Maxwell Theory and Galilean Relativity

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This paper revisits the relationship between Maxwell's Electromagnetic Theory (EMT) and coordinate transformations that can be implemented via tensors. It is well known that under Lorentz transformation, Maxwell's equations are form-invariant (although of course not invariant in numerical values, except for a few one-dimensional constructs). That fact means Maxwell's EMT fits well with Einstein's Special Relativity Theory (SRT). This paper shows that the situation is almost the same under Galilean transformations, and possibly other plausible transformations that may be considered in the future. The only difference is that some constructs that are number-invariant under Lorentz transformation become only form-invariant under Galilean transformation. Thus the issue of form invariance for Maxwell's equations is not a strong indicator in favor of any particular kind of coordinate transformation. That fact means Maxwell's EMT fits well with just about any reasonable variant of SRT. **Key Words:** electrodynamics of moving bodies, special relativity theory.

1. Introduction

It is generally believed that Maxwell's EMT is not invariant in form under Galilean coordinate transformations [1-5], and since Maxwell's EMT *is* demonstrably form-invariant under Lorentz transformations, that it must be uniquely linked to Einstein's SRT. The present paper shows that this assessment is not in fact justified.

The technique for showing form-invariance of Maxwell's equations under Galilean transformation is tensor analysis, as displayed in 4×4 matrix algebra. The early investigations of Maxwell's equations under Galilean transformation did not use this approach, and missed a detail that it manages very well for the user; namely, the distinction between 'covariant' and 'contravariant' behavior, and its realization in the appearance of + and – signs.

Section 2 reviews current beliefs about the status of Maxwell's equations in relation to Galilean coordinate transformations; namely, that Maxwell's equations are not form-invariant under Galilean transformations. Weaknesses in the evidence for that claim are pointed out.

Section 3 comes to the crux of the present argument; namely, demonstrating that Maxwell's equations are in fact form-invariant under Galilean transformation of coordinates. Section 3, supplemented by the Appendix, generalizes the demonstration for quite arbitrary transformations. Section 4 draws conclusions about where we stand now.

2. Some Current Beliefs About Maxwell's Equations

It is widely believed [1-5] that Maxwell's EMT represented a real break with Newtonian physics. For one thing, Newton's gravitational theory was a point-particle model with instantaneous action at a distance, whereas Maxwell's EMT is a field theory with finite signal propagation speed. For another thing, Newton's theory had a gravitational 'potential' that created a force that, when integrated around a closed path, yielded zero, whereas Maxwell's EMT has fields, and hence forces, with 'curl'; *i.e. non*-zero closed path integrals, mandating some subtle change in the definition of the word 'potential'. Most importantly, it seems, Newton's equations were clearly form-invariant under Galilean coordinate transformation, whereas Maxwell's equations are everywhere said to lack form-invariance under Galilean transformation.

Refs. [1-5] provide a fair sampling of what is still current belief about Maxwell's equations *vis a vis* Galilean coordinate transformations. Møller [1] said "…the velocity of light must have the same constant value c in all systems of inertia,…This is obviously in conflict with the usual kinematical concepts [Galilean] …" Feynman [2] said, "…according to Galilean transformation the apparent speed of…light [for a moving observer]…should not be c …" and "…Lorentz noticed a remarkable and curious thing…Maxwell's equations remain in the same form when [Lorentz] transformation is applied to them!" Jackson [3] said, "The form of the wave equation is not invariant under Galilean transformations. Furthermore, no kinematic transformation can [preserve the form]…" Phipps [4] said, "…the [Galilean] invariance of Maxwell's equation [involving curl $\mathbf{E}$] is spoiled by a[n extra] term…" Gray [5] said, "Maxwell derived his equations with respect to a particular reference frame…so that they are not invariant with respect to the full Galilean group."

Among the five authors cited, only Jackson [3] and Phipps [4] displayed some mathematics to justify their claim of non-invariance for Maxwell's equations under Galilean transformation. Jackson looked at the D'Alembertian operator $\Box$ in a wave equation $\Box\,\psi = 0$:

$$\Box = \sum_i \partial^2 / \partial x_i'^2 - \frac{1}{c^2}\partial^2 / \partial t'^2 = \nabla'^2 - \frac{1}{c^2}\partial^2 / \partial t'^2$$

He said that applying Galilean transformation with velocity $\mathbf{v}$ produces

$$\Box' = \nabla^2 - \frac{1}{c^2}\partial^2 / \partial t^2 - \frac{2}{c^2}\mathbf{v}\boldsymbol{\cdot}\nabla\frac{\partial}{\partial t} - \frac{1}{c^2}\mathbf{v}\boldsymbol{\cdot}\nabla\mathbf{v}\boldsymbol{\cdot}\nabla$$

and that the terms $-(2/c^2)\mathbf{v}\boldsymbol{\cdot}\nabla\,\partial/\partial t - (1/c^2)\mathbf{v}\boldsymbol{\cdot}\nabla\mathbf{v}\boldsymbol{\cdot}\nabla$ spoil the form invariance of the wave equation. But is this right? Jackson used the chain rule $\partial/\partial t' = \partial/\partial t + (1/c)\mathbf{v}\bullet\nabla$ to form the operator $\partial^2/\partial t'^2$ from $\partial^2/\partial t^2$. But one could reasonably argue that he should have discussed forming ∇'^2 from ∇^2 too; *i.e.*, he should have explicitly formed *both* operators in primed coordinates. Indeed, one could argue that a *mixed* expression, with ∇^2 and $\partial^2/\partial t'^2$, just doesn't have meaning. And what of the ψ function on which the D'Alembertian operates? Is that a function of the four variables ct, x, y, z individually, or of some overall characteristic of those variables, such as squared length?

Phipps looked at the two Maxwell equations involving curl. He first looked at

$$\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}=\nabla'\times\mathbf{E}'+\frac{1}{c}\left(\frac{\partial}{\partial t'}+\mathbf{v}'\bullet\nabla'\right)\mathbf{B}'$$
$$=\nabla'\times\mathbf{E}'+\frac{1}{c}\frac{\partial}{\partial t'}\mathbf{B}'+\frac{1}{c}\left(\mathbf{v}'\bullet\nabla'\right)\mathbf{B}'=0$$

in which $\mathbf{v}'$ means $-\mathbf{v}$ [see his Eq. (4.14)]. He identified the term $\frac{1}{c}\left(\mathbf{v}'\bullet\nabla'\right)\mathbf{B}'$ as a form spoiler. He also looked at

$$\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}-\frac{4\pi}{c}\mathbf{j}_s=\nabla'\times\mathbf{B}'-\frac{1}{c}\left(\frac{\partial}{\partial t'}+\mathbf{v}'\bullet\nabla'\right)\mathbf{E}'-\frac{4\pi}{c}\left(\mathbf{j}'_s-\rho'\mathbf{v}'\right)$$
$$=\nabla'\times\mathbf{B}'-\frac{1}{c}\frac{\partial}{\partial t'}\mathbf{E}'-\frac{4\pi}{c}\mathbf{j}'_s+\left\{-\frac{1}{c}(\mathbf{v}'\bullet\nabla')\mathbf{E}'+\frac{4\pi}{c}\rho'\mathbf{v}'\right\}=0$$

in which $\mathbf{j}_s$ is a source current and ρ is a charge density. He identified the terms displayed in curly brackets as form spoilers. But is all this actually valid? It comes from using the chain rule to form $\partial/\partial t'$ from $\partial/\partial t$. Should something similar be done for ∇'? And what of the $\mathbf{E}$ and $\mathbf{B}$? Is it critical that Phipps liked to regard those as invariant?

Both the Jackson argument and the Phipps argument for non-invariance of Maxwell's equations under Galilean transformation leave so many questions because they are so focused on isolated parts of the problem. The questions move one to look for a more global approach to the problem as a whole.

Note that both the Jackson argument and the Phipps argument just rely on the chain rule $(1/c)\partial/\partial t' = (1/c)\partial/\partial t + (V/c)\nabla$ familiar from differential calculus, and so could have been constructed very early, even before the development of Einstein's Special Relativity Theory (SRT). Indeed, the believed non-invariance of Maxwell's EMT under Galilean transformation, coupled with its demonstrable invariance under Lorentz transformation, was weighty evidence in favor of dropping the idea of Galilean invariance, assuming Lorentz invariance in the 'Principle of Relativity', and allowing the modifications to Newton's laws that SRT mandates.

SRT then produced a big gift for physics: an additional mathematical technique for use in physics. We received tensor analysis into our routine tool kit. That technique is useful for making a more global approach to the Maxwell invariance problem. The next Section shows how this works out.

3. The Tensor / Matrix Approach to Maxwell's EMT

One extremely important idea from tensor analysis is that, under coordinate transformation, different mathematical objects can exhibit different behaviors. One behavior is called 'covariant', meaning that the object transforms in the same way as space-time coordinates. Here the language is treacherous, since the word 'covariant' is also commonly used to mean 'form-invariant', and the word 'invariant', unmodified, is often used to mean 'number-invariant', which is much stronger. Another tensor behavior is called 'contravariant', meaning the object transforms in a contrary, inverse, way. Here, too, the language will turn out to be treacherous, as the present analysis will show a bit further on.

The distinction between 'covariant' and 'contravariant' behaviors requires attention in applying coordinate transformations to the fields and differential operators in Maxwell's equations. This is well known in the case of Lorentz transformations, but it has apparently not been tried out in the case of Galilean transformations. The following arguments will show that proper attention results in form invariance for Maxwell's equations under Galilean transformations.

Maxwell theory never seems to need tensors with more than two tensor indices, which means that simple matrices can be used to display everything explicitly. For tensors, the distinction between covariant and contravariant is expressed by position of indices: up or down. For matrices, it is often revealed in the difference between row vectors and column vectors.

SRT invites us to think about coordinates ct, x, y, z as a 4-dimensional column vector, that undergoes transformation by a 4×4 matrix representing Lorentz transformation:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}=\frac{1}{\sqrt{1-v^2/c^2}}\begin{bmatrix} 1 & -v/c & 0 & 0 \\ -v/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

The v is the speed of the new coordinate frame, here taken to be along the x direction. The minus signs mean the new coordinate frame is moving forward along the x direction. The coordinate column vector, and other 4-dimensional column vectors that transform in the same way, are called 'covariant'. Observe that the Lorentz transformation matrix is symmetric.

SRT further invites us to think about row vectors with reversed sign on space coordinates. Such constructs are called 'contravariant'. The Lorentz transformation of contravariant coordinates goes:

$$\begin{bmatrix} ct' & -x' & -y' & -z' \end{bmatrix}=$$
$$\begin{bmatrix} ct & -x & -y & -z \end{bmatrix}\frac{1}{\sqrt{1-v^2/c^2}}\begin{bmatrix} 1 & +v/c & 0 & 0 \\ +v/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observe that the Lorentz-transformation matrices for covariant and contravariant objects are reversed in the sign of v, and inversely related in multiplication:

$$\frac{1}{\sqrt{1-v^2/c^2}}\begin{bmatrix} 1 & +v/c & 0 & 0 \\ +v/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\times \frac{1}{\sqrt{1-v^2/c^2}}\begin{bmatrix} 1 & -v/c & 0 & 0 \\ -v/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\equiv\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The 'inverse' property assures that the four-vector square length $s^2 = c^2t^2 - x^2 - y^2 - z^2$ is number-invariant under Lorentz transformation:

$$s^2 = \begin{bmatrix} ct & -x & -y & -z \end{bmatrix}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \to s'^2 = \begin{bmatrix} ct' & -x' & -y' & -z' \end{bmatrix}\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} \equiv s^2$$

The corresponding Galilean transformation of covariant coordinates is

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -V/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} .$$

Observe that for Galilean transformations, no square root is needed to normalize the transformation matrix. Observe also that the Galilean V (unlimited) replaces the Einsteinian v (limited to light speed c) inside the transformation matrix. Observe finally that only one off-diagonal term is needed in the transformation matrix. Thus the Galilean transformation matrix is *not* symmetric, like the Lorentz transformation matrix was.

The asymmetry of the Galilean transformation matrix creates two new possibilities where the Lorentz transformation created only one. The transformation of contravariant coordinates could go:

$$\begin{bmatrix} ct' & -x' & -y' & -z' \end{bmatrix} = \begin{bmatrix} ct & -x & -y & -z \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ +V/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

Or it could go:

$$\begin{bmatrix} ct' & -x' & -y' & -z' \end{bmatrix} = \begin{bmatrix} ct & -x & -y & -z \end{bmatrix}\begin{bmatrix} 1 & +V/c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

With the first option, the transformation matrices for contravariant and covariant coordinates are inverses, as in the Lorentz case. But the contravariant transformation yields $ct' = ct - xV/c$, which differs from the covariant $ct' = ct$, which expresses the universal time that one expects of a Galilean transformation. Furthermore, it yields contravariant $-x' = -x$, which conflicts with $x' = x - Vt$, which defines a Galilean transformation.

With the second option, the transformation of contravariant coordinates matches the covariant $ct' = ct$ and $x' = x - Vt$. But the transformation matrices for contravariant and covariant coordinates are not inverses, as in the Lorentz case. Instead, they multiply to

$$\begin{bmatrix} 1 & +V/c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -V/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1-V^2/c^2 & +V/c & 0 & 0 \\ -V/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

This product matrix is unimodular, as in the Lorentz case, but since it is not an identity matrix, it changes the coordinate four-vector square length. For example, the time-only square length $s^2 = c^2t^2$ transforms to the square length $s'^2 = c^2t'^2 - V^2t'^2 = (c^2 - V^2)t'^2$. This is in fact just what one should expect of a Galilean transformation. So for coordinates, we take the second option, and we have form invariance of the square length, but not number invariance.

Again, the language currently available is a little bit treacherous: we presently have only the *one* word 'contravariant' to refer to the *two* candidate transformation behaviors; This language deficiency is unfortunate, and needs remedy. Since the second transformation behavior involves a matrix transposition, let us invent the new name 'trans-contravariant' for it. This detail about matrix transposition never comes up with Lorentz transformations, because Lorentz transformation matrices are symmetric. It is something newly revealed because we are looking at Galilean transformations.

We leave the name 'contravariant', or 'plain contravariant' for the first transformation behavior. The difference between 'contravariant' and 'transcontravariant' is just this: 'transcontravariance' is about 'form invariance', whereas 'plain contravariance' is about number invariance. Form invariance, and hence transcontravariance, is what we need for coordinates.

An instance requiring number invariance comes up with differential operators. Consider the D'Alembertian operator $\Box = -c^{-2}\partial^2/\partial t^2 + \nabla^2$ that Jackson [3] invoked. Imagine it factored as

$$\Box = -\begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}\begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{bmatrix}^{\mathrm{tr}}$$

(Here 'tr' means 'transpose', an operation most handy for saving display space!).

By definition of the word 'covariant', the covariant differential-operator column vector

$$\begin{bmatrix} c^{-1}\partial/\partial t & -\partial/\partial x & -\partial/\partial y & -\partial/\partial z \end{bmatrix}^{\mathrm{tr}}$$

must transform like the covariant coordinate column vector:

$$\begin{bmatrix} c^{-1}\partial/\partial t' \\ -\partial/\partial x' \\ -\partial/\partial y' \\ -\partial/\partial z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -V/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c^{-1}\partial/\partial t \\ -\partial/\partial x \\ -\partial/\partial y \\ -\partial/\partial z \end{bmatrix} \quad .$$

But how should the contravariant differential-operator row vector

$$\begin{bmatrix} c^{-1}\partial/\partial t' & \partial/\partial x' & \partial/\partial y' & \partial/\partial z' \end{bmatrix}$$

be treated?

To answer this question, we can invoke a logical requirement: *all* products of differential operators with their own variables of differentiation have to be number invariant. So we *have* to have

$$\begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t'} & \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'} \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} \equiv$$

$$\begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ +V/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -V/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$$\equiv \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \equiv 4$$

In short, the contravariant differential-operator row vector *has* to have the *plain* contravariant behavior: the transformation matrix for the contravariant differential-operator row vector has to be the inverse of the covariant coordinate transformation matrix.

The plain contravariant transformation matrix assures that under Galilean transformation the complete D'Alembertian operator is invariant, just like the number 4 is invariant:

$$\Box' = -\begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ +V/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ -V/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c^{-1}\partial/\partial t \\ -\partial/\partial x \\ -\partial/\partial y \\ -\partial/\partial z \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{bmatrix}^{\mathrm{tr}}$$

$$= -\begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{bmatrix}^{\mathrm{tr}} \equiv \Box$$

Thus under Galilean transformation, just as under Lorentz transformation, the D'Alembertian is well and truly invariant. So the Jackson argument goes away.

Both Jackson [3] and Phipps [4] relied upon the chain rule familiar from differential calculus, saying that $c^{-1}\partial/\partial t' = c^{-1}\partial/\partial t + (V/c)\partial/\partial x$. This familiar chain rule emerges as the contravariant $c^{-1}\partial/\partial t'$. But another less familiar complementing chain rule emerges from the covariant $-\partial/\partial x'$; *i.e.* $-\partial/\partial x' = -\partial/\partial x - (V/c^2)\partial/\partial t$. Forming the D'Alembertian then creates canceling cross terms $(V/c^2)\partial^2/\partial x\partial t$ from $c^{-2}\partial^2/\partial t'^2$ and $-(V/c^2)\partial^2/\partial t\partial x$ from $-\partial^2/\partial x'^2$.

The *two* complementing chain rules *both* emerge in the present system of mathematics, because the system is representing Maxwell's equations, which have only *one* parameter c, which for a wave is equal to $\lambda\nu$, wavelength times frequency. So a change in ν (*i.e.* the change from $\partial/\partial t$ to $\partial/\partial t'$) has to be balanced by a change in wavelength λ, which is the inverse of wave number k, (*i.e.* the change from $\partial/\partial x$ to $\partial/\partial x'$).

The familiar chain rule is saying that a single sensor moving through a stationary spatial field pattern sees temporal changes because of its motion. The unfamiliar chain rule is saying that a field pattern passing an array of sensors looks distorted. The reason for this is finite light speed: information about different spatial points in the pattern arrives to one perception point at different times.

Using only the first chain rule, without the second one, is what led to the conclusions that both Jackson [3] and Phipps [4] drew. Now let us examine in detail the curl equations that Phipps [4] considered,

$$\nabla\times\mathbf{E} + \frac{1}{c}\partial\mathbf{B}/\partial t = 0 \quad \text{and} \quad \nabla\times\mathbf{B} - \frac{1}{c}\partial\mathbf{E}/\partial t - \frac{4\pi}{c}\mathbf{j}_s = 0 \quad .$$

These equations can be put in matrix form by introducing the field matrix and its dual (Ref. [3], Sect. 11.9).

$$\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & -E_y & -E_x & 0 \end{bmatrix} \quad .$$

and recalling the contravariant differential operator $\begin{bmatrix} c^{-1}\partial/\partial t & \partial/\partial x & \partial/\partial y & \partial/\partial z \end{bmatrix}$.

The first curl equation, $\nabla\times\mathbf{E} + c^{-1}\partial\mathbf{B}/\partial t = 0$, amounts to the spatial part of the row vector created by

$$\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}0 & -B_x & -B_y & -B_z\\ B_x & 0 & E_z & -E_y\\ B_y & -E_z & 0 & E_x\\ B_z & E_y & -E_x & 0\end{bmatrix}=0$$

The leftover time part is that other Maxwell equation, $\nabla \bullet \mathbf{B}=0$.

The second curl equation,

$$\nabla\times\mathbf{B}-c^{-1}\partial\mathbf{E}/\partial t-(4\pi/c)\mathbf{j}_\mathrm{s}=0 \quad ,$$

amounts to the spatial part of the row vector resulting from

$$\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{bmatrix}$$

$$-4\pi\begin{bmatrix}\rho & \frac{j_x}{c} & \frac{j_y}{c} & \frac{j_z}{c}\end{bmatrix}^{\mathrm{tr}}=0$$

The leftover time part is that other Maxwell equation, $\nabla\bullet\mathbf{E}=4\pi\rho$.

Now how do all these things transform under Galilean transformation? (The equations below get quite long, so I use the Russian convention of repeating the symbols $=$, $+$, $-$ and $\times$ from one line to the next line.)

1. We already know

$$\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t'} & \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'}\end{bmatrix}=\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ +V/c & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

2. The $\begin{bmatrix}\rho & j_x/c & j_y/c & j_z/c\end{bmatrix}$ transforms like $\begin{bmatrix}ct & x & y & z\end{bmatrix}$; *i.e.*,

$$\begin{bmatrix}\rho' & \frac{j'_x}{c} & \frac{j'_y}{c} & \frac{j'_z}{c}\end{bmatrix}=\begin{bmatrix}\rho & \frac{j_x}{c} & \frac{j_y}{c} & \frac{j_z}{c}\end{bmatrix}\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

3. The two field matrices transform as

$$\begin{bmatrix}0 & -E'_x & -E'_y & -E'_z\\ E'_x & 0 & -B'_z & B'_y\\ E'_y & B'_z & 0 & -B'_x\\ E'_z & -B'_y & B'_x & 0\end{bmatrix}=$$

$$=\begin{bmatrix}1 & 0 & 0 & 0\\ -V/c & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{bmatrix}\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

and

$$\begin{bmatrix}0 & -B'_x & -B'_y & -B'_z\\ B'_x & 0 & E'_z & -E'_y\\ B'_y & -E'_z & 0 & E'_x\\ B'_z & E'_y & -E'_x & 0\end{bmatrix}=$$

$$=\begin{bmatrix}1 & 0 & 0 & 0\\ -V/c & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -B_x & -B_y & -B_z\\ B_x & 0 & E_z & -E_y\\ B_y & -E_z & 0 & E_x\\ B_z & E_y & -E_x & 0\end{bmatrix}\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

When all these transformed vectors and matrices are put together, we have

$$\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t'} & \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'}\end{bmatrix}\begin{bmatrix}0 & -B'_x & -B'_y & -B'_z\\ B'_x & 0 & E'_z & -E'_y\\ B'_y & -E'_z & 0 & E'_x\\ B'_z & E'_y & -E'_x & 0\end{bmatrix}=$$

$$=\left[\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ +V/c & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right]\times$$

$$\times\left[\begin{bmatrix}1 & 0 & 0 & 0\\ -V/c & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -B_x & -B_y & -B_z\\ B_x & 0 & E_z & -E_y\\ B_y & -E_z & 0 & E_x\\ B_z & E_y & -E_x & 0\end{bmatrix}\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right]=$$

$$=\left[\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}0 & -B_x & -B_y & -B_z\\ B_x & 0 & E_z & -E_y\\ B_y & -E_z & 0 & E_x\\ B_z & E_y & -E_x & 0\end{bmatrix}\right]\times$$

$$\times\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=0\times\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=0$$

And we have

$$\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t'} & \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'}\end{bmatrix}\begin{bmatrix}0 & -E'_x & -E'_y & -E'_z\\ E'_x & 0 & -B'_z & B'_y\\ E'_y & B'_z & 0 & -B'_x\\ E'_z & -B'_y & B'_x & 0\end{bmatrix}-$$

$$-4\pi\begin{bmatrix}\rho' & \frac{j'_x}{c} & \frac{j'_y}{c} & \frac{j'}{c}_z\end{bmatrix}=$$

$$=\left[\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ +V/c & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right]\times$$

$$\times\left[\begin{bmatrix}1 & 0 & 0 & 0\\ -V/c & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{bmatrix}\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right]-$$

$$-4\pi\begin{bmatrix}\rho & \frac{j_x}{c} & \frac{j_y}{c} & \frac{j_z}{c}\end{bmatrix}\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=$$

$$=\left[\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{bmatrix}-\right.$$

$$\left.-4\pi\begin{bmatrix}\rho & \frac{1}{c}j_x & \frac{1}{c}j_y & \frac{1}{c}j_z\end{bmatrix}\right]\times\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=$$

$$=0\times\begin{bmatrix}1 & -V/c & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=0$$

So the transformed vectors and matrices fit together just like the untransformed ones did: as tight as Lego blocks. There can be no doubt that the vector / matrix equations retain their form under Galilean transformation. This suggests that that under Galilean transformation, the two Maxwell curl equations, like the square line element, are form-invariant, although not number-invariant. Indeed, all four Maxwell equations are form-invariant under Galilean transformation. What changes status are derived constructs like $\Phi^2 - A^2$, or $E^2 - c^2B^2$, which, like the line element $s^2 = c^2t^2 - x^2 - y^2 - z^2$, are number-invariant under Lorentz transformation, but only form-invariant under Galilean transformations. The things remaining number-invariant are parameters like e and c. Such parameters are simply numbers, and not the result of some vector inner product, or matrix product / tensor contraction, any of which can change from number-invariant under Lorentz transformation to form-invariant under Galilean transformations.

4. More General Velocity Transformations

As a matter of fact, it is possible to state the matrix relationships of Sect. 5 in a more abstract and general way that accommodates Lorentz transformations $\left[L\right]$, Galilean transformations $\left[G\right]$, and indeed any other arbitrary velocity transformations $\left[A\right]$ that may at some future time be of interest. The most useful relationships go:

1.

$$\begin{bmatrix}ct' & x' & y' & z'\end{bmatrix}^{\mathrm{tr}}=\left[A\right]\begin{bmatrix}ct & x & y & z\end{bmatrix}^{\mathrm{tr}},$$

$$\begin{bmatrix}ct' & -x' & -y' & -z'\end{bmatrix}=\begin{bmatrix}ct & -x & -y & -z\end{bmatrix}\left[\left[A\right]^{-1}\right]^{\mathrm{tr}}$$

2.

$$\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t'} & -\frac{\partial}{\partial x'} & -\frac{\partial}{\partial y'} & -\frac{\partial}{\partial z'}\end{bmatrix}^{\mathrm{tr}}=\left[A\right]\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t} & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial z}\end{bmatrix}^{\mathrm{tr}}$$

$$\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t'} & \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'}\end{bmatrix}=\begin{bmatrix}\frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\left[A\right]^{-1}$$

3.

$$\begin{bmatrix}\rho' & j'_x & j'_y & j'_z\end{bmatrix}^{\mathrm{tr}}=\left[A\right]\begin{bmatrix}\rho & j_x & j_y & j_z\end{bmatrix}^{\mathrm{tr}}$$

$$\begin{bmatrix}\rho' & j'_x & j'_y & j'_z\end{bmatrix}=\begin{bmatrix}\rho & j_x & j_y & j_z\end{bmatrix}\left[A\right]^{\mathrm{tr}}$$

4.

$$\begin{bmatrix}0 & -E'_x & -E'_y & -E'_z\\ E'_x & 0 & -B'_z & B'_y\\ E'_y & B'_z & 0 & -B'_x\\ E'_z & -B'_y & B'_x & 0\end{bmatrix}=\left[A\right]\begin{bmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{bmatrix}\left[A\right]^{\mathrm{tr}}$$

$$\begin{bmatrix}0 & -B'_x & -B'_y & -B'_z\\ B'_x & 0 & E'_z & -E'_y\\ B'_y & -E'_z & 0 & E'_x\\ B'_z & E'_y & -E'_x & 0\end{bmatrix}=\left[A\right]\begin{bmatrix}0 & -B_x & -B_y & -B_z\\ B_x & 0 & E_z & -E_y\\ B_y & -E_z & 0 & E_x\\ B_z & E_y & -E_x & 0\end{bmatrix}\left[A\right]^{\mathrm{tr}}$$

The only requirement here is that the transformation matrix $\left[A\right]$ be invertible; *i.e.* be not singular; *i.e.* not have its determinant be equal to zero.

A further trivial extension also accommodates simple rotations; see Appendix 1.

5. Conclusions

This author hopes that readers are now thoroughly disabused of the 20th century belief that Maxwell's equations imply that Galilean transformations are obsolete. The key to recognizing this fact is to focus on the distinction between *form*-invariance and *number*-invariance. The Lorentz transformations that replace Galilean transformations in Einstein's SRT leave lots of things form-invariant, but not number-invariant. Galilean transformations just leave a few *more* things form-invariant, but not number-invariant. That does not mean that form-invariance is at all spoiled. Only those few things that were number-invariant in the Lorentz case are affected at all, and they only to the extent of losing their number-invariance, and falling back to form-invariance.

The truth apparently is this: Maxwell's equations are simply *indifferent* about coordinate transformations. Expressed in tensor

/ matrix form, they retain that same form under any physically plausible kind of coordinate transformation. Only the specific numbers in the matrices change with the kind of coordinate transformation done. So Galilean coordinate transformation does not in fact spoil the form of Maxwell's equations, as is commonly believed.

Indeed, it has been brought to my attention that this truth is already known among the mathematical cognoscenti; see [6]. So, what should the broader physics community now do with the truth? It seems obvious that we should, first of all, stop repeating the erroneous idea that Galilean coordinate transformations are incompatible with Maxwell's equations. Perpetuating a revealed error is corrosive to the whole idea of science, so let us stop it! Moreover, I think we should not just let such errors quietly die (which they don't); we should call them out into the open. Only then will future generations not repeat the same errors. Science is supposed to be a self-correcting enterprise; let it actually be so.

And we can use this truth as a 'teaching moment'. It suggests a more generally useful exploration technique. One can call it 'active anachronism', meaning the deliberate use of a more recently adopted mathematical technique on a problem that was much earlier pronounced 'completely settled'. This technique should be very useful for future investigations.

Let me conclude by noting that this particular issue about Galilean invariance of Maxwell's equations is one of several, which when taken together, open the door to study of alternatives to Einstein's SRT. This author's particular interest is in an alternative [7] that can be founded in the computable behavior of Maxwell's coupled differential equations for electric and magnetic fields, rather than in Einstein's Second Postulate, which by comparison seems rather *ad hoc*. This new foundation leads to an extended version of SRT that is similar to Einstein's SRT for problems where Einstein's SRT has traditionally been applied and tested, but also somewhat expanded in scope, so that it performs in areas where SRT has previously been thought to offer only an incomplete description of the physics – such as inside atoms, where quantization emerges. In [7] I did not have the name for it that I like to use today: 'Maxwell Relativity'. Maxwell Relativity is wider in scope that Einstein's SRT, and in fact includes Galilean Relativity, as well as other alternatives yet to be named.

Acknowledgments

I am thankful for many chances to discuss this research, and argue about it, with authors and commentators to Galilean Electrodynamics, where I have written much, and with members of the Natural Philosophy Alliance, where I have presented often. My special thanks go to Thomas E. Phipps, Jr., who has been a stalwart in both of those venues.

Appendix 1. Simple Rotations

A general coordinate transformation can include not only a velocity change, but also a simple rotation, which leaves the time-like coordinate alone. The transformation matrix for simple rotation is unitary $\left[U\right]$, meaning its transpose is the same thing as its inverse. Only one of those operations, and it doesn't matter which one, need be used. The reason for this is the change in coordinate-frame 'handedness' in changing from $+x, +y, +z$ to $-x, -y, -z$. The most useful transformations go:

1.
$$\left[ct' \quad x' \quad y' \quad z'\right]^{\text{tr}} = \left[U\right]\left[ct \quad x \quad y \quad z\right]^{\text{tr}} ,$$

$$\left[ct' \quad -x' \quad -y' \quad -z'\right] = \left[ct \quad -x \quad -y \quad -z\right]\left[U\right]^{\text{tr}} .$$

2.
$$\left[\frac{1}{c}\frac{\partial}{\partial t'} \quad \frac{\partial}{\partial x'} \quad \frac{\partial}{\partial y'} \quad \frac{\partial}{\partial z'}\right] = \left[\frac{1}{c}\frac{\partial}{\partial t} \quad \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}\right]\left[U\right]^{\text{tr}} ,$$

$$\left[\frac{1}{c}\frac{\partial}{\partial t'} \quad -\frac{\partial}{\partial x'} \quad -\frac{\partial}{\partial y'} \quad -\frac{\partial}{\partial z'}\right]^{\text{tr}} = \left[U\right]\left[\frac{1}{c}\frac{\partial}{\partial t'} \quad -\frac{\partial}{\partial x'} \quad -\frac{\partial}{\partial y'} \quad -\frac{\partial}{\partial z'}\right]^{\text{tr}} .$$

3.
$$\left[\rho' \quad \frac{j'_x}{c} \quad \frac{j'_y}{c} \quad \frac{j'_z}{c}\right]^{\text{tr}} = \left[U\right]\left[\rho \quad \frac{j_x}{c} \quad \frac{j_y}{c} \quad \frac{j_z}{c}\right]^{\text{tr}} ,$$

$$\left[\rho' \quad \frac{j'_x}{c} \quad \frac{j'_y}{c} \quad \frac{j'_z}{c}\right] = \left[\rho \quad \frac{j_x}{c} \quad \frac{j_y}{c} \quad \frac{j_z}{c}\right]\left[U\right]^{\text{tr}} .$$

4.
$$\begin{bmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{bmatrix} = \left[U\right] \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \left[U\right]^{\text{tr}} ,$$

$$\begin{bmatrix} 0 & -B'_x & -B'_y & -B'_z \\ B'_x & 0 & E'_z & -E'_y \\ B'_y & -E'_z & 0 & E'_x \\ B'_z & E'_y & -E'_x & 0 \end{bmatrix} = \left[U\right] \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix} \left[U\right]^{\text{tr}} .$$

The most general arbitrary coordinate transformation can be written as a product, either $\left[P\right] = \left[U\right]\left[A\right]$, or $\left[P\right] = \left[A'\right]\left[U\right]$, as the user chooses, with $\left[A'\right] = \left[U\right]\left[A\right]\left[U\right]^{\text{tr}}$.

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