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RESEARCH NOTE

Ray-theory Green's function reciprocity and ray-centred coordinates in anisotropic media

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SUMMARY

Lighthill and others have expressed the ray-theory limit of Green's function for a point source in a homogeneous anisotropic medium in terms of the slowness-surface Gaussian curvature. Using this form we are able to match with ray theory for inhomogeneous media so that the final solution does not depend on arbitrarily chosen 'ray coordinates' or 'ray parameters' (e.g. take-off angles at the source). The reciprocity property is clearly displayed by this 'ray-coordinate-free' solution. The matching can be performed straightforwardly using global Cartesian coordinates. However, the 'ray-centred' coordinate system (not to be confused with 'ray coordinates') is useful in analysing diffraction problems because it involves 2×2 matrices not 3×3 matrices. We explore ray-centred coordinates in anisotropic media and show how the usual six characteristic equations for three dimensions can be reduced to four, which in turn can be derived from a new Hamiltonian. The corresponding form of the ray-theory Green's function is obtained. This form is applied in a companion paper.

Key words: body-wave seismology, geometrical spreading, point sources, reciprocity, ray-centred coordinates.

1 INTRODUCTION

In scattering or diffraction problems, it is sometimes necessary to know the zeroth-order ray approximation to Green's function and to exploit the reciprocity property of this approximation. See, for example, Coates & Chapman (1990) and Thomson, Kendall & Guest (1992, this issue). The reciprocity of point-source geometrical spreading functions is at the heart of the issue and Coates & Chapman (1990) refer to Richards (1971, section 3.5) for the proof. The discussion in Richards (1971) applies to multilayered media, but anisotropy and the relevance of symmetry in the geometrical spreading equations are not considered. Here, the approximate Green's function is found by matching zeroth-order ray theory with the exact point-source solution for a homogenous anisotropic medium obtained by Lighthill (1960), Buchwald (1959) and Burridge (1967). This point-source solution is expressed in terms of the Gaussian curvature of the slowness surface at the source and it matches elegantly with the geometrical spreading function in global Cartesian coordinates. Arbitrarily chosen ray coordinates or parameters at the source do not appear in the resulting Green's function and the reciprocity is evident in this 'coordinate-free' solution.

Ray-centred coordinates (as opposed to ray coordinates or parameters) are useful for local analysis problems such as that in Thomson *et al.* (1991). This is partly because the 6×6 system of geometrical spreading equations in Cartesian coordinates reduces to a 4×4 system in ray-centred coordinates. Popov & Pšenčík (1976) have described the latter for isotropic media and deduced the corresponding ray and geometrical spreading equations from a Lagrangian argument involving stationarity of ray traveltime. Here we obtain the ray and geometrical spreading equations for anisotropic media by starting from the eikonal or phase equation expressed in ray-centred coordinates and applying the method of characteristics. This is quite analogous to the procedure used with Cartesian coordinates and six ray equations are still obtained for a 3-D space. However, the properties of ray-centred coordinates allow this system to be reduced to four equations. We find these reduced equations may be considered to come from a Hamiltonian (other than the eikonal equation). This Hamiltonian turns out to be related to one component of slowness in the ray-centred system, as noted by Farra & Madariaga (1987) for the isotropic equations of Popov & Pšenčík (1976). As the reduced system of ray equations may be derived from a Hamiltonian, the corresponding 4×4 system of geometrical spreading equations has the symplectic property. Point-source reciprocity and other symmetries follow straight away.

The forms of the eikonal and other equations in ray-centred coordinates are generally quite complicated. Fortunately though, explicit forms are not needed to derive the results being sought. As a rule it seems ray-centred coordinates are useful for proving local analytical results, as in Thomson *et al.* (1992), but the Cartesian equations are simpler for numerical implementations. We should note that the theory of narrow beams in inhomogeneous anisotropic media relies on ray-centred coordinates and that Hanyga (1982, 1986) has also considered the geometrical spreading equations in this system. It is hoped that the discussion of these coordinates, reciprocity and other properties given here, coupled with the companion paper Thomson *et al.* (1992), will be helpful in realizing their wider potential in seismological scattering problems.

2 MATCHING WITH A POINT SOURCE AND RECIPROCITY IN ANISOTROPIC RAY THEORY

2.1 Point source in a uniform anisotropic medium

The field due to a point force in a uniform anisotropic medium has been considered by, among others, Buchwald (1959), Lighthill (1960), Duff (1960) and Burridge (1967).

In our notation, the appropriate form of the Green's function given by the latter author is

$$G_{pq}(\mathbf{x}, \omega) = \frac{1}{4\pi\rho K^{1/2}(\mathbf{p}_0)\,|\mathbf{x}|}\frac{g_p g_q}{|\mathbf{v}|}e^{i\omega\mathbf{p}_0\cdot\mathbf{x}} \tag{1}$$

[Burridge (1967), equation (6.8)]. This is the p-component of displacement due to the q-component of a unit body force per unit volume at the origin [not per unit mass as in Burridge (1967), note]. The displacement eigenvectors g_p are defined in the usual way [Kendall & Thomson (1989), equation (4)] and are normalized to one. Vector $\mathbf{v}$ is the total/group/ray velocity and $\mathbf{x}$ is the receiver position corresponding to slowness $\mathbf{p}_0$ at the origin. The quantity $K(\mathbf{p}_0)$ is the Gaussian curvature of the slowness surface at $\mathbf{p}_0$. Lighthill (1960) and Buchwald (1959) give the following expression for this curvature, adapted to our notation,

$$K = \frac{1}{|\mathbf{v}|^4}\sum_{ijk} H_i^2(H_{jj}H_{kk} - H_{jk}^2) + 2H_jH_k(H_{ij}H_{ik} - H_{ii}H_{jk}), \tag{2a}$$

where the ray theory Hamiltonian $H(\mathbf{p}, \mathbf{x}) = H(\mathbf{p})$ in the uniform medium, $H = 0$ defines the slowness surface and $H_i = \partial H/\partial p_i = v_i$. The summation in (2a) is wrt cyclical permutation of i, j and k. The following alternative form of the curvature is useful

$$K = \frac{1}{|\mathbf{v}|^4}\sum_{ijk} H_i(\epsilon_{lmn}H_lH_{jm}H_{kn}), \tag{2b}$$

where the ijk summation is still cyclical and lmn are summed in the usual manner. This form makes the relationship to the wavefront vector-area easier to recognize.

For S-waves, it is possible that the slowness surface is not convex in the direction of $\mathbf{p}_0$ and a form other than (1) should be used. In some directions the usual ray approximation does not apply at all for S-waves. Complications such as these and the affects of caustics far from the source will not be considered in the present work.

The ray theory solution in an inhomogeneous medium may be written

$$u_p(\mathbf{x}, \omega) = \frac{\psi_0}{(\rho v_n J)^{1/2}} g_p(\mathbf{x}) e^{i\omega\tau(\mathbf{x})} \tag{3}$$

[Červený (1972), following equation (36)]. The function ψ_0 is to be found by matching (3) with (1) as the source is approached. Matching the phase is simple since $\tau(\mathbf{x}) \to |\mathbf{x}|/|\mathbf{v}| = \mathbf{p}_0\cdot\mathbf{v}\,|\mathbf{x}|/|\mathbf{v}| = \mathbf{p}_0\cdot\mathbf{x}$, where we have used the fact that $\mathbf{p}_0\cdot\mathbf{v} = 1$. The geometrical spreading in (3) is defined by [Červený (1972), equation (27)]

$$v_nJ = v_n\,|\partial_{q_1}\mathbf{x} \times \partial_{q_2}\mathbf{x}| = |\mathbf{v}\cdot(\partial_{q_1}\mathbf{x} \times \partial_{q_2}\mathbf{x})|, \tag{4}$$

where v_n is the normal velocity. The parameters q_1 and q_2 uniquely define a ray leaving the source. The derivatives in (4) are at constant τ and so the vector product defines an area in the wavefront. There is some freedom in the choice of the ray parameters or coordinates. The final solution (3) cannot depend on this choice and so the unknown ψ_0 must be a function of q_1 and q_2 in such a way as to remove the dependence on how these two are chosen. It should be possible to rewrite the ray solution (3) in terms of unique quantities, such as $\mathbf{p}$ and $\mathbf{v}$ at the source, rather than arbitrary parameters on which these quantities have been made to depend simply for convenience. This 'ray-coordinate free' form of the solution will also make the reciprocity relation apparent.

2.2 Geometrical spreading function near the source

It is appropriate to consider the geometrical spreading equations in the Cartesian form used by Thomson & Chapman (1985).

The geometrical spreading equations are

$$\frac{d\mathbf{y}}{d\tau} = \mathbf{A}\mathbf{y}, \tag{5}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{T}^{\mathrm{T}} & \mathbf{B} \\ \mathbf{C} & -\mathbf{T} \end{pmatrix}, \qquad C_{ij} = -\partial_{x_i}\partial_{x_j}H, \qquad B_{ij} = \partial_{p_i}\partial_{p_j}H, \qquad T_{ij} = \partial_{x_i}\partial_{p_j}H.$$

For present purposes, the elements of vector $\mathbf{y}$ may be considered to be $(\partial_{q_l}\mathbf{x}^{\mathrm{T}}, \partial_{q_l}\mathbf{p}^{\mathrm{T}})^{\mathrm{T}}$, where $q_l = q_1$ or q_2. At the point source, only the latter three elements of $\mathbf{y}$ are non-zero and so we have for small τ

$$\partial_{q_l}x_i(\tau) = B_{ij}(0)(\partial_{q_l}p_j)\tau + O(\tau^2). \tag{6}$$

Substituting this into (4) yields the leading-order form

$$v_nJ = \epsilon_{ijk}v_iB_{jl}B_{km}q_l^{(1)}q_m^{(2)}\tau^2, \quad \text{where} \quad q_j^{(l)} = \partial_{q_l}p_j \tag{7}$$

at the source. In order to bring this into a form such as (2b), first note the equivalent form

$$v_nJ = -\epsilon_{ijk}v_iB_{jl}B_{km}q_m^{(1)}q_l^{(2)}\tau^2, \tag{8}$$

from the properties of ϵ_{ijk}. Adding (7) and (8) leads to

$$v_nJ = \tfrac{1}{2}\epsilon_{ijk}v_iB_{jl}B_{km}(q_l^{(1)}q_m^{(2)} - q_m^{(1)}q_l^{(2)})\tau^2. \tag{9}$$

At this stage the summation in (9) takes place over all l and all m. However, the case $l = m$ clearly gives zero contribution. Moreover, the contributions for the pair lm and the pair ml are equal. This follows from noting that when l and m are interchanged (i) the bracketed ( ) term changes sign and (ii) the effect on the B_{ij} terms is effectively to interchange j and k and hence the sign of ϵ_{ijk}. Lastly we note that the term in brackets is parallel to the p-component of $\mathbf{v}$ ($p \neq l$, $p \neq m$). Thus we have the leading approximation near the source

$$v_nJ = C_0\tau^2 \sum_{plm} \epsilon_{ijk}v_iB_{jl}B_{km}v_p, \tag{10}$$

with cyclical summation on p, l and m and

$$C_0 = \frac{|\partial_{q_1}\mathbf{p} \times \partial_{q_2}\mathbf{p}|}{|\mathbf{v}|} \tag{11}$$

at the source.

Comparing (2b) and (10) shows that the matching is now straightforward and one obtains for inhomogeneous media the leading approximation

$$G_{pq}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{4\pi}\left(\frac{C_0}{\rho_0\rho v_nJ}\right)^{1/2} g_p(\mathbf{x})g_q(\mathbf{x}_0)e^{i\omega\tau(\mathbf{x},\mathbf{x}_0)}, \tag{12}$$

where ρ_0 is the density at the source.

2.3 Reciprocity and geometrical spreading

Now consider the propagator matrix $\mathbf{W}(\tau, 0)$ of the system of geometrical spreading equations (5). The ray path along which this system is integrated is written $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{p}_0, \tau)$, $\mathbf{p} = \mathbf{p}(\mathbf{x}_0, \mathbf{p}_0, \tau)$ and the elements of the propagator are the same as the elements of the determinant

$$\frac{\partial(x_1, x_2, x_3, p_1, p_2, p_3)}{\partial(x_{0_1}, x_{0_2}, x_{0_3}, p_{0_1}, p_{0_2}, p_{0_3})}.$$

It is important to note that the x_{0_i} and p_{0_i} are considered independent for the partial derivatives in $\mathbf{W}$. The vector solution to a specific problem, $\mathbf{y}$ in (5), is given by $\mathbf{y} = \mathbf{W}\mathbf{c}$, where $\mathbf{c}$ is an initial vector that imparts the interdependences of the x_{0_i} and the p_{0_i} for the problem at hand. Thomson & Chapman (1985) show that the propagator satisfies

$$\mathbf{W}^{-1}(\tau, 0) = \mathbf{W}(0, \tau) = \mathbf{J}^{\mathrm{T}}\mathbf{W}^{\mathrm{T}}\mathbf{J}, \qquad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{13}$$

where **I** is the 3×3 identity matrix. The matrix **W** is said to be symplectic, a property which follows from the structure of the matrix of Hamiltonian derivatives **A** (5).

The propagator is composed of its 3×3 blocks according to

$$\mathbf{W} = \begin{pmatrix} \mathbf{O} & \mathbf{R} \\ \mathbf{Q} & \mathbf{P} \end{pmatrix} \tag{14a}$$

and for point-source problems the block $R_{ij} = \partial x_i / \partial p_{0_j}$ is important ('R' for Reciprocity). It follows from (13) and the definition of a propagator (Gilbert & Backus 1966) that

$$\mathbf{R}(\tau, 0) = -\mathbf{R}^{\mathrm{T}}(0, \tau). \tag{14b}$$

The generalization of equation (7) to large τ is

$$v_n J = \epsilon_{ijk} v_i R_{jl} R_{km} q_l^{(1)} q_m^{(2)}, \qquad q_j^{(l)} = \partial_{q_l} p_{0_j}, \tag{15}$$

where we now use subscript 0 to distinguish values at the source. Steps analogous to those leading to (10) now lead to its generalization

$$v_n J = C_0 \sum_{plm} \epsilon_{ijk} v_i R_{jl} R_{km} v_{0_p}, \tag{16}$$

where C_0 is still given by (11). Equation (16) is almost the result we seek. Note that it allows us to replace $v_n J / C_0$ in the matched ray theory result (12) with the sum on the rhs of (16). This eliminates the dependence of the solution (12) on the arbitrary parameters $q_{1,2}$.

In order to establish reciprocity we must replace the cyclical sum in (16) with a complete sum as follows. For each cyclic pair lm in the sum

$$\mathcal{R} = \sum_{plm} \epsilon_{ijk} v_i R_{jl} R_{km} v_{0_p} \tag{17a}$$

interchanging l and m is equivalent to interchanging j and k. Hence, from the properties of ϵ_{ijk},

$$\mathcal{R} = -\sum_{plm} \epsilon_{ijk} v_i R_{jm} R_{kl} v_{0_p}$$

and

$$\mathcal{R} = \frac{1}{2} \sum_{plm} \epsilon_{ijk} v_i (R_{jl} R_{km} - R_{jm} R_{kl}) v_{0_p} = \tfrac{1}{2} \epsilon_{ijk} v_i \epsilon_{plm} R_{jl} R_{km} v_{0_p}, \tag{17b}$$

where now the summation of plm is in the conventional manner repeated subscripts. Now the ray theory Green's function may be written

$$G_{pq}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{4\pi} \frac{g_p(\mathbf{x}) g_q(\mathbf{x}_0)}{(\rho_0 \rho \mathcal{R})^{1/2}} e^{i\omega\tau(\mathbf{x}, \mathbf{x}_0)}, \tag{18}$$

with $\mathcal{R}$ given by (17). Once again we note that modifications due to the presence of caustics are not considered and hence $\mathcal{R}$ is assumed to be positive.

Reciprocity in the geometrical spreading contribution to (18) is recognized by noting that interchanging the source and receiver in (17) is equivalent to replacing $(\mathbf{R})_{jl}$ with $(\mathbf{R}^{\mathrm{T}})_{jl}$. The sum $\mathcal{R}$ is left unchanged, though, because of (14b).

Lastly, it turns out to be useful for Thomson *et al.* (1992) if we note that the adjugate (sometimes 'adjoint') of the matrix **R** can be written

$$(\operatorname{adj} \mathbf{R})_{si} = \det |\mathbf{R}| \, (R^{-1})_{si} = \sum_{plm} \epsilon_{ijk} R_{jl} R_{km} \delta_{ps}$$

and hence an alternative form of (17) is

$$\mathcal{R} = \mathbf{v}_0^{\mathrm{T}} \operatorname{adj} \mathbf{R} \mathbf{v} = \det |\mathbf{R}| \, \mathbf{v}_0^{\mathrm{T}} \mathbf{R}^{-1} \mathbf{v}. \tag{19}$$

3 RAY-CENTRED COORDINATES IN THE ANISOTROPIC CASE

The ray or Hamilton equations are usually expressed in the 3-D Cartesian coordinates used for the wave and eikonal equations (e.g. Červený 1972; Thomson & Chapman 1985; Kendall & Thomson 1989). This leads to a sixth-order system for the equations of variation or the geometrical spreading or the dynamic ray-tracing equations. Sometimes it is convenient to use ray-centred coordinates, as for the proof in Appendix B of Thomson *et al.* (1992), because these lead to a lower order system. Popov & Pšenčík (1976) describe ray-centred coordinates for isotropic media and motivate an appropriate form of the

Hamiltonian by considering stationarity of the traveltime along a ray. Farra & Madariaga (1987) and Coates & Chapman (1990) use the isotropic Hamiltonian of Popov & Pšenčík (1976) as a starting point, but they do not indicate how it is motivated.

Here we take as our starting point the anisotropic eikonal equation in 3-D ray-centred coordinates and consider how the corresponding characteristic equations (Courant & Hilbert 1962, p. 78) may be reduced in number. It is found that these reduced ray equations may be considered to have come from a certain Hamiltonian [which is essentially that of Popov & Pšenčík (1976) in isotropic media]. The ray-centred coordinates themselves have been considered for the anisotropic case by Hanyga (1982, 1986), where a novel alternative approach to the geometrical spreading problem, not involving Hamiltonians, may be found. We use the same notation for the coordinates. They are not necessarily orthogonal and there is some arbitrariness in the way they are chosen (see Section 3.2).

3.1 General theory

The ray-centred coordinates (t, y_1, y_2) are defined relative to the reference ray $\mathbf{x}_0(t)$. Coordinate t is time along the reference ray and y_1, y_2 lie in the plane tangential to the wavefront at $\mathbf{x}_0(t)$. A general position is written

$$x_i(t, y_1, y_2) = x_{0i}(t) + Y_{ia}(t)y_a, \qquad a = 1, 2, \tag{20}$$

where $Y_{ia}(t)$ depends on precisely how the y_a are chosen. See Hanyga [1982, equations (21)–(23)] and Section 3.2 for examples. The eikonal or phase function is written $\tau(\mathbf{x}) = \bar{\tau}(t, y_1, y_2)$ and by the chain rule

$$\left(\frac{\partial\bar{\tau}}{\partial t}\right)_{y_a} = \left(\frac{\partial\tau}{\partial x_i}\right)\left(\frac{\partial x_i}{\partial t}\right)_{y_a}, \qquad \left(\frac{\partial\bar{\tau}}{\partial y_1}\right)_{t,y_2} = \left(\frac{\partial\tau}{\partial x_i}\right)\left(\frac{\partial x_i}{\partial y_1}\right)_{t,y_2}. \tag{21}$$

In equations (21), $\partial x_i/\partial t$ and $\partial x_i/\partial y_a$ are found from (20). We note that on the reference ray itself the $\partial\bar{\tau}/\partial y_a = 0$ and $\partial\bar{\tau}/\partial t = 1$, though on a general ray these vary. The role of 'momenta' in the ray-centred coordinates will be played by

$$P_t = \left(\frac{\partial\bar{\tau}}{\partial t}\right)_{y_a}, \qquad P_1 = \left(\frac{\partial\bar{\tau}}{\partial y_1}\right)_{t,y_2}, \qquad P_2 = \left(\frac{\partial\bar{\tau}}{\partial y_2}\right)_{t,y_1}, \tag{22a}$$

and the three equations (21) may be written

$$P_b = T_{bi}(t, y_1, y_2)p_i, \qquad p_i = \bar{T}_{ib}(t, y_1, y_2)P_b, \qquad b = t, 1, 2. \tag{22b}$$

The relations (22b) may be inverted to give the Cartesian components of slowness p_i in terms of the new momenta and as a result of these transformations the original eikonal equation $H(\mathbf{x}, \mathbf{p}) = 0$ is transformed into the new equation

$$\bar{H}(t, y_1, y_2, P_t, P_1, P_2) = 0. \tag{23}$$

The method of characteristics applied to (23) yields the system of equations

$$\frac{dt}{dv} = \bar{H}_{P_t}, \quad \frac{dy_1}{dv} = \bar{H}_{P_1}, \quad \frac{dy_2}{dv} = \bar{H}_{P_2}, \quad \frac{dP_t}{dv} = -\bar{H}_t, \quad \frac{dP_1}{dv} = -\bar{H}_{y_1}, \quad \frac{dP_2}{dv} = -\bar{H}_{y_2}, \quad \frac{d\bar{\tau}}{dv} = P_t\bar{H}_{P_t} + P_1\bar{H}_{P_1} + P_2\bar{H}_{P_2}, \tag{24}$$

where $\bar{H}_{P_t} = \partial\bar{H}/\partial P_t$, etc. (Courant & Hilbert 1962, p. 78). The variable v may be, for example, time or arclength along the ray through (t, y_1, y_2). On the reference ray itself we expect that $dt/dv \neq 0$ (since the wavespeed of the medium is finite). Hence in some vicinity of the reference ray we may divide the last six of equations (24) by the first and take t to be the independent variable. These six new equations may be integrated to yield y_1, y_2, P_t, P_1, P_2 and $\bar{\tau}(t, y_1, y_2)$ at a given time t along the reference ray. However, of these new equations the one for P_t is redundant since for given t, y_1, y_2, P_1 and P_2 one may in principle obtain P_t from (23) (one of the characteristic equations is always redundant in view of the original eikonal equation). Therefore instead of the first six of equations (24) it is sufficient to consider only the four

$$\frac{dy_1}{dt} = \bar{H}_{P_1}/\bar{H}_{P_t}, \qquad \frac{dy_2}{dt} = \bar{H}_{P_2}/\bar{H}_{P_t}, \qquad \frac{dP_1}{dt} = -\bar{H}_{y_1}/\bar{H}_{P_t}, \qquad \frac{dP_2}{dt} = -\bar{H}_{y_2}/\bar{H}_{P_t}, \tag{25}$$

plus the equation for $\bar{\tau}$.

Equations (25) are in Hamilton form for the time-dependent Hamiltonian $\bar{P} = -P_t(t, y_1, y_2, P_1, P_2)$. This may be recognized by differentiating (23) at constant t to obtain, for example,

$$-\bar{P}_{P_1} = \left(\frac{\partial P_t}{\partial P_1}\right)_{y_1,y_2,P_2,t} = -\left(\frac{\partial\bar{H}}{\partial P_1}\right)_{y_1,y_2,P_t,P_2,t} \bigg/ \left(\frac{\partial\bar{H}}{\partial P_t}\right)_{y_1,y_2,P_1,P_2,t}, \tag{26}$$

which is just the negative rhs of the first of equations (25). Thus (25) are in the Hamilton form

$$\frac{dy_1}{dt} = \bar{P}_{P_1}, \qquad \frac{dy_2}{dt} = \bar{P}_{P_2}, \qquad \frac{dP_1}{dt} = -\bar{P}_{y_1}, \qquad \frac{dP_2}{dt} = -\bar{P}_{y_2}. \tag{27}$$

The Hamiltonian of Popov & Pšenčík [1976, equation (3.14)] is essentially $\tilde{P}$ for the isotropic case, as was noted by Farra & Madariaga (1987). The only difference is that they use arclength rather than time along the reference ray as one of the coordinates.

The geometrical spreading equations may be obtained by considering the two rays through (t, y_1, y_2) and $(t, y_1 + \delta y_1, y_2 + \delta y_2)$. On these two rays the ray-centred coordinate momenta are (P_1, P_2) and $(P_1 + \delta P_1, P_2 + \delta P_2)$. The leading-order equations for $\mathbf{y} = (\delta y_1, \delta y_2, \delta P_1, \sigma P_2)$ are obtained as usual by differentiating the system (27). As (27) is a Hamiltonian system the resulting geometrical spreading equations display the symplectic property, (Thomson & Chapman 1985). This fourth-order geometrical spreading system is written

$$\frac{d\mathbf{y}}{dt} = \tilde{\mathbf{A}}\mathbf{y} \tag{28}$$

where

$$\tilde{\mathbf{A}} = \begin{pmatrix} \tilde{\mathbf{T}}^{\mathrm{T}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & -\tilde{\mathbf{T}} \end{pmatrix}, \qquad \tilde{C}_{ij} = -\partial_{y_i}\partial_{y_j}\tilde{P}, \qquad \tilde{B}_{ij} = \partial_{P_i}\partial_{P_j}\tilde{P}, \qquad \tilde{T}_{ij} = \partial_{y_i}\partial_{P_j}\tilde{P}, \qquad i, j = 1, 2. \tag{29}$$

Equations (28) describe the geometrical spreading along a general ray, provided it is close enough to the reference ray for the ray-centred coordinates to be useful. The initial conditions may be of point-source or plane-wave type and the propagator-matrix solution of (28), $\tilde{\mathbf{W}}(t, 0)$, has the same elements as the Jacobian

$$\frac{\partial(y_1, y_2, P_1, P_2)}{\partial(y_{10}, y_{20}, P_{10}, P_{20})},$$

where subscript 0 indicates values at $t = 0$. For use in Section 3.3 and Thomson *et al.* (1992), we define the 2×2 blocks of this propagator according to

$$\tilde{\mathbf{W}} = \begin{pmatrix} \tilde{\mathbf{O}} & \tilde{\mathbf{R}} \\ \tilde{\mathbf{Q}} & \tilde{\mathbf{P}} \end{pmatrix} \tag{30}$$

[*cf.* (14a)].

The solution of the transport equation of ray theory involves the Jacobian $\partial(x_1, x_2, x_3)/\partial(t, q_1, q_2)$, where $q_{1,2}$ are two parameters which define a single ray. From the chain rule for Jacobians

$$\frac{\partial(x_1, x_2, x_3)}{\partial(t, q_1, q_2)} = \frac{\partial(x_1, x_2, x_3)}{\partial(t, y_1, y_2)}\frac{\partial(t, y_1, y_2)}{\partial(t, q_1, q_2)} = \frac{\partial(x_1, x_2, x_3)}{\partial(t, y_1, y_2)}\frac{\partial(y_1, y_2)}{\partial(q_1, q_2)}. \tag{31}$$

The first Jacobian on the rhs is that corresponding to the transformation to ray-centred coordinates [see matrix $\mathbf{T}$ in (22b)] and the second-order Jacobian is obtained from the solution of the geometrical spreading equations (28). Note that in (31) we are using the time t along the reference ray on the lhs, as opposed to the time or arclength along the actual ray in question. The latter is perhaps more usual in geometrical spreading considerations and if it were used instead, a slightly longer argument is needed in order to reduce the results to a second-order Jacobian on the right. See Popov & Pšenčík [1976, equations (3.8) to (3.11), which are essentially unchanged in the anisotropic case]. In practice, it is the spreading along the reference ray itself which is of main interest.

3.2 Particular ray-centred coordinates

The slowness $\mathbf{p}_0$ and total velocity $\mathbf{v}_0$ on the reference ray may be used to define two orthogonal vectors $\mathbf{m}$ and $\mathbf{n}$ in the wavefront according to

$$m_k = \epsilon_{klm} p_{0l} H_{p_m}, \qquad n_i = \epsilon_{ijk} p_{0j} m_k = (\delta_{ij} - p_{0j} H_{p_i}) p_{0j} = P^i_j(t) p_{0j}, \tag{32}$$

where the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$ has been used. These definitions break down in the isotropic case, when $\mathbf{p}_0$ and $\mathbf{v}_0$ are parallel. However, we note that the operator P^i_j on the rhs of (32) has the property $p_{0_i} P^i_j = p_{0_j} - p_{0_j}(p_{0_i} v_{0_i}) = 0$, since $p_{0_i} v_{0_i} = 1$ by definition. Thus, for fixed j, P^i_j is a vector in the wavefront in either the isotropic or anisotropic case. The operators P^i_j were introduced by Hanyga (1982) (hence we use the same symbol). There are three such operators P^i_1, P^i_2 and P^i_3, two of which suffice to define two directions in the wavefront. Hanyga [1982, equation (23)] has chosen the defining relation for the ray-centred coordinates as

$$x_i(t) = x_{0_i}(t) + P^i_a(t) y_a, \qquad a = 1, 2, \tag{33}$$

and then we have the Jacobian

$$\frac{\partial(x_1, x_2, x_3)}{\partial(t, y_1, y_2)} = \det|H_{p_i}, P^i_1, P^i_2| = H_{p_3} \quad \text{at} \quad y_a = 0 \tag{34}$$

[Hanyga (1982), equation (24)]. If $H_{p_s} = 0$ another choice of two from the P^i_j must be used to obtain a well-defined transformation.

Evidently a choice yielding a non-zero value for the transformation Jacobian on the lhs of (34) at one point on the ray may not do so at another point on the ray. It is of interest to enquire if a linear combination of the P^i_j can be taken so that the Jacobian is never zero. However, such a combination has not been found.

Directions in the wavefront can be defined in other ways. For example, by taking the vector product of $\mathbf{p}_0$ with a unit vector along a Cartesian coordinate axis. The three wavefront vectors obtained this way have much the same properties as the P^i_j. A third alternative is to start from the normal and binormal for the ray (Mathews & Walker 1964, p. 409; Popov & Pšenčík 1976). The projections of these two vectors in the wavefront can be constructed easily and intuitively it seems unlikely that these projections could be colinear in any reasonable situation.

The choice of ray-centred coordinates will not be considered further at this time and we return to the point source problem.

3.3 Matching with a point source in ray-centred coordinates

Two possible approaches to this problem will be described. It is understood that from now on all quantities are evaluated on the reference ray itself and that subscript zero indicates values taken at the point $t = 0$ on the reference ray. We take P_{01} and P_{02} as the parameters defining the rays leaving the source point [i.e. as q_1 and q_2 in (31) above] and introduce the notation [see (31), (22b) and (30)]

$$\tilde{J} = \frac{\partial(x_1, x_2, x_3)}{\partial(t, P_{01}, P_{02})} = \det|\mathbf{T}| \frac{\partial(y_1, y_2)}{\partial(P_{01}, P_{02})} = \det|\mathbf{T}| \det|\tilde{\mathbf{R}}|. \tag{35}$$

Note that the matrix $\tilde{\mathbf{R}}$ is the 2×2 counterpart of the 3×3 submatrix $\mathbf{R}$ of equation (14a), and that $\tilde{\mathbf{R}}$ also has the reciprocity property (14b).

Method I

Here the aim is to exploit equation (18) of Section 2.3, the result of matching in Cartesian coordinates. The task reduces to finding the relationship between the elements of the 3×3 Cartesian spreading matrix $R_{ij} = \partial x_i / \partial p_{0_j}$ and the elements of the Jacobian $\tilde{J}$ (35). From the chain rule and transformation (22b) at the source point

$$\frac{\partial x_i}{\partial P_{0a}} = R_{ij} \bar{T}_{0ja}, \qquad a = 1, 2, \tag{36}$$

where $\bar{T}_{0ja} = \bar{T}_{ja}(0, 0, 0)$. Hence, by steps analogous to those leading to equation (10) of Section 2.2, we have at time t

$$\begin{aligned} \tilde{J} &= \det|\mathbf{v}, R_{jm}\bar{T}_{0m1}, R_{kn}\bar{T}_{0n2}| = \epsilon_{ijk} v_i R_{jm} R_{kn} \bar{T}_{0m1} \bar{T}_{0n2} = \tfrac{1}{2} \epsilon_{ijk} v_i R_{jm} R_{kn} (\bar{T}_{0m1}\bar{T}_{0n2} - \bar{T}_{0n1}\bar{T}_{0m2}) \\ &= \sum_{plm} \epsilon_{ijk} v_i R_{jm} R_{kn} t_{0p} \quad \text{where} \quad t_{0p} = \bar{T}_{0m1}\bar{T}_{0n2} - \bar{T}_{0n1}\bar{T}_{0m2} \end{aligned} \tag{37}$$

and the summation over pmn is cyclic over the three Cartesian components. The vector t_{0p} has a simple interpretation obtained as follows: by taking the cross product of the second two columns of the defining relationship $\mathbf{I} = \mathbf{T}_0 \bar{\mathbf{T}}_0$ one finds

$$\delta_{i1} = \epsilon_{ijk} T_{0jm} T_{0kn} \bar{T}_{0m1} \bar{T}_{0n2} = \epsilon_{ijk} T_{0jm} T_{0kn} (\bar{T}_{0m1}\bar{T}_{0n2} - \bar{T}_{0n1}\bar{T}_{0m2}) = (\operatorname{adj} \mathbf{T})_{0p_i} t_{0p} = \det|\mathbf{T}_0| \, \bar{T}_{0p_i} t_{0p} \tag{38a}$$

and hence

$$t_{0p} = \det|\mathbf{T}_0|^{-1} \, T_{01p} = \det|\mathbf{T}_0|^{-1} \, v_{0p}, \tag{38b}$$

since the first row of $\mathbf{T}_0$ is just $\mathbf{v}_0$ (20, 22b). As a result (37) and (35) lead to

$$\det|\mathbf{T}_0| \, \tilde{J} = \det|\mathbf{T}_0| \det|\mathbf{T}| \det|\tilde{\mathbf{R}}| = \sum_{plm} \epsilon_{ijk} v_i R_{jm} R_{kn} v_{0p} = \mathcal{R}, \tag{39}$$

where $\mathcal{R}$ was previously defined in equation (17a). Thus, $\tilde{J}$ and $\tilde{\mathbf{R}}$ may be inserted in equation (18) of Section 2.3, giving the ray solution at large distances in the ray-centred coordinates. Note that reciprocity is apparent in the second part of equation (39), in view of the way $\mathbf{T}_0$ and $\mathbf{T}$ appear and the properties of $\tilde{\mathbf{R}}$.

Method II

In this method the matching with the point source solution [Section 2.1, equation (1)] is performed directly in the ray-centred coordinates. The ray theory solution at a general point is written [see (3)]

$$u_p = \frac{\tilde{\psi}_0}{(\rho \tilde{J})^{1/2}} g_p(\mathbf{x}) e^{i\omega\tau(\mathbf{x})} \tag{40}$$

and the aim is to determine $\tilde{\psi}_0$.

As the source is approached one obtains from (29) that

$$\tilde{R}_{ij} = (\partial_{P_i}\partial_{P_j}\tilde{P})t + O(t^2), \qquad i, j = 1, 2, \tag{41}$$

where the derivatives of Hamiltonian $\tilde{P}$ are evaluated at the source. The three second derivatives of $\tilde{P}$ are found in terms of the six derivatives of $\tilde{H}$ (23) by differentiation of (26). This yields

$$\frac{\partial^2 \tilde{P}}{\partial P_i\,\partial P_j} = [\tilde{H}_{ij}\tilde{H}_t^2 - \tilde{H}_t(\tilde{H}_{ti}\tilde{H}_j + \tilde{H}_{tj}\tilde{H}_i) + \tilde{H}_{tt}\tilde{H}_i\tilde{H}_j]/\tilde{H}_t^3, \tag{42}$$

where a subscript notation $\tilde{H}_i$, $i = t, 1, 2$, is used for derivatives. With (41) and (42) the Jacobian det $|\tilde{\mathbf{R}}|$ may be written in terms of derivatives of $\tilde{H}$ at the source. After some cancellations and rearrangement one finds that

$$\det|\tilde{\mathbf{R}}| = \tilde{H}_t^{-4}\Big(\sum_{plm} \epsilon_{ijk}\tilde{H}_i\tilde{H}_{lj}\tilde{H}_{mk}\tilde{H}_p\Big)t^2 + O(t^3) = (\tilde{H}_t)^{-4}\tilde{H}_i(\mathrm{adj}\,\tilde{\mathbf{H}})_{pi}\tilde{H}_p t^2 + O(t^3)$$
$$= (\tilde{H}_t)^{-4}\tilde{H}_i \det|\tilde{\mathbf{H}}|\,(\tilde{\mathbf{H}}^{-1})_{pi}\tilde{H}_p t^2 + O(t^3), \tag{43}$$

where the summation on plm is cyclical over t, 1 and 2 and $\tilde{\mathbf{H}}$ represents the matrix of second derivatives of $\tilde{H}$. In order to relate (43) to the Gaussian curvature of the slowness surface at the source (equations 1, 2) we must pull back from the P_i to the p_i using (22b). From that equation at the source

$$\tilde{H}_i = H_m\bar{T}_{0mi}, \qquad \tilde{H}_{ij} = H_{mn}\bar{T}_{0mi}\bar{T}_{0nj}, \quad \text{i.e. } \bar{\mathbf{T}}_0^{\mathrm{T}}\mathbf{H}\bar{\mathbf{T}}_0,$$

and then from the last form of (43) it is apparent that

$$\det|\tilde{\mathbf{R}}| = \det|\mathbf{T}_0|^{-2}\frac{H_i\det|\mathbf{H}|\,(\mathbf{H}^{-1})_{pi}H_p}{(\bar{T}_{mt}v_{0m})^4}t^2 + O(t^3). \tag{44a}$$

The denominator here may be shown to be just unity by forming the product of (38b) with $\bar{T}_{0pt}$. Hence,

$$\det|\tilde{\mathbf{R}}| = \det|\mathbf{T}_0|^{-2}\Big(\sum_{pmn} H_i\epsilon_{ijk}H_{jm}H_{kn}H_p\Big)t^2 + O(t^3), \tag{44b}$$

H_i, etc. evaluated at the source. Combining (44b) with (35) and then matching (40) with the point source solution (1) shows that $\bar{\psi}_0^2$ is just $(4\pi)^{-2}\rho_0^{-1}\det|\mathbf{T}_0|^{-1}$ and we recover the same results as method I.

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