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Hyperbolic geometry as an alternative to perspective for constructing drawings of visual space

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Abstract. It is demonstrated by the use of drawings: (i) that two-point perspective drawings based on projective geometry possess systematic distortions, and (ii) that these distortions can be eliminated in drawings based on hyperbolic geometry.

The purpose of this communication is to establish drawings based on hyperbolic geometry as an alternative to perspective drawings based on projective geometry for evoking the visual experience of depth, or more precisely the visual experience of receding squares. This is accomplished in several steps. First, the distortions possessed by two-point perspective drawings are brought to the reader's attention. An explanation is given as to why these distortions are inherent and cannot be eliminated in any two-point perspective drawing. Second, it is demonstrated in a drawing of a hyperbola of revolution of one sheet that these distortions can be greatly reduced. A discussion and a second drawing show how hyperbolic geometry can lead to the construction of drawings that elicit a visual experience of receding squares, which possesses no distortions. The difference between the visual experience evoked by the hyperbolic and the perspective drawing is related to their perspective underlying mathematical concepts of space and the shape of the eye.

Pirenne (1970) and Graham (1951, 1965) give a discussion and bibliography of visual perceptionists and others who have been interested in the curvature of visual space. This curvature is experienced as physically straight lines that are seen as curved, or as physically curved lines that are seen as straight. Most pertinent to this discussion is Luneburg's (1947) *Mathematical Analysis of Binocular Vision.* This mathematical analysis led Luneburg to conclude that visual experience is more correctly described by the concepts of space as expressed by hyperbolic geometry than by Euclidean or elliptic geometry. Using this concept of space, several investigators (Ittleson 1952; Hardy 1953) have constructed rooms. These rooms, which have walls the shape of a hyperbola of revolution of one sheet, are seen as rectangular in shape by observers. On the basis of the principles developed by these authors, it is possible to construct drawings of hyperbolas of revolution of one sheet that evoke the visual experience of receding squares. This visual experience is similar to the visual experience elicited by a two-point perspective drawing.

The perspective method of drawing was developed in the Renaissance by Alberti and later amended by da Vinci, Durer, and Viator (Ivins 1973; Spencer 1966; Giosseffi 1966a, 1966b). It has been accepted to the present as the correct and only method of constructing drawings that evoke the visual experience of receding squares. In the two-point perspective drawing (figure 1), the quadrilaterals, general four-sided forms, in the center may be seen as receding squares. But, as the eye reaches the extremes on the far right and left sides near the horizon, the quadrilaterals become flat and rectangular in shape. This change in perception is a distortion which conflicts with our perception of physical space. A square grid in physical space would not elicit this experience. This flattening of the receding squares on the sides is the result of the two-point perspective method of constructing a drawing.

A perspective drawing is a doubled ruled surface. Two sets of intersecting straight lines are generated from three graphical elements. They are the foreground line, which is divided into equal segments, and the two points on the horizon line. Each segment on the foreground line is connected by two straight lines, one each to the two points on the horizon line. To obtain all the possible variations of a two-point perspective drawing, it is only necessary to vary the length of the segments, the distance between the two points on the horizon line, and the distance between the foreground line and the horizon line. There is no variation that will eliminate the distortions pointed out in figure 1.

In figure 2, the quadrilaterals in the center of the hyperbolic drawing may be seen as receding squares. As the eye reaches the extremes on the far right and left sides near the horizon, the quadrilaterals appear as approximately square (see area indicated by arrow in figure 2). The degree of distortion, by inspection, is much less than in the two-point perspective drawing. As in figure 1, all the quadrilaterals in figure 2 do not evoke the visual response of a square. For figure 2, those distortions are the result of emphasizing similarities and differences in the visual experience elicited by the two drawing methods so they can be compared.

In order to understand the construction method of figure 2, first consider figure 3, which is also a drawing of a hyperbola of revolution of one sheet. To construct this figure, two circles are divided into any number of equal-length arcs. The arcs on the two circles are then connected by two sets of straight lines. Like a perspective drawing, it is also a doubled ruled surface.

The quadrilaterals between the heavy lines in figure 3, may be seen approximately as receding squares on a curved surface. As the eye reaches the far right or left sides, denoted by the heavy lines, the quadrilaterals near the horizon lines appear as elongated squares in the bottom half and as flattened squares in the top half of this drawing. By varying the shape of the circles (ellipses), their distance apart, the

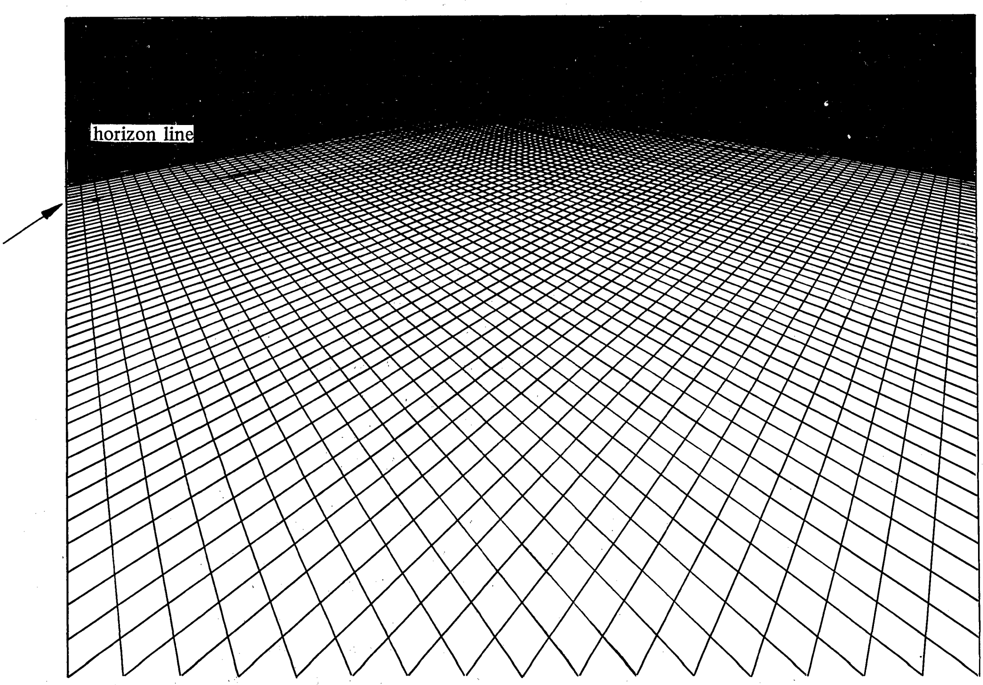

Figure 1. Two-point perspective drawing. Arrow indicates area of greatest distortion.

number of arcs, and the degree of rotation of the straight lines connecting the two ellipses, it is possible to alter the shape of the quadrilaterals. In particular, this allows a choice in construction, so that the quadrilaterals from the center to the sides, near the heavy lines, may be seen as squares.

Figure 2 is a drawing of a hyperbola of revolution of one sheet like figure 3, but only the area between the heavy lines is shown. It is a bottom half of the figure 3 drawing.

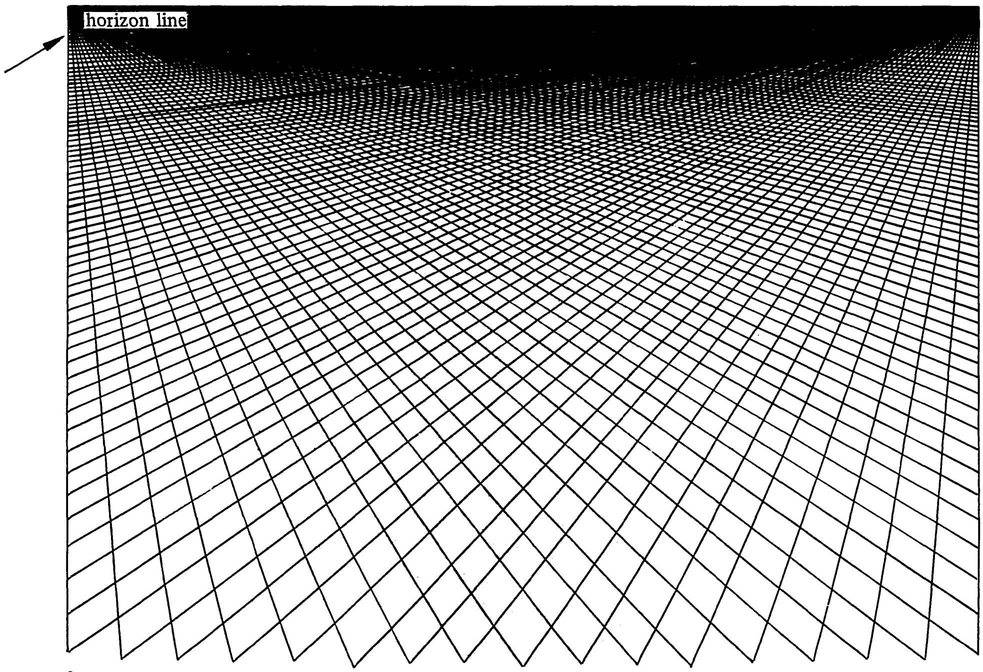

Figure 2. Hyperbolic drawing. Arrow indicates area of greatest distortion in the two-point perspective drawing.

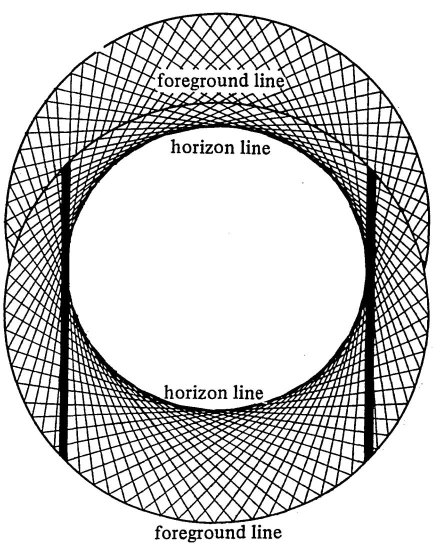

Figure 3. Drawing of a hyperbola of revolution of one sheet.

I believe that the difference in visual experience between the hyperbolic drawing and the perspective drawing is directly related to the differences in construction discussed above and the shape of the eye as defined by these two different mathematical concepts of space. Perspective drawing, developed by Alberti in 1435, defines the eye as a point. A mathematical point has no shape or volume. For the hyperbolic drawing, the shape of the eye is defined as a sphere in elliptic geometry, and visual space concurs with hyperbolic geometry (Luneburg 1947). A sphere is a curved surface, and the space we perceive is slightly curved. This curvature of visual space is demonstrated in figure 2 by the black area extending towards the foreground line in a rounded shape. Conversely, in the perspective drawing based on Euclidean projective geometry, the horizon line consists of two straight lines forming a 'V'. This explains the difference in visual experience evoked by these two drawings as a physiological mechanism rather than as a cognitive strategy in our perception of space (Gregory 1973). From the evidence presented here, this distinction cannot be made. For contrasting views see Pirenne (1970) and Platt (1960).

In summary, it has been demonstrated that perspective drawings, based on concepts of space as defined by Euclidean projective geometry, possess distortions in evoking the visual experience of receding squares. These systematic distortions are the result of the construction method and cannot be eliminated from any two-point perspective drawing. Figures 2 and 3 demonstrate that hyperbolic drawings possess sufficient freedom in their construction method to eliminate this systematic distortion. It is suggested that the differences between the two drawing methods is directly related to their two different mathematical concepts of space in defining the shape of the eye. Figure 2 adds support to the experimental evidence and perceptual concepts of Helmholtz (1924), Blumenfeld (1913), Luneburg (1947), Ittleson (1952), Hardy (1953), Shipley (1957), and Blank (1961, 1962) that visual space is not a flat plane and does not concur with Euclidean projective geometry, but possesses a slight curvature and concurs with hyperbolic geometry.

Future communications are planned which will present larger hyperbolic drawings without the distortions found in figure 2, and the hyperbolic equivalent of a one-point perspective drawing.

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