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LXVII. *On a new Determination of the Lengths of Waves of Light, and on a Method of determining, by Optics, the Translatory Motion of the Solar System.* By Å. J. Angström*.

[With a Plate.]

I.

In the Note on Fraunhofer's lines which I had the honour of communicating to the Royal Academy in October 1861, I spoke of my intention of revising the lengths of luminous waves, as determined by Fraunhofer†, and of extending these determinations to all the remarkable lines of the spectrum, in order with their help to obtain the wave-lengths for the metal-spectra.

The weather last summer was, on the whole, scarcely favourable to such experiments on the solar spectrum, nor are these experiments by any means complete. Nevertheless, since my measurements of the principal lines of Fraunhofer are sufficiently numerous and self-accordant to secure my results from any essential change, I have deemed it of some interest to examine whether, and to what extent, these new determinations agree with those obtained by Fraunhofer himself—the more so because no new measurements on the wave-lengths of light have, to my knowledge, been made since Fraunhofer closed his wonderful investigations.

I employed in my experiments an optical theodolite constructed by Pistor and Martins in Berlin, and a glass grating made by the optician Nobert in Barth. The theodolite was provided with two telescopes, the second of which served as a sight-indicator (*Sehzeichen*). In reading off, two microscopes were used, and one division of the micrometer corresponded to an angle of $2''{\cdot}1$.

The eyepiece is also provided with a micrometer arrangement: the screw-head is divided into 100 parts; and when the telescope

* From Poggendorff's *Annalen*, vol. cxxiii. p. 489; to which journal the paper was communicated by the Author after its publication in the *Oefversigt af K. Vet. Akad. Förh.* 1863, No. 2.

† Poggendorff's *Annalen*, vol. cxvii. p. 290.

is adjusted on an infinitely distant object, every scale-division corresponds to $1''\cdot308$.

The glass grating prepared by Nobert is particularly well constructed. In a space 9·0155 Par. lines broad, there are 4501 lines drawn by a diamond. Errors of division, as tested by Nobert with a microscope which magnified 800 times, lie below 0·00002 of a Par. line.

The breadth, as given by Nobert, was obtained by comparison with a standard prepared by the mechanician Baumann of Berlin, and which was a copy of the one made by the same artist for Bessel.

As a proof of the excellence of this glass grating, I may state that Fraunhofer's lines can be seen therewith in the third and fourth spectrum, and that in distinctness and richness of detail these lines far exceed those which are obtained by the refraction of light through a flint-glass prism.

During the observations the grating was always placed perpendicularly to the incident rays. This was accomplished, *first*, by always giving to the unscratched side of the grating a position such that the image of the heliostat-aperture reflected by it coincided with the aperture itself; *secondly*, by adjusting on the heliostat-aperture the moveable telescope used in the observations; and *thirdly*, by fixing the axis of the second telescope so as to coincide with the prolongation of the optic axis of the first.

The scratched side of the glass grating was always turned from the incident light and towards the moveable telescope, being placed in the middle over the rotation-axis of the instrument.

The observations were calculated according to the known formula

$$e \sin \Theta = m\lambda,$$

where e, or the distance between two scratches on the grating, had, according to the above remark, the value

$$e = 0\cdot000166954 \text{ of a Par. inch},$$

λ denotes the required wave-length, Θ the observed angle, and m the order of the spectrum.

As the values of λ thus obtained have reference to air, they must be dependent upon its temperature and barometric pressure; I have consequently always noted these two elements, although under ordinary circumstances their influence on the measurements was found to be inappreciable. The changes in the temperature of the grating itself exercise a somewhat more important action; nevertheless since, at the time the observations were made (September and commencement of October), the temperature of the room only oscillated between 13° and 18° C., I have likewise omitted this correction.

That no appreciable errors can have thereby arisen in the mean values thus obtained—values which may be regarded as true for 15° C. and the mean barometric pressure—is readily seen on calculating the magnitudes of these corrections.

Assuming the refraction-coefficient of air to be

$$n = 1{\cdot}000294,$$

$\frac{n-1}{d}$ to be a constant magnitude, independent of temperature and pressure, and the value of e, moreover, to hold for 15° C., w . obtain the following corrected value:—

$$\log\lambda = \log\frac{e\sin\Theta}{m} - 0{\cdot}36\,(t_e{}^\circ - 15^\circ) + 0{\cdot}09(t_n{}^\circ - 15^\circ) - 0{\cdot}04(\mathrm{H} - 0^{\mathrm{m}}{\cdot}76),$$

whence we conclude that the correction for $\log\lambda$ amounts to

$$+0{\cdot}45\,(t^\circ - 15^\circ) - 0{\cdot}14\,(\mathrm{H} - 0^{\mathrm{m}}{\cdot}76),$$

expressed in units of the fifth decimal place.

Accordingly a change of 2 degrees in temperature produces a change of 2″ in the value of the angle Θ, if Θ be assumed equal to 25°; this error is comparable with the error of adjustment itself. For smaller values of Θ the error will of course be smaller.

The angle Θ is also subject to a correction dependent upon the absolute motion of the instrument in the direction of the path of the incident ray; this correction, however, is almost inappreciable for the observations upon which the numerical values in the following Table are founded.

The wave-lengths are, like those of Fraunhofer, expressed in units whose magnitude is equal to 0·00000001 of a Par. inch.

Table I.—Wave-lengths, in $\frac{1}{100{,}000{,}000}$ths of a Paris inch.

B.	Spectrum.	C.	Spectrum.	D.	Spectrum.	E.	Spectrum.	*b.*	Spectrum.	F.	Spectrum.	G.	Spectrum.	H.	Spectrum.	H_1.	Spectrum.
2539·91	1	2426·50	1	2178·69	3	1948·25	1	1916·51	1	1797·38	1	1592·32	2	1467·19	1	1454·88	4
2539·54	2	2426·28	2	2178·53	3	1948·21	3	1916·64	3	1797·37	3	1592·53	1	1467·58	4	1453·39	3
2539·76	3	2426·23	2	2178·62	2	1948·24	2	1916·46	2	1797·21	3	1592·22	2	1467·32	3	1453·74	2
		2426·33	3	2178·64	1	1948·20	1	1916·53	4	1797·27	2	1592·16	2	1466·66	2	1453·89	1
		2426·25	1	2178·57	4	1948·25	3	1916·56	1	1797·05	1	1592·50	2	1467·12	1		
		2426·27	2	2178·61	4	1948·24	3	1916·49	4	1797·20	3	1592·32	2	1467·34	4		
				2178·56	2	1948·23	3	1916·43	4	1797·11	2	......		1466·98	3		
				2178·48	4	1948·32	4	1916·47	4	1797·55	4						
2539·73		2426·29		2178·59		1948·24		1916·50		1797·27		1592·34		1467·18		1453·98	

The difference of the wave-lengths corresponding to the two D lines, as measured in the third and in the fourth spectrum, amounts to 2·226,—that between the wave-lengths corresponding to the two E lines being only 0·395, as measured in the third spectrum.

Fraunhofer has given two different series of values for the wave-lengths of light. The first series was obtained by measurements with wire gratings, and it is upon this that Cauchy founded his calculations in the *Mémoire sur la Dispersion*. It contains the following numerical values (β) :—

B.	C.	D.	E.	F.	G.	H.
2541,	2425,	2175,	1943,	1789,	1585,	1451.

Comparing these values with the corresponding ones in the foregoing Table, which I will call the series (α), the following differences ($\alpha-\beta$) are obtained :—

−1·3, +1·3, +3·6, +5·2, +8·3, +7·4, +16·2.

The differences increase, as will be seen, towards the violet end of the spectrum, and are there very considerable. This arises from the difficulty, when using gratings so coarse as those employed by Fraunhofer, of accurately distinguishing the dark lines at the violet end of the spectrum.

The best of all the gratings employed by Fraunhofer is, without doubt, that which he denoted as No. 4, and with which he observed the line E even in the thirteenth spectrum. This grating gives, in general, values which agree better with my own. For the lines C, D, and E the agreement is nearly perfect. The grating in question gave, in fact, the values

B.	C.	D.	E.	F.	G.	H.
2542,	2426,	2178,	1947,	1794,	1586,	1457.

I conclude from this that the disagreement between the series (α) and (β) must arise principally from errors of observation, which, with the wire gratings used by Fraunhofer, were unavoidable.

The other series of values of wave-lengths given by Fraunhofer is of a somewhat later date. It will be found in Gilbert's *Annalen der Physik*, vol. lxxiv., as well as in Herschel's 'Optics,' Schwerd's *Beugungs-Erscheinungen*, and other works. This series, on account of its exactitude, appears to have been held by Fraunhofer in greater esteem than the older ones.

It contains the following values (γ) :—

C.	D.	E.	F.	G.	H.
2422,	2175,	1945,	1794,	1587,	1464;

and gives, when compared with the series (α), the differences ($\alpha-\gamma$): +4·3, +3·6, +3·2, +3·3, +5·4, +3·1.

The values of the wave-lengths contained in the series (γ) depend on measurements of the first interference-spectrum of a *glass grating* which was considerably finer than the one I employed. According to Fraunhofer's statement, in fact,

$$e = 0{\cdot}0001223 \text{ of a Par. inch.}$$

Since, however, the number of marks in this grating of Fraunhofer's amounted only to 3601, the breadth reduces itself to

$$5{\cdot}2833 \text{ Par. lines;}$$

and consequently it must have been considerably less luminous than that of Nobert. In another respect, too, Fraunhofer's grating, although an excellent one, appears to me to have been inferior to that of Nobert; for the line B could not be measured even in the first spectrum, and the lines from C to G were not visible in any of the spectra beyond the second.

Nevertheless, since almost all the differences ($\alpha-\gamma$) have the same value, a constant error appears to be indicated, either in my measurements or in those of Fraunhofer. That an error of this character cannot have affected the value of Θ in *my measurements*, is evident from the fact that the value of this angle was obtained from mutually agreeing observations on four different spectra. The introduction of such an error into Fraunhofer's measurements is equally inadmissible, since on calculating the wave-lengths of the lines from C to G (which Fraunhofer also observed in the second interference-spectrum, though he did not introduce them into his calculation), the following mutually according values are obtained from the two spectra:—

	C.	D.	E.	F.	G.
First spectrum .	2422·00,	2174·58,	1944·81,	1793·98,	1586·89;
Second spectrum.	2421·54,	2174·36,	1944·63,	1793·92,	1588·07.

It is only for the line G that the difference is somewhat greater.

The reason of the differences ($\alpha-\gamma$), therefore, must arise from an erroneous determination of the value of e; which latter may have been caused either by a wrong enumeration of the lines in one of the two gratings, or by an incorrect estimation of their breadth. In order to make the two values of the wave-lengths for the line D agree, in the series (α) and (γ), by altering the value of e, the breadth of Nobert's grating would have to be *diminished* by

$$0{\cdot}0123 \text{ of a Par. line} = 0{\cdot}001025 \text{ of a Par. inch,}$$

or the number of lines in the grating *increased* by 6.

The same object would be attained by *increasing* the breadth of Fraunhofer's grating by

0·00061 of a Par. inch,

or by *diminishing* the number of lines by 5.

That the second decimal is wrong in the above breadth (=9·0155 lines) of Nobert's grating is not probable; far more so is the supposition of an error of about half this magnitude in the estimation of the breadth of Fraunhofer's grating, especially since the microscope, forty years ago, had not reached its present high degree of perfection. Fraunhofer, moreover, was compelled to *strengthen the extreme lines* of his grating, in order to see them more distinctly when measuring, a circumstance which may possibly have affected the positions of these two lines.

Besides the fact that my measurements agree with the results which Fraunhofer obtained by means of the grating No. 4, there is another reason in favour of the assumption that the differences $(\alpha-\gamma)$ arise from an incorrect value of e in Fraunhofer's glass grating. For the above-cited memoir of Fraunhofer's contains measurements made with another glass grating for which e had the considerably greater value of

0·0005919 of a Paris inch.

Fraunhofer made no use of these measurements, probably because this grating proved to be far less perfect, the spectra on one side of the axis being twice as intense as those on the other. On calculating these measurements, however, we obtain the following values corresponding to the lines from D to G:—

D.	E.	F.	G.	Spectrum.
2177·25	1947·21	....	....	5
2177·48	1947·18	1796·10	....	4
2177·64	1947·23	1796·09	1590·90	3
2176·80	1946·63	1795·99	1591·07	2
2177·55	1947·25	1796·39	1590·16	1
2177·34	1947·10	1796·14	1590·71	

These values, compared with the series (α), indicate a constant difference; here, however, the differences amount only to

1·25, 1·14, 1·13, 1·63,

or to about one-third of those last given.

Now, since this last grating was nearly five times as coarse as the former, and probably also broader, it must have been easier to determine accurately its corresponding e. This circumstance tends to increase the probability of the existence of an error in the value of *e* corresponding to the finer grating.

The values of the wave-lengths obtained by means of Nobert's grating, therefore, appear to me to merit a greater confidence than that which Fraunhofer's can justly claim.

II.

As already stated at the commencement of this paper, I have not limited my measurements to the principal lines of Fraunhofer. I have measured, with the circle, the angle Θ for all the stronger lines at a distance from each other of from $10'$ to $20'$, and determined with the eyepiece-micrometer the positions of the remaining intermediate lines. The measurements, moreover, were repeated in the second, third, and fourth spectra, in order to verify their exactitude.

The following Table contains some of these results, those wave-lengths alone being given which correspond to the strongest and most prominent lines of the solar spectrum. Most of these lines belong to iron or to lime, and have consequently a double interest, since they present themselves also in the gas-spectra of these substances. In order to give the reader a visible image of the position and breadth of these lines in the solar spectrum, I have added a figure (Plate III. fig. 1), which correctly shows their respective positions as presented by a prism of sulphide of carbon having an angle of 60°. An arc of $2'$ corresponds in the figure to a length of one millimetre.

Table II.—Wave-lengths, in hundred millionths $(=\frac{1}{10^8})$ of a Paris inch.

Line.	Wave-length.	Spectra in which corresponding lines are observed.	Remarks.
A	2812		
B	2539·7		
C	2426·29		
α	2312·2	Earth's atmosphere ...	Strong line.
	2287·3		
	2279·6		
	2276·8		
	2269·4	Iron and calcium	Group of strong lines.
	2267·7		
	2262·1		
	2255·1		
D	2179·70		
	2177·48		
1	2076·1	Iron.	
2	2071·3	Iron	Two groups of lines.
3	2069·7		
4	2068·3		
5	2065·4		

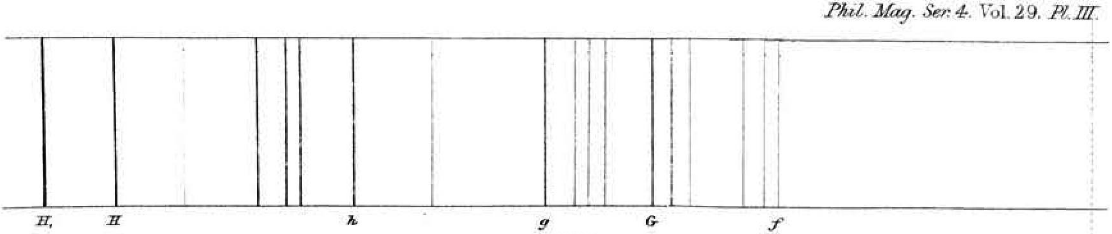

Fig. 1.

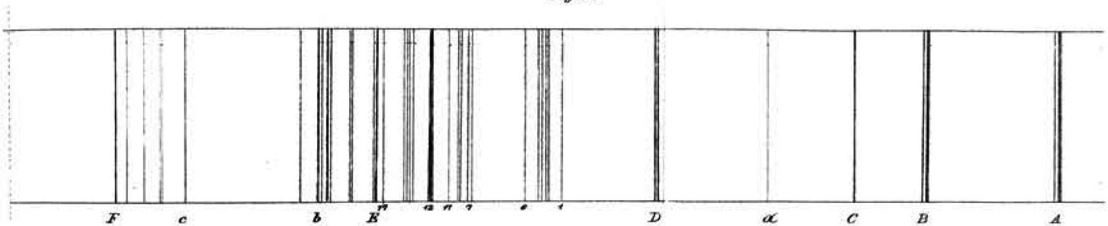

Fig. 2.

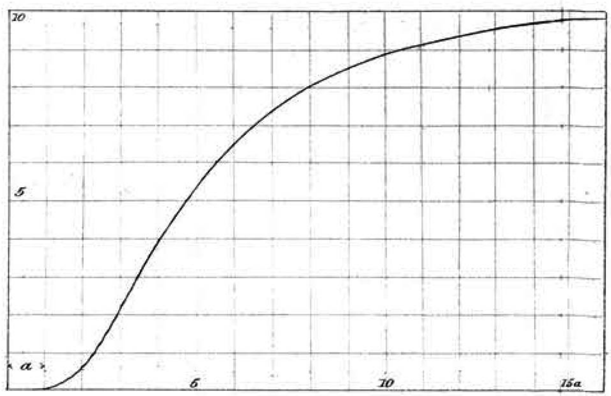

Fig. 4.

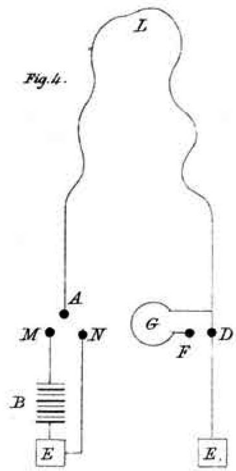

Fig. 3.

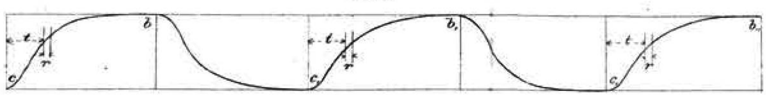

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TABLE II. (*continued*).

Line.	Wave-length.	Spectra in which corresponding lines are observed.	Remarks.
6	2060·1	Iron.	
7	2016·9	„	
8 {	2013·6	„	
	2013·1	„	
9	2007·3	„	
10	2005·3	„	
11 {	1998·4	„	
	1997·9	„	
12	1985·8	„	Strong line.
	1985·3 }		
	1984·2 }	„	Weak.
	1983·5 }		
13	1974·2	„	Double line, like E.
14	1969·6	„	
15	1968·1	„	
16	1965·3	„	
17	1953·2	„	
E {	1948·44 }	Iron and calcium.	
	1948·04 }		
	1946·8	„ „	Double line, like E.
	1934·6	Iron.	
	1936·4	„	
	1919·6	„	Double line, like E.
b	1916·50	Magnesium.	
$b_{,}$	1912·39	„	
$b_{,,}$ {	1911·10	Iron and magnesium.	
	1910·49	Iron.	
	1903·4	„	
c	1832·70	„	
	1819·1 }		
	1818·4 }	„	Iron when weakly incandescent gave but one of these lines; when strongly incandescent, however, a third was visible.
	1808·3	„	Double.
	1801·1	„	
F	1797·27	Hydrogen.	
f	1632·2	Iron.	
	1628·5	„	
	1620·4	„	
	1604·3	Hydrogen.	
	1598·8	Iron.	
G	1592·34	„	
	1579·1	„	
	1574·7	„	
	1571·2	„	
g	1562·4	Calcium	Double line.
	1532·0	Iron	Double line; several weak lines were also visible between *g* and *h*.
h	1515·9	Unknown.	Very strong line.
	1505·3	Iron	Strong line.
	1502·0	„	„
	1495·2	„	„
	1480·4	Unknown	„
H	1467·2	Calcium.	
H	1454·0	„	

III.

In a lecture given on October 6, 1860, to the Royal Scientific Society of Upsala, I explained a method of determining the motion of the solar system by observations on the interference-bands of a glass grating. I then showed that if we assume the propagation of the undiffracted rays, passing through the openings of the grating, to be uninfluenced by the motion of the instrument, the same must be true of the formation of the interference-bands on both sides; consequently, also, that when a telescope is used in the observations the customary aberration must ensue, and be proportional to the ratio between the motion of the telescope, in a direction perpendicular to its axis, and the velocity of light along this axis.

Hence, the velocity of light being taken as the unit, if h be the velocity of the instrument in the direction of the incident light, then for an angle Θ, under which, *e. g.*, the D line in an interference-spectrum is observed, the velocity of the telescope perpendicular to this direction will be

$$h \sin \Theta,$$

which accordingly must be the expression for the aberration.

If the angle Θ were observed for two positions of the instrument in which the velocities in the path of the incident rays were h and h', we should then have

$$\Delta\Theta = (h - h') \sin \Theta, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, since 2Θ is the angle immediately given by observation,

$$\Delta \, . \, 2\Theta = 2(h - h') \sin \Theta.$$

Putting $h(= -h')$ equal to the velocity of the earth in its orbit, this equation gives

$$\Delta \, . \, 2\Theta = 81''{\cdot}6 \sin \Theta;$$

and since, for the double line D in the fourth spectrum,

$$2\Theta = 62^\circ \; 55' \; 44''{\cdot}2,$$

we deduce

$$\Delta \, . \, 2\Delta = 42''{\cdot}6,$$

a magnitude capable of being readily observed.

Two questions have here to be answered by observation. The one has reference to the actual existence of the phenomenon, and may be most readily answered by applying the method to the known orbital motion of the earth; the other has reference to the employment of the method, when proved to be accurate, to the determination of the translatory motion of the solar system.

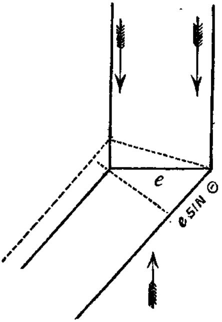
The experiments hitherto made cannot in any respect be considered as quite decisive. Last midsummer the weather was unfavourable to my observations, and at the end of October the latter were not sufficiently numerous to furnish an answer even to the first of the above questions.

I should not in fact have alluded to the subject had not M. Babinet, in the Academy of Sciences, proposed a method of determining the translatory motion of the solar system identical with the one which, two years ago, I submitted to the Royal Scientific Society of Upsala.

A small difference exists, however, in our calculations. I had assumed the motion of the grating to have no influence on the angle Θ, whereas Babinet introduces, on this account, the correction

$$h(1-\cos\Theta)\tan\Theta.$$

The truth of this formula may in fact be readily established by help of the adjoining figure, in which $e\sin\Theta$ denotes the distance traversed by light during the time that the grating describes the distance $-he\sin\Theta$ in a direction contrary to that of the incident rays. The difference of path for the two interfering waves will consequently, through the motion of the grating, be diminished by



$$he(1-\cos\Theta)\sin\Theta,$$

a magnitude which, when equated to*

$$-e\cos\Theta\, d\Theta,$$

gives

$$d\Theta=-h(1-\cos\Theta)\tan\Theta.$$

The value of $d\Theta$ will, of course, be positive when the instrument moves in the same direction as the light.

The expression thus obtained, added to the one in the formula (1), gives for the total variation of the angle Θ the value

$$\Delta\Theta=(h-h')\tan\Theta;$$

and if, moreover, $h=-h'=20''\cdot4$, and

$$2\Theta=62^\circ\, 55'\, 41'',$$

then will

$$\Delta 2\Theta=49''\cdot8.$$

Hence in the special case under consideration, the variation of the angle 2Θ is increased by $7''\cdot2$ in consequence of the motion of the grating.

The observations on which the numerical values of Table I.

are based were all (with a few exceptions) made at or near mid-day. On this account I thought that the corrections due to the motion of the instrument might be neglected in calculating the results, since in the final mean such corrections must, for the most part, disappear.

In proof of the accuracy of the theory here established, I will give a few of the observations made last year at the commencement of October. They have reference to the double line D in the fourth interference spectrum. The light was always incident from *south* to *north*. The second telescope and the grating were readjusted every day.

TABLE III.

Time of observation, in 1862.		$2\Theta_4=\phi$.	Remarks.
	h	° ′ ″	
Oct. 5	11·4 A.M.	62 55 38	Mean of three observations.
	3·58 P.M.	62 55 53	
	5 P.M.	62 56 7	Mean of six observations.
Oct. 9, 10	3·74 P.M.	62 56 0	Mean of six observations.
Oct. 11	9·5 A.M.	62 55 51	Mean of two observations.
	1 P.M.	62 55 58	
	3·75 P.M.	62 56 7	

From the mean value of the wave-lengths corresponding to the line D given in Table I. we deduce

$$2\Theta_4=62^\circ\ 55'\ 41''\cdot2=\phi_0;$$

and since this value must be very nearly free from any error due to the motion of the instrument, it ought to agree with that furnished by the observations in Table III., after applying to the latter the corrections due to the motion of the instrument.

If X be the velocity of the solar system in a direction determined by the coordinates of the equator,

$$D=34^\circ\cdot5 \text{ and } A=259^\circ\cdot8,$$

the magnitude of the motion of the instrument *from north to south,* due to the motion of the solar system, will be

$$X\cos b=X[\cos D\sin P\cos(A-*)-\sin D\cos P],$$

where P denotes the altitude of the pole, and $*$ the sidereal time of the observation.

For Upsala, therefore, we shall have the formula

$$X[0\cdot713\cos(259^\circ\cdot8-*)-0\cdot284].$$

The velocity of the instrument, in the above direction, due to the earth's *annual motion* is equal to

$$h \cos b_1 = h \{ \cos D_1 \sin P \sin[\odot - *] - \sin D_1 \cos P \},$$

where

$$-\sin D_1 = \sin 23^\circ\ 38' \cos \odot.$$

In this formula $\odot$ denotes the right ascension of the sun, P and $*$ the same magnitudes as before, and $h = 20''\cdot4$ the velocity of the earth expressed by the angle it subtends at the centre of a circle whose radius is the velocity of light. The total correction of the angle ϕ, therefore, will be

$$\Delta\phi = 24''\cdot9 [\cos b_1 + n \cos b],$$

since

$$X = nh \text{ and } 40''\cdot8 \tan \Theta = 24''\cdot9.$$

If by means of this last formula, and under different assumptions for the value of n, we calculate the correction for each angle ϕ in Table III., and afterwards add these corrections to their respective angles, the resulting values of $\phi + \Delta\phi$, subtracted from the assumed true value of $2\Theta_4$, that is to say, from

$$\phi_0 = 62^\circ\ 55'\ 41'',$$

will give the following:

TABLE IV.

$\phi_0 - \phi$.	$\phi_0 - (\phi + \Delta\phi)$.			
	$n=0$.	$n=\frac{1}{3}$.	$n=\frac{1}{2}$.	$n=1$.
$+3''$	$+3''$	$+4''$	$+4''$	$+7''$
-11	$+9$	$+5$	$+3$	-2
-26	-3	-6	-7	-13
-19	$+2$	-1	-3	-9
-10	-17	-14	-12	-8
-18	-7	-10	-10	-12
-26	-5	-10	-10	-16

The sums of the squares of the differences are respectively 2267, 462, 419, 427, 719.

So far as we can conclude from the above observations, *the influence of the earth's annual motion* appears to be verified; that of the motion of the solar system is less perceptible. Nevertheless it is obvious that if we were to assume that motion to be zero, or to be equal to that of the earth in its orbit, the agreement between the observations would be worse than under the assumption that the magnitude of the motion in question is somewhat more than one-third of that of the earth. Between this result, and what we already know of the motion of the solar system through astronomy, there is no great divergence.

I hope during the present year, however, to be able to continue my spectrum-experiments, and to have a better opportunity of determining, numerically, the magnitude of the motion of the solar system. In the present paper my object has merely been to show the possibility of solving, optically, this interesting problem in physical astronomy.

LXVIII. *On the Intersections of a Pencil of four Lines by a Pencil of two Lines.* *By* Professor Cayley, *F.R.S.**

PLÜCKER has considered ("Analytisch-geometrische Aphorismen," *Crelle*, vol. xi. (1834) pp. 26–32) the theory of the eight points which are the intersections of a pencil of four lines by any two lines, or say the intersections of a pencil of *four* lines by a pencil of *two* lines: viz., the eight points may be connected two together by twelve new lines; the twelve lines meet two together in forty-two new points; and of these, six lie on a line through the centre of the two-line pencil, twelve lie four together on three lines through the centre of the four-line pencil, and twenty-four lie two together on twelve lines, also through the centre of the four-line pencil.

The first and third of these theorems, viz. (1) that the six points lie on a line through the centre of the two-line pencil, and (3) that the twenty-four points lie two together on twelve lines through the centre of the four-line pencil, belong to the more simple theory of the intersections of a pencil of *three* lines by a pencil of *two* lines; the second theorem, viz. (2) the twelve points lie four together on three lines through the centre of the four-line pencil, is the only one which properly belongs to the theory of the intersections of a pencil of *four* lines by a pencil of *two* lines. The theorem in question (proved analytically by Plücker) may be proved geometrically by means of two fundamental theorems of the geometry of position: these are the theorem of two triangles in perspective, and Pascal's theorem for a line-pair. I proceed to show how this is.

Consider a pencil of two lines meeting a pencil of four lines in the eight points (a, b, c, d), (a', b', c', d'); so that the two lines are $abcd$, $a'b'c'd'$ meeting suppose in Q; and the four lines are aa', bb', cc', dd' meeting suppose in P; then the twelve points are

* Communicated by the Author.