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Dedicated to

Arlene, for her patience

Gerard, for his existence

In appreciation

to Dr. James Anderson,

for his assistance

Abstract

RELATIVISTIC RIGID BODY MOTION

Robert Bennett

Dr. J. L. Anderson

May, 1971

The definition of rigidity as invariant distance between body points leads in classical theory to 6 degrees of freedom, but in relativistic theory leads only to further restrictions on the motion or to contradictions. As the criterion for rigid body motion this paper proposes a relativistically invariant 6 parameter group of motions defined with respect to the instantaneous rest frame of the object's EMS tensor. The 6 arbitrary constants arising from motional freedom are used to specify the center and axes of rotation.

For the free body in special relativity, the velocity and mass density are found to be constant, while the rotational motion satisfies the Euler equations. It is shown that the often-discussed spiral motion is absent, principally due to the choice of center of mass. Generalization to external fields and forces is made, and the special case of the electromagnetic field is solved in detail.

The rigid body definition is formulated in generalized coordinates and the motion obtained by expansion of these coordinates about the body center. The expansion is terminated by neglecting field derivatives beyond a certain order, and not by assumptions concerning the body's internal structure. For completeness the approximations are carried out to second order.

In zero and first order, the center of mass follows the usual geodesic path, but differs in the second order due to terms coupling derivatives of the Riemann

tensor to the spin.

For the rotational motion in first approximation a new spin equation is obtained. When applied to planetary motion, this equation reduces to the classical Euler equation with an external gravitational torque and correctly predicts the earth's astronomical precession. Finally it is proven that the relativistic correction to precession of order $1/c^2$ for an earth satellite provides a possible test of the validity of this proposed relativistic spin equation.

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SECTION 1

Introduction and Definitions

1.1 Historical Review

The definition of a rigid body and the description of its motion within the framework of relativity theory has plagued physics since the turn of the century. This paper describes an attempt to resolve this chronic problem.

In the classical mechanics of Newton a rigid body is one in which the distance between two body points is constant for any time t, distance and time being measured in a Galilean coordinate system. Since time is absolute, this definition amounts to a constraint on the motion,

$$(r_i - r_j)^2 = c_{ij}^2 \tag{1.1}$$

the r_i being the spatial body coordinates and the c_{ij} a set of constants. If this constraint is to be satisfied for arbitrary body motion, then it is necessary that six independent coordinates representing three translational and three rotational degrees of freedom be specified (G1). The location of a point in the body will then be given by

$$x^m(t,a^b) = z^m(t) + R^m{}_b(t)a^b$$

$$x^0(t,a^b) = t \tag{1.2}$$

In 1910 Born (B1) attempted to bring this classical concept into the theory of relativity. We will present here an outline of the Born rigidity condition as given by Anderson (A1).

If a^b ($b=1,2,3$) are the Lagrangian coordinates of a body element, then

$$x^\mu = x^\mu(\lambda,a^b) \tag{1.3}$$

are the points lying along a streamline. The displacements between streamlines with coordinates a^b and

$a^b + da^b$ are

$$\delta x^\mu \equiv da^b \frac{\partial x^\mu}{\partial a^b} \tag{1.4}$$

The velocity u^μ of a body element is given by

$$u^\mu \equiv \frac{\partial x^\mu}{\partial \lambda} = \eta^{\mu\nu} u_\nu$$

$$u^\mu u_\mu = 1 \tag{1.5}$$

It follows that

$$\frac{\partial}{\partial \lambda} \delta x^\mu = da^b \frac{\partial}{\partial a^b}\left(\frac{\partial x^\mu}{\partial \lambda}\right) = da^b \frac{\partial u^\mu}{\partial a^b}$$

$$= da^b \frac{\partial u^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial a^b} = \delta x^\nu u^\mu{}_{,\nu} \tag{1.6}$$

and

$$\frac{\partial u^\mu}{\partial \lambda} = u^\mu{}_{,\nu} u^\nu \tag{1.7}$$

The projection operator onto the space normal to a streamline is

$$P_{\mu\nu} = \eta_{\mu\nu} - u_\mu u_\nu$$

or

$$P^\mu{}_\nu = \delta^\mu{}_\nu - u^\mu u_\nu \tag{1.8}$$

where

$$P_{\mu\nu} u^\nu = 0 \,, \qquad P^\mu{}_\nu P^\nu{}_\rho = P^\mu{}_\rho \tag{1.9}$$

The projection of δx^μ onto this space is

$$\bar{\delta} x^\mu \equiv P^\mu{}_\nu \delta x^\nu \tag{1.10}$$

and the projection of its velocity is

$$\bar{\delta} v^\mu \equiv P^\mu{}_\nu \frac{\partial}{\partial \lambda} \bar{\delta} x^\nu = P^\mu{}_\nu \left(P^\nu{}_\rho \frac{\partial}{\partial \lambda} \delta x^\rho + \delta x^\rho \frac{\partial}{\partial \lambda} P^\nu{}_\rho\right)$$

$$= P^\mu{}_\rho \delta x^\nu u^\rho{}_{,\nu} + P^\mu{}_\nu\left(-\frac{\partial u^\nu}{\partial \lambda} u_\rho - \cancel{u^\nu \frac{\partial u_\rho}{\partial \lambda}}\right)\delta x^\rho$$

$$= P^\mu{}_\rho \delta x^\nu u^\rho{}_{,\nu} - P^\mu{}_\nu u^\nu{}_{,\kappa} u^\kappa u_\rho \delta x^\rho$$

$$= \mathcal{P}^{\mu}{}_{\rho}\,\delta x^{\nu} u^{\rho}{}_{,\kappa}\,\delta^{\kappa}{}_{\nu} - \mathcal{P}^{\mu}{}_{\rho}\,\delta x^{\nu} u^{\rho}{}_{,\kappa} u^{\kappa} u_{\nu}$$

$$= \mathcal{P}^{\mu}{}_{\rho}\,\delta x^{\nu} u^{\rho}{}_{,\kappa}\,\mathcal{P}^{\kappa}{}_{\nu}$$

$$= \bar{\delta} x^{\nu}\,\mathcal{P}^{\mu}{}_{\rho}\, u^{\rho}{}_{,\nu} = \delta x^{\sigma}\,\mathcal{P}^{\nu}{}_{\sigma}\,\mathcal{P}^{\mu}{}_{\rho}\, u^{\rho}{}_{,\nu}$$

(1.11)

Eqs. (1.10) and (1.11) represent the position and velocity of $a^b + da^b$ relative to a^b, since

$$\bar{\delta} x^{\mu} u_{\mu} = \bar{\delta} v^{\mu} u_{\mu} = 0 . \tag{1.12}$$

$\bar{\delta} v^{\mu}$ may be rewritten as

$$\bar{\delta} v^{\mu} = \bar{\delta} x^{\sigma} \Big\{ \Big[\frac{\mathcal{P}^{\nu}{}_{\sigma}\mathcal{P}^{\mu\rho}}{2} (u_{\rho,\nu} - u_{\nu,\rho}) \Big]$$

$$+ \Big[\mathcal{P}^{\nu}{}_{\sigma}\mathcal{P}^{\mu\rho} \Big(\frac{u_{\rho,\nu} + u_{\nu,\rho}}{2} - \frac{u^{\kappa}{}_{,\kappa}}{3}\mathcal{P}_{\rho\nu} \Big) \Big] + \Big[\frac{u^{\kappa}{}_{,\kappa}}{3}\,\delta^{\mu}{}_{\sigma} \Big] \Big\}$$

$$\bar{\delta} v^{\mu} \equiv \bar{\delta} x^{\nu} \Big\{ \omega^{\mu}{}_{\nu} + \sigma^{\mu}{}_{\nu} + \frac{\Theta}{3}\,\delta^{\mu}{}_{\nu} \Big\}$$

(1.13)

where the quantities in brackets are the rotation $(\omega^{\mu}{}_{\nu})$, shear $(\sigma^{\mu}{}_{\nu})$ and expansion (Θ) of the element.

If the infinitesimal distance between two streamlines measured in the space orthogonal to them is

$$d\xi^2 = \eta_{\mu\nu}\,\bar{\delta} x^{\mu} \bar{\delta} x^{\nu} = \mathcal{P}_{\mu\nu}\,\delta x^{\mu} \delta x^{\nu} \tag{1.14}$$

then the Born rigid body condition requires that this orthogonal interval between neighboring streamlines remain constant for all motions of the body,

$$\frac{\partial}{\partial \lambda}\, d\xi = 0 \tag{1.15}$$

From Eq. (1.14) we find that

$$\frac{\partial}{\partial\lambda} d\bar{s} = \frac{1}{d\bar{s}} P_{\mu\nu} \bar{\delta} x^\mu \frac{\partial}{\partial\lambda} \bar{\delta} x^\nu = \frac{1}{d\bar{s}} P_{\mu\nu} \bar{\delta} x^\mu \bar{\delta} v^\nu$$

$$= \frac{1}{d\bar{s}} P_{\mu\nu} \bar{\delta} x^\mu \bar{\delta} x^\rho \left\{ \omega^\nu{}_\rho + \sigma^\nu{}_\rho + \frac{\theta}{3} \delta^\nu{}_\rho \right\} = 0$$

(1.16)

Thus, arbitrary motion of the Born rigid body is characterized as being free of shear and expansion,

$$\sigma^\mu{}_\rho = \theta = 0 \, . \qquad (1.7)$$

Unfortunately this condition was later shown by Herglotz (H1) and Nöther (N1) to restrict the motion to three degrees of freedom. (Both authors catalogued the permissible groups of motion and Herglotz demonstrated that if one point of the body has a prescribed motion, the entire body motion is completely determined.)

Recent authors have used a discrete set of points in their definition (M1, E1) or confined the rigidity to the surface (HM1).

In classical mechanics, the success of the rigid body abstraction is based on its correct predictions for the motion of many solid objects. Any proposed relativistic generalization of this ideal construct must satisfy the same dynamical test.

1.2 Definition of Rigidity

We begin by generalizing the classical representation of a point in the rigid body, Eq. (1.2), to

$$x^\mu(\lambda,\vec{a}) = z^\mu(\lambda) + h^\mu{}_b(\lambda)a^b \tag{2.1}$$

The $x^\mu(\lambda,\vec{a})$ are the curvilinear coordinates of an arbitrary streamline in the body, where

λ is a monotonic parameter labeling the path,

$z^\mu(\lambda)$ represents a translation to the streamline from the origin,

$h^\mu{}_b(\lambda)$ are a triad of space-like vectors orthonormal to the streamline, describing the orientation with respect to the $\{x^\mu\}$,

$\{a^b\}$ are three distance parameters on a λ = constant hypersurface Σ.

If we define the streamline's tangent vector at a^b = 0 as

$$\frac{dz^\mu}{d\lambda} = \dot{z}^\mu \equiv v^\mu \equiv h^\mu{}_0 \tag{2.2}$$

then the set $\{v^\mu, h^\mu{}_a\}$ constitute an orthonormal tetrad $\{h^\mu{}_\alpha\}$ satisfying

$$h^\mu{}_\alpha h^\beta{}_\mu = \delta^\beta{}_\alpha, \qquad h^\mu{}_\alpha h^{\alpha\nu} = \eta^{\mu\nu} = (1,-1,-1,-1),$$

$$h^\mu{}_\alpha h^\alpha{}_\nu = \delta^\mu{}_\nu = v^\mu v_\nu + h^\mu{}_a h^a{}_\nu,$$

$$h^\beta{}_\mu = \eta^{\beta\gamma}\eta_{\mu\nu}h^\nu{}_\gamma \tag{2.3}$$

By differentiating with respect to λ,

$$\dot{h}^\mu{}_\alpha h^\beta{}_\mu + h^\mu{}_\alpha \dot{h}^\beta{}_\mu = 0,$$

$$h^\mu{}_\alpha \dot{h}^\beta{}_\mu \equiv \Omega_\alpha{}^\beta = -\Omega^\beta{}_\alpha \tag{2.4}$$

These relations may be used to prove that the distance from the origin on Σ ,

$$x^\mu(\lambda,\vec{a}) - z^\mu(\lambda) \equiv R^\mu(\lambda,a^b) = h^\mu{}_b(\lambda)a^b \tag{2.5}$$

represents a rotation. Since

$$\frac{dR^2}{d\lambda} = \frac{d}{d\lambda}(h^\mu{}_b a^b h^\nu{}_c a^c \eta_{\mu\nu}) = \frac{d}{d\lambda}(a^b a^c \delta_{bc}) = 0 \tag{2.6}$$

R^μ is constant in length. Angles are preserved, because

$$R^\mu(\lambda,\vec{a})\, R'^\nu(\lambda,\vec{a}')\eta_{\mu\nu} = |R||R'|\cos\theta,$$

$$h^\mu{}_b a^b h^\nu{}_c a'^c \eta_{\mu\nu} = \delta_{bc} a^b a'^c = |a||a'|\cos\theta,$$

$$\rightarrow \dot{\theta} = 0 \tag{2.7}$$

Thus, the geometric character of any λ = constant hypersurface Σ is that of a plane, since any vector R^μ in the surface is normal to the time-like vector v_μ,

$$R^\mu(\lambda,\vec{a})\, v_\mu(\lambda) = h^\mu{}_b v_\mu a^b = 0 \tag{2.8}$$

Eq. (2.1) may be expressed in differential form as

$$\frac{d^2 x^\sigma}{da^b da^c} = x^\sigma{}_{,b,c} = 0. \tag{2.9}$$

In the case of an arbitrary metric $g_{\mu\nu}(x)$, the parallel transport of the basis vectors $x^\sigma{}_{,b}$ in Σ results in

$$x^\sigma{}_{,b,c} + \Gamma^\sigma_{\mu\nu}(x)\, x^\nu{}_{,b}\, x^\mu{}_{,c} = 0, \tag{2.10}$$

which expresses the rigid body constraint in generalized coordinates.

The dynamics of the body are given by the set of 16 basis vectors $\{h^\mu{}_\alpha\}$. But there are 10 orthonormality conditions, Eq. (2.3), expressing relations among the $\{h^\mu{}_\alpha\}$, and so there are only six independent descriptors of the motion, as was also true classically. We may take these to be the 3 $v^m(\lambda)$ of translation and the 3 Euler angles of rotation implicitly contained in $h^\mu{}_a[\theta(\lambda), \phi(\lambda), \psi(\lambda)]$.

1.3 Instantaneous Rest Frame

We specify the structure of the body with respect to an inertial frame of reference $\{x^\mu\}$ by means of a spatially bounded energy-momentum-stress (EMS) tensor $T^{\mu\nu}(x)$ and reserve the last letters of the Greek alphabet for indices in this system. The first letters of the alphabet will denote indices in the body frame, so that $\bar{T}^{\alpha\beta}(\xi)$ and $\xi^\beta=(\lambda, a^b)$ will represent the body's EMS tensor and internal coordinates.

In the body system of coordinates we define the instantaneous rest frame (IRF) as

$$\bar{T}^{0a} = 0 \,, \qquad (3.1)$$

This definition corresponds to (but does not coincide with) the derivation of a fluid's "proper" frame given explicitly by Landau and Lifschitz, (LLI), and implicitly assumed by other authors (B2, F1, MF1).

We may formulate this definition in a geometrically invariant way by transforming to the space frame:

$$\bar{T}^{0b} = 0 \quad \rightarrow \quad \bar{T}^{0\beta} - \delta^\beta{}_0 \bar{T}^{00} = 0 \qquad (3.2)$$

$$\frac{\partial\lambda}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu} T^{\mu\nu} - \delta^\beta{}_0 \frac{\partial\lambda}{\partial x^\mu}\frac{\partial\lambda}{\partial x^\nu} T^{\mu\nu} = 0 \qquad (3.3)$$

Multiply by $\dfrac{\partial x^\rho}{\partial \xi^\beta}$;

$$\frac{\partial\lambda}{\partial x^\mu} T^{\mu\rho} - \frac{\partial x^\rho}{\partial\lambda}\frac{\partial\lambda}{\partial x^\mu}\frac{\partial\lambda}{\partial x^\nu} T^{\mu\nu} = 0 \qquad (3.4)$$

Note that

$$n_\mu \equiv \frac{\partial\lambda}{\partial x^\mu} \qquad (3.5)$$

is a time-like vector normal to a λ = constant hypersurface, and

$$u^\mu \equiv \frac{\partial x^\mu}{\partial\lambda} \qquad (3.6)$$

is the tangent vector to any streamline passing through the hypersurface. Therefore,

$$n_\mu T^{\mu\rho} = u^\rho n_\nu n_\mu T^{\mu\nu} = u^\rho \bar{T}^{00} \qquad (3.7)$$

This equation states that the operator $u^{\rho} n_{\nu}$ possesses a time-like eigenvector, $n_{\mu}T^{\mu\nu}$, with eigenvalue 1.

Because

$$n_{\nu} n_{\mu} T^{\mu\nu} = T^{00} \tag{3.8}$$

the contraction of the body's EMS tensor with the surface normal moves along the streamlines.

Note that

a) the analysis is independent of any rigid constraints on the body,

b) one can always find an IRF for any EMS tensor of matter,

c) physically the IRF refers to a reference frame in which there is no flow of energy or momentum flux.

1.4 Consequences of the Six Degrees of Freedom.

If four arbitrary constants are added to z^μ, the motion is unaffected,

$$\frac{d}{d\lambda}(z^\mu + c^\mu_T) = \frac{dz^\mu}{d\lambda} = v^\mu \tag{4.1}$$

C^0_T relates to a temporal shift of origin, while C^m_T represents the arbitrariness of spatial origin in the body system $\{\xi^\beta\}$ arising from translational freedom. The indefinite placement of origin may be removed (indeed, must be removed to obtain mathematical completeness and physical solution of any motion) in a natural way by letting the first moment of the energy density vanish in the body frame,

$$I^b(\lambda) = \int \epsilon\, a^b\, d^3a = 0, \quad \epsilon \equiv T^{00} \tag{4.2}$$

These three conditions specify the world line of the "center of energy" and the origin of the tetrad.

We also observe that the triad $\{h^\mu{}_a\}$ is defined modulo a rotation. That is, consider an associated triad $\{h'^\mu{}_a\}$ which is rotated around the original $\{h^\mu{}_a\}$,

$$h'^\mu{}_a(\lambda) = h^\mu{}_b(\lambda) R^b{}_a(\lambda), \quad R^b{}_a R^c{}_b = \delta^c_a \tag{4.3}$$

Then

$$h'^\mu{}_a h'^\nu{}_b \eta_{\mu\nu} = h^\mu{}_c R^c{}_a h^\nu{}_d R^d{}_b \eta_{\mu\nu} = \delta_{cd} R^c{}_a R^d{}_b = \delta_{ab},$$

$$h'^\mu{}_a v_\mu = h^\mu{}_b R^b{}_a v_\mu = 0 \tag{4.4}$$

The orientation of the triad $\{h^\mu{}_a\}$ with respect to the body can be set unambiguously by choosing the principal axes which diagonalize the second moments of the energy density,

$$I^{bc}(\lambda) = \int \epsilon\, a^b a^c\, d^3a = \begin{pmatrix} I^{11} & 0 & 0 \\ 0 & I^{22} & 0 \\ 0 & 0 & I^{33} \end{pmatrix} = I^{bc}\delta^{bc} \quad \text{(no sum)}$$

Since ϵ is positive definite, the three off-diagonal terms can always be made to vanish for a particular value of λ and the choice can be maintained throughout the motion by applying the appropriate rotation.

Thus a set of triads can always be found which satisfies the orthogonality conditions and points along the principal axes, regardless of the body motion.

1.5 Kinematics - Functional Representation of the Vierbein.

We obtain an explicit form for the $\{h^{\mu}{}_{\alpha}\}$, based on the work of Nodvick (N2), which will be required for calculations in general relativity.

I. The inertial space frame S.

Measured with respect to S, the coordinates of a point in the rigid body are

$$x^{\mu} = \{x^0, x^m\} = \{ct, z^m(t) + h^m{}_b(t)a^b\} \tag{5.1}$$

and the world line of the energy center in S is given by

$$z^{\mu}(t) = (ct, z^m(t)) \tag{5.2}$$

and

$$v^{\mu}(t) = \frac{dz^{\mu}}{d\lambda} = \left(\gamma, \gamma \frac{v^m(t)}{c}\right)$$

$$\gamma(t) \equiv \frac{dt}{d\lambda} = \left(1 - \frac{v_m v_m}{c^2}\right)^{-1/2} \tag{5.3}$$

are the components of a unit 4-vector tangent to the world line at time t. The 4-vectors $\hat{\lambda}^{(m)}$ are unit vectors along the space axes x^m, and $\hat{\lambda}^{(0)}$ is a unit 4-vector directed along the time axis $x^0 = ct$. The $\hat{\lambda}^{(\mu)}$ are orthonormal,

$$\hat{\lambda}^{(\mu)} \cdot \hat{\lambda}^{(\nu)} = \delta^{\mu\nu} \tag{5.4}$$

II. The rest system S'.

The energy system is at rest in S'; for arbitrary body motion this is a non-inertial system which consists of a sequence of inertial systems, the members of the sequence being in a one-to-one correspondence with the time parameter t. For a fixed value of t, S' coincides with the inertial frame which is related to the space frame S by a Lorentz transformation,

$$L^{\mu}{}_{\nu}(t) = \begin{pmatrix} \gamma & -\gamma \frac{v_m}{c} \\ \gamma \frac{v^m}{c} & \delta^m{}_n - \frac{v^m v_n(\gamma-1)}{v^2} \end{pmatrix} \tag{5.5}$$

The unit vectors in S', $\hat{s}'^{(\mu)}$, are related to $\hat{s}^{(\mu)}$ by

$$\hat{s}'^{(\mu)} = L^{\mu}{}_{\nu}\,\hat{s}^{(\nu)} \tag{5.6}$$

The $\hat{s}^{(\mu)}$ are constant, but the $\hat{s}'^{(\mu)}$ vary with t; by differentiating (5.6), we find

$$\frac{d}{dt}\hat{s}'^{(\mu)} = \Lambda^{\mu}{}_{\nu}\,\hat{s}'^{(\nu)} = L^{\mu}{}_{\rho}\dot{L}^{\rho}{}_{\nu}\hat{s}'^{(\nu)} = -\Lambda_{\nu}{}^{\mu}\hat{s}'^{(\nu)} \tag{5.7}$$

$$\Lambda^{m}{}_{n} = \frac{\gamma-1}{v^2}\left(v^m \dot{v}_n - v_n \dot{v}^m\right) \tag{5.8}$$

$$\Lambda^{m}{}_{o} = \frac{-\gamma}{c}\left[\dot{v}^m + \frac{\gamma-1}{v^2}\left(v^m v_s \dot{v}_s\right)\right] \tag{5.9}$$

In general the system S' has a "spatial component of rotation", $\Lambda^m{}_n$. This is the Thomas effect, resulting from the fact that two successive non-collinear Lorentz transformations are equivalent to a single Lorentz transformation plus a spatial rotation.

To see the effect of this transformation on the tetrad $\{h^{\mu}{}_{\alpha}\}$, return to the derivative of the orthogonality conditions, Eq. (3.8),

$$\dot{h}^{\mu}{}_{\alpha}h^{\alpha}{}_{\nu} + h^{\mu}{}_{\alpha}\dot{h}^{\alpha}{}_{\nu} = 0,$$

and multiply by $h^{\nu}{}_{\beta}$,

$$\dot{h}^{\mu}{}_{\beta} = -h^{\mu}{}_{\alpha}\dot{h}^{\alpha}{}_{\nu}h^{\nu}{}_{\beta} = -(h^{\mu}{}_{o}\dot{h}^{o}{}_{\nu} + h^{\mu}{}_{a}\dot{h}^{a}{}_{\nu})h^{\nu}{}_{\beta} \tag{5.10}$$

When $\beta = 0$,

$$\dot{h}^{\mu}{}_{o} = \dot{v}^{\mu} = -(v^{\mu}\dot{v}_{\nu} + h^{\mu}{}_{a}\dot{h}^{a}{}_{\nu})v^{\nu} \tag{5.11}$$

The v^{μ} have been normalized by choice of path parameter, so the first term is zero. Thus the acceleration

$$\dot{v}^\mu = a^\mu = -(\dot{h}^a{}_\nu v^\nu)\, h^\mu{}_a \tag{5.12}$$

is always in the space-like hypersurface spanned by the triad $h^\mu{}_a$.

However, when $\beta = b$,

$$\dot{h}^\mu{}_b = -(v^\mu \dot{v}_\nu + h^\mu{}_a \dot{h}^a{}_\nu)\, h^\nu{}_b$$

$$= -v^\mu(\dot{v}_\nu h^\nu{}_b) - h^\mu{}_a \Lambda^a{}_b \tag{5.13}$$

so the change in the triad $\{h^\mu{}_b\}$ contains "spatial components of rotation", (second term), as well as components in the direction of motion (v^μ).

We may now employ the fact that the triad is defined by the orthogonality conditions modulo a rotation (previously discussed in Chap 1.4) to remove the $\Lambda^m{}_n$.

III. The rest system T.

The energy center is also at rest in system T. This system, like S', is a non-inertial system made up of a sequence of inertial systems; the time axis of T is parallel to that of S' for all t, but the space axes rotate relative to the space axes of S'. The orientation of the unit vectors in T, $\hat{t}^{(\alpha)}$, relative to $\hat{s}'^{(\mu)}$, is specified by the Euler angles $(\theta_T, \phi_T, \psi_T)$ contained in the transformation

$$\hat{t}'^{(\alpha)} = T^\alpha{}_\mu\, \hat{s}'^{(\mu)} \tag{5.14}$$

$$T^\alpha{}_\mu = \begin{pmatrix} 1 & 0 \\ 0 & T^a{}_m \end{pmatrix} \tag{5.15}$$

$$T^a{}_m = \begin{pmatrix} \cos\phi_T\cos\psi_T - \cos\theta_T\sin\phi_T\sin\psi_T & \sin\phi_T\cos\psi_T + \cos\theta_T\cos\phi_T\sin\psi_T & \sin\theta_T\sin\psi_T \\ -\cos\phi_T\sin\psi_T - \cos\theta_T\sin\phi_T\cos\psi_T & -\sin\phi_T\sin\psi_T + \cos\theta_T\cos\phi_T\cos\psi_T & \sin\theta_T\cos\psi_T \\ \sin\theta_T\sin\phi_T & -\sin\theta_T\cos\phi_T & \cos\theta_T \end{pmatrix}$$

$$\dot{T}^\alpha{}_\mu = T^\alpha{}_\rho\,\Omega_T{}^\rho{}_\mu, \quad \Omega_T{}^\rho{}_\mu = T^\rho_\alpha\,\dot{T}^\alpha{}_\mu = -\Omega_T{}_\mu{}^\rho \tag{5.16}$$

$$\Omega_T{}^\rho{}_o = \Omega_T{}^o{}_\mu = 0 \tag{5.17}$$

These angles will be required to change in the course of the motion in such a way that they compensate for the $\Lambda^m{}_n$ of system S'; with this requirement the system T will have no components proportional to $\{h^\mu{}_a\}$, but then the space axes of T will precess relative to S (Thomas precession).

According to the preceding,

$$\hat{\lambda}^{(\mu)} = L^\mu{}_\nu T^\nu{}_\alpha \hat{t}^{(\alpha)} = h^\mu{}_\alpha \hat{t}^{(\alpha)} \tag{5.18}$$

$$h^\nu{}_a = L^\mu{}_\nu T^\nu{}_a, \qquad T^\nu{}_a = (T_a{}^\nu)^{-1} = (T_a{}^\nu)^T$$
$$\dot{h}^\nu{}_a = \dot{L}^\mu{}_\nu T^\nu{}_a + L^\mu{}_\nu \dot{T}^\nu{}_a$$
$$(h^\mu{}_a)^{-1} = (L^\mu{}_\nu T^\nu{}_a)^{-1} = T_a{}^\nu L_\nu{}^\mu \tag{5.19}$$

From Eq. (5.13) the spatial component of rotation in $\dot{h}^\mu{}_b$ is

$$\dot{h}^a{}_\nu h^\nu{}_b = (\dot{T}^a{}_\rho L^\rho{}_\nu + T^a{}_\rho \dot{L}^\rho{}_\nu)(L^\nu{}_\sigma T^\sigma{}_b)$$
$$= \dot{T}^a{}_\rho T^\rho{}_b + T^a{}_\rho \Lambda^\rho{}_\sigma T^\sigma{}_b$$
$$= \dot{T}^a{}_r T^r{}_b + T^a{}_r \Lambda^r{}_s T^s{}_b \tag{5.20}$$

We choose $T^{\alpha}{}_{\nu}$ so that $\dot{h}^{a}{}_{\nu} h^{\nu}{}_{b} = 0$ and multiply by $T^{b}{}_{m} T^{n}{}_{a}$:

$$0 = \dot{T}^{a}{}_{m} T^{n}{}_{a} + \Lambda^{n}{}_{m} \Rightarrow \Omega_{T}{}^{n}{}_{m} = -\Lambda^{n}{}_{m} \tag{5.21}$$

When this constraint is satisfied,

$$\dot{v}^{\mu} = -h^{\mu}{}_{a} (\dot{h}^{a}{}_{\nu} v^{\nu})$$

$$\dot{h}^{\mu}{}_{b} = -v^{\mu} (\dot{v}_{\nu} h^{\nu}{}_{b}) \tag{5.22}$$

These equations do not depend on the particular space frame S from which we started and have the same form for all inertial systems (relativistically covariant). All inertial observers will agree that $\dot{h}^{\mu}{}_{b}$ in system T has no component normal to v^{μ} if the constraint, Eq. (5.21), is satisfied.

IV. The body system B

The space axes of B are fixed in the body and the energy center is at rest. The orientation of the $\hat{b}^{(\gamma)}$ relative to the $\hat{t}^{(\alpha)}$ is given by the Euler angles (θ, ϕ, ψ) contained in the rotation $R^{\gamma}{}_{\alpha}$,

$$\hat{b}^{(\gamma)} = R^{\gamma}{}_{\alpha} \hat{t}^{(\alpha)} \tag{5.23}$$

Since $R^{\gamma}{}_{\alpha}$ is a rotation similar in form to $T^{\alpha}{}_{\mu}$, it will obey the Eqs. (5.15) to (5.17) if T is replaced by R and the subscript $_T$ on the Euler angles is dropped.

Changes in these angles are determined by the dynamical equations of motion. If R^{μ} is the displacement from the energy center to a fixed element in the rigid body, then R^{0} is zero and R^{m} remain constant throughout the motion (cf. Sect. 3, Eqs. 9 to 11).

V. Summary.

The passage from the space frame S to the body frame B is accomplished through the vierbein representation

$$h^{\mu}{}_{\beta} = L^{\mu}{}_{\nu} T^{\nu}{}_{\alpha} R^{\alpha}{}_{\beta} \tag{5.24}$$

$$\dot{h}^{\mu}{}_{\beta} = \dot{L}^{\mu}{}_{\nu} T^{\nu}{}_{\alpha} R^{\alpha}{}_{\beta} + L^{\mu}{}_{\nu} \dot{T}^{\nu}{}_{\alpha} R^{\alpha}{}_{\beta} + L^{\mu}{}_{\nu} T^{\nu}{}_{\alpha} \dot{R}^{\alpha}{}_{\beta} \tag{5.25}$$

$$\dot{h}^{\mu}{}_{0} = \dot{v}^{\mu} = \dot{L}^{\mu}{}_{0}$$

$$\dot{h}^{\mu}{}_{b} = L^{\mu}{}_{m} T^{m}{}_{a} \dot{R}^{a}{}_{b} + L^{\mu}{}_{0} L^{0}{}_{k} \dot{L}^{k}{}_{m} T^{m}{}_{a} R^{a}{}_{b} \tag{5.26}$$

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma \frac{v_m}{c} \\ \gamma \frac{v^m}{c} & \delta^m{}_n - \frac{v^m v_n}{v^2}(\gamma - 1) \end{pmatrix} \tag{5.27}$$

$$T^{\nu}{}_{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & T^{m}{}_{a} \end{pmatrix} \qquad T_{a}{}^{m} = \left(T_a{}^m\right)^{-1} = \left(T_a{}^m\right)^{T} \tag{5.28}$$

$$\Omega_T{}^{m}{}_{r} = T^{m}{}_{a} \dot{T}^{a}{}_{r} = -\Lambda^{m}{}_{r} = \frac{\gamma - 1}{v^2}\left(\dot{v}^m v_r - \dot{v}_r v^m\right) \tag{5.29}$$

$$\Omega_T{}^{m}{}_{r} = \left(\begin{array}{c|c|c} 0 & \dot{\psi}_T \cos\theta_T + \dot{\phi}_T & -\dot{\theta}_T \sin\phi_T + \dot{\psi}_T \sin\theta_T \cos\phi_T \\ -\dot{\psi}_T \cos\theta_T - \dot{\phi}_T & 0 & \dot{\theta}_T \cos\phi_T + \dot{\psi}_T \sin\theta_T \sin\phi_T \\ \dot{\theta}_T \sin\phi_T - \dot{\psi}_T \sin\theta_T \cos\phi_T & -\dot{\theta}_T \cos\phi_T - \dot{\psi}_T \sin\theta_T \sin\phi_T & 0 \end{array}\right) \tag{5.30}$$