\quad = -\frac{ie}{2} B^0 \int d^4x\, \bar{\psi}(x)\, \gamma_\mu\, \psi(x). \end{aligned}$$

Thus we obtain the second term of (13.67). In the next chapter we shall show that this result is also obtained by requiring that the S-matrix be unitary. The third term (...) in (13.62) gives no contribution to (13.67) on account of the field equation (3.33).

The contribution of the diagram Fig. 13.10(b) to the S-matrix element can be written as

$$i e B \int d^4x\, \bar{\psi}(x)\, \gamma_\mu\, \psi(x)\, A^e_\mu(x) + \Lambda\,, \tag{13.69a}$$

where Λ is (SCHWINGER [1949])

$$\left.\begin{aligned} \Lambda = &-\frac{e}{3\pi\varkappa^2}\frac{e^2}{4\pi}\left(\log\frac{\varkappa}{2k_0}+\frac{11}{24}\right)\int d^4x\, \bar{\psi}\,\gamma_\mu\,\psi\,\Box\, A^e_\mu \\ &+\frac{e}{2\pi\varkappa}\left(\frac{e^2}{4\pi}\right)\frac{1}{8}\int d^4x\, \bar{\psi}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\,\psi(\partial_\mu A^e_\nu-\partial_\nu A^e_\mu) \\ &+\dots\,. \end{aligned}\right\} \tag{13.69b}$$

Here B is an infinite constant, while (...) denotes the finite terms containing higher derivatives (of order $\geqslant 3$) operating on A^e_μ. Equation (13.69b) can be derived by means of the power series expansion with respect to the momentum change of the external field A^e_μ. We have assumed in (13.69b) that the energies k of photons in virtual states have a minimum value k_0 ($k \geqslant k_0$). In the limit $k_0 \to 0$, (13.69b) gives the infra-red catastrophe. However, since, as shown in the last example, this is not the essential difficulty, we can continue our discussion without touching on this point.

By calculation we can show that

$$B^0 = B. \tag{13.70}$$

Therefore, the S-matrix element given by Fig. 13.10(a), (b) is

$$\left.\begin{aligned} &\frac{e}{2}\,A^0 \int d^4x_1\, d^4x\, \{\bar{\psi}(x_1)\, S_F(x_1-x)\, \gamma_\mu\, \psi(x)\, A^e_\mu(x) \\ &\quad + \bar{\psi}(x_1)\, \gamma_\mu\, S_F(x_1-x)\, \psi(x)\, A^e_\mu(x)\} + \Lambda. \end{aligned}\right\} \tag{13.71}$$

We shall now consider Fig. 13.13 (i.e. the self-energy graph of a photon). The S-matrix element given by this diagram can be written as

$$\left.\begin{aligned} &-i\int d^4x\, A_\mu(x)\, \Pi^0_{\mu\nu}\, A_\nu(x) \\ &= -\frac{e^2}{8}\int d^4x_1 \int d^4x_2\, \mathrm{S_P}(S_F(x_1-x_2)\,\gamma_\nu\, S_F(x_2-x_1)\,\gamma_\mu)\, A_\mu(x_1)\, A_\nu(x_2), \end{aligned}\right\} \tag{13.72}$$

because $\varepsilon = -1$ and $g = 2$ (cf. (13.25)). By expanding as a power

series of terms that are powers of the momentum of the photon we have

$$\Pi^0_{\mu\nu} = \tfrac{1}{2}\left\{ D\delta_{\mu\nu} + C^0 \Box\, \delta_{\mu\nu} - \frac{1}{15\pi\varkappa^2}\frac{e^2}{4\pi}\Box^2\, \delta_{\mu\nu} + \ldots, \right\} \qquad (13.73)$$

where ... is a finite term containing a factor $\Box^3$, and D and C^0 are quadratically and logarithmically diverging constants (DIRAC [1934], HEISENBERG [1934], PEIERLS [1934]).

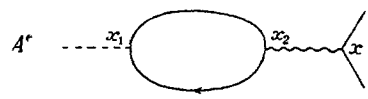

Fig. 13.12

Fig. 13.13

By using a similar discussion to that employed for the electron self-energy, we can see that D is the self-energy of a photon due to the proper electron field around the photon. However, a non-zero D is not compatible with the fact that the photon mass is zero. Moreover, since the first term of (13.73) gives the non-gauge invariant term in (13.72), a non-zero D seems to be incompatible with the requirement of gauge-invariance. However, we obtain various results (distributed from zero to infinity) for D by using different methods of calculation. Moreover, even if we want to use the method of calculation which gives $D=0$, this method gives contradictory results when it is applied to other calculations (KATAYAMA [1950], FUKUDA and KINOSHITA [1950]) [1]). It is for this reason that the problem of the self-energy of the photon has been discussed by so many authors. We shall take up this problem in the next example.

Since the photon mass must be zero ($D=0$) and the results must be gauge-invariant, the best method of approach in problems that can be separated from the problem of the self-energy of the photon seems to be to assume provisionally that $D=0$.

The difficulties occasioned by the infinite values of D and C^0 are called those of vacuum polarisation. By calculation we can show that (cf. next example)

$$C^0 = -\frac{2}{3\pi}\frac{e^2}{4\pi}\left\{\log\frac{P}{\varkappa}\right\}_{P\to\infty} + \text{finite constant.} \qquad (13.74)$$

[1]) See also TAKAHASHI [1954].

The contribution of the vacuum polarisation to the S-matrix is given by Fig. 13.12 as

$$\left.\begin{aligned} & i\,e\int d^4x_1 \int d^4x\, \bar{\psi}(x)\,\gamma_\mu\,\psi(x)\,\Delta_F(x-x_1)\,\Pi^0_{\mu\nu}\,A^e_\nu(x_1) \\ & \quad = -\,e\,C^0 \int d^4x\, \bar{\psi}(x)\,\gamma_\mu\,A^e_\mu(x)\,\psi(x) \\ & \qquad + \frac{e}{15\pi\varkappa^2}\,\frac{e^2}{4\pi}\int d^4x\,\bar{\psi}(x)\,\gamma_\mu\,\psi(x)\,\Box\,A^e_\mu(x) + \ldots, \end{aligned}\right\} \tag{13.75}$$

because $\varepsilon = -1$ and $g = 1$. In (13.75) ... is a finite term containing a factor $\Box^2$ operating on A^e_μ. This shows that an electron in an external field is observed as a charged particle with the charge $e\,(1+C^0)$. The quantity eC^0 is called the self-charge. The difficulty of the concept of self-charge is illustrated by the fact that C^0 is infinity. As shown by (13.74), the effect of the electric charge is weakened by the negative self-charge C^0. This fact can be intuitively understood from hole theory. In hole theory we must consider an electron in polarisable matter, i.e. in an ocean of vacuum electrons and, therefore, the effect of the electric charge of the electrons must be weakened on account of the Lenz law. Since this explanation does not depend on the spin of the charged particle, we may expect that the self-charge δe of any charged particle due to the vacuum polarisation effect is negative. In fact we can prove this fact without using the perturbation approximation (UMEZAWA and KAMEFUCHI [1951]) [1].

Thus, we have seen that the infinities due to the effects of the proper field of an electron appear in the self-mass and self-charge.

When the momenta of the electron in its initial and final states are not large,

$$|\mathbf{p}| \ll \varkappa, \quad |\mathbf{q}| \gg \varkappa, \tag{13.76}$$

we can obtain non-relativistic results for Λ (cf. (13.69b)) by neglecting the higher degree ($\geqslant 3$) terms of $|\mathbf{P}|$ and $|\mathbf{q}|$. Thus

$$\left.\begin{aligned} \Lambda = & -\frac{i}{3\pi\varkappa^2}\,\frac{e^2}{4\pi}\left\{\log\frac{\varkappa}{2k_0}+\frac{11}{24}\right\}\int d^4x\,\psi^*(x)\,\Delta\,\phi(x)\,\psi(x) \\ & -\frac{i}{4\pi\varkappa}\,\frac{e^2}{4\pi}\int d^4x\,\psi^*\,\gamma_k\,\psi(x)\,\partial_k\,\phi(x). \end{aligned}\right\} \tag{13.77}$$

[1] However, when the order of infinity of δe is high, its values depend on the method of calculation and sometimes give results which are incompatible with the general proof. In fact, we can obtain positive δe of a charged vector particle by using a special calculation method (KATAYAMA [1949], MCCONNELL [1951].

Here we have assumed a static potential $\phi(x)$ instead of $A^e_\mu(x)$ (i.e. we have taken A^e_μ as $(0, 0, 0, i\phi)$ in (13.69b)).

EXAMPLE 6. VACUUM POLARISATION

We shall consider the current δJ_μ induced by an external electromagnetic field A^e_μ in the e^2-approximation. From (10.77) and (13.3) we have

$$\delta J_\mu(x) = i e^2 \int_{-\infty}^{\sigma(x)} d^4x' \,([j_\mu(x), j_\nu(x')])_0 A^e_\nu(x') + e^2 \,(j_{\mu\nu}[x:n])_0 A^e_\nu(x), \quad (13.78)$$

where $(\ )_0$ indicates vacuum expectation values.

By calculation we can obtain

$$\left.\begin{aligned}\delta J_\mu(x) = -\int d^4x' \,\{D A^e_\mu(x') + C^0 (\Box' \,\delta_{\mu\nu} - \partial'_\mu \partial'_\nu) A^e_\nu(x') \\ + f(\Box')\,(\Box' \,\delta_{\mu\nu} - \partial'_\mu \partial'_\nu)\, A^e_\nu(x')\}\,\delta(x-x').\end{aligned}\right\} \quad (13.79)$$

Now (10.74) shows that the first term of (13.79) can be regarded as the current derived from the interaction Lagrangian $L' = -(\frac{1}{2})\, D A^e_\mu A^e_\mu$. Since the Lagrangian $L+L'$, where L is the free Lagrangian (7.51) of the electromagnetic field, leads to the equation

$$(\Box - D)\, A_\mu = 0$$

$D^{1/2}$ may be regarded as the self-mass of the photon. The gauge-invariance of the theory requires that δJ_μ should be invariant under the transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$, i.e. that

$$D = 0. \quad (13.80)$$

We can express this requirement of gauge-invariance in another way. Equation (13.78) must be written in the form

$$\delta J_\mu(x) = -4e^2 \int d^4x' \, K_{\mu\nu}(x-x')\, A_\nu(x') \quad (13.81)$$

on account of the Lorentz invariance. Gauge-invariance requires that $K_{\mu\nu}$ should satisfy the equation

$$\partial_\nu K_{\mu\nu}(x) = 0. \quad (13.82a)$$

Comparing (13.79) and (13.81), we have

$$\partial_\nu K_{\mu\nu}(x) = D \partial_\mu \delta(x). \quad (13.82b)$$

Calculation gives the following $K_{\mu\nu}$ for various charged fields:—

$$\left.\begin{aligned}K^{(s)}_{\mu\nu} = -\tfrac{1}{4}\{\partial_\mu \overline{\Delta} \cdot \partial_\nu \Delta^{(1)} + \partial_\nu \overline{\Delta} \cdot \partial_\mu \Delta^{(1)} - \overline{\Delta} \cdot \partial_\mu \partial_\nu \Delta^{(1)} \\ - \Delta^{(1)} \cdot \partial_\mu \partial_\nu \overline{\Delta} + \delta_{\mu\nu} (\Delta^{(1)} \,\Box\, \overline{\Delta} + \overline{\Delta} \,\Box\, \Delta^{(1)} - 2\varkappa^2 \Delta^{(1)} \overline{\Delta})\}\end{aligned}\right\} \quad (13.83)$$

for the charged field of spin zero,

$$K_{\mu\nu}^{(f)} = \partial_\mu \bar{\Delta} \cdot \partial_\nu \Delta^{(1)} + \partial_\nu \bar{\Delta} \cdot \partial_\mu \Delta^{(1)} - \delta_{\mu\nu} (\partial_\varrho \bar{\Delta} \cdot \partial_\varrho \Delta^{(1)} + \varkappa^2 \bar{\Delta} \Delta^{(1)} \quad (13.83f)$$

for spin 1/2,

$$\left.\begin{aligned} K_{\mu\nu}^{(v)} = & -\tfrac{3}{4} \{\partial_\mu \bar{\Delta} \cdot \partial_\nu \Delta^{(1)} + \partial_\nu \bar{\Delta} \cdot \partial_\mu \Delta^{(1)} - \bar{\Delta} \cdot \partial_\mu \partial_\nu \Delta^{(1)} - \Delta^{(1)} \cdot \partial_\mu \partial_\nu \bar{\Delta} \\ & + \delta_{\mu\nu} (\Delta^{(1)} \Box \bar{\Delta} + \bar{\Delta} \Box \Delta^{(1)} - 2\varkappa^2 \Delta^{(1)} \bar{\Delta})\} \\ & - \frac{1}{2\varkappa^2} \{\partial_\mu \partial_\nu \Delta^{(1)} \cdot \Box \bar{\Delta} + \Box \Delta^{(1)} \cdot \partial_\mu \partial_\nu \bar{\Delta} - \partial_\mu \partial_\varrho \Delta^{(1)} \cdot \partial_\nu \partial_\varrho \bar{\Delta} \\ & - \partial_\nu \partial_\varrho \Delta^{(1)} \cdot \partial_\mu \partial_\varrho \bar{\Delta} + \delta_{\mu\nu} (\partial_\varrho \partial_\sigma \Delta^{(1)} \cdot \partial_\varrho \partial_\sigma \bar{\Delta} - \Box \Delta^{(1)} \cdot \Box \bar{\Delta})\} \end{aligned}\right\} \quad (13.83v)$$

for spin 1, where $\Delta^{(1)}$- and $\bar{\Delta}$-functions are those for the charged fields (FELDMAN [1949]). From these relations we can derive (PAULI and VILLARS [1949], SAKATA and UMEZAWA [1950])

$$\partial_\nu K_{\mu\nu}(x) = \frac{n}{2} \delta(x) \partial_\mu \Delta^{(1)}(x), \quad (13.84a)$$

with

$$n = \left\{\begin{array}{rl} 1 & \text{for spin } 0 \\ -2 & \text{for spin } \tfrac{1}{2} \\ 3 & \text{for spin } 1. \end{array}\right. \quad (13.84b)$$

As shown by (8.25a), $\Delta^{(1)}$ has a strong singularity near the light cone. Indeed,

$$\Delta^{(1)}(x) = \frac{1}{2\pi^2 (x_\mu x_\mu)} + \frac{\varkappa^2}{8\pi^2} \log |(x_\mu x_\mu)| + \Delta_{\text{reg}}^{(1)},$$

where $\Delta_{reg}^{(1)}$ is a regular function of $(x_\mu x_\mu)$. From this, we have

$$\partial_\mu \Delta^{(1)}(x) = -\frac{x_\mu}{\pi^2 (x_\mu x_\mu)^2} + \frac{\varkappa^2}{4\pi^2} \frac{x_\mu}{x_\mu x_\mu} + 2 x_\mu \frac{d}{d(x_\mu x_\mu)} \Delta_{\text{reg}}^{(1)}.$$

This leads to

$$\partial_\nu K_{\mu\nu}(x) = -\frac{n}{2} \frac{x_\mu}{\pi^2 (x_\mu x_\mu)^2} \delta(x) + \frac{n}{2} \frac{\varkappa^2}{4\pi^2} \frac{x_\mu}{(x_\mu x_\mu)} \delta(x), \quad (13.85)$$

which is quite indefinite. Indeed, practical calculation shows that D is not always zero, but depends on the method of calculation (WENTZEL [1948]). This difficulty is called the problem of the self-energy of the photon. Equation (13.82b) and (13.85) show that the self-mass of the

photon is made up of two terms [1]); one is proportional to n, the other to $n\varkappa^2$.

We shall assume that there exist k charged fields $U^1 \ldots U^k$ of spin 0, l charged fields $\psi^1 \ldots \psi^l$ of spin $\frac{1}{2}$, and m charged fields $U_\mu^1 \ldots U_\mu^m$ of spin 1 with masses $(\varkappa_1^{(s)} \ldots \varkappa_k^{(s)})$, $(\varkappa_1^{(f)} \ldots \varkappa_l^{(f)})$, $(\varkappa_1^{(v)} \ldots \varkappa_m^{(v)})$; also that

$$k - 2l + 3m = 0$$

$$\sum_{i=1}^{k} (\varkappa_i^{(s)})^2 - 2 \sum_{i=1}^{l} (\varkappa_i^{(f)})^2 + 3 \sum_{i=0}^{m} (\varkappa_i^{(v)})^2 = 0.$$

Then, (13.85) shows that the self-energy of the photon becomes zero without any ambiguity due to the method of calculation (RAYSKI [1948], UMEZAWA, YUKAWA and YAMADA [1948]). Indeed, (13.85) leads to

$$\partial_\nu K_{\mu\nu}(x) = 0.$$

However, there is not universal agreement about these assumptions. PAULI and VILLARS [1949] have developed a formalistic mixture theory, the regularisation theory, from a wider viewpoint. There, some of fields U^i, ψ^i, U_μ^i are temporarily regarded as auxiliary fields whose masses must be taken to be infinite at the end of the calculations in order to make them inaccessible to experimental observation.

For C^0 and $f(\Box)$ in (13.79) we obtain (FELDMAN [1949], UMEZAWA and KAWABE [1949*b*])

$$C^0 = \int_0^1 dv\, G(v)$$

$$= \left\{ \begin{array}{ll} \dfrac{e^2}{4\pi} \left\{ -\dfrac{1}{12\pi} \log \dfrac{2v}{\varkappa^2} + \dfrac{1}{18\pi} \right\}_{v\to\infty} & \text{for the scalar type} \\ \dfrac{e^2}{4\pi} \left\{ -\dfrac{1}{3\pi} \log \dfrac{2v}{\varkappa^2} + \dfrac{5}{9\pi} \right\}_{v\to\infty} & \text{for the spinor type} \\ \dfrac{e^2}{4\pi} \left\{ -\dfrac{1}{6\pi} \dfrac{v}{2\varkappa^2} + \dfrac{1}{4\pi} \log \dfrac{2v}{\varkappa^2} \right\}_{v\to\infty} & \text{for the vector type,} \end{array} \right. \tag{13.86}$$

---

[1]) This fact can be understood intuitively as follows: the energies of the vacuum particles in an external field $A_\mu^e(x)$ can be roughly written as

$$\frac{n}{(2\pi)^3} \int d^3p\, [\{(\mathbf{p} - e\,\mathbf{A}^e)^2 + \varkappa^2\}^{1/2} - e\,A_0^e].$$

Expanding this as a power series in e, we find the zero point energies of the vacuum as the first term and (D/2) $|A^e|^2$ as the second term. The latter term may be regarded as the self-energy term of the photon. In fact the coefficient D in this term, which is proportional to n, just agrees with D in (13.79). It must be noted that the factor n in D comes from the weights of the various charged particles in the zero point energy.

$$f(\Box) = \frac{1}{4\varkappa^2}\int_0^1 dv\, G(v)\,\frac{1-v^2}{1-\frac{\Box}{4\varkappa^2}(1-v^2)}$$

$$= \begin{cases} \frac{e^2}{4\pi}\left\{-\frac{1}{120\pi\varkappa^2}-\frac{\Box}{1630\pi\varkappa^4}+\ldots\right\} & \text{for the scalar type} \\ \frac{e^2}{4\pi}\left\{-\frac{1}{15\pi\varkappa^2}-\frac{\Box}{140\pi\varkappa^4}+\ldots\right\} & \text{for the spinor type} \\ \frac{e^2}{4\pi}\left[-\left\{\frac{1}{12\pi\varkappa^2}\log\frac{2v}{\varkappa}-\frac{71}{360\pi\varkappa^2}\right\}_{v\to\infty}\right. & \\ \qquad \left.-\frac{17}{1680\pi\varkappa^4}\Box+\ldots\right] & \text{for the vector type} \end{cases} \tag{13.87}$$

where

$$G(v) \equiv \begin{cases} -\frac{e^2}{4\pi}\frac{v^4}{6\pi(1-v^2)} & \text{for the scalar type} \\ -\frac{e^2}{4\pi}\frac{v^2}{\pi(1-v^2)}\left(1-\frac{v^2}{3}\right) & \text{for the spinor type} \\ -\frac{e^2}{4\pi}\left\{\frac{v^4}{2\pi(1-v^2)}+\frac{2v^4}{3\pi(1-v^2)^2}\right\} & \text{for the vector type.} \end{cases} \tag{13.88}$$

These results can be obtained by substituting (8.32) and (8.33) into $(13.83)_s$, $(13.83)_f$, $(13.83)_v$ and changing the variables of integration [1] $(\alpha, \beta \to v, w)$ according to

$$\alpha = \frac{w}{2\varkappa^2}(1-v), \qquad \beta = \frac{w}{2\varkappa^2}(1+v)$$

(SCHWINGER [1949]). The $G(v)$ in (13.88) are obtained as a result of the integration with respect to w.

The second lines in (13.86) and (13.87) give the second and third terms of the vacuum polarisation effects (13.73) of the electron field. In the case of a charged vector field, not only C^0 but also $f(\Box)$ diverges. As shown in Ch. XV the latter infinity is a serious difficulty for the renormalisation theory.

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[1] On the other hand, in the non-covariant perturbation theory (cf. § 2) v is related with the scalar product $(p_\mu p'_\mu)$ of the energy-momenta p, p' $(p_\mu p_\mu + \varkappa^2 = p'_\mu p'_\mu + \varkappa^2 = 0)$ of a charged particle and its antiparticle in a virtual state as follows:

$$2\{p_\mu p'_\mu + \varkappa^2\} = \frac{4\varkappa^2}{1-v^2}$$

(UMEZAWA and KAWABE [1949b]).

EXAMPLE 7. THE SELF-STRESS OF AN ELEMENTARY PARTICLE

We shall consider the expectation value of the energy-momentum tensor $T_{\mu\nu}$ in a state of an elementary particle with energy-momentum p_μ and mass μ ($p_\mu p_\mu + \mu^2 = 0$). In relativistic field theory the energy-momentum tensor $T_{\mu\nu}$ must obey the transformation law of the tensor of the second rank, namely [1])

$$\left.\begin{aligned} E_p &\equiv (p \,|\int d^3x'\, T'_{44}(x')\,|\, p) \\ &= \frac{1}{\sqrt{(1-\beta^2)}}\,(\mathbf{p}=0,\,\mu\,|\int d^3x\, T_{44}(x) - \beta^2 \int d^3x\, T_{11}(x)\,|\,\mathbf{p}=0,\,\mu). \end{aligned}\right\} \tag{13.89}$$

Here $T_{\mu\nu}$ and $T'_{\mu\nu}$ are the energy-momentum tensors in the rest coordinate system (x_μ) of the particle and in the co-ordinate system (x'_μ) moving in the x_1-direction with a velocity $v = \beta = p_1/p_0$. In (13.89) $(p_\mu|$ and $|p_\mu)$ denote the state of the particle with the energy-momentum p_μ. On the other hand E must have the transformation property of the energy of the particle,

$$E_p = \frac{\mu}{\sqrt{(1-\beta^2)}} = \frac{1}{\sqrt{(1-\beta^2)}}\,(\mathbf{p}=0,\,\mu\,|\int d^3x\, T_{44}(x)\,|\,\mathbf{p}=0,\,\mu). \tag{13.90}$$

Transformations (13.89) and (13.90) show that

$$(\mathbf{p}=0,\,\mu|\int d^3x\, T_{11}(x)|\mathbf{p}=0,\,\mu),$$

called the self-stress of the particle, must be zero.

We can prove that this self-stress is zero in the following formal way (TAKAHASHI and UMEZAWA [1952]). First, we shall consider a spinor field ψ interacting with some spinless fields $U^{(\alpha)}(\alpha = 1, 2, \ldots)$. The Lagrangian is

$$L = -\bar{\psi}(\gamma_\mu \partial_\mu + \mu)\psi - \tfrac{1}{2}\sum_\alpha (\partial_\nu U^{(\alpha)} \cdot \partial_\nu U^{(\alpha)} + (\varkappa^{(\alpha)})^2\, U^{(\alpha)}\, U^{(\alpha)}) + L' \tag{13.91a}$$

$$L' = \sum_\alpha f^{(\alpha)}\, \bar{\psi}\, O^{(\alpha)}\, \psi\, U^{(\alpha)} + \sum_\alpha g^{(\alpha)}\, \bar{\psi}\, O^{(\alpha)}_\mu\, \psi\, \partial_\mu U^{(\alpha)}, \tag{13.91b}$$

where $O^{(\alpha)}$ and $O^{(\alpha)}_\mu$ are products of γ-matrices and $g^{(\alpha)}$ are coupling constants *having the dimension of length*: $[g^{(\alpha)}] = [L]$. By means of field equations for $U^{(\alpha)}$ and ψ we have, from (7.3*b*);

$$T_{\mu\mu} = \mu\, \bar{\psi}\, \psi + \sum_\alpha \{(\varkappa^{(\alpha)})^2\, U^{(\alpha)}\, U^{(\alpha)} - f^{(\alpha)}\, \bar{\psi}\, O^{(\alpha)}\, \psi\, U^{(\alpha)}\} + T, \tag{13.92}$$

$$T \equiv \sum_\alpha (\partial_\mu U^{(\alpha)} \cdot \partial_\mu U^{(\alpha)} + (\varkappa^{(\alpha)})^2\, U^{(\alpha)}\, U^{(\alpha)}). \tag{13.93}$$

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[1]) Here we make use of the well known relation for the Lorentz contraction:

$$d^3x' = \sqrt{1-\beta^2}\, d^3x.$$

The energy-momentum tensor in the interaction representation can be obtained from $T_{\mu\nu}$ by means of the unitary transformation $S[\sigma]$. Such an energy-momentum tensor in the interaction representation has some additive terms to (13.92). However, as shown in § 5, we can disregard them by means of the calculation method of the P^*-symbol. Then, $(\mathbf{p}=0, \mu|T_{\mu\mu}|\mathbf{p}=0, \mu)$ can be calculated by substituting (13.92) into $F[x:n]$ of (13.18) and regarding ψ and $U^{(\alpha)}$ as the operators in the interaction representation. Although we have only three dimensional integration d^3x in (13.89), we may integrate it over all four dimensional space and finally omit the time integral, because we must pick up the matrix element (of $\int d^3x T_{\mu\mu}$) in which the energy is conserved:

$$(p'|\int_{-\infty}^{\infty} d^4x\, T_{\mu\mu}(x)|p) = 2\pi(\mathbf{p}'|\int d^3x\, T_{\mu\mu}(x)|\mathbf{p})\ \delta(p_0'-p_0)\ \delta_{\mathbf{p}'\mathbf{p}}.$$

The expectation value

$$(\mathbf{p}=0, \mu\,|\int_{-\infty}^{\infty} d^4x\, T|\,\mathbf{p}=0, \mu)$$

can be calculated by substituting the term [1])

$$2\int_{-\infty}^{\infty} d^4x\, \tfrac{1}{4}\{\partial_\mu \Delta_F^{(\alpha)}(x'-x)\cdot\partial_\mu \Delta_F^{(\alpha)}(x-x'') + (\varkappa^{(\alpha)})^2 \Delta_F^{(\alpha)}(x'-x)\cdot\Delta_F^{(\alpha)}(x-x'')$$

into an arbitrarily chosen α-Boson line $(1/2)\ \Delta_F^{(\alpha)}(x'-x'')$ in the self-energy graph of the Fermion ψ (cf. Fig. 13.14a).

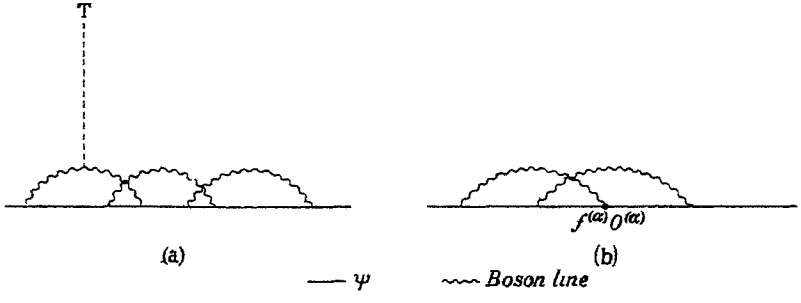

Fig. 13.14

From (8.37) we have

$$\left.\begin{aligned} &2\int_{-\infty}^{\infty} d^4x\, \tfrac{1}{4}\{\partial_\mu \Delta_F^{(\alpha)}(x'-x)\cdot\partial_\mu \Delta_F^{(\alpha)}(x-x'') + (\varkappa^{(\alpha)})^2 \Delta_F^{(\alpha)}(x'-x)\,\Delta_F^{(\alpha)}(x-x'')\} \\ &\quad = -2i\,(\tfrac{1}{2})\,\Delta_F^{(\alpha)}(x'-x''). \end{aligned}\right\} \quad (13.94)$$

[1]) $\Delta_F^{(\alpha)}$ is the Δ_F-function of the $U^{(\alpha)}$-field.

Thus, we see that one insertion of the operator T gives $-2\delta\mu$, where $\delta\mu$ is the self-energy of the Fermion.

Since there are $n/2$ positions at which this insertion in the nth order self-energy graph of the Fermion can be made, it follows that

$$(\mathbf{p}=0,\,\mu\,|\int d^3x\,T\,|\,\mathbf{p}=0,\,\mu)=-\sum_{n=0}^{\infty} n\,\delta\mu^{(n)}. \qquad (13.95a)$$

Here $\delta\mu^{(n)}$ means the nth order term of the self-energy $\delta\mu$ in the perturbation calculation, and therefore has the form

$$\delta\mu^{(n)}=\sum^{(\alpha)}\{\prod_{\alpha}(g^{(\alpha)})^{m_\alpha}(f^{(\alpha)})^{l_\alpha}\}\,F(m_1\dots m_\alpha\dots;\,l_1\dots l_\alpha\dots), \quad (13.95b)$$

where $\sum^{(\alpha)}$ means to sum over all possible m_α and l_α under the condition

$$\sum_{\alpha}(m_\alpha+l_\alpha)=n.$$

The functions $F(\dots;\dots)$ are independent of coupling constants. Then, from (13.95a, b) we have

$$(\mathbf{p}=0,\,\mu\,|\int d^4x\,T\,|\,\mathbf{p}=0,\,\mu)=-\sum_{\alpha}\Big\{f^{(\alpha)}\frac{\partial}{\partial f^{(\alpha)}}\delta\mu+g^{(\alpha)}\frac{\partial}{\partial g^{(\alpha)}}\delta\mu.\Big\} \quad (13.96)$$

The term

$$(\mathbf{p}=0,\,\mu|f^{(\alpha)}\textstyle\int d^3x\,\bar\psi O^{(\alpha)}\psi\,U^{(\alpha)}|\mathbf{p}=0,\,\mu)$$

can also be calculated by means of the self-energy graph of the Fermion, in which any vertex of the interaction $f^{(\alpha)}\bar\psi O^{(\alpha)}\psi U^{(\alpha)}$ is regarded as that due to $f^{(\alpha)}\bar\psi O^{(\alpha)}\psi U^{(\alpha)}$ in $T_{\mu\mu}$ (cf. Fig. 13.14b). Since each term of (13.95b) in the perturbation calculation is made up of m_α such vertices, we have

$$\left.\begin{aligned}&\sum_{(\alpha)}(\mathbf{p}=0,\,\mu\,|\,f^{(\alpha)}\textstyle\int d^3x\,\bar\psi\,O^{(\alpha)}\,\psi\,U^{(\alpha)}\,|\,\mathbf{p}=0,\,\mu)\\&\quad=\sum^{(\alpha)}m_\alpha\,(f^{(\alpha)})^{m_\alpha}\,(g^{(\alpha)})^{l_\alpha}\,F(\dots;\dots)=-\sum_{\alpha}f^{(\alpha)}\frac{\partial}{\partial f^{(\alpha)}}\,\delta\mu.\end{aligned}\right\} \quad (13.9)$$

On the other hand, from (8.37), and (13.41), we have

$$\mu\frac{\partial}{\partial\mu}S_F(x-x')=-\frac{i}{2}\,\mu\int d^4x''\,S_F(x-x'')\,S_F(x''-x') \quad (13.97a)$$

$$\varkappa^{(\alpha)}\frac{\partial}{\partial\varkappa^{(\alpha)}}\Delta_F^{(\alpha)}(x-x')=-\,i\,\varkappa^2\int d^4x''\,\Delta_F^{(\alpha)}(x-x'')\,\Delta_F^{(\alpha)}(x''-x'). \quad (13.97b)$$

This shows that the operation $\mu\partial/\partial\mu$ on an internal Fermion line or the operation $\varkappa^{(\alpha)}\partial/\partial\varkappa^{(\alpha)}$ on an α-Boson line are equivalent to the insertion of $-i\mu\bar{\psi}\psi$ and $-i(\varkappa^{(\alpha)})^2\, U^{(\alpha)}U^{(\alpha)}$ respectively. Thus, we have

$$\left.\begin{aligned}&(\mathbf{p}=0,\mu\,|\,\mu\int d^3x\;\bar{\psi}\,\psi\,|\,\mathbf{p}=0,\mu)=\mu\frac{\partial}{\partial\mu}\,\delta\mu\\&(\mathbf{p}=0,\mu\,|\,(\varkappa^{(\alpha)})^2\int d^3x\;U^{(\alpha)}\,U^{(\alpha)}\,|\,\mathbf{p}=0,\mu)=\varkappa^{(\alpha)}\frac{\partial}{\partial\varkappa^{(\alpha)}}\,\delta\mu.\end{aligned}\right\}\tag{13.98}$$

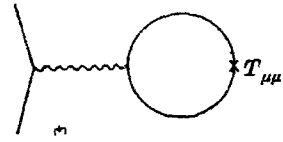

Fig. 13.15

From (13.92),

$$(\mathbf{p}=0,\mu\,|\int d^3x\,T_{\mu\mu}\,|\mathbf{p}=0,\mu)=\Big(\mu\frac{\partial}{\partial\mu}+\sum_\alpha\varkappa^{(\alpha)}\frac{\partial}{\partial\varkappa^{(\alpha)}}-\sum_\alpha g^{(\alpha)}\frac{\partial}{\partial g^{(\alpha)}}\Big)\,\delta\mu+\mu.\tag{13.99}$$

Since the energy operator $\int d^3x T_{44}$ must give the self-energy $\delta\mu$ according to

$$(\mathbf{p}=0,\mu|\int d^3x\,T_{44}|\mathbf{p}=0,\mu)=\mu+\delta\mu$$

we have [1])

$$(\mathbf{p}=0,\mu\,|\int d^3x\,T_{kk}\,|\mathbf{p}=0,\mu)=\Big(\mu\frac{\partial}{\partial\mu}+\sum_\alpha\varkappa^{(\alpha)}\frac{\partial}{\partial\varkappa^{(\alpha)}}-\sum_\alpha g^{(\alpha)}\frac{\partial}{\partial g^{(\alpha)}}-1\Big)\,\delta\mu.\tag{13.100}$$

Since we must have

$$\begin{aligned}(\mathbf{p}=0,\mu|\int d^3x\,T_{11}|\mathbf{p}=0,\mu)&=(\mathbf{p}=0,\mu|\int d^3x\,T_{22}|\mathbf{p}=0,\mu)\\&=(\mathbf{p}=0,\mu|\int d^3x\,T_{33}|\mathbf{p}=0,\mu)\end{aligned}$$

in the rest system of the Fermi particle ($\mathbf{p}=0$, $p_0=\mu$), we have from (13.100):

$$\left.\begin{aligned}&(\mathbf{p}=0,\mu\,|\int d^3x\,T_{11}\,|\,\mathbf{p}=0,\mu)\\&\quad=\frac{1}{3}\Big(\mu\frac{\partial}{\partial\mu}+\sum_\alpha\varkappa^{(\alpha)}\frac{\partial}{\partial\varkappa^{(\alpha)}}-\sum_\alpha g^{(\alpha)}\frac{\partial}{\partial g^{(\alpha)}}-1\Big)\,\delta\mu.\end{aligned}\right\}\tag{13.101}$$

---

[1]) In the calculation of $(T_{\mu\mu})$ there appear some diagrams in which the operator $T_{\mu\mu}$ operates on a closed loop, which is connected with the other part of the diagram through a single line with zero momentum (cf. Fig. 13.15). The contribution of these diagrams is called the vacuum polarisation. In the usual calculation of the self-energy $\delta\mu$ this contribution is neglected. However, $\delta\mu$ in (13.100) and (13.101) must include them. In quantum electrodynamics they are zero on account of the Furry's theorem (SAWADA [1950], EPSTEIN [1951], TAKAHASHI and UMEZAWA [1952].

Similar results can be obtained in more general case. Suppose that there are fields $Q^{(\alpha)}$ ($\alpha = 1, 2, \ldots$) interacting with each other through interactions $H^{(a)}$ ($a = 1, 2, \ldots$), the coupling constants $g^{(a)}$ of which have dimensions $[g^{(a)}] = [L^{n^{(a)}}]$. Then, we have (TAKAHASHI and UMEZAWA [1952]):

$$\left.\begin{aligned}&(\mathbf{p}=0, \varkappa^{(1)}|\int d^3x\, T_{11}|\mathbf{p}=0, \varkappa^{(1)})\\ &\quad=\frac{1}{3}\Big[\sum_{\alpha}\varkappa^{(\alpha)}\frac{\partial}{\partial\varkappa^{(\alpha)}}-\sum_{a}n^{(a)}g^{(a)}\frac{\partial}{\partial g^{(a)}}-1\Big]\delta\varkappa^{(1)}\end{aligned}\right\} \tag{13.102}$$

where $\varkappa^{(\alpha)}$ denote the mass of the $Q^{(\alpha)}$-field, $\delta\varkappa^{(1)}$ is the self-energy of the particle ($\alpha = 1$) and $(\mathbf{p}=0, \varkappa^{(1)}|$ indicates the expectation value in the state of the particle ($\alpha = 1$) at rest. This relation was first discovered by PAIS and EPSTEIN [1949] in the simple case of the electron and the electromagnetic field, for which $g^{(a)} = 0$, and $\varkappa^{(\alpha)} = 0$ and m for photon and electron, respectively.

Since $\delta\varkappa^{(1)}$ has dimensions $[\delta\varkappa^{(1)}] = [L^{-1}]$, it must have the form

$$\delta\varkappa^{(1)}=\sum_{m_\alpha, l_a} a(m_\alpha, l_a)\prod_{\alpha, a}(\varkappa^{(\alpha)})^{m_\alpha}(g^{(a)})^{l_a} \tag{13.103}$$

with

$$\sum_{a} n^{(a)} l_a - \sum_{\alpha} m_\alpha = -1.$$

Here the dimensionless coefficients $a(m_\alpha, l_a)$ can depend on $\log(\varkappa^{(\alpha)}/\varkappa^{(\alpha')})$. By substituting (13.103) into (13.102), we have

$$(\mathbf{p}=0, \varkappa^{(1)}|\int d^3x\, T_{11}(x)|\mathbf{p}=0, \varkappa^{1}) = 0. \tag{13.104}$$

However, practical calculation sometimes lead to non-zero self-stress. This fact can also be understood from (13.102). When the self-energy $\delta\varkappa^{(1)}$ is infinite, it must have, instead of (13.103), the form

$$\delta\varkappa^{(1)} = \lim_{P\to\infty}\delta\varkappa^{(1)}(P). \tag{13.105}$$

In other words, there appears a new constant $P(\to\infty)$, whose dimension is $[L^{-1}]$. Here P denotes the upper limit of the integration appearing in the calculation of the self-energy. Thus, above dimensional consideration without taking into account P cannot hold and we find a non-zero self-stress. This difficulty is called the problem of the self-stress.

For example, the self-stress of an electron interacting with an

electromagnetic field can be calculated from its self-energy (13.63)

$$\delta\varkappa = \lim_{P\to\infty}\left[a\varkappa\log\frac{P}{\varkappa} - b\varkappa\right].$$

Here a and b are dimensionless finite numbers ($a = 3e^2/8\pi^2$, $b = 3e^2/48\pi^2$). Since the charge e is dimensionless, (13.102) gives:

$$(\mathbf{p}=0,\varkappa|\int d^3x T_{11}|\mathbf{p}=0,\varkappa) = \frac{1}{3}\left(\varkappa\frac{\partial}{\partial\varkappa} - 1\right)\delta\varkappa = -\frac{a}{3}\varkappa = -\frac{e^2}{2\pi(4\pi)}\varkappa. \quad (13.106)$$

This agrees with the results of the practical calculations. However, if $\delta\varkappa$ *were finite*, its only possible form would be

$$\delta\varkappa = c\varkappa \quad (13.107)$$

with a dimensionless constant c, because $\varkappa$ is the only constant with dimension in quantum electrodynamics. If we substitute this into (13.102), we have

$$(\mathbf{p}=0,\varkappa|\int d^3x\, T_{11}|\mathbf{p}=0,\varkappa) = \frac{1}{3}\left(\varkappa\frac{\partial}{\partial\varkappa} - 1\right)\delta\varkappa = 0.$$

Equation (13.104) shows why Pauli's regularisation method (RÖHRLICH [1950]) or the mixture theory (YUKAWA and UMEZAWA [1949]) succeeded in giving a zero-self-stress, for those theories lead to a finite self-energy of the electron. On the other hand, the difficulty of the self-stress cannot be settled by the renormalisation theory, in which, as shown in the next Chapter, the self-energy itself is not finite.

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CHAPTER XIV

RENORMALISATION THEORY IN QUANTUM ELECTRODYNAMICS

§ 1. Renormalisation Theory

In the present quantum field theory serious difficulties arise from the existence of the proper fields. However, since the infra-red catastrophe can only be eliminated by taking account of the proper field, we cannot disregard it. In such a case it would seem difficult to understand the experimental success of quantum electrodynamics. The resolution of this difficulty may be provided by searching for the observable effects of proper fields.

First, we shall again consider the ultra-violet catastrophe discussed in the last Chapter. In the problem of the elastic scattering of an electron in the e^2-approximation, the effects of the proper electromagnetic fields (i.e. the so-called radiative corrections) give infinite values for the modifications $\delta\varkappa$ and δe of the electron mass and charge. Since we observe not the bare electron but the electron with the proper field, the observed values of the mass and charge must be equal to $\varkappa' \equiv \varkappa + \delta\varkappa$ and $e_1 \equiv e + \delta e$.

The mass term of the Lagrangian can be written as

$$-\varkappa\bar{\psi}\psi = -\varkappa'\bar{\psi}\psi + \delta\varkappa\bar{\psi}\psi. \tag{14.1}$$

When we take an interaction representation in which ψ satisfies the wave equation of the free electron of mass $\varkappa'(\varkappa \to \varkappa'$ in (7.110)), the second term of (14.1) gives the interaction Hamiltonian

$$H'_{\text{m}}(x) = -\delta\varkappa\bar{\psi}(x)\psi(x). \tag{14.2}$$

Taking $\delta\varkappa = A^0$, we can eliminate the effect of (13.64). Similarly, writing the electromagnetic interaction as

$$e\bar{\psi}\gamma_\mu\psi A_\mu = e_1\bar{\psi}\gamma_\mu\psi A_\mu - \delta e\bar{\psi}\gamma_\mu\psi A_\mu \tag{14.3}$$

with

$$e_1 \equiv e + \delta e \tag{14.4}$$

and regarding $e_1^2/4\pi$ as the observed value 1/137 of the elementary charge, we can eliminate the first term in (13.75) by means of the

second term in (14.3) (with $\delta e = eC^0$). These procedures are called mass and charge renormalisations (ITO, KOBA and TOMONAGA [1947], SCHWINGER [1948], LEWIS [1948], EPSTEIN [1948]). Since we can obtain definite finite answers in the e^2-approximation by means of these procedures we can compare them with the experimental data.

For example, this theory gives for an electron in a static magnetic field an anomalous magnetic moment 0.00117 $(-e/2m)$ due to its proper field (SCHWINGER [1948]). Here, $-e/2m$ is the Bohr magneton for an electron. In the same way one would expect the energy levels of an electron in the Coulomb potential of the hydrogen atom to be influenced by its proper field: the $2^2S_{1/2}$-level energy (which must be the same as the $2^2P_{1/2}$-level energy when we do not take into account radiative corrections) comes out larger by about 1000 Mc than the latter level energy. This latter fact had been previously suggested by some experiments (PASTERNACK [1938]).

Conclusive experimental data on the energy levels of the hydrogen atom and the anomalous magnetic moment of the electrons were given by LAMB and RETHERFORD [1947] and RABI's group [1947]; these results have been successfully explained by the renormalisation theory.

This fact shows that the concept of the proper field has a real meaning, and that renormalisation theory is valid at least in some circumstances. However, in order to show that the interpretation is consistent, we must prove two points: First, that in the renormalisation theory of quantum electrodynamics there are no infinities in any approximation of the perturbation series; second, that this perturbation series converges to a definite result. The first problem was analysed in detail by DYSON [1949]. We shall consider this problem in the following paragraphs.

§ 2. Primitively Divergent Diagrams

We shall now consider how many kinds of infinity appear in quantum electrodynamics. A diagram is called primitively divergent (DYSON [1949]) when its S-matrix element is infinite but the infinity arises only in the last 4-momentum integration; i.e. the result of the $n-1$ (n = number of internal lines) 4-momentum integrations which we choose to perform first is finite. In other words, a diagram obtained by cutting any internal line of a primitively divergent diagram gives a finite S-matrix element.

Let us ask how many sorts of primitively divergent diagrams there are in quantum-electrodynamics.

It is easily seen that for a primitively divergent diagram, we must have the relation

$$N = (\text{highest order of products of independent variables in the numerator}) - (\text{those in the denominator}) \geqslant 0. \tag{14.5}$$

The independent variables are here selected as follows: Some integration variables are eliminated by the δ-function [1]) representing the energy-momentum conservation law at each vertex, and the remaining variables are regarded as the independent variables.

The inequality (14.5) must be interpreted as a condition for divergence only when the denominators of the integrands give rise to no singularities. In fact, we can rewrite the denominators in a positive definite form by changing the contour of the k_0-integration from the real axis to the imaginary axis [2]). As shown in Fig. 13.4 this can be done without crossing undisplaced poles. Moreover, the contribution of the displaced poles is finite, because the momenta integration domain at the displaced poles is limited by (13.39).

We shall consider a primitively divergent diagram in which the numbers of external and internal electron lines are E_e and I_e respectively, in which those of the external and internal photon lines are E_p and I_p respectively and the number of vertices is n (from which n_s correspond to the mass type interaction (14.2)). Equations (8.35) and (13.24a) show that each electron or photon internal line contributes 1 or 2 to the second term in (14.5). Moreover, we have $4I$ integration variables ($d^4k_1 \ldots d^4k_I$, $I \equiv I_e + I_p$) in the numerator. However, $4(n-1)$ of them are eliminated on account of the δ-functions at each vertex. On the other hand, taking into account the fact that three lines are inserted at each vertex for the interaction (14.3), we can easily derive the relation [3])

$$\left.\begin{aligned} 2I_p + E_p &= n - n_s \\ 2I_e + E_e &= 2n \end{aligned}\right\} \tag{14.6}$$

---

[1]) This δ-function comes from the integration $\int_{-\infty}^{\infty} d^4x$ at each vertex.

[2]) Then $k_\mu^2 = k_1^2 + k_2^2 + k_3^2 - k_0^2$ changes into $k_1^2 + k_2^2 + k_3^2 + |k_0|^2$.

[3]) Each vertex corresponds to one external photon line or a half of internal photon line.

Thus, we have

$$\left.\begin{aligned} N &= 4\,(I-n+1)-I_e-2I_p \\ &= 4-\tfrac{1}{2}E_e-E_p-n_s. \end{aligned}\right\} \tag{14.7}$$

Then, the condition (14.5) for a diagram to be primitively divergent can be written as

$$4-\frac{3}{2}E_e-E_p-n_s \geqslant 0 \tag{14.8}$$

DYSON [1949]). The number N is the highest degree of the infinities of the primitively divergent diagrams, it being understood that $N=0$ implies logarithmic divergence.

There are six cases satisfying (14.8):

(a) $E_p=0$, $E_e=2$, (b) $E_p=1$, $E_e=2$, (c) $E_p=2$, $E_e=0$,
(d) $E_p=4$, $E_e=0$, (e) $E_p=3$, $E_e=0$, (f) $E_p=1$, $E_e=0$.

However, the S-matrix elements given by (e) or (f) are zero by Furry's theorem (cf. Example 2 of Ch. XIII); (d) leads to a logarithmically divergent S-matrix of the form $A_\mu A_\nu A_\varrho A_\sigma$. We shall take this as zero on account of the gauge-invariance of the theory, in a similar way to the zero self-energy of a photon.

The diagrams (a), (b) and (c) of Fig. 14.1 are called photon self-energy, vertex part and electron self-energy diagrams, respectively. Then, the above result can be expressed as follows: any primitively divergent diagrams must be (a), (b) or (c) [1].

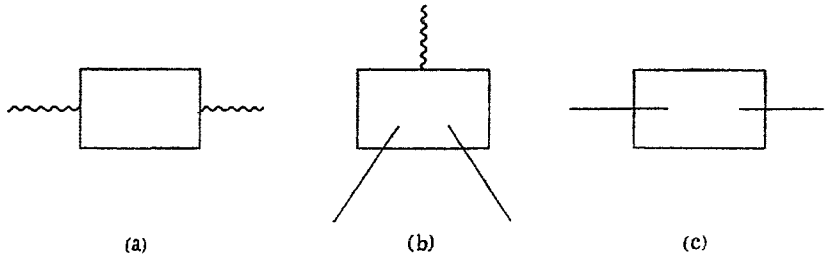

Fig. 14.1

§ 3. Separation of Infinities

We shall consider a primitively divergent diagram $G(E_p, E_e)$ with the Ep electron and Ee photon external lines. When this primitively divergent diagram is contained in a Feynman diagram, the momenta

[1] However, divergences of (a), (b) and (c) are not always primitive.

$t^{(m)}$ $(m=1 \ldots n)$ of its external lines do not necessarily satisfy the free field equation. By denoting all the external momenta by the symbol t, we can write the S-matrix element given by $G(E_p, E_e)$ as

$$\int d^4k\, R(k, t). \tag{14.9}$$

Since the diagram is primitively divergent, $R(k, t)$ contains no infinities. The integration (14.9) is called the final integration.

Let us introduce momenta $t^{0(m)}$ (corresponding to $t^{(m)}$) of the free electrons and photons, by the definitions

$$\left.\begin{aligned} i\gamma_\mu t^{0(m)} + \varkappa' = 0 \quad & \text{when } t^{(m)} \text{ is the momentum of the electron line} \\ t_\mu^{0(m)}\, t_\mu^{0,m)} = 0 \quad & \text{when } t^{(m)} \text{ is the momentum of the photon line.} \end{aligned}\right\} \tag{14.10}$$

By denoting $(t^{0(1)} \ldots t^{0(n)})$ by t^0 we can write (14.9) as

$$\left.\begin{aligned} & \int d^4k\, R(k, t) = \int d^4k\, R(k, t^0) \\ & \sum_m (t_\mu^{(m)} - t_\mu^{0(m)}) \int d^4k \left[\frac{\partial}{\partial t_\mu^{(m)}} R(k, t)\right]_{t=t^0} \\ & \sum_{m,l} (t_\mu^{(m)} - t_\mu^{0(m)})\,(t_\nu^{l} - t_\nu^{0(l)}) \int d^4k \left[\frac{\partial^2}{\partial t_\mu^{(m)}\, \partial t_\mu^{(l)}} R(k, t)\right]_{t=t_0} \\ & \ldots + \frac{1}{N!} \sum (t_{\mu_1}^{(m_1)} - t_{\mu_1}^{0(m_1)}) \ldots (t_{\mu_N}^{(m_N)} - t_{\mu_N}^{0(m_N)}) \int d^4k \left[\frac{\partial}{\partial t_{\mu_1}^{(m_1)}} \ldots \frac{\partial}{\partial t_{\mu_N}^{(m_N)}} R(k, t)\right]_{t=t^0} \\ & \qquad + (\text{a finite function of } t). \end{aligned}\right\} \tag{14.11}$$

(DYSON [1949]) It can be easily seen that the highest degrees of infinities of the first, second, ... terms of (14.11) are N, $N-1$, ... (N is defined by (14.5)) respectively. Thus, we can separate the infinite terms from the finite terms by means of (14.11) (cf. (13.62)). For large t (i.e. large $t^{(1)} \ldots t^{(n)}$ in comparison with masses of particles), the last term in (14.11) may be approximately written as a sum of terms whose forms are products $(t^{(m_1)} \ldots t^{(m_{N'})})$ multiplied by constant factors.

The $(l+1)$th term is called $G(Ep, Ee, l)$-term. The highest degree of infinities of the $G(Ep, Ee, l)$-term is $(N-l)$, it being understood that $N-l=0$ implies logarithmic divergence.

It is clear that every diagram that gives rise to a divergent S-matrix element contains some primitively divergent diagrams.

§ 4. Irreducible Diagrams

The diagram which is obtained by replacing all the self-energy and vertex parts of a given diagram by lines and vertices containing

no radiative corrections is called a skeleton. A diagram is called irreducible, when it coincides with its skeleton.

There are no proper irreducible diagrams of the electron and photon self-energies except those shown in Fig. 14.2.

A diagram is called a proper one when it cannot be divided into two parts by cutting only one internal line.

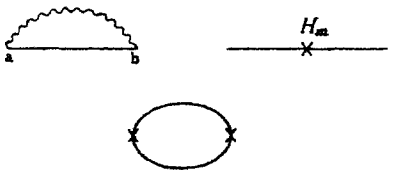

Fig. 14.2

§ 5. Renormalisation Constants

Let us denote all contributions of the electron and photon self-energies and vertex parts diagrams by $S'_F(t)$, $\Delta'_F(t)$ and $\Gamma'_\mu(t^1, t^2)$ respectively; t is the energy-momentum of the external line of the self-energies diagrams, and (t^1, t^2) are those of the external electron lines of the vertex part. All contributions of the proper diagrams of the electron and photon self-energies are denoted by $\bar{\psi}\Sigma^*\psi$ and $A_\mu \Pi^*_{\mu\nu} A_\nu$ respectively. They are obtained by summing the contributions of all possible diagrams given by inserting internal lines in the respective irreducible diagrams (Fig. 14.2). In other words, they are obtained from the S-matrix elements of the respective irreducible diagrams by replacing S_F, Δ_F and γ_μ at the points a of Fig. 14.2 by S'_F, Δ'_F and Γ'_μ. It must be noted that Γ_μ must not replace γ_μ at the points b of Fig. 14.2. In fact, the diagram given by inserting an internal line "around" the point b, i.e. the vertex at b, is equivalent to the diagram given by inserting an internal line "around" the point a, i.e. the vertex at a, (cf. Fig. 14.3).

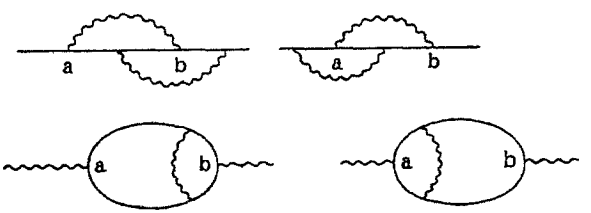

Fig. 14.3

Thus we see that Σ^* and $\Pi^*_{\mu\nu}$ can be written as

$$\Sigma^* = e^2 \int d^4k\, \gamma_\mu\, S'_F(t-k)\, \Delta'_F(k)\, \Gamma_\mu(t-k, t) - 2\pi i \delta\varkappa \qquad (14.12a)$$

$$\Pi^*_{\mu\nu} = e^2 \int d^4k\, S_F(S'_F(t-k)\, \gamma^\mu\, S'(k)\, \Gamma_\nu(t-k, t)). \qquad (14.12b)$$

Since any self-energy diagram is obtained by connecting a self-energy diagram with a proper self-energy diagram, we can write (DYSON [1949])

$$S'_F = S_F + S_F \Sigma^* S'_F \qquad (14.13a)$$

$$\Delta'_F = \Delta_F + \Delta_F \Pi^* \Delta'_F, \qquad (14.13b)$$

where it is to be understood that all quantities on both sides refer to the same momentum. $S_F(k)$ and $\Delta_F(k)$ are the Fourier amplitudes of $S_F(x)$ and $\Delta_F(x)$ respectively:

$$\left.\begin{aligned} \Delta_F(k) &= \frac{1}{2\pi i}\,\frac{1}{k_\mu k_\mu} \\ S_F(k) &= \frac{1}{2\pi i}\,\frac{i\gamma_\mu k_\mu - \varkappa'}{(k_\mu k_\mu + \varkappa'^2)} \end{aligned}\right\} \qquad (14.13c)$$

Quantities S'_F and Δ'_F can be calculated by (14.13a, b).

We shall now assume that S'_F, Δ'_F and Γ'_μ have the forms

$$\left.\begin{aligned} S'_F &= Z_2 S'_{F1}(e_1) \\ \Delta'_F &= Z_3 \Delta'_{F1}(e_1) \\ \Gamma_\mu &= Z_1^{-1}\, \Gamma_{\mu 1}(e_1). \end{aligned}\right\} \qquad (14.14)$$

Here Z_1, Z_2 and Z_3 are constants which must be determined such that S'_{F1}, Δ'_{F1} and $\Gamma_{\mu 1}$ contain no infinities. The observable charge e_1 is finite and such that

$$e_1 = fe \qquad (14.15)$$

(with a constant f). This equation can be regarded as an extended expression of the charge renormalisation (14.4).

Substituting (14.14) into (14.12a, b),

$$\Sigma^* = Z_1^{-1} Z_2 Z_3\, e^2 \int d^4k\, \gamma_\mu S'_{F1}(e_1)\, \Delta'_{F1}(e_1)\, \Gamma_{\mu 1}(e_1) - 2\pi i \delta\varkappa \qquad (14.16a)$$

$$\Pi^*_{\mu\nu} = Z_2^2 Z_1^{-1} e^2 \int d^4k\, S_F(S'_{F1}(e_1)\, \gamma_\mu S'_{F1}(e_1)\, \Gamma_{\mu 1}(e_1)). \qquad (14.16b)$$

Equation (14.13) can be written by using the rules of the operational calculus as

$$S'_{F1}(e_1) = \frac{S_F}{Z_2 - Z_2 S_F \Sigma^*} \qquad (14.17a)$$

$$\Delta'_{F1}(e_1) = \frac{\Delta_F}{Z_3 - Z_3 \Delta_F \Pi^*}. \qquad (14.17b)$$

On the other hand, we shall prove in the next paragraph that

$$\left.\begin{aligned}&\int d^4k\,\gamma_\mu S'_{F1}(e_1)\,D'_{F1}(e_1)\,\Gamma_{\mu 1}(e_1)\\&=Z_1^{-1}\{A(e_1)-iB(e_1)(\gamma_\mu\partial_\mu+\varkappa')+\ldots\}\end{aligned}\right\} \tag{14.18}$$

$$\int d^4k\,S_P(S'_{F1}(e_1)\,\gamma_\mu S'_{F1}(e_1)\,\Gamma_{\mu 1}(e_1))=Z_1^{-1}\{-C(e_1)\,\Box+\ldots\} \tag{14.19}$$

$$\Gamma_\mu=\gamma_\mu+Z_1^{-1}\{L(e_1)\,\gamma_\mu+\ldots\}, \tag{14.20}$$

where $A(e_1)$, $B(e_1)$, $C(e_1)$ and $L(e_1)$ are some constants. Terms ... in (14.18) (14.19) and (14.20) are finite and contain the factors $(\gamma_\mu\partial_\mu+\varkappa')$, $\Box^2$ and ∂_ϱ respectively. From (14.16a, b) we have

$$Z_2 S_F\,\Sigma^*=(Z_1^{-1}Z_2Z_3^{1/2}e)^2\left\{A(e_1)S_F+\frac{1}{2\pi}B(e_1)+\ldots\right\}-2\pi i Z_2\,\delta\varkappa S_F \tag{1421}$$

$$Z_3\Delta_F\Pi^*=(Z_1^{-1}Z_2Z_3^{1/2}e)^2\left\{\frac{1}{2\pi}C(e_1)+\ldots\right\}. \tag{14.22}$$

Substituting these relations into (14.17a, b), and taking into account the conditions $S'_{F1}\to S_F$, $\Delta'_{F1}\to\Delta_F$, $\Gamma'_{\mu 1}\to\gamma_\mu$ for $e_1\to 0$ (cf. (14.13c)), we see that S'_{F1}, Δ'_{F1} and $\Gamma_{\mu 1}$ contain no infinities only when the constants are taken to be

$$\left.\begin{aligned}\delta\varkappa&=\frac{1}{2\pi i}Z_2^{-1}e_1^2A(e_1)\\Z_1&=1-L(e_1)\\Z_2&=1+\frac{1}{2\pi}e_1^2B(e_1)\\Z_3&=1+\frac{1}{2\pi}e_1^2C(e_1)\\f&=Z_1^{-1}Z_2Z_3^{1/2}.\end{aligned}\right\} \tag{14.23}$$

These are the "renormalisation constants".

§ 6. Final Integration

We shall now establish the equations (14.18, 19, 20). Any proper diagram ((14.5) defines a number N for this diagram) of the electron self-energy may contain primitively divergent diagrams called G_1, G_2 ... ((14.5) gives numbers N_1, N_2 ... for them). Infinities of G can be separated by using (14.11). Omitting these infinite terms, we can obtain the finite term. This term is approximately proportional to the N_ith power of the external momenta t_μ of the primitively divergent diagram G_i, when the latter momenta are large. These

momenta become internal ones for a primitively divergent diagram in which the abovementioned finite terms for all G_i are substituted into the internal line or the vertex, corresponding to the former primitively divergent diagrams (i.e. G_i, $(i=1, 2, \ldots)$. Separating infinite terms of this integration by (14.11), omitting them and repeating the procedure, we obtain the "final integration" of the electron self-energy diagram as

$$\int d^4k\, \gamma_\mu\, S'_{F1}(e)\, \Delta'_{F1}(e)\, \Gamma_{\mu 1}(e) \tag{14.24}$$

on account of (14.12a).

This successive separation of infinities is discussed in detail in § 9. We can say here that it determines $(S'_{F1}, \Delta'_{F1}, \Gamma_{\mu 1})$ and (A, B, C, L); the latter constants then determine the renormalisation constants $(\delta\varkappa, Z_1, Z_2, Z_3, f)$ by (14.23).

It must be noted that (14.24) is not yet written in terms of e_1 but e. We see from the above discussion that the highest degree of the infinities in (14.24) is N, given by (14.5) (and therefore by (14.7)). Taking $E_p = 0$ and $E_e = 2$, we see that (14.24) contains linear and logarithmic infinities. Therefore, using (14.11) we can write (14.24) as

$$\int d^4k S'_{F1}(e)\gamma_\mu \Delta'_{F1}(e)\, \Gamma_{\mu 1}(e) = Z_1^{-1}\{A(e) - iB(e)(\gamma_\mu \delta_\mu + \varkappa') + \ldots\}. \tag{14.25}$$

Here $A(e)$ and $B(e)$ are constants diverging at most linearly and logarithmically, respectively [1]).

The reason for the Z_1^{-1} factor in (14.25) is the following one. Although Γ_μ has replaced γ_μ in the vertex a and not in the vertex b, we must expect the final result to be symmetric with respect to the two vertices a and b. Then, (14.14) shows that we also have the factor Z_1^{-1} at the vertex b. The proof of this fact, which we shall give in § 9, was given by SALAM [1951]. The infinite Z_1^{-1} at the vertex b is called the b-divergence.

Replacing e in (14.25) by e_1 we obtain (14.18). Equations (14.19) and (14.20) can be established in a similar way.

We can obtain finite results for the S-matrix by using this procedure of successive separation of infinities.

§ 7. Renormalisation of External Lines

We shall denote the sums of all possible ψ-, $\bar{\psi}$- and A_μ-external lines with many internal lines by ψ', $\bar{\psi}'$ and A'_μ respectively. When

[1]) The electron self-energy can be proved to be logarithmically divergent. (WEISSKOPF [1939]).

ψ, $\bar{\psi}$ and A_μ are operators of free fields we have

$$\psi' = Z_2^{1/2}\psi, \quad \bar{\psi}' = Z_2^{1/2}\bar{\psi}, \quad A'_\mu = Z_3^{1/2} A_\mu. \tag{14.26}$$

In fact, in the diagram of the $S^{-1}[\infty]\,S[\infty]$ (which is equal to unity on account of the unitarity of $S[\infty]$), some external lines of $S[\infty]$ and $S^{-1}[\infty]$ are connected with each other to give internal lines. These units occur with the infinite coefficient given by (14.14). In other words, (14.26) gives the normalisation of ψ', $\bar{\psi}'$ and A'_μ such that $S[\infty]$ is unitary.

In fact, using the equation similar to (14.13a, b) with ψ', $\bar{\psi}'$ and A'_μ:

$$\psi' = \psi + \psi\,\Sigma^*\,S'_F$$
$$A'_\mu = A_\mu + A_\nu\,\Pi^*_{\nu\mu}\,\Delta'_F$$

we can calculate these infinite coefficients. However, these results cannot be unique. We therefore must use the unitarity of $S[\infty]$ to determine these coefficients uniquely. This situation was exemplified by the determination of B^0 in (13.67) (cf. discussions following (13.67)). As shown there, the coefficients in (14.26) can also be uniquely determined by making the assumption of adiabatic switching on and off of the interaction constant (DYSON [1951], LÜDERS [1952]).

§ 8. Renormalisation

We have seen in § 4 that the S-matrix element can be obtained by replacing S_F, Δ_F and γ_μ corresponding to the electron and photon internal lines and vertices of all possible irreducible diagrams, by S'_F, Δ'_F and Γ_μ respectively. Then, all infinities can be written as the factor $Z_1^{-1}\,Z_2\,Z_3^{1/2}$ at each vertex (cf. Fig. 14.4) where Z_2 and $Z_3^{1/2}$ are given by two electron lines and one photon line entering the vertex respectively (cf. (14.14)), and Z_1^{-1} appears on account of the replacement $\gamma_\mu \to \Gamma_\mu$. However, this infinite factor can be amalgamated into the observed charge $e_1 = fe$ (cf. (14.23). Thus, we see that there are no infinities in the S-matrix of the perturbation theory of the quantum electrodynamics. Since we can prove the following relation

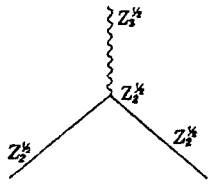

Fig. 14.4

$$Z_1 = Z_2 \tag{14.27a}$$

(WARD [1951]), we have

$$e_1 = Z_3^{1/2} e, \quad f = Z_3^{1/2}. \tag{14.28}$$

The equation (14.27a) corresponds to (13.70) in the e^2-approximation.

A proof of (14.27a) is as follows. On account of the relation

$$\frac{\partial}{\partial k_\mu} S_F(k) = -2\pi S_F(k)\, \gamma_\mu S_F(k) \tag{14.27b}$$

we see that the derivation with respect to k_μ corresponds to adding a photon line of zero energy-momentum to the diagram. Therefore, we have

$$\Gamma_\mu = \gamma_\mu - \frac{1}{2\pi} \frac{\partial}{\partial k_\mu} \Sigma^*(k) \quad \text{for } l=0, \tag{14.27c}$$

where l is the energy-momentum of the photon represented by the external line of the vertex part. Substituting (14.16a) and (14.20) into (14.27c) we have

$$-Z_1^{-1} L(e_1) = \frac{1}{2\pi} Z_2^{-1} e_1^2 B(e_1)$$

which gives

$$Z_1^{-1}(1-Z_1) = Z_2^{-1}(1-Z_2)$$

on account of (14.23). This gives (14.27a). The property (14.27b) of S_F used in this proof is based on the gauge-invariancy of the theory, which implies that the electromagnetic interaction i.e. $-ie\bar{\psi}\gamma_\mu A_\mu \psi$ occurs only in the combination $\bar{\psi}(\gamma_\mu, \partial_\mu - ieA_\mu)\ \psi$.

Equation (14.28) shows that the charge renormalisation depends only on the factor Z_3 of Δ'_F (photon internal line). This result is valid not only in the system of the electromagnetic interaction of the electron but that of any charged field, because (14.27) is a consequence of gauge invariance. (WARD [1951], KAMEFUCHI and UMEZAWA [1952]). On the other hand, as shown in Example 6 of Ch. XIII, the deviation δe of the charge is induced by the polarization of the vacuum particles of all charged fields due to an incident photon. Thus, Z_3 must include the contributions of all charged fields and therefore does not depend on the property of the respective charged field. This fact is compatible with the fact that every charged particle has the elementary charge $e_1/\sqrt{2\pi} = 1/\sqrt{137}$.

§ 9. Separation of Infinities

There are various methods to separate successively infinities given by any diagram. (WARD [1951], SALAM [1951]). We shall now give the method due to Salam. First, let us consider the diagram of Fig.

14.5. This is a part of the electron self-energy diagrams in the e^4-approximation.

The S-matrix element given by this diagram has the integral

$$M = \int d^4k' \int d^4k'' \, F(p, k') \, G(p, k', k'') \, H(p, k''). \tag{14.29}$$

The d^4k' integration leads to a logarithmic infinity in the vertex part at the point a and the d^4k'' integration to a similar infinity at point b. Since $G(p, k', k'')$ does not have the form of a product of a function of k' and a function of k'', we cannot write M as a product of two diverging integrations; for this reason such a diagram is said to be "overlapping divergent".

In general, the integration with respect to $(k^{(1)}, \ldots, k^{(m)})$, selected from the variables $(k^{(1)}, \ldots, k^{(n)})$, is called the $(k^{(1)}, \ldots, k^{(m)})$-subintegration. In the $(k^{(1)}, \ldots, k^{(m)})$-subintegration, the integration is carried out for any factor containing a variable $(k^{(1)} \ldots k^{(m)})$ in the integrand. We shall denote the integration of the above factor, in

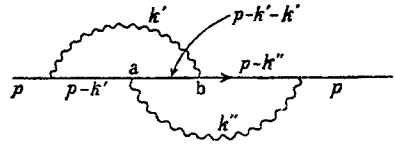

Fig. 14.5

which variables other than $(k^{(1)}, \ldots, k^{(m)})$ are replaced by momenta of the free fields (we shall denote them by the symbols with superscripts 0, i.e. $A_{\alpha\beta}(it^0_\mu) = 0$ in (8.2a), by $D(k^{(1)}, \ldots k^{(m)})$. For example, from (14.29) we have

$$\left.\begin{aligned} D(k') &= \int d^4k' \, F(p^0, k') \, G(p^0, k', k^{0\prime\prime}) \\ D(k'') &= \int d^4k'' \, G(p^0, k^{0\prime}, k'') \, H(p^0, k''). \end{aligned}\right\} \tag{14.30}$$

Whence

$$\left.\begin{aligned} M = D(k') \int d^4k'' \, H(p, k'') + D(k'') \int d^4k' \, F(p, k') \\ + \int d^4k' \int d^4k'' \, R(p, k', k''). \end{aligned}\right\} \tag{14.31}$$

where

$$\left.\begin{aligned} &R(p, k', k'') \equiv \\ &\{F(p, k') \, G(p, k', k'') - F(p^0, k') \, G(p^0, k', k^{0\prime\prime})\} \, H(p, k'') \\ &- F(p, k') \, G(p^0, k^{0\prime}, k'') \, H(p^0, k''). \end{aligned}\right\} \tag{14.32}$$

The infinite quantities in the third term of (14.31) are separated by means of (14.11) as follows:

$$\int d^4k' \int d^4k'' \, R(p, k', k'') = T(p) + I_e(p) \tag{14.33}$$

where

$$T(p) \equiv \int d^4k' \int d^4k'' \{R(p^0, k', k'') + (p_\mu - p^0_\mu)\left(\frac{\partial}{\partial p_\mu} R(p, k', k'')\right)_{p=p^0} \quad (14.34)$$

and $I_c(p)$ is finite.

The proof of (14.33) is given as follows: The dk'-integration of the first bracket in (14.32) does not diverge and its dk''-integration gives the linearity at most diverging terms (cf. discussion of N following (14.24)) which can be separated in the form of (14.33) by means of (14.11). The integration of the last term of (14.32) can be written as follows:

$$\begin{aligned}
&-\int d^4k' \int d^4k''\, F(p, k')\, G(p^0, k^{0\prime}, k'')\, H(p^0, k'') \\
&= -\int d^4k' \left\{ F(p, k') - F(p^0, k') - (p_\mu - p^0_\mu)\left(\frac{\partial}{\partial p_\mu} F(p, k')\right)_{p=p^0} \right\} \\
&\qquad \int d^4k''\, G(p^0, k^{0\prime}, k'')\, H(p^0, k'') \\
&\quad - \int d^4k'\, F(p^0, k') \int d^4k''\, G(p^0, k^{0\prime}, k'')\, H(p^0, k'') \\
&\quad - (p_\mu - p^0_\mu) \int d^4k' \left(\frac{\partial}{\partial p_\mu} F(p, k')\right)_{p=p^0} \int d^4k''\, G(p^0, k^{0\prime}, k'')\, H(p^0, k'').
\end{aligned}$$

The first term of this does not diverge in the dk'-integration and diverges logarithmically in the dk''-integration; its infinity can be separated in the form of (14.33). The second and third terms of the above integration have the forms of the first and second terms of (14.34) respectively.

Thus, we see that M can be written as follows:

$$\left.\begin{aligned} M = {} & T(k') \int d^4k''\, H(p, k'') + T(k'') \int d^4k'\, F(p, k') \\ & + T(k', k'') + I_c(p), \end{aligned}\right\} \quad (14.35)$$

where

$$T(k) \equiv D(k),\ T(k', k'') \equiv T(p). \quad (14.36)$$

The first term of (14.35) is the product of the infinite constant $T(k')$ and the integration given by the k'-reduced graph which is defined as that obtained by omitting the k'-internal line in Fig. 14.5. As shown by Fig. 14.5, $T(k')$ is equal to Z_1^{-1}; it can be eliminated by the renormalisation (the second line of (14.23)) of the vertex part of the point a. In a similar way, the second term of (14.35) has the form: $T(k'') \times k''$-reduced graph and is cancelled out by the renormalisation of the vertex part of the point b. We see that $T(k'')$ is just the b-divergence Z_1^{-1} of (14.25). This fact shows that, while Fig. 14.5 is constructed by inserting a line stepping over only one point of Fig. 14.2, the final

integration contains two Z_1^{-1} of two points and therefore it is symmetric with respect to the right and left sides. This fact can be established in any order of the perturbation series [1]). The third term of (14.35) must be eliminated by the mass renormalization $\delta\varkappa$ and Z_2-renormalization in the e^4-approximation, and then we obtain the finite answer I_c.

This method can be extended to determine $Z_1, Z_2, Z_3, \delta\varkappa$ in any order of the perturbation series as follows (MATHEWS and SALAM [1951]). We shall denote a S-matrix element of a diagram of a self-energy or a vertex part by M, in which the independent variables of integration are $(k^{(1)}, \ldots, k^{(n)})$. The graph given by omitting the internal lines of momenta $(k^{(1)}, \ldots, k^{(m)})$ is called $(k^{(1)}, \ldots k^{(m)})$-reduced graph. Separate from M (by means of (14.11)) all possible $T(k^{(i)}) \times k^{(i)}$-reduced graph, where $T(k^{(i)})$ are the infinite constants of the $k^{(i)}$-subintegrations $(i = 1, \ldots, n)$. The remaining parts do not diverge with respect to $k^{(i)}$-integration. Then, make the $(k^{(i)}k^{(j)})$ $(i = 1, \ldots n, j = 1, \ldots, n, i \neq j)$-subintegrations of the latter part.

These integrations give the infinite terms $T(k^{(i)}k^{(j)}) \times (k^{(i)}k^{(j)})$-reduced graph, where $T(k^{(i)}k^{(j)})$ are the infinite constants given by the $(k^{(i)}, k^{(j)})$-subintegrations. Repeating similar methods of separating and omitting infinities we can obtain a finite answer. The infinite constant $T(k^{(1)}, \ldots, k^{(m)})$ which must be subtracted at the last stage of elimination of the infinities of the $(k^{(1)}, \ldots, k^{(m)})$-subintegration is called the true divergence of the $(k^{(1)}, \ldots, k^{(m)})$-subintegration.

Since the divergences $T(k^{(1)}, \ldots, k^{(m)})$ in the successive procedures above appear as factors of $(k^{(l)} \ldots k^{(m)})$-reduced graphs, they are cancelled by the Z_1, Z_2, Z_3 and $\delta\varkappa$ of the lower order approximation except when $T(k^{(1)} \ldots, k^{(m)})$ is the true divergence obtained in the last stage of the procedure. This true divergence must be eliminated by Z_1, Z_2, Z_3 and $\delta\varkappa$ in the same order of approximation as the present graph. Then we obtain the infinite result I_c

$$I_c = [1 - T(k^{(1)} \ldots k^{(n)})]\Phi \tag{14.37}$$

where

$$\left.\begin{aligned} \Phi \equiv\ & [1 - T(k^{(1)} \ldots k^{(n-1)}) - T(k^{(2)} \ldots k^{(n)}) - \ldots] \\ & [1 - T(k^{(1)} \ldots k^{(n-2)}) - T(k^{(2)} \ldots k^{(n-1)}) - \ldots] \ldots \\ & [1 - T(k^{(1)}) - T(k^{(2)}) - \ldots] M. \end{aligned}\right\} \tag{14.38}$$

[1]) In this proof Salam introduced a concept "category". This concept has been extended to other cases (TAKEDA [1952]).

"$-T(k^1 \dots k^m)$" means the operation of omitting the divergence $T(k^1 \dots k^m)$.

§ 10. Examples

EXAMPLE 1. LAMB SHIFT AND WELTON IMAGE

Let us denote the electric intensity of the electromagnetic field (i.e. the proper field) around an electron by $\mathbf{E}(x)$. Newton's equation of the motion of the electron is

$$m \frac{d^2}{dt^2} \mathbf{x} = e \mathbf{E}(x), \tag{14.39}$$

where $\mathbf{x}$ is the three dimensional vector of the position of the electron.

Using the Fourier amplitude $\mathbf{E}(\mathbf{l})$ of $\mathbf{E}(x)$:

$$\mathbf{E}(x) = \sum_{\mathbf{l}} \mathbf{E}(\mathbf{l}) e^{i(\mathbf{l}\cdot\mathbf{x} - lt)} \quad (l \equiv |\mathbf{l}|), \tag{14.40}$$

we can write the solution of (14.39) (if $\mathbf{x}=0$ for $\mathbf{E}=0$) as

$$\mathbf{x} = \sum_{\mathbf{l}} \frac{e}{m} \frac{1}{l^2} \mathbf{E}(\mathbf{l}) e^{i(\mathbf{l}\cdot\mathbf{x} - lt)}. \tag{14.41}$$

In other words, the position of the electron fluctuates by an amount $\mathbf{x}$ due to the proper electric field of the electron itself. In the following, we shall write $\mathbf{x}$ by $\delta\mathbf{x}$ because it describes the fluctuation of the position.

In the lowest order of the perturbation approximation we can regard $\mathbf{E}$ as the electric intensity of the electromagnetic field made from $A_\mu^{(in)}(x)$ of (12.29). Then, we can obtain using (7.64) and (9.50)

$$((\mathbf{E}(\mathbf{l})\cdot\mathbf{E}(\mathbf{l}')))_0 = \frac{1}{2lV}\{(\mathbf{l},\mathbf{l}) - 3l^2\}\delta_{\mathbf{l}\mathbf{l}'} = -\frac{l}{V}\delta_{\mathbf{l}\mathbf{l}'}. \tag{14.42}$$

Here the symbol $(\ )_0$ means the expectation value with respect to a state containing no photon.

Substituting (14.41) into (14.42) we have

$$(\delta\mathbf{x}\cdot\delta\mathbf{x})_0 = \frac{1}{V}\sum_{\mathbf{l}} \frac{e^2}{m^2}\frac{1}{l^2} = \frac{e^2}{(2\pi)^3 m^2}\int d^3l \frac{1}{l^3} = \frac{e^2}{2\pi^2 m^2}\int \frac{dl}{l}. \tag{14.43}$$

When the electron with the electric proper field moves in an external potential $\phi(x)$, its motion is made up of the superposition of the slow motion due to the external field and the rapid fluctuation

$\delta\mathbf{x}$ due to the radiative reaction. If we can neglect the higher power (>2) of small $\delta\mathbf{x}$, we have

$$\phi(x+\delta x) \approx \phi(x) + \delta x_k \frac{\partial}{\partial x_k}\phi(x) + \tfrac{1}{2}\,\delta x_k\,\delta x_l \frac{\partial^2}{\partial x_k \partial x_l}\phi(x).$$

Since the three directions ($k=1, 2, 3$) of $\delta\mathbf{x}$ cannot be distinguished physically, we have

$$(\phi(x+\delta x))_0 \approx \phi(x) + \tfrac{1}{2}(\delta\bar{x})^2\,\Delta\phi(x) \tag{14.44}$$

where

$$(\delta\bar{x})^2 = \tfrac{1}{3}((\delta\mathbf{x}\cdot\delta\mathbf{x}))_0.$$

In the problem of an electron of a hydrogen atom, $\phi(x)$ is the Coulomb potential due to a proton, or

$$\phi(x) = -\frac{e^2}{4\pi r} \qquad (r \equiv |\mathbf{x}|), \tag{14.45}$$

where the origin is taken to be the position of the proton. Then we see that the change of the energy due to the radiative reaction is

$$\Delta E = (\psi^*\{\phi(x+\delta x)-\phi(x)\}\psi)_0 = \tfrac{1}{2}\,(\psi^*(\delta\bar{x})^2\psi\cdot\Delta\phi) = \tfrac{1}{2}\,(\delta\bar{x})^2|\psi(0)|^2 \tag{14.46}$$

on account of the relation

$$\Delta\frac{1}{4\pi r} = -\,\delta^3(x).$$

Substituting (14.43) into (14.46) we have

$$\Delta E = \frac{e^4}{12\pi^2 m^2}\int\frac{dl}{l}\cdot|\psi(0)|^2. \tag{14.47}$$

On the other hand, we know that

$$|\psi(0)|^2 = \begin{cases} \dfrac{1}{\pi n^3 a^3} & \text{for the electron of } S\text{-state} \\ 0 & \text{for the electron of } P\text{-state} \end{cases} \tag{14.48a}$$

where

$$a = \frac{4\pi}{me^2} \tag{14.48b}$$

and n is the principal quantum number. Thus, for an electron in a $2S$-state we have

$$\Delta E = \frac{e^4}{96\pi^2 m^2 a^3}\int\frac{dl}{l}. \tag{14.49}$$

(WELTON [1948]).

Although there are many high energy photons in the electromagnetic proper field, we can adopt the non-relativistic calculation

in the view-point of the renormalisation that the main contribution of the high energy region can be amalgamated into the observed mass (i.e. the mass renormalisation). Therefore, we shall restrict the region of the integration of the momenta l in (14.49) by the condition

$$l \leqslant m. \tag{14.50a}$$

It is easily seen that (14.49) leads to an infra-red catastrophe. However, since we have seen in Example 4 of Ch. XIII that this is not the essential difficulty of the quantum field theory, we shall proceed as follows.

As is well known in the non-relativistic quantum mechanics of the electron, the electron in the $2S$-state of the hydrogen atom has the energy $(2\pi Ry/4)$ (Ry is the Rydberg constant $me^4/(4\pi)^3$). In other words this electron has the frequency $(Ry/4)$. Therefore, we can assume that the effect of the fluctuation due to photons of the smaller momenta than $2\pi Ry/4$ can be neglected. Then, the momenta l in (14.49) are such that

$$l \geqslant 2\pi Ry/4. \tag{14.50b}$$

Then

$$\Delta E = \frac{e^4}{96\pi^3 m^2 a^3} \log\left(\frac{4m}{2\pi Ry}\right) = \frac{m}{6\pi}\left(\frac{e^2}{4\pi}\right) \log\{8(4\pi)^2/e^2\}. \tag{14.51}$$

This equation gives $\Delta E = 1600\ Mc$. Taking into account the fact that the present calculation is a rough estimation based on an intuitive image of the proper field, we can say that this result is in rough agreement with the experimental result 1057.77 Mc (LAMB and RETHERFORD [1947]).

EXAMPLE 2. LAMB SHIFT AND ANOMALOUS MAGNETIC MOMENT OF ELECTRON

We shall now make the more accurate calculation of the Lamb shift. The Schrödinger equation in the interaction representation is

$$i\frac{\delta}{\delta\sigma(x)}\Psi[\sigma] = \{H'[x:n] + \psi^*(x)\,\phi(x)\,\psi(x)\}\,\Psi[\sigma],$$

where $H'[x:n]$ is the sum of the well known electromagnetic interaction of the electron field and the mass- and charge-counter terms, and $\psi^*\phi\psi$ is the external potential.

Introducing the transformation

$$\Psi[\sigma] = v^{-1}[\sigma]\,S^{-1}[\sigma]\,\Psi[\sigma]$$

with $S[\sigma]$ and $v[\sigma]$ satisfying the relations

$$i\frac{\delta}{\delta\sigma(x)}S[\sigma]=H'[x:n]\,S[\sigma]$$

$$i\frac{\delta}{\delta\sigma(x)}v[\sigma]=\psi^*(x)\,\phi(x)\,\psi(x)\,v[\sigma],$$

we have

$$i\frac{\delta}{\delta\sigma(x)}\Psi[\sigma]=\{\psi^*(x)\,\phi(x)\,\psi(x)-\varphi^*(x)\,\phi(x)\,\varphi(x)\}\Psi[\sigma]. \quad (14.52a)$$

Operators ψ, φ and $\mathbf{A}_\mu$ are defined by

$$\psi(x)=v^{-1}[\sigma]\,S^{-1}[\sigma]\,\psi(x)\,S[\sigma]\,v[\sigma],$$
$$\mathbf{A}_\mu(x)=v^{-1}[\sigma]\,S^{-1}[\sigma]\,A_\mu(x)\,S[\sigma]\,v[\sigma],$$
$$\varphi(x)=v^{-1}[\sigma]\,\psi(x)\,v[\sigma].$$

It can be easily proved by means of (6.5a) that φ satisfies the equations

$$\left.\begin{aligned}&\{\gamma_\mu\,\partial_\mu+m+\phi(x)\,\gamma_4\}\,\varphi(x)=0\\&\partial_\mu\,\bar\varphi(x)\,\gamma_\mu-\bar\varphi(x)\,\{m+\gamma_4\,\phi(x)\}=0.\end{aligned}\right\} \quad (14.52b)$$

Since (14.52a) has the form of Schrödinger equation with the interaction Hamiltonian

$$\psi^*(x)\,\phi(x)\,\psi(x)-\varphi^*(x)\,\phi(x)\,\varphi(x),$$

the energy operator T_4^h in the Heisenberg representation may be written as

$$T_4^h=U^{-1}[\sigma](T_4+\int d^3x\,\{\psi^*(x)\,\phi(x)\,\psi(x)-\varphi^*(x)\,\phi(x)\varphi(x)\})\,U[\sigma]. \quad (14.52c)$$

Here T_4 is the energy operator (defined as the displacement operator for ψ, $\mathbf{A}_\mu$) in the Ψ-representation and $U[\sigma]$ is the transformation connecting the Heisenberg representation with the Ψ-representation. The relation (14.52c) can be obtained by a consideration similar to that used in the deduction of (10.32).

Introducing the eigenvector Φ of T_4, we can write the perturbed energy ΔE due to the external potential as

$$\Delta E=(U^{-1}[\sigma]\,[T_4+\textstyle\int_\sigma d^3x\{\psi^*(x)\phi(x)\psi(x)-\varphi^*(x)\phi(x)\varphi(x)\}]\,U[\sigma]-T_4)$$

where ( ) means the expectation value with respect to the unperturbed state Φ. The equation for $U[\sigma]$ is derived from (14.52a) to be

$$i\frac{\delta}{\delta\sigma(x)}U[\sigma]=\{\psi^*(x)\,\phi(x)\,\psi(x)-\varphi^*(x)\,\phi(x)\,\varphi(x)\}\,U[\sigma].$$

In just the same way as in the proof (13.8), we can prove that

$$\left.\begin{aligned}\Delta E = \sum_{n,m=0}^{\infty} \frac{1}{n+m+1} \{U^{(m)}[-\infty, \sigma] \textstyle\int_\sigma d^3x \\ \{\psi^*(x)\,\phi(x)\,\psi(x) - \varphi^*(x)\,\phi(x)\,\varphi(x)\}\, U^{(n)}[\sigma, -\infty]\},\end{aligned}\right\} \quad (14.53a)$$

where $U^{(l)}[\sigma, -\infty]$ is the term of the l-th order power of the external potential ϕ in $U[\sigma]$; or

$$U[\sigma] = \sum_l U^{(l)}[\sigma, -\infty].$$

Equation (14.53a) can be written as (cf. (13.11))

$$\Delta E = (i\,|\textstyle\int_\sigma d^3x\, V^e(x)\,|\,i) + \sum_i \frac{(i|\int d^3x V^e(x)|v)\,(v|\int d^3x V^e(x)|i)}{w_i - w_v} \quad (14.53b)$$

where w_i and w_v are energies of the initial state Φ_i and virtual state Φ_v respectively and

$$\left.\begin{aligned}V^e(x) &= \psi^*(x)\,\phi(x)\,\psi(x) - \varphi^*(x)\,\phi(x)\,\varphi(x) \\ &= v^{-1}[\sigma]\,\{S^{-1}[\sigma]\,\varphi^*(x)\,\phi(x)\,\varphi(x)\,S[\sigma] - \varphi^*(x)\,\phi(x)\,\psi(x)\}\,v[\sigma].\end{aligned}\right\} \quad (14.54)$$

The operator in the above bracket is just the radiative correction in the transition matrix of the elastic scattering of the electron ψ. Thus, its e^2-term is equal to the sum of (13.71), (13.75) (cf. Example 5 of Ch. XIII)) and the contribution of the counter mass and charge terms (cf. (14.2) and (14.3)). The latter counter terms cancel the infinite terms (i.e. A^0 and C^0). The transformation operator $v[\sigma]$ can be eliminated by the replacement $\psi \to \varphi$. Thus, we have [1])

$$-i\textstyle\int_{-\infty}^{\infty} d^4x\, V^{(2)}(x) = \Lambda + \frac{i}{15\pi m^2}\cdot\frac{e^2}{4\pi}\int d^4x\, \varphi^*(x)\,\Box\,\phi(x)\,\varphi(x) + \ldots \quad (14.55)$$

for the e^2-term $V^{(2)}(x)$ in $V^e(x)$. When the external potential $\phi(x)$ is constant in time, we have, in the non-relativistic approximation,

$$\left.\begin{aligned}\textstyle\int_{-\infty}^{\infty} d^3x\, V^{(2)}(x) = \frac{1}{3\pi m^2}\frac{e^2}{4\pi}\left\{\log\frac{m}{2k_0} + \frac{11}{24} - \frac{1}{5}\right\}\int d^3x\, \varphi^*\varphi\,\Delta\phi \\ + \frac{1}{4\pi m}\frac{e^2}{4\pi}\int d^3x\,(\varphi^*\boldsymbol{\gamma}\varphi\cdot\boldsymbol{\partial}\phi)\end{aligned}\right\} \quad (14.56)$$

(cf. (13.77)).

The e-term $V^{(1)}(x)$ of (14.54) contributes to the Lamb shift an amount

$$\Delta E^{(1)} = \sum_v \frac{(i|\int d^3x V^{(1)}(x)|v)\,(i|\int d^3x V^{(1)}(x)|v)}{w_i - w_v} \quad (14.57)$$

[1]) The potential $\phi\ (x)$ corresponds to (14.45).

with

$$\left.\begin{aligned} w_i &= E_i \\ w_v &= E_v + k \end{aligned}\right\} \tag{14.58}$$

Here E_v and k are energies of an electron and a photon in a virtual state, and E_i is the energy of the electron in the initial state.

The $V^{(1)}(x)$ is obtained by (13.58a), and is, in the non-relativistic approximation,

$$\left.\begin{aligned} V^{(1)}(x) &= v^{-1}[\sigma]\left\{\frac{e}{m}\sum_{\mathbf{k},r}\frac{1}{k\sqrt{2kV}}\,\psi^*(x)\,\psi(x)\,(\mathbf{e}^{(r)}\cdot\partial\phi(x))\right\}v[\sigma] \\ &= \frac{e}{m}\sum_{\mathbf{k},r}\frac{1}{k\sqrt{2kV}}\,\varphi^*(x)\,\varphi(x)\,(\mathbf{e}^{(r)}\cdot\partial\phi(x)). \end{aligned}\right\} \tag{14.59}$$

Since each photon has two polarization directions ($r=1, 2$) we can replace $\sum_r (\mathbf{e}^r, \partial\phi)^2$ in (14.57) by (2/3) $(\partial\phi\cdot\partial\phi)$. By substituting (14.59) into (14.57) we have

$$\left.\begin{aligned} &\Delta E^{(1)} = -\frac{2}{3}\frac{e^2}{m^2}\sum_v\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2k^3(k+E_v-E_i)} \\ &(i|\int d^3x\,\varphi^*(x)\,\varphi(x)\,\partial_k\phi(x)|v)\,(v|\int d^3x\,\varphi^*(x)\,\varphi(x)\,\partial_k\phi(x)|i). \end{aligned}\right\} \tag{14.60}$$

This can be calculated as follows: The relations in (14.52b) lead to

$$\left.\begin{aligned} &(v|\int d^3x\,\varphi^*\,\varphi\,\partial_k\phi|i) \\ &\quad= -(v|\int d^3x\,\bar\varphi(\gamma_\mu\,\partial_\mu+m+\phi\,\gamma_4)\,\partial_k\varphi|i) \\ &\quad= (v|\int d^3x\,[\{\partial_\mu\,\bar\varphi\,\gamma_\mu-\bar\varphi(m+\phi\,\gamma_4)\}\,\partial_k\varphi \\ &\qquad-\partial_4\,\bar\varphi\,\gamma_4\,\partial_k\varphi-\bar\varphi\,\gamma_4\,\partial_4\,\partial_k\varphi]|i) \\ &\quad= i\frac{\partial}{\partial t}(v|\int d^3x\,\bar\varphi^*\,\partial_k\varphi|i) \\ &\quad= (E_i-E_v)\,(v|d^3x\,\varphi^*\,\partial_k\varphi|i). \end{aligned}\right\} \tag{14.61}$$

Substituting (14.61) into (14.60) we obtain

$$\Delta E^{(1)} = -\sum_v\sum_{l=1,2,3}\frac{e^2}{6\pi^2m^2}\int_{k_0}^{\infty}\frac{dk}{k}\,\frac{(E_i-E_v)^2}{(k+E_v-E_i)}\,|(v|\int d^3x\,\varphi^*\,\partial_l\varphi|i)|^2 \tag{14.62}$$

$$= -\frac{e^2}{6\pi^2m^2}\sum_v\sum_{l=1,2,3}(E_v-E_i)\,|(v|\int d^3x\,\varphi^*\,\partial_l\varphi|i)|^2\log\frac{E_v-E_i}{k_0}. \tag{14.63}$$

We shall rewrite (14.63) by means of the substitution

$$\left.\begin{aligned} &\sum_v(E_v-E_i)\,|(v|\int d^3x\,\varphi^*\,\partial_k\varphi|i)|^2\log(E_v-E_i) \\ &\quad= \log(E_v-E_i)_{Av}\sum_v(E_v-E_i)\,|(v|\int d^3x\,\varphi^*\,\partial_k\varphi|i)|^2 \end{aligned}\right\} \tag{14.64}$$

(BETHE [1947]).

The average value $\log (E_v - E_i)_{Av}$ is 17.8 Ry when the i-state is a $2S$-state. Using the relation

$$\sum_v \sum_l (E_v - E_i)\, |(v|\int d^3x\, \varphi^* \partial_l \varphi |i)|^2 = \tfrac{1}{2}\, (i|\int d^3x\, \varphi^* \varphi \Delta \varphi |i) \quad (14.65)$$

(see BETHE [1947]), we can write (14.63) as

$$\Delta E^{(1)} = -\frac{e}{12\pi^2 m^2}\, (i|\int d^3x\, \varphi^* \varphi\, \Delta\, \varphi |i)\, \log \frac{(E_v - E_i)_{Av}}{k_0}. \quad (14.66)$$

The Lamb shift ΔE is given in the e^2-approximation by $\Delta E^{(1)} + \Delta E^{(2)}$ where

$$\Delta E^{(2)} = (i \mid \int d^3x\; V^{(2)}(x) \mid i). \quad (14.67)$$

Using (14.56) we obtain

$$\left.\begin{aligned} \Delta E = \frac{1}{3\pi m^2}\frac{e^2}{4\pi}\left\{\log \frac{m}{2(E_v - E_i)_{Av}} + \frac{11}{24} - \frac{1}{5}\right\} (i|\int d^3x\, \varphi^* \varphi\, \Delta \phi |i) \\ + \frac{1}{4\pi m}\frac{e^2}{4\pi}\, (i|\int d^3x\, (\varphi^* \boldsymbol{\gamma}\, \varphi \cdot \boldsymbol{\partial}\, \phi)|i). \end{aligned}\right\} \quad (14.68)$$

It must be noted that (14.68) has no infra-red catastrophe for $k_0 \to 0$. This fact gives an example of the general discussion in Example 5 of Ch. XIII.

The last term in (14.68) can be calculated as follows:

$$\left.\begin{aligned} &(i|\int d^3x\, (\varphi^* \boldsymbol{\gamma}\, \varphi \cdot \boldsymbol{\partial}\, \phi)|i) \\ &= \frac{1}{2m}\, (i|\int d^3x\, \bar\varphi \{\gamma_4 \gamma_k (m + \gamma_4 \phi) - (m + \gamma_4 \phi)\, \gamma_k \gamma_4\}\, \varphi \cdot \partial_k \phi |i) \\ &= -\frac{1}{2m}\, (i|\int d^3x \{\partial_\mu \bar\varphi\, \gamma_\mu \gamma_k \gamma_4\, \varphi + \bar\varphi\, \gamma_4 \gamma_k \gamma_\mu \partial_\mu \varphi\}\, \partial_k \phi |i) \\ &= \frac{1}{2m}\, \partial_4\, (i|\int d^3x\, \bar\varphi\, \gamma_k \varphi\, \partial_k \phi |i) \\ &\quad - \frac{1}{2m}\, (i|\int d^3x\, (\partial_l \varphi^*\, \gamma_l \gamma_k \varphi + \varphi^* \gamma_k \gamma_l\, \partial_l \varphi) \cdot \partial_k \phi |i) \\ &= \frac{1}{2m}\, (i|\int d^3x\, \varphi^* \varphi \Delta \varphi - 2i \int d^3x\, \varphi^*\, (\boldsymbol{\sigma} \cdot [\mathbf{grad}\, \phi \wedge \mathbf{grad}])\, \varphi |i) \end{aligned}\right\} \quad (14.69)$$

on account of (14.52b), (3.49) and the relation

$$\partial_4\, (i|\int d^3x\, \bar\varphi\, \gamma_k \varphi \cdot \partial_k \phi |i) = (E_i - E_i)\, (i|\int d^3x\, \bar\varphi\, \gamma_k \varphi \cdot \partial_k \phi |i) = 0. \quad (14.70)$$

When ϕ is a central potential we have

$$[\mathbf{grad}\, \phi \wedge \mathbf{grad}] = i\, \frac{1}{r}\, \frac{\partial \phi}{\partial r}\, \mathbf{L} \quad (14.71)$$

with

$$\mathbf{L} = [\mathbf{r} \wedge \mathbf{grad}] \qquad \text{(angular momentum operator)}$$

and $r = |\mathbf{x}|$.

Substituting (14.69) into (14.68), we obtain

$$\left.\begin{aligned}\Delta E = \frac{1}{3\pi m^2}\frac{e^2}{4\pi}\left\{\log\frac{m}{2(E_v-E_i)_{Av}}+\frac{5}{6}-\frac{1}{5}\right\}(i|\int d^3x\,\varphi^*\varphi\cdot\Delta\phi|i)\\ +\frac{1}{4\pi m^2}\frac{e^2}{4\pi}\left(i\left|\int d^3x\,\varphi^*(\boldsymbol{\sigma},\mathbf{L})\,\varphi\cdot\frac{1}{r}\frac{\partial\phi}{\partial r}\right|i\right).\end{aligned}\right\} \tag{14.72}$$

Now $(\boldsymbol{\sigma},\mathbf{L})$ can be written as

$$(\boldsymbol{\sigma}\cdot\mathbf{L}) = (\mathbf{L}+\tfrac{1}{2}\boldsymbol{\sigma})^2-\mathbf{L}^2-\tfrac{1}{4}\boldsymbol{\sigma}^2.$$

Denoting the total and orbital angular momenta of the i state by j and l respectively, we can calculate $q\equiv(\boldsymbol{\sigma},\mathbf{L})$ to be

$$\left.\begin{aligned}q &= j(j+1)-l(l+1)-\tfrac{1}{2}(\tfrac{1}{2}+1)\\ &= \begin{cases} l & \text{for}\quad j=l+\tfrac{1}{2}\\ -l-1 & \text{for}\quad j=l-\tfrac{1}{2}.\end{cases}\end{aligned}\right\} \tag{14.73}$$

In the problem of an electron in a hydrogen atom $\phi(x)$ is Coulomb potential (14.45) and therefore

$$\left.\begin{aligned}\Delta E = \frac{e^2}{3\pi m^2}\frac{e^2}{4\pi}\left\{\log\frac{m}{2(E_v-E_i)_{Av}}+\frac{5}{6}-\frac{1}{5}\right\}|\varphi_i(0)|^2\\ +\frac{e^2}{16\pi^2m^2}\frac{e^2}{4\pi}q\int d^3x\,\frac{1}{r^3}\,|\varphi_i(x)|^2,\end{aligned}\right\} \tag{14.74}$$

where φ_i is the wave function of an electron in the i-state.

When the i-state is the S-state, $q=0$ and therefore ΔE is given by the first term of (14.74). On the other hand, when the i-state is not an S-state, we have $|\varphi_i(0)|=0$ and therefore ΔE is given by the second term of (14.74). In the latter case we have

$$\Delta E = \frac{e^2}{16\pi^2m^2}\frac{e^2}{4\pi}\frac{q}{n^3a^3l(l+1)(l+\frac{1}{2})} \tag{14.75}$$

where a is defined by (14.48*b*).

For an S-state (14.74) gives

$$\Delta E = \frac{e^4}{12\pi^2m^2}\frac{1}{\pi n^3a^3}\left\{\log\frac{m}{2(E_v-E_i)_{Av}}+\frac{5}{6}-\frac{1}{5}\right\}. \tag{14.76}$$

The difference of level shifts between the $2S$ and $2P_{1/2}$-levels is

$$\left.\begin{aligned}&\Delta E\,(n=2,l=0)-\Delta E\,(n=2,l=1,j=l-\tfrac{1}{2})\\ &=\frac{m}{6\pi}\left(\frac{e^2}{4\pi}\right)^5\left\{\log\frac{m}{2(E_v-E_i)_{Av}}+\frac{5}{6}-\frac{1}{5}+\frac{1}{8}\right\},\end{aligned}\right\} \tag{14.77}$$

or, numerically (1050) Mc. This result is in excellent agreement with experimental results.

We shall now consider the anomalous magnetic moment of an electron. The second term of (13.69*b*) shows that the radiative correction gives, in the e^2-approximation, the anomalous magnetic moment (cf. (7.114))

$$\delta\mu = \frac{1}{2\pi}\left(\frac{e^2}{4\pi}\right)\mu \tag{14.78}$$

where

$$\mu = -\frac{e}{2m} \tag{14.79}$$

Here μ is the well known proper magnetic moment of an electron. The result (14.78) is in good agreement with the experimental results $\delta\mu/\mu = 0.00114$.

On account of these successes of renormalisation theory, it is fair to say that proper fields have real observable effects and that renormalisation is probably a step in the direction that will some day lead to a coherent theory of elementary particles.

§ 11. Various Problems in Renormalisation Theory

The renormalisation method can remove all the infinities of the S-matrix of quantum electrodynamics. Nevertheless, in order to regard experimental successes of the quantum electrodynamics as that of this renormalisation theory, we must have a consistent quantum field theory based on the renormalisation theory.

We have seen that the renormalisation procedure leads to the gauge-invariant quantum electrodynamics and that the observed charge has a definite value e_1 irrespective of the type of the charged fields, when we start with the common value e of a mechanical (not observable) charge of all charged fields (cf. § 8).

In order that the theory shall be completely rational, we require that the perturbation series should converge to a finite result. Let $N_M(n)$ be the number of Feynman diagrams in the n-th order approximation which describe a certain transition M. In general, it is found [1]) that $f_M(n) = N_M(n+2)/N_M(n)$ increases rapidly with increasing n. Therefore, it is not impossible that the higher order terms for which $f_M(n) > 137$ make larger contributions to the S-matrix. Thus it seems very doubtful whether the perturbation series of the renormalisation theory will converge. These doubts have been expressed by many

[1]) See on the calculation of $N_M(n)$ the reference HURST [1952].

writers (DYSON [1952], KATAYAMA and YAMAZAKI [1952]). Moreover, others have shown that the perturbation series of the renormalisation theory diverges even in the simple case of the scalar field U with the interaction of the form λU^3 (and in the energy domain which is not enough to produce free particles) [1]) (HURST [1952], THIRRING [1953], UTIYAMA and IMAMURA [1953], PETERMANN [1953]).

If this is true also in quantum electrodynamics, so that the perturbation series of this theory does not converge, it is again necessary to explain the excellent agreement between experimental results and the lower order calculation of quantum electrodynamics. We have no proof, but it is possible that this agreement comes from the fact that the perturbation expansion of the S-matrix is an asymptotic series in the coupling constant e_1, which is small to make the error of the lower order calculation small. If so, the point $e_1 = 0$ would be an essential singularity of the S-matrix and the error would be decreased to an order of approximation and then begin to increase again.

It would seem unreasonable to expect that renormalisation theory should be based only on the perturbation expansion. Many attempts have been made to formulate the renormalisation theory independently of the perturbation calculation (KÄLLEN [1952], LEHMAN [1954]). The theory of propagators of the Heisenberg representation which we shall discuss in Ch. XVIII also suggests interesting possibilities of reformulating renormalisation in a manner independent of the perturbation calculation.

Even if we could formulate the renormalisation theory without any approximation, it would normally be necessary to introduce some approximation in order to apply such a renormalisation theory to complicated systems such as quantum electrodynamics, meson and nucleon fields etc. Thus, it may be worthwhile to find a model which is simple enough for renormalisation to be applied without any approximation. Such a simple model has been proposed by LEE [1954]. The renormalisation of this model leads to a difficulty, which is discussed in detail in Ch. XVIII. It is shown in Ch. XVIII that such a difficulty disappears if we cut off the effects of high energy particles in the proper fields to make renormalisation constants finite,

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[1]) This restriction is introduced in order to avoid the contribution of the interference threshold (cf. Example 1 of Ch. XIII).

and if the coupling constants are suitably small. Thus, although the renormalisation theory has been introduced to eliminate the ultraviolet catastrophes, we must make renormalisation constants finite in order to obtain a consistent renormalisation. If we should have a similar difficulty in quantum electrodynamics to that in the case of Lee's model, it would again be necessary to explain the well known success of the renormalisation theory in the quantum electrodynamics. One possible explanation is as follows: It is obvious that, in the high energy domain, effects of various particles are mixed up and therefore quantum electrodynamics cannot really be regarded as a closed system. If the various particles play the role of cohesive fields, i.e. of reducing the number of high energy particles in the proper fields, the renormalisation constants may be finite. Then the difficulty which we find in the case of Lee's model may not appear in the quantum electrodynamics, on account of the smallness of the electric charge.

In the present Chapter we have considered only quantum electrodynamics. We shall turn to the renormalization theory of various fields in the next Chapter.

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Note added in proof

Readers are recommended to compare this section (§ 11) with Note added in proof in Ch. XVIII, where discussions in the end of § 11 are revised by taking into account the recent progress of the renormalisation theory.

CHAPTER XV

RENORMALISATION THEORY IN GENERAL CASE

§ 1. Applicability of Renormalisation Theory

Based on the use of the relativistically covariant formalism of the quantum theory of wave fields, renormalisation theory aims at reformulating the theory in its equivalent non-singular form [1]). The effects of the reaction of proper fields are divided into two parts, the observable and unobservable parts [2]), and divergences appearing in the latter are removed by the procedure of renormalisation to give definite finite results that can be compared with experiment. In this context the excellent agreement of the results given by the lower order approximation of the perturbation theory with experimental data can only be understood if renormalisation theory leads to the non-singular form of the quantum field theory, not only for quantum electrodynamics but also for those theories which describe the behaviour of other elementary particles. Hence, there arises an important problem of the applicability of the renormalization theory.

There is an ambiguous point in the renormalisation theory which seems to stem from its transitional character. In the present quantum theory we start with interactions (i.e. the primary interactions) which are important in the higher order terms of the perturbation theory; and the latter effects are observed as if these were induced by the direct effects of some interactions (i.e. the secondary interactions) without being mediated by virtual states [3]). In a renormalisation theory these secondary interactions are in part amalgamated with some of the primary interactions to give finite results. At present there is no criterion for deciding what kinds of interactions are primary and what kinds of interactions are the consequences of them. Thus,

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[1]) We shall refer to "non-singular theory" or "closed theory" when all divergens thereof can be removed by the finite number of counter interactions.

[2]) For example, the self-mass $\delta\varkappa$ is amalgamated into the observable mass $\varkappa' = \varkappa + \delta\varkappa$. It is the interesting problem whether the experimental observation of $\delta\varkappa$ is in principle impossible or not.

[3]) Remember the mass term (14.2) due to the self-energy effect.

for example, the anomalous magnetic moment of the electron could be explained by assuming the existence of a primary interaction of the $F_{\mu\nu}$-type, even though it might not be explicable in terms of the renormalisation theory based on a primary interaction of the A_μ-type. This ambiguous property of renormalisation arises from the fact that all the relativistic invariant, gauge-invariant interactions are compatible with present quantum theory. In other words, this problem is connected with the problem of explaining "why these particular elementary particles with these particular interactions are selected to exist in nature". So far no sufficient explanation exists. For the present we shall investigate the applicability condition of the renormalisation theory for different kinds of interaction.

§ 2. Dimensional Analysis

Suppose the interaction Lagrangians $g_l \mathsf{L}_l$ of the fields $Q_\alpha^{(a)}$ (where $a = 1, 2, \ldots$, denote field and $\alpha = 1, 2, \ldots$ its components) are such that

$$L' = \sum_l g_l \mathsf{L}_l \tag{15.1}$$

The $Q_\alpha^{(a)}$ satisfy the commutation relations (8.14a, b). Let us denote the derivation operators of the commutation relations of the a-field by $d^{(a)}(\partial)$. The highest degree $b^{(a)}$ of the derivation operator $d^{(a)}(\partial)$ is determined by the spin of the a-field as shown by (8.52a, b).

For large momentum $k \geqslant \varkappa^{(a)}$ it is approximately true that

$$d^{(a)}(ik) \propto k^{b^{(a)}} \tag{15.2}$$

The interaction Lagrangians $g\mathsf{L}_l$ contain $Q^{(a)}(x)$ in the expression [1]), $D^{(a)}(\partial)Q^{(a)}(x)$ where $D^{(a)}(\partial)$ are derivation operators. Using the true orders $t^{(a)}$ of $D^{(a)}(\partial)$ (cf. Example 6 of Ch. VIII) we can write approximately (for a large k)

$$D^{(a)}(ik)\, D^{(a)}(ik)\, d^{(a)}(ik) \propto k^{(b^{(a)} + 2t^{(a)})} \tag{15.3}$$

For convenience in the following discussions, we shall now introduce several quantities which characterize the interaction form $g_l \mathsf{L}_l$. These are

$$\left.\begin{aligned} A_l &= \sum_a t^{(a)}\, \lambda_l^{(a)} \\ B_l &= \sum_a b^{(a)}\, \lambda_l^{(a)} \\ C_l &= \sum_a \lambda_l^{(a)} \end{aligned}\right\} \tag{15.4}$$

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[1]) The sum over a is not taken.

where $\lambda_l^{(a)}$ means the number of the field operators of a $Q^{(a)}$-field in L_l. In other words, A_l is the total sum of the true degrees of the derivation operators in L_l, B_l is the total sum of b's of field operators in L_l and C_l is the number of the field operators in L_l.

Using (8.52a) and (8.52b) we can write B_l in terms of the spin $S^{(a)}$ of field operators; that is

$$B_l = \sum_{a'} 2S^{(a')}\lambda_l^{(a')} + \sum_{a''} \lambda_l^{(a'')} \tag{15.5}$$

Here the (a') denote fields of the non-zero masses $\varkappa^{(a')} \neq 0$ and the spins $S^{(a')}$ and the a'' denote fields of the zero masses $\varkappa^{(a'')} = 0$ and the half integer spins respectively. Since every interaction contains an even number of field operators of half-integer spin, we see from (15.5) that B_l is always even.

We introduce the constants

$$\eta_l = A_l + \frac{B_l}{2} + C_l - 4 \tag{15.6}$$

(The meaning of η_l will become apparent in the next paragraph). We can assume that the normalization of the field quantities is so performed that $d^{(a)}$ have the forms

$$d^{(a)}(\partial) = a_0\partial^{b^{(a)}} + a_1\varkappa^{(a)}\partial^{(b^{(a)}-1)} + \dots, \tag{15.7}$$

where $a_0, a_1, \dots$ are dimensionless constants [1]. In this representation $d^{(a)}(ik)$ contain no quantities of dimension [2] $[L^n]$ $(n \neq 0)$ except k_μ in problems of large momentum $k \gg \varkappa^{(a)}$. We shall also define the coupling constants g_l in such a way that the coefficients of the highest derivation operators of $D^{(a)}(\partial)$ in

$$\mathsf{L}_l = \{\text{product of } D^{(a)}(\partial)\, Q^{(a)}(x)\} \tag{15.8}$$

are dimensionless. Then there are no non-zero dimensional quantities, except g_l and k_μ, in the S matrix when this momentum is large.

§ 3. Dimensions of Coupling Constants

We shall now prove that the dimensions of the coupling constants g_l are $[L^{\eta_l}]$.

Since $g_l \int d^3x\, \mathsf{L}_l(x)$ has the dimension of energy (i.e. $[L^{-1}]$), the

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[1] For example, in the case of a vector field U_μ (for which $d_{\mu\nu} = \delta_{\mu\nu} + - (1/\varkappa^2)\delta_\mu\delta_\nu$), by using the field operator $V_\mu \equiv \varkappa U_\mu$ instead of U_μ we can obtain $d_{\mu\nu} = \varkappa^2\delta_{\mu\nu} - \partial_\mu\partial_\nu$ which has the form of (15.7).

[2] The symbol $[L^n]$ means the dimension of the nth power of length.

dimension of $\mathsf{L}_l(x)$ is $[L^{-4}]$. On the other hand, (15.3) and (8.14a, b) show that the dimensions of the $D^{(a)}(\partial)\, Q^{(a)}(x)$ are

$$[L^{-\{t^{(a)}+(b^{(a)}/2)+1\}}] \tag{15.9}$$

Thus we see that

$$[g_l] = [L^{\eta_l}]. \tag{15.10}$$

Therefore the η_l represent the dimensions of the coupling constants.

§ 4. S-Matrix

Because we started with the interaction Lagrangian, the S-matrix given by (13.35) (and not (13.14)) is convenient for our present purpose. It must be noted that although (13.35) was derived from (13.14), it is not clear that the interaction Hamiltonian $H'[x : n]$ exists when the interaction Lagrangian contains derivatives of high degree. Nevertheless we shall adopt (13.35), which has been used by many workers even when the interaction Lagrangian contains derivatives of high degree.

§ 5. The Condition for Primitively Divergent Diagrams

Let us consider a Feynman diagram. It is easily seen from (15.3) or (15.9) that each $D^{(a)}(\partial)\, Q^{(a)}(x)$ contributes $t^{(a)}+(b^{(a)}/2)+1$ to the N defined by (14.5).

We can calculate N by using a method similar to that used for deriving (14.5). By performing the integration over internal momenta this diagram can be expressed in terms of secondary interaction L_s, which resembles an interaction giving rise, in a first order term of the perturbation expansion, to the corresponding transition. We now show that L_s is given a form of

$$L_s = \sum_i g_s^{(i)}\, \mathsf{L}_s^{(i)}$$

by using (14.11). The characteristic constant $\eta_s^{(i)}$ of the interaction $\mathsf{L}_s^{(i)}$ is, from (15.6), given by

$$\eta_s^{(i)} = A_s^{(i)} + \frac{B_s}{2} + C_s - 4 \tag{15.11a}$$

where

$$\left.\begin{aligned} A_s^{(i)} &= \sum_a (N^{(a)} + M_i^{(a)})\, E^{(a)} \\ B_s &= \sum_a b^{(a)} E^{(a)} \\ C_s &= \sum_a E^{(a)} \end{aligned}\right\} \tag{15.11b}$$

Here $E^{(a)}$ denotes the number of external lines of the a-fields and $N^{(a)}$ and $M_s^{(a)}$ are defined as follows. Each external operator $Q^{(a)}$ contains derivation operators of the true degree $t^{(a)}$. Therefore, we can specify the Feynman diagram as $G(E^{(a)}, N^{(a)})$, in which $N^{(a)}$ denotes the sum $\Sigma t^{(a)}$ of true degrees of the external a-field operators. Furthermore, as a result of the integration over internal momenta, $G(E^{(a)}, N^{(a)})$ splits into several independent diagrams, denoted by $G(E^{(a)}, N^{(a)}+M_s^{(a)})$. These contain new derivation operators on $Q^{(a)}$ of total number $M_s^{(a)}=0, 1, 2, \ldots$. This decomposition of $G(E^{(a)}, N^{(a)})$ into $G(E^{(a)}, N^{(a)}+M_s^{(a)})$ is obtained by using the expansion of the integral (14.11) in which the order of the product of $t^{(m)}$ contributes to $M_s^{(a)}$. It is easily seen that the infinities of the highest degrees occur when $M_s^{(a)}=0$. Each $G(E^{(a)}, N^{(a)}+M_s^{(a)})$ corresponds to a secondary interaction $\mathsf{L}_s^{(s)}$; its characteristic constant $\eta_s^{(s)}$ is given by (15.6), which leads to (15.11a). Divergent integral appears to be a factor in the coupling constant $g_s^{(s)}$. Since $g_l \int d^4x \mathsf{L}_l(x)$ is dimensionless and the dimension of g_l is $[L^{\eta_l}]$, each $\int d^4x \mathsf{L}_l(x)$ has the dimension $[L^{-\eta_l}]$ and therefore contributes an amount η_l to the highest degree of the momenta in $S[\infty]$. Thus, the highest degree of momenta (including the product of d^4k) in $S^m[\infty]$ is $\Sigma n_l \eta_l$ where n_l is the number of vertices of the interaction L_l. Since the dimension of $g_s^{(s)}$ is $[L^{\eta_s^{(s)}}]$ the highest degree of the internal momenta in $S^{(m)}[\infty]$ is

$$N = \sum_l n_l \eta_l - \eta_s^{(s)}. \tag{15.12}$$

Now, from (15.12), the menber of primitive divergent diagrams must satisfy the condition

$$\sum_l n_l \eta_l \geqslant \eta_s^{(s)} \tag{15.13}$$

(SAKATA, UMEZAWA and KAMEFUCHI [1952], STUECKELBERG and PETERMANN [1951]). It must be noted that this condition is written in terms of η_l and $\eta_s^{(s)}$. The degree of the divergence of such a diagram is N defined by (15.12) (with the rule that $N=0$ means the logarithmic divergence).

§ 6. Applicability Condition of Renormalisation

The last equation shows that the condition that the number of primitive divergent diagrams $G(E^{(a)}, N^{(a)}+M_s^{(a)})$ should remain finite in any order of the perturbation approximation is

$$\eta_l \leqslant 0 \qquad \text{(for all } l\text{)}. \tag{15.14}$$

We shall first consider the case for which (15.14) is satisfied. Then, since $b^{(a)} \geqq 0$, (15.11b) shows that all the diagrams satisfying the condition

$$4 < C_s \tag{15.15}$$

have no infinities. We can therefore introduce a definite number j, which depends only on the number of the fields participating in the interaction under consideration [1]), in such a way that the number of primitively divergent diagrams cannot exceed j. The divergences in $G(E^a, N^a + M_i^a)$ can be eliminated by introducing $-g_s^{(i)} \mathsf{L}_s^{(i)}$ into the Lagrangian at the starting point. It must be noted that on account of (15.13), (15.14) $\eta_s^{(i)} \leqq 0$ and therefore $-g_s^{(i)} \mathsf{L}_s^{(i)}$ can be introduced without violating the condition (15.14). Thus we see that, when (15.14) is satisfied, all the divergences can be eliminated by introducing a finite number ($\leqq j$) of counter interactions. The theories in which (15.14) is satisfied are called "renormalisable". The discussions on the methods of separation of infinities and the methods of renormalisation are just the same as those of the quantum electrodynamics (in Ch. XIV).

On the other hand, when there is at least one interaction for which

$$\eta_l > 0, \tag{15.16}$$

any $G(E^a, N^a + M_i^a)$ diverges if n_l is sufficiently large in the high order terms of perturbation series. It is impossible, (except in some cases which we discuss in the next chapter), to eliminate this infinite number of divergences by introducing a finite number of counter terms because, as we proceed to the higher order, new diverging terms which are topologically independent, appear one after another. Therefore, it is impossible to remove all the divergences by introducing only a finite number of local interactions. Such a theory is called non-renormalisable (SAKATA, UMEZAWA and KAMEFUCHI [1952]).

§ 7. Classification of Interactions

We have seen in the last section that the condition of the renormalisability is

$$\left.\begin{array}{ll} \eta_l \leqq 0 & \text{for all interactions} \rightarrow \text{renormalisable} \\ \eta_l > 0 & \text{for at least one interaction} \rightarrow \text{unrenormalisable.} \end{array}\right\} \tag{15.17}$$

[1]) For example, we can take the number of diagrams having less than 5 external operators as j.

We shall call interactions which satisfy (15.14) and (15.16) interactions of the first and the second kinds respectively.

In the case when all the interactions realized in nature are of the first kind, renormalisation may form a closed non-singular theory in the framework of the present quantum field theory.

We shall now consider what types of fields may lead to the interactions of the first kind (UMEZAWA [1952]). Since $A_l \geqslant 0$ and $C_l \geqslant 0$, we see that a field for which

$$b^{(a)} > 4 \tag{15.18}$$

cannot lead to interactions of the first kind. Taking into account (8.52*a*), we see that *the fields with spin* $S \geqslant 2$ *and the mass* $\varkappa \neq 0$ *can never have interactions of the first kind.*

For the field $U_{\mu\nu}$ with $S=2$ and $\varkappa \neq 0$ the interaction with the smallest η_l is

$$\mathsf{L}_l = U_{\mu\nu} B_{\mu\nu} \qquad (\eta_l = 0), \tag{15.19}$$

where $B_{\mu\nu}$ is the field with $S=2$ and $\varkappa=0$. All other interactions are of the second kind.

For the field $\varkappa_\mu$ with $S=3/2$ and $\varkappa \neq 0$ the interaction with the smallest η_l is

$$\mathsf{L}_l = \varkappa_\mu \omega_\mu \qquad (\eta_l = 0), \tag{15.20}$$

where ω_μ is the field with $S=3/2$ and $\varkappa=0$. All other interactions are of the second kind.

For the field U_μ with $S=1$, and $\varkappa=0$ the interactions of the first kind are

$$F_{\mu\nu} F'_{\mu\nu} \ , \ U_\mu U'_\mu \ , \ U_\mu \partial_\mu U' \qquad (\eta_l = 0) \tag{15.21}$$

$$U_\mu \, C^{(1)}_{\mu\nu} \ C^{(2)}_{\nu} \qquad (\eta_l = 0), \tag{15.22}$$

where U'_μ and U' are fields with $S=1$ and $S=0$ respectively. Here $C^{(1)}_{\mu\nu\ldots}$ and $C^{(2)}_{\nu\ldots}$ are fields with $\varkappa=0$. It must be noted that each interaction in (15.21) consists of only two field operators. For the real vector U_μ, the vector couplings with a spinor field ψ ($S=1/2$) and those with a scalar field U are also of the first kind, namely

$$g_l \mathsf{L}_l = i g_l (\partial_\mu U^* \cdot U - U^* \cdot \partial_\mu U) U_\mu - g_l^2 U_\mu U_\mu U^* U \quad (\eta_l = 0) \tag{15.23}$$

$$g_l \mathsf{L}_l = i g_l \bar{\psi} \gamma_\mu \psi \cdot U_\mu \qquad (\eta_l = 0) \tag{15.24}$$

At first sight (15.23) and (15.24) seem to be interactions of the second kind because $b^{(a)}=2$ for the field U_μ. However, as shown in Ch. XI,

the B-component of $U_\mu = A_\mu + (1/\varkappa)\partial_\mu B$ has no physical effect, and we must take the $b^{(a)}$ of A_μ not as $b^{(a)}=2$ but $b^{(a)}=0$. Therefore we find $\eta_l = 0$ for (15.23) or (15.24).

As shown in Ch. XI, this depends on the gauge-invariance (cf. (7.101)). It is an example of the fact that invariances of the theory lead to relations between some of the diagrams of such a kind that their contributions nullify each other. Another example of this situation is given in the discussion of true degrees in Example 6 of Ch. VIII, where the relativistic invariance of the theory may make the true degree smaller and hence the degrees of the divergences smaller.

The fields with $S<1$ have various interactions of the first kind.

The interactions of the A_μ-type or $F_{\mu\nu}$-type between the electromagnetic field and a complex field ψ with $S=1/2$ belong to the first or the second kind respectively.

The interaction of the A_μ-type between the electromagnetic field and a complex field U with $S=0$ is of the first kind. In this case there appear primitively divergent diagrams with four external U-field operators, which lead to the secondary interaction $\mathsf{L}_s^{(4)} = U^* U U^* U$. We must therefore introduce the interaction $-g_s^{(4)} \mathsf{L}_s^{(4)}$ in the total Lagrangian to cancel this divergence (MATHEWS [1950, 1951], SALAM [1951], RÖHRLICH [1950]).

The scalar or pseudoscalar interactions between a spinor field and a field U with $S=0$ are of the first kind ($\eta_l = 0$). In such cases we must also introduce the counter interactions gU^3 ($\eta_l = -1$) and gU^4 ($\eta_l = 0$) in order to obtain a non-singular theory. It is easily seen by taking into account the charge conservation and the relativistic invariance of the theory that gU^3 must be introduced only when U is a real scalar. Thus, for the spin 0 field U, interactions of the first kind are made up of

$$\left.\begin{array}{l} UU' \ (\eta_l=-2) \ , \ UU'U'' \ (\eta_l=-1) \ , \ UU'U''U''' \ (\eta_l=0) \\ \partial_\mu U \partial_\mu U' \ (\eta_l=0) \ , \ \bar{\psi} O \psi' U \ (\eta_l=0) \end{array}\right\} \quad (15.25)$$

and the vector coupling (15.23) with a real vector field U_μ (including the electromagnetic field). Here U', U'', U''' denote fields of spin 0 and ψ and ψ' denote fields of spin 1/2.

The vector or pseudovector interactions between a spinor field and a scalar field or a pseudoscalar field are of the second kind ($\eta_l = 1$) [1].

[1] However, as shown in Ch. XI, the vector interaction between a real scalar and a spinor field has no physical effects.

The direct interactions $(\bar{\psi}^a O \psi^b)\,(\bar{\psi}^c O' \psi^d)$ (O and O' are products of γ_μ's) between spinor fields belong to the second kind ($\eta_l = 2$).

For the field ψ of spin 1/2 interactions of the first kind are made up of

$$\bar{\psi}' O \psi \quad (\eta_l = -1), \quad \bar{\psi}' O \psi U \quad (\eta_l = 0) \tag{15.26}$$

together with the vector coupling (15.24) with a real vector field U_μ (including the electromagnetic field).

Table I shows the properties of important interactions between common fields.

TABLE I

	U		U_μ		ψ	
	Charged	Neutral	Charged	Neutral	Charged	Neutral
A_μ	V (0) 1st $[U^* U U^* U(0)]$		V (1) 2nd T (2) 2nd		V (0) 1st T (1) 2nd	
ψ	S (0) 1st $[U^* U U^* U(0)]$	S (0) 1st $[U^4(0), U^3(0)]$	V (1) 2nd	V (0) for v 1st V (1) for pv 2nd	Direct Interaction (2) 2nd	
	V (1) 2nd	V (1) 2nd for ps	T (1) 2nd	T (1) 2nd		

S, V and T denote the scalar, vector and tensor couplings respectively. The values in brackets show η_l for respective interactions. The counter interactions required for the renormalisation are written in [ ]. The symbols s, ps, v and pv denote the scalar, pseudoscalar, vector and pseudovector fields respectively.

§ 8. The Physical Meaning of the Classification of Interactions

We shall now give a discussion based on a dimensional analysis which shows the physical meaning of the classification of interactions given in the last paragraph (SAKATA, UMEZAWA and KAMEFUCHI [1952]).

In high energy processes, in which the "high energies" are expressed roughly by $(1/\lambda)$ (λ = wave length), the ratio of the n-th and $(n+1)$th terms ($S^{(n)}$, $S^{(n+1)}$) of the S-matrix can be written as

$$\frac{S^{(n+1)}}{S^{(n)}} \propto \frac{g_l}{\lambda^{\eta_l}}. \tag{15.29}$$

For the dimension of g_l is $[L^{\eta_l}]$. Therefore, we see that, when $\eta_l > 0$, higher order terms of the S-matrix make the large contributions to

the high energy processes ($\lambda^{\eta_l} \ll g_l$) even when the coupling constant g_l is very small.

This shows that the reaction of the self-field becomes very important when $\eta_l > 0$. For this reason it has often been said (HEISENBERG [1936, 1938, 1939*a*, *b*]), that there exists a fundamental length r_0 and that (15.29) must be changed in the energy region $\lambda < r_0$. From this point of view r_0 can be regarded as the radius of the elementary particle and $\lambda \approx r_0$ marks the limit of the applicability of the present quantum field theory. This is lent support by the following discussion. If we try to obtain a non-singular theory of the interactions of the second kind by using the renormalisation method, it is necessary that there should exist simultaneously an infinite number of interactions (i.e. the counter interactions) which are also of the second kind. Such an assembly of interactions is equivalent to a non-local interaction corresponding to an extended model of the elementary particles [1]). However, there are no known phenomena which are established to depend essentially on r_0. The phenomenon which most strongly suggests an interaction of the second kind is that of β-decay, although it is possible that β-decay may be explained by interactions of the first kind (UMEZAWA [1952], TANIKAWA [1953], TANAKA and ITO [1953]). The perturbation theory cannot account for meson phenomena (e.g. meson production by π-, γ- or nucleon-nucleon collision, meson-nucleon scattering and so on), which suggest that the coupling constant of the meson-nucleon interaction is not small enough for perturbation theory to be valid. Furthermore, recent experiments have sometimes suggested the existence of isobaric levels (of the nucleon-meson system), which has been expected by the strong (WENTZEL [1940]) and the intermediate (TOMONAGA [1946]) coupling theories. This makes it difficult to decide the type of the meson-nucleon interaction. Since the higher order effects have larger contributions in the case of an interaction of the second kind, we can expect the simultaneous production of many mesons by a nucleon-nucleon collision. The occurrence of multiple production is supported by recent experiments. However, from these experiments we cannot yet conclude that the meson-nucleon interaction belongs to the second kind.

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[1]) The interactions between field operators at different points are called non-local interactions.

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CHAPTER XVI

DAMPING THEORY

§ 1. Damping Equations

In damping theory some of the higher order terms of the perturbation expansion are taken account of in the same way as the damping effect is in the lower approximation.

For simplicity, we shall now take [1]) a flat surface at the time t. We obtain the evolution operator $\bar{S}[t, t_1]$ in the Schrödinger representation from that in the interaction representation by means of the relation (cf. (10.56))

$$\bar{S}(t, t_1) = \exp\,[-iT_4^0 t]\, S[\sigma(t), \sigma(t_1)]\, \exp\,[iT_4^0 t_1].$$

On account of (10.57) we obtain

$$i\frac{\partial}{\partial t}\bar{S}(t, t_1) = (H_0 + H')\,\bar{S}(t, t_1) \quad \text{for } t > t_1, \tag{16.1}$$

where $H_0 + H'$ is the total energy operator in the Schrödinger representation, and is given by (10.58) and (10.59).

The operator $\bar{S}(t, t_1)$ can be obtained as a solution of (16.1) with the initial condition

$$\bar{S}(t, t_1) = 1 \quad \text{for } t = t_1. \tag{16.2}$$

We write $\bar{S}(t, t_1)$ as a Fourier expansion

$$\bar{S}(t, t_1) = \int_{-\infty}^{\infty} dE\, S(E, t, t_1) \tag{16.3}$$

with $S(E, t, t_1)$ whose form is

$$S(E, t, t_1) = \exp\,\{iE(t - t_1)\} \cdot S(E).$$

Then, (16.1) leads to

$$\int_{-\infty}^{\infty} dE\, (E - H^0 - H')\, S(E, t, t_1) = 0 \quad \text{for } t > t_1. \tag{16.4}$$

We shall consider the case in which the interaction Hamiltonian H' is made up of two parts, or

$$H' = H'' + \phi, \tag{16.5}$$

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[1]) For the covariant formulation of the damping theory see FUKUDA and MIYAZIMA [1950].

and adopt the representation in which

$$H = H^0 + \phi \tag{16.6}$$

is diagonal [1]).

It is easily seen that (16.4) is satisfied by the solution of the equation

$$(E - H - H'')\, S(E, t, t_1) = -\frac{i}{2\pi} \exp\{-iE(t-t_1)\}. \tag{16.7}$$

We now assume that $S(E, t, t_1)$ has the form

$$S(E, t, t_1) = \{1 - 2\pi i \delta_+(E-H)U(E)\}\lambda(E, t, t_1). \tag{16.8}$$

(ARNOUS and ZIENAU [1951]). Then, the left hand side of (16.7) can be written, by means of (8.30a), as

$$\left.\begin{aligned} &(E-H-H'')\{1-2\pi i\delta_+(E-H)U(E)\}\lambda(E, t, t_1) \\ &= [E - H + U(E) - \{H'' - 2\pi i H''\delta_+(E-H)U(E)\}]\,\lambda(E, t, t_1). \end{aligned}\right\} \tag{16.9}$$

By assuming that $U(E)$ is a matrix whose diagonal elements are zero, we obtain, from (16.7), the equations

$$U(E) = \{H'' - 2\pi i H''\delta_+(E-H)U(E)\}_{\text{n.d}} \tag{16.10}$$

$$\lambda(E, t, t_1) = -\frac{i}{2\pi}\{E - H + \frac{i}{2}\Gamma(E)\}^{-1} \exp\{-iE(t-t_1)\} \tag{16.11}$$

$$\Gamma(E) \equiv 2i\{H'' - 2\pi i H''\delta_+(E-H)U(E)\}_{\text{d}}. \tag{16.12}$$

Here the matrix $\{\ \}_{\text{d}}$ is obtained from the matrix $\{\ \}$ by putting all non-diagonal elements $= 0$, and $\{\ \}_{\text{n.d}}$ is obtained by putting all diagonal elements 0.

Thus, we see that (16.10) and (16.11) are the equations from which $S(t, t_1)$ must be determined. Equation (16.10) is called the Heitler damping equation (HEITLER [1941], WILSON [1941]).

Equation (16.3) shows that

$$\left.\begin{aligned} \bar{S}(t, t_1) &= -\frac{i}{2\pi}\int_{-\infty}^{\infty} dE\,\{1 - 2\pi i\delta_+(E-H)\,U(E)\}\left\{E - H + \frac{i}{2}\Gamma(E)\right\}^{-1} \\ &\qquad \exp\{-iE(t-t_1)\} \\ &= -\int_{-\infty}^{\infty} dE\ \delta_+(E-H)\ \exp\{-iE(t-t_1)\} \\ &\quad -\int_{-\infty}^{\infty} dE\ \delta_+(E-H)\left\{U(E) - \frac{i}{2}\Gamma(E)\right\}\left\{E - H + \frac{i}{2}\Gamma(E)\right\}^{-1} \\ &\qquad \exp\{-iE(t-t_1)\} \end{aligned}\right\} \tag{16.13}$$

[1]) For example, in the problem of the electron in a hydrogen atom, ϕ can be regarded as the Coulomb potential due to the proton.

When $t>t_1$, we can integrate the first term on a large semi-circle below the real axis in the complex E-plane (cf. Fig. 16.1), and obtain

$$\left.\begin{aligned}\bar{S}(t,t_1) &= e^{-iH(t-t_1)} \\ &\quad -\int_{-\infty}^{\infty} dE\ \delta_+(E-H)\left\{U(E)-\frac{i}{2}\Gamma(E)\right\}\left\{E-H+\frac{i}{2}\Gamma(E)\right\}^{-1} \\ &\qquad \exp\{-iE(t-t_1)\} \quad \text{for } t>t_1.\end{aligned}\right\} \tag{16.14}$$

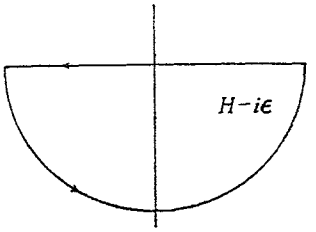

Fig. 16.1

We can show (ARNOUS and ZIENAU [1951]) that if $\bar{S}(t,t_1)$ is given by (16.14), it satisfies the initial condition (16.2).

In the calculation of the transition probability w_{if} between two eigenstates (i,f) of the unperturbed energy operator H, the use of $\bar{S}(t,t_1)$ may be replaced by that of $S'(t,t_1)$ given by

$$S'(t,t_1)=e^{iH(t-t_1)}\bar{S}(t,t_1). \tag{16.15}$$

In fact when E_i and E_f are the eigenvalues of the i- and f-states of H we have

$$\left.\begin{aligned}w_{if} &= |(f\,|\,\bar{S}(t,t_1)\,|\,i)|^2 \\ &= |(f\,|\,S'(t,t_1)\,|\,i)|^2\end{aligned}\right\} \tag{16.16}$$

Substituting (16.14) into (16.15), we have

$$\left.\begin{aligned}S'(t,t_1) &= 1 \\ &\quad -\int_{-\infty}^{\infty} dE\ \exp\{-i(E-H)(t-t_1)\}\cdot \\ &\quad \cdot\delta_+(E-H)\left\{U(E)-\frac{i}{2}\Gamma(E)\right\}\left\{E-H+\frac{i}{2}\Gamma(E)\right\}^{-1}\end{aligned}\right\} \tag{16.17}$$

It can be proved (ARNOUS and ZIENAU [1951]) that when $\phi=0$ and therefore $H=T_4^0$, the power series expansion of (16.17) agrees with the Dyson S-matrix (13.14).

Equation (16.17) gives

$$S'[t,-\infty]=1+\int_{-\infty}^{\infty} dE\,\delta(E-H)\left\{U(E)-\frac{i}{2}\Gamma(E)\right\}\left\{E-H+\frac{i}{2}\Gamma(E)\right\}^{-1} \tag{16.18}$$

on account of the relation

$$\left.\begin{aligned}-\lim_{t\to-\infty} e^{iat}\,\delta_+(a) &= -\lim_{t\to-\infty}\frac{1}{2\pi}\int_0^{\infty} d\beta\, e^{ia(\beta+t)} \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\beta\, e^{ia\beta}=\delta(a)\end{aligned}\right\} \tag{16.19}$$

It must be noted that the equality in (16.19) is only valid in the integration

$$-\lim_{t\to-\infty}\int_{-\infty}^{\infty} da\, e^{iat}\,\delta_+(a)f(a)=f(0)$$

when $f(a)$ is an analytic function.

From (16.18) we have

$$(f\,|\,S'(t,-\infty)\,|\,i)=\frac{(f\,|\,U(E_f)\,|\,i)}{E_f-E_i+\frac{i}{2}\Gamma_0(E_f)}\qquad\text{for } f\neq i \tag{16.20}$$

$$(i|S'(t,-\infty)|i)=0\qquad(\text{when } \Gamma_0(E_i)\neq 0), \tag{16.21}$$

where

$$\Gamma_0(E)\equiv(i|\,\Gamma(E)|i). \tag{16.22}$$

The probability that the f-state lies in the range (E_f, E_f+dE_f) is

$$w_{if}\,dE_f=\frac{|\,(f\,|\,U(E_f)\,|\,i)\,|^2}{(E_f-E_i-\frac{1}{2}\Gamma_0^{(\mathrm{I})}(E_f))^2+\frac{1}{4}(\Gamma_0^{(\mathrm{R})}(E_f))^2}\,dE_f, \tag{16.23}$$

where $\Gamma_0^{(\mathrm{R})}$ and $\Gamma_0^{(\mathrm{I})}$ are the real and imaginary parts of Γ_0, so that

$$\Gamma_0(E)=\Gamma_0^{(\mathrm{R})}(E)+i\Gamma_0^{(\mathrm{I})}(E). \tag{16.24}$$

§ 2. Damping Effects

First, we shall consider the physical meaning of $\Gamma_0^{(\mathrm{I})}$. Equation (16.23) shows that the energy of the final state has a maximum probability for $E_f=E_i+(1/2)\Gamma_0^{(\mathrm{I})}(E_f)$. In other words, the system which was initially in the i-state changes its energy by $(1/2)\Gamma_0^{(\mathrm{I})}(E_f)$ on account of the disturbance due to the various states connected with the i-state by the interaction. For example, when $\phi=0$ and the i-state corresponds to a free particle, $(1/2)\Gamma_0^{(\mathrm{I})}(E_f)$ is equal to the self-energy. Then, although it is infinite, we must take the observed value of the energy of a free particle as $E_f=E_i+(1/2)\Gamma_0^{(\mathrm{I})}(E_f)$ in the renormalised theory. When the i-state corresponds to an electron in a hydrogen atom, $(1/2)\Gamma_0^{(\mathrm{I})}(E_f)$ is the deviation of the energy levels due to radiative corrections. The difference between the latter deviation and the self-energy of an electron gives the Lamb-shift.

We shall now consider the physical meaning of $\Gamma_0^{(\mathrm{R})}$.

In the classical theory of the electron an accelerated electron loses energy by radiation. This results in a reaction force $(2/3)e^2(d^3/dt^3)x$ acting on the electron itself. Therefore the equation of motion of an

electron oscillating with the natural frequency ν_0 due to an external force is

$$m\frac{d^2}{dt^2}x = -m\nu_0^2 x + \frac{2}{3}e^2\frac{d^3}{dt^3}x\,, \tag{16.25}$$

where the terms with higher time derivatives ($\geqslant 3$) and depending on the electron radius are omitted.

Since, when $\nu_0 > 1/r_0$ (r_0 is the classical electron radius e^2/m), the wave length is smaller than the electron radius and we cannot use (16.25) in which the terms depending on the structure of the electron do not occur. Therefore, it is necessary that the condition

$$\nu_0 r_0 \ll 1$$

should be satisfied. This condition leads to

$$\nu_0 \gg \gamma \tag{16.26}$$

with

$$\gamma \equiv \frac{2e^2}{3m}\nu_0^2 = \frac{2}{3}\nu_0^2 r_0\,.$$

Under the condition (16.26), the solution of (16.25) can be written approximately as

$$x \approx x_0\, e^{-\frac{\gamma}{2}t}\, e^{i\nu_0 t}\,. \tag{16.27}$$

This shows that the intensity $\mathbf{E}$ of the electromagnetic field produced by the electron is also damped by the factor $\exp(-\gamma t/2)$:

$$\mathbf{E} = \mathbf{E}_0\, e^{-\frac{\gamma}{2}t}\, e^{i\nu_0 t}\,. \tag{16.28}$$

The Fourier representation of (16.28) is

$$\mathbf{E} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, \mathbf{E}(\nu)\, e^{i\nu t}$$

$$\mathbf{E}(\nu) = \frac{i}{2\pi}\mathbf{E}_0\frac{1}{(\nu_0-\nu)+i(\gamma/2)}\,. \tag{16.29}$$

Thus we obtain the radiation intensity

$$I = I_0\,\frac{\gamma}{2\pi}\,\frac{1}{(\nu-\nu_0)^2+(\gamma^2/4)}\,, \tag{16.30}$$

where I_0 is a constant which is independent of ν. The term $(\gamma^2/4)$ in the denominator comes from the radiative correction (the second term of (16.25)) and leads to the width of $\gamma/2$ around the maximum $\nu=\nu_0$ in the frequency distribution of the radiation intensity. The quantity $\gamma/2$ is called the natural line width.

We see that (16.23) and (16.30) have very similar forms and that $(1/2)\Gamma_0^{(\mathrm{R})}(E_f)$ can be regarded as the line width of the probability distribution (16.23).

From (16.12) we have

$$\Gamma^{(\mathrm{R})}(E) = 2\pi(U^*(E)\,\delta(E-H)\,U(E))_{\mathrm{d}}\,. \tag{16.31}$$

In fact, (16.12) and (16.10) give

$$\begin{aligned}
&\Gamma^{(\mathrm{R})}(E) = \tfrac{1}{2}(\Gamma(E)+\Gamma^*(E))\\
&= 2\pi\{H''\delta_+(E-H)U(E) + U^*(E)\delta_-(E-H)H''\}_{\mathrm{d}}\\
&= 2\pi\{H''_{n\mathrm{d}}\delta_+(E-H)U(E) + U^*(E)\delta_-(E-H)H''_{n\mathrm{d}}\}_{\mathrm{d}}\\
&= 2\pi\{U^*(E)\delta_+(E-H)U(E) + U^*(E)\delta_-(E-H)U(E)\}_{\mathrm{d}}\\
&= 2\pi(U^*(E)\delta(E-H)U(E))_{\mathrm{d}}.
\end{aligned}$$

On the other hand, from (16.17) we obtain

$$\begin{aligned}
&\frac{d}{dt}(f\,|\,S'(t,t_1)\,|\,i)\\
&\quad = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dE\; e^{-\frac{i}{\hbar}(E-E_f)(t-t_1)}\,\frac{(f|\,U(E)\,|\,i)}{E-E_i+\frac{i}{2}\Gamma_0(E)} \qquad \text{for } f\neq i.
\end{aligned}$$

The relation (16.23) shows that, if $\Gamma_0^{(\mathrm{R})}$ is small, for most final states we have

$$E_f \approx E_i + \tfrac{1}{2}\Gamma_0^{(\mathrm{I})}(E_f)\,.$$

In such a case we obtain

$$\begin{aligned}
&\frac{d}{dt}(f\,|\,S'(t,t_1)\,|\,i)\\
&\quad \approx -i\int_{-\infty}^{\infty} dE\; e^{\frac{i}{\hbar}(E-E_f)(t_1-t)}\,\delta_+(E-E_f)\,(f|\,U(E)\,|\,i).
\end{aligned}$$

From this we obtain

$$\begin{aligned}
\frac{d}{dt}(f\,|\,S'(t,-\infty)\,|\,i) &= i\int_{-\infty}^{\infty} dE\;\delta(E-E_f)\,(f\,|\,U(E)\,|\,i)\\
&= i(f|\,U(E_f)\,|\,i) \qquad \text{for } f\neq i.
\end{aligned} \tag{16.32}$$

on account of (16.19).

Moreover (16.20) leads to

$$(f\,|\,S'(t,-\infty)\,|\,i) \approx -2\pi i(f|\,U(E_f)\,|\,i)\delta_+(E_f-E_i-\tfrac{1}{2}\Gamma_0^{(\mathrm{I})}(E_f)).$$

By substituting this and (16.32) into (13.4), we find that the transition probability per unit time is

$$\frac{d}{dt}w_{fi} \approx 2\pi\,|\,(f|\,U(E_f)\,|\,i)\,|^2\,\delta\,(E_f-E_i-\tfrac{1}{2}\Gamma_0^{(\mathrm{I})}(E_f)) \tag{16.33}$$

Comparing (16.31) and (16.33), we have

$$\Gamma_0^{(R)}(E_f) \approx \sum_f \frac{d}{dt} w_{if} . \tag{16.34}$$

In other words, $\Gamma_0^{(R)}(E_f)$ is roughly equal to the sum of the transition probabilities (per unit time) from the i-state to the other states. Therefore $1/\Gamma_0^{(R)}(E_f)$ can be regarded as the life time of the i-state. This is in agreement with the discussion in Ch. XIII which showed that the imaginary parts of the self-energies give the reciprocal of the life-times of natural decay. We see from (16.23) and (16.34) that the probability of transition to any f-state is decreased because of the transition effects ($\Gamma_0^{(R)}$) from the i-state to the other f-states.

This fact, that the width of spectral lines are larger when the transition probabilities from them are larger, shows that the levels of unperturbed states are disturbed by states strongly connected with them by the perturbing interaction energy.

As an example we shall consider the two levels a, b of an electron in an atom. Here a is the ground state and therefore its width is zero. Since an electron in the level b can make a transition to the level a by γ-ray emission, b has a large width. Moreover, if different atoms collide with each other frequently (as in a gas), the width of the b-level becomes still wider on account of the increase in the transition probabilities induced by the collisions. Therefore, experiments which set out to determine natural line widths must be carried out in gases at low pressure. These results for the line widths, first given by WEISSKOPF and WIGNER [1930], have been confirmed experimentally.

It must be noted that for interactions of the second kind, although the numerator of (16.23) becomes very large for high energy processes, the $\Gamma_0^{(R)}$ of the denominator cancels this effect and it may be expected that the transition probability cannot become extremely large. This result is very reasonable on account of the unitarity of the S-matrix.

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CHAPTER XVII

THEORY OF S-MATRIX

§ 1. Observable Quantities

Since the present quantum field theory is beset with difficulties concerning the infinite value of certain physical quantities, it must be regarded as a provisional theory; we still lack a correct theory of the elementary particles. In an attempt to formulate a theory HEISENBERG [1943] discussed the phenomenological relations between observable quantities in the expectation that there would be some correspondence between the present theory and a correct one.

The observable quantities adopted by Heisenberg as fundamental to the theory of elementary particles are the energy-momenta of free particles and the quantities

(1) the discrete energy eigenvalues for the bound states,

(2) the asymptotic form (phase shift) of the incident and scattered waves at points very far from the scattering centres.

Since in scattering problems the cross-sections can be determined only by asymptotic forms of wave functions at points very far from points of interactions, (2) can be regarded as observable quantities.

Since bound states can be regarded as the case of zero diverging wave of a stationary scattering, we can expect intimate relations between (1) and (2).

The asymptotic forms of diverging waves can be determined at a time very long after the interactions and therefore determined by the S-matrix. For this reason, Heisenberg investigated whether (1) and (2) may both be determined by means of the S-matrix (HEISENBERG [1943]).

§ 2. S-Matrix

For simplicity we adopt flat surfaces $\sigma(t)$ at time t. From (10.1) and (10.50) we have

$$\Psi[\sigma]=S[\sigma]\Psi[-\infty] \tag{17.1}$$

$$S[-\infty]=1 \tag{17.2}$$

$$S[\sigma]=1-i\int_{-\infty}^{t}dt'\,H'[\sigma(t')]\,S[\sigma(t')]. \tag{17.3}$$

The probability of finding a state Φ_b at $t=\infty$ when the state vector at $t=-\infty$ is Φ_a is [1])

$$w_{b,a}=|(\Phi_b, S[\infty]\Phi_a)|^2 \equiv |S_{ba}|^2. \tag{17.4}$$

We introduce the R-matrix by

$$S=1+R. \tag{17.5}$$

Here S is the matrix $[S_{ba}]$. Since S is unitary we have

$$R^*R=-(R+R^*). \tag{17.6}$$

Then (17.5) and (17.4) give, for $a \neq b$,

$$w_{b,a}=|R_{ba}|^2. \tag{17.7}$$

On the other hand, (17.3) gives

$$\left.\begin{aligned} R_{ba} &= -i\int_{-\infty}^{\infty} dt\,(\Phi_b, H'[\sigma(t)]\,S[\sigma(t)]\,\Phi_a) \\ &= -i\int_{-\infty}^{\infty} dt\,(\Phi_b, e^{iH^0t}H'e^{-iH^0t}S[\sigma(t)]\,\Phi_a), \end{aligned}\right\} \tag{17.8}$$

where H^0 is the free energy operator (10.59) and H' the interaction Hamiltonian (10.58) in the Schrödinger representation. Assuming that the interaction is switched on and off at $t=-\infty$ and $t=+\infty$ respectively, we can take the eigenfunctions of H^0 as Φ_a and Φ_b

$$\left.\begin{aligned} H^0\Phi_a &= E_a\Phi_a \\ H^0\Phi_b &= E_b\Phi_b \end{aligned}\right\}. \tag{17.9}$$

We must then introduce a factor $\exp(-\epsilon|t|)$ in H' (cf. (13.10)), where ϵ is an infinitesimal positive constant. Then, (17.8) can be written as

$$R_{ba}=-i(\Phi_b, H'\,\Psi_a^{(+)}(E_b)), \tag{17.10}$$

where

$$\Psi_a^{(+)}(E)=\int_{-\infty}^{\infty} dt\,e^{i(E-H^0)t}e^{-\epsilon|t|}S[\sigma(t)]\,\Phi_a. \tag{17.11}$$

Using (8.30a), (17.3) and (17.11) we see that $\Psi_a^+(E)$ satisfies the equation [2])

$$\Psi_a^+(E)=2\pi\delta(E-E_a)\Phi_a+\frac{1}{E-H^0+i\varepsilon}H'\Psi_a^{(+)}(E) \tag{17.12}$$

Introducing Ψ_a^+ by

$$\Psi_a^{(+)}(E)=2\pi\delta(E-E_a)\Psi_a^+ \tag{17.13}$$

---

[1]) $(\Phi_a, A\,\Phi_b)\equiv\Phi_a^*\,A\,\Phi_b$.

[2]) Here, the following relation is used

$$\int_{-\infty}^{\infty} dt\int_{-\infty}^{t} dt'\,f(t,t')=\int_{-\infty}^{\infty} dt'\int_{t'}^{\infty} dt\,f(t,t').$$

we have, from (17.12),

$$\Psi_a^+ = \Phi_a + \frac{1}{E_a - H^0 + i\varepsilon} H' \Psi_a^+. \tag{17.14}$$

The probability per unit time of transition to a state Φ_b at time t from a state Φ_a at $t = -\infty$ is (cf. (13.4))

$$\frac{d}{dt} w_{b,a} = \frac{d}{dt} |(\Phi_b, S[\sigma(t)]\, \Phi_a)|^2. \tag{17.15}$$

For $b \neq a$, by using (17.3), this gives

$$\left.\begin{aligned} \frac{d}{dt} w_{b.a} &= i\,(H'[(\dot{\sigma}(t)]\, S[\sigma(t)]\, \Phi_a, \Phi_b)\,(\Phi_b, S[\sigma(t)]\, \Phi_a) + \text{C.C.} = \\ &= (e^{i(E_b - H^0)t} S[\sigma(t)]\, \Phi_a, H'\Phi_b) \int_{-\infty}^{t} dt' (\Phi_b, H' e^{i(E_b - H^0)t'} S[\sigma(t')]\, \Phi_a) \\ &\qquad + \text{C.C.} \end{aligned}\right\} \tag{17.16}$$

where C.C. means the complex conjugate quantity of the preceding term.

Using (17.10), (17.13) and the relation

$$e^{-iH^0 t} S[\sigma(t)]\, \Phi_a = e^{-iE_a t} \Psi_a^+ ,$$

which is derived from (17.11), we can write (17.16) as

$$\left.\begin{aligned} \frac{d}{dt} w_{ba} &= |\mathbf{R}_{ba}|^2 \int_{-\infty}^{t} dt'\, e^{i(E_a - E_b)(t - t')} + \text{C.C.} \\ &= 2\pi\, |\mathbf{R}_{ba}|^2\, \delta(E_a - E_b). \end{aligned}\right\} \tag{17.17}$$

Here $\mathbf{R}_{ba}$ is defined as

$$R_{ba} = -2\pi i \delta(E_a - E_b) \mathbf{R}_{ba}. \tag{17.18a}$$

From (17.18) we obtain (LIPPMAN and SCHWINGER [1950])

$$\mathbf{R}_{ba} = (\Phi_b, H' \Psi_a^+). \tag{17.18b}$$

On the other hand, (17.14) can be written, by means of (8.30a) as

$$\Psi_a^+ = \Phi_a - 2\pi i \delta_+(E_a - H^0) H' \Psi_a^+. \tag{17.14b}$$

As shown by the δ_+-function, (17.14b) has the form of a solution of a scattering problem with the "ausstrahlung" boundary condition [1]).

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[1]) In general, when Φ_a describes a state of many particles, the spacial representation corresponding to $\delta_+(E - H_0)$ can be written asymptotically as a product of their outgoing waves.

Since the first term of (17.14*b*) has the form of an incoming wave, the second term can be regarded as the scattered wave.

Let us introduce a matrix $\Psi=[\Psi_{ba}]$

$$\Psi_{ba} \equiv (b|\Psi|a) \equiv (\Phi_b, \Psi_a^+).$$

Then, (17.14*b*) can be written as

$$\Psi = 1 + \Psi^{sc} \tag{17.14c}$$

with

$$\left.\begin{aligned} \Psi_{ab}^{sc} &= -2\pi i\delta_+(E_a-E_b)\,(\Phi_b, H'\Psi_a^+) \\ &= -2\pi i\delta_+(E_a-E_b)\mathbf{R}_{ba}. \end{aligned}\right\} \tag{17.19}$$

The quantity Ψ^{sc} is the matrix of the scattered wave (MØLLER [1945, 1946]).

Now we shall give an important theorem in wave theory.

Equation (17.6) can be written in terms of $\mathbf{R}$ as

$$4\pi^2 \sum_b \delta(E_a-E_b)\mathbf{R}_{ba}^*\,\delta\,(E_b-E_c)\mathbf{R}_{bc} = 2\pi\, i\delta(E_a-E_c)\,(\mathbf{R}_{ac}-\mathbf{R}_{ca}^*).$$

Omitting $\delta(E_a-E_c)$ on both sides and taking $a=c$, we obtain

$$\sum_b \frac{d}{dt} w_{ba} = -2I_m\,(\mathbf{R}_{aa}). \tag{17.20}$$

The left hand side of (17.20) can be regarded as the total transition probability from an a-state to all possible b-states ($b\neq a$). Therefore, (17.20) shows that this total probability can be given by the "forward scattering" (i.e. the scattering from the a-state to itself). In other words, the decrease in the intensity of a wave propagating through a medium is the result of interference between the incident wave and the scattered wave (LAX [1950]).

§ 3. Scattering Problems

We shall apply the fundamental equation (17.14*c*) to a scattering process involving two spin 0 particles (masses $\varkappa_1$, $\varkappa_2$). Let us take the coordinates and the energy-momenta of the two particles as $(\mathbf{r}^{(1)}, \mathbf{k}^{(1)})$ and $(\mathbf{r}^{(2)}, \mathbf{k}^{(2)})$ respectively. Then the total and relative momenta $\mathbf{K}$ and $\mathbf{k}$ are

$$\left.\begin{aligned} \mathbf{K} &= \mathbf{k}^{(1)} + \mathbf{k}^{(2)} \\ \mathbf{k} &= \tfrac{1}{2}(\mathbf{k}^{(1)}-\mathbf{k}^{(2)}) \end{aligned}\right\} \tag{17.21}$$

and the total energy E is

$$E = \sqrt{\varkappa_1^2+(\tfrac{1}{2}\mathbf{K}-\mathbf{k},\, \tfrac{1}{2}\mathbf{K}-\mathbf{k})} + \sqrt{\varkappa_2^2+(\tfrac{1}{2}\mathbf{K}+\mathbf{k},\, \tfrac{1}{2}\mathbf{K}+\mathbf{k})}. \tag{17.22}$$

Any state of the two particles can be determined by a set $(\mathbf{K}, E, \zeta, \varphi)$ of variables, where ζ and φ are

$$\zeta = \cos\theta = \frac{k_3}{k}, \quad \varphi = \tan^{-1}\frac{k_2}{k_1}. \tag{17.23}$$

When $(\mathbf{K}, E, \eta)$ is a set of "scattering constants", (17.14c) can be written in the mixed representation of $(\mathbf{K}, E, \zeta, \varphi)$ and $(\mathbf{K}, E, \eta)$ as

$$\left.\begin{aligned}&(E', \zeta', \varphi'|\Psi|E, \eta) = \delta(E-E')\,(\zeta', \varphi'|\eta)\\ &+\delta_+(E-E')\,(E', \zeta', \varphi'|-2\pi i\mathbf{R}|E, \eta)\end{aligned}\right\} \tag{17.24a}$$

where the factor δ representing the conservation of total momentum is omitted. Therefore, (17.24a) is true only for $\mathbf{K}=\mathbf{K}'$.

The last equation can be written in the form

$$\left.\begin{aligned}&(E', \zeta', \varphi'|\Psi|E, \eta) = \delta_-(E-E')\,(\zeta', \varphi'|\eta)\\ &+\delta_+(E-E')\,(E', \zeta', \varphi'|A|E, \eta)\end{aligned}\right\} \tag{17.24b}$$

where

$$(E', \zeta', \varphi'|A|E, \eta) = (\zeta', \varphi'|\eta) + (E', \zeta', \varphi'|-2\pi i\mathbf{R}|E, \eta). \tag{17.25a}$$

From (17.25a) we obtain

$$\left.\begin{aligned}(E, \zeta', \varphi' \mid A \mid E, \eta) &= \int d\zeta''\, d\varphi''\,\{\delta(\zeta'-\zeta'')\,\delta(\varphi'-\varphi'')\\ &\quad + (E, \zeta', \varphi' \mid -2\pi i\,\mathbf{R} \mid E, \zeta'', \varphi'')\}\,(\zeta'', \varphi'' \mid E, \eta)\\ &= S^0\,(\zeta', \varphi' \mid \eta) \qquad \text{for } (E'=E).\end{aligned}\right\} \tag{17.25b}$$

where S^0 is determined in such a way that $S^0\delta(E'-E)$ is an eigenvalue of S corresponding to a state $(\mathbf{K}, E, \zeta', \varphi')$.

We shall now transform (17.24b) into the coordinate representation. In the centre of mass system $(\mathbf{K}=0)$ we can take the angular momenta l and their components m in the x_3-direction as the constants η. When the incoming wave is $\exp(i\mathbf{k}\cdot\mathbf{r})$ we obtain

$$\left.\begin{aligned}\Psi(\mathbf{r}) = \frac{\sqrt{\pi}}{ikr}\sum_l \sqrt{2l+1}\; i^l\{&-e^{-i(kr-\frac{\pi}{2}l)}\\ &+e^{i(kr-\frac{\pi}{2}l)}\,S^0\}\,Y_l^0(\cos\theta),\ k \equiv |\mathbf{k}|,\ r = |\mathbf{r}|,\end{aligned}\right\} \tag{17.26a}$$

where θ is the angle of $\mathbf{r}$ with respect to the direction of the momentum $\mathbf{k}$.

The last equation can be obtained as follows: The coordinate representation of Ψ is given, first by changing the representation from (E', ζ', φ') to $\mathbf{k}'$ in (17.24b) and next by integrating over d^3k'

multiplied by $(1/2\pi)^{3/2} \exp(i\,\mathbf{k}'\cdot\mathbf{r}) d^3k'$. The change of the representation from (E', ζ', φ') to $(\mathbf{k}')$ can be done by multiplying Ψ by a normalisation factor $1/\sqrt{\Delta}$. This factor is required to preserve the normalisation of the wave function under the change of the representation and therefore is defined as

$$\int \frac{1}{\Delta} dE'\, d\zeta'\, d\varphi'\, e^{i(\mathbf{k}'\cdot\mathbf{r})} = \int d^3k'\, e^{i(\mathbf{k}',\mathbf{r})} = \int k'^2\, dk'\, d\zeta'\, d\varphi'\, e^{i(\mathbf{k}',\mathbf{r})}.$$

In other words, $k'^2\Delta$ is the Jacobian [1]) $\Delta = \partial E'/\partial k'$.

Thus. the coordinate representation of Ψ is

$$\Psi(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{l,m} \int \frac{dE'}{\sqrt{\Delta}} d\zeta'\, d\varphi'\, e^{i(\mathbf{k}'\cdot\mathbf{r})}\, (E', \zeta', \varphi' \mid \Psi \mid E, l, m)$$
$$= \left(\frac{1}{2\pi}\right)^{3/2} \sum_{l,m} \int \frac{dE'}{\sqrt{\Delta}} \int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\theta'\, d\theta'\, e^{ik'r\cos\theta'}\, (E', \zeta', \varphi' \mid \Psi \mid E, l, m),$$

where (φ', θ') are the angles of k' with respect to the direction $\mathbf{r}$. Integrating over $d\varphi' d\theta'$ and neglecting the higher power terms ($\geqslant 2$) of $(1/k'r)$, we obtain the asymptotic form

$$\left.\begin{aligned}\Psi(\mathbf{r}) \approx \left(\frac{1}{2\pi}\right)^{3/2} \sum_{l,m} \int dE' \frac{2\pi}{i\sqrt{\Delta}\, k'r} \{ & e^{ik'r}\, (E', \zeta' = 1 \mid \Psi \mid E, l, m) \\ & - e^{-ik'r}\, (E', \zeta' = -1 \mid \Psi \mid E, l, m)\}\,.\end{aligned}\right\} \quad (17.26b)$$

On the other hand, we know that

$$\int_{-\infty}^{\infty} dE'\, \delta_-(E - E')\, e^{-ik'r} f(k') = e^{-ikr} f(k)$$
$$\int_{-\infty}^{\infty} dE'\, \delta_+(E - E')\, e^{ik'r} f(k') = e^{ikr} f(k)$$
$$\int_{-\infty}^{\infty} dE'\, \delta_-(E - E')\, e^{ik'r} f(k') = 0$$
$$\int_{-\infty}^{\infty} dE'\, \delta_+(E_0 - E')\, e^{-ik'r} f(k') = 0$$

for any regular function $f(k)$. The first equation is derived by writing

$$\int_{-\infty}^{\infty} dE'\, \delta_-(E - E')\, e^{-ik'r} f(k')$$
$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dE' \frac{1}{E - E' - i\epsilon} e^{-ik'r} f(k')$$
$$= \frac{1}{2\pi i} \int_C dE' \frac{1}{E - E' - i\epsilon} e^{-ik'r} f(k')$$

[1]) For example, when $\varkappa_1 = \varkappa_2 = \varkappa$, Δ is given by (17.22) and (17.23) as

$$\Delta = \frac{2}{k'\sqrt{\varkappa^2 + k'^2}}.$$

where the contour C is made up of the real axis and the semi-circle below the real axis (cf. Fig. 17.1). The residual at the pole $E' = E - i\epsilon$ is $(-)\, e^{-ikr} f(k)$. Thus we have

$$\int_{-\infty}^{\infty} dE'\ \delta_-(E - E')\, e^{-ik'r} f(k') = e^{-ikr} f(k)\,.$$

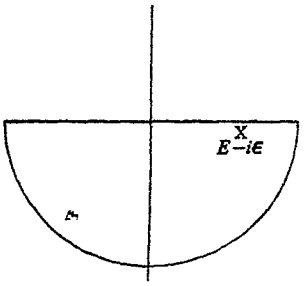

Fig. 17.1

The other equations can be proved in the same way.

Then, substituting (17.24b) and (17.25b) into (17.26b) we obtain

$$\begin{aligned}\Psi(\mathbf{r}) &= \left(\frac{1}{2\pi\Delta}\right)^{1/2} \frac{1}{ikr}\{-e^{-ikr}(\zeta = -1 \mid \eta) + e^{ikr}(E, \zeta = 1 \mid A \mid E, \eta)\} \\ &= \left(\frac{1}{2\pi\Delta}\right)^{1/2} \frac{1}{ikr}\{-e^{-ikr}(\zeta = -1 \mid \eta) + S^0 e^{ikr}(\zeta = 1 \mid \eta)\}.\end{aligned}$$

On the other hand, the incoming plane wave Ψ_{in} can be developed in spherical harmonics, so that

$$\left.\begin{aligned}\Psi_{in} = e^{i(\mathbf{k}\cdot\mathbf{r})} \approx \frac{\sqrt{\pi}}{ikr} \sum_l \sqrt{2l+1}\, i^l \{-e^{-i(kr - \frac{\pi}{2}l)} \\ + e^{i(kr - \frac{\pi}{2}l)}\}\, Y_l^0(\cos\theta) \qquad \text{for large } r.\end{aligned}\right\} \tag{17.27}$$

Since $\Psi(\mathbf{r})$ must reduce to the incoming wave when $S^0 = 1$, we see that

$$\left.\begin{aligned}\left(\frac{1}{2\pi\Delta}\right)^{1/2}(\zeta = 1 \mid l, m) &= \sqrt{\pi} \cdot \sqrt{2l+1}\, e^{i\frac{\pi}{2}l}\, Y_l^0(\cos\theta) \\ \left(\frac{1}{2\pi\Delta}\right)^{1/2}(\zeta = -1 \mid l, m) &= \sqrt{\pi} \cdot \sqrt{2l+1}\, e^{-i\frac{\pi}{2}l}\, Y_l^0(\cos\theta)\end{aligned}\right\} \tag{17.28}$$

for $m = 0$, and $(\zeta = 1 | l, m) = (\zeta = -1 | l, m) = 0$ for $m \neq 0$. These relations lead to (17.26a).

The amplitude of the scattered wave $(1/r)f(\theta)$ obtained as the difference of (17.27) and (17.26a) is

$$\frac{1}{r} f(\theta) = \frac{\sqrt{\pi}}{kr} \sum_l i^{l-1} \sqrt{2l+1}\, R^0\, Y_l^0(\cos\theta)\, e^{i(kr - \frac{\pi}{2}l)} \tag{17.29a}$$

where

$$R^0 \equiv 1 - S^0. \tag{17.29b}$$

It follows that the differential and total cross-sections are

$$d\sigma(\theta) = |f(\theta)|^2 d\Omega \tag{17.30a}$$

$$\sigma = \int d\Omega \, d\sigma(\theta) = \frac{\pi}{k^2} \sum (2l+1) \, |R^0|^2. \tag{17.30b}$$

The first of those can be obtained as follows:

The number of the incoming particles in a unit volume is unity. For

$$\int d^3x |\Psi_{\text{in}}(\mathbf{r})|^2 = \int d^3x = 1$$

where the domain of integration has unit volume (cf. (17.27)). Therefore, the number of incoming particles through unit surface in unit time is equal to v (v is the velocity of incoming particles). On the other hand, the number of outgoing particles through a small surface ds at a position $\mathbf{r}$ is $vds r^{-2}|f(\theta)|^2$. Therefore, the cross-section is

$$d\sigma(\theta) = vds r^{-2}|f(\theta)|^2/v = |f(\theta)|^2 d\Omega.$$

Since S is unitary, it can be written as $\exp(i\eta)$ in which the hermitean matrix η is called the phase matrix.

We shall consider a case in which the only possible processes are scattering (no absorption). Then S^0 must satisfy

$$|S^0|^2 = 1. \tag{17.31}$$

Thus S^0 can be written as

$$S^0 = e^{2i\delta} \tag{17.32}$$

where δ is a c-number.

Substituting (17.32) into (17.26a), we see that 2δ is the phase difference of the converging waves. Furthermore, it is easily seen that the incoming wave (17.27) and outgoing wave (17.29a) have the phase difference δ.

Using (17.32) we can write (17.30b) as

$$\sigma = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l. \tag{17.30c}$$

§ 4. Bound States

We shall now discuss the relations between the S-matrix and bound states, taking an S-state ($l=0$) as an example. $S(k)$, an eigen-

value S^0 of the state (k, l) with $l=0$, can be defined in the complex k-plane by ananlytic continuation.

For $k=-i|k|$ the terms in the bracket of (17.26a) can be written as

$$\Psi_{-i|k|}(\mathbf{r}) \propto \{-e^{-|k|r} + e^{|k|r} S(-i|k|)\}. \tag{17.33}$$

The first and second of these vanish and become infinite respectively at $r \to \infty$. Therefore, when $\Psi_{-i|k|}(\mathbf{r})$ is a bound state, $S(-i|k|)$ must be zero. Thus, we have

$$S(k)=0, \quad k=-i|k|, \tag{17.34}$$

for a bound state, the energy of which is

$$E=\frac{1}{2\varkappa}k^2=-\frac{|k|^2}{2\varkappa} \tag{17.35a}$$

in the non-relativistic approximation.

We shall now consider whether or not (17.34) can always give rise to bound states. The Schrödinger equation of a particle in a central potential field $\phi(r)$ is

$$\Delta\Psi(\mathbf{r})+2\varkappa\{E-\phi(r)\}\,\Psi(\mathbf{r})=0. \tag{17.36a}$$

If $l=0$ we introduce φ by $r\Psi(\mathbf{r})=\varphi(r)$ and can write (17.36a) as

$$\varphi''(r)+k^2\varphi(r)=\phi(r)\varphi(r) \tag{17.36b}$$

where

$$\left.\begin{aligned} k^2 &= 2\varkappa E \\ \varphi'' &= \frac{d^2}{dr^2}\varphi. \end{aligned}\right\} \tag{17.35b}$$

First we shall assume the potential (BETHE and BACHER [1936])

$$\phi(r)=-2\varkappa\phi_0 e^{-\frac{r}{a}} \tag{17.37}$$

where ϕ_0 and a are positive. When $k^2>0$, the solutions of (17.36b) can be written in terms of Bessel functions

$$J_{\pm i\varrho}(\alpha \exp(-r/2a)).$$

Here

$$\varrho=2ak, \quad \alpha=2a\sqrt{2\pi\phi_0}.$$

The solution, which is zero at $r=0$, is

$$\left.\begin{aligned} \varphi_k(r) = {} & \frac{i}{\sqrt{2\pi}}\,|\Gamma(i\varrho+1)/J_{i\varrho}(\alpha)| \\ & \{J_{-i\varrho}(\alpha)\,J_{i\varrho}(\alpha\exp(-r/2a))-J_{i\varrho}(\alpha)\,J_{-i\varrho}(\alpha\exp(-r/2a))\}. \end{aligned}\right\} \tag{17.38a}$$

The asymptotic form of (17.38a) at $r \to \infty$ is

$$\left.\begin{aligned}\varphi_k(r) \approx \frac{i}{\sqrt{2\pi}} \left|\Gamma(i\varrho+1)/J_{i\varrho}(\alpha)\right| \\ \left\{\frac{J_{-i\varrho}(\alpha)(\alpha/2)^{i\varrho}}{\Gamma(i\varrho+1)} e^{-ikr} - \frac{J_{i\varrho}(\alpha)(\alpha/2)^{-i\varrho}}{\Gamma(-i\varrho+1)} e^{ikr}\right\}.\end{aligned}\right\} \tag{17.38b}$$

Comparing (17.38b) with (17.26a), we obtain S^0 as

$$S(k) = \frac{J_{i\varrho}(\alpha)\,\Gamma(i\varrho+1)}{J_{-i\varrho}(\alpha)\,\Gamma(-i\varrho+1)} \left(\frac{\alpha}{2}\right)^{-2i\varrho} \tag{17.39a}$$

This shows that $S(k)$ is zero when (i) $J_{i\varrho}(\alpha)=0$ or (ii) $\Gamma(-i\varrho+1)$ is infinite. Since all the zero points of $J_{i\varrho}(\alpha)$ below the real axes appear on the imaginary axes, the condition of (i) can be written as

$$J_{|\varrho|}(\alpha)=0. \tag{17.40}$$

At these zero points $k=k_n$ ($\varrho=\varrho_n$) the wave function (17.38a) is equal to

$$\varphi_n(r) = C_n \frac{1}{\sqrt{2\pi}} \Gamma(|\varrho_n|+1)\,(2/\alpha)^{|\varrho_n|} J_{|\varrho_n|}(\alpha \exp(-r/2a)) \tag{17.38c}$$

where C_n is a normalisation constant (BETHE and BACHER [1936]).

Condition (ii) implies that

$$\Gamma(-i\varrho+1)=\infty \tag{17.41a}$$

which is satisfied for

$$i\varrho=|\varrho|=1,\ 2,\ 3,\ \ldots. \tag{17.41b}$$

However, the wave functions for these ϱ's are zero on account of the relation

$$J_m(\alpha)=(-1)^n J_{-n}(\alpha) \quad \text{for } n=1,\ 2,\ \ldots.$$

Thus, we see that the zero points given by (17.41a) do not correspond to bound states. They are called the "redundant zeros" which were first found by MA [1946, 1947a, b].

We shall now consider the redundant zeros in detail. Let us assume that the central potential satisfies the condition

$$\int_0^\infty dr|\phi(r)| = \text{finite}. \tag{17.42}$$

There are two independent solutions of the equation (17.36b) $f(k, r)$ and $f(-k, r)$, (JOST [1947]),

$$\left.\begin{aligned}\lim_{r\to\infty} e^{ikr} f(k, r) = 1 \\ \lim_{r\to\infty} e^{-ikr} f(-k, r) = 1.\end{aligned}\right\} \tag{17.43}$$

We introduce $f(k)$ and $f(-k)$ defined by

$$f(k) \equiv f(k, 0), \qquad f(-k) \equiv f(-k, 0). \tag{17.44}$$

If (17.42) is satisfied it can be proved that $f(k, r)$ and $f(-k, r)$ are complex conjugates for real non-zero k and continuous with respect to k and r; also that $f(k) \neq 0$ and $\lim_{k\to\infty} f(k) = 1$. Furthermore, when the potential satisfies the stronger condition (than (17.42))

$$\int_0^\infty dr\, r|\phi(r)| = \text{finite}, \tag{17.45}$$

it can be proved that $f(k, r)$ is continuous even for $k = 0$. We can write the solution of (17.36b), which is zero at $r = 0$, as

$$\varphi(r) = \frac{i}{2|f(k)|}\{f(-k)\, f(k, r) - f(k)\, f(-k, r)\} \tag{17.46}$$

with the omission of a constant factor.

Then, (17.26a) shows that

$$S(k) = e^{2i\delta(k)} = f(k)/f(-k). \tag{17.47}$$

Since, for $k = -i|k|$, $f(k, r)$ vanishes exponentially when $r \to \infty$, it describes a bound state. Therefore, the condition that (17.46) describes a bound state can be written as

$$f(k) = 0 \quad \text{for } k = -i|k|. \tag{17.48}$$

It must be noted that the bound states at $k = 0$ (i.e. $E = 0$) cannot be determined by $f(0) = 0$, because $f(k, r)$ and $f(-k, r)$ are equal and therefore not independent at $k = 0$. We must find another independent solution in order to construct the solution φ at $k = 0$. However, it can be proved (BARGMANN [1949]) that when the condition (17.45) is satisfied, there is no bound state of zero energy ($E = 0$).

Summarising, these results, we see that, when (17.45) is satisfied, the phase shifts $\delta(k)$ and the discrete energies of the bound states are determined by $f(k)$. However, there is no converse theorem: i.e., $\delta(k)$ cannot determine $f(k)$ but only $S(k)$. Moreover, $S(k) = 0$ does not always correspond to bound states, because $f(-k) = \infty$ also gives $S(k) = 0$.

We shall give a concrete example of these results by using our previous example (17.37). In this example, $f(k, r)$ and $f(k)$ are

$$f(k, r) = \left(\frac{\alpha}{2}\right)^{-i\varrho} \Gamma(i\varrho + 1)\, J_{i\varrho}(\alpha \exp(-r/2a)) \tag{17.49a}$$

$$f(k) = \left(\frac{\alpha}{2}\right)^{-i\varrho} \Gamma(i\varrho + 1)\, J_{i\varrho}(\alpha). \tag{17.49b}$$

The condition (i) (i.e. (17.40)) is equivalent to (17.48) which leads to bound states. However, the condition (ii) (i.e. (17.41)) gives $f(-k)=\infty$ which implies redundant zeros.

These results suggest that there may be many potentials which give the same phase shifts and different bound state energies. In fact, an example of this fact was given by BARGMANN [1949] who considered on the two potentials

$$\phi_1(r)=\frac{\varrho\sigma\{4\varrho\sigma+(\varrho-\sigma)^2\cosh((\varrho+\sigma)r-2\theta)-(\varrho+\sigma)^2\cosh((\varrho-\sigma)r)\}}{\{\sigma\sinh(\varrho r-\theta)-\varrho\sinh(\sigma r-\theta)\}^2} \tag{17.50a}$$

$$\phi_2(r)=\frac{\varrho\sigma\{4\varrho\sigma+(\varrho-\sigma)^2\cosh((\varrho+\sigma)r)-(\varrho+\sigma)^2\cosh((\varrho-\sigma)r+2\theta)\}}{\{\sigma\sinh(\varrho r+\theta)-\varrho\sinh(\sigma r+\theta)\}^2} \tag{17.50b}$$

where $\varrho>\sigma>0$ and $\theta>0$.

Let us denote the $f(k)$'s given by ϕ_1 and ϕ_2 by $f_1(k)$ and $f_2(k)$ respectively. Then $f_1(k)$ is

$$f_1(k)=\frac{2k+i(\varrho+\sigma)}{2k-i(\varrho-\sigma)}, \tag{17.51}$$

which gives $k=-(i/2)(\varrho+\sigma)$ for the bound state and therefore its energy is

$$E_1=-\frac{1}{8\varkappa}(\varrho+\sigma)^2. \tag{17.52}$$

On the other hand, $f_2(k)$ is given by

$$f_2(k)=\frac{2k+i(\varrho-\sigma)}{2k-i(\varrho+\sigma)} \tag{17.53}$$

which gives $k=-(i/2)(\varrho-\sigma)$ for the bound state. The energy of this is therefore

$$E_2=-\frac{1}{8\varkappa}(\varrho-\sigma)^2. \tag{17.54}$$

However (17.47), (17.51) and (17.53) show that ϕ_1 and ϕ_2 give the same $S(k)$; i.e.,

$$S_1(k)=S_2(k)=\frac{(2k+i(\varrho-\sigma))(2k+i(\varrho+\sigma))}{(2k-i(\varrho-\sigma))(2k-i(\varrho+\sigma))}. \tag{17.55}$$

Furthermore, since $\phi_1(r)$ depends on θ, and $f_1(k)$ and $S_1(k)$ do not depend on θ, many potentials continuous with respect to θ give the same phase shift (i.e., S). This result is true also for ϕ_2.

It follows from these conclusions that the potential cannot be uniquely determined merely by means of phase shifts (or $S(k)$). In

fact, it can be proved that the difference of two potentials which give the same phase shifts and same eigenvalues of k can be written as.

$$\phi_1(r) - \phi_2(r) = \sum_m a_m Z_m(r), \tag{17.56}$$

where a_m are constants and the Z_m are defined by $Z_m = \varphi_m^{(1)}\ \varphi_m^{(2)}$. The functions $\varphi_m^{(1)}$ (or $\varphi_m^{(2)}$) are solutions of (17.36b) with $k_m = -i|k|$ for the case of the potential ϕ_1 (or ϕ_2) (HOLMBERG [1952]).

However, it can be proved (BARGMANN [1949]) that when two potentials ϕ_1 and ϕ_2 which satisfy the condition (17.45) and

$$\phi(r) + \frac{l(l+1)}{r^2} \geqslant 0 \qquad \text{(for a certain } l \text{ and any } r)$$

give the same phase shifts for "every" angular momentum l, these two potentials must be equal [1]) (i.e. $\phi_1(r) = \phi_2(r)$).

We shall now discuss the conditions under which there are no redundant zeros. Since redundant zeros come from $f(-k) = \infty$ we have no redundant zeros when $f(k, r)$ is regular in the whole domain of the complex k-plane. In such a case the same $S(k)$ gives the same bound state energies.

It can be proved that a sufficient condition for the potential to give regular $f(k, r)$ is (see BARGMANN [1949])

$$I(\alpha) \equiv \int_0^\infty e^{\alpha r} |\phi(r)| dr = \text{finite} \qquad \text{(for all positive } \alpha). \tag{17.57}$$

However, this condition is too strong for present quantum theory; for example, the Yukawa potential (12.25) cannot satisfy this condition.

We shall demonstrate the above fact by using the previous example (17.37). First, we shall modify the potential (17.37) at $r = R$ so that it satisfies (17.57) by putting (MA [1946, 1947])

$$\phi(r) \begin{cases} = -2\varkappa\phi_0 e^{-\frac{r}{a}} & \text{for } 0 < r < R \\ = 0 & \text{for } r > R. \end{cases} \tag{17.58}$$

The solution $\varphi(r)$ of (17.36b) is proportional to $\sin(kr + \delta)$ for $r > R$ and to (17.38a) for $r < R$ respectively. Their coefficients must be

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[1]) An excellent method to construct potentials by means of phase shifts was given by JOST and KOHN [1952].

determined such that φ and its first derivative are continuous at $r=R$. Then, we obtain $S(k)$ as

$$k) = -e^{-2ikR}\frac{J_{-iq}(\alpha)J_{iq+1}(\alpha\exp(-R/2a))+J_{iq}(\alpha)J_{-iq-1}(\alpha\exp(-R/2a))}{J_{-iq}(\alpha)J_{iq-1}(\alpha\exp(-R/2a))+J_{iq}(\alpha)J_{-iq+1}(\alpha\exp(-R/2a))}. \quad (17.39b$$

The condition $S(k)=0$ gives

$$J_{-iq}(\alpha)J_{iq+1}(\alpha\exp(-R/2a))+J_{iq}(\alpha)J_{-iq-1}(\alpha\exp(-R/2a))=0, \quad (17.59)$$

which is equivalent to the condition (17.40) at $R\to\infty$. Thus, we see that the redundant zeros are dispensed with calculating the discrete energies from a potential which vanishes for $r>R$ and then allowing R to approach ∞.

A more detailed discussion on the relation between $S(k)$ and bound states was given by VAN KAMPEN [1951]. Let us define $f(k)$ and $S(k)$ in the complex E-plane by analytic continuation. We shall call Γ the whole domain of the complex E-plane without the real axes and denote $f(k,r)$ in Γ by $f(E,r)$ (its asymptotic form is $\exp(iE^{\frac{1}{2}}r)$; $E^{\frac{1}{2}}$ is defined so that $I_m(E^{\frac{1}{2}})>0$ (cf. (17.35a) in order that $\lim_{r\to\infty} f(E,r)\to 0$). Then it can be proved that $f(E,r)$ is a single valued function in Γ, and that $f(E)\equiv f(E,0)$ is analytic in Γ; also that its zero points are on the real "negative" axes of E (see VAN KAMPEN [1951]). $S(E)$ can be written as

$$S(E)=\frac{f(E_+)}{f(E_-)} \qquad \text{for real } E. \quad (17.60)$$

where $f(E_\pm)$ means the value of $f(E)$ obtained as its limit in the upper or lower sides of the cut "real" positive axes. For example in the lower domain of the real axes, $I_m(E_-)<0$ and therefore $(E)^{\frac{1}{2}}$ is not $k+i\epsilon$ but $-k+i\epsilon$ $(k>0)$. Thus, we see that $f(E_-)=f(-k)$. In a similar way we have $f(E_+)=f(k)$. From (17.47) we obtain (17.60).

From (17.60) we obtain

$$\log f(E)=\frac{1}{2\pi i}\int_0^\infty dE'\,\frac{\log S(E')}{E'-E}+\sum_m \log\left(1-\frac{E_m}{E}\right) \quad (17.61)$$

where E_m are zero points of $f(E)$ and the logarithms are defined by the appropriate branches.

The proof of (17.61) is as follows: Since the function

$$g(E)\equiv \log\left\{f(E)\prod_m \frac{E}{E-E_m}\right\}=\log f(E)-\sum_m \log\left(1-\frac{E_m}{E}\right)$$

is analytic in the domain Γ, Cauchy's theorem gives

$$g(E) = \frac{1}{2\pi i} \int_L dE' \, \frac{g(E')}{E' - E},$$

where L is the contour shown in Fig. 17.2.

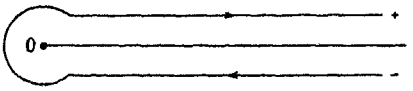

Fig. 17.2

It follows that

$$\begin{aligned} g(E) &= \frac{1}{2\pi i} \int_0^\infty dE' \, \frac{g(E'_+) - g(E'_-)}{E' - E} \\ &= \frac{1}{2\pi i} \int_0^\infty dE' \, \frac{1}{E' - E} \log \frac{f(E_+)}{f(E_-)} \\ &= \frac{1}{2\pi i} \int_0^\infty dE' \, \frac{\log S(E')}{E' - E} \end{aligned}$$

which leads to (17.61).

This equation shows that $f(E)$ cannot be determined only by means of $S(E)$ on the real positive E but requires the values of E_m; and that even when $S(E)$ is given, E_m can be made arbitrary by modification of $f(E)$ which do not violate (17.61) (see also WILDERMUTH [1950] and HU [1948]).

Summarising the above results, we can say that the S-matrix can determine the energies of bound states only when it has no redundant zero and the discrimination of the redundant zeros requires the more detailed information of the wave function than its asymptotic form.

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CHAPTER XVIII

THEORY OF PROPAGATORS

§ 1. Introduction

In Ch. XII we considered a theory of the Heisenberg representation in which the basic quantities were the free field $Q_\alpha^{(\mathrm{in})}(x)$ describing incoming particles. However, such a theory does not exactly correspond to the real situation where incoming and outgoing particles carry their proper fields. It might be expected that, in a theory in which no bare particles appear, the renormalisation of mass and coupling constants might be automatically performed. In the following paragraphs we will develop a first approach to such a theory. It has been shown in Ch. XIV that, in the renormalisation theory of quantum electrodynamics, the normalisation of the external operators of the electron and photon fields must be changed by the factors Z_2 and Z_3 respectively. This is in contrast to the normalisation of internal operators, for which the constants Z_2 and Z_3 can be amalgamated into the coupling constant (cf. Fig. 4 in Ch. XIV). However, there cannot be any essential difference between external and internal operators; in fact, incoming and outgoing particles may be observed by means of some electromagnetic interactions, and so the change in normalisation of external operators may be cancelled out by the renormalisation of coupling constant of the external electromagnetic interactions. In other words, by writing the external electromagnetic interaction, $ie\bar{\psi}\gamma_\mu A_\mu\psi$ for the incoming electron as $\lambda\bar{\psi}\eta$ $(\lambda\eta = ie\gamma_\mu A_\mu\psi)$ we can regard the change of the normalisation of ψ as a change of the strength of the external coupling constant λ. Following this consideration, we shall discuss the renormalisation in a form independent of perturbation expansion. It will be shown in § 6 that the renormalised theory is connected with the unrenormalised theory through a transformation, i.e. renormalisation transformation.

Discussions in § 8 are concerned with more general cases than the quantum electrodynamics; — with some general features of the renormalisation of the one-body-propagators. The renormalisation method given in Ch. XIV and in § 7 of this Chapter are developed for cases

where particles of each field have one stable mass state. However, it sometimes happens that the proper field leads to more than one mass state of the particle. The renormalisation in latter cases is discussed in § 8.

§ 2. Formulation of the Theory

Let us consider [1]) an electron-positron field $\psi(x)$, $\bar{\psi}(x)$ interacting with an electromagnetic field $A_\mu(x)$.

The external sources of these fields we will denote by η, $\bar{\eta}$ and J_μ respectively. [2]) We shall introduce a representation, the source free representation, in which the field equations are

$$\mathsf{F}(x)\,\psi(x) \equiv (\gamma_\mu \partial_\mu + \varkappa - i e \gamma_\mu A_\mu(x))\,\psi(x) = 0 \tag{18.1a}$$

$$\bar{\mathsf{F}}(x)\,\bar{\psi}(x) \equiv (\gamma_\mu^T \partial_\mu - \varkappa + i e \gamma_\mu^T A_\mu(x))\,\bar{\psi}(x) = 0 \tag{18.1b}$$

$$\Box A_\mu(x) = -j_\mu(x) \equiv -i\frac{e}{2}\bar{\psi}(x)\,\gamma_\mu\psi(x) + h.c. \tag{18.1c}$$

If we denote $\boldsymbol{\psi}$, $\boldsymbol{\psi}$. $\mathbf{A}_\mu$ the same fields in the Heisenberg representation, their field equations are given by

$$\mathsf{F}(x)\,\boldsymbol{\psi}(x) = \lambda\eta(x) \tag{18.2a}$$

$$\bar{\mathsf{F}}(x)\,\bar{\boldsymbol{\psi}}(x) = -\lambda\bar{\eta}(x) \tag{18.2b}$$

$$\Box\,\mathbf{A}_\mu(x) = -\mathbf{j}_\mu(x) - \lambda' J_\mu(x) \tag{18.2c}$$

where λ and λ' are the coupling constants of the external sources. This corresponds to an interaction Lagrangian due to the sources (SCHWINGER [1951]):

$$\mathbf{L}_s(x) = \lambda\,\bar{\boldsymbol{\psi}}(x)\,\eta(x) + \lambda\,\bar{\eta}(x)\,\boldsymbol{\psi}(x) + \lambda' J_\mu(x)\,\mathbf{A}_\mu(x). \tag{18.3}$$

As stated in Example 4 of Ch. X, each term in the interaction Hamiltonian density must include an even number of operators, which anticommute with any given field operator. By taking $\boldsymbol{\psi}$, $\mathbf{A}_\mu$, η and J_μ successively as test operators, we deduce from (18.3)

$$[\eta(x), \bar{\boldsymbol{\psi}}(x')]_+ = [\eta(x), \boldsymbol{\psi}(x')]_+ = [\eta(x), \eta(x')]_+ = 0 \tag{18.4a}$$

$$[\eta(x), \mathbf{A}_\mu(x')]_- = [\eta(x), J_\mu(x')]_- = 0 \tag{18.4b}$$

$$[J_\mu(x), \bar{\boldsymbol{\psi}}(x')]_- = [J_\mu(x), \boldsymbol{\psi}(x')]_- = 0 \tag{18.4c}$$

$$[J_\mu(x), \mathbf{A}_\mu(x')]_- = [J_\mu(x), J_\nu(x')]_- = 0. \tag{18.4d}$$

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[1]) Here we restrict ourselves to quantum electrodynamics. On propagators in meson theory, see EDWARDS [1953].

[2]) The theory of propagators has also been formulated in the interaction representation by NISHIJIMA [1954].

These commutation relations are proved to be valid only when the four vector $x-x'$ is space-like. We can, however, assume them to obtain for any two points x, x' without contradicting the field equations, since, then they commute with the Hamiltonian at arbitrary points.

The unitary transformation U connecting the state vectors Ψ, $\mathbf{\Psi}$ in source free and Heisenberg representations gives

$$\Psi[\sigma] = U[\sigma, \sigma_1, \eta, J]\, \mathbf{\Psi}[\sigma_1] \tag{18.5a}$$

with

$$U[\sigma_1, \sigma_1, \eta, J] = 1 \tag{18.5b}$$

and the field operators in the two representations are connected by the relations

$$\psi(x) = U[\sigma(x), \sigma_1, \eta, J]\, \boldsymbol{\psi}(x)\, U^{-1}[\sigma(x), \sigma_1, \eta, J]$$
$$A_\mu(x) = U[\sigma(x), \sigma_1, \eta, J]\, \mathbf{A}_\mu(x)\, U^{-1}[\sigma(x), \sigma_1, \eta, J].$$

It can be proved that the evolution operator U satisfies

$$i \frac{\delta}{\delta\sigma(x)} U[\sigma, \sigma_1, \eta, J] = H_s(x)\, U[\sigma, \sigma_1, \eta, J]. \tag{18.6a}$$

Here the interaction Hamiltonian (in the source free representation) is

$$H_s(x) = -\lambda\bar{\psi}(x)\, \eta(x) - \lambda\bar{\eta}(x)\, \psi(x) - \lambda' J_\mu(x)\, A_\mu(x). \tag{18.6b}$$

In fact, from (6.5a), (18.2a) and (18.6a) we obtain

$$\begin{aligned}
\gamma_\mu \partial_\mu \psi(x) &= \gamma_\mu \frac{\delta}{\delta\sigma(x)} \int_\sigma d\sigma'_\mu\, U[\sigma, \sigma_1, \eta, J]\, \boldsymbol{\psi}(x')\, U^{-1}[\sigma, \sigma_1, \eta, J] \\
&= U[\sigma, \sigma_1, \eta, J]\{\gamma_\mu \partial_\mu \boldsymbol{\psi}(x) - \lambda\gamma_\mu \int_\sigma d\sigma'_\mu [\mathbf{\Psi}(x), \boldsymbol{\psi}(x')]_+ \eta(x)\} \\
&\quad U^{-1}[\sigma, \sigma_1, \eta, J] \\
&= U[\sigma, \sigma_1, \eta, J]\{\gamma_\mu \partial_\mu \boldsymbol{\psi}(x) - \lambda\eta(x)\}\, U^{-1}[\sigma, \sigma_1, \eta, J] \\
&= ie\gamma_\mu A_\mu(x)\, \psi(x) - \varkappa\psi(x),
\end{aligned}$$

which leads to (18.1a). In a similar way, it can be proved that the transformation U converts (18.2c) into (18.1c).

If the energy operator in the source-free representation is denoted by T_4, its eigenstates Ψ_n are defined by

$$T_4 \Psi_n = E_n \Psi_n. \tag{18.7}$$

It is an advantage of the present representation that the states Ψ_n

are independent of the sources. The vacuum Ψ_0 is defined [1]) as the state of lowest energy E_0.

The transition matrix between states $\Psi[\sigma_2]$, $\Psi[\sigma_1]$ on two spacelike surfaces σ_2 and σ_1 is given by the operator $U[\sigma_2, \sigma_1, \eta, J]$. This operator may be expanded in the functional Taylor series. In particular, its vacuum expectation value can be written as

$$\left.\begin{aligned}&(U[\sigma_2, \sigma_1, \eta, J])_0 = \\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{i^{m+n+l}}{m!\,n!\,l!}\int \bar\eta(\xi_1)\dots\bar\eta(\xi_n)\, G^{(n,m)}_{\mu_1\dots\mu_l}(\xi_1\dots\xi_n, \xi'_1\dots\xi'_m, \zeta_1\dots\zeta_l)\\ &\eta(\xi'_1)\dots\eta(\xi'_m)\, J_{\mu_1}(\zeta_1)\dots J_{\mu_l}(\zeta_l).\end{aligned}\right\} \quad (18.8)$$

Here and in the following we integrate over all Greek variables (ξ, ζ). The symbol $(\,)_0$ means the vacuum expectation value. Since each incoming (outgoing) particle must go through the external fields on the surface σ_1 (σ_2) for determining its state, the coefficients $G^{(n,m)}_{\mu\dots}$ in (18.8) may lead to the transition matrix elements: — the propagators

$$G^{(n,m)}_{\mu_1\dots\mu_l\mu'_1\dots\mu'_{l'}}(x_1\dots x_n, x'_1\dots x'_m, z_1\dots z_l\, z'_1\dots z'_{l'})$$

(where $x_1\dots x_n$, $z_1\dots z_l$ are on σ_2, $x'_1\dots x'_m$, $z'_1\dots z'_{l'}$ on σ_1) are proportional to the transition matrix elements between n electrons on σ_2 and m electrons on σ_1, accompanied by the emission of l photons on σ_2 and the absorption of l' photons. The quantities $G^{(n,m)}_{\mu\dots}$ are called propagators and we proceed to give some of their properties.

The $G^{(n,m)}_{\mu\dots}$ can be obtained from the quantities [2])

$$\left.\begin{aligned}&\mathfrak{J}^{(n,m)}_{\mu_1\dots\mu_l}[\eta, J; x_1\dots x_n, x'_1\dots x'_m, z_1\dots z_l]\\ &\equiv (i)^{-n-l-m}\left(\frac{\delta^{n+m+l}}{\delta\bar\eta(x_1)\dots\delta\bar\eta(x_n)\,\delta\eta(x'_1)\dots\delta\eta(x'_m)\,\delta J_{\mu_1}(z_1)\dots\delta J_{\mu_l}(z_l)}\,U[\sigma_2, \sigma_1, \eta, J]\right)_0\end{aligned}\right\} \quad (18.9)$$

by taking limits $\eta \to 0$, $\bar\eta \to 0$, $J_\mu \to 0$.

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[1]) It must be noted that the vacuum state Ψ_0 is defined as the lowest energy state of the interacting fields, not that of the free fields. GELL-MANN and LOW [1951] have derived the relation between the vacuum states (Ψ_0, Φ_0) of interacting and free fields.

[2]) Functional derivatives are defined by

$$\begin{aligned}F[\eta+\delta\eta, \bar\eta+\delta\bar\eta, J+\delta J] &= F[\eta, \bar\eta, J] + \int d^4\xi\,\frac{\delta}{\delta\eta(\xi)}F[\eta, \bar\eta, J]\,\delta\eta(\xi)\\ &+ \int d^4\xi\,\delta\bar\eta(\xi)\,\frac{\delta}{\delta\bar\eta(\xi)}F[\eta, \bar\eta, J] + \int d^4\zeta\,\frac{\delta}{\delta J_\mu(\zeta)}F[\eta, \bar\eta, J]\,\delta J_\mu(\zeta)\end{aligned}$$

with infinitesimal increments.

The equation (18.6a) can be replaced by the integral equation

$$U[\sigma_2, \sigma_1, \eta, J] = 1 - i \int_{\sigma_1}^{\sigma_2} d^4x'\, H_s(x')\, U[\sigma', \sigma_1, \eta, J], \qquad (18.10)$$

where σ' passes through the point x'. We change H_s by an infinitesimal amount δH and ask for the corresponding change δU.

Quite generally, when two operators U and U_1 satisfy the operator equations

$$U = 1 + (K_1 + K_2)U$$
$$U_1 = 1 + K_1 U_1,$$

we can prove

$$U - U_1 = U_1 K_2 U.$$

From this follows, neglecting higher order terms in δH,

$$\delta U[\sigma_2, \sigma_1, \eta, J] = -i \int_{\sigma_1}^{\sigma_2} d^4x'\, U[\sigma_2, \sigma', \eta, J]\, \delta H(x')\, U[\sigma', \sigma_1, \eta, J]. \qquad (18.11)$$

We now define δH as a change induced by a variation of the sources:

$$\delta H(x) = -\lambda(\bar{\psi}(x)\, \delta\eta(x) + \delta\bar{\eta}(x)\, \psi(x)) - \lambda' \delta J_\mu(x)\, A_\mu(x),$$

where $\delta\eta$, $\delta\bar{\eta}$, δJ_μ have the same commutation properties as η, $\bar{\eta}$, J_μ. Then, (18.11) leads to

$$\frac{1}{i} \frac{\delta}{\delta\eta(x')} (U[\sigma_2, \sigma_1, \eta, J])_0 = \lambda(\sigma_2, \bar{\psi}(x'), \sigma_1)_0 \qquad (18.12a)$$

$$\frac{1}{i} \frac{\delta}{\delta\bar{\eta}(x)} (U[\sigma_2, \sigma_1, \eta, J])_0 = \lambda(\sigma_2, \psi(x), \sigma_1)_0 \qquad (18.12b)$$

$$\frac{1}{i} \frac{\delta}{\delta J_\mu(z)} (U[\sigma_2, \sigma_1, \eta, J])_0 = \lambda'(\sigma_2, A_\mu(z), \sigma_1)_0 \qquad (18.12c)$$

where the symbol can be understood from the example: —

$$\left.\begin{aligned} &(\sigma_2, A(x)\, B(x'), \sigma_1)_0 \\ &= (U[\sigma_2, \sigma(x), \eta, J] A(x) U[\sigma(x), \sigma(x'), \eta, J] B(x') U[\sigma(x'), \sigma_1, \eta, J])_0 \end{aligned}\right\} \qquad (18.13)$$

with any operator $A(x)$, $B(x)$. From (18.4a), (18.12a) and (18.13) we obtain

$$\left.\begin{aligned} &\frac{1}{i^2} \frac{\delta^2}{\delta\bar{\eta}(x)\delta\eta(x')} (U[\sigma_2, \sigma_1, \eta, J])_0 \\ &= \begin{cases} \lambda^2(\sigma_2, \psi(x)\, \bar{\psi}(x'), \sigma_1) & \text{for } \sigma(x) > \sigma(x') \\ -\lambda^2(\sigma_2, \bar{\psi}(x')\, \psi(x), \sigma_1) & \text{for } \sigma(x) < \sigma(x') \end{cases} \\ &= \varepsilon(x, x')\, \lambda^2(\sigma_2, P[\psi(x)\, \bar{\psi}(x')], \sigma_1). \end{aligned}\right\} \qquad (18.14)$$

In this way we can derive the relations

$$\left.\begin{aligned} &G^{(n,m)}_{\mu_1 \dots \mu_l}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m, z_1 \dots z_l] \\ &\quad = \lambda^{n+m} \lambda'^l \varepsilon(x_1 \dots x_n, x'_1 \dots x'_m) \\ &\qquad (\sigma_2, P[\psi(x_1) \dots \psi(x_n) \bar{\psi}(x'_1) \dots \bar{\psi}(x'_m) A_{\mu_1}(z_1) \dots A_{\mu_l}(z_l)], \sigma_1)_0, \end{aligned}\right\} \quad (18.15)$$

where P is the chronological operator (time increasing from right to left) and $\varepsilon(x_1 \dots x_n, x'_1 \dots x'_m)$ is the product of all $\varepsilon(x_i, x_j)$ $(i>j)$, $\varepsilon(x'_i, x'_j)$ $(i>j)$ and $\varepsilon(x_i, x'_j)$. As examples of (18.15), we obtain

$$G^{(0,1)}[\eta, J; x'] = \lambda(\sigma_2, \bar{\psi}(x'), \sigma_1)_0 \quad (18.16a)$$

$$G^{(1,0)}[\eta, J; x] = \lambda(\sigma_2, \psi(x), \sigma_1)_0 \quad (18.16b)$$

$$G_\mu[\eta, J; z] = \lambda'(\sigma_2, A_\mu(z), \sigma_1)_0 \quad (18.16c)$$

$$\left.\begin{aligned} &G^{(n,m)}[\eta, J; x_1 \dots x_n, x'_1 \dots x'_m] \\ &= \lambda^{n+m} \varepsilon(x_1 \dots x_n, x'_1 \dots x'_m) (\sigma_2, P[\psi(x_1) \dots \psi(x_n) \bar{\psi}(x'_1) \dots \bar{\psi}(x'_m)], \sigma_1)_0 \end{aligned}\right\} \quad (18.17a)$$

$$G_{\mu_1 \dots \mu_l}[\eta, J; z_1 \dots z_l] = \lambda'^l (\sigma_2, P[A_{\mu_1}(z_1) \dots A_{\mu_l}(z_l)], \sigma_1)_0. \quad (18.17b)$$

We shall denote the limit of a functional $F[\eta, J; x \dots]$ for $\bar{\eta} = \eta = 0$ or $J_\mu = 0$ by

$$F[J; x \dots] \equiv F[\eta = 0, J; x \dots] \quad (18.18a)$$

$$F[\eta; x \dots] \equiv F[\eta, J = 0; x \dots] \quad (18.18b)$$

$$F(x \dots) \equiv F[\eta = 0, J = 0; x \dots]. \quad (18.18c)$$

Since the Lagrangian with $\eta = \bar{\eta} = 0$ is invariant under the transformation $\psi \to e^{i\alpha}\psi$, the functionals $G^{(n,m)}$ and $G^{(n,m)}_{\mu \dots}$ must also be invariant under this transformation, therefore $G^{(m,n)}[J; \dots]$ and $G^{(m,n)}_{\mu \dots}[J; \dots]$ with different numbers of ψ's and $\bar{\psi}$'s, i.e. for $m \neq n$, are zero.

Furthermore, for $\eta = \bar{\eta} = J_\mu = 0$, the Lagrangian is invariant under charge conjugation of the ψ-field together with $A_\mu \to -A_\mu$. Therefore the function $G_{\mu \dots}(\dots)$ made up of an odd number of A_μ's must be zero [1]).

We note further that the operator $U[\sigma_2, \sigma_1, \eta, J]$ is the generating operator of the propagators as shown by (18.9) (UMEZAWA and VISCONTI [1955a]).

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[1]) It must be noted that this is just Furry's theorem (cf. Example 2, Ch. XIII).

§ 3. Equations for the Propagators. Normalised Propagators

From (18.12a, b, c) [1]) we can derive the relations (UMEZAWA and VISCONTI [1955a], SYMANZIK [1954])

$$\left.\begin{aligned}\left(\gamma_\mu \partial_\mu + \varkappa - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)}\right) \frac{\delta}{\delta \bar{\eta}(x)} (U[\sigma_2, \sigma_1, \eta, J])_0 \\ = i\lambda^2 \eta(x)\, (U[\sigma_2, \sigma_1, \eta, J])_0\end{aligned}\right\} \tag{18.19a}$$

$$\left.\begin{aligned}\Box \frac{\delta}{\delta J_\mu(z)} (U[\sigma_2, \sigma_1, \eta, J])_0 = i\lambda'^2 J_\mu(z)\, (U[\sigma_2, \sigma_1, \eta, J])_0 \\ + i\,(\lambda'/\lambda)^2 e_1 \,\mathrm{Sp}\, (G^{(1,1)}[J; z, z]\, \gamma_\mu)_0 .\end{aligned}\right\} \tag{18.19b}$$

Here

$$e_1 = e/\lambda'. \tag{18.20}$$

The derivation of (18.19a) is as follows; from (18.12a) and (6.5a) we obtain

$$\begin{aligned}&(\gamma_\mu \partial_\mu) \frac{\delta}{\delta \bar{\eta}(x)} (U[\sigma_2, \sigma_1, \eta, J])_0 \\ &\quad = i\lambda \left(\gamma_\mu \frac{\delta}{\delta \sigma(x)}\right) \int_{\sigma(x)} d\sigma'_\mu (U[\sigma_2, \sigma(x), \eta, J]\, \psi(x')\, U[\sigma(x), \sigma_1, \eta, J])_0 \\ &\quad = i\lambda\, (U[\sigma_2, \sigma(x), \eta, J]\, \gamma_\mu \partial_\mu \psi(x)\, U[\sigma(x), \sigma_1, \eta, J])_0 \\ &\quad + i\lambda^2 (U[\sigma_2, \sigma(x), \eta, J] \int_{\sigma(x)} [\psi(x'), \bar{\psi}(x)]_+ d\sigma'_\mu\, U[\sigma(x), \sigma_1, \eta, J])_0\, \eta(x)\end{aligned}$$

which is equal to

$$\begin{aligned}&= -i\lambda\varkappa (U[\sigma_2, \sigma(x), \eta, J]\, \psi(x)\, U[\sigma(x), \sigma_1, \eta, J])_0 \\ &\quad - \lambda e \gamma_\mu (U[\sigma_2, \sigma(x), \eta, J]\, A_\mu(x)\, \psi(x)\, U[\sigma(x), \sigma_1, \eta, J])_0 \\ &\quad + i\lambda^2 (U[\sigma_2, \sigma_1, \eta, J])_0\, \eta(x)\end{aligned}$$

on account of (18.1a). This leads to (18.19a); (18.19b) can be proved in a similar way.

Equations (18.19a) and (18.19b) are the generating equations for the many-body propagators; by operating successively with $\delta/\delta\eta$, $\delta/\delta\bar{\eta}$ and $\delta/\delta J_\mu$, we can derive equations for all possible propagators. This is done in the following for the one-electron, one-photon, and two-electron propagators.

Taking the derivative of (18.19a) with respect to $\eta(x')$, we obtain the equation for $G^{(1,1)}[\eta, J; x, x']$:

$$\left.\begin{aligned}i\left(\gamma_\mu \partial_\mu + \varkappa - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)}\right) G^{(1,1)}[\eta, J; x, x'] \\ = \lambda^2 \delta(x - x')\, (U[\sigma_2, \sigma_1, \eta, J])_0 + i\lambda^2 \eta(x) \frac{\delta}{\delta \eta(x')} (U[\sigma_2, \sigma_1, \eta, J])_0 .\end{aligned}\right\} \tag{18.21a}$$

[1]) An attempt for solving these equations has been given by SYMANZIK [1954].

Let us define the normalised propagators by dividing them by the vacuum expectation value of U, e.g.

$$G_N^{(1,1)}[\eta, J; x, x'] \equiv \frac{G^{(1,1)}[\eta, J; x, x']}{(U[\sigma_2, \sigma_1, \eta, J])_0}. \tag{18.22}$$

It must be noted that the propagators are not changed by the above normalisation when η and J are zero, because of

$$(U[\sigma_2, \sigma_1, \eta=0, J=0])_0 = 1.$$

Putting $\eta=0$ in (18.21a) we have [1] [2]

$$\left.\begin{aligned} & i\left(\gamma_\mu \partial_\mu + \varkappa - i e_1 \gamma_\mu G_{\mu,N}[J; x] - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)}\right) G_N^{(1,1)}[J; x, x'] \\ & \quad = \lambda^2 \delta(x-x'). \end{aligned}\right\} \tag{18.21b}$$

Operating with $\delta^3/\delta\bar{\eta}(x_2)\,\delta\eta(x_1')\,\delta\eta(x_2')$ on (18.19a) gives the equation for the two-electron propagator (with $\eta=0$)

$$\left.\begin{aligned} & i\left(\gamma_\mu \partial_\mu + \varkappa - i e_1 \gamma_\mu G_{\mu,N}[J; x] - e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)}\right) G_N^{(2,2)}[J; x\, x_2, x_1'\, x_2'] \\ & \quad = \lambda^2 \delta(x-x_1')\, G_N^{(1,1)}[J; x_2, x_2'] - \lambda^2 \delta(x-x_2')\, G_N^{(1,1)}[J; x_2, x_1']. \end{aligned}\right\} \tag{18.23}$$

In the same manner, we can deduce equations for the one-photon propagator from (18.19b).

§ 4. Vertex Part, Mass Operator and Free Dressed Particle

Since we always consider normalised propagators in the rest of this Chapter, we shall drop the index N. For the derivative of a composite functional we have (SCHWINGER [1951])

$$\left.\begin{aligned} -i \frac{\delta}{\delta J_\mu(z)} G^{(1,1)}[J; x, x'] &= -i \int \frac{\delta}{\delta G_\nu[J; \zeta]} G^{(1,1)}[J; x, x'] \\ \cdot \frac{\delta}{\delta J_\mu(z)} G_\nu[J; \zeta] &= \int \bar{G}_{\nu\mu}[J; \zeta\, z] \frac{\delta}{\delta G_\nu[J; \zeta]} G^{(1,1)}[J; x, x'] \end{aligned}\right\} \tag{18.24}$$

where

$$\bar{G}_{\nu\mu}[J; zz'] \equiv G_{\nu\mu}[J; zz'] - G_\nu[J; z]\, G_\mu[J; z']. \tag{18.25}$$

It must be noted that $\bar{G}_{\nu\mu}$ is equal to the one-photon-propagator $G_{\nu\mu}$ if $J=0$.

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[1] $$\frac{\delta}{\delta J_\mu(z)} F_N[\eta, J] \equiv \frac{\delta}{\delta J_\mu(z)} \frac{F[\eta, J]}{(U[\sigma_2, \sigma_1, \eta, J])_0} = \frac{1}{(U[\sigma_2, \sigma_1, \eta, J])_0} \frac{\delta}{\delta J_\mu(z)} F[\eta, J] - i G_{\mu N}[\eta, J; z] \cdot F_N[\eta, J].$$

[2] An interesting method for solving (18.21b) has been presented by EDWARDS and PEIERLS [1954].

In order to express $\delta G^{(1,1)}/\delta G_\nu$ by the vertex part Γ_ν, we assume the existence of the inverse of the propagator $G^{(1,1)}[J; x, x']$ satisfying

$$\int G^{(1,1)-1}[J; x, \xi]\, G^{(1,1)}[J; \xi, x'] = \delta(x-x') \tag{18.26}$$

and define the vertex part by

$$\Gamma_\nu[J; z, x, x'] \equiv \frac{1}{e_1}\frac{\delta}{\delta G_\nu[J; z]} G^{(1,1)-1}[J; x, x'] \tag{18.27a}$$

$$= -\int G^{(1,1)-1}[J; x, \xi] \cdot \frac{1}{e_1}\frac{\delta}{\delta G_\nu[J; z]} G^{(1,1)}[J; \xi, \xi']\, G^{(1,1)-1}[J; \xi', x'] \tag{18.27b}$$

(SCHWINGER [1951]). Equation (18.27*b*) leads to

$$\left.\begin{aligned} &\frac{1}{e_1}\frac{\delta}{\delta G_\nu[J; z]} G^{(1,1)}[J; x, x'] \\ &\quad = -\int G^{(1,1)}[J; x, \xi]\, \Gamma_\nu[J; z, \xi, \xi']\, G^{(1,1)}[J; \xi', x']. \end{aligned}\right\} \tag{18.28}$$

Finally, we have from (18.24)

$$\left.\begin{aligned} &i\frac{\delta}{\delta J_\mu(z)} G^{(1,1)}[J; x, x'] \\ &\quad = e_1 \int G^{(1,1)}[J: x, \xi]\, \Gamma_\nu[J; \zeta, \xi, \xi']\, G^{(1,1)}[J; \xi', x']\, \bar{G}_{\nu\mu}[J; \zeta z]. \end{aligned}\right\} \tag{18.29}$$

Equations (18.28) and (18.29) have the graphical interpretations given in Fig. 18.1, where solid and dotted lines correspond to one-electron and one-photon propagators, respectively. The derivative operator $\delta/\delta J_\mu(z)$ means the creation of a photon (at a point z) by the source $J_\mu(z)$. This photon must be absorbed at a point in the

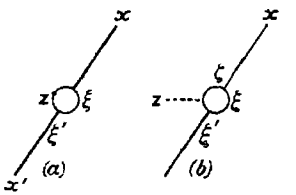

Fig. 18.1

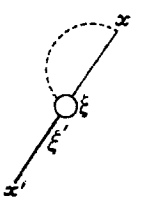

Fig. 18.2

diagram, because we always consider vacuum expectation values of operators. Therefore, it is apparent from (18.24) that $\delta/\delta G_\mu[J; z]$ means to insert a photon line at z (cf. Fig. 18.1). If $z=x$, Fig. 18.1*b* reduces to Fig. 18.2. Corresponding to Fig. 18.2, we introduce the mass operator by

$$\mathsf{M}[J; x, x'] = i e_1^2 \int \gamma_\mu\, G^{(1,1)}[J; x, \xi]\, \Gamma_\nu[J; \zeta, \xi, x']\, \bar{G}_{\mu\nu}[J; x, \zeta] \tag{18.30}$$

(SCHWINGER [1951]). Equation (18.29) goes over into

$$-e_1 \gamma_\mu \frac{\delta}{\delta J_\mu(x)} G^{(1,1)}[J; x, x'] = \int \mathsf{M}[J; x, \xi]\, G^{(1,1)}[J; \xi, x']. \quad (18.31)$$

Then (18.21*b*) can be written as follows

$$\left.\begin{aligned} i(\gamma_\mu \partial_\mu - i e_1 \gamma_\mu G_\mu[J; x] + \varkappa) \cdot G^{(1,1)}[J; x, x'] \\ + i \int \mathsf{M}[J; x, \xi]\, G^{(1,1)}[J; \xi, x'] = \lambda^2 \delta(x - x'). \end{aligned}\right\} \quad (18.32)$$

For $J_\mu = 0$ and $t > t'$, equation (18.32) becomes

$$i(\gamma_\mu \partial_\mu + \varkappa)\, G^{(1,1)}(x, x') + i \int \mathsf{M}(x, \xi)\, G^{(1,1)}(\xi, x') = 0. \quad (18.33a)$$

Since the theory is invariant under an inhomogeneous Lorentz transformation, the propagator $G^{(1,1)}(x, x')$ and the mass operator $\mathsf{M}(x, x')$ must be functions of $x - x'$. Their Fourier components depend on one momentum p, only. In particular, the Fourier component of M — this being a scalar — must be a function of $\gamma_\mu p_\mu$; we write it $\mathsf{M}(-i\gamma p)$

Equation (18.33*a*) reads then, for $t' = -\infty$,

$$\{i \gamma_\mu p_\mu + \varkappa + \mathsf{M}(-i \gamma p)\}\, G_i^{(1,1)}(p) = 0. \quad (18.33b)$$

Here $G_i^{(1,1)}(p)$ is the Fourier amplitude of the function $G^{(1,1)}(x, x')$ with $t' = -\infty$.

On the other hand, according to (18.17*a*), we have

$$G^{(1,1)}(x, x') = (\mathrm{vac} \,|\, \psi(x)\, \bar{\psi}(x') \,|\, \mathrm{vac}) \qquad \text{for } t > t'.$$

This leads to

$$G(x, x') = \sum_{(a)} (\mathrm{vac} \,|\, \psi(x) \,|\, p^{(a)})\, (p^{(a)} \,|\, \bar{\psi}(x') \,|\, \mathrm{vac}) \qquad \text{for } t > t'$$

(LEHMANN [1954]). Here $|\, p^{(a)})$ denotes an eigenstate of the total Hamiltonian and the summation is taken over all such eigenstates. By making use of the Fourier components of ψ and $\bar{\psi}$,

$$\psi(x) = \sum_p \frac{1}{\sqrt{V}} \psi(p)\, e^{i p_\mu x_\mu} \text{ etc.},$$

(V: the fundamental volume) we obtain

$$G(x, x') = \sum_{(a)} (\mathrm{vac} \,|\, \psi(p^{(a)}) \,|\, p^{(a)})\, (p^{(a)} \,|\, \bar{\psi}(p^{(a)}) \,|\, \mathrm{vac})\, e^{i(p^{(a)}, x - x')}. \quad (18.34)$$

We shall write the energy eigenvalues $p_0^{(a)}$ in the form of

$$p_0^{(a)} = \sqrt{\mathbf{p} \cdot \mathbf{p} + c^{(a)2}}$$

Comparing (18.34) with (18.33a), we see that the relation

$$-\alpha+\varkappa+\mathsf{M}(\alpha)=0 \tag{18.35a}$$

can hold only for $\alpha=c^{(a)}$. In other words, roots of (18.35a) appear at the points $\alpha=m^{(1)}, m^{(2)}, \ldots, m^{(n)}$. $\mathsf{M}(\alpha)$ is complex for $\alpha \geqslant c$. Here the particle is assumed to have observable mass values $m^{(1)}, \ldots, m^{(n)}$ and the constant c is defined in such a way that $\alpha \geqslant c$ corresponds to states of more than one particle. The quantities

$$m^{(i)}-\varkappa=\mathsf{M}(m^{(i)}) \quad (i=1, \ldots, n) \tag{18.35b}$$

are called self-energies.

When we take the point x' in $G^{(1,1)}(x, x')$ on the initial surface $\sigma'=-\infty$, it follows from (18.33b) and (18.35b) that the incoming particles described by the propagator $G^{(1,1)}(x, x')$ has masses $m^{(1)}, \ldots, m^{(n)}$. We call these the free dressed particles. This is true also for outgoing particles (i.e. particles arriving on the surface σ_2).

§ 5. Two-Body-Propagators

We shall now show that the equation (18.23) of the two-electron-propagator is equivalent to the following set (UMEZAWA and VISCONTI [1955a]).

$$\left.\begin{aligned} G^{(2,2)}[J; x_1 x_2, x_1' x_2'] &= G^{(1,1)}[J; x_1, x_1']\, G^{(1,1)}[J; x_2, x_2'] \\ &- i e_1 \int G^{(1,1)}[J; x_1, \xi]\, v_\varrho[J; \zeta, \xi, \xi']\, I_\varrho[J; \zeta, \xi' x_2, x_1' x_2'] \\ &- (x_1' \leftrightarrow x_2') \end{aligned}\right\} \tag{18.36}$$

where

$$v_\mu[J; z, x, x'] \equiv i\, \Gamma_\varrho[J; \zeta, x, \xi]\, G^{(1,1)}[J; \xi, x']\, \bar{G}_{\varrho\mu}[J; \zeta z] \tag{18.37}$$

and I_μ satisfies the equation

$$\left.\begin{aligned} I_\mu[J; z, x_1 x_2, x_1' x_2'] &= \delta(x_1-x_1')\, \frac{\delta}{\delta G_\mu[J; z]}\, G^{(1,1)}[J; x_2, x_2'] \\ &+ e_1 \int V_{\mu\varrho}[J; z, x_1, \zeta, \xi_1]\, I_\varrho[J; \zeta, \xi_1 x_2, x_1' x_2'] \end{aligned}\right\} \tag{18.38}$$

with

$$V_{\mu\varrho}[J; z, x, z', x'] \equiv v_\varrho[J; z', x, x']\, \frac{\delta}{\delta G_\mu[J; z]}. \tag{18.39}$$

In (18.36) the bracket $(x_1' \leftrightarrow x_2')$ means the addition of all the terms obtained by exchanging x_1' and x_2'. We call these the "renormalised equations" because it can be shown (cf. § 6 and § 7) that they can yield the renormalised solutions.

Before deducing (18.36) let us consider its physical meaning.

Successive iteration of (18.38) leads to

$$\left.\begin{aligned} &I_\mu[J; z, x_1 x_2, x_1' x_2'] = \\ &- e_1 \delta(x_1 - x_1') \int G^{(1,1)}[J; x_2, \xi_2]\, \Gamma_\mu[J; z, \xi_2, \xi_2']\, G^{(1,1)}[J; \xi_2', x_2'] \\ &+ \sum_{n=1}^{\infty} (-e_1)^{n+1} \int V_{\mu\varrho_n}[J; z, x_1, \zeta_n, \xi_n]\, V_{\varrho_n\varrho_{n-1}}[J; \zeta_n, \xi_n, \zeta_{n-1}, \xi_{n-1}] \\ &\ldots V_{\varrho_2\varrho_1}[J; \zeta_2, \xi_2, \zeta_1, x_1']\, G^{(1,1)}[J; x_2, \xi']\, \Gamma_{\varrho_1}[J; \zeta_1, \xi', \xi]\, G^{(1,1)}[J; \xi, x_2']. \end{aligned}\right\} \quad (18.40)$$

By substituting this expansion into (18.36) we obtain the two-body propagator which can be graphically expressed by Fig. 18.3.

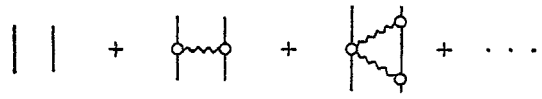

Fig. 18.3

The nth term of this expansion can be represented by Fig. 18.4, in which a photon created at a vertex Γ_n can be absorbed by the electron lines, photon lines and vertices only in the region above the

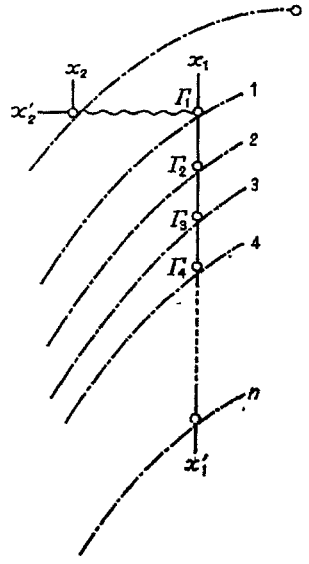

Fig. 18.4

boundary represented by the dotted curve $n-2$. It can be seen from Fig. 18.4 that we have no self-energy and corrected vertex diagrams. Therefore, this diagram is made up of all possible irreducible diagrams (i.e. skeletons). This result agrees with that given in Ch. XIV, when

we identify $G^{(1,1)}$, $G_{\mu\nu}$ and Γ_μ with $S'_{F,1}$, Δ'_{F1} and $\Gamma_{\mu 1}$. However, it must be noted that the present proof for this fact does not depend on the perturbation approximation.

The derivation of equation (18.36) is as follows: Equation (18.21*b*) gives

$$\left.\begin{aligned}\left(\gamma_\varrho \partial_\varrho + \varkappa - i\, e_1 \gamma_\varrho\, G_\varrho[J;x] - e_1 \gamma_\varrho \frac{\delta}{\delta J_\varrho(x)}\right) \frac{\delta}{\delta J_\mu(z)} G^{(1,1)}[J;x,x'] \\ = e_1 \gamma_\varrho\, \bar{G}_{\varrho\mu}[J;x\,z]\, G^{(1,1)}[J;x,x'],\end{aligned}\right\} \tag{18.41}$$

which leads, using equations (18.29), to

$$\left.\begin{aligned}&\lambda^2 e_1 \int \Gamma_\varrho[J;\zeta,x,\xi]\, \bar{G}_{\varrho\mu}[J;\zeta\,z]\, G^{(1,1)}[J;\xi,x'] = \\ &\quad - e_1 \gamma_\varrho\, \bar{G}_{\varrho\mu}[J;x\,z]\, G^{(1,1)}[J;x,x'] \\ &\quad + i e_1^2 \gamma_\varrho \int G^{(1,1)}[J;x,\xi] \frac{\delta}{\delta J_\varrho(x)} \{\Gamma_\sigma[J;\zeta,\xi,\xi']\, G^{(1,1)}[J;\xi',x']\, \bar{G}_{\sigma\mu}[J;\zeta\,z]\}.\end{aligned}\right\} \tag{18.42}$$

Figure 18.5 gives the graphical expression of (18.42). In the second graph on the right hand side the photon created at x must be absorbed by any part of the diagram except the electron line connecting x and ξ.

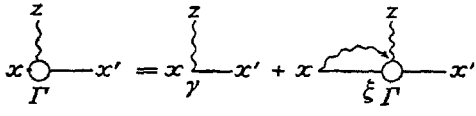

Fig. 18.5

We can show by means of (18.38) and (18.42) that (18.36) satisfies (18.23) as follows: by substituting (18.36) into (18.23) we have

$$\left.\begin{aligned}&i\, e_1 \gamma_\mu\, G^{(1,1)}[J;x_1,x'_1] \frac{\delta}{\delta J_\mu(x_1)} G^{(1,1)}[J;x_2,x'_2] \\ &\quad + e_1 \lambda^2 \int \bar{G}_{\nu\mu}[J;\zeta,\zeta']\, \Gamma_\mu[J;\zeta',x_1,\xi_1]\, G^{(1,1)}[J;\xi_1,\xi'_1] \\ &\quad I_\nu[J;\zeta,\xi'_1\,x_2,x'_1\,x'_2] + i\, e_1 \gamma_\mu \int G^{(1,1)}[J;x_1,\xi_1] \frac{\delta}{\delta J_\mu(x_1)} \\ &\quad \{\Gamma_\nu[J;\zeta,\xi_1,\xi'_1]\, \bar{G}_{\varrho\nu}[J;\zeta',\zeta] G^{(1,1)}[J;\xi'_1,\xi''_1]\, I_\varrho[J;\zeta',\xi''_1\,x_2,x'_1\,x'_2]\} \\ &\quad + (x'_1 \leftrightarrow x'_2) = 0\end{aligned}\right\} \tag{18.43}$$

which, on account of (18.42), may be written

$$\left.\begin{aligned}&i\, e_1 \gamma'_\mu\, G^{(1,1)}[J;x_1,x'_1] \frac{\delta}{\delta J_\mu(x_1)} G^{(1,1)}[J;x_2,x'_2] \\ &\quad + e_1 \gamma_\varrho \int \bar{G}_{\varrho\mu}[J;x_1,\zeta']\, G^{(1,1)}[J;x_1,\xi_1]\, I_\mu[J;\zeta',\xi_1\,x_2,x'_1\,x'_2] \\ &\quad + i\, e_1 \gamma_\mu \int G^{(1,1)}[J;x_1,\xi_1]\, \Gamma_\mu[J;\zeta,\xi_1,\xi'_1]\, \bar{G}_{\varrho\nu}[J;\zeta',\zeta] \\ &\quad G^{(1,1)}[J;\xi'_1,\xi''_1] \frac{\delta}{\delta J_\mu(x_1)} I_\nu[J;\zeta',\xi''_1,x_2,x'_1\,x'_2] = 0.\end{aligned}\right\} \tag{18.44}$$

This can be satisfied if (18.38) is true. We can calculate two-electron propagators by substituting solutions of (18.38) into (18.36).

We now see that the equation for the two-body propagator can be written in a form of

$$\left.\begin{aligned} G^{(2,2)}(x_1\,x_2,\,x_1'\,x_2') &= G^{(1,1)}(x_1,\,x_1')\,G^{(1,1)}(x_2,\,x_2') \\ &- G^{(1,1)}(x_1,\,x_2')\,G^{(1,1)}(x_2,\,x_1') \\ &+ \int G^{(1,1)}(x_1,\,\xi_1)\,G^{(1,1)}(x_2,\,\xi_2)\,M(\xi_1\,\xi_2,\,\xi_1'\,\xi_2')\,G^{(2,2)}(\xi_1'\,\xi_2',\,x_1'\,x_2'). \end{aligned}\right\} \tag{18.45}$$

Here the two-body mass operator $M(x_1, x_2, x_1', x_2')$ is given by the semi-proper irreducible diagrams for the Møller scattering, where semi-proper diagram means graphs which cannot be cut into two separate parts by cutting two-electron lines. Indeed, the ladders made up of the semi-proper irreducible diagrams give all possible irreducible diagrams for the Møller scattering (cf. Fig. 18.6).

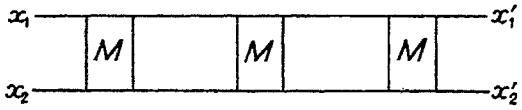

Fig. 18.6

An equation of the form (18.45) is called the Bethe-Salpeter equation [1]) (SALPETER and BETHE [1951]; GELL-MANN and LOW [1951]; NAMBU [1950], KITA [1952]).

Other many-body propagators may be treated in a similar way. For instance, the propagator for the one-electron – one-photon system (Compton scattering) can be derived from (18.19a) as follows

$$\left.\begin{aligned} &i\left(\gamma_\varrho\,\partial_\varrho + \varkappa - i\,e_1\,\gamma_\varrho\,G_\varrho[J;x] - e_1\,\gamma_\varrho\,\frac{\delta}{\delta J_\varrho(x)}\right) G_{\mu\nu}^{(1,1)}[J;x,x',z_1\,z_2] \\ &= \lambda^2\,\delta(x-x')\,(\bar{G}_{\mu\nu}[J;z_1\,z_2] + G_\mu[J;z_1]\,G_\nu[J;z_2]). \end{aligned}\right\} \tag{18.46}$$

We can rewrite (18.46) by means of (18.42) in the following way

$$\left.\begin{aligned} &G_{\mu\nu}^{(1,1)}[J;x,x',z_1\,z_2] \\ &= G^{(1,1)}[J;x,x']\,\{\bar{G}_{\mu\nu}[J;z_1\,z_2] + G_\mu[J;z_1]\,G_\nu[J;z_2]\} \\ &+ e_1 \int G^{(1,1)}[J;x,\xi]\,v_\sigma[J;\zeta,\xi',\xi]\,I_{\sigma\mu\nu}[J;\zeta,\xi,x',z_1\,z_2], \end{aligned}\right\} \tag{18.47}$$

---

[1]) By using a simple example for the kernel M, WICK [1954] and SALAM and KEMMER [1955] clarified the properties of the Bethe-Salpeter equation for bound states of two-Fermion system.

where $I_{\sigma\mu\nu}$ is a solution of

$$\left.\begin{aligned} I_{\sigma\mu\nu}[J; z, x, x', z_1 z_2] = \delta(x-x') \left\{ \frac{\delta}{\delta G_\sigma[J; z]} \bar{G}_{\mu\nu}[J; z_1 z_2] \right. \\ + \delta(z-z_1)\, \delta_{\sigma\mu}\, G_\nu[J; z_2] + \delta(z-z_2)\, \delta_{\sigma\nu}\, G_\mu[J; z_1] \Big\} \\ + e_1 \int V_{\sigma\varrho}[J; z, x, \zeta, \xi]\, I_{\varrho\mu\nu}[J; \zeta, \xi, x', z_1 z_2]. \end{aligned}\right\} \quad (18.48)$$

By applying the successive substitutions to (18.48), we can show that $G^{(1,1)}_{\mu\nu}(x, x', z_1, z_2)$ is given by all possible irreducible diagrams for the Compton scattering.

§ 6. Renormalisation Invariance

The transformation

$$\lambda \to C\lambda, \qquad \lambda' \to C'\lambda' \quad (18.49)$$

is equivalent to

$$e_1 \to e_1/C' \quad (18.50)$$

and induces the following changes (of the one-body propagator)

$$G^{(1,1)}(x, x') \to C^2\, G^{(1,1)}(x, x'), \quad \bar{G}_{\mu\nu}(z, z') \to C'^2\, \bar{G}_{\mu\nu}(z, z') \quad (18.51)$$

and analogous changes for the many-body propagators. We call (18.49) the *renormalisation transformation* because it just changes the normalisation factors of the propagators.

We note that equations (18.36), (18.38), (18.46) and (18.48) contain neither λ nor λ' and are therefore invariant under the transformation (18.49); they are *renormalisation invariant* (UMEZAWA and VISCONTI [1955a]). It is physically reasonable that the value of the charge depends on the normalisation of the propagators; indeed if the strength of the electromagnetic field is multiplied by C' the charge must be divided by the same constant. When $\lambda' = 1$ and $\lambda = 1$, we obtain the unrenormalised theory. However, there is one condition to be fulfilled by the constants λ, λ': the normalisations of the propagators for the free dressed electron and photon must be the same as those of S_F and Δ_F respectively. To determine λ, λ' by means of this condition is called renormalisation.

§ 7. The Normalisation Constants

Let us consider the equation (18.32) of the one-electron propagator, for $J = 0$ Its Fourier transform has the form

$$i\,(i\,\gamma_\mu\, p_\mu + \varkappa + \mathsf{M}(-i\,\gamma\, p))\, G^{(1,1)}(p) = \lambda^2. \quad (18.52)$$

The causality principle requires that $G(x, x')$ contains only positive

(negative) frequency parts for $t>t'$ ($t<t'$) (cf. § 2, Ch. VIII):

$$G^{(1,1)}(x, x') = \begin{cases} G^{(1,1)-}(x, x') & \text{for } t < t' \\ G^{(1,1)+}(x, x') & \text{for } t > t'. \end{cases} \tag{18.53}$$

Such a propagator is obtained from (18.52) as follows:

$$G^{(1,1)}(x, x') = \frac{-i\lambda^2}{(2\pi)^4} \int d^3p \int_L dp_0 \, e^{i(p, x-x')} \frac{1}{i\gamma_\mu p_\mu + \varkappa + \mathsf{M}(-i\gamma p)}. \tag{18.54}$$

The integration path L is defined so as to give (18.53).

In the following discussion we shall assume that the electron has only one mass level m. From (18.35a) we see that the denominator can be written in the form

$$a(-i\gamma p)(i\gamma_\mu p_\mu + m) \tag{18.55}$$

where $a(\alpha)$ is a function without zero in the domain $\alpha<c$. To the eigenvalue m of the mass we define a constant Z_2 by

$$1/Z_2 \equiv \frac{\partial}{\partial\alpha}(\alpha - \mathsf{M}(\alpha))\,|_{\alpha=m} \tag{18.56}$$

$$= a(m). \tag{18.57}$$

Then (18.54) can be written in the form

$$\left.\begin{aligned} G^{(1,1)}(x, x') = \\ -\frac{i\lambda^2}{(2\pi)^4} Z_2 \lim_{\epsilon\to 0} \int d^3p \int_L dp_0 \, e^{i(p, x-x')} \frac{1}{i\gamma_\mu p_\mu + m - i\epsilon} \frac{a(m)}{a(-i\gamma p)} \end{aligned}\right\} \tag{18.58a}$$

$$= \frac{i\lambda^2}{(2\pi)^4} Z_2 \lim_{\epsilon\to 0} \int d^3p \int dp_0 \, e^{i(p, x-x')} \frac{i\gamma_\mu p_\mu - m}{p_\mu p_\mu + m^2 - i\epsilon} \frac{a(m)}{a(-i\gamma p)}. \tag{18.58b}$$

For $t>t'$, we can carry out the integration over p_0 in (18.58b) by taking the path of integration as the half-circle below the real axis. Thus we see that singular points appearing under the real axis can contribute to (18.58b) (for $t>t'$), i.e. $p_0 = p_0^{(m)} \equiv \sqrt{\mathbf{p}\cdot\mathbf{p} + m^2}$ and interference effects in the domain of $p_0 \geqq \sqrt{\mathbf{p}\cdot\mathbf{p} + c^2}$. The pole at $p_0 = p_0^{(m)}$ contributes to (18.58b) through an oscillating term with frequency $p_0^{(m)}$. This term can be calculated by using the residue of (18.58b) at $p_0 = p_0^{(m)}$ to give

$$\left.\begin{aligned} -\frac{\lambda^2 Z_2}{(2\pi)^3} \int d^3p \frac{i\gamma_\mu p_\mu^{(m)} - m}{2p_0^{(m)}} e^{i(p^{(m)}, x-x')} \\ = \frac{\lambda^2 Z_2}{(2\pi)^3} \int d^3p \frac{p_0^{(m)} + H}{2p_0^{(m)}} \overset{*}{\gamma}_4 \, e^{i(p^{(m)}, x-x')} \end{aligned}\right\} \tag{18.58c}$$

with

$$p_\mu^{(m)} \equiv (\mathbf{p}, p_0^{(m)}).$$

Here we have used the relation

$$(i\gamma_\mu p_\mu^{(m)} - m)\,(i\gamma_\mu p_\mu^{(m)} + m)^n = 0 \qquad (n \text{ integer})$$

and therefore

$$\frac{(i\gamma_\mu p_\mu^{(m)} - m)}{a(-i\gamma p^{(m)})} = \frac{(i\gamma_\mu p_\mu^{(m)} - m)}{a(-i\gamma p^{(m)} - m + m)} = \frac{1}{a(m)}\,(i\gamma_\mu p_\mu^{(m)} - m).$$

The operator H in (18.59a) is defined by

$$H = i\gamma_4(\boldsymbol{\gamma}\cdot\mathbf{p}) + m\gamma_4 = (\boldsymbol{\alpha}\cdot\mathbf{p}) + m\beta.$$

Since $(p_0^{(m)} + H)/2p_0^{(m)} = 1$ for positive energy states, (18.58c) shows that the propagation function of the free dressed particle appears with the normalisation factor $\lambda^2 Z_2$. Then, for the renormalisation of the propagation function of the free dressed electron we have to take

$$\lambda^2 = 1/Z_2. \tag{18.59}$$

From (18.55) and (18.57) we obtain

$$i\gamma_\mu p_\mu + \varkappa + \mathsf{M}(-i\gamma p) = \frac{1}{Z_2}\,(i\gamma_\mu p_\mu + m)\,(1 + \mathsf{M}_1(-i\gamma p)), \tag{18.60a}$$

where, according to (18.55),

$$\mathsf{M}_1(-i\gamma p) = \frac{a(-i\gamma p)}{a(m)} - 1 \tag{18.60b}$$

so that

$$\mathsf{M}_1(m) = 0.$$

From (18.54) and (18.60a) we obtain

$$G^{(1,1)}(x, x') = \frac{1}{(2\pi)^4}\int_L d^4p\, e^{i(p.x - x')}\,\frac{1}{i\gamma_\mu p_\mu + m}\,\frac{1}{1 + \mathsf{M}_1(-i\gamma p)}. \tag{18.61}$$

The one-photon propagator can be discussed in a similar way. In particular, the normalisation constant λ' can be determined in a similar way to λ.

From (18.61) we can derive

$$(i\gamma_\mu p_\mu + m)\,(1 + \mathsf{M}_1(-i\gamma p))\,G^{(1,1)}(p) = 1 \tag{18.62}$$

where m is the experimental mass. Now (18.60a) shows that $\mathsf{M}_1(-i\gamma p)$ and Z_2 correspond respectively to the finite part of the mass-operator Σ^*, and to Z_2, in the Dyson theory. Therefore, we see that $G^{(1,1)}(p)$ is the renormalised propagator.

By using the formulae (18.27a) we can show that Γ_μ corresponds to the renormalised vertex part, $\Gamma_{\mu 1}$ defined in Ch. XIV. Since we showed in § 5 that propagators can be given only by *irreducible diagrams* made up of $G^{(1,1)}$, $G_{\mu\nu}$ and vertices (with the observable charge e_1), we see that the theory is renormalised.

§ 8. Renormalisation and Mass Levels

Discussions in the last paragraph were concerned with cases where particles of each field have one mass state. We shall now discuss some features of the one-body-propagator of a spinor particle which has a finite number of different stable states of mass

$$m^{(1)}, \ldots, m^{(n)} \quad (m^{(1)} < m^{(2)} < \ldots < m^{(n)}).$$

In the following discussion we don't restrict ourselves to the quantum electrodynamics.

The equation for the one-body-propagator may have the form [1])

$$i(i\gamma_\mu p_\mu + \varkappa + \mathsf{M}(-i\gamma p))\, G(p) = 1. \tag{18.63}$$

The mass values of the particle are given by the discrete roots $\alpha = m^{(1)}, \ldots, m^{(n)}$ of the equation (18.35a), i.e.

$$\alpha - \varkappa - \mathsf{M}(\alpha) = 0 \tag{18.64}$$

in the domain

$$\alpha < c. \tag{18.65}$$

We shall assume, for simplicity, that these mass values correspond to simple roots of (18.64). Thus, we can write the operator

$$h(-i\gamma p) \equiv (-i\gamma_\mu p_\mu - \varkappa - \mathsf{M}(-i\gamma p)) \tag{18.66}$$

in the form

$$h(-i\gamma p) = a(-i\gamma p) \prod_{j=1}^{n} (-i\gamma_\mu p_\mu - m^{(j)}). \tag{18.67}$$

Here $a(\alpha)$ is a function without any zero for $\alpha < c$.

Corresponding to each mass value $m^{(j)}$ we define constants $Z^{(j)}$ by

$$\frac{1}{Z^{(j)}} \equiv \frac{\partial}{\partial \alpha} (\alpha - \mathsf{M}(\alpha))\big|_{\alpha = m^{(j)}} \tag{18.68}$$

$$= a(m^{(j)}) \prod_{k \neq j} (m^{(j)} - m^{(k)}). \tag{18.69}$$

[1]) Here λ is taken to be 1.

Then, it can be shown from (18.63) that (UMEZAWA and VISCONTI [1955b])

$$G(p) = \sum_{(j)} Z^{(j)} \frac{i}{-i\gamma_\mu p_\mu - m^{(j)} + i\epsilon} \frac{a(m^{(j)})}{a(-i\gamma p)}. \tag{18.70}$$

The relation (18.70) shows that $Z^{(j)}$ is just the normalization factor of the propagator of the particle whose mass is $m^{(j)}$. The definition (18.68) can be regarded as the extension of (18.56) for Z_2 to the case of many mass levels.

In order to clarify the physical meaning of the constants $Z^{(j)}$, we shall now calculate the propagator

$$\left.\begin{aligned} G(x, x') &= \frac{1}{(2\pi)^4} \int d^4p\, G(p)\, e^{i(p_\mu, x_\mu - x'_\mu)} \\ &= \lim_{\epsilon \to 0} \sum_{(j)} \frac{iZ^{(j)}}{(2\pi)^4} \int d^4p \frac{1}{-i\gamma_\mu p_\mu - m^{(j)} + i\epsilon} \frac{a(m^{(j)})}{a(-i\gamma p)} e^{i(p_\mu, x_\mu - x'_\mu)} \\ &= \lim_{\epsilon \to 0} \sum_{(j)} \frac{iZ^{(j)}}{(2\pi)^4} \int d^4p \frac{i\gamma_\mu p_\mu - m^{(j)}}{p_\mu p_\mu + m^{(j)2} - i\epsilon} \frac{a(m^{(j)})}{a(-i\gamma p)} e^{i(p_\mu, x_\mu - x'_\mu)}. \end{aligned}\right\} \tag{18.71}$$

for $t > t'$. Since we take $t > t'$, we can carry out the integration over p_0 in (18.71) by taking the path of integration as the half-circle below the real axis. Thus we see that singular points appearing under the real axis can contribute to (18.71), i.e. $p_0 = p_0^{(j)} \equiv \sqrt{\mathbf{p}^2 + m^{(j)2}}$ for all j and interference effects in the domain of $p_0 \geqslant \sqrt{\mathbf{p} \cdot \mathbf{p} + c^2}$. The pole at $p_0 = p_0^{(j)}$ contributes to (18.71) through an oscillating term with frequency $p_0^{(j)}$. This term can be calculated by using the residue of (18.71) at $p_0^{(j)}$ to give

$$\left.\begin{aligned} &-\frac{Z^{(j)}}{(2\pi)^3} \int d^3p \frac{i\gamma_\mu p_\mu^{(j)} - m^{(j)}}{2p_0^{(j)}} e^{i(\mathbf{p}^{(j)}, x - x')} \\ &= \frac{Z^{(j)}}{(2\pi)^3} \int d^3p \frac{p_0^{(j)} + H^{(j)}}{2p_0^{(j)}} \gamma_4\, e^{i(p^{(j)}, x - x')} \end{aligned}\right\} \tag{18.72}$$

with

$$p_\mu^{(j)} \equiv (\mathbf{p}, p_0^{(j)})$$

and

$$H^{(j)} = i\gamma_4(\boldsymbol{\gamma} \cdot \mathbf{p}) + m^{(j)} \gamma_4 = (\boldsymbol{\alpha} \cdot \mathbf{p}) + m^{(j)} \beta.$$

On the other hand, from (18.34) we obtain

$$\left.\begin{aligned} G(x, x')\gamma_4 = &\sum_{(j)} \int d^3p\, (\mathrm{vac}|\psi(p^{(j)})|p^{(j)})(p^{(j)}|\psi^*(p^{(j)})|\mathrm{vac})\, e^{i(p_\mu^{(j)}, x_\mu - x'_\mu)} \\ &+ \int_c^\infty dp_0 \int d^3p\, \varrho(\mathbf{p}, p_0)\, e^{i(p_\mu, x_\mu - x'_\mu)} \qquad \text{for } t > t'. \end{aligned}\right\} \tag{18.73}$$

The last term of (18.73) denotes the contribution from the eigenstates in the domain $\alpha \geqslant c$. Comparing (18.73) with (18.72), and taking account of the fact that $(p_0^{(j)}+H^{(j)})/2p_0^{(j)}=1$ for positive energy states, we have

$$Z^{(j)} = (\mathrm{vac}\,|\,\psi(p^{(j)})\,|\,p^{(j)})\,(p^{(j)}\,|\,\psi^*(p^{(j)})\,|\,\mathrm{vac}). \tag{18.74}$$

Since the function $a(\alpha)$ has no zeros for $\alpha<c$ it has no zeros for $\alpha<m^{(n)}$. Thus, provided it has no singularities in this domain of α, the function $a(\alpha)$ is either positive or negative definite. For a moment we will suppose that this condition is satisfied. Then, it follows from (18.69) that the sign of $Z^{(j)}$ alternates with increasing j (UMEZAWA and VISCONTI [1955b]):

$$Z^{(j)}/Z^{(j+2m+1)} < 0 \qquad (m\text{: integer}). \tag{18.75}$$

In other words, half of the $Z^{(j)}$ are negative, and the others are positive. Taking into account (18.74), we have the curious situation that the matrix element $(\mathrm{vac}\,|\psi|\,p^{(j)})$ is the complex conjugate of $(p^{(j)}|\psi^*|\,\mathrm{vac})$ for those j only whose $Z^{(j)}$ is positive. Thus, the interaction Hamiltonian $H(x)$ containing ψ cannot be Hermitian;— if the matrix element $(a|H(x)|p^{(j)})$ ($(a|$ denotes an eigenstate of the total Hamiltonian) is the complex conjugate of $(p^{(j)}|H(x)|a)$ for $Z^{(j)}>0$ ($Z^{(j)}<0$), then it is not so for $Z^{(j)}<0$ ($Z^{(j)}>0$). Thus, we can expect that the S-matrix is not unitary.

This situation gives rise to the following difficulty: When we have processes where the spinor particle transmutes into another particle through vertex Γ (cf. Fig. 18.7), which is assumed to have common renormalization factor for all mass states of the spinor particle, the observable coupling constants $g^{(j)}$ satisfy relations (cf. Fig. 14.4)

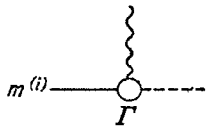

Fig. 18.7

$$\frac{g^{(i)}}{g^{(j)}} = \left(\frac{Z^{(i)}}{Z^{(j)}}\right)^{1/2} \tag{18.76}$$

Here the coupling constant $g^{(i)}$ corresponds to the transmutation process from the $m^{(i)}$-state of the spinor particle. Thus, we may find that some ratios of the observable coupling constants are imaginary, and therefore the S-matrix is not unitary.

We may obtain a unitary S-matrix only when all of $Z^{(j)}$ are positive. The relation (18.69) shows that all of the $Z^{(j)}$ may appear to be positive only when the function $a(\alpha)$ has a very special distribution of singu-

larities; — $a(\alpha)$ has at least one singular point in each domain of $m^{(1)}<\alpha<m^{(2)}$, $m^{(2)}<\alpha<m^{(3)}$, ..., $m^{(n-1)}<\alpha<m^{(n)}$, as Fig. 18.8 shows.

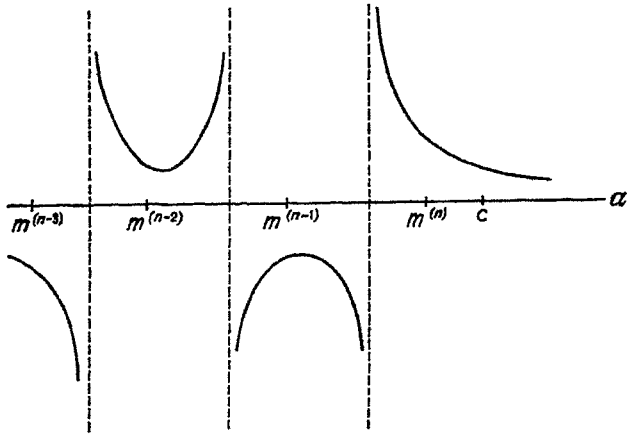

Fig. 18.8

§ 9. A Simple Model

Since it is difficult to calculate rigorously the one-body-propagators and vertex Γ_μ in the quantum electrodynamics, we shall consider a simple model, to which the discussions in § 7 and § 8 can be applied without using any approximation method.

We shall start from the Hamiltonian

$$\bar{H}=\int d^3x\, H \tag{18.77a}$$

$$\left.\begin{aligned} H=\varkappa\bar{\psi}\psi+M\bar{\varphi}\varphi+\tfrac{1}{2}\{\partial_k U\cdot\partial_k U+\mu^2U^2\}\\ +g(\bar{\psi}\varphi U^{+}+\bar{\varphi}\psi U^{-})\end{aligned}\right\} \tag{18.77b}$$

for two spinor fields ψ, φ and one scalar field U (LEE [1954]). In (18.77*b*) $\varkappa$, M and μ are the mechanical masses of the ψ-, φ- and U-particles, respectively, and U^{+} and U^{-} are the positive- and negative-frequency parts of the U-field operator.

Denoting the field operators in the interaction representation by bold letters, the field equations are

$$(\partial_4+\varkappa)\boldsymbol{\psi}=0$$
$$(\partial_4+M)\boldsymbol{\varphi}=0$$

for the $\boldsymbol{\psi}$- and $\boldsymbol{\varphi}$-fields respectively. It follows that $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are just

the positive-frequency parts, i.e. the annihilation operators. Thus, $\overline{\psi}$ and $\overline{\varphi}$ are the creation operators.

It is clear from (18.77*b*) that the φ-particles (U-particles) cannot create the U-particle (pairs of the spinor particles). Thus, the mass operators of the φ- and U-particles are zero;— one-φ-particle-propagator is just the S_F-function satisfying the relation

$$(p_0 - M)\, S_F(p) = 1, \tag{18.78}$$

and one-U-particle-propagator is the Δ_F defined in Ch. VIII.

On the other hand, the mass operator $\mathsf{M}(p_0)$ for the ψ-particle can be obtained by calculating the process corresponding to Fig. 18.9. There is no other diagram which contributes to $\mathsf{M}(p_0)$. By using (18.78) and (8.35), $\mathsf{M}(p_0)$ can be given as

$$\mathsf{M}(p_0) = -\frac{g^2}{(2\pi)^3}\int_0^\infty dk\, \frac{k^2}{\omega(\omega - p_0 + M)} \tag{18.79}$$

with

$$\omega = \sqrt{k^2+\mu^2}.$$

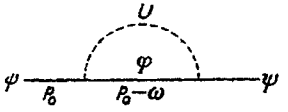

Fig. 18.9

Corresponding to (18.63), we have the equation for the one-ψ-particle-propagator $G(p)$:

$$i(-p_0 + \varkappa + \mathsf{M}(p_0))\, G(p) = 1.$$

We shall denote the function $(p_0 - \varkappa - \mathsf{M}(p_0))$ by $h(p_0)$.

According to the discussion in § 8, the stable mass levels can appear for $p_0 < c = M + \mu$. Since the mass values m_i are zeros of $h(p_0)$:

$$h(m_i) = 0 \tag{18.80}$$

we obtain

$$\left.\begin{aligned} h(p_0) &= h(p_0) - h(m_i) \\ &= p_0 - m_i - \mathsf{M}(p_0) + \mathsf{M}(m_i) \\ &= (p_0 - m_i)\left\{1 + \frac{g^2}{(2\pi)^3}\int_0^\infty dk\, \frac{k^2}{\omega(\omega - p_0 + M)\,(\omega - m_i + M)}\right\}. \end{aligned}\right\} \tag{18.81}$$

By using (18.68), we define constants Z_i by

$$\left.\begin{aligned} \frac{1}{Z_i} &= \frac{\partial}{\partial p_0}\,(p_0 - \mathsf{M}(p_0))_{p_0 = m_i} \\ &= 1 + \frac{g^2}{(2\pi)^3}\int_0^\infty dk\, \frac{k^2}{\omega(\omega - m_i + M)^2}. \end{aligned}\right\} \tag{18.82}$$

It can be easily shown that the vertex Γ is given by $\Gamma = 1$. Thus,

the observable charge g_i, corresponding to the process ψ-particle in the m_i mass state $\rightarrow \varphi$-particle + U-particle is given by

$$g_i^2 = g^2 Z_i . \tag{18.83}$$

From (18.82) and (18.83), we obtain

$$\frac{1}{g_i^2} = \frac{1}{g^2} + \frac{1}{(2\pi)^2} \int_0^\infty dk \, \frac{k^2}{\omega(\omega - m_i + M)^2} . \tag{18.84}$$

This shows that g^2 must be negative in order that g_i^2 is finite.

When g^2 is negative, the Hamiltonian (18.77b) cannot be hermitian. Moreover, we can show that, when g^2 is negative, there appear two mass values and one of two Z-factors (Z_1, Z_2) is negative. The relation (18.81) leads to the condition for two mass values m_1 and m_2:

$$1 + \frac{g^2}{(2\pi)^2} \int_0^\infty dk \, \frac{k^2}{\omega(\omega - m_1 + M)(\omega - m_2 + M)} = 0. \tag{18.85}$$

We shall show later that (18.85) can be satisfied and, therefore, that there exist two mass levels.

By using (18.81) and (18.85), $h(p_0)$ can be written in the form

$$h(p_0) = (p_0 - m_1)(p_0 - m_2) a(p_0) \tag{18.86}$$

with

$$a(p_0) = \frac{g^2}{(2\pi)^2} \int_0^\infty dk \, \frac{k^2}{\omega(\omega - p_0 + M)(\omega - m_1 + M)(\omega - m_2 + M)} . \tag{18.87}$$

It is clear from (18.87) that $a(p_0)$ has no zeros for $p_0 < M + \mu$, where it is negative. It follows that $a(p_0)$ is not the function of the property shown by Fig. 18.8. Then, according to the discussions in § 8, one of Z_i-factors ($i = 1, 2$) must be negative. Indeed, using (18.69) we obtain

$$\frac{1}{Z_1} = a(m_1)(m_1 - m_2) \tag{18.88a}$$

$$\frac{1}{Z_2} = a(m_2)(m_2 - m_1). \tag{18.88b}$$

These relations show that

$$Z_1 / Z_2 < 0.$$

It follows from (18.83) that one of two observable coupling constants g_1, g_2 must be imaginary.

It is clear from (18.87) that both of $a(m_1)/g^2$ and $a(m_2)/g^2$ are finite, and therefore Z_1/Z_2 is finite. Thus, we can take both g_1 and g_2 to be finite.

As an example of the scattering problem, we shall calculate the phase shift δ for the process $\varphi + U \to \varphi' + U'$. The momenta of U and U' are $\mathbf{k}$ and $\mathbf{k}'$ ($|\mathbf{k}| = |\mathbf{k}'|$) respectively. There is only one diagram (Fig. 18.10) for this process. The S-matrix element is given by

$$(\varphi', U'|S|\varphi, U) = \lim_{\epsilon \to 0} \frac{g^2}{2V\omega} G(M+\omega+i\epsilon)$$
$$= \lim_{\epsilon \to 0} \frac{1}{2V\omega} \sum_{j=1,2} g_j^2 a(m_j) \frac{1}{(\omega+M-m_j+i\epsilon)} \frac{1}{a(\omega+M+i\epsilon)}$$

with

$$\omega = \sqrt{\mathbf{k}\cdot\mathbf{k}+\mu^2} = \sqrt{\mathbf{k}'\cdot\mathbf{k}'+\mu^2}.$$

On the other hand, (18.88a, b) leads to

$$g_1^2 a(m_1) + g_2^2 a(m_2) = 0.$$

The phase shift δ can be calculated from the relation

$$\left.\begin{aligned} \tan\delta &= \frac{1}{i}\frac{(\varphi', U'|S|\varphi, U) - (\varphi', U'|S|\varphi, U)^*}{(\varphi', U'|S|\varphi, U) + (\varphi', U'|S|\varphi, U)^*} \\ &= \pi|\mathbf{k}| \Big/ \Big\{ (\omega+M-m_1)(\omega+M-m_2) \\ &\qquad P\int_0^\infty dk' \frac{k'^2}{\omega'(\omega'-\omega)(\omega'+M-m_1)(\omega'+M-m_2)} \Big\} \end{aligned}\right\} \quad (18.89)$$

(cf. (17.32)). It is not suprising that the right hand side of (18.89) does not depend explicitly on either g_1 or g_2, because the latter constants depend uniquely on the mass values m_1 and m_2. There are two diagrams (Fig. 18.11) for the scattering process $\psi + U \to \psi' + U'$. Here ψ and ψ' can be in either of the two mass states.

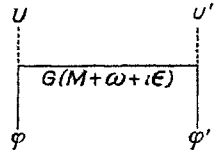

Fig. 18.10

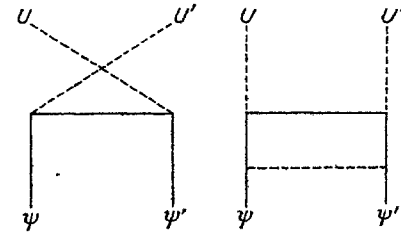

Fig. 18.11

We shall now examine the condition (18.85). Equation (18.82) leads to

$$Z_i = 1 - \frac{g_i^2}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{\omega(\omega-m_i+M)^2}. \quad (18.90)$$

The relation (18.85) can now be expressed in terms of g_1 by

$$1-\frac{g_1^2}{(2\pi)^2}(m_1-m_2)\int_0^\infty dk\,\frac{k^2}{\omega(\omega-m_1+M)^2(\omega-m_2+M)}=0 \quad (18.91a)$$

on account of the relation

$$\begin{aligned} Z_1\left(1+\frac{g^2}{(2\pi)^2}\int_0^\infty dk\,\frac{k^2}{\omega(\omega-m_1+M)(\omega-m_2+M)}\right) \\ = Z_1+\frac{g_1^2}{(2\pi)^2}\int_0^\infty dk\,\frac{k^2}{\omega(\omega-m_1+M)(\omega-m_2+M)} \\ = 1-\frac{g_1^2}{(2\pi)^2}(m_1-m_2)\int_0^\infty dk\,\frac{k^2}{\omega(\omega-m_1+M)^2(\omega-m_2+M)}. \end{aligned}$$

We can rewrite (18.85) also in terms of g_2:

$$1-\frac{g_2^2}{(2\pi)^2}(m_2-m_1)\int_0^\infty dk\,\frac{k^2}{\omega(\omega-m_1+M)(\omega-m_2+M)^2}=0. \quad (18.91b)$$

It is interesting to note that (18.91a, b) can also be derived from (18.83) by using (18.88a, b).

We shall now consider the condition (18.91a, b) in the two cases

(i) $g_1^2>0$ and therefore $g_2^2<0$, and

(ii) $g_1^2<0$ and therefore $g_2^2>0$.

In the case (i), (18.91a, b) leads to $m_2<m_1$. Then (18.88a, b) shows that $Z_1<0$, $Z_2>1$ and therefore $g^2<0$. By changing (g_1, Z_1, m_1) into (g_2, Z_2, m_2) in this discussion, we can obtain the result for the case (ii). These results are summarised in Table I. In the following discussion, we shall denote the higher and lower mass by m_1 and m_2 respectively.

TABLE I

	higher mass	lower mass
Z_i-factor	<0	>1
g_i	real	imaginary
g^2	negative	

We shall now restrict the domain of integration over k by $0<k<K(K\gg\mu)$. Then, the condition (18.91a, b) is

$$1-\frac{g_1^2}{(2\pi)^2}(m_1-m_2)\int_0^K dk\,\frac{k^2}{\omega(\omega-m_1+M)^2(\omega-m_2+M)}. \quad (18.92)$$

By using the relations

$$A(m_1, m_2) \equiv (m_1 - m_2) \int_0^K dk \frac{k^2}{\omega(\omega - m_1 + M)^2 (\omega - m_2 + M)} > 0$$

$$\frac{\partial}{\partial m_2} A(m_1, m_2) < 0$$

$$\lim_{m_1 \to -\infty} A(m_1, m_2) = \int_0^K dk \frac{k^2}{\omega(\omega - m_1 + M)^2},$$

it can be shown that there is a lower limit for g_1. i e

$$g_{\text{crit}} = (2\pi)^2 \Big/ \int_0^K dk \frac{k^2}{\omega(\omega - m_1 + M)^2},$$

satisfying (18.92). This result has been given by KÄLLEN and PAULI [1955] in the case $m_1 = M$. It can be shown from (18.90), (18.91*b*) and (18.92) that, when the lower mass level goes to $-\infty$ and the higher mass value remains to be fixed, the observable coupling constants, Z-factors and the unrenormalised coupling constant g behave as shown in Table II. When g_1 becomes smaller than g_{crit}, Z_1 turns out to be positive on account of (18.90), and g^2 becomes also positive; — all difficulties disappear with the lower mass level.

TABLE II

	higher mass	lower mass
g_i^2	$\to g_{\text{crit}}^2 > 0$	$\to -\infty$
Z_i-factor	$0 \leftarrow$	$\to 1$
g^2	$\to -\infty$	

$\to a$ means to decrease to a
$a \leftarrow$ means to increase to a.

Summarizing the results, we see that, by cutting off the integrations in the momentum space and by taking g^2 positive, we obtain Z-factor to be positive and finite and the observable coupling constant to be real and smaller than g_{crit}. In other words, the renormalisation can lead to reasonable results, only when the theory has no ultra-violet catastrophe before the renormalisation is performed.

It is interesting to reconsider the renormalisation scheme in quantum electrodynamics. Although there has been no complete calculation of the renormalisation factor Z_3 of the photon propagator, the lowest order approximation of the perturbation calculation has always shown that $Z_3 = -\infty$. If Z_3 would be negative, it would be a little

difficult to understand the successes of the renormalisation procedure in quantum electrodynamics. One possible answer to this question can be obtained from the results of the previous paragraphs. In the high energy domain the effects of various other charged particles may intervene and so we cannot treat the system of electron and electromagnetic field as an isolated system. If we suppose that such effects could lead to a finite mass operator of the electron and photon, then there might appear a critical value of the coupling constant such that for values of the electric charge smaller than this critical value there is only one mass level. In that case Z-factors would be positive. Since we have no precise notions about such additional high energy interactions we have no way of checking this suggestion. However, it might be possible to apply such considerations to a simplified model:— instead of introducing a cut off in Lee's case, some cohesive field could be introduced to make the mass operator of the ψ-field finite.

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NOTES ADDED IN PROOF

NOTE ADDED IN PROOF, 1

We note here that the solution of the fundamental equations (18.19a, b) can be written in a form similar to Feynman's path integral formula. By using the functional Fourier transform, we shall assume the solution of (18.19a, b) in the form:

$$\begin{aligned}(U[\sigma_2, \sigma_1, \eta, J])_0 = \int \delta\bar{a}\ \delta a\ \delta b_1 \ldots \delta b_4\, u[a, \bar{a}, b, \sigma_2, \sigma_1] \cdot \\ \exp\{-i \int_{\sigma_1}^{\sigma_2} [\bar{\eta}(\xi)a(\xi) + \bar{a}(\xi)\eta(\xi) + b_\mu(\xi)J_\mu(\xi)]d\xi\}.\end{aligned} \tag{1}$$

Here b_μ is treated as a c-number field and any two of a, $\bar{a}$, η and $\bar{\eta}$ anticommute with each other. To substitute (1) into (18.19a, b) leads to

$$(\gamma_\mu \partial_\mu + \varkappa - ie_1\gamma_\mu b_\mu(x))a(x)u[a, \bar{a}, b, \sigma_2, \sigma_1] = i\lambda^2 \frac{\delta}{\delta\bar{a}(x)} u[a, \bar{a}, b, \sigma_2, \sigma_1],$$

$$i \Box b_\mu(z)u[a, \bar{a}, b, \sigma_2, \sigma_1] = \lambda'^2 \frac{\delta}{\delta b_\mu(z)} u[a, \bar{a}, b, \sigma_2, \sigma_1] -$$

$$-(\lambda'/\lambda)^2 e_1 \operatorname{Sp}(\bar{a}(z)\gamma_\mu a(z))u[a, \bar{a}, b, \sigma_2, \sigma_1].$$

These equations can be solved to give

$$u[a, \bar{a}, b, \sigma_2, \sigma_1] = \frac{1}{N} \exp\{i \int_{\sigma_1}^{\sigma_2} d^4\xi\, L[\xi]\} \tag{2}$$

with

$$L[\xi] = -\frac{1}{\lambda^2}\bar{a}(\xi)\{\gamma_\mu\partial_\mu + \varkappa - ie_1\gamma_\mu b_\mu(\xi)\}a(\xi) + \frac{1}{2\lambda'^2} b(\xi) \Box b(\xi). \tag{3}$$

It must be noted that (3) has the form of the Lagrangian density for the electromagnetic field $A_\mu = b_\mu/\lambda'$ and the spinor field $\psi = a/\lambda$ with the charge $e = e_1\lambda'$. Since $U[\sigma_2, \sigma_1, \eta = 0, J = 0] = 1$, we obtain N as

$$N = \int \delta\bar{a}\ \delta a\ \delta b_1 \ldots \delta b_4\, u[a, \bar{a}, b, \sigma_2, \sigma_1]. \tag{4}$$

By substituting the generating operator (1) with (2) and (4) into (18.9) can be obtained the many-body propagators.

NOTE ADDED IN PROOF, 2

The relation (18.74) says that the renormalisation factor Z is, in general, positive when the interaction Hamiltonian is hermitian. On the other hand, we have pointed out in Example 5 of Ch. XIII that $Z_3 < 1$ for the one-photon-propagator (UMEZAWA and KAMEFUCHI,

[1951]). The latter feature has been extended to the renormalisation factors in general cases (KÄLLEN [1952], LEHMANN [1954]). As an example, we shall consider a spinless field $U(x)$. The one-body-propagator of this field can be derived in a way similar to that of the electro-magnetic field and is (cf. (18.17b))

$$G(x, x') = \lambda^2 (P[U(x), U^*(x')])_0 \tag{5}$$

This is the renormalised propagator when $\lambda^2 = 1/Z$ (cf. § 7). Taking advantage of the causality condition (cf. § 2, Ch. VIII), Lehmann proved that the propagator can be written in the form

$$G(x, x') = \tfrac{1}{2} \int d\alpha^2 \varrho(\alpha^2) \Delta_F(x, x', \alpha) \tag{6}$$

Here $\Delta_F(x, x', \alpha)$ is the Δ_F-function for the mass α. Then, from (8.51) and (8.47) we obtain the relation

$$G(x, x') = \frac{1}{(2\pi)^3} \int d^4k \left(\frac{1+\varepsilon(k)}{2}\right) \varrho(-k^2)\, e^{ik_\mu x_\mu}. \tag{7}$$

By introducing the eigenstates $\Phi(k)$ of the total energy-momentum operator in the source free representation, we can expand the state $U^*(x)\Phi_0$ in the following form (cf. (18.7)):

$$U^*(x)\Phi_0 = \sum_k \Phi(k) e^{ik_\mu x_\mu}. \tag{8}$$

By comparing (7) with (8) under the condition $t > t'$, we see that $Z\varrho(\alpha^2)$ is the probability of the states $\Phi(k)$ with $k^2 = -\alpha^2$:

$$Z\varrho(\alpha^2) = \sum \Phi^*(k)\, \Phi(k), \tag{9}$$

where the summation is taken over all states with $k^2 = -\alpha^2$. The relation (9) says that $\varrho(\alpha^2)$ is positive definite when the interaction Hamiltonian is hermitian. In summation (8) the one-particle state appears with the observable mass m and many-particles states with $k^2 > c^2$ (cf. § 4). According to the discussions (in § 7) for the normalisation of the propagation function of the free dressed particle, the one-particle state should appear with the unit normalisation:

$$\varrho(\alpha^2) = \delta(\alpha^2 - m^2) + \sigma(\alpha^2) \tag{10}$$

with

$$\sigma(\alpha^2) = 0 \text{ for } \alpha^2 < c^2.$$

The relations (9) and (11) show that the renormalisation factor Z is the probability of the one-particle state (with the mass m) appearing

in the state (8). On the other hand, the canonical commutation relation leads to

$$\lim_{t\to t'}\left\{\frac{d}{dt}G(x,x')_{t>t'}-\frac{d}{dt}G(x,x')_{t<t'}\right\}=\frac{1}{Z}\left(\left[\frac{d}{dt}U(\mathbf{x},t),\,U^*(\mathbf{x}',t)\right]\right)_0 \\ =-i\frac{1}{Z}\delta(\mathbf{x}-\mathbf{x}'). \tag{11}$$

To substitute (6) into the left hand side of (11) gives the relation

$$Z\int d\alpha^2\,\varrho(\alpha^2)=1. \tag{12}$$

This with (10) says that $Z\leqslant 1$, when the interaction Hamiltonian is hermitian. Thus, in this case, we have

$$0\leqslant Z\leqslant 1. \tag{13}$$

In a similar way we can prove the relation (13) for the spinor fields and the photon field. There is an exceptional case; the renormalisation factors of the charged fields depend on the gauge of the electromagnetic field and therefore do not necessarily satisfy (13).

We shall further give a useful theorem. This is that, by writing the renormalised propagator $G(k)$ (for the scalar field) or $G(p)$ (for the spinor field) as

$$G(k)=\Delta_F(k)\frac{1}{Z^R(-k^2,g_1)+iZ^I(-k^2,g_1)}, \tag{14}$$

$$G(p)=S_F(k)\frac{1}{Z^R(-i\gamma p,g_1)+iZ^I(-i\gamma p,g_1)}, \tag{15}$$

we can prove that

$$\lim_{\Lambda\to\infty}Z^R(\Lambda^2,g_1)=Z \tag{16a}$$

$$Z^I(\alpha^2,g_1)=0\quad\text{for}\quad\alpha\leqslant c \tag{17a}$$

$$\lim_{\Lambda\to\infty}Z^R(\Lambda,g_1)=Z \tag{16b}$$

$$Z^I(\alpha,g_1)=0\quad\text{for}\quad\alpha\leqslant c \tag{17b}$$

The constant Z is the renormalisation factor, Z^R and Z^I are real functions and g_1 is the renormalised coupling constant. When the many-particles states $\Phi(k)$ $(-k^2\geqslant c^2)$ are of positive probabilities (i.e. $\sigma(\alpha^2)\geqslant 0$), comparison of the imaginary part of (14) with that of (6) leads to the fact that the function Z^I is positive definite. It can be shown that the $Z^R(\Lambda^2,g_1)$ and $Z^R(\Lambda,g_1)$ are the renormalisation factors obtained by a cut-off method (with the cut-off energy Λ).

The Z^I comes from the contributions of displaced poles in the self-energy diagrams (or interference effects of many-particles processes to the one-particle process, cf. Example 1, Ch. XIII and § 2, Ch. XIV) and therefore is of no divergencies. It should be noted that theorems (14), (15), (16a, b) hold only in cases of interactions of the 1st kind (cf. Ch. XV).

NOTE ADDED IN PROOF, 3

We shall develop discussions here as an addendum to Ch. I. They may also be regarded as addenda to Ch. XIV and Ch. XVIII.

Although the renormalisation theory has made a great success in the realm of quantum electrodynamics, there still remain open questions as to the internal consistency and the prediction as to high-energy phenomena. It is assumed in the renormalisation theory that the renormalised coupling constant g_1 is equal to the observed value g_{ob} of the coupling constant. Suppose that, when the unrenormalised coupling constant g runs over all possible values for which the interaction Hamiltonian is hermitian, the renormalised coupling constant g_1, being a function of g, runs over a certain domain $N(g_1)$. This domain is called the normal zone. If g_{ob} does not lie in the normal zone, to regard g_1 as g_{ob} is equivalent to assume the interaction Hamiltonian to be non-hermitian. Such a situation happened in the Lee example. There, $N(g_1)$ contains only $g_1=0$ and g_{ob} being out of $N(g_1)$ leads to appearance of the negative-probability state (KÄLLEN and PAULI [1955]). Quite generally, the renormalisation factor Z is a function $Z(g_1)$ of g_1. However, in the renormalisation theory, g_1 in $Z(g_1)$ is replaced by g_{ob}. It is now clear that, when the condition (13) in the preceding note is not fulfilled by $Z(g_{ob})$, e.g. $Z(g_{ob})<0$, g_{ob} is out of $N(g_1)$. In this connection a question immediately arises as to whether the coupling constants realized in nature really lie in their normal zones. In the quantum electrodynamics, it has been shown in various ways that the observed value of the electric charge, $e_{ob}=1/\sqrt{137}$, is out of its normal zone $N(e_1)$ (LANDAU *et al.* [1954], UMEZAWA and KAMEFUCHI [1956], TAYLOR [1956]). In the Lee example, it was shown that there exists a critical cut-off energy λ; g_{ob} can lie in $N(g_1)$ and the renormalisation factor can be positive only when we disregard effects of states with energy larger than λ. Similar critical constants λ also appear in the quantum electrodynamics. CHEW and LOW [1955] made a reliable estimation of the coupling constant of the meson-

nucleon interaction in problems of low energy meson-nucleon scattering and photon-meson production phenomena and obtained the result $g_{ob}^2 = 0.08$. In their analysis, use is made of the cut-off method; the cut-off energy is about $5m(\pi)$; ($m(\pi)$: the π-meson mass). In this theory it seems that the critical cut-off constant λ is about the nucleon mass. This λ is fortunately not smaller than the cut-off energy used in calculations.

A peculiar feature of cases where g_{ob} is out of $N(g_1)$ is appearance of states of negative probabilities. However, it is clear that the latter states never appear in observations. Thus, it seems better to search for observable peculiarities that appear in cases where g_{ob} does not lie in the normal zone. To do this it may be useful to remember the fact that all transition matrix elements are built with the renormalised constants g_1 and with the renormalised propagators $G(g_1)$ and vertices $\Gamma(g_1)$. In the renormalisation theory, g_1 is replaced by g_{ob}. Therefore, it is important to find essential differences between $G(g_{ob})$ with g_{ob} in $N(g_1)$ and the one with g_{ob} out of $N(g_1)$. Such a difference really appears in the high energy region. When g_{ob} is out of $N(g_1)$ and the renormalisation factor is negative, the function $Z^R(-k^2, g_{ob})$, defined by (14)] in the preceding note, turns out to be negative in the high energy domain $-k^2 > \lambda^2$, where λ is the critical cut-off constant. This can be easily found from the fact that $Z^R(\Lambda^2, g_{ob})$ is the renormalisation factor obtained by the cut-off method with the cut-off energy Λ. The sign of Z^R is important in problems of high energy phenomena. In fact, it can be shown that the sign of the scattering phase shift in the Lee Example depends essentially on that of Z^R. Furthermore, in quantum electrodynamics, the Coulomb potential in the renormalisation theory turns out to be attractive in the scattering processes of charged particles with energy transfer larger than λ.

When g_{ob} is out of the normal zone, it may be reasonable to expect that some physical effects have been out of consideration and that to take account of such effects may change essentially the renormalised propagator in the high energy domain ($-k^2 > \lambda^2$). Since interactions due to particles with masses larger than λ may affect the propagator $G(k)$ with $-k^2 > \lambda^2$, it may be of importance to examine how the effects of the heavy particles have influences on the normal zone $N(g_1)$. The critical [constant λ] in quantum electrodynamics is about the value of $m \exp(3\pi \times 137) \approx 10^{23} m$ (m: the electron mass). This is much larger than any of the masses of known particles. There-

fore, it seems that the failure of quantum electrodynamics can never be removed taking into account effects of other fields, if they are subject to interactions of the first kind. The final solution may therefore be obtained by considering the structure of space-time itself or by introducing a new fundamental length. We can expect that the fundamental length may be introduced into quantum electrodynamics through the intermediary of interactions of the second kind (cf. Ch. XV) in which the electron participates, that is, the Fermi interaction ($\mu-e$ decay, β-decay etc.) and the gravitational interaction. From the consideration of order of magnitude they are supposed to be effective in the energy regions $\gtrsim 5\times10^5\,m$ and $\gtrsim 10\times10^{22}\,m$, respectively. As these values are much larger than any of the masses of known particles and are small in comparison with the critical cut-off value $\lambda\approx10^{23}\,m$, their contribution may play an essential role in the discussion of the normal zone of the electric charge.

In the meson theory, we may expect that λ is about the nucleon mass and therefore that the renormalisation theory for the system consisting only of meson and nucleon will immediately lead to contradictions in the observable energy region. The renormalised propagators of the π-meson and the nucleon for high energy-momenta depend on effects due to heavy mesons and hyperons, and therefore we should carefully take into account the latter effects in order to obtain the normal zone for the meson-nucleon interaction.

It was pointed out in Ch. I that the fundamental problems in the quantum field theory, e.g. problems of divergencies, mass and coupling values of particles, limit of applicability of the theory and structure of the particles etc., have found the realistic footing for their further investigations in the observable effects of the proper field, which are estimated by the renormalisation theory. We now see that observed values of coupling constants should be limited to their normal zones and that the high energy behaviour of renormalised fields is important for further clarification of the fundamental problems.

As was pointed out before, the low energy meson-nucleon phenomena have been analysed to give low energy meson-nucleon interaction. The analysis of the nuclear potential has also made great progress; it has been confirmed that the present meson theory, in the 2nd and 4th order approximations of the perturbation theory, can predict the properties of the long-range part (range $\gtrsim 0.6/m(\pi)$) of the nuclear

potential (TAKETANI *et al.* [1952], [1954]). The Brookhaven-cosmotron experiment (WALKER *et al.* [1954], FOWLER *et al.* [1954)] has succeeded to obtain the multiple production of mesons and found a peculiar angular distribution of produced mesons. It is clearly important to construct a synthetical view to explain both low energy and high energy meson-nucleon phenomena. Existence of the negative proton has been confirmed by the Berkeley-accelerator experiment (1956). This is in harmony with the present theory where charged particles are always assumed to exist in positive and negative forms. The Berkeley-bevatron experiment (CHUPP, GOLDHABER *et al.* [1955]) succeeded to obtain the beam of K-particles. This will stimulate the study of heavy mesons to clarify their properties.

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