ON THE APPLICATION OF INTERFERENCE PHENOMENA TO THE SOLUTION OF VARIOUS PROBLEMS OF SPECTROSCOPY AND METROLOGY.[1]

By A. PEROT and CHARLES FABRY.

INTERFERENCE phenomena permit us to refer determinations of length to a very small unit (of the order of ½ micron), the wave-length of a luminous radiation; for this reason the use of these phenomena is at once suggested when it is a question of measuring very small thicknesses or very small changes or differences in thickness. To make evident the services which interference methods may render in this direction, it suffices to mention the investigations of Fizeau on expansion, those of M. Cornu on the elastic change of figure of solid bodies, and the methods devised by M. Laurent for testing optical surfaces.

The extreme minuteness of the wave-length introduces certain difficulties when it is desired to apply interference methods to greater lengths. The measurement will involve at the outset the determination of the number of times the length to be measured contains the wave-length, *i. e.*, the integral part of the number which represents the quantity to be measured in terms of the chosen unit; in practice, this measurement will consist in the determination of the *order* of a fringe. Although a *whole number* is in question, its determination may give rise to some difficulty if it amounts to some tens or hundreds of thousands. This operation being supposed completed (and it must involve no error), the measure will be susceptible of extreme precision, on account of the very minuteness of the chosen unit; it will only remain to determine a fraction of a wave-length, and even if this fraction is found with a rather rough degree of approximation, the quantity to be measured will be known with very great precision. It will often happen that it is double the length

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to be measured which is determined directly in wave-lengths, which will double the precision of the measures.

Knowing how to measure a given distance in wave-lengths, we can compare any two lengths by measuring them successively in terms of this unit. We can also compare with a high degree of precision the wave-lengths of two given radiations, by comparing the same length with each of these wave-lengths. Finally, we can distinguish very small differences of wave-length, and consequently separate close lines in the spectrum, thus permitting the spectroscopic study of a group of lines.

These various applications of interference phenomena presuppose the employment of a light-source such that interference can be obtained with great differences of path; this requires that the light be strictly monochromatic, corresponding to a single and well-defined vibrational motion. It is clear, moreover, that if this condition were not satisfied no precise measurement in wave-lengths would be possible, since the light employed would not have a *single* well-determined wave-length. At the present time it is easy to produce almost absolutely monochromatic[1] radiations, thanks to the light-sources brought into use by Professors Michelson and Morley, which consist, as is well known, of metallic vapors rendered luminous by the discharge of an induction coil.

Further, a suitable interference apparatus is required.

We propose to give a brief account of a part of the investigations which we have made for the purpose of solving the various problems just enumerated, employing an interference apparatus having special properties which will be described at the outset.

I. FRINGES FROM SILVERED PLATES.

The ordinary forms of interference apparatus divide each incident wave into two waves capable of interfering. Each point

[1] The radiations employed up to the present time, excepting the red line of cadmium, are not single, but are composed of several closely grouped lines, one of which is much brighter than the rest; the numbers given for the wave-lengths refer to these predominant lines.

in the focal plane of the observing telescope—the imaginary observation screen—thus receives, from each point of the light-source, two vibratory motions having a difference of path Δ. In order that the phenomenon may be distinct, it is necessary

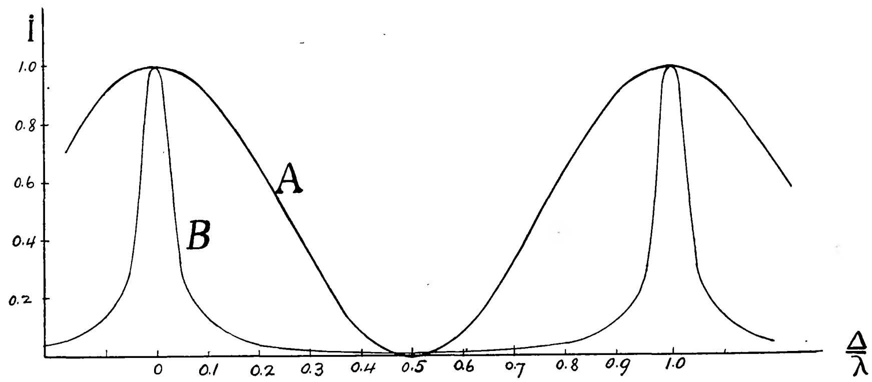

FIG. 1.

that Δ have a single value at every point of the screen, *i. e.*, that every pair of waves reaching a given point have the same difference of path. Supposing this *condition of perfect distinctness* to be satisfied, the luminous intensity will vary from one point to another in the focal plane; it is a function of Δ alone: the curves of equal luminosity are represented by the general equation $\Delta =$ a constant. The maxima are defined by $\Delta = K\lambda$, and the minima by $\Delta = K\lambda + \frac{\lambda}{2}$, λ being the wave-length of the light employed and K a whole number. If we suppose that the two interfering waves have the same intensity (as is ordinarily the case), the minima are zero; the curve which gives the luminous intensity I as a function of Δ is a sinusoid (Fig. 1, curve A). The fringes consequently have the appearance of bright bands, separated by dark bands with ill-defined edges; the passage from maximum brightness to the neighboring minimum is gradual and without abrupt change.

The phenomenon assumes a wholly different aspect if the apparatus, instead of dividing each wave into two, separates it into a very great number having differences of path which are in arithmetical progression, such that the differences of path with respect to one of them are Δ, 2Δ, 3Δ. . . . . A grating is a familar example of such an apparatus, in which the effect of the superposition of all these waves is known. When Δ is a whole number of wave-lengths, there is accordance of *all* the interfering waves, and consequently a light maximum; but if $\frac{\Delta}{\lambda}$ differs ever so little from a whole number, among all these waves there are some whose difference of path as compared with the first is far from a whole number, and which consequently considerably diminish the resulting intensity. The intensity thus falls off very rapidly away from the maximum, and the phenomenon consists of bright lines which are very narrow as compared with the dark interval which separates two successive maxima. The fineness of these bright lines will be the greater as the number of interfering waves increases.

A phenomenon of this character may arise in certain kinds of interference apparatus, on account of the multiple reflections

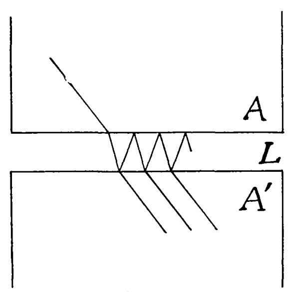

FIG. 2.

which the light may experience. For example, let L be a thin film of air bounded by two transparent surfaces A and A' (Fig. 2). An incident wave will give rise to an infinite number of emergent waves which have respectively undergone 0, 2, 4 . . . ., $2n$, ... reflections, and which have, with respect to the first, differences of path 0, 2Δ, ... , $n\Delta$, ... ($\Delta = 2e\cos i$, i being the angle of incidence in the layer of air and e the thickness of this layer). In the case where the surfaces A and A' are simple surfaces of glass, the intensities of these waves decrease very rapidly, on account of the small reflecting power of glass (about $\frac{1}{20}$); beyond the third they are wholly negligible. The case is different if the glass surfaces A and A' are lightly silvered; by this means it is possible to give them a very high reflecting power, at the same time leaving them a sufficient degree of transparence to permit an appreciable quantity of light to traverse the system. The intensities of the successive waves then decrease in a geometrical progression, the ratio of which does not differ much from unity, and the superposition of an infinite number of these waves gives a result analogous to that obtained with a grating.

Further calculation of the luminous intensity resulting from the superposition of all these waves leads to the expression

$$I = I_0 \frac{1}{1 + \frac{4f}{(1-f)^2} \sin^2 \pi \frac{\Delta}{\lambda}},$$

I_0 being a constant (intensity of the maxima), and f the reflective power of each of the surfaces A and A'. If f differs but little from 1, $\frac{4f}{(1-f)^2}$ is very great; for example, if $f = 0.8$ this fraction is equal to 80, and the expression for I becomes

$$I = \frac{I_0}{1 + 80 \sin^2 \pi \frac{\Delta}{\lambda}}.$$

When $\frac{\Delta}{\lambda}$ is a whole number we have $I = I_0$; but if $\frac{\Delta}{\lambda}$ differs ever so little from a whole number, I becomes almost equal to zero, on account of the term $80 \sin^2 \pi \frac{\Delta}{\lambda}$ in the denominator The curve B (Fig. 1) represents the law of variation of I as a function of Δ.

Thus a layer of air bounded by two lightly silvered surfaces gives, when examined *by transmission* in monochromatic light, a system of fringes in which the bright part is very narrow as compared with the dark interval which separates two consecutive fringes; the small quantity of light which the system allows to pass is distributed in very narrow bright lines. This effect is the more pronounced as the reflecting power f becomes more nearly equal to unity; now the reflecting power of silvered glass increases with the thickness of the silver film, and approaches that of the compact metal, which is about 0.90; but at the same time the quantity of light absorbed by the silver films increases. If this absorption did not exist, the intensity I_0 of the maxima would be always equal to that of the incident light; the existence of the absorption limits the thickness of the silver films that can be employed, and this thickness must depend upon the intensity of the light at command. In fact, when the light is fairly intense, it is possible to obtain fringes, the bright part of which does not occupy at the most more than $\frac{1}{20}$ of the interval which separates two consecutive fringes.

In addition to the characteristics described above, fringes from silver films possess the properties of fringes from ordinary isotropic films, and can be examined under the same conditions. It is always necessary to respect the condition of perfect distinctness, *i. e.*, that the value of Δ must be invariable for every point in the observation plane. The observation can be made in two simple ways, the choice of which will be governed by circumstances.

1. *In parallel light normal to the film;* where $i=0$ and $\Delta = 2e$. A system of fringes is obtained which describe the curves of equal thickness of the film and whose form essentially depends upon the form of the limiting surfaces. This mode of observation is very convenient when the thickness e is small; it then suffices to have the beam utilized approximately parallel and normal to the film in order to obtain a system of fringes localized in the film (Newton's rings; fringes of thin plates). When great differences of path are reached it is necessary to employ a rigorously parallel beam, without which the variously inclined waves would give scattered fringes, and the phenomenon would be rendered indistinct. The second mode of observation is free from all difficulties of this kind.

2. *In convergent light*, the layer being limited by two plane parallel surfaces. The thickness e is then perfectly constant,

and the difference of path $\Delta = 2\,e \cos i$ depends only on the angle of incidence i. The fringes are observed by means of a telescope focused for parallel rays. To every point in the focal plane of the telescope there thus corresponds a single value of i, and consequently a single value of Δ; the conditions of perfect distinctness are thus realized, and a system of rings centered on the normal to the layer is obtained. The expression for Δ may be written, when it is remembered that the field of the telescope is of small extent and that consequently i is small,

$$\Delta = 2\,e - e\,i^2.$$

Δ decreases proportionally to i^2; the diameters of these rings obey the same law as those of Newton's rings, but with the difference that it is at the center ($i = 0$) that Δ attains a maximum. Moreover, these rings at infinity have the appearance of very fine lines, commonly seen in fringes from silvered films.

This second mode of observation is especially advantageous in the case of great differences of path. In order to observe these fringes, two plates of glass, each having a silvered plane face, should be employed. These surfaces, which face one another, must be made exactly parallel; it is convenient to be able to change their distance apart without destroying this parallelism, so that, without readjustment, it may be possible to observe the rings corresponding to various values of the difference of path.

The apparatus employed, which we call an *interference spectroscope*, essentially consists of two plane plates of silvered glass placed vertically. One of these plates, L, carried by an old theodolite, can be given a wide range of displacement in azimuth, and small parallel displacements by means of a strong iron stirrup which can be bent by distending a rubber bag filled with water placed inside the stirrup; this bag is connected, like the two others referred to below, by a rubber tube to a vessel filled with water, the level of which can be varied; it is thus possible to produce a displacement of a few microns by a motion as slow as may be desired, and without lost motion.

The other plate, L', can be given small adjustments in azimuth and large parallel displacements. It is carried normally at the end of a horizontal strip of steel, 5mm in diameter and 10cm long, rigidly fixed at the other end, against which two bags filled with water press in two directions at right angles to one another; it is thus possible to obtain through flexure of the steel strip, very small angular displacements which are produced without in any wise disturbing the apparatus. The steel strip is supported by a carriage having the form of a triangular prism with horizontal edges; two strips of St. Gobain glass are cemented to the two lower faces, which meet at a right angle; these rest on ways also made of glass strips cemented to a wood base. The carriage may thus be given a parallel displacement, in which it will be perfectly guided if not subjected to any lateral pressure. To effect this the carriage can be pushed, in either direction, by two points attached to two other auxiliary carriages; the principal carriage has a little play between the two points, and consequently can be pushed by only one of them. The two auxiliary carriages can be moved together by means of a screw connected with them, which passes through a nut that can be turned by hand when a rapid displacement is desired, and by means of a tangent screw when a slower motion is needed. A single turn of this screw corresponds to a displacement of $3\,\mu$ or $4\,\mu$. The whole apparatus, carried on a thick plank, is suspended in the air by rubber rings to protect it from vibration, which would render the fringes invisible.

The adjustment of the parallelism of the plates is effected by moving the plate L for approximate adjustment, and L', through flexure of its support, for the final adjustment. Large parallel displacements are produced by moving the nut on the thread, and small ones by bending the stirrup which carries the plate L. Accidental displacements, caused by imperfections in the guides along a distance of several centimeters, are so small that the rings do not disappear, and they can be brought back to their normal appearance by bending the support of L'.

A divided scale, attached to the plate-carriage, and read by a fixed microscope, gives a means of measuring approximately the displacement of the carriage, and consequently the distance between the silvered surfaces.

II. PHENOMENA PRODUCED WHEN THE INCIDENT LIGHT IS COMPOSED OF TWO MONOCHROMATIC RADIATIONS.

Thanks to the fineness of the bright fringes, if several radiations simultaneously enter an interference apparatus with silvered plates, the systems of fringes corresponding to these several radiations are not confused, but may be seen in juxtaposition. Let us consider what occurs when only two radiations are employed; later certain applications of these phenomena will be given.

Let there be a first radiation, red, for example, of wave-length λ. Consider a point in the observation plane for which the difference of path of the first two interfering waves is Δ. The quotient $p = \frac{\Delta}{\lambda}$ is the difference of path expressed in wave-lengths. This quotient plays a fundamental part in the phenomenon: each integral value of p corresponds to a bright fringe, of which this number expresses the *order*. To avoid circumlocution, we will call p the *order of interference* corresponding to the difference of path Δ; for the radiation λ a definite value of p corresponds to every point in the observation plane.

If we have in addition a second radiation of wave-length λ', yellow, let us say, $(\lambda' < \lambda)$, it will also give a system of fringes, in which the bright fringes will be defined by the integral values of the quotient $p' = \frac{\Delta}{\lambda'}$. These two systems of bright lines belong, moreover, to the same family of curves, the general equation of which is $\Delta =$ const., but the yellow correspond to values of Δ which are multiples of λ', and the red to multiples of λ. In passing from one red fringe to the next, Δ increases by λ; it increases only by λ' in passing from a yellow fringe to the following one, which may be expressed by saying that the yellow fringes are more closely crowded together than the red.

If the position of the two systems of fringes is marked on a straight line *Ox* (supposing that the point *O* corresponds to $\Delta = 0$, and that the values of Δ increase along *Ox* in proportion to the distance from the point *O*), a figure like the following will result (Fig. 3):[1] the two systems of fringes, at first confused,

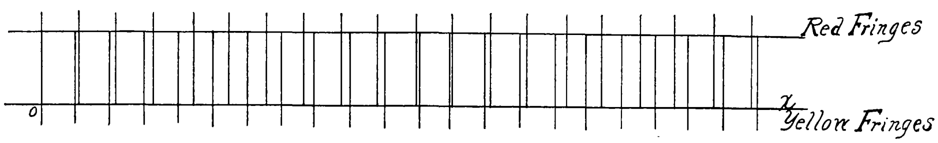

FIG. 3.

separate little by little; then a yellow fringe falls about half way between two red ones; again there is approximate coincidence of the two systems of fringes, or, more correctly, two consecutive red fringes are found, which comprise not one only, but two yellow fringes. Further on separation again occurs, then a new coincidence, etc. Let us find when this phenomenon of coincidence is produced. Let *A* and A_1 (Fig. 4)[2] be two consecutive red fringes, of order *K* and $K+1$, between which are found two

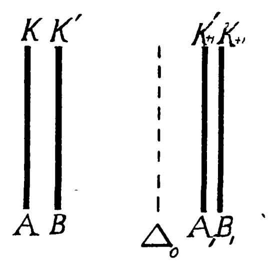

FIG. 4.

yellow fringes *B* and B_1 of order K' and $K'+1$, and let $K' - K = m$. In *B*, p' is equal to K', while *p* is a little greater than *K*; we therefore have $p' - p < K' - K$, or $p' - p < m$; similarly in B_1, $p' - p > m$. As, moreover, $p' - p$ increases in proportion to Δ, we find between *A* and *B* a certain value Δ_0 of the difference of path for which

$$p' - p = m. \tag{1}$$

[1] To distinguish the two systems of fringes the red have been prolonged toward the top and the yellow toward the bottom.

[2] In Fig. 4, interchange $A_1 B_1$ to read $B_1 A_1$.

The difference of path thus defined will be called the difference of path of coincidence; it is defined by equation (1), which may be written

$$\frac{\Delta_\circ}{\lambda'} - \frac{\Delta_\circ}{\lambda} = m, \text{ or } \Delta_\circ = m\frac{\lambda\lambda'}{\lambda - \lambda'}.$$

It is evident that the phenomenon is periodic, repeating itself for values of Δ which are multiples of a length

$$\Pi = \frac{\lambda\lambda'}{\lambda - \lambda'},$$

which we will call the *period*. This quantity may be expressed in wave-lengths from any radiating source; expressed as a function of λ, it will have the value

$$\omega = \frac{\lambda'}{\lambda - \lambda'},$$

and as a function of λ'

$$\omega' = \frac{\lambda}{\lambda - \lambda'} = \omega + 1.$$

The m^{th} coincidence occurs when the order of interference corresponding to the yellow radiation exceeds by the integral number m that which corresponds to the red radiation; it is repeated for the difference of path having the value $\Delta = m\,\Pi$; $m\,\omega$ and $m\,\omega'$ are the corresponding values of the order of interference. If $m\omega$ is an integral number K there will be *exact coincidence* between the K^{th} red fringe and the $(K+m)^{\text{th}}$ yellow fringe. In the general case, $m\,\omega$ will be a fraction $K+\theta$ $(0 < \theta < 1)$, and the appearance will be that shown in Fig. 4. The fraction θ characterizes the appearance of this *approximate coincidence*, for, if we designate by a and a_1 the distances AB and B_1A_1, we have

$$\theta = \frac{a}{a + a_1}.$$

Observation furnishes a check on the value of θ, at least when the two radiations employed differ sufficiently in wave-length.

Discordances are similarly defined by the condition that the difference $p' - p$ shall be equal to an integral number plus ½; the values of Δ corresponding to the discordances are

$$\Delta = (m + \tfrac{1}{2})\,\Pi.$$

They are recognized by the fact that a fringe of one kind occurs practically in the center of the dark interval which separates two fringes of the other kind

It is evident that all of these phenomena depend on the *period*, which may be calculated when the two wave-lengths are known. If the two radiations are far apart in the spectrum, as for instance, one in the green and one in the red, the period is short (for example, 4 or 5 fringes); the separation of the red fringes is sensibly greater than that of the green; the coincidences are repeated at short intervals, and they are recognized the more easily as the colors of the two systems of rings differ more widely, reducing the possibility of any confusion between them. It is even easy to distinguish the exact from the inexact coincidences, and observation gives an indication of the value of θ which establishes the inexactness of the coincidence.

If, on the contrary, the two radiations differ but little in wave-length, as, for example, the two sodium lines, the period is long (about 1000 in this case); the position of coincidence can be determined by observation only within a few fringes; the two kinds of rings have almost exactly the same color, and their separation, which gives rise to no lack of symmetry in color, becomes appreciable only a certain number of fringes before or after coincidence, by the widening of the rings.

Such are the phenomena which arise from the superposition of two systems of fringes. We have made no hypothesis regarding the method of observing them; the phenomena will remain precisely the same whether it is a question of *fringes of thin films* observed in parallel light, or whether the fringes produced by a layer of air of uniform thickness are observed in convergent light.

III. APPLICATION TO SPECTROSCOPY.

It follows from what has been stated that the observation of a system of fringes from silvered plates furnishes a means of separating two radiations of different wave-length, and consequently of making a *spectroscopic study* of a mixture of radiations. It is easy to see that the resolving power of the apparatus increases with the order of the fringes observed. Consider two radiations of nearly equal wave-length, λ and $\lambda' = \lambda + \epsilon$. Under what conditions can they be separated? Experience shows that the separation of the two systems of rings is always clearly visible when the distance between rings of different kinds is $\frac{1}{5}$ the distance between consecutive rings of the same kind. It will suffice, to effect separation, to reach the fringe whose order is $\frac{1}{5}$ the period, or of the order $p = \frac{1}{5}\frac{\lambda}{\epsilon}$, corresponding to a distance between the silvered surfaces $e = p\frac{\lambda}{2} = \frac{\lambda}{10} - \frac{\lambda}{\epsilon}$. Suppose, for example, that a distance between the silvered surfaces $e = 5\text{cm}$ has been reached; the corresponding order will be, supposing $\lambda = 0.5\mu$, $p = 200{,}000$, and two radiations such that $\frac{\epsilon}{\lambda} = \frac{1}{1000000}$, where the distance apart in the spectrum is only $\frac{1}{1000}$ of the distance between the D lines, can be separated. Beyond the distance $e = 5\text{mm}$ ($p = 20{,}000$) it is possible to resolve two radiations whose distance apart is less than $\frac{1}{100}$ that of the D lines; *i. e.*, the power of the apparatus is already comparable with that of the best spectroscopes having prisms or gratings.

In order to effect such a result, it is necessary to be able to obtain very sharp fringes with very great differences of path; the employment of fringes in convergent light is thus indicated. We use the apparatus permitting a parallel displacement, which has already been described. Suppose that it is desired to study with this apparatus an approximately monochromatic light. It is illuminated with this light, and the distance between the two silvered surfaces is gradually increased. If the radiation under examination is multiple, each ring will be seen to separate successively into several others; each of these partial rings corresponds to a monochromatic radiation, and the more refrangible radiations are on the inside. Each fringe constitutes a veritable spectrum of the luminous source; the apparatus is thus similar to a grating, the resolving power of which is certainly small, but with which it is possible to observe spectra of very high order, which permits its resolving power to be increased almost indefinitely.

When the light is complex it is easy to obtain a precise measure of the ratio of the wave-lengths of the radiations which constitute it; let there be two radiations of nearly equal wave-length, λ and $\lambda + \epsilon$. The distance between the silvered surfaces is increased until the discordance between the two systems of rings is complete. Then, if e is the distance between the surfaces (which is given with sufficient accuracy by the micrometer), we have $\frac{\epsilon}{\lambda} = \frac{\lambda}{4e}$.

We have studied in this way a certain number of radiations emitted by metallic vapors illuminated by an induction discharge (mercury, cadmium, thallium).[1] Our results are not identical with those deduced by Professor Michelson from his investigations on the visibility of fringes; but it should not be forgotten that Professor Michelson's method does not permit the complete determination of the constitution of a group of lines; an infinite number of hypotheses on the constitution of the group can be made to correspond to a single visibility curve, and the result is therefore in large degree arbitrary. In fact, our results would lead to visibility curves identical with those found by Professor Michelson; far from contradicting the experimental results of this investigator, our researches confirm them completely.

This method is also readily adapted to the study of the change of wave-length of a given line, on condition that the radiation be sufficiently near monochromatic; in such a case a comparison can be made of two sources emitting, for instance, in the one case the altered radiation, and in the other the normal radiation, attention being directed to the change in the appearance of the rings produced by the two sources successively.

[1] In certain cases, it is necessary to separate out radiations which would be troublesome on account of a complication of colors, or even, if the wave-lengths differ but little, because of the confusion of the rings; we have employed for this purpose either tanks of absorbing liquids, or one or more carbon bisulphide prisms, used in the ordinary way.

IV. DETERMINATION OF THE ORDER OF A FRINGE.

This problem presents itself in all determinations of length by interference methods. It is not ordinarily practicable to count the fringes beginning with which corresponds to zero difference of path, either because the zero fringe is not accessible, or simply on account of the difficulty of counting a number of fringes which may attain hundreds of thousands, if a measurement of a length of several centimeters is involved.

Our method is based on the observation of coincidences of fringes produced when the incident light contains two monochromatic radiations. It has been seen that this phenomenon is periodic, so that there exist certain fringes, of which a list may be made, which are, so to speak, characterized by the same distinctive sign. If the two radiations differ greatly, the fringes thus characterized are distinguishable without difficulty, but they are numerous and make a long list. On the contrary, when the two radiations differ but little, the fringes characterized by a distinctive mark succeed one another at long intervals; they are few in number and it is not difficult, when one is seen, to find it in the table. But observation will not suffice to designate the particular fringe with precision; it can only be said that it occurs in a certain region. By a careful choice of radiations combined in pairs, it is possible to determine the exact number of a fringe, provided its roughly approximate value is already known.

The method of coincidences is applicable whatever mode of observing the fringes be employed. In the case of the lower orders, they can be observed in parallel light, in the form of fringes from thin films. It then suffices to have the radiations employed approximately monochromatic; those given by alkaline salts in the flame serve perfectly. Thus in the verification of the order of the fringes furnished by our *standard films* (see below), we have employed the radiations of sodium and lithium in the flame of a Bunsen burner or an oxy-hydrogen blowpipe.

If it is desired to pass to fringes of higher orders, interference is produced in *convergent light*, and it is necessary to have recourse to the monochromatic radiations from the induction discharge in a metallic vapor. We have employed the brightest and most nearly monochromatic lines available in order to render possible the production and enumeration of fringes of a high order. These are the red and green lines of cadmium, the two yellow lines and the green line of mercury. To produce them the two tubes containing metallic vapors (Michelson tubes) are placed in series in the secondary of an induction coil. They occupy the foci of two convex lenses whose axes meet in a right angle. At the point of intersection is placed a lightly silvered glass plate, or a pile of plates which is traversed by one of the beams, while the other is reflected; we thus obtain the complete superposition of two beams, as though they came from the same light-source. Two movable screens, which the observer controls by means of cords without moving from his seat, permit the light from either tube to be cut off. Small tanks of colored liquids, which are placed directly before the eye, are used to cut out superfluous radiations.

In what follows we will refer everything to the fringes given by the green light of cadmium; the periods of coincidence will be expressed in terms of the wave-length of this radiation.

We group the radiations as follows:

1. The two yellow lines of mercury

$$\lambda = 0.57906593\ \mu, \qquad \lambda = 0.57695984\ \mu.$$

These two lines are close together in the spectrum (about three times the distance of the D lines); their period of coincidence is 311.9 (expressed in terms of the wave-length of the green cadmium line). We observe the coincidences, which can be determined within about twenty fringes, *i. e.*, in the twenty fringes which precede or follow the coincidence the separation of the two systems of rings is not appreciable.

2. The green line of cadmium ($\lambda = 0.50858240\ \mu$) and the green line of mercury ($\lambda = 0.54607424\ \mu$) have a period of 14.56515. We observe the discordances, and the observation determines without question the fringe for which this phenomenon is produced.

The green and red ($\lambda = 0.64384722\ \mu$) radiations of cadmium, which differ widely, have a period of 4.759901. We observe the coincidences, and this observation is greatly facilitated by the great difference in color of the two systems of rings. Observation gives to within about 0.1 the fraction which denotes the exactness of the discordance.

We now come to the determination of the order of a fringe.

The divided scale attached to the carriage of the interference apparatus, gives the distance between the silvered surfaces within a few hundredths of a millimeter. This measurement suffices to determine between what coincidences of the two yellow lines the observed fringes lie. Moreover, observation of these coincidences themselves gives a determination of the reading which corresponds to zero distance, and calibrates the scale with a sufficient degree of precision.

This being understood, as the coincidences of the red and green radiations of cadmium occur at short intervals, one of them is always in the field of the telescope; let us consider one of the green rings which encloses this coincidence, and let K be its order, which it is desired to determine.

Cutting out the red radiation, we superpose the green radiations of cadmium and mercury; then, slowly varying the distance between the silvered plates by means of the arrangement for producing flexure, we count the number of cadmium fringes, starting from fringe K, which must be caused to pass in order to produce discordance between the two systems of green rings. Call C this number, which cannot exceed fourteen, because coincidence of the two green lines occurs every fourteen fringes, and which is even less than seven, if care is taken to produce the displacement in the most favorable direction.

We continue to change the distance until a coincidence of the two yellow lines of mercury is reached, and during this motion we count the number C' of coincidences of the two green lines which pass across the field; the motion need not be very slow, since we no longer count the *fringes*, but the *coincidences*, which are fourteen times less numerous. The number of green fringes of cadmium which have passed during this motion is about $C' \times 14.57$.

Finally, let m be the number of the coincidence of the two yellow lines which has been reached, a number known from the approximate measurement of the thickness by means of the divided scale. It suffices to know the three integral numbers C, C', m, in order to solve the problem.

Suppose, to make the matter clear, that the two motions thus effected have resulted in bringing the plates nearer together. The m^{th} coincidence of the two yellow lines occurs when the number of the green cadmium fringe is $311.9 \times m$. To go from this to the discordance of the two green lines which was observed near K, $14.57 \times C'$ fringes had to pass; as the coincidence of two yellow fringes is observed to only about ± 20 fringes, the number of the discordance fringe of the two green lines will be

$$311.9 \times m + 14.57 \times C' \pm 20.$$

The number of discordance fringes in this interval is next calculated; there will be three or four at most, among them the one which has been observed. Further, on adding the number C to the number of this discordance fringe, we should encounter a coincidence of the green and red lines of cadmium; having calculated a table of coincidences, a choice will be made without hesitation. An important check will be given by the fact that observation gives with a precision of 0.1 the fraction which denotes the inexactness of coincidence of the two cadmium lines; the observed fraction should agree with its calculated value.

It is evident that the direct result of observation is reduced to three integral numbers, one of which is given directly by reading a divided scale, the two others being each less than 15. This method is applicable up to thicknesses of 4 cm or 5 cm, and consequently renders possible the rigorous determination of numbers of fringes which may reach as high as 200,000, by counting only the coincidences, the numbers to be counted being less than 10, and the quickly-obtained result being easily found anew for the purpose of verification. The application of the method requires the use of no special measuring instrument, such as a micrometer, compensator, etc.; it is sufficient to be able to observe the few fringes which lie near those which it is desired to study.

V. COMPARISON OF WAVE-LENGTHS.

The method just described requires that the ratios of the wave-lengths of the radiations employed be exactly known. A precision of $\frac{1}{1000000}$ is not sufficient.

For the lines of cadmium, the ratios of the wave-lengths are known with a precision which leaves nothing to be desired, thanks to the beautiful investigations of Professor Michelson. The same is not true of the mercury lines, which are known only from old measures made with gratings, the precision of which is far from sufficient. We have, therefore, compared the wave-lengths of these radiations with those of cadmium, by the observation of interference phenomena. This comparison can be made by means of observations identical with those which serve for the determination of the order of fringes, provided that fringes of a low order are used at the outset, and subsequently those of higher and higher orders.

The old measures give a first approximation of the ratios sought, with which the approximate values of the periods of coincidence of these radiations among themselves, or with the cadmium lines, can be calculated. It is consequently possible to calculate approximate tables of the coincidences, and these tables will contain only small errors, such that the order number of the fringes will reach only a moderate value (*e. g.*, a few thousands); moreover, the coincidences of the cadmium fringes among themselves can be exactly calculated by using the values obtained by Professor Michelson.

Setting the two surfaces of the interference apparatus a short distance apart (1 mm for example), the observations of coincidences are made just as though it were merely a question of determining the order of a fringe. In comparing the observations with the approximate tables of coincidences, it will be found that only a single hypothesis regarding the order of the observed fringes will permit the observed phenomena to be brought into close accordance with the results of calculation. If the accurate identification is not readily made it is because the errors of the tables of coincidences are still too great for the orders of the observed fringes, and it will be necessary to repeat the observation on fringes of lower orders. The more inexact the values used in the preliminary calculations, the lower the order of fringes that must be chosen to avoid all doubt.

Only one hypothesis being admissible, the small discrepancy remaining between observation and calculation indicates that the values used for the wave-lengths are not quite exact, and permits them to be corrected.

These new values render possible the calculation of more exact tables of coincidences, by means of which the same operations can be repeated with fringes of a higher order, double the previous one, for example. This new observation furnishes the means of again correcting the wave-length tables, and continuing thus, we obtain more and more precise values as fringes of higher orders are observed.

In brief, the experiment consists in observing the reciprocal position of two systems of rings, using for this purpose coincidences or discordances; an observation of this character fixes the reciprocal position of two fringes within at least $\frac{1}{20}$ of a fringe. In comparing two wave-lengths, λ and λ', the relative error possible for the second, supposing the first known, will be, if the observations are discontinued at fringes of order p,

$$\frac{d\lambda'}{\lambda'} = \pm \frac{1}{20\,p}.$$

A very high degree of precision is soon attained; for example, let $p = 10{,}000$, which corresponds to a distance of less than 3 mm between the silvered surfaces, the possible relative error will be $\frac{1}{200000}$, a precision which it would doubtless be difficult to surpass with gratings.[1]

[1] This is particularly true of lines widely separated in the spectrum, since in this case the errors of the divided circles will enter.

Our measures have been carried to a separation of the silvered surfaces amounting to 32 mm (order about 125,000 for the green cadmium fringes). The observations once completed, it is desirable to utilize all of them for the definitive calculation, applying the method of least squares to the series of equations which they furnish.

The values below, based on Professor Michelson's values of the wave-lengths of the cadmium lines, are referred to the latter in *air* at 15°, under a pressure of 760 mm; the ratios of these numbers remain sensibly constant under ordinary atmospheric conditions. The probable error is 5 units in the last place, or $\frac{1}{100000000}$ in relative value.

It is evident that interference methods permit wave-lengths to be compared with a remarkable degree of precision. These methods have the advantage of being based directly on the *definition* of the quantity measured, and of being free from all systematic error due to delicate instruments (gratings, divided circles, etc.), which are encountered in other methods.

We have thus found

$$\text{Yellow lines of mercury} \begin{cases} \lambda_1 = 0.57906593\,\mu \\ \lambda_2 = 0.57695984\,\mu \end{cases}$$

$$\text{Green line of mercury } \lambda = 0.54607424\,\mu$$

VI. MEASUREMENT OF LENGTHS.

The determination of the order of a fringe makes known the whole number of wave-lengths contained in a given length. In order to have a measure expressed in wave-lengths, it only remains to determine a fraction, which does not need to be known with a very great relative precision on account of the extreme minuteness of the wave-length.

Let it be required to determine the thickness, at a given point, of a *thin film* of air between silvered surfaces. By illuminating the system with a parallel beam of monochromatic light, a system of fringes will be obtained; the position of the given point with reference to the two fringes which encircle it is determined, and finally the order of one of these fringes is sought.

In the case of a film with parallel faces giving rings at infinity, the order K of the ring immediately surrounding the center will be found. The order of interference corresponding to the center of the system is a little greater than K, say

$$K+\eta, \qquad (0<\eta<1);$$

and the distance between the two silvered surfaces will be $(K+\eta)\,\frac{\lambda}{2}$; this will be known if the fraction η is determined. The simplest means of doing this is to find the angular diameter of the ring K, using for this purpose, for example, an eyepiece micrometer. Call this diameter $2\,i$. We have $K=\frac{2\,e\cos i}{\lambda}$, whence $e=K\frac{\lambda}{2}\frac{1}{\cos i}$. The fraction η, which it is really unnecessary to calculate, would have the value

$$\eta=K\frac{1-\cos i}{\cos i}=K\frac{i^2}{2},$$

remembering that i is very small.

The use of these methods of measuring thicknesses presupposes the possibility of applying the method already described for the determination of the order of a fringe; for this a certain number of conditions must be met: among others, it is necessary to be able to examine a certain number of fringes in the neighborhood of the one whose order is required. This condition greatly limits the cases in which these measuring processes can be applied. An optical method by which it would be possible to establish the equality of two thicknesses would evidently possess great interest; it would thus be possible to compare the length to be measured with a thickness measured beforehand, or also to copy a given length by means of a system which is kept under conditions favorable to purposes of measurement. We have succeeded in solving this problem, thanks to the use of fringes in white light, the theory of which we will give. Our method rendering it possible to double, triple, etc., a thickness will permit the measurement of lengths much greater than those which can be determined directly by interference methods.

Superposition fringes.—These fringes are produced when a beam of white light traverses successively two air films bounded by silvered surfaces, A and A', of suitable thicknesses. An incident ray can, in fact, give rise to two emergent rays, one of which has passed directly through A and has been twice reflected in A', while the other has passed directly through A' and has been twice reflected on the surfaces of A. The difference of path of these two rays is $\Delta - \Delta'$, if Δ and Δ' are the differences of path corresponding to each of the films for the ray considered. It is zero if $\Delta = \Delta'$. We will thus have, if the two differences of path are nearly alike, and consequently the thicknesses of the two films not far from equal, a system of fringes in which the central white fringe marks out the position of points such that $\Delta = \Delta'$. This central fringe is bordered by brilliant colors.

Fringes in white light appear not only when the two thicknesses are nearly equal, but also when they stand in a simple ratio; they are then due to the interference of rays which have undergone an unequal number of reflections. The systems of fringes corresponding to $\frac{\Delta}{\Delta'} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{2}, \frac{4}{3} \ . \ . \ .$ can be easily observed.

However, in proportion as this ratio becomes less simple the fringes are fainter, since they are due to the interference of rays which have undergone a greater and greater number of reflections, and since an increasingly important fraction of the light cannot interfere and produces white light, which diminishes the contrast of the fringes.

As for the manner of observing these phenomena, this can be varied according to circumstances. In the case of small thicknesses they are observed in parallel rays; with thick layers and uniform thicknesses, the fringes will be observed at infinity in convergent light.

Superposition fringes in parallel light; measurement of small thicknesses.—The beam of white light in this case passes normally through the two thin films A and A', which will be placed directly in line with one another, or better *superposed optically*, by projecting on the second, by means of an optical system, an image of the first. The fringes are then localized in this plane, which contains at once the second film and the image of the first. At one point in this plane the thicknesses e and e' correspond for the two films; the corresponding differences of path are $\Delta = 2e$, $\Delta' = 2e'$. If in a certain region $\frac{e}{e'}$ approximates the commensurable ratio $\frac{p}{q}$ a system of fringes in which the central fringe is defined by $\frac{e}{e'} = \frac{p}{q}$ is obtained. Or again, if in a certain region the two thicknesses differ but little, we obtain a system of fringes in which the central fringe outlines the region of points such that $e = e'$.

From this we have a means of establishing the equality in thickness of two thin films at given points: the image of one is projected upon the other, so that the two given points correspond. If the thickness at these points is equal they must occur on the central fringe.

On this consideration we have based the construction and use of *standard films* for the almost instantaneous measurement of small thicknesses; it is clear, in fact, that if the film A' has been calibrated, *i. e.*, if its thickness has been determined at different points, the point in this film where the thickness is equal to that which it is wished to measure may be sought. The measurement is extremely rapid, the standard plate taking the place of a divided scale, the graduation of which can be controlled, when desired, by means of fringes in monochromatic light.

We will also point out, as an application of these superposition fringes, the solution of the following problem which may arise in the construction of various measuring instruments. Two parts of an apparatus are susceptible of small displacements with respect to one another; it is desired to supply the system with a reference mark so that it can be brought back to the same relative position. Ordinarily a microscope attached to one of the parts, and focused on an index carried by the other, is employed for this purpose. The use of superposition fringes renders it possible to fix the reference position within a few thousandths of a micron.[1]

Superposition fringes in convergent light; measurement of great thicknesses.—Great difficulties will be encountered if it is desired to obtain by the preceding methods superposition fringes with moderately large thicknesses. It is then advisable to employ plates of uniform thickness and to observe the fringes in convergent light with a telescope focused for parallel rays.

Let L and L' be the two plates (Fig. 5); both of them have parallel faces, but the faces of the first make a small angle

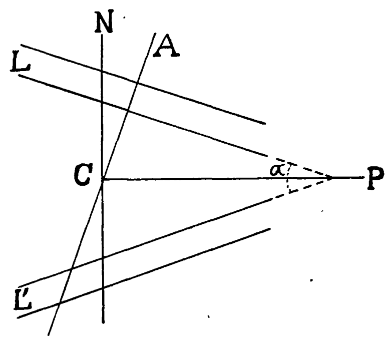

FIG. 5.

a with those of the second. The plane of the figure is normal to the two systems of faces; their bisecting plane is directed along CP, and the normal to this plane is CN. Suppose at the outset that the thicknesses e and e' of the two plates are nearly equal; there may then be interference between the wave which has passed directly through L and has been twice reflected on the faces of L' with that which has pursued the reverse course. Consider a direction making a small angle with CN, and which is projected on the plane of the figure in CA. To this direction

[1] See *Annales de Chimie et de Physique.*

An effective application of this method has been made by us. ("Sur un nouveau voltmètre électrostatique interférentiel" *Jour. de Phys.*, November 1898.)

corresponds a single value of the difference of path between the two interfering waves, the value of which is

$$\Delta = 2\,e \cos i - 2\,e' \cos i',$$

i and i' being the angles of incidence of the given direction in the two films. Remembering that i and i' are very small, and that e' differs but very little from e, this expression may be written

$$\Delta = 2\,(e - e') + e\,(i'^2 - i^2),$$

or by a simple transformation

$$\Delta = 2\,(e - e') + 2\,e\,a\,\theta,$$

θ being the angle made by CN with the projection CA of the direction considered.

It is seen that Δ varies proportionally with θ. We will thus have in a telescope focused for parallel rays a system of rectilinear fringes, equidistant and perpendicular to the plane of the figure, *i. e.*, parallel to the intersection of the faces of the two plates. The angular distance of two consecutive fringes is $\frac{\lambda}{2\,e\,a}$, and the central fringe is defined by $\theta = \frac{e - e'}{e\,a}$.

The entire system of fringes is displaced parallel to itself if one of the thicknesses is varied; the central fringe moves toward the point in the field which corresponds to the direction CN ($\theta = 0$) when $e = e'$. Moreover, the fringes broaden if the angle a is diminished, and for $a = 0$ Δ approaches the constant value $2\,(e - e')$; the system of fringes tends to acquire a uniform color, which is white if $e = e'$.

From this we derive a method of establishing with a high degree of precision, the equality of two thicknesses each corresponding to the distance between two plane parallel surfaces of silvered glass. If there is a very small difference between the two thicknesses, it can be measured with precision; finally if one of the thicknesses is susceptible of slight variation, it can be brought into exact equality with the other, and thus a given thickness can be *copied*.

Phenomena entirely similar to those just mentioned will be obtained if the thickness e', instead of being equal to e, is, for example, one half or one third of it. From this is derived a means for exactly multiplying a given length by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . . or by 2, 3, 4, . . . As we are dealing here with phenomena in *white light*, we can apply these methods even to very great thicknesses, and although the adjustments become more difficult in proportion as the distances increase, it seems possible, under good conditions, to observe superposition fringes with thicknesses of at least 1 m.

Moreover, the distance between two parallel silvered surfaces can be measured directly in wave-lengths, provided that it does not exceed 4 cm or 5 cm and is susceptible of slight variation. Combining this method of measuring with the use of superposition fringes, it becomes possible to measure the given and invariable distance between two parallel silvered surfaces, even if it greatly exceeds the limit just given.

Let it be required to measure the thickness of the layer of air L (Fig. 5) which we will suppose at first to be less than 5 cm. Employing the superposition fringes, this thickness is copied by means of the film L', the thickness of which can be varied at will; then this latter is measured by the methods indicated. It should be remarked that the operation can be, so to speak, instantaneous, and gives the measure of the desired thickness at a given instant, without requiring that this distance be invariable.

If the thickness of L exceeds 5 cm the same method is followed, but instead of copying it, the half, third or quarter is taken, which will render possible the measurement in wave-lengths of thicknesses up to 20 cm in a single operation.

Finally, for greater thicknesses, the same procedure may be followed, by taking a certain number of intermediate standards. Let it be required to measure the thickness of the layer L_1, too great to permit the preceding method to be applied. By means of a layer L_2 of variable thickness, we can, for instance, take a quarter of it, then a quarter of this latter by means of the layer L_3, and so on, until a layer L_n of thickness less than 5 cm, directly measurable in wave-lengths, is reached. By means of easily devised arrangements the whole of these operations can be rendered almost instantaneous, thus removing all danger of error arising from any change occurring during measurement.

All the lengths to be measured by these methods must be represented by the distance between two plane parallel silvered surfaces placed facing each other. But it is easy to pass from this case to that where it is required to measure the thickness of a solid with polished and sensibly plane parallel surfaces, like the thickness of a parallelopiped of glass. It suffices to place this solid between the parallel silvered faces of a suitable system, which will play the part of *calipers;* the distance of the surfaces is adjusted in such a manner that there remain only very small thicknesses of air between them and the faces of the solid. The use of our *standard films* (see above) permits this last measurement to be made rapidly.

We have applied these processes to the measurement in wave-lengths of the thickness of a glass cube 3 cm on an edge. The measurement was made with precision, in spite of the very defective conditions under which it was effected; almost the entire apparatus was constructed of wood; on account of the vibration of the soil it had to be suspended by means of rubber rings; no precaution was taken to avoid temperature variations. The success of the experiment under such unfavorable conditions was evidently due to the fact that the measurement was instantaneous, and consequently free from any error arising from a modification of the apparatus.

The same methods would evidently be applicable to much greater thicknesses. We may hope to be able to measure in wave-lengths, with no microscope settings, the length of a "mètre à bouts." The conditions under which our experiments have been made did not permit us to attempt so delicate an application of the method with any chance of success; our purpose was only to render evident the possibility of such a measurement.

Such are, in brief, the principal applications that we have made of interference phenomena given by silvered plates. The greater part of the problems that we have attacked have already received other solutions ; our methods are notable for the simplicity of the apparatus, the work of the constructor being reduced to the figuring of two plane surfaces. This is evidently a good means of avoiding systematic errors, and of utilizing as completely as possible the remarkable power of interference methods. Particularly for the measurement of lengths and all allied problems, these methods can render effective service with the ordinary resources of a laboratory. The difficulties resulting from the extreme minuteness of the wave-length are largely offset by the precision of the measures and the certainty offered by the standard employed.