Experimental Test of Luneburg's Theory. Horopter and Alley Experiments*

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Binocular visual space is, according to Luneburg, a hyperbolic space; two personal constants determine the geometry of subjective space of individual observers. Personal constants σ (the degree of depth perception) and K (the curvature of binocular space) were obtained by Luneburg's 3 and 4 point tests and served as predictors for frontal plane horopter and alley experiments. In the result, the observed distances of the straight-line horopters agreed with values predicted within certain limits of confidence. The predictions concerning the shapes of concave and convex horopters and the shapes of alleys tended to hold in the case of observers of good depth perception and a relatively low absolute value of K. Three different alley experiments yielded values of personal constants agreeing with values obtained by the 3 and 4 point tests again only within certain limits of confidence. Good agreement with prediction for all observers was obtained from an alley experiment in which physiological and technical difficulties were reduced. While a study of the constancy of σ and improvement of experimental conditions are needed, the measure of agreement obtained in the experiments reported gives considerable support for the hypothesis that binocular visual space is metric and hyperbolic.

INTRODUCTION

THE specific and unmistakable sensation of straightness of a line or planeness of a surface are prerequisites which permitted Luneburg to approach the elusive world of visual appearance in an attempt to describe subjective visual space in terms of Riemannian geometry of spaces of constant curvature.[1–3] The concept of identifying visually straight lines by the equations of geodesics and of considering as geodesic the distances visually assessed, is founded on the existence of the primary sensation of straightness or of definite convictions based on purely visual comparisons of distances.

For the unique, completely satisfying sensation of straightness an observer has to arrange the stimuli himself. Otherwise he will miss the specific purely visual sensation and will only classify an object as belonging to a class of objects known to be physically straight.

In the frontal plane horopter experiment the observer arranges the stimuli, seen against a uniform background in the horizontal plane of his eyes, until he attains the sensation of the stimuli lying on a straight line parallel to his forehead. The experiment demonstrates that the stimuli are found to lie on curves concave towards the observer if arranged at a near distance and on convex curves if presented far from the eyes. Only at some intermediate distance, which differs with different observers, the frontal plane horopters are straight in the vicinity of the median (Fig. 1).

In the Blumenfeld alleys[4] experiment the observer is faced with two rows of lights in the horizontal plane of the eyes extending from himself into depth on either side of the median. A pair of lights is fixed symmetrically at the ends of the two rows and the observer has to perform two tasks. First, he arranges the two rows of lights into two lines which appear to him straight and parallel. In the second experiment he concentrates on the task of forming two lines at a constant distance from each other. In both experiments the lights are found to lie on curves, but the results of the two experiments differ; the curves of the parallel alley converge more towards the eyes of the observer, and are found to lie inside the curves of the distance alley (Fig. 2).

The striking feature of these experiments is that the apparently parallel lines do not necessarily appear equidistant and that the curves built as equidistant do not appear to be straight and parallel—a paradox from the point of view of Euclidean geometry.

This issue emerges within the framework of Luneburg's theory as a demonstration that binocular space is non-Euclidean. He shows that, if the visually straight lines are defined as geodesics, and if the visually constant distance is expressed as constant geodesic distance, and if further an adequate choice is made

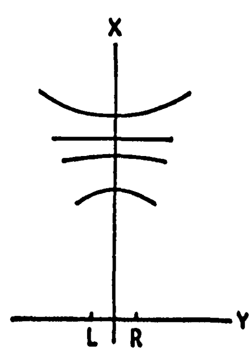

FIG. 1. Frontal plane horopters at different distances from the eyes (L,R).

* This paper reports part of a more general study of Luneburg's theory carried out for the degree of Ph.D. in Psychology at the University of London.

[1] R. K. Luneburg, *Mathematical Analysis of Binocular Vision* (Princeton University Press, Princeton, 1947).

[2] R. K. Luneburg, "Metric Methods in Binocular Visual Perception," in *Studies and Essays, Courant Anniversary Volume* (Interscience Publishers, Inc., New York, 1948).

[3] R. K. Luneburg, J. Opt. Soc. Am. 40, 627 (1950).

[4] W. Blumenfeld, Z. Psychol. u. Physiol. d. Sinnesorgane 65, 241 (1913).

amongst geodesics to represent the parallel curves, the parallel alley is found inside the distance alley (where it is found experimentally) only if the geometry of visual space is hyperbolic. Moreover, in accordance with this theory, the more the experimental parallel curves depart from the distance curves, the more an observer's space departs from Euclidean space and is curved—a hypothesis which can be tested quantitatively.

While investigating Luneburg's theory, we must be very clear about its scope of application. This theory claims to describe solely the binocular vision based on the convergence of the optic axes in the absence of monocular and experiential factors. Such conditions are approached, for example, when the observers pass judgements concerning shapes, distance between, or relative localization of small light points presented to them in darkness.

The theory asserts, first, that binocular visual space is a Riemannian space of constant negative curvature. Second, it attempts to measure visual space in terms of coordinates of stimulating points by means of a relationship between convergence angle and visual distance of points from oneself. The Riemannian metric, expressed in terms of coordinates of stimulating points, applies to every normally sighted observer; the sole factors which differ from individual to individual are the values of two personal constants which can be determined by experiment. One of these constants is σ, the degree of depth perception which is greater for observers of good depth perception and smaller for those less sensitive to the differences between convergence angles. The second constant is K, the curvature of visual space, a value which influences the shapes of the geodesics and indicates by how much the visual space of an observer departs from Euclidean space.

Luneburg's coordinates, as reported in his last paper,[3] are used in this study. For the description of experimental situations in *physical space, bipolar coordinates* are used [Fig. 3(A)]. These are for the horizontal plane:

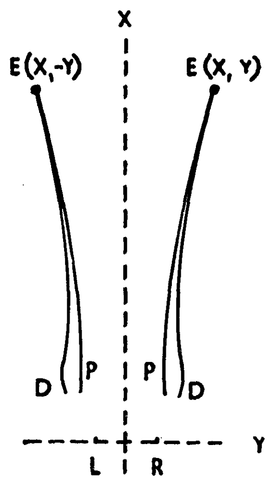

FIG. 2. Parallel (P) and distance (D) alleys passing through common points (E).

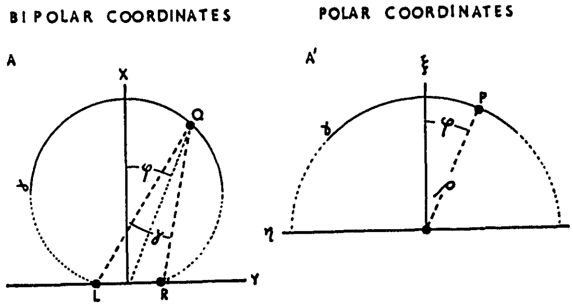

FIG. 3. The coordinates. Bipolar coordinates serve for physical space, polar coordinates for visual space.

angle γ (the bipolar parallax, approximating the convergence angle) and angle φ (the bipolar latitude). The axes of reference are y-axis passing through the rotation centers of the eyes (oriented positively to the right) and x-axis, orthogonal to y-axis at the midpoint between the rotation centers of the eyes. Angle γ, in terms of Cartesian coordinates, can be expressed with sufficient accuracy for our purpose as

$$\gamma=\frac{pd\cos^2\varphi}{x}, \tag{1}$$

where pd is the interpupillary distance measured in centimeters, and φ is given by

$$\tan\varphi=y/x. \tag{2}$$

A different set of coordinates is used for *visual space* assumed to be Riemannian. The positions of points in visual space will be specified by the coordinates of corresponding points on a conformal Euclidean model of visual space. The distances on this model are distorted: the connection between the distance ρ of a point from the origin of the coordinate system on the model and the geodesic distance S of the corresponding point in Riemannian space is expressed by $S=2/\sqrt{-K}$ arc $\tanh[(-K)^{\frac{1}{2}}\rho/2]$. For the horizontal plane either *polar coordinates* ρ and φ (the polar latitude) or the Cartesian coordinates ξ and η are used [Fig. 3(A′)]. The origin of the coordinate system is assumed to coincide with the apparent center of observation, the ξ and η axes correspond to subjective frontal and lateral directions. The familiar transformation formulas connect the Cartesian (ξ,η) and the polar coordinates:

$$\begin{aligned}\xi&=\rho\cos\varphi\\ \eta&=\rho\sin\varphi.\end{aligned} \tag{3}$$

While it is assumed that the physical axes x, y and the axes of reference for visual space ξ, η coincide directly in both spaces, and that the bipolar φ equals the polar φ, the theory reduces the *relation between physical and visual space* to the relation between γ and ρ. It is laid down first that the stimulus points subtending $\gamma=$constant, in physical space, are seen at a distance corresponding to $\rho=$constant, in visual space

TABLE I. Values of σ and K obtained by the 3 and 4 point tests for 30 observers. Daggers indicate eleven observers who took part in the frontal plane horopter and alley experiments.

Observer	Sex	Age years,months	σ	K
A.S.	m	6,10	10.8	−0.71
S.H.B.	f	7, 1	11.8	−0.94
M.F.	m	7, 6	8.3	−0.23
A.B.	f	8, 0	13.8	−0.91
M.E.	f	8, 6	10.7	−0.29
J.B.	m	11, 3	9.1	−0.40
S.B.	f	12, 9	7.4	−0.75
G.B.	f	13, 1	7.0	−0.62
B.W.	m	14, 2	9.9	−0.30
J.W.	f	16,11	11.9	−0.47
J.P.	m	18, 0	12.4	−0.52
E.K.	f	18, 6	14.8	−1.06
†B.A.	m	25	11.9	−0.47
S.P.	m	26	7.8	−0.48
†T.K.	m	29	11.7	−0.60
M.S.	f	30	6.8	−0.99
†A.J.	m	32	10.7	−0.45
†M.K.	f	37	8.7	−0.68
S.H.	f	38	8.5	−0.97
†M.V.S.	f	38	7.5	−0.79
†H.S.	f	38	9.7	−0.98
I.M.	f	39	0.0	−1.00
†K.D.	m	44	12.9	−0.66
M.H.B.	m	44	8.9	−0.79
†W.K.	m	47	16.9	−0.55
K.K.	m	48	5.3	−0.72
E.F.	f	48	6.4	−0.33
†M.B.S.	m	53	5.4	−0.57
†K.G.	m	59	8.6	−0.65
†S.V.S.	m	72	3.3	−0.05

(Fig. 3). Further, the relation between ρ and γ has been specified as

$$\rho = 2e^{-\sigma\gamma}, \tag{4}$$

where ρ is expressed as a function of convergence angle γ and σ constant for an observer, his degree of depth perception. The relationship (4) expresses the specific distortion of binocular space in comparison to physical space which is independent from curvature of visual space; it expresses, for instance, that the horopters or alleys are curves even for $K=0$ and Euclidean geometry. In addition to σ it is the value of K which determines the precise shape of horopter and alley curves.

Luneburg's theory implies that once the personal constants of an observer are known, the whole of the geometry of visual space for this individual is determined. The aim of this study was an experimental test of this contention. The constants σ and K were obtained for a group of 30 observers by Luneburg's 3 and 4 point tests described in his paper "The Metric of Binocular Visual Space,"† formulas (11.7) and (11.9) of this paper were used for computation of the values of σ and K. Table I lists the results.[5]

The sign of the obtained values of K and the interval $0>K>-1$ in which they are on the whole included supports the hypothesis of hyperbolic visual space. The objective of the present study was to test the significance of the numerical values of the personal constants obtained. Eleven observers out of the former group were available for more extensive studies. The values of constants σ and K obtained for them from the 3 and 4 point tests served as predictors of the results to be expected in the frontal plane horopter and in the parallel and distance alleys experiments. All experiments were carried out under darkroom conditions.

THE FRONTAL PLANE HOROPTERS

1. Experimental Situation

The principle of Helmholtz's[6] three-rod experiment served for the frontal plane horopter experiment. Three light points were presented to the observer. Two of these, $Q_1(\gamma_1, -\varphi)$ and $Q_2(\gamma_1, +\varphi)$, were fixed symmetrically on either side of the x-axis (Fig. 4). The third light point $Q_0(\gamma_0, 0°)$ was moved along the same axis by the experimenter, under the directions of the observer, who attempted to bring this third point into such a position that the three points appeared to him to lie on a straight line.

2. The Frontal Plane Geodesics

The theory identifies frontal plane horopters with frontal plane geodesics in visual space. The equation of these curves is

$$\frac{K}{4}(\xi^2+\eta^2)-1=C\xi, \tag{5}$$

where C is constant. K can be expressed by the use of another quantity:

$$\mu=\frac{\ln(1/-K)}{2\sigma}, \tag{6}$$

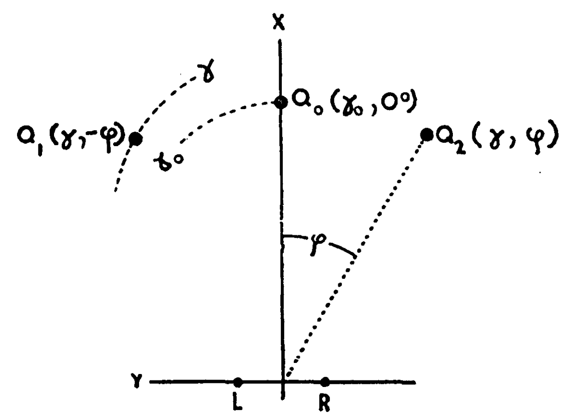

FIG. 4. Experimental situation in Helmholtz's three-rod experiment. $Q(\gamma, \pm\varphi)$ are the fixed lights; $Q_0(\gamma_0, 0°)$ is the light movable along x-axis.

† See reference 3, pp. 637 and 638.

[5] A. Zajaczkowska, Quart. J. Exptl. Psychol. VIII, 66 (1956). Table I is taken from this paper and lists the results of described experiments carried out with $(\gamma_1-\gamma_0)\sim 0.01$, $\gamma_1 \sim 0.06$, and values of $(\varphi_1-\varphi_0)$ ranging from 9.93° to 20.77° in the 3 point test, $(\gamma_1-\gamma_0)\sim 0.04$, $\gamma_1\sim 0.06$, and values of $(\varphi_1-\varphi_0)$ ranging from 2.29° to 8.03° in the 4 point test.

[6] H. L. F. Von Helmholtz, *Physiological Optics*, edited by J. P. C. Southall (Optical Society of America, Rochester, New York, 1925), Vol. 3, p. 319.

and the constant C is given for each observer by the coordinates of the intersection point of the horopter with x-axis, $(\gamma=\gamma_0)$, $(\varphi=0°)$. Hence by aid of (3) and (4), Eq. (5) may for our purpose be conveniently written in terms of bipolar coordinates and values of personal constants

$$\frac{\cosh[\sigma(\gamma+\mu)]}{\cosh[\sigma(\gamma_0+\mu)]}=\cos\varphi. \tag{7}$$

The curves computed from this equation approximate, in general outline, the curves obtained in actual experiments; at a certain distance they are straight; they become concave as they approach the eyes, and convex as they recede (Fig. 5). The distance from the eyes of the straight line horopter, denoted as a point X_0 on the x-axis, is a function of the constants σ and K; for any given pair of these values, X_0 can be found by solving for γ_0 the transcendental equation

$$\tan\frac{\gamma_0}{2}\tanh[\sigma(\gamma_0+\mu)]-\frac{1}{4\sigma}=0, \tag{8}$$

where μ is defined by (6). Then

$$X_0=\frac{pd}{\gamma_0},$$

where pd is the interpupillary distance.

3. Aim of the Experiment

(1) Our purpose was to compare the observed X_0 (the distance from the eyes of the straight line horopter) with the value X_0 predicted by aid of (8) on the basis of constants σ and K observed in the 3 and 4 point tests with the same observers.

(2) A further aim was to compare predictions concerning the shapes of the concave and convex horopters following from Eq. (7), with data observed in the experiment.

4. Experimental Conditions

Eight pairs of fixed lights were presented successively at distances from the eyes $x=50$, 65, 83, 108, 139, 180, 232, and 300 cm, respectively (i.e., at eight stations). The first five pairs were fixed at $\varphi=\pm10°$, the remaining three at $\varphi=\pm9.46°$, $\pm7.37°$, and $\pm5.71°$, respectively. All stimuli were presented at eye level.

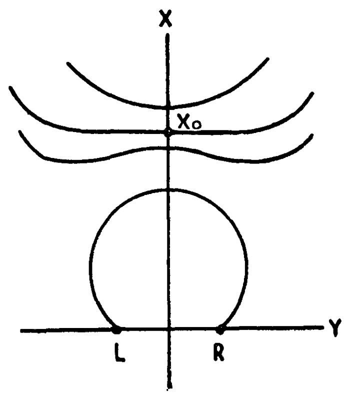

FIG. 5. Frontal plane horopters derived by Luneburg from Eq. (7).

TABLE II. The distance X_0 of the straight line frontal plane horopter from the eyes, predicted on the basis of the 3 and 4 point tests, and observed in the horopter experiment.

Observer	The 3 and 4 point tests σ	K	pd (cm)	X_0 predicted from Eq. (8) (cm)	X_0 observed in horopter experiment (cm)
A.J.	10.68	−0.446	6.48	107.9	108
T.K.	11.69	−0.600	6.70	114.9	106
M.K.	8.67	−0.680	6.40	79.2	108
M.V.S.	7.50	−0.785	6.25	64.6	121
B.A.	11.87	−0.470	6.40	117.3	137
S.V.S.	3.32	−0.046	6.60	42.5	44
H.S.	9.70	−0.980	6.30	79.6	78
K.G.	8.58	−0.650	6.90	85.3	82
W.K.	16.90	−0.550	5.80	146.5	145

5. Procedure

As in all experiments described in this study, the observer, while his head was being arranged in the headrest, was specifically asked to use his eyes freely and actively and to converge on each light in turn. It was convenient to separate this instruction from the formulation of the task. Each observer then directed the experimenter in adjusting the three lights on each of the stations to the same brightness. A number of training trials were carried out.

The test consisted of six determinations on each station; two by gradually increasing the distance of the central light from the observer, two by decreasing this distance, and two from an unspecified position between the two extremes. The six variants were presented in random order. The instruction was as follows:

"Place the central light exactly on a straight line between the two fixed lights."

6. The Measurements

To evaluate experimental values of X_0, the deviations of the central light from the straight line joining the two lateral fixed lights were expressed in centimeters. Positive values were used for increments of X with respect to the eyes (concave horopters), negative values for decrements (convex horopters). The six determinations were averaged and means of deviations for each of eight stations were plotted against the distance from the eyes. The best fitting curve was drawn between the experimental points. The intersection of the curve with the x-axis determined the value X_0.

7. Results

Nine observers were tested. Table II compares the values for X_0 predicted from Eq. (8) and the values X_0 observed. The characteristic patterns of the curves obtained are illustrated in Fig. 6.

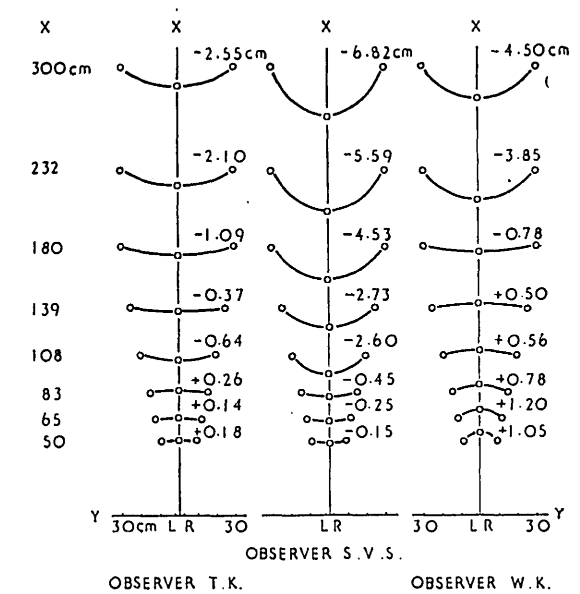

FIG. 6. Results of the frontal plane horopter experiment for Observer T.K., $\sigma=11.7$, $K=-0.60$; Observer S.V.S., $\sigma=3.3$, $K=-0.05$; and Observer W.K., $\sigma=16.9$, $K=-0.55$ (the 3 and 4 point tests values of constants). Distances from the eyes marked on the left. Deviations of the central point from the straight line are not drawn to scale; exact values in cm marked on each curve. The smaller is the value of σ (the poorer is depth perception) the nearer to the eyes is the straight line horopter.

(1) The scatter diagram of values X_0 predicted (assuming the 3 and 4 point test data to be free from error) and the values X_0 observed is given in Fig. 7. On this diagram the regression line and 0.95 confidence limits are marked. We can read from this diagram, for example, that if it has been predicted on the basis of the 3 and 4 point tests, that the distance of the straight-line horopter for an observer should be 75 cm, we may expect with some confidence to obtain for the same observer in the horopter experiment a value between 67 cm and 108 cm. The product moment correlation coefficient, for the group of nine, between the predicted and observed values is $r=0.77$ (significant 0.02 level).

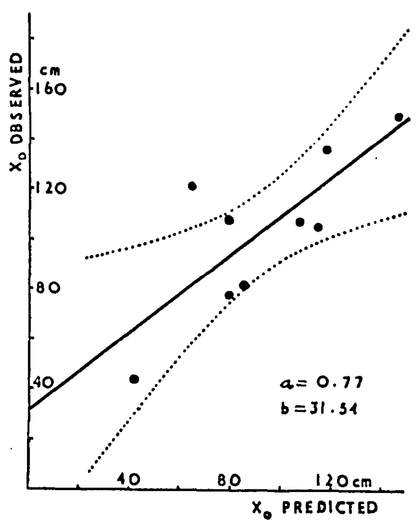

FIG. 7. Scatter diagram of values of X_0 predicted on the basis of the 3 and 4 point tests and observed in the horopter experiment. Heavy line is the regression line (assuming the 3 and 4 point test data as free from error); dotted lines indicate the confidence limits 0.95 level; values a and b denote the slope and the intercept of the regression line.

(2) Analysis of other than straight-line horopters at distances from the eyes other than X_0 carried out by the aid of equation of frontal geodesics (7) indicated discrepancies between observed and predicted values in particular for distances far from the eyes. The predicted and observed values of the deviations from straight line 300 cm distant from the eyes are listed in Table III; the discrepancies are particularly pronounced for small

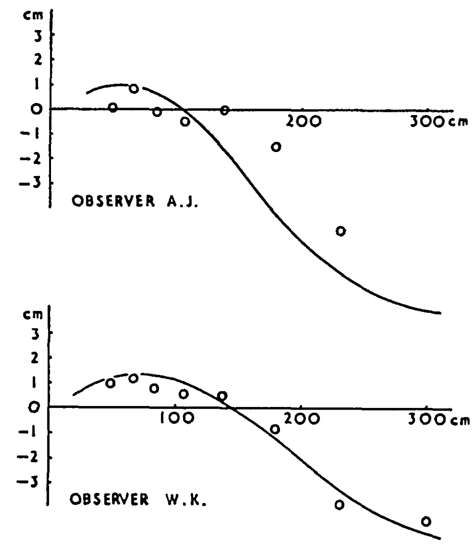

FIG. 8. Deviations of the frontal plane horopters from straight line plotted against distance from the eyes. Data for Observer A.J., $\sigma=10.7$, $K=-0.45$, and Observer W.K., $\sigma=16.9$, $K=-0.55$ (the 3 and 4 point tests values of constants). Distances of the central point $(x,0)$ from a straight line passing through lateral points $(x,\pm y)$ of a horopter are marked on the vertical axis; positive values denote the concave horopters, negative values the convex horopters with respect to the eyes. Distances x of the lateral points $(x,\pm y)$ are marked along the horizontal axis. Circles indicate experimental values; curves correspond to the prediction derived on the basis of the 3 and 4 point tests.

values of σ (poor depth perception) and large absolute values of K. On the other hand, diagrams in Fig. 8 illustrate adequate predictions with regard to eight distances from the eyes in the horopter experiment for two observers having good depth perception and relatively low absolute values of K; the experimental values are found on these diagrams scattered around curves corresponding to the predicted deviations of the horopter curves from a straight line.

THE PARALLEL AND DISTANCE ALLEYS

8. Experimental Situation

In the *parallel alley* experiment the observer was confronted with two rows of lights lying in the horizon-

TABLE III. Deviations of frontal plane horopters from a straight line. Distance d (in cm) of the central point $(x, 0)$ of the horopter from the straight line passing through the lateral points (300 cm, ± 30 cm), predicted from the 3 and 4 point tests by the aid of Eq. (7), and observed in the horopter experiment. The negative sign denotes convex horopters with respect to the eyes.

0	A.J.	T.K.	M.K.	M.V.S.	B.A.	S.V.S.	H.S.	K.G.	W.K.
d predicted	−8.2	−8.7	−18.0	−28.5	−7.4	−17.9	−27.7	−15.4	−5.1
d observed	−9.3	−2.6	−5.7	−3.5	−1.8	−6.8	−8.4	−5.3	−4.5

tal plane of his eyes. The two end lights $E(x_0, \pm y_0)$, the farthermost from the eyes, were fixed. Pairs of lights $Q_n(x_n, \pm y_n)$ were movable, sideways, both at the same distance x from the eyes (Fig. 9). The task of the observer was to select a position for the movable lights such as to obtain an impression of two parallel rows of lights.

In the *distance alley* experiment, in addition to the end lights, of the same coordinates as in the first experiment, only one more pair of movable lights $Q_n(x_n, \pm y_n)$ was presented at a time. The observer had to decide the position of the movable pair so that the interval between the two lights appeared to him equal to that between the fixed end lights. A number of such movable pairs were presented *successively*.

9. Luneburg's Theory of Alleys

The physical *parallel curves* extend from the end points $E(x_0, \pm y_0)$ towards the eyes of the observer. In visual space straight and parallel lines pass through a pair of apparent points $P(\xi_0, \pm\eta_0)$ symmetrically with regard to the median. In Riemannian space the geodesic lines correspond to straight lines, but in this geometry the concept of parallelism in the Euclidean sense is missing. To define parallel alleys, out of an infinite number of geodesics passing through points $P(\xi_0, \pm\eta_0)$ without intersecting, Luneburg selected those which

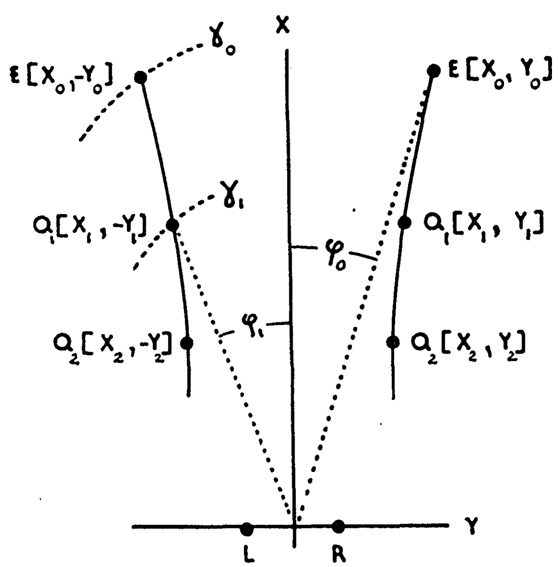

FIG. 9. Experimental situation in the alley experiment. $E(x_0, \pm y_0)$ are the fixed lights; $Q_n(x_n, \pm y_n)$ are the lights which are movable laterally (x = constant).

share with Euclidean lines parallel to ξ axis the property of being normal to η axis. The equation of these geodesics is

$$\frac{K}{4}(\xi^2+\eta^2)-1=C\eta, \tag{9}$$

where C is a constant. In terms of bipolar coordinates and the personal constants, using (3), (4), and (6) the same equation can be written

$$\cosh[\sigma(\gamma+\mu)]=C\sin\varphi. \tag{10}$$

The *distance curves* are given in Luneburg's system by the equation of lines of constant geodesic distance, which has the form

$$\frac{K}{4}(\xi^2+\eta^2)+1=C\eta, \tag{11}$$

or expressed in bipolar coordinates

$$\sinh[\sigma(\gamma+\mu)]=C\sin\varphi. \tag{12}$$

The constants C in the above four equations are determined by the coordinates of the points common both to parallel and distance alleys.

The personal constants σ and K can be determined by the alley experiment. If we denote the y-axis intercepts of the tangents to a parallel and a distance curve at their common point as b_P and b_D, respectively (Fig. 10), and express these intercepts by derivatives $d\varphi/d\gamma$, it can be found by differentiating Eqs. (10) and (12) of the two curves expanded into series that

$$\sigma=\frac{1}{pd\tan\varphi_0}(b_Pb_D)^{\frac{1}{2}}, \tag{13}$$

$$K=e^{2\sigma\gamma_0}\frac{\left[\dfrac{b_P}{b_D}\right]^{\frac{1}{2}}-1}{\left[\dfrac{b_P}{b_D}\right]^{\frac{1}{2}}+1}, \tag{14}$$

where φ_0 and γ_0 are the bipolar coordinates of the points common to both alleys and pd is the interpupillary distance.

10. Aim of the Experiment

(1) Detailed predictions concerning the shapes of the whole parallel and distance curves were worked

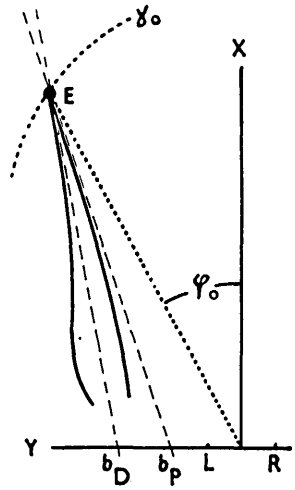

FIG. 10. The y-axis intercepts b of the tangents to alley curves at the end points E. The theoretical intercepts increase with larger values of φ_0.

out for several observers from Eqs. (10) and (12) on the basis of the 3 and 4 point tests constants for the purpose of comparing them with experimental alley curves.

(2) Another objective has been to compare values of σ and K observed in the 3 and 4 point tests with the values for the same constants derived from the alley experiments.

11. Experimental Conditions

In the classic alley experiments the fixed pair of stimuli at the far ends of an alley were usually four meters distant from the eyes of the observer and ± 30 cm distant from the median, so that in terms of bipolar coordinates, angle φ_0 was small (4.3°). It is readily seen from (13) that for small values of σ (poor depth perception) the observer is expected, in accordance with Luneburg's theory, to build both parallel and distance alleys strongly convergent towards his eyes (small values for b_P and b_D). It is also seen from (14) that in the case of values of K approaching -1, the parallel alleys are expected to approach the φ_0 direction lines of their end points (and b_P the value zero). To meet this situation, the observer must encounter difficulties arising from the presence of double images of the light points. On the other hand, the interposition of the images may reduce the experiment to the condition that, while the left side of the alley is observed by the right eye only, the right side may be seen only by the left eye. It was to be expected that this condition would be automatically avoided by the observers, who in the case of small σ would build both alleys bending away from the x-axis (spuriously increasing the value of σ), while in the case of large absolute values of K the curves of parallel alley would bend away from x-axis (decreasing the value of K). At the same time, it again follows from the equations of the alleys that the observers of good depth perception and small absolute value of K would not encounter these difficulties and might be expected to meet prediction. Three different experimental conditions were therefore used: (1) *Classic alleys*, reproducing conditions similar to classic experiments; (2) *intermediate alleys*, where the bipolar latitude φ_0 of the end points was increased; and (3) the *broad alleys*, in which φ_0 was considerably increased in an attempt to approach (within the limits of available laboratory space and apparatus) conditions equivalent for all observers. The conditions were as follows.

(1) In the *classic alleys* the position of the end points was E(420 cm, ± 30 cm), $\varphi_0 = 4.1°$. Eight pairs of movable lights, denoted as stations 1 to 8, were used. These were movable along lines, $x=$ constant, with x successively equal to 300, 232, 180, 139, 108, 83, 65, and 50 cm.

(2) In the *intermediate alleys* experiment, the position of the end points was E(300 cm, ± 30 cm), $\varphi_0 = 5.75°$. Six pairs of movable lights, denoted as stations 1 to 6, were used, with x successively equal to 180, 139, 108, 83, 65, and 50 cm.

(3) In the *broad al'eys* experiment the position of the end points was E(139 cm, ± 28 cm), $\varphi_0 = 11.4°$. Four pairs of movable lights, denoted as stations 1 to 4, were used. These corresponded to x equal successively to 108, 83, 65, and 50 cm.

12. Apparatus

The alley board reproduced Blumenfeld's[4] arrangements. The stimuli used were 6.3-volt lamps with rheostat control for each light separately. The filaments of lights were not shown: to counteract interposition of the light points narrow turrets were built over the lamps and the apertures 0.8 mm drilled on the top of the turrets were illuminated by reflected light. The stimuli were shown strictly at eye level.

13. Procedure

In the *parallel alley* experiment all lights were first arranged in two physically parallel rows. The observer, when his head was already in the headrest, directed the experimenter in bringing all lights to the same apparent brightness. The fact that the lights could be moved was then demonstrated and the fixed end lights were pointed out. A code was established to refer to lights by number and to use words "nearer" and "farther" with respect to the median. The instruction to build parallel alleys was as follows:

"Out of these lights form two parallel lines. They should neither converge nor diverge. Pay attention to the lines as a whole. The lines should be straight, of course, but first of all they must be parallel. Make them really parallel, dead parallel."

If questions occurred, they were cut short to eliminate the attitude of reasoning. The trials followed beginning with the two lights at the furthermost station 1. Great care was taken in determining their position; the observer was given ample time to satisfy himself that the segments he formed were parallel. Questions: "Are the two (specified) line segments parallel?" and reminders "They must be parallel" were repeated. The position of station 1 was further carefully checked in conjunction with that of station 2 and again with station 3. Then, excepting for the first, which was the training trial, the observer was encouraged to act rather quickly with respect to the remaining lights and to be led by instantaneous judgment. Particularly, he was not allowed to exhaust himself on the pairs of lights nearest to the eyes where it was expected that he would have to struggle with physiologic diplopia. Corrections were carried out, readjusting those segments of the lines which were reported by the observer as not parallel, by beginning again with the furthermost point of the segment or rebuilding the whole curve again. It was noted that, if single points were corrected, the observer often found that he had irretrievably spoiled his alley. In addition to the training trial, from two to four test trials were carried out.

After a rest interval, the observer was asked to build the distance alley. In addition to the end lights, the same as in former experiment, only one other pair of lights was shown at a time. The instruction was:

"Now, forget completely about parallel lines or segments directed towards you. Look at the two end lights. How far is it from the left light to the right light? Then look at the new lights. How far is it from the left light to the right light? Make these two distances equal. Do not think about anything else. Think only: From here to there; from here to there. It must be equal."

The consistency of the results in the distance alley experiment was from the start remarkable and no training was required. At the end of the experiment, all lights which were adjusted pair by pair were lit up, and the observer was asked whether the lines he successively formed appeared to him to be straight and parallel.

It will be noted that in the first instruction there was no emphasis on the straightness of lines, and there was no mention of symmetry with respect to the x-axis in both experiments. This was done in view of the fact that the observers were spontaneously inclined to attend to straightness and symmetry to the detriment of the main task.

14. Measurements

The experimental values of y for given values of x were averaged for all settings of the alley and were again averaged for the left and right sides of an alley. Thus each experimental y is a mean of four to eight determinations. On the graphs one only (the left) side of a parallel or distance alley is given in terms of these averages, and the mirror image of the curve, symmetrical with respect to the x-axis, is to be imagined on the right-hand side.

The intercepts of the tangents at the end point to the experimental curves with the y-axis were computed from the formula obtained by passing a straight line through the end point and the nearest experimental point:

$$b=y-\frac{y-y_1}{x-x_1}x. \tag{15}$$

This expression was used for both b_P and b_D; the values y and x are the coordinates of the end point of the alleys and x_1 equals 300, 180, and 108 cm for the classic, intermediate, and broad alleys, respectively; in each case, y_1 is the corresponding experimental value. The values of σ and K were computed by introducing the experimental values of the intercepts b_P and b_D into Eqs. (13) and (14). (In the case of broad alleys, these last two formulas had to be replaced by a more exact method.)

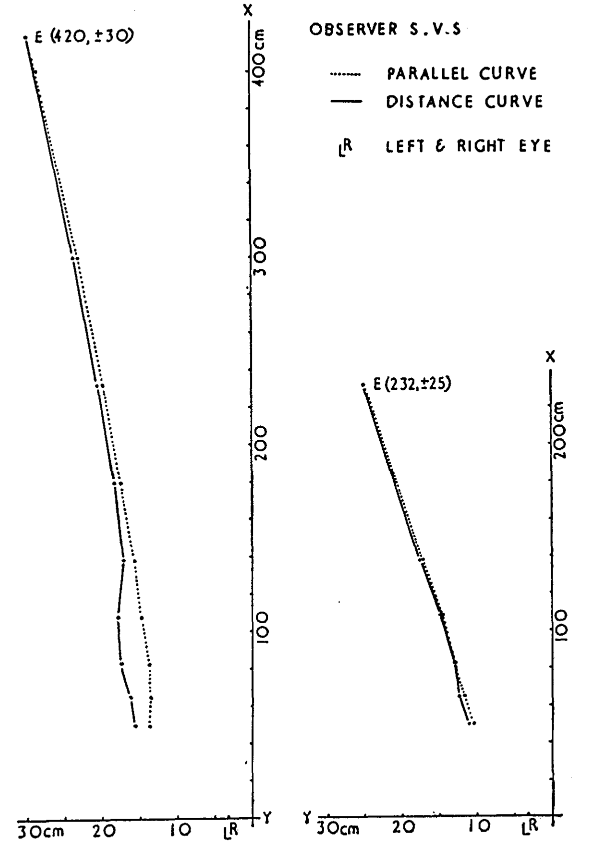

FIG. 11. The results of two alley experiments for Observer S.V.S. having small absolute value of K. Values of personal constants observed in the 3 and 4 point tests: $\sigma=3.3$, $K=-0.05$. Left side of each alley presented as an average.

15. Results

Eleven observers took part in the alley experiments. Twenty experimental series were carried out; in all these, the parallel alleys were found to lie inside the distance alleys. Individual differences in the patterns of parallel and distance alleys appeared very distinctly, as examples in Figs. 11 and 12 indicate. In the distance

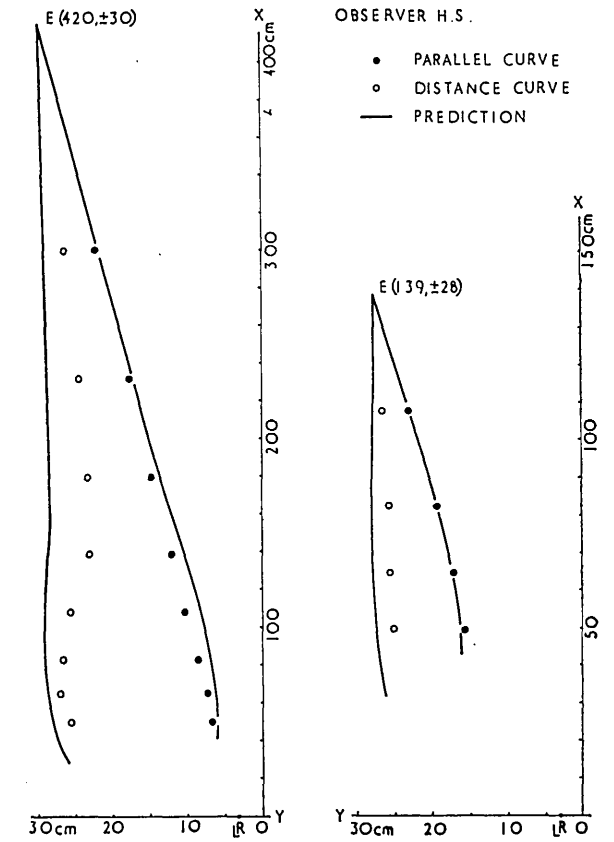

FIG. 12. Predicted and observed alley curves for Observer H.S. having large absolute value of K. Prediction based on the 3 and 4 point tests values of constants: $\sigma=9.7$, $K=-0.98$. Left side of each alley presented as an average.

alley experiment, all observers stated that, when the alley was illuminated after having been built up pair by pair, the lines they had so formed appeared to them to be neither straight nor parallel.

(1) Presentation of the experimental data in conjunction with the curves computed from the equations of parallel and distance alleys for the 3 and 4 point tests constants (Figs. 12, 13, and 14) indicates fair agreement between data predicted and observed. It must be stressed, however, that illustration by computed curves is given only for those observers whose constants permitted the expectation of adequate performance in the alley experiments. Comparison of the predicted and the observed curves shows that those parts of the curves remote from the eyes correspond best to prediction. Further, it is noticeable that in the classic alley condition the experimental parallel alleys bend away from the x-axis, corroborating the reasoning which suggested the broad alleys condition. Lastly, Figs. 12 and 14 illustrate remarkable agreement between the curves observed and predicted in the case of the broad alleys experiment.

(2) The values obtained by the 3 and 4 point tests are compared in Table IV with the values yielded by the 3 alley experiments for the personal constants. The product moment correlation coefficients between the 3 and 4 point tests and the alley values for personal constants are given below together with their significance level:

	σ	K
Classic alleys (9 observers)	0.75(0.02)	0.84(0.005)
Intermediate alleys (6 observers)	0.72(–)	0.92(0.01)
Broad alleys (5 observers)	0.91(0.05)	0.84(–).

The smallness of our samples does not permit the drawing of conclusions from this high and significant correlation.

The scatter diagrams of the values for σ and K obtained by the 3 and 4 point tests and by the alleys are given in Fig. 15. On each of the diagrams the regression line (assuming, again, the 3 and 4 point tests data as free from error) and 0.95 confidence limits are indicated. We can read, for example, from the diagrams in Fig. 15 that for values of σ equal to 8 obtained by the 3 and 4 point tests, we can expect to obtain, by the classic type of alleys, values in the interval between 9.4 and 16.4, while by the broad alleys we can expect values ranging from 5.5 to 12.1.

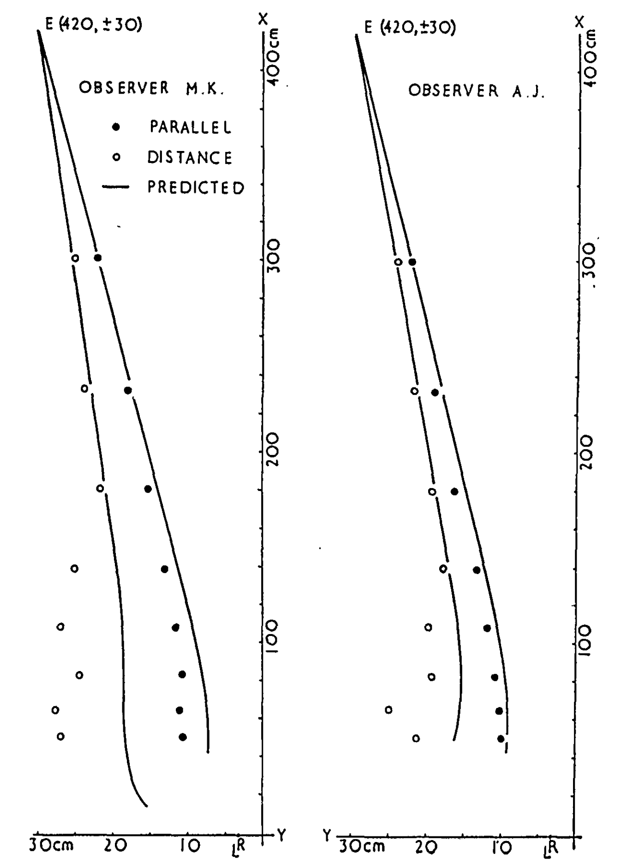

FIG. 13. Predicted and observed alley curves for Observer M.K., $\sigma=8.7$, $K=-0.68$ and Observer A.J., $\sigma=10.7$, $K=-0.45$ (the 3 and 4 point tests values of constants).

The confidence limits are rather wide, in particular for the broad alleys, as very small samples of observers were used. For much larger samples greater precision can be expected. Nevertheless, the data indicate that prediction with the aid of Luneburg's theory is within certain limits possible.

Direct inspection of the values of σ and K in Table IV and of the scatter diagrams indicates:

(a) In comparison with the 3 and 4 point tests results the values of σ are considerably increased by the classic alleys, while in this last condition the values of K are diminished.

(b) It is seen from Fig. 15 that the values of σ obtained by the classic alleys are on the average 5 units larger (b=5.025) than the values obtained by the 3 and the 4 point tests. This error may be associated with the difficulties in fusion and arising from the interposition of images which have already been discussed. In the broad alleys, the error expressed as the intercept of the regression line is smaller (b=2.127).

(c) With respect to K, however, we observe that the highest correlation, r=0.92, was obtained for the intermediate alleys, a type near to the classic alleys, and that the error expressed by the intercept of the regression line is absent with respect to K in both the intermediate and in the classic alleys. This indicates that greater distance from the eyes favors reliable determination of K.

(d) It is probable that broad and long alleys would provide the best conditions for the determination of both personal constants.

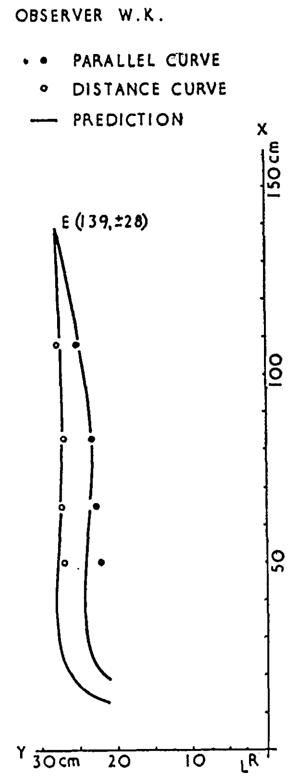

FIG. 14. The broad alley experiment with Observer W.K.; Values of constants observed in the 3 and 4 point tests: σ=16.9, K=−0.55; Values of constants observed in the alley experiment: σ=17.8, K=−0.57.

DISCUSSION

To sum up our results:

(1) The predictions concerning the *distance from the eyes of the straight line frontal plane horopter* were confirmed by the values observed within certain limits of confidence.

(2) In the case of observers having good depth perception and relatively low absolute value of K, the observed shapes of the *concave and convex frontal plane horopters* were nearer to those expected than in the case of other observers.

(3) With the classic type of alleys, it was possible to predict the *shapes of parallel and distance curves* again only for observers with good depth perception and not very high absolute value of K. With the broad alleys condition, when the difficulties arising from physiological and technical reasons were reduced, the agreement between the curves predicted and observed was on the whole close for all observers.

(4) The *alley* experiments indicated that predictions with regard to the *values of constants σ and K* are possible within certain limits of confidence.

TABLE IV. The values of constants σ and K, obtained by the 3 and 4 point tests, compared with the values obtained by the alley experiments for three types of alleys.

			Parallel and distance alleys					
	3 and 4 point tests		Classic $E(420, \pm 30)$ $\varphi_0=4.1°$		Intermediate $E(300, \pm 30)$ $\varphi_0=5.75°$		Broad $E(139, \pm 28)$ $\varphi_0=11.4°$	
O	σ	K	σ	K	σ	K	σ	K
A.J.	10.7	−0.45	11.3	−0.40				
T.K.	11.7	−0.60	20.2	−0.29				
M.K.	8.7	−0.68	10.6	−0.57				
M.V.S.	7.5	−0.79	9.0	−0.65				
M.B.S.	5.4	−0.57	10.9	−0.28				
K.D.	12.9	−0.66			16.7	−0.28		
B.A.	11.9	−0.47			10.6	−0.37	9.1	−0.26
S.V.S.	3.3	−0.05	13.9	−0.08	9.6	−0.05	5.6	−0.18
H.S.	9.7	−0.98	14.9	−0.69	9.0	−0.60	9.5	−0.72
K.G.	8.6	−0.65	16.1	−0.26	12.2	−0.47	10.3	−0.34
W.K.	16.9	−0.55	26.4	−0.29	15.9	−0.30	17.8	−0.57

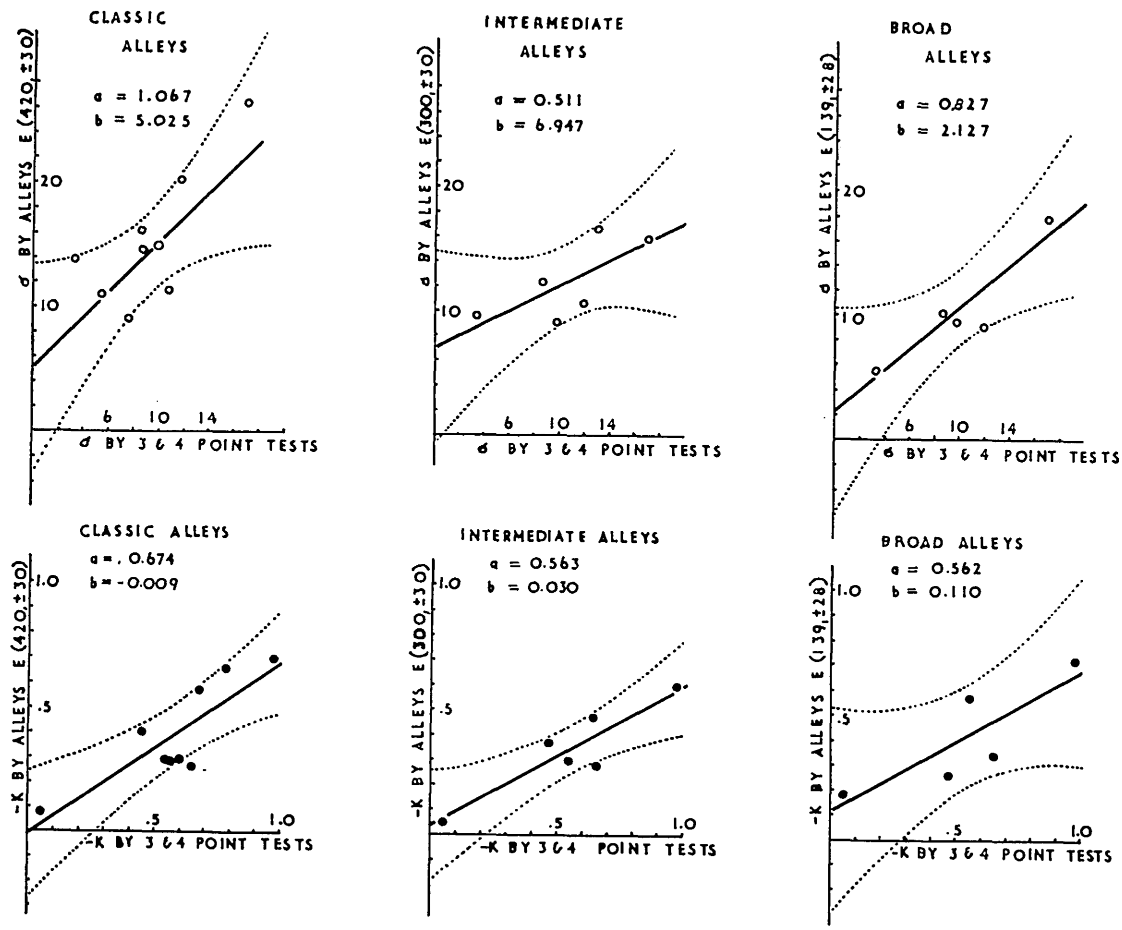

FIG. 15. Scatter diagrams of values of σ and K obtained by the 3 and 4 point tests and of the values for these constants obtained from the alley experiments. Heavy lines are the regression lines (assuming the 3 and 4 point test data as free from error); dotted lines indicate the confidence limits 0.95 level; values a and b denote the slope and the intercept of the regression line.

(5) The observed values of σ were found nearest to the values predicted in the broad alleys experiment.

To appreciate the significance of these results let us consider again the assumptions of the theory, the experimental conditions and the method of evaluation of findings.

16. Subjective Criteria

Visual straightness, equality or inequality of visual distances or subjective parallelism allowed Luneburg the building up of a mathematical construct implying an intuitive framework of visual measurements. (The experiment also indicates existence of a criterion which could be interpreted as subjective orthogonality expressing itself as a need for a symmetric arrangement.) Visual straightness obviously differs from the "straightness" we attribute, in a total act of perception, to objects manufactured with increasing perfection in our physical surroundings. The sensation of striking purely visual straightness with regard to a major portion of binocular space can be obtained in successful horopter or parallel alley experiment; *can*, but not always *is* obtained. The observers are very often dissatisfied with their efforts; some never reach during the experiment the sensation which is unfamiliar in ordinary surroundings. Yet this criterion has to be reached in all its authenticity to permit equations of geodesics to be used. Other subjective criteria must also be genuinely attained to allow for conclusions on the metric of visual space.

17. Effective Convergence

Assuming that nonbinocular and experiential factors are eliminated from the enquiry, the theory is concerned with subjective counterparts of convergence. But surely it assumes effective convergence. The difficulties due to unfusable disparities and the resulting lack of subjective localization in the case of points lying near the line, φ= constant, but subtending different convergence angles, have already been discussed in conjunction with appropriate conditions for the alley experiments. Another limitation exists with regard to the lateral extent of effective convergence. The field of effective binocular vision is narrow with some observers. The departures from theoretically expected parallel and distance alleys in those parts close to the eyes (see Fig. 13, for example) can be attributed to the fact that they lie outside of the field of effective convergence. Yet the theory must be tested within the field to which it claims to apply.

18. Mathematical Conditions

Further, the provisions of experimental arrangements are assumed to reproduce abstract mathematical conditions. But the images of the light points differ grossly from abstract points. The task of squeezing a number of sizeable images into a narrow angular sector as theoretically expected in the classic parallel alleys is practically impossible. More adequate or fewer stimuli are required. Or instead of *simultaneous* parallel alleys (of which the principle consists in illuminating the whole alleys throughout the experiment) *successive* parallel alleys could be used (in which only one pair of points is adjusted at a time by the observer). At best only one pair of points to be adjusted could suffice for derivation of personal constants from alleys with accuracy and economy in stress.

19. Eye Defects

The theory claims to apply to normal sight only. Yet hardly any of our observers was undisturbed by the asymmetric appearance of the lateral fixed lights in the horopter experiment. On the other hand, while freedom to satisfy the need for symmetry in the alleys experiments was given, hardly any physical curves were found symmetric. Aniseikonia was also a disturbing factor in the 3 and 4 point experiments. The theory is based on the assumption that the size of frontal images on the two retinas is the same. The distortion of the perceived shapes in the case where one retinal image is larger due to an eye defect, or induced artifically by an afocal magnification lens, can be predicted from the theory. For example, a prediction was derived for a stimulus configuration in the 3 point test that with a 5% magnification lens in front of the right eye of an observer having $\sigma=10$, $K=-1$, an approximate value of $\sigma\sim 30$, should be obtained from the experiment as an artifact. An actual experiment with observer H.S., $\sigma=9.7$, $K=-0.98$, yielded value of $\sigma\sim 28$. The opposite effect is observed with the same stimulus configuration if the size lens is placed in front of the left eye.

20. Lack of Training

A point was made in this enquiry of using uniformed observers; any measure of agreement in their case means more than agreement in the case of the observers acquainted with the theory. But most of our observers were also untrained. The use of the angle of convergence is replaced to a large extent in adult life by reliance on learned visual clues. However, effective use of convergence can be restored by training. Possibly this mechanism, singled out by Luneburg, served us better at one time as experiments with children indicate. Both the experiments with those of our observers who acquired considerable training and the experimental results of the highly trained observers of the Knapp Memorial Laboratories[7] are in very close agreement with expectation based on Luneburg's theory. A great value of this theory may therefore lie in the rehabilitation of active convergence as against the so-called visual cues which are more conceptual than visual.

21. Experimental Arrangements

In comparison with the earlier study of the alleys by Hardy, Rand, and Rittler[8] several alterations have been introduced in our experiments. The stimuli were presented strictly at eye level (they were presented 5.5 cm below eye level in the study quoted), no lights were used to mark the median, some changes have been introduced into the spacing of the intervals along the x-axis, the instruction was made shorter and more emphatic and a start was made in exploring new conditions in order to reduce physiological obstacles (broad alleys). In the result, while the conclusion from the earlier study quoted was that "the alley experiment presents insuperable difficulties to practical application" and that this experiment "is inadequate as a test of the personal constants σ and K,"‡ the experiments reported in the present paper indicate that the alley experiment is in fact promising in this respect. The need for further careful study of correct experimental conditions cannot be overemphasised.

22. The Predictors

The standard errors, expressed as percentages of values of σ and K, observed in the 3 and 4 point tests are estimated as equal to 20 and 17%, respectively. Apart from the consistency of measurements the constant error due to asymmetries between the eyes affects the results of these experiments. Further, the 3 point test is very sensitive to lack of training, in particular in the case of observers having poor depth perception; the performance of these observers is probably more influenced by experiential factors than that of keen-sighted observers. It is difficult to assess validly the low degree of depth perception; therefore, the relatively high absolute values of K following for small values of σ as a numerical result from the 4 point test are unreliable. In the instance, for example, of the discrepancies between the observed and predicted deviations of the frontal plane horopters from straight line 300 cm distant from the eyes (Table III), no

[7] Hardy, Rand, Rittler, and Blank, "The geometry of binocular space perception" (Final Report to U. S. Office of Naval Research from the Knapp Memorial Laboratory of Physiological Optics, Columbia University College of Physicians and Surgeons, New York, 1953).

[8] Hardy, Rand, and Rittler, Arch. Ophthalmol. (Chicago) 45, 53 (1951).

‡ See reference 8, p. 63.

TABLE V. Values of ω predicted from Eq. (18) for the 3 and 4 point test values of σ and K, compared with experimental values of ω derived by the method of Hardy and his group from the classic ($\gamma_0\sim0.015$) and broad ($\gamma_0\sim0.045$) alley experiments.

O	γ_0 (Classic alleys)	γ_0 (Broad alleys)	ω Values predicted from Eq. (18) in accordance with Luneburg.		ω Values obtained from alley experiments by the method of Hardy *et al.*	
H.S.	0.0149	0.0436	2.560	1.547	2.162	1.387
K.G.	0.0163	0.0477	1.736	1.194	1.329	1.045
W.K.	0.0137	0.0401	1.349	0.792	1.014	0.797
B.A.	0.0211[a]	0.0442	1.189	0.859	0.959	0.899

[a] Observer B.A. was tested using intermediate alleys condition, ($\gamma_0=0.0211$).

satisfactory evidence is at hand to conclude whether these results invalidate Eq. (5) or (4) or whether the 3 and 4 point tests failed to determine the values of σ and K adequately.

23. The Mapping Function

Considering the present stage of experimental exploration we have to recall that the theory itself was left by its author at a stage of theoretical exploration. The major contentions of the hypothesis are that visual space is capable of mathematical description, that the appropriate geometry is hyperbolic, and that it is the angle of convergence which relates physical to visual space. The form, however, of the relationship between the convergence angle and ρ, the mapping function used throughout this study: $\rho=2e^{-\sigma\gamma}$(4), was only tentatively suggested by Luneburg. A mapping function is indispensable for a quantitative approach to visual space and the evaluation of experimental data.

A different mapping function, namely relationship $r=r(\Gamma)$, was introduced into Luneburg's theory by Blank and Hardy.[9,7] Here r is the radial coordinate replacing ρ; $\Gamma=\gamma-\gamma_0$; and γ_0 is the convergence angle which the eyes subtend with the furthermost point within the configuration seen at one time. The relationship between Γ and r can be determined only by experiment. The value of r for $\Gamma=0$ is constant for a given observer,

$$r=r(0)=\omega. \tag{16}$$

The relationship between r and ρ is given by

$$r=2\text{ arc tanh}\left[\frac{\sqrt{-K}}{2}\rho\right]; \tag{17}$$

(the radial coordinate r is the geodesic distance of a point from the origin multiplied by $\sqrt{-K}$ constant for an observer). Expressing r by Luneburg's $\rho=2e^{-\sigma\gamma}$ we have

$$\begin{aligned} r&=2\text{ arc tanh}[(-K)^{\frac{1}{2}}e^{-\sigma\gamma}]\\ &=2\text{ arc tanh}[(-K)^{\frac{1}{2}}e^{-\sigma\Gamma}e^{-\sigma\gamma_0}]\end{aligned}$$

and

$$\omega=r(0)=2\text{ arc tanh}[(-K)^{\frac{1}{2}}e^{-\sigma\gamma_0}]. \tag{18}$$

It is seen from the last equation that in accordance with Luneburg, the value ω *depends on the value of* γ_0, the convergence angle subtended by the furthermost point of the stimulus configuration. According to Hardy and Blank, the value of ω *does not depend on* γ_0.

An opportunity to test the two views empirically is given by our two alley experiments: the classic alleys with the end points subtending convergence angles $\gamma_0\sim0.015$ and the broad alleys with $\gamma_0\sim0.045$. Predictions concerning values of ω consistent with Luneburg's view, were computed from formula (18) for 4 observers, on the basis of their constants obtained by the 3 and 4 point tests. The eight values of ω predicted with regard to the γ_0 of the two alley experiments are listed in columns 4 and 5 of Table V. We note that larger values of ω are obtained for smaller γ_0.

On the other hand the measurements taken in the actual alley experiments yielded eight experimental values of ω computed by using the method of Hardy and his group for derivation of the constant ω from alleys.§ The experimental values of ω from our alley experiments are given in columns 6 and 7 of Table V. It is seen that:

(a) Two values of ω obtained for each observer for two different γ_0 differ, while in accordance with Hardy and his group, the same two values should be obtained for each observer.

(b) The experimental values of ω in all four cases are found to be larger for smaller γ_0, in accordance with prediction derived by the aid of Luneburg's mapping function.

(c) The difference between each of the two experimental values of ω is generally smaller than the differ-

[9] A. A. Blank, J. Opt. Soc. Am. 43, 717 (1953).

§ See reference 7, pp. 50, 51, 52.

ence predicted on the basis of the values of σ and K derived from the 3 and 4 point tests.

In the light of these results neither Luneburg's mapping function nor that of Hardy and Blank can be considered as final. The value of γ_0 seems to be of greater significance than assumed by Hardy and his group and of lesser significance than assumed by Luneburg. The concept of the degree of depth perception identified by Luneburg as a constant multiplicative factor of $d\gamma$ may require revision.

CONCLUSION

Verification of the theory centered on the mechanism of effective convergence, as apart from monocular and experiential factors, calls for greater control over experimental conditions than was obtained in the experiments reported. The adaptation of axiomatic geometry for description of this form of observation is still explorative. A measure of agreement with the theory within the whole sample of our observers, and agreement obtained in the case of observers having good depth perception, gives considerable support to Luneburg's hypothesis that binocular space is metric and hyperbolic.

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