Gravity with a Dynamical Spinning Aether

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Einstein-aether theory is extended by allowing for spinning degrees of freedom of the aether. In addition to the acceleration, shear, expansion, and vorticity of the aether velocity field, a spin rotation describing the dynamics of a classical intrinsic angular momentum of the aether is introduced as a kinematic quantity. The action of Einstein-aether theory is augmented by a term quadratic in the spin rotation and by coupling terms with the vorticity and the acceleration. Besides breaking the Lorentz boost invariance, the theory breaks the invariance under spatial rotations in the direction of the aether velocity. In the weak field limit, there is a linear relationship between the spin rotation, the vorticity, and the acceleration. Linearized wave solutions correspond to the ones of Einstein-aether theory where the speeds of the spin 0 and spin 1 mode are modified. The extension of Einstein-aether theory has a natural formulation in the framework of a teleparallel geometry where the kinematic quantities become torsion fields.

I. INTRODUCTION

Einstein-aether theory is a theory of gravity where a dynamical unit timelike vector field — the aether — is coupled to general relativity [1–3]. The vector field can be regarded as a kind of preferred frame violating local Lorentz invariance of the theory in that the symmetry under local Lorentz boosts is broken while the symmetry under local spatial rotations is retained. In its original formulation, the interpretation of the vector field as a four-velocity in Einstein-aether theory is achieved by a unit norm constraint imposed by a Lagrange multiplier in the action functional thus implying a spontaneous Lorentz boost symmetry breaking. If, however, the vector field in the action is implicitly assumed to be timelike and of unit norm, the approach represents a semi-tetrad formulation and the symmetry under Lorentz boosts is explicitly broken.

The timelike vector field can be seen as a fluid existing everywhere in space-time. In this work, we will consider an extension of Einstein-aether theory that is motivated by the physics of spin-fluids [4, 5]. The description of spin in this paper is purely classical. It is considered an intrinsic degree of freedom that is not quantized. The mathematical formulation of the spinning aether makes it necessary to introduce a spatial triad with respect to which the spin is fixed. In the same way as the aether in Einstein-aether theory is solely given by its four-velocity, the spinning aether will additionally be described by its triad orthonormal to the velocity. Hence, the spinning aether is described by a tetrad.

The invariance of the theory under local spatial rotations will be retained as far as possible. For this reason, we avoid the appearance of derivatives of the triad in the spatial directions in the action functional. Only derivatives of the triad in the direction of the aether velocity will be allowed which means that only the invariance under spatial rotations in this direction will be broken. In this way, isotropy of space is preserved since there is no preferred spatial direction.

In a formulation within Riemannian geometry, the aether is not a pure geometrical object. The geometrical objects of Riemannian geometry — the metric, the Christoffel symbols, and the curvature tensor — are Lorentz invariant; the Lorentz invariance breaking aether is an additional degree of freedom. Nevertheless, a pure geometrical formulation of gravity with an aether is possible using non-Riemannian geometry. A preferred frame defines a distant parallelism in that vectors with constant components with respect to the frame are considered parallel. This motivates the formulation of theories of gravity with preferred frames using teleparallel gravity. If the spatial orientation of the frames are not fixed, the natural geometry corresponds to a partial parallelization of space-time.

This paper is organized as follows. In section II, the extended version of Einstein-aether theory is introduced. The kinematic quantities of the spinning aether are defined and the action functional is chosen. The field equations are derived in section III and it is shown in which way they can be simplified using constraint equations. Section IV examines the weak field limit of the field equations. The aether excitations are explored using analogies with Maxwell's equations. In section V, the extended Einstein-aether theory is formulated as a (semi-)teleparallel theory of gravity. The geometry is defined and a procedure to find action functionals is described.

We use the following conventions: Greek indices $\mu, \nu, \rho, ...$ with the range $0, 1, 2, 3$ denote space-time indices. Latin indices $a, b, c, ...$ with the range $0, 1, 2, 3$ are internal indices. Spatial indices in the range $1, 2, 3$ are denoted by latin

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indices from the middle of the alphabet, $i, j, k, \ldots$. The metric signature is $(+,-,-,-)$. Symmetrization is denoted by round brackets, antisymmetrization by square brackets. The totally antisymmetric pseudotensor is $\varepsilon_{\mu\nu\rho\sigma}$. Units are chosen in which $c = 1$.

II. ACTION FUNCTIONAL

We assume that the preferred frame is given by a tetrad $e_a{}^\mu$ where the four velocity of the aether is $u^\mu = e_0{}^\mu$. The tetrad is assumed to be orthonormal which means that the space-time metric is given by $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$ where $e^a{}_\mu$ is the cobasis defined by $e_a{}^\mu e^a{}_\nu = \delta^\mu_\nu$ and $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowksi metric.

The spatial projection tensor determined by the tetrad is given by

$$h^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu = e_i{}^\mu e^i{}_\nu. \tag{1}$$

The covariant derivative corresponding to the metric $g_{\mu\nu}$ will be denoted by D_μ. Using the covariant derivative, we can compute the space-time components of the Ricci rotation coefficients [6],

$$C_{\mu\nu\rho} = e^a{}_\mu D_\rho e_{a\nu}, \tag{2}$$

which measure the deviation of the tetrad from being inertial.

Given the velocity field, kinematic quantities can be defined which are projections of $C_{\mu\nu\rho}$. The acceleration a^μ is given by

$$a^\mu = D_u u^\mu = u^\rho D_\rho u^\mu = -C^\mu{}_{\nu\rho} u^\nu u^\rho. \tag{3}$$

The expansion θ is defined as

$$\theta = D_\rho u^\rho = C_{\rho\mu\nu} u^\rho h^{\mu\nu}. \tag{4}$$

The shear tensor $\sigma_{\mu\nu}$ being the trace-free part of the deformation tensor is given by

$$\begin{aligned}\sigma_{\mu\nu} &= h^\rho_\mu h^\sigma_\nu D_{(\rho} u_{\sigma)} - \frac{1}{3} h_{\mu\nu}\theta = D_{(\mu} u_{\nu)} - u_{(\mu} a_{\nu)} - \frac{1}{3} h_{\mu\nu}\theta \\ &= C_{\rho\sigma\tau} u^\rho \left(h^\sigma_{(\mu} h^\tau_{\nu)} - \frac{1}{3} h_{\mu\nu} h^{\sigma\tau} \right).\end{aligned} \tag{5}$$

The vorticity tensor $\omega_{\mu\nu}$ is defined as

$$\begin{aligned}\omega_{\mu\nu} &= h^\rho_\mu h^\sigma_\nu D_{[\rho} u_{\sigma]} = \partial_{[\mu} u_{\nu]} - u_{[\mu} a_{\nu]} \\ &= -C_{\rho\sigma\tau} u^\rho h^\sigma_{[\mu} h^\tau_{\nu]}.\end{aligned} \tag{6}$$

The kinematic quantities associated with the four velocity u^μ can be combined in the tensor $D_\mu u_\nu$ which can be irreducibly decomposed according to

$$D_\mu u_\nu = u_\mu a_\nu + \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3} h_{\mu\nu}\theta = C_{\rho\nu\mu} u^\rho. \tag{7}$$

As an additional kinematic quantity, associated with the spatial triad $e_i{}^\mu$, we define the spin rotation $\kappa_{\mu\nu}$ by

$$\kappa_{\mu\nu} = e^i{}_\nu \overset{F}{D}_u e_{i\mu} \tag{8}$$

where $\overset{F}{D}_u$ denotes the Fermi derivative in the direction of u^μ given by

$$\overset{F}{D}_u X^\mu = D_u X^\mu + X^\rho a_\rho u^\mu - X^\rho u_\rho a^\mu \tag{9}$$

for a vector field X^μ. The spin rotation can then be written as

$$\kappa_{\mu\nu} = e^i{}_\nu D_u e_{i\mu} + u_\mu a_\nu = -C_{\sigma\tau\rho} h^\sigma_{[\mu} h^\tau_{\nu]} u^\rho. \tag{10}$$

If the aether spin vector is assumed to be fixed with respect to the triad, the spin rotation measures the deviation (a spatial rotation) of the spin from a Fermi-Walker transported spin in the direction of the aether velocity.

In order to understand the role of the spin rotation $\kappa_{\mu\nu}$ as a kinematic quantity, we note that the tensor $C_{\mu\nu\rho}$ can be decomposed according to

$$C_{\mu\nu\rho} = Q_{\mu\nu\rho} + S_{\mu\nu\rho} \tag{11}$$

into its spatial projection

$$Q_{\mu\nu\rho} = h^{\lambda}_{\mu} h^{\sigma}_{\nu} h^{\tau}_{\rho} C_{\lambda\sigma\tau} = h^{\sigma}_{\nu} h^{\tau}_{\rho} e^{i}{}_{\mu} D_{\tau} e_{i\sigma} \tag{12}$$

and its time-space components $S_{\mu\nu\rho}$. Equation (11) can be solved for $S_{\mu\nu\rho}$ yielding

$$\begin{aligned} S_{\mu\nu\rho} &= u_{\mu} D_{\rho} u_{\nu} - u_{\nu} D_{\rho} u_{\mu} - \kappa_{\mu\nu} u_{\rho} \\ &= 2u_{[\mu} a_{\nu]} u_{\rho} + \frac{2}{3} u_{[\mu} h_{\nu]\rho} \theta + 2u_{[\mu} \sigma_{\nu]\rho} - 2u_{[\mu} \omega_{\nu]\rho} - \kappa_{\mu\nu} u_{\rho}. \end{aligned} \tag{13}$$

The tensor $S_{\mu\nu\rho}$ can be seen as the generalization of the expression $D_{\mu} u_{\nu}$ in Equation (7). Equation (13) corresponds to its irreducible decomposition where the spin rotation $\kappa_{\mu\nu}$ naturally appears as an irreducible part.

The vorticity tensor and the spin rotation can be represented by spatial vectors ω_{μ} and κ_{μ} defined by

$$\omega_{\mu} = \frac{1}{2} \varepsilon_{\mu\nu\rho} \omega^{\nu\rho} \tag{14}$$

and a similar equation for κ_{μ}. Here, the totally antisymmetric spatial pseudotensor $\varepsilon_{\mu\nu\rho}$ is given by $\varepsilon_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} u^{\sigma}$. We will also introduce an antisymmetric acceleration tensor $a_{\mu\nu}$ defined by

$$a_{\mu\nu} = -\varepsilon_{\mu\nu\rho} a^{\rho}. \tag{15}$$

In Einstein-Aether theory, the action functional is quadratic in the expansion, shear, vorticity, and acceleration. We will add to this action a term quadratic in the spin rotation, $\kappa^2 = \kappa_{\mu\nu}\kappa^{\mu\nu}$, and a term that couples the spin rotation and the vorticity, $\omega \cdot \kappa = \omega_{\mu\nu}\kappa^{\mu\nu}$. We will also include a parity violating term that couples the spin rotation and the acceleration, $\kappa \cdot a = \kappa_{\mu\nu} a^{\mu\nu}$. A similar term $\omega \cdot a$ that couples the vorticity and the acceleration is a total derivative (see Equation (33) below). We will thus consider the following action functional:

$$S[e_a{}^{\mu}] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R + \mathcal{L}_e) \tag{16}$$

where R is the Ricci scalar and

$$\mathcal{L}_e = \frac{1}{3} c_{\theta} \theta^2 + c_{\sigma} \sigma^2 + c_{\omega} \omega^2 + 2c_{\omega\kappa} \omega \cdot \kappa + c_{\kappa} \kappa^2 + 2c_{\kappa a} \kappa \cdot a - c_a a^2 \tag{17}$$

with $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}$, $\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}$, and $a^2 = a_{\mu\nu} a^{\mu\nu}$ and where c_{θ}, c_{σ}, c_{ω}, $c_{\omega\kappa}$, c_{κ}, $c_{\kappa a}$, and c_a are dimensionless coupling constants. Further action terms involving matter and other fields can be added to the action (16) which we will not do in this paper. Using the tensor $S_{\mu\nu\rho}$, $\mathcal{L}_e$ can be written in the form

$$\mathcal{L}_e = K^{\kappa\lambda\mu\rho\nu\sigma} S_{\kappa\nu\mu} S_{\lambda\sigma\rho} \tag{18}$$

where the supermetric $K^{\kappa\lambda\mu\rho\nu\sigma}$ depends algebraically on $e_a{}^{\mu}$. Since we implicitly assume that $e_a{}^{\mu}$ is orthonormal, the form of $K^{\kappa\lambda\mu\rho\nu\sigma}$ is not unique. We here choose

$$\begin{aligned} K^{\kappa\lambda\mu\rho\nu\sigma} &= u^{\kappa} u^{\lambda} \Big(c_1 g^{\mu\rho} g^{\nu\sigma} + c_2 h^{\mu\nu} h^{\rho\sigma} + c_3 h^{\mu\sigma} h^{\rho\nu} + c_4 u^{\mu} u^{\rho} g^{\nu\sigma} \Big) \\ &+ c_{\omega\kappa} \left(u^{\kappa} u^{\rho} h^{\sigma[\mu} h^{\nu]\lambda} + u^{\lambda} u^{\mu} h^{\nu[\rho} h^{\sigma]\kappa} \right) - c_{\kappa} u^{\mu} u^{\rho} h^{\kappa\sigma} h^{\lambda\nu} + 2c_{\kappa a} u^{\mu} u^{\rho} \varepsilon^{\nu\sigma[\kappa} u^{\lambda]} \end{aligned} \tag{19}$$

where

$$c_1 = \frac{c_{\sigma} + c_{\omega}}{2}, \tag{20}$$

$$c_2 = \frac{c_{\theta} - c_{\sigma}}{3}, \tag{21}$$

$$c_3 = \frac{c_{\sigma} - c_{\omega}}{2}, \tag{22}$$

$$c_4 = 2c_a - \frac{c_{\sigma} + c_{\omega}}{2}. \tag{23}$$

Due to the presence of the aether velocity field u^μ, the action (16) is not invariant under local Lorentz boosts. Under local spatial rotations of the tetrad field, that is, $e^i{}_\mu \to \Lambda^i{}_j e^j{}_\mu$ where $\Lambda^i{}_j$ is a rotation matrix, the kinematic quantities a_μ, $\omega_{\mu\nu}$, $\sigma_{\mu\nu}$, and θ are invariant since they do not depend on $e^i{}_\mu$. However, the spin rotation transforms according to

$$\kappa_{\mu\nu} \to \kappa_{\mu\nu} - e_{i\mu} e^j{}_\nu \Lambda_k{}^i \partial_u \Lambda^k{}_j. \tag{24}$$

From this equation follows that if terms of the form κ^2, $\omega \cdot \kappa$ or $\kappa \cdot a$ are present in the action (16), there is only invariance under spatial rotations if $\partial_u \Lambda^i{}_j = 0$. This means that the presence of the spin rotation breaks the invariance under spatial rotations in the direction of the aether velocity field.

III. FIELD EQUATIONS

The field equations following from the action (16) are obtained by variation of the action with respect to $e_a{}^\mu$. After contraction with $e_{a\nu}$ and taking the symmetric and antisymmetric part, the field equations read

$$G_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{L}_e - D_\rho \left(J^\rho{}_{(\mu\nu)} - J_{(\mu\nu)}{}^\rho \right) - \left(J^{\rho\sigma}{}_{(\mu} - J_{(\mu}{}^{\sigma\rho} \right) S_{\nu)\rho\sigma} - J^\rho{}_{(\mu}{}^\sigma Q_{|\rho\sigma|\nu)} - \frac{1}{2} N^{\alpha\beta\lambda\rho\sigma\tau}{}_{(\mu\nu)} S_{\alpha\sigma\lambda} S_{\beta\tau\rho}, \tag{25}$$

$$D_\rho J_{[\mu}{}^\rho{}_{\nu]} - \left(J^{\rho\sigma}{}_{[\mu} - J_{[\mu}{}^{\sigma\rho} \right) S_{\nu]\rho\sigma} - J^\rho{}_{[\mu}{}^\sigma Q_{|\rho\sigma|\nu]} + \frac{1}{2} N^{\alpha\beta\lambda\rho\sigma\tau}{}_{[\mu\nu]} S_{\alpha\sigma\lambda} S_{\beta\tau\rho} = 0 \tag{26}$$

where $G_{\mu\nu}$ is the Einstein tensor,

$$J^{\kappa\mu\nu} = K^{\kappa\lambda\mu\rho\nu\sigma} S_{\lambda\sigma\rho}, \tag{27}$$

and

$$N^{\alpha\beta\lambda\rho\sigma\tau}{}_{\mu\nu} = \frac{\delta K^{\alpha\beta\lambda\rho\sigma\tau}}{\delta e_a{}^\mu} e_{a\nu}. \tag{28}$$

The “momentum” $J^{\rho\mu\nu}$ is explicitly given by

$$\begin{aligned} J^{\rho\mu\nu} = \frac{1}{3} c_\theta u^\rho h^{\mu\nu} \theta + c_\sigma u^\rho \sigma^{\mu\nu} + c_\omega u^\rho \omega^{\mu\nu} + c_{\omega\kappa} \left(u^\rho \kappa^{\mu\nu} + u^\mu \omega^{\nu\rho} \right) + c_\kappa u^\mu \kappa^{\nu\rho} \\ + c_{\kappa a} u^\mu \left(a^{\nu\rho} - 2 u^\rho \kappa^\nu \right) + 2 c_a u^\rho u^\mu a^\nu. \end{aligned} \tag{29}$$

Equations (25) represent the Einstein field equations with the aether stress-energy tensor on the right hand side. Equations (26) are the field equations of the aether field. In the case $c_\kappa = c_{\omega\kappa} = c_{\kappa a} = 0$, Equations (25) and (26) are equivalent with the field equations of Einstein-aether theory.

If we are working at the level of the kinematic quantities — and not at the level of the tetrad — the field equations have to be supplemented by the constraint and evolution equations for the kinematic quantities which follow from the Ricci identity

$$D_\rho D_\nu e^a{}_\mu - D_\nu D_\rho e^a{}_\mu = R^\sigma{}_{\mu\nu\rho} e^a{}_\sigma \tag{30}$$

where $R_{\sigma\rho\mu\nu}$ is the Riemann curvature tensor (see also [7]). This equation can be solved for $R_{\sigma\rho\mu\nu}$ in terms of the tensor $C_{\mu\nu\rho}$ yielding

$$R_{\sigma\mu\nu\rho} = D_\rho C_{\sigma\mu\nu} - D_\nu C_{\sigma\mu\rho} + C_{\sigma\lambda\rho} C^\lambda{}_{\mu\nu} - C_{\sigma\lambda\nu} C^\lambda{}_{\mu\rho}. \tag{31}$$

The first Bianchi identity then leads to the following 16 constraint equations for $C_{\mu\nu\rho}$.

$$\mathcal{C}^\tau{}_\sigma = \varepsilon^{\mu\nu\rho\tau} \left(D_\rho C_{\sigma\mu\nu} + C_{\sigma\lambda\rho} C^\lambda{}_{\mu\nu} \right) = 0. \tag{32}$$

Only the projections $\mathcal{C}^\tau{}_\sigma u^\sigma$ contain solely the kinematic quantities. These four equations are the constraint and evolution equations for the vorticity,

$$D_\rho \omega^\rho = \omega \cdot a, \tag{33}$$

$$D_u \omega^\rho = \frac{1}{2} D_\sigma a^{\sigma\rho} + \omega^\sigma \sigma_\sigma{}^\rho - \frac{2}{3} \theta \omega^\rho. \tag{34}$$

Furthermore, from Equation (31), we can compute the time-time and time-space projections of the Ricci tensor $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ which can be expressed by the kinematic quantities,

$$R_{\mu\nu}u^\mu u^\nu = -\partial_u\theta - \frac{1}{3}\theta^2 + D_\rho a^\rho - \sigma^2 + \omega^2, \tag{35}$$

$$R_{\mu\nu}u^\mu h^\nu_\rho = D_\sigma \sigma^\sigma{}_\rho + u_\rho\sigma^2 + \sigma_{\rho\sigma}a^\sigma - D_\sigma\omega^\sigma{}_\rho - u_\rho\omega^2 - \omega_{\rho\sigma}a^\sigma - \frac{2}{3}\partial_\rho\theta + \frac{2}{3}u_\rho\partial_u\theta. \tag{36}$$

By projecting the field equations (25) onto the aether velocity and its orthogonal spatial directions using u^μ and h^μ_ν, they can be split into three groups of equations consisting of a temporal-temporal equation $[E_{00}]$ obtained by contraction with $u^\mu u^\nu$, temporal-spatial equations $[E_{0i}]$ obtained by contraction with $u^\mu h^\nu_\rho$, and spatial-spatial equations $[E_{ij}]$ as a result of a contraction with $h^\mu_\rho h^\nu_\sigma$. We first note that by taking the trace of Equation (25), the curvature scalar is given by

$$R = c_\theta\left(\partial_u\theta + \frac{2}{3}\theta^2\right) - c_\sigma\sigma^2 - c_\omega\omega^2 - 2c_{\omega\kappa}\omega\cdot\kappa - c_\kappa\kappa^2 + 2c_{\kappa a}\left(D_\rho\kappa^\rho - \kappa\cdot a\right) - c_a\left(2D_\rho a^\rho - a^2\right). \tag{37}$$

The temporal part of Equation (25), which contains $G_{\mu\nu}u^\mu u^\nu = R_{\mu\nu}u^\mu u^\nu - \frac{1}{2}R$, can then be expressed solely in terms of the kinematic quantities by using Equations (35) and (37). The result is

$$\begin{aligned}[E_{00}] \qquad & \left(1+\frac{c_\theta}{2}\right)\left(\partial_u\theta + \frac{1}{3}\theta^2\right) + (1-c_\sigma)\,\sigma^2 - (1-c_\omega+2c_{\omega\kappa})\,\omega^2 - c_\kappa\left(\kappa^2 + 2\omega\cdot\kappa\right) \\ & -c_{\kappa a}\left[D_\rho\left(\omega^\rho+\kappa^\rho\right) + (\omega+\kappa)\cdot a\right] - (1-c_a)\,D_\rho a^\rho = 0. \end{aligned} \tag{38}$$

In a similar way, the temporal-spatial part of the Ricci tensor $R_{\mu\nu}u^\mu h^\nu_\rho$ can be eliminated from $[E_{0i}]$ using Equation (36) resulting again in equations containing only the kinematic quantities. The Ricci tensor is only present in the spatial-spatial part $[E_{ij}]$ in the form $R_{\mu\nu}h^\mu_\rho h^\nu_\sigma$. The long equations $[E_{0i}]$ and $[E_{ij}]$ are given in Appendix A.

The antisymmetric aether field equations (26) can be analogously split in two groups $[A_{0i}]$ and $[A_{ij}]$ by projections using u^μ and h^μ_ν. The spatial-spatial equations, which are empty in Einstein-aether theory, can be written in the form

$$\begin{aligned}[A_{ij}] \qquad & c_\kappa\left(D_u\kappa^\mu + \theta\kappa^\mu - \frac{1}{2}u^\mu\kappa\cdot a\right) + c_{\omega\kappa}\left(D_u\omega^\mu - \kappa^\mu{}_\rho\omega^\rho + \theta\omega^\mu - \frac{1}{2}u^\mu\omega\cdot a\right) \\ & +c_{\kappa a}\left(D_u a^\mu - \kappa^\mu{}_\rho a^\rho + \theta a^\mu - \frac{1}{2}u^\mu a^2\right) = 0. \end{aligned} \tag{39}$$

These equations relate the time evolution of κ_μ, a_μ, and ω_μ. The long equations $[A_{0i}]$ are given in Appendix A.

Due to the constraint (35), only the spatial part $R_{\mu\nu}h^{\mu\nu}$ of the curvature scalar appears in the gravitational action besides the kinematic quantities. This can be made more transparent by expressing R by $C_{\mu\nu\rho}$ using Equation (31). To this end, the spatial projection $Q_{\mu\nu\rho}$ of $C_{\mu\nu\rho}$ can be decomposed in the following way. Since $Q_{\mu\nu\rho}$ has only spatial components, we can define a second order tensor $Q_{\mu\nu}$ by

$$Q_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\rho\sigma}Q^{\rho\sigma}{}_\nu. \tag{40}$$

This tensor can be irreducibly decomposed into its antisymmetric part $\Omega_{\mu\nu}$, its trace Θ which is the totally antisymmetric part of $Q_{\mu\nu\rho}$, and its symmetric trace-free part $\Sigma_{\mu\nu}$:

$$\Omega_{\mu\nu} = Q_{[\mu\nu]} = \frac{1}{2}\varepsilon_{\rho\sigma[\mu}Q^{\rho\sigma}{}_{\nu]}, \tag{41}$$

$$\Theta = Q^\rho{}_\rho = \frac{1}{2}\varepsilon_{\mu\nu\rho}Q^{\mu\nu\rho}, \tag{42}$$

$$\Sigma_{\mu\nu} = Q_{(\mu\nu)} - \frac{1}{3}h_{\mu\nu}\Theta = \frac{1}{2}\varepsilon_{\rho\sigma(\mu}Q^{\rho\sigma}{}_{\nu)} - \frac{1}{3}h_{\mu\nu}\Theta. \tag{43}$$

Defining the vector

$$\Omega_\mu = \frac{1}{2}\varepsilon_{\mu\rho\sigma}\Omega^{\rho\sigma} = \frac{1}{2}Q_{\mu\rho}{}^\rho, \tag{44}$$

the irreducible decomposition of $Q_{\mu\nu\rho}$ reads

$$Q_{\mu\nu\rho} = 2\Omega_{[\mu}h_{\nu]\rho} - \varepsilon_{\mu\nu\sigma}\Sigma^\sigma{}_\rho - \frac{1}{3}\varepsilon_{\mu\nu\rho}\Theta. \tag{45}$$

Using Equation (31), the Ricci scalar is then given by

$$R = -\frac{2}{3}\Theta^2 + \Sigma^2 - \Omega^2 + 2\Omega \cdot a + \frac{2}{3}\theta^2 - \sigma^2 + \omega^2 + 2\omega \cdot \kappa - 2D_\rho \left(2\Omega^\rho + u^\rho \theta - a^\rho\right) \tag{46}$$

where $\Sigma^2 = \Sigma_{\mu\nu}\Sigma^{\mu\nu}$, $\Omega^2 = \Omega_{\mu\nu}\Omega^{\mu\nu}$, and $\Omega \cdot a = \Omega_{\mu\nu}a^{\mu\nu}$. Inserting this equation into the Lagrangian $\mathcal{L} = R + \mathcal{L}_e$, we obtain

$$\mathcal{L} = -\frac{2}{3}\Theta^2 + \Sigma^2 - \Omega^2 + 2\Omega \cdot a + \frac{2}{3}\left(1 + \frac{c_\theta}{2}\right)\theta^2 - \left(1 - c_\sigma\right)\sigma^2 + \left(1 + c_\omega\right)\omega^2 + 2\left(1 + c_{\omega\kappa}\right)\omega \cdot \kappa + c_\kappa \kappa^2 + 2c_{\kappa a}\kappa \cdot a - c_a a^2 \tag{47}$$

where total derivatives have been omitted. While the quantities Ω_μ, $\Sigma_{\mu\nu}$, and Θ appear together with the kinematic quantities in the decomposition of $C_{\mu\nu\rho}$, they have a different status than the kinematic quantities since they depend on the arbitrary orientation of the triad in space. The kinematic quantities can be seen as field strengths with the tetrad as potentials.

Note that while the quantities Ω_μ, $\Sigma_{\mu\nu}$, and Θ are not invariant under space dependent spatial rotations, the special combination $-\frac{2}{3}\Theta^2 + \Sigma^2 - \Omega^2 + 2\Omega \cdot a$ is invariant under such transformations up to a total derivative. By adding general terms quadratic in Ω_μ, $\Sigma_{\mu\nu}$, and Θ as well as cross terms with the kinematic quantities to the Lagrangian, the invariance under the full Lorentz group could be broken which we will not do in this paper.

IV. WEAK FIELDS

In this section, we will study the linearized version of the field equations (25) and (26). In the weak field approximation, the vectors ${e_a}^\mu$ and ${e^a}_\mu$ can be expanded around Minkowski space-time up to first order terms according to

$${e_a}^\mu = \delta^\mu_a + {\chi_a}^\mu, \tag{48}$$
$${e^a}_\mu = \delta^a_\mu + {\psi^a}_\mu \tag{49}$$

where indices of ${\chi_a}^\mu$ and ${\psi^a}_\mu$ are raised and lowered with the Minkowski metric. In the following, we will therefore not differentiate between internal and spacetime indices of first order fields. The condition $\delta^\mu_\nu = {e_a}^\mu {e^a}_\nu$ leads to the relation $\chi_{\mu\nu} = -\psi_{\nu\mu}$ which allows to use only $\psi_{\mu\nu}$ in the following. $\psi_{\mu\nu}$ can be decomposed into its symmetric and antisymmetric part according to

$$\psi_{\mu\nu} = \frac{1}{2}\gamma_{\mu\nu} + \zeta_{\mu\nu} \tag{50}$$

where $\gamma_{\mu\nu} = \psi_{\mu\nu} + \psi_{\nu\mu}$ and $\zeta_{\mu\nu} = \psi_{[\mu\nu]}$. The metric tensor is then given up to first order by

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}. \tag{51}$$

Under infinitesimal coordinate transformations $x^\mu \to x^\mu + \xi^\mu$, the tetrad transforms as

$$\delta {e_a}^\mu = \mathcal{L}_\xi {e_a}^\mu = \xi^\nu D_\nu {e_a}^\mu - {e_a}^\nu D_\nu \xi^\mu = -\partial_a \xi^\mu. \tag{52}$$

This leads to the following gauge transformations of $\gamma_{\mu\nu}$ and $\zeta_{\mu\nu}$.

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{53}$$
$$\zeta_{\mu\nu} \to \zeta_{\mu\nu} - \partial_{[\mu}\xi_{\nu]}. \tag{54}$$

In order to simplify the notation, in the following all fields will be first order quantities in this section. The tensor $C_{\mu\nu\rho}$ in first order approximation is

$$C_{\mu\nu\rho} = \partial_\rho \zeta_{\mu\nu} + \partial_{[\mu}\gamma_{\nu]\rho}. \tag{55}$$

It can be checked that $C_{\mu\nu\rho}$ is gauge invariant implying that all kinematic quantities in first order approximation are gauge invariant.

The field equations (25) and (26) in first order are given by

$$G_{\mu\nu} = \partial_\rho {J_{(\mu\nu)}}^\rho - \partial_\rho {J^\rho}_{(\mu\nu)}, \tag{56}$$
$$\partial_\rho {J_{[\mu}}^\rho{}_{\nu]} = 0. \tag{57}$$

Since the first order approximation of the kinematic quantities as well as the first order curvature tensor are gauge invariant, it is convenient to write the first order field equations in terms of the kinematic quantities. The first order approximations of the aether field equations $[A_{ij}]$ and $[A_{0i}]$ are

$$[A_{ij}] \qquad c_{\omega\kappa}\dot{\boldsymbol{\omega}} + c_\kappa \dot{\boldsymbol{\kappa}} + c_{\kappa a}\dot{\boldsymbol{a}} = \mathbf{0}, \tag{58}$$

$$[A_{0i}] \qquad \frac{1}{3}c_\theta \boldsymbol{\nabla}\theta - c_\sigma \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - c_\omega \boldsymbol{\nabla}\times\boldsymbol{\omega} - c_{\omega\kappa}\boldsymbol{\nabla}\times\boldsymbol{\kappa} - 2c_{\kappa a}\dot{\boldsymbol{\kappa}} + 2c_a\dot{\boldsymbol{a}} = \mathbf{0}. \tag{59}$$

Here and in the following, we use vector notation. For example $(\boldsymbol{a})_i = a_i$, $(\boldsymbol{\nabla}\times\boldsymbol{\omega})_i = \epsilon_{ijk}\partial_j\omega_k = \partial_j\omega_{ji}$, $\boldsymbol{\nabla}\cdot\boldsymbol{\kappa} = \partial_i\kappa_i$, $\dot{\theta} = \partial_0\theta$, $(\boldsymbol{\sigma})_{ij} = \sigma_{ij}$, $(\boldsymbol{R})_{ij} = R_{ij}$, $(\mathbf{1})_{ij} = \delta_{ij}$. ($\epsilon_{ijk}$ is the totally antisymmetric symbol with $\epsilon_{123} = 1$.) The first order approximations of the field equations $[E_{00}]$ and $[E_{0i}]$ lead to

$$[I_0] \qquad c_{\kappa a}\boldsymbol{\nabla}\cdot\boldsymbol{\kappa} + (1-c_a)\,\boldsymbol{\nabla}\cdot\boldsymbol{a} = -\left(1+\frac{c_\theta}{2}\right)\dot{\theta}, \tag{60}$$

$$[I_i] \qquad \left(c_\omega + (1+c_{\omega\kappa})\,\frac{c_\sigma}{1-c_\sigma}\right)\boldsymbol{\nabla}\times\boldsymbol{\omega} + \left(c_{\omega\kappa} + c_\kappa\frac{c_\sigma}{1-c_\sigma}\right)\boldsymbol{\nabla}\times\boldsymbol{\kappa}$$
$$+\, 2c_{\kappa a}\dot{\boldsymbol{\kappa}} + c_{\kappa a}\frac{c_\sigma}{1-c_\sigma}\boldsymbol{\nabla}\times\boldsymbol{a} - 2c_a\dot{\boldsymbol{a}} = \frac{2c_\sigma + c_\theta}{3\,(1-c_\sigma)}\boldsymbol{\nabla}\theta \tag{61}$$

where in Equation (61) the field equation $[A_{0i}]$ has been substituted in order to eliminate $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}$. Finally, the weak field version of the field equation $[E_{ij}]$ is

$$[I_{ij}] \qquad \boldsymbol{R} = -\frac{1}{6}c_\theta\mathbf{1}\dot{\theta} - c_\sigma\dot{\boldsymbol{\sigma}} + c_{\kappa a}\mathbf{1}\boldsymbol{\nabla}\cdot\boldsymbol{\kappa} - c_a\mathbf{1}\boldsymbol{\nabla}\cdot\boldsymbol{a}. \tag{62}$$

Since we are working at the level of the kinematic quantities, we have to take the corresponding constraint equations into account. The linear approximations of the constraint equations (33) and (34) are

$$[H_0] \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0, \tag{63}$$

$$[H_i] \qquad \boldsymbol{\nabla}\times\boldsymbol{a} + 2\dot{\boldsymbol{\omega}} = \mathbf{0}. \tag{64}$$

The spin rotation $\boldsymbol{\kappa}$ can be eliminated from the weak field equations using Equation (58). For this, we can split the fields into time independent and time varying parts. The static part corresponds to the zero frequency Fourier mode of the fields while the dynamic part represents the finite frequency contribution. We here consider only the dynamic case. Equation (58) can then be integrated resulting in

$$\boldsymbol{\kappa} = -\frac{c_{\omega\kappa}}{c_\kappa}\boldsymbol{\omega} - \frac{c_{\kappa a}}{c_\kappa}\boldsymbol{a}. \tag{65}$$

Substituting this equation into the field equations $[I_0]$, $[I_i]$, and $[I_{ij}]$, we obtain

$$[I_0] \qquad \boldsymbol{\nabla}\cdot\boldsymbol{a} = -\frac{1+\frac{c_\theta}{2}}{1-\bar{c}_a}\dot{\theta}, \tag{66}$$

$$[I_i] \qquad \frac{1}{2\bar{c}_a}\left(\bar{c}_\omega + \frac{c_\sigma}{1-c_\sigma}\right)\boldsymbol{\nabla}\times\boldsymbol{\omega} - \dot{\boldsymbol{a}} = \frac{c_\sigma + \frac{c_\theta}{2}}{3\bar{c}_a\,(1-c_\sigma)}\boldsymbol{\nabla}\theta, \tag{67}$$

$$[I_{ij}] \qquad \boldsymbol{R} = -\frac{1}{3}\left[\frac{c_\theta}{2} - \frac{3\bar{c}_a\left(1+\frac{c_\theta}{2}\right)}{1-\bar{c}_a}\right]\mathbf{1}\dot{\theta} - c_\sigma\dot{\boldsymbol{\sigma}} \tag{68}$$

where

$$\bar{c}_\omega = c_\omega - \frac{c_{\omega\kappa}^2}{c_\kappa}, \tag{69}$$

$$\bar{c}_a = c_a + \frac{c_{\kappa a}^2}{c_\kappa}. \tag{70}$$

Furthermore, combining Equations (59) and (67) yields

$$[A_{0i}] \qquad \boldsymbol{\nabla}\times\boldsymbol{\omega} = \frac{2}{3}\left(1+\frac{c_\theta}{2}\right)\boldsymbol{\nabla}\theta + (1-c_\sigma)\,\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}. \tag{71}$$

Comparing Equations (66)-(68) with the limiting case $c_\kappa = c_{\omega\kappa} = c_{\kappa a} = 0$ of Equations (60)-(62), that is, Einstein-aether theory, we conclude that the introduction of the spin rotation amounts in the dynamic case to the substitutions $c_\omega \to \bar{c}_\omega$ and $c_a \to \bar{c}_a$.

Equations (63), (64), (66), and (67) have a close resemblance with Maxwell's equations where $\boldsymbol{a}$ and $2\boldsymbol{\omega}$ play the role of electric and magnetic fields, respectively, and where the electric charge density ρ and current $\boldsymbol{j}$ are given by

$$\rho = -\frac{1+\frac{c_\theta}{2}}{1-\bar{c}_a}\dot{\theta}, \tag{72}$$

$$\boldsymbol{j} = \frac{c_\sigma + \frac{c_\theta}{2}}{3\bar{c}_a\left(1-c_\sigma\right)}\boldsymbol{\nabla}\theta. \tag{73}$$

A further useful equation involving the kinematic quantities follows from the first order limit of the contracted second Bianchi identity and the field equation $[I_{ij}]$ (see Appendix B for a derivation):

$$\Delta\boldsymbol{\sigma} - \left(1-c_\sigma\right)\ddot{\boldsymbol{\sigma}} - 2\left(\nabla\left(\nabla\cdot\boldsymbol{\sigma}\right)\right)_{\text{sym}} - \frac{1}{3}\mathbf{1}\Delta\theta + \frac{1}{3}\left(1+\frac{c_\theta}{2}\right)\left(1-\frac{3\bar{c}_a}{1-\bar{c}_a}\right)\mathbf{1}\ddot{\theta} - \frac{1}{3}\nabla\nabla\theta + \left(\nabla\dot{\boldsymbol{a}}\right)_{\text{sym}} = \mathbf{0} \tag{74}$$

where the suffix sym denotes symmetrization.

The linearized wave solutions of Einstein-aether theory were given in [8]. Since the weak field equations of the extended theory are effectively the same as in Einstein-aether theory, the wave solutions are the same but with different wave speeds and field content. The analogy of the weak field equations with Maxwell's equations suggests the existence of spin 1 acceleration-vorticity waves similar to electromagnetic waves. Indeed, in the case $\theta = 0$ Equations (64) and (67) lead to the wave equations

$$\ddot{\boldsymbol{a}} - s_{a\omega}^2\Delta\boldsymbol{a} = \mathbf{0}, \qquad \ddot{\boldsymbol{\omega}} - s_{a\omega}^2\Delta\boldsymbol{\omega} = \mathbf{0} \tag{75}$$

with the wave speed $s_{a\omega}$ given by

$$s_{a\omega}^2 = \frac{1}{4\bar{c}_a}\left(\bar{c}_\omega + \frac{c_\sigma}{1-c_\sigma}\right). \tag{76}$$

From Equations (63), (66), and (64) follows that $\boldsymbol{a}$ and $\boldsymbol{\omega}$ are transverse and perpendicular to each other. Equation (65) with Equations (75) leads to a wave equation for $\boldsymbol{\kappa}$ with a polarization determined by the coupling constants $c_{\omega\kappa}$, $c_{\kappa a}$, and c_κ.

Furthermore, the conservation equation $\dot{\rho} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$ for the charge density (72) and the current (73) corresponds to a wave equation for spin 0 expansion waves,

$$\ddot{\theta} - s_\theta^2\Delta\theta = 0 \tag{77}$$

where the wave speed s_θ is given by

$$s_\theta^2 = \frac{1-\bar{c}_a}{3\bar{c}_a}\left(\frac{\frac{c_\theta}{2}}{1+\frac{c_\theta}{2}} + \frac{c_\sigma}{1-c_\sigma}\right). \tag{78}$$

From Equation (66) follows that these waves are accompanied by longitudinal acceleration waves. Equation (64) then shows that for these solutions $\boldsymbol{\omega} = \mathbf{0}$. From Equation (65) follows that the spin rotation waves are also longitudinal.

Finally, in the case $\theta = 0$, $\boldsymbol{a} = \boldsymbol{\omega} = \boldsymbol{\kappa} = \mathbf{0}$ it follows from Equation (71) that $\nabla\cdot\boldsymbol{\sigma} = \mathbf{0}$ and Equation (74) leads to a wave equation for spin 2 transverse shear waves,

$$\ddot{\boldsymbol{\sigma}} - s_\sigma^2\Delta\boldsymbol{\sigma} = \mathbf{0} \tag{79}$$

with the wave speed squared

$$s_\sigma^2 = \frac{1}{1-c_\sigma}. \tag{80}$$

The plane wave solutions for the three cases are given in Appendix C.

If the weak field equations are expressed by the tetrad variables, we have to take the gauge transformations (53) and (54) into account. In order to identify the independent wave modes, a gauge fixing has to be applied. In the present case, we use the gauge fixing conditions

$$\psi_{i0} = 0, \tag{81}$$

$$\partial_i\psi_{0i} = 0. \tag{82}$$

Given a field configuration $\psi_{\mu\nu}$, these conditions can be reached by applying the gauge transformation

$$\xi_0 = \int \frac{d^3x'}{4\pi|x-x'|}\partial_i\psi_{0i}, \tag{83}$$

$$\xi_i = -\int dt'\psi_{i0}. \tag{84}$$

In terms of the decomposition (50), the gauge fixing reads

$$\zeta_{i0} = -\frac{1}{2}\gamma_{i0}, \tag{85}$$

$$\partial_i\zeta_{0i} = 0, \quad \partial_i\gamma_{0i} = 0. \tag{86}$$

In the following, we will again use vector notation to simplify equations. We define $(\boldsymbol{A})_i = 2\zeta_{i0} = -\gamma_{i0}$, $(\boldsymbol{\zeta})_i = -\frac{1}{2}\epsilon_{ijk}\zeta_{jk}$, $\gamma = \frac{1}{2}\gamma_{kk}$, $(\boldsymbol{\gamma})_{ij} = \frac{1}{2}\gamma_{ij} - \frac{1}{3}\delta_{ij}\gamma$, $\phi = \frac{1}{2}\gamma_{00}$. The gauge fixing conditions can then be written as a kind of Coulomb gauge,

$$\nabla\cdot\boldsymbol{A} = 0. \tag{87}$$

Taking the gauge fixing into account, the kinematic quantities in terms of the tetrad variables are

$$\boldsymbol{a} = -\dot{\boldsymbol{A}} - \nabla\phi, \tag{88}$$

$$\boldsymbol{\omega} = \frac{1}{2}\nabla\times\boldsymbol{A}, \tag{89}$$

$$\boldsymbol{\kappa} = -\dot{\boldsymbol{\zeta}} - \frac{1}{2}\nabla\times\boldsymbol{A}, \tag{90}$$

$$\boldsymbol{\sigma} = \dot{\boldsymbol{\gamma}}, \tag{91}$$

$$\theta = -\dot{\gamma}. \tag{92}$$

The tetrads for plane waves are given in Appendix C.

V. GEOMETRICAL CONSIDERATIONS

In this section, we resume the discussion of the nonlinear theory of the spinning aether in sections II and III. The formulation given there is entirely within Riemannian geometry, that is, gravity is described by the metric part of the tetrad. The aether, in contrast, is described by the full tetrad. Moreover, the symmetries of the aether — the broken Lorentz invariance — are imposed at the dynamical level through the action functional. In the following, we will argue for a pure geometrical formulation of the aether which implements the gravitational nature of the aether and its symmetries already at the kinematical level.

For a start, we observe that we can define a covariant derivative $\overset{*}{D}_\mu$ by

$$\overset{*}{D}_\mu X^\nu = D_\mu X^\nu + S^\nu{}_{\rho\mu}X^\rho \tag{93}$$

where X^μ is a vector field and $S_{\mu\nu\rho}$ is given by Equation (13). Since $S_{\mu\nu\rho}$ is antisymmetric in the first two indices, the connection corresponding to $\overset{*}{D}_\mu$ is metric compatible, that is, $\overset{*}{D}_\mu g_{\nu\rho} = 0$. With the help of Equations (7), (10), and (13), we can show that for a preferred frame $e_a{}^\mu$

$$\overset{*}{D}_\mu u^\nu = 0, \tag{94}$$

$$\overset{*}{D}_u e_i{}^\mu = 0. \tag{95}$$

These equations may be viewed as trivial rearrangements of Equations (7) and (10). However, they can also be interpreted as defining a non-Riemannian geometry in which the aether velocity u^μ is a parallel vector field and the triad $e_i{}^\mu$ is parallel in the direction of u^μ. In this geometry, $S_{\mu\nu\rho}$ is the contortion tensor with the torsion tensor

$$T_{\mu\nu\rho} = S_{\mu\rho\nu} - S_{\mu\nu\rho}. \tag{96}$$

Thus, in this geometry, the kinematic quantities are torsion fields which realizes mathematically the interpretation of the kinematic quantities as field strengths.

In order to gain insight into the geometry of $\overset{*}{D}_\mu$, we can compute the corresponding spin connection. The spin connection $\overset{*}{\omega}{}^a{}_{b\mu}$ is related to the connection coefficients $\overset{*}{\Gamma}{}^\mu{}_{\nu\rho}$ of the derivative $\overset{*}{D}_\mu$ by

$$\overset{*}{\omega}{}^a{}_{b\mu} = e^a{}_\nu \partial_\mu e_b{}^\nu + e^a{}_\nu \, \overset{*}{\Gamma}{}^\nu{}_{\lambda\mu} e_b{}^\lambda = e^a{}_\nu \, \overset{*}{D}_\mu e_b{}^\nu. \tag{97}$$

This equation corresponds to a local linear transformation of the connection coefficients from a holonomic (coordinate) basis to an anholonomic basis given by the tetrad. In the case of the derivative (93), it can be shown that

$$\overset{*}{\omega}{}^i{}_{0\mu} = 0, \tag{98}$$

$$\overset{*}{\omega}{}^i{}_{j\rho} u^\rho = 0, \tag{99}$$

$$\overset{*}{\omega}{}^i{}_{j\rho} h^\rho_\mu = \omega^i{}_{j\rho} h^\rho_\mu. \tag{100}$$

where $\omega^a{}_{b\mu}$ is the torsion-free Levi-Civita connection. Equations (98)-(100) can be summarized as

$$\overset{*}{\omega}{}^a{}_{b\mu} = -Q^a{}_{b\mu} \tag{101}$$

where $Q^a{}_{b\mu} = e^a{}_\rho e_b{}^\sigma Q^\rho{}_{\sigma\mu}$. Equation (98) means that the connection $\overset{*}{\omega}{}^a{}_{b\mu}$ is a $SO(3)$ connection. Equation (99) means additionally that $\overset{*}{\omega}{}^a{}_{b\mu}$ is a trivial $\mathbf{1}$ connection along u^μ. According to Equation (100), the spatial geometry is Riemannian. The connection (98) was used in [9] to formulate a version of Einstein-aether theory employing Weinberg's quasi-Riemannian gravity. The full connection (98)-(100) was derived from the symmetries of the spinning aether in [10]. The corresponding geometry was referred to as semi-teleparallel since only the temporal part of space-time is parallelized.

From Equation (98) follows that the time-space components of the curvature tensor $R^{ab}{}_{\mu\nu} = 2\partial_{[\mu}\omega^{ab}{}_{\nu]} + 2\omega^a{}_{c[\mu}\omega^{cb}{}_{\nu]}$ vanish,

$$\overset{*}{R}{}^{i0}{}_{\rho\sigma} = 2\partial_{[\rho} \, \overset{*}{\omega}{}^{i0}{}_{\sigma]} + 2 \, \overset{*}{\omega}{}^i{}_{j[\rho} \, \overset{*}{\omega}{}^{j0}{}_{\sigma]} = 0. \tag{102}$$

Using space-time indices, Equation (102) means

$$u_\mu \overset{*}{R}{}^{\mu\nu}{}_{\rho\sigma} = 0. \tag{103}$$

From the Ricci identity for the derivative (93),

$$\overset{*}{D}_\rho \overset{*}{D}_\nu e^a{}_\mu - \overset{*}{D}_\nu \overset{*}{D}_\rho e^a{}_\mu + T^\sigma{}_{\rho\nu} \overset{*}{D}_\sigma e^a{}_\mu = \overset{*}{R}{}^\sigma{}_{\mu\nu\rho} e^a{}_\sigma, \tag{104}$$

follows a relation between the curvature tensors of the derivatives $\overset{*}{D}_\mu$ and D_μ,

$$\overset{*}{R}_{\sigma\mu\nu\rho} = R_{\sigma\mu\nu\rho} - D_\rho S_{\sigma\mu\nu} + D_\nu S_{\sigma\mu\rho} - S_{\sigma\lambda\rho} S^\lambda{}_{\mu\nu} + S_{\sigma\lambda\nu} S^\lambda{}_{\mu\rho}. \tag{105}$$

Using Equations (103) and (105), the relations (35) and (36) can be derived which can be seen to be a direct consequence of the semi-teleparallel geometry. Moreover, the first Bianchi identity for the semi-teleparallel connection is equivalent with the constraint equations (32).

The action functional (16) can be directly interpreted in terms of the semi-teleparallel geometry. A more general action, which is not equivalent with (16), can be obtained if we relax the implicit assumption that the spatial geometry is Riemannian, that is, has vanishing torsion. In this case, however, we go beyond the formulation of Einstein-aether theory within Riemannian geometry in that we start with a Riemann-Cartan geometry. In this geometry, we use a general Lorentz connection

$$\tilde{\omega}^a{}_{b\mu} = \omega^a{}_{b\mu} + \tilde{K}^a{}_{b\mu} \tag{106}$$

where $\tilde{K}^a{}_{b\mu} = e^a{}_\rho e_b{}^\nu \tilde{K}^\rho{}_{\nu\mu}$ is the contortion tensor. In order to formulate a theory of gravity with a spinning aether using Riemann-Cartan geometry, the action (16) has to be extended to an Einstein-Cartan theory. The natural way is to define the kinematic quantities with respect to the connection $\tilde{\omega}^a{}_{b\mu}$ and to use the curvature scalar belonging to

$\tilde{\omega}^a{}_{b\mu}$ in the Lagrangian. The kinematic quantities corresponding to the derivative $\tilde{D}_\mu$ are (see also [11])

$$\tilde{a}_\mu = a_\mu + \tilde{K}_{\mu\rho\sigma} u^\rho u^\sigma, \tag{107}$$

$$\tilde{\omega}_{\mu\nu} = \omega_{\mu\nu} + \tilde{K}^\rho{}_{[\mu\nu]} u_\rho - u_{[\mu} \tilde{K}_{\nu]\rho\sigma} u^\rho u^\sigma, \tag{108}$$

$$\tilde{\theta} = \theta - \tilde{K}_\rho u^\rho, \tag{109}$$

$$\tilde{\sigma}_{\mu\nu} = \sigma_{\mu\nu} - \tilde{K}^\rho{}_{(\mu\nu)} u_\rho + u_{(\mu} \tilde{K}_{\nu)\rho\sigma} u^\rho u^\sigma + \frac{1}{3} h_{\mu\nu} \tilde{K}_\rho u^\rho, \tag{110}$$

$$\tilde{\kappa}_{\mu\nu} = \kappa_{\mu\nu} + \tilde{K}_{\mu\nu\rho} u^\rho + 2u_{[\mu} \tilde{K}_{\nu]\rho\sigma} u^\rho u^\sigma \tag{111}$$

where $\tilde{K}^\mu = \tilde{K}^{\mu\rho}{}_\rho$ is the trace of the contortion tensor. Furthermore, the curvature scalar corresponding to $\tilde{D}_\mu$ is

$$\tilde{R} = R + 2D_\rho \tilde{K}^\rho - \tilde{K}_\rho \tilde{K}^\rho + \tilde{K}_{\sigma\lambda\rho} \tilde{K}^{\rho\lambda\sigma}. \tag{112}$$

The generalization of the action (16) to an Einstein-Cartan-aether theory thus is

$$\begin{aligned} S[e_a{}^\mu, \tilde{\omega}^a{}_{b\mu}] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \big(& R - \tilde{K}_\rho \tilde{K}^\rho + \tilde{K}_{\sigma\lambda\rho} \tilde{K}^{\rho\lambda\sigma} \\ & + \frac{1}{3} \tilde{c}_\theta \tilde{\theta}^2 + \tilde{c}_\sigma \tilde{\sigma}^2 + \tilde{c}_\omega \tilde{\omega}^2 + 2\tilde{c}_{\omega\kappa} \tilde{\omega} \cdot \tilde{\kappa} + \tilde{c}_\kappa \tilde{\kappa}^2 + 2\tilde{c}_{\kappa a} \tilde{\kappa} \cdot \tilde{a} - \tilde{c}_a \tilde{a}^2 \big) \end{aligned} \tag{113}$$

up to a surface term.

In order to get an action functional for a semi-teleparallel geometry, simply replacing the Lorentz connection in the action for Einstein-Cartan theory (113) by a semi-teleparallel connection is not suitable since the corresponding kinematic quantities (107)-(111) are zero in this case and the scalar curvature contains only the spatial part of the curvature tensor. A method to obtain an action functional for a semi-teleparallel geometry was proposed in [10]. It was shown there that starting from a general Lorentz connection $\tilde{\omega}^a{}_{b\mu}$, there is a unique decomposition

$$\tilde{\omega}^a{}_{b\mu} = \overset{*}{\omega}{}^a{}_{b\mu} + H^a{}_{b\mu} \tag{114}$$

where $\overset{*}{\omega}{}^a{}_{b\mu}$ is a semi-teleparallel connection with respect to a tetrad $e_a{}^\mu$ and where $H^a{}_{b\mu}$ is a tensor field satisfying $H_{ab\mu} = -H_{ba\mu}$ and which has vanishing spatial components, $H_{ij\mu} h^\mu_\nu = 0$. Actually, $H^a{}_{b\mu}$ is the difference between the contortion tensors of $\tilde{\omega}^a{}_{b\mu}$ and $\overset{*}{\omega}{}^a{}_{b\mu}$,

$$H^\mu{}_{\nu\rho} = \tilde{K}^\mu{}_{\nu\rho} - \overset{*}{K}{}^\mu{}_{\nu\rho}. \tag{115}$$

The basic idea in constructing an action for the semi-teleparallel geometry is the following:

1. Start with an action functional $S[e_a{}^\mu, \tilde{\omega}^a{}_{b\mu}]$ that depends on the tetrad and an arbitrary Lorentz connection $\tilde{\omega}^a{}_{b\mu}$.
2. Insert the decomposition (114) to obtain an action functional $S[e_a{}^\mu, \overset{*}{\omega}{}^a{}_{b\mu}, H^a{}_{b\mu}]$.
3. Find a stationary point of this action with respect to $H^a{}_{b\mu}$ resulting in an action functional $S'[e_a{}^\mu, \overset{*}{\omega}{}^a{}_{b\mu}]$ that is defined for a semi-teleparallel connection $\overset{*}{\omega}{}^a{}_{b\mu}$.

In the present case, we start with the action (113). Insertion of the decomposition (114) and variation with respect to $H_{\mu\nu\rho}$ leads to algebraic equations for $H_{\mu\nu\rho}$. Solving these equations and inserting $H_{\mu\nu\rho}$ back into the action (113) results in a new action of the form

$$\begin{aligned} S'[e_a{}^\mu, \overset{*}{\omega}{}^a{}_{b\mu}] = \frac{1}{16\pi G} \int d^4x \Big[& R + \frac{1}{3} c_\theta \theta^2 + c_\sigma \sigma^2 + c_\omega \omega^2 + 2c_{\omega\kappa} \omega \cdot \kappa + c_\kappa \kappa^2 \\ & + \left(2a_\rho + c_{\omega Z} \omega_\rho + c_{\kappa Z} \kappa_\rho\right) Z^\rho + c_Z Z_\rho Z^\rho + Z_{\sigma\lambda\rho} Z^{\rho\lambda\sigma} \Big] \end{aligned} \tag{116}$$

where

$$Z^{\mu\nu\rho} = h^\mu_\sigma h^\nu_\tau h^\rho_\lambda \overset{*}{K}{}^{\sigma\tau\lambda} \tag{117}$$

is the spatial part of the contortion tensor of the semi-teleparallel geometry and $Z^\mu = Z^{\mu\rho}{}_\rho$ is its trace. The coupling constants c_σ and c_θ in (116) are given by

$$c_\sigma = \frac{\tilde{c}_\sigma}{\tilde{c}_\sigma + 1}, \tag{118}$$

$$c_\theta = \frac{2\tilde{c}_\theta}{2 - \tilde{c}_\theta}. \tag{119}$$

The remaining coupling constants c_ω, $c_{\omega\kappa}$, c_κ, $c_{\omega Z}$, $c_{\kappa Z}$, and c_Z are functions of the coupling constants $\tilde{c}_\omega$, $\tilde{c}_{\omega\kappa}$, $\tilde{c}_\kappa$, $\tilde{c}_{\kappa a}$, $\tilde{c}_a$.

The action (116) differs from the original action (16) in the presence of terms quadratic in the spatial torsion and coupling terms of the acceleration, vorticity, and spin rotation with the trace of the spatial torsion. Furthermore, there are no terms of the form $c_{\kappa a}\kappa \cdot a$ and $c_a a^2$ in the action. Nevertheless, in the source-free case, the dynamics following from action (116) is effectively the same as the one following from the action (16). To see this, we first note that — analogously to the tensor $Q_{\mu\nu\rho}$ in Equation (45) — the spatial contortion tensor $Z_{\mu\nu\rho}$ can be decomposed into irreducible parts according to

$$Z_{\mu\nu\rho} = Z_{[\mu}h_{\nu]\rho} - \varepsilon_{\mu\nu\sigma}Z^\sigma{}_\rho - \frac{1}{3}\varepsilon_{\mu\nu\rho}Z. \tag{120}$$

where Z is a scalar and $Z_{\mu\nu}$ a symmetric trace-free tensor. Variation of the action (116) with respect to the semi-teleparallel connection $\overset{*}{\omega}{}^a{}_{b\mu}$ leads to the field equations

$$Z = 0, \qquad Z_\mu = -\frac{1}{1+2c_Z}\left(2a_\mu + c_{\omega Z}\omega_\mu + c_{\kappa Z}\kappa_\mu\right), \qquad Z_{\mu\nu} = 0. \tag{121}$$

Inserting these field equation back into the action (116) leads to an action of the form (16).

Although the coupling constants in the action (116) are by construction not independent, the action can be used as a starting point for a semi-teleparallel theory of gravity with independent coupling constants. Moreover, it is possible to add terms of the form θZ and $\sigma_{\mu\nu}Z^{\mu\nu}$, which couple the spatial torsion with the expansion and the shear, as well as a further term quadratic in the spatial torsion of the form $Z_{\mu\nu\rho}Z^{\mu\nu\rho}$. Since $\overset{*}{K}{}_{\mu\nu\rho} = Z_{\mu\nu\rho} + S_{\mu\nu\rho}$, the general action has the form

$$S[e_a{}^\mu, \overset{*}{\omega}{}^a{}_{b\mu}] = \frac{1}{16\pi G}\int d^4x \left(R + \mathcal{K}^{\kappa\lambda\mu\rho\nu\sigma}\ \overset{*}{K}{}_{\kappa\nu\mu}\ \overset{*}{K}{}_{\lambda\sigma\rho}\right) \tag{122}$$

with a suitable choice of the supermetric $\mathcal{K}^{\kappa\lambda\mu\rho\nu\sigma}$.

VI. CONCLUSIONS

In this paper, an extension of Einstein-aether theory has been proposed which incorporates internal rotational degrees of freedom of the aether. The main objectives of the paper are

(1) The introduction of the spin rotation $\kappa_{\mu\nu}$ as an additional kinematic quantity. The fact that this tensor naturally appears besides the acceleration, vorticity, shear, and expansion in the time-space decomposition of the Ricci rotation coefficients gives this approach a certain completeness.

(2) The formulation of a theory of gravitation that incorporates couplings of the spin rotation with the vorticity and the acceleration.

(3) The study of the theory in the weak field limit. In the case of dynamical fields, the spin rotation, the vorticity, and the acceleration are linearly related which allows to eliminate one of them from the field equations. The field equations acquire the simplest form if the spin rotation is eliminated. In that case, the linearized theory has the form of Einstein-aether theory with rescaled coupling constants.

(4) The formulation of the theory as a (semi-)teleparallel theory of gravitation. The geometry in this approach is adapted to the symmetries of the spinning aether. As a result, the kinematic quantities are part of the torsion tensor. In the matter-free case, the spatial torsion is algebraically related to the kinematic quantities which makes the approach effectively equivalent to the Riemannian formulation. However, if matter fields are present, the semi-teleparallel formulation may be different from the Riemannian one.

There are many open questions connected with the Einstein-spin-aether theory which concern the physical implications of the spin rotation. These could be elucidated by studying exact solutions of the nonlinear theory such as black hole solutions and cosmological solutions. It is to be expected that the simple relation between the kinematic quantities found in the linearized theory will no longer hold in solutions of the nonlinear theory. Further open questions concern the coupling to matter, in particular to spinning matter, where a semi-teleparallel formulation may lead to different interactions than the Riemannian formulation, the Hamiltonian formulation of the extended Einstein-aether theory, and the study of the observational constraints on the coupling constants.

Appendix A: Field Equations

In this appendix we will list the projections $[E_{0i}]$, $[E_{ij}]$, and $[A_{0i}]$ of the field equations (25), (26) not given in Section III.

$$
\begin{aligned}
[E_{0i}] \quad & \frac{1}{3}\left(\frac{1}{2}c_\theta+2\right)(\partial_\mu\theta-u_\mu\partial_u\theta)+\left(\frac{1}{2}c_\sigma-1\right)\left(D_\rho\sigma^\rho{}_\mu+u_\mu\sigma^2+\sigma_{\mu\rho}a^\rho\right)-\left(\frac{1}{2}c_\omega-1\right)\left(D_\rho\omega^\rho{}_\mu+u_\mu\omega^2+\omega_{\mu\rho}a^\rho\right)\\
&+c_{\omega\kappa}\left[\frac{1}{2}D_\rho\left(2\omega^\rho{}_\mu-\kappa^\rho{}_\mu\right)+\frac{1}{2}u_\mu\omega\cdot(2\omega-\kappa)-\frac{1}{2}\kappa_{\mu\rho}a^\rho+\frac{1}{2}Q_{\rho\sigma\mu}\omega^{\rho\sigma}\right]+c_\kappa\left(D_\rho\kappa^\rho{}_\mu+u_\mu\omega\cdot\kappa+\frac{1}{2}Q_{\rho\sigma\mu}\kappa^{\rho\sigma}\right)\\
&+c_{\kappa a}\left[D_u\left(2\omega_\mu+\kappa_\mu\right)+\frac{2}{3}\theta\left(2\omega_\mu+\kappa_\mu\right)-\sigma_\mu{}^\rho\left(2\omega_\rho+\kappa_\rho\right)+\frac{1}{2}u_\mu\left(2\omega-\kappa\right)\cdot a+3\omega_{\mu\rho}\kappa^\rho+\frac{1}{2}Q_{\rho\sigma\mu}a^{\rho\sigma}\right]\\
&-c_a\left(D_ua_\mu-\frac{1}{2}u_\mu a^2-\sigma_{\mu\rho}a^\rho+3\omega_{\mu\rho}a^\rho+\frac{2}{3}a_\mu\theta\right)=0 \qquad \text{(A1)}
\end{aligned}
$$

$$
\begin{aligned}
[E_{ij}] \quad & h^\rho_\mu h^\sigma_\nu R_{\rho\sigma}=\frac{1}{6}c_\theta h_{\mu\nu}\left(\partial_u\theta+\theta^2\right)-c_\sigma\left(D_u\sigma_{\mu\nu}+2u_{(\mu}\sigma_{\nu)\rho}a^\rho+\sigma_{\mu\nu}\theta+2\sigma_{(\mu}{}^\rho\omega_{\nu)\rho}\right)-2c_\omega\omega_{\rho\mu}\omega^\rho{}_\nu\\
&+2c_{\omega\kappa}\left(\omega_{\rho(\mu}\sigma^\rho{}_{\nu)}+\omega_{\rho\mu}\omega^\rho{}_\nu-\omega_{\rho(\mu}\kappa^\rho{}_{\nu)}\right)+2c_\kappa\left(\kappa_{\rho(\mu}\sigma^\rho{}_{\nu)}+\kappa_{\rho(\mu}\omega^\rho{}_{\nu)}\right)\\
&+2c_{\kappa a}\left[\frac{1}{2}h_{\mu\nu}D_\rho\left(\omega^\rho+\kappa^\rho\right)+a_{(\mu}\left(\omega_{\nu)}+\kappa_{\nu)}\right)+\sigma_{\rho(\mu}a^\rho{}_{\nu)}\right]-c_a\left(h_{\mu\nu}D_\rho a^\rho+2a_\mu a_\nu\right) \qquad \text{(A2)}
\end{aligned}
$$

$$
\begin{aligned}
[A_{0i}] \quad & \frac{1}{6}c_\theta\left(\partial_\mu\theta-u_\mu\partial_u\theta\right)+\frac{1}{2}c_\sigma\left(D_\rho\sigma^\rho{}_\mu+u_\mu\sigma^2+\sigma_{\mu\rho}a^\rho\right)+\frac{1}{2}c_\omega\left(D_\rho\omega^\rho{}_\mu+u_\mu\omega^2+\omega_{\mu\rho}a^\rho\right)\\
&+\frac{1}{2}c_{\omega\kappa}\left(D_\rho\kappa^\rho{}_\mu+u_\mu\omega\cdot\kappa+\kappa_{\mu\rho}a^\rho-2\omega_{\mu\rho}a^\rho+Q_{\rho\sigma\mu}\omega^{\rho\sigma}\right)-c_\kappa\left(\kappa_{\mu\rho}a^\rho-\frac{1}{2}Q_{\rho\sigma\mu}\kappa^{\rho\sigma}\right)\\
&-c_{\kappa a}\left(D_u\kappa_\mu-\frac{1}{2}u_\mu\kappa\cdot a-\sigma_{\mu\rho}\kappa^\rho-\omega_{\mu\rho}\kappa^\rho+\frac{2}{3}\kappa_\mu\theta-\frac{1}{2}Q_{\rho\sigma\mu}a^{\rho\sigma}\right)\\
&+c_a\left(D_ua_\mu-\frac{1}{2}u_\mu a^2-\sigma_{\mu\rho}a^\rho-\omega_{\mu\rho}a^\rho+\frac{2}{3}a_\mu\theta\right)=0 \qquad \text{(A3)}
\end{aligned}
$$

Appendix B: Derivation of Equation (74)

Starting with the second Bianchi identity,

$$D_\mu R_{\nu\sigma\rho\tau}+D_\rho R_{\nu\sigma\tau\mu}+D_\tau R_{\nu\sigma\mu\rho}=0, \qquad \text{(B1)}$$

contracting with $g^{\sigma\tau}$ leads to

$$D_\mu R_{\nu\rho}+D_\sigma R^\sigma{}_{\nu\rho\mu}-D_\rho R_{\mu\nu}=0. \qquad \text{(B2)}$$

For weak fields, it follows

$$R_{ij,0}=R_{0i,j}+R_{i0j0,0}-R_{j0ik,k} \qquad \text{(B3)}$$

where commas denote partial derivatives. The terms on the right hand side can be obtained from the Ricci identity (30) in the following way. Using the first order form of Equation (36) yields

$$R_{0i,j}=-\frac{2}{3}\theta_{,ij}-\sigma_{ki,jk}+\omega_{ki,jk}. \qquad \text{(B4)}$$

Furthermore, from Equation (31), we obtain

$$R_{i0j0,0} = \frac{1}{3}\delta_{ij}\theta_{,00} - \sigma_{ij,00} + \omega_{ij,00} + a_{i,j0}, \tag{B5}$$

$$R_{j0ik,k} = -\frac{1}{3}\theta_{,ij} + \frac{1}{3}\delta_{ij}\theta_{kk}\sigma_{kj,ik} - \sigma_{ij,kk} + \omega_{kj,ik} - \omega_{ij,kk}. \tag{B6}$$

Inserting Equations (B4) - (B6) into Equation (B3) and taking the symmetric part yields

$$\dot{\boldsymbol{R}} = (\nabla\dot{\boldsymbol{a}})_{\rm sym} - 2\left(\nabla\left(\nabla\cdot\boldsymbol{\sigma}\right)\right)_{\rm sym} + \Delta\boldsymbol{\sigma} - \ddot{\boldsymbol{\sigma}} - \frac{1}{3}\nabla\nabla\theta - \frac{1}{3}\boldsymbol{1}\left(\Delta\theta - \ddot{\theta}\right) \tag{B7}$$

where vector notation has been used. The time derivative of $\boldsymbol{R}$ can also be obtained from the field equation (68) resulting in

$$\dot{\boldsymbol{R}} = -\frac{1}{3}\left[\frac{c_\theta}{2} - \frac{3\bar{c}_a\left(1+\frac{c_\theta}{2}\right)}{1-\bar{c}_a}\right]\boldsymbol{1}\ddot{\theta} - c_\sigma\ddot{\boldsymbol{\sigma}}. \tag{B8}$$

Equating (B7) and (B8) yields Equation (74).

Appendix C: Linearized plane wave solutions

We assume that the kinematic quantities and the tetrad in first order approximation are given by plane waves,

$$S_{\mu\nu\rho} = \hat{S}_{\mu\nu\rho}e^{ik_\sigma x^\sigma}, \tag{C1}$$

$$\psi_{\mu\nu} = \hat{\psi}_{\mu\nu}e^{ik_\sigma x^\sigma} \tag{C2}$$

where k_μ is the wavevector with $(\boldsymbol{k})_i = k_i$ and $\boldsymbol{k} = \boldsymbol{n}k$. The spatial vector $\boldsymbol{n}$ is the normal of the wave front and $s = k_0/k$ is its speed. In the following, we will give the solutions to the field equations which are obtained by insertion of (C1) and (C2) into the field equations and Equation (74).

1. Spin 0 expansion waves

We assume that the expansion is given by $\theta = \hat{\theta}e^{ik_\sigma x^\sigma}$. The other kinematic quantities are then given by

$$\boldsymbol{a} = -s_\theta\frac{1+\frac{c_\theta}{2}}{1-\bar{c}_a}\theta\boldsymbol{n}, \tag{C3}$$

$$\boldsymbol{\omega} = \boldsymbol{0}, \tag{C4}$$

$$\boldsymbol{\kappa} = s_\theta\frac{c_{\kappa a}}{c_\kappa}\frac{1+\frac{c_\theta}{2}}{1-\bar{c}_a}\theta\boldsymbol{n}, \tag{C5}$$

$$\boldsymbol{\sigma} = -\frac{1+\frac{c_\theta}{2}}{1-c_\sigma}\theta\left(\boldsymbol{nn} - \frac{1}{3}\boldsymbol{1}\right). \tag{C6}$$

Using the tetrad, the spatial trace of the first order metric is given by $\gamma = \hat{\gamma}e^{ik_\sigma x^\sigma}$. The other components of $\psi_{\mu\nu}$ in the gauge (87) then are

$$\boldsymbol{A} = \boldsymbol{0}, \tag{C7}$$

$$\phi = -s_\theta^2\frac{1+\frac{c_\theta}{2}}{1-\bar{c}_a}\gamma, \tag{C8}$$

$$\boldsymbol{\zeta} = s_\theta\frac{c_{\kappa a}}{c_\kappa}\frac{1+\frac{c_\theta}{2}}{1-\bar{c}_a}\gamma\boldsymbol{n}, \tag{C9}$$

$$\boldsymbol{\gamma} = \frac{1+\frac{c_\theta}{2}}{1-c_\sigma}\gamma\left(\boldsymbol{nn} - \frac{1}{3}\boldsymbol{1}\right). \tag{C10}$$

2. Spin 1 acceleration-vorticity waves

If the acceleration is given by $\boldsymbol{a} = \hat{\boldsymbol{a}} e^{ik_\sigma x^\sigma}$ with the transversality condition $\boldsymbol{n} \cdot \boldsymbol{a} = 0$, the kinematic quantities are

$$\theta = 0, \tag{C11}$$

$$\boldsymbol{\omega} = -\frac{1}{2s_{a\omega}} \boldsymbol{n} \times \boldsymbol{a}, \tag{C12}$$

$$\boldsymbol{\kappa} = -\frac{c_{\kappa a}}{c_\kappa} \boldsymbol{a} + \frac{c_{\omega\kappa}}{2s_{a\omega} c_\kappa} \boldsymbol{n} \times \boldsymbol{a}, \tag{C13}$$

$$\boldsymbol{\sigma} = \frac{1}{s_{a\omega}\left(1 - c_\sigma\right)} \left(\boldsymbol{n}\boldsymbol{a}\right)_{\mathrm{sym}} . \tag{C14}$$

Analogously to electromagnetic waves, the condition of transversality reduces the number of independent modes to two. In terms of the tetrad, the potential $\boldsymbol{A}$ is given by $\boldsymbol{A} = \hat{\boldsymbol{A}} e^{ik_\sigma x^\sigma}$ with the gauge condition $\boldsymbol{n} \cdot \boldsymbol{A} = 0$. The other components of the first order tetrad are then given by

$$\phi = 0, \tag{C15}$$

$$\boldsymbol{\zeta} = \frac{1}{2s_{a\omega}} \left(\frac{c_{\omega\kappa}}{c_\kappa} - 1 \right) \boldsymbol{n} \times \boldsymbol{A} - \frac{c_{\kappa a}}{c_\kappa} \boldsymbol{A}, \tag{C16}$$

$$\boldsymbol{\gamma} = -\frac{1}{s_{a\omega}\left(1 - c_\sigma\right)} \left(\boldsymbol{n}\boldsymbol{A}\right)_{\mathrm{sym}} , \tag{C17}$$

$$\gamma = 0. \tag{C18}$$

3. Spin 2 shear waves

The shear waves are given by $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}} e^{ik_\sigma x^\sigma}$ with the transversality condition $\boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{0}$ and all other kinematic quantities vanishing, $\boldsymbol{a} = \boldsymbol{\omega} = \boldsymbol{\kappa} = \boldsymbol{0}$, $\theta = 0$. Since $\boldsymbol{\sigma}$ has five independent components and the transversality condition consists of three equations, there are two independent shear modes. If the tetrad is used, the solutions are the gravitational waves of Einstein gravity with $\boldsymbol{\gamma} = \hat{\boldsymbol{\gamma}} e^{ik_\sigma x^\sigma}$ and $\boldsymbol{n} \cdot \boldsymbol{\gamma} = 0$. All other components of the first order tetrad are zero, $\phi = \gamma = 0$, $\boldsymbol{A} = \boldsymbol{\zeta} = \boldsymbol{0}$.

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