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Technical Report 32-1527

Mathematical Formulation of the Double-Precision Orbit Determination Program (DPODP)

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Preface

The work described in this report was performed by the Mission Analysis Division of the Jet Propulsion Laboratory.

Abstract

This report documents the complete mathematical model for the Double-Precision Orbit Determination Program (DPODP), a third-generation program which has recently been completed at the Jet Propulsion Laboratory. The DPODP processes earth-based doppler, range, and angular observables of the spacecraft to determine values of the parameters that specify the spacecraft trajectory for lunar and planetary missions. The program was developed from 1964 to 1968; it was first used operationally for the *Mariner* VI and VII spacecraft which encountered Mars in August of 1969.

The DPODP has more accurate mathematical models, a significant increase in numerical precision, and more flexibility than the second-generation Single-Precision Orbit Determination Program (SPODP). Doppler and range observables are computed to accuracies of 10^{-5} m/s and 0.1 m, respectively, exclusive of errors in the tropospheric, ionospheric, and space plasma corrections.

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Mathematical Formulation of the Double-Precision Orbit Determination Program (DPODP)

I. Introduction

This technical report documents the mathematical model for the Double-Precision Orbit Determination Program (DPODP), a third-generation program that has recently been completed at the Jet Propulsion Laboratory (JPL). The DPODP will be used to determine values of the parameters that specify the spacecraft trajectory for lunar and planetary missions; it will be used for both real-time and post-flight reduction of tracking data. The DPODP differentially corrects *a priori* estimates of injection parameters, physical constants, maneuver parameters, and station locations to minimize the sum of weighted squares of residual errors between observed and computed quantities.

The analysis began in 1964, and coding for the IBM 7094 computer was initiated the next year. The program was completed and fully checked out at the end of 1968; it was first used operationally for the *Mariner* VI and VII spacecraft, which encountered Mars in August, 1969. Conversion of the program to the Univac 1108 computer was completed early in 1970.

The DPODP has more accurate mathematical models, significantly more numerical precision, and more flexibility than the second-generation Single-Precision Orbit Determination Program (SPODP). The basic limitations on the accuracy of computed observables are the inaccuracies in the troposphere and ionosphere corrections. Before these corrections are added, the computed values of the doppler and range observables have accuracies of 10^{-5} m/s and 0.1 m, respectively. The parameters whose values may be estimated by the DPODP are:

(1) Injection parameters. Rectangular components of the spacecraft position and velocity vectors at the injection epoch.

(2) Reference parameters. Parameters that affect the relative position and velocity of the sun, planets, and the moon:

A_E = the number of kilometers per astronomical unit (AU). This parameter converts the precomputed heliocentric ephemerides of eight planets and the earth–moon barycenter from astronomical units to kilometers.

R_E = scaling factor for lunar ephemeris, km/fictitious earth radius. This factor converts the precomputed geocentric lunar ephemeris from fictitious earth radii to kilometers.

E = osculating orbital elements for the precomputed ephemeris of a planet, earth–moon barycenter, or the moon. The estimated correction ΔE is used to differentially correct position and velocity obtained from the precomputed ephemeris.

μ_E, μ_M = gravitational constants for the earth and moon, km^3/s^2. These parameters affect the location of the earth–moon barycenter.

(3) Gravitational constants. The constant μ_i is the gravitational constant for body i, such as the sun, a planet, or the moon. (Note that μ_E and μ_M are also listed under reference parameters.)

(4) Harmonic coefficients. The harmonic coefficients J_n, C_{nm}, S_{nm}, along with the gravitational constant μ, describe the gravitational field of a planet or the moon.

(5) Parameters affecting the acceleration of the spacecraft due to solar radiation pressure.

(6) Coefficients of quadratic for small acceleration acting along each spacecraft axis. These quadratics are used to represent gas leaks and small forces arising from operation of the attitude control system.

(7) Parameters affecting spacecraft motor burns.

(8) Parameters affecting the transformation from universal time to ephemeris time.

(9) Coefficients of quadratics which represent the departure of atomic time at each tracking station from broadcast UTC time.

(10) Station parameters. (1) Radius, (2) latitude, and (3) longitude or (1) distance from spin axis, (2) height above equator, and (3) longitude for each tracking station and a landed spacecraft on a planet or the moon. For a tracking ship: (1) spherical coordinates at an epoch, (2) velocity, and (3) azimuth.

(11) Speed of light. An adopted constant which defines the light-second as the basic length unit; it is not normally included in the solution vector.

(12) Constant bias for range observables.

(13) Spacecraft transmitter frequency for one-way doppler.

(14) Biases affecting observed angles.

(15) Relativity parameter γ. This parameter will be added to the program. It is equal to $(1 + \omega)/(2 + \omega)$ where ω is the coupling constant of the scalar field, a free parameter of the Brans-Dicke theory of gravitation.

Given the *a priori* estimate of the parameter vector q, the program integrates the spacecraft acceleration using the second-sum numerical integration method to give position and velocity at any desired time. Using the spacecraft ephemeris along with the precomputed ephemerides for the other bodies within the solar system, and the parameter vector q, the program computes values for each observed quantity (normally doppler, range, or angles) and forms the *observed* minus *computed* (O − C) residuals.

In addition to integrating the acceleration of the spacecraft to obtain the spacecraft ephemeris, the program integrates the partial derivative of the spacecraft acceleration with respect to (wrt) the parameter vector q using the second-sum numerical integration procedure to give the partial derivative of the spacecraft state vector X (position and velocity components) wrt the parameter vector q, $\partial X/\partial q$. Using $\partial X/\partial q$, the program computes the partial derivative of each computed observable quantity z wrt q, $\partial z/\partial q$. Given the O − C residuals, $\partial z/\partial q$, and the weights applied to each residual along with the *a priori* parameter vector and its covariance matrix, the program computes the differential correction Δq to the parameter vector. Starting with $q + \Delta q$, the program computes a new spacecraft ephemeris, residuals, and partial derivatives and obtains a second differential correction Δq. This process is repeated until convergence is obtained and the sum of weighted squares of residual errors between observed and computed quantities is minimized.

The DPODP formulation was heavily influenced by the general theory of relativity. Section II gives the equations from general relativity, which are the basis of the DPODP formulation, and also the principal relativistic equations contained in the formulation. The derivations of three of these equations are given in Appendixes A, B, and C.

The time transformations used throughout the program and the formulation for computing the relative position, velocity, acceleration, and jerk of any two celestial bodies (sun, moon, or planets) are described in Sections III and IV, respectively. The equations for the acceleration of the spacecraft relative to the center of integration (any planet, the sun, or the moon) are given in Section V.

The first step in the computation of all observable quantities is the light time solution, which is described in Section VI. The formulation for computing the geocentric inertial position and velocity of a tracking station is presented in Section VII. The computation of doppler, range, and angular observables is described in Sections VIII–X.

A forthcoming change to the formulation will be to compute doppler observables from differenced range observables divided by the count time, with partial derivatives of the doppler observables with respect to estimated parameters obtained from differenced range partial derivatives. The formulation necessary to implement this change is given in Section XI.

Corrections to the observables due to antenna motion, the troposphere, and the ionosphere are described in Section XII. The variational equations for the spacecraft trajectory and the partial derivatives of the observables with respect to the estimated parameters are described in Sections XIII and XIV.

In the original version of the DPODP, the parameter estimate was obtained from the normal equations, which are documented in Section XV. In the latest version of the program, this formulation has been replaced by the square root form of the normal equations, which is described in Section XVI. The square root formulation is theoretically equivalent to the normal equations but is numerically superior. The superior numerical techniques of the square root formulation were first applied to the linear least squares problem by R. J. Hanson and C. L. Lawson[1] (Ref. 1).

II. Relativistic Terms of DPODP Formulation

The general theory of relativity is basically a geometrical theory. The geometry is embodied in the components of the symmetrical metric tensor g_{pq}:

$$g_{pq} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix} \tag{1}$$

The subscripts 1, 2, 3, and 4 correspond to the space–time coordinates x^1, x^2, x^3, and x^4, which are associated with a particular space–time frame of reference. Usually the frame of reference is nonrotating and centered at the barycenter of the system of masses considered. Then x^1, x^2, and x^3 are position coordinates and $x^4 = ct$, where c is the speed of light and t is coordinate time, a uniform system of time which exists throughout the frame of reference; it is synonymous with ephemeris time. The components of the metric tensor g_{pq} are obtained from a solution of Einstein's field equations. The solution depends upon the distribution of matter and the system of coordinates selected.

The invariant interval ds between two points with differences in their space and time coordinates of dx^1, dx^2, dx^3, and dx^4 is given by

$$ds^2 = g_{pq} dx^p dx^q \tag{2}$$

where, using the Einstein convention, the repeated indices p and q are summed over the integers 1 through 4.

In an infinitesimally small region surrounding an observer, the components of the metric tensor are constant and the expression for the interval ds can be transformed to the special relativity form

$$ds^2 = c^2 d\tau^2 - dX^2 - dY^2 - dZ^2 \tag{3}$$

where τ is proper time recorded on the observer's atomic clock and X, Y, and Z are components of observed position referred to the observer's local frame of reference. Since the atomic clock is fixed relative to the observer, the interval ds corresponding to an observed interval of proper time $d\tau$ is

$$ds = c\,d\tau \tag{4}$$

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[1] Jet Propulsion Laboratory, Computation and Analysis Section.

or

$$d\tau = \frac{ds}{c} \tag{5}$$

Hence Eq. (2) relates an observed interval of proper time $d\tau$ to the changes in the space and time coordinates of the clock.

The space–time coordinates are used to represent the motion of particles, bodies, and light. The coordinates have no physical significance and are not observable. Furthermore, the choice of coordinates is completely arbitrary. The solution of the field equations for g_{pq} varies with the coordinates selected in such a manner that the value of an observed interval of proper time computed from Eqs. (2) and (5) is independent of the coordinates selected to represent the motion of the atomic clock.

The field equations have been solved exactly for the case of a massless particle moving under the influence of a single spherically symmetric massive body located at the origin of a nonrotating system of coordinates. The solution of this 1-body problem was first obtained by Schwarzschild and is given in Ref. 2, p. 85, Eq. (38.8). A simple transformation in the radial coordinate gives the "1-body" solution in isotropic spherical coordinates (Ref. 2, p. 93, Eq. 43.2):

$$ds^2 = \frac{\left(1 - \dfrac{\mu}{2c^2r}\right)^2}{\left(1 + \dfrac{\mu}{2c^2r}\right)^2} c^2dt^2 - \left(1 + \frac{\mu}{2c^2r}\right)^4 (dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2) \tag{6}$$

where

μ = gravitational constant of nonrotating spherically symmetric massive body located at origin of nonrotating frame of reference, km^3/s^2. The constant μ is equal to the product of the universal gravitational constant G and the rest mass m of the body.

c = speed of light

r, ϕ, θ = spherical coordinates. The spherical and rectangular coordinates of a particle P are shown in Fig. 1.

t = coordinate time

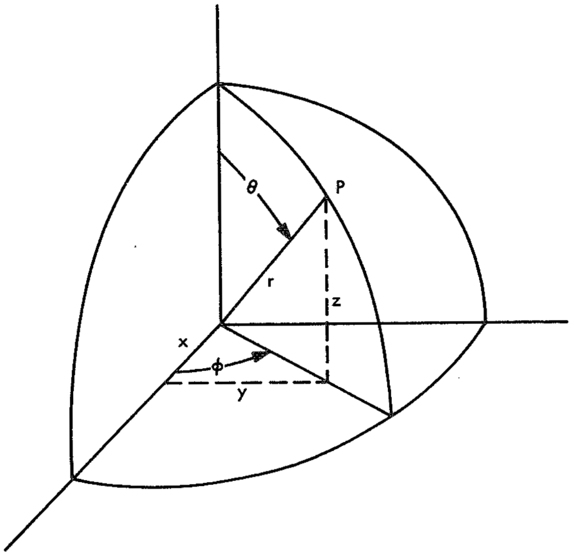

Fig. 1. Spherical and rectangular coordinates

Expanding and retaining all terms of order $1/c^2$ gives

$$ds^2 = \left(1 - \frac{2\mu}{c^2r} + \frac{2\mu^2}{c^4r^2}\right) c^2dt^2 - \left(1 + \frac{2\mu}{c^2r}\right)(dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2) \tag{7}$$

In isotropic rectangular coordinates,

$$ds^2 = \left(1 - \frac{2\mu}{c^2r} + \frac{2\mu^2}{c^4r^2}\right) c^2dt^2 - \left(1 + \frac{2\mu}{c^2r}\right)(dx^2 + dy^2 + dz^2) \tag{8}$$

where

$$r = [x^2 + y^2 + z^2]^{1/2} \tag{9}$$

Fock (Ref. 3) and Yilmaz (Ref. 4) differ from Einstein and obtain metrics that differ from Eq. (6). However, when expanded, their metrics are identical with Eq. (8) to order $1/c^2$. The small departures of the components of the metric tensor in Eq. (8) from the unity values of special relativity in Eq. (3) represent the "curvature" of space–time due to the mass of the central body.

The trajectory of a massless particle in the gravitational field of a massive body is a geodesic curve which extremizes the integral of ds between two points:

$$\delta \int ds = 0 \tag{10}$$

In order to obtain the equations of motion with coordinate time t as independent variable, Eq. (10) is written as

$$\delta \int L \, dt = 0 \tag{11}$$

where the Lagrangian L is given by

$$L = \frac{ds}{dt} \tag{12}$$

and L^2 is obtained from the expression for ds^2 by replacing differentials of the space coordinates by derivatives of the coordinates with respect to t multiplied by dt. The Lagrangian L may be obtained by expanding the square root of L^2 in powers of $1/c^2$. Given L, the equations of motion that extremize the integral (11) are the Euler–Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad x \to y, z \tag{13}$$

where

$$\dot{x} = \frac{dx}{dt} \qquad x \to y, z \tag{14}$$

A simpler procedure for obtaining the equations of motion directly from derivatives of L^2 is developed as follows. The Euler–Lagrange equations (Eq. 13) are unchanged by multiplying both terms by L:

$$L\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - L\frac{\partial L}{\partial x} = 0 \qquad x \to y, z \tag{15}$$

Differentiating $L\,(\partial L/\partial \dot{x})$ with respect to t gives

$$\frac{d}{dt}\left(L\frac{\partial L}{\partial \dot{x}}\right) = \left(\frac{\dot{L}}{L}\right)\left(L\frac{\partial L}{\partial \dot{x}}\right) + L\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \qquad x \to y, z \tag{16}$$

where

$$\dot{L} = \frac{dL}{dt} \tag{17}$$

The equations of motion are obtained by substituting the last term of Eq. (16) into Eq. (15):

$$\frac{d}{dt}\left(L\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\dot{L}}{L}\right)\left(L\frac{\partial L}{\partial \dot{x}}\right) - \left(L\frac{\partial L}{\partial x}\right) = 0 \qquad x \to y, z \tag{18}$$

The derivatives $L\,(\partial L/\partial x)$ and $L\,(\partial L/\partial \dot{x})$ are obtained by direct differentiation of L^2. For the usual situation where only the $1/c^2$ terms of the relativistic perturbative acceleration are required,

$$\frac{\dot{L}}{L} = \frac{L\dot{L}}{L^2} \approx \frac{L\dot{L}}{c^2} \tag{19}$$

where L^2 has been replaced by its leading term c^2 and $L\dot{L}$ is obtained by differentiating a simplified expression for L^2 containing terms to $1/c^0$ only. Computation of the equations of motion from Eqs. (18) and (19) is simpler than taking the square root of L^2 and using Eq. (13).

From Eqs. (8), (12), (18), and (19), the relativistic perturbative acceleration of a massless particle moving in the gravitational field of one body is given by

$$\ddot{\mathbf{r}} = \frac{\mu}{c^2 r^3}\left[\left(4\frac{\mu}{r} - \dot{s}^2\right)\mathbf{r} + 4\,(\mathbf{r}\cdot\dot{\mathbf{r}})\,\dot{\mathbf{r}}\right] \tag{20}$$

where the dots indicate differentiation with respect to coordinate time t. The position, velocity, and acceleration vectors are given by

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \dot{\mathbf{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}; \quad \ddot{\mathbf{r}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \tag{21}$$

and

$$\dot{s} = \text{magnitude of } \dot{\mathbf{r}}$$

An approximate solution to the field equations for the case of a massless particle moving in the gravitational field of n massive bodies was first obtained by J. Droste in 1916 (Ref. 5). In that same year, W. deSitter extended the work of Droste to account for the mass of the body whose motion is desired (Ref. 6). However, he made a theoretical error in the calculation of one of his terms, which was corrected by Eddington and Clark in 1938 (Ref. 7). The components of the Droste/deSitter/Eddington and Clark metric are given by (Ref. 7, Eqs. 3.1, 3.2, and 3.6)

$$g_{11} = g_{22} = g_{33} = -\left(1 + \frac{2}{c^2}\sum_{j \neq i}\frac{\mu_j}{r_{ij}}\right) \tag{22}$$

$$g_{pq} = 0\,(p, q = 1, 2, 3; p \neq q) \tag{23}$$

$$g_{14} = g_{41} = \frac{4}{c^3} \sum_{j \neq i} \frac{\mu_j \dot{x}_j}{r_{ij}} \tag{24}$$

$$g_{24} = g_{42} = \frac{4}{c^3} \sum_{j \neq i} \frac{\mu_j \dot{y}_j}{r_{ij}} \tag{25}$$

$$g_{34} = g_{43} = \frac{4}{c^3} \sum_{j \neq i} \frac{\mu_j \dot{z}_j}{r_{ij}} \tag{26}$$

$$\begin{aligned} g_{44} = 1 &- \frac{2}{c^2} \sum_{j \neq i} \frac{\mu_j}{r_{ij}} \\ &+ \frac{2}{c^4} \left[\sum_{j \neq i} \frac{\mu_j}{r_{ij}} \right]^2 - \frac{3}{c^4} \sum_{j \neq i} \frac{\mu_j \dot{s}_j^2}{r_{ij}} \\ &+ \frac{2}{c^4} \sum_{j \neq i} \frac{\mu_j}{r_{ij}} \sum_{k \neq j} \frac{\mu_k}{r_{jk}} \\ &- \frac{1}{c^4} \sum_{j \neq i} \mu_j \frac{\partial^2 r_{ij}}{\partial t^2} \end{aligned} \tag{27}$$

where the indices j and k refer to the n bodies and k includes body i whose motion is desired. Also,

μ_j = gravitational constant for body j

= Gm_j, where G is the universal gravitational constant and m_j is the rest mass of body j.

x, y, z;
$\dot{x}, \dot{y}, \dot{z}$;
$\ddot{x}, \ddot{y}, \ddot{z}$ = rectangular components of position, velocity, and acceleration ($\dot{x} = dx/dt$, etc.) relative to a nonrotating frame of reference centered at the barycenter of the system of n bodies. The position, velocity, and acceleration vectors are given by Eq. (21); they and their components are identified by the subscript i, j, or k.

r_{ij} = coordinate distance between bodies i and j

$= [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2}$

$\dot{s}^2$ = square of velocity $= \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

The second partial derivative of r_{ij} with respect to t in Eq. (27) is obtained holding $\mathbf{r}_i$ fixed:

$$r_{ij}^2 = (\mathbf{r}_j - \mathbf{r}_i) \cdot (\mathbf{r}_j - \mathbf{r}_i) \tag{28}$$

$$\frac{\partial r_{ij}}{\partial t} = \frac{(\mathbf{r}_j - \mathbf{r}_i) \cdot \dot{\mathbf{r}}_j}{r_{ij}} \tag{29}$$

$$\frac{\partial^2 r_{ij}}{\partial t^2} = \frac{(\mathbf{r}_j - \mathbf{r}_i) \cdot \ddot{\mathbf{r}}_j}{r_{ij}} + \frac{\dot{s}_j^2}{r_{ij}} - \frac{[(\mathbf{r}_j - \mathbf{r}_i) \cdot \dot{\mathbf{r}}_j]^2}{r_{ij}^3} \tag{30}$$

Since terms of order greater than $1/c^2$ will not be retained in the expression for the acceleration of body i, the acceleration $\ddot{\mathbf{r}}_j$ in Eq. (30) may be evaluated from Newtonian theory:

$$\ddot{\mathbf{r}}_j = \sum_{k \neq j} \frac{\mu_k (\mathbf{r}_k - \mathbf{r}_j)}{r_{jk}^3} \tag{31}$$

The summation over $k \neq j$ includes body i. The four space–time coordinates associated with the n-body metric are

$$\left. \begin{aligned} x^1 &= x_i \\ x^2 &= y_i \\ x^3 &= z_i \\ x^4 &= ct \end{aligned} \right\} \tag{32}$$

Hence, from Eq. (2) and Eqs. (22–27), the expression for ds^2 is

$$\begin{aligned} ds^2 = c^2 g_{44}\, dt^2 &+ g_{11} (dx_i^2 + dy_i^2 + dz_i^2) \\ &+ 2cg_{14}\, dx_i\, dt + 2cg_{24}\, dy_i\, dt + 2cg_{34}\, dz_i\, dt \end{aligned} \tag{33}$$

Dividing by dt^2 gives the expression for L^2:

$$\begin{aligned} L^2 = c^2 g_{44} &+ g_{11} (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) \\ &+ 2cg_{14}\dot{x}_i + 2cg_{24}\dot{y}_i + 2cg_{34}\dot{z}_i \end{aligned} \tag{34}$$

The equations of motion for body i are obtained from Eqs. (18) and (19) with x and $\dot{x}$ replaced by x_i and $\dot{x}_i$. However, in carrying out the required differentiations of Eq. (34), the contribution to the field from the mass of body i must be held fixed.

Specifically, the Newtonian potential at each perturbing body j in the fifth term of g_{44} (Eq. 27) must be considered to be a function of time only. The potential at body j due to body i, μ_k/r_{jk} with k set equal to i, must not be differentiated with respect to x_i, y_i, and z_i.

The last term of g_{44} is evaluated with Eq. (30), which contains the acceleration of body j given by Eq. (31). The acceleration of body j, $\ddot{\mathbf{r}}_j$, must also be considered to be a function of coordinate time t only. The contribution from the mass of body i must not be differentiated with respect to x_i, y_i, and z_i. (I am indebted to two relativists at JPL, Dr. Frank B. Estabrook and Dr. Hugo Wahlquist, for pointing out these special conditions.)

The details of the derivation of the expression for the acceleration of body i are given in Appendix A. The final expression for the acceleration of body i relative to the barycenter of the system of n bodies with rectangular components referred to a nonrotating coordinate system is given by

$$\begin{aligned}\ddot{\mathbf{r}}_i = \sum_{j \neq i} \frac{\mu_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \Bigg\{ & 1 - \frac{4}{c^2} \sum_{l \neq i} \frac{\mu_l}{r_{il}} - \frac{1}{c^2} \sum_{k \neq j} \frac{\mu_k}{r_{jk}} + \left(\frac{\dot{s}_i}{c}\right)^2 + 2\left(\frac{\dot{s}_j}{c}\right)^2 - \frac{4}{c^2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j - \frac{3}{2c^2} \left[\frac{(\mathbf{r}_i - \mathbf{r}_j) \cdot \dot{\mathbf{r}}_j}{r_{ij}}\right]^2 \\ & + \frac{1}{2c^2} (\mathbf{r}_j - \mathbf{r}_i) \cdot \ddot{\mathbf{r}}_j \Bigg\} \\ & + \frac{1}{c^2} \sum_{j \neq i} \frac{\mu_j}{r_{ij}^3} [(\mathbf{r}_i - \mathbf{r}_j) \cdot (4\dot{\mathbf{r}}_i - 3\dot{\mathbf{r}}_j)] (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) + \frac{7}{2c^2} \sum_{j \neq i} \frac{\mu_j \ddot{\mathbf{r}}_j}{r_{ij}} \end{aligned} \tag{35}$$

where $\ddot{\mathbf{r}}_j$ is computed from Eq. (31) and the summation over $k \neq j$ in Eqs. (31) and (35) includes body i. Note that the first term of Eq. (35) is the Newtonian acceleration of body i. The effect of the mass of body i on its own acceleration is contained in its contribution to the Newtonian potential at each perturbing body j (term 3) and in its contribution to the Newtonian acceleration of each body j (terms 8 and 10).

A method for obtaining the motion of a system of n heavy bodies directly from the field equations, without recourse to additional equations such as those of a geodesic, was obtained for the first time by Einstein, Infeld, and Hoffman in 1938 (Ref. 8). The method, referred to as the EIH approximation method, was subsequently perfected from the mathematical viewpoint in Refs. 9 and 10. The EIH method is illustrated in Ref. 8 by obtaining the equations of motion for two bodies. The equations for the motion of a system of n bodies were obtained from a later (1960) work of Infeld and Plebański (Ref. 11). According to Bažański (Ref. 12), the EIH approximation method is, in principle, the only tool in the problem of the motion of heavy bodies in the general theory of relativity.

After deriving the n-body relativistic equations of motion, Infeld and Plebański noticed that these equations could be put into the form of a Lagrangian L with the equations of motion following from the Euler–Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad x \rightarrow y, z \tag{36}$$

where i refers to the body whose motion is desired. The Infeld Lagrangian is given in Ref. 11, p. 112, Eq. (3.3.37) or p. 128, Eq. (4.2.25). The same Lagrangian may be found on p. 149 of Ref. 13.

In the notation used for the de Sitter n-body metric (except that the index i, as well as j and k, now refers to the n bodies), the Lagrangian is given by

$$\begin{aligned} L = {} & \frac{1}{2} \sum_i \mu_i \dot{s}_i^2 + \frac{1}{8c^2} \sum_i \mu_i (\dot{s}_i^2)^2 + \frac{3}{4c^2} \sum_i \sum_{j \neq i} \frac{\mu_i \mu_j}{r_{ij}} (\dot{s}_i^2 + \dot{s}_j^2) - \frac{2}{c^2} \sum_i \sum_{j \neq i} \frac{\mu_i \mu_j}{r_{ij}} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j \\ & - \frac{1}{4c^2} \sum_i \sum_{j \neq i} \mu_i \mu_j \frac{\partial^2 r_{ij}}{\partial \xi_i^c \partial \xi_j^b} \dot{\xi}_j^b \dot{\xi}_i^c + \frac{1}{2} \sum_i \sum_{j \neq i} \frac{\mu_i \mu_j}{r_{ij}} - \frac{1}{4c^2} \sum_i \sum_{j \neq i} \frac{\mu_i \mu_j (\mu_i + \mu_j)}{r_{ij}^2} \\ & - \frac{1}{6c^2} \sum_i \sum_{j \neq i} \sum_{k \neq j, i} \mu_i \mu_j \mu_k \left(\frac{1}{r_{ij} r_{jk}} + \frac{1}{r_{jk} r_{ki}} + \frac{1}{r_{ki} r_{ij}} \right) \end{aligned} \tag{37}$$

where

ξ = position vector with components $\xi^1 = x$, $\xi^2 = y$, and $\xi^3 = z$. A repeated superscript implies a summation over the values 1, 2, and 3.

Carrying out the partial derivatives in term five gives two terms, one of which combines with term four. Also, the last term contains three identical subterms; two of them may be deleted and the coefficient of the remaining term multiplied by three. With these changes, the expression for L becomes

$$
\begin{aligned}
L = {} & \frac{1}{2}\sum_{i} \mu_i \dot{s}_i^2 + \frac{1}{8c^2}\sum_{i} \mu_i \left(\dot{s}_i^2\right)^2 + \frac{3}{4c^2}\sum_{i}\sum_{j\neq i} \frac{\mu_i\mu_j}{r_{ij}}\left(\dot{s}_i^2 + \dot{s}_j^2\right) - \frac{7}{4c^2}\sum_{i}\sum_{j\neq i}\frac{\mu_i\mu_j}{r_{ij}}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j \\
& - \frac{1}{4c^2}\sum_{i}\sum_{j\neq i}\frac{\mu_i\mu_j}{r_{ij}^3}\left[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j\right]\left[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i\right] + \frac{1}{2}\sum_{i}\sum_{j\neq i}\frac{\mu_i\mu_j}{r_{ij}} - \frac{1}{4c^2}\sum_{i}\sum_{j\neq i}\frac{\mu_i\mu_j\left(\mu_i + \mu_j\right)}{r_{ij}^2} \\
& - \frac{1}{2c^2}\sum_{i}\sum_{j\neq i}\sum_{k\neq j,i}\frac{\mu_i\mu_j\mu_k}{r_{ij}r_{jk}}
\end{aligned}
\tag{38}
$$

This equation, expressed in a slightly different form, may be found in Ref. 14, p. 372.

The equations of motion (Eq. 36) involve the partial derivatives of the Lagrangian L with respect to the position and velocity coordinates of the particular body whose motion is desired. Hence, Eq. (38) will be rewritten with the index i referring to the particular body (body i) whose motion is desired and the indices j and k referring to the n other bodies (perturbing bodies). For the single-summation terms of Eq. (38), the transformation consists simply of removal of the i summation. Since all double-summation terms are unchanged by interchanging the indices i and j, they are transformed by removing the i summation and multiplying by two. Terms of the triple summation with the index i or k referring to the specific body i are transformed to the original triple-summation term multiplied by two with the i summation removed. Terms with the index j referring to the specific body i are transformed to

$$
-\frac{1}{2c^2}\sum_{j\neq i}\sum_{k\neq j,i}\frac{\mu_i\mu_j\mu_k}{r_{ij}r_{ik}}
$$

After transformation, the gravitational constant μ_i may be deleted from each term. Thus, with i now referring to the specific body i whose motion is desired, and j and k referring to the other bodies, the Lagrangian L is given by

$$
\begin{aligned}
L = {} & \frac{1}{2}\dot{s}_i^2 + \frac{1}{8c^2}\left(\dot{s}_i^2\right)^2 + \frac{3}{2c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\left(\dot{s}_i^2 + \dot{s}_j^2\right) - \frac{7}{2c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j \\
& - \frac{1}{2c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\left[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j\right]\left[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i\right] + \sum_{j\neq i}\frac{\mu_j}{r_{ij}} - \frac{1}{2c^2}\sum_{j\neq i}\frac{\mu_j\left(\mu_i + \mu_j\right)}{r_{ij}^2} \\
& - \frac{1}{c^2}\sum_{j\neq i}\sum_{k\neq j,i}\frac{\mu_j\mu_k}{r_{ij}r_{jk}} - \frac{1}{2c^2}\sum_{j\neq i}\sum_{k\neq j,i}\frac{\mu_j\mu_k}{r_{ij}r_{ik}}
\end{aligned}
\tag{39}
$$

The expression for the acceleration of body i is obtained by applying the Euler–Lagrange equation, Eq. (36), to Eq. (39) for L. The details are given in Appendix A. The resulting n-body equations of motion, derived from the Infeld Lagrangian, are identical to Eqs. (35) derived from the Droste/de Sitter/Eddington and Clark metric.

The equations of motion for a massless particle moving in the field of one massive body may be obtained by simplifying the n-body equations of motion (Eq. 35). With one perturbing body, its position, velocity, and acceleration are zero. Also, with the mass of body i, whose motion is desired, set equal to zero, the Newtonian potential at the perturbing body j is zero. With these simplifications, the n-body acceleration (Eq. 35) reduces exactly to the acceleration (Eq. 20) obtained from the 1-body isotropic metric (Eq. 8). Of course, the components of the n-body metric tensor (Eqs. 22–27) reduce to those of the 1-body isotropic metric (Eq. 8). Some of the relativity terms of the DPODP formulation are derived from the 1-body metric, whereas others are obtained from the n-body metric. The 1-body isotropic metric was selected since it is a special case of the n-body de Sitter metric, or equivalently the n-body Infeld Lagrangian. The choice of coordinates in general relativity is arbitrary, but the same coordinates must be used in all computations.

The general theory of relativity has been generalized by C. Brans and R. H. Dicke (Ref. 15). Supposedly, their theory is more in accord with Mach's principle than the general theory of relativity. According to Mach's principle, the inertial forces experienced in an accelerated laboratory are gravitational, having their origin in the distant matter of the universe accelerated relative to the laboratory. Brans and Dicke (Ref. 15) state that "locally observed inertial reactions depend upon the mass distribution of the universe about the point of observation and consequently the quantitative aspects of locally observed physical laws (as expressed in the physical "constants") are position dependent."

The Eötvös experiment was recently repeated at Princeton University by Dicke et al. and showed that all bodies fall with the same acceleration to an accuracy of 1 part in 10^{11}. Brans and Dicke concluded from this result that the only physical "constant" of their theory (Ref. 15) whose value needs to vary with position in the universe is the universal constant of gravitation G (see Ref. 16, p. 7–8). In order to obtain this variation, they added a scalar gravitational field to the tensor field of general relativity. The gravitational constant G varies with the strength of the scalar field. However, it can be considered constant in the small region of the universe known as the solar system.

In the Brans–Dicke scalar–tensor theory of gravity, the attraction between two particles of matter is due partly to the tensor field and partly to the scalar field. The fraction of the gravitational attraction due to the scalar field is given by

$$\frac{1}{4 + 2\omega}$$

where ω is the coupling constant of the scalar field, a free parameter of their theory. It is shown below that $\omega \gtrsim 6$. For $\omega = 6$, 1/16 of the force of gravity is derived from the scalar field and 15/16 is due to the tensor field.

Because of the expansion of the universe, the strength of the scalar field (if it exists) is changing, and G should decrease by roughly 1–3 parts in 10^{11} per year (Ref. 16, p. 107). The variation in G is inconsistent with the strong principle of equivalence, which is one of the postulates of the general theory of relativity. According to this principle, in a freely falling, nonrotating laboratory, the form of the locally determined laws of physics and the values of the dimensionless physical constants appearing therein do not vary with the position of the laboratory in space and time.

Nutku (Ref. 17) has obtained the post-Newtonian equations of hydrodynamics for a nonviscous fluid in the scalar–tensor theory of Brans and Dicke. From these equations, Estabrook (Ref. 18) has obtained the n-body metric tensor, the n-body Lagrangian, and the resulting n-body equations of motion. These equations contain exactly the same terms as the corresponding equations of general relativity; however, the coefficients of these terms, which were constant in general relativity, are functions of the free parameter ω, the coupling constant of the scalar field. The value of ω must be positive, and, as the value of ω approaches infinity, the equations of the Brans–Dicke theory revert to the corresponding equations of general relativity.

From Ref. 15, the relativistic perihelion rotation rate $\dot{\theta}$ of a planetary orbit is

$$\dot{\theta} = \left[\frac{4 + 3\omega}{6 + 3\omega}\right] \times \text{[value from general relativity]} \tag{40}$$

For Mercury, the predicted value from general relativity is 208 μrad (43 arc-seconds)/century, which agrees with the observed value. However, the solar oblateness recently observed by Dicke (Ref. 19) would produce an advance of Mercury's perihelion of 16 μrad (3.4 arc seconds)/century, leaving only 192 μrad (39.6 arc seconds)/century to be attributed to relativity. The Brans–Dicke theory will produce this perihelion rotation rate for a value of ω approximately equal to 6. Since the true solar oblateness lies somewhere between zero and approximately the value observed by Dicke, $\omega \geqq 6$ (approximately).

The basic equations of the Brans–Dicke theory are given below with coefficients expressed as functions of the parameter γ, where

$$\gamma = \frac{1 + \omega}{2 + \omega} \tag{41}$$

As ω increases from zero to infinity, γ increases from 1/2 to unity (its general relativity value).

The DPODP will be modified so that the value of the parameter γ may be estimated. The constant coefficients of all existing DPODP relativity terms, derived from the general theory of relativity, will be changed to the functions of γ specified in this report. Also, the partial derivatives of the observables with respect to γ specified in Section XIV will be added to the program. This will enable the value of γ to be obtained by fitting the theory to observation. Given γ, the corresponding value of ω is given by (see Eq. 41)

$$\omega = \frac{2\gamma - 1}{1 - \gamma} \tag{42}$$

It will be seen that the relativity terms of the DPODP formulation which are functions of γ vary linearly with γ. Also, it will be seen that the only components of the 1-body isotropic metric tensor that are functions of ω are $g_{11} = g_{22} = g_{33}$. The departure of this coefficient from unity is proportional to the function $(1 + \omega)/(2 + \omega)$. This is the source of the change of variable to γ (Eq. 41). The parameter γ was first used at JPL by Anderson (Ref. 20).

From Estabrook (Ref. 18), the components of the n-body metric tensor (written here as functions of γ) are

$$g_{11} = g_{22} = g_{33} = -\left(1 + \frac{2\gamma}{c^2} \sum_{j \neq i} \frac{\mu_j}{r_{ij}}\right) \tag{43}$$

$$g_{pq} = 0 \, (p, q = 1, 2, 3; p \neq q) \tag{44}$$

$$g_{14} = g_{41} = \frac{2 + 2\gamma}{c^3} \sum_{j \neq i} \frac{\mu_j \dot{x}_j}{r_{ij}} \tag{45}$$

$$g_{24} = g_{42} = \frac{2 + 2\gamma}{c^3} \sum_{j \neq i} \frac{\mu_j \dot{y}_j}{r_{ij}} \tag{46}$$

$$g_{34} = g_{43} = \frac{2 + 2\gamma}{c^3} \sum_{j \neq i} \frac{\mu_j \dot{z}_j}{r_{ij}} \tag{47}$$

$$\begin{aligned} g_{44} = 1 &- \frac{2}{c^2} \sum_{j \neq i} \frac{\mu_j}{r_{ij}} + \frac{2}{c^4} \left[\sum_{j \neq i} \frac{\mu_j}{r_{ij}}\right]^2 \\ &- \frac{1 + 2\gamma}{c^4} \sum_{j \neq i} \frac{\mu_j \dot{s}_j^2}{r_{ij}} \\ &+ \frac{2}{c^4} \sum_{j \neq i} \frac{\mu_j}{r_{ij}} \sum_{k \neq j} \frac{\mu_k}{r_{jk}} \\ &- \frac{1}{c^4} \sum_{j \neq i} \mu_j \frac{\partial^2 r_{ij}}{\partial t^2} \end{aligned} \tag{48}$$

where $\partial^2 r_{ij}/\partial t^2$ is given by Eqs. (30) and (31). The coefficients 2γ, $2 + 2\gamma$, and $1 + 2\gamma$ appearing in Eqs. (43–48) above appear as $(2 + 2\omega)/(2 + \omega)$, $(6 + 4\omega)/(2 + \omega)$, and $(4 + 3\omega)/(2 + \omega)$, respectively, in Ref. 18. With γ equal to unity (its general relativity value), the equations above are identical to Eqs. (22–27), derived from general relativity.

If the mass of body i is reduced to zero and the number of perturbing bodies is reduced to one, the n-body metric tensor reduces to the following diagonal 1-body metric:

$$g_{11} = g_{22} = g_{33} = -\left(1 + \frac{2\gamma\mu}{c^2 r}\right) \tag{49}$$

$$g_{44} = 1 - \frac{2\mu}{c^2 r} + \frac{2\mu^2}{c^4 r^2} \tag{50}$$

using the notation listed after Eq. (6). In spherical coordinates, the expression for the interval is

$$\begin{aligned} ds^2 = &\left(1 - \frac{2\mu}{c^2 r} + \frac{2\mu^2}{c^4 r^2}\right) c^2 \, dt^2 \\ &- \left(1 + \frac{2\gamma\mu}{c^2 r}\right)(dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2) \end{aligned} \tag{51}$$

Setting γ equal to unity gives the general relativity expression (Eq. 7).

Estabrook (Ref. 18) also gives an expression for the *n*-body Lagrangian *L* in the Brans–Dicke theory. Changing the coefficients to functions of γ (using Eq. 42) and also changing the form of his equation slightly gives

$$\begin{aligned} L = {} & \frac{1}{2}\sum_{i} \mu_i \dot{s}_i^2 + \frac{1}{8c^2}\sum_{i} \mu_i (\dot{s}_i^2)^2 + \frac{1+2\gamma}{4c^2}\sum_{i}\sum_{j\neq i}\frac{\mu_i\mu_j}{r_{ij}}(\dot{s}_i^2 + \dot{s}_j^2) - \frac{3+4\gamma}{4c^2}\sum_{i}\sum_{j\neq i}\frac{\mu_i\mu_j}{r_{ij}}\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j \\ & - \frac{1}{4c^2}\sum_{i}\sum_{j\neq i}\frac{\mu_i\mu_j}{r_{ij}^3}[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j][(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i] + \frac{1}{2}\sum_{i}\sum_{j\neq i}\frac{\mu_i\mu_j}{r_{ij}} - \frac{1}{4c^2}\sum_{i}\sum_{j\neq i}\frac{\mu_i\mu_j(\mu_i+\mu_j)}{r_{ij}^2} \\ & - \frac{1}{2c^2}\sum_{i}\sum_{j\neq i}\sum_{k\neq j,i}\frac{\mu_i\mu_j\mu_k}{r_{ij}r_{jk}} \end{aligned} \tag{52}$$

The corresponding equation from the general theory of relativity is Eq. (38); for $\gamma = 1$, the two expressions are identical.

Transforming Eq. (52) so that the index *i* refers to the particular body *i* whose motion is desired and the indices *j* and *k* refer to the *n* other bodies gives

$$\begin{aligned} L = {} & \frac{1}{2}\dot{s}_i^2 + \frac{1}{8c^2}(\dot{s}_i^2)^2 + \frac{1+2\gamma}{2c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}(\dot{s}_i^2 + \dot{s}_j^2) - \frac{3+4\gamma}{2c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j \\ & - \frac{1}{2c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}[(\mathbf{r}_j - \dot{\mathbf{r}}_i)\cdot\mathbf{r}_j][(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i] + \sum_{j\neq i}\frac{\mu_j}{r_{ij}} - \frac{1}{2c^2}\sum_{j\neq i}\frac{\mu_j(\mu_i+\mu_j)}{r_{ij}^2} \\ & - \frac{1}{c^2}\sum_{j\neq i}\sum_{k\neq j,i}\frac{\mu_j\mu_k}{r_{ij}r_{jk}} - \frac{1}{2c^2}\sum_{j\neq i}\sum_{k\neq j,i}\frac{\mu_j\mu_k}{r_{ij}r_{ik}} \end{aligned} \tag{53}$$

The corresponding equation from general relativity is Eq. (39).

In Appendix A, the *n*-body equations of motion are derived from the *n*-body metric tensor (Eqs. 43–48) and from the *n*-body Lagrangian (Eq. 53). The result (also given in Ref. 18) is

$$\begin{aligned} \ddot{\mathbf{r}}_i = {} & \sum_{j\neq i}\frac{\mu_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}\Bigg\{1 - \frac{2(1+\gamma)}{c^2}\sum_{l\neq i}\frac{\mu_l}{r_{il}} - \frac{1}{c^2}\sum_{k\neq j}\frac{\mu_k}{r_{jk}} + \gamma\left(\frac{\dot{s}_i}{c}\right)^2 + (1+\gamma)\left(\frac{\dot{s}_j}{c}\right)^2 - \frac{2(1+\gamma)}{c^2}\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j \\ & - \frac{3}{2c^2}\left[\frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot\dot{\mathbf{r}}_j}{r_{ij}}\right]^2 + \frac{1}{2c^2}(\mathbf{r}_j - \mathbf{r}_i)\cdot\ddot{\mathbf{r}}_j\Bigg\} \\ & + \frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\{[\mathbf{r}_i - \mathbf{r}_j]\cdot[(2+2\gamma)\dot{\mathbf{r}}_i - (1+2\gamma)\dot{\mathbf{r}}_j]\}(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) + \frac{3+4\gamma}{2c^2}\sum_{j\neq i}\frac{\mu_j\ddot{\mathbf{r}}_j}{r_{ij}} \end{aligned} \tag{54}$$

where $\ddot{\mathbf{r}}_j$ is given by Eq. (31) and the summation over $k \neq j$ in Eqs. (31) and (54) includes body *i*. With $\gamma = 1$, Eq. (54) is identical to Eq. (35), derived from general relativity. Simplifying Eq. (54) to the case of a massless particle moving in the field of one massive body gives the following relativistic perturbative acceleration:

$$\ddot{\mathbf{r}} = \frac{\mu}{c^2r^3}\left\{\left[2(1+\gamma)\frac{\mu}{r} - \gamma\dot{s}^2\right]\mathbf{r} + 2(1+\gamma)(\mathbf{r}\cdot\dot{\mathbf{r}})\dot{\mathbf{r}}\right\} \tag{55}$$

For $\gamma = 1$, this equation is identical to Eq. (20), derived from general relativity.

The ephemerides of the moon, sun, and planets could be obtained by a simultaneous numerical integration using Eq. (54). Using these precomputed n-body ephemerides, the DPODP could generate the spacecraft ephemeris using Eq. (54) to calculate the point-mass gravitational accelerations of the spacecraft and the body which is the center of integration.

However, a number of the relativistic perturbative acceleration terms (the $1/c^2$ terms) would be insignificant. For instance, for the heliocentric ephemeris of a planet other than the earth, only the perturbative acceleration of the planet due to the mass of the sun, computed from Eq. (55), need be considered. Equation (54) is required only when a planet or moon is nearby; that is, when one is computing the acceleration of the earth, the moon, or the spacecraft when it is near the earth and moon or a planet.

The relativistic perturbative acceleration terms required are specified in Sections IV and V, which describe the precomputed n-body ephemerides and the spacecraft ephemeris. A more detailed discussion of the required terms and their effect on the various ephemerides may be found in Refs. 21 and 22.

A brief summary of the effect of general relativity on the various ephemerides is as follows. For the orbit of a planet, the mean distance a is about 1.5 km less than the Newtonian value. Periodic variations in position are proportional to the eccentricity and range from about 0.2 km for Venus and Neptune to about 6 km for Mercury and Pluto. Periodic variations in velocity are proportional to the product of the mean motion and the eccentricity. The largest variation is 4 mm/s for Mercury; the variations for the remaining planets are less than 0.25 mm/s, which is the value for Mars.

The primary terms of the periodic variations in position and velocity have periods equal to the orbital period and one-third the orbital period. The only significant secular variation in the orbital elements is the advance of perihelion, which amounts to the well known value of 208 μrad (43 arc seconds)/century for Mercury.

For the orbit of the moon relative to the earth, the mean distance is about 8 m less than the Newtonian value (using the same values for the gravitational constants of the earth and moon). Maximum values of the periodic variations in position and velocity are less than 10 m and 10^{-5} m/s. The differential solar relativistic acceleration produces a secular variation in the moon's perigee of 10 μrad (2 arc seconds)/century.

For the *Pioneer* VI, *Mariner* IV, and *Mariner* V spacecraft, the periodic variations in position and velocity are in the ranges of 3 to 5 km and 0.7 to 1.1 mm/s. The major terms of these variations have periods equal to the orbital period and one-third of the orbital period. For an earth orbiter with a perigee of 7000 km and an eccentricity of 0.2, the advance of perigee is 39 μrad (8 arc seconds)/year.

The ephemerides for the planets, the earth–moon barycenter, the moon, and the spacecraft give the position coordinates (and their derivatives with respect to coordinate time) as a function of coordinate time t. For a given proper time τ at some point on earth, the time transformation $t - \tau$ is thus required to interpolate the ephemerides.

The time transformation may be derived from the expression for the interval which relates an observed interval of proper time τ to the changes in the space and time coordinates of the atomic clock. Substituting the components of the n-body metric tensor (Eqs. 43 to 48) into Eq. (33) for the interval and retaining terms to order $(1/c)^0$ gives

$$ds^2 = \left(1 - \frac{2\phi}{c^2}\right) c^2\,dt^2 - (dx^2 + dy^2 + dz^2) \tag{56}$$

where x, y, and z may be interpreted as heliocentric coordinates of the atomic clock, although strictly speaking they are referred to the barycenter, and ϕ is the Newtonian potential at the clock given by

$$\phi = \sum_j \frac{\mu_j}{r_j} \tag{57}$$

where r_j is the coordinate distance from the clock to body j. Expressing the second term of Eq. (56) as the square of the heliocentric velocity of the clock $\dot{s}$ multiplied by dt^2 and using Eq. (5) gives

$$\frac{d\tau}{dt} = \left[1 - \frac{2\phi}{c^2} - \left(\frac{\dot{s}}{c}\right)^2\right]^{1/2} \tag{58}$$

Since $1/c^4$ terms are ignored,

$$\frac{d\tau}{dt} \approx 1 - \frac{\phi}{c^2} - \frac{1}{2}\left(\frac{\dot{s}}{c}\right)^2 \tag{59}$$

Equation (59) relates an interval of proper time τ (obtained from the observer's atomic clock) to the corresponding interval of coordinate time t, the Newtonian potential at the clock, and the heliocentric velocity of the clock.

Coordinate time t may be considered to be a uniform system of time that pervades the nonrotating heliocentric frame of reference. For a fixed atomic clock at infinite distance from the sun, $\phi = \dot{s} = 0$ and $d\tau = dt$. That is, the atomic clock runs at the rate of a coordinate clock (a clock yielding coordinate time t). This condition and the length of the coordinate time second fixes the conversion factor (n cycles/second) used to convert cycles or ticks from the observer's atomic clock to seconds of proper time τ. From Eq. (59), the rate of an atomic clock decreases as the Newtonian potential at the clock and the heliocentric velocity of the clock increase.

For a fixed atomic clock on earth, $d\tau < dt$, and proper time τ falls behind coordinate time t. However, by the simple expedient of choosing a different number of cycles from the observer's atomic clock per second of proper time, the latter can be made to agree on the average with coordinate time t. Equation (59) may be written as

$$\frac{d\tau}{dt} = 1 - \frac{\bar{\phi}}{c^2} - \frac{1}{2}\frac{\overline{\dot{s}^2}}{c^2} - \frac{\phi - \bar{\phi}}{c^2} - \frac{1}{2}\frac{\dot{s}^2 - \overline{\dot{s}^2}}{c^2} \tag{60}$$

where

$\bar{\phi}$ = time average of ϕ

$\overline{\dot{s}^2}$ = time average of $\dot{s}^2$

Ignoring $1/c^4$ terms, this may be written as

$$\frac{d\tau}{dt\left(1 - \frac{\bar{\phi}}{c^2} - \frac{1}{2}\frac{\overline{\dot{s}^2}}{c^2}\right)} = 1 - \frac{\phi - \bar{\phi}}{c^2} - \frac{1}{2}\frac{\dot{s}^2 - \overline{\dot{s}^2}}{c^2} \tag{61}$$

Note that $d\tau$ is obtained as dN cycles from the observer's atomic clock divided by the conversion factor n cycles/s. If the conversion factor is changed to n^*, where

$$n^* = n\left(1 - \frac{\bar{\phi}}{c^2} - \frac{1}{2}\frac{\overline{\dot{s}^2}}{c^2}\right) \tag{62}$$

and proper time is obtained as dN/n^* and denoted by $d\tau^*$, then Eq. (61) may be written as

$$\frac{d\tau^*}{dt} = 1 - \frac{\phi - \bar{\phi}}{c^2} - \frac{1}{2}\frac{\dot{s}^2 - \overline{\dot{s}^2}}{c^2} \tag{63}$$

Thus proper time τ^* obtained from the observer's atomic clock using the conversion factor n^* cycles/s will, on the average, agree with coordinate time t. Periodic variations in τ^* from t are due to variations in ϕ and $\dot{s}^2$ from their average values.

Coordinate time t is the independent variable for the equations of motion and is commonly referred to as ephemeris time ET. The A1 atomic time scale on earth is based upon oscillations of a cesium atomic clock. The adopted length of the A1 second is $f_{\text{cesium}} = 9{,}192{,}631{,}770$ cycles of cesium, which is the current experimentally determined average length of the ET second.[2] In the DPODP, the true average length of the ET second is represented by $f_{\text{cesium}} + \Delta f_{\text{cesium}}$ cycles of cesium. The quantity Δf_{cesium} is a solve-for parameter; its value is probably no more than two or three cycles. The quantity $f_{\text{cesium}} + \Delta f_{\text{cesium}}$ is the length of the τ^* second and hence

$$\frac{d\text{A1}}{d\tau^*} = \frac{f_{\text{cesium}} + \Delta f_{\text{cesium}}}{f_{\text{cesium}}} = 1 + \frac{\Delta f_{\text{cesium}}}{f_{\text{cesium}}}$$

The quantity $d\text{A1}/d\text{ET}$ is the product of this equation and Eq. (63), which is given to sufficient accuracy by

$$\frac{d\text{A1}}{d\text{ET}} = 1 - \frac{\phi - \bar{\phi}}{c^2} - \frac{1}{2}\frac{\dot{s}^2 - \overline{\dot{s}^2}}{c^2} + \frac{\Delta f_{\text{cesium}}}{f_{\text{cesium}}} \tag{64}$$

where ϕ is the Newtonian potential at a particular A1 atomic clock and $\dot{s}$ is the heliocentric velocity of the atomic clock.

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[2] Interpolation of the lunar ephemeris with an observed longitude of the moon gives the value of the independent variable, ET. The value of Δf_{cesium} given above was determined by counting cycles of a cesium atomic clock between two observations of the moon separated by 10 years and dividing the observed number of cycles by the "observed" ET interval.

In Appendix B, equations are generated for the departure of ϕ and $\dot{s}^2$ from their average values, and Eq. (64) is integrated to give an expression for ET − A1. The initial conditions were evaluated by considering the method by which the A1 atomic time scale was set up. The master A1 clock was set equal to UT2[3] on January 1, 1958, 0^h UT2. The A1 clocks at other locations are synchronized with the master clock by means of radio signals, accounting for the propagation delay, or by means of a traveling clock, or by other methods. Hence, the average offset between A1 time and ET is the same for all A1 clocks. The resulting expression for ET − A1 (in units of seconds) is

$$
\begin{aligned}
\mathrm{ET} - \mathrm{A1} = {} & \Delta T_{1958} \\
& - (t - 252{,}460{,}800)\,\frac{\Delta f_{\mathrm{cesium}}}{f_{\mathrm{cesium}}} \\
& + 1.658 \times 10^{-3} \sin E \\
& + 0.317679 \times 10^{-9}\, u \sin(\mathrm{UT} + \lambda) \\
& + 5.341 \times 10^{-12}\, u \sin(\mathrm{UT} + \lambda - M) \\
& + 1.01 \times 10^{-13}\, u \sin(\mathrm{UT} + \lambda - 2M) \\
& - 1.3640 \times 10^{-11}\, u \sin(\mathrm{UT} + \lambda + 2L) \\
& - 2.27 \times 10^{-13}\, u \sin(\mathrm{UT} + \lambda + 2L + M) \\
& + 1.672 \times 10^{-6} \sin D \\
& + 1.38 \times 10^{-13}\, u \sin(\mathrm{UT} + \lambda - D) \qquad (65)
\end{aligned}
$$

where

ΔT_{1958} = ET − UT2 on January 1, 1958, $0^h\,0^m\,0^s$ UT2 minus the periodic terms of Eq. (65) evaluated at this epoch using u and λ of the master A1 clock. The master A1 clock was set equal to UT2 on this date. The parameter ΔT_{1958} may be estimated by the DPODP

f_{cesium} = 9,192,631,770 cycles of cesium atomic clock per second of A1 time (definition). This adopted length of the A1 second is the current experimentally determined average length of the ET second

$f_{\mathrm{cesium}} + \Delta f_{\mathrm{cesium}}$ = cycles of cesium atomic clock per ephemeris second. The parameter $\Delta f_{\mathrm{cesium}}$ may be estimated by the DPODP; its current nominal value is zero

t = seconds past January 1, 1950, 0^h

252,460,800 = seconds from January 1, 1950, 0^h to January 1, 1958, 0^h

M = mean anomaly of heliocentric orbit of earth–moon barycenter

E = eccentric anomaly of heliocentric orbit of earth–moon barycenter

L = geometric mean longitude of the sun, referred to mean equinox and ecliptic of date

D = ☾ − L = mean elongation of the moon from the sun, where

☾ = mean longitude of the moon, measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit, and then along the orbit

u = distance of atomic clock from earth's spin axis, km

λ = east longitude of atomic clock

UT = universal time, hours past midnight, converted to radians. It is computed from

$$
\mathrm{UT} = 2\pi \left[\frac{\mathrm{UT1}}{86{,}400}\right]_{\text{decimal part}} \qquad (66)
$$

where UT1 = seconds of UT1[4] time past January 1, 1950, 0^hUT1. The angles M, L, and D in radians are given by

$$
M = 6.248291 + 1.99096871 \times 10^{-7}\, t \qquad (67)
$$

$$
L = 4.888339 + 1.99106383 \times 10^{-7}\, t \qquad (68)
$$

$$
D = 2.518410 + 2.462600818 \times 10^{-6}\, t \qquad (69)
$$

To a sufficient degree of accuracy, the eccentric anomaly E is given by

$$
E \approx M + e \sin M \qquad (70)
$$

where

e = eccentricity of heliocentric orbit of earth-moon barycenter = 0.01672

---

[3] The UT2 time scale is described in Section III.

[4] The UT1 time scale is described in Section III.

Term 4 of Eq. (65) is the sum of two terms with coefficients of 0.318549 and −0.000870. The larger term arises from the daily variation in the heliocentric velocity of the atomic clock, while the smaller term accounts for the diurnal variation in potential. The expression for ET − A1 used in the current version of the DPODP consists of the first three terms of Eq. (65) and the following term derived by J. D. Anderson (Ref. 20):

$$2.03 \times 10^{-6} \cos\phi \sin(\mathrm{UT} + \lambda)$$

where ϕ is the latitude of the atomic clock. Anderson's term is the fourth term of Eq. (65) with the coefficient of 0.318549 mentioned above and r_s set equal to 6372 km $\cos\phi$.

Changing Anderson's diurnal term to the fourth term of Eq. (65) and addition of the last six terms of Eq. (65) is required to implement the change to the current version of the program specified in Section XI, namely, the computation of doppler observables from differenced range observables divided by the count time. The contribution to "differenced-range" doppler from a term of ET − A1 is approximately equal to the second time derivative of the term multiplied by the spacecraft range. All terms affecting "differenced-range" doppler by more than 2×10^{-7} m/s per astronomical unit of distance from the tracking station to the spacecraft were retained in Eq. (65). Terms of ET − A1 which could be derived from the $1/c^4$ terms of $d\tau/dt$ would be at least eight orders of magnitude smaller than the terms of Eq. (65). Their contribution to differenced-range doppler would be several orders of magnitude less than the criterion above. Hence, there is no requirement for $1/c^4$ terms in the expression for ET − A1.

In order to compute doppler, range, and angular observables, the time for light to travel from the transmitting station on earth to the spacecraft, and from there to the receiving station on earth, must be computed. Thus, an equation is required which relates the position coordinates of two points to the coordinate time t for light to travel from one of the points to the other. This equation will be referred to as the light time equation. It will be derived from Eq. (51), the 1-body expression for the interval in the Brans–Dicke theory. Thus, the effects of the masses of the planets and the moon on the propagation time are neglected.

A massless particle moves on a geodesic curve in the four-dimensional geometry of space–time, which is determined by the distribution of matter and the system of coordinates selected. This is also true for light with the additional condition that $ds = 0$. Thus, light moves along a null geodesic.

The equations of a geodesic are the Euler–Lagrange equations which extremize the integral of ds between two points. When Eq. (10) is written as Eqs. (11) and (12), the Euler–Lagrange Eq. (13) or (18) gives the second-order differential equations for the three position coordinates with coordinate time t as independent variable. However, if proper time s is taken as the independent variable, equations are obtained for the three position coordinates and also for coordinate time t. The equation for the fourth coordinate is required in the derivation of the light time equation. Eq. (10) may be expressed as

$$\delta \int \mathcal{L}\, ds = 0 \tag{71}$$

where

$$\mathcal{L} = \frac{ds}{ds} = 1 \tag{72}$$

From Eq. (51),

$$\mathcal{L}^2 = \left(1 - \frac{2\mu}{c^2 r} + \frac{2\mu^2}{c^4 r^2}\right) c^2 \left(\frac{dt}{ds}\right)^2 - \left(1 + \frac{2\gamma\mu}{c^2 r}\right) \times \left[\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{ds}\right)^2\right] \tag{73}$$

The Euler–Lagrange equations for $q = r, \theta, \phi$, or t are

$$\frac{d}{ds}\left[\frac{\partial \mathcal{L}}{\partial\left(\dfrac{dq}{ds}\right)}\right] - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{74}$$

The equation for θ is

$$r\frac{d^2\theta}{ds^2} + 2\frac{dr}{ds}\frac{d\theta}{ds}\left(1 - \frac{\gamma\mu}{c^2 r}\right) - r\left(\frac{d\phi}{ds}\right)^2 \sin\theta\cos\theta = 0 \tag{75}$$

If coordinates are chosen so that a particle moves initially in the plane $\theta = \pi/2$, then $d\theta/ds = 0$ and Eq. (75) gives the result that $d^2\theta/ds^2 = 0$. Thus, in the 1-body problem, the motion of particles and of light is planar, and the equations may be simplified by setting

$$\theta = \pi/2 \tag{76}$$

Since $\mathcal{L}$ is explicitly independent of t and ϕ, first integrals of Eq. (74) for $q = t$ and ϕ are given by $\partial\mathcal{L}/\partial\,(dt/ds) = \text{constant}$, and $\partial\mathcal{L}/\partial\,(d\phi/ds) = \text{constant}$.

Differentiating Eq. (73) accordingly with $\theta = \pi/2$ and making use of the fact that $\mathcal{L} = 1$ gives

$$\frac{dt}{ds} = \frac{\text{constant}}{1 - \dfrac{2\mu}{c^2 r} + \dfrac{2\mu^2}{c^4 r^2}} \tag{77}$$

and

$$\frac{d\phi}{ds} = \frac{\text{constant}}{r^2\left(1 + \dfrac{2\gamma\mu}{c^2 r}\right)} \tag{78}$$

Dividing Eq. (77) by Eq. (78) and ignoring $1/c^4$ terms gives

$$\frac{dt}{d\phi} = r^2\left[1 + \frac{2(1+\gamma)\mu}{c^2 r}\right]\text{constant} \tag{79}$$

Setting $ds = 0$ and $\theta = \pi/2$ in Eq. (51) gives

$$\left(1 - \frac{2\mu}{c^2 r} + \frac{2\mu^2}{c^4 r^2}\right) c^2 dt^2 = \left(1 + \frac{2\gamma\mu}{c^2 r}\right)(dr^2 + r^2 d\phi^2) \tag{80}$$

Substituting dt from Eq. (79) into Eq. (80), setting $dr/d\phi = 0$ when $r = R$ (the minimum value of r on the light path), and ignoring $1/c^4$ terms gives

$$d\phi = \pm\frac{\left[R + \dfrac{(1+\gamma)\mu}{c^2}\right]dr}{r\left[r^2 + \dfrac{2(1+\gamma)\mu}{c^2}r - \left(R^2 + \dfrac{2(1+\gamma)\mu}{c^2}R\right)\right]^{1/2}} \tag{81}$$

Integrating between limits of (r, ϕ) and $(R, 0)$ and ignoring $1/c^4$ terms gives

$$\begin{aligned}\phi &= \pm\left\{\frac{\pi}{2} - \sin^{-1}\left[\frac{R + \dfrac{(1+\gamma)\mu}{c^2}}{r} - \frac{(1+\gamma)\mu}{c^2 R}\right]\right\} \\ &= \pm\cos^{-1}\left[\frac{R + \dfrac{(1+\gamma)\mu}{c^2}}{r} - \frac{(1+\gamma)\mu}{c^2 R}\right]\end{aligned} \tag{82}$$

where the plus sign applies for increasing r and the minus sign applies for decreasing r. When r approaches ∞ in Eq. (82), the angle ϕ will approach one of the two asymptotic values:

$$\phi = \pm\left[\frac{\pi}{2} + \frac{(1+\gamma)\mu}{c^2 R}\right] \tag{83}$$

The angle between the incoming and outgoing asymptotes is thus

$$\Delta\phi = \frac{2(1+\gamma)\mu}{c^2 R} \tag{84}$$

For general relativity, $\gamma = 1$ and $\Delta\phi = 4\mu/c^2 R$, which has a maximum value of 8.48 μrad (1.75 arc seconds) when R is set equal to the radius of the sun, 695,500 km. Figure 2 shows the curved path of a photon passing the sun S. Light is moving in the positive y direction and the point of closest approach occurs at $x = R$, $y = 0$. The polar coordinates (r, ϕ) and rectangular coordinates (x, y) of two points on the light path are shown along with the straight line path (of length r_{12}) joining these two points. The y intercept was obtained from Eq. (82) by setting $\cos\phi$ equal to zero; the x intercept of the asymptotes follows from the y intercept and the angle of the asymptote.

Given that light moves in a plane along the curved path (Eq. 82), the light time equation may be derived by two alternative methods. The first method consists of substituting $d\phi$ from Eq. (79) into Eq. (80), giving a relation between dr and dt. Integration gives the light time equation. The second method is a direct integration of the differential of coordinate distance divided by the coordinate speed of light v_c along the light path between two points. For planar motion, the space coordinates of a photon change by dr and $d\phi$ in coordinate time dt. Hence, an expression for the square of the coordinate velocity v_c is

$$v_c^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 \tag{85}$$

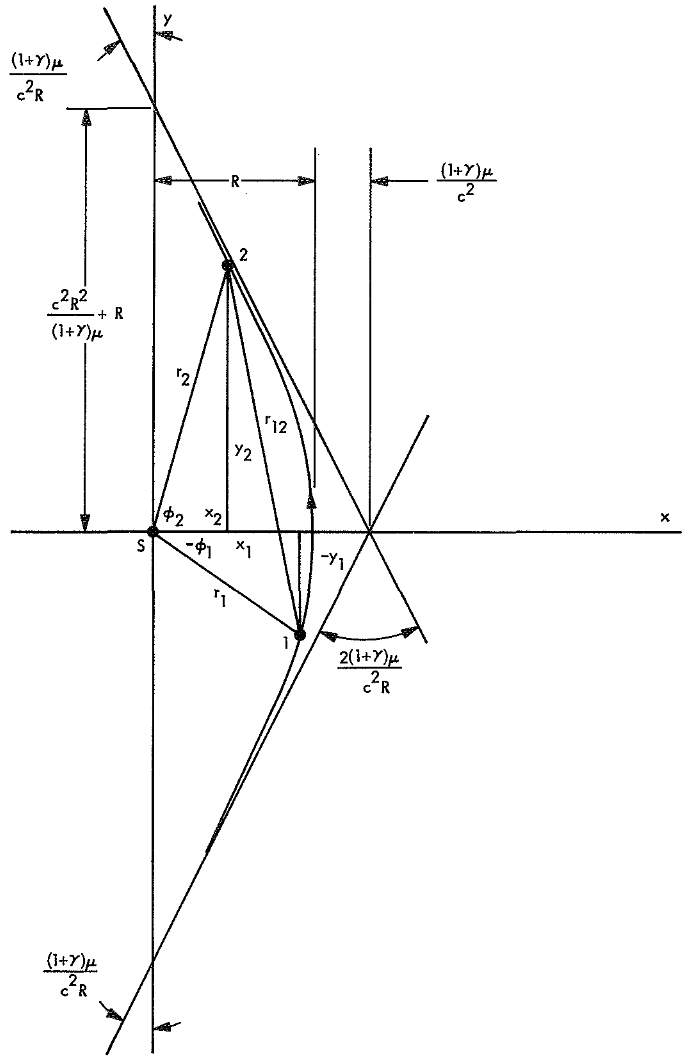

Fig. 2. Light path

Dividing Eq. (80) by dt^2, substituting Eq. (85), and ignoring $1/c^4$ terms gives

$$v_c \approx c\left[1 - \frac{(1+\gamma)\mu}{c^2r}\right] \tag{86}$$

The coordinate speed of light v_c decreases slightly as the photon approaches the sun. The Newtonian light time between two points is the straight-line coordinate distance between them, divided by the speed of light c. However, since $v_c < c$, the actual light time will be longer; the additional time is of order $1/c^3$.

The direct effect of the bending of light upon the light time is the increase in the path length divided by the nominal velocity c. The maximum angle between the straight line path between two points and the curved geodesic path is the bending, $2(1+\gamma)\mu/c^2R$. If the nominal length of the light path is l, the difference in length between the curved and straight line paths will satisfy the inequality

$$\Delta l < \frac{l}{\cos\dfrac{2(1+\gamma)\mu}{c^2R}} - l \approx \frac{l}{2}\left[\frac{2(1+\gamma)\mu}{c^2R}\right]^2 \tag{87}$$

which is of order $1/c^4$. Thus, the direct effect of the bending of light on the light time is an additional term of order $1/c^5$.

The indirect effect of the bending of light is to alter the value of r used in Eq. (86) by a term of order $1/c^2$. The coordinate velocity divided by c along the curved geodesic path will differ from the corresponding value along the straight line path by a term of order $1/c^4$. Thus, the indirect effect of the bending of light upon the light time is the same order as the direct effect, namely $1/c^5$.

Since all terms of order $1/c^5$ and greater are ignored in the light time equation, it is obtained by integrating the differential of coordinate distance divided by v_c along the straight line path joining two points.

Both of the above-mentioned derivations of the light time equation are given in Appendix C. In either case, the resulting light time equation is

$$t_j - t_i = \frac{r_{ij}}{c} + \frac{(1+\gamma)\mu}{c^3}\ln\left(\frac{r_i + r_j + r_{ij}}{r_i + r_j - r_{ij}}\right) \tag{88}$$

where light travels from point i at coordinate time (ephemeris time) t_i to point j at coordinate time t_j, and

$$r_{ij} = \|\mathbf{r}_j^S(t_j) - \mathbf{r}_i^S(t_i)\|$$
$$r_i = \|\mathbf{r}_i^S(t_i)\|$$
$$r_j = \|\mathbf{r}_j^S(t_j)\|$$

$\mathbf{r}_i^S(t_i), \mathbf{r}_j^S(t_j)$ = heliocentric position vectors of point i at transmission time t_i (ET) and point j at reception time t_j (ET), respectively, with rectangular components referred to a nonrotating frame of reference

μ = gravitational constant of sun, km^3/s^2

This form for the relativistic perturbation of the light time equation was derived independently and introduced to the Jet Propulsion Laboratory by Holdridge (Ref. 23).

However, it had been derived a year earlier by Tausner (Ref. 24, Eq. 6-105). For two alternative forms, see Appendix C.

As discussed in detail in Section IX, the relativistic correction to the light time becomes as large as 36 km/c when the spacecraft approaches superior conjunction and the minimum distance from the light path to the surface of the sun becomes very small. This effect is seen directly in range observables and is the only really large effect of general relativity on earth-based tracking data.

The most accurate observables computed by the DPODP and observed by the Deep Space Network are round-trip range and two-way doppler data. The remainder of this section will summarize briefly the procedure for computation of these observables from a relativistic point of view.

The observables are defined as follows. A signal is transmitted from the tracking station at coordinate time t_1 (proper time τ_1), received and retransmitted by the spacecraft at coordinate time t_2, and received by the tracking station at coordinate time t_3 (proper time τ_3). The range observable is the elapsed round-trip proper time $\tau_3 - \tau_1$. For purposes of this discussion, two-way doppler may be considered to be the ratio of the received frequency f_R to the transmitted frequency f_T. In actuality, it is the average value of $1 - (f_R/f_T)$ over a period of time called the count time.

As previously mentioned, the precomputed ephemerides for the planets, the earth–moon barycenter, and the moon are obtained, in principle, by a simultaneous numerical integration using Eq. (54). Given the estimated values of the spacecraft injection conditions and other parameters, the spacecraft ephemeris is integrated numerically using Eq. (54) to compute the point mass gravitational accelerations. These ephemerides give the position coordinates and their derivatives with respect to coordinate time as a function of coordinate time t. Given the ephemerides, the first step in the computation of each observable quantity is the solution of the light time problem. Equation (65) is used to convert the reception time τ_3 for each observable to coordinate time (ephemeris time) t_3, and the heliocentric position and velocity of the tracking station are computed at this epoch.

Solution of the light time equation (Eq. 88) for the down leg of the light path gives the spacecraft time t_2 and its heliocentric position and velocity at t_2. Similarly, solution of the light time equation for the up leg of the light path gives the transmission time t_1 and the heliocentric position and velocity of the tracking station at t_1.

For the range observable, Eq. (65) is used to convert the round-trip light time from an accurate value of the coordinate time interval $(t_3 - t_1)$ to the observed proper time interval $\tau_3 - \tau_1$. The doppler observable is

$$\frac{f_R}{f_T} = \frac{dn}{d\tau_3} \cdot \frac{d\tau_1}{dn} = \frac{d\tau_1}{d\tau_3} \tag{89}$$

where dn cycles are transmitted in the interval of proper time $d\tau_1$ and received in the interval $d\tau_3$. The ratio of received to transmitted frequency is computed from

$$\frac{f_R}{f_T} = \frac{\left(\dfrac{d\tau}{dt}\right)_1}{\left(\dfrac{d\tau}{dt}\right)_3} \frac{dt_1}{dt_2} \frac{dt_2}{dt_3} \tag{90}$$

The ratios dt_1/dt_2 and dt_2/dt_3 are obtained by differentiation of the light time equations for the up and down legs, respectively, of the light path. The $d\tau/dt$ ratio is evaluated at t_1 and t_3 from Eq. (59).

All observable quantities are functions of intervals of the observer's proper coordinates associated with his local space–time frame of reference. The range and two-way doppler observables are functions of intervals of proper time τ only, namely τ_3–τ_1 and $d\tau_1/d\tau_3$, respectively. Thus, the computation of observables will always involve a transformation from the space and time coordinates of the frame of reference in which the motion of bodies and of light is represented mathematically to the observer's proper coordinates.

Theoretically, the frame of reference and the coordinates selected are arbitrary. The relativistic terms in the equations of motion (Eq. 54), the light time equation (Eq. 88), and the transformation from coordinate time to proper time (Eq. 65) will vary with the frame of reference and system of coordinates selected. In general, the numerical values of the various constants, obtained by fitting the theory to observations, will also vary. However, the numerical values of the computed observables are independent of the frame of reference and system of coordinates selected.

III. Time Transformations

This section describes the systems of time used in the DPODP and gives the formulas for transforming between these time scales.

A. Systems of Time

The DPODP uses the five systems of time discussed below.

1. Ephemeris time (ET). This is a uniform measure of time which is synonymous with coordinate time t of the general theory of relativity. It is the independent variable for the motion of bodies and of light rays in the barycentric space–time frame of reference. The represented motion is strictly mathematical in the sense that the three position coordinates and their independent variable (coordinate time) are not observable. However, the values of observable quantities computed using these coordinates are invariant with the selection of coordinates. Thus, the selection is arbitrary. Ephemeris time differs from the other four time scales of the DPODP since it is an abstract, unobservable time scale.

2. Atomic time (A1). This is derived from oscillations of a cesium atomic clock. The value of A1 was set equal to UT2 on January 1, 1958, $0^h0^m0^s$ UT2. The adopted length of the A1 second is 9,192,631,770 cycles of cesium, which is the current experimentally determined average length of the ET second.

3. Universal time (UT) (specifically UT0, UT1, or UT2). This is the measure of time which is the basis for all civil time-keeping. Universal time is defined in Ref. 25, p. 73 (the differences between UT0, UT1, and UT2 will be described below) as 12 h plus the Greenwich hour angle of a point on the true equator whose right ascension measured from the mean equinox of date is:

$$R_U(\mathrm{UT}) = 18^h38^m45\overset{s}{.}836 + 8{,}640{,}184\overset{s}{.}542\,T_U + 0\overset{s}{.}0929\,T_U^2 \tag{91}$$

where

T_U = number of Julian centuries of 36,525 days of UT elapsed since January 0, 1900, 12^hUT

The Greenwich hour angle of this point is $\theta_M - R_U(\mathrm{UT})$, where

θ_M = Greenwich mean sidereal time, the Greenwich hour angle of the mean equinox of date

Hence, UT is a function only of θ_M:

$$\theta_M = \mathrm{UT} + R_U(\mathrm{UT}) - 12^h \qquad 0 \leqq \theta_M, \mathrm{UT} \leqq 24^h \tag{92}$$

(Note that any integer multiple of 24 h may be added to the right-hand side, and hence the -12^h term could also be written as $+12^h$.)

Universal time is obtained from meridian transits of stars, observed in practice with a photographic zenith tube (PZT). At the instant of meridian transit, the right ascension of the observing station is equal to that of the observed star, relative to the true equator and equinox of date. Subtracting the east longitude of the observing station gives the true Greenwich sidereal time θ at the instant of observation:

θ = true Greenwich sidereal time, the Greenwich hour angle of the true equinox of date

Subtracting the nutation in right ascension (Ref. 25, p. 43) gives Greenwich mean sidereal time θ_M. Solving Eq. (92) gives the value of UT at the instant of observation. Each observing station has a nominal value of longitude used for computing UT; if this nominal value is used, the resulting UT is labeled UT0. Because the pole wanders, the latitude and longitude of a fixed point on the earth are a function of time.[5] Using the true longitude of the observing station at the observation time, the resulting UT is labeled UT1. There are fairly predictable seasonal fluctuations in UT1; if the adopted seasonal correction is added to UT1, the resulting time is labeled UT2.

The DPODP uses only UT1. It takes the value of UT1 supplied by the U.S. Naval Observatory and computes θ_M from Eq. (92). Adding the nutation in right ascension gives θ, which is used to compute the position of a tracking station relative to the true equator and equinox of the date of observation.

4. Broadcast Universal time (UTC). This is Greenwich civil time, which is an approximation of UT2; UTC is derived from oscillations of a cesium atomic clock. It is broadcast from several stations of the National Bureau of Standards such as WWVL, WWV, and WWVH. The seconds pulses are the length of 9,192,631,770 $(1 - S)$ cycles of cesium.

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[5] See Subsection VII-B-1.

The value of the frequency offset S is adopted annually by international agreement. Since 1964, the value of S must be a positive or negative integral multiple of 50×10^{-10} (Ref. 26, p. 306). For the years 1960 to 1969, the annual values of S were -150, -150, -130, -130, -150, -150, -300, -300, -300, and -300×10^{-10}, respectively. At 0^h UTC on the first day of any month, UTC may be advanced or retarded by exactly 0.100 s (Ref. 26, p. 307). These step adjustments to broadcast UTC are announced in advance. The frequency offsets and step adjustments are selected so that broadcast UTC will deviate from UT2 by no more than a few tenths of a second.

*5. **Station time (ST).*** This is the operational time scale at each tracking station derived from oscillations of a rubidium atomic clock. The ST second is ideally equal to the UTC second. Also, the ST clocks are stepped along with the step adjustments in UTC. Currently, ST at each tracking station departs from UTC by less than 100 μs and is known to 10–20 μs. The value of the UTC–ST offset is determined by using a traveling UTC clock (previously synchronized with the National Bureau of Standards) or by transmitting a timing signal (derived from the master UTC clock of the DSN) from the Deep Space Communications Complex at Goldstone, Calif., to a particular tracking station via moon bounce (accounting for the fairly well known propagation delay). The traveling clock provides UTC–ST to 5μs or better, while the moon bounce currently provides an accuracy of about 20 μs.

In the DPODP, time is represented as double-precision seconds past January 1, 1950, 0^h. On the IBM 7094 computer, double precision is 54 bits or slightly more than 16 decimal digits; from 1967 to 1984, time is represented to 0.6×10^{-7} s. If UTC is 600,000,000 s past January 1, 1950, 0^h UTC, and ET $-$ UTC $= 40$ s, then ET is 600,000,040 s past January 1, 1950, 0^h ET.

B. Transformations Between Time Scales

The complete transformation between A1 time and ET is given by Eq. (65). The terms of Eq. (65) are defined in detail after that equation. The first term, ΔT_{1958}, is the constant part of the offset between A1 time and ET. The second term accounts for a possible difference in the average length of the ET second ($9{,}192{,}631{,}770 + \Delta f_{\text{cesium}}$ cycles of cesium) and the length of the A1 second (9,192,631,770 cycles of cesium). The nominal values of ΔT_{1958} and Δf_{cesium} are 32.15 s and 0, respectively; both are solve-for parameters.

The remaining terms of Eq. (65) arise from general relativity; they represent periodic variations in proper time on earth (namely the A1, UTC, and ST atomic time scales) relative to uniform coordinate time t (ephemeris time ET). These variations in proper time relative to coordinate time are due to variations in the Newtonian potential at the atomic clock and in the heliocentric velocity of the atomic clock (see Eq. 64).

In the computation of the range observables used to compute differenced-range doppler (see Section XI), the complete expression for ET $-$ A1 (Eq. 65) is required to accurately transform round-trip ephemeris time from the light time solution to observed round-trip station time. However, in the general time transformation subroutine of the DPODP, only the annual relativity term of ET$-$A1 has been retained. The expression, giving ET $-$ A1 in seconds, is

$$\mathrm{ET} - \mathrm{A1} = \Delta T_{1958} - (t - 252{,}460{,}800)\frac{\Delta f_{\text{cesium}}}{9{,}192{,}631{,}770} + 1.658 \times 10^{-3} \sin E \qquad (93)$$

where E is computed from Eqs. (67) and (70).

The largest terms of ET $-$ A1 neglected in Eq. (93) are the 2-μs daily term (the fourth term of Eq. 65) and the 1.7-μs monthly term. Also, there are long period variations of the same approximate magnitude due to periodic variations in the heliocentric orbital elements of the earth–moon barycenter arising from perturbations from the other planets. Thus, the accuracy of ET $-$ A1 computed from Eq. (93) in the general time transformation subroutine is about 10^{-5} s.

The remaining transformations between the various time scales are specified by linear or quadratic functions of time t. The coefficients of these polynomials are specified by time block and the argument t is seconds past the start of the time block. Thus

$$\mathrm{UTC} - \mathrm{ST} = a + bt + ct^2 \qquad (94)$$

$$\mathrm{A1} - \mathrm{UTC} = d + et \qquad (95)$$

$$\mathrm{A1} - \mathrm{UT1} = f + gt + ht^2 \qquad (96)$$

Equations (93–96) are used to transform in either direction, the right-hand side being evaluated with the known time. For instance, Eq. (95) is evaluated with UTC when transforming from a UTC epoch to the corresponding A1 epoch. Alternatively, it is evaluated with A1 time when transforming from an A1 epoch to a UTC epoch.

As previously indicated, observed values of UTC − ST are available for each tracking station. Values of a, b, and c may be obtained by fitting to these data. The value of UTC − ST is typically less than 100 μs and is known to 10–20 μs. The coefficients a, b, and c are solve-for parameters; however, it is doubtful if the estimated values of a, b, and c would yield UTC − ST more accurately than the observed accuracy of 10–20 μs.

The U.S. Naval Observatory supplies values of A1 − UTC and A1 − UT1 to the nearest 0.1 ms. Curve-fitting techniques are used to obtain the polynomial coefficients d through h by time block, normally of 1 month's duration. Real-time reduction of tracking data is accomplished by using extrapolated polynomials for the current month.

The fitted expressions for A1 − UTC are probably accurate to about 2×10^{-5} s. A more accurate expression could be obtained by fitting to the data published by the National Bureau of Standards (to the nearest μs) or, better yet, by computing the expression directly from the known frequency offsets and step adjustments. The published data are obtained in this manner.

A small error is incurred in the evaluation of Eqs. (93) to (96) since each may be evaluated with either of the two time scales which it relates. The largest error occurs in the evaluation of Eq. (95) or (96) where e and g are about 0.3×10^{-7}, h is about 10^{-15}, and t may be as large as 3×10^{6} s. Since t varies by about 8 s, depending upon whether it is evaluated with A1 or UT, the resulting uncertainty in A1 − UTC or A1 − UT1 is about 2 to 3×10^{-7} s.

The observables are recorded in ST. In order to obtain the computed values of the observables, the ephemerides of the spacecraft, planets, and moon which affect the observables must be interpolated at the ET value of the epoch of observation, obtained from the ST epoch by using Eqs. (93–95). Since Eq. (93) could be in error by 10 μs and each of Eqs. (94) and (95) could be in error by 20 μs, the ET value of the epoch of observation could be in error by as much as 5×10^{-5} s.

The error in the computed value of a range observable due to an error of 5×10^{-5} s in the ET epoch at which it is evaluated is the spacecraft range rate multiplied by 5×10^{-5} s. For a typical range rate of 30 km/s, the error in computed range is 1.5 m, which is close to the desired accuracy of 0.1 m. The largest conceivable range rate is about 1000 km/s, which can occur for the spacecraft on a hyperbola grazing the solar surface. For this extreme case, the error in computed range is an acceptable 50 m. Thus, an accuracy of about 10^{-5} s in the individual time transformations is acceptable for the accurate computation of range observables.

The maximum error in the computed value of a doppler observable due to an error of 5×10^{-5} s in the ET epoch at which it is evaluated is the acceleration of the spacecraft relative to the tracking station multiplied by 5×10^{-5} s. During heliocentric cruise, this acceleration is less than 0.1 m/s^2, and the error in computed doppler is less than 5×10^{-6} m/s. This compares favorably with the desired accuracy of 10^{-5} m/s.

However, for a grazing encounter with Venus or Jupiter, or an approach to within 1 solar radius of the sun's surface, the accelerations are 9 m/s^2, 25 m/s^2, and 70 m/s^2, respectively. For a 5×10^{-5} s timing error, the errors in computed doppler observables are 5×10^{-4} m/s, 1.3×10^{-3} m/s, and 3.5×10^{-3} m/s, respectively. These doppler residuals are one to two orders of magnitude larger than desired. With good tracking data, doppler residuals are often obtained with a maximum value of about 10^{-4} m/s. Thus, during heliocentric cruise, a timing accuracy of 5×10^{-5} s is adequate for the accurate computation of doppler observables. But, when the spacecraft is near a planet or the sun, this timing accuracy is only marginally acceptable.

When the offset from UTC to ST at each tracking station is known to significantly better than the current accuracy of 10–20 μs, one of the two previously indicated methods for increasing the accuracy of the A1 − UTC time transformation should be implemented. The next step in increasing the accuracy of the time transformations would be to add the 2-μs daily term and the 1.7-μs monthly term to the expression for ET − A1 used in the general time transformation subroutine. Evaluation of the daily term would require that each A1 and UTC epoch be associated with a particular tracking station and that the location of the station be input to the subroutine. However, there is no point in attempting to obtain time transformations much more accurate than the microsecond level, because of the unknown long period fluctuations of order 10^{-6} s in ET − A1.

The value of A1 − UT1 computed from Eq. (96) at any instant defines the location of the 0° meridian on earth at that instant. Over a short period of time from this epoch (a few weeks or months), the angular position of this meridian computed from Eqs. (96) and (92) will depart in a random manner from its actual position by an angle equivalent to an error of 5–8 ms (1 sigma) in UT1. In addition to this random error in UT1, there may be a secular error of a few milliseconds per year. The geocentric velocity of a tracking station on the equator is 465 m/s. Hence, the random error in UT1 of 5–8 ms (1 sigma) produces fluctuations in the computed right ascensions of tracking stations of 2–4 m (1 sigma).[6] A secular error in UT1 of 2 ms per year would cause the estimated station longitudes to drift by about 1 m per year.[6] These errors are large in relation to the current goal of obtaining station locations to an accuracy of 1 m. Currently, the uncertainties in the estimated tracking station locations are about 5 m (see Mottinger, Ref. 27).

For further details on the subject of timing, see Trask and Muller (Ref. 28) and Ref. 29, Sections II-E and II-F.

IV. *n*-Body Ephemerides

Section IV-A describes the precomputed *n*-body ephemerides for the celestial bodies of the solar system and the manner in which they were generated. Section IV-B describes the method by which these ephemerides are differentially corrected within the DPODP and gives the formulation for obtaining corrected position, velocity, acceleration, and jerk from any ephemeris. Section C gives the formulas for combining these quantities to obtain the relative position, velocity, acceleration, and jerk between any two celestial bodies of the solar system.

Acceleration and jerk are required to compute doppler observables. Acceleration is also used in the computation of partial derivatives of the observables with respect to the estimated parameters.

A. Description of Precomputed *n*-Body Ephemerides

The DPODP uses the following precomputed position and velocity ephemerides for the celestial bodies of the solar system: (1) heliocentric ephemerides for eight planets and the earth–moon barycenter and (2) the geocentric lunar ephemeris. The lunar ephemeris is obtained by a numerical integration fit to a corrected version of the Improved Brown Lunar Theory, as will be described in detail below. Given the precomputed ephemeris of the moon, the planetary ephemerides are obtained by a simultaneous numerical integration performed by the SSDPS (Solar System Data Processing System).

Values of a number of parameters are differentially corrected to produce a least-squares fit to observed angular data for all of the planets and the sun, radar range data for Mercury, Venus, and Mars, and ranging data to a spacecraft when it is in the vicinity of a planet. The parameters whose values may be estimated are (1) osculating orbital elements for each ephemeris, (2) osculating orbital elements for the trajectory of the spacecraft relative to the planet it is passing, (3) masses of the planets, (4) radii of planets which have been tracked by radar ranging, (5) right ascension and declination limb biases for Mercury and Venus, and (6) the astronomical unit.

The equations of motion are Newton's equations plus relativistic perturbative accelerations derived from the 1-body metric of the Brans–Dicke theory. When the solve-for parameter γ approaches unity, this metric reduces to the 1-body isotropic metric of general relativity. Development Ephemeris 69 (DE69), which is the latest export ephemeris produced at JPL, is the first to be based upon isotropic relativistic coordinates. Previous ephemerides were based upon the Schwarzschild coordinates of general relativity. This permanent change was made so that the precomputed *n*-body ephemerides would be compatible with the DPODP, which is based upon isotropic coordinates.

The ephemeris DE69 is based upon a 60-year backward integration from the epoch of August 2, 1970, 0^h ET to 1910. The observations consist of over 34,000 optical observations of the planets (except Pluto) and the sun obtained from the 150-mm and 230-mm transit circles of the U.S. Naval Observatory for 1910–1968, radar range data for Mercury, Venus, and Mars for 1964–1968, and range observables for the *Mariner* V spacecraft near its encounter with Venus (data for June 21–November 12, 1967). After being fitted to these data, the ephemerides were integrated forward from the 1970 epoch to 1976. The ephemeris DE69 consists of the latter portion of the 60-year integration from October 28, 1961, to the 1970 epoch and the forward integration from this epoch to January 23, 1976. The lunar ephemeris contained in DE69 is Lunar Ephemeris 16 (LE16), described below; DE69 is described in Ref. 30.

An easy way to describe LE16 is to consider the evolution of LE4 (Ref. 31) through LE6 (Ref. 32) to LE16

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[6] The angular error multiplied by the distance of the tracking station from the earth's spin axis.

(Ref. 33). The Improved Lunar Ephemeris (ILE) (Ref. 34) is the result of removing certain deficiencies in the original Brown Lunar Theory (Refs. 35 and 36). Brown's solution for the motion of the moon was obtained in rotating rectangular coordinates and then transformed to spherical coordinates. Because precise observations were not available in his time, Brown evaluated this coordinate transformation with less accuracy than he used in his solution for the moon's motion.

These coordinate transformations have recently been recomputed to a higher precision by Eckert, Walker, and Eckert (Ref. 37). Eckert and Smith (Ref. 38) have obtained a numerical general theory for the motion of the moon that is independent of the Brown Lunar Theory. From a comparison of the two theories, Eckert has recommended that the ILE be augmented by the longitude correction

$$0\overset{''}{.}072 \sin(2F - 2l)$$

Positions for LE4 were obtained by evaluating the ILE with aberration terms removed to make the ephemeris strictly geometric, addition of the transformation corrections of Eckert et al. (Ref. 37) and the longitude correction of Eckert and Smith (Ref. 38), and addition of corrections to effectively change the constants of the theory to those adopted by the International Astronomical Union (IAU) in 1964 (Ref. 26, pp. 594–5), except for the value of the second zonal harmonic J_2 for the earth. Numerical differentiation of these positions gave the velocities for LE4. Addition to LE4 of correction terms to account for the modern value of J_2 gave LE6.

Van Flandern has obtained corrections to certain constants of the ILE from a reduction of meridian circle observations of the moon and a few grazing occultations in the period 1956–1966 (Refs. 39 and 40). The latter observations are particularly accurate in declination. The observations were referred to the moon's center of mass by the use of Watts' limb corrections (Ref. 41). These charts indicate that the geometric center moves relative to the center of mass with a maximum amplitude of 7.3 μrad (1.5 arc seconds) (Ref. 39).

Van Flandern's corrections to the constants of the ILE essentially change the equinox from Brown's equinox (close to Newcomb's equinox) to the FK4 equinox, which is the basis of modern observations and the planetary ephemerides. Correction terms were added to LE6 to change certain of the constants in the theory to those obtained by Van Flandern (Ref. 40). A numerically integrated lunar ephemeris was obtained by fitting to this version of the lunar theory. Addition of corrections to account for certain observable but currently unmodelable terms of the lunar motion gave LE16.

In Refs. 21 and 22, it is shown that the significant part of the relativistic perturbative acceleration for the heliocentric ephemeris of a planet or the earth–moon barycenter is the direct perturbative acceleration due to the sun, the indirect perturbative acceleration of the sun due to the other bodies of the solar system being negligible.

In the general theory of relativity, the perturbative acceleration of a body i due to the sun is given by Eq. (35) with the Newtonian term and the j summation removed and the index j referring to the sun. In Ref. 21, pp. 49–51, it is shown that all terms containing the sun's barycentric velocity, the sun's acceleration, or the Newtonian potential at the sun are insignificant and hence that the relativistic inertial acceleration (relative to the barycenter of the solar system) of a body due to the sun, denoted $\ddot{\mathbf{r}}(S)$, may be computed from

$$\ddot{\mathbf{r}}(S) = \frac{\mu_S}{c^2 r^3}[(4\phi - \dot{s}^2)\,\mathbf{r} + 4(\mathbf{r}\cdot\dot{\mathbf{r}})\,\dot{\mathbf{r}}] \tag{97}$$

where

μ_S = gravitational constant of sun, km^3/s^2

c = speed of light

$\mathbf{r}, \dot{\mathbf{r}}$ = heliocentric position and velocity vectors of body, with rectangular components referred to the mean earth equator and equinox of 1950.0

$r, \dot{s}$ = magnitudes of $\mathbf{r}$ and $\dot{\mathbf{r}}$, respectively

ϕ = Newtonian potential at body (positive sign convention)

In the Brans–Dicke theory, Eqs. (35) and (97) are replaced by Eq. (54) and the following equation:

$$\ddot{\mathbf{r}}(S) = \frac{\mu_S}{c^2 r^3}\{[2(1+\gamma)\,\phi - \gamma\dot{s}^2]\,\mathbf{r} + 2(1+\gamma)(\mathbf{r}\cdot\dot{\mathbf{r}})\,\dot{\mathbf{r}}\} \tag{98}$$

where γ (or ω; see Eq. 41) is the free parameter of the Brans–Dicke theory whose value is to be estimated by fitting the theory to observation.

As γ approaches unity, its general relativity value, Eq. (98) approaches Eq. (97) of general relativity. If ϕ

in Eqs. (97) and (98) were replaced by the potential due to the sun, μ_S/r, these equations would be identical to the corresponding 1-body equations, namely Eqs. (20) and (55), respectively.

For the heliocentric ephemeris of a planet, the relativistic perturbative acceleration is given by Eq. (98). However, the only significant term of ϕ is μ_S/r and thus, for this application, Eq. (98) reduces to the corresponding 1-body equation, Eq. (55). For the heliocentric ephemeris of the earth–moon barycenter, the perturbative acceleration is computed from[7]

$$\ddot{\mathbf{r}} = \frac{\mu}{1+\mu}\ddot{\mathbf{r}}_E(S) + \frac{1}{1+\mu}\ddot{\mathbf{r}}_M(S) \tag{99}$$

where

$$\mu = \frac{\mu_E}{\mu_M} \tag{100}$$

and

μ_E, μ_M = gravitational constants of the earth and moon, respectively, km^3/s^2

The perturbative accelerations of the earth and moon due to the sun are computed from Eq. (98) with the potentials at these two bodies given by

$$\phi_E = \frac{\mu_S}{r_{SE}} + \frac{\mu_M}{r_{EM}} \tag{101}$$

$$\phi_M = \frac{\mu_S}{r_{SM}} + \frac{\mu_E}{r_{EM}} \tag{102}$$

where r_{ij} is the coordinate distance from body i to body j. The formulas above are used in the SSDPS to compute the relativistic perturbative acceleration for each planetary ephemeris.

From Ref. 22, Table 3, the maximum amplitude of the periodic variations in position for a planetary ephemeris, arising from Eq. (98), is about 6 km. It is shown in Ref. 21, p. 51, that the ratio of terms of Eq. (54) not included in Eq. (98) to the acceleration computed from Eq. (98) has a maximum value of 10^{-3}. Thus the above-mentioned position variations are computed to an accuracy of at least 6 m. The relativistic acceleration of the earth–moon barycenter computed from Eq. (99) should also contain the terms

$$\frac{\mu}{1+\mu}\ddot{\mathbf{r}}_E(M) + \frac{1}{1+\mu}\ddot{\mathbf{r}}_M(E)$$

where the mutual accelerations of the earth and moon are computed from Eq. (54). However, it is shown in Ref. 21, p. 53, that the periodic variations in the position of the earth–moon barycenter due to these terms are more than three orders of magnitude smaller than the relativistic variations due to the sun, which, from Table 3 of Ref. 22, have a magnitude of about 400 m. Thus, the variations in position of the earth–moon barycenter due to the mutual accelerations of the earth and moon have an amplitude of less than 1 m. The errors in the planetary ephemerides due to neglecting the contribution to the Newtonian potential ϕ in Eq. (98) from the other planets are less than 10 m for the inner planets and 100 m for the outer planets.

The relativistic acceleration due to a planet or the moon is significant, relative to the solar relativistic acceleration, in only a small region surrounding the body (small in relation to the scale of the solar system). For simplicity, this region is taken to be a sphere, termed the relativity sphere, whose center is at the center of mass of the body. The relativistic acceleration due to a planet or the moon should be computed only within that body's relativity sphere. The radius of the relativity sphere for each body of the solar system is given in Ref. 21, Table 5. Since no planet is within the relativity sphere of another planet, the relativistic acceleration of a planet or the earth–moon barycenter due to another planet is negligible. It has been estimated (Ref. 21, p. 53) that neglecting the indirect relativistic acceleration of the sun produces periodic errors in position of less than 1 m for the inner planets and less than 1 km for the outer planets.

Considering all of the errors mentioned above, the planetary ephemerides produced by the SSDPS contain periodic errors of up to 20 m for the inner planets and up to 1 km for the outer planets due to neglected terms in the specified formulation for the relativistic perturbative acceleration.

Fragmentary evidence indicates that LE16 may be as accurate as 100 m. The maximum effect of general relativity on the geocentric lunar ephemeris is less than 10 m in position and 10^{-5} m/s in velocity (Ref. 22, p. 4). Thus, it

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[7] The notation $\ddot{\mathbf{r}}_i(j)$ is the relativistic perturbative acceleration of body i due to body j.

is not important which relativity terms were included in the numerical integration fitted to the lunar theory, which produced LE16.

However, in the future when the lunar ephemeris is obtained by a numerical integration fitted to observations, as is currently done for the planetary ephemerides and the spacecraft ephemeris, the relativistic perturbative acceleration of the moon relative to the earth should be computed from

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_M(S) - \ddot{\mathbf{r}}_E(S) + \ddot{\mathbf{r}}_M(E) - \ddot{\mathbf{r}}_E(M) \tag{103}$$

The first two terms are evaluated with Eqs. (98), (101), and (102). The last two terms are evaluated with Eq. (54), with the Newtonian term and the j summation removed and the index j referring to the body producing the acceleration. All velocities appearing in Eq. (54) are barycentric but may be evaluated with heliocentric values. The acceleration of the perturbing body may be evaluated with Newtonian theory, Eq. (31). The Newtonian potentials at bodies i and j may be evaluated with Eqs. (101) and (102). The sum of terms 1 and 2 of Eq. (103) is about 10^{-14} km/s^2, whereas the individual terms are one order of magnitude larger. The magnitudes of terms 3 and 4 are about 10^{-13} and 10^{-15} km/s^2, respectively. The total acceleration computed from Eq. (103) is accurate to three or four figures.

B. Obtaining Corrected Position, Velocity, Acceleration, and Jerk From Each Ephemeris

1. Uncorrected position and velocity. As previously mentioned, the n-body ephemeris consists of (1) heliocentric ephemerides for eight planets and the earth–moon barycenter and (2) the geocentric lunar ephemeris. These ephemerides are in the so-called type-50 format; they contain modified second and fourth central differences of position and velocity. Interpolation with the fifth-order Everett's formula gives rectangular components of position and velocity referred to the mean earth equator and equinox of 1950.0 (commonly referred to as 1950.0 coordinates). Positions and velocities from the planetary ephemerides are expressed in astronomical units AU and AU/day, respectively, while data from the lunar ephemeris are expressed in "fictitious earth radii" and "fictitious earth radii"/day.

The conversion factors used to convert the length units to kilometers are A_E km per AU and R_E km per fictitious earth radius. The scaling factors A_E and R_E are related to other solve-for parameters by the so-called solar and lunar constraints, respectively. These constraints and the recommended values of the scaling factors are given in the following section.

2. Solar and lunar constraints. The solar constraint is an exact relation between the estimated value of

A_E = the number of kilometers per astronomical unit

and the estimated value of

μ_S = gravitational constant of the sun, km^3/s^2

The relation is

$$\mu_S = \frac{k^2 A_E^3}{(86{,}400)^2} \tag{104}$$

where

k = the Gaussian gravitational constant
= 0.01720209895 AU$^{3/2}$/day (exactly)

The gravitational constant of the sun k^2 expressed in astronomical units cubed per day squared is a mathematical constant which defines the length of 1 AU. The solar constraint is simply a conversion of the sun's gravitational constant from AU^3/day^2 to km^3/s^2.

From Ref. 29, p. 35, Table 17, the values of μ_S and A_E currently adopted by JPL are

$$\mu_S = 1.32712499 \times 10^{11}\,\text{km}^3/\text{s}^2$$
$$A_E = 149{,}597{,}893\,\text{km/AU}$$

These values satisfy the solar constraint (Eq. 104) to the stated accuracy of nine figures. The value of A_E is the recommended scaling factor for the planetary ephemerides of DE69.

One of the constants of the lunar theory is

$\sin \pi_{☾}$ = the constant of sine parallax for the moon
= the ratio of a fictitious mean equatorial radius of the earth (the length unit of the lunar ephemeris) to the perturbed mean distance of the moon. The constant $\sin \pi_{☾}$ is dimensionless; however, it is usually expressed in seconds of arc by multiplying by the number of seconds of arc in one radian.

The value of $\sin \pi_{☾}$ adopted by the IAU in 1964 (Ref. 26) and used in the construction of LE4 and succeeding lunar ephemerides is 3,422.451 arc seconds. The mean distance to the moon in terms of fictitious earth radii is given by

$$a_M = \frac{1}{\sin \pi_{☾}\,(\text{dimensionless})} = \frac{206{,}264.80625}{\sin \pi_{☾}\,(\text{arc seconds})} \tag{105}$$

where

a_M = perturbed mean distance of moon (the perturbation is due to the sun), fictitious earth radii

The value of a_M in kilometers is

$$R_E a_M$$

where

R_E = scaling factor for the lunar ephemeris, km/fictitious earth radius

The value of $R_E a_M$ is obtained from a modified version of Kepler's third law:

$$n_M^2 a_M^3 R_E^3 = F_2^3 (\mu_E + \mu_M) \tag{106}$$

where

n_M = sidereal mean motion of moon (1900)
= $2.661699489 \times 10^{-6}$ rad/s

F_2 = 0.999093141975298 (as computed by E. W. Brown in 1897)
= ratio of perturbed mean distance of moon to 2-body mean distance (sun not present and mean motion remains constant)

μ_E, μ_M = gravitational constants of earth and moon, respectively, km^3/s^2.

Solving for R_E gives

$$R_E = C\,(\mu_E + \mu_M)^{1/3} \tag{107}$$

where

$$C = \frac{F_2}{n_M^{2/3}\, a_M} \tag{108}$$

For $\sin \pi_{☾} = 3{,}422.451$ arc seconds, the numerical value of C is 86.3135017.

Equation (107) is the so-called lunar constraint. The value of a_M in Eq. (108) is computed from the value of $\sin \pi_{☾}$ used to generate the lunar ephemeris. Either a_M or $\sin \pi_{☾}$ may be considered to be a defined constant of the lunar theory. Hence, the accuracy of C is that of n_M, namely about 10 figures. On the other hand, $\mu_E + \mu_M$ is known to only about seven figures. Hence, for all practical purposes, the lunar constraint, Eq. (107), is an exact relation which must be satisfied by the estimated values of μ_E, μ_M, and R_E.

The lunar ephemeris LE16 is based upon values of μ_E and μ_M adopted by the IAU in 1964, namely

$$\mu_E = 398{,}603 \text{ km}^3/\text{s}^2$$

and

$$\mu = \mu_E/\mu_M = 81.30$$

which gives

$$\mu_M = 4{,}902.87 \text{ km}^3/\text{s}^2$$

Substituting these values into Eq. (107) gives

$$R_E = 6{,}378.160 \text{ km/fictitious earth radius}$$

which is the value of the mean equatorial radius of the earth adopted by the IAU in 1964.

However, since 1964, more accurate values of μ_E and μ_M have been adopted by JPL (Ref. 29, p. 35, Table 16):

$$\mu_E = 398{,}601.2 \text{ km}^3/\text{s}^2$$

and

$$\mu = 81.3010$$

which gives

$$\mu_M = 4{,}902.78 \text{ km}^3/\text{s}^2$$

The corresponding value of R_E is

$$R_E = 6{,}378.1492 \text{ km/fictitious earth radius}$$

Strictly speaking, the lunar ephemeris should be corrected for these more modern values of μ_E and μ_M as was done in the generation of LE4 where Brown's constants were corrected to those adopted by the IAU in 1964. However, the major part of this correction can be obtained by scaling the lunar ephemeris with $R_E = 6{,}378.1492$ km rather than the value of 6,378.160 km.

3. *Corrected position and velocity.* Each of the precomputed ephemerides may be differentially corrected with conic formulas. Position and velocity are interpolated from the ephemeris at an epoch of osculation specified by the user and are converted to orbital elements, specifically the Brouwer and Clemence Set III (Ref. 42, pp. 241–242). The elliptical orbit with these elements agrees exactly with the precomputed ephemeris at the osculation epoch and approximately at other epochs. The orbital elements of the precomputed ephemeris at the osculation epoch are solve-for parameters. Partial derivatives of position and velocity from the ephemeris with respect to these orbital elements are approximated by those from the osculating elliptical orbit. These partial derivatives are used to determine corrections in the osculating orbital elements and, given these corrections, to apply a linear differential correction to the ephemeris.

The actual parameters whose values are estimated are six parameters which represent corrections $\Delta \mathbf{E}$ to the osculating orbital elements $\mathbf{E}$. The corrections are

$$\Delta \mathbf{E} = \begin{bmatrix} \Delta a/a \\ \Delta e \\ \Delta M_0 + \Delta w \\ \Delta p \\ \Delta q \\ e\Delta w \end{bmatrix} \tag{109}$$

where

a = semimajor axis of osculating elliptical orbit

e = eccentricity

M_0 = value of mean anomaly at osculation epoch, t_0 (ET)

$\Delta p, \Delta q, \Delta w$ = right-handed rotations of the orbit about the **P**, **Q**, and **W** axes, respectively, where **P** is directed from the focus to perifocus, **Q** is $\pi/2$ rad ahead of **P** in the orbital plane, and $\mathbf{W} = \mathbf{P} \times \mathbf{Q}$

Let $\Delta \mathbf{E}_1$ equal the estimated value of $\Delta \mathbf{E}$ obtained from the first iteration of the orbit determination process (see Section I). The second iteration will produce an *additional* correction $\Delta \mathbf{E}_2$ or a total correction $\Delta \mathbf{E}_1 + \Delta \mathbf{E}_2$. Let the contribution to $\Delta \mathbf{E}$ obtained from the *i*th iteration be denoted as $\Delta \mathbf{E}_i$. With this notation, the correction $\Delta \mathbf{E}(n)$ used to correct the ephemeris for the *n*th iteration consists of the accumulated correction obtained from the previous $n - 1$ iterations:

$$\Delta \mathbf{E}(n) = \sum_{i=1}^{n-1} \Delta \mathbf{E}_i = \sum_{i=1}^{n-1} \begin{bmatrix} (\Delta a | a)_i \\ (\Delta e)_i \\ (\Delta M_0 + \Delta w)_i \\ (\Delta p)_i \\ (\Delta q)_i \\ (e\Delta w)_i \end{bmatrix} \qquad n > 1 \tag{110}$$

If the correction process is convergent, $\Delta \mathbf{E}_n$ will be less than $\Delta \mathbf{E}_{n-1}$ and the accumulated correction will approach a limit.

Given 1950.0 position $\mathbf{r}$ (AU) and velocity $\dot{\mathbf{r}}$ (AU/s) obtained from a planetary ephemeris (at any time) in units of AU and AU/s (the interpolated value in AU/day divided by 86,400), corrected position and velocity for the *n*th iteration, expressed in km and km/s, are computed from

$$\mathbf{r}_n(\text{km}) = A_E \mathbf{r}(\text{AU}) + \frac{\partial \mathbf{r}}{\partial \mathbf{E}} \Delta \mathbf{E}(n) \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{111}$$

where $\Delta \mathbf{E}(n)$ is given by Eq. (110). For the lunar ephemeris,

$$\mathbf{r}_n(\text{km}) = R_E \mathbf{r}\,(\text{fictitious earth radii}) + \frac{\partial \mathbf{r}}{\partial \mathbf{E}} \Delta \mathbf{E}(n) \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{112}$$

In these equations,

$$\frac{\partial \mathbf{r}}{\partial \mathbf{E}} = \left[\frac{\partial \mathbf{r}}{\partial \left(\frac{\Delta a}{a}\right)}, \frac{\partial \mathbf{r}}{\partial (\Delta e)}, \frac{\partial \mathbf{r}}{\partial (\Delta M_0 + \Delta w)}, \frac{\partial \mathbf{r}}{\partial (\Delta p)}, \frac{\partial \mathbf{r}}{\partial (\Delta q)}, \frac{\partial \mathbf{r}}{\partial (e\Delta w)} \right] \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{113}$$

where

$$\frac{\partial \mathbf{r}}{\partial E_i} = \begin{bmatrix} \dfrac{\partial x}{\partial E_i} \\ \dfrac{\partial y}{\partial E_i} \\ \dfrac{\partial z}{\partial E_i} \end{bmatrix} \qquad \begin{matrix} \mathbf{r} \rightarrow \dot{\mathbf{r}} \\ x, y, z \rightarrow \dot{x}, \dot{y}, \dot{z} \end{matrix} \tag{114}$$

where x, y, and z are the rectangular components of $\mathbf{r}$ referred to the mean earth equator and equinox of 1950.0. The formulation for computing $\partial\mathbf{r}/\partial\mathbf{E}$ and $\partial\dot{\mathbf{r}}/\partial\mathbf{E}$ is given in the next section.

4. Partial derivatives of position and velocity with respect to orbital elements. In order to compute $\partial\mathbf{r}/\partial\mathbf{E}$ and $\partial\dot{\mathbf{r}}/\partial\mathbf{E}$ for any of the precomputed ephemerides, position and velocity at the osculation epoch must be converted to orbital elements $\mathbf{E}$. Let

$\mathbf{r}_0$ = 1950.0 position interpolated from ephemeris at osculation epoch t_0 (ET) in AU or fictitious earth radii and converted to km by multiplying by A_E or R_E.

$\dot{\mathbf{r}}_0$ = 1950.0 velocity interpolated from ephemeris at osculation epoch t_0 (ET) in AU/day or fictitious earth radii/day and converted to km/s by multiplying by A_E or R_E and dividing by 86,400.

For the heliocentric ephemeris of a planet, the parameter μ is computed from

$$\mu\,(\text{planet}) = \mu_S + \mu_P \tag{115}$$

For the heliocentric ephemeris of the earth–moon barycenter, μ is given by

$$\mu\,(\text{earth–moon barycenter}) = \mu_S + \mu_E + \mu_M \tag{116}$$

For the geocentric lunar ephemeris,

$$\mu\,(\text{moon}) = \mu_E + \mu_M \tag{117}$$

where

μ_S, μ_E, μ_M, μ_P = gravitational constants for the sun, the earth, the moon, and a planet, km^3/s^2

Given $\mathbf{r}_0$ (km), $\dot{\mathbf{r}}_0$ (km/s), and μ, the required orbital elements are computed as follows:

$$r_0 = (\mathbf{r}_0 \cdot \mathbf{r}_0)^{1/2} \tag{118}$$

The semimajor axis a is given by

$$a = \frac{1}{\dfrac{2}{r_0} - \dfrac{\dot{\mathbf{r}}_0 \cdot \dot{\mathbf{r}}_0}{\mu}} \tag{119}$$

The mean motion n is computed from

$$n = \frac{\mu^{1/2}}{a^{3/2}} \tag{120}$$

and the following computations are made:

$$e \cos E_0 = 1 - \frac{r_0}{a} \tag{121}$$

$$e \sin E_0 = \frac{\mathbf{r}_0 \cdot \dot{\mathbf{r}}_0}{(\mu a)^{1/2}} \tag{122}$$

$$e = [(e \cos E_0)^2 + (e \sin E_0)^2]^{1/2} \tag{123}$$

$$\cos E_0 = \frac{(e \cos E_0)}{e} \tag{124}$$

$$\sin E_0 = \frac{(e \sin E_0)}{e} \tag{125}$$

The unit vectors $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{W}$ are computed from

$$\mathbf{P} = \frac{\cos E_0}{r_0}\mathbf{r}_0 - \left(\frac{a}{\mu}\right)^{1/2} \sin E_0\, \dot{\mathbf{r}}_0 \tag{126}$$

$$\mathbf{W} = \frac{\mathbf{r}_0 \times \dot{\mathbf{r}}_0}{[\mu a (1 - e^2)]^{1/2}} \tag{127}$$

$$\mathbf{Q} = \mathbf{W} \times \mathbf{P} \tag{128}$$

The partial derivatives $\partial\mathbf{r}/\partial\mathbf{E}$ and $\partial\dot{\mathbf{r}}/\partial\mathbf{E}$ are computed from the orbital elements a, e, n, $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{W}$, which are computed once, and from the following quantities, which are computed at each time t that the partials are evaluated:

$\mathbf{r}$, $\dot{\mathbf{r}}$ = 1950.0 position and velocity interpolated from the ephemeris at time t (ephemeris time) and converted to units of km and km/s as indicated previously for $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$.

$$r = (\mathbf{r} \cdot \mathbf{r})^{1/2} \tag{129}$$

$$r\dot{r} = \mathbf{r} \cdot \dot{\mathbf{r}} \tag{130}$$

$$\dot{r} = \frac{(r\dot{r})}{r} \tag{131}$$

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{132}$$

From Ref. 42, p. 241, the partial derivatives of $\mathbf{r}$ at ephemeris time t with respect to each element of $\Delta\mathbf{E}$ used in Eq. (113), are given by

$$\frac{\partial \mathbf{r}}{\partial \left(\frac{\Delta a}{a}\right)} = \mathbf{r} - \frac{3}{2}(t - t_0)\,\dot{\mathbf{r}} \tag{133}$$

$$\frac{\partial \mathbf{r}}{\partial (\Delta e)} = H_1 \mathbf{r} + K_1 \dot{\mathbf{r}} \tag{134}$$

where the quantities H_1 and K_1, which are functions of t, are given by (Ref. 42, p. 237)

$$H_1 = \frac{r - a(1 + e^2)}{ae(1 - e^2)} \tag{135}$$

$$K_1 = \frac{r\dot{r}}{a^2 n^2 e}\left[1 + \frac{r}{a(1 - e^2)}\right] \tag{136}$$

$$\frac{\partial \mathbf{r}}{\partial (\Delta M_0 + \Delta w)} = \frac{\dot{\mathbf{r}}}{n} \tag{137}$$

$$\frac{\partial \mathbf{r}}{\partial (\Delta p)} = \mathbf{P} \times \mathbf{r} \tag{138}$$

$$\frac{\partial \mathbf{r}}{\partial (\Delta q)} = \mathbf{Q} \times \mathbf{r} \tag{139}$$

$$\frac{\partial \mathbf{r}}{\partial (e\Delta w)} = \frac{1}{e}\left(\mathbf{W} \times \mathbf{r} - \frac{\dot{\mathbf{r}}}{n}\right) \tag{140}$$

Differentiating Eqs. (133–140) with respect to ephemeris time gives the partial derivatives of $\dot{\mathbf{r}}$ at ephemeris time t with respect to each element of $\Delta\mathbf{E}$:[8]

$$\frac{\partial \dot{\mathbf{r}}}{\partial \left(\frac{\Delta a}{a}\right)} = -\frac{1}{2}\dot{\mathbf{r}} - \frac{3}{2}(t - t_0)\,\ddot{\mathbf{r}} \tag{141}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial (\Delta e)} = H_2 \mathbf{r} + K_2 \dot{\mathbf{r}} \tag{142}$$

where

$$H_2 = \frac{\dot{r}}{ae(1 - e^2)}\left\{1 - \frac{a}{r}\left[1 + \frac{a}{r}(1 - e^2)\right]\right\} \tag{143}$$

$$K_2 = \frac{1}{e(1 - e^2)}\left(1 - \frac{r}{a}\right) \tag{144}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial (\Delta M_0 + \Delta w)} = \frac{\ddot{\mathbf{r}}}{n} \tag{145}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial (\Delta p)} = \mathbf{P} \times \dot{\mathbf{r}} \tag{146}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial (\Delta q)} = \mathbf{Q} \times \dot{\mathbf{r}} \tag{147}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial (e\Delta w)} = \frac{1}{e}\left(\mathbf{W} \times \dot{\mathbf{r}} - \frac{\ddot{\mathbf{r}}}{n}\right) \tag{148}$$

5. Acceleration and jerk. Acceleration and jerk vectors from each ephemeris are computed from corrected position and velocity vectors using 2-body formulas. Given the corrected position and velocity vectors, denoted here as $\mathbf{r}$ and $\dot{\mathbf{r}}$, compute a corrected value of r from Eq. (129), the acceleration vector $\ddot{\mathbf{r}}$ from Eq. (132), and the jerk vector $\dddot{\mathbf{r}}$ from

$$\dddot{\mathbf{r}} = \frac{3\mu(\mathbf{r} \cdot \dot{\mathbf{r}})}{r^5}\mathbf{r} - \frac{\mu}{r^3}\dot{\mathbf{r}} \tag{149}$$

where μ is given by Eq. (115), (116), or (117).

C. Position, Velocity, Acceleration, and Jerk of One Celestial Body Relative to Another

Section IV-B gave the formulation for computing the corrected position, velocity, acceleration, and jerk of a planet P or the earth–moon barycenter B relative to the sun S or of the moon M relative to the earth E:

$$\mathbf{r}_P^S, \mathbf{r}_B^S, \mathbf{r}_M^E \qquad \mathbf{r} \to \dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}$$

The position, velocity, acceleration, and jerk of the moon relative to the earth–moon barycenter and of the barycenter relative to the earth are computed from

$$\mathbf{r}_M^B = \frac{\mu}{1 + \mu}\mathbf{r}_M^E \qquad \mathbf{r} \to \dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}} \tag{150}$$

and

$$\mathbf{r}_B^E = \frac{1}{1 + \mu}\mathbf{r}_M^E \qquad \mathbf{r} \to \dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}} \tag{151}$$

where

$$\mu = \frac{\mu_E}{\mu_M} \tag{152}$$

---

[8] The velocity partials were first derived by P. R. Peabody, formerly of the Jet Propulsion Laboratory.

Listed below are sums of the above-mentioned position vectors which give the position vectors of the earth, moon, sun, and a planet relative to each of the following bodies:

(1) Earth = reference body

$$\mathbf{r}_M^E = \mathbf{r}_M^E$$

$$\mathbf{r}_S^E = \mathbf{r}_B^E - \mathbf{r}_B^S$$

$$\mathbf{r}_P^E = \mathbf{r}_B^E - \mathbf{r}_B^S + \mathbf{r}_P^S$$

(2) Moon = reference body

$$\mathbf{r}_E^M = -\mathbf{r}_M^E$$

$$\mathbf{r}_S^M = -\mathbf{r}_M^B - \mathbf{r}_B^S$$

$$\mathbf{r}_P^M = -\mathbf{r}_M^B - \mathbf{r}_B^S + \mathbf{r}_P^S$$

(3) Sun = reference body

$$\mathbf{r}_E^S = \mathbf{r}_B^S - \mathbf{r}_B^E$$

$$\mathbf{r}_M^S = \mathbf{r}_B^S + \mathbf{r}_M^B$$

$$\mathbf{r}_P^S = \mathbf{r}_P^S$$

(4) Planet = reference body

$$\mathbf{r}_E^P = -\mathbf{r}_P^S + \mathbf{r}_B^S - \mathbf{r}_B^E$$

$$\mathbf{r}_M^P = -\mathbf{r}_P^S + \mathbf{r}_B^S + \mathbf{r}_M^B$$

$$\mathbf{r}_S^P = -\mathbf{r}_P^S$$

$$\mathbf{r}_{P'}^P = -\mathbf{r}_P^S + \mathbf{r}_{P'}^S$$

where P and P' represent two different planets. All of the sums above apply when $\mathbf{r}$ is replaced by $\dot{\mathbf{r}}$, $\ddot{\mathbf{r}}$, or $\dddot{\mathbf{r}}$.

The solve-for parameters which affect the relative position and velocity between two celestial bodies are the scaling factor A_E for the heliocentric ephemerides; the scaling factor R_E for the lunar ephemeris; corrections to osculating orbital elements ΔE for any of the ephemerides; and the gravitational constants of the earth and moon, μ_E and μ_M. These are known as reference parameters.

V. Spacecraft Trajectory

A. General Description

The acceleration of the spacecraft relative to the center of integration consists of:

(1) The Newtonian point mass acceleration relative to the center of integration.

(2) The perturbative acceleration from general relativity.

(3) The direct acceleration of the spacecraft due to the oblateness of a near planet or the moon.

(4) The indirect acceleration of the center of integration (if it is the earth or the moon) due to the oblateness of the earth and the moon.

(5) The acceleration due to solar radiation pressure.

(6) The acceleration due to small forces originating in the spacecraft, such as from operation of the attitude control system and from gas leaks.

(7) The acceleration due to motor burns.

Section V-B contains the formulation for computation of each of these terms of the spacecraft acceleration.

The total acceleration is integrated numerically to give the spacecraft ephemeris, with ephemeris time (ET) as the independent variable. The acceleration is computed at each integration step and is used to produce three sum and difference (s. a. d.) arrays (one for each rectangular component of position). Each s. a. d. array contains two sums and ten differences of an acceleration component. The arrays may be interpolated at any ET epoch to give the rectangular components of position, velocity, acceleration, and jerk of the spacecraft relative to the current center of integration. The rectangular components are referred to the mean earth equator and equinox of 1950.0. The x axis is directed along the mean equinox of 1950.0, the z axis is normal to the mean earth equator of 1950.0, directed north, and the y axis completes the right-handed system.

The center of integration is located at the center of mass of the sun, the moon, or one of the nine planets. It may be specified as one of these bodies, or it may be allowed to change as the spacecraft passes through the sphere of influence of a planet (relative to the sun) or of the moon (relative to the earth). For this case, the center of integration will be that body within whose sphere of influence the spacecraft lies. At a change in center of integration, the position and velocity of the spacecraft relative to the old center of integration are incremented by the position and velocity, respectively, of the old center relative to the new center (computed from the formulation of Section IV).

The 1950.0 rectangular components of the spacecraft position and velocity vectors at the injection epoch are solve-for parameters and may be referenced to any body (not necessarily the center of integration). The injection epoch must be specified in the A1, UTC, or ST time scales and transformed to ET. The time transformation and the ET value of the epoch will vary from iteration to iteration of the orbit determination process if ΔT_{1958} or Δf_{cesium} is an estimated parameter. The injection position and velocity vectors are transformed to values relative to the initial center of integration (using the formulation of Section IV) and are used to start the s. a. d. arrays.

A motor burn of short duration or a spring separation may be represented as an instantaneous change in the position and velocity vectors of the spacecraft. The estimated parameters are the burn time t_b and the rectangular components of the velocity increment $\Delta\dot{\mathbf{r}}$. At the epoch of the motor burn, the velocity is incremented by $\Delta\dot{\mathbf{r}}$ and the position is incremented by

$$\Delta\mathbf{r} = \frac{1}{2}\Delta\dot{\mathbf{r}}t_b$$

B. Spacecraft Acceleration

The equations for computing each term of the total spacecraft acceleration relative to the center of integration are given below.

1. Point-mass gravitational acceleration. The point-mass gravitational acceleration of the spacecraft (S/C) relative to the center of integration (C) includes all gravitational accelerations except those arising from the oblateness of the various bodies. The point-mass acceleration is given by

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{S/C} - \ddot{\mathbf{r}}_C \tag{153}$$

where

$\ddot{\mathbf{r}}_{S/C}, \ddot{\mathbf{r}}_C$ = inertial gravitational acceleration of spacecraft and center of integration, respectively, computed by treating each body of the solar system as a point mass. These inertial accelerations are relative to the barycenter of the solar system and have rectangular components referred to the mean earth equator and equinox of 1950.0.

Each of these accelerations is computed from Eq. (54). The $1/c^0$ term is the Newtonian acceleration and the remaining $1/c^2$ terms are relativistic perturbative accelerations derived from the Brans–Dicke theory (these terms revert to those of general relativity, Eq. (35), when $\gamma \rightarrow 1$). The summation over $j \neq i$ includes the sun, the nine planets, and the moon. For each of these perturbing bodies, the user has the option of

(1) Computing the Newtonian acceleration and the relativistic perturbative acceleration.

(2) Computing the Newtonian acceleration only.

(3) Ignoring the acceleration due to that body.

The acceleration $\ddot{\mathbf{r}}_j$ of each perturbing body in Eq. (54) is computed from the Newtonian expression, Eq. (31). The summation over $k \neq j$ in Eqs. (31) and (54) and over $l \neq i$ in Eq. (54) includes all bodies of the solar system which are "turned on" (treated as (1) or (2) above and included in the j summation of Eq. 54). The velocities in Eq. (54) are heliocentric.

2. Direct acceleration of spacecraft due to oblateness. The acceleration of the spacecraft relative to the center of integration due to the oblateness of the bodies of the solar system consists of the direct acceleration of the spacecraft minus the indirect acceleration of the center of integration. Currently, the oblateness for only the earth, the moon, and Mars is considered. However, the capability for accounting for the oblateness of the remaining planets and the sun will be added in the near future. The direct acceleration of the spacecraft due to the oblateness of a body is computed only when the spacecraft is within the so-called harmonic sphere for the body. The radii of the harmonic spheres may be changed by input; the nominal values for the earth, Mars, and the moon are 2.5×10^6 km, 1.0×10^6 km, and 2×10^5 km, respectively. The formulation for computing the direct acceleration of the spacecraft due to the oblateness of a body is given in this section. The indirect acceleration of the center of integration due to oblateness, computed only when the center of integration is the earth or the moon, accounts for the oblateness of each of these two bodies. The formulation is given in Section V-B-3.

The direct acceleration of the spacecraft due to the oblateness of a body is derived from the generalized potential function (Ref. 43, pp. 173–174) for that body:

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{a_p}{r}\right)^n P_n^m(\sin\phi) \times (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right] \tag{154}$$

where

μ = gravitational constant of body, km^3/s^2

r, ϕ, λ = radius, latitude, and longitude (positive east of prime meridian) of spacecraft relative to body

a_p = mean equatorial radius of body (an adopted constant used for U)

$P_n^m(\sin\phi)$ = associated Legendre function of the first kind. The argument $\sin\phi$ will be omitted here.

C_{nm} and S_{nm} = numerical coefficients (tesseral harmonic coefficients). The values may be estimated by the DPODP.

The associated Legendre function P_n^m is defined by

$$P_n^m = \cos^m\phi \frac{d^m}{(d\sin\phi)^m} P_n \tag{155}$$

where

P_n = Legendre polynomial of degree n in $\sin\phi$

The zonal harmonic coefficient J_n is defined as

$$J_n = -C_{no} \tag{156}$$

Equation (154) may be written as the sum of three terms corresponding to the potential of a point mass, zonal harmonics J_n, and tesseral harmonics C_{nm} and S_{nm} $(m \neq 0)$:

$$U = \frac{\mu}{r} + U(J) + U(C,S) \tag{157}$$

where

$$U(J) = -\frac{\mu}{r}\sum_{n=1}^{\infty} J_n \left(\frac{a_p}{r}\right)^n P_n \tag{158}$$

$$U(C,S) = \frac{\mu}{r}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\left(\frac{a_p}{r}\right)^n P_n^m (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda) \tag{159}$$

The inertial acceleration of the spacecraft is computed in a rectangular coordinate system $(x'y'z')$ with the x' axis directed outward along the instantaneous radius to the spacecraft, the y' axis directed east, and the z' axis directed north. Figure 3 shows these axes relative to body-fixed axes $x_b y_b z_b$, where x_b is along the intersection of the prime meridian and equator of the body, z_b is directed north along the axis of rotation of the body, and y_b completes the right-handed system. The transformation from body-fixed coordinates $\mathbf{r}_b = (x_b, y_b, z_b)^T$ to $\mathbf{r}' = (x', y', z')^T$ coordinates is given by

$$\mathbf{r}' = R\,\mathbf{r}_b \tag{160}$$

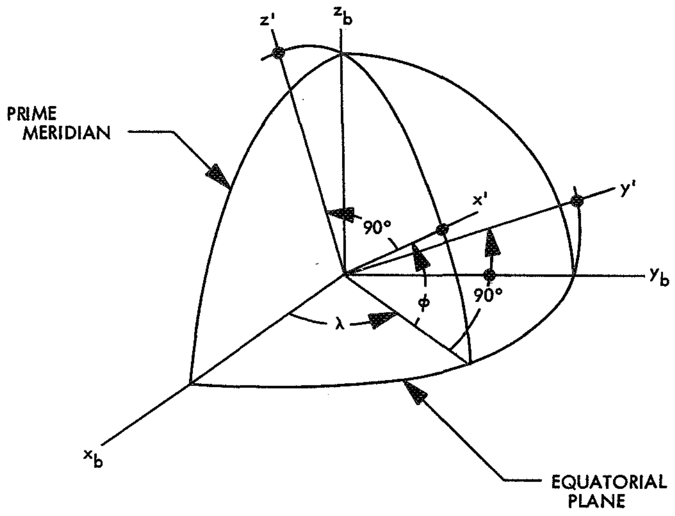

Fig. 3. x', y', and z' axes relative to body-fixed x_b, y_b, and z_b axes

where

$$R = \begin{bmatrix} \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \end{bmatrix} \tag{161}$$

The position vector of the spacecraft relative to the body (denoted as body i) with rectangular components referred to the mean earth equator and equinox of 1950.0 is $\mathbf{r} - \mathbf{r}_i^C$ where

$\mathbf{r}$ = position vector of spacecraft relative to center of integration with rectangular components referred to the mean earth equator and equinox of 1950.0, i.e., the "1950.0" position vector

$\mathbf{r}_i^C$ = 1950.0 position vector of body i relative to the center of integration C

The transformation from these 1950.0 body-centered coordinates to body-fixed coordinates $\mathbf{r}_b$ is denoted as

$$\mathbf{r}_b = T^T(\mathbf{r} - \mathbf{r}_i^C) \tag{162}$$

The overall transformation from $(\mathbf{r} - \mathbf{r}_i^C)$ to $\mathbf{r}'$ is thus

$$\mathbf{r}' = RT^T(\mathbf{r} - \mathbf{r}_i^C) \equiv G(\mathbf{r} - \mathbf{r}_i^C) \tag{163}$$

The inverse transformation is

$$(\mathbf{r} - \mathbf{r}_i^C) = G^T\mathbf{r}' = TR^T\mathbf{r}' \tag{164}$$

Using $\mathbf{r}_b$ from Eq. (162), the sines and cosines of ϕ and λ and the angle λ are computed from

$$\sin\phi = \frac{z_b}{r} \tag{165}$$

$$\cos\phi = \frac{(x_b^2 + y_b^2)^{1/2}}{r} \tag{166}$$

$$\sin\lambda = \frac{y_b}{(x_b^2 + y_b^2)^{1/2}} \tag{167}$$

$$\cos\lambda = \frac{x_b}{(x_b^2 + y_b^2)^{1/2}} \tag{168}$$

The transformation T is currently specified in the DPODP for the earth, the moon, and Mars. These and most of the other coordinate transformations of the DPODP were specified by F. M. Sturms. The formulation for T for the earth is specified in Section VII. Sturms' formulation for T for the moon and for Mars are specified in JPL internal publications.[9,10] He has specified modifications to the existing transformations and specified transformations for the remaining planets and for the sun in another internal publication.[11] Sturms also plans to publish this formulation in a JPL Technical Report.

Let $\ddot{\mathbf{r}}'$ denote the inertial acceleration of the spacecraft due to the oblateness of any body with rectangular components along the instantaneous directions of the x', y', and z' axes. This acceleration can be broken down into $\ddot{\mathbf{r}}'(J)$ due to the zonal harmonics J_n and $\ddot{\mathbf{r}}'(C,S)$ due to the tesseral harmonics C_{nm} and S_{nm}. Given these terms, the direct acceleration of the spacecraft due to the oblateness of any body, with rectangular components referred to the mean earth equator and equinox of 1950.0, is given by

$$\ddot{\mathbf{r}} = G^T\ddot{\mathbf{r}}' = G^T\,[\ddot{\mathbf{r}}'(J) + \ddot{\mathbf{r}}'(C,S)] \tag{169}$$

The components of $\ddot{\mathbf{r}}'(J)$ and $\ddot{\mathbf{r}}'(C,S)$ are given by

$$\ddot{x}'(J) = \frac{\partial U(J)}{\partial r} \qquad (J) \rightarrow (C,S) \tag{170}$$

$$\ddot{y}'(J) = \frac{1}{r\cos\phi}\frac{\partial U(J)}{\partial\lambda} \qquad (J) \rightarrow (C,S) \tag{171}$$

$$\ddot{z}'(J) = \frac{1}{r}\frac{\partial U(J)}{\partial\phi} \qquad (J) \rightarrow (C,S) \tag{172}$$

Carrying out these differentiations gives

$$\ddot{\mathbf{r}}'(J) = \frac{\mu}{r^2}\sum_{n=1}^{n_1} J_n\left(\frac{a_p}{r}\right)^n \begin{bmatrix} (n+1)\,P_n \\ 0 \\ -\cos\phi\,P_n' \end{bmatrix} \tag{173}$$

$$\ddot{\mathbf{r}}'(C,S) = \frac{\mu}{r^2}\sum_{n=1}^{n_2}\sum_{m=1}^{n}\left(\frac{a_p}{r}\right)^n \times \begin{bmatrix} -(n+1)\,P_n^m\,\{C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\} \\ m\sec\phi\,P_n^m\,\{-C_{nm}\sin m\lambda + S_{nm}\cos m\lambda\} \\ \cos\phi\,P_n^{m\prime}\,\{C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\} \end{bmatrix} \tag{174}$$

where the primes indicate derivatives with respect to $\sin\phi$. Currently, n_1 has a maximum value of 15 and n_2 has a maximum value of 8. These limits will undoubtedly be increased in a future version of the program.

The Legendre polynomial P_n is computed recursively from (Ref. 44, p. 308, Eq. II)

$$P_n = \frac{2n-1}{n}\sin\phi\,P_{n-1} - \left(\frac{n-1}{n}\right)P_{n-2} \tag{175}$$

starting with

$$P_o = 1 \tag{176}$$

$$P_1 = \sin\phi \tag{177}$$

The derivative of P_n with respect to $\sin\phi$, denoted P_n', is given by (Ref. 44, p. 308, Eq. I)

$$P_n' = \sin\phi\,P_{n-1}' + n\,P_{n-1} \tag{178}$$

starting with

$$P_1' = 1 \tag{179}$$

The function $\sec\phi\,P_n^m$ is computed by first generating

$$\sec\phi\,P_n^n = (2n-1)\cos\phi\,(\sec\phi\,P_{n-1}^{n-1}) \tag{180}$$

starting with

$$\sec\phi\,P_1^1 = 1 \tag{181}$$

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[9] Warner, M. R., et al., *Double Precision Orbit Determination Program*, Vol. III, TRAJ Segment, EPD 426 (JPL Internal Report), June 15, 1967.

[10] Witt, J., *User's Guide for TRIC*, 900-168 (JPL Internal Report), Oct. 20, 1968.

[11] Sturms, F. M., *New Coordinate Transformations for DPTRAJ*, RFP 392-16 (JPL Internal Report), Dec. 16, 1969.

and continuing until $n = n_2$, and then generating

$$\sec\phi\, P_n^m = \left(\frac{2n-1}{n-m}\right)\sin\phi\,(\sec\phi\, P_{n-1}^m) - \left(\frac{n+m-1}{n-m}\right)(\sec\phi\, P_{n-2}^m) \qquad (182)$$

For each value of m between 1 and n_2, n is varied from $m+1$ to n_2. The general term P_a^b is zero if $b > a$. Equation (180) may be obtained by successive differentiation of Eq. (175) with respect to $\sin\phi$ and substitution into Eq. (155). Equation (182) was obtained from Ref. 45, p. 161, Eq. 12. The function P_n^m is obtained by multiplying $(\sec\phi\, P_n^m)$ by $\cos\phi$.

The function $\cos\phi\, P_n^{m\prime}$, where $P_n^{m\prime}$ is the derivative of P_n^m with respect to $\sin\phi$, is computed from (Ref. 45, p. 161, Eq. 19)

$$\cos\phi\, P_n^{m\prime} = -n\sin\phi\,(\sec\phi\, P_n^m) + (n+m)(\sec\phi\, P_{n-1}^m) \qquad (183)$$

3. ***Indirect acceleration of center of integration due to oblateness.*** As previously mentioned, the indirect oblateness acceleration of the spacecraft relative to the center of integration is the negative of the acceleration of the center of integration due to oblateness. It is computed only when the center of integration is the earth or moon and accounts for the oblateness of both of these bodies.

The force of attraction between the earth and moon consists of

(1) The force of attraction between the point-mass earth and point-mass moon.

(2) The force of attraction between the oblate part of the earth and the point-mass moon.

(3) The force of attraction between the oblate part of the moon and the point-mass earth.

(4) The force of attraction between the oblate part of the earth and the oblate part of the moon.

The force (1) is accounted for in Subsection V-B-1. The formulation of this section will account for the forces (2) and (3), but will ignore the force (4).

Let

$\ddot{\mathbf{r}}_M(E)$ = inertial acceleration of point-mass moon due to the oblateness of the earth

$\ddot{\mathbf{r}}_E(M)$ = inertial acceleration of point-mass earth due to the oblateness of the moon

These accelerations, with rectangular components referred to the mean earth equator and equinox of 1950.0, may be computed from the formulation of Subsection V-B-2. In the computation of $\ddot{\mathbf{r}}_M(E)$, the moon is treated as the spacecraft of Subsection V-B-2, and $\mathbf{r} - \mathbf{r}_i^c$ in Eq. (162) is replaced by $\mathbf{r}_M^E$. Similarly, in the computation of $\ddot{\mathbf{r}}_E(M)$, the earth is treated as the spacecraft and $\mathbf{r} - \mathbf{r}_i^c$ is replaced by $\mathbf{r}_E^M$.

Consider the force of attraction between the earth and moon due to the oblateness of the earth, assuming the moon to be a point mass. This force produces $\ddot{\mathbf{r}}_M(E)$ and also

$\ddot{\mathbf{r}}_E(E)$ = inertial acceleration of the earth due to the force of attraction between the oblate part of the earth and the point-mass moon

Since these two accelerations are derived from equal and opposite forces,

$$\ddot{\mathbf{r}}_E(E) = -\frac{\mu_M}{\mu_E}\ddot{\mathbf{r}}_M(E) \qquad (184)$$

Similarly, consider the force of attraction between the earth and moon due to the oblateness of the moon, considering the earth to be a point mass. This force produces $\ddot{\mathbf{r}}_E(M)$ and also

$\ddot{\mathbf{r}}_M(M)$ = inertial acceleration of the moon due to the force of attraction between the oblate part of the moon and the point-mass earth

Since these two accelerations are derived from equal and opposite forces,

$$\ddot{\mathbf{r}}_M(M) = -\frac{\mu_E}{\mu_M}\ddot{\mathbf{r}}_E(M) \qquad (185)$$

The acceleration of the earth due to the oblateness of the earth and moon is

$$\begin{aligned}\ddot{\mathbf{r}}_E &= \ddot{\mathbf{r}}_E(M) + \ddot{\mathbf{r}}_E(E)\\ &= \ddot{\mathbf{r}}_E(M) - \frac{\mu_M}{\mu_E}\ddot{\mathbf{r}}_M(E) \qquad (186)\end{aligned}$$

Similarly,

$$\begin{aligned}\ddot{\mathbf{r}}_M &= \ddot{\mathbf{r}}_M(E) + \ddot{\mathbf{r}}_M(M)\\ &= \ddot{\mathbf{r}}_M(E) - \frac{\mu_E}{\mu_M}\ddot{\mathbf{r}}_E(M) \qquad (187)\end{aligned}$$

Note that $\ddot{\mathbf{r}}_E(M)$ is proportional to μ_M and $\ddot{\mathbf{r}}_M(E)$ is proportional to μ_E. The contribution to the spacecraft acceleration relative to the center of integration is the negative of the acceleration of the center of integration, or

$$\ddot{\mathbf{r}} = \pm \mu_i \left[\frac{1}{\mu_E}\ddot{\mathbf{r}}_M(E) - \frac{1}{\mu_M}\ddot{\mathbf{r}}_E(M)\right] \tag{188}$$

where

If earth = center of integration, $\pm\mu_i = +\mu_M$

If moon = center of integration, $\pm\mu_i = -\mu_E$

Sturms' algorithm for computation of this acceleration accounts for J_2, C_{22}, and S_{22} of the earth and moon. Equation (188), evaluated with these harmonic coefficients, is equivalent to Sturms' formulation. An earlier version of his formulation, which is based upon the principal moments of inertia A, B, and C for the earth and moon, is given in Ref. 46.

*4. **Acceleration of spacecraft due to solar radiation pressure and small forces originating in spacecraft.*** This section gives the model for representing the acceleration of the spacecraft due to solar radiation pressure and to small forces originating in the spacecraft, such as those from operation of the attitude control system (particularly if it uses uncoupled attitude control jets) and from gas leaks. The model applies to any spacecraft which has one axis (the roll axis) continuously oriented toward the sun and utilizes a star or planet tracker to orient the spacecraft about the roll axis. The various *Mariner* spacecraft are of this type.

The solar radiation pressure model accounts for the acceleration of the spacecraft due to solar radiation pressure acting along three mutually perpendicular spacecraft axes, one of which is the roll axis. Normally, the solar panels are oriented normal to the roll axis so that the largest component of the force due to solar radiation pressure is along the roll axis. However, the model can also account for the small forces acting along the other two spacecraft axes and arising from departures of the spacecraft shape from rotational symmetry about the roll axis.

The small force model accounts in a crude fashion for the acceleration arising from small forces originating in the spacecraft. The component of this acceleration along each spacecraft axis is represented as a quadratic. This model is currently being expanded to allow this acceleration to be represented alternatively as an exponential decay with components along each spacecraft axis.

The acceleration of the spacecraft due to solar radiation pressure and small forces originating in the spacecraft is represented by

$$\begin{aligned}\ddot{\mathbf{r}} = &\Big\{[a_r + b_r(t - T_{AC1}) + c_r(t - T_{AC1})^2]\,[u(t - T_{AC1}) - u(t - T_{AC2})] \\ &+ \Delta a_r + \frac{C_1A_p}{mr_{SP}^2}[G_r + G'_r(EPS) + \Delta G_r]\,u^*(t - T_{SRP})\Big\}\,\mathbf{U}_{SP} \\ &+ \Big\{[a_x + b_x(t - T_{AC1}) + c_x(t - T_{AC1})^2]\,[u(t - T_{AC1}) - u(t - T_{AC2})] \\ &+ \Delta a_x + \frac{C_1A_p}{mr_{SP}^2}[G_x + G'_x(EPS) + \Delta G_x]\,u^*(t - T_{SRP})\Big\}\,\mathbf{X}^* \\ &+ \Big\{[a_y + b_y(t - T_{AC1}) + c_y(t - T_{AC1})^2]\,[u(t - T_{AC1}) - u(t - T_{AC2})] \\ &+ \Delta a_y + \frac{C_1A_p}{mr_{SP}^2}[G_y + G'_y(EPS) + \Delta G_y]\,u^*(t - T_{SRP})\Big\}\,\mathbf{Y}^*\end{aligned} \tag{189}$$

The terms in this equation are defined as

$\mathbf{U}_{SP}$ = unit vector from sun to spacecraft

$\mathbf{X}^*, \mathbf{Y}^*$ = unit vectors along spacecraft x and y axes ($\mathbf{X}^* \times \mathbf{Y}^* = \mathbf{U}_{SP}$) (defined below)

a_i, b_i, c_i where $i = r, x$, or y = solve-for coefficients of acceleration polynomials, km/s^2, km/s^3, km/s^4

t = ephemeris time

T_{AC1}, T_{AC2} = epochs at which the acceleration polynomials are turned on and off, respectively. The epochs may be specified in the UTC, ST, or A1 time scales. They must be transformed to ET for use in Eq. (189). The transformation will be different for each iteration of the orbit determination process if the values of ΔT_{1958} or Δf_{cesium} are estimated.

$u(t - T_{AC1})$ = 1 for $t \geqq T_{AC1}$, 0 for $t < T_{AC1}$
$T_{AC1} \rightarrow T_{AC2}$

$\Delta a_r, \Delta a_x, \Delta a_y$ = input acceleration (not solve-for), km/s^2. The value for each Δa_i is obtained by linear interpolation between input points specified in any time scale. The acceleration is started at the epoch of the first point and ended at the epoch of the last point.

$$C_1 = \frac{JA_E^2}{c} \times \frac{1\,\text{km}^2}{10^6 m^2} = 1.010 \times 10^8 \frac{\text{km}^3\text{kg}}{\text{s}^2 m^2}$$

where

J = solar radiation constant

= 1.3525×10^3 W/m^2 (Ref. 47)[12]

= 1.3525×10^3 kg/s^3

$A_E = 1.496 \times 10^8$ km

$c = 2.997925 \times 10^5$ km/s

A_p = nominal area of spacecraft projected onto plane normal to sun–spacecraft line, m^2

m = instantaneous mass of spacecraft, kg

r_{SP} = distance from sun to spacecraft, km

T_{SRP} = epoch at which acceleration due to solar radiation pressure is turned on (epoch of solar panel unfolding). The epoch may be specified in the UTC, ST, or A1 time scales and must be transformed to ET for use in Eq. (189).

$u^*(t - T_{SRP})$ = 1 for $t \geqq T_{SRP}$ if spacecraft in sunlight, 0 for $t < T_{SRP}$ or if spacecraft in shadow of a planet or the moon

G_r = solve-for effective area for acceleration of spacecraft in radial direction due to solar radiation pressure, divided by nominal area A_p

G_x = solve-for effective area for acceleration of spacecraft in the direction of its positive x axis (along $\mathbf{X}^*$ vector) divided by A_p

G_y = solve-for effective area for acceleration of spacecraft in the direction of its positive y axis (along $\mathbf{Y}^*$ vector) divided by A_p

G'_r, G'_x, G'_y = solve-for derivatives of G_r, G_x, G_y with respect to earth–spacecraft–sun angle, *EPS*

EPS = earth–spacecraft–sun angle, rad

$\Delta G_r, \Delta G_x, \Delta G_y$ = increments to G_r, G_x, and G_y obtained by linear interpolation of input points specified in any time scale. The value of ΔG_i is computed at each integration step contained between the epoch of the first point and the epoch of the last point.

The term $G'_i(EPS)$ along each spacecraft axis was included so that the model would be compatible with the *Mariner* II spacecraft, which contained a high-gain antenna that moved continuously with respect to the spacecraft axes and always pointed toward the earth. These terms account for the variation in G_r, G_x, and G_y due to this moving antenna.

The *Mariner* IV spacecraft contained movable attitude control vanes situated at the end of each solar panel. Movement of these vanes caused G_r, G_x, and G_y to fluctuate with time. The ΔG_i terms account for these fluctuations.

The unit sun–spacecraft vector $\mathbf{U}_{SP}$ is computed from

$$\mathbf{U}_{SP} = \frac{\mathbf{r} - \mathbf{r}_S^C}{\|\mathbf{r} - \mathbf{r}_S^C\|} \tag{190}$$

where

$\mathbf{r}$ = position vector of spacecraft relative to center of integration with rectangular components referred to the mean earth equator and equinox of 1950.0

$\mathbf{r}_S^C$ = 1950.0 position vector of sun relative to center of integration C

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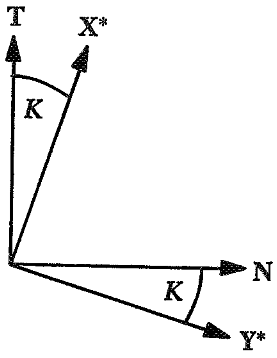
[12] On July 20, 1970, the author of Ref. 47 stated that a more accurate reduction of the data gave a value of 1.348×10^3 W/m^2.

The spacecraft X* and Y* unit vectors are obtained as a rotation of the tangential T and normal N vectors through the angle K:



$$\begin{bmatrix} \mathbf{X}^* \\ \mathbf{Y}^* \end{bmatrix} = \begin{bmatrix} \cos K & \sin K \\ -\sin K & \cos K \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \end{bmatrix} \tag{191}$$

The angle K is an input (non-solve-for) constant. Computation of the unit vectors T and N requires the unit vector $\mathbf{U}_R$

$\mathbf{U}_R$ = unit vector from spacecraft to reference body which orients the spacecraft about the roll axis (sun–spacecraft line). The reference body may be a star, a planet, or the moon.

If the reference body is a star,

$$\mathbf{U}_R = \begin{bmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{bmatrix} \tag{192}$$

where the right ascension α and declination δ of the star are referred to the mean earth equator and equinox of 1950.0. If the reference body B is a planet or the moon (normally the earth),

$$\mathbf{U}_R = \frac{\mathbf{r}_B^C - \mathbf{r}}{\| \mathbf{r}_B^C - \mathbf{r} \|} \tag{193}$$

where

$\mathbf{r}_B^C$ = 1950.0 position vector of reference body B relative to center of integration C

The unit normal vector N (normal to sun–spacecraft–reference body plane) is computed from

$$\mathbf{N} = \frac{\mathbf{U}_R \times \mathbf{U}_{SP}}{\| \mathbf{U}_R \times \mathbf{U}_{SP} \|} \tag{194}$$

The unit tangential vector T (tangent to sun–spacecraft–reference body plane) is

$$\mathbf{T} = \mathbf{N} \times \mathbf{U}_{SP} \tag{195}$$

Given T and N, the vectors X* and Y* are given by Eqs. (191). The angle K may be selected to achieve a specific orientation of X* and Y* relative to the spacecraft.

The EPS angle may be computed from

$$\cos \mathrm{EPS} = -\mathbf{U}_{SP} \cdot \mathbf{U}_R' \qquad 0 < EPS < 180^\circ \tag{196}$$

where

$\mathbf{U}_R'$ is computed from Eq. (193) using B = earth.

5. ***Acceleration due to motor burn.*** The acceleration of the spacecraft due to a motor burn is represented by

$$\ddot{\mathbf{r}} = a\mathbf{U}\,[u(t - T_o) - u(t - T_f)] \qquad \mathrm{km/s^2} \tag{197}$$

where

a = magnitude of $\ddot{\mathbf{r}}$

$\mathbf{U}$ = unit vector in direction of $\ddot{\mathbf{r}}$

T_o = effective start time of motor, the ET value of the solve-for epoch, which may be specified in the UTC, ST, or A1 time scales

T_f = effective stop time of motor, ET

t = ephemeris time

$$u(t - T_o) = \begin{cases} 1 \text{ for } t \geqq T_o \\ 0 \text{ for } t < T_o \end{cases} \qquad T_o \rightarrow T_f$$

The effective stop time T_f is given by

$$T_f = T_o + T \tag{198}$$

where

T = solve-for burn time of motor, ET seconds

The acceleration magnitude a is given by

$$a = \frac{F(t)}{m(t)} C = \frac{F_o + F_1\bar{t} + F_2\bar{t}^2 + F_3\bar{t}^3 + F_4\bar{t}^4}{m_o - \dot{M}_o\bar{t} - \frac{1}{2}\dot{M}_1\bar{t}^2 - \frac{1}{3}\dot{M}_2\bar{t}^3 - \frac{1}{4}\dot{M}_3\bar{t}^4} C \tag{199}$$

where

$F(t)$ = magnitude of thrust at time t. The polynomial coefficients of $F(t)$ are solve-for parameters

$\bar{t} = t - T_o$, seconds

$m(t)$ = spacecraft mass at time t

m_o = spacecraft mass at T_o

$\dot{M}_o, \dot{M}_1, \dot{M}_2, \dot{M}_3$ = polynomial coefficients of propellant mass flow rate (positive) at time t: $\dot{M}(t) = \dot{M}_o + \dot{M}_1\bar{t} + \dot{M}_2\bar{t}^2 + \dot{M}_3\bar{t}^3$ (not solve-for parameters)

C = 0.001 for F in newtons and m in kg. For F in lb and m in lbm, $C = 0.00980665$

The unit vector $\mathbf{U}$ in the direction of thrust is given by

$$\mathbf{U} = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix} \tag{200}$$

where

α, δ = right ascension and declination, respectively, of $\mathbf{U}$, referred to the mean earth equator and equinox of 1950.0

given by

$$\alpha = \alpha_o + \alpha_1\bar{t} + \alpha_2\bar{t}^2 + \alpha_3\bar{t}^3 + \alpha_4\bar{t}^4 \tag{201}$$

$$\delta = \delta_o + \delta_1\bar{t} + \delta_2\bar{t}^2 + \delta_3\bar{t}^3 + \delta_4\bar{t}^4 \tag{202}$$

where the polynomial coefficients of Eqs. (201) and (202) are solve-for parameters.

VI. Light Time Solution

This section gives the formulation and procedure for solution of the light time problem, which is the first step in the computation of all observable quantities.

A. Introduction

An electromagnetic signal is transmitted from a tracking station on earth at time t_1. This signal is received by the spacecraft (either a free spacecraft or a landed spacecraft on the moon or on one of the planets) and retransmitted at time t_2, arriving at the same or a different tracking station on earth at time t_3. Alternatively, the signal may be transmitted directly by the spacecraft at time t_2. All observables are related to characteristics of this electromagnetic radiation, i.e., the angle of the incoming ray, the ratio of received to transmitted frequency, or the round-trip transit time. The transmitting station, the spacecraft, and the receiving station are referred to as direct participants, and t_1, t_2, and t_3, respectively, are their epochs of participation. The solution of the light time problem consists of these epochs of participation and the heliocentric position, velocity, acceleration, and jerk of each direct participant evaluated at its epoch of participation. The rectangular components of these vectors are referred to the mean earth equator and equinox of 1950.0. Sections VIII–XI give the formulations for computing doppler, range, and angular observables, starting with the solution to the light time problem.

The solution to the light time problem is obtained by solving the light time equation for each leg of the path of electromagnetic radiation from the transmitting to the receiving station. The light time equation relates the light time between two points to the heliocentric positions of each of the two participants evaluated at their epochs of participation. Starting with the known reception time t_3, the light time equation is solved by an iterative technique for the down leg of the light path to give the epoch of participation for the spacecraft, t_2. Given t_2, the light time equation is solved iteratively for the up leg of the light path to give the transmission time t_1.

Section VI-B gives the formulation for solution of the light time problem; the detailed procedure is given in Section VI-C.

B. Formulation

Let the subscripts i or j equal 1, 2, or 3 where

1 refers to the transmitting station on earth at the transmission time t_1

2 refers to the spacecraft (free or landed) at the reflection time t_2

3 refers to the receiving station on earth at the reception time t_3

The time for light to travel from point i at ephemeris time (coordinate time) t_i to point j at ephemeris time t_j is given by Eq. (88), repeated here:

$$t_j - t_i = \frac{r_{ij}}{c} + \frac{(1+\gamma)\,\mu_S}{c^3}\ln\left(\frac{r_i + r_j + r_{ij}}{r_i + r_j - r_{ij}}\right) \tag{203}$$

where

$$r_{ij} = \|\mathbf{r}_{ij}\| = \|\mathbf{r}_j^S(t_j) - \mathbf{r}_i^S(t_i)\|$$

$$r_i = \|\mathbf{r}_i^S(t_i)\|$$

$$r_j = \|\mathbf{r}_j^S(t_j)\|$$

$\mathbf{r}_i^S(t_i), \mathbf{r}_j^S(t_j)$ = heliocentric position vector of point i at transmission time t_i and point j at reception time t_j, with rectangular components referred to the mean earth equator and equinox of 1950.0

c = speed of light, km/s

μ_S = gravitational constant of sun, km^3/s^2

γ = solve-for free parameter of Brans–Dicke theory of relativity. The parameter γ is related to ω, the coupling constant of the scalar field, through Eq. (41).

Equation (203), which is referred to as the light time equation, relates the light time in ephemeris time for a given leg of the light path to the heliocentric position vectors of the two participants evaluated at their epochs of participation. The light time equation applies to the down leg of the light path when $i = 2$ and $j = 3$; when $i = 1$ and $j = 2$, it applies to the up leg.

Let

$\mathbf{r}_j^i = \mathbf{r}_{ij}$ = position vector of point j relative to point i, with rectangular components referred to the mean earth equator and equinox of 1950.0.

With this notation, the heliocentric position vectors of the transmitter, spacecraft, and receiver at their epochs of participation are computed from the following equations. For the transmitter,

$$\mathbf{r}_1^S(t_1) = \mathbf{r}_1^E\,[t_1(\mathrm{UT1}), t_1(\mathrm{ET})] + \mathbf{r}_E^S[t_1(\mathrm{ET})] \tag{204}$$

where S = sun and E = earth. Similarly, for the receiver,

$$\mathbf{r}_3^S(t_3) = \mathbf{r}_3^E\,[t_3(\mathrm{UT1}), t_3(\mathrm{ET})] + \mathbf{r}_E^S[t_3(\mathrm{ET})] \tag{205}$$

For a free spacecraft S/C, with center of integration C,

$$\mathbf{r}_2^S(t_2) = \mathbf{r}_{S/C}^C\,[t_2(\mathrm{ET})] + \mathbf{r}_C^S[t_2(\mathrm{ET})] \tag{206}$$

For a landed spacecraft on body B,

$$\mathbf{r}_2^S(t_2) = \mathbf{r}_2^B\,[t_2(\mathrm{ET})] + \mathbf{r}_B^S[t_2(\mathrm{ET})] \tag{207}$$

Each of these 1950.0 vector sums applies with $\mathbf{r}$ replaced by $\dot{\mathbf{r}}$, $\ddot{\mathbf{r}}$, and $\dddot{\mathbf{r}}$. The heliocentric position, velocity, acceleration, and jerk of the earth, as well as the center of integration or the body upon which the spacecraft has landed are obtained as indicated in Section IV. The position, velocity, acceleration and jerk of the spacecraft relative to the center of integration are obtained by interpolation of the spacecraft ephemeris sum and difference arrays. The formulation for computing the 1950.0 position, velocity, acceleration, and jerk of a tracking station relative to the earth or of a landed spacecraft relative to the body B on which it is located is given in Section VII. The geocentric 1950.0 position and higher derivatives for a tracking station are primarily functions of the UT1 value of the epoch, although the ET value is also required.

Solution of the light time equation (Eq. 203) for a given leg of the light path gives the transmission time t_i for that leg. The time t_i is used to compute $\mathbf{r}_i^S(t_i)$ in the evaluation of the right-hand side of the light time equation and also appears explicitly in the left-hand side. The light time equation must be solved for t_i by an iterative technique. The DPODP uses the Newton–Raphson method. Let the function f whose value is to be minimized be the left-hand side of the light time equation minus the right-hand side:

$$f = t_j - t_i - \frac{r_{ij}}{c} - \frac{(1+\gamma)\mu_S}{c^3}\ln\left(\frac{r_i + r_j + r_{ij}}{r_i + r_j - r_{ij}}\right) \tag{208}$$

When the relativity term is ignored, the partial derivative of f with respect to t_i is

$$\frac{\partial f}{\partial t_i} = -1 + \frac{1}{c}\frac{\mathbf{r}_{ij}}{r_{ij}} \cdot \dot{\mathbf{r}}_i^S(t_i) \tag{209}$$

Let $\Delta(t_i)$ equal the linear differential correction to the estimate of t_i. Then

$$\frac{\partial f}{\partial t_i}\Delta(t_i) = -f \tag{210}$$

Substituting Eqs. (208) and (209) into Eq. (210) gives

$$\Delta(t_i) = \frac{t_j - t_i - \dfrac{r_{ij}}{c} - \dfrac{(1+\gamma)\mu_S}{c^3}\ln\left(\dfrac{r_i + r_j + r_{ij}}{r_i + r_j - r_{ij}}\right)}{1 - \dfrac{1}{c}\dfrac{\mathbf{r}_{ij}}{r_{ij}} \cdot \dot{\mathbf{r}}_i^S(t_i)} \tag{211}$$

The procedure for using this iterative formula for obtaining the transmission time t_2 for the down leg and the transmission time t_1 for the up leg is given in the following section.

C. Procedure

The procedure is as follows:

(1) Convert the observation time t_3 (ST) to t_3 (UTC), t_3 (A1), t_3 (UT1), and t_3 (ET) using the time transformations of Section III. Compute $\mathbf{r}_3^S(t_3)$ from Eq. (205). Compute also $\dot{\mathbf{r}}_3^S(t_3), \ddot{\mathbf{r}}_3^S(t_3), \dddot{\mathbf{r}}_3^S(t_3)$.

(2) Obtain the first estimate for t_2 (ET) as:

  (a) For the first observation of the spacecraft on a pass of the spacecraft relative to the receiving station, $t_2 = t_3$.

  (b) For the remaining observations of the pass, $t_2 = t_3$ minus the converged light time for the down leg of the previous observation.

(3) Given the estimate for t_2 (ET), compute $\mathbf{r}_2^S(t_2), \dot{\mathbf{r}}_2^S(t_2)$, $\ddot{\mathbf{r}}_2^S(t_2)$, and $\dddot{\mathbf{r}}_2^S(t_2)$ from Eq. (206) or (207) and $\Delta(t_2)$ from Eq. (211). The next estimate for t_2 is $t_2 + \Delta(t_2)$. Repeat step 3 until $\Delta(t_2) < 10^{-7}$ s. (On the IBM 7094 computer, time is represented as double-precision seconds past January 1, 1950, 0^h to a precision of 0.6×10^{-7} s from 1967 to 1984.)

(4) Obtain the first estimate for t_1 (ET) as t_2 minus the converged light time for the down leg of the current observable.

(5) Convert the estimate for t_1 (ET) to t_1 (A1), t_1 (UTC), t_1 (UT1), and t_1 (ST). Compute $\mathbf{r}_1^S(t_1)$, $\dot{\mathbf{r}}_1^S(t_1)$, $\ddot{\mathbf{r}}_1^S(t_1)$, and $\dddot{\mathbf{r}}_1^S(t_1)$ from Eq. (204) and $\Delta(t_1)$ from Eq. (211). The next estimate for t_1 is $t_1 + \Delta(t_1)$. Repeat step 5 until $\Delta(t_1) < 10^{-7}$ s.

Most of the intermediate quantities used in the computation of the heliocentric position, velocity, acceleration, and jerk of each participant at its epoch of participation are saved and used in the computation of the observable and the partial derivatives of the observable with respect to the estimated parameters.

VII. Body-Centered 1950.0 Position, Velocity, Acceleration, and Jerk of Tracking Station and Landed Spacecraft

A. Introduction

This section gives the formulation for computation of the position, velocity, acceleration, and jerk of a tracking station relative to the center of the earth or of a landed spacecraft relative to the center of the body on which it is located, with rectangular components referred to the mean earth equator and equinox of 1950.0. In addition to a fixed tracking station, a model is included for representing the motion of a tracking ship.

The first step in the computation of 1950.0 position, velocity, acceleration, and jerk is to obtain the "body-fixed" position $\mathbf{r}_b$ (and also velocity, acceleration, and jerk in the case of a tracking ship), where x_b is along the intersection of the prime meridian (passing through the instantaneous axis of rotation) and the instantaneous equator, where z_b is along the instantaneous axis of rotation, directed north, and where y_b completes the right-handed rectangular coordinate system.

Given $\mathbf{r}_b$ (and higher derivatives for a tracking ship), the 1950.0 position, velocity, acceleration, and jerk are obtained from the transformation matrix T (which relates these two coordinate systems) and from $\dot{T}$, $\ddot{T}$, and $\dddot{T}$. As mentioned in Section V, these transformations are currently specified for the earth, the moon, and Mars. The transformations for the remaining planets and for the sun have been specified by F. M. Sturms and will be added to the program in the near future.

The location of a fixed tracking station on earth is specified by its spherical or cylindrical coordinates relative to the mean pole, equator, and prime meridian of 1903.0. These station coordinates are solve-for parameters. Because the pole (axis of rotation) wanders relative to the earth, the "body-fixed" coordinate system moves relative to the earth and the "body-fixed" position $\mathbf{r}_b$ of a fixed tracking station on earth is a variable quantity. It is computed from the time-varying coordinates of the true pole of date relative to the mean pole of 1903.0 supplied by the B.I.H.[13] The location of a landed spacecraft on a planet or the moon is specified by constant spherical or cylindrical coordinates (solve-for parameters) relative to the body-fixed coordinate system. The body-fixed position of a tracking ship is specified by its spherical coordinates

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[13] Bureau International de l'Heure.

at an arbitrary epoch, and by its azimuth and velocity; the values of these five parameters may be estimated. The value of the geocentric radius to the ship is constant.

Section VII-B gives the formulation for computing body-fixed position (and higher derivatives for a tracking ship). Section VII-C gives the general formulation for transforming these quantities to 1950.0 position, velocity, acceleration, and jerk using the transformation matrices T, $\dot{T}$, $\ddot{T}$, and $\dddot{T}$. These matrices are specified for the earth in Section VII-D.

B. Body-Fixed Rectangular Coordinates

1. Fixed tracking station or landed spacecraft. For a tracking station on earth or a landed spacecraft on the moon or a planet, the spherical coordinates referred to the $x_b y_b z_b$ "body-fixed" coordinate system are

r = radius from center of body, km

ϕ = body-centered latitude measured from true equator (plane normal to instantaneous axis of rotation and containing center of mass)

λ = longitude measured east from prime meridian (passing through instantaneous axis of rotation)

The cylindrical coordinates are

u = distance from spin axis (instantaneous axis of rotation), km

$= r \cos \phi$

v = height above true equator, km

$= r \sin \phi$

λ = longitude measured east from prime meridian (passing through instantaneous axis of rotation)

For spherical coordinates, the body-fixed rectangular coordinates are

$$\mathbf{r}_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} r \cos \phi \cos \lambda \\ r \cos \phi \sin \lambda \\ r \sin \phi \end{bmatrix} \tag{212}$$

For cylindrical coordinates,

$$\mathbf{r}_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} u \cos \lambda \\ u \sin \lambda \\ v \end{bmatrix} \tag{213}$$

For a landed spacecraft on a planet or the moon, the spherical or cylindrical coordinates are constant and are solve-for parameters. For a tracking station on earth, the solve-for parameters are the spherical or cylindrical coordinates relative to the mean pole, equator, and prime meridian of 1903.0. The spherical coordinates are denoted by r, ϕ_0, and λ_0; the cylindrical coordinates are denoted by u_0, v_0, and λ_0. The transformations from these 1903.0 coordinates to those referred to the "body-fixed" coordinate system are

$$\phi = \phi_0 + \Delta\phi \tag{214}$$

$$\lambda = \lambda_0 + \Delta\lambda \tag{215}$$

$$u = u_0 + \Delta u \tag{216}$$

$$v = v_0 + \Delta v \tag{217}$$

The formulas for computing the corrections $\Delta\phi$, $\Delta\lambda$, Δu, and Δv are derived below. Given the body-fixed spherical or cylindrical coordinates, the rectangular components of $\mathbf{r}_b$ are computed from Eq. (212) or (213).

Figure 4 shows the latitude ϕ_0 and longitude λ_0 of a tracking station S relative to the mean pole of 1903.0 (P_0), and the instantaneous latitude ϕ and longitude λ relative to the true pole of date (P). The pole P_0 and associated grid of equator and meridians is rotated through the angle ω carrying P_0 to P. The angular coordinates of P

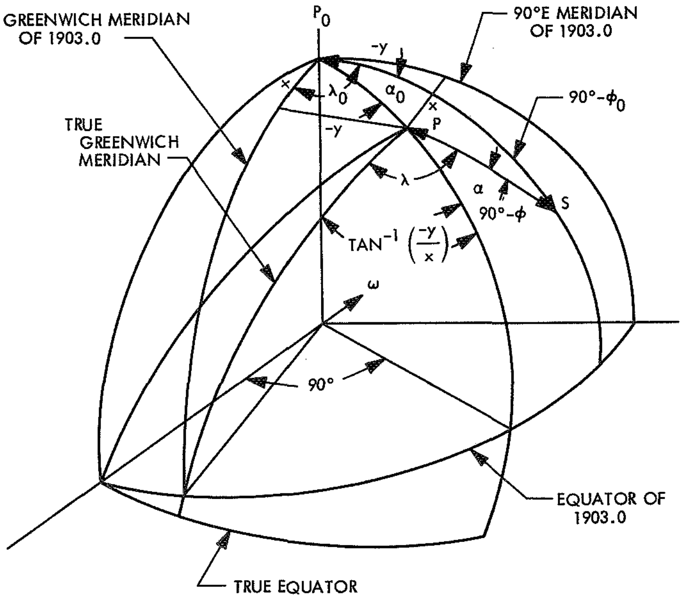

Fig. 4. Latitude and longitude relative to mean pole of 1903.0 and true pole of date

relative to P_0 are x measured south along the Greenwich meridian of 1903.0 (strictly the 1903.0 meridian of zero longitude) and y measured south along the 90°W meridian of 1903.0. Values of x and y are obtained from the B.I.H. They are represented by linear polynomials:

$$x = l + mt \tag{218}$$

$$y = p + qt \tag{219}$$

The coefficients l, m, p, and q are specified by time block, usually of one month's duration, and t is in seconds past the start of the time block. Since the angles x and y correspond to a displacement along the earth's surface of only a few meters (to date the maximum value has been about 10 m), an approximate expression for $\Delta\phi = \phi - \phi_0$ is

$$\Delta\phi = x \cos\lambda_0 - y \sin\lambda_0 \tag{220}$$

Noting α_0 and α on Fig. 4, one obtains

$$\lambda_0 = \alpha_0 + \tan^{-1}\left(\frac{-y}{x}\right) \tag{221}$$

$$\lambda = \alpha + \tan^{-1}\left(\frac{-y}{x}\right) \tag{222}$$

Thus,

$$\Delta\lambda = \lambda - \lambda_0 = \alpha - \alpha_0 \tag{223}$$

From the spherical triangle $P P_0 S$,

$$\frac{\sin\alpha_0}{\cos\phi} = \frac{\sin\alpha}{\cos\phi_0} \tag{224}$$

Cross multiplying and using Eqs. (223) and (214) gives

$$\sin\alpha_0 \cos\phi_0 = \sin(\alpha_0 + \Delta\lambda)\cos(\phi_0 + \Delta\phi) \tag{225}$$

Expanding, noting that $\Delta\lambda$ and $\Delta\phi$ are very small angles, and ignoring the higher-order term containing $\Delta\lambda\,\Delta\phi$ gives

$$\Delta\lambda = \tan\alpha_0 \tan\phi_0 \,\Delta\phi \tag{226}$$

From Eq. (221),

$$\tan\alpha_0 = \frac{\tan\lambda_0 + \dfrac{y}{x}}{1 - \dfrac{y}{x}\tan\lambda_0} \tag{227}$$

Substituting Eqs. (220) and (227) into Eq. (226) gives

$$\Delta\lambda = \tan\phi_0 (x \sin\lambda_0 + y \cos\lambda_0) \tag{228}$$

The cylindrical coordinates relative to the pole of 1903.0 and the true pole of date are

$$u_0 = r\cos\phi_0 \qquad u = r\cos\phi = r\cos(\phi_0 + \Delta\phi) \tag{229}$$

$$v_0 = r\sin\phi_0 \qquad v = r\sin\phi = r\sin(\phi_0 + \Delta\phi) \tag{230}$$

Solving for $\Delta u = u - u_0$ and $\Delta v = v - v_0$ gives

$$\Delta u = -v_0\,\Delta\phi \tag{231}$$

$$\Delta v = u_0\,\Delta\phi \tag{232}$$

where $\Delta\phi$ is given by Eq. (220). Using cylindrical coordinates, $\Delta\lambda$ is computed from

$$\Delta\lambda = \frac{v_0}{u_0}(x \sin\lambda_0 + y \cos\lambda_0) \tag{233}$$

The "body-fixed" position $\mathbf{r}_b$ of a fixed tracking station on earth varies with the motion of the pole, and hence the body-fixed velocity $\dot{\mathbf{r}}_b$ is non-zero. However, its maximum magnitude is about 2×10^{-6} m/s, which is less than the desired accuracy of 10^{-5} m/s for computed doppler observables. Hence $\dot{\mathbf{r}}_b$ is taken to be zero.

For a description of the wandering of the earth's axis of rotation, see Ref. 48.

2. Moving tracking ship. The ship is assumed to move on a sphere of radius r at constant azimuth A measured east of north, and at constant speed v. The ship passes through the point with latitude ϕ_0 and longitude λ_0 at time t_0 (UTC). All quantities are referenced to the $x_b\,y_b\,z_b$ body-fixed coordinate system defined in Section VII-A. The parameters r, ϕ_0, λ_0, v, and A are solve-for parameters.

The velocity along the meridian is given by

$$r\dot{\phi} = v\cos A \tag{234}$$

Thus the latitude may be expressed as

$$\phi = \phi_0 + \frac{v\cos A}{r}\left[t\,(\mathrm{UTC}) - t_0\,(\mathrm{UTC})\right] \tag{235}$$

The velocity normal to the meridian is given by

$$r\cos\phi\dot{\lambda} = v\sin A \tag{236}$$

Equation (236) can be integrated by replacing dt in the integral of $\dot{\lambda}\,dt$ by $rd\phi/v\cos A$ from Eq. (234). The result is

$$\lambda = \lambda_0 + \tan A \ln\left[\frac{\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\phi_0}{2}\right)}\right] \qquad A \neq 90\ \text{deg}, 270\ \text{deg} \tag{237}$$

This expression is indeterminate for $A = 90$ or 270 deg. For these cases, compute

$$\lambda = \lambda_0 \pm \frac{v}{r\cos\phi_0}\left[t\,(\mathrm{UTC}) - t_0\,(\mathrm{UTC})\right] \qquad \begin{matrix} + \ \text{for}\ A = 90\ \text{deg} \\ - \ \text{for}\ A = 270\ \text{deg} \end{matrix} \tag{238}$$

Given ϕ from Eq. (235) and λ from Eq. (237) or (238), $\mathbf{r}_b$ is given by Eq. (212), repeated here:

$$\mathbf{r}_b = \begin{bmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{bmatrix} r \tag{239}$$

Differentiation with respect to time using Eqs. (234) and (236) gives

$$\dot{\mathbf{r}}_b = \begin{bmatrix} -\cos A\sin\phi\cos\lambda - \sin A\sin\lambda \\ -\cos A\sin\phi\sin\lambda + \sin A\cos\lambda \\ \cos A\cos\phi \end{bmatrix} v \tag{240}$$

Similarly, differentiation of this equation gives

$$\ddot{\mathbf{r}}_b = \begin{bmatrix} -\left(\cos^2 A\cos\phi + \dfrac{\sin^2 A}{\cos\phi}\right)\cos\lambda + (\sin A\cos A\tan\phi)\sin\lambda \\ -\left(\cos^2 A\cos\phi + \dfrac{\sin^2 A}{\cos\phi}\right)\sin\lambda - (\sin A\cos A\tan\phi)\cos\lambda \\ -\cos^2 A\sin\phi \end{bmatrix} \frac{v^2}{r} \tag{241}$$

Equation (241) would be simpler if the tracking ship were moving along a great circle (at varying azimuth A). The transformation from body-fixed position, velocity, and acceleration to 1950.0 position, velocity, acceleration, and jerk is given in the next section. The body-fixed jerk ($\dddot{\mathbf{r}}_b$) is ignored since its maximum contribution of about 10^{-5} m/s to computed doppler is considerably smaller than the accuracy of tracking-ship data.

C. Transformation of Body-Fixed Rectangular Coordinates to 1950.0 Position, Velocity, Acceleration, and Jerk

Let the 1950.0 position, velocity, acceleration, and jerk of a fixed tracking station, a moving tracking ship, or a landed spacecraft relative to the center of the body i on which it is located be denoted by $\mathbf{r}_{50}$, $\dot{\mathbf{r}}_{50}$, $\ddot{\mathbf{r}}_{50}$, and $\dddot{\mathbf{r}}_{50}$. The transformation from the body-fixed position vector $\mathbf{r}_b$ to the 1950.0 position vector $\mathbf{r}_{50}$ is given by

$$\mathbf{r}_{50} = T_i\mathbf{r}_b \tag{242}$$

where T_i is the 3×3 transformation matrix for the body i in question.

For a fixed tracking station on earth or a landed spacecraft on a planet or the moon, $\dot{\mathbf{r}}_b$ is negligibly small and is taken to be zero. Thus,

$$\dot{\mathbf{r}}_{50} = \dot{T}_i\mathbf{r}_b \tag{243}$$

$$\ddot{\mathbf{r}}_{50} = \ddot{T}_i\mathbf{r}_b \tag{244}$$

$$\dddot{\mathbf{r}}_{50} = \dddot{T}_i \mathbf{r}_b \tag{245}$$

For a moving tracking ship, $\dot{\mathbf{r}}_b$ and $\ddot{\mathbf{r}}_b$ are nonzero and $\dddot{\mathbf{r}}_b$ is ignored. Thus,

$$\mathbf{r}_{50} = T_E \mathbf{r}_b \tag{246}$$

$$\dot{\mathbf{r}}_{50} = \dot{T}_E \mathbf{r}_b + T_E \dot{\mathbf{r}}_b \tag{247}$$

$$\ddot{\mathbf{r}}_{50} = \ddot{T}_E \mathbf{r}_b + 2\dot{T}_E \dot{\mathbf{r}}_b + T_E \ddot{\mathbf{r}}_b \tag{248}$$

$$\dddot{\mathbf{r}}_{50} \approx \dddot{T}_E \mathbf{r}_b + 3\ddot{T}_E \dot{\mathbf{r}}_b + 3\dot{T}_E \ddot{\mathbf{r}}_b \tag{249}$$

where $T_E \dddot{\mathbf{r}}_b$ has been ignored in Eq. (249).

The formulation for computing the transformation matrices T_i, $\dot{T}_i$, $\ddot{T}_i$, and $\dddot{T}_i$ for the earth ($i = E$) is given in the next section.

D. Body-Fixed to Space-Fixed Transformation for the Earth

For the earth, the transformation T is given by the product of three 3×3 matrices:

$$T_E = (BNA)^T \tag{250}$$

Substituting Eq. (250) into Eq. (242) gives

$$\mathbf{r}_{50} = T_E \mathbf{r}_b = (BNA)^T \mathbf{r}_b \tag{251}$$

or

$$\mathbf{r}_b = T_E^T \mathbf{r}_{50} = BNA\, \mathbf{r}_{50} \tag{252}$$

The matrices A, N, and B are defined as

$A =$ precession matrix, transforming from coordinates referred to the mean earth equator and equinox of 1950.0 to coordinates referred to the mean earth equator and equinox of date

$N =$ nutation matrix, transforming from coordinates referred to the mean earth equator and equinox of date to coordinates referred to the true earth equator and equinox of date

$B =$ rotation from coordinates referred to the true earth equator and equinox of date to body-fixed coordinates $\mathbf{r}_b = (x_b, y_b, z_b)^T$, where x_b is along the intersection of the prime meridian (passing through the instantaneous axis of rotation) and the instantaneous equator, z_b is along the instantaneous axis of rotation, directed north, and y_b completes the right-handed rectangular coordinate system.

The matrix B is given by

$$B = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{253}$$

where

$\theta =$ apparent (true) sidereal time = Greenwich hour angle of true equinox of date

The derivative of T_E with respect to ephemeris time $\dot{T}_E$ is given by

$$\dot{T}_E = (\dot{B}NA + B\dot{N}A + BN\dot{A})^T \tag{254}$$

The formulation for computation of the precession matrix A, the nutation matrix N, and their derivatives with respect to ephemeris time, $\dot{A}$ and $\dot{N}$, is given in a JPL internal publication.[14] Differentiation of B with respect to ephemeris time gives

$$\dot{B} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} \tag{255}$$

where $\dot{\theta}$ is the derivative of θ with respect to ephemeris time.

The contribution to the "space-fixed" velocity of the tracking station relative to the center of the earth, $\dot{\mathbf{r}}_{50}$, from the precession and nutation rates is a maximum of about 10^{-4} m/s. Since doppler observables are computed to an accuracy of 10^{-5} m/s, these terms are included in Eq. (254). The computation of doppler observables also requires the acceleration and jerk of each participant; however, only approximate values are needed. Thus, $\ddot{T}_E$ and $\dddot{T}_E$ are obtained by differentiation of $\dot{T}_E \approx (\dot{B}NA)^T$ holding N and A constant:

$$\ddot{T}_E \approx (\ddot{B}NA)^T \tag{256}$$

$$\dddot{T}_E \approx (\dddot{B}NA)^T \tag{257}$$

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[14] Warner, M. R., et al., *Double Precision Orbit Determination Program*, Vol. III, TRAJ Segment, EPD 426 (JPL Internal Report), June 15, 1967.

The second and third derivatives of B with respect to ephemeris time are obtained by successive differentiation of Eq. (255). However, the sidereal rate $\dot{\theta}$ in Eq. (255) is an extremely constant quantity and is held fixed during this differentiation. The resulting expressions are:

$$\ddot{B} \approx \begin{bmatrix} -\cos\theta & -\sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}^2 \qquad (258)$$

$$\dddot{B} \approx \begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}^3 \qquad (259)$$

The neglected terms of $\ddot{T}_E$ and $\dddot{T}_E$ contribute less than 10^{-5} m/s to the computed doppler observables.

The true sidereal time θ and true sidereal rate $\dot{\theta}$ are computed from the following formulation (where dots indicate differentiation with respect to ephemeris time). Let

θ_M = mean sidereal time = Greenwich hour angle of mean equinox of date

$\delta\psi$ = nutation in longitude = longitude of mean equinox relative to true equinox

$\delta\epsilon$ = nutation in obliquity

ϵ = true obliquity of ecliptic

$\bar{\epsilon}$ = mean obliquity of ecliptic

Then,

$$\theta = \theta_M + \delta\psi\cos\epsilon \qquad \text{(rad)} \qquad (260)$$

$$\dot{\theta} = \dot{\theta}_M + \delta\dot{\psi}\cos\epsilon - \dot{\epsilon}\delta\psi\sin\epsilon \qquad \text{(rad/s)} \qquad (261)$$

$$\epsilon = \bar{\epsilon} + \delta\epsilon \qquad \text{(rad)} \qquad (262)$$

$$\dot{\epsilon} = \dot{\bar{\epsilon}} + \delta\dot{\epsilon} \qquad \text{(rad/s)} \qquad (263)$$

From Ref. 25, p. 98,

$$\bar{\epsilon} = \frac{A + BT + CT^2 + DT^3}{206{,}264.80625} \quad \text{(rad)} \qquad (264)$$

where

$A = 23°27'8''.26 = 84{,}428''.26$

$B = -46''.845$

$C = -0''.0059$

$D = 0''.00181$

T = Julian centuries of 36,525 ephemeris days elapsed since January 0, 1900, 12$^{\rm h}$ ET

The quantity T is computed from

$$T = \frac{\text{JED} - 241\ 5020}{36{,}525} = 0.5 + \frac{\text{ET}}{86{,}400 \times 36{,}525} \qquad (265)$$

where

JED = Julian ephemeris date

ET = seconds of ephemeris time from January 1, 1950, 0$^{\rm h}$ ET

Differentiation of Eq. (264) with respect to ET gives

$$\dot{\bar{\epsilon}} = \frac{B + 2CT + 3DT^2}{86{,}400 \times 36{,}525 \times 206{,}264.80625} \quad \text{(rad/s)} \qquad (266)$$

The nutations $\delta\psi$ and $\delta\epsilon$ and their derivatives $\delta\dot{\psi}$ and $\delta\dot{\epsilon}$ are contained on the n-body ephemeris tapes (described in Section IV). The nutations $\delta\psi$ and $\delta\epsilon$ are based upon the theory of E. W. Woolard (Ref. 49). The derivatives $\delta\dot{\psi}$ and $\delta\dot{\epsilon}$ are obtained by numerical differentiation.

Mean sidereal time θ_M is a function of universal time. The expression for θ_M is obtained by substituting R_U (UT) from Eq. (91) into Eq. (92). Since θ_M is the hour angle of the mean equinox of date measured from the 0° meridian passing through the instantaneous axis of rotation, it should be computed specifically from UT1 (see Section III). Thus, from Eqs. (91) and (92),

$$\theta_M = \text{UT1} + J + KT_U + LT_U^2 \quad (\text{angular seconds, } ^{\rm s}) \qquad (267)$$

where[15]

UT1 = seconds of UT1 time past January 1, 1950, 0$^{\rm h}$ UT1

$J = 6^{\rm h}38^{\rm m}45^{\rm s}.836 = 23{,}925^{\rm s}.836$

$K = 8{,}640{,}184^{\rm s}.542$

$L = 0^{\rm s}.0929$

T_U = number of Julian centuries of 36,525 days of UT1 elapsed since January 0, 1900, 12$^{\rm h}$ UT1

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[15] Note that 1 second of UT1 time is the time for the angle UT1 (see Section III) to change by 1 angular second (86,400 angular seconds = 2π radians).

The quantity T_U is computed from

$$T_U = \frac{\text{JD}(\text{UT1}) - 241\,5020}{36{,}525} = 0.5 + \frac{\text{UT1}}{86{,}400 \times 36{,}525} \tag{268}$$

where

JD (UT1) = Julian date computed from UT1

Substituting Eq. (267) into Eq. (260), and removing multiples of 2π so that $0 < \theta < 2\pi$ gives

$$\theta = \left[\left(\frac{\text{UT1} + J + KT_U + LT_U^2}{86{,}400} + \frac{\delta\psi \cos\epsilon}{2\pi}\right)_{\text{decimal part}}\right] 2\pi \qquad \text{(rad)} \tag{269}$$

The quantities UT1/86,400 and $KT_U/86{,}400$ currently have magnitudes of about 7,000 revolutions (1 revolution of $\theta = 2\pi$ radians of θ) and 70 revolutions, respectively. Thus, when taking the decimal part of θ expressed as revolutions, four decimal digits are lost. Since double precision on the IBM 7094 is about 16 decimal digits, θ is represented to a precision of about 12 figures or $2\pi \times 10^{-12}$ rad. For a tracking station with spin axis distance u of 6×10^6 m, its longitudinal position is represented to a precision of about 4×10^{-5} m.

Differentiating Eq. (267) with respect to ET gives

$$\dot{\theta}_M = \frac{d\text{UT1}}{d\text{ET}} \left(1 + \frac{K + 2LT_U}{36{,}525 \times 86{,}400}\right) \frac{\pi}{43{,}200} \qquad \text{(radian/ephemeris second)} \tag{270}$$

From Section III,

$$\text{UT1} = \text{ET} - (\text{ET} - \text{A1}) - (\text{A1} - \text{UT1}) \tag{271}$$

and

$$\frac{d\text{UT1}}{d\text{ET}} = 1 + \frac{\Delta f_{\text{cesium}}}{9{,}192{,}631{,}770} - g - 2ht \tag{272}$$

where

t = seconds past start of current time block for polynomial coefficients f, g, and h of Eq. (96).

Substituting Eq. (272) into Eq. (270) gives

$$\dot{\theta}_M = \left(1 + \frac{K + 2LT_U}{36{,}525 \times 86{,}400}\right)\left(1 + \frac{\Delta f_{\text{cesium}}}{9{,}192{,}631{,}770} - g - 2ht\right) \frac{\pi}{43{,}200} \qquad \text{(radian/ephemeris second)} \tag{273}$$

Given $\dot{\theta}_M$, $\dot{\theta}$ is computed from Eq. (261).

The term $g + 2ht$ in Eq. (273) has a typical magnitude of 3×10^{-8} and affects the geocentric tracking station velocity by about 10^{-5} m/s, which is the accuracy of computed doppler observables. Since Δf_{cesium} is probably no more than 5, the term $\Delta f_{\text{cesium}}/9{,}192{,}631{,}770$ is probably not significant. In the derivation of Eq. (272), the annual relativity term of ET − A1 (Eq. 93) was not differentiated. The derivative of this term has a maximum magnitude of about 3×10^{-10}, which is not significant. Equation (65) is a more accurate expression for (ET − A1) than Eq. (93) used in the general time transformation subroutine. The time derivatives of the additional relativity terms of Eq. (65) are 1.5×10^{-10} or smaller.

VIII. Doppler Observables

This section gives the formulation for computation of doppler observables, namely, 1-way doppler, 2-way doppler, and 3-way doppler.

A. Introduction

For 1-way doppler, an electromagnetic signal is transmitted continuously from the spacecraft and received by a tracking station on earth. For 2-way doppler, the signal is transmitted continuously from a tracking station on earth, received and retransmitted by the spacecraft, and received continuously by the same tracking station. The signal may also be received by a different tracking station; in this case, the resulting observable is 3-way doppler. For each of these cases, the frequency of the received signal differs from that of the transmitted signal because of the doppler shift. The observable is the average value of this frequency shift over a period of time called the count time or count interval T_c. It is proportional to the average range rate along the light path from the transmitter to the receiver during T_c or, more accurately, to the change in range along this light path during T_c. The count intervals for successive observables are contiguous.

The expression for computing each of these observables is obtained by expressing the frequency shift in a Taylor series, with coefficients evaluated at the midpoint of the count interval, and integrating term by term. The odd derivatives of the frequency shift vanish and the fourth and higher even derivatives are ignored. Thus, doppler observables are computed from the frequency shift and its second time derivative evaluated along the light path whose reception time at the receiving station is the midpoint of the count interval.

For observables computed to an accuracy of 10^{-5} m/s, truncation of the Taylor series limits the count time to values as low as 1–10 s when the spacecraft is very near the earth or another planet. When the spacecraft is in heliocentric cruise, count times as large as 1,000 s may be used. In each of these cases, however, larger count times may be used if the observable is computed from the subinterval doppler formulation. For this case, the count interval is divided into m subintervals, each of which is short enough so that the Taylor series truncation error is negligible. The observable is the sum of the observables computed for each subinterval divided by m.

In a future version of the DPODP, the Taylor-series doppler formulation will be replaced or supplemented by the differenced-range doppler formulation described in Section XI. The primary advantage of differenced-range doppler is that there is no upper limit to the count time.

The formulation for computation of 1-way, 2-way, and 3-way doppler from the frequency shift and its second time derivative is given in Section VIII-B, and the formulas for computing these two quantities are given in Sections VIII-C and -D. The equation for computing each doppler observable contains a correction term Δ, which accounts for the effects of the troposphere, the ionosphere, and the motion of the tracking point on the antenna during the count time. The computation of Δ is described in Section XII.

B. General Expressions

An intermediate output from the electronic equipment at the receiving station on earth is a signal whose frequency in cycles per second of station time (ST) is denoted by f. This signal contains the doppler frequency shift[16] and a bias frequency whose primary purpose is to keep f positive when the spacecraft range rate is negative. For 1-way, 2-way, and 3-way doppler, the expressions for f are

$$f_1 = C_1 - C_2 f_{S/C}\left(\frac{f_R}{f_T}\right) \tag{274}$$

$$f_2 = C_3\left[f_q(t_3) - f_q(t_1)\left(\frac{f_R}{f_T}\right)\right] + C_4 \tag{275}$$

$$f_3 = C_1 - C_5 f_q(t_1)\left(\frac{f_R}{f_T}\right) \tag{276}$$

where C_1 to C_5 are constants, defined below, and

$f_{S/C}$ = spacecraft auxiliary transponder oscillator frequency, cycles per UTC second [9,192,631,770 $(1-S)$ cycles[17] of imaginary cesium atomic clock carried by spacecraft]

The quantity $f_{S/C}$ is the frequency of the signal transmitted by the spacecraft for 1-way doppler. It is represented by

$$f_{S/C} = f_{T_0} + \Delta f_{T_0} + f_{T_1}(t_2 - t_0) + f_{T_2}(t_2 - t_0)^2 \tag{277}$$

where

f_{T_0} = nominal value of $f_{S/C}$

$\Delta f_{T_0}, f_{T_1}, f_{T_2}$ = solve-for parameters, specified by time block

t_0 = UTC epoch at start of time block

t_2 = UTC value of spacecraft transmission time

The remaining quantities in Eqs. (274–276) are defined as

f_R/f_T = ratio of received to transmitted frequency (for unity frequency multiplication at spacecraft). The received frequency f_R is measured in cycles per second of station time ST derived from the

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[16] The transmitted frequency minus the received frequency.

[17] See Subsection III-A-4.

atomic frequency standard at the receiving station. For 2-way or 3-way doppler, the transmitted frequency f_T is measured in cycles per second of ST derived from the atomic frequency standard at the transmitting station. For 1-way doppler, f_T is measured in cycles per UTC second (9,192,631,770 (1 − S) cycles[17] of imaginary cesium atomic clock at spacecraft).

$f_q(t_1), f_q(t_3)$ = reference oscillator frequency at transmitting station, cycles per second of ST (derived from transmitter atomic frequency standard), evaluated at transmission time t_1 and reception time t_3, respectively.[18] The frequency f_q is reset periodically but remains constant between settings. The doppler formulation presumes that $f_q(t_3)$ is constant over the reception interval T_c for 2-way doppler and that $f_q(t_1)$ is constant over the transmission interval. If these intervals overlap for 2-way doppler, $f_q(t_1)$ must equal $f_q(t_3)$.

The doppler tracking equipment originally operated in the L-band frequency range.[19] Later, the system was changed to operate in the S-band range.[20] In the interim period, some tracking data were obtained in the so-called L–S configuration (modified L-band tracking stations with an S-band transponder on the spacecraft). The DPODP has the capability of processing doppler tracking data from each of these frequency bands. The only change in the doppler formulation due to changing the frequency band is the change in the values of the coefficients C_1 through C_5:

$$C_1 = \begin{cases} 930.15 \times 10^6 & \text{L-band} \\ 9.375 \times 10^6 + 30K_1(t_3) & \text{L–S band} \\ 96\left(\dfrac{240}{221}\right) K_s(t_3) + 10^6 & \text{S-band} \end{cases}$$

$$C_2 = \begin{cases} \dfrac{31}{32} & \text{L-band} \\ \dfrac{30}{96} & \text{L–S band} \\ 1 & \text{S-band} \end{cases}$$

$$C_3 = \begin{cases} 30\left(\dfrac{96}{89}\right) & \text{L-band} \\ 96\left(\dfrac{240}{221}\right) & \text{S-band} \end{cases}$$

$$C_4 = \begin{cases} 10^5 & \text{L-band} \\ 10^6 & \text{S-band} \end{cases}$$

$$C_5 = \begin{cases} 30\left(\dfrac{31}{32}\right)\left(\dfrac{96}{89}\right) & \text{L-band} \\ 30\left(\dfrac{240}{221}\right) & \text{L–S band} \\ 96\left(\dfrac{240}{221}\right) & \text{S-band} \end{cases}$$

where

$K_1(t_3)$ = receiver reference oscillator (synthesizer) frequency at reception time t_3 for L–S band doppler. The frequency $K_1(t_3)$ is different from $f_q(t_1)$.

$K_S(t_3)$ = receiver reference oscillator frequency at reception time t_3 for S-band doppler. The receiver and transmitter reference oscillators are physically the same and operate at the same nominal frequency.

As with f_q, both of these frequencies are reset periodically but remain constant between settings. The doppler formulation presumes that $K_1(t_3)$ and $K_S(t_3)$ are constant over the reception interval T_c. Two-way L–S band doppler is computed from the 3-way formulation. Hence, L–S band values of C_3 and C_4 do not exist.

The second term of Eqs. (274), (275), and (276) is the frequency of the received signal (relative to ST at the receiving station). The first term (plus C_4 for 2-way doppler) is the frequency of a reference signal derived from the receiver atomic frequency standard.

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[18] Note that $f_q(t_3)$ applies only for 2-way doppler.
[19] 390–1550 MHz.
[20] 1550–5200 MHz.

For 2-way doppler, the reference frequency and received frequency are derived from the same atomic frequency standard. Hence 2-way doppler gives the most accurate measure of the doppler frequency shift and thus the range rate from the tracking station to the spacecraft.

For 1-way and 3-way doppler, the reference signal and received signal are derived from different atomic frequency standards. Hence, these data types are less accurate than 2-way doppler. Furthermore, for 1-way doppler, the signal transmitted from the spacecraft is currently derived from a crystal oscillator. Because of the large drift in frequency of this type of oscillator, 1-way doppler is very inaccurate and is rarely used in the determination of accurate spacecraft trajectories.

For $f_R/f_T = 1$, that is, for a spacecraft range-rate of zero, the values of f_1, f_2, and f_3 are 10^5 Hz for L-band and L–S band operation and 10^6 Hz for S-band operation. These biases are included so that the frequency f will remain positive for negative spacecraft range rates ($f_R/f_T > 1$).

For the existing S-band doppler system, the transmitted frequency is 96 times the transmitter reference oscillator frequency. The spacecraft transponder multiplies the frequency of the received signal by 240/221 before retransmitting. The reference oscillator frequency is approximately 22 MHz and hence the frequency of the signal received at the tracking station on earth is about 2300 MHz plus the effect of the doppler frequency shift. For 1-way doppler, the frequency of the signal transmitted by the spacecraft is also about 2300 MHz. For 1-way, 2-way, or 3-way doppler, the frequency of the reference signal at the receiving station is 96(240/221) times the receiver reference oscillator frequency plus the 1-MHz bias. For 2-way doppler, of course, the receiver reference oscillator is the transmitter reference oscillator.

Noting the S-band values for C_1, C_3, C_4, and C_5, one can see that the expressions for f_2 and f_3 are identical for S-band operation. The only differences are that two physically different atomic frequency standards are used for 3-way doppler and that the frequency shifts are based upon different light paths.

Equations (274–276) for f may be written as

$$f_1 = C_1 - C_2 f_{S/C} + C_2 f_{S/C}\left(1 - \frac{f_R}{f_T}\right) \tag{278}$$

$$f_2 = C_3\left[f_q(t_3) - f_q(t_1)\right] + C_4 + C_3 f_q(t_1)\left(1 - \frac{f_R}{f_T}\right) \tag{279}$$

$$f_3 = C_1 - C_5 f_q(t_1) + C_5 f_q(t_1)\left(1 - \frac{f_R}{f_T}\right) \tag{280}$$

Part or all of the constant part of each expression for f is designated as f_{bias}:

$$f_{1_{\text{bias}}} = C_1 - C_2 f_{T_0} \tag{281}$$

$$f_{2_{\text{bias}}} = C_3\left[f_q(t_3) - f_q(t_1)\right] + C_4 \tag{282}$$

$$f_{3_{\text{bias}}} = C_1 - C_5 f_q(t_1) \tag{283}$$

Hence,

$$f_1 - f_{1_{\text{bias}}} = C_2 f_{S/C}\left(1 - \frac{f_R}{f_T}\right) - C_2\left[\Delta f_{T_0} + f_{T_1}(t_2 - t_0) + f_{T_2}(t_2 - t_0)^2\right] \tag{284}$$

$$f_2 - f_{2_{\text{bias}}} = C_3 f_q(t_1)\left(1 - \frac{f_R}{f_T}\right) \tag{285}$$

$$f_3 - f_{3_{\text{bias}}} = C_5 f_q(t_1)\left(1 - \frac{f_R}{f_T}\right) \tag{286}$$

The signal with frequency f is input to an electronic counter whose register is incremented by 1 each time the magnitude of the signal changes from minus to plus. A total of N cycles are counted during the count time T_c. The doppler observable F which the data editing program passes on to the orbit determination program is:[21]

$$F = \frac{N}{T_c} - f_{\text{bias}} \tag{287}$$

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[21] In addition to the integer cycle count, the time from the start of the count interval to the first positive zero crossing is observed. Multiplying this time by N cycles per T_c seconds gives an estimate of the fraction of one cycle not counted at the beginning of T_c. One minus this quantity for the next observable is the fraction of one cycle not counted at the end of T_c. Adding these two fractions of 1 cycle to the integer cycle count gives N used in Eq. (287).

where f_{bias} is computed from Eq. (281), (282), or (283). Since N is the integral of f over the count time T_c,

$$F = \frac{1}{T_c} \int_{t_{3_m}(\text{ST})-(1/2)T_c}^{t_{3_m}(\text{ST})+(1/2)T_c} (f - f_{\text{bias}})\, dt_3(\text{ST}) \qquad (288)$$

where

$t_3(\text{ST})$ = station time (ST) at receiving station, derived from station atomic frequency standard

$t_{3_m}(\text{ST})$ = epoch at midpoint of count interval T_c

Equations (284), (285), and (286) for $f - f_{\text{bias}}$ are substituted into Eq. (288). For 1-way doppler, the variations in $f_{S/C}$ and the second term of Eq. (284) over the count interval are ignored. In each of these three equations, the quantity $[1 - (f_R/f_T)]$ is expanded in a Taylor series, with the reception time $t_3(\text{ST})$ minus the epoch $t_{3_m}(\text{ST})$ as the argument. The coefficients of each Taylor series are the derivatives of $[1 - (f_R/f_T)]$ with respect to $t_3(\text{ST})$, evaluated along the light path with reception time $t_{3_m}(\text{ST})$. A term-by-term integration of each of these equations gives the desired expressions for the computation of 1-way doppler ($F1$), 2-way doppler ($F2$), and 3-way doppler ($F3$).

In carrying out the integrations, the odd derivatives of $[1 - (f_R/f_T)]$ with respect to $t_3(\text{ST})$ vanish, and the fourth and higher even derivatives are ignored. The resulting expressions are

$$F1 = C_2 f_{S/C}\left(1 - \frac{f_R}{f_T}\right)^* - C_2\left[\Delta f_{T_0} + f_{T_1}(t_2 - t_0) + f_{T_2}(t_2 - t_0)^2\right] \qquad (289)$$

$$F2 = C_3 f_q(t_1)\left(1 - \frac{f_R}{f_T}\right)^* \qquad (290)$$

$$F3 = C_5 f_q(t_1)\left(1 - \frac{f_R}{f_T}\right)^* \qquad (291)$$

where

$$\left(1 - \frac{f_R}{f_T}\right)^* = \left(1 - \frac{f_R}{f_T}\right) + \frac{T_c^2}{24}\left(1 - \frac{f_R}{f_T}\right)^{\cdot\cdot} \qquad (292)$$

The quantities $[1 - (f_R/f_T)]$, $[1 - (f_R/f_T)]^{\cdot\cdot}$, and t_1 are evaluated along the light path whose reception time at the receiving station, $t_3(\text{ST})$, is the midpoint $t_{3_m}(\text{ST})$ of the count interval T_c (station time). The quantity $[1 - (f_R/f_T)]^{\cdot\cdot}$ is the second derivative of $[1 - (f_R/f_T)]$ with respect to $t_3(\text{ST})$. The first term that has been truncated in Eq. (292) is $(1/1{,}920)(T_c^4)\ [1 - (f_R/f_T)]^{\text{iv}}$ where iv indicates the fourth derivative with respect to $t_3(\text{ST})$. For 1-way doppler, $f_{S/C}$ and the second term of Eq. (289) are evaluated with the spacecraft transmission time t_2 for the above-mentioned light path.

For 2-way or 3-way doppler, the definition of f_R/f_T is

$$\frac{f_R}{f_T} = \frac{dn}{dt_3(\text{ST})} \cdot \frac{dt_1(\text{ST})}{dn} = \frac{dt_1(\text{ST})}{dt_3(\text{ST})} \qquad (293)$$

where

dn = infinitesimal number of cycles transmitted at time t_1. The dn cycles travel at constant phase from the transmitter to the receiver and are received at time t_3. The propagation speed is the phase velocity, which is greater than c in the presence of charged particles.

$dt_1(\text{ST})$ = infinitesimal period (of station time ST) for transmission of dn cycles from transmitting station at time t_1.

$dt_3(\text{ST})$ = infinitesimal period (of station time ST) for reception of dn cycles at receiving station at time t_3.

Equation (293) may be written as

$$\frac{f_R}{f_T} = \frac{\left(\dfrac{d\text{ST}}{d\text{UTC}}\right)_1 \dfrac{d\text{UTC}}{d\tau}\left(\dfrac{d\tau}{dt}\right)_1}{\left(\dfrac{d\text{ST}}{d\text{UTC}}\right)_3 \dfrac{d\text{UTC}}{d\tau}\left(\dfrac{d\tau}{dt}\right)_3} \frac{dt_1}{dt_2}\,\frac{dt_2}{dt_3} \qquad (294)$$

where

dt_1, dt_2, dt_3 = ephemeris time (ET) value of transmission interval $[dt_1(\text{ST})]$, reflection interval at the spacecraft, and reception interval $[dt_3(\text{ST})]$.

The ratios dt_1/dt_2 and dt_2/dt_3 will be obtained by differentiation of the light time equations for the up and down legs of the light path. The factors $(d\tau/dt)$ at t_1 and t_3 transform dt_1 and dt_3 from ephemeris time to proper time τ[22] obtained from imaginary ideal atomic

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[22] This time scale was defined in Section II after Eq. (58).

clocks at the transmitting and receiving stations; they are computed from Eq. (58) using the Newtonian potential at each tracking station and the heliocentric velocity of each tracking station.

The factor $d\text{UTC}/d\tau$ converts the transmission and reception intervals from seconds of atomic time τ to seconds of UTC atomic time. These two atomic time scales differ only in the length of the second (the number of cycles defined equal to 1 s).

The factors $d\text{ST}/d\text{UTC}$ at t_1 and t_3 convert the transmission and reception intervals from UTC seconds obtained from ideal atomic clocks to seconds of station time ST obtained from the actual atomic clocks at the transmitting and receiving stations (the same station and clock for 2-way doppler). The transformation from UTC to ST at each tracking station is specified by Eq. (94), repeated here:

$$\text{UTC} - \text{ST} = a + bt + ct^2 \tag{295}$$

where a, b, and c are specified by time block and t is in seconds past the start of the time block. Let the coefficients of Eq. (295) which apply for the receiving station at t_3 and for the transmitting station at t_1 be denoted by subscripts R and T respectively. Also, define F by:

$$1 + F = \frac{\left(\dfrac{d\text{ST}}{d\text{UTC}}\right)_1}{\left(\dfrac{d\text{ST}}{d\text{UTC}}\right)_3} \tag{296}$$

Then, since $d\text{ST}/d\text{UTC}$ is extremely close to unity,

$$F \approx b_R(t_3) - b_T(t_1) + 2t_3c_R(t_3) - 2t_1c_T(t_1) \tag{297}$$

where the transmission and reception times t_1 and t_3 are expressed as seconds past the start of the time blocks for a, b, and c used at t_1 and t_3, respectively. Also, define F_R/F_T by

$$\frac{F_R}{F_T} = \frac{\left(\dfrac{d\tau}{dt}\right)_1}{\left(\dfrac{d\tau}{dt}\right)_3} \frac{dt_1}{dt_2} \frac{dt_2}{dt_3} \tag{298}$$

Then, substituting Eqs. (296) and (298) into Eq. (294) gives

$$\left(1 - \frac{f_R}{f_T}\right) = (1 + F)\left(1 - \frac{F_R}{F_T}\right) - F \tag{299}$$

The effect on 2-way doppler of the variation in F during the count interval T_c is about 10^{-9} m/s, which is completely negligible. The corresponding effect on 3-way doppler is about 10^{-5} m/s, which is the desired accuracy for computed doppler observables. However, the error in 3-way doppler due to the unknown difference in frequency of the two atomic frequency standards ($\Delta f/f \approx 2 \times 10^{-11}$) is a few mm/s, which probably cannot be reduced to the 10^{-5}-m/s level by estimating the b and c coefficients of UTC − ST for the transmitting and receiving stations. Thus, the variation in F during the count interval T_c is ignored and

$$\left(1 - \frac{f_R}{f_T}\right)^{\cdot\cdot} \approx (1 + F)\left(1 - \frac{F_R}{F_T}\right)^{\cdot\cdot} \tag{300}$$

Substituting Eqs. (299) and (300) into Eq. (292) gives

$$\left(1 - \frac{f_R}{f_T}\right)^* = \left(1 - \frac{F_R}{F_T}\right)^* - F\left[1 - \left(1 - \frac{F_R}{F_T}\right)^*\right] \tag{301}$$

where

$$\left(1 - \frac{F_R}{F_T}\right)^* = \left(1 - \frac{F_R}{F_T}\right) + \frac{T_c^2}{24}\left(1 - \frac{F_R}{F_T}\right)^{\cdot\cdot} \tag{302}$$

Substitution of Eqs. (301) and (302) into Eqs. (290) and (291) gives 2-way and 3-way doppler as a function of $[1 - (F_R/F_T)]$, $[1 - (F_R/F_T)]^{\cdot\cdot}$, and F.

For 1-way doppler, the definition of f_R/f_T is

$$\frac{f_R}{f_T} = \frac{dn}{dt_3(\text{ST})} \cdot \frac{dt_2(\text{UTC})}{dn} = \frac{dt_2(\text{UTC})}{dt_3(\text{ST})} \tag{303}$$

since $f_{S/C}$ is referenced to an imaginary UTC atomic clock on board the spacecraft. This equation may be written as

$$\frac{f_R}{f_T} = \frac{\dfrac{d\text{UTC}}{d\tau}\left(\dfrac{d\tau}{dt}\right)_2}{\left(\dfrac{d\text{ST}}{d\text{UTC}}\right)_3 \dfrac{d\text{UTC}}{d\tau}\left(\dfrac{d\tau}{dt}\right)_3} \frac{dt_2}{dt_3} \tag{304}$$

As in Eq. (296), define F_1 by

$$1 + F_1 = \frac{1}{\left(\dfrac{d\text{ST}}{d\text{UTC}}\right)_3} \tag{305}$$

Then,

$$F_1 \approx b_R(t_3) + 2t_3c_R(t_3) \tag{306}$$

where t_3 is expressed as seconds past the start of the time block for a, b, and c used at t_3. Also, define F_R/F_T for 1-way doppler by

$$\frac{F_R}{F_T} = \frac{\left(\dfrac{d\tau}{dt}\right)_2}{\left(\dfrac{d\tau}{dt}\right)_3} \frac{dt_2}{dt_3} \tag{307}$$

Substituting Eqs. (305) and (307) into Eq. (304) gives Eqs. (299–302) with F replaced by F_1 and F_R/F_T defined by Eq. (307).

Substituting Eq. (301) into Eqs. (289),[23] (290), and (291) gives the final expressions for the computation of 1-way doppler ($F1$), 2-way doppler ($F2$), and 3-way doppler ($F3$). Each of these expressions contains an additive correction Δ, which accounts for the effects of the troposphere, the ionosphere, and the motion of the tracking point on the transmitting and receiving antennas during T_c. The computation of Δ is described in Section XII. The expressions for $F1$, $F2$, and $F3$ are

$$F1 = C_2 f_{S/C} \left\{\left(1 - \frac{F_R}{F_T}\right)^* - F_1\left[1 - \left(1 - \frac{F_R}{F_T}\right)^*\right] + \Delta\right\}$$

$$- C_2\left[\Delta f_{T_0} + f_{T_1}(t_2 - t_0) + f_{T_2}(t_2 - t_0)^2\right] \tag{308}$$

$$F2 = C_3 f_q(t_1) \left\{\left(1 - \frac{F_R}{F_T}\right)^* - F\left[1 - \left(1 - \frac{F_R}{F_T}\right)^*\right] + \Delta\right\} \tag{309}$$

$$F3 = C_5 f_q(t_1) \left\{\left(1 - \frac{F_R}{F_T}\right)^* - F\left[1 - \left(1 - \frac{F_R}{F_T}\right)^*\right] + \Delta\right\} \tag{310}$$

where $[1 - (F_R/F_T)]^*$ is given by Eq. (302) in terms of $[1 - (F_R/F_T)]$ and its second derivative with respect to t_3 (ST), $[1 - (F_R/F_T)]^{\cdot\cdot}$, evaluated along the light path whose reception time at the receiving station, t_3 (ST), is the midpoint t_{3_m} (ST) of the count interval T_c. Expressions for these quantities are derived in Sections VIII-C and -D respectively, starting from Eq. (298) for F_R/F_T for 2-way and 3-way doppler and Eq. (307) for 1-way doppler. The quantities $f_{S/C}$, F, and F_1 are computed from Eqs. (277), (297), and (306), respectively. The quantities $[1 - (F_R/F_T)]$, $[1 - (F_R/F_T)]^{\cdot\cdot}$, F, F_1, $f_{S/C}$, t_2, t_1, and Δ are evaluated with quantities obtained from the light time solution for the above-mentioned light path (see Section VI).

Equations (308), (309), and (310) are used to compute 1-way, 2-way, and 3-way doppler using either the L-band, L–S band, or S-band values of the coefficients C_2, C_3, and C_5. In the L–S band configuration, the so-called 2-way doppler observable is actually 3-way doppler (from the electronics point of view) obtained using the same tracking station as the transmitter and the receiver. This data type is computed from the 3-way formula, Eq. (310).

Another data type not previously mentioned is coherent 3-way doppler, which is essentially 2-way doppler obtained from two different tracking stations. The two stations are only a few kilometers apart and the reference frequency $f_q(t_3)$ is beamed from the transmitter to the receiver via microwave link. Coherent 3-way doppler is computed from the 2-way formula, Eq. (309).

The term in Eq. (308) containing F_1 and the term in Eq. (309) containing F are not included in the current DPODP formulation. The latter will be added at the earliest convenience, and the former will be added when $f_{S/C}$ is derived from an atomic frequency standard on board the spacecraft instead of the currently used crystal oscillator.

Because of truncation of the fourth and higher even derivatives of $[1 - (F_R/F_T)]$ in Eq. (302), the doppler observables are limited to count times as low as 1–10 s when the spacecraft is near a planet and no more than roughly 1,000 s in heliocentric cruise. However, larger count times may be used if the subinterval doppler formulation is utilized. With this method, the count time T_c is divided into m subintervals of length T_c/m. For each subinterval, a light time solution is obtained for the light path with reception time t_3 (ST) equal to the midpoint of the subinterval, and a doppler observable F (1-way, 2-way, or 3-way doppler) is computed using T_c/m in place of T_c in Eq. (302).

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[23] With F replaced by F_1.

Let the observable computed for subinterval i be denoted as F_i. Then, the observable for the overall count interval T_c is given by

$$F = \frac{1}{m} \sum_{i=1}^{m} F_i \tag{311}$$

This follows directly from Eq. (287).

Predicted values of the number of cycles N which a station will observe in a given count interval T_c are computed from

$$N = (F + f_{\text{bias}}) T_c \tag{312}$$

where $F = F1$, $F2$, or $F3$ and f_{bias} is the corresponding bias frequency from Eq. (281), (282), or (283). Equation (312) follows directly from Eq. (287).

C. Doppler Frequency Shift

The expression for $[1 - (F_R/F_T)]$ used to compute 2-way and 3-way doppler and also the expression used to compute 1-way doppler are derived in this section. The definitions of F_R/F_T are Eq. (298) for 2-way and 3-way doppler and Eq. (307) for 1-way doppler, evaluated along the light path whose reception time at the receiving station, t_3 (ST), is the midpoint of the count interval T_c. The expressions for $[1 - (F_R/F_T)]$ are obtained as expansions in powers of $1/c$. In order to obtain the desired accuracy of 10^{-5} m/s for computed doppler, all terms to order $1/c^3$ are retained.

The terms dt_1/dt_2 and dt_2/dt_3 are obtained by differentiation of the light time equations for the up and down legs of the light path. The light time equation for a given leg of the light path is Eq. (88) or (203). For the up and down legs, it is given by

$$t_2 - t_1 = \frac{r_{12}}{c} + \frac{(1+\gamma)\mu_S}{c^3} \ln\left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}}\right) \tag{313}$$

and

$$t_3 - t_2 = \frac{r_{23}}{c} + \frac{(1+\gamma)\mu_S}{c^3} \ln\left(\frac{r_2 + r_3 + r_{23}}{r_2 + r_3 - r_{23}}\right) \tag{314}$$

Solution of these equations (see Section VI) gives the following quantities:

t_1, t_2, t_3 = ephemeris time (ET) values of transmission time at tracking station on earth, reflection time at spacecraft (or transmission time for 1-way doppler), and reception time at tracking station on earth, respectively. The station time (ST) value of t_3 is the midpoint of the count interval T_c.

$\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ = heliocentric position vectors of transmitting station on earth at t_1, spacecraft at t_2, and receiving station on earth at t_3, respectively, with rectangular components referred to the mean earth equator and equinox of 1950.0.

$\dot{\mathbf{r}}_i, \ddot{\mathbf{r}}_i, \dddot{\mathbf{r}}_i$ = heliocentric velocity, acceleration, and jerk vectors of participant i at its epoch of participation t_i (i = 1, 2, or 3). The dots indicate differentiation of $\mathbf{r}_i$ with respect to ephemeris time.

The quantities on the right-hand sides of Eqs. (313) and (314) are

$$r_{12} = [(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{r}_2 - \mathbf{r}_1)]^{1/2} \tag{315}$$

$$r_{23} = [(\mathbf{r}_3 - \mathbf{r}_2) \cdot (\mathbf{r}_3 - \mathbf{r}_2)]^{1/2} \tag{316}$$

$$r_1 = (\mathbf{r}_1 \cdot \mathbf{r}_1)^{1/2} \tag{317}$$

$$r_2 = (\mathbf{r}_2 \cdot \mathbf{r}_2)^{1/2} \tag{318}$$

$$r_3 = (\mathbf{r}_3 \cdot \mathbf{r}_3)^{1/2} \tag{319}$$

c = speed of light, km/s

μ_S = gravitational constant of sun, km³/s²

γ = solve-for free parameter of the Brans–Dicke theory of relativity. The parameter γ is related to ω, the coupling constant of the scalar field, through Eq. (41).

Differentiation of Eq. (313) with respect to t_2 gives

$$1-\frac{dt_1}{dt_2}=\frac{1}{c}\left(\frac{\partial r_{12}}{\partial t_2}+\frac{\partial r_{12}}{\partial t_1}\frac{dt_1}{dt_2}\right)$$

$$+\frac{(1+\gamma)\,\mu_S}{c^3}\cdot\frac{\dfrac{dr_1}{dt_1}\dfrac{dt_1}{dt_2}+\dfrac{dr_2}{dt_2}+\dfrac{\partial r_{12}}{\partial t_2}+\dfrac{\partial r_{12}}{\partial t_1}\dfrac{dt_1}{dt_2}}{r_1+r_2+r_{12}}$$

$$-\frac{(1+\gamma)\,\mu_S}{c^3}\cdot\frac{\dfrac{dr_1}{dt_1}\dfrac{dt_1}{dt_2}+\dfrac{dr_2}{dt_2}-\dfrac{\partial r_{12}}{\partial t_2}-\dfrac{\partial r_{12}}{\partial t_1}\dfrac{dt_1}{dt_2}}{r_1+r_2-r_{12}}$$

$$\begin{matrix}1\rightarrow 2\\ 2\rightarrow 3\end{matrix}\qquad (320)$$

The derivative of Eq. (314) with respect to t_3 is obtained from Eq. (320) by replacing the subscripts 1 and 2 by 2 and 3, respectively. The expression for dt_1/dt_2 obtained from Eq. (320) is unity plus terms of order $1/c$ and greater arising from the $1/c$ (Newtonian) term of Eq. (313) plus a term of order $1/c^3$ arising from the $1/c^3$ (relativity) term of Eq. (313).

Since terms of order greater than $1/c^3$ are not retained in dt_1/dt_2 from Eq. (320), the factor dt_1/dt_2 appearing in the $1/c^3$ terms may be approximated by unity. The derivatives appearing in Eq. (320) and combinations of them are given by

$$\dot{r}_1=\frac{dr_1}{dt_1}=\frac{\mathbf{r}_1}{r_1}\cdot\dot{\mathbf{r}}_1 \qquad (321)$$

$$\dot{r}_2=\frac{dr_2}{dt_2}=\frac{\mathbf{r}_2}{r_2}\cdot\dot{\mathbf{r}}_2 \qquad (322)$$

$$\dot{r}_3=\frac{dr_3}{dt_3}=\frac{\mathbf{r}_3}{r_3}\cdot\dot{\mathbf{r}}_3 \qquad (323)$$

Using the notation

$$\mathbf{r}_{ij}=\mathbf{r}_j-\mathbf{r}_i$$

$$\dot{\mathbf{r}}_{ij}=\dot{\mathbf{r}}_j-\dot{\mathbf{r}}_i$$

the remaining terms are

$$\frac{\partial r_{12}}{\partial t_2}=\frac{\mathbf{r}_{12}}{r_{12}}\cdot\dot{\mathbf{r}}_2 \qquad (324)$$

$$\frac{\partial r_{12}}{\partial t_1}=-\frac{\mathbf{r}_{12}}{r_{12}}\cdot\dot{\mathbf{r}}_1 \qquad (325)$$

$$\dot{r}_{12}=\frac{\partial r_{12}}{\partial t_2}+\frac{\partial r_{12}}{\partial t_1}=\frac{\mathbf{r}_{12}}{r_{12}}\cdot\dot{\mathbf{r}}_{12} \qquad (326)$$

$$\frac{\partial r_{23}}{\partial t_3}=\frac{\mathbf{r}_{23}}{r_{23}}\cdot\dot{\mathbf{r}}_3 \qquad (327)$$

$$\frac{\partial r_{23}}{\partial t_2}=-\frac{\mathbf{r}_{23}}{r_{23}}\cdot\dot{\mathbf{r}}_2 \qquad (328)$$

$$\dot{r}_{23}=\frac{\partial r_{23}}{\partial t_3}+\frac{\partial r_{23}}{\partial t_2}=\frac{\mathbf{r}_{23}}{r_{23}}\cdot\dot{\mathbf{r}}_{23} \qquad (329)$$

Substituting these expressions into Eq. (320) and using $dt_1/dt_2 = 1$ in the $1/c^3$ terms gives

$$\frac{dt_1}{dt_2}=\frac{1-\dfrac{1}{c}\dfrac{\mathbf{r}_{12}}{r_{12}}\cdot\dot{\mathbf{r}}_2+\dfrac{(1+\gamma)\,\mu_S}{c^3}\,\epsilon_{12}}{1-\dfrac{1}{c}\dfrac{\mathbf{r}_{12}}{r_{12}}\cdot\dot{\mathbf{r}}_1}\qquad \begin{matrix}1\rightarrow 2\\ 2\rightarrow 3\end{matrix} \qquad (330)$$

where

$$\epsilon_{12}=\frac{\dot{r}_1+\dot{r}_2-\dot{r}_{12}}{r_1+r_2-r_{12}}-\frac{\dot{r}_1+\dot{r}_2+\dot{r}_{12}}{r_1+r_2+r_{12}}\qquad \begin{matrix}1\rightarrow 2\\ 2\rightarrow 3\end{matrix} \qquad (331)$$

The first term of Eq. (331) approaches $0 \div 0$ as the distance from the light path to the center of the sun approaches zero. However, because of the finite radius of the sun (700,000 km), the limiting indeterminacy will not occur. For a light ray grazing the surface of the sun and $r_1 = r_2 = 50$ AU, the sum $r_1 + r_2 - r_{12}$ is about 65 km. Since $(r_1 + r_2)$ and r_{12} are 100 AU, which is represented to 10^{-5} km on the 16-decimal-digit IBM 7094 computer, the 65-km difference is represented to 7 decimal digits.

For any case where the light path grazes the surface of the sun and $r_1 + r_2 \approx r_{12}$, the contribution to the spacecraft range rate from the first term of Eq. (331) is a maximum of about 0.5 m/s (for a spacecraft velocity of 100 km/s). Since the denomination of this term is represented to at least 7 decimal digits, the contribution of 0.5 m/s is accurate to at least 10^{-7} m/s, which is smaller than the desired accuracy of 10^{-5} m/s for computed doppler. Thus, the numerical difficulties associated with the first term of Eq. (331) are not significant.

Let

$$\dot{p}_{12} = \frac{\mathbf{r}_{12}}{r_{12}} \cdot \dot{\mathbf{r}}_1 \tag{332}$$

and, for the down leg,

$$\dot{p}_{23} = \frac{\mathbf{r}_{23}}{r_{23}} \cdot \dot{\mathbf{r}}_2 \tag{333}$$

Substituting Eq. (332) into the reciprocal of the denominator of Eq. (330) and expanding gives

$$\left(1 - \frac{1}{c}\frac{\mathbf{r}_{12}}{r_{12}} \cdot \dot{\mathbf{r}}_1\right)^{-1} = 1 + \left(\frac{\dot{p}_{12}}{c}\right) + \left(\frac{\dot{p}_{12}}{c}\right)^2 + \left(\frac{\dot{p}_{12}}{c}\right)^3 \qquad \begin{matrix} 1 \rightarrow 2 \\ 2 \rightarrow 3 \end{matrix} \tag{334}$$

Multiplying by the numerator of Eq. (330) and retaining terms to order $1/c^3$ gives

$$\frac{dt_1}{dt_2} = 1 - \frac{\dot{r}_{12}}{c} - \frac{\dot{r}_{12}\dot{p}_{12}}{c^2} + \frac{1}{c^3}\left[(1+\gamma)\,\mu_S\epsilon_{12} - \dot{r}_{12}\dot{p}_{12}^2\right] \qquad \begin{matrix} 1 \rightarrow 2 \\ 2 \rightarrow 3 \end{matrix} \tag{335}$$

Multiplying dt_1/dt_2 by dt_2/dt_3 gives

$$\frac{dt_1}{dt_2} \cdot \frac{dt_2}{dt_3} = 1 - \frac{1}{c}(\dot{r}_{12} + \dot{r}_{23}) + \frac{1}{c^2}(\dot{r}_{12}\dot{r}_{23} - \dot{r}_{12}\dot{p}_{12} - \dot{r}_{23}\dot{p}_{23}) + \frac{1}{c^3}\left[(1+\gamma)\,\mu_S(\epsilon_{12} + \epsilon_{23}) + \dot{r}_{12}\dot{r}_{23}(\dot{p}_{12} + \dot{p}_{23}) - \dot{r}_{12}\dot{p}_{12}^2 - \dot{r}_{23}\dot{p}_{23}^2\right] \tag{336}$$

From Eq. (58), the quantities $(d\tau/dt)_1$, $(d\tau/dt)_2$, and $(d\tau/dt)_3$ are given by

$$\left(\frac{d\tau}{dt}\right)_i = \left[1 - \frac{2\phi_i}{c^2} - \left(\frac{\dot{s}_i}{c}\right)^2\right]^{1/2} \qquad i = 1, 2, \text{or } 3 \tag{337}$$

where

ϕ_i = Newtonian potential at participant i at its epoch of participation t_i

$\dot{s}_i$ = heliocentric velocity of participant i at its epoch of participation t_i

The potential ϕ_i is given by

$$\phi_i = \sum_j \frac{\mu_j}{r_{ij}} \tag{338}$$

where the summation over j includes the sun, all of the planets, and the moon, and r_{ij} is the coordinate distance from the participant i to the center of the body j. The velocity $\dot{s}_i$ is obtained from

$$\dot{s}_i^2 = \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \tag{339}$$

Since terms of order greater than $1/c^3$ are not retained, Eq. (337) may be approximated by

$$\left(\frac{d\tau}{dt}\right)_i = 1 - \frac{\phi_i}{c^2} - \frac{1}{2}\left(\frac{\dot{s}_i}{c}\right)^2 \tag{340}$$

For 2-way or 3-way doppler,

$$\frac{\left(\frac{d\tau}{dt}\right)_1}{\left(\frac{d\tau}{dt}\right)_3} = 1 + \frac{1}{c^2}\left[(\phi_3 - \phi_1) + \frac{1}{2}(\dot{s}_3^2 - \dot{s}_1^2)\right] \tag{341}$$

where terms of order $1/c^4$ have been ignored. Similarly, for 1-way doppler,

$$\frac{\left(\frac{d\tau}{dt}\right)_2}{\left(\frac{d\tau}{dt}\right)_3} = 1 + \frac{1}{c^2}\left[(\phi_3 - \phi_2) + \frac{1}{2}(\dot{s}_3^2 - \dot{s}_2^2)\right] \tag{342}$$

Substituting Eqs. (341) and (336) into Eq. (298) and retaining all terms to order $1/c^3$ gives the desired expression for $[1 - (F_R/F_T)]$ for 2-way or 3-way doppler:

$$\begin{aligned}\left(1 - \frac{F_R}{F_T}\right) = {} & \frac{1}{c}(\dot{r}_{12} + \dot{r}_{23}) + \frac{1}{c^2}\left[\dot{r}_{12}\dot{p}_{12} + \dot{r}_{23}\dot{p}_{23} - \dot{r}_{12}\dot{r}_{23} + (\phi_1 - \phi_3) + \frac{1}{2}(\dot{s}_1^2 - \dot{s}_3^2)\right] \\ & + \frac{1}{c^3}\left\{\dot{r}_{12}\dot{p}_{12}^2 + \dot{r}_{23}\dot{p}_{23}^2 - \dot{r}_{12}\dot{r}_{23}(\dot{p}_{12} + \dot{p}_{23}) \right. \\ & \left. - (\dot{r}_{12} + \dot{r}_{23})\left[(\phi_1 - \phi_3) + \frac{1}{2}(\dot{s}_1^2 - \dot{s}_3^2)\right] - (1 + \gamma)\,\mu_S(\epsilon_{12} + \epsilon_{23})\right\}\end{aligned} \tag{343}$$

This quantity is used in Eq. (302), which is substituted into Eq. (309) or (310) to compute 2-way or 3-way doppler. Similarly, substituting Eq. (342) and dt_2/dt_3 obtained from Eq. (335) into Eq. (307) gives the expression for $[1 - (F_R/F_T)]$ for 1-way doppler:

$$\begin{aligned}\left(1 - \frac{F_R}{F_T}\right) = {} & \frac{1}{c}(\dot{r}_{23}) + \frac{1}{c^2}\left[\dot{r}_{23}\dot{p}_{23} + (\phi_2 - \phi_3) + \frac{1}{2}(\dot{s}_2^2 - \dot{s}_3^2)\right] \\ & + \frac{1}{c^3}\left\{\dot{r}_{23}\dot{p}_{23}^2 - \dot{r}_{23}\left[(\phi_2 - \phi_3) + \frac{1}{2}(\dot{s}_2^2 - \dot{s}_3^2)\right] - (1 + \gamma)\,\mu_S\epsilon_{23}\right\}\end{aligned} \tag{344}$$

Equation (344) is used in the computation of 1-way doppler from Eqs. (302) and (308). Note that setting all up-leg factors equal to zero in Eq. (343) and changing ϕ_1 and $\dot{s}_1$ to ϕ_2 and $\dot{s}_2$ gives Eq. (344).

For 2-way or 3-way doppler, ϕ_1 is very nearly equal to ϕ_3. The contribution to $(\phi_1 - \phi_3)$ from the other planets and from the moon affects the observable by less than 10^{-5} m/s and hence can be ignored. Thus, only the potential from the sun and from the earth needs to be considered, and ϕ_1 and ϕ_3 are given accordingly by

$$\phi_1 = \frac{\mu_S}{r_1} + \frac{\mu_E}{r_1^E} \tag{345}$$

$$\phi_3 = \frac{\mu_S}{r_3} + \frac{\mu_E}{r_3^E} \tag{346}$$

where r_1^E and r_3^E are the geocentric radii of the transmitting and receiving stations, respectively. The second terms of Eqs. (345) and (346) are required for the computation of 3-way doppler but cancel in $(\phi_1 - \phi_3)$ used for 2-way doppler.

For 1-way doppler, ϕ_2 and ϕ_3 are computed from Eq. (338) as indicated after that equation.

D. Second Derivative of Doppler Frequency Shift

The computation of doppler observables requires an expression for $[1 - (F_R/F_T)]^{\cdot\cdot}$, which is the second derivative of $[1 - (F_R/F_T)]$ with respect to the reception time t_3 (ST), evaluated along the light path whose reception time is the midpoint of the count interval T_c. The expression for $[1 - (F_R/F_T)]^{\cdot\cdot}$ for 2-way and 3-way doppler and also the expression for 1-way doppler are derived in this section. They are obtained by differentiation of the corresponding expressions for $[1 - (F_R/F_T)]$ obtained from Section VIII-C.

In order to limit the doppler truncation error (due to ignoring the fourth and higher even derivatives of the frequency shift in Eq. 302) to 10^{-5} m/s or less, count times as low as 1–10 s must be used when the spacecraft is very near one of the celestial bodies of the solar system;

alternatively, when the spacecraft is in heliocentric cruise, count times as large as 1,000 s may be used.

For either of these situations, the $1/c^3$ terms of $[1 - (F_R/F_T)]^{\cdot\cdot}$ affect doppler observables by less than 10^{-5} m/s. Hence, the expressions for $[1 - (F_R/F_T)]^{\cdot\cdot}$ are obtained by differentiating Eqs. (343) and (344), ignoring the $1/c^3$ terms. For 2-way or 3-way doppler, the variations in $(\phi_1 - \phi_3)/c^2$ and $(\dot{s}_1^2 - \dot{s}_3^2)/2c^2$ over the count interval affect the observable by less than 10^{-5} m/s; hence these terms are also ignored. For 1-way doppler, the corresponding terms and their variations are quite large. However, they have not been included in the expression that is differentiated because of the inaccuracy of 1-way doppler obtained by using a crystal oscillator on board the spacecraft.

In the future, when 1-way doppler derived from an atomic frequency standard becomes available, it will be mandatory that $[1 - (F_R/F_T)]^{\cdot\cdot}$ include the derivatives of $(\phi_2 - \phi_3)/c^2$ and $(\dot{s}_2^2 - \dot{s}_3^2)/2c^2$. For 2-way or 3-way doppler, $[1 - F_R/F_T]^{\cdot\cdot}$ is obtained from

$$\left(1 - \frac{F_R}{F_T}\right) \approx \frac{1}{c}(\dot{r}_{12} + \dot{r}_{23}) + \frac{1}{c^2}(\dot{r}_{12}\dot{p}_{12} + \dot{r}_{23}\dot{p}_{23} - \dot{r}_{12}\dot{r}_{23}) \tag{347}$$

For 1-way doppler, the corresponding expression is

$$\left(1 - \frac{F_R}{F_T}\right) \approx \frac{1}{c}(\dot{r}_{23}) + \frac{1}{c^2}(\dot{r}_{23}\dot{p}_{23}) \tag{348}$$

The terms in Eqs. (347) and (348) are functions of the heliocentric position and velocity vectors of the transmitter, spacecraft, and receiver at their epochs of participation. Since the time unit for the velocity, acceleration, and jerk vectors of each participant is ephemeris time (ET), the derivatives of $[1 - (F_R/F_T)]$, which are obtained naturally, are the first and second derivatives with respect to t_3 (ET). Given these quantities, the second derivative with respect to t_3 (ST) is

$$\frac{d^2}{dt_3\,(\mathrm{ST})^2}\left(1 - \frac{F_R}{F_T}\right) = \left[\frac{d^2}{dt_3\,(\mathrm{ET})^2}\left(1 - \frac{F_R}{F_T}\right)\right]\left[\frac{dt_3\,(\mathrm{ET})}{dt_3\,(\mathrm{ST})}\right]^2 + \left[\frac{d}{dt_3\,(\mathrm{ET})}\left(1 - \frac{F_R}{F_T}\right)\right]\frac{d^2t_3\,(\mathrm{ET})}{dt_3\,(\mathrm{ST})^2} \tag{349}$$

The second term of Eq. (349) affects doppler observables by less than 10^{-10} m/s and hence can be ignored. The second derivative of $[1 - (F_R/F_T)]$ with respect to t_3 (ET) contains $1/c$ and $1/c^2$ terms and hence is accurate to about 8 figures. The multiplicative factor in the first term of Eq. (349) is unity to about this many figures; hence, it may be ignored. Thus, $[1 - (F_R/F_T)]^{\cdot\cdot}$ is computed from

$$\left(1 - \frac{F_R}{F_T}\right)^{\cdot\cdot} = \frac{d^2}{dt_3\,(\mathrm{ST})^2}\left(1 - \frac{F_R}{F_T}\right) \approx \frac{d^2}{dt_3\,(\mathrm{ET})^2}\left(1 - \frac{F_R}{F_T}\right) \tag{350}$$

In terms of first and second derivatives of the terms of Eq. (347) with respect to t_3 (ET), denoted as t_3,

$$\begin{aligned}\left(1 - \frac{F_R}{F_T}\right) &= \frac{1}{c}\frac{d^2\dot{r}_{12}}{dt_3^2}\left[1 + \frac{1}{c}(\dot{p}_{12} - \dot{r}_{23})\right] \\ &+ \frac{1}{c}\frac{d^2\dot{r}_{23}}{dt_3^2}\left[1 + \frac{1}{c}(\dot{p}_{23} - \dot{r}_{12})\right] \\ &+ \frac{1}{c^2}\frac{d^2\dot{p}_{12}}{dt_3^2}\dot{r}_{12} + \frac{1}{c^2}\frac{d^2\dot{p}_{23}}{dt_3^2}\dot{r}_{23} \\ &+ \frac{2}{c^2}\left(\frac{d\dot{r}_{12}}{dt_3}\frac{d\dot{p}_{12}}{dt_3}\right. \\ &+ \left.\frac{d\dot{r}_{23}}{dt_3}\frac{d\dot{p}_{23}}{dt_3} - \frac{d\dot{r}_{12}}{dt_3}\frac{d\dot{r}_{23}}{dt_3}\right)\end{aligned} \tag{351}$$

which applies for 2-way or 3-way doppler. Similarly, from Eq. (348),

$$\begin{aligned}\left(1 - \frac{F_R}{F_T}\right)^{\cdot\cdot} &= \frac{1}{c}\frac{d^2\dot{r}_{23}}{dt_3^2}\left(1 + \frac{\dot{p}_{23}}{c}\right) + \frac{1}{c^2}\frac{d^2\dot{p}_{23}}{dt_3^2}\dot{r}_{23} \\ &+ \frac{2}{c^2}\frac{d\dot{r}_{23}}{dt_3}\frac{d\dot{p}_{23}}{dt_3}\end{aligned} \tag{352}$$

which applies for 1-way doppler.

The quantities $\dot{r}_{12}$, $\dot{r}_{23}$, $\dot{p}_{12}$, and $\dot{p}_{23}$ are functions of t_1 (ET), t_2 (ET), and t_3 (ET) which will be denoted as t_1, t_2, and t_3, respectively, in the remainder of this section.

In order to obtain derivatives of these quantities with respect to t_3, the following subpartial derivatives are required:

$$\frac{dt_2}{dt_3} = 1 - \frac{\dot{r}_{23}}{c} \tag{353}$$

$$\frac{dt_1}{dt_3} = 1 - \frac{1}{c}(\dot{r}_{12} + \dot{r}_{23}) \tag{354}$$

The terms above are derived from Eqs. (313) and (314), ignoring the $1/c^3$ relativity terms.

The first and second derivatives of $\dot{r}_{12}$, $\dot{r}_{23}$, $\dot{p}_{12}$, and $\dot{p}_{23}$ with respect to t_3 are functions of the following partial derivatives, whose sums are denoted as:

$$\dot{r}_{12} = \frac{\partial r_{12}}{\partial t_2} + \frac{\partial r_{12}}{\partial t_1} \qquad \begin{matrix} 1 \rightarrow 2 \\ 2 \rightarrow 3 \end{matrix} \tag{355}$$

$$\ddot{r}_{12} = \frac{\partial \dot{r}_{12}}{\partial t_2} + \frac{\partial \dot{r}_{12}}{\partial t_1} \qquad \begin{matrix} 1 \rightarrow 2 \\ 2 \rightarrow 3 \end{matrix} \tag{356}$$

$$\dddot{r}_{12} = \frac{\partial \ddot{r}_{12}}{\partial t_2} + \frac{\partial \ddot{r}_{12}}{\partial t_1} \qquad \begin{matrix} 1 \rightarrow 2 \\ 2 \rightarrow 3 \end{matrix} \tag{357}$$

$$\dot{p}_{12} = -\frac{\partial r_{12}}{\partial t_1} \qquad \begin{matrix} 1 \rightarrow 2 \\ 2 \rightarrow 3 \end{matrix} \tag{358}$$

$$\ddot{p}_{12} = \frac{\partial \dot{p}_{12}}{\partial t_2} + \frac{\partial \dot{p}_{12}}{\partial t_1} = -\frac{\partial \dot{r}_{12}}{\partial t_1} \qquad \begin{matrix} 1 \rightarrow 2 \\ 2 \rightarrow 3 \end{matrix} \tag{359}$$

$$\dddot{p}_{12} = \frac{\partial \ddot{p}_{12}}{\partial t_2} + \frac{\partial \ddot{p}_{12}}{\partial t_1} = -\frac{\partial \ddot{r}_{12}}{\partial t_1} \qquad \begin{matrix} 1 \rightarrow 2 \\ 2 \rightarrow 3 \end{matrix} \tag{360}$$

where the previously defined quantities $\dot{r}_{12}$ and $\dot{p}_{12}$ have been included for completeness. Equation (358) follows from Eqs. (332), (333), (325), and (328). Substituting Eq. (358) into the first form of Eq. (359), changing the order of differentiation in the second mixed partial derivative, and substituting Eq. (355) gives the second form of Eq. (359). Similarly, the second form of Eq. (360) follows from the second form of Eq. (359) and from Eq. (356).

Using (353) and (354), one obtains

$$\begin{aligned} \frac{d\dot{r}_{12}}{dt_3} &= \frac{\partial \dot{r}_{12}}{\partial t_2}\frac{dt_2}{dt_3} + \frac{\partial \dot{r}_{12}}{\partial t_1}\frac{dt_1}{dt_3} \\ &= \frac{\partial \dot{r}_{12}}{\partial t_2}\left(1 - \frac{\dot{r}_{23}}{c}\right) + \frac{\partial \dot{r}_{12}}{\partial t_1}\left[1 - \frac{1}{c}(\dot{r}_{12} + \dot{r}_{23})\right] \end{aligned} \tag{361}$$

Substituting Eqs. (356) and (359) gives

$$\frac{d\dot{r}_{12}}{dt_3} = \ddot{r}_{12} + \frac{1}{c}(\dot{r}_{12}\ddot{p}_{12} - \dot{r}_{23}\ddot{r}_{12}) \tag{362}$$

Similarly,

$$\frac{d\dot{r}_{23}}{dt_3} = \ddot{r}_{23} + \frac{1}{c}(\dot{r}_{23}\ddot{p}_{23}) \tag{363}$$

The following derivatives are required to order $1/c^0$:

$$\frac{d\dot{p}_{12}}{dt_3} = \ddot{p}_{12} \tag{364}$$

$$\frac{d\dot{p}_{23}}{dt_3} = \ddot{p}_{23} \tag{365}$$

Differentiating Eq. (362) with respect to t_3, using Eqs. (353) and (354), gives

$$\begin{aligned} \frac{d^2\dot{r}_{12}}{dt_3^2} &= \frac{\partial \ddot{r}_{12}}{\partial t_2}\left(1 - \frac{\dot{r}_{23}}{c}\right) + \frac{\partial \ddot{r}_{12}}{\partial t_1}\left[1 - \frac{1}{c}(\dot{r}_{12} + \dot{r}_{23})\right] \\ &\quad + \frac{1}{c}(\dot{r}_{12}\dddot{p}_{12} + \ddot{r}_{12}\ddot{p}_{12} - \dot{r}_{23}\dddot{r}_{12} - \ddot{r}_{23}\ddot{r}_{12}) \end{aligned} \tag{366}$$

Since $1/c^2$ terms are ignored, the $1/c$ terms were differentiated by inspection using Eqs. (353) and (354) equal to unity. Substituting Eqs. (357) and (360) gives

$$\frac{d^2\dot{r}_{12}}{dt_3^2} = \dddot{r}_{12} + \frac{1}{c}[2(\dot{r}_{12}\dddot{p}_{12} - \dot{r}_{23}\dddot{r}_{12}) + \ddot{r}_{12}(\ddot{p}_{12} - \ddot{r}_{23})] \tag{367}$$

Similarly,

$$\frac{d^2\dot{r}_{23}}{dt_3^2} = \dddot{r}_{23} + \frac{1}{c}(2\dot{r}_{23}\dddot{p}_{23} + \ddot{r}_{23}\ddot{p}_{23}) \tag{368}$$

and, to order $1/c^0$,

$$\frac{d^2\dot{p}_{12}}{dt_3^2} = \dddot{p}_{12} \tag{369}$$

$$\frac{d^2\dot{p}_{23}}{dt_3^2} = \dddot{p}_{23} \tag{370}$$

Substituting Eqs. (362–365) and Eqs. (367–370) into Eqs. (351) and (352) gives, for 2-way and 3-way doppler,

$$\left(1-\frac{F_R}{F_T}\right)^{\cdot\cdot}=\frac{1}{c}(\dddot{r}_{12}+\dddot{r}_{23})+\frac{1}{c^2}[\dddot{r}_{12}(\dot{p}_{12}-3\dot{r}_{23})+\dddot{r}_{23}(\dot{p}_{23}-\dot{r}_{12})+3(\dot{r}_{12}\dddot{p}_{12}+\dot{r}_{23}\dddot{p}_{23}+\ddot{r}_{12}\ddot{p}_{12}+\ddot{r}_{23}\ddot{p}_{23}-\ddot{r}_{12}\ddot{r}_{23})] \tag{371}$$

and for 1-way doppler,

$$\left(1-\frac{F_R}{F_T}\right)^{\cdot\cdot}=\frac{1}{c}(\dddot{r}_{23})+\frac{1}{c^2}[\dddot{r}_{23}\dot{p}_{23}+3(\dot{r}_{23}\dddot{p}_{23}+\ddot{r}_{23}\ddot{p}_{23})] \tag{372}$$

The quantities in Eqs. (371) and (372) are defined in Eqs. (355–360). They are computed from:

$$\dot{r}_{12}=\frac{\mathbf{r}_{12}}{r_{12}}\cdot\dot{\mathbf{r}}_{12} \qquad \begin{matrix}1\rightarrow 2\\ 2\rightarrow 3\end{matrix} \tag{373}$$

$$\ddot{r}_{12}=\frac{\mathbf{r}_{12}\cdot\ddot{\mathbf{r}}_{12}+\dot{\mathbf{r}}_{12}\cdot\dot{\mathbf{r}}_{12}-\dot{r}_{12}^2}{r_{12}} \qquad \begin{matrix}1\rightarrow 2\\ 2\rightarrow 3\end{matrix} \tag{374}$$

$$\dddot{r}_{12}=\frac{\mathbf{r}_{12}\cdot\dddot{\mathbf{r}}_{12}+3\dot{\mathbf{r}}_{12}\cdot\ddot{\mathbf{r}}_{12}-3\dot{r}_{12}\ddot{r}_{12}}{r_{12}} \qquad \begin{matrix}1\rightarrow 2\\ 2\rightarrow 3\end{matrix} \tag{375}$$

$$\dot{p}_{12}=\frac{\mathbf{r}_{12}}{r_{12}}\cdot\dot{\mathbf{r}}_1 \qquad \begin{matrix}1\rightarrow 2\\ 2\rightarrow 3\end{matrix} \tag{376}$$

$$\ddot{p}_{12}=\frac{\mathbf{r}_{12}\cdot\ddot{\mathbf{r}}_1+\dot{\mathbf{r}}_1\cdot\dot{\mathbf{r}}_{12}-\dot{p}_{12}\dot{r}_{12}}{r_{12}} \qquad \begin{matrix}1\rightarrow 2\\ 2\rightarrow 3\end{matrix} \tag{377}$$

$$\dddot{p}_{12}=\frac{\mathbf{r}_{12}\cdot\dddot{\mathbf{r}}_1+2\dot{\mathbf{r}}_{12}\cdot\ddot{\mathbf{r}}_1+\dot{\mathbf{r}}_1\cdot\ddot{\mathbf{r}}_{12}-2\dot{r}_{12}\ddot{p}_{12}-\dot{p}_{12}\ddot{r}_{12}}{r_{12}} \qquad \begin{matrix}1\rightarrow 2\\ 2\rightarrow 3\end{matrix} \tag{378}$$

where

$$\mathbf{r}_{ij}=\mathbf{r}_j-\mathbf{r}_i \qquad \mathbf{r}\rightarrow\dot{\mathbf{r}},\ddot{\mathbf{r}},\dddot{\mathbf{r}}$$

Equations (373) and (376) are Eqs. (326), (329), (332), and (333). The remaining equations follow by successive differentiation according to Eqs. (356), (357), (359), and (360).

IX. Range Observables

This section gives the formulation for computation of range observables.

A. Introduction

There are several different range tracking systems. However, all of them are conceptually the same. For each system, an electromagnetic signal is transmitted from a tracking station on earth at time t_1, received and retransmitted by the spacecraft at time t_2, and received by the same tracking station at time t_3. The mathematical representation of the range observable ρ is

$$\rho=(t_3-t_1)_{ST}\,F,\text{ modulo } M$$

where

$(t_3-t_1)_{ST}$ = round-trip time of the signal in seconds of station time ST (derived from the atomic frequency standard at the tracking station)

F = conversion factor from seconds of station time ST to the units of the range observable

M = modulo number. The largest integer multiple of M which is less than $(t_3 - t_1)_{ST} F$ is removed from this quantity, leaving the observable ρ, which is less than M. This operation on a number n will be referred to as "modding" n by M.

The conversion factor F and modulo number M for each ranging system are given in Section IX-C.

The first step in obtaining the computed value of a range observable is to solve the light time equations for the down and up legs of the light path, whose reception time t_3 (ST) is the observation time. This light time solution, described in Section VI, gives the quantities used to compute a precision value of the round-trip light time in seconds of ephemeris time. This precision value is converted to seconds of station time by using the time transformations of Sections II and III. Corrections are added to account for the effects of the troposphere, the ionosphere, and the offset of the tracking point on the antenna from the earth-fixed "station location" on the antenna mount. In addition, the estimated value of a range bias is added. This sum for the round-trip station time is multiplied by F and modded by M, as indicated above. The expression for computing the range observable ρ is given in Section IX-B. The computation of the troposphere, ionosphere, and antenna corrections is described in Section XII.

Section XI contains the formulation for computation of doppler observables from differenced range observables divided by the count time T_c. The required changes to the range observable formulation of this section, which are minor, are described in Section XI.

B. Formulation

The range observable ρ, obtained from any of the tracking systems described in Section IX-C, is computed from:

$$\begin{aligned}\rho = \Bigg\{ & \frac{r_{12}}{c} + \frac{(1+\gamma)\mu_S}{c^3} \ln\left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}}\right) + \frac{r_{23}}{c} + \frac{(1+\gamma)\mu_S}{c^3} \ln\left(\frac{r_2 + r_3 + r_{23}}{r_2 + r_3 - r_{23}}\right) \\ & - (\mathrm{ET} - \mathrm{A1})_{t_3} \qquad + (\mathrm{ET} - \mathrm{A1})_{t_1} \\ & - \delta(\mathrm{ET} - \mathrm{A1})_{t_3} \qquad + \delta(\mathrm{ET} - \mathrm{A1})_{t_1} \\ & - (\mathrm{A1} - \mathrm{UTC})_{t_3} \qquad + (\mathrm{A1} - \mathrm{UTC})_{t_1} \\ & - (\mathrm{UTC} - \mathrm{ST})_{t_3} \qquad + (\mathrm{UTC} - \mathrm{ST})_{t_1} \\ & + \frac{R_c + \Delta_A\rho(t_3) + \Delta_T\rho(t_3) + \Delta_I\rho(t_3) + \Delta_A\rho(t_1) + \Delta_T\rho(t_1) + \Delta_I\rho(t_1)}{10^3 c} \Bigg\} F, \qquad \text{modulo } M \qquad (379)\end{aligned}$$

Equation (379) is evaluated with quantities obtained from the light time solution for the observable, listed after Eq. (314). The epochs of participation t_1, t_2, and t_3 are available in the ET, A1, UTC, UT1, and ST time scales. The quantities r_{12}, r_{23}, r_1, r_2, and r_3 are computed from Eqs. (315–319). The definitions of c, μ_S, and γ follow Eq. (319). The time transformations (ET − A1), (A1 − UTC), and (UTC − ST) are given by Eqs. (93), (95), and (94), respectively. The quantity δ (ET − A1), to be discussed below, represents additional relativity terms of (ET − A1) not contained in Eq. (93), which is used in the general time transformation subroutine of the DPODP.

Each of the four time transformations of Eq. (379) is evaluated with the transmission time t_1 and with the reception time t_3, expressed in one of the two time scales related by the transformation. Either time scale may be used, but the same time scale *must* be used at both t_1 and t_3. The remaining terms of Eq. (379) are

R_c = estimated constant range bias (specified by time block for each station)

$\Delta_i\rho(j)$ = range correction in meters due to $i = A$ (antenna offset), T (troposphere), or I (ionosphere) for down leg ($j = t_3$) or for up leg ($j = t_1$)

The sum of the first two terms of Eq. (379) is the right-hand side of the light time equation for the up leg of the light path (Eq. 313). Similarly, the sum of terms 3 and 4 is the right-hand side of the light time equation for the down leg of the light path (Eq. 314). The sum of these four terms is an accurate expression for the round-trip ephemeris time. The largest error in the computation of this quantity arises from truncation of the epochs of participation beyond a precision[24] of 10^{-7} s.

The maximum conceivable heliocentric velocity of the spacecraft is 1,000 km/s. For this velocity, the maximum error in the computed round-trip ephemeris time due to truncation of the epochs of participation is 1.4×10^{-9} s. The corresponding error in range is 0.4 m round trip or 0.2 m one way. The typical errors are at least one order of magnitude lower than these figures.

An alternative method for obtaining the round-trip ephemeris time would be to subtract the ET values of the epochs of participation t_3 and t_1. However, this difference could be in error by as much as 2×10^{-7} s. The corresponding range error would be 60 m round trip or 30 m one way, which would be unacceptable.

The time transformations of Eq. (379) convert the precision round-trip light time from an interval of ephemeris time to an interval of station time ST. The remaining terms of Eq. (379) account for the effects of the troposphere and the ionosphere, the offset of the tracking point on the antenna from the earth-fixed "station location," and a constant range bias R_c, whose value may be estimated.

Section XI contains the differenced-range doppler formulation, i.e., the formulation for computing doppler observables from differenced range observables divided by the count time. The required analytical change to the range observable formulation consists of a more accurate expression for the (ET − A1) time transformation used to transform the round-trip light time from ephemeris time to station time. The required expression is Eq. (65), which is derived in Appendix B.

The (ET − A1) time transformations in Eq. (379) are evaluated with the general time transformation subroutine of the DPODP. This subroutine computes (ET − A1) from Eq. (93), which consists of the first three terms of Eq. (65). Currently, δ(ET − A1) in Eq. (379) consists only of term 4 of Eq. (65). The following listing gives the maximum contributions to 1-way range ($\rho/2$) from each of terms 3–10 of Eq. (65):

Term No.	Contribution to 1-way range (m/AU of 1-way range)
3	50
4	22
5	0.4
6	0.007
7	1
8	0.02
9	0.6
10	0.01

The observables obtained from the Tau or Mu ranging systems described in Section IX-C have a potential accuracy of about 1 m or slightly better. In order to obtain the maximum benefits from these accurate data types, the computed range observables should have an accuracy of about 0.1 m. For the forthcoming Grand Tour missions to the outer planets, the range to the spacecraft will be several tens of AUs, and all of the relativity terms of Eq. (65) will contribute more than 0.1 m to it (see the listing above). Therefore, terms 5 through 10 of Eq. (65) should be added to δ (ET − A1). There is a small monthly variation in (ET − A1), which is not included in Eq. (65) since it does not significantly affect differenced-range doppler. However, it does affect 1-way range by about 0.05 m/AU. Hence, an expression for computing this term should be derived and added to δ (ET − A1).

The second and fourth terms of Eq. (379) are the relativistic corrections to the light time for the up and down legs of the light path. These terms become very large when the spacecraft approaches superior conjunction and the minimum distance from the up and down legs of the light path to the surface of the sun becomes very small. For this situation with the light ray grazing the sun of radius R and with the earth and the spacecraft at the same distance r from the center of the sun, the relativistic correction to the light time for each leg of the light path is given approximately by

$$\frac{(1+\gamma)\,\mu_S}{c^3} \ln\left[4\left(\frac{r}{R}\right)^2\right]$$

With $\gamma = 1$, its general relativity value, $r = 1\,\text{AU} = 150 \times 10^6$ km, and $R = 0.7 \times 10^6$ km, this 1-way light time correction amounts to 36 km/c. The round-trip range observable is affected by 72 km/c or 240 μs. This is the

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[24] On the 16-decimal digit IBM 7094 computer, time is represented as seconds past January 1, 1950, 0^h to a precision of 0.6×10^{-7} s from 1967 to 1984.

only really large effect of general relativity on earth-based tracking data. Fitting to tracking data obtained from a spacecraft which is in the vicinity of superior conjunction provides this so-called fourth check of general relativity. Presuming that the *observed* minus *computed* range residuals will be vastly smaller when the second and fourth terms of Eq. (379) are turned on, fitting to these tracking data should provide an estimate of the parameter γ and hence, from Eq. (42), the coupling constant ω of the scalar field of the Brans–Dicke theory of gravitation.

C. Ranging Systems

To date, range tracking data have been obtained from five different range tracking systems: the Air Force Eastern Test Range (AFETR) pulse-radar ranging system, the Mark 1 and Mark 1A lunar ranging systems, and the Tau and Mu planetary ranging systems. The latter four systems have operated at tracking stations of the Deep Space Network. The lunar ranging systems are used for lunar missions and during the early phases of planetary missions. The planetary ranging systems are used for all deep space applications. The Mark 1 system has been replaced by the Mark 1A system. The Mu system is the latest research and development planetary ranging system. Both the Tau and Mu ranging systems have a potential accuracy of a few meters and possibly as low as 1 m or slightly better. Table 1 gives the values of the conversion factor F and the modulo number M for each of these systems, where

c = speed of light, km/s

$f_q(t_1)$ = reference oscillator frequency at transmitting station, cycles per second of station time ST (derived from transmitter atomic frequency standard), evaluated at transmission time t_1

n = number of components of ranging code used with Mu ranging system

The frequency $f_q(t_1)$ is the same quantity used in the computation of doppler observables. The number n associated with the Mu system varies from a typical value of 10 to the maximum system capability of 18.

The units of the Mark 1 and 1A observables are referred to as "range units," RU. The length of 1 RU is $10^3 c/(2F)$ meters of range from the tracking station to the spacecraft. For the Mark 1A system, 1 RU $\approx$ 1.04 m. Using the nominal value of $f_q(t_1) = 22 \times 10^6$ Hz gives approximately the same value for the Mark 1 system. The units of the Tau and Mu observables are round-trip nanoseconds and microseconds, respectively.

Table 1. Conversion factors and modulo numbers for ranging systems

Ranging system	Conversion factor F	Modulo number M
AFETR	$\frac{c}{2}$	None
Mark 1	$\frac{1440}{221} f_q(t_1)$	785,762,208
Mark 1A	$96 \times 1{,}496{,}500$	785,762,208
Tau	10^9	$\frac{1.00947}{1.0002} \times 10^9$
Mu	10^6	$\frac{64 \times 2^n}{3 f_q(t_1)} \times 10^6$

The Mark 1 and 1A range observables are modded by approximately 800,000 km in 1-way range to the spacecraft. The corresponding figure for the Tau system is 150,000 km. Using the maximum value of $n = 18$ and $f_q(t_1) = 22 \times 10^6$ Hz, the Mu range observables are modded by about 38,000 km, one way.

The AFETR range observables computed by the DPODP are expressed in one-way kilometers and are not modded; they are used primarily for study purposes.

For all practical purposes, all of the range tracking systems except the Mu system provide a continuous measure of the range observable ρ given by Eq. (379). The Mu system provides one range observable each time the ranging system is initialized during the pass of the spacecraft over the tracking station. Also, it provides a direct measure of the correction to all 2-way doppler observables obtained during the pass due to charged particles of the ionosphere and interplanetary medium.

The output from the Mu ranging system at time t is $\rho_0(t)$, given by

$$\rho_0(t) = \rho(t) - \int_{t_0}^{t} \dot{\rho}\, dt$$

The ranging system is initialized at some epoch t_0 during the pass of the spacecraft over the tracking station. The first term is the range observable ρ obtained at time t. The second term is counted doppler from the epoch t_0 to t. It is the 2-way doppler observable $F2$ of Section VIII multiplied by the count time T_c, which extends from t_0 to t, and with the units converted from those of $F2$ to those of ρ.

In the absence of charged particles, counted doppler is equivalent to differenced range:[25]

$$\int_{t_0}^{t} \dot{\rho}\, dt = \rho(t) - \rho(t_0)$$

and the output from the Mu ranging system would be

$$\rho_0(t) = \rho(t) - [\rho(t) - \rho(t_0)] = \rho(t_0)$$

That is, the output would be constant and equal to the range ρ at the initialization epoch t_0. With charged particles present,

$$\rho(t)_{\text{corrected}} = \rho(t) + \Delta_c\rho(t)$$

and

$$\left[\int_{t_0}^{t} \dot{\rho}\, dt\right]_{\text{corrected}} = [\rho(t) - \rho(t_0)] - [\Delta_c\rho(t) - \Delta_c\rho(t_0)]$$

where $\Delta_c\rho(t)$ is the correction to $\rho(t)$ due to charged particles. The effect of charged particles on counted doppler is the negative of the correction to the corresponding differenced range observables. Thus, when doppler observables are computed from differenced range observables, the sign of the charged particle correction to each range observable must be changed. From the two equations above, the output of the Mu ranging system with charged particles present is

$$\begin{aligned}\rho_0(t) &= \rho(t) + \Delta_c\rho(t)\\ &\quad - [\rho(t) - \rho(t_0)] + [\Delta_c\rho(t) - \Delta_c\rho(t_0)]\\ &= \rho(t_0) + 2\Delta_c\rho(t) - \Delta_c\rho(t_0)\end{aligned}$$

The output at $t = t_0$ is

$$\rho_0(t_0) = \rho(t_0) + \Delta_c\rho(t_0)$$

This quantity is equal to ρ computed from Eq. (379), with the reception time t_3 equal to the initialization epoch t_0. The charged particle correction $\Delta_c\rho(t_0)$ is the round-trip ionospheric correction $[\Delta_I\rho(t_3) + \Delta_I\rho(t_1)]\, F/10^3c$ of Eq. (379). The output $\rho_0(t)$ for $t > t_0$ is not a true range observable with reception time t_0 because the charged particle correction $2\Delta_c\rho(t) - \Delta_c\rho(t_0)$ does not equal the correction $\Delta_c\rho(t_0)$ for a range observable. Thus, the output from the Mu ranging system is a range observable ρ corresponding to Eq. (379) only at an initialization epoch.

The output $\rho_0(t)$ of the Mu ranging system evaluated at an epoch t_2 minus the value at an epoch t_1 is

$$\rho_0(t_2) - \rho_0(t_1) = 2[\Delta_c\rho(t_2) - \Delta_c\rho(t_1)]$$

This quantity is an observed value of twice the charged particle correction to the 2-way doppler observable whose count time T_c extends from t_1 to t_2.

X. Angular Observables

This section gives the formulation for computing angular observables, which are of two types: (1) directly observed angles of the incoming radiation relative to the tracking station's earth-fixed reference coordinate system, and (2) optical angles—topocentric right ascension α and declination δ—obtained from reduction of photographic plates. As opposed to directly observed angles, optical angles do not contain effects due to stellar aberration and atmospheric refraction (to first order).

The directly observed angle pairs are: (1) hour angle HA and declination δ—most DSN stations; (2) azimuth σ and elevation γ—AFETR stations and some DSN stations; (3) X, Y angles—Manned Space Flight Network (MSFN stations); and (4) X', Y' angles—MSFN stations.

The topocentric coordinate systems and unit vectors associated with each directly observed angle pair are described in Section X-A. The formulation for computing the direction of the incoming radiation and each pair of angular observables is given in Section X-B. Corrections to the directly observed angles due to small solve-for rotations of the earth-fixed reference coordinate system are given in Section X-C. Partial derivatives of the angular observables with respect to the heliocentric positions of the spacecraft and the tracking station are given in Section X-D. These will be used in Section XIV to form the partial derivatives of the angular observables with respect to the solve-for parameters.

A. Coordinate Systems and Unit Vectors

The reference coordinate system at each tracking station is rigidly fixed to the earth, and its orientation relative to the true pole, equator, and prime meridian varies with

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[25] However, differences can arise from sources other than charged particles, such as from variations in the electrical path length through the range tracking system which differ from those of the doppler tracking system.

the motion of the pole (see Section VII). The maximum excursion of the earth's axis of rotation from its mean position is about 10 m, and since the latitudes of all tracking stations are low (less than 45 deg), the maximum change in the orientation of the reference coordinate system from its mean orientation relative to the true pole, equator, and prime meridian is about 1 arc second. The maximum attainable accuracy for directly observed angles is about 0.002–0.003 deg or 7–11 arc seconds, and thus the 1-arc second variations due to polar motion may be neglected. Therefore, the computation of directly observed angles is based upon a fixed orientation of the reference coordinate system relative to the true pole, equator, and prime meridian.

*1. **Right ascension, hour angle, and declination.*** Figure 5 shows a rectangular coordinate system centered at the tracking station on earth. The x- and y-axes are parallel to the earth's true equator; the x-axis is toward the true vernal equinox, and the z-axis is parallel to the true axis of rotation of the earth, directed north.

The unit vector **L** is directed from the tracking station at the reception time t_3 to the spacecraft (a free spacecraft or a station on some celestial body other than earth) at its transmission time t_2. The angles α and δ are the right ascension and declination of the spacecraft. The observer's meridian contains the unit vectors **P** and **Q** and makes an angle $(\theta + \lambda)$ with the vernal equinox, where

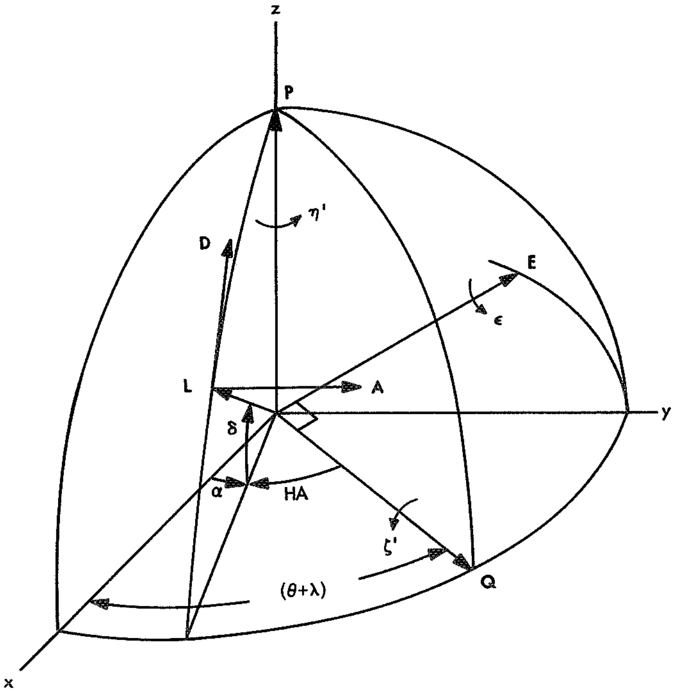

Fig. 5. Right ascension, hour angle, and declination

θ = true sidereal time = Greenwich hour angle of true equinox at reception time t_3

λ = east longitude of tracking station, relative to true pole

The sidereal time θ is computed from Eq. (269) and associated equations, using t_3(UT1) and t_3(ET). The unit vector **E** is normal to **P** and **Q**. The angle HA is the hour angle of the spacecraft.

Nominal computed values of directly observed HA and δ are based upon the geometry of Fig. 5. However, the reference coordinate system **QEP** may be rotated through the small angles ζ' about **Q**, ϵ about **E**, and η' about **P**, thus changing the angle HA in the **QE** plane and the angle δ normal to it. Corrections to the nominal computed values of HA and δ due to the solve-for rotations ζ', ϵ, and η' are given in Section X-C.

The unit vectors **D** and **A** in the directions of increasing declination and right ascension are used in computing the partial derivatives of α, δ, and HA with respect to the estimated parameters. The vector **A** is normal to **L** and **D**. The rectangular components of **D** and **A** along x, y, and z are

$$\mathbf{D} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} -\sin\delta\cos\alpha \\ -\sin\delta\sin\alpha \\ \cos\delta \end{bmatrix} \qquad (380)$$

$$\mathbf{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \qquad (381)$$

*2. **The north–east–zenith coordinate system.*** Figure 6 shows a rectangular coordinate system whose origin coincides with the center of the earth. The x- and y-axes are in the earth's true equator with the x-axis directed toward the true vernal equinox and the z-axis along the instantaneous axis of rotation, directed north. The unit vectors N, E, and Z originate at the tracking station S, whose meridian makes an angle $(\theta + \lambda)$ with the x-axis. The zenith vector Z is contained in the meridian plane and makes an angle ϕ_g with the true equatorial plane, where ϕ_g is the computed geodetic latitude. The north vector N and east vector E are normal to Z and are directed to the north and to the east, respectively. The angle pairs σ–γ, X–Y, and X'–Y' are referred to the rectangular NEZ

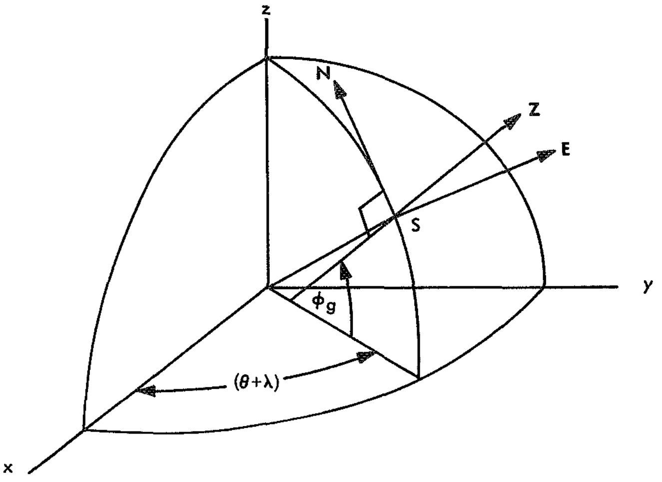

Fig. 6. The north–east–zenith coordinate system

coordinate system at the tracking station. The rectangular components of these unit vectors along x, y, and z are

$$\mathbf{N} = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix} = \begin{bmatrix} -\sin\phi_g \cos(\theta+\lambda) \\ -\sin\phi_g \sin(\theta+\lambda) \\ \cos\phi_g \end{bmatrix} \qquad (382)$$

$$\mathbf{E} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} -\sin(\theta+\lambda) \\ \cos(\theta+\lambda) \\ 0 \end{bmatrix} \qquad (383)$$

$$\mathbf{Z} = \begin{bmatrix} Z_x \\ Z_y \\ Z_z \end{bmatrix} = \begin{bmatrix} \cos\phi_g \cos(\theta+\lambda) \\ \cos\phi_g \sin(\theta+\lambda) \\ \sin\phi_g \end{bmatrix} \qquad (384)$$

The geodetic latitude ϕ_g of the tracking station is computed from

$$\phi_g = \phi + (\phi_g - \phi) \qquad (385)$$

where

ϕ = solve-for geocentric latitude of tracking station, referred to true pole and equator

and $(\phi_g - \phi)$ is computed from

$$(\phi_g - \phi) \approx \frac{e^2 a_e}{r} \sin\phi \cos\phi \times \left[1 + \frac{e^2 a_e}{r} - e^2\left(\frac{2a_e}{r} - \frac{1}{2}\right)\sin^2\phi\right] \qquad (386)$$

where

e = eccentricity of reference spheroid

r = solve-for geocentric radius of tracking station

a_e = mean equatorial radius of earth = 6,378.160 km

The eccentricity e can be computed from the flattening f, using a nominal value of 1/298.25, as

$$e^2 = 2f - f^2 \qquad (387)$$

3. Azimuth and elevation. Figure 7 shows the unit vector **L** in the NEZ coordinate system centered at the tracking station S. The angles σ and γ are the azimuth and elevation, respectively. The reference coordinate system may be rotated through the small angles η about N, ϵ about E, and ζ about Z. Section X-C gives corrections to the computed values of σ and γ as a function of the solve-for rotations η, ϵ, and ζ.

The unit vectors $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{A}}$ (normal to **L**) in the directions of increasing γ and σ, respectively, are used in computing the partial derivatives. The components of $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{A}}$ along N, E, and Z are

$$\tilde{\mathbf{D}} = \begin{bmatrix} \tilde{D}_N \\ \tilde{D}_E \\ \tilde{D}_Z \end{bmatrix} = \begin{bmatrix} -\sin\gamma\cos\sigma \\ -\sin\gamma\sin\sigma \\ \cos\gamma \end{bmatrix} \qquad (388)$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{A}_N \\ \tilde{A}_E \\ \tilde{A}_Z \end{bmatrix} = \begin{bmatrix} -\sin\sigma \\ \cos\sigma \\ 0 \end{bmatrix} \qquad (389)$$

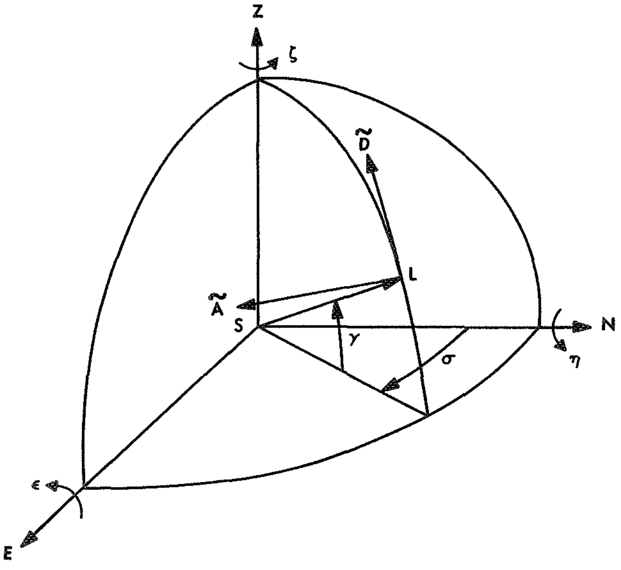

Fig. 7. Azimuth and elevation

Using Eqs. (382–384), the rectangular components of $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{A}}$ referred to the true earth equator and equinox are

$$\tilde{\mathbf{D}} = \begin{bmatrix} \tilde{D}_x \\ \tilde{D}_y \\ \tilde{D}_z \end{bmatrix} = \begin{bmatrix} \sin\gamma\,[\cos\sigma\sin\phi_g\cos(\theta+\lambda) + \sin\sigma\sin(\theta+\lambda)] + \cos\gamma\cos\phi_g\cos(\theta+\lambda) \\ \sin\gamma\,[\cos\sigma\sin\phi_g\sin(\theta+\lambda) - \sin\sigma\cos(\theta+\lambda)] + \cos\gamma\cos\phi_g\sin(\theta+\lambda) \\ -\sin\gamma\cos\sigma\cos\phi_g + \cos\gamma\sin\phi_g \end{bmatrix} \tag{390}$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{A}_x \\ \tilde{A}_y \\ \tilde{A}_z \end{bmatrix} = \begin{bmatrix} \sin\sigma\sin\phi_g\cos(\theta+\lambda) - \cos\sigma\sin(\theta+\lambda) \\ \sin\sigma\sin\phi_g\sin(\theta+\lambda) + \cos\sigma\cos(\theta+\lambda) \\ -\sin\sigma\cos\phi_g \end{bmatrix} \tag{391}$$

4. The X and Y angles for MSFN stations with 6-m (20-ft) antenna. Figure 8 shows the angles X and Y referred to the NEZ reference coordinate system at the tracking station.

The unit vectors $\mathbf{D}'$ and $\mathbf{A}'$ (normal to $\mathbf{L}$) are in the directions of increasing Y and X, respectively. The components of $\mathbf{D}'$ and $\mathbf{A}'$ along the N, E, and Z axes are

$$\mathbf{D}' = \begin{bmatrix} D'_N \\ D'_E \\ D'_Z \end{bmatrix} = \begin{bmatrix} \cos Y \\ -\sin Y\sin X \\ -\sin Y\cos X \end{bmatrix} \tag{392}$$

$$\mathbf{A}' = \begin{bmatrix} A'_N \\ A'_E \\ A'_Z \end{bmatrix} = \begin{bmatrix} 0 \\ \cos X \\ -\sin X \end{bmatrix} \tag{393}$$

Using Eqs. (382–384), the rectangular components of $\mathbf{D}'$ and $\mathbf{A}'$ referred to the true earth equator and equinox are

$$\mathbf{D}' = \begin{bmatrix} D'_x \\ D'_y \\ D'_z \end{bmatrix} = \begin{bmatrix} \sin Y\,[\sin X\sin(\theta+\lambda) - \cos X\cos\phi_g\cos(\theta+\lambda)] - \cos Y\sin\phi_g\cos(\theta+\lambda) \\ -\sin Y\,[\sin X\cos(\theta+\lambda) + \cos X\cos\phi_g\sin(\theta+\lambda)] - \cos Y\sin\phi_g\sin(\theta+\lambda) \\ \cos Y\cos\phi_g - \sin Y\cos X\sin\phi_g \end{bmatrix} \tag{394}$$

$$\mathbf{A}' = \begin{bmatrix} A'_x \\ A'_y \\ A'_z \end{bmatrix} = \begin{bmatrix} -\sin X\cos\phi_g\cos(\theta+\lambda) - \cos X\sin(\theta+\lambda) \\ -\sin X\cos\phi_g\sin(\theta+\lambda) + \cos X\cos(\theta+\lambda) \\ -\sin X\sin\phi_g \end{bmatrix} \tag{395}$$

5. The X′ and Y′ angles for MSFN stations with 26-m (85-ft) antenna. Figure 9 shows the angles X' and Y' referred to the NEZ reference coordinate system at the tracking station.

The unit vectors $\mathbf{D}''$ and $\mathbf{A}''$ (normal to $\mathbf{L}$) are in the directions of increasing Y' and X', respectively. The components of $\mathbf{D}''$ and $\mathbf{A}''$ along the N, E, and Z axes are

$$\mathbf{D}'' = \begin{bmatrix} D''_N \\ D''_E \\ D''_Z \end{bmatrix} = \begin{bmatrix} \sin Y'\sin X' \\ \cos Y' \\ -\sin Y'\cos X' \end{bmatrix} \tag{396}$$

$$\mathbf{A}'' = \begin{bmatrix} A''_N \\ A''_E \\ A''_Z \end{bmatrix} = \begin{bmatrix} -\cos X' \\ 0 \\ -\sin X' \end{bmatrix} \tag{397}$$

The rectangular components of $\mathbf{D}''$ and $\mathbf{A}''$ referred to the true earth equator and equinox are

$$\mathbf{D}'' = \begin{bmatrix} D''_x \\ D''_y \\ D''_z \end{bmatrix} = \begin{bmatrix} -\sin Y' \left[\sin X' \sin \phi_g \cos(\theta + \lambda) + \cos X' \cos \phi_g \cos(\theta + \lambda)\right] - \cos Y' \sin(\theta + \lambda) \\ -\sin Y' \left[\sin X' \sin \phi_g \sin(\theta + \lambda) + \cos X' \cos \phi_g \sin(\theta + \lambda)\right] + \cos Y' \cos(\theta + \lambda) \\ \sin Y' (\sin X' \cos \phi_g - \cos X' \sin \phi_g) \end{bmatrix} \tag{398}$$

$$\mathbf{A}'' = \begin{bmatrix} A''_x \\ A''_y \\ A''_z \end{bmatrix} = \begin{bmatrix} \cos X' \sin \phi_g \cos(\theta + \lambda) - \sin X' \cos \phi_g \cos(\theta + \lambda) \\ \cos X' \sin \phi_g \sin(\theta + \lambda) - \sin X' \cos \phi_g \sin(\theta + \lambda) \\ -\cos X' \cos \phi_g - \sin X' \sin \phi_g \end{bmatrix} \tag{399}$$

B. Computation of Angular Observables

The computation of each pair of angular observables requires the following quantities from the light time solution (see Section VI):

$\mathbf{r}_3, \dot{\mathbf{r}}_3$ = heliocentric position and velocity vectors of tracking station at reception time t_3, with rectangular components referred to mean earth equator and equinox of 1950.0

$\mathbf{r}_E$ = heliocentric position vector of earth at reception time t_3, with rectangular components referred to mean earth equator and equinox of 1950.0

$\mathbf{r}_2$ = heliocentric position vector of spacecraft at transmission time t_2, with rectangular components referred to mean earth equator and equinox of 1950.0

t_3 (ET),
t_3 (UT1) = ET and UT1 values of reception time t_3.

The true sidereal time θ at the reception time t_3 is computed from Eq. (269) and associated equations, using t_3 (UT1) and t_3 (ET).

1. Computation of unit vector **L**. The unit vector **L** will be computed by one procedure for the directly observed angles (hour angle–declination, azimuth–elevation, X–Y, and X'–Y') and will be computed by a second procedure for optical right ascension–declination obtained from the reduction of photographic plates.

a. Directly observed angles. The unit vector **L** is directed from the heliocentric position of the tracking station at the reception time t_3 to the heliocentric position of the spacecraft (a free spacecraft or a station on some celestial body other than the earth) at its transmission time t_2. This vector, with rectangular components referred to the mean earth equator and equinox of 1950.0, is denoted as $\mathbf{L}_{50}$. It is computed from

$$\mathbf{L}_{50} = -\frac{\mathbf{r}_{23}}{r_{23}} \tag{400}$$

where

$$\mathbf{r}_{23} = \mathbf{r}_3 - \mathbf{r}_2 \tag{401}$$

$$r_{23} = (\mathbf{r}_{23} \cdot \mathbf{r}_{23})^{1/2} \tag{402}$$

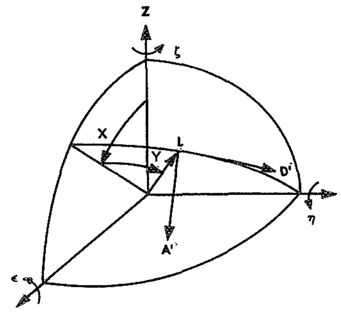

Fig. 8. X and Y angles

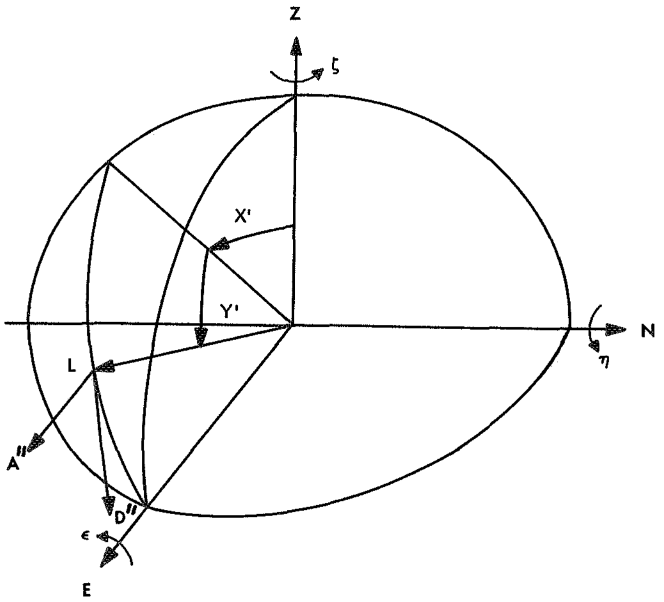

Fig. 9. X' and Y' angles

The unit vector $\mathbf{L}_{50}$ is directed from the station to the spacecraft in the heliocentric space–time frame of reference. In the observer's topocentric space–time frame of reference, the direction to the spacecraft is $\mathbf{L}_{50} + \Delta\mathbf{L}_{50}$, where $\Delta\mathbf{L}_{50}$ can be derived from the Lorentz transformation of special relativity. The following first-order expression for $\Delta\mathbf{L}_{50}$ is the same as that due to the stellar aberration of light, the change in the direction of incoming light due to the heliocentric motion of the tracking station:

$$\Delta\mathbf{L}_{50} = \frac{1}{c}\left[\dot{\mathbf{r}}_3 - (\dot{\mathbf{r}}_3 \cdot \mathbf{L}_{50})\,\mathbf{L}_{50}\right] \qquad (403)$$

where

c = speed of light

The unit vector $\mathbf{L}$ with rectangular components referred to the true equator and equinox of the reception time t_3 is denoted as $\mathbf{L}_{\text{true}}$. It is given by

$$\mathbf{L}_{\text{true}} = N(t_3)\,A(t_3)\,(\mathbf{L}_{50} + \Delta\mathbf{L}_{50}) \qquad (404)$$

where

$A(t_3)$ = precession matrix, transforming rectangular components of a vector referred to the mean earth equator and equinox of 1950.0 to components referred to the mean earth equator and equinox of t_3.

$N(t_3)$ = nutation matrix, transforming rectangular components of a vector referred to the mean earth equator and equinox of t_3 to components referred to the true earth equator and equinox of t_3.

The A and N matrices are a function of ephemeris time and hence are computed from t_3 (ET).

The direction $\mathbf{L}_{\text{true}}$ from Eq. (404) does not account for the bending of the incoming ray due to atmospheric refraction, which increases the elevation angle γ of the incoming ray by $\Delta_r\gamma$. Referring to Fig. 7, the change in $\mathbf{L}$ due to atmospheric refraction is $\Delta_r\gamma\tilde{\mathbf{D}}$. Thus, the unit vector from the observer outward along the incoming ray is given by

$$\mathbf{L}_{\text{true}} = \frac{N(t_3)\,A(t_3)\,(\mathbf{L}_{50} + \Delta\mathbf{L}_{50}) + \Delta_r\gamma\tilde{\mathbf{D}}}{\|N(t_3)\,A(t_3)\,(\mathbf{L}_{50} + \Delta\mathbf{L}_{50}) + \Delta_r\gamma\tilde{\mathbf{D}}\|} \qquad (405)$$

This vector has been normalized since the value of the vector in the numerator is slightly greater than unity. In order to compute $\tilde{\mathbf{D}}$ and $\Delta_r\gamma$, the azimuth σ and elevation γ are required. They are obtained from Eqs. (423–425) using $\mathbf{L}_{\text{true}}$ from Eq. (404). Given σ and γ, the rectangular components of $\tilde{\mathbf{D}}$ referred to the true earth equator and equinox of t_3 are computed from Eq. (390). The refraction correction is computed as a function of the elevation angle γ from the formulation of D. Cain (Ref. 50, pp. 21–22):

$$\Delta_r\gamma = \frac{N_S}{340.0}\,b_1 b_2, \qquad \gamma < 0.17\text{ rad} \qquad (406)$$

$$\Delta_r\gamma = \frac{N_S \times 10^{-6}}{\tan\gamma}, \qquad \gamma \geqq 0.17\text{ rad} \qquad (407)$$

where

N_S = surface refractivity at tracking station (see Subsections XII-B-2-a and -b).

$$b_1 = 1.0 - (1.216 \times 10^5\, b_3\, \gamma_{\text{rad}}) - (51.0 - 300.0\,\gamma_{\text{rad}})\,(b_3)^{1/2} \qquad (408)$$

$$b_2 = \frac{7.0 \times 10^{-4}}{0.0589 + \gamma_{\text{rad}}} - 1.26 \times 10^{-3} \qquad (409)$$

$$b_3 = \frac{1}{10^3\,(r - a_e)} \qquad (410)$$

γ_{rad} = elevation angle, rad

a_e = mean equatorial radius of earth = 6378.160 km

r = geocentric radius to the spacecraft, km $= \|\mathbf{r}_2 - \mathbf{r}_E\|$

b. Optical right ascension and declination. Optical right ascension and declination obtained from the reduction of photographic plates are referred to the mean or true earth equator and equinox of a date t_R, which generally is not equal to the observation time (the reception time t_3). The unit vector $\mathbf{L}$ with rectangular components referred to the mean or true equator and equinox of t_R is computed from

$$\mathbf{L}_{\text{opt (mean)}} = \frac{A(t_R)\,[\mathbf{L}_{50} + A(t_3)^T N(t_3)^T \Delta_r\gamma\tilde{\mathbf{D}}]}{\|A(t_R)\,[\mathbf{L}_{50} + A(t_3)^T N(t_3)^T \Delta_r\gamma\tilde{\mathbf{D}}]\|} \qquad (411)$$

or

$$\mathbf{L}_{\text{opt (true)}} = \frac{N(t_R)\,A(t_R)\,[\mathbf{L}_{50} + A(t_3)^T N(t_3)^T \Delta_r\gamma\tilde{\mathbf{D}}]}{\|N(t_R)\,A(t_R)\,[\mathbf{L}_{50} + A(t_3)^T N(t_3)^T \Delta_r\gamma\tilde{\mathbf{D}}]\|} \qquad (412)$$

where $A(t_R)$ and $N(t_R)$ are the precession and nutation matrices evaluated at the reference time t_R. The vector

$\tilde{\mathbf{D}}$ is computed from Eq. (390) with θ evaluated at t_3 and σ and γ computed from Eqs. (423–425) using $\mathbf{L}$ equal to

$$\mathbf{L}_{\text{true}} = N(t_3)\,A(t_3)\,\mathbf{L}_{50} \tag{413}$$

The right ascension and declination of a *star* obtained from the reduction of photographic plates are free from the effects of stellar aberration and refraction at least to first order. If a second-order plate reduction method is used, the effects of refraction can be removed completely. However, the right ascension and declination of a spacecraft obtained from the reduction of photographic plates are affected to a small extent by refraction because the spacecraft is much nearer than the background stars. The expression for the correction to the computed elevation angle $\Delta_r\gamma$ due to this effect has been derived by D. Cain (Ref. 50, p. 22). However, the sign of the correction is wrong and should be *negative.* The corrected expression is

$$\Delta_r\gamma = -\tan^{-1}\left(\frac{b_4}{r_{23} - b_5}\right) \tag{414}$$

where

$$b_4 = \frac{0.00211}{(\gamma_{\text{rad}} + 0.0598)^{2.42}} \tag{415}$$

$$b_5 = (b_6^2 - a_e^2\cos^2\gamma)^{1/2} - a_e\sin\gamma \tag{416}$$

$$b_6 = a_e + 51.2064 \tag{417}$$

The right ascension and declination of a spacecraft or star obtained from the reduction of photographic plates are not affected by stellar aberration; hence, $\Delta\mathbf{L}_{50}$ does not appear in Eqs. (411–413).

2. ***Computation of observed angles.*** The directly observed angles are computed from $\mathbf{L}$ given by Eq. (405). Optical right ascension and declination are computed from $\mathbf{L}$ given by Eq. (411) or (412). In either case, the rectangular earth equatorial components of $\mathbf{L}$ are denoted below by

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_x \\ \mathbf{L}_y \\ \mathbf{L}_z \end{bmatrix} \tag{418}$$

a. Right ascension and declination. Referring to Fig. 5, compute declination δ from[26]

$$\sin\delta = L_z, \qquad -90\,\text{deg} \leq \delta \leq 90\,\text{deg} \tag{419}$$

and compute right ascension from

$$\sin\alpha = \frac{L_y}{\cos\delta}, \qquad 0\,\text{deg} \leq \alpha \leq 360\,\text{deg} \tag{420}$$

$$\cos\alpha = \frac{L_x}{\cos\delta} \tag{421}$$

b. Hour angle and declination. Compute α and δ from Eqs. (419–421). Compute HA from (see Fig. 5)

$$HA = (\theta + \lambda) - \alpha, \qquad 0\,\text{deg} \leq HA \leq 360\,\text{deg} \tag{422}$$

where

θ = true sidereal time at reception time t_3

λ = east longitude of tracking station, relative to true pole

c. Azimuth and elevation. Compute the unit vectors $\mathbf{N}$, $\mathbf{E}$, and $\mathbf{Z}$ for the reception time t_3 from Eqs. (382–384). Compute the elevation angle γ from (see Fig. 7)

$$\sin\gamma = \mathbf{L}\cdot\mathbf{Z}, \qquad 0\,\text{deg} \leq \gamma \leq 90\,\text{deg} \tag{423}$$

and compute the azimuth σ from

$$\sin\sigma = \frac{\mathbf{L}\cdot\mathbf{E}}{\cos\gamma}, \qquad 0\,\text{deg} \leq \sigma \leq 360\,\text{deg} \tag{424}$$

$$\cos\sigma = \frac{\mathbf{L}\cdot\mathbf{N}}{\cos\gamma} \tag{425}$$

Note that σ is indeterminate for $\gamma = 90\,\text{deg}$.

d. X and Y angles for MSFN stations with a 9-m (30-ft) antenna. Referring to Fig. 8, compute the angle Y from

$$\sin Y = \mathbf{L}\cdot\mathbf{N}, \qquad -90\,\text{deg} \leq Y \leq 90\,\text{deg} \tag{426}$$

and compute the angle X from

$$\sin X = \frac{\mathbf{L}\cdot\mathbf{E}}{\cos Y}, \qquad -90\,\text{deg} \leq X \leq 90\,\text{deg} \tag{427}$$

Note that X is indeterminate for $Y = \pm 90$ deg, which can occur only when the spacecraft is on the horizon.

e. X′ and Y′ angles for MSFN stations with a 26-m antenna. Referring to Fig. 9, compute the angle Y' from

$$\sin Y' = \mathbf{L}\cdot\mathbf{E}, \qquad -90\,\text{deg} \leq Y' \leq 90\,\text{deg} \tag{428}$$

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[26] The angular observables are measured in degrees.

and compute the angle X' from

$$\sin X' = -\frac{\mathbf{L} \cdot \mathbf{N}}{\cos Y'}, \quad -90\,\text{deg} \leq X' \leq 90\,\text{deg} \tag{429}$$

Note that X' is indeterminate for $Y' = \pm 90$ deg, which can occur only when the spacecraft is on the horizon.

C. Corrections Due to Small Rotations of Reference Coordinate System at Tracking Station

The computed angles may not agree with the observed angles because the mathematical representation of the orientation of the reference coordinate system at the tracking station differs from the actual orientation of the coordinate system. The difference in orientation is due to two errors: (1) errors in the mathematical model (primarily the difference between the actual plumb bob direction and the geodetic plumb bob direction computed from a reference ellipsoid of revolution), and (2) errors in orientation of the instrument axes (e.g., alignment of the vertical axis with the plumb bob direction for the azimuth–elevation system).

Formulas are developed for corrections to the computed angles as linear functions of the small rotations of the computed reference coordinate system about each of its three mutually perpendicular axes.

This type of correction does not apply for right ascension and declination obtained from the reduction of photographic plates.

1. Hour angle–declination. Referring to Fig. 5, the reference coordinate system is **QEP** and the rotations are ζ' about the **Q** axis, ϵ about the **E** axis, and η' about the **P** axis. All rotations are in the positive direction, using the right-hand rule.

The dot products of **L** with **Q**, **E**, and **P** are

$$\mathbf{L} \cdot \mathbf{Q} = \cos\delta \cos HA \tag{430}$$

$$\mathbf{L} \cdot \mathbf{E} = -\cos\delta \sin HA \tag{431}$$

$$\mathbf{L} \cdot \mathbf{P} = \sin\delta \tag{432}$$

In terms of the rotations, the variations in the unit vectors are

$$\Delta\mathbf{Q} = \eta'\mathbf{E} - \epsilon\mathbf{P} \tag{433}$$

$$\Delta\mathbf{E} = \zeta'\mathbf{P} - \eta'\mathbf{Q} \tag{434}$$

$$\Delta\mathbf{P} = \epsilon\mathbf{Q} - \zeta'\mathbf{E} \tag{435}$$

The variation in δ due to the variation in **P** is obtained from Eq. (432) as

$$(\cos\delta)\,\Delta\delta = \mathbf{L} \cdot \Delta\mathbf{P} \tag{436}$$

Substituting Eqs. (435), (430), and (431) gives

$$\Delta\delta = \zeta' \sin HA + \epsilon \cos HA \tag{437}$$

From Eq. (430), the variation in HA is given by

$$(\cos\delta \sin HA)\,\Delta HA = -\mathbf{L} \cdot \Delta\mathbf{Q} - (\sin\delta \cos HA)\,\Delta\delta \tag{438}$$

Substituting Eqs. (433), (431), (432), and (437) and simplifying gives

$$\Delta HA = \eta' + \tan\delta\,(\epsilon \sin HA - \zeta' \cos HA) \tag{439}$$

This same equation may be obtained by differentiating Eq. (431).

The meridian plane is determined by the vector **P** to the pole and by the plumb bob line. If the plumb bob is displaced to the west through the angle θ_w, the meridian plane is displaced to the east through the angle

$$\eta' = \frac{\theta_w}{\cos\phi_g}$$

If θ_w is known, this equation provides an *a priori* estimate of η'.

2. Azimuth–elevation. Referring to Fig. 7, the reference coordinate system is **NEZ** and the rotations are η about **N**, ϵ about **E**, and ζ about **Z**.

The variations in the unit vectors due to the rotations are

$$\Delta\mathbf{N} = \epsilon\mathbf{Z} - \zeta\mathbf{E} \tag{440}$$

$$\Delta\mathbf{E} = \zeta\mathbf{N} - \eta\mathbf{Z} \tag{441}$$

$$\Delta\mathbf{Z} = \eta\mathbf{E} - \epsilon\mathbf{N} \tag{442}$$

The variations in elevation γ and azimuth σ due to the variations in the unit vectors are obtained from Eqs. (423–425). The results are

$$\Delta\gamma = \eta \sin\sigma - \epsilon \cos\sigma \tag{443}$$

$$\Delta\sigma = \zeta - \tan\gamma\,(\eta\cos\sigma + \epsilon\sin\sigma) \tag{444}$$

3. Angles X, Y. Referring to Fig. 8, the reference coordinate system and rotations are the same as for the azimuth–elevation system. Using Eqs. (426), (427), (440), and (441), and Fig. 8,

$$\Delta Y = -\zeta\sin X + \epsilon\cos X \tag{445}$$

$$\Delta X = -\eta + \tan Y\,(\epsilon\sin X + \zeta\cos X) \tag{446}$$

4. Angles X′, Y′. The azimuth–elevation reference coordinate system and rotations are also used for the $X'Y'$ system. Using Eqs. (428), (429), (440), and (441), and Fig. 9,

$$\Delta Y' = -\zeta\sin X' - \eta\cos X' \tag{447}$$

$$\Delta X' = -\epsilon + \tan Y'\,(\zeta\cos X' - \eta\sin X') \tag{448}$$

D. Partial Derivatives of Angular Observables With Respect to Heliocentric 1950.0 Position Vectors of Spacecraft and Tracking Station

This section gives the partial derivatives of each angular observable with respect to the rectangular components of the heliocentric position vectors of the spacecraft and tracking station, referred to the mean equator and equinox of 1950.0. These subpartial derivatives will be used in Section XIV to form the partial derivative of each angular observable with respect to the total parameter vector $\mathbf{q}$.

The partial derivatives of the observed angles with respect to $\mathbf{r}_2$, obtained from an examination of Figs. 5 and 7–9 are given below. In these expressions, a subscript 50 after a unit vector indicates that the rectangular components of the vector are referred to the mean earth equator and equinox of 1950.0.

$$\begin{aligned}\frac{\partial\alpha}{\partial\mathbf{r}_2} &= \left(\frac{\partial\alpha}{\partial x_2}, \frac{\partial\alpha}{\partial y_2}, \frac{\partial\alpha}{\partial z_2}\right) = \frac{\mathbf{A}_{50}^T}{r_{23}\cos\delta}\\ &= \left(\frac{A_{x50}}{r_{23}\cos\delta}, \frac{A_{y50}}{r_{23}\cos\delta}, \frac{A_{z50}}{r_{23}\cos\delta}\right)\end{aligned} \tag{449}$$

$$\frac{\partial\delta}{\partial\mathbf{r}_2} = \frac{\mathbf{D}_{50}^T}{r_{23}} \tag{450}$$

$$\frac{\partial HA}{\partial\mathbf{r}_2} = -\frac{\partial\alpha}{\partial\mathbf{r}_2} \tag{451}$$

$$\frac{\partial\sigma}{\partial\mathbf{r}_2} = \frac{\tilde{\mathbf{A}}_{50}^T}{r_{23}\cos\gamma} \tag{452}$$

$$\frac{\partial\gamma}{\partial\mathbf{r}_2} = \frac{\tilde{\mathbf{D}}_{50}^T}{r_{23}} \tag{453}$$

$$\frac{\partial X}{\partial\mathbf{r}_2} = \frac{\mathbf{A}_{50}'^T}{r_{23}\cos Y} \tag{454}$$

$$\frac{\partial Y}{\partial\mathbf{r}_2} = \frac{\mathbf{D}_{50}'^T}{r_{23}} \tag{455}$$

$$\frac{\partial X'}{\partial\mathbf{r}_2} = \frac{\mathbf{A}_{50}''^T}{r_{23}\cos Y'} \tag{456}$$

$$\frac{\partial Y'}{\partial\mathbf{r}_2} = \frac{\mathbf{D}_{50}''^T}{r_{23}} \tag{457}$$

For any of these angles,

$$\frac{\partial\,\text{angle}}{\partial\mathbf{r}_3} = -\frac{\partial\,\text{angle}}{\partial\mathbf{r}_2} \tag{458}$$

For the directly observed angles, compute $\mathbf{D}, \mathbf{A}$ from Eqs. (380) and (381), $\tilde{\mathbf{D}}, \tilde{\mathbf{A}}$ from Eqs. (390) and (391), $\mathbf{D}', \mathbf{A}'$ from Eqs. (394) and (395), and $\mathbf{D}'', \mathbf{A}''$ from Eqs. (398) and (399). These unit vectors all have rectangular components referred to the true equator and equinox of the reception time t_3. Transform the rectangular components of each of these vectors to the mean equator and equinox of 1950.0 as

$$\mathbf{D}_{50} = A^T(t_3)\,N^T(t_3)\,\mathbf{D} \qquad \mathbf{D} \rightarrow \mathbf{A}, \tilde{\mathbf{D}}, \tilde{\mathbf{A}}, \mathbf{D}'', \mathbf{A}'', \tag{459}$$

For optical right ascension and declination, compute $\mathbf{D}, \mathbf{A}$ from Eqs. (380) and (381). For angles referred to the true equator and equinox of the date t_R, transform the rectangular components of $\mathbf{D}$ and $\mathbf{A}$ to the mean earth equator and equinox of 1950.0 as

$$\mathbf{D}_{50} = A^T(t_R)\,N^T(t_R)\,\mathbf{D} \qquad \mathbf{D} \rightarrow \mathbf{A} \tag{460}$$

For angles referred to the mean equator and equinox of the date t_R,

$$\mathbf{D}_{50} = A^T(t_R)\,\mathbf{D} \qquad \mathbf{D} \rightarrow \mathbf{A} \tag{461}$$

Note that the partial derivatives are computed using angles affected by refraction. Strictly, these angles should not include refraction, and the refraction correction should also be differentiated with respect to the position of the spacecraft. Because of the approximations made, the partial derivatives of the angular observables with respect to the positions of the spacecraft and tracking station are accurate to roughly five significant figures for L directed near the zenith and three significant figures for L directed toward the horizon. These figures apply for directly observed angles. For optical angles obtained from the reduction of photographic plates, the secondary refraction correction and hence the error in the partial derivatives approaches zero with increasing range.

XI. Differenced-Range Doppler

A. Introduction

This section gives the formulation for the computation of 1-way, 2-way, and 3-way doppler observables from the difference of two range observables whose reception times are the end and start of the count interval T_c. The computation of accurate doppler observables with this differenced-range doppler formulation requires a computer with a large word length. On the Univac 1108 computer with a double-precision word length of 60 bits or 18 decimal digits, the formulation for the computation of 2-way and 3-way differenced-range doppler is accurate to about 10^{-5} m/s for all count times above a lower limit which varies from about 0.1 to 1.5 s. This formulation was made possible by the derivation (in Appendix B) of an accurate expression (Eq. 65) for the transformation from coordinate time (ephemeris time ET) to proper time on earth (atomic time A1). The computation of accurate 1-way differenced-range doppler requires a similar expression for ET minus A1 obtained from an atomic clock on board the spacecraft. This expression does not exist and the resulting 1-way formulation is accurate to only about 10^{-3} m/s for count times ranging from about 10 s when the spacecraft passes by a planet or the moon at very low altitude to about 1,000 s when the spacecraft is in heliocentric cruise.

The primary advantage of the differenced-range doppler formulation is that there is no upper limit to the count time for 2-way or 3-way doppler, whereas count times used with the current Taylor series formulation (Section VIII) are limited due to truncation of the fourth and higher even derivatives of the doppler frequency shift in the Taylor series expansion. For an accuracy of 10^{-5} m/s, the maximum allowable count time for the Taylor series formulation varies from 1–10 s when the spacecraft passes by a planet or the moon at very low altitude to about 1,000 s when the spacecraft is in heliocentric cruise. The computation of doppler observables to an accuracy of 10^{-5} m/s with the Taylor series formulation thus requires the computation of 43 observables for a 1/2-day pass of the spacecraft over a tracking station during heliocentric cruise. However, preliminary considerations indicate that the information content of a pass of tracking data during heliocentric cruise is not significantly reduced if the count time is increased to about 8,640 s, which requires the computation of only five observables. The use of the differenced-range doppler formulation will allow these very large count times to be used and greatly reduce the number of observables which must be computed and hence the running time of the DPODP. Furthermore, the formulation is much simpler, which further reduces the running time and also decreases the size of the program. The differenced-range doppler formulation will be added to the Univac 1108 version of the DPODP, either as a replacement for or alternative option to the existing Taylor series formulation.

Reference 51 demonstrates the 10^{-5} m/s accuracy of 2-way differenced-range doppler. However, in order to obtain this accuracy for 2-way and also for 3-way doppler, a number of changes to the range observable formulation of Section IX are required. The primary analytical change is the use of the more accurate expression (Eq. 65) for the relativistic transformation from coordinate time (ephemeris time ET) to proper time (atomic time A1). Currently, only the first four terms of this equation are used. The increase in numerical precision from the 16 decimal digits of the IBM 7094 to the 18 decimal digits of the Univac 1108 is required; also, the precision of representation of time must be increased from double- to triple-precision seconds past January 1, 1950,0^h. Alternatively, time could be represented as one single-precision word (8-decimal digits) for the Julian day number plus one double-precision word (18-decimal digits) for seconds past the beginning of the day. It is also recommended that the current type-50 *n*-body ephemeris be replaced by the more accurate type-66 ephemeris or the equivalent.

The expressions for the computation of 1-way, 2-way, and 3-way doppler observables from differenced 1-way, 2-way, and 3-way range observables are derived in Section XI-B. Section XI-C gives the numerical and analytical modifications to the 2-way range observable formulation of Section IX required for the computation of 2-way and 3-way differenced-range doppler. Also, the formulation is modified for an approximate computation of the change in 1-way range during the count time, used to compute 1-way differenced-range doppler.

B. Equivalence of Doppler Observables and Differenced Range Observables

The doppler observables are defined by Eq. (288), repeated here:

$$F = \frac{1}{T_c}\int_{t_{3_m}(\mathrm{ST})-(1/2)T_c}^{t_{3_m}(\mathrm{ST})+(1/2)T_c} (f - f_{\mathrm{bias}})\, dt_3(\mathrm{ST}) \tag{462}$$

The notation is that of Section VIII. Equations (284–286) give the expressions for $f - f_{\mathrm{bias}}$ for 1-way doppler ($F1$), 2-way doppler ($F2$), and 3-way doppler ($F3$), respectively. Substituting these equations into Eq. (462) gives

$$F1 = \frac{C_2 f_{S/C}}{T_c} I - C_2\left[\Delta f_{T_0} + f_{T_1}(t_2 - t_0) + f_{T_2}(t_2 - t_0)^2\right] \tag{463}$$

$$F2 = \frac{C_3 f_q(t_1)}{T_c} I \tag{464}$$

$$F3 = \frac{C_5 f_q(t_1)}{T_c} I \tag{465}$$

where

$$I = \int_{t_{3_m}(\mathrm{ST})-(1/2)T_c}^{t_{3_m}(\mathrm{ST})+(1/2)T_c} \left(1 - \frac{f_R}{f_T}\right) dt_3(\mathrm{ST}) \tag{466}$$

For 2-way or 3-way doppler, f_R/f_T is given by Eq. (293) and

$$I = \int_{t_{3_m}(\mathrm{ST})-(1/2)T_c}^{t_{3_m}(\mathrm{ST})+(1/2)T_c} \left[1 - \frac{dt_1(\mathrm{ST})}{dt_3(\mathrm{ST})}\right] dt_3(\mathrm{ST}) \tag{467}$$

The count time T_c is an interval of reception time; the corresponding transmission interval is denoted by T'_c and has midpoint t_{1_m}. Thus,

$$I = \int_{t_{3_m}(\mathrm{ST})-(1/2)T_c}^{t_{3_m}(\mathrm{ST})+(1/2)T_c} dt_3(\mathrm{ST}) - \int_{t_{1_m}(\mathrm{ST})-(1/2)T'_c}^{t_{1_m}(\mathrm{ST})+(1/2)T'_c} dt_1(\mathrm{ST})$$

$$= T_c - T'_c \tag{468}$$

The epochs corresponding to the start and end of the reception and transmission intervals T_c and T'_c are denoted as

$t_{3_e}(\mathrm{ST})$ = end of reception interval T_c

$t_{3_s}(\mathrm{ST})$ = start of reception interval T_c

$t_{1_e}(\mathrm{ST})$ = end of transmission interval T'_c

$t_{1_s}(\mathrm{ST})$ = start of transmission interval T'_c

Also, define 2-way range ρ_2 and 3-way range ρ_3 as

$$\rho_2 = t_3(\mathrm{ST}) - t_1(\mathrm{ST}) \qquad 2 \rightarrow 3 \tag{469}$$

where

$t_1(\mathrm{ST})$ = transmission time of the crest of a wave at the transmitting station (station time at transmitter)

$t_3(\mathrm{ST})$ = reception time of same crest at receiving station (station time at receiver)

Then, the range ρ with reception time equal to the end of T_c is

$$\rho_{2_e} = t_{3_e}(\mathrm{ST}) - t_{1_e}(\mathrm{ST}) \qquad 2 \rightarrow 3 \tag{470}$$

and the range ρ with reception time equal to the start of T_c is

$$\rho_{2_s} = t_{3_s}(\mathrm{ST}) - t_{1_s}(\mathrm{ST}) \qquad 2 \rightarrow 3 \tag{471}$$

Thus,

$$I = T_c - T'_c = [t_{3_e}(\mathrm{ST}) - t_{3_s}(\mathrm{ST})] - [t_{1_e}(\mathrm{ST}) - t_{1_s}(\mathrm{ST})]$$

$$= \rho_{2_e} - \rho_{2_s} \qquad 2 \rightarrow 3 \tag{472}$$

For 1-way doppler, f_R/f_T is given by Eq. (303) and

$$I = \int_{t_{3_m}(\mathrm{ST})-(1/2)T_c}^{t_{3_m}(\mathrm{ST})+(1/2)T_c} \left[1 - \frac{dt_2(\mathrm{UTC})}{dt_3(\mathrm{ST})}\right] dt_3(\mathrm{ST}) \tag{473}$$

The transmission interval at the spacecraft in UTC seconds (9,192,631,770 $(1 - S)$ cycles[27] of an imaginary cesium atomic clock at the spacecraft) is denoted by T'_c and has midpoint t_{2_m}. Thus,

$$I = \int_{t_{3_m}(\mathrm{ST})-(1/2)T_c}^{t_{3_m}(\mathrm{ST})+(1/2)T_c} dt_3(\mathrm{ST}) - \int_{t_{2_m}(\mathrm{UTC})-(1/2)T'_c}^{t_{2_m}(\mathrm{UTC})+(1/2)T'_c} dt_2(\mathrm{UTC})$$

$$= T_c - T'_c \tag{474}$$

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[27] See Subsection III-A-4.

The epochs corresponding to the start and end of the transmission interval T'_c are denoted as

t_{2_e} (UTC) = end of transmission interval T'_c

t_{2_s} (UTC) = start of transmission interval T'_c

Also, define 1-way range ρ_1 as

$$\rho_1 = t_3\,(\mathrm{ST}) - t_2\,(\mathrm{UTC}) \tag{475}$$

Then,

$$\rho_{1_e} = t_{3_e}\,(\mathrm{ST}) - t_{2_e}\,(\mathrm{UTC}) \tag{476}$$

$$\rho_{1_s} = t_{3_s}\,(\mathrm{ST}) - t_{2_s}\,(\mathrm{UTC}) \tag{477}$$

Thus,

$$\begin{aligned} I = T_c - T'_c &= [t_{3_e}\,(\mathrm{ST}) - t_{3_s}\,(\mathrm{ST})] \\ &\quad - [t_{2_e}\,(\mathrm{UTC}) - t_{2_s}\,(\mathrm{UTC})] \\ &= \rho_{1_e} - \rho_{1_s} \end{aligned} \tag{478}$$

Substituting Eq. (478) into Eq. (463), and Eq. (472) evaluated with ρ_2 and ρ_3 into Eqs. (464) and (465), respectively, gives

$$\begin{aligned} F1 = {} & C_2 f_{S/C} \frac{\rho_{1_e} - \rho_{1_s}}{T_c} \\ & - C_2\,[\Delta f_{T_0} + f_{T_1}(t_2 - t_0) + f_{T_2}(t_2 - t_0)^2] \end{aligned} \tag{479}$$

$$F2 = C_3 f_q(t_1) \frac{\rho_{2_e} - \rho_{2_s}}{T_c} \tag{480}$$

$$F3 = C_5 f_q(t_1) \frac{\rho_{3_e} - \rho_{3_s}}{T_c} \tag{481}$$

In the computation of differenced-range doppler, the epochs at the end and start of the count interval T_c are converted from ST to ET and used to start the light time solutions for ρ_e and ρ_s. This conversion is accomplished using the general time transformation subroutine of the DPODP. This subroutine evaluates (ET − A1) from Eq. (93), which consists of the first three terms of the complete expression for ET − A1 (Eq. 65). The converted epochs t_{3_e} (ET) and t_{3_s} (ET) are in error by $-\delta\,(\mathrm{ET} - \mathrm{A1})_{t_{3_e}}$ and $-\delta\,(\mathrm{ET} - \mathrm{A1})_{t_{3_s}}$, respectively, where

δ (ET − A1) = the last seven terms of Eq. (65). That is, δ (ET − A1) consists of the terms of (ET − A1) not included in the general time transformation subroutine of the DPODP.

The resulting error in differenced-range doppler (DRD) expressed in 1-way meters/second is

$$\begin{aligned} \delta\,\mathrm{DRD} \approx {} & \dot{\rho} \left[\frac{\delta\,(\mathrm{ET} - \mathrm{A1})_{t_{3_s}} - \delta\,(\mathrm{ET} - \mathrm{A1})_{t_{3_e}}}{T_c} \right] \\ & - \frac{1}{2} \ddot{\rho} \left[\delta\,(\mathrm{ET} - \mathrm{A1})_{t_{3_s}} + \delta\,(\mathrm{ET} - \mathrm{A1})_{t_{3_e}} \right] \end{aligned} \tag{482}$$

where $\dot{\rho}$ is the 1-way tracking-station-to-spacecraft range-rate evaluated at the midpoint of the count interval and $\ddot{\rho}$ is the time derivative of $\dot{\rho}$, assumed constant over T_c.

The second term of Eq. (482) has been discussed in Section III. It represents the time derivative of the observable multiplied by the error in the time at which it is evaluated. The largest terms of δ (ET − A1) are the 2-μs daily term and the 1.7-μs monthly term. Furthermore, there are unknown long-period variations in (ET − A1) of the same approximate magnitude due to periodic variations in the heliocentric orbital elements of the earth–moon barycenter arising from perturbations from the other planets. Hence, Eq. (93) for (ET − A1) used in the general time transformation subroutine may be in error by as much as 10^{-5} s. For a spacecraft acceleration of 25 m/s^2 in the vicinity of Jupiter, the resulting error in doppler observables can be as large as 2.5×10^{-4} m/s.

The first term of Eq. (482) is due primarily to neglecting the 2-μs daily term of (ET − A1) in the general time transformation subroutine and has a typical value of about 10^{-5} m/s. It can be eliminated in favor of a much smaller error by a simple modification of T_c used in Eqs. (479–481). If the epochs t_{3_e} (ET) and t_{3_s} (ET), obtained using Eq. (93), are transformed back to ST using Eq. (65) and subtracted, the result is a computed count time given by

$$T_c\,(\mathrm{computed}) = T_c + \delta\,(\mathrm{ET} - \mathrm{A1})_{t_{3_s}} - \delta\,(\mathrm{ET} - \mathrm{A1})_{t_{3_e}} \tag{483}$$

The computation of differenced-range doppler using T_c (computed) rather than T_c in Eqs. (479–481) eliminates the error given by the first term of Eq. (482). However, the computed observable is based upon a count time of T_c (computed) rather than the correct value of T_c. Fortunately, doppler observables vary slowly with T_c and the maximum error is about 10^{-7} m/s, which is negligible.

Thus, differenced-range doppler observables are computed from

$$F1 = C_2 f_{S/C}\left[\frac{\rho_{1_e} - \rho_{1_s}}{T_c\,(\text{computed})}\right] - C_2\left[\Delta f_{T_0} + f_{T_1}(t_2 - t_0) + f_{T_2}(t_2 - t_0)^2\right] \tag{484}$$

$$F2 = C_3 f_q(t_1)\left[\frac{\rho_{2_e} - \rho_{2_s}}{T_c\,(\text{computed})}\right] \tag{485}$$

$$F3 = C_5 f_q(t_1)\left[\frac{\rho_{3_e} - \rho_{3_s}}{T_c\,(\text{computed})}\right] \tag{486}$$

where T_c (computed) is given by Eq. (483). The formulation for computing the 1-way, 2-way, and 3-way range observables at the end and start of the count interval is given in Section XI-C. As in the Taylor series formulation, the variation in $f_{S/C}$ over the transmission interval T'_c for 1-way doppler is ignored. It is computed from Eq. (277) using t_2 equal to the average of t_{2_e} (UTC) and t_{2_s} (UTC) obtained from the light time solutions for ρ_{1_e} and ρ_{1_s}, respectively. This value of t_2 is also used in the second term of Eq. (484). As in Section VIII, the doppler formulation is valid only when $f_q(t_1)$ is constant over T'_c and $f_q(t_3)$, $K_1(t_3)$, and $K_S(t_3)$ are constant over T_c. Also, if T'_c overlaps T_c, $f_q(t_3)$ must equal $f_q(t_1)$. It is recalled that the doppler observable which the data editing program passes on to the orbit determination program is given by Eq. (287), which uses f_{bias} computed from $f_q(t_1)$, $f_q(t_3)$, $K_1(t_3)$, and $K_S(t_3)$ using Eqs. (281–283).

C. Modified Range Observable Formulation

1. Numerical considerations. Each of the computed range observables used to form differenced-range doppler contains random errors due to truncation of time and position beyond the double-precision word length of the computer being used.

The range observables computed by the IBM 7094 version of the DPODP contain a random error of a few millimeters due to truncation of time (seconds past 1950) beyond 16 decimal digits.[28] The corresponding error in differenced-range doppler is a maximum of 3×10^{-3} m divided by the count time (Ref. 51). However, the differenced-range doppler formulation will be added to the Univac 1108 version of the DPODP, which has a double-precision word length of 18 decimal digits (60 bits). The increase in the word length from 54 to 60 bits increases the precision of representation of time from 0.6×10^{-7} s to 10^{-9} s in the interval 1967–1984. This should decrease the time truncation error of differenced-range doppler to about 5×10^{-5} m divided by the count time.

For the desired accuracy of 10^{-5} m/s, the minimum allowable count time is 5 s. Since count times as low as 0.1 s are sometimes used, it is recommended that the representation of time be changed from double-precision to triple-precision seconds past January 1, 1950, 0^h or double-precision seconds past midnight with one single-precision word used for the Julian day number. This will, for all practical purposes, completely eliminate the time truncation error, and allow count times as low as 0.1 s to be used.

In order to utilize the increased precision for representation of time, the accuracy of the light time solution for the epochs of participation of the transmitter and the spacecraft must be increased from the current value of 10^{-7} s to 10^{-12} s. For the maximum conceivable spacecraft velocity of 1,000 km/s, the maximum error in computed range due to an error of 10^{-12} s in the epoch of participation of the spacecraft is 10^{-6} m. The maximum corresponding error in differenced-range doppler is $2 \times 10^{-6}\,\text{m}/T_c$, allowing an accuracy of 10^{-5} m/s to be obtained for all count times above 0.2 s.

On the forthcoming Grand Tour missions to the outer planets, the tracking-station-to-spacecraft range will approach the 50-AU radius of the solar system. For ranges of 29–57 AU, the computed round-trip range (ρ_2 or ρ_3) of 57–114 AU will be represented to a precision of 1.5×10^{-5} m on the 60-bit Univac 1108 computer. Differenced-range doppler may be in error by $3 \times 10^{-5}\,\text{m}/T_c$ (round-trip) or $1.5 \times 10^{-5}\,\text{m}/T_c$ (one way), allowing the desired accuracy of 10^{-5} m/s to be obtained for all count times above 1.5 s. For ranges of 3.5–7 AU, the round-trip range of 7–14 AU is represented to 2×10^{-6} m, and differenced-range doppler may be in error by as much as $2 \times 10^{-6}\,\text{m}/T_c$ (one way). For the desired accuracy of 10^{-5} m/s, count times as low as 0.2 s may be used.

The precomputed n-body ephemeris tapes used by the DPODP are of the so-called type-50 format. They contain modified second and fourth central differences of position

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[28] Time is represented as double-precision (54 bits on the IBM 7094 computer) seconds past January 1, 1950,0^h. From 1967 to 1984, the value of the last bit is 0.6×10^{-7} s. The transmission time, reflection time at the spacecraft, and reception time (in ephemeris time) obtained from the light time solution may be in error by about this amount. Hence, for a spacecraft range rate of 30 km/s, the error in computed range will be about 30 km/s $\times\ 10^6$ mm/km $\times\ 0.6 \times 10^{-7}$ s = 1.8 mm.

and velocity. Interpolation is obtained by the fifth-order Everett's formula. Both the velocity interpolation error, which affects doppler observables computed from the Taylor series formulation, and the differenced position interpolation error divided by the count time, which affects differenced-range doppler, can approach 10^{-5} m/s. This small error could be eliminated by converting to the type-66 *n*-body ephemeris tape format, which contains the full sum and difference array (on acceleration) used to generate the ephemeris. The heliocentric velocity of the spacecraft is affected by errors in interpolation of the heliocentric ephemeris of the center of integration for the spacecraft trajectory, while errors in interpolation of the heliocentric ephemeris of the earth–moon barycenter affect the heliocentric velocity of the tracking station.

*2. **Formulation.*** This section gives the modifications to the 2-way range observable formulation of Section IX which are necessary for the computation of 1-way range ρ_1, 2-way range ρ_2, and 3-way range ρ_3 used in Eqs. (484–486), respectively, to compute 1-way, 2-way, and 3-way differenced-range doppler.

The range observable ρ_{i_e} (where $i = 1$, 2, or 3) is computed from a light time solution with reception time $t_3\,(\text{ST})$ equal to

$$t_3\,(\text{ST}) = t_{3_m}(\text{ST}) + \frac{1}{2}T_c \qquad (487)$$

where

$t_{3_m}(\text{ST})$ = "time tag" for doppler observable

= midpoint of count interval T_c, station time

Similarly, the range observable ρ_{i_s} (where $i = 1$, 2, or 3) is computed from a light time solution with reception time equal to

$$t_3\,(\text{ST}) = t_{3_m}(\text{ST}) - \frac{1}{2}T_c \qquad (488)$$

The 1-way range observables are based upon a 1-leg light time solution, and the 2-way and 3-way range observables are based upon a 2-leg light time solution. As indicated in Subsection XI-C-1, the iteration for the epochs of participation for the spacecraft and transmitter must be continued until the indicated correction to the epoch is less than 10^{-12} s. Aside from this change, the light time solution for each range observable is identical to that described in Section VI.

Since the count intervals for successive doppler observables are contiguous, each light time solution and range observable is used twice: once as ρ_e for the preceding doppler observable and the second time as ρ_s for the succeeding doppler observable.

a. Two-way range ρ_2 and three-way range ρ_3. The 2-way range observables of Section IX are computed from Eq. (379). Considering this equation and the definition (Eq. 469) for 2-way range ρ_2 and 3-way range ρ_3 used to compute differenced-range doppler, it is evident that ρ_2 and ρ_3 may be computed from Eq. (379) using $F = 1$ and $M = \infty$.

The (ET − A1) time transformation in Eq. (379) is evaluated with the general time transformation subroutine of the DPODP using Eq. (93), which consists of the first three terms of Eq. (65). Currently, $\delta\,(\text{ET} - \text{A1})$ in Eq. (379) consists of an approximation of term 4 of Eq. (65) (see Section II after Eq. 70). In order to compute accurate differenced-range doppler, $\delta\,(\text{ET} - \text{A1})$ must be computed from the last seven terms of Eq. (65) so that $(\text{ET}-\text{A1})+\delta\,(\text{ET}-\text{A1})$ will equal Eq. (65) for (ET−A1). This expression was derived in Appendix B specifically for the purpose of computing accurate differenced-range doppler. However, it was shown in Section IX that all of the terms of Eq. (65) are also required in order to compute the range observables to the desired accuracy of 0.1 m.

In the computation of ρ_3 from Eq. (379), evaluation of $\delta\,(\text{ET} - \text{A1})$ at t_1 and t_3 is accomplished using the longitude and spin axis distance of the transmitting and receiving stations, respectively. Similarly, (UTC − ST) is evaluated at t_1 and t_3 using coefficients which apply for the transmitter and receiver, respectively. Since the constant range bias R_c cannot affect differenced-range doppler, it is set equal to zero in the computation of ρ_2 or ρ_3.

The range observables of Section IX represent the time for a signal to travel from the transmitter to the receiver at the group velocity ($\leqq c$). On the other hand, the range observables used to compute differenced-range doppler represent the time for the crest of a wave to travel from the transmitter to the receiver at the phase velocity ($\geqq c$). In the presence of charged particles, the departure of each of these velocities from c is equal in magnitude but opposite in sign. Hence the ionospheric range corrections $\Delta_I\rho\,(t_1)$ and $\Delta_I\rho\,(t_3)$ in Eq. (379) for true range observables will be equal in magnitude but opposite in sign to those for range observables used to compute differenced-range doppler. The corrections for the true range observables will be positive.

Each periodic relativity term of (ET − A1) is evaluated at t_{3_e} and t_{1_e} in the computation of ρ_{2_e} or ρ_{3_e} from Eq. (379) and also at t_{3_s} and t_{1_s} in the computation of ρ_{2_s} or ρ_{3_s} from Eq. (379). The effect of these four values of a periodic term of (ET − A1) on 2-way differenced-range doppler computed from Eq. (485) is

$$\delta\dot{\rho} \leq \frac{2Mc}{T_c}\sin\left(\frac{2\pi}{P}\cdot\frac{T_c}{2}\right)\sin\left(\frac{2\pi}{P}\cdot\frac{\rho}{c}\right) \tag{489}$$

where

$\delta\dot{\rho}$ = effect on $F2$, expressed as 1-way m/s

M = amplitude of periodic term of (ET − A1), s

c = speed of light, m/s

T_c = count time, s

P = period of periodic term of (ET − A1), s

ρ = one-way range to spacecraft, m

The periodic terms of (ET − A1) have periods of 1 day, 1 month, and 1 year. Since the minimum value of P is 1 day and the maximum possible value of T_c is normally about 1/2 day, the argument of the first sine term of Eq. (489) will rarely exceed $\pi/2$. Hence, a rough approximation for this term is its small angle approximation, which gives

$$\delta\dot{\rho} < Mc\left(\frac{2\pi}{P}\right)\sin\left(\frac{2\pi}{P}\cdot\frac{\rho}{c}\right) \tag{490}$$

For a daily term of (ET − A1) and a count time of 1/2 day, the right-hand side of Eq. (490) is 57% greater than that of Eq. (489). However, for a count time of about 1/10 day, which probably will be used with differenced-range doppler, the difference between Eqs. (490) and (489) is negligible.

Equation (490) gives the contribution to 2-way differenced-range doppler from a daily, monthly, or annual term of (ET − A1). It also gives the contribution to 3-way differenced-range doppler from a monthly or annual term of (ET − A1). The contribution from a daily term is given by

$$\delta\dot{\rho} < Mc\left(\frac{2\pi}{P}\right)\sin\left(\frac{2\pi}{P}\cdot\frac{\rho}{c} + \frac{\Delta\lambda}{2}\right) \tag{491}$$

where

$\Delta\lambda$ = east longitude of receiving station minus that of transmitting station

For a daily term of (ET − A1), the argument of the sine term of Eq. (490) approaches $\pi/2$ as ρ approaches the 40–50 AU radius of the solar system. The argument of the sine term of Eq. (491) can also approach $\pi/2$. However, the range at which this occurs depends upon the separation in longitude $\Delta\lambda$ of the receiving and transmitting stations. The maximum effect of a diurnal term of (ET − A1) on 2-way or 3-way differenced-range doppler is thus

$$\delta\dot{\rho} < M\left(\frac{2\pi}{P}\right)c \tag{492}$$

The maximum effect from the 2-μs daily term of ET − A1 (term 4 of Eq. 65) is 0.05 m/s.

For a monthly or annual term of (ET − A1), the argument of the sine term in Eq. (490) is very small. Hence, this term may be replaced by its small angle approximation, and Eq. (490) becomes

$$\delta\dot{\rho} < M\left(\frac{2\pi}{P}\right)^2 \rho \tag{493}$$

For a range of 50 AU, the maximum effect of the monthly term of Eq. (65) (term 9) on 2-way or 3-way differenced-range doppler is about 7.5×10^{-5} m/s; the annual term (term 3) contributes about 5×10^{-4} m/s. The contribution from the 2-μs daily term of Eq. (65), computed from Eq. (493), is 0.08 m/s, whereas the actual upper limit computed from Eq. (492) is 0.05 m/s. The ratio 0.05/0.08 is $(\sin x)/x$ evaluated at $x = \pi/2$. For a range of 10 AU or less, Eq. (493) is a fairly accurate representation of the contribution from a daily term of (ET − A1) to 2-way differenced-range doppler.

In Appendix B, Eq. (493) is used to determine which terms should be retained in the final expression for ET − A1 (Eq. 65). All terms affecting 2-way differenced-range doppler by more than 2×10^{-7} m/s/AU of range to the spacecraft are retained. Several terms of this magnitude are neglected, and the resulting error in differenced-range doppler is no more than 10^{-6} m/s/AU or 5×10^{-5} m/s at 50 AU (using Eq. 493).

b. One-way range ρ_1. From the definition (Eq. 475) for 1-way range ρ_1, it may be obtained from Eq. (379) (used to compute the range observables of Section IX) by removing the terms associated with the up leg of the light path, evaluating the time transformations with subscript

t_1 at the spacecraft transmission time t_2, deleting the resulting term $(\mathrm{UTC} - \mathrm{ST})_{t_2}$, and by setting $R_c = 0$, $F = 1$, and $M = \infty$. The result is

$$\begin{aligned}\rho_1 = {} & \frac{r_{23}}{c} + \frac{(1+\gamma)\,\mu_S}{c^3}\ln\left(\frac{r_2 + r_3 + r_{23}}{r_2 + r_3 - r_{23}}\right)\\ & - (\mathrm{ET} - \mathrm{A1})_{t_3} + (\mathrm{ET} - \mathrm{A1})_{t_2}\\ & - \delta\,(\mathrm{ET} - \mathrm{A1})_{t_3}\\ & - (\mathrm{A1} - \mathrm{UTC})_{t_3} + (\mathrm{A1} - \mathrm{UTC})_{t_2}\\ & - (\mathrm{UTC} - \mathrm{ST})_{t_3}\\ & + \frac{\Delta_A\rho\,(t_3) + \Delta_T\rho\,(t_3) + \Delta_I\rho\,(t_3)}{10^3 c} \qquad (494)\end{aligned}$$

The $(\mathrm{ET} - \mathrm{A1})$ time transformation at the reception time t_3, i.e., $(\mathrm{ET} - \mathrm{A1})_{t_3} + \delta\,(\mathrm{ET} - \mathrm{A1})_{t_3}$, relates A1 time at the tracking station to ET. It is evaluated with Eq. (65), which applies for A1 time derived from any fixed atomic clock on earth. However, an expression is not available for evaluating $(\mathrm{ET} - \mathrm{A1})_{t_2}$, which relates A1 time obtained from an atomic clock on board the spacecraft (9,192,631,770 cycles from a cesium atomic clock equals one A1 second) to ET. The differential equation relating these two time scales is Eq. (64). With a slight change in notation,

$$\begin{aligned}\frac{d\mathrm{A1}}{d\mathrm{ET}} = 1 & - \frac{\phi_{S/C} - \bar{\phi}_E}{c^2}\\ & - \frac{1}{2}\,\frac{\mathring{s}^2_{S/C} - \bar{\mathring{s}}^2_E}{c^2} + \frac{\Delta f_{\mathrm{cesium}}}{f_{\mathrm{cesium}}} \qquad (495)\end{aligned}$$

where

$\phi_{S/C}$ = Newtonian potential at spacecraft

$\mathring{s}_{S/C}$ = heliocentric velocity of spacecraft

$\bar{\phi}_E$ = average value of Newtonian potential at a fixed point on earth

$\bar{\mathring{s}}^2_E$ = average value of square of heliocentric velocity of a fixed point on earth

It would be extremely difficult to integrate Eq. (495) to obtain an expression for $\mathrm{ET} - \mathrm{A1}$ obtained from the spacecraft atomic clock which would be valid for the trajectory of any spacecraft. From Eq. (64), the average rate of an A1 clock on earth is equal to the rate of an ET clock (if $\Delta f_{\mathrm{cesium}} = 0$). However, from Eq. (495), the rate of an A1 clock on board a spacecraft will be significantly different from the rate of an ET clock if the heliocentric distance and velocity of the spacecraft are significantly different from 1 AU and 30 km/s, respectively. Under these conditions, the term $(\mathrm{ET} - \mathrm{A1})_{t_{2_e}}$ of ρ_{1_e} will differ significantly from the term $(\mathrm{ET} - \mathrm{A1})_{t_{2_s}}$ of ρ_{1_s}. The remainder of this section gives an approximate formulation for computing the difference between these two terms and also the range change $\rho_{1_e} - \rho_{1_s}$ used in Eq. (484) for 1-way differenced-range doppler.

Define a modified 1-way range ρ_1^* as

$$\rho_1^* = \rho_1 - (\mathrm{ET} - \mathrm{A1})_{t_2} \qquad (496)$$

It is computed from Eq. (494) for ρ_1 with the term $(\mathrm{ET} - \mathrm{A1})_{t_2}$ omitted. Then,

$$\rho_{1_e} - \rho_{1_s} = \rho_{1_e}^* - \rho_{1_s}^* + (\mathrm{ET} - \mathrm{A1})_{t_{2_e}} - (\mathrm{ET} - \mathrm{A1})_{t_{2_s}} \qquad (497)$$

or

$$\begin{aligned}\rho_{1_e} - \rho_{1_s} = {} & \rho_{1_e}^* - \rho_{1_s}^*\\ & + [t_{2_e}(\mathrm{ET}) - t_{2_s}(\mathrm{ET})]\\ & - [t_{2_e}(\mathrm{A1}) - t_{2_s}(\mathrm{A1})] \qquad (498)\end{aligned}$$

The last two terms represent the transmission interval T'_c at the spacecraft in the ET and A1 time scales, respectively. The last term is evaluated as the product of the next-to-last term and an approximation to the average value of $d\mathrm{A1}/d\mathrm{ET}$ from Eq. (495) over T'_c. The light time solutions for $\rho_{1_e}^*$ and $\rho_{1_s}^*$ allow the computation of the Newtonian potential at the spacecraft $\phi_{S/C}$ and the square of the heliocentric velocity of the spacecraft $\mathring{s}^2_{S/C}$ at the epochs t_{2_e} and t_{2_s}. The potential $\phi_{S/C}$ is computed from Eq. (338) as indicated after that equation. Assuming a linear variation in these quantities over T'_c, their average values are

$$\phi_{S/C} = \frac{1}{2}\left[(\phi_{S/C})_{t_{2_e}} + (\phi_{S/C})_{t_{2_s}}\right] \qquad (499)$$

$$\mathring{s}^2_{S/C} = \frac{1}{2}\left[(\mathring{s}^2_{S/C})_{t_{2_e}} + (\mathring{s}^2_{S/C})_{t_{2_s}}\right] \qquad (500)$$

Substituting these quantities into Eq. (495) gives the approximation to the average value of $d\mathrm{A1}/d\mathrm{ET}$ over T'_c. Using $d\mathrm{A1}/d\mathrm{ET}$ as indicated above to evaluate the last term of Eq. (498) gives

$$\rho_{1_e} - \rho_{1_s} = \rho_{1_e}^* - \rho_{1_s}^* + [t_{2_e}(\mathrm{ET}) - t_{2_s}(\mathrm{ET})]\left(1 - \frac{d\mathrm{A1}}{d\mathrm{ET}}\right) \qquad (501)$$

Substituting Eq. (495) gives

$$\rho_{1_e} - \rho_{1_s} = \rho^*_{1_e} - \rho^*_{1_s} + \left[t_{2_e}(\mathrm{ET}) - t_{2_s}(\mathrm{ET})\right] \left\{\frac{1}{c^2}\left[\left(\phi_{S/C} + \frac{1}{2}\dot{s}^2_{S/C}\right) - \left(\bar{\phi}_E + \frac{1}{2}\overline{\dot{s}^2_E}\right)\right] - \frac{\Delta f_{\mathrm{cesium}}}{f_{\mathrm{cesium}}}\right\} \tag{502}$$

Since the mean distance of an inner planet from the earth is about 1 AU and the mean distance of an outer planet from the earth is approximately equal to the semi-major axis of its heliocentric orbit, the average value of ϕ_E is given approximately by

$$\bar{\phi}_E \approx \frac{1}{A_E}\left(\mu_S + \mu_{Me} + \mu_V + \frac{\mu_{Ma}}{1.5} + \frac{\mu_J}{5.203} + \frac{\mu_{Sa}}{9.54} + \frac{\mu_U}{19} + \frac{\mu_N}{30}\right) + \frac{\mu_M}{384{,}400\,\mathrm{km}} + \frac{\mu_E}{r} \tag{503}$$

where

$\mu_S, \mu_{Me}, \mu_V, \mu_{Ma}, \mu_J,$
$\mu_{Sa}, \mu_U, \mu_N, \mu_M, \mu_E =$ gravitational constants for the sun, Mercury, Venus, Mars, Jupiter, Saturn, Uranus, Neptune, the moon, and the earth, respectively, km^3/s^2:

$$\mu_S = 1{,}327.1250 \times 10^8$$

$$\mu_{Me} = 0.0002 \times 10^8$$

$$\mu_V = 0.0032 \times 10^8$$

$$\mu_{Ma} = 0.0004 \times 10^8$$

$$\mu_J = 1.267 \times 10^8$$

$$\mu_{Sa} = 0.379 \times 10^8$$

$$\mu_U = 0.058 \times 10^8$$

$$\mu_N = 0.069 \times 10^8$$

$$\mu_M = 4{,}902.78$$

$$\mu_E = 398{,}601.2$$

$A_E =$ the number of kilometers per astronomical unit AU

$= 149{,}597{,}900$ km

$r =$ geocentric radius of tracking station, km

In Eq. (503), the gravitational constant of each outer planet is divided by the semimajor axis of its heliocentric orbit expressed in AU, and the gravitational constant of the moon is divided by the mean distance to the moon. Substituting numerical values gives

$$\bar{\phi}_E \approx 887.336 + \frac{398601}{r} \qquad \mathrm{km^2/s^2} \tag{504}$$

From Eq. (B-14) and associated equations of Appendix B, the average value of $\dot{s}^2_E$ is given approximately by

$$\overline{\dot{s}^2_E} \approx \frac{\mu_S + \mu_E}{A_E} + u^2\dot{\theta}^2_M \tag{505}$$

where

$u =$ distance of tracking station from earth's spin axis, km

$\dot{\theta}_M =$ mean sidereal rate (see Eq. 273)

$= 0.729{,}212 \times 10^{-4}$ rad/s

Substituting numerical values gives

$$\overline{\dot{s}^2_E} \approx 887.131 + 0.532 \times 10^{-8}\, u^2 \qquad \mathrm{km^2/s^2} \tag{506}$$

Dividing Eq. (506) by 2 and adding the result to Eq. (504) gives

$$\bar{\phi}_E + \frac{1}{2}\overline{\dot{s}^2_E} \approx 1330.90 + \frac{398601}{r} + 0.266 \times 10^{-8}\, u^2 \qquad \mathrm{km^2/s^2} \tag{507}$$

This equation is accurate to 0.01 km^2/s^2, a value that affects the spacecraft range rate by 3×10^{-5} m/s.

The range change $\rho_{1_e} - \rho_{1_s}$ used in Eq. (484) to compute 1-way differenced-range doppler is given by Eq. (502) using $\rho^*_{1_e}$ and $\rho^*_{1_s}$ computed from Eq. (494) with the term $(\mathrm{ET} - \mathrm{A1})_{t_2}$ omitted, $\phi_{S/C}$ from Eq. (499), $\dot{s}^2_{S/C}$ from Eq. (500), and $(\bar{\phi}_E + \frac{1}{2}\overline{\dot{s}^2_E})$ from Eq. (507). The times t_{2_e}(ET) and t_{2_s}(ET) are available from the light time solutions for $\rho^*_{1_e}$ and $\rho^*_{1_s}$, respectively.

The 1-way differenced-range doppler formulation is based upon the assumption that $(\phi_{S/C} + \frac{1}{2}\dot{s}^2_{S/C})$ varies linearly over the transmission interval T'_c. The resulting error in the observable varies directly with the departure from linearity (the second derivative of $\phi_{S/C} + \frac{1}{2}\dot{s}^2_{S/C}$) and with the square of T'_c. An accuracy of at least 10^{-3} m/s can be achieved if the count time T_c does not exceed approximately 10 s when the spacecraft passes by a planet or the moon at extremely small altitudes or 1000 s in heliocentric cruise. This is approximately the range of count times used with the Taylor series formulation of Section VIII. The 1-mm/s accuracy for computed 1-way doppler is acceptable, since this data type is currently derived from a crystal oscillator on board the spacecraft rather than an atomic frequency standard.

XII. Antenna, Troposphere, and Ionosphere Corrections to Observables

Section XII-A defines the correction terms for the range, doppler, and angular observables which account for the effects of (1) the offset of the tracking point on the moving antenna from the earth-fixed "station location" (see Section VII), (2) the troposphere, and (3) the ionosphere. The evaluation of these corrections is described in Section XII-B. Expressions are given for the antenna and the troposphere corrections. The general procedure for obtaining the ionosphere corrections is summarized.[29]

A. Definitions of Correction Terms

1. Range observables. The range observables (see Section IX) are computed from Eq. (379). The quantity in braces represents the time for the signal (ranging code) to travel from the tracking station to the spacecraft and return, in seconds of station time. In the presence of charged particles, this signal travels at the group velocity ($<c$). The range corrections $\Delta_A\rho$, $\Delta_T\rho$, and $\Delta_I\rho$ in meters divided by 10^3c (where c is the speed of light in km/s) represent the time delay in seconds due to the antenna offset, the troposphere, and the ionosphere, respectively. Each type of correction $\Delta_i\rho$ has a value $\Delta_i\rho(t_3)$ for the down leg of the light path and a value $\Delta_i\rho(t_1)$ for the up leg.

The antenna corrections $\Delta_A\rho(t_1)$ and $\Delta_A\rho(t_3)$ represent the distance along the light path from the "station location" to the actual tracking point on the antenna at the transmission time t_1 and reception time t_3, respectively. Addition of these corrections changes the round-trip light time based upon transmission and reception at the station location to the light time based upon transmission and reception at the actual tracking point on the antenna.

The troposphere corrections $\Delta_T\rho(t_1)$ and $\Delta_T\rho(t_3)$ account for the increase in round-trip light time due to the reduction in propagation speed below c and the increase in path length due to bending when passing through the troposphere.

The ionosphere corrections $\Delta_I\rho(t_1)$ and $\Delta_I\rho(t_3)$ account for the increase in light time due to propagation through the charged particles of the ionosphere at the group velocity, which is less than c.

2. Doppler observables. Equations (308), (309), and (310) for 1-way, 2-way, and 3-way doppler observables contain a term Δ which accounts for the effects of antenna offsets, the troposphere, and the ionosphere. The expression for Δ is obtained by comparing these equations to the equivalent differenced-range doppler formulation of Section XI, which contains correction terms for these effects.

Differenced-range doppler is computed from the difference of two range observables whose reception times are the end and start of the count interval T_c. Each of these range observables represents the time for the crest of a wave to travel from the transmitter to the receiver. In the presence of charged particles, the propagation speed for the crest of a wave is the phase velocity, which is greater than c.

As with the true range observables of Section IX, the range corrections $\Delta_A\rho$, $\Delta_T\rho$, and $\Delta_I\rho$ in meters divided by 10^3c represent the time delay in seconds due to the antenna offset, the troposphere, and the ionosphere, respectively. For 2-way and 3-way range used to compute 2-way and 3-way differenced-range doppler, respectively, each of these corrections has a value $\Delta_i\rho(t_3)$ for the down leg of the light path and a value $\Delta_i\rho(t_1)$ for the up leg. For 1-way range used to compute 1-way differenced-range doppler, there are no up-leg corrections.

The antenna and troposphere corrections are the same as those described in Subsection XII-A-1 above for the true range observables of Section IX. The ionosphere corrections have the same magnitude as those for true range observables but with the opposite sign, because charged particles cause the phase velocity to increase above c by the same amount that the group velocity decreases below c. Hence, charged particles of the ionosphere cause the range code for true range observables to arrive late by $[\Delta_I\rho(t_3) + \Delta_I\rho(t_1)]/(10^3c)$ seconds and the crest of a wave transmitted and received by the doppler tracking equipment to arrive early by the same amount. Thus, the ionosphere corrections for range observables used to compute differenced-range doppler are negative.

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[29] Details are available in Ref. 59.

Comparing the correction terms of the differenced-range doppler formulation (Section XI) to the correction term Δ of the Taylor series doppler formulation (Eqs. 308–310) gives, for 2-way or 3-way doppler,

$$\Delta = \frac{1}{10^3 c\, T_c}\left[\Delta\rho(t_{3_e}) + \Delta\rho(t_{1_e}) - \Delta\rho(t_{3_s}) - \Delta\rho(t_{1_s})\right] \tag{508}$$

where

c = speed of light, km/s

T_c = count interval, s

t_{3_e} = epoch at end of reception interval T_c

t_{3_s} = epoch at start of reception interval T_c

t_{1_e} = epoch at end of transmission interval T'_c

t_{1_s} = epoch at start of transmission interval T'_c

and

$\Delta\rho(t)$ = sum of range corrections in meters due to the antenna offset, the troposphere, and the ionosphere for up leg with transmission time t or for down leg with reception time t

That is,

$$\Delta\rho(t) = \Delta_A\rho(t) + \Delta_T\rho(t) + \Delta_I\rho(t) \tag{509}$$

As mentioned above, the antenna and troposphere corrections are the same as those used for a range observable; the ionosphere correction has the same magnitude but the opposite sign (negative in Eqs. 508–509) as that used for a range observable. For 1-way doppler, the light path consists of a down leg only and

$$\Delta = \frac{1}{10^3 c\, T_c}\left[\Delta\rho(t_{3_e}) - \Delta\rho(t_{3_s})\right] \tag{510}$$

Given the midpoint t_{3_m} of the reception interval T_c in any time scale, the epochs t_{3_e} and t_{3_s} in the same time scale are given to sufficient accuracy by

$$t_{3_e} = t_{3_m} + \frac{1}{2} T_c \tag{511}$$

$$t_{3_s} = t_{3_m} - \frac{1}{2} T_c \tag{512}$$

where T_c is given in seconds of station time (ST). The light time solution for the doppler observable has a reception time t_{3_m} and a transmission time t_{1_m} which is the midpoint of the transmission interval T'_c. Given t_{1_m} in any time scale, t_{1_e} and t_{1_s} in the same time scale are given approximately by

$$t_{1_e} \approx t_{1_m} + \frac{1}{2} T_c \tag{513}$$

$$t_{1_s} \approx t_{1_m} - \frac{1}{2} T_c \tag{514}$$

3. *Angular observables.* The formulation of Section X for computing directly observed angles contains an expression for the increase in the elevation angle $\Delta_r\gamma$ of the incoming ray due to bending of the ray by the troposphere. Specifically, $\Delta_r\gamma$ is the elevation angle of the incoming ray minus the elevation angle of the straight line path from the tracking station to the spacecraft.

B. Evaluation of One-Leg Range Corrections

This section gives the formulation for computation of corrections to the 1-way range from the tracking station to the spacecraft due to (1) the offset of the tracking point on the antenna from the station location, $\Delta_A\rho$; (2) the troposphere, $\Delta_T\rho$; and (3) the ionosphere, $\Delta_I\rho$. As descibed in Subsection XII-A-1, the range observable formulation includes these corrections for the up and down legs of the light path. From Subsection XII-A-2, the doppler observable formulation includes these corrections for the up and down legs of the light paths whose reception times are the end and start of the reception interval T_c.

1. *Antenna correction.* The antennas at the tracking stations of the DSN, MSFN, and AFETR have four different types of mounts: (1) hour angle and declination (HA-dec); (2) azimuth and elevation (az-el); (3) X and Y angles (MSFN); and (4) X' and Y' angles (MSFN). These angles are defined in Section X, Figs. 5–9. For the 26-m (85-ft) HA-dec, az-el, and X'–Y' antennas, the two mutually perpendicular axes do not intersect. The offset between the two axes (the perpendicular distance between them) is denoted by b and ranges from about 1 to 7 m. The axis which has a fixed position relative to the earth will be denoted as the primary axis (the HA, az, or X' axis). Due to the offset b between the two axes, rotation of the antenna about the primary axis causes the secondary axis to move relative to the earth.

Figure 10 shows the two mutually perpendicular axes of a HA-dec, az-el, or X'–Y' antenna. The primary axis

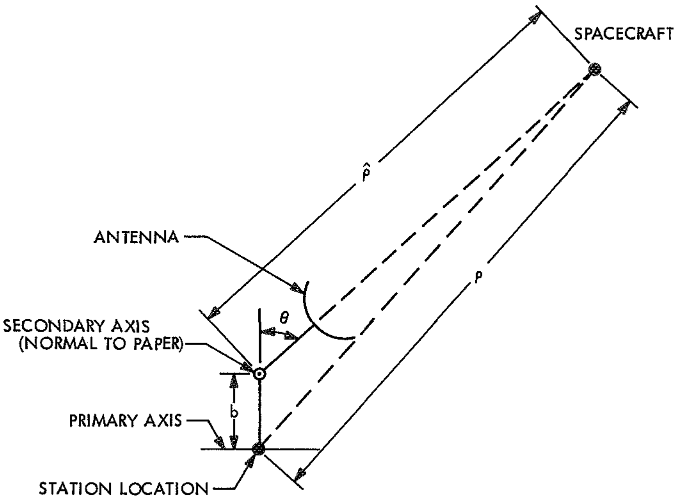

Fig. 10. Antenna correction

(HA, az, or X') is in the plane of the paper, and the secondary axis (dec, el, or Y') is normal to it. The offset between the two axes is b. The positions of the station location and spacecraft are indicated. The secondary angle (dec, el, or Y') is indicated by θ.

Each range tracking system is calibrated so that the tracking point lies on the secondary (moving) axis. That is, the calibrated range observable obtained from the tracking station corresponds to a 1-way range $\hat{\rho}$ measured from the secondary axis to the spacecraft. However, the computed range observable is based upon the 1-way range ρ (i.e.,r_{12} or r_{23} of Eq. 379) measured from a specific point on the antenna which is fixed relative to the earth. This point is called the station location. From Section VII, its geocentric position is represented by spherical or cylindrical coordinates, which are solve-for parameters. For all antennas, the station location is the intersection of the primary axis with the plane perpendicular to it which contains the secondary axis.

From Eq. (379), the computed range for the up or down leg of the light path is r_{12} or r_{23} (denoted as ρ in Fig. 10) plus $\Delta_A\rho$ for that leg. The sum $\rho + \Delta_A\rho$ must equal $\hat{\rho}$. Hence, the antenna correction $\Delta_A\rho$ is given by

$$\Delta_A\rho = \hat{\rho} - \rho \tag{515}$$

The maximum displacement of the secondary axis from the tracking station to spacecraft line is less than 10 m. The maximum effect of this transverse displacement upon $\hat{\rho} - \rho$ is about 0.5×10^{-3} m (for a spacecraft range of 10^5 m) which is insignificant. Thus, the significant part of $\hat{\rho} - \rho$ is due to the component of b along the direction to the spacecraft. Since $b < 10$ m and $\rho > 10^5$ m,

$$\hat{\rho} \approx \rho - b\cos\theta \tag{516}$$

to an accuracy of better than 10^{-3} m and

$$\Delta_A\rho = -b\cos\theta \tag{517}$$

From Eq. (508), the doppler observable formulation includes antenna corrections for the up and down legs of the light paths which have reception times equal to the end and start of the reception interval T_c. The tracking point for doppler observables is located along the spacecraft to secondary axis line at a constant distance r_c from this axis. Hence, each of the four antenna corrections is given by Eq. (517) plus the constant r_c. However, since the round-trip range correction at the beginning of the count interval T_c is subtracted from the corresponding correction at the end of T_c, the effect of r_c on Δ and hence on doppler observables is zero. Hence, Eq. (517) applies also for doppler observables.

For the 26-m HA-dec antennas of the DSN,

$$\Delta_A\rho = -b\cos\delta \tag{518}$$

where δ is the observed declination of the spacecraft and $b = 6.706$ m. These antennas are located at DSN Deep Space Stations 11, 12, 41, 42, 51, 61, and 62.

For the 26-m az-el antenna at Deep Space Station 13,

$$\Delta_A\rho = -b\cos\gamma \tag{519}$$

where γ is the observed elevation of the spacecraft and $b = 0.9144$ m.

For the 26-m X'–Y' antennas of the MSFN,

$$\Delta_A\rho = -b\cos Y' \tag{520}$$

where Y' is the observed angle Y' to the spacecraft and $b = 1.2192$ m. These antennas are located at station MAD at Madrid, Spain; DRA at Canberra, Australia; and ODS at Goldstone, California.

The axis offset b is zero for the 64-m (210-ft) az-el antenna at Deep Space Station 14, the 9-m (30-ft) X–Y antennas of the MSFN, and all antennas of the AFETR (station numbers 73–77, 79–84, and 87). Hence there are no antenna corrections for these stations.

The antenna correction for the up leg of a light path is based upon the antenna type of the transmitting station and the value of the angle δ, γ, or Y' to the spacecraft at the transmission time for that leg. Similarly, the antenna correction for the down leg of a light path is based upon the antenna type of the receiving station and the value of δ, γ, or Y' at the reception time for that leg. For 3-way doppler, the antenna types at the transmitter and receiver may be different.

The maximum transverse displacement of the secondary axis from the tracking station to spacecraft line is less than 10 m, which affects directly observed angles by less than 20 arc seconds at the minimum spacecraft range of 100 km. Since such small ranges are rarely encountered and the maximum attainable accuracy for directly observed angles is only 7–11 arc seconds, the computed angular observables are not corrected for this effect.

*2. **Troposphere and ionosphere corrections.*** Discussed below are ray path equations, troposphere corrections, and ionosphere corrections.

a. Ray path equations. The speed of propagation of the doppler or ranging signal through the troposphere is given by

$$v_p = \frac{c}{n} \tag{521}$$

where

c = speed of light in vacuum

n = index of refraction of troposphere

From Ref. 52, p. 9, or Ref. 53 or 54,

$$n = 1 + 10^{-6} N \tag{522}$$

where

N = refractivity

given by

$$N = N_0 e^{-Bh} \tag{523}$$

where

N_0 = refractivity at mean sea level

B = reciprocal of scale height of troposphere, km^{-1}

h = altitude above mean sea level, km

The speed of propagation through the ionosphere is given by Eq. (521) using the following index of refraction:

$$n = 1 \pm \frac{40.3}{f^2} N_e \tag{524}$$

where

N_e = electron density

= number of electrons/m^3

f = transmitted frequency for up or down leg of light path (see Section VIII), Hz

For range observables, the range code travels at the group velocity, which is less than c, and hence the positive sign of Eq. (524) applies. For doppler observables, the doppler signal (the crest of a wave) travels at the phase velocity, which is greater than c, and hence the negative sign applies. The electron density vs altitude profile is assumed to be that of the Chapman model:

$$N_e = N_{\max} e^{(1/2)(1-u-e^{-u})} \tag{525}$$

where

$N_{\max}$ = maximum value of N_e

$u = (h - h_{\max})/B$

h = altitude above mean sea level, km

$h_{\max}$ = altitude of $N_{\max}$

B = scale height of ionosphere, km

The doppler and ranging signals travel on a curved path C through the troposphere and ionosphere. The time for the signal to travel between the tracking station and spacecraft along C is given by

$$T = \int_C \frac{ds}{v_p} = \frac{1}{c} \int_C n \, ds \tag{526}$$

where ds is an increment of distance along C. The path C follows from the condition that the propagation time T is a minimum (Fermat's principle). Since n is a function of altitude only, the path is planar and may be described by its geocentric radius r and geocentric angle ϕ from the tracking station. Hence Eq. (526) can be written as

$$T = \frac{1}{c} \int_C n(r) \left[1 + r^2 \left(\frac{d\phi}{dr} \right)^2 \right]^{1/2} dr \tag{527}$$

where n is indicated as a function of r. The differential equation of the path which extremizes the integral (Eq. 527) is the Euler–Lagrange equation of the calculus of variations applied to the integrand of Eq. (527).

The equations for the path C were developed by D. L. Cain and A. Liu and were documented by A. Liu in Ref. 55. Equation (14) gives the total bending of the path and Eqs. (17) and (18) give the range correction. Use of the index of refraction given by Eqs. (522) and (523) gives the bending and range correction $\Delta_T\rho$ due to the troposphere. Use of n given by Eqs. (524) and (525) gives the bending and range correction $\Delta_I\rho$ due to the ionosphere.

Given the observed value of the elevation angle, these corrections are obtained by a quadrature integration from the position of the tracking station to that of the spacecraft (assumed at infinite distance from the earth). Equations (14), (17), and (18) give computed minus observed values of the corrections. However, observed minus computed corrections are added to the computed values of the angular, range, and doppler observables. For this purpose, the sign of Eq. (14) and of each term of Eq. (17) must be changed. Furthermore the factor $1/C_1$ must be added to Eq. (18). In the derivation of Eq. (14), the term $-E_0$ was omitted in Eqs. (11) and (12).

Given N_0 and B for the troposphere and $N_{\max}$, $h_{\max}$, and B for the ionosphere in the vicinity of a tracking station, Eqs. (14), (17), and (18) of Ref. 55, as modified above, give the elevation angle correction used in the computation of directly observed angles and the tropospheric and ionospheric range corrections used in the computation of range and doppler observables.

b. Troposphere corrections. The expression that will be given below for the tropospheric range correction $\Delta_T\rho$ was obtained by a procedure equivalent to the following: For selected values of the observed elevation angle γ_0 between 0 and $\pi/2$ rad, the ray tracing formulation described in Subsection XII-B-2-a above was used to compute the elevation angle correction $\Delta_r\gamma$ and the range correction $\Delta_T\rho$ for a spacecraft at infinite distance from the earth. Subtraction of $\Delta_r\gamma$ from γ_0 gave the corresponding computed elevation angle γ based upon a straight-line light path from the tracking station to the spacecraft. The corrections were computed using a sea level refractivity N_0 of 340.0 and a scale height of 7 km or inverse scale height B of 0.142 km^{-1}. The range corrections $\Delta_T\rho$ were plotted vs the computed elevation angle γ.

The range correction was assumed to be of the form

$$\Delta_T\rho = \frac{A}{(\sin\gamma + B)^C} \tag{528}$$

where A, B, and C are constants. Fitting this expression to the tabular data above gave

$$\Delta_T\rho = \frac{1.8958\,\text{m}}{(\sin\gamma + 0.06483)^{1.4}} \tag{529}$$

which was originally obtained by D. L. Cain.

Let

N_S = surface refractivity at tracking station

which ideally could be computed from Eq. (523) using the altitude h of the tracking station. The range correction $\Delta_T\rho$ varies directly with N_S and since Eq. (529) was obtained using $N_S = 340.0$, the general result is

$$\Delta_T\rho = \frac{1.8958\,\text{m}}{(\sin\gamma + 0.06483)^{1.4}} \cdot \frac{N_S}{340.0} \tag{530}$$

Recommended values of N_S for the various tracking stations are given in Ref. 56.

For elevation angles above 15 deg, where most tracking data are taken, the maximum difference between the model (Eq. 530) and the tabular data obtained from the ray tracing formulation is 1–2 m, which is quite large. Hence Eq. (528) was fitted to the tabular data for $15 < \gamma < 90$ deg, giving

$$\Delta_T\rho = \frac{2.6\,\text{m}}{\sin\gamma + 0.015} \cdot \frac{N_S}{340.0} \tag{531}$$

For $\gamma > 15$ deg, the maximum difference between this model and the tabular ray tracing data is less than ¼ m.

The models (Eqs. 530 and 531) are based upon an average value of the surface refractivity N_S at each tracking station and a global average value of the scale height. The daily departures of these parameters from the constant values used are currently not accounted for. The resulting errors in $\Delta_T\rho$ from Eq. (530) or (531) are less than 10% for about 90% of the time, with a maximum possible error of about 15%. The following listing shows the approximate range corrections for elevation angles of 90, 15, and 0 deg and the corresponding 10% errors:

Elevation angle, deg	Range correction, m	10% error, m
90	2.5	0.25
15	9.5	0.95
0	87	8.7

Reference 57 describes the daily variations in the parameters of the troposphere and the resulting variations in the range correction $\Delta_T \rho$.

Equations (530) and (531) are based upon the assumption that the spacecraft is at an infinite distance from the earth. The error due to this assumption increases as the topocentric range to the spacecraft decreases. The maximum error is about 5 m, which occurs at an altitude of 100 km and an elevation angle of 0 deg.

An expression should be found which approximates with negligible error for all elevation angles the range correction $\Delta_T \rho$ obtained from the ray tracing formulation. Furthermore, a correction factor should be added which accounts for the noninfinite range to the spacecraft.

The change in the elevation angle due to tropospheric refraction, $\Delta_r \gamma$, which affects directly observed angles, is computed from Eq. (406) or (407) of Section X. Equation (407), which applies for high elevation angles, is the standard flat-earth textbook equation (see Ref. 58, p. 61, Eq. 6). Equation (406), which applies for low elevation angles, was obtained by D. L. Cain by fitting to values obtained from the ray tracing formulas. The factor b_2 is the total bending of the path, which equals $\Delta_r \gamma$ for a spacecraft at infinite distance from the earth. The factor b_1 accounts for the noninfinite range to the spacecraft.

c. Ionosphere corrections. The earth's ionosphere is caused by ultraviolet light from the sun ionizing the upper atmosphere. The maximum electron density is in the general direction of the sun. Hence the density of charged particles above a tracking station increases and decreases with a diurnal period. A given tracking station passes under the point of maximum electron density between 12 p.m. and 3 p.m. local time (1:30 p.m., average). The electron density is a minimum and fairly constant throughout the night. The electron density also varies with the magnetic latitude. It varies from essentially zero at a magnetic latitude of ±90 deg to a maximum in the general vicinity of the magnetic equator.

The time for the doppler or ranging signal to travel between the tracking station and the spacecraft is given by Eq. (526). Since the index of refraction of the ionosphere is given by Eq. (524), the effect of the ionosphere on the propagation time is given by

$$\delta T = \pm \frac{1}{c} \frac{40.3}{f^2} \int_\sigma N_e \, ds \tag{532}$$

Since δT is expressed as the so-called ionospheric range correction $\Delta_I \rho$ divided by c,

$$\Delta_I \rho = \pm \frac{40.3}{f^2} \int_\sigma N_e \, ds \tag{533}$$

For the ionosphere, the effect of the bending is negligible and the integral can be evaluated along the straight line path from the tracking station to the spacecraft. The propagation speed for the doppler signal is the phase velocity, which is greater than c, and hence the negative signs of Eqs. (532) and (533) apply. The ranging signal propagates at the group velocity, which is less than c, and hence the positive signs apply. The integral

$$\int_\sigma N_e \, ds$$

evaluated along a particular (straight-line) light path is referred to as the electron content for that path. It is a function of:

(1) Time of day. For an elevation angle γ of 90 deg, the maximum electron content occurs between 12 p.m. and 3 p.m. local time. The minimum electron content occurs at night.

(2) Elevation angle. As the elevation angle decreases, the path length through the ionosphere and the electron content increase.

(3) Geomagnetic latitude. The electron density approaches zero as the geomagnetic latitude approaches ±90 deg.

Unfortunately, the ionosphere is a very dynamic entity. There are models that describe the properties of the ionosphere, but the parameters of the model vary greatly with the position in the ionosphere and with time for a fixed position in the ionosphere. The large and unpredictable variations in these parameters preclude the computation of $\Delta_I\rho$ from a model. Thus, the only way of determining $\Delta_I\rho$ is by making direct measurements of the ionosphere.

These measurements may be obtained from a measuring station which is within a few hundred kilometers of the tracking station. However, ideally, they would be made at the tracking station along the actual ray path to the spacecraft. Reference 59 describes the computation of $\Delta_I\rho$ for the *Mariner* Mars 1969 mission and discusses the various types of ionospheric measurements and the procedure used to map measurements obtained from stations near a tracking station to the actual ray path to the spacecraft. The mapping is also discussed in Ref. 60.

Some types of ionospheric measurements that can be made are:

(1) Dual frequency. Two different frequencies (one of which is an exact integer multiple of the other) are transmitted in phase. Since the phase velocity for the charged particles of the ionosphere and space plasma is frequency-dependent, the two carrier signals will be out of phase when received. This phase shift gives the total electron content along the ray path, using Eq. (532).

(2) Group velocity vs phase velocity. As discussed in Section XI, doppler observables are equivalent to differenced range observables whose reception times are the end and start of the count interval. However, the ionospheric corrections for these pseudo-range-observables are the negative of the corrections for true range observables at the same epochs. Thus, a comparison of doppler observables with differenced true range observables yields twice the correction to doppler observables due to the charged particles of the ionosphere and space plasma. The doppler correction represents the change in the electron content along the round-trip light path during the count interval and is used to correct the computed values of doppler observables. This Differenced-Range Versus Integrated Doppler (DRVID) technique does not provide the absolute value of the electron content necessary to correct range observables.

(3) Faraday rotation. If the radio wave is linearly polarized, the plane of polarization will rotate as the signal passes through the earth's ionosphere because of the presence of the earth's magnetic field. This is the Faraday effect. Since the earth's magnetic field is known, the polarization of the received signal minus that of the transmitted signal can be used to compute the electron content along the ray path due to the ionosphere.

(4) Ionosonde. A radio signal is transmitted vertically, reflected by the ionosphere, and received by the transmitting station. The height of reflection h is the observed round-trip time multiplied by $c/2$. The electron density at this height is given by (Ref. 59)

$$N_e = 1.24 \times 10^{-2} f^2 \tag{534}$$

where

N_e = electron density, electrons/m^3

f = transmitted frequency, Hz

As the frequency is increased, the density N_e and altitude h increase until the critical frequency is reached where the signal pierces the ionosphere. Substituting this frequency into Eq. (534) gives the maximum electron density $N_{\max}$. The plot of h vs f gives the corresponding altitude $h_{\max}$. Assuming that the electron density N_e vs altitude profile is given by the Chapman model (Eq. 525), the vertical electron content E_c is given by

$$E_c = \int_0^\infty N_e(h)\,dh = BN_{\max}(2\pi e)^{1/2} \tag{535}$$

A comparison of E_c from Eq. (535) with Faraday rotation data gives the scale height B. Since B is fairly constant, a constant value is usually used at each ionosonde station to compute E_c from Eq. (535).

Currently, there is no means for directly measuring the electron content along the ray path from any tracking station to the spacecraft. However, plans for converting the tracking stations to the dual frequency (S-band and X-band) mode of operation are under consideration. Unfortunately, it will probably be for only the down leg of the light path. Implementation of such a system for both the up and down legs of the light path would provide a direct measure of the round-trip electron content, which would be used to compute the charged particle (ionosphere and space plasma) corrections for computed doppler and range observables.

The DRVID technique is currently being used at Deep Space Station 14 (Goldstone) to provide the charged-particle corrections for doppler observables. It will be available at other tracking stations when the Mu ranging system (see Section IX) is installed. Faraday rotation equipment is also available at Goldstone. However, most of the spacecraft to date (and probably those forthcoming) have not had the linearly polarized antennas that are required in order to use this equipment. Furthermore, this equipment does not measure the electron content due to space plasma.

For the *Mariner* Mars 1969 spacecraft, Faraday rotation data from tracking of a geostationary satellite and/or ionosonde data were obtained from measuring stations which were within a few hundred kilometers of some of the tracking stations. These measurements gave the electron content along ray paths differing from that of the spacecraft.

However, the physical separation between these paths and the difference in the measurement times were small enough so that the parameters of the ionosphere could be presumed to be the same for both paths. This enabled the electron content to be mapped from the measured ray path to that of the spacecraft, accounting for the differences in time of day, elevation angle, and geomagnetic latitude. The details of this mapping are given in Ref. 59 and are summarized in the following paragraphs.

The Faraday rotation and ionosonde data were taken at a constant elevation angle γ (90 deg for the ionosonde data). The ray path for each of these measurements pierces the ionosphere (assumed to be at an altitude of 400 km) at east longitude λ_M. At an observation time t (UTC), the ray path to the spacecraft pierces the ionosphere at east longitude λ_0. Then the electron content for the spacecraft ray path must be obtained by correcting the electron content for the measured ray path that has the measurement time

$$t\,(\mathrm{UTC}) + \frac{\lambda_0\,(\mathrm{deg}) - \lambda_M\,(\mathrm{deg})}{15\,\mathrm{deg/h}}$$

The measurement ray path at this time and the ray path to the spacecraft at time t (UTC) pierce the (space-fixed) ionosphere at the same right ascension.

However, the spacecraft ray path has a different elevation angle (γ_0) than the measurement ray path (γ_M). Hence, the measurement must be multiplied by the ratio of the electron content at elevation angle γ_0 to the electron content at elevation angle γ_M. This correction factor is computed from an approximate formula which agrees very well with the ratio obtained from the ray tracing formulation (Ref. 55 and Subsection XII-B-2-a above) using $h_{\max} = 300$ km and a scale height B of 39 km.

The approximate correction is based upon a uniform electron distribution between the altitudes of 206.5 and 441.5 km. The elevation angle correction factor is the straight-line distance through this uniform ionosphere at the spacecraft elevation angle γ_0 divided by the distance at the measurement elevation angle γ_M.

Finally, the electron content must be multiplied by a correction factor which accounts for the difference in the geomagnetic latitudes of the points where the spacecraft and measurement ray paths pierce the ionosphere (ϕ_0 and ϕ_M respectively). This correction factor is $(90\,\mathrm{deg} - \phi_0)/(90\,\mathrm{deg} - \phi_M)$.

Given the corrected electron content, the ionospheric range correction $\Delta_I\rho$ is given by Eq. (533).

XIII. Variational Equations

This section gives the formulation for the solution of the variational equations. The partial derivative of the spacecraft acceleration vector with respect to the solve-for parameter vector q is integrated numerically by the second-sum procedure to give the partial derivatives of the spacecraft velocity and position vectors with respect to q. These subpartial derivatives are used in Section XIV to form the partial derivatives of the doppler, range, and angular observables with respect to q.

The partial derivatives specified in this section are obtained by differentiation of the formulation of Section V for the acceleration of the spacecraft relative to the center of integration. However, the relativity terms and the indirect acceleration of the center of integration due to the oblateness of the earth and moon are ignored. The notation is that of Section V.

A. Variational Equations and Method of Integration

The formulation for the acceleration of the spacecraft relative to the center of integration is given in Section V. In functional form, it is given by

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}(\mathbf{r}, \dot{\mathbf{r}}, \mathbf{q}) \tag{536}$$

where

$\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}$ = position, velocity, and acceleration vectors of spacecraft relative to center of integration with rectangular components x, y, and z referred to the mean earth equator and equinox of 1950.0. The argument is ephemeris time

$\mathbf{q}$ = solve-for parameter vector

$$= \left[\begin{array}{c} \mathbf{X}_0^B \\ \hline \mathbf{a} \end{array}\right]$$

where

$\mathbf{X}_0^B = \left[\begin{array}{c} \mathbf{r}_0^B \\ \hline \dot{\mathbf{r}}_0^B \end{array}\right]$ = state vector (position and velocity vectors) of spacecraft relative to body B (not necessarily the center of integration C) at injection epoch t_0

$\mathbf{a}$ = dynamic constants affecting spacecraft trajectory

The state vector of the spacecraft relative to the center of integration at the injection epoch, $\mathbf{X}_0$, is given by

$$\mathbf{X}_0 = \mathbf{X}_0^B + (\mathbf{X}_B^C)_0 \tag{537}$$

where

$(\mathbf{X}_B^C)_0$ = state vector of reference body B relative to center of integration C at injection epoch

Differentiating Eq. (536) with respect to $\mathbf{q}$ gives

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{q}} = \frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} \frac{\partial \mathbf{r}}{\partial \mathbf{q}} + \frac{\partial \ddot{\mathbf{r}}}{\partial \dot{\mathbf{r}}} \frac{\partial \dot{\mathbf{r}}}{\partial \mathbf{q}} + \left.\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{q}}\right|_{\mathbf{r}, \dot{\mathbf{r}} = \text{constant}} \tag{538}$$

Let

$$A = \frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} = \begin{bmatrix} \dfrac{\partial \ddot{x}}{\partial x} & \dfrac{\partial \ddot{x}}{\partial y} & \dfrac{\partial \ddot{x}}{\partial z} \\ \dfrac{\partial \ddot{y}}{\partial x} & \dfrac{\partial \ddot{y}}{\partial y} & \dfrac{\partial \ddot{y}}{\partial z} \\ \dfrac{\partial \ddot{z}}{\partial x} & \dfrac{\partial \ddot{z}}{\partial y} & \dfrac{\partial \ddot{z}}{\partial z} \end{bmatrix} \tag{539}$$

$$B = \frac{\partial \ddot{\mathbf{r}}}{\partial \dot{\mathbf{r}}} \tag{540}$$

$$C = \left.\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{q}}\right|_{\mathbf{r}, \dot{\mathbf{r}} = \text{constant}} \tag{541}$$

$$Z = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} = \left[\frac{\partial \mathbf{r}}{\partial q_1}, \frac{\partial \mathbf{r}}{\partial q_2}, \ldots, \frac{\partial \mathbf{r}}{\partial q_n}\right] \tag{542}$$

$$\dot{Z} = \frac{\partial \dot{\mathbf{r}}}{\partial \mathbf{q}} \tag{543}$$

$$\ddot{Z} = \frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{q}} \tag{544}$$

Then,

$$\ddot{Z} = AZ + B\dot{Z} + C \tag{545}$$

where the first six columns of C corresponding to the injection conditions $\mathbf{X}_0^B$ are zero.

The variational equation (Eq. 545) is integrated numerically by the second-sum method to give Z and $\dot{Z}$ as functions of ephemeris time t. The partial derivative of the spacecraft state vector.

$$\mathbf{X} = \left[\begin{array}{c} \mathbf{r} \\ \hline \dot{\mathbf{r}} \end{array}\right] \tag{546}$$

with respect to $\mathbf{q}$ at any time t is

$$\frac{\partial \mathbf{X}}{\partial \mathbf{q}} = \left[\begin{array}{c} Z \\ \hline \dot{Z} \end{array}\right] = [U \,\vdots\, V] \tag{547}$$

where

$$U = \frac{\partial \mathbf{X}}{\partial \mathbf{X}_0^B} = \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_0)} \equiv U(t, t_0) \tag{548}$$

$$V = \frac{\partial \mathbf{X}}{\partial \mathbf{a}} \tag{549}$$

For each parameter q_i, three sum and difference numerical integration arrays, having two sums and 10 differences of $\partial\ddot{x}/\partial q_i$, $\partial\ddot{y}/\partial q_i$, and $\partial\ddot{z}/\partial q_i$, respectively, are generated. These three sum and difference arrays may be interpolated at any time t to give $\partial x, \dot{x}/\partial q_i$, $\partial y, \dot{y}/\partial q_i$, and $\partial z, \dot{z}/\partial q_i$, respectively, which are the elements of the q_i column of U or V.

When the injection conditions are referred to the center of integration, the initial value of $\partial\mathbf{X}/\partial\mathbf{q}$ at the injection epoch is

$$\left[\frac{\partial \mathbf{X}}{\partial \mathbf{q}}\right]_0 = [I \,\vdots\, 0] \tag{550}$$

where I is a 6×6 identity matrix. When the injection conditions are referred to a body B other than the center of integration C,

$$\left[\frac{\partial \mathbf{X}}{\partial \mathbf{q}}\right]_0 = \left[I \,\vdots\, \frac{\partial (\mathbf{X}_B^C)_0}{\partial \mathbf{a}}\right] \tag{551}$$

The eighteen sum and difference arrays for the six injection parameters are started at the injection epoch t_0, with $\partial\mathbf{X}/\partial\mathbf{X}_0^B = I\,(6 \times 6)$ as initial conditions. For reasons that will become evident later, these sum and difference arrays are restarted with initial values I at a number of intermediate epochs $t_1, t_2, t_3, \cdots, t_n$. The U matrix of Eq. (548) is then formed by the chain rule as:

$$U = \frac{\partial \mathbf{X}}{\partial \mathbf{X}_0^B} = \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_0)} = \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_n)} \frac{\partial \mathbf{X}(t_n)}{\partial \mathbf{X}(t_{n-1})} \cdots \frac{\partial \mathbf{X}(t_1)}{\partial \mathbf{X}(t_0)}$$

$$= U(t, t_n)\, U(t_n, t_{n-1}) \cdots U(t_1, t_0) \qquad (552)$$

Similarly, a U matrix from any intermediate epoch t_i to any time t is formed by

$$U(t, t_i) = U(t, t_n)\, U(t_n, t_{n-1}) \cdots U(t_{i+1}, t_i) \qquad (553)$$

For a dynamic parameter a_i, if the corresponding column of the C matrix is always nonzero, the three sum and difference arrays are started at the injection epoch and continued for the duration of the mission. For each of these parameters, the elements of the column of the V matrix can be obtained by interpolation of the three sum and difference arrays at the desired time t.

For certain other parameters a_i, the column of the C matrix is nonzero only for $t_a < t < t_b$, and the sum and difference arrays are generated only for this interval of time. For $t < t_a$, $\partial \mathbf{X}/\partial a_i = 0$. For $t_a < t < t_b$, the column of V is obtained by interpolation of the sum and difference arrays. For $t > t_b$,

$$\frac{\partial \mathbf{X}(t)}{\partial a_i} = \frac{\partial \mathbf{X}(t)}{\partial \mathbf{X}(t_b)} \frac{\partial \mathbf{X}(t_b)}{\partial a_i} = U(t, t_b) \frac{\partial \mathbf{X}(t_b)}{\partial a_i} \qquad (554)$$

where $\partial \mathbf{X}(t_b)/\partial a_i$ is obtained from the sum and difference arrays at the stop time t_b, and $U(t, t_b)$ is obtained from Eq. (553) using $t_i = t_b$.

Some parameters have an initial value $\partial \mathbf{X}(t_b)/\partial a_i$ at a discontinuity epoch t_b, and the column of the C matrix is zero for all time. For this case, $\partial \mathbf{X}(t)/\partial a_i$ is computed directly from Eq. (554); no sum and difference arrays are generated for this type of parameter.

Some parameters are a combination of the two previous cases. A period of time $t_a < t < t_b$ exists when the column of C is nonzero and sum and difference arrays are generated; also there are several epochs where discontinuities to the partial derivatives occur. At each discontinuity epoch or stop time for sum and difference arrays, the increment to the partial derivative is added to the accumulated partial and mapped to the next discontinuity epoch or start time for sum and difference arrays (using Eqs. 553 and 554). If the discontinuity occurs during the integration of the sum and difference arrays, they must be restarted using the incremented partial derivatives as initial values. For $t_a < t < t_b$, $\partial \mathbf{X}/\partial a_i$ may be obtained directly by interpolation. Otherwise, the accumulated value of $\partial \mathbf{X}/\partial a_i$ at the last discontinuity epoch or stop time for sum and difference arrays is mapped to the current time, using Eq. (554).

B. Computation of *A* Matrix

The A matrix is defined by Eq. (539). The terms of the spacecraft acceleration vector (considered in this section) which are a function of the spacecraft position vector are:

(1) The direct Newtonian point mass acceleration due to each celestial body i (nine planets, sun, and moon).

(2) The direct Newtonian acceleration due to oblateness for each oblate body j.

(3) The acceleration due to the solar radiation pressure (SRP) and small force (SF) models.

The A matrix is computed from the following sum of terms:

$$A = \sum_i \frac{\partial \ddot{\mathbf{r}}^{(i)}_{\text{Newtonian}}}{\partial \mathbf{r}} + \sum_j \frac{\partial \ddot{\mathbf{r}}^{(j)}_{\text{oblate}}}{\partial \mathbf{r}} + \frac{\partial \ddot{\mathbf{r}}(\text{SRP} - \text{SF})}{\partial \mathbf{r}} \qquad (555)$$

The formulation for computing each of these terms is given in the following sections. The notation used is that of Section V. All vectors appearing in the formulation are column vectors.

1. Contribution from Newtonian point mass acceleration. The direct Newtonian acceleration of the spacecraft due to body i treated as a point mass is given by

$$\ddot{\mathbf{r}} = -\frac{\mu_i (\mathbf{r} - \mathbf{r}_i^C)}{\|\mathbf{r} - \mathbf{r}_i^C\|^3} \qquad (556)$$

where

$\mathbf{r}$ = position vector of spacecraft relative to center of integration with rectangular components x, y, and z referred to the mean earth equator and equinox of 1950.0

$\mathbf{r}_i^C$ = 1950.0 position vector of body i relative to center of integration

μ_i = gravitational constant of body i, km³/s²

Differentiating Eq. (556) with respect to $\mathbf{r}$ gives

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} = \frac{3\mu_i (\mathbf{r} - \mathbf{r}_i^C)(\mathbf{r} - \mathbf{r}_i^C)^T}{\|\mathbf{r} - \mathbf{r}_i^C\|^5} - \frac{\mu_i I}{\|\mathbf{r} - \mathbf{r}_i^C\|^3} \tag{557}$$

where I is a 3×3 identity matrix.

2. ***Contribution from oblateness acceleration.*** The direct acceleration of the spacecraft due to the oblateness of one celestial body is given by Eqs. (169), (173), (174) and associated equations of Section V. Differentiating Eq. (169) with respect to $\mathbf{r}$, using Eq. (163), gives

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} = \left[\frac{\partial G^T}{\partial x}\ddot{\mathbf{r}}', \frac{\partial G^T}{\partial y}\ddot{\mathbf{r}}', \frac{\partial G^T}{\partial z}\ddot{\mathbf{r}}'\right] + G^T\left[\frac{\partial \ddot{\mathbf{r}}'(J)}{\partial \mathbf{r}'} + \frac{\partial \ddot{\mathbf{r}}'(C,S)}{\partial \mathbf{r}'}\right]G \tag{558}$$

where

$\dfrac{\partial G^T}{\partial x} = G^T$ with each term differentiated with respect to x $x \rightarrow y, z$

$\dfrac{\partial G^T}{\partial x}\ddot{\mathbf{r}}' =$ first column of first term of Eq. (558)

$$\ddot{\mathbf{r}}' = \ddot{\mathbf{r}}'(J) + \ddot{\mathbf{r}}'(C,S) \tag{559}$$

$$\frac{\partial \ddot{\mathbf{r}}'(J)}{\partial \mathbf{r}'} = \left[\frac{\partial \ddot{\mathbf{r}}'(J)}{\partial x'}, \frac{\partial \ddot{\mathbf{r}}'(J)}{\partial y'}, \frac{\partial \ddot{\mathbf{r}}'(J)}{\partial z'}\right] = \left[\frac{\partial \ddot{\mathbf{r}}'(J)}{\partial r}, \frac{1}{r\cos\phi}\frac{\partial \ddot{\mathbf{r}}'(J)}{\partial \lambda}, \frac{1}{r}\frac{\partial \ddot{\mathbf{r}}'(J)}{\partial \phi}\right] \qquad (J) \rightarrow (C,S) \tag{560}$$

Differentiating Eqs. (173) and (174) with respect to r, ϕ, and λ and using Eq. (560) gives

$$\frac{\partial \ddot{\mathbf{r}}'(J)}{\partial x'} = -\frac{1}{r}\sum_{n=1}^{n_1}(n+2)\ddot{\mathbf{r}}'(J_n) \tag{561}$$

$$\frac{\partial \ddot{\mathbf{r}}'(J)}{\partial y'} = 0 \tag{562}$$

$$\frac{\partial \ddot{\mathbf{r}}'(J)}{\partial z'} = \frac{\mu}{r^3}\sum_{n=1}^{n_1} J_n \left(\frac{a_p}{r}\right)^n \begin{bmatrix} (n+1)\cos\phi\, P_n' \\ 0 \\ \sin\phi\, P_n' - \cos^2\phi\, P_n'' \end{bmatrix} \tag{563}$$

$$\frac{\partial \ddot{\mathbf{r}}'(C,S)}{\partial x'} = -\frac{1}{r}\sum_{n=1}^{n_2}(n+2)\sum_{m=1}^{n}[\ddot{\mathbf{r}}'(C_{nm}) + \ddot{\mathbf{r}}'(S_{nm})] \tag{564}$$

$$\frac{\partial \ddot{\mathbf{r}}'(C,S)}{\partial y'} = \frac{\mu}{r^3}\sum_{n=1}^{n_2}\sum_{m=1}^{n} m\left(\frac{a_p}{r}\right)^n \begin{bmatrix} (n+1)\sec\phi\, P_n^m (C_{nm}\sin m\lambda - S_{nm}\cos m\lambda) \\ m\sec^2\phi\, P_n^m (-C_{nm}\cos m\lambda - S_{nm}\sin m\lambda) \\ P_n^{m\prime} (-C_{nm}\sin m\lambda + S_{nm}\cos m\lambda) \end{bmatrix} \tag{565}$$

$$\frac{\partial \ddot{\mathbf{r}}'(C,S)}{\partial z'} = \frac{\mu}{r^3}\sum_{n=1}^{n_2}\sum_{m=1}^{n}\left(\frac{a_p}{r}\right)^n \begin{bmatrix} -(n+1)\cos\phi\, P_n^{m\prime} (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda) \\ m(\sin\phi\sec^2\phi\; P_n^m + P_n^{m\prime})(-C_{nm}\sin m\lambda + S_{nm}\cos m\lambda) \\ (\cos^2\phi\, P_n^{m\prime\prime} - \sin\phi\, P_n^{m\prime})(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda) \end{bmatrix} \tag{566}$$

where $\ddot{\mathbf{r}}'(J_n)$, $\ddot{\mathbf{r}}'(C_{nm})$, and $\ddot{\mathbf{r}}'(S_{nm})$ are the terms of $\ddot{\mathbf{r}}'$ due to J_n, C_{nm}, and S_{nm}, respectively. The primes above the terms P_n and P_n^m indicate derivatives with respect to $\sin\phi$.

From Eq. (164),

$$G^T = T R^T \tag{567}$$

The body-fixed to space-fixed transformation T, defined by Eq. (162), is a function of time only. The matrix R, given by Eq. (161), transforms from body-fixed to up–east–north coordinates and is a function of the spacecraft position. Thus,

$$\frac{\partial G^T}{\partial x} = T\frac{\partial R^T}{\partial x} \qquad x \rightarrow y, z \tag{568}$$

Let

$$R^T = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \tag{569}$$

Then,

$$\frac{\partial R^T}{\partial x} = \begin{bmatrix} \frac{\partial e_{11}}{\partial x} & \frac{\partial e_{12}}{\partial x} & \frac{\partial e_{13}}{\partial x} \\ \frac{\partial e_{21}}{\partial x} & \frac{\partial e_{22}}{\partial x} & \frac{\partial e_{23}}{\partial x} \\ \frac{\partial e_{31}}{\partial x} & \frac{\partial e_{32}}{\partial x} & \frac{\partial e_{33}}{\partial x} \end{bmatrix} \quad x \to y, z \tag{570}$$

$$\frac{\partial e_{ij}}{\partial \mathbf{r}} = \left(\frac{\partial e_{ij}}{\partial x}, \frac{\partial e_{ij}}{\partial y}, \frac{\partial e_{ij}}{\partial z} \right) = \frac{\partial e_{ij}}{\partial \mathbf{r}'} G \tag{571}$$

and

$$\frac{\partial e_{ij}}{\partial \mathbf{r}'} = \left(\frac{\partial e_{ij}}{\partial r}, \frac{1}{r \cos \phi} \frac{\partial e_{ij}}{\partial \lambda}, \frac{1}{r} \frac{\partial e_{ij}}{\partial \phi} \right) \tag{572}$$

Differentiating each e_{ij} gives

$$\frac{\partial e_{11}}{\partial \mathbf{r}'} = \left[0, -\frac{\sin \lambda}{r}, -\frac{\sin \phi \cos \lambda}{r} \right] \tag{573}$$

$$\frac{\partial e_{21}}{\partial \mathbf{r}'} = \left[0, \frac{\cos \lambda}{r}, -\frac{\sin \phi \sin \lambda}{r} \right] \tag{574}$$

$$\frac{\partial e_{31}}{\partial \mathbf{r}'} = \left[0, 0, \frac{\cos \phi}{r} \right] \tag{575}$$

$$\frac{\partial e_{12}}{\partial \mathbf{r}'} = \left[0, -\frac{\cos \lambda}{r \cos \phi}, 0 \right] \tag{576}$$

$$\frac{\partial e_{22}}{\partial \mathbf{r}'} = \left[0, -\frac{\sin \lambda}{r \cos \phi}, 0 \right] \tag{577}$$

$$\frac{\partial e_{32}}{\partial \mathbf{r}'} = [0, 0, 0] \tag{578}$$

$$\frac{\partial e_{13}}{\partial \mathbf{r}'} = \left[0, \frac{\sin \phi \sin \lambda}{r \cos \phi}, -\frac{\cos \phi \cos \lambda}{r} \right] \tag{579}$$

$$\frac{\partial e_{23}}{\partial \mathbf{r}'} = \left[0, -\frac{\sin \phi \cos \lambda}{r \cos \phi}, -\frac{\cos \phi \sin \lambda}{r} \right] \tag{580}$$

$$\frac{\partial e_{33}}{\partial \mathbf{r}'} = \left[0, 0, -\frac{\sin \phi}{r} \right] \tag{581}$$

Substituting Eqs. (573–581) into Eq. (571) gives the 27 terms of $\partial R^T/\partial x$, $\partial R^T/\partial y$, and $\partial R^T/\partial z$ used in Eq. (568) to give $\partial G^T/\partial x$, $\partial G^T/\partial y$, and $\partial G^T/\partial z$, which are used in Eq. (558).

The formulation for computation of Legendre polynomials for the oblateness acceleration terms was given in Section V. The following is an extension of that formulation used to compute the acceleration terms and the corresponding partial derivatives:

(1) Legendre polynomials and derivatives

$$(n = 1, 2, 3, \cdots, n_1)$$

The Legendre polynomial P_n is computed recursively from

$$P_n = \frac{2n-1}{n} \sin \phi \, P_{n-1} - \left(\frac{n-1}{n} \right) P_{n-2} \tag{582}$$

beginning with

$$P_0 = 1 \tag{583}$$

$$P_1 = \sin \phi \tag{584}$$

The first derivative of P_n with respect to $\sin \phi$, denoted P'_n, is given by

$$P'_n = \sin \phi \, P'_{n-1} + n \, P_{n-1} \tag{585}$$

starting with

$$P'_1 = 1 \tag{586}$$

Differentiation of Eq. (585) with respect to $\sin \phi$ gives

$$P''_n = \sin \phi \, P''_{n-1} + (n+1) \, P'_{n-1} \tag{587}$$

beginning with

$$P''_1 = 0 \tag{588}$$

(2) Associated Legendre functions and derivatives ($m = 2, 3, \cdots, n$; $n = 2, 3, \cdots, n_2$). Compute $\sec^2 \phi \, P_n^m$ by first generating

$$\sec^2 \phi \, P_n^n = (2n-1) \cos \phi \, (\sec^2 \phi \, P_{n-1}^{n-1}) \tag{589}$$

for $n = 2, 3, \cdots, n_2$, starting with

$$\sec^2 \phi \, P_2^2 = 3 \tag{590}$$

and then generating

$$\sec^2 \phi \, P_n^m = \left(\frac{2n-1}{n-m} \right) \sin \phi \, (\sec^2 \phi \, P_{n-1}^m) - \left(\frac{n+m-1}{n-m} \right) (\sec^2 \phi \, P_{n-2}^m) \tag{591}$$

For each value of m between 2 and n_2, n is varied from $m+1$ to n_2. The general term P_a^b is zero if $b > a$. Multiply $\sec^2\phi\, P_n^m$ by $\cos\phi$ and $\cos^2\phi$ to give $\sec\phi\, P_n^m$ and P_n^m, respectively.

The derivative of P_n^m with respect to $\sin\phi$, denoted $P_n^{m\prime}$, is computed from

$$P_n^{m\prime} = -n\sin\phi\,(\sec^2\phi\, P_n^m) + (n+m)\,(\sec^2\phi\, P_{n-1}^m) \tag{592}$$

Multiplying Eq. (592) by $\cos^2\phi$ and differentiating with respect to $\sin\phi$ gives

$$\cos^2\phi\, P_n^{m\prime\prime} = -(n-2)\sin\phi\, P_n^{m\prime} + (n+m)\,P_{n-1}^{m\prime} - nP_n^m \tag{593}$$

(3) Associated Legendre functions and derivatives

$$(m = 1; n = 1, 2, 3, \cdots, n_2)$$

Compute

$$\sec\phi\, P_n^1 = \left(\frac{2n-1}{n-1}\right)\sin\phi\,(\sec\phi\, P_{n-1}^1) - \left(\frac{n}{n-1}\right)(\sec\phi\, P_{n-2}^1) \tag{594}$$

starting with

$$\sec\phi\, P_1^1 = 1 \tag{595}$$

Multiply Eq. (594) by $\cos\phi$ and $1/\cos\phi$ to give P_n^1 and $\sec^2\phi\, P_n^1$, the latter of which is indeterminate for $\phi = 90$ deg.

Compute

$$\cos\phi\, P_n^{1\prime} = -n\sin\phi\,(\sec\phi\, P_n^1) + (n+1)\,(\sec\phi\, P_{n-1}^1) \tag{596}$$

Multiplication by $1/\cos\phi$ gives $P_n^{1\prime}$, which is indeterminate for $\phi = 90$ deg.

The following sums (derived from Eq. 155) are not indeterminate when $\phi = 90$ deg, although their individual terms are

$$(\sin\phi\sec^2\phi\, P_n^1 + P_n^{1\prime}) = \sec\phi\, P_n^2 \tag{597}$$

$$(\cos^2\phi\, P_n^{1\prime\prime} - \sin\phi\, P_n^{1\prime}) = -P_n^1 - 3\sin\phi\,(\sec\phi\, P_n^2) + P_n^3 \tag{598}$$

The Legendre functions that are indeterminate for $\phi = 90$ deg appear in Eq. (565).

3. Contribution from solar radiation pressure and small force models. The acceleration of the spacecraft due to the solar radiation pressure and small force models is given by Eq. (189). The spacecraft position vector $\mathbf{r}$ affects r_{SP}, EPS, and the unit vectors $\mathbf{U}_{SP}$, $\mathbf{X}^*$, and $\mathbf{Y}^*$. Hence,

$$\begin{aligned}\frac{\partial\ddot{\mathbf{r}}}{\partial\mathbf{r}} = &-\frac{2}{r_{SP}}\ddot{\mathbf{r}}_{SRP}\,\mathbf{U}_{SP}^T + (\ddot{\mathbf{r}}\cdot\mathbf{U}_{SP})\frac{\partial\mathbf{U}_{SP}}{\partial\mathbf{r}} + (\ddot{\mathbf{r}}\cdot\mathbf{X}^*)\frac{\partial\mathbf{X}^*}{\partial\mathbf{r}} \\ &+ (\ddot{\mathbf{r}}\cdot\mathbf{Y}^*)\frac{\partial\mathbf{Y}^*}{\partial\mathbf{r}} + \frac{C_1A_p}{mr_{SP}^2}(G_r'\,\mathbf{U}_{SP} + G_x'\,\mathbf{X}^* \\ &+ G_y'\,\mathbf{Y}^*)\frac{\partial EPS}{\partial\mathbf{r}}\,\mu^*(t - T_{SRP})\end{aligned} \tag{599}$$

where

$\ddot{\mathbf{r}}$ = acceleration of spacecraft due to solar radiation pressure and small forces (Eq. 189)

$\ddot{\mathbf{r}}_{SRP}$ = acceleration of spacecraft due to solar radiation pressure (terms of Eq. 189 proportional to A_p)

From Eq. (190),

$$\frac{\partial\mathbf{U}_{SP}}{\partial\mathbf{r}} = \left[\frac{\partial\mathbf{U}_{SP}}{\partial x}, \frac{\partial\mathbf{U}_{SP}}{\partial y}, \frac{\partial\mathbf{U}_{SP}}{\partial z}\right] = \frac{1}{r_{SP}}[I - \mathbf{U}_{SP}\mathbf{U}_{SP}^T] \tag{600}$$

where I is a 3×3 identity matrix. From Eq. (191),

$$\begin{bmatrix}\partial\mathbf{X}^*/\partial\mathbf{r} \\ \partial\mathbf{Y}^*/\partial\mathbf{r}\end{bmatrix} = \begin{bmatrix}\cos K & \sin K \\ -\sin K & \cos K\end{bmatrix}\begin{bmatrix}\partial\mathbf{T}/\partial\mathbf{r} \\ \partial\mathbf{N}/\partial\mathbf{r}\end{bmatrix} \tag{601}$$

From Eq. (194),

$$\frac{\partial\mathbf{N}}{\partial\mathbf{r}} = \frac{1}{\|\mathbf{U}_R \times \mathbf{U}_{SP}\|}[I - \mathbf{N}\mathbf{N}^T]\left\{\left[\mathbf{U}_R \times \frac{\partial\mathbf{U}_{SP}}{\partial x}; \mathbf{U}_R \times \frac{\partial\mathbf{U}_{SP}}{\partial y}; \mathbf{U}_R \times \frac{\partial\mathbf{U}_{SP}}{\partial z}\right] + \left[\frac{\partial\mathbf{U}_R}{\partial x} \times \mathbf{U}_{SP}; \frac{\partial\mathbf{U}_R}{\partial y} \times \mathbf{U}_{SP}; \frac{\partial\mathbf{U}_R}{\partial z} \times \mathbf{U}_{SP}\right]\right\} \tag{602}$$

If the reference body B is a star,

$$\frac{\partial \mathbf{U}_R}{\partial \mathbf{r}} = \left[\frac{\partial \mathbf{U}_R}{\partial x}, \frac{\partial \mathbf{U}_R}{\partial y}, \frac{\partial \mathbf{U}_R}{\partial z}\right] = 0 \tag{603}$$

If the reference body B is a planet or the moon, we obtain from Eq. (193),

$$\frac{\partial \mathbf{U}_R}{\partial \mathbf{r}} = \left[\frac{\partial \mathbf{U}_R}{\partial x}, \frac{\partial \mathbf{U}_R}{\partial y}, \frac{\partial \mathbf{U}_R}{\partial z}\right] = -\frac{1}{\|\mathbf{r}_B^C - \mathbf{r}\|}[I - \mathbf{U}_R \mathbf{U}_R^T] \tag{604}$$

From Eq. (195),

$$\frac{\partial \mathbf{T}}{\partial \mathbf{r}} = \left[\frac{\partial \mathbf{N}}{\partial x} \times \mathbf{U}_{SP}; \frac{\partial \mathbf{N}}{\partial y} \times \mathbf{U}_{SP}; \frac{\partial \mathbf{N}}{\partial z} \times \mathbf{U}_{SP}\right] + \left[\mathbf{N} \times \frac{\partial \mathbf{U}_{SP}}{\partial x}; \mathbf{N} \times \frac{\partial \mathbf{U}_{SP}}{\partial y}; \mathbf{N} \times \frac{\partial \mathbf{U}_{SP}}{\partial z}\right] \tag{605}$$

From Eq. (196),

$$\frac{\partial EPS}{\partial \mathbf{r}} = \frac{1}{\sin EPS}\left[\mathbf{U}_R'^T \frac{\partial \mathbf{U}_{SP}}{\partial \mathbf{r}} + \mathbf{U}_{SP}^T \frac{\partial \mathbf{U}_R'}{\partial \mathbf{r}}\right] \tag{606}$$

where $\partial \mathbf{U}_R'/\partial \mathbf{r}$ is computed from Eq. (604) using $\mathbf{U}_R'$ instead of $\mathbf{U}_R$ and $B =$ earth.

C. Computation of B Matrix

The terms of the spacecraft acceleration vector considered in this section are not a function of the spacecraft velocity. Hence, currently,

$$B = 0 \tag{607}$$

where the zero indicates a 3 × 3 null matrix.

D. C Matrix and Integration Tables for Each Parameter

In this section, the remaining partial derivatives necessary to generate the three sum and difference arrays for each parameter will be specified. These include the column of the C matrix and the initial values for $\partial \mathbf{X}/\partial q_i$ and discontinuities to it for each parameter q_i.

1. Injection parameters. The method of generating the 18 sum and difference arrays for the six injection parameters has been specified in Section XIII-A. Given these sum and difference arrays, the U matrix from the injection epoch is computed from Eq. (552) and the U matrix from each discontinuity epoch or stop time for sum and difference arrays is computed from Eq. (553). These latter mapping matrices are used in the computation of the column of the V matrix for some of the dynamic parameters a_i.

2. Reference parameters. The reference parameters $\mathbf{f}$ consist of

$A_E =$ the number of kilometers per AU

$R_E =$ the scaling factor for the lunar ephemeris, km/fictitious earth radius

$\mathbf{E} =$ osculating orbital elements for the heliocentric ephemeris of a planet or the earth–moon barycenter or for the geocentric lunar ephemeris

$\mu_E, \mu_M =$ gravitational constants for the earth and moon, respectively, km^3/s^2

They affect the position vector $\mathbf{r}_i^C$ of each perturbing body i (a planet, the sun, or the moon) relative to the center of integration, and hence affect the Newtonian point mass and oblate acceleration of the spacecraft due to these bodies. The partial derivative of the spacecraft acceleration $\ddot{\mathbf{r}}$ with respect to the reference parameters $\mathbf{f}$ (due to moving the perturbing bodies) is given by

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{f}} = \sum_i \frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}_i^C} \frac{\partial \mathbf{r}_i^C}{\partial \mathbf{f}} \tag{608}$$

where

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}_i^C} = \mu_i \left[\frac{3\mathbf{r}_i^C \mathbf{r}_i^{CT}}{\|\mathbf{r}_i^C\|^5} - \frac{I}{\|\mathbf{r}_i^C\|^3}\right] - \mu_i \left[\frac{3(\mathbf{r} - \mathbf{r}_i^C)(\mathbf{r} - \mathbf{r}_i^C)^T}{\|\mathbf{r} - \mathbf{r}_i^C\|^5} - \frac{I}{\|\mathbf{r} - \mathbf{r}_i^C\|^3}\right] - \frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} \text{(from Eq. 558)} \tag{609}$$

The first two terms of Eq. (609) are the derivatives of the indirect and direct terms, respectively, of the Newtonian point mass acceleration with respect to $\mathbf{r}_i^C$. The last term is the derivative of the direct acceleration due to the oblateness of body i with respect to $\mathbf{r}_i^C$. This acceleration is a function of $(\mathbf{r} - \mathbf{r}_i^C)$. Hence $\partial \ddot{\mathbf{r}}/\partial \mathbf{r}_i^C = -\partial \ddot{\mathbf{r}}/\partial \mathbf{r}$ computed from Eq. (558).

The acceleration due to solar radiation pressure and small forces (SRP–SF) is affected by the positions of the sun, earth, and reference body (for roll control) relative to the center of integration. However, the partial derivative of $\ddot{\mathbf{r}}$ (SRP–SF) with respect to $\mathbf{r}_i^C$ is about four orders of magnitude less than the first two terms of Eq. (609) and can safely be neglected in the partial derivatives. Hence, these terms do not appear in Eq. (609).

Section IV gives the formulas for computing corrected position and velocity vectors for the heliocentric ephemeris of a planet or the earth–moon barycenter or for the geocentric lunar ephemeris. Also, the corrected position and velocity vectors of the moon relative to the earth are broken down into the position and velocity vectors of the barycenter relative to the earth and of the moon relative to the barycenter. The relative position or velocity vector between two bodies (a planet, sun, or moon) is computed as a sum of the above vectors (see listing in Section IV-C). Correspondingly, the partial derivative of the relative position or velocity between two bodies with respect to $\mathbf{f}$ may be computed as the sum of partial derivatives of each subvector with respect to $\mathbf{f}$. The partial derivatives of each basic position or velocity vector with respect to the reference parameters that affect it are obtained from the derivatives of Eqs. (111), (112), (150), and (151):

$$\frac{\partial \mathbf{r}_P^S}{\partial A_E} = \frac{\mathbf{r}_P^S}{A_E} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{610}$$

$$\frac{\partial \mathbf{r}_M^E}{\partial R_E} = \frac{\mathbf{r}_M^E}{R_E} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{611}$$

The partial derivatives

$$\frac{\partial \mathbf{r}_P^S}{\partial \mathbf{E}_P} \quad \text{and} \quad \frac{\partial \mathbf{r}_M^E}{\partial \mathbf{E}_M} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}}$$

are computed from the formulation of Subsections IV-B-3 and -4.

$$\frac{\partial \mathbf{r}_M^B}{\partial R_E} = \frac{\mathbf{r}_M^B}{R_E} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{612}$$

$$\frac{\partial \mathbf{r}_M^B}{\partial \mathbf{E}_M} = \frac{\mu}{1+\mu} \frac{\partial \mathbf{r}_M^E}{\partial \mathbf{E}_M} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{613}$$

$$\frac{\partial \mathbf{r}_M^B}{\partial \mu_E} = \frac{\mathbf{r}_M^E}{(1+\mu)^2 \mu_M} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{614}$$

$$\frac{\partial \mathbf{r}_M^B}{\partial \mu_M} = -\frac{\mu \mathbf{r}_M^E}{(1+\mu)^2 \mu_M} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{615}$$

and

$$\frac{\partial \mathbf{r}_B^E}{\partial R_E} = \frac{\mathbf{r}_B^E}{R_E} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{616}$$

$$\frac{\partial \mathbf{r}_B^E}{\partial \mathbf{E}_M} = \frac{1}{1+\mu} \frac{\partial \mathbf{r}_M^E}{\partial \mathbf{E}_M} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{617}$$

$$\frac{\partial \mathbf{r}_B^E}{\partial \mu_E} = -\frac{\mathbf{r}_M^E}{(1+\mu)^2 \mu_M} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{618}$$

$$\frac{\partial \mathbf{r}_B^E}{\partial \mu_M} = \frac{\mu \mathbf{r}_M^E}{(1+\mu)^2 \mu_M} \qquad \mathbf{r} \rightarrow \dot{\mathbf{r}} \tag{619}$$

where

$$\mu = \frac{\mu_E}{\mu_M}$$

and all position and velocity vectors are corrected values. These equations are used to compute $\partial \mathbf{r}_i^C / \partial \mathbf{f}$ in Eq. (608).

The sum and difference arrays for A_E, R_E, and $\mathbf{E}$ for each ephemeris to be corrected are started at the injection epoch T_{inj} and continued for the duration of the mission. The initial values of $\partial \mathbf{X}/\partial q_i$ are obtained from Eq. (550) or (551), as appropriate. The columns of the C matrix are obtained from Eq. (608). At a change of phase (change of center of integration), $\partial \mathbf{X}/\partial q_i$ must be incremented by the following value, which necessitates a restart of the sum and difference arrays:

$$\Delta \frac{\partial \mathbf{X}}{\partial q_i} = \frac{\partial \mathbf{X}_0^N}{\partial q_i} \tag{620}$$

where

$\mathbf{X}_0^N$ = state vector of old center of integration relative to new center of integration at time of phase change with rectangular components referred to mean earth equator and equinox of 1950.0.

3. ***Gravitational constants*** μ_j. The gravitational constants μ_j for the planets, sun, and moon affect the Newtonian point mass and oblate acceleration terms directly. The constants μ_E and μ_M are reference parameters, and hence may also affect these acceleration terms indirectly. Also, they may produce nonzero initial values at the injection epoch and discontinuities to the partials at phase changes.

The sum and difference arrays for the μ_j are started at the injection epoch with initial values given by Eq. (550) or (551). They may be nonzero for μ_E or μ_M if the injection conditions are not referred to the center of integration. The column of the C matrix for μ_j is given by

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \mu_j} = \frac{\ddot{\mathbf{r}}(\mu_j)}{\mu_j} + \sum_i \frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}_i^C} \frac{\partial \mathbf{r}_i^C}{\partial \mu_j} \tag{621}$$

where $\ddot{\mathbf{r}}(\mu_j)/\mu_j$ is the sum of the direct and indirect Newtonian point mass accelerations and the direct oblate acceleration due to body j, computed with $\mu_j = 1$. The second term of Eq. (621) is Eq. (608) and may be nonzero for $\mu_j = \mu_E$ or μ_M. At a change of phase, $\partial \mathbf{X}/\partial \mu_E$ and $\partial \mathbf{X}/\partial \mu_M$ must be incremented by Eq. (620), necessitating a restart of the sum and difference arrays.

4. ***Harmonic coefficients*** J_n, C_{nm}, S_{nm}. The acceleration terms due to the harmonic coefficients J_n, C_{nm}, S_{nm} of an oblate body are computed only when the distance of the spacecraft from the center of the body is less than a value specified by the user (radius of the "harmonic sphere" for that body). Thus, the three sum and difference arrays for each harmonic coefficient are started when the spacecraft enters the harmonic sphere (or at injection) and are terminated when the spacecraft leaves the harmonic sphere. The initial value of $\partial \mathbf{X}/\partial q_i$ is zero. The column of the C matrix for J_n, C_{nm}, and S_{nm} is computed from

$$\frac{\partial \ddot{\mathbf{r}}}{\partial J_n} = \frac{\ddot{\mathbf{r}}(J_n)}{J_n} \tag{622}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial C_{nm}} = \frac{\ddot{\mathbf{r}}(C_{nm})}{C_{nm}} \tag{623}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial S_{nm}} = \frac{\ddot{\mathbf{r}}(S_{nm})}{S_{nm}} \tag{624}$$

The acceleration due to each J_n, C_{nm}, and S_{nm} may be obtained from Eqs. (169), (173), and (174). However, instead of dividing the acceleration term by the coefficient (which may have an *a priori* value of zero), the acceleration term is simply computed using a value of unity for the coefficient.

5. ***Coefficients of solar radiation pressure and small force model.*** The acceleration of the spacecraft due to the SRP–SF model is given by Eq. (189). The columns of the C matrix for the 15 parameters of the model are

$$\frac{\partial \ddot{\mathbf{r}}}{\partial a_r} = \mathbf{U}_{SP} \tag{625}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial a_x} = \mathbf{X}^* \tag{626}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial a_y} = \mathbf{Y}^* \tag{627}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial b_i} = \frac{\partial \ddot{\mathbf{r}}}{\partial a_i}(t - T_{AC1}) \qquad i = r, x, \text{or } y \tag{628}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial c_i} = \frac{\partial \ddot{\mathbf{r}}}{\partial a_i}(t - T_{AC1})^2 \qquad i = r, x, \text{or } y \tag{629}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial G_r} = \frac{C_1 A_p}{m r_{SP}^2} \mathbf{U}_{SP}\, \mu^*(t - T_{SRP}) \tag{630}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial G_x} = \frac{C_1 A_p}{m r_{SP}^2} \mathbf{X}^*\, \mu^*(t - T_{SRP}) \tag{631}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial G_y} = \frac{C_1 A_p}{m r_{SP}^2} \mathbf{Y}^*\, \mu^*(t - T_{SRP}) \tag{632}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial G_i'} = \frac{\partial \ddot{\mathbf{r}}}{\partial G_i}(EPS) \qquad i = r, x, \text{or } y \tag{633}$$

The three sum and difference arrays for each of the nine SF parameters are started at T_{AC1}, using zero for the initial values of the partial derivatives, and continued to the epoch T_{AC2}. The three sum and difference arrays for each of the six SRP parameters are started at T_{SRP}, using zero for the initial values of the partial derivatives, and continued for the remainder of the mission. Each time the spacecraft passes into or out of a shadow, all sum and difference arrays must be restarted.

6. ***Coefficients of finite burn motor model.*** The acceleration due to a finite motor burn is given by Eq. (197). The columns of the C matrix for the polynomial coefficients F_i, α_i, and δ_i $(i = 0, 1, 2, 3, \text{or } 4)$ are

$$\frac{\partial \ddot{\mathbf{r}}}{\partial F_i} = \frac{\bar{t}^i C}{m(t)} \mathbf{U} \tag{634}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \alpha_i} = a \begin{bmatrix} -\cos\delta \sin\alpha \\ \cos\delta \cos\alpha \\ 0 \end{bmatrix} \bar{t}^i \tag{635}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \delta_i} = a \begin{bmatrix} -\sin\delta \cos\alpha \\ -\sin\delta \sin\alpha \\ \cos\delta \end{bmatrix} \bar{t}^i \tag{636}$$

The three sum and difference arrays for each of these parameters are started at T_0 with zero initial conditions and terminated at $T_f = T_0 + T$.

The three sum and difference arrays for T_0 (specified in the UTC, ST, or A1 time scales) are started at T_0 using as initial conditions:

$$\frac{\partial \mathbf{X}(T_0)}{\partial T_0} = -\begin{bmatrix} 0\,(3 \times 1) \\ \hdashline a(T_0)\,\mathbf{U}(T_0) \end{bmatrix} \tag{637}$$

The column of the C matrix for T_0 is

$$\frac{\partial \mathring{\mathbf{r}}}{\partial T_0} = -\,[\mathring{a}\mathbf{U} + a\mathring{\mathbf{U}}] \tag{638}$$

where

$$\mathring{a} = \frac{F_1 + 2F_2\bar{t} + 3F_3\bar{t}^2 + 4F_4\bar{t}^3}{m\,(t)}C + \frac{a}{m\,(t)}(\mathring{M}_0 + \mathring{M}_1\bar{t} + \mathring{M}_2\bar{t}^2 + \mathring{M}_3\bar{t}^3) \tag{639}$$

$$\mathring{\mathbf{U}} = \begin{bmatrix} -\mathring{\alpha}\cos\delta\sin\alpha - \mathring{\delta}\sin\delta\cos\alpha \\ \mathring{\alpha}\cos\delta\cos\alpha - \mathring{\delta}\sin\delta\sin\alpha \\ \mathring{\delta}\cos\delta \end{bmatrix} \tag{640}$$

and

$$\mathring{\alpha} = \alpha_1 + 2\alpha_2\bar{t} + 3\alpha_3\bar{t}^2 + 4\alpha_4\bar{t}^3 \tag{641}$$

$$\mathring{\delta} = \delta_1 + 2\delta_2\bar{t} + 3\delta_3\bar{t}^2 + 4\delta_4\bar{t}^3 \tag{642}$$

The sum and difference arrays for T_0 are terminated at $T_f = T_0 + T$, and the following increment to the partial derivatives is added:

$$\Delta\frac{\partial \mathbf{X}(T_f)}{\partial T_0} = \begin{bmatrix} 0\,(3\times1) \\ \hdashline a\,(T_f)\,\mathbf{U}\,(T_f) \end{bmatrix} \tag{643}$$

The initial value of the partial derivative with respect to T occurs at T_f:

$$\frac{\partial \mathbf{X}(T_f)}{\partial T} = \begin{bmatrix} 0\,(3\times1) \\ \hdashline a\,(T_f)\,\mathbf{U}\,(T_f) \end{bmatrix} \tag{644}$$

There are no sum and difference arrays for the parameter T.

7. Parameters for instantaneous burn motor model. A motor burn of short duration or a spring separation may be represented as a discontinuity to the spacecraft trajectory. The rectangular components of the velocity increment $\Delta\mathring{\mathbf{r}}$ and the burn time t_b are the solve-for parameters. The increment to the spacecraft position at the maneuver epoch T_M is computed as

$$\Delta\mathbf{r} = \frac{1}{2}\Delta\mathring{\mathbf{r}}t_b \tag{645}$$

There are no sum and difference arrays for these four parameters. However, the initial values of the partial derivatives at T_M are

$$\left[\frac{\partial \mathbf{X}}{\partial\Delta\mathring{x}}, \frac{\partial \mathbf{X}}{\partial\Delta\mathring{y}}, \frac{\partial \mathbf{X}}{\partial\Delta\mathring{z}}, \frac{\partial \mathbf{X}}{\partial t_b}\right] = \begin{bmatrix} \tfrac{1}{2}t_b & 0 & 0 & \tfrac{1}{2}\Delta\mathring{x} \\ 0 & \tfrac{1}{2}t_b & 0 & \tfrac{1}{2}\Delta\mathring{y} \\ 0 & 0 & \tfrac{1}{2}t_b & \tfrac{1}{2}\Delta\mathring{z} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{646}$$

8. Parameters affecting transformation from atomic time to ephemeris time. The parameters ΔT_{1958} and Δf_{cesium} affect the ET values of the following epochs, specified in a known time scale (A1, UTC, UT1, or ST, but usually UTC) and represented as seconds past January 1, 1950, 0^h:

1. (1) Injection, T_{inj}
2. (2) Unfolding of solar panels, T_{SRP}
3. (3) Start and stop times for small force polynomials, T_{AC1} and T_{AC2}
4. (4) Epoch of instantaneous maneuver, T_M
5. (5) Start and stop times of finite burn motor, T_0 and T_f

Since the acceleration versus time curve for the finite burn motor is shifted in ET, sum and difference arrays for ΔT_{1958} and Δf_{cesium} must be generated from T_0 to $T_f = T_0 + T$. Also, discontinuities to the partial derivatives with respect to ΔT_{1958} and Δf_{cesium} occur at T_{inj}, T_{SRP}, T_{AC1}, T_{AC2}, T_M, T_0, and T_f.

The partial derivatives of the ET value of an epoch T specified in the A1, UTC, UT1, and ST time scales with respect to ΔT_{1958} and Δf_{cesium} are (see Eq. 93)

$$\frac{\partial T\,(\text{ET})}{\partial\Delta T_{1958}} = 1 \tag{647}$$

$$\frac{\partial T\,(\text{ET})}{\partial\Delta f_{\text{cesium}}} = -\frac{T - 252{,}460{,}800}{9{,}192{,}631{,}770} \tag{648}$$

An infinitesimal change in the ET value of the injection epoch, $dT_{\text{inj}}\,(\text{ET})$, holding the injection state vector $\mathbf{X}_0^B$ fixed, is equivalent to the following changes in the injection position $\mathbf{r}_0$ and velocity $\dot{\mathbf{r}}_0$ relative to the center of integration at the nominal epoch T_{inj} (ET):

$$d\mathbf{r}_0 = -\dot{\mathbf{r}}_0^B \, dT_{\mathrm{inj}}\,(\mathrm{ET}) \tag{649}$$

$$d\dot{\mathbf{r}}_0 = -\ddot{\mathbf{r}}_0^B \, dT_{\mathrm{inj}}\,(\mathrm{ET}) \tag{650}$$

where

$\dot{\mathbf{r}}_0^B$ = solve-for velocity of spacecraft relative to body B at injection epoch

$\ddot{\mathbf{r}}_0^B$ = acceleration of spacecraft relative to body B at injection epoch

Hence, the initial values of the partial derivatives of $\mathbf{X}$ with respect to ΔT_{1958} and $\Delta f_{\mathrm{cesium}}$ at the injection epoch are

$$\frac{\partial \mathbf{X}(T_{\mathrm{inj}})}{\partial \Delta T_{1958}} = -\begin{bmatrix} \dot{\mathbf{r}}_0^B \\ \ddot{\mathbf{r}}_0^B \end{bmatrix} \tag{651}$$

$$\frac{\partial \mathbf{X}(T_{\mathrm{inj}})}{\partial \Delta f_{\mathrm{cesium}}} = \frac{T_{\mathrm{inj}} - 252{,}460{,}800}{9{,}192{,}631{,}770} \begin{bmatrix} \dot{\mathbf{r}}_0^B \\ \ddot{\mathbf{r}}_0^B \end{bmatrix} \tag{652}$$

Similarly, at the epoch T_M of an instantaneous maneuver, $\partial\mathbf{X}/\partial\,(\Delta T_{1958})$ and $\partial\mathbf{X}/\partial\,(\Delta f_{\mathrm{cesium}})$ must be incremented by

$$\Delta \frac{\partial \mathbf{X}(T_M)}{\partial \Delta T_{1958}} = \begin{bmatrix} -\Delta\dot{\mathbf{r}} \\ \hline \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 \end{bmatrix} \tag{653}$$

$$\Delta \frac{\partial \mathbf{X}(T_M)}{\partial \Delta f_{\mathrm{cesium}}} = \frac{T_M - 252{,}460{,}800}{9{,}192{,}631{,}770} \begin{bmatrix} \Delta\dot{\mathbf{r}} \\ \hline \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 \end{bmatrix} \tag{654}$$

where

$\ddot{\mathbf{r}}_2 = \ddot{\mathbf{r}}$ at T_M after $\Delta\mathbf{r}, \Delta\dot{\mathbf{r}}$ have been added

$\ddot{\mathbf{r}}_1 = \ddot{\mathbf{r}}$ at T_M before $\Delta\mathbf{r}, \Delta\dot{\mathbf{r}}$ have been added

At the epoch T_{SRP} where the solar panels are unfolded, the partial derivatives must be incremented by

$$\Delta \frac{\partial \mathbf{X}(T_{SRP})}{\partial \Delta T_{1958}} = -\begin{bmatrix} 0\,(3 \times 1) \\ \hline \ddot{\mathbf{r}}_{SRP}\,(T_{SRP}) \end{bmatrix} \tag{655}$$

$$\Delta \frac{\partial \mathbf{X}(T_{SRP})}{\partial \Delta f_{\mathrm{cesium}}} = \frac{T_{SRP} - 252{,}460{,}800}{9{,}192{,}631{,}770} \begin{bmatrix} 0\,(3 \times 1) \\ \hline \ddot{\mathbf{r}}_{SRP}\,(T_{SRP}) \end{bmatrix} \tag{656}$$

Since the small force accelerations are extremely small, the increments to the partial derivatives at T_{AC1} and T_{AC2} have been ignored.

The three sum and difference arrays for ΔT_{1958} and for $\Delta f_{\mathrm{cesium}}$ are started at T_0 for the finite burn motor and terminated at $T_f = T_0 + T$. The initial values of the partial derivatives are given by

$$\frac{\partial \mathbf{X}(T_0)}{\partial \Delta T_{1958}} = -\begin{bmatrix} 0\,(3 \times 1) \\ \hline a\,(T_0)\,\mathbf{U}\,(T_0) \end{bmatrix} \tag{657}$$

$$\frac{\partial \mathbf{X}(T_0)}{\partial \Delta f_{\mathrm{cesium}}} = \frac{T_0 - 252{,}460{,}800}{9{,}192{,}631{,}770} \begin{bmatrix} 0\,(3 \times 1) \\ \hline a\,(T_0)\,\mathbf{U}\,(T_0) \end{bmatrix} \tag{658}$$

The increments to the partial derivatives occurring prior to T_0 are mapped to T_0 and added to the above values. The columns of the C matrix are computed from

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \Delta T_{1958}} = -\,[\mathring{a}\,\mathbf{U} + a\,\dot{\mathbf{U}}] \tag{659}$$

$$\frac{\partial \ddot{\mathbf{r}}}{\partial \Delta f_{\mathrm{cesium}}} = \frac{T_0 - 252{,}460{,}800}{9{,}192{,}631{,}770}\,[\mathring{a}\,\mathbf{U} + a\,\dot{\mathbf{U}}] \tag{660}$$

At T_f, the partial derivatives must be incremented by

$$\Delta \frac{\partial \mathbf{X}(T_f)}{\partial \Delta T_{1958}} = \begin{bmatrix} 0\,(3 \times 1) \\ \hline a\,(T_f)\,\mathbf{U}\,(T_f) \end{bmatrix} \tag{661}$$

$$\Delta \frac{\partial \mathbf{X}(T_f)}{\partial \Delta f_{\mathrm{cesium}}} = -\frac{T_0 - 252{,}460{,}800}{9{,}192{,}631{,}770} \begin{bmatrix} 0\,(3 \times 1) \\ \hline a\,(T_f)\,\mathbf{U}\,(T_f) \end{bmatrix} \tag{662}$$

E. Summary

The solution to the variational equations is the matrix $\partial\mathbf{X}/\partial\mathbf{q}$ given by Eq. (547). The U matrix is computed using Eq. (552). Each column of the V matrix corresponding to parameter q_i is obtained by one or more of the following methods:

(1) Interpolation of the three sum and difference arrays for q_i.

(2) Mapping forward the final partial derivatives from discontinued sum and difference arrays, using Eq. (554).

(3) Mapping forward a discontinuity to the partial derivatives using Eq. (554).

XIV. Regression Partial Derivatives

This section gives the formulation for computing the regression partial derivatives which are the partial derivatives of each observable quantity with respect to the solve-for parameter vector q. Section XIII gave the formulation for integrating the variational equations which yield the partial derivatives of the spacecraft state vector at any time t with respect to q. These subpartial derivatives are required to form the regression partial derivatives.

The general expression for computing the partial derivatives of doppler and angular observables with respect to q is given in Section XIV-A. The formulation for computing the various terms appearing therein is given in Sections XIV-A to -E. The formulations for computing the partial derivatives of range observables and differenced-range doppler observables with respect to q are given in Sections XIV-F and -G respectively.

The DPODP currently does not have the capability for solving for the relativity parameter γ. However, approximate partial derivatives of the observables with respect to γ are included in Sections XIV-E and -F. These partial derivatives are based solely upon the variation of the relativity term of the light time equation (Eq. 203) with γ. A solution for γ using these approximate partial derivatives should converge when the spacecraft passes through superior conjunction and the relativistic correction to the light time becomes very large. The partial derivatives do not account for the smaller effect of γ on the ephemerides of the celestial bodies and the spacecraft.

A. General Expression for Partial Derivatives of Doppler and Angular Observables With Respect to q

Let

z = an observable quantity (doppler or angles)

q = solve-for parameter vector

$$\mathbf{X}_i^S(t_i) = \left[\begin{array}{c}\mathbf{r}_i^S(t_i)\\ \hdashline \dot{\mathbf{r}}_i^S(t_i)\end{array}\right]$$ = heliocentric state vector (position and velocity vectors) of direct participant i at its epoch of participation t_i, with rectangular components referred to the mean earth equator and equinox of 1950.0. The units are km and km/ephemeris second.

$$\dot{\mathbf{X}}_i^S(t_i) = \left[\begin{array}{c}\dot{\mathbf{r}}_i^S(t_i)\\ \hdashline \ddot{\mathbf{r}}_i^S(t_i)\end{array}\right]$$

The direct participants i are

1 = transmitter on earth at transmission time t_1.

2 = spacecraft at intermediate time t_2 (or transmission time t_2 for angular observables or 1-way doppler).

3 = receiver on earth at reception time t_3. For doppler, t_3 is the midpoint of the count interval T_c.

For purposes of obtaining partial derivatives of an observable z with respect to the parameter vector q,

$$z = z\left[\mathbf{X}_3^S(t_3, \mathrm{q}), \mathbf{X}_2^S(t_2, \mathrm{q}), \mathbf{X}_1^S(t_1, \mathrm{q}), \mathrm{q}\right] \tag{663}$$

Differentiating Eq. (663) with respect to q gives

$$\begin{aligned}\frac{\partial z}{\partial \mathrm{q}} = {} & \frac{\partial z}{\partial \mathbf{X}_3^S(t_3)}\left[\frac{\partial \mathbf{X}_3^S(t_3)}{\partial \mathrm{q}}\right]_{t_3=\mathrm{constant}} \\ & + \frac{\partial z}{\partial \mathbf{X}_2^S(t_2)}\left\{\left[\frac{\partial \mathbf{X}_2^S(t_2)}{\partial \mathrm{q}}\right]_{t_2=\mathrm{constant}} + \dot{\mathbf{X}}_2^S(t_2)\left[\frac{\partial t_2(\mathrm{ET})}{\partial \mathrm{q}}\right]_{c,\ \mathrm{time\ transformations}=\mathrm{constant}}\right\} \\ & + \frac{\partial z}{\partial \mathbf{X}_1^S(t_1)}\left\{\left[\frac{\partial \mathbf{X}_1^S(t_1)}{\partial \mathrm{q}}\right]_{t_1=\mathrm{constant}} + \dot{\mathbf{X}}_1^S(t_1)\left[\frac{\partial t_1(\mathrm{ET})}{\partial \mathrm{q}}\right]_{c,\ \mathrm{time\ transformations}=\mathrm{constant}}\right\} \\ & + \left[\frac{\partial z}{\partial \mathrm{q}}\right]_{c,\ \mathrm{time\ transformations}=\mathrm{variable}} \\ & + \left[\frac{\partial z}{\partial \mathrm{q}}\right]_{\mathbf{X}_3^S(t_3),\, \mathbf{X}_2^S(t_2),\, \mathbf{X}_1^S(t_1)=\mathrm{constant}}\end{aligned} \tag{664}$$

The terms of Eq. (664) are of two basic types. The last term gives the direct variation of the observable z due to a variation in q, holding the state vectors of each participant constant. The remaining terms give the variation in the observable due to variations in the state vectors of each participant. The term on line 1 and the first terms on lines 2 and 3 give the direct variation in the state vector (and hence z) with respect to a variation in q holding the epochs of participation constant. Since the state vectors are a function of q, the epochs of participation t_2 (ET) and t_1 (ET) obtained from the solution of the light-time problem will also vary with q. The second terms on lines 2 and 3 account for this effect. The epochs of participation may also vary due to variations in c, the speed of light,[30] and the parameters affecting the time transformations which are used in the light-time solution. The partial derivative of z with respect to q due to these effects is indicated on line 4.

The partial derivatives of the heliocentric state vectors with respect to the parameter vector q, holding the epochs of participation constant, are given by the following sums:

$$\frac{\partial \mathbf{X}_3^S(t_2)}{\partial \mathbf{q}} = \frac{\partial \mathbf{X}_3^E(t_3)}{\partial \mathbf{q}} + \frac{\partial \mathbf{X}_E^S(t_3)}{\partial \mathbf{q}} \tag{665}$$

$$\frac{\partial \mathbf{X}_2^S(t_2)}{\partial \mathbf{q}} = \frac{\partial \mathbf{X}_2^{B2}(t_2)}{\partial \mathbf{q}} + \frac{\partial \mathbf{X}_{B2}^S(t_2)}{\partial \mathbf{q}} \tag{666}$$

$$\frac{\partial \mathbf{X}_1^S(t_1)}{\partial \mathbf{q}} = \frac{\partial \mathbf{X}_1^E(t_1)}{\partial \mathbf{q}} + \frac{\partial \mathbf{X}_E^S(t_1)}{\partial \mathbf{q}} \tag{667}$$

where

$B2$ = center of integration for free spacecraft or body on which a landed spacecraft is resting

E = earth

S = sun

The partial derivatives $\partial z/\partial \mathbf{X}_3^S(t_3)$, $\partial z/\partial \mathbf{X}_2^S(t_2)$, and $\partial z/\partial \mathbf{X}_1^S(t_1)$ for doppler and angular observables are given in Section XIV-B.

For a free spacecraft, $\partial \mathbf{X}_2^{B2}(t_2)/\partial \mathbf{q}$ is obtained from the solution of the variational equations (Section XIII).

In Section XIV-C, the formulation is given for computing $\partial \mathbf{X}_3^E(t_3)/\partial \mathbf{q}$, $\partial \mathbf{X}_1^E(t_1)/\partial \mathbf{q}$, and $\partial \mathbf{X}_2^{B2}(t_2)/\partial \mathbf{q}$ (if point 2 is a landed spacecraft on a celestial body).

The partial derivatives $\partial \mathbf{X}_E^S(t_3)/\partial \mathbf{q}$, $\partial \mathbf{X}_{B2}^S(t_2)/\partial \mathbf{q}$, and $\partial \mathbf{X}_E^S(t_1)/\partial \mathbf{q}$ are computed from the following (where E = earth, M = moon, B = earth–moon barycenter, P = planet, S = sun):

$$\frac{\partial \mathbf{X}_E^S}{\partial \mathbf{q}} = \frac{\partial \mathbf{X}_B^S}{\partial \mathbf{q}} - \frac{\partial \mathbf{X}_B^E}{\partial \mathbf{q}} \tag{668}$$

$$\frac{\partial \mathbf{X}_M^S}{\partial \mathbf{q}} = \frac{\partial \mathbf{X}_B^S}{\partial \mathbf{q}} + \frac{\partial \mathbf{X}_M^B}{\partial \mathbf{q}} \tag{669}$$

$$\frac{\partial \mathbf{X}_P^S}{\partial \mathbf{q}} = \frac{\partial \mathbf{X}_P^S}{\partial \mathbf{q}} \tag{670}$$

where the right-hand terms are obtained from Eqs. (610–619) (Eq. 610 applies also for $P = B$). The columns of Eqs. (668–670) are nonzero only for the reference parameters A_E, R_E, μ_E, μ_M, and osculating orbital elements **E** for the ephemeris of a planet, the earth–moon barycenter, or the moon.

The derivatives ∂t_2 (ET)/∂q and ∂t_1 (ET)/∂q are obtained by differentiating the light time equations (Eqs. 313 and 314), ignoring the relativity terms. Using the notation of Section VIII-C, the results are

$$\frac{\partial t_2\,(\mathrm{ET})}{\partial \mathbf{q}} = \frac{1}{c}\,\frac{\mathbf{r}_{23}^T}{r_{23}}\left[\frac{\partial \mathbf{r}_2^S(t_2)}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_3^S(t_3)}{\partial \mathbf{q}}\right]\left(1 + \frac{\dot{p}_{23}}{c}\right) \tag{671}$$

$$\frac{\partial t_1\,(\mathrm{ET})}{\partial \mathbf{q}} = \frac{\partial t_2\,(\mathrm{ET})}{\partial \mathbf{q}}\left(1 - \frac{\dot{r}_{12}}{c}\right) + \frac{1}{c}\,\frac{\mathbf{r}_{12}^T}{r_{12}}\left[\frac{\partial \mathbf{r}_1^S(t_1)}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_2^S(t_2)}{\partial \mathbf{q}}\right]\left(1 + \frac{\dot{p}_{12}}{c}\right) \tag{672}$$

The partial derivatives of $\mathbf{r}_3^S(t_3)$, $\mathbf{r}_2^S(t_2)$, and $\mathbf{r}_1^S(t_1)$ with respect to q are simply the first three rows of Eqs. (665–667), respectively.

The last two terms of Eq. (664) are evaluated in Sections XIV-D and -E, respectively.

For angular observables, Eq. (664) is evaluated with the state vector X of each participant taken to be its position vector only. Since angular observables and 1-way doppler involve only two participants, the third line of Eq. (664) is omitted for these data types.

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[30] The speed of light is an adopted constant that defines the light-second as the basic length unit. It is not normally included in the solution vector.

From Eq. (311), the partial derivative of a subinterval doppler observable z with respect to $\mathbf{q}$ is given by

$$\frac{\partial z}{\partial \mathbf{q}} = \frac{1}{m} \sum_{i=1}^{m} \frac{\partial z_i}{\partial \mathbf{q}}$$

where z_i is the doppler observable computed for subinterval i and $\partial z_i / \partial \mathbf{q}$ is computed from Eq. (664) and associated equations using a count time of T_c/m.

B. Partial Derivatives of Doppler and Angular Observables With Respect to State Vectors of Each Direct Participant

1. Doppler observables. One-way, 2-way, and 3-way doppler observables are computed from Eqs. (302), (308), (309), (310), (343), (344), (371), (372) and auxiliary equations of Section VIII. The partial derivatives of these observables with respect to the heliocentric state vectors of the three participants are obtained by a straightforward differentiation of these formulas, ignoring the terms of Eqs. (308), (309), and (310) which contain the very small factors F_1, F, and Δ.

The $1/c$ and $1/c^2$ terms of Eqs. (343) and (344) for $[1 - (F_R/F_T)]$ were differentiated. For 2-leg doppler (2-way or 3-way doppler), however, the relativistic terms $(\phi_1 - \phi_3)$ and $\frac{1}{2}(\dot{s}_1^2 - \dot{s}_3^2)$ were ignored. The potential term contributes a maximum of only about 10^{-3} m/s to range rate. The velocity term has a maximum value of about 0.1 m/s, but its variation due to a variation in the parameter vector $\mathbf{q}$ is very small. For 1-way doppler, these relativistic terms can be very large, and hence were included in the differentiation. For this purpose, the potential was assumed to be due to the sun only, a reasonable assumption for the inner part of the solar system.

Only the $1/c$ terms of Eqs. (371) and (372) for $[1 - (F_R/F_T)]^{\cdot\cdot}$ were differentiated.

Near earth, with a range $\rho = r_{12}, r_{23} = 100$ km, and a count time $T_c = 10$ s, the partial derivatives from $[1 - (F_R/F_T)]^{\cdot\cdot}$ are the same order of magnitude as those from $[1 - (F_R/F_T)]$. Since the $1/c^2$ terms of $[1 - (F_R/F_T)]^{\cdot\cdot}$ were not differentiated, the partial derivatives for this extreme near-earth case are good to about four figures.

The ratio of the partial derivatives derived from $[1 - (F_R/F_T)]^{\cdot\cdot}$ to those derived from $[1 - (F_R/F_T)]$ is proportional to $(T_c/\rho)^2$. For ρ increasing from 100 km (with $T_c = 10$ s) to 10^6 km (with $T_c = 1000$ s), the factor $(T_c/\rho)^2$ reduces by four orders of magnitude, and the partials from $[1 - (F_R/F_T)]^{\cdot\cdot}$ are the same order of magnitude as those from the $1/c^2$ terms of $[1 - (F_R/F_T)]$. Hence, when the spacecraft is far from the earth (and other bodies), the partial derivatives are accurate to seven or eight figures.

In deriving Eq. (664), the dependency of the observable z on the acceleration and jerk vectors[31] of each participant was ignored. This omission limits the accuracy of the partial derivatives to four or five significant figures for $\rho = 100$ km, and seven or eight significant figures for $\rho = 10^6$ km or more. This limitation on accuracy is the same as that resulting from truncating the doppler formulas before differentiating.

The partial derivative of 1-way doppler ($F1$), 2-way doppler ($F2$), or 3-way doppler ($F3$) with respect to the heliocentric state vector $\mathbf{X}_i^S(t_i)$ of the ith direct participant at its epoch of participation t_i is

$$\frac{\partial z}{\partial \mathbf{X}_i^S(t_i)} = C_6 \left[\frac{\partial \left(1 - \frac{F_R}{F_T}\right)^*}{\partial \mathbf{r}_i^S(t_i)} \;\vdots\; \frac{\partial \left(1 - \frac{F_R}{F_T}\right)^*}{\partial \dot{\mathbf{r}}_i^S(t_i)} \right] \qquad i = 1, 2, \text{or } 3 \tag{673}$$

where

$$z = F1, F2, \text{or } F3$$

and

$$C_6 = \begin{cases} C_2 f_{S/C} & \text{for } F1 \\ C_3 f_q(t_1) & \text{for } F2 \\ C_5 f_q(t_1) & \text{for } F3 \end{cases}$$

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[31] The jerk vector is the time derivative of the acceleration vector.

The partial derivatives of $[1 - (F_R/F_T)]^*$ (see Eq. 302) with respect to the heliocentric position and velocity vectors of each participant are functions of the following quantities (see Section VIII for definitions of terms):

$$D = \frac{1}{c}\left[-\frac{(\dot{\mathbf{r}}_{12})^T}{r_{12}} + \frac{\dot{r}_{12}}{r_{12}}\frac{(\mathbf{r}_{12})^T}{r_{12}}\right]\left[1 + \frac{1}{c}(\dot{p}_{12} - \dot{r}_{23})\right]$$

$$+ \frac{1}{c^2}\left[-\frac{(\dot{\mathbf{r}}_1)^T}{r_{12}} + \frac{\dot{p}_{12}}{r_{12}}\frac{(\mathbf{r}_{12})^T}{r_{12}}\right]\dot{r}_{12}$$

$$+ \frac{T_c^2}{24cr_{12}}\left[\left(\dddot{r}_{12} - 6\frac{\ddot{r}_{12}\dot{r}_{12}}{r_{12}} + 6\frac{\dot{r}_{12}^3}{r_{12}^2}\right)\frac{(\mathbf{r}_{12})^T}{r_{12}}\right.$$

$$\left. + 3\left(\ddot{r}_{12} - 2\frac{\dot{r}_{12}^2}{r_{12}}\right)\frac{(\dot{\mathbf{r}}_{12})^T}{r_{12}} + 3\frac{\dot{r}_{12}}{r_{12}}(\ddot{\mathbf{r}}_{12})^T - (\dddot{\mathbf{r}}_{12})^T\right] \tag{674}$$

$D\begin{pmatrix}12 \leftrightarrow 23\\ 1 \rightarrow 2\end{pmatrix}$ = Eq. (674) with subscript 12 changed to 23 and vice versa, and subscript 1 changed to 2.

$D^* = D$ with the $\dot{r}_{23}$ term removed

$$E = -\frac{1}{c}\frac{(\mathbf{r}_{12})^T}{r_{12}}\left[1 + \frac{1}{c}(\dot{p}_{12} - \dot{r}_{23})\right]$$

$$+ \frac{T_c^2}{8cr_{12}}\left[\left(\ddot{r}_{12} - 2\frac{\dot{r}_{12}^2}{r_{12}}\right)\frac{(\mathbf{r}_{12})^T}{r_{12}}\right.$$

$$\left. + 2\frac{\dot{r}_{12}}{r_{12}}(\dot{\mathbf{r}}_{12})^T - (\ddot{\mathbf{r}}_{12})^T\right] \tag{675}$$

$E\,(12 \leftrightarrow 23)$ = Eq. (675) with subscript 12 changed to 23 and vice versa.

$E^* = E$ with the $\dot{r}_{23}$ term removed

$$\Delta E = \frac{1}{c^2}\frac{(\mathbf{r}_{12})^T}{r_{12}}\dot{r}_{12} \tag{676}$$

For 2-way or 3-way doppler,

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \mathbf{r}_1^S(t_1)} = D \tag{677}$$

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \mathbf{r}_2^S(t_2)} = D\begin{pmatrix}12 \leftrightarrow 23\\ 1 \rightarrow 2\end{pmatrix} - D \tag{678}$$

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \mathbf{r}_3^S(t_3)} = -D\begin{pmatrix}12 \leftrightarrow 23\\ 1 \rightarrow 2\end{pmatrix} \tag{679}$$

For 1-way doppler,

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \mathbf{r}_2^S(t_2)} = D^*\begin{pmatrix}12 \leftrightarrow 23\\ 1 \rightarrow 2\end{pmatrix} - \frac{1}{c^2}\frac{\mu_S}{(r_2)^3}(\mathbf{r}_2)^T \tag{680}$$

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \mathbf{r}_3^S(t_3)} = -D^*\begin{pmatrix}12 \leftrightarrow 23\\ 1 \rightarrow 2\end{pmatrix} + \frac{1}{c^2}\frac{\mu_S}{(r_3)^3}(\mathbf{r}_3)^T \tag{681}$$

For 2-way or 3-way doppler,

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \dot{\mathbf{r}}_1^S(t_1)} = E + \Delta E \tag{682}$$

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \dot{\mathbf{r}}_2^S(t_2)} = E\,(12 \leftrightarrow 23) + \Delta E\,(12 \leftrightarrow 23) - E \tag{683}$$

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \dot{\mathbf{r}}_3^S(t_3)} = -E\,(12 \leftrightarrow 23) \tag{684}$$

For 1-way doppler,

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \dot{\mathbf{r}}_2^S(t_2)} = E^*(12 \leftrightarrow 23) + \Delta E\,(12 \leftrightarrow 23) + \frac{1}{c^2}(\dot{\mathbf{r}}_2)^T \tag{685}$$

$$\frac{\partial\left(1 - \frac{F_R}{F_T}\right)^*}{\partial \dot{\mathbf{r}}_3^S(t_3)} = -E^*(12 \leftrightarrow 23) - \frac{1}{c^2}(\dot{\mathbf{r}}_3)^T \tag{686}$$

2. ***Angular observables.*** The partial derivatives of angular observables with respect to the heliocentric position vectors of the spacecraft and tracking station are given by Eqs. (449–461) and auxiliary formulas of Section X.

C. Partial Derivatives of Body-Centered State Vector of Tracking Station or Ship or Landed Spacecraft With Respect to Parameter Vector

*1. **General formulas.*** This section gives the partial derivatives of $\mathbf{X}_3^E(t_3)$ and $\mathbf{X}_1^E(t_1)$ for a fixed tracking station or a moving tracking ship and $\mathbf{X}_2^{B2}(t_2)$ for a landed spacecraft on a planet or the moon with respect to the parameter vector $\mathbf{q}$. The parameters that affect these state vectors are the three spherical or cylindrical coordinates of a fixed tracking station or landed spacecraft and spherical coordinates at an epoch plus velocity and azimuth for a tracking ship. Each of the state vectors above, with rectangular components of position and velocity referred to the mean earth equator and equinox of 1950.0, will be denoted here by

$$\mathbf{X}_{50} = \begin{bmatrix} \mathbf{r}_{50} \\ \hdashline \dot{\mathbf{r}}_{50} \end{bmatrix} \tag{687}$$

From Eqs. (242), (243), (246), and (247),

$$\mathbf{r}_{50} = T_i \mathbf{r}_b \tag{688}$$

$$\dot{\mathbf{r}}_{50} = \dot{T}_i \mathbf{r}_b + T_i \dot{\mathbf{r}}_b \tag{689}$$

where $\mathbf{r}_b$ is the body-fixed position vector of the station (fixed tracking station, tracking ship, or landed spacecraft) defined in Section VII-A, $\dot{\mathbf{r}}_b$ is the body-fixed velocity vector (nonzero for the tracking ship only), T_i is the 3×3 transformation matrix for body i which transforms body-fixed rectangular components of a vector to rectangular components referred to the mean earth equator and equinox of 1950.0, and $\dot{T}_i$ is the derivative of T_i with respect to ephemeris time. The body-fixed to space-fixed transformations T_i and $\dot{T}_i$ are not functions of solve-for parameters. The one exception to this is $\dot{T}_E$ for the earth, which is a function of Δf_{cesium}. However, it affects the magnitude of $\dot{\mathbf{r}}_{50}$ by less than 10^{-5} m/s, and hence is ignored in the partial derivatives. Thus,

$$\frac{\partial \mathbf{X}_{50}}{\partial \mathbf{q}} = \begin{bmatrix} T_i \dfrac{\partial \mathbf{r}_b}{\partial \mathbf{q}} \\ \hdashline \dot{T}_i \dfrac{\partial \mathbf{r}_b}{\partial \mathbf{q}} + T_i \dfrac{\partial \dot{\mathbf{r}}_b}{\partial \mathbf{q}} \end{bmatrix} \tag{690}$$

*2. **Partial derivatives of body-fixed position and velocity vectors with respect to parameter vector.*** Discussed below are the cases for the fixed tracking station (or landed spacecraft) and the tracking ship.

a. Fixed tracking station or landed spacecraft. The partial derivatives of $\mathbf{r}_b$ with respect to spherical station coordinates are obtained by differentiating Eq. (212):

$$\frac{\partial \mathbf{r}_b}{\partial r} = \begin{bmatrix} \cos\phi \cos\lambda \\ \cos\phi \sin\lambda \\ \sin\phi \end{bmatrix} = \frac{\mathbf{r}_b}{r} \tag{691}$$

$$\frac{\partial \mathbf{r}_b}{\partial \phi} = \begin{bmatrix} -r \sin\phi \cos\lambda \\ -r \sin\phi \sin\lambda \\ r \cos\phi \end{bmatrix} \tag{692}$$

$$\frac{\partial \mathbf{r}_b}{\partial \lambda} = \begin{bmatrix} -r \cos\phi \sin\lambda \\ r \cos\phi \cos\lambda \\ 0 \end{bmatrix} \tag{693}$$

The partial derivatives of $\mathbf{r}_b$ with respect to cylindrical station coordinates are obtained by differentiating Eq. (213):

$$\frac{\partial \mathbf{r}_b}{\partial u} = \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix} \tag{694}$$

$$\frac{\partial \mathbf{r}_b}{\partial v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{695}$$

$$\frac{\partial \mathbf{r}_b}{\partial \lambda} = \begin{bmatrix} -u \sin\lambda \\ u \cos\lambda \\ 0 \end{bmatrix} \tag{696}$$

These partial derivatives are evaluated with coordinates referenced to the true pole, equator, and prime meridian of date (the body-fixed coordinate system). They apply for the solve-for coordinates of a landed spacecraft, which are referenced to this coordinate system, and also for the solve-for coordinates of a tracking station on earth, which are referenced to the mean pole, equator, and prime meridian of 1903.0. For this latter case, the partial derivatives are accurate to approximately seven significant figures.

b. Tracking ship. The formulation for computing $\mathbf{r}_b$ for a tracking ship is given in Subsection VII-B-2. The solve-for parameters of this model are the constant geocentric radius r, the latitude ϕ_0, and longitude λ_0 at the epoch t_0(UTC), the constant speed v, and the constant azimuth A. From Eq. (239), the partial derivative of $\mathbf{r}_b$ with respect to each of these parameters, denoted as q_i, is computed from

$$\frac{\partial \mathbf{r}_b}{\partial q_i} = \frac{\partial \mathbf{r}_b}{\partial \phi}\frac{\partial \phi}{\partial q_i} + \frac{\partial \mathbf{r}_b}{\partial \lambda}\frac{\partial \lambda}{\partial q_i} + \left.\frac{\partial \mathbf{r}_b}{\partial q_i}\right|_{\phi,\lambda=\text{fixed}} \tag{697}$$

Similarly, from Eq. (240),

$$\frac{\partial \dot{\mathbf{r}}_b}{\partial q_i} = \frac{\partial \dot{\mathbf{r}}_b}{\partial \phi}\frac{\partial \phi}{\partial q_i} + \frac{\partial \dot{\mathbf{r}}_b}{\partial \lambda}\frac{\partial \lambda}{\partial q_i} + \left.\frac{\partial \dot{\mathbf{r}}_b}{\partial q_i}\right|_{\phi,\lambda=\text{fixed}} \tag{698}$$

The partial derivatives of $\mathbf{r}_b$ with respect to ϕ and λ are given by Eqs. (692) and (693). The last term of Eq. (697) is nonzero only for $q_i = r$; it is given by Eq. (691). The partial derivatives of $\dot{\mathbf{r}}_b$ with respect to ϕ and λ are obtained by differentiating Eq. (240):

$$\frac{\partial \dot{\mathbf{r}}_b}{\partial \phi} = -\begin{bmatrix} \cos\phi\cos\lambda \\ \cos\phi\sin\lambda \\ \sin\phi \end{bmatrix} v\cos A \tag{699}$$

$$\frac{\partial \dot{\mathbf{r}}_b}{\partial \lambda} = \begin{bmatrix} \cos A\sin\phi\sin\lambda - \sin A\cos\lambda \\ -\cos A\sin\phi\cos\lambda - \sin A\sin\lambda \\ 0 \end{bmatrix} v \tag{700}$$

The last term of Eq. (698) is nonzero for $q_i = A$ and v:

$$\left.\frac{\partial \dot{\mathbf{r}}_b}{\partial A}\right|_{\phi,\lambda=\text{fixed}} = \begin{bmatrix} \sin A\sin\phi\cos\lambda - \cos A\sin\lambda \\ \sin A\sin\phi\sin\lambda + \cos A\cos\lambda \\ -\sin A\cos\phi \end{bmatrix} v \tag{701}$$

$$\left.\frac{\partial \dot{\mathbf{r}}_b}{\partial v}\right|_{\phi,\lambda=\text{fixed}} = \frac{\dot{\mathbf{r}}_b}{v} \tag{702}$$

which should be computed from Eq. (240) with $v = 1$.

The partial derivatives of ϕ with respect to the solve-for parameters are obtained by differentiating Eq. (235):

$$\frac{\partial \phi}{\partial r} = -\frac{(\phi - \phi_0)}{r} \tag{703}$$

$$\frac{\partial \phi}{\partial \phi_0} = 1 \tag{704}$$

$$\frac{\partial \phi}{\partial \lambda_0} = 0 \tag{705}$$

$$\frac{\partial \phi}{\partial A} = -\frac{v\sin A}{r}\,[t\,(\text{UTC}) - t_0\,(\text{UTC})] \tag{706}$$

$$\frac{\partial \phi}{\partial v} = \frac{\cos A}{r}\,[t\,(\text{UTC}) - t_0\,(\text{UTC})] \tag{707}$$

The partial derivatives of λ with respect to the solve-for parameters are obtained by differentiating Eq. (237) using Eqs. (703–707) for $A \neq 90$ deg or 270 deg and by differentiating Eq. (238) for $A = 90$ deg or 270 deg. For $A \neq 90$ deg or 270 deg,

$$\frac{\partial \lambda}{\partial r} = -\frac{v\sin A}{r^2\cos\phi}\,[t\,(\text{UTC}) - t_0\,(\text{UTC})] \tag{708}$$

$$\frac{\partial \lambda}{\partial \phi_0} = \tan A\left(\frac{1}{\cos\phi} - \frac{1}{\cos\phi_0}\right) \tag{709}$$

$$\frac{\partial \lambda}{\partial \lambda_0} = 1 \tag{710}$$

$$\frac{\partial \lambda}{\partial A} = \sec^2 A\ln\left[\frac{\tan\left(\dfrac{\pi}{4} + \dfrac{\phi}{2}\right)}{\tan\left(\dfrac{\pi}{4} + \dfrac{\phi_0}{2}\right)}\right] - \frac{v\sin A\tan A}{r\cos\phi}\,[t\,(\text{UTC}) - t_0\,(\text{UTC})] \tag{711}$$

$$\frac{\partial \lambda}{\partial v} = \frac{\sin A}{r\cos\phi}\,[t\,(\text{UTC}) - t_0\,(\text{UTC})] \tag{712}$$

For $A = 90$ deg or 270 deg, with the top sign applying for $A = 90$ deg and the bottom sign applying for $A = 270$ deg,

$$\frac{\partial \lambda}{\partial r} = \mp\frac{v}{r^2\cos\phi_0}\,[t\,(\text{UTC}) - t_0\,(\text{UTC})] \tag{713}$$

$$\frac{\partial \lambda}{\partial \phi_0} = (\lambda - \lambda_0)\tan\phi_0 \tag{714}$$

$$\frac{\partial \lambda}{\partial \lambda_0} = 1 \tag{715}$$

$$\frac{\partial \lambda}{\partial A} = 0 \tag{716}$$

$$\frac{\partial \lambda}{\partial v} = \pm\frac{1}{r\cos\phi_0}\,[t\,(\text{UTC}) - t_0\,(\text{UTC})] \tag{717}$$

D. Partial Derivatives of Doppler and Angular Observables With Respect to Speed of Light and Parameters Affecting Time Transformations

This section gives the partial derivatives indicated on line 4 of Eq. (664). These are the partial derivatives of doppler and angular observables z with respect to $\mathbf{q}$ due to (1) variation of the speed of light c in the light time solution only, and (2) variation of the time transformations used in the light time solution. The parameters affecting the time transformations are ΔT_{1958} and Δf_{cesium}, which affect (ET − A1), and the polynomial coefficients a, b, and c (specified by time block) of (UTC − ST) for the receiving station. Additional terms for $\partial z / \partial \Delta T_{1958}$, Δf_{cesium} arise from the variational equations. These partials are substituted into the first term of Eq. (666) and hence appear on line 2 of Eq. (664). The polynomial coefficients b and c and the speed of light appear explicitly in the doppler equations; hence, line 5 of Eq. (664) contains additional terms for the partial derivatives of z with respect to these parameters (see Section XIV-E below).

1. Speed of light c. Given the reception time t_3 (ET), the solution of the light time problem for t_2 (ET) and t_1 (ET) is affected by the value used for c. For angular observables, however, a change in c of 3 km/s along with a spacecraft velocity of 300 km/s will produce a maximum change of only 0.002 arc seconds. Hence, this partial derivative is ignored for angular observables. The partial derivative of a doppler observable z with respect to a change in c in the light time equation is given by

$$\frac{\partial z}{\partial c} = \frac{\partial z}{\partial \mathbf{X}_2^S(t_2)} \dot{\mathbf{X}}_2^S(t_2) \frac{\partial t_2(\mathrm{ET})}{\partial c} + \frac{\partial z}{\partial \mathbf{X}_1^S(t_1)} \dot{\mathbf{X}}_1^S(t_1) \frac{\partial t_1(\mathrm{ET})}{\partial c} \tag{718}$$

Differentiating the light time equations for the down and up legs (Eqs. 314 and 313, respectively) with respect to c gives

$$\frac{\partial t_2(\mathrm{ET})}{\partial c} = \frac{t_3(\mathrm{ET}) - t_2(\mathrm{ET})}{c}\left(1 + \frac{\dot{p}_{23}}{c}\right) \tag{719}$$

$$\frac{\partial t_1(\mathrm{ET})}{\partial c} = \frac{\partial t_2(\mathrm{ET})}{\partial c}\left(1 - \frac{\dot{r}_{12}}{c}\right) + \frac{t_2(\mathrm{ET}) - t_1(\mathrm{ET})}{c}\left(1 + \frac{\dot{p}_{12}}{c}\right) \tag{720}$$

Substituting Eqs. (719) and (720) into Eq. (718) gives

$$\begin{aligned}\frac{\partial z}{\partial c} = {} & \frac{\partial z}{\partial \mathbf{X}_2^S(t_2)} \dot{\mathbf{X}}_2^S(t_2) \frac{t_3(\mathrm{ET}) - t_2(\mathrm{ET})}{c}\left(1 + \frac{\dot{p}_{23}}{c}\right) \\ & + \frac{\partial z}{\partial \mathbf{X}_1^S(t_1)} \dot{\mathbf{X}}_1^S(t_1) \left\{\frac{t_3(\mathrm{ET}) - t_2(\mathrm{ET})}{c}\left[1 + \frac{1}{c}(\dot{p}_{23} - \dot{r}_{12})\right] + \frac{t_2(\mathrm{ET}) - t_1(\mathrm{ET})}{c}\left(1 + \frac{\dot{p}_{12}}{c}\right)\right\}\end{aligned} \tag{721}$$

For 1-way doppler, omit the second term.

2. Parameters affecting (ET − A1) time transformation: ΔT_{1958} and Δf_{cesium}. The observables are computed from the state vectors

$$\mathbf{X}_3^S(t_3) = \mathbf{X}_3^E(t_3) + \mathbf{X}_E^S(t_3) \tag{722}$$

$$\mathbf{X}_2^S(t_2) = \mathbf{X}_2^{B2}(t_2) + \mathbf{X}_{B2}^S(t_2) \tag{723}$$

$$\mathbf{X}_1^S(t_1) = \mathbf{X}_1^E(t_1) + \mathbf{X}_E^S(t_1) \tag{724}$$

The state vectors $\mathbf{X}_E^S(t_3)$, $\mathbf{X}_{B2}^S(t_2)$, $\mathbf{X}_E^S(t_1)$, and $\mathbf{X}_2^{B2}(t_2)$ are functions of ephemeris time. From Section VII-D, the state vectors $\mathbf{X}_3^E(t_3)$ and $\mathbf{X}_1^E(t_1)$ are functions of both the UT1 and ET values of the epoch. However, the variation in $\mathbf{X}_i^E(t_i)$ due to δt_i (ET) is insignificant compared with the variation due to δt_i (UT1). Thus, for purposes of taking partial derivatives, $\mathbf{X}_3^E(t_3)$ and $\mathbf{X}_1^E(t_1)$ are considered to be functions of UT1 only.

Given the data time tag t_3 (ST) (the midpoint of the count interval for doppler observables), the values of t_3 (ET) and t_3 (UT1) are computed from

$$\begin{aligned}t_3(\mathrm{ET}) = {} & t_3(\mathrm{ST}) + (\mathrm{UTC} - \mathrm{ST})_{t_3} \\ & + (\mathrm{A1} - \mathrm{UTC})_{t_3} + (\mathrm{ET} - \mathrm{A1})_{t_3}\end{aligned} \tag{725}$$

$$t_3\,(\mathrm{UT1}) = t_3\,(\mathrm{ST}) + (\mathrm{UTC} - \mathrm{ST})_{t_3} + (\mathrm{A1} - \mathrm{UTC})_{t_3} - (\mathrm{A1} - \mathrm{UT1})_{t_3} \qquad (726)$$

Using Eqs. (93–96) for these time transformations,

$$\frac{\partial t_3\,(\mathrm{ET})}{\partial \Delta T_{1958}} = 1 \qquad (727)$$

$$\frac{\partial t_3\,(\mathrm{ET})}{\partial \Delta f_{\mathrm{cesium}}} = -\frac{t_3 - 252{,}460{,}800}{9{,}192{,}631{,}770} \qquad (728)$$

$$\frac{\partial t_3\,(\mathrm{UT1})}{\partial \Delta T_{1958}} = \frac{\partial t_3\,(\mathrm{UT1})}{\partial \Delta f_{\mathrm{cesium}}} = 0 \qquad (729)$$

Differentiating the light time equation for the down leg (Eq. 314) with respect to ΔT_{1958} and $\Delta f_{\mathrm{cesium}}$ gives

$$\frac{\partial t_2\,(\mathrm{ET})}{\partial \Delta T_{1958}} = \left(1 - \frac{\mathring{r}^*_{23}}{c}\right)\frac{\partial t_3\,(\mathrm{ET})}{\partial \Delta T_{1958}}$$

$$\Delta T_{1958} \rightarrow \Delta f_{\mathrm{cesium}} \qquad (730)$$

where

$$\mathring{r}^*_{23} = \frac{\mathbf{r}_{23}}{r_{23}} \cdot [\dot{\mathbf{r}}^S_E\,(t_3) - \dot{\mathbf{r}}^S_2\,(t_2)] \qquad (731)$$

Differentiating the light time equation for the up leg (Eq. 313) and

$$t_1\,(\mathrm{UT1}) = t_1\,(\mathrm{ET}) - (\mathrm{ET} - \mathrm{A1})_{t_1} - (\mathrm{A1} - \mathrm{UT1})_{t_1} \qquad (732)$$

with respect to ΔT_{1958} and $\Delta f_{\mathrm{cesium}}$ and solving simultaneously for the partial derivatives of t_1 (ET) and t_1 (UT1) with respect to ΔT_{1958} and $\Delta f_{\mathrm{cesium}}$ gives

$$\frac{\partial t_1\,(\mathrm{ET})}{\partial \Delta T_{1958}} = \left[1 - \frac{1}{c}\,(\mathring{r}^*_{12} + \mathring{r}^*_{23})\right]\frac{\partial t_3\,(\mathrm{ET})}{\partial \Delta T_{1958}}$$

$$\Delta T_{1958} \rightarrow \Delta f_{\mathrm{cesium}} \qquad (733)$$

where

$$\mathring{r}^*_{12} = \frac{\mathbf{r}_{12}}{r_{12}} \cdot [\dot{\mathbf{r}}^S_2\,(t_2) - \dot{\mathbf{r}}^S_E\,(t_1)] \qquad (734)$$

and

$$\frac{\partial t_1\,(\mathrm{UT1})}{\partial \Delta T_{1958}} = -\frac{1}{c}\,(\mathring{r}^*_{12} + \mathring{r}^*_{23}) \qquad (735)$$

$$\frac{\partial t_1\,(\mathrm{UT1})}{\partial \Delta f_{\mathrm{cesium}}} = -\frac{(t_3 - t_1)}{9{,}192{,}631{,}770} + \frac{1}{c}\,(\mathring{r}^*_{12} + \mathring{r}^*_{23})\,\frac{t_3 - 252{,}460{,}800}{9{,}192{,}631{,}770} \qquad (736)$$

where terms of order $(v/c)^2$ or terms having that magnitude have been ignored in Eq. (733).

Using the above partial derivatives, the partial derivative of a 2-way or 3-way doppler observable with respect to a variation of ΔT_{1958} or $\Delta f_{\mathrm{cesium}}$ in the light time solution is:

$$\frac{\partial z}{\partial \Delta T_{1958}} = \left\{\frac{\partial z}{\partial \mathbf{X}^S_3\,(t_3)}\,\dot{\mathbf{X}}^S_E\,(t_3) + \frac{\partial z}{\partial \mathbf{X}^S_2\,(t_2)}\,\dot{\mathbf{X}}^S_2\,(t_2)\left(1 - \frac{\mathring{r}^*_{23}}{c}\right) + \frac{\partial z}{\partial \mathbf{X}^S_1\,(t_1)}\,\dot{\mathbf{X}}^S_E\,(t_1)\left[1 - \frac{1}{c}\,(\mathring{r}^*_{12} + \mathring{r}^*_{23})\right]\right\}\frac{\partial t_3\,(\mathrm{ET})}{\partial \Delta T_{1958}}$$

$$+ \frac{\partial z}{\partial \mathbf{X}^S_1\,(t_1)}\,\dot{\mathbf{X}}^E_1\,(t_1)\,\frac{\partial t_1\,(\mathrm{UT1})}{\partial \Delta T_{1958}}$$

$$\Delta T_{1958} \rightarrow \Delta f_{\mathrm{cesium}} \qquad (737)$$

where use is made of Eqs. (727), (728), (735), and (736). For one-way doppler or angular observables, there are only two participants and the terms containing the factor $\partial z/\partial \mathbf{X}^S_1\,(t_1)$ are omitted.

3. ***Parameters affecting (UTC − ST) time transformation: a, b, and c.*** The transformation from station time ST at the receiving station to UTC is given by

$$(\mathrm{UTC} - \mathrm{ST}) = a_R + b_R t + c_R t^2 \qquad (738)$$

where the coefficients a_R, b_R, and c_R are specified by time block (the subscript R denoting the receiving station for the observable), and the argument t is seconds past the start of the time block.

The values of a_R, b_R, and c_R affect the transformation from the data time tag t_3 (ST) to all other time scales equally, and hence also affect the values of t_2 and t_1 from the light time solution. From Eqs. (353), (354), (725), (726), and (738), the derivative of 2-way or 3-way doppler observables with respect to a variation of a_R in the light time solution is given by

$$\frac{\partial z}{\partial a_R} = \frac{\partial z}{\partial \mathbf{X}_3^S(t_3)} \dot{\mathbf{X}}_3^S(t_3) + \frac{\partial z}{\partial \mathbf{X}_2^S(t_2)} \dot{\mathbf{X}}_2^S(t_2)\left(1 - \frac{\dot{r}_{23}}{c}\right)$$
$$+ \frac{\partial z}{\partial \mathbf{X}_1^S(t_1)} \dot{\mathbf{X}}_1^S(t_1)\left[1 - \frac{1}{c}(\dot{r}_{12} + \dot{r}_{23})\right] \tag{739}$$

For 1-way doppler or angular observables, omit the last term. For all doppler and angular observables,

$$\frac{\partial z}{\partial b_R} = \frac{\partial z}{\partial a_R} t_3 \tag{740}$$

and

$$\frac{\partial z}{\partial c_R} = \frac{\partial z}{\partial a_R} t_3^2 \tag{741}$$

where t_3 is the reception time of the signal measured in seconds past the start of the current time block for a_R, b_R, and c_R.

E. Partial Derivatives of Doppler and Angular Observables With Respect to Parameter Vector, Holding State Vectors Fixed

This section gives the partial derivatives indicated on line 5 of Eq. (664). They are the partial derivatives of doppler and angular observables with respect to the parameters that affect the data directly, holding the state vectors of each participant constant. The parameters in this category that significantly affect the observables are the speed of light c and the polynomial coefficients b and c of (UTC − ST) appearing in the doppler formulation, the polynomial coefficients Δf_{T_0}, f_{T_1}, and f_{T_2} of the 1-way doppler transmitter frequency, the small rotations η', ϵ, ζ' or η, ϵ, ζ of the reference coordinate system at the tracking station for angular observables, and the parameter γ of the Brans–Dicke theory of relativity.

The unit vectors $\mathbf{P}, \mathbf{Q}, \mathbf{E}$ to which the angles HA, δ are referenced, and $\mathbf{N}, \mathbf{E}, \mathbf{Z}$ to which σ, γ; X, Y; and X', Y' are referenced, are functions of the station coordinates. However, for a 100-m change in station location, the maximum change in orientation of any of these unit vectors is only 3 arc seconds, which is less than the 7- to 11-arc-second accuracy of directly observed angular position. Thus, the partial derivatives of angular observables with respect to this particular effect of change in station coordinates are ignored.

1. Speed of light c. Doppler observables are computed from Eqs. (302), (308), (309), (310), (343), (344), (371), and (372) and associated equations of Section VIII. The speed of light c appears explicitly in the latter four expressions, which may be written as

$$\left(1 - \frac{F_R}{F_T}\right) = \frac{D_1}{c} + \frac{D_2}{c^2} + \cdots \tag{742}$$

$$\left(1 - \frac{F_R}{F_T}\right)^{\cdot\cdot} = \frac{\ddot{D}_1}{c} + \frac{\ddot{D}_2}{c^2} \tag{743}$$

The derivative of a doppler observable z with respect to c appearing explicitly in the formulation is given approximately by

$$\frac{\partial z}{\partial c} = -\frac{C_6}{c}\left[\left(\frac{D_1}{c} + 2\frac{D_2}{c^2}\right) + \frac{T_c^2}{24}\left(\frac{\ddot{D}_1}{c} + 2\frac{\ddot{D}_2}{c^2}\right)\right] \tag{744}$$

where C_6 is defined after Eq. (673).

2. Polynomial coefficients b and c of (UTC − ST). The polynomial coefficients b_R and c_R used to compute UTC − ST at the receiving station at t_3 affect the factor F_1 of 1-way doppler and the factor F of 2-way and 3-way doppler. The polynomial coefficients b_T and c_T used to compute UTC − ST at the transmitting station at t_1 also affect the factor F. From Eqs. (297), (306), and (308–310), the partial derivatives of 1-way, 2-way, or 3-way doppler observables (denoted as z) with respect to b_R and c_R used at t_3 (and appearing explicitly in the formulation) are

$$\frac{\partial z}{\partial b_R(t_3)} = -C_6\left[1 - \left(1 - \frac{F_R}{F_T}\right)^*\right] \tag{745}$$

$$\frac{\partial z}{\partial c_R(t_3)} = 2t_3 \frac{\partial z}{\partial b_R(t_3)} \tag{746}$$

where the reception time t_3 is expressed as seconds past the start of the time block for a, b, and c used at t_3. The partial derivatives of 2-way or 3-way doppler with respect to b_T and c_T used at t_1 (and appearing explicitly in the formulation) are

$$\frac{\partial z}{\partial b_T(t_1)} = C_6\left[1 - \left(1 - \frac{F_R}{F_T}\right)^*\right] \tag{747}$$

$$\frac{\partial z}{\partial c_T(t_1)} = 2t_1 \frac{\partial z}{\partial b_T(t_1)} \tag{748}$$

where the transmission time t_1 is expressed as seconds past the start of the time block for a, b, and c used at t_1.

For 2-way doppler, the transmitting station is also the receiving station and the same set of coefficients a, b, and c is usually used at t_3 and t_1. For this case, $b_R\,(t_3)$ is the same parameter as $b_T\,(t_1)$ and $\partial F2/\partial b$ is the sum of Eqs. (745) and (747), which is zero. However, from Eqs. (746) and (748), $\partial F2/\partial c \neq 0$.

3. ***Polynomial coefficients*** Δf_{T_0}, f_{T_1}, ***and*** f_{T_2} ***of 1-way doppler transmitter frequency.*** The transmitter frequency for 1-way doppler is represented by Eq. (277), where the solve-for coefficients Δf_{T_0}, f_{T_1}, and f_{T_2} are specified by time block. From Eqs. (277) and (308), the partial derivatives of 1-way doppler ($F1$) with respect to the specific coefficients used to compute the observable are

$$\frac{\partial F1}{\partial \Delta f_{T_0}} = -\,C_2\left[1-\left(1-\frac{F_R}{F_T}\right)^{*}\right] \tag{749}$$

$$\frac{\partial F1}{\partial f_{T_1}} = (t_2 - t_0)\,\frac{\partial F1}{\partial \Delta f_{T_0}} \tag{750}$$

$$\frac{\partial F1}{\partial f_{T_2}} = (t_2 - t_0)^2\,\frac{\partial F1}{\partial \Delta f_{T_0}} \tag{751}$$

where t_2 and t_0 are defined after Eq. (277).

4. ***Rotations*** η', ϵ, ζ' ***or*** η, ϵ, ζ ***of reference coordinate system for angular observables.*** Eqs. (437), (439) and (443–448) give corrections to the computed values of the directly observed angles as linear functions of the small rotations of the reference coordinate system about each of its three mutually perpendicular axes. The coefficients of the rotations in each equation are the partial derivatives of the angular observable with respect to the rotations affecting it.

5. ***Parameter*** γ ***of the Brans–Dicke theory of relativity.*** From Eqs. (308–310), (343), and (344), the partial derivative of a doppler observable with respect to γ appearing explicitly in the formulation is given by

$$\frac{\partial z}{\partial \gamma} = -\,C_6\,\frac{\mu_S}{c^3}\,(\epsilon_{12} + \epsilon_{23}) \tag{752}$$

where ϵ_{12} is omitted for 1-way doppler.

F. Partial Derivatives of Range Observables With Respect to Parameter Vector

Range observables are computed from Eq. (379) of Section IX. The partial derivative of a range observable ρ with respect to the solve-for parameter vector q is the sum of the several terms given below.

The sum of the first four terms of Eq. (379) is an accurate expression for the round-trip ephemeris time: $t_3\,(\mathrm{ET}) - t_1\,(\mathrm{ET})$. The terms r_{12}/c and r_{23}/c of $t_3(\mathrm{ET}) - t_1\,(\mathrm{ET})$ vary directly with q and also indirectly with the resulting variations in $t_2\,(\mathrm{ET})$ and $t_1\,(\mathrm{ET})$. The partial derivative of ρ with respect to q due to both of these effects is given by

$$\frac{\partial \rho}{\partial \mathrm{q}} = -\,F\,\frac{\partial t_1\,(\mathrm{ET})}{\partial \mathrm{q}} \tag{753}$$

where $\partial t_1\,(\mathrm{ET})/\partial \mathrm{q}$ is computed from Eqs. (665–667), (671), and (672).

The speed of light c affects the solution of the light time problem for the epochs of participation and hence affects line 1 of Eq. (379) which represents $t_3\,(\mathrm{ET}) - t_1\,(\mathrm{ET})$. It also appears explicitly in line 6 of Eq. (379). However, the variations of the terms of Eq. (379) containing the antenna, troposphere, and ionosphere corrections due to a variation in c are negligible. Thus,

$$\frac{\partial \rho}{\partial c} = -\,F\left[\frac{\partial t_1\,(\mathrm{ET})}{\partial c} + \frac{R_c}{10^3 c^2}\right] \tag{754}$$

where $\partial t_1(\mathrm{ET})/\partial c$ is given by Eqs. (719) and (720). For normal values of the range bias R_c, the second term of Eq. (754) is negligible.

The range observable given by Eq. (379) is the round-trip station time $t_3\,(\mathrm{ST}) - t_1\,(\mathrm{ST})$ multiplied by the conversion factor F. The reception time $t_3\,(\mathrm{ST})$ is given and $t_1\,(\mathrm{ST})$ varies with q. In addition to the partial derivatives above, $t_1\,(\mathrm{ST})$ varies with variations in ΔT_{1958}, $\Delta f_{\mathrm{cesium}}$, and the polynomial coefficients a, b, and c of (UTC − ST) in the light time solution. The variations of $t_1\,(\mathrm{A1})$, $t_1\,(\mathrm{UT1})$, $t_1\,(\mathrm{UTC})$, and $t_1\,(\mathrm{ST})$ due to variations of ΔT_{1958} and $\Delta f_{\mathrm{cesium}}$ in the light time solution are identical. Hence, from Eqs. (735) and (736),

$$\frac{\partial t_1\,(\mathrm{ST})}{\partial \Delta T_{1958}} = -\,\frac{1}{c}\,(\mathring{r}^{*}_{12} + \mathring{r}^{*}_{23}) \tag{755}$$

$$\frac{\partial t_1\,(\mathrm{ST})}{\partial \Delta f_{\mathrm{cesium}}} = -\frac{(t_3 - t_1)}{9{,}192{,}631{,}770} + \frac{1}{c}\left(\dot{r}^{*}_{12} + \dot{r}^{*}_{23}\right)\frac{t_3 - 252{,}460{,}800}{9{,}192{,}631{,}770} \tag{756}$$

From Eqs. (94) and (354), the partial derivatives of $t_1\,(\mathrm{ST})$ with respect to the polynomial coefficients a, b, and c of (UTC − ST) used at t_3 in the light time solution are given by

$$\frac{\partial t_1\,(\mathrm{ST})}{\partial a\,(t_3)} = 1 - \frac{1}{c}\left(\dot{r}_{12} + \dot{r}_{23}\right) \tag{757}$$

$$\frac{\partial t_1\,(\mathrm{ST})}{\partial b\,(t_3)} = t_3\left[1 - \frac{1}{c}\left(\dot{r}_{12} + \dot{r}_{23}\right)\right] \tag{758}$$

$$\frac{\partial t_1\,(\mathrm{ST})}{\partial c\,(t_3)} = t_3^2\left[1 - \frac{1}{c}\left(\dot{r}_{12} + \dot{r}_{23}\right)\right] \tag{759}$$

where t_3 is the reception time of the signal measured in seconds past the start of the time block for a, b, and c containing t_3. Since

$$t_1\,(\mathrm{ST}) = t_1\,(\mathrm{ET}) - (\mathrm{ET} - \mathrm{A1})_{t_1} - (\mathrm{A1} - \mathrm{UTC})_{t_1} - (\mathrm{UTC} - \mathrm{ST})_{t_1} \tag{760}$$

the partial derivatives of $t_1\,(\mathrm{ST})$ with respect to the polynomial coefficients a, b, and c used at t_1 in the light time solution are given by

$$\frac{\partial t_1\,(\mathrm{ST})}{\partial a\,(t_1)} = -1 \tag{761}$$

$$\frac{\partial t_1\,(\mathrm{ST})}{\partial b\,(t_1)} = -t_1 \tag{762}$$

$$\frac{\partial t_1\,(\mathrm{ST})}{\partial c\,(t_1)} = -t_1^2 \tag{763}$$

where t_1 is the transmission time of the signal measured in seconds past the start of the time block for a, b, and c containing t_1. In general, t_3 and t_1 will fall within the same time block, so that the partial derivatives given by Eqs. (757–759) are associated with the same parameters as those in Eqs. (761–763).

Letting $p_i = \Delta T_{1958}$, $\Delta f_{\mathrm{cesium}}$, or a, b, and c used at t_3 and t_1, $\partial\rho/\partial p_i$ due to a variation of p_i in the time transformations of the light time solution is given by

$$\frac{\partial \rho}{\partial p_i} = -F\frac{\partial t_1\,(\mathrm{ST})}{\partial p_i} \tag{764}$$

where $\partial t_1\,(\mathrm{ST})/\partial p_i$ is given by Eqs. (755–759) and (761–763).

The partial derivative of ρ with respect to R_c is given by

$$\frac{\partial \rho}{\partial R_c} = \frac{F}{10^3 c} \tag{765}$$

The partial derivative of ρ with respect to the parameter γ of the Brans–Dicke theory of relativity appearing explicitly in Eq. (379) is

$$\frac{\partial \rho}{\partial \gamma} = F\frac{\mu_S}{c^3}\left[\ln\frac{(r_1 + r_2 + r_{12})}{(r_1 + r_2 - r_{12})} + \ln\frac{(r_2 + r_3 + r_{23})}{(r_2 + r_3 - r_{23})}\right] \tag{766}$$

The partial derivative $\partial\rho/\partial\mathbf{q}$ for range observables is computed as the sum of the terms given above, where each term $\partial\rho/\partial q_i$ must be placed in the proper column of $\partial\rho/\partial\mathbf{q}$.

G. Partial Derivatives of Differenced-Range Doppler Observables With Respect to Parameter Vector

From Eqs. (484–486), the partial derivatives of 1-way, 2-way, and 3-way differenced-range doppler observables with respect to the estimated parameter vector $\mathbf{q}$ are given by

$$\frac{\partial F1}{\partial \mathbf{q}} = \frac{C_2 f_{S/C}}{T_c\,(\mathrm{computed})}\left(\frac{\partial \rho_{1_e}}{\partial \mathbf{q}} - \frac{\partial \rho_{1_s}}{\partial \mathbf{q}}\right) \tag{767}$$

$$\frac{\partial F2}{\partial \mathbf{q}} = \frac{C_3 f_q\,(t_1)}{T_c\,(\mathrm{computed})}\left(\frac{\partial \rho_{2_e}}{\partial \mathbf{q}} - \frac{\partial \rho_{2_s}}{\partial \mathbf{q}}\right) \tag{768}$$

$$\frac{\partial F3}{\partial \mathbf{q}} = \frac{C_5 f_q\,(t_1)}{T_c\,(\mathrm{computed})}\left(\frac{\partial \rho_{3_e}}{\partial \mathbf{q}} - \frac{\partial \rho_{3_s}}{\partial \mathbf{q}}\right) \tag{769}$$

Thus, the partial derivative of a differenced-range doppler observable with respect to q is computed from the difference of the partial derivatives of the two range observables with respect to q. The subscripts e and s denote the range observables whose reception times are the end and start, respectively, of the count interval T_c. The partial derivatives of $F1$ with respect to Δf_{T_0}, f_{T_1}, and f_{T_2} appearing in the second term of Eq. (484) and in $f_{S/C}$ in the first term must be added to Eq. (767). They are given by Eqs. (749–751) with the term $[1 - (F_R/F_T)]^*$ replaced by $[(\rho_{1_e} - \rho_{1_s})/T_c\,(\text{computed})]$.

The partial derivatives of the DPODP 2-way range observables with respect to q (excluding R_c) are given by Eqs. (753–759), (761–764), and (766). These equations are used to compute the partial derivatives of ρ_{2_e} and ρ_{2_s} with respect to q in Eq. (768); however, from Eqs. (379) and (469), the factor F in Eqs. (753), (754), (764), and (766) must be set equal to unity.

These equations are also used to compute the partial derivatives of ρ_{3_e} and ρ_{3_s} with respect to q in Eq. (769). However, it must be remembered that the coefficients a, b, and c used at t_3 are not the same parameters as those used at t_1; for two-way range, the same parameters are usually used at both t_1 and t_3.

In order to compute the partial derivatives of ρ_{1_e} and ρ_{1_s} with respect to q in Eq. (767), the following additional changes are necessary. In Eq. (753), $\partial t_1\,(\text{ET})/\partial \text{q}$ computed from Eqs. (671) and (672) should be replaced by $\partial t_2\,(\text{ET})/\partial \text{q}$ from Eq. (671). Similarly, in Eq. (754), $\partial t_1\,(\text{ET})/\partial c$ computed from Eqs. (719) and (720) should be replaced by $\partial t_2\,(\text{ET})/\partial c$ from Eq. (719); also, set $R_c = 0$. In Eqs. (755) and (756), $\dot{r}^*_{12}$ should be deleted. Also in Eq. (756), replace t_1 by t_2. Similarly, $\dot{r}_{12}$ should be deleted in Eqs. (757–759). Equations (761–763) do not apply for ρ_1. Thus, Eq. (764) is used to compute partial derivatives of ρ_1 with respect to ΔT_{1958}, Δf_{cesium}, and a, b, and c used at t_3. In Eq. (766), the first natural logarithm term should be deleted.

The principal terms of $\partial \rho_{i_e}/\partial \text{q}$ and $\partial \rho_{i_s}/\partial \text{q}$ (where $i = 1$, 2, or 3) in Eqs. (767–769) are computed from Eq. (753), which contains terms of relative order $(v/c)^0$ and $(v/c)^1$, where v is the tracking-station-to-spacecraft range-rate. Since $v/c \approx 10^{-4}$, and terms of relative order $(v/c)^2$ have been ignored, $\partial \rho_{i_e}/\partial \text{q}$ and $\partial \rho_{i_s}/\partial \text{q}$ are accurate to about eight significant figures.

Let N_0, N_1, and N_2 denote the number of leading digits of the $(v/c)^0$, $(v/c)^1$, and the omitted $(v/c)^2$ terms, respectively, of $\partial \rho_{i_e}/\partial \text{q}$ which are identical to the corresponding terms of $\partial \rho_{i_s}/\partial \text{q}$. Then, on the Univac 1108 computer, which has a double-precision word length of 18 decimal digits, the $(v/c)^0$, $(v/c)^1$, and $(v/c)^2$ terms of $\partial \rho_{i_e}/\partial \text{q}$ minus those of $\partial \rho_{i_s}/\partial \text{q}$ will contribute $18 - N_0$, $14 - N_1$, and $10 - N_2$ significant digits, respectively, to the partial derivatives of differenced-range doppler observables with respect to q (denoted as $\partial z/\partial \text{q}$) computed from Eqs. (767–769). Since the $(v/c)^2$ terms are omitted, $\partial z/\partial \text{q}$ will be accurate to $(18 - N_0) - (10 - N_2)$ or $8 + N_2 - N_0$ decimal digits.

When the light time solutions for ρ_{i_e} and ρ_{i_s} are similar, the $(v/c)^0$, $(v/c)^1$, and $(v/c)^2$ terms of $\partial \rho_{i_e}/\partial \text{q}$ will be similar to those of $\partial \rho_{i_s}/\partial \text{q}$; the parameters N_0, N_1, and N_2 will be nonzero and should be approximately equal to each other. Hence, $8 + N_2 - N_0 \approx 8$ and the partial derivatives of differenced-range doppler observables with respect to q computed from Eqs. (767–769) should be accurate to approximately the 8-decimal-digit accuracy of $\partial \rho_{i_e}/\partial \text{q}$ and $\partial \rho_{i_s}/\partial \text{q}$.

However, in order to obtain this accuracy, no more than the first 10 digits of $\partial \rho_{i_e}/\partial \text{q}$ may equal those of $\partial \rho_{i_s}/\partial \text{q}$; that is, N_0 must not exceed 10. In order to obtain this much cancellation, the count time T_c would have to be 0.01 s or smaller, which is an order of magnitude below the probable lower limit of 0.1 s for usable count times.

The probable accuracy of 8 decimal digits (or close to it) for the partial derivatives of differenced-range doppler observables with respect to q compares favorably with the accuracy of the integrated doppler partial derivatives: 8 decimal digits in heliocentric cruise and 4 decimal digits near earth (see Subsection XIV-B-1).

XV. Normal-Equations Form of Estimation Formulas

A. Introduction

This section gives the normal-equations form of the estimation formulas which yield the estimate of the parameter vector q and the statistics of the estimate; namely, the covariance matrix for q. This formulation was used in the original version of the DPODP. However, it has been replaced by the square-root form of the normal equations in the latest version of the program. The square-root formulation is theoretically equivalent to the normal-equations formulation but is numerically superior; it is documented in Section XVI.

The estimate for q minimizes the sum of weighted squares of residual errors between observed and computed quantities where *a priori* parameter estimates are treated as observed quantities. The parameter vector q is partitioned into a "solve-for" parameter vector x and a "consider" parameter vector y. The values of the solve-for parameters are adjusted to minimize the sum of squares. The *a priori* estimates of the consider parameters are not changed; however, the effects of errors in the consider parameters on the estimates of the solve-for parameters are "considered" when computing the covariance matrix for the solve-for parameters.

A given quantity of tracking data can be processed in one batch or divided into a number of sub-batches which are processed sequentially. That is, processing of the first batch yields an estimate of the parameter vector q and a corresponding covariance matrix, which are used as *a priori* information for processing the second batch, etc. As currently programmed, processing of each sub-batch requires a separate run of the DPODP. The one-batch solution is identical to the multiple sub-batch solution using the formulation of this section (or the equivalent formulation of Section XVI).

When the *a priori* covariance matrix is not obtained from a previous reduction of tracking data, the *a priori* cross-covariance between solve-for and consider parameters ($\tilde{\Gamma}_{xy}$) must be zero. After a batch of tracking data is processed, the cross-covariance between the estimate of the solve-for parameters and the consider parameters is computed and used as *a priori* information for processing the next batch of data.

There may be functional relations (constraints) between the members of q. These constraints may be applied by an exact procedure or an inexact procedure. For the exact treatment, the estimates of the parameters related by the constraint are required to satisfy the constraint. For the inexact treatment, the estimates of the parameters related by the constraint are allowed to deviate from values that would satisfy the constraint. This deviation contributes a weighted residual error to the sum of squares, which is minimized by the parameter estimate.

The formulation is given for mapping the covariance matrix from the injection epoch to any other epoch. The parameter vector corresponding to the mapped covariance matrix is q, with the injection position and velocity components (referred to a selected body, not necessarily the center of integration) replaced by the position and velocity components relative to the center of integration or any other specified body (planet, moon, or sun) at the map time. Frequently, the covariance matrix computed after processing a batch of data is mapped to a new epoch and used as *a priori* information for processing the next batch of data. The position and velocity components of the spacecraft at the new epoch (the solve-for injection conditions for processing the next batch of data) may be referred to a different body than that used for the previous batch of data.

The formulation of this section is a variation of the formulation originally derived by J. D. Anderson and used in the Single-Precision Orbit Determination Program (SPODP)[32]. The DPODP formulation was obtained from Anderson's formulation by adding the *a priori* correlation between solve-for and consider parameters and a method of treating inexact constraints. The formulation of this section includes a modification recently obtained by C. F. Peters[33] and is equivalent to the square-root formulation of Section XVI. The DPODP formulation does not contain Peters' modification and is documented in Ref. 62; it follows from the derivation of this section if the zero residual ($\tilde{y} - y$) is deleted from the residual vector **R** given by Eq. (780).

Peters' modification does not affect the parameter estimate and associated covariance matrix obtained from processing one batch of tracking data if $\tilde{\Gamma}_{xy} = 0$. However, his modification is required in order to obtain the correct estimate when processing the tracking data sequentially in batches.

B. Categorization of Parameters and Constraints

The parameter vector q consists of those parameters required to compute observable quantities; it is composed of three subvectors:

$$q = \begin{bmatrix} x \\ \hdashline y \\ \hdashline s \end{bmatrix} \qquad (770)$$

where

x = solve-for parameters: those parameters whose estimates are obtained from the least squares fit

y = "consider" parameters: those whose *a priori* estimates are not corrected, but whose errors are considered when computing the covariance matrix for q

s = exactly "constrained" parameters: parameters that are functionally related to the ($x \mid y$) parameters; one parameter from each exact constraint is placed in s

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[32] A simplified version of his formulation without consider parameters and exact constraints is given in Ref. 61.

[33] Peters, C. F., *The Consider Option Reconsidered,* JPL Section 391 Technical Memorandum 86 (JPL Internal Report), Mar. 27, 1970.

The parameter estimation formulation allows constraints (functional relations between the parameters) to be treated as exact or inexact. A constraint may be represented by

$$f_i(\mathrm{x}, \mathrm{y}, \mathrm{s}, \mathrm{N}_i) = 0 \qquad i = 1, 2, \cdots, n \tag{771}$$

where N_i = vector of constants which appear in the ith constraint. A constraint is considered to be exact or inexact as N_i is considered to be exact or inexact. Estimates for parameters related by a constraint that is considered to be exact are required to satisfy the constraint. Estimates for parameters related by a constraint that is considered to be inexact are allowed to deviate from values that would satisfy the constraint.

One parameter from each exact constraint is designated as a constrained parameter and is placed in s. The exactly constrained parameter vector is given by

$$\mathrm{s} = \mathrm{s}(\mathrm{x}, \mathrm{y}) = \begin{bmatrix} s_1(\mathrm{x}, \mathrm{y}) \\ s_2(\mathrm{x}, \mathrm{y}) \\ \vdots \\ s_i(\mathrm{x}, \mathrm{y}) \\ \vdots \\ s_n(\mathrm{x}, \mathrm{y}) \end{bmatrix} \tag{772}$$

where $s_i(\mathrm{x}, \mathrm{y})$ represents the solution of the ith exact constraint for the constrained parameter as a function of the estimates of the related parameters of the constraint. The derivative of s with respect to $(\mathrm{x} \mid \mathrm{y})$ is denoted by

$$\left[\frac{\partial \mathrm{s}}{\partial \mathrm{x}} \;\vdots\; \frac{\partial \mathrm{s}}{\partial \mathrm{y}}\right] = \left[\begin{array}{cccc:cccc} \dfrac{\partial s_1}{\partial x_1} & \cdots & \cdots & \dfrac{\partial s_1}{\partial x_n} & \dfrac{\partial s_1}{\partial y_1} & \cdots & \cdots & \dfrac{\partial s_1}{\partial y_n} \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ \dfrac{\partial s_n}{\partial x_1} & \cdots & \cdots & \dfrac{\partial s_n}{\partial x_n} & \dfrac{\partial s_n}{\partial y_1} & \cdots & \cdots & \dfrac{\partial s_n}{\partial y_n} \end{array}\right]$$

$$\equiv [\mathrm{S}_x \mid \mathrm{S}_y] \tag{773}$$

The ith exact constraint may be specified as either of the following:

(1) The functional relation $s_i = s_i(\mathrm{x}, \mathrm{y})$, in which case the *a priori* value of s_i is $s_i(\bar{\mathrm{x}}, \bar{\mathrm{y}})$, where $\bar{\mathrm{x}}$ is the *a priori* estimate of x.

(2) The derivative of the constrained parameter with respect to each related parameter of the constraint; i.e., specifying the ith row of Eq. (773). In this case, the *a priori* value of s_i must be given.

As opposed to the exact treatment of constraints, all parameters related by an inexact constraint are members of $(\mathrm{x} \mid \mathrm{y})$. One parameter from each inexact constraint is designated a constrained parameter and will have a "computed" value and an "observed" value. The computed value is the estimate for the parameter; the observed value is calculated from the constraint as a function of the estimates of the related parameters. Since parameter estimates are obtained by minimizing weighted squares of residual errors between observed and computed quantities, the degree to which each inexact constraint is satisfied will depend upon the weight applied to the constraint (see Section XV-C).

The observed inexactly constrained parameter vector is given by

$$\mathrm{c}(\mathrm{x}, \mathrm{y}) = \begin{bmatrix} c_1(\mathrm{x}, \mathrm{y}) \\ c_2(\mathrm{x}, \mathrm{y}) \\ \vdots \\ c_i(\mathrm{x}, \mathrm{y}) \\ \vdots \\ c_n(\mathrm{x}, \mathrm{y}) \end{bmatrix} \tag{774}$$

where $c_i(\mathrm{x}, \mathrm{y})$ represents the solution of the ith inexact constraining equation for the constrained parameter as a function of the estimates of the related parameters of the constraint. For a given constraint, the functions $s_i(\mathrm{x}, \mathrm{y})$ of Eq. (772) and $c_i(\mathrm{x}, \mathrm{y})$ of Eq. (774) are the same. The symbol used depends on whether the constraint is treated as exact or inexact.

The computed inexactly constrained parameter vector is given by

$$\mathrm{c}^{(\mathrm{x},\mathrm{y})} = \begin{bmatrix} c_1^{(\mathrm{x},\mathrm{y})} \\ c_2^{(\mathrm{x},\mathrm{y})} \\ \vdots \\ c_i^{(\mathrm{x},\mathrm{y})} \\ \vdots \\ c_n^{(\mathrm{x},\mathrm{y})} \end{bmatrix} = [D'_x \mid D'_y] \left[\begin{array}{c} \mathrm{x} \\ \hdashline \mathrm{y} \end{array}\right] \tag{775}$$

The element of the ith row of $[D'_x \mid D'_y]$ corresponding to the member of $(\mathrm{x} \mid \mathrm{y})$ which is the constrained parameter for the ith inexact constraint is unity, and the remaining elements of that row are zero.

The derivative of $\mathbf{c}(\mathbf{x},\mathbf{y})$ with respect to $(\mathbf{x} \mid \mathbf{y})$ is denoted by

$$\left[\frac{\partial \mathbf{c}(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}} \;\Big|\; \frac{\partial \mathbf{c}(\mathbf{x},\mathbf{y})}{\partial \mathbf{y}}\right] = [D_x \mid D_y] \tag{776}$$

From Eq. (775),

$$\left[\frac{\partial \mathbf{c}^{(\mathbf{x},\mathbf{y})}}{\partial \mathbf{x}} \;\Big|\; \frac{\partial \mathbf{c}^{(\mathbf{x},\mathbf{y})}}{\partial \mathbf{y}}\right] = [D'_x \mid D'_y] \tag{777}$$

The differences of the matrices in Eqs. (776) and (777) are denoted by

$$\Delta D_x = D'_x - D_x \tag{778}$$

$$\Delta D_y = D'_y - D_y \tag{779}$$

There are two constraints stored within the DPODP: they are the so-called solar and lunar constraints described in Subsection IV-B-2. The user specifies whether each of these constraints is to be applied, and also specifies the exact or inexact treatment. The user may also apply exact constraints by supplying the information listed under item 2 after Eq. (773).

C. Error Function

Let $\mathbf{R}$ denote a column vector containing all of the observed minus computed residuals associated with the processing of one batch of data:

$$\mathbf{R} = \begin{bmatrix} \hat{\mathbf{z}} - \mathbf{z} \\ \hline \tilde{\mathbf{x}} - \mathbf{x} \\ \hline \tilde{\mathbf{y}} - \mathbf{y} \\ \hline \mathbf{c}(\mathbf{x},\mathbf{y}) - \mathbf{c}^{(\mathbf{x},\mathbf{y})} \end{bmatrix} \tag{780}$$

where

$\hat{\mathbf{z}}$ = column vector of observables (doppler, range, angles, etc.)

$\mathbf{z} = \mathbf{z}(\mathbf{x},\mathbf{y},\mathbf{s}) = \mathbf{z}(\mathbf{x},\mathbf{y})$ = vector of computed observables

$\tilde{\mathbf{x}}$ = column vector of *a priori* estimates of solve-for parameters

$\tilde{\mathbf{y}}$ = column vector of *a priori* estimates of consider parameters

$\mathbf{x}$ = column vector of estimated values of solve-for parameters

$\mathbf{y}$ = column vector of estimated values of consider parameters

$= \tilde{\mathbf{y}}$

In addition to the actual observed quantities which produce the residuals $\hat{\mathbf{z}} - \mathbf{z}$, the *a priori* estimates of the solve-for parameters are treated as observables and produce the residuals $\tilde{\mathbf{x}} - \mathbf{x}$. The zero residuals $\tilde{\mathbf{y}} - \mathbf{y}$ are retained in Eq. (780) because the estimates $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are correlated; this will become clear below. Also, Eq. (780) contains the residual between the "observed" and "computed" values of the constrained parameter for each inexact constraint. The sum of weighted squares of residual errors between observed and computed quantities is given by

$$Q = \mathbf{R}^T W_T \mathbf{R} \tag{781}$$

where the weighting matrix W_T is given by

$$W_T = \begin{bmatrix} W & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\left[\begin{array}{c|c} \tilde{\Gamma}_x & \tilde{\Gamma}_{xy} \\ \hline \tilde{\Gamma}^T_{xy} & \tilde{\Gamma}_y \end{array}\right]^{-1}$}} & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & W_c \end{bmatrix} \tag{782}$$

where

W = data weighting matrix (diagonal); the weight for each observable is 1 divided by the input variance for the observable

$\tilde{\Gamma}_x$ = covariance matrix for $\tilde{\mathbf{x}}$

$\tilde{\Gamma}_y$ = covariance matrix for $\tilde{\mathbf{y}}$

$\tilde{\Gamma}_{xy}$ = cross-covariance matrix for $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$

W_c = diagonal weighting matrix for inexact constraints

$$W_c = \begin{bmatrix} \dfrac{1}{\sigma^2 c_1(\mathbf{x},\mathbf{y})} & 0 \cdots\cdots\cdots & & 0 \\ 0 & \dfrac{1}{\sigma^2 c_2(\mathbf{x},\mathbf{y})} & 0 \cdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 \cdots\cdots\cdots & & 0 & \dfrac{1}{\sigma^2 c_n(\mathbf{x},\mathbf{y})} \end{bmatrix} \tag{783}$$

where $\sigma^2 c_i(\mathbf{x},\mathbf{y})$ is the input variance of $c_i(\mathbf{x},\mathbf{y})$ calculated from the variances and covariances of the constants N_i of the ith inexact constraining equation and the *a priori* estimates of x and y. In Eq. (782), the inverse of the covariance matrix for $(\tilde{\mathbf{x}} \,\vdots\, \tilde{\mathbf{y}})$ is denoted by

$$\left[\begin{array}{c:c} \tilde{\Gamma}_x & \tilde{\Gamma}_{xy} \\ \hdashline \tilde{\Gamma}_{xy}^T & \tilde{\Gamma}_y \end{array}\right]^{-1} = \left[\begin{array}{c:c} W_x & W_{xy} \\ \hdashline W_{xy}^T & W_y \end{array}\right] \qquad (784)$$

It will be seen that only W_x is required. It can be obtained by inverting the complete covariance matrix for $(\tilde{\mathbf{x}} \,\vdots\, \tilde{\mathbf{y}})$ or from

$$W_x = [\tilde{\Gamma}_x - \tilde{\Gamma}_{xy}\,\tilde{\Gamma}_y^{-1}\,\tilde{\Gamma}_{xy}^T]^{-1} \qquad (785)$$

Substituting Eqs. (780), (782), and (784) into Eq. (781) and deleting all terms containing the zero residual $\tilde{\mathbf{y}} - \mathbf{y}$ gives

$$Q = (\hat{\mathbf{z}} - \mathbf{z})^T W (\hat{\mathbf{z}} - \mathbf{z}) + (\tilde{\mathbf{x}} - \mathbf{x})^T W_x (\tilde{\mathbf{x}} - \mathbf{x}) + [\mathbf{c}(\mathbf{x},\mathbf{y}) - \mathbf{c}^{(\mathbf{x},\mathbf{y})}]^T W_c [\mathbf{c}(\mathbf{x},\mathbf{y}) - \mathbf{c}^{(\mathbf{x},\mathbf{y})}] \qquad (786)$$

D. Parameter Estimation Formula

Since $\mathbf{s} = \mathbf{s}(\mathbf{x},\mathbf{y})$ and $\mathbf{y} =$ constant, the sum of squares Q is a function of x only:

$$Q = Q(\mathbf{x}) \qquad (787)$$

The estimate of x is the vector that minimizes Q. If Q is a minimum,

$$\frac{\partial Q}{\partial \mathbf{x}} = \left[\frac{\partial Q}{\partial x_1}\,\frac{\partial Q}{\partial x_2}\cdots\cdots\frac{\partial Q}{\partial x_n}\right] = 0 \qquad (788)$$

From Eq. (786),

$$\frac{\partial Q}{\partial \mathbf{x}} = -2\left\{(\hat{\mathbf{z}} - \mathbf{z})^T W \frac{\partial \mathbf{z}}{\partial \mathbf{x}} + (\tilde{\mathbf{x}} - \mathbf{x})^T W_x + [\mathbf{c}(\mathbf{x},\mathbf{y}) - \mathbf{c}^{(\mathbf{x},\mathbf{y})}]^T W_c \left[\frac{\partial \mathbf{c}^{(\mathbf{x},\mathbf{y})}}{\partial \mathbf{x}} - \frac{\partial \mathbf{c}(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}}\right]\right\} = 0 \qquad (789)$$

The partial derivative of z with respect to q is designated the A matrix:

$$A = \frac{\partial \mathbf{z}}{\partial \mathbf{q}} = \left[\left.\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\right|_{\mathbf{s}\,\text{fixed}} \,\vdots\, \left.\frac{\partial \mathbf{z}}{\partial \mathbf{y}}\right|_{\mathbf{s}\,\text{fixed}} \,\vdots\, \frac{\partial \mathbf{z}}{\partial \mathbf{s}}\right] \equiv [A_x \,\vdots\, A_y \,\vdots\, A_s] \qquad (790)$$

But $\mathbf{s} = \mathbf{s}(\mathbf{x},\mathbf{y})$ and hence, using Eq. (773),

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = A_x + A_s S_x \qquad (791)$$

Similarly (for use in computing the covariance matrix for q),

$$\frac{\partial \mathbf{z}}{\partial \mathbf{y}} = A_y + A_s S_y \qquad (792)$$

Substituting Eqs. (776–778) and (791) into Eq. (789) gives

$$(\hat{\mathbf{z}} - \mathbf{z})^T W (A_x + A_s S_x) + (\tilde{\mathbf{x}} - \mathbf{x})^T W_x + [\mathbf{c}(\mathbf{x},\mathbf{y}) - \mathbf{c}^{(\mathbf{x},\mathbf{y})}]^T W_c \Delta D_x = 0 \qquad (793)$$

which is a row vector with the number of elements equal to the number of solve-for parameters. Let the transpose of this vector (a column vector) be denoted by $F(\mathbf{x})$:

$$F(\mathbf{x}) = (A_x + A_s S_x)^T W (\hat{\mathbf{z}} - \mathbf{z}) + W_x (\tilde{\mathbf{x}} - \mathbf{x}) + \Delta D_x^T W_c [\mathbf{c}(\mathbf{x}, \mathbf{y}) - \mathbf{c}^{(\mathbf{x},\mathbf{y})}] = 0 \qquad (794)$$

The estimate of the solve-for parameter vector $\mathbf{x}$ must satisfy this equation. Assuming the partial derivatives are constant, the derivative of $F(\mathbf{x})$ with respect to $\mathbf{x}$ is

$$\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}} \equiv F'(\mathbf{x}) = -[(A_x + A_s S_x)^T W (A_x + A_s S_x) + W_x + \Delta D_x^T W_c \Delta D_x] \qquad (795)$$

Defining

$$J = (A_x + A_s S_x)^T W (A_x + A_s S_x) \qquad (796)$$

and

$$K = \Delta D_x^T W_c \Delta D_x \qquad (797)$$

Eq. (795) becomes

$$F'(\mathbf{x}) = -[J + W_x + K] \qquad (798)$$

The solution of Eq. (794) for the estimate of $\mathbf{x}$ is obtained by using Newton–Raphson iteration:

$$\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)} = -\{F'[\mathbf{x}^{(n)}]\}^{-1} F[\mathbf{x}^{(n)}] \qquad (799)$$

where $\mathbf{x}^{(n)}$ is the nth estimate of $\mathbf{x}$, and $\mathbf{x}^{(n+1)}$ is the $n + 1$st estimate. Substituting Eqs. (794) and (798) into Eq. (799) gives the parameter estimation formula

$$\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)} = [J + W_x + K]^{-1} [(A_x + A_s S_x)^T W \{\hat{\mathbf{z}} - \mathbf{z}[\mathbf{x}^{(n)}, \tilde{\mathbf{y}}]\} + W_x [\tilde{\mathbf{x}} - \mathbf{x}^{(n)}] + \Delta D_x^T W_c \{\mathbf{c}[\mathbf{x}^{(n)}, \tilde{\mathbf{y}}] - \mathbf{c}^{[\mathbf{x}^{(n)}, \tilde{\mathbf{y}}]}\}] \qquad (800)$$

For exact constraints specified by a functional relation,

$$s_i^{(n+1)} = s_i [\mathbf{x}^{(n+1)}, \tilde{\mathbf{y}}] \qquad (801)$$

For exact constraints specified by a row of Eq. (773) and the *a priori* estimate of the exactly constrained parameter,

$$s_i^{(n+1)} = s_i^{(n)} + \mathrm{S}_{x_i} [\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}] \qquad (802)$$

where S_{x_i} is the ith row of S_x.

E. Covariance Matrix

This section gives the formulation for computation of the covariance matrix Γ_q for the estimate of the parameter vector $\mathbf{q}$. Let the error in an estimate of $\mathbf{q}$ be denoted by

$$\delta\mathbf{q} = \left[\begin{array}{c} \delta\mathbf{x} \\ \hdashline \delta\mathbf{y} \\ \hdashline \delta\mathbf{s} \end{array}\right] \qquad (803)$$

where $\delta\mathbf{y} = \delta\tilde{\mathbf{y}}$, the *a priori* error. Then, the covariance matrix is given by

$$\Gamma_q = \overline{\delta\mathbf{q}\,\delta\mathbf{q}^T} = \left[\begin{array}{c:c:c} \overline{\delta\mathbf{x}\delta\mathbf{x}^T} & \overline{\delta\mathbf{x}\delta\mathbf{y}^T} & \overline{\delta\mathbf{x}\delta\mathbf{s}^T} \\ \hdashline \overline{\delta\mathbf{y}\delta\mathbf{x}^T} & \overline{\delta\mathbf{y}\delta\mathbf{y}^T} & \overline{\delta\mathbf{y}\delta\mathbf{s}^T} \\ \hdashline \overline{\delta\mathbf{s}\delta\mathbf{x}^T} & \overline{\delta\mathbf{s}\delta\mathbf{y}^T} & \overline{\delta\mathbf{s}\delta\mathbf{s}^T} \end{array}\right] \qquad (804)$$

where a bar indicates the ensemble average or expected value of the function. This may be written as

$$\Gamma_q = \left[\begin{array}{c:c:c} \Gamma_x & \Gamma_{xy} & \Gamma_{xs} \\ \hdashline \Gamma_{xy}^T & \Gamma_y & \Gamma_{ys} \\ \hdashline \Gamma_{xs}^T & \Gamma_{ys}^T & \Gamma_s \end{array}\right] \qquad (805)$$

If the true value of the solve-for parameter vector $\mathbf{x}$ were substituted into the parameter estimation formula (Eq. 800) along with true values for $\hat{\mathbf{z}}$, $\tilde{\mathbf{y}}$, $\tilde{\mathbf{x}}$, and the constants in the inexact constraints, we would have

$$\hat{\mathbf{z}} = \mathbf{z}[\mathbf{x}^{(n)}, \tilde{\mathbf{y}}]$$

$$\tilde{\mathbf{x}} = \mathbf{x}^{(n)}$$

$$\mathbf{c}[\mathbf{x}^{(n)}, \tilde{\mathbf{y}}] = \mathbf{c}^{[\mathbf{x}^{(n)}, \tilde{\mathbf{y}}]}$$

and $\mathbf{x}^{(n+1)}$ would equal $\mathbf{x}^{(n)}$, the input true value. However, the vectors $\hat{\mathbf{z}}$, $\tilde{\mathbf{x}}$, and $\tilde{\mathbf{y}}$ are in error by $\delta\hat{\mathbf{z}}$, $\delta\tilde{\mathbf{x}}$, and $\delta\tilde{\mathbf{y}}$, and the errors in the constants of the inexact constraints give an error in $\mathbf{c}(\mathbf{x}, \mathbf{y})$ of $\delta\mathbf{c}_N$. Substituting the true value of x into Eq. (800) and using Eqs. (776), (777), (779) and (792) gives an erroneous correction $\delta\mathbf{x} = \mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}$, which is the error of the estimate:

$$\delta\mathbf{x} = [J + W_x + K]^{-1} \{(A_x + A_s S_x)^T W [\delta\hat{\mathbf{z}} - (A_y + A_s S_y)\, \delta\tilde{\mathbf{y}}] + W_x \delta\tilde{\mathbf{x}} + \Delta D_x^T W_c\ [\delta\mathbf{c}_N - \Delta D_y\, \delta\tilde{\mathbf{y}}]\} \tag{806}$$

Let

$$L = (A_x + A_s S_x)^T W (A_y + A_s S_y) + \Delta D_x^T W_c \Delta D_y \tag{807}$$

Then

$$\delta\mathbf{x} = [J + W_x + K]^{-1} [(A_x + A_s S_x)^T W \delta\hat{\mathbf{z}} + W_x \delta\tilde{\mathbf{x}} - L\delta\tilde{\mathbf{y}} + \Delta D_x^T W_c \delta\mathbf{c}_N] \tag{808}$$

In order to derive the submatrices of Eq. (805), the assumption is made that the data covariance matrix Γ_z is the inverse of the data weighting matrix:

$$\Gamma_z = \overline{\delta\hat{\mathbf{z}}\, \delta\hat{\mathbf{z}}^T} = W^{-1} \tag{809}$$

Also, from the definition of the weighting matrix W_c for inexact constraints,

$$\overline{\delta\mathbf{c}_N\, \delta\mathbf{c}_N^T} = W_c^{-1} \tag{810}$$

Postmultiplying Eq. (808) by its transpose and averaging, using Eqs. (809) and (810), gives

$$\Gamma_x = [J + W_x + K]^{-1} [J + W_x \tilde{\Gamma}_x W_x + K + L\, \tilde{\Gamma}_y L^T - W_x \tilde{\Gamma}_{xy} L^T\ - L\, \tilde{\Gamma}_{xy}^T\, W_x]\ [J + W_x + K]^{-1} \tag{811}$$

From Eq. (785),

$$\tilde{\Gamma}_x = W_x^{-1} + \tilde{\Gamma}_{xy} \tilde{\Gamma}_y^{-1} \tilde{\Gamma}_{xy}^T \tag{812}$$

Substituting this into Eq. (811) gives

$$\Gamma_x = [J + W_x + K]^{-1} + [J + W_x + K]^{-1} [W_x \tilde{\Gamma}_{xy}\ \tilde{\Gamma}_y^{-1} \tilde{\Gamma}_{xy}^T\ W_x + L\, \tilde{\Gamma}_y L^T - W_x \tilde{\Gamma}_{xy} L^T\ - L\, \tilde{\Gamma}_{xy}^T\ W_x]\ [J + W_x + K]^{-1} \tag{813}$$

Postmultiplying Eq. (808) by $\delta\tilde{\mathbf{y}}^T$ and averaging gives

$$\Gamma_{xy} = [J + W_x + K]^{-1} [W_x \tilde{\Gamma}_{xy} - L\tilde{\Gamma}_y] \tag{814}$$

Since the estimate of the consider parameters is not changed,

$$\Gamma_y = \tilde{\Gamma}_y \tag{815}$$

The remaining submatrices of Eq. (805) are Γ_s, Γ_{xs}, and Γ_{ys}. From Eqs. (772) and (773),

$$\delta\mathbf{s} = S_x\, \delta\mathbf{x} + S_y\, \delta\mathbf{y} \tag{816}$$

Postmultiplying $\delta\mathbf{s}$ by $\delta\mathbf{s}^T$ and averaging gives

$$\Gamma_s = S_x \Gamma_x S_x^T + S_y \Gamma_y S_y^T + S_x \Gamma_{xy} S_y^T + S_y \Gamma_{xy}^T S_x^T \tag{817}$$

Postmultiplying $\delta\mathbf{x}$ and $\delta\mathbf{y}$ by $\delta\mathbf{s}^T$ and averaging gives

$$\Gamma_{xs} = \Gamma_x S_x^T + \Gamma_{xy} S_y^T \tag{818}$$

$$\Gamma_{ys} = \Gamma_{xy}^T S_x^T\ + \Gamma_y S_y^T \tag{819}$$

The covariance matrix for q is evaluated from Eq. (805) using the submatrices given by Eqs. (813–815) and (817–819).

The following paragraphs relate the various terms of Eq. (813) for Γ_x to the various error sources which affect the estimates of the solve-for parameters obtained from Eq. (800).

If the *a priori* parameter estimate and covariance matrix are not obtained by processing previous batches of data, $\tilde{\Gamma}_{xy}$ must be zero. For this case, Eq. (785) gives

$$W_x = \tilde{\Gamma}_x^{-1}$$

Substituting this and $\tilde{\Gamma}_{xy} = 0$ into Eqs. (813) and (814) gives

$$\Gamma_x = [J + \tilde{\Gamma}_x^{-1} + K]^{-1} + [J + \tilde{\Gamma}_x^{-1} + K]^{-1} \, [L \tilde{\Gamma}_y L^T] \, [J + \tilde{\Gamma}_x^{-1} + K]^{-1} \tag{820}$$

and

$$\Gamma_{xy} = -[J + \tilde{\Gamma}_x^{-1} + K]^{-1} L \tilde{\Gamma}_y \tag{821}$$

The contributions to Γ_x from the information matrix J, the *a priori* covariance matrix $\tilde{\Gamma}_x$, and the matrix K in the first term of Eq. (820) account for errors in the tracking data being processed, errors in the *a priori* parameter estimate, and errors in the constants of the inexact constraints applied to the solution. The second term of Eq. (820) accounts for the effect on the estimate of errors in the consider parameters.

The complete expression of Eq. (820) is referred to as the consider covariance matrix since errors in the "consider" parameters are considered. The first term is referred to as the nonconsider covariance matrix $\Gamma_{x_{NC}}$:

$$\Gamma_{x_{NC}} = [J + \tilde{\Gamma}_x^{-1} + K]^{-1} \tag{822}$$

Using Eqs. (821) and (822), Eq. (820) may be expressed as

$$\Gamma_x = \Gamma_{x_{NC}} + \Gamma_{xy} \tilde{\Gamma}_y^{-1} \Gamma_{xy}^T \tag{823}$$

For the case where the tracking data are processed sequentially in batches, Γ_x and Γ_{xy} are computed from Eqs. (813) and (814) after processing each batch of data and are used as *a priori* information for processing the next batch of data.[34] It can be shown that Γ_x and Γ_{xy} obtained after processing the last batch of data are identical to the results that would be obtained from Eqs. (820) and (821) if all of the data were processed in one batch.[35] The equality of Eqs. (813) and (820) applies to each term. Hence, the first term of Eq. (813) is the nonconsider covariance matrix:

$$\Gamma_{x_{NC}} = [J + W_x + K]^{-1} \tag{824}$$

The information matrix J accounts for the errors in the current batch of data. The matrix K accounts for errors in the constants of the inexact constraints applied to the processing of the current batch of data. If the current batch contained no data, J and K would be zero and $\Gamma_{x_{NC}}$ would equal its *a priori* value $\tilde{\Gamma}_{x_{NC}}$, giving the relation

$$\tilde{\Gamma}_{x_{NC}} = W_x^{-1} \tag{825}$$

Thus, the quantity W_x is the inverse of the *a priori* nonconsider covariance matrix. It accounts for errors in previously reduced batches of data, errors in the constants of the inexact constraints applied to each previous batch reduction of data, and the input errors for the initial estimates of the solve-for parameters (prior to reduction of the first batch of data).

The second term of Eq. (813) contains a sum of four terms. The first of these, namely $W_x \tilde{\Gamma}_{xy} \tilde{\Gamma}_y^{-1} \tilde{\Gamma}_{xy}^T W_x$, accounts for the effects of errors in the consider parameters on the previous batch reductions of data. The remaining three terms of the sum contain the matrix L and account for the effects of errors in the consider parameters on the reduction of the current batch of data.

Equation (813) for Γ_x may be expressed as Eq. (823) using $\Gamma_{x_{NC}}$ given by Eq. (824) and Γ_{xy} given by Eq. (814). Equation (823) also applies for the previous batch of data; that is,

$$\tilde{\Gamma}_x = \tilde{\Gamma}_{x_{NC}} + \tilde{\Gamma}_{xy} \tilde{\Gamma}_y^{-1} \tilde{\Gamma}_{xy}^T \tag{826}$$

Substituting $\tilde{\Gamma}_x$ from Eq. (826) into Eq. (785) for W_x gives the result that

$$W_x = \tilde{\Gamma}_{x_{NC}}^{-1} \tag{827}$$

which is identical to Eq. (825).

The sensitivity matrix S_{xy} is defined as

$$S_{xy} = \frac{\partial \mathrm{x}}{\partial \mathrm{y}} \tag{828}$$

That is, S_{xy} is the partial derivative of the estimate of the solve-for parameter vector with respect to the consider parameter vector. The sensitivity matrix is a very useful quantity since it relates errors in the consider parameters to errors in the estimates of the solve-for parameters.

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[34] Often, Γ_x and Γ_{xy} are mapped to a new epoch (see Section XV-F) and then used as *a priori* information for the next batch of data.

[35] The constraint weighting matrix W_c (and hence K) used to process each batch of data is different from the matrix W_c (and hence K) required to process all of the data in one batch.

The error $\delta \mathrm{x}$ in the estimate of the solve-for parameter vector is given by

$$\delta \mathrm{x} = S_{xy}\, \delta \tilde{\mathrm{y}} + \cdots \tag{829}$$

The dots represent the contributions due to errors in the processed observables, errors in the *a priori* estimates of the solve-for parameters (uncorrelated with $\delta \tilde{\mathrm{y}}$ since $\tilde{\Gamma}_{xy} = 0$ prior to processing observables), and errors in the constants of the inexact constraints. Since these errors are uncorrelated with $\delta \tilde{\mathrm{y}}$, postmultiplication of Eq. (829) by $\delta \tilde{\mathrm{y}}^T$ and averaging gives

$$\Gamma_{xy} = S_{xy}\, \tilde{\Gamma}_y \tag{830}$$

and

$$S_{xy} = \Gamma_{xy}\, \tilde{\Gamma}_y^{-1} \tag{831}$$

Substituting Γ_{xy} given by Eq. (814) gives

$$S_{xy} = [J + W_x + K]^{-1} [W_x \tilde{\Gamma}_{xy} \tilde{\Gamma}_y^{-1} - L] \tag{832}$$

If $\tilde{\Gamma}_{xy} = 0$,

$$S_{xy} = -[J + \tilde{\Gamma}_x^{-1} + K]^{-1} L \tag{833}$$

For a given amount of tracking data processed sequentially in batches, the sensitivity matrix is computed from Eq. (832) after processing the last batch. If this same amount of tracking data is processed in one batch, $\tilde{\Gamma}_{xy} = 0$ before processing data and S_{xy} is computed from Eq. (833). It can be shown that these two sensitivity matrices are identical.

The consider covariance matrix (Eq. 813) can be expressed as Eq. (823), which is identical to

$$\Gamma_x = \Gamma_{x_{NC}} + (\Gamma_{xy} \tilde{\Gamma}_y^{-1})\, \tilde{\Gamma}_y\, (\tilde{\Gamma}_y^{-1} \Gamma_{xy}^T) \tag{834}$$

Substituting Eq. (831) gives

$$\Gamma_x = \Gamma_{x_{NC}} + S_{xy}\, \tilde{\Gamma}_y\, S_{xy}^T \tag{835}$$

F. Mapping Covariance Matrix to New Epoch

This section gives the formulation for mapping the covariance matrix for the parameter vector q from the injection epoch to any other epoch. Subsection XV-F-1 gives the mapped covariance matrix relative to the center of integration at the map time. Subsection XV-F-2 gives the covariance matrix relative to any specified body other than the center of integration at the map time. There is a slight difference in either formulation, depending on whether the map epoch is specified in the ET time scale or in another time scale (A1, UTC, UT1, or ST).

1. *Mapped covariance matrix relative to center of integration.*

The solve-for parameter vector q will be denoted as q_0 in this section. However, for purposes of mapping the covariance matrix to a new epoch, dividing the parameters into the solve-for, consider, and exactly constrained categories is inappropriate. The appropriate categories are

X_0^B = spacecraft state vector (1950.0 earth equatorial rectangular position and velocity components) at injection epoch t_0 (specified in a time scale other than ephemeris time ET) relative to a specified body B which is not necessarily the center of integration at t_0

a = all parameters that affect the spacecraft state vector X relative to the center of integration C (except X_0^B)

b = all parameters that affect observables but do not affect X

Thus, the solve-for parameter vector q_0 is given by

$$\mathrm{q}_0 = \begin{bmatrix} \mathrm{x} \\ \hdashline \mathrm{y} \\ \hdashline \mathrm{s} \end{bmatrix} = \begin{bmatrix} \mathrm{X}_0^B \\ \hdashline \mathrm{a} \\ \hdashline \mathrm{b} \end{bmatrix} \tag{836}$$

where the second form of the vector is obtained from the first by reordering the elements.

The mapped parameter vector relative to the center of integration C at the map time is denoted by q^C:

$$\mathrm{q}^C = \begin{bmatrix} \mathrm{X} \\ \hdashline \mathrm{a} \\ \hdashline \mathrm{b} \end{bmatrix} \tag{837}$$

where

X = spacecraft state vector relative to center of integration C at map time.

The mapped covariance matrix Γ_q^C relative to the center of integration at the map time corresponds to the mapped parameter vector q^C.

The injection covariance matrix is given by Eq. (805). Reordering the rows and columns and partitioning according to the second vector of Eq. (836) gives, suppressing the subscript and superscript on X_0^B,

$$\Gamma_{q_0} = \left[\begin{array}{c|c|c} \Gamma_X & \Gamma_{Xa} & \Gamma_{Xb} \\ \hline \Gamma_{Xa}^T & \Gamma_a & \Gamma_{ab} \\ \hline \Gamma_{Xb}^T & \Gamma_{ab}^T & \Gamma_b \end{array}\right] \tag{838}$$

The state vector of the spacecraft relative to the center of integration is a function of ET and the parameter vector $\mathbf{q}_0$:

$$\mathbf{X} = \mathbf{X}(\mathrm{ET}, \mathbf{q}_0) \tag{839}$$

The variation in X at an ET epoch is given by

$$\delta \mathbf{X} = U\, \delta \mathbf{X}_0^B + V\, \delta \mathbf{a} \tag{840}$$

where the U and V matrices are obtained from the solution of the variational equations (see Section XIII-A). If the epoch is specified in a time scale other than ET (i.e., A1, UTC, UT1, or ST), the variation in X is

$$\delta \mathbf{X} = U\, \delta \mathbf{X}_0^B + V\, \delta \mathbf{a} + \dot{\mathbf{X}}\, \delta \mathrm{ET} \tag{841}$$

where, from Eq. (93),

$$\delta \mathrm{ET} = \delta\,(\Delta T_{1958}) - \frac{t - 252{,}460{,}800}{9{,}192{,}631{,}770}\, \delta\,(\Delta f_{\mathrm{cesium}}) \tag{842}$$

Thus, Eq. (841) may be written as

$$\delta \mathbf{X} = U\, \delta \mathbf{X}_0^B + V^*\, \delta \mathbf{a} \tag{843}$$

where V^* is the V matrix with the ΔT_{1958} column incremented by

$$\dot{\mathbf{X}}$$

and the $\Delta f_{\mathrm{cesium}}$ column incremented by

$$- \frac{t - 252{,}460{,}800}{9{,}192{,}631{,}770}\, \dot{\mathbf{X}}$$

where $\dot{\mathbf{X}}$ is evaluated at the map epoch t.

Using Eq. (840), the variation of $\mathbf{q}^C$ at an ET map epoch due to a variation in $\mathbf{q}_0$ is given by

$$\delta \mathbf{q}^C = \left[\begin{array}{c} \delta \mathbf{X} \\ \hline \delta \mathbf{a} \\ \hline \delta \mathbf{b} \end{array}\right] = \left[\begin{array}{c|c|c} U & V & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right] \left[\begin{array}{c} \delta \mathbf{X}_0^B \\ \hline \delta \mathbf{a} \\ \hline \delta \mathbf{b} \end{array}\right] \equiv M \delta \mathbf{q}_0 \tag{844}$$

where 0 and I represent null and identity matrices. The variation of $\mathbf{q}^C$ at a map epoch specified in a time scale other than ET is given by Eq. (844) with V replaced by V^*.

Using Eq. (844), the mapped covariance matrix for $\mathbf{q}^C$ is given by

$$\Gamma_q^C = \overline{\delta \mathbf{q}^C\, [\delta \mathbf{q}^C]^T} = M\, \Gamma_{q_0} M^T \tag{845}$$

Substituting Eq. (838) and M from Eq. (844) into Eq. (845) gives

$$\Gamma_q^C = \left[\begin{array}{c|c|c} U\Gamma_X U^T + V\Gamma_{Xa}^T U^T + U\Gamma_{Xa} V^T + V\Gamma_a V^T & U\Gamma_{Xa} + V\Gamma_a & U\Gamma_{Xb} + V\Gamma_{ab} \\ \hline \Gamma_{Xa}^T U^T + \Gamma_a V^T & \Gamma_a & \Gamma_{ab} \\ \hline \Gamma_{Xb}^T U^T + \Gamma_{ab}^T V^T & \Gamma_{ab}^T & \Gamma_b \end{array}\right] \tag{846}$$

This matrix may be simplified by using the following combined submatrices of Eq. (838):

$$\Gamma_{X,ab} \equiv [\Gamma_{Xa} \,|\, \Gamma_{Xb}] \tag{847}$$

$$\Gamma_{a,ab} \equiv [\Gamma_a \,|\, \Gamma_{ab}] \tag{848}$$

$$\Gamma_{ab,ab} \equiv \left[\begin{array}{c|c} \Gamma_a & \Gamma_{ab} \\ \hline \Gamma_{ab}^T & \Gamma_b \end{array}\right] \tag{849}$$

With these definitions, Eq. (846) simplifies to

$$\Gamma_q^C = \left[\begin{array}{c:c} U\Gamma_X U^T + V\Gamma_{Xa}^T U^T + U\Gamma_{Xa} V^T + V\Gamma_a V^T & U\Gamma_{X,ab} + V\Gamma_{a,ab} \\ \hdashline [U\Gamma_{X,ab} + V\Gamma_{a,ab}]^T & \Gamma_{ab,ab} \end{array}\right] \tag{850}$$

This equation gives the mapped covariance matrix relative to the center of integration for an ET map epoch. If the map epoch is specified in any other time scale, Eq. (850) is used with the V matrix replaced by the V^* matrix.

2. *Mapped covariance matrix relative to body other than center of integration.* The mapped parameter vector relative to the center of integration at the map time is given by Eq. (837). The mapped parameter vector relative to a body R other than the center of integration at the map time is given by

$$\mathbf{q}^R = \left[\begin{array}{c} \mathbf{X}^R \\ \hdashline \mathbf{a} \\ \hdashline \mathbf{b} \end{array}\right] = \left[\begin{array}{c} \mathbf{X} - \mathbf{X}_R^C \\ \hdashline \mathbf{a} \\ \hdashline \mathbf{b} \end{array}\right] \tag{851}$$

where

$\mathbf{X}^R$ = spacecraft state vector relative to body R at map time

$\mathbf{X}_R^C$ = state vector of body R relative to center of integration C at map time

The variation in $\mathbf{X}$ is given by Eq. (840) for an ET map epoch and by Eq. (841) or (843) for a non-ET map epoch (epoch specified in the A1, UTC, UT1, or ST time scale). Since ET is the independent variable for the spacecraft ephemeris and the precomputed n-body ephemerides, the state vector $\mathbf{X}_R^C$ is a function of ET and the dynamic parameters $\mathbf{a}$:

$$\mathbf{X}_R^C = \mathbf{X}_R^C(\text{ET}, \mathbf{a}) \tag{852}$$

However, the dependence upon $\mathbf{a}$ is limited to the reference parameters: A_E, R_E, E for each precomputed ephemeris, μ_E and μ_M. The variation in $\mathbf{X}_R^C$ at an ET map epoch is given by[36]

$$\delta\mathbf{X}_R^C = \frac{\partial \mathbf{X}_R^C}{\partial \mathbf{a}}\,\delta\mathbf{a} \tag{853}$$

For a non-ET map epoch,

$$\delta\mathbf{X}_R^C = \frac{\partial \mathbf{X}_R^C}{\partial \mathbf{a}}\,\delta\mathbf{a} + \dot{\mathbf{X}}_R^C\,\delta\text{ET} \tag{854}$$

where δET is given by Eq. (842). Thus, for an ET map epoch,

$$\delta\mathbf{X}^R = \delta\mathbf{X} - \delta\mathbf{X}_R^C = U\,\delta\mathbf{X}_0^B + \left(V - \frac{\partial \mathbf{X}_R^C}{\partial \mathbf{a}}\right)\delta\mathbf{a} \tag{855}$$

For a non-ET map epoch,

$$\delta\mathbf{X}^R = U\,\delta\mathbf{X}_0^B + \left(V - \frac{\partial \mathbf{X}_R^C}{\partial \mathbf{a}}\right)\delta\mathbf{a} + (\dot{\mathbf{X}} - \dot{\mathbf{X}}_R^C)\,\delta\text{ET} \tag{856}$$

where δET is given by Eq. (842).

A comparison of Eq. (855) to Eq. (840) and of Eq. (856) to Eqs. (841) and (843) shows that the mapped covariance matrix relative to a body R other than the center of integration C at the map time can be computed from Eq. (850) by using a modified V matrix. For an ET map epoch, the matrix V is replaced by

$$V - \frac{\partial \mathbf{X}_R^C}{\partial \mathbf{a}}$$

For a non-ET map epoch, the matrix V is replaced by

$$V^* - \frac{\partial \mathbf{X}_R^C}{\partial \mathbf{a}}$$

where V^* is computed as indicated after Eq. (843) except that $\dot{\mathbf{X}}$ is replaced by $\dot{\mathbf{X}} - \dot{\mathbf{X}}_R^C = \dot{\mathbf{X}}_{S/C}^C - \dot{\mathbf{X}}_R^C = \dot{\mathbf{X}}_{S/C}^R$, where S/C refers to the spacecraft.

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[36] The partial derivatives are computed as indicated in Subsection XIII-D-2.

XVI. Square-Root Form of Estimation Formulas

A. Introduction

This section gives the square-root formulation of the estimation formulas which yield the estimate of the parameter vector q and the statistics of the estimate; namely, the covariance matrix for q. The square-root formulation is used in the latest version of the DPODP; it replaces the normal-equations formulation of the estimation formulas (Section XV) used in the original version of the program.

The square-root formulation is theoretically equivalent to the normal-equations method but is numerically superior. The normal-equations formulation requires the inverse of the normal matrix A^TWA (see Section XV), which is frequently ill-conditioned and influenced greatly by round-off errors. Instead of forming A^TWA, the square-root formulation utilizes Householder transformations to convert the A matrix to the triangular matrix R, whose order is the same as that of A^TWA. To obtain the parameter estimate requires the inversion of R rather than that of A^TWA. Since the condition number[37] of R is the square root of the condition number of A^TWA, the inverse of R can be obtained with less numerical error than the inverse of A^TWA. This is the primary advantage of the square-root formulation.

The superior numerical techniques of the square-root formulation were first applied to the linear least-squares problem by R. J. Hanson and C. L. Lawson[38] (Ref. 1). Using these techniques, the DPODP square root formulation was written by P. Dyer[39] (Ref. 63); however, many of the details are due to T. Starbird.[40]

Section XV-A applies also to this section. However, the application of constraints to the parameter estimate is limited to the "solar" and "lunar" constraints (described in Subsection IV-B-2), treated as exact constraints. The treatment of these constraints as inexact relations between the estimated parameters has been discontinued along with the user input differential exact constraints (see item 2 after Eq. 773).

Section XVI-B-1 gives the derivation of the parameter estimation formula, which requires the inverse of the triangular matrix R. Subsection XVI-B-2 describes the singular-value decomposition method of inverting R and also the alternative "mass below the diagonal" technique. Also, the partial-step algorithm for obtaining the parameter estimate in the presence of significant nonlinearities is given; this algorithm was originated by D. Boggs.[41] The formulation for the covariance matrix of the parameter estimate is given in Section XVI-C. Section XVI-D gives the formulation for mapping the square root of the covariance matrix (or its inverse) from the injection epoch to any other epoch.

B. Parameter Estimation Formula

The parameter estimation formula is derived in Subsection XVI-B-1. Three different numerical techniques for evaluating this equation are given in Subsection XVI-B-2.

1. Equations. The parameter vector q is given by

$$\mathrm{q} = \begin{bmatrix} \mathrm{x} \\ \hdashline \mathrm{y} \end{bmatrix} \tag{857}$$

where

x = solve-for parameter vector. The estimates of these parameters are obtained from the least-squares fit.

y = "consider" parameter vector. The *a priori* estimates of these parameters are not corrected. However, the errors in these parameters are considered when computing the covariance matrix for the solve-for parameters.

The user may specify that the parameter estimate must satisfy the solar constraint (Eq. 104) and/or the lunar constraint (Eq. 107). The solar constraint relates the gravitational constant of the sun μ_S and the scaling factor A_E for the heliocentric ephemerides of the planets and the earth–moon barycenter. The lunar constraint relates the gravitational constants of the earth and moon, μ_E and μ_M, and the scaling factor R_E for the geocentric lunar ephemeris.

If either of these constraints is applied, the estimates of the parameters related by the constraint must satisfy

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[37] Ratio of largest to smallest singular value.

[38] JPL Computation and Analysis Section.

[39] Formerly, JPL Tracking and Orbit Determination Section.

[40] Formerly, JPL Flight Operations and DSN Programming Section.

[41] Boggs, D., "The Partial-Step Estimation Algorithms and Their Application to Mariner '71", pp. 4-74 to 4-90 of Project Document 610-33, *Preliminary Orbit Determination Strategy and Accuracy, Mariner Mars 1971,* Ed. by S. K. Wong and G. W. Reynolds (JPL Internal Report), Aug. 15, 1970.

the constraint. This is accomplished by designating one parameter from each applied constraint as a constrained parameter which is placed in the exactly constrained parameter vector s given by Eq. (772). In the square-root formulation, the number n of exactly constrained parameters can be 0, 1, or 2. The quantity $s_i(\mathbf{x}, \mathbf{y})$ represents the solution of the ith exact constraint for the estimate of the constrained parameter as a function of the estimates of the related parameters of the constraint. If the lunar constraint is labeled as the first constraint and if R_E is designated as the constrained parameter, then $s_1(\mathbf{x}, \mathbf{y})$ is the estimate of R_E, which is equal to the right-hand side of Eq. (107) computed from the estimates of μ_E and μ_M, which are members of q given by Eq. (857).

Application of an exact constraint replaces the constrained parameter, wherever it appears in the DPODP formulation, by a function of the related parameters of the constraint, namely $s_i(\mathbf{x}, \mathbf{y})$. For instance, application of the lunar constraint with R_E as the constrained parameter replaces R_E by $86.3135017\,(\mu_E + \mu_M)^{1/3}$. Hence, specification of the constraints to be applied and the corresponding constrained parameters effectively eliminates the constrained parameters from the formulation. As a result, the constrained parameters are not included in the parameter vector q given by Eq. (857), and the elements of q are independent parameters.

Let $\mathbf{R}$ denote a column vector containing all of the observed minus computed residuals associated with the processing of one batch of data:

$$\mathbf{R} = \begin{bmatrix} \hat{\mathbf{z}} - \mathbf{z} \\ \hdashline \tilde{\mathbf{x}} - \mathbf{x} \\ \hdashline \tilde{\mathbf{y}} - \mathbf{y} \end{bmatrix} \tag{858}$$

where

$\hat{\mathbf{z}}$ = column vector of observables (doppler, range, angles, etc.)

$\mathbf{z} = \mathbf{z}(\mathbf{x}, \mathbf{y})$ = column vector of computed observables

$\tilde{\mathbf{x}}$ = column vector of *a priori* estimates of solve-for parameters

$\tilde{\mathbf{y}}$ = column vector of *a priori* estimates of consider parameters

$\mathbf{x}$ = column vector of estimated values of solve-for parameters

$\mathbf{y}$ = column vector of estimated values of consider parameters $= \tilde{\mathbf{y}}$

The zero residual vector $\tilde{\mathbf{y}} - \mathbf{y}$ is retained because the estimates $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are correlated. The sum of weighted squares of residual errors between observed and computed quantities is given by

$$Q = \mathbf{R}^T W_T \mathbf{R} \tag{859}$$

The weighting matrix W_T is given by

$$\begin{aligned} W_T &= \begin{bmatrix} W & 0 \\ 0 & \tilde{\Gamma}_q^{-1} \end{bmatrix} \\ &\equiv \begin{bmatrix} W & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} \tilde{\Gamma}_x & \tilde{\Gamma}_{xy} \\ \tilde{\Gamma}_{xy}^T & \tilde{\Gamma}_y \end{bmatrix}^{-1}$}} \\ 0 & & \end{bmatrix} \\ &\equiv \begin{bmatrix} W & 0 & 0 \\ 0 & W_x & W_{xy} \\ 0 & W_{xy}^T & W_y \end{bmatrix} \end{aligned} \tag{860}$$

where

W = data weighting matrix (diagonal); the weight for each observable is 1 divided by the input variance for the observable

$\tilde{\Gamma}_x$ = covariance matrix for $\tilde{\mathbf{x}}$

$\tilde{\Gamma}_y$ = covariance matrix for $\tilde{\mathbf{y}}$

$\tilde{\Gamma}_{xy}$ = cross-covariance matrix for $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$

When the *a priori* covariance matrix $\tilde{\Gamma}_q$ is not obtained from a previous reduction of tracking data, $\tilde{\Gamma}_{xy}$ must be zero. The matrix W_x is given by Eq. (785), repeated here:

$$W_x = [\tilde{\Gamma}_x - \tilde{\Gamma}_{xy} \tilde{\Gamma}_y^{-1} \tilde{\Gamma}_{xy}^T]^{-1} \tag{861}$$

The sum of squares Q given by Eq. (859) is a function of the solve-for parameter vector x; the estimate of x is that vector which minimizes Q.

Before proceeding, the square root of a matrix must be defined. The relation between a symmetric positive-definite matrix M and its square root $M^{1/2}$ is

$$M = (M^{1/2})^T M^{1/2} \tag{862}$$

The square root is not unique; the form used in the DPODP is upper triangular. The square root of the inverse is denoted as

$$M^{-1/2} \equiv (M^{-1})^{1/2} \tag{863}$$

Also,

$$(M^{-1})^{1/2} = [(M^{1/2})^{-1}]^T = [(M^{1/2})^T]^{-1} \tag{864}$$

The latter form follows since the transpose and inverse can always be interchanged.

Since the parameter estimate minimizes the sum of squares Q, it also minimizes the square root of Q. From Eqs. (859) and (862),

$$Q = (W_T^{1/2}\mathbf{R})^T (W_T^{1/2}\mathbf{R}) = \|W_T^{1/2}\mathbf{R}\|^2 \tag{865}$$

where the bars indicate the magnitude of the vector $W_T^{1/2}\mathbf{R}$. Hence,

$$Q^{1/2} = \|W_T^{1/2}\mathbf{R}\| \tag{866}$$

From the first form of Eq. (860), the square root of W_T is given by

$$W_T^{1/2} = \left[\begin{array}{c:c} W^{1/2} & 0 \\ \hdashline 0 & \tilde{\Gamma}_q^{-1/2} \end{array}\right] \tag{867}$$

Using the partitioned form of $\tilde{\Gamma}_q$ from Eq. (860),

$$\tilde{\Gamma}_q^{-1/2} = \left[\begin{array}{c:c} W_x^{1/2} & -W_x^{1/2}\tilde{\Gamma}_{xy}\tilde{\Gamma}_y^{-1} \\ \hdashline 0 & \tilde{\Gamma}_y^{-1/2} \end{array}\right] \tag{868}$$

where W_x is computed from $\tilde{\Gamma}_x$, $\tilde{\Gamma}_{xy}$, and $\tilde{\Gamma}_y$ using Eq. (861).

If the *a priori* parameter estimate is obtained from the processing of previous batches of data, $\tilde{\Gamma}_q$ and its submatrices $\tilde{\Gamma}_x$, $\tilde{\Gamma}_{xy}$, and $\tilde{\Gamma}_y$ are obtained by mapping the covariance matrix obtained from processing the last batch of data to the "injection" epoch (the epoch for the solve-for spacecraft state vector) for the current batch of data. If there is no previous data, $\tilde{\Gamma}_x$ and $\tilde{\Gamma}_y$ are input and $\tilde{\Gamma}_{xy}$ must be zero. In either case, W_x could be computed from Eq. (861) and substituted into Eq. (868) to give $\tilde{\Gamma}_q^{-1/2}$. Then $W_T^{1/2}$ is given by Eq. (867).

In the equivalent formulation of Section XV, the quantity W_x was identified as the inverse of the *a priori* nonconsider covariance matrix:

$$W_x = \tilde{\Gamma}_{x_{NC}}^{-1} \tag{869}$$

where, from Eq. (826),

$$\tilde{\Gamma}_{x_{NC}} = \tilde{\Gamma}_x - \tilde{\Gamma}_{xy}\tilde{\Gamma}_y^{-1}\tilde{\Gamma}_{xy}^T \tag{870}$$

Substituting Eq. (869) into Eq. (868) gives

$$\tilde{\Gamma}_q^{-1/2} = \left[\begin{array}{c:c} \tilde{\Gamma}_{x_{NC}}^{-1/2} & -\tilde{\Gamma}_{x_{NC}}^{-1/2}\tilde{\Gamma}_{xy}\tilde{\Gamma}_y^{-1} \\ \hdashline 0 & \tilde{\Gamma}_y^{-1/2} \end{array}\right] \tag{871}$$

The DPODP computes $W_T^{1/2}$ from Eqs. (867) and (871), using the input quantities $\tilde{\Gamma}_{x_{NC}}$, $\tilde{\Gamma}_{xy}$, and $\tilde{\Gamma}_y$. However, the available quantities obtained from processing previous batches of data are $\tilde{\Gamma}_x$, $\tilde{\Gamma}_{xy}$, and $\tilde{\Gamma}_y$. The required input $\tilde{\Gamma}_{x_{NC}}$ can be computed, external to the DPODP, from Eq. (870). If $\tilde{\Gamma}_{xy} = 0$, $\tilde{\Gamma}_{x_{NC}} = \tilde{\Gamma}_x$, and no extra calculation is required.

The program should be modified so that $\tilde{\Gamma}_x$, $\tilde{\Gamma}_{xy}$, and $\tilde{\Gamma}_y$ are input and $W_T^{1/2}$ is computed from Eqs. (861), (868), and (867). Furthermore, the option should be added for mapping $\tilde{\Gamma}_q^{-1/2}$ obtained from processing a previous batch of data to the "injection" epoch for the current batch of data and putting this quantity directly into Eq. (867). The formulation for mapping $\tilde{\Gamma}_q^{-1/2}$ is included in Section XVI-D.

Processing of a batch of data requires several iterations of the orbit determination process. For the first iteration, the initial estimate of the solve-for parameter vector $\mathbf{x}$ is usually taken to be the *a priori* estimate $\tilde{\mathbf{x}}$. Given $\mathbf{x} = \tilde{\mathbf{x}}$ and the *a priori* estimate of the consider parameter vector $\tilde{\mathbf{y}}$ (which is not corrected), the orbit determination process consists of computing the spacecraft ephemeris, the vector $\mathbf{z}$ of computed observables, the observed minus computed residual vector $\hat{\mathbf{z}} - \mathbf{z}$, and the partial derivatives of $\mathbf{z}$ with respect to $\mathbf{q}$.

Substituting these quantities along with the data weighting matrix W, the *a priori* estimate $\tilde{\mathbf{x}}$, and the *a priori* covariance matrix $\tilde{\Gamma}_q$ into the parameter estimation formula (to be developed below) gives the differential correction $\delta\mathbf{x}$ to the solve-for parameter vector. Because $\mathbf{z}$ does not vary linearly with $\mathbf{x}$, the orbit determination process is repeated using $\mathbf{x} + \delta\mathbf{x}$ as the initial estimate of the solve-for parameter vector. After several iterations, the estimate for $\mathbf{x}$ will converge and $Q^{1/2}$ given by Eq. (866) (or Q given by Eq. 859) will be minimized.

Let $\mathbf{q}$ given by Eq. (857) be the parameter estimate at the beginning of an iteration of the orbit determination process and $\mathbf{R}(\mathbf{q})$ from Eq. (858) be the corresponding residual vector. The differential correction produced by the iteration is

$$\delta\mathbf{q} = \left[\begin{array}{c} \delta\mathbf{x} \\ \hdashline 0 \end{array}\right] \tag{872}$$

The correction $\delta y = 0$ since the *a priori* estimate is not corrected. The expected residual after correcting q is $\mathbf{R}(\mathbf{q}) + (\partial \mathbf{R}/\partial \mathbf{q})\,\delta \mathbf{q}$. From Eq. (858), $\partial \mathbf{R}/\partial \mathbf{q} = -A_T$ where

$$A_T = \begin{bmatrix} \dfrac{\partial \mathbf{z}}{\partial \mathbf{q}} \\ \hdashline \dfrac{\partial \mathbf{x}}{\partial \mathbf{q}} \\ \hdashline \dfrac{\partial \mathbf{y}}{\partial \mathbf{q}} \end{bmatrix} = \left[\begin{array}{c:c} \dfrac{\partial \mathbf{z}}{\partial \mathbf{x}} & \dfrac{\partial \mathbf{z}}{\partial \mathbf{y}} \\ \hdashline I & 0 \\ \hdashline 0 & I \end{array}\right] \tag{873}$$

The partial derivatives $\partial \mathbf{z}/\partial \mathbf{x}$ and $\partial \mathbf{z}/\partial \mathbf{y}$ account for the variations in the exactly constrained parameter vector $\mathbf{s} = \mathbf{s}(\mathbf{x}, \mathbf{y})$ with variations in x and y. However, the DPODP computes the partial derivatives of the observables with respect to the solve-for, consider, and exactly constrained parameters treated as independent variables:

$$\left.\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\right|_{\mathbf{s}\ \text{fixed}} \equiv A_x \tag{874}$$

$$\left.\frac{\partial \mathbf{z}}{\partial \mathbf{y}}\right|_{\mathbf{s}\ \text{fixed}} \equiv A_y \tag{875}$$

$$\frac{\partial \mathbf{z}}{\partial \mathbf{s}} \equiv A_s \tag{876}$$

From Eqs. (772) and (773),

$$\frac{\partial \mathbf{s}}{\partial \mathbf{x}} \equiv S_x \tag{877}$$

$$\frac{\partial \mathbf{s}}{\partial \mathbf{y}} \equiv S_y \tag{878}$$

In terms of the quantities above, the desired partial derivatives are given by

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = A_x + A_s S_x \tag{879}$$

$$\frac{\partial \mathbf{z}}{\partial \mathbf{y}} = A_y + A_s S_y \tag{880}$$

Substituting into Eq. (873) gives

$$A_T = \left[\begin{array}{c:c} A_x + A_s S_x & A_y + A_s S_y \\ \hdashline I & 0 \\ \hdashline 0 & I \end{array}\right] \tag{881}$$

The expected value of $Q^{1/2}$ after correcting q is given by

$$Q^{1/2} = \|W_T^{1/2}\mathbf{R}(\mathbf{q}) - W_T^{1/2} A_T\,\delta \mathbf{q}\| \tag{882}$$

An orthogonal matrix P is found such that

$$P W_T^{1/2} A_T = \underbrace{\begin{bmatrix} R \\ \hdashline 0 \end{bmatrix}}_{n} \begin{array}{l} \}\, n = x + y \\ \}\, z \end{array} = \begin{array}{c} \left[\begin{array}{c:c} R_x & R_{xy} \\ \hdashline 0 & R_y \\ \hdashline 0 & 0 \end{array}\right] \begin{array}{l} \}\, x \\ \}\, y \\ \}\, z \end{array} \\ \underbrace{\hphantom{R_x}}_{x}\ \underbrace{\hphantom{R_{xy}}}_{y} \end{array} \tag{883}$$

where the matrix R is upper triangular. The dimension n is the number of parameters in q, x is the number of solve-for parameters, y is the number of consider parameters, and z is the number of true observables. The matrix P is a product of n Householder orthogonal transformations. The formation of P is described in detail in Ref. 1.

Since P is orthogonal,

$$Q^{1/2} = \|P W_T^{1/2}\mathbf{R}(\mathbf{q}) - P W_T^{1/2} A_T\,\delta \mathbf{q}\| \tag{884}$$

The first term of the vector in Eq. (884) is formed and denoted by

$$P W_T^{1/2}\mathbf{R}(\mathbf{q}) \equiv \delta \mathbf{z}' = \begin{bmatrix} \delta \mathbf{z}'_x \\ \hdashline \delta \mathbf{z}'_y \\ \hdashline \delta \mathbf{z}'_z \end{bmatrix} \begin{array}{l} \}\, x \\ \}\, y \\ \}\, z \end{array} \tag{885}$$

Substituting Eqs. (872), (883), and (885) into Eq. (884) gives

$$Q^{1/2} = \left\| \begin{bmatrix} \delta \mathbf{z}'_x \\ \hdashline \delta \mathbf{z}'_y \\ \hdashline \delta \mathbf{z}'_z \end{bmatrix} - \left[\begin{array}{c:c} R_x & R_{xy} \\ \hdashline 0 & R_y \\ \hdashline 0 & 0 \end{array}\right] \begin{bmatrix} \delta \mathbf{x} \\ \hdashline 0 \end{bmatrix} \right\| \tag{886}$$

or

$$Q^{1/2} = \left\| \begin{array}{c} \delta \mathbf{z}'_x - R_x\,\delta \mathbf{x} \\ \hdashline \delta \mathbf{z}'_y \\ \hdashline \delta \mathbf{z}'_z \end{array} \right\| \tag{887}$$

The quantity $Q^{1/2}$ is minimized if

$$\delta \mathbf{z}'_x = R_x\,\delta \mathbf{x} \tag{888}$$

Hence, the parameter estimation formula is given by

$$\delta \mathbf{x} = R_x^{-1}\,\delta \mathbf{z}'_x \tag{889}$$

This equation gives the linear differential correction $\delta \mathbf{x}$ to the solve-for parameter vector $\mathbf{x}$ produced by one iteration of the orbit determination process. The *a priori* estimate $\bar{\mathbf{y}}$ of the consider parameter vector is not corrected. The exactly constrained parameter vector $\mathbf{s}$ is computed from $\mathbf{x} + \delta \mathbf{x}$ and $\bar{\mathbf{y}}$ using Eq. (772). Evaluation of $\delta \mathbf{x}$ from Eq. (889) requires the computation of $W_T^{1/2}$ from Eq. (867) and associated equations, A_T from Eq. (881), $\mathbf{R}(\mathbf{q})$ from Eq. (858), R_x from Eq. (883), and $\delta \mathbf{z}'_x$ from Eq. (885). After several iterations of the orbit determination process, the quantity $Q^{1/2}$ should approach

$$Q^{1/2} = \left\| \begin{array}{c} \delta \mathbf{z}'_y \\ \hdashline \delta \mathbf{z}'_x \end{array} \right\|$$

2. ***Numerical techniques.*** Evaluation of Eq. (889), the parameter estimation formula, requires the inverse of the upper triangular matrix R_x. Subsection XVI-B-2-a describes the singular value decomposition method of inverting R_x. The alternative "mass below the diagonal" technique for inverting R_x is described in Subsection XVI-B-2-b. The partial-step algorithm for obtaining the parameter estimate in the presence of significant nonlinearities is described in Subsection XVI-B-2-c; it is a constrained evaluation of Eq. (889).

a. Singular-value decomposition of R_x. The x by x matrix[42] R_x is nearly always of rank x. However, its condition number (ratio of largest to smallest singular value) is often very large. When this occurs, the product $R_x^{-1}\,\delta \mathbf{z}'_x$ in Eq. (889) can greatly magnify errors in the residual vector $\delta \mathbf{z}'_x$. The error in the computed differential correction $\delta \mathbf{x}$ can be reduced by using an r-rank approximation to R_x^{-1} in place of R_x^{-1} in Eq. (889).

The first step is to find x by x orthogonal matrices U and V and a diagonal matrix S such that

$$U^T R_x V = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_x) \equiv S \tag{890}$$

The elements of S are, by definition, the singular values of R_x ordered from largest to smallest. Solving for R_x gives

$$R_x = USV^T \tag{891}$$

This expression for R_x is called the singular-value decomposition of the matrix R_x. Inverting Eq. (891) gives

$$R_x^{-1} = VS^{-1}U^T \tag{892}$$

Let the ith columns of U and V be denoted by $\mathbf{u}_i$ and $\mathbf{v}_i$, respectively. Then

$$U = [\mathbf{u}_1 \,\vdots\, \mathbf{u}_2 \,\vdots\, \cdots \,\vdots\, \mathbf{u}_x] \tag{893}$$

$$V = [\mathbf{v}_1 \,\vdots\, \mathbf{v}_2 \,\vdots\, \cdots \,\vdots\, \mathbf{v}_x] \tag{894}$$

Substituting Eqs. (893), (894), and S from Eq. (890) into Eq. (892) gives

$$R_x^{-1} = \sum_{i=1}^{x} \frac{\mathbf{v}_i \mathbf{u}_i^T}{\lambda_i} \tag{895}$$

Substituting Eq. (895) into Eq. (889) gives

$$\delta \mathbf{x} = \sum_{i=1}^{x} \frac{\mathbf{v}_i (\mathbf{u}_i \cdot \delta \mathbf{z}'_x)}{\lambda_i} \equiv \sum_{i=1}^{x} \delta \mathbf{x}_i \tag{896}$$

where the dot indicates a dot product. If a singular value λ_i is very small, the error in $\delta \mathbf{x}_i$ due to the error in $\delta \mathbf{z}'_x$ can be very large. That is, errors in $\delta \mathbf{z}'_x$ lying along eigendirections associated with smaller singular values are magnified more than error components corresponding to larger singular values. The quantities $\mathbf{v}_i$, $\mathbf{u}_i$, and λ_i of Eq. (896) are generally computed to a greater accuracy (more significant figures) than the components of $\delta \mathbf{z}'_x$; hence the errors in these quantities do not contribute significantly to $\delta \mathbf{x}$.

In Eq. (889), R_x^{-1} is replaced by its r-rank approximation, which consists of the first r terms of Eq. (895):

$$(R_x^{-1})_r = \sum_{i=1}^{r} \frac{\mathbf{v}_i \mathbf{u}_i^T}{\lambda_i} \tag{897}$$

The integer r is called the pseudorank of the matrix R_x. It must be chosen so that the error due to truncating the last $(x - r)$ terms of Eq. (895) is less than the error that would be incurred by retaining them. The pseudorank r is the largest integer such that

$$\frac{\lambda_r}{\lambda_1} > \epsilon \tag{898}$$

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[42] The term x represents the number of solve-for parameters.

where ϵ is a small input positive number. Equivalently, r is the largest integer such that the rank deficient condition number $\lambda_1/\lambda_r < 1/\epsilon$.

Let

$$(S^{-1})_r \equiv \operatorname{diag}(1/\lambda_1, 1/\lambda_2, \cdots, 1/\lambda_r) \qquad (899)$$

$$U_r \equiv [\mathbf{u}_1 \,\vdots\, \mathbf{u}_2 \,\vdots\, \cdots \,\vdots\, \mathbf{u}_r] \qquad (x \text{ by } r) \qquad (900)$$

$$V_r \equiv [\mathbf{v}_1 \,\vdots\, \mathbf{v}_2 \,\vdots\, \cdots \,\vdots\, \mathbf{v}_r] \qquad (x \text{ by } r) \qquad (901)$$

Then,

$$(R_x^{-1})_r = V_r (S^{-1})_r U_r^T \qquad (x \text{ by } x) \qquad (902)$$

Equation (889) is evaluated with R_x^{-1} replaced by $(R_x^{-1})_r$ from Eq. (902).

b. "Mass below the diagonal" technique. The title refers to the method used to determine the pseudorank r of the matrix R_x. The parameter estimation formula used with this method gives linear differential corrections for the r most significant solve-for parameters; the $x - r$ least significant solve-for parameters are not corrected.

Let the element which is in the ith row and jth column of the matrix R_x be denoted as r_{ij} and let ϵ be a small input positive number. Then the pseudorank of the x by x matrix R_x is the smallest integer r for which

$$\sum_{i=r+1}^{x} r_{ij}^2 \leq \epsilon^2 \sum_{i=1}^{r} r_{ij}^2 \qquad j = r+1, \cdots, x \qquad (903)$$

Given the pseudorank r, partition the vectors $\mathbf{x}$ and $\delta\mathbf{z}'_x$ and the matrix R_x as follows:

$$\mathbf{x} = \left.\begin{bmatrix} \left.\mathbf{x}_1\right\} r \\ \hdashline \mathbf{x}_2 \end{bmatrix}\right\} x \qquad (904)$$

$$\delta\mathbf{z}'_x = \left.\begin{bmatrix} \left.\delta\mathbf{z}'_{x_1}\right\} r \\ \hdashline \delta\mathbf{z}'_{x_2} \end{bmatrix}\right\} x \qquad (905)$$

$$R_x = r\Big\{ \underbrace{\overbrace{\begin{bmatrix} R_{11} & \vdots & R_{12} \\ \hdashline 0 & \vdots & R_{22} \end{bmatrix}}^{r}}_{x} \Big\} x \qquad (906)$$

The parameter vector $\mathbf{x}_1$ is estimated but $\mathbf{x}_2$ is not. Thus,

$$\delta\mathbf{x} = \begin{bmatrix} \delta\mathbf{x}_1 \\ \hdashline 0 \end{bmatrix} \qquad (907)$$

Substituting Eqs. (905–907) into Eq. (887) gives

$$Q^{1/2} = \left\| \begin{matrix} \begin{bmatrix} \delta\mathbf{z}'_{x_1} \\ \hdashline \delta\mathbf{z}'_{x_2} \end{bmatrix} - \begin{bmatrix} R_{11} & \vdots & R_{12} \\ \hdashline 0 & \vdots & R_{22} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_1 \\ \hdashline 0 \end{bmatrix} \\ \hdashline \delta\mathbf{z}'_y \\ \hdashline \delta\mathbf{z}'_z \end{matrix} \right\| \qquad (908)$$

or

$$Q^{1/2} = \left\| \begin{matrix} \delta\mathbf{z}'_{x_1} - R_{11}\,\delta\mathbf{x}_1 \\ \hdashline \delta\mathbf{z}'_{x_2} \\ \hdashline \delta\mathbf{z}'_y \\ \hdashline \delta\mathbf{z}'_z \end{matrix} \right\| \qquad (909)$$

The quantity $Q^{1/2}$ is minimized if

$$\delta\mathbf{z}'_{x_1} = R_{11}\,\delta\mathbf{x}_1 \qquad (910)$$

Hence, the parameter estimation formula is given by

$$\delta\mathbf{x}_1 = R_{11}^{-1}\,\delta\mathbf{z}'_{x_1} \qquad (911)$$

After several iterations of the orbit determination process, the quantity $Q^{1/2}$ should approach

$$Q^{1/2} = \left\| \begin{matrix} \delta\mathbf{z}'_{x_2} \\ \hdashline \delta\mathbf{z}'_y \\ \hdashline \delta\mathbf{z}'_z \end{matrix} \right\|$$

c. Partial-step algorithm. The partial-step algorithm is a modification of the singular-value decomposition method for obtaining the parameter estimate (Subsection XVI-B-2-a). The singular-value decomposition of the matrix R_x is given by Eq. (891). Substituting the inverse of this expression (Eq. 892) into the parameter estimation formula (Eq. 889) gives Eq. (896) for the differential correction to the solve-for parameter vector. The singular-value decomposition method deletes terms of Eq. (896) which correspond to small singular values since they magnify errors in the residual vector $\delta\mathbf{z}'_x$. However, this magnification error affects only the magnitude of a term $\delta\mathbf{x}_i$ of Eq. (896); its direction is that of the vector $\mathbf{v}_i$, which is computed to sufficient accuracy. The partial-step algorithm computes each term $\delta\mathbf{x}_i$ of Eq. (896). Then, a weighted length of each term is computed:

$$\|\delta\mathbf{x}_i\|_W = (\delta\mathbf{x}_i^T\,\tilde{\Gamma}_{x_i}^{-1}\,\delta\mathbf{x}_i)^{1/2} \qquad (912)$$

where $\widetilde{\Gamma}_{x_i}$ is an input realistic *a priori* covariance matrix for the solve-for parameter vector x. If $\|\delta \mathbf{x}_i\|_W$ is greater than an input number QB, the correction vector $\delta \mathbf{x}_i$ is scaled to $\delta \mathbf{x}_i$ (adjusted) so that $\|\delta \mathbf{x}_i \text{ (adjusted)}\|_W$ is equal to a second input number QC, which is usually smaller than QB. The differential correction $\delta \mathbf{x}$ is then given by

$$\delta \mathbf{x} = \sum_{i=1}^{x} \delta \mathbf{x}_i \text{ (adjusted)} \tag{913}$$

The scaling process reduces the magnitude of any correction vector $\delta \mathbf{x}_i$ which is unrealistically large in relation to the *a priori* uncertainty in the solve-for parameter vector x. Since unrealistically large corrections result from magnification errors associated with small singular values, the scaling process places an upper limit on errors of this type. The partial-step algorithm produces a more accurate parameter estimate than the singular-value decomposition method since some of the information contained in the terms of Eq. (896) associated with small singular values is retained.

If the relation between the computed observables and the solve-for parameters is extremely nonlinear, many iterations of the orbit determination process will be required in order to obtain convergence of the parameter estimate. For this nonlinear problem, small errors in the computed correction $\delta \mathbf{x}$ can eliminate the small amount of convergence obtained on one iteration of the orbit determination process. Hence, the high accuracy of the partial-step algorithm is particularly suited to the nonlinear estimation problem.

C. Covariance Matrix

This section gives the formulation for computation of the covariance matrix for the estimate of the parameter vector q. Let the error in the estimate of q be denoted by[43]

$$\delta \mathbf{q} = \left[\begin{array}{c} \delta \mathbf{x} \\ \hdashline \delta \mathbf{y} \end{array}\right] \tag{914}$$

where $\delta \mathbf{y} = \delta \tilde{\mathbf{y}}$, the *a priori* error. Then, the covariance matrix is given by

$$\Gamma_q = \overline{\delta \mathbf{q}\, \delta \mathbf{q}^T} = \left[\begin{array}{c:c} \overline{\delta \mathbf{x}\, \delta \mathbf{x}^T} & \overline{\delta \mathbf{x}\, \delta \mathbf{y}^T} \\ \hdashline \overline{\delta \mathbf{y}\, \delta \mathbf{x}^T} & \overline{\delta \mathbf{y}\, \delta \mathbf{y}^T} \end{array}\right] \tag{915}$$

where a bar indicates the ensemble average or expected value of the function. The covariance matrix for the solve-for parameter vector is given by

$$\Gamma_x = \overline{\delta \mathbf{x}\, \delta \mathbf{x}^T} \tag{916}$$

Similarly, for the consider parameter vector,

$$\Gamma_y = \overline{\delta \mathbf{y}\, \delta \mathbf{y}^T} \tag{917}$$

The cross-covariance matrix for the solve-for and consider parameter vectors is given by

$$\Gamma_{xy} = \overline{\delta \mathbf{x}\, \delta \mathbf{y}^T} \tag{918}$$

Substituting Eqs. (916–918) into Eq. (915) gives

$$\Gamma_q = \left[\begin{array}{c:c} \Gamma_x & \Gamma_{xy} \\ \hdashline \Gamma_{xy}^T & \Gamma_y \end{array}\right] \tag{919}$$

In order to compute the submatrices of Eq. (919), an expression is required for the error $\delta \mathbf{x}$ in the estimate of the solve-for parameter vector x. Equation (889) gives the linear differential correction to x obtained from each iteration of the orbit determination process. On the last iteration, the linear differential correction is very small and the neglected nonlinear terms are negligible.

The matrix R_x of Eq. (889), computed from the weighting matrix W_T and the A_T matrix of partial derivatives, can be considered to be correct. The residual vector $\delta \mathbf{z}'_x$ of Eq. (889) is computed from W_T and the residual vector $\mathbf{R}$ given by Eq. (858). The error in the estimate x is due entirely to the error in $\mathbf{R}$ given by

$$\delta \mathbf{R} = \left[\begin{array}{c} \delta \hat{\mathbf{z}} \\ \hdashline \delta \tilde{\mathbf{x}} \\ \hdashline \delta \tilde{\mathbf{y}} \end{array}\right] - \left[\begin{array}{c} (A_y + A_s\, S_y)\, \delta \tilde{\mathbf{y}} \\ \hdashline 0 \\ \hdashline \delta \tilde{\mathbf{y}} \end{array}\right] \equiv \delta \mathbf{R}_1 + \delta \mathbf{R}_2 \tag{920}$$

The quantities $\delta \hat{\mathbf{z}}$, $\delta \tilde{\mathbf{x}}$, and $\delta \tilde{\mathbf{y}}$ are errors in $\hat{\mathbf{z}}$, $\tilde{\mathbf{x}}$, and $\tilde{\mathbf{y}}$, respectively. The quantity $(A_y + A_s\, S_y)\, \delta \mathbf{y}$ is the error in z due to the error in $\tilde{\mathbf{y}}$ used to compute it (see Eq. 880).

From Eq. (885), the error $\delta \mathbf{R}_1$ in $\mathbf{R}$ will produce errors $\delta \mathbf{z}'_1$, $\delta \mathbf{z}'_{x1}$, $\delta \mathbf{z}'_{y1}$, and $\delta \mathbf{z}'_{z1}$ in the residual vectors $\delta \mathbf{z}'$, $\delta \mathbf{z}'_x$, $\delta \mathbf{z}'_y$, and $\delta \mathbf{z}'_z$, respectively. Similarly, the error $\delta \mathbf{R}_2$ will produce

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[43] In Section XVI-B, the differential correction to the estimate of the solve-for parameter vector was denoted by $\delta \mathbf{x}$. Here, the same notation will be used for the error in the differential correction.

errors $\delta \mathbf{z}'_2$, $\delta \mathbf{z}'_{x2}$. $\delta \mathbf{z}'_{y2}$, and $\delta \mathbf{z}'_{z2}$. Substituting $\delta \mathbf{R}_1$ from Eq. (920) into Eq. (885) gives

$$PW_T^{1/2}\begin{bmatrix} \delta \hat{\mathbf{z}} \\ \hdashline \delta \tilde{\mathbf{x}} \\ \hdashline \delta \tilde{\mathbf{y}} \end{bmatrix} \equiv \delta \mathbf{z}'_1 = \begin{bmatrix} \delta \mathbf{z}'_{x1} \\ \hdashline \delta \mathbf{z}'_{y1} \\ \hdashline \delta \mathbf{z}'_{z1} \end{bmatrix} \tag{921}$$

Substituting $\delta \mathbf{R}_2$ from Eq. (920) into Eq. (885) and using Eqs. (881) and (883) gives

$$-PW_T^{1/2}\begin{bmatrix} (A_y + A_s S_y)\,\delta \tilde{\mathbf{y}} \\ \hdashline 0 \\ \hdashline \delta \tilde{\mathbf{y}} \end{bmatrix} = -PW_T^{1/2} A_T \begin{bmatrix} 0 \\ \hdashline \delta \tilde{\mathbf{y}} \end{bmatrix} \}\, x = -\begin{bmatrix} R_x & \vdots & R_{xy} \\ \hdashline 0 & \vdots & R_y \\ \hdashline 0 & \vdots & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \hdashline \delta \tilde{\mathbf{y}} \end{bmatrix} = -\begin{bmatrix} R_{xy}\,\delta \tilde{\mathbf{y}} \\ \hdashline R_y\,\delta \tilde{\mathbf{y}} \\ \hdashline 0 \end{bmatrix} \equiv \begin{bmatrix} \delta \mathbf{z}'_{x2} \\ \hdashline \delta \mathbf{z}'_{y2} \\ \hdashline \delta \mathbf{z}'_{z2} \end{bmatrix} \tag{922}$$

This result could have been obtained directly by inspection of Eq. (886) (with the null correction vector replaced by $\delta \mathbf{y}$). The total error in the residual vector $\delta \mathbf{z}'_x$ computed from Eq. (885) is $\delta \mathbf{z}'_{x1}$ from Eq. (921) plus $\delta \mathbf{z}'_{x2}$ from Eq. (922) or

$$\delta \mathbf{z}'_{x1} - R_{xy}\,\delta \tilde{\mathbf{y}}$$

Substituting this error into Eq. (889) gives the required expression for the error $\delta \mathbf{x}$ in the estimate of $\mathbf{x}$:

$$\delta \mathbf{x} = R_x^{-1}\,(\delta \mathbf{z}'_{x1} - R_{xy}\,\delta \tilde{\mathbf{y}}) \tag{923}$$

Substituting Eq. (923) into Eq. (916) gives

$$\Gamma_x = R_x^{-1}\,\overline{(\delta \mathbf{z}'_{x1} - R_{xy}\,\delta \tilde{\mathbf{y}})\,(\delta \mathbf{z}'^T_{x1} - \delta \tilde{\mathbf{y}}^T\; R^T_{xy})}\,(R_x^{-1})^T \tag{924}$$

or

$$\Gamma_x = R_x^{-1}\,(\overline{\delta \mathbf{z}'_{x1}\,\delta \mathbf{z}'^T_{x1}} + R_{xy}\,\tilde{\Gamma}_y\,R^T_{xy}$$
$$- R_{xy}\,\overline{\delta \tilde{\mathbf{y}}\,\delta \mathbf{z}'^T_{x1}} - \overline{\delta \mathbf{z}'_{x1}\,\delta \tilde{\mathbf{y}}^T}\,R^T_{xy})\,(R_x^{-1})^T \tag{925}$$

Postmultiplying $\delta \mathbf{z}'_1$ from Eq. (921) by its transpose and averaging gives

$$\overline{\delta \mathbf{z}'_1\,\delta \mathbf{z}'^T_1} = PW_T^{1/2}\begin{bmatrix} \overline{\delta \hat{\mathbf{z}}\,\delta \hat{\mathbf{z}}^T} & \vdots & 0 & \vdots & 0 \\ \hdashline 0 & \vdots & \tilde{\Gamma}_x & \vdots & \tilde{\Gamma}_{xy} \\ \hdashline 0 & \vdots & \tilde{\Gamma}^T_{xy} & \vdots & \tilde{\Gamma}_y \end{bmatrix}(W_T^{1/2})^T\,P^T \tag{926}$$

But $\overline{\delta \hat{\mathbf{z}}\,\delta \hat{\mathbf{z}}^T}$ is the data covariance matrix Γ_z which is presumed to be the inverse of the diagonal data weighting matrix W:

$$\overline{\delta \hat{\mathbf{z}}\,\delta \hat{\mathbf{z}}^T} = \Gamma_z = W^{-1} \tag{927}$$

Hence, the partitioned matrix in Eq. (926) is W_T^{-1} (see Eq. 860) and

$$\overline{\delta \mathbf{z}'_1\,\delta \mathbf{z}'^T_1} = PW_T^{1/2}\,W_T^{-1}\,(W_T^{1/2})^T\,P^T \tag{928}$$

Using Eqs. (862) and (864), this reduces to

$$\overline{\delta \mathbf{z}'_1\,\delta \mathbf{z}'^T_1} = PIP^T = PP^T = I \tag{929}$$

Since Eq. (929) is an identity matrix, its submatrix $\overline{\delta \mathbf{z}'_{x1}\,\delta \mathbf{z}'^T_{x1}}$ (see Eq. 921) is also an identity matrix:

$$\overline{\delta \mathbf{z}'_{x1}\,\delta \mathbf{z}'^T_{x1}} = I \qquad (x \text{ by } x) \tag{930}$$

The matrix $\overline{\delta \tilde{\mathbf{y}}\,\delta \mathbf{z}'^T_{x1}}$ is also required in Eq. (925). Using Eq. (921), it can be expressed as

$$\overline{\delta \tilde{\mathbf{y}}\,\delta \mathbf{z}'^T_{x1}} = \overline{\delta \tilde{\mathbf{y}}\,\delta \mathbf{z}'^T_{x1}}\,R_x\,R_x^{-1}$$

$$= \overline{\delta \tilde{\mathbf{y}}\,\delta \mathbf{z}'^T_1}\begin{bmatrix} R_x \\ \hdashline 0 \\ \hdashline 0 \end{bmatrix}\begin{array}{l} \\ \}\, y\; R_x^{-1} \\ \}\, z \end{array}$$

$$= (\overline{\delta \mathbf{z}'_1\,\delta \tilde{\mathbf{y}}^T})^T\begin{bmatrix} R_x \\ \hdashline 0 \\ \hdashline 0 \end{bmatrix} R_x^{-1} \tag{931}$$

Postmultiplying Eq. (921) by $\delta \tilde{\mathbf{y}}^T$ and averaging gives

$$\overline{\delta \mathbf{z}'_1\,\delta \tilde{\mathbf{y}}^T} = PW_T^{1/2}\begin{bmatrix} 0 \\ \hdashline \tilde{\Gamma}_{xy} \\ \hdashline \tilde{\Gamma}_y \end{bmatrix}\begin{array}{l} \}\, z \\ \\ \\ \end{array} \tag{932}$$

Substituting $W_T^{1/2}$ from Eqs. (867) and (868) and using Eqs. (862) and (864) gives

$$\overline{\delta \mathbf{z}_1' \, \delta \tilde{\mathbf{y}}^T} = P \left[\begin{array}{c} 0 \\ \hdashline 0 \\ \hdashline \tilde{\Gamma}_y^{1/2} \end{array}\right] \begin{array}{l} \}\, z \\ \}\, x \\ \end{array} \tag{933}$$

Substituting $W_T^{1/2}$ from Eqs. (867) and (868) and A_T from Eq. (881) into Eq. (883) and retaining the first x columns only gives

$$P \left[\begin{array}{c} W^{1/2}(A_x + A_s S_x) \\ \hdashline W_x^{1/2} \\ \hdashline 0 \end{array}\right] \begin{array}{l} \}\, z \\ \}\, x \\ \}\, y \end{array} = \left[\begin{array}{c} R_x \\ \hdashline 0 \\ \hdashline 0 \end{array}\right] \begin{array}{l} x \\ y \\ z \end{array} \tag{934}$$

Substituting Eqs. (933) and (934) into Eq. (931) gives

$$\overline{\delta \tilde{\mathbf{y}} \, \delta \mathbf{z}_{x1}'^T} = [0 \,\vdots\, 0 \,\vdots\, (\tilde{\Gamma}_y^{1/2})^T] \, P^T P \left[\begin{array}{c} W^{1/2}(A_x + A_s S_x) \\ \hdashline W_x^{1/2} \\ \hdashline 0 \end{array}\right] R_x^{-1}$$

$$= [0] \qquad (y \text{ by } x) \tag{935}$$

Substituting Eqs. (930) and (935) into Eq. (925) gives

$$\Gamma_x = R_x^{-1} (R_x^{-1})^T + R_x^{-1} R_{xy} \tilde{\Gamma}_y R_{xy}^T (R_x^{-1})^T \tag{936}$$

Postmultiplying Eq. (923) by $\delta \tilde{\mathbf{y}}^T$, averaging, and substituting Eqs. (918) and (935) gives

$$\Gamma_{xy} = -R_x^{-1} R_{xy} \tilde{\Gamma}_y \tag{937}$$

Of course,

$$\tilde{\Gamma}_y = \tilde{\Gamma}_y \tag{938}$$

If the parameter estimate is obtained by using the singular value decomposition method (Section XVI-B-2-a) to invert the matrix R_x, that is, if the parameter estimate is obtained from Eq. (889) with R_x^{-1} replaced by its r-rank approximation $(R_x^{-1})_r$ computed from Eq. (902), then this substitution is also made in Eqs. (936) and (937).

If the "mass below the diagonal" technique (Section XVI-B-2-b) is used to determine the pseudorank r of the matrix R_x, estimates for the r most significant solve-for parameters are obtained from Eq. (911). For this case, R_x^{-1} in Eqs. (936) and (937) is replaced by

$$\begin{array}{c} \left[\begin{array}{c:c} R_{11}^{-1} & 0 \\ \hdashline 0 & 0 \end{array}\right] \\ \underbrace{\phantom{R_{11}^{-1}}}_{r} \underbrace{}_{x-r} \end{array} \begin{array}{l} \}\, r \\ \}\, x - r \\ \end{array}$$

This produces zeroes in the rows and columns of Γ_x for the $x - r$ least significant solve-for parameters whose values are not estimated. It also produces zeroes in the rows of Γ_{xy} for these parameters.

If the parameter estimate is obtained by using the partial-step algorithm (Subsection XVI-B-2-c), Γ_x and Γ_{xy} are not currently computed by the DPODP.

Given Γ_x, Γ_{xy}, and Γ_y from Eqs. (936–938) and the matrices S_x and S_y defined by Eq. (773) or (877–878), the covariance matrix for the exactly constrained parameter vector **s** is computed from Eq. (817). The square roots of the diagonal elements are the standard deviations for the exactly constrained parameters (0, 1, or 2 parameters). The matrices Γ_{xs} and Γ_{ys} given by Eqs. (818) and (819) are not computed.

The covariance matrix Γ_q given by Eqs. (919), (936), (937), and (938) can be expressed as

$$\Gamma_q = S^T S \tag{939}$$

From the definition of Eq. (862), the square root of Γ_q is the matrix S. It is given by

$$\Gamma_q^{1/2} = S = \left[\begin{array}{c:c} (R_x^{-1})^T & 0 \\ \hdashline -\tilde{\Gamma}_y^{1/2} R_{xy}^T (R_x^{-1})^T & \tilde{\Gamma}_y^{1/2} \end{array}\right] \tag{940}$$

The matrix $(R_x^{-1})^T$ is lower triangular while $\tilde{\Gamma}_y^{1/2}$ is upper triangular. The inverse of Eq. (939) is

$$\Gamma_q^{-1} = (S^{-1})(S^{-1})^T \tag{941}$$

The square root of Γ_q^{-1} is given by

$$(\Gamma_q^{-1})^{1/2} \equiv \Gamma_q^{-1/2} = (S^{-1})^T = (S^T)^{-1}$$

$$= \left[\begin{array}{c:c} R_x & R_{xy} \\ \hdashline 0 & \tilde{\Gamma}_y^{-1/2} \end{array}\right] \tag{942}$$

The following will show that Eqs. (936) and (937) for computing Γ_x and Γ_{xy}, respectively, from the square-root formulation are identical to the corresponding equations (Eqs. 813 and 814) of the normal-equations formulation (if the inexact constraints of the latter formulation are not applied). Substituting $W_T^{1/2}$ from Eqs. (867) and (868) and A_T from Eq. (881) into Eq. (883) gives

$$P\left[\begin{array}{c:c} W^{1/2}(A_x + A_s S_x) & W^{1/2}(A_y + A_s S_y) \\ \hdashline W_x^{1/2} & -W_x^{1/2}\widetilde{\Gamma}_{xy}\widetilde{\Gamma}_y^{-1} \\ \hdashline 0 & \widetilde{\Gamma}_y^{-1/2} \end{array}\right] = \left[\begin{array}{c:c} R_x & R_{xy} \\ \hdashline 0 & R_y \\ \hdashline 0 & 0 \end{array}\right] \tag{943}$$

Premultiplying each side of this equation by its transpose gives

$$\left[\begin{array}{c:c} (A_x + A_s S_x)^T W (A_x + A_s S_x) + W_x & (A_x + A_s S_x)^T W (A_y + A_s S_y) - W_x \widetilde{\Gamma}_{xy} \widetilde{\Gamma}_y^{-1} \\ \hdashline (A_y + A_s S_y)^T W (A_x + A_s S_x) - \widetilde{\Gamma}_y^{-1} \widetilde{\Gamma}_{xy}^T W_x & (A_y + A_s S_y)^T W (A_y + A_s S_y) + \widetilde{\Gamma}_y^{-1} \widetilde{\Gamma}_{xy}^T W_x \widetilde{\Gamma}_{xy} \widetilde{\Gamma}_y^{-1} + \widetilde{\Gamma}_y^{-1} \end{array}\right] =$$

$$\left[\begin{array}{c:c} R_x^T R_x & R_x^T R_{xy} \\ \hdashline R_{xy}^T R_x & R_{xy}^T R_{xy} + R_y^T R_y \end{array}\right] \tag{944}$$

Equating the upper left-hand submatrices gives

$$R_x^T R_x = (A_x + A_s S_x)^T W (A_x + A_s S_x) + W_x \tag{945}$$

and equating the upper right-hand submatrices gives

$$R_x^T R_{xy} = (A_x + A_s S_x)^T W (A_y + A_s S_y) - W_x \widetilde{\Gamma}_{xy} \widetilde{\Gamma}_y^{-1} \tag{946}$$

Substituting Eq. (796) into Eq. (945) gives

$$R_x^T R_x = J + W_x \tag{947}$$

Inverting this equation gives

$$R_x^{-1} (R_x^{-1})^T = (J + W_x)^{-1} \tag{948}$$

If inexact constraints are not applied to the estimation of the parameter vector with the normal-equations formulation, the matrix L given by Eq. (807) reduces to

$$L = (A_x + A_s S_x)^T W (A_y + A_s S_y) \tag{949}$$

Substituting this expression for L into Eq. (946) gives

$$R_x^T R_{xy} = L - W_x \widetilde{\Gamma}_{xy} \widetilde{\Gamma}_y^{-1} \tag{950}$$

In order to facilitate the substitution of Eqs. (948) and (950) into Eqs. (936) and (937), the latter are written as

$$\Gamma_x = R_x^{-1} (R_x^{-1})^T + R_x^{-1} (R_x^{-1})^T R_x^T R_{xy} \widetilde{\Gamma}_y R_{xy}^T R_x R_x^{-1} (R_x^{-1})^T \tag{951}$$

and

$$\Gamma_{xy} = - R_x^{-1} (R_x^{-1})^T R_x^T R_{xy} \widetilde{\Gamma}_y \tag{952}$$

Substituting Eqs. (948), (950), and the transpose of Eq. (950) into Eqs. (951) and (952) gives

$$\Gamma_x = (J + W_x)^{-1} + (J + W_x)^{-1} \, (W_x \tilde{\Gamma}_{xy} \tilde{\Gamma}_y^{-1} \tilde{\Gamma}_{xy}^T W_x + L \tilde{\Gamma}_y L^T - W_x \tilde{\Gamma}_{xy} L^T - L \tilde{\Gamma}_{xy}^T W_x) \, (J + W_x)^{-1} \qquad (953)$$

and

$$\Gamma_{xy} = (J + W_x)^{-1} (W_x \tilde{\Gamma}_{xy} - L \tilde{\Gamma}_y) \qquad (954)$$

In the normal-equations formulation, Γ_x and Γ_{xy} are computed from Eqs. (813) and (814), respectively. If there are no inexact constraints, the matrix K is removed from Eqs. (813) and (814), the matrix L given by Eq. (807) reduces to Eq. (949), and Eqs. (813) and (814) reduce to Eqs. (953) and (954), respectively. Thus, in the absence of inexact constraints, Eqs. (813) and (814) for computing Γ_x and Γ_{xy} with the normal-equations formulation are equal to the corresponding equations (Eqs. 936 and 937) of the square-root formulation. Furthermore, the equality of Eqs. (936) and (813) applies to each of the two terms.

From the equality of Eqs. (813) and (936) and the discussion of Section XV-E, which relates the various terms of Eq. (813) to the various error sources which affect the estimate of the solve-for parameter vector, the following conclusions can be drawn. The first term of Eq. (936) is the nonconsider covariance matrix:

$$\Gamma_{x_{NC}} = R_x^{-1} (R_x^{-1})^T \qquad (955)$$

It accounts for errors in all of the processed tracking data (the current batch of data and all previously reduced batches of data) and the error in the *a priori* parameter estimate for the first batch of data. The second term of Eq. (936) accounts for errors in the consider parameters. Because of its presence, Γ_x computed from Eq. (936) is referred to as the consider covariance matrix. Substituting Eqs. (937) and (955) into Eq. (936) gives

$$\Gamma_x = \Gamma_{x_{NC}} + \Gamma_{xy} \tilde{\Gamma}_y^{-1} \Gamma_{xy}^T \qquad (956)$$

The sensitivity matrix S_{xy} is defined by Eq. (828) and related to Γ_{xy} by Eq. (831). Substituting Eq. (937) into Eq. (831) gives

$$S_{xy} = - R_x^{-1} R_{xy} \qquad (957)$$

Substituting Eqs. (955) and (957) into Eq. (936) gives

$$\Gamma_x = \Gamma_{x_{NC}} + S_{xy} \tilde{\Gamma}_y S_{xy}^T \qquad (958)$$

D. Mapping Covariance Matrix to New Epoch

This section gives the formulation for mapping the covariance matrix for the parameter vector q, Γ_q, or the square root of its inverse, $\Gamma_q^{-1/2}$, from the injection epoch[44] to any other epoch. The parameter vector corresponding to the mapped covariance matrix is q with the spacecraft injection position and velocity components replaced by the position and velocity components of the spacecraft relative to a specified body R at the map epoch.

The mapping formulation is used to map Γ_q to a new epoch for statistical purposes or to map Γ_q or $\Gamma_q^{-1/2}$ to the injection epoch for a new batch of data, where it is used as *a priori* information ($\tilde{\Gamma}_q$ or $\tilde{\Gamma}_q^{-1/2}$). The processing of a batch of data requires the square root of the weighting matrix, $W_T^{1/2}$, which can be computed directly from Eq. (867) if $\tilde{\Gamma}_q^{-1/2}$ is available. However, if only $\tilde{\Gamma}_q$ (containing submatrices $\tilde{\Gamma}_x$, $\tilde{\Gamma}_{xy}$, and $\tilde{\Gamma}_y$) is available, $\tilde{\Gamma}_q^{-1/2}$ can be computed from Eqs. (861) and (868).

*1. **General mapping formulas.*** The solve-for parameter vector q, given by Eq. (857), will be denoted as q_0 in this section. It can be re-ordered and partitioned as

$$q_0 = \begin{bmatrix} X_0^B \\ \hdashline a \\ \hdashline b \end{bmatrix} \qquad (959)$$

The spacecraft state vector X_0^B, dynamic parameter vector a, and observational parameter vector b are defined before Eq. (836).[45] The mapped parameter vector is given by

$$q^R = \begin{bmatrix} X^R \\ \hdashline a \\ \hdashline b \end{bmatrix} \qquad (960)$$

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[44] The epoch of the solve-for spacecraft state vector.

[45] The parameter vector q_0 given by Eq. (836) contains the exactly constrained parameter vector s, whereas q_0 given by Eq. (959) does not.

where

X^R = spacecraft state vector relative to body R at map time

The injection covariance matrix is the covariance matrix for q_0 and will be denoted as Γ_{q_0}. The mapped covariance matrix is the covariance matrix for q^R and will be denoted by Γ.

The injection covariance matrix (Eq. 919) with rows and columns ordered according to Eq. (857) is given by Eq. (939). This same matrix with rows and columns ordered according to Eq. (959) is given by

$$\Gamma_{q_0} = TS^TST^T \tag{961}$$

where T is an (orthogonal) permutation matrix. Premultiplication of S^TS by T re-orders the rows, while postmultiplication by T^T re-orders the columns. Eq. (961) can be rewritten as

$$\Gamma_{q_0} = (ST^T)^T (ST^T) \tag{962}$$

The matrix ST^T is the matrix S given by Eq. (940) with the columns re-ordered according to Eq. (959). Let it be denoted by

$$S_C \equiv ST^T \tag{963}$$

Then,

$$\Gamma_{q_0} = S_C^T S_C \tag{964}$$

Let the mapping matrix M be defined by

$$\delta q^R = M \, \delta q_0 \tag{965}$$

The formulation for computing M is given in Subsection XVI-D-2. From Eq. (965), the mapped covariance matrix is given by

$$\begin{aligned} \Gamma &= \overline{\delta q^R \, \delta q^{RT}} \\ &= M \, \overline{\delta q_0 \delta q_0^T} \, M^T \\ &= M \, \Gamma_{q_0} M^T \\ &= M \, S_C^T S_C \, M^T \\ &= (S_C M^T)^T (S_C M^T) \end{aligned} \tag{966}$$

Hence, the square root of Γ is

$$\Gamma^{1/2} = S_C M^T \tag{967}$$

This equation maps the square root of Γ_{q_0}, namely S_C, to the square root of Γ. Given $\Gamma^{1/2}$ from Eq. (967), the mapped covariance matrix is given by

$$\Gamma = \Gamma^{1/2T} \Gamma^{1/2} \tag{968}$$

Inverting Eq. (967) gives

$$(\Gamma^{1/2})^{-1} = (M^{-1})^T S_C^{-1} \tag{969}$$

But, from Eqs. (864) and (863),

$$(\Gamma^{1/2})^{-1} = [(\Gamma^{-1})^{1/2}]^T \equiv (\Gamma^{-1/2})^T \tag{970}$$

Hence, the square root of the inverse of Γ is given by

$$\Gamma^{-1/2} = (S_C^{-1})^T M^{-1} \tag{971}$$

From Eq. (963),

$$(S_C^{-1})^T = (S^{-1})^T T^T \tag{972}$$

which is the square root of the inverse of Γ_{q_0} given by Eq. (942), with the columns re-ordered according to Eq. (959). Thus, Eq. (971) maps the square root of the inverse of Γ_{q_0} to the square root of the inverse of Γ.

2. *Mapping matrix.* This section gives the formulation for the mapping matrix M and its inverse M^{-1}. The former is used in Eq. (967) to map the square root of the covariance matrix, while the latter is used in Eq. (971) to map the square root of the inverse of the covariance matrix.

The dynamic parameter vector a of Section XV will be denoted here as a′; it includes all of the dynamic parameters which affect the spacecraft trajectory. Among them are μ_E, μ_M, and R_E, which are related by the lunar constraint, and μ_S and A_E, which are related by the solar constraint. One parameter from each of these constraints is placed in the exactly constrained parameter vector s. The remaining parameters of a′ are members of q given by Eq. (857) or Eq. (959). Hence, a′ can be partitioned as

$$a' = \begin{bmatrix} a \\ \hdashline s \end{bmatrix} \tag{973}$$

where a is the dynamic parameter vector of Eq. (959). It includes all of the dynamic solve-for and consider parameters except the components of X_0^B.

From the formulation of Subsections XV-F-1 and -2 and Eq. (973), the variation in $\mathbf{X}^R$ of Eq. (960) at an ET map epoch is given by

$$\delta\mathbf{X}^R = U\,\delta\mathbf{X}_0^B + \left(V - \frac{\partial\mathbf{X}_R^C}{\partial\mathbf{a}'}\right)\delta\mathbf{a}' \qquad (974)$$

The U and V matrices are obtained from the solution of the variational equations (Section XIII). For a non-ET map epoch (A1, UTC, UT1, or ST), the matrix V is replaced by V^*, which is the V matrix with the ΔT_{1958} column incremented by

$$\dot{\mathbf{X}}_{S/C}^R$$

and the Δf_{cesium} column incremented by

$$-\frac{t - 252{,}460{,}800}{9{,}192{,}631{,}770}\,\dot{\mathbf{X}}_{S/C}^R$$

where $\dot{\mathbf{X}}_{S/C}^R$ is evaluated at the map epoch t. Let

$$V' \equiv V - \frac{\partial\mathbf{X}_R^C}{\partial\mathbf{a}'} \qquad (975)$$

where V is replaced by V^* for a non-ET map epoch. Partitioning the columns of V' according to Eq. (973) gives

$$V' = [V'_a \,\vdots\, V'_s] \qquad (976)$$

The matrix V'_a gives partial derivatives of $\mathbf{X}^R$ with respect to solve-for and consider parameters and V'_s gives partial derivatives of $\mathbf{X}^R$ with respect to the exactly constrained parameters. Substituting Eqs. (973), (975), and (976) into Eq. (974) gives

$$\delta\mathbf{X}^R = U\,\delta\mathbf{X}_0^B + V'_a\,\delta\mathbf{a} + V'_s\,\delta\mathbf{s} \qquad (977)$$

From Eqs. (772) and (773), the variation in the exactly constrained parameter vector s is

$$\delta\mathbf{s} = [S_x \,\vdots\, S_y]\begin{bmatrix}\delta\mathbf{x}\\ \hdashline \delta\mathbf{y}\end{bmatrix} \qquad (978)$$

Repartitioning according to Eq. (959) gives

$$\delta\mathbf{s} = [S_X \,\vdots\, S_a \,\vdots\, S_b]\begin{bmatrix}\delta\mathbf{X}_0^B\\ \hdashline \delta\mathbf{a}\\ \hdashline \delta\mathbf{b}\end{bmatrix} \qquad (979)$$

where S_X, S_a, and S_b are the partial derivatives of s with respect to $\mathbf{X}_0^B$, a, and b, respectively. However, all of the parameters of the solar and lunar constraints are dynamic parameters. Thus, S_X and S_b are null matrices and

$$\delta\mathbf{s} = S_a\,\delta\mathbf{a} \qquad (980)$$

Substituting Eq. (980) into Eq. (977) gives

$$\delta\mathbf{X}^R = U\,\delta\mathbf{X}_0^B + (V'_a + V'_s S_a)\,\delta\mathbf{a} \qquad (981)$$

From Eqs. (959), (960), (965), and (981),

$$\delta\mathbf{q}^R = \begin{bmatrix}\delta\mathbf{X}^R\\ \hdashline \delta\mathbf{a}\\ \hdashline \delta\mathbf{b}\end{bmatrix} = \left[\begin{array}{c:c:c} U & (V'_a + V'_s S_a) & 0\\ \hdashline 0 & I & 0\\ \hdashline 0 & 0 & I\end{array}\right]\begin{bmatrix}\delta\mathbf{X}_0^B\\ \hdashline \delta\mathbf{a}\\ \hdashline \delta\mathbf{b}\end{bmatrix} = M\,\delta\mathbf{q}_0 \qquad (982)$$

Thus,

$$M = \left[\begin{array}{c:c:c} U & (V'_a + V'_s S_a) & 0\\ \hdashline 0 & I & 0\\ \hdashline 0 & 0 & I\end{array}\right] \qquad (983)$$

Inverting this equation gives

$$M^{-1} = \left[\begin{array}{c:c:c} U^{-1} & -\,U^{-1}(V'_a + V'_s S_a) & 0\\ \hdashline 0 & I & 0\\ \hdashline 0 & 0 & I\end{array}\right] \qquad (984)$$

The formulation of Section XV maps the covariance matrix for the parameter vector q, which contains the solve-for, consider, and exactly constrained parameters. The submatrix of the mapped covariance matrix which corresponds to the solve-for and consider parameters is identical to the covariance matrix for solve-for and consider parameters mapped with the formulation of this section. The constraints are used in Section XV to compute the rows and columns of the injection covariance matrix which correspond to the exactly constrained parameters; in this section, they are used to combine partial derivatives in the mapping matrix.

3. ***Multiplication of matrices.*** This section shows how to form $\Gamma^{1/2}$ from Eq. (967), Γ from Eq. (968), and $\Gamma^{-1/2}$ from Eq. (971) as products of partitioned matrices. The purpose of partitioning is to take advantage of the large number of null and identity matrices in M and M^{-1}.

In order to compute $\Gamma^{1/2}$ from Eq. (967), partition the columns of the matrix S_C according to Eq. (959):

$$S_C \equiv \underbrace{[n_6}_{6}\,|\,\underbrace{n_a}_{a}\,|\,\underbrace{n_b]}_{b} \tag{985}$$

where a is the number of dynamic parameters in **a** and b is the number of observational parameters in **b**. Also, in Eq. (983), define

$$V_A \equiv V_a' + V_s' S_a \tag{986}$$

Then, substituting Eqs. (983), (985), and (986) into Eq. (967) gives

$$\Gamma^{1/2} = S_C M^T = [n_6\,|\,n_a\,|\,n_b]\left[\begin{array}{c|c|c} U^T & 0 & 0 \\ \hline V_A^T & I & 0 \\ \hline 0 & 0 & I \end{array}\right] = [n_6\,|\,n_a]\left[\begin{array}{c|c|c} U^T & 0 & 0 \\ \hline V_A^T & I & 0 \end{array}\right] + [n_b]\,[0\,|\,0\,|\,I] = \Bigg[\underbrace{[n_6\,|\,n_a]\left[\begin{array}{c} U^T \\ \hline V_A^T \end{array}\right]}_{6}\,\Bigg|\,\underbrace{n_a}_{a}\,\Bigg|\,\underbrace{n_b}_{b}\Bigg] \tag{987}$$

After computing $\Gamma^{1/2}$ from the last form of Eq. (987), partition as

$$\Gamma^{1/2} = [\underbrace{A}_{6}\,|\,\underbrace{B}_{q'}] \tag{988}$$

where $q' = a + b$. Substituting Eq. (988) into Eq. (968) gives

$$\Gamma = \left[\begin{array}{c|c} A^T A & A^T B \\ \hline (A^T B)^T & B^T B \end{array}\right] \tag{989}$$

This is the covariance matrix for

$$\mathbf{q}^R = \left[\begin{array}{c} \mathbf{X}^R \\ \hline \mathbf{q}' \end{array}\right] \tag{990}$$

where

$$\mathbf{q}' = \left[\begin{array}{c} \mathbf{a} \\ \hline \mathbf{b} \end{array}\right] \tag{991}$$

The rows and columns of Eq. (989) are partitioned according to Eq. (990). The covariance matrix for $\mathbf{X}^R$ is $A^T A$, the cross-covariance matrix for $\mathbf{X}^R$ and $\mathbf{q}'$ is $A^T B$, and the covariance matrix for $\mathbf{q}'$ is $B^T B$. However, this latter submatrix of Eq. (989) need not be computed, since *it is identical to the corresponding submatrix of the injection covariance matrix.*

In order to compute $\Gamma^{-1/2}$ from Eq. (971), partition the columns of $(S_C^{-1})^T$ according to Eq. (959):

$$(S_C^{-1})^T \equiv \underbrace{[d_6}_{6}\,|\,\underbrace{d_a}_{a}\,|\,\underbrace{d_b]}_{b} \tag{992}$$

Substituting Eqs. (984), (986), and (992) into Eq. (971) gives

$$\begin{aligned} \Gamma^{-1/2} = (S_C^{-1})^T M^{-1} &= [d_6\,|\,d_a\,|\,d_b]\left[\begin{array}{c|c|c} U^{-1} & -U^{-1} V_A & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right] \\ &= [d_6]\,[U^{-1}\,|\,-U^{-1} V_A\,|\,0] \\ &\quad + [d_a]\,[0\,|\,I\,|\,0] + [d_b]\,[0\,|\,0\,|\,I] \\ &= [d_6 U^{-1}\,|\,-d_6 U^{-1} V_A + d_a\,|\,d_b] \end{aligned} \tag{993}$$

The mapped covariance matrix Γ computed from Eq. (989) has its rows and columns ordered according to the ordering of parameters in Eq. (960). If Γ is used as *a priori* information for processing a new batch of data, i.e., if it is used as $\tilde{\Gamma}_q$, the rows and columns must be re-ordered according to the ordering of parameters in Eq. (857) for the new batch of data. Formally, this re-ordering is obtained by premultiplying Γ by T^T and postmultiplying it by T. After re-ordering, the submatrices $\tilde{\Gamma}_x$, $\tilde{\Gamma}_{xy}$, and $\tilde{\Gamma}_y$ can be extracted.

Similarly, $\Gamma^{-1/2}$ computed from Eq. (993) is the square root of the inverse of the mapped covariance matrix whose rows and columns are ordered according to the ordering of parameters in Eq. (960). If $\Gamma^{-1/2}$ is used as *a priori* information for processing a new batch of data, i.e., if it is used as $\tilde{\Gamma}_q^{-1/2}$ in Eq. (867), it must be postmultiplied by T. This changes the ordering of the columns of $\Gamma^{-1/2}$ from the ordering of the parameters in Eq. (960) to the ordering of parameters in Eq. (857) for the new batch of data.

Glossary

The meaning of the symbols used frequently throughout the text are given below. In order to prevent the notation from becoming excessively complex, some of the symbols have more than one meaning; the correct meaning can easily be determined from the context. The symbols are also defined in the text. There are many localized departures from the meaning of the symbols given here.

Subscripts and Superscripts

E	earth
M	moon
B	earth–moon barycenter
P	planet
S	sun
C	center of integration for spacecraft trajectory ($C = E, M, S,$ or P)
S/C	spacecraft
1	transmitter (transmitting station on earth)
2	spacecraft (a free spacecraft or a landed spacecraft on a planet or the moon)
3	receiver (receiving station on earth)
t_1	transmission time at point 1
t_3	reflection time or transmission time at point 2
t_2	reception time at point 3

Position and Velocity Vectors

In the following, a "1950.0" position vector has rectangular components referred to the mean earth equator and equinox of 1950.0.

$\mathbf{r}_i^j, \mathbf{r}_{ji}$	1950.0 position vector of point i relative to point j
$\mathbf{r}_i$	1950.0 position vector of point i relative to the sun S. That is, $\mathbf{r}_i = \mathbf{r}_i^S$.
$\mathbf{r}$	1950.0 position vector of spacecraft relative to the center of integration C
$\mathbf{r}_b$	body-fixed position vector of tracking station, landed spacecraft, or free spacecraft, with rectangular components referred to the equator and prime meridian
$\mathbf{r}'$	body-fixed position vector of spacecraft with rectangular components referred to the up–east–north coordinate system
$\mathbf{r}_{50}$	1950.0 position vector of tracking station or landed spacecraft relative to body on which located

For any of the position vectors above, $\mathbf{r} \rightarrow \dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}$, where the dots denote differentiation with respect to ephemeris time.

r, r_i, r_{ji}	magnitudes of $\mathbf{r}$ or $\mathbf{r}_b$, $\mathbf{r}_i$, and $\mathbf{r}_i^j = \mathbf{r}_{ji}$, respectively
$\dot{s}_i, \dot{s}_{ij}$	magnitudes of $\dot{\mathbf{r}}_i$ and $\dot{\mathbf{r}}_{ij}$, respectively
$x, y, z,$ $\dot{x}, \dot{y}, \dot{z},$	rectangular components of position vector and velocity vector (may have same indices as vectors)

$$\mathbf{X} = \begin{bmatrix} \mathbf{r} \\ \hdashline \dot{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \text{state vector (all may have indices)}$$

In the relativistic 1-body problem (Section II),

r, θ, ϕ	spherical coordinates (see Fig. 1) relative to the body (the sun in all DPODP applications)
$\mathbf{r}, \dot{\mathbf{r}}$	position and velocity vectors relative to the body with rectangular components referred to a nonrotating coordinate system (1950.0 components in DPODP applications)
$\dot{s}$	magnitude of $\dot{\mathbf{r}}$

In the relativistic n-body problem (Section II),

$\mathbf{r}_i, \dot{\mathbf{r}}_i$	position and velocity vectors of point i relative to the barycenter of the solar system, with rectangular components referred to a nonrotating coordinate system

Station Location Parameters

r, ϕ, λ	body-centered radius, latitude, and longitude (measured east from prime meridian) of tracking station, landed spacecraft, or free spacecraft

u, v	distance from spin axis and height above equator of a tracking station or landed spacecraft

For a tracking station, a subscript 0 refers the values above to the mean pole, equator, and prime meridian of 1903.0. Otherwise, the quantities are referred to the true pole, equator, and prime meridian of date.

ϕ_g	geodetic latitude of tracking station

Physical Constants

μ_i	gravitational constant of body i, km^3/s^2 $= Gm_i$, where $G =$ universal constant of gravitation and m_i is the mass of body i
μ	$\dfrac{\mu_E}{\mu_M}$ = ratio of mass of earth to mass of moon
μ	gravitational constant in the relativistic 1-body problem, km^3/s^2 (Section II and Appendix C)
A_E	the number of kilometers per astronomical unit AU = scaling factor for heliocentric ephemerides of the planets and earth–moon barycenter
R_E	the number of kilometers per fictitious earth radius = scaling factor for geocentric lunar ephemeris
$\mathbf{E}$	osculating orbital elements for the heliocentric ephemeris of a planet or the earth–moon barycenter or for the geocentric lunar ephemeris
$\Delta\mathbf{E}$	estimated correction to $\mathbf{E}$
c	speed of light, km/s

Time

ET	ephemeris time = coordinate time t of general relativity, the independent variable for the ephemerides
A1	atomic time derived from oscillations of a cesium atomic clock. One A1 second is 9,192,631,770 cycles of cesium.
UTC	broadcast universal time
UT1	observed universal time, corrected for polar motion
ST	station time = time derived from an atomic clock at each tracking station

An epoch is expressed as double-precision seconds past January 1, 1950, 0^h and is denoted as $t(i)$ or i, where i is the symbol for the time scale (ET, A1, UTC, UT1, or ST). The symbol t indicates (1) ephemeris time or (2) time in any time scale. The epoch $t(i)$ or t may be subscripted as indicated under "Subscripts and Superscripts".

τ, τ^*	proper time recorded on the observer's atomic clock. The length of the τ second is chosen so that at zero Newtonian potential and zero barycentric velocity, $d\tau = dt$ (ephemeris time). The length of the τ^* second is chosen so that τ^* on earth agrees on the average with ephemeris time.
T_U	number of Julian centuries of 36,525 days of UT1 elapsed since January 0, 1900, 12^h UT1
T	number of Julian centuries of 36,525 ephemeris days elapsed since January 0, 1900, 12^h ET
JD	Julian date
$a, b, c, d,$ e, f, g, h	polynomial coefficients for time transformations

$$\mathrm{UTC} - \mathrm{ST} = a + bt + ct^2$$

$$\mathrm{A1} - \mathrm{UTC} = d + et$$

$$\mathrm{A1} - \mathrm{UT1} = f + gt + ht^2$$

The polynomial coefficients are specified by time block and t is seconds past the start of the time block. It is evaluated with one of the two times related by the transformation.

ΔT_{1958}	the constant part of the (ET − A1) time transformation = 32.15 s (adopted)
f_{cesium}	conversion factor from cycles obtained from a cesium atomic clock to seconds of A1 time = 9,192,631,770 cycles per second
$f_{\mathrm{cesium}} + \Delta f_{\mathrm{cesium}}$	cycles of cesium atomic clock per ephemeris second (average). The parameter $\Delta f_{\mathrm{cesium}}$ may be estimated by the DPODP; its current nominal value is zero.

Glossary (contd)

Miscellaneous

θ true sidereal time = Greenwich hour angle of true equinox of date

$\dot{\theta}$ derivative of θ with respect to ephemeris time

A precession matrix, transforming from coordinates referred to the mean earth equator and equinox of 1950.0 to coordinates referred to the mean earth equator and equinox of date

N nutation matrix, transforming from coordinates referred to the mean earth equator and equinox of date to coordinates referred to the true earth equator and equinox of date

ϕ Newtonian potential (positive)

$$\gamma = \frac{1 + \omega}{2 + \omega}$$

where ω = the coupling constant of the scalar field, a free parameter of the Brans–Dicke theory of gravitation

$\equiv$ defined equal to

Appendix A

Derivation of *n*-Body Relativistic Equations of Motion

This appendix gives two derivations of Eq. (54), the n-body equations of motion in the Brans–Dicke theory. The derivations also apply for the corresponding equations of general relativity (Eq. 35) if the parameter γ of the Brans–Dicke theory is set equal to unity. In Section I, the equations are derived from the n-body Lagrangian (Eq. 53), while in Section II they are derived from the n-body metric tensor (Eqs. 43–48 and 30–31).

I. Derivation From *n*-Body Lagrangian

The n-body Lagrangian L of the Brans–Dicke theory (Eq. 52) may be expressed as Eq. (53), where the index i refers to the particular body i whose motion is desired and the indices j and k refer to the n other bodies. The n-body equations of motion are the Euler–Lagrange equations:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0 \qquad x \rightarrow y, z \tag{A-1}$$

In Eq. (53),

$$\dot{s}_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \tag{A-2}$$

and

$$\frac{\partial \dot{s}_i^2}{\partial \dot{x}_i} = 2\,\dot{x}_i \tag{A-3}$$

Differentiating L (Eq. 53) with respect to $\dot{x}_i$ gives

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}_i} = {} & \left(1 + \frac{1}{2c^2}\dot{s}_i^2 + \frac{1+2\gamma}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\right)\dot{x}_i \\ & - \frac{3+4\gamma}{2c^2}\sum_{j\neq i}\frac{\mu_j\,\dot{x}_j}{r_{ij}} \\ & - \frac{1}{2c^2}\sum_{j\neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^3}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j]\end{aligned} \tag{A-4}$$

In order to differentiate Eq. (A-4) with respect to coordinate time t for use in Eq. (A-1), the derivatives of $\dot{s}_i^2$ and r_{ij} with respect to t are required:

$$\dot{s}_i^2 = \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \tag{A-5}$$

$$\frac{d\dot{s}_i^2}{dt} = 2\dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i \tag{A-6}$$

Since $\dot{s}_i^2$ appears in a term of order $1/c^2$ in Eq. (A-4), the Newtonian expression for $\ddot{\mathbf{r}}_i$ may be used:

$$\ddot{\mathbf{r}}_i = \sum_{j\neq i}\frac{\mu_j\,(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \tag{A-7}$$

Substituting Eq. (A-7) into Eq. (A-6) gives

$$\frac{d\dot{s}_i^2}{dt} = 2\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\,(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i \tag{A-8}$$

The coordinate distance r_{ij} is given by

$$r_{ij}^2 = (\mathbf{r}_j - \mathbf{r}_i)\cdot(\mathbf{r}_j - \mathbf{r}_i) \tag{A-9}$$

Hence,

$$\dot{r}_{ij} = \frac{1}{r_{ij}}(\mathbf{r}_j - \mathbf{r}_i)\cdot(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i) \tag{A-10}$$

Using Eqs. (A-8) and (A-10), a straightforward differentiation of Eq. (A-4) with respect to t is given by

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = {} & \left(1 + \frac{1}{2c^2}\dot{s}_i^2 + \frac{1+2\gamma}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\right)\ddot{x}_i + \frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i]\,\dot{x}_i - \frac{1+2\gamma}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i)]\,\dot{x}_i \\ & - \frac{3+4\gamma}{2c^2}\sum_{j\neq i}\frac{\mu_j\,\ddot{x}_j}{r_{ij}} + \frac{3+4\gamma}{2c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i)]\,\dot{x}_j - \frac{1}{2c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j]\,(\dot{x}_j - \dot{x}_i) \\ & + \frac{3}{2c^2}\sum_{j\neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^5}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j]\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i)] - \frac{1}{2c^2}\sum_{j\neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^3}\,[(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i)\cdot\dot{\mathbf{r}}_j] \\ & - \frac{1}{2c^2}\sum_{j\neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^3}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\ddot{\mathbf{r}}_j]\end{aligned} \tag{A-11}$$

Evaluating $\ddot{x}_i$ in the $1/c^2$ terms from Eq. (A-7), combining like terms, and changing the sign of the equation gives

$$
\begin{aligned}
-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = & -\ddot{x}_i + \frac{1}{c^2}\sum_{j \neq i}\frac{\mu_j}{r_{ij}^3}\left\{[\mathbf{r}_i - \mathbf{r}_j]\cdot\left[(2+2\gamma)\,\dot{\mathbf{r}}_i - \left(\frac{1}{2}+2\gamma\right)\dot{\mathbf{r}}_j\right]\right\}\dot{x}_i \\
& -\frac{1}{c^2}\sum_{j \neq i}\frac{\mu_j}{r_{ij}^3}\left\{[\mathbf{r}_i - \mathbf{r}_j]\cdot\left[\left(\frac{3+4\gamma}{2}\right)\dot{\mathbf{r}}_i - (1+2\gamma)\,\dot{\mathbf{r}}_j\right]\right\}\dot{x}_j \\
& +\sum_{i \neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^3}\left\{-\frac{(1+2\gamma)}{c^2}\sum_{l \neq i}\frac{\mu_l}{r_{il}} - \frac{1}{2}\left(\frac{\dot{s}_i}{c}\right)^2 + \frac{1}{2}\left(\frac{\dot{s}_j}{c}\right)^2\right. \\
& -\frac{1}{2c^2}\,\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j - \frac{3}{2c^2}\left[\frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot\dot{\mathbf{r}}_j}{r_{ij}}\right]^2 + \frac{1}{2c^2}\,(\mathbf{r}_j - \mathbf{r}_i)\cdot\ddot{\mathbf{r}}_j \\
& \left. +\frac{3}{2c^2\,r_{ij}^2}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j]\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i]\right\} + \frac{3+4\gamma}{2c^2}\sum_{j \neq i}\frac{\mu_j\,\ddot{x}_j}{r_{ij}}
\end{aligned}
\qquad \text{(A-12)}
$$

In order to differentiate L with respect to x_i, the following subpartial derivatives are required:

$$
r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \qquad \text{(A-13)}
$$

$$
\frac{\partial r_{ij}}{\partial x_i} = \frac{x_i - x_j}{r_{ij}} \qquad \text{(A-14)}
$$

$$
\frac{\partial}{\partial x_i}\left(\frac{1}{r_{ij}}\right) = \frac{x_j - x_i}{r_{ij}^3} \qquad \text{(A-15)}
$$

$$
\frac{\partial}{\partial x_i}\left(\frac{1}{r_{ij}^2}\right) = \frac{2\,(x_j - x_i)}{r_{ij}^4} \qquad \text{(A-16)}
$$

$$
\frac{\partial}{\partial x_i}\left(\frac{1}{r_{ij}^3}\right) = \frac{3\,(x_j - x_i)}{r_{ij}^5} \qquad \text{(A-17)}
$$

Using these equations, a straightforward differentiation of L (Eq. 53) with respect to x_i gives

$$
\begin{aligned}
\frac{\partial L}{\partial x_i} = & \frac{1+2\gamma}{2c^2}\sum_{j \neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^3}\,(\dot{s}_i^2 + \dot{s}_j^2) - \frac{3+4\gamma}{2c^2}\sum_{j \neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^3}\,(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j) \\
& +\frac{1}{2c^2}\sum_{j \neq i}\frac{\mu_j}{r_{ij}^3}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i]\,\dot{x}_j + \frac{1}{2c^2}\sum_{j \neq i}\frac{\mu_j}{r_{ij}^3}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j]\,\dot{x}_i \\
& -\frac{3}{2c^2}\sum_{j \neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^5}\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j]\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i] + \sum_{j \neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^3} - \frac{1}{c^2}\sum_{j \neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^3}\left(\frac{\mu_i + \mu_j}{r_{ij}}\right) \\
& -\frac{1}{c^2}\sum_{j \neq i}\frac{\mu_j\,(x_j - x_i)}{r_{ij}^3}\sum_{k \neq j,\,i}\frac{\mu_k}{r_{jk}} - \frac{1}{2c^2}\sum_{j \neq i}\sum_{k \neq j,\,i}\frac{\mu_j\,\mu_k\,(x_j - x_i)}{r_{ij}^3\,r_{ik}} - \frac{1}{2c^2}\sum_{j \neq i}\sum_{k \neq j,\,i}\frac{\mu_j\,\mu_k\,(x_k - x_i)}{r_{ij}\,r_{ik}^3}
\end{aligned}
\qquad \text{(A-18)}
$$

Note that interchanging the j and k indices in the last term of Eq. (A-18) gives the next-to-last term. Hence the last term may be deleted and the next-to-last term doubled. With this change, a combination of like terms in Eq. (A-18) gives

$$\begin{aligned}\frac{\partial L}{\partial x_i} = \sum_{j\neq i} \frac{\mu_j (x_j - x_i)}{r_{ij}^3} \Bigg\{ 1 - \frac{1}{c^2}\sum_{l\neq i}\frac{\mu_l}{r_{il}} - \frac{1}{c^2}\sum_{k\neq j}\frac{\mu_k}{r_{jk}} + \frac{1+2\gamma}{2c^2}(\mathring{s}_i^2 + \mathring{s}_j^2) - \frac{3+4\gamma}{2c^2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j \\ - \frac{3}{2c^2 r_{ij}^2}[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j]\,[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_i] \Bigg\} + \frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\left\{[\mathbf{r}_i - \mathbf{r}_j]\cdot\left[-\frac{1}{2}\dot{\mathbf{r}}_j\right]\right\}\mathring{x}_i \\ - \frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\left\{[\mathbf{r}_i - \mathbf{r}_j]\cdot\left[\frac{1}{2}\dot{\mathbf{r}}_i\right]\right\}\mathring{x}_j \qquad \text{(A-19)}\end{aligned}$$

Adding Eqs. (A-19) and (A-12) and setting the result equal to zero as indicated in Eq. (A-1) gives Eq. (54) for the acceleration of one of the bodies relative to the barycenter of the system of n bodies, with rectangular components referred to a nonrotating coordinate system. The acceleration of body j appearing in the $1/c^2$ terms is evaluated with the Newtonian expression (Eq. 31), and the summations over $k \neq j$ in Eqs. (31) and (54) include body i.

II. Derivation From *n*-body Metric

The components of the n-body metric tensor in the Brans–Dicke theory are given by Eqs. (43–48) and (30–31). Substituting these components into Eq. (34) gives the following expression for L^2 (where $L = ds/dt$):

$$\begin{aligned}L^2 = c^2 - 2\sum_{j\neq i}\frac{\mu_j}{r_{ij}} + \frac{2}{c^2}\left[\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\right]^2 + \frac{2}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\sum_{k\neq j}\frac{\mu_k}{r_{jk}} \\ - \frac{1}{c^2}\sum_{j\neq i}\mu_j\left\{(2+2\gamma)\frac{\mathring{s}_j^2}{r_{ij}} - \frac{[(\mathbf{r}_j - \mathbf{r}_i)\cdot\dot{\mathbf{r}}_j]^2}{r_{ij}^3} + \frac{(\mathbf{r}_j - \mathbf{r}_i)\cdot\ddot{\mathbf{r}}_j}{r_{ij}}\right\} \\ - \left(1 + \frac{2\gamma}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\right)(\mathring{x}_i^2 + \mathring{y}_i^2 + \mathring{z}_i^2) + \frac{4+4\gamma}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j) \qquad \text{(A-20)}\end{aligned}$$

The equations of motion are given by Eqs. (18) and (19), a form of the Euler–Lagrange equations, which are repeated here with x and $\mathring{x}$ replaced by x_i and $\mathring{x}_i$, respectively:

$$\frac{d}{dt}\left(L\frac{\partial L}{\partial \mathring{x}_i}\right) - \left(\frac{\mathring{L}}{L}\right)\left(L\frac{\partial L}{\partial \mathring{x}_i}\right) - \left(L\frac{\partial L}{\partial x_i}\right) = 0 \qquad x \rightarrow y, z \qquad \text{(A-21)}$$

where

$$\frac{\mathring{L}}{L} = \frac{L\mathring{L}}{L^2} \approx \frac{L\mathring{L}}{c^2} \qquad \text{(A-22)}$$

The quantity $L\mathring{L}$ is obtained by differentiating a simplified expression for L^2 containing terms to order $1/c^0$ only. The derivative $L\,\partial L/\partial x_i$ is obtained from Eq. (A-20) by considering the Newtonian potential at each perturbing body j (in term 4 of Eq. A-20) and the acceleration of body j, computed from Eq. (31), to be functions of coordinate time t only, as indicated after Eq. (34).

Differentiating Eq. (A-20) with respect to $\dot{x}_i$ gives

$$L\frac{\partial L}{\partial \dot{x}_i} = -\left(1 + \frac{2\gamma}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}}\right)\dot{x}_i + \frac{2+2\gamma}{c^2}\sum_{j\neq i}\frac{\mu_j \dot{x}_j}{r_{ij}} \qquad \text{(A-23)}$$

Differentiating with respect to coordinate time t using Eq. (A-10) and evaluating $\ddot{x}_i$ in a $1/c^2$ term with Eq. (A-7) gives

$$\frac{d}{dt}\left(L\frac{\partial L}{\partial \dot{x}_i}\right) = -\ddot{x}_i + \sum_{j\neq i}\frac{\mu_j(x_j - x_i)}{r_{ij}^3}\left\{-\frac{2\gamma}{c^2}\sum_{l\neq i}\frac{\mu_l}{r_{il}}\right\} + \frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\{[\mathbf{r}_i - \mathbf{r}_j]\cdot[2\gamma\dot{\mathbf{r}}_i - 2\gamma\dot{\mathbf{r}}_j]\}\,\dot{x}_i$$

$$-\frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\{[\mathbf{r}_i - \mathbf{r}_j]\cdot[(2+2\gamma)\dot{\mathbf{r}}_i - (2+2\gamma)\dot{\mathbf{r}}_j]\}\,\dot{x}_j + \frac{2+2\gamma}{c^2}\sum_{j\neq i}\frac{\mu_j\ddot{x}_j}{r_{ij}} \qquad \text{(A-24)}$$

where $\ddot{x}_j$ is given by Eq. (31) with the summation over $k \neq j$ including body i.

To terms of order $1/c^0$, L^2 is given by

$$L^2 \approx c^2 - 2\sum_{j\neq i}\frac{\mu_j}{r_{ij}} - \dot{s}_i^2 \qquad \text{(A-25)}$$

Differentiation with respect to t using Eqs. (A-8) and (A-10) and substitution into Eq. (A-22) gives $\dot{L}/L$ to order $1/c^2$. Substitution of this expression and Eq. (A-23) to order $1/c^0$ into the second term of Eq. (A-21) gives the following expression, containing all terms to order $1/c^2$:

$$-\left(\frac{\dot{L}}{L}\right)\left(L\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\{[\mathbf{r}_i - \mathbf{r}_j]\cdot[2\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j]\}\,\dot{x}_i \qquad \text{(A-26)}$$

Differentiating Eq. (A-20) with respect to x_i using Eqs. (A-15) and (A-17) and considering the Newtonian potential at each perturbing body j and the Newtonian acceleration of each perturbing body j to be functions of coordinate time t only gives

$$-L\frac{\partial L}{\partial x_i} = \sum_{j\neq i}\frac{\mu_j(x_j - x_i)}{r_{ij}^3}\left\{1 - \frac{2}{c^2}\sum_{l\neq i}\frac{\mu_l}{r_{il}} - \frac{1}{c^2}\sum_{k\neq j}\frac{\mu_k}{r_{jk}} + \gamma\left(\frac{\dot{s}_i}{c}\right)^2 + (1+\gamma)\left(\frac{\dot{s}_j}{c}\right)^2 - \frac{2(1+\gamma)}{c^2}\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_j\right.$$

$$\left.-\frac{3}{2c^2}\left[\frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot\dot{\mathbf{r}}_j}{r_{ij}}\right]^2 + \frac{1}{2c^2}(\mathbf{r}_j - \mathbf{r}_i)\cdot\ddot{\mathbf{r}}_j\right\} - \frac{1}{c^2}\sum_{j\neq i}\frac{\mu_j}{r_{ij}^3}\,[(\mathbf{r}_i - \mathbf{r}_j)\cdot\dot{\mathbf{r}}_j]\,\dot{x}_j - \frac{1}{2c^2}\sum_{j\neq i}\frac{\mu_j\ddot{x}_j}{r_{ij}}$$

$$\text{(A-27)}$$

where $\ddot{\mathbf{r}}_j$ and $\ddot{x}_j$ are obtained from Eq. (31) and the summation over $k \neq j$ in Eq. (31) and in Eq. (A-27) includes body i. Adding Eqs. (A-24), (A-26), and (A-27) and setting equal to zero according to Eq. (A-21) gives Eq. (54) for the acceleration of one of the bodies relative to the barycenter of the system of n bodies.

Appendix B

Derivation of ET — A1 Time Transformation

I. Introduction

This appendix gives the derivation of the periodic relativity terms in the ET − A1 time transformation (Eq. 65). The nomenclature used in this appendix and the numerical values of the constants used are given in Section II. The expression for ET − A1 includes all terms which affect "differenced-range" doppler (see Section XI) by more than 2×10^{-7} m/s per AU of range to the spacecraft. Using this criterion, minimum values for the coefficients of the retained daily, monthly, and annual terms of $d\text{A1}/d\text{ET}$ are generated in Section III. The terms of $d\text{A1}/d\text{ET}$ which must be retained are identified in Section IV. Expressions for the magnitude and orientation of the velocity vector obtained from the elliptical orbit of the earth–moon barycenter relative to the sun, which are required in the derivation of Eq. (65), are derived in Section V. The derivations of the retained terms of $d\text{A1}/d\text{ET}$ are given in Section VI. In Section VII, these terms are summed and integrated to give the final expression for ET–A1.

II. Nomenclature and Numerical Values of Constants

In the following, a dot indicates the derivative of the quantity with respect to coordinate time t (synonymous with ephemeris time ET).

μ_S, μ_E, μ_M = gravitational constants for sun, earth, and moon, respectively:

$$\mu_S = 1.32712499 \times 10^{11}\,\text{km}^3/\text{s}^2$$

$$\mu_E = 398{,}601.2\,\text{km}^3/\text{s}^2$$

$$\mu_M = 4{,}902.78\,\text{km}^3/\text{s}^2$$

$$\mu = \mu_E/\mu_M = 81.301$$

c = speed of light = 299,792.5 km/s

A_E = the number of kilometers per AU

= 149,597,893.0 km/AU

a = semimajor axis of heliocentric orbit of earth–moon barycenter

= $1.00000023\,A_E$ = 149,597,927 km ≈ 149,597,900 km

e = eccentricity of heliocentric orbit of earth–moon barycenter

= 0.01672. From 1950 to 2000, the last figure changes from 3 to 1.

M = mean anomaly of heliocentric orbit of earth–moon barycenter (Eq. 67)

E = eccentric anomaly of heliocentric orbit of earth–moon barycenter

v = true anomaly of heliocentric orbit of earth–moon barycenter

L = geometric mean longitude of the sun, referred to the mean equinox and ecliptic of date (Eq. 68).

l = true longitude of the sun, referred to the mean equinox and ecliptic of date

r = radial coordinate of earth–moon barycenter from sun

$\dot{s}_S$ = velocity of earth–moon barycenter relative to sun

$\dot{s}_c$ = circular orbit velocity of earth–moon barycenter relative to sun

$$= \left(\frac{\mu_S + \mu_E + \mu_M}{a}\right)^{1/2} = 29.784741\,\text{km/s}$$

γ = elevation angle of heliocentric velocity vector of earth–moon barycenter from the transverse direction

= $\tan^{-1}[\dot{r}/(r\dot{v})]$

ϵ = mean obliquity of the ecliptic

$\cos\epsilon = 0.91746$. From 1950 to 2000, the last figure changes from 4 to 8.

a_M = semimajor axis of geocentric orbit of the moon

= 384,399.285 km. The term a_M is computed from the lunar constraint, Eqs. (107) and (108), using the values of μ_E and μ_M given above

e_M = eccentricity of geocentric orbit of moon

= 0.0549 (not used in expression for ET–A1)

☾ = mean longitude of the moon, measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit, and then along the orbit

$D = \text{☾} - L$ = mean elongation of the moon from the sun (Eq. 69)

$\dot{s}_M$ = circular orbit velocity of moon

$$= \left(\frac{\mu_E + \mu_M}{a_M}\right)^{1/2} = 1.024549\,\text{km/s}$$

☊ = longitude of the mean ascending node of the lunar orbit on the ecliptic, measured from the mean equinox of date

i = inclination of the lunar orbit to the ecliptic plane

$\cos i = 0.99597$

θ_M = mean sidereal time = Greenwich hour angle of mean equinox of date

$\dot{\theta}_M$ = mean sidereal rate (Eq. 273)

$= 0.729212 \times 10^{-4}$ rad/s

UT = universal time UT1, hours past midnight, converted to radians (computed from Eq. 66)

$$\dot{\text{UT}} = \frac{d\text{UT}}{dt} = \frac{2\pi\,\text{rad}}{86{,}400\,\text{s}} = 0.7272205 \times 10^{-4}\,\text{rad/s}$$

(To this accuracy, this UT derivative equals the desired ET derivative.)

λ = east longitude of tracking station at which A1 atomic clock is located

u = distance of tracking station at which A1 atomic clock is located from earth's spin axis, km

v = height of tracking station above earth's equator, km

$\dot{s}_{STN}$ = geocentric velocity of tracking station

$\mathbf{r}_a^b, \dot{\mathbf{r}}_a^b$ = position and velocity vectors of point a relative to point b

where the indices a and b may be

B = earth–moon barycenter

E = earth

M = moon

STN = tracking station on earth (location of A1 atomic clock)

S = sun

$\dot{s}_a^b$ = magnitude of $\dot{\mathbf{r}}_a^b$

ϕ_a = Newtonian potential at point a due to the sun

ϕ = Newtonian potential at tracking station due to the sun

$\dot{s}$ = heliocentric velocity of tracking station

The angles M, L, and D are computed from Eqs. (67), (68), and (69). These linear representations are tangent to the quadratic or cubic expressions of Ref. 25, pp. 98 and 107, for $T = 0.7$ Julian centuries past January 0, 1900, 12^h ET.

The values of the gravitational constants, c, and A_E were obtained from Ref. 29; the remaining constants were obtained from Ref. 25.

III. Criteria for Retention of Periodic Variations in ϕ and $\dot{s}^2$ in Expression for $dA1/dET$

An accurate expression for the ET − A1 time transformation is required to implement the forthcoming program change specified in Section XI, namely the computation of doppler observables from differenced range observables.

In the derivation of the expression for ET − A1, all terms affecting "differenced-range" doppler by more than 2×10^{-7} m/s per AU of distance to the spacecraft were retained. Several terms of this magnitude were neglected and the resulting error in differenced-range doppler is about 10^{-6} m/s/AU or 10^{-5} m/s for a spacecraft range of 10 AU. The figure of 10^{-5} m/s is the accuracy of the current doppler observable.

The contribution to differenced-range doppler (DRD) from each term of ET − A1 satisfies the inequality[46]

$$|\delta DRD|\,(\text{m/s}) \leq \left|\frac{d^2}{dt^2}(\text{ET} - \text{A1})\right| \rho \qquad \text{(B-1)}$$

The absolute value of the contribution, $|\delta DRD|$, is expressed in 1-way m/s, and (ET − A1) represents a periodic relativity term of ET − A1. Since the range ρ to the spacecraft is the range in AU times 1.5×10^{11} m/AU,

$$|\delta DRD|\,(\text{m/s/AU}) \leq \left|\frac{d^2}{dt^2}(\text{ET} - \text{A1})\right| (1.5 \times 10^{11}\,\text{m/AU}) \qquad \text{(B-2)}$$

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[46] See Subsection XI-C-2-a.

Since all retained terms of ET − A1 contribute more than 2×10^{-7} m/s/AU, they satisfy the inequality

$$\left| \frac{d^2}{dt^2} (\mathrm{ET} - \mathrm{A1}) \right| > 1.33 \times 10^{-18}/\mathrm{s} \qquad \text{(B-3)}$$

or

$$\left| \frac{d}{dt} \left(\frac{d\mathrm{A1}}{d\mathrm{ET}} \right) \right| > 1.33 \times 10^{-18}/\mathrm{s} \qquad \text{(B-4)}$$

where $d\mathrm{A1}/d\mathrm{ET}$ represents a periodic relativity term of $d\mathrm{A1}/d\mathrm{ET}$.

The expression for $d\mathrm{A1}/d\mathrm{ET}$ is Eq. (64), repeated here:

$$\frac{d\mathrm{A1}}{d\mathrm{ET}} = 1 - \frac{\phi - \bar{\phi}}{c^2} - \frac{1}{2} \frac{\dot{s}^2 - \bar{\dot{s}}^2}{c^2} + \frac{\Delta f_{\mathrm{cesium}}}{f_{\mathrm{cesium}}} \qquad \text{(B-5)}$$

Thus, the periodic variations in ϕ and $\dot{s}^2$ retained in Eq. (B-5) must satisfy

$$\left| \frac{d\phi}{dt} \right| > 1.20 \times 10^{-7}\, \mathrm{km}^2/\mathrm{s}^3 \qquad \text{(B-6)}$$

$$\left| \frac{d}{dt} \left(\frac{1}{2} \dot{s}^2 \right) \right| > 1.20 \times 10^{-7}\, \mathrm{km}^2/\mathrm{s}^3 \qquad \text{(B-7)}$$

If the periodic variations in ϕ and $(1/2)\,\dot{s}^2$ have a magnitude of $M\,(\mathrm{km}^2/\mathrm{s}^2)$ and a period P, the inequalities of Eqs. (B-6) and (B-7) become

$$\frac{2\pi}{P} |M| > 1.20 \times 10^{-7}\, \mathrm{km}^2/\mathrm{s}^3 \qquad \text{(B-8)}$$

The variations in ϕ and $\dot{s}^2$ have periods of about 1 day, 1 synodic month, and 1 year. Let the retained terms satisfy

$$|M| > a \qquad \text{(B-9)}$$

$$\left| \frac{M}{c^2} \right| > b \qquad \text{(B-10)}$$

The values of a and b for each of the above-mentioned periods are shown in Table B-1.

It will be seen that the daily variations in $d\mathrm{A1}/d\mathrm{ET}$ are proportional to u, the distance of the A1 atomic clock from the earth's spin axis in kilometers. Since the maximum value of u is 6,378 km, all daily terms of $d\mathrm{A1}/d\mathrm{ET}$ whose coefficient $M/(c^2 u)$ is greater than 2.8×10^{-18}/km should be retained.

Table B-1. Values of *a* and *b*

Period *P*	Value of *a*, km²/s²	Value of *b*, dimensionless
1 day = 0.864×10^5 s	1.6×10^{-3}	1.8×10^{-14}
1 synodic month = 2.55×10^6 s	4.8×10^{-2}	5.4×10^{-13}
1 year = 3.16×10^7 s	0.60	6.7×10^{-12}

IV. Identification of Significant Terms of *d*A1/*d*ET

The terms of $\phi - \bar{\phi}$ which must be retained in Eq. (B-5) may easily be identified by consideration of Eq. (B-6). The potential at the A1 clock on earth due to a specific body j is

$$\phi^j = \frac{\mu_j}{r_j} \qquad \text{(B-11)}$$

where μ_j is the gravitational constant of body j and r_j is the coordinate distance from the A1 clock to body j. The periodic variation in ϕ^j must be retained if

$$\left| \dot{\phi}^j \right| = \left| \frac{\mu_j}{r_j^2} \dot{r}_j \right| > 1.2 \times 10^{-7}\, \mathrm{km}^2/\mathrm{s}^3 \qquad \text{(B-12)}$$

The maximum value of $\dot{\phi}^j$ from a planet is about $10^{-8}\,\mathrm{km}^2/\mathrm{s}^3$. This value is obtained from either Venus or Jupiter and is less than the criterion of $1.2 \times 10^{-7}\,\mathrm{km}^2/\mathrm{s}^3$. Hence, the variation in the potential ϕ due to the planets may be ignored in Eq. (B-5). The peak value of $\dot{\phi}^j$ from the moon is about $2 \times 10^{-8}\,\mathrm{km}^2/\mathrm{s}^3$. Hence the lunar potential may also be ignored in Eq. (B-5).

The solar potential at the A1 clock has an annual, monthly, and daily variation. The coefficient of the monthly term of $\dot{r}_S$ in Eq. (B-12) is about 0.012 km/s, and the monthly component of $\dot{\phi}^S$ has a maximum value of about $7 \times 10^{-8}\,\mathrm{km}^2/\mathrm{s}^3$. Thus, the monthly variation in the solar potential may be ignored. The coefficients of the annual and daily terms of $\dot{r}_S$ are both about 0.5 km/s, and the corresponding values of $\dot{\phi}^S$ are about $3 \times 10^{-6}\,\mathrm{km}^2/\mathrm{s}^2$, which is significant. Hence, in Eq. (B-5), the only significant variations in the Newtonian potential ϕ at the location of the A1 clock on earth are the annual and diurnal variations in the solar potential.

The expression for the square of the heliocentric velocity $\dot{s}$ of the tracking station at which the A1 atomic clock is located, used in Eq. (B-5), is given by

$$\dot{s}^2 = [\dot{\mathbf{r}}_B^S + \dot{\mathbf{r}}_E^B + \dot{\mathbf{r}}_{STN}^E] \cdot [\dot{\mathbf{r}}_B^S + \dot{\mathbf{r}}_E^B + \dot{\mathbf{r}}_{STN}^E] \qquad \text{(B-13)}$$

or

$$\mathring{s}^2 = (\mathring{s}_B^S)^2 + (\mathring{s}_E^B)^2 + (\mathring{s}_{STN}^E)^2$$
$$+ 2\,\mathring{\mathbf{r}}_{STN}^E \cdot \mathring{\mathbf{r}}_B^S + 2\,\mathring{\mathbf{r}}_E^B \cdot \mathring{\mathbf{r}}_B^S + 2\,\mathring{\mathbf{r}}_{STN}^E \cdot \mathring{\mathbf{r}}_E^B \qquad \text{(B-14)}$$

The terms of $\mathring{s}^2 - \overline{\mathring{s}^2}$ which must be retained in Eq. (B-5) consist of the periodic variation in the first term of Eq. (B-14) and the last three terms. Referring to Section III, the value of M for the second term of Eq. (B-14) is one-half of the magnitude of the periodic variation of the term or about $10^{-5}\,\text{km}^2/\text{s}^2$, which is less than the minimum value of $4.8 \times 10^{-2}\,\text{km}^2/\text{s}^2$ for a retained monthly term. The geocentric velocity of a tracking station is an extremely constant quantity, and hence the variation in the third term of Eq. (B-14) is also insignificant.

Section VI will give the derivations of the periodic terms of $d\text{A1}/d\text{ET}$ arising from the following:

(1) The annual variation in ϕ and $\mathring{s}^2$ of the tracking station (term AT)

(2) The daily variation in potential at the tracking station (term DP)

(3) The product of the daily and annual velocity components in $\mathring{s}^2$ (Eq. B-14, term 4) (term DA)

(4) The product of the monthly and annual velocity components in $\mathring{s}^2$ (Eq. B-14, term 5) (term MA)

(5) The product of the daily and monthly velocity components in $\mathring{s}^2$ (Eq. B-14, term 6) (term DM)

The expression for $d\text{A1}/d\text{ET}$ which contains these terms is

$$\frac{d\text{A1}}{d\text{ET}} = 1 + \left(\frac{d\text{A1}}{d\text{ET}}\right)_{AT} + \left(\frac{d\text{A1}}{d\text{ET}}\right)_{DP} + \left(\frac{d\text{A1}}{d\text{ET}}\right)_{DA}$$
$$+ \left(\frac{d\text{A1}}{d\text{ET}}\right)_{MA} + \left(\frac{d\text{A1}}{d\text{ET}}\right)_{DM} + \frac{\Delta f_{\text{cesium}}}{f_{\text{cesium}}} \qquad \text{(B-15)}$$

where

$$\left(\frac{d\text{A1}}{d\text{ET}}\right)_{AT} = -\frac{1}{c^2}\left\{\left[\phi_B + \frac{1}{2}(\mathring{s}_B^S)^2\right] - \overline{\left[\phi_B + \frac{1}{2}(\mathring{s}_B^S)^2\right]}\right\} \qquad \text{(B-16)}$$

$$\left(\frac{d\text{A1}}{d\text{ET}}\right)_{DP} = -\frac{1}{c^2}(\phi_{STN} - \phi_E) \qquad \text{(B-17)}$$

$$\left(\frac{d\text{A1}}{d\text{ET}}\right)_{DA} = -\frac{1}{c^2}(\mathring{\mathbf{r}}_{STN}^E \cdot \mathring{\mathbf{r}}_B^S) \qquad \text{(B-18)}$$

$$\left(\frac{d\text{A1}}{d\text{ET}}\right)_{MA} = -\frac{1}{c^2}(\mathring{\mathbf{r}}_E^B \cdot \mathring{\mathbf{r}}_B^S) \qquad \text{(B-19)}$$

$$\left(\frac{d\text{A1}}{d\text{ET}}\right)_{DM} = -\frac{1}{c^2}(\mathring{\mathbf{r}}_{STN}^E \cdot \mathring{\mathbf{r}}_E^B) \qquad \text{(B-20)}$$

V. Velocity Vector of Earth–Moon Barycenter Orbit

In the derivation of an integrable expression for $d\text{A1}/d\text{ET}$ in Section VI, expressions are required for the magnitude and orientation of the velocity vector from the elliptical orbit of the earth–moon barycenter relative to the sun. These expressions are derived in this section.

The square of the velocity $\mathring{s}_S$ is given by

$$\mathring{s}_S^2 = (\mu_S + \mu_E + \mu_M)\left(\frac{2}{r} - \frac{1}{a}\right) \qquad \text{(B-21)}$$

where

$$r = a(1 - e\cos E) \qquad \text{(B-22)}$$

Dividing by ra gives

$$\frac{1}{r} = \frac{1}{a} + \frac{e}{r}\cos E \qquad \text{(B-23)}$$

Substituting Eq. (B-23) into Eq. (B-21) gives

$$\mathring{s}_S^2 = \frac{\mu_S + \mu_E + \mu_M}{a}\left(1 + \frac{2ae}{r}\cos E\right) \qquad \text{(B-24)}$$

The expression for $\mathring{s}_S$ is obtained by expanding the square root in powers of e, retaining all terms to order e^2. Then, using Eq. (B-23) to eliminate a/r and using trigonometric identities gives

$$\mathring{s}_S = \left(\frac{\mu_S + \mu_E + \mu_M}{a}\right)^{1/2}\left(1 + \frac{1}{4}e^2 + e\cos E + \frac{1}{4}e^2\cos 2E\right) \qquad \text{(B-25)}$$

Since terms of order greater than e^2 are ignored, E is given by

$$E = M + e\sin M + \cdots \qquad \text{(B-26)}$$

Substituting Eq. (B-26) into Eq. (B-25) and retaining terms to order e^2 gives the required expression for $\mathring{s}_S$:

$$\mathring{s}_S = \mathring{s}_c\left(1 - \frac{1}{4}e^2 + e\cos M + \frac{3}{4}e^2\cos 2M\right) \qquad \text{(B-27)}$$

where

$$\mathring{s}_c = \left(\frac{\mu_S + \mu_E + \mu_M}{a}\right)^{1/2} \qquad \text{(B-28)}$$

In Section VI, the orientation of the velocity vector of the sun relative to the earth–moon barycenter is specified by the angle

$$l + 90\,\text{deg} - \gamma$$

An expression will be developed for

$$l - \gamma = L + (v - M) - \gamma \qquad \text{(B-29)}$$

From Ref. 58, p. 120,

$$v - M = 2e\sin M + \frac{5}{4}e^2\sin 2M + \cdots \qquad \text{(B-30)}$$

The elevation angle γ is given by

$$\gamma = \tan^{-1}\left(\frac{\mathring{r}}{r\mathring{v}}\right) \qquad \text{(B-31)}$$

The expressions for $\mathring{r}$ and $r\mathring{v}$ are

$$\mathring{r} = \left(\frac{\mu_S + \mu_E + \mu_M}{p}\right)^{1/2} e\sin v \qquad \text{(B-32)}$$

$$r\mathring{v} = \frac{[(\mu_S + \mu_E + \mu_M)\,p]^{1/2}}{r} \qquad \text{(B-33)}$$

where

$$p = a(1 - e^2) \qquad \text{(B-34)}$$

Thus,

$$\gamma = \tan^{-1}\left(\frac{r}{p}e\sin v\right) \qquad \text{(B-35)}$$

Ignoring terms of order greater than e^2,

$$\gamma \approx \frac{r}{a}e\sin v \qquad \text{(B-36)}$$

Using Eqs. (B-22), (B-26), and (B-30), and retaining terms to e^2 gives the desired expression for γ:

$$\gamma = e\sin M + \frac{1}{2}e^2\sin 2M \qquad \text{(B-37)}$$

Substituting Eqs. (B-30) and (B-37) into Eq. (B-29) gives

$$l - \gamma = L + e\sin M + \frac{3}{4}e^2\sin 2M \qquad \text{(B-38)}$$

VI. Derivation of Integrable Expression for *d*A1/*d*ET

Integrable forms for the five terms of dA1/dET specified by Eqs. (B-16) to (B-20) are obtained in the five subsections below. A number of terms are obtained as expansions in powers of the eccentricity of the heliocentric orbit of the earth–moon barycenter or the geocentric orbit of the moon. The required order of e for each term is stated before the term is derived. It will be seen that all of the derived terms of dA1/dET are larger than the minimum values for retained terms specified in Section III, and that using the next order of e in each expansion would yield terms which are smaller than these criteria.

A. Term *AT*: Annual Variation in ϕ and $\mathring{s}^2$

Repeating Eq. (B-16),

$$\left(\frac{d\text{A1}}{d\text{ET}}\right)_{AT} = -\frac{1}{c^2}\left\{\left[\phi_B + \frac{1}{2}(\mathring{s}^S_B)^2\right] - \overline{\left[\phi_B + \frac{1}{2}(\mathring{s}^S_B)^2\right]}\right\}$$

$$= -\frac{1}{c^2}\left\{[\phi_B - \overline{\phi_B}] + \frac{1}{2}\,[(\mathring{s}_S)^2 - \overline{(\mathring{s}_S)^2}]\right\} \qquad \text{(B-39)}$$

Since $\phi_B = \mu_S/r$, Eq. (B-23) gives

$$\phi_B - \overline{\phi_B} = \frac{\mu_S e}{r}\cos E \qquad \text{(B-40)}$$

From Eq. (B-24),

$$\frac{1}{2}[(\mathring{s}_S)^2 - \overline{(\mathring{s}_S)^2}] \approx \frac{\mu_S e}{r}\cos E \qquad \text{(B-41)}$$

Thus,

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{AT} = -\frac{2\mu_S e}{c^2 r}\cos E$$

$$= -\frac{2\mu_S e}{c^2 a}\left(\frac{a}{r}\right)\cos E \qquad \text{(B-42)}$$

Inserting numerical values from Section II gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{AT} = -0.330074 \times 10^{-9}\left(\frac{a}{r}\right)\cos E \qquad \text{(B-43)}$$

However, for the purpose of integrating Eq. (B-15), a slight variation of this equation is required. Since

$$\overset{\bullet}{E} = \frac{dE}{dt} = \frac{1}{r}\left(\frac{\mu_S + \mu_E + \mu_M}{a}\right)^{1/2} \approx \frac{1}{r}\left(\frac{\mu_S}{a}\right)^{1/2} \qquad \text{(B-44)}$$

where $t = \mathrm{ET}$, Eq. (B-42) may be expressed as

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{AT} = -\frac{2(\mu_S a)^{1/2} e}{c^2}(\cos E)\overset{\bullet}{E} \qquad \text{(B-45)}$$

Inserting numerical values gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{AT} = -(1.658 \times 10^{-3}\,\mathrm{s})(\cos E)\overset{\bullet}{E} \qquad \text{(B-46)}$$

When Eq. (B-15) is multiplied by $d\mathrm{ET} = dt$, this term is exactly integrable.

Since e is constant to approximately four figures from 1950 to 2000, the coefficient of Eq. (B-46) is given to that many figures. A variation of one digit in the fourth figure changes the magnitude of the term by 2×10^{-13}, which is less than the retention criterion of 6.7×10^{-12} for an annual term (Section III). The approximation of the factor $(\mu_S + \mu_E + \mu_M)$ by μ_S above is valid since these two quantities differ in the seventh significant figure.

B. Term DP: Daily Variation in Solar Potential

Repeating Eq. (B-17),

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DP} = -\frac{1}{c^2}(\phi_{STN} - \phi_E) \qquad \text{(B-47)}$$

This term may be derived assuming that the earth moves in a circular orbit with radius a in the ecliptic plane. The distance from the sun to the tracking station where the A1 clock is located is denoted as r_{STN}. Then,

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DP} \approx -\frac{\mu_S}{c^2}\left(\frac{1}{r_{STN}} - \frac{1}{a}\right) \approx \frac{\mu_S}{c^2 a^2}(r_{STN} - a) \qquad \text{(B-48)}$$

where

$$r_{STN} \approx a + \frac{\mathbf{r}^E_{STN} \cdot \mathbf{r}^S_B}{a} = a - \frac{\mathbf{r}^E_{STN} \cdot \mathbf{r}^B_S}{a} \qquad \text{(B-49)}$$

The two vectors, with rectangular components referred to the mean earth equator and equinox of date, are given by

$$\mathbf{r}^E_{STN} = \begin{bmatrix} u\cos(\theta_M + \lambda) \\ u\sin(\theta_M + \lambda) \\ v \end{bmatrix} \qquad \text{(B-50)}$$

$$\mathbf{r}^B_S = \begin{bmatrix} \cos L \\ \sin L \cos\epsilon \\ \sin L \sin\epsilon \end{bmatrix} a \qquad \text{(B-51)}$$

Substituting Eqs. (B-49), (B-50), and (B-51) into Eq. (B-48) gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DP} = -\frac{\mu_S u}{c^2 a^2}[\cos(\theta_M + \lambda)\cos L + \sin(\theta_M + \lambda)\sin L \cos\epsilon] - \frac{\mu_S v}{c^2 a^2}\sin\epsilon \sin L \qquad \text{(B-52)}$$

The last term has a maximum magnitude of about 2×10^{-13}, which is less than the retention criterion of 6.7×10^{-12} for an annual term. Ignoring this term and using trigonometric identities gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DP} = -\frac{\mu_S u}{2c^2 a^2}[(1 + \cos\epsilon)\cos(\theta_M - L + \lambda) + (1 - \cos\epsilon)\cos(\theta_M + L + \lambda)] \qquad \text{(B-53)}$$

Both terms have a period of about 1 day. The maximum magnitude of the second term is about 1.8×10^{-14}, which is the retention criterion for a daily term. Since several terms with only a slightly smaller magnitude are neglected, this term is ignored also.

In order to evaluate the surviving term of Eq. (B-53), the definition of universal time UT (which means specifically UT1) must be considered. From Eq. (92) or Ref. 25, pp. 73–74,

$$\mathrm{UT} = \theta_M - R_U + 12\,\mathrm{h} \tag{B-54}$$

where all quantities are expressed in hours and R_U is the right ascension, measured from the mean equinox of date, of a fictitious point on the equator. The adopted expression for R_U is (Ref. 25, p. 73)

$$R_U = 18^{\mathrm{h}}38^{\mathrm{m}}45^{\mathrm{s}}.836 + 8{,}640{,}184^{\mathrm{s}}.542\, T_U + 0^{\mathrm{s}}.0929\, T_U^2 \tag{B-55}$$

where

T_U = number of Julian centuries of 36,525 days of UT elapsed since January 0, 1900, 12^{h} UT.

Changing units in Eq. (B-55) gives

$$R_U = 279^\circ 41' 27''.54 + 129{,}602{,}768''.13\, T_U + 1''.3935\, T_U^2 \tag{B-56}$$

This expression for R_U is almost identical to the following expression for the geometric mean longitude of the sun, L, referred to the mean equinox and ecliptic of date (Ref. 25, p. 98):

$$L = 279^\circ 41' 48''.04 + 129{,}602{,}768''.13\, T + 1''.089\, T^2 \tag{B-57}$$

where

T = number of Julian centuries of 36,525 ephemeris days elapsed since January 0, 1900, 12^{h} ET.

The constant term of R_U is $20''.5$ smaller than the corresponding term of L since R_U is corrected for stellar aberration. The derivative of the quadratic term of Eq. (B-56) is the linear term in the precession rate in right ascension. That is, the point described by R_U moves at a uniform rate with respect to a fixed equinox, whereas the mean sun does not. The difference in the quadratic terms of R_U and L will amount to only about 0.3 arc seconds by the end of the century. For a fixed epoch, the contribution to $L - R_U$ due to computing the former from the ET value of the epoch and the latter from the UT value of the epoch is in the range of 1 to 2 arc seconds. Thus, for the remainder of the century, L and R_U will differ by no more than 23 arc seconds. Because of this small difference, R_U in Eq. (B-54) is approximated by L, giving

$$\mathrm{UT} \approx \theta_M - L + 12\,\mathrm{h} \tag{B-58}$$

or, in units of radians,

$$\theta_M - L \approx \mathrm{UT} - \pi \tag{B-59}$$

The following argument will show that this approximation is sufficiently accurate for all daily terms of $d\mathrm{A1}/d\mathrm{ET}$. The largest daily term is the first term of Eq. (B-70) of Section C, which has a maximum value of

$$-1.5 \times 10^{-10} \cos(\mathrm{UT} + \lambda)$$

Because of the approximation above, the variable UT is in error by a nearly constant value of 23 arc seconds or 1.1×10^{-4} rad. Assuming this error is constant, the error in $d\mathrm{A1}/d\mathrm{ET}$ is

$$1.7 \times 10^{-14} \sin(\mathrm{UT} + \lambda)$$

The magnitude of this neglected term is not greater than the retention criterion of 1.8×10^{-14} for a daily term. Hence, the assumption that $R_U = L$ in Eq. (B-54) is valid.

Substituting $(\theta_M - L)$ from Eq. (B-59) into the first term of Eq. (B-53) gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DP} = \frac{\mu_S u}{2c^2a^2}(1 + \cos\epsilon)\cos(\mathrm{UT} + \lambda) \tag{B-60}$$

Substituting numerical values gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DP} = 0.6326 \times 10^{-16}\, u \cos(\mathrm{UT} + \lambda) \tag{B-61}$$

This term is retained since $6.3 \times 10^{-17} > 2.8 \times 10^{-18}$, as specified in Section III.

C. Term *DA*: Product of Daily and Annual Velocity Components in $\mathring{s}^2$

Repeating Eq. (B-18),

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DA} = -\frac{1}{c^2}\,(\mathring{\mathbf{r}}^{E}_{STN} \cdot \mathring{\mathbf{r}}^{S}_{B})$$

$$= \frac{1}{c^2}\,(\mathring{\mathbf{r}}^{E}_{STN} \cdot \mathring{\mathbf{r}}^{B}_{S}) \qquad \text{(B-62)}$$

The two velocity vectors, with rectangular components referred to the mean earth equator and equinox of date, are given by

$$\mathring{\mathbf{r}}^{E}_{STN} = \begin{bmatrix} -\sin(\theta_M + \lambda) \\ \cos(\theta_M + \lambda) \\ 0 \end{bmatrix} \mathring{s}_{STN} \qquad \text{(B-63)}$$

$$\mathring{\mathbf{r}}^{B}_{S} = \begin{bmatrix} -\sin(l - \gamma) \\ \cos(l - \gamma)\cos\epsilon \\ \cos(l - \gamma)\sin\epsilon \end{bmatrix} \mathring{s}_{S} \qquad \text{(B-64)}$$

Substituting Eqs. (B-63) and (B-64) into Eq. (B-62) gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DA} = \frac{\mathring{s}_{STN}\,\mathring{s}_S}{c^2}\,[\sin(\theta_M + \lambda)\sin(l - \gamma) + \cos(\theta_M + \lambda)\cos(l - \gamma)\cos\epsilon] \qquad \text{(B-65)}$$

Using trigonometric identities gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DA} = \frac{\mathring{s}_{STN}\,\mathring{s}_S}{2c^2}\,[(1 + \cos\epsilon)\cos(\theta_M + \lambda - l + \gamma) - (1 - \cos\epsilon)\cos(\theta_M + \lambda + l - \gamma)] \qquad \text{(B-66)}$$

Substituting $l - \gamma$ from Eq. (B-38), eliminating θ_M in favor of UT by using Eq. (B-59), using Eq. (B-27) for $\mathring{s}_S$ and evaluating $\mathring{s}_{STN}$ from

$$\mathring{s}_{STN} = u\,\mathring{\theta}_M \qquad \text{(B-67)}$$

gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DA} = -\frac{\mathring{s}_c\,\mathring{\theta}_M}{2c^2}(1 + \cos\epsilon)\,u\left(1 - \frac{1}{4}e^2 + e\cos M + \frac{3}{4}e^2\cos 2M\right)\cos\left(\mathrm{UT} + \lambda - e\sin M - \frac{3}{4}e^2\sin 2M\right)$$

$$+ \frac{\mathring{s}_c\,\mathring{\theta}_M}{2c^2}(1 - \cos\epsilon)\,u\,(1 + e\cos M)\cos(\mathrm{UT} + \lambda + 2L + e\sin M) \qquad \text{(B-68)}$$

where terms to order e^2 are retained in the term proportional to $(1 + \cos\epsilon)$ and terms to order e only are retained in the smaller term, which is proportional to $(1 - \cos\epsilon)$. Expanding and retaining terms to these orders of e gives

$$\left(\frac{d\mathrm{A1}}{d\mathrm{ET}}\right)_{DA} = -\frac{\mathring{s}_c\,\mathring{\theta}_M}{2c^2}(1 + \cos\epsilon)\,u\left[\left(1 - \frac{1}{2}e^2\right)\cos(\mathrm{UT} + \lambda) + e\cos(\mathrm{UT} + \lambda - M)\right.$$

$$\left. + \frac{9}{8}e^2\cos(\mathrm{UT} + \lambda - 2M) - \frac{1}{8}e^2\cos(\mathrm{UT} + \lambda + 2M)\right]$$

$$+ \frac{\mathring{s}_c\,\mathring{\theta}_M}{2c^2}(1 - \cos\epsilon)\,u\,[\cos(\mathrm{UT} + \lambda + 2L) + e\cos(\mathrm{UT} + \lambda + 2L + M)] \qquad \text{(B-69)}$$

Substituting numerical values gives

$$\begin{aligned}\left(\frac{dA1}{dET}\right)_{DA} = & -2.316550 \times 10^{-14} u \cos(UT + \lambda) \\ & -3.8738 \times 10^{-16} u \cos(UT + \lambda - M) \\ & -7.287 \times 10^{-18} u \cos(UT + \lambda - 2M) \\ & +8.096 \times 10^{-19} u \cos(UT + \lambda + 2M) \\ & +0.997334 \times 10^{-15} u \cos(UT + \lambda + 2L) \\ & +1.6675 \times 10^{-17} u \cos(UT + \lambda + 2L + M)\end{aligned} \tag{B-70}$$

From Section III, any diurnal term of $dA1/dET$ with a coefficient (exclusive of the value of u) of 2.8×10^{-18} or less may be deleted. Thus, the fourth term of Eq. (B-70) will be deleted. The first three terms of Eq. (B-70) are terms of order e^0, e^1, and e^2 of the expansion of the first term of Eq. (B-68).

If e^3 terms were retained in this expansion, the maximum value of the numerical coefficient would be about 1.4×10^{-19}, which is not significant. Similarly, the last two terms of Eq. (B-70) are terms of order e^0 and e^1 in the expansion of the second term of Eq. (B-68). If e^2 terms were retained, the maximum value of the numerical coefficient would be about 3×10^{-19}, which also is not significant.

D. Term *MA*: Product of Monthly and Annual Velocity Components in $\mathring{s}^2$

Repeating Eq. (B-19),

$$\begin{aligned}\left(\frac{dA1}{dET}\right)_{MA} &= -\frac{1}{c^2}\left(\mathring{\mathbf{r}}_E^B \cdot \mathring{\mathbf{r}}_B^S\right) \\ &= -\frac{1}{c^2(1+\mu)}\left(\mathring{\mathbf{r}}_M^E \cdot \mathring{\mathbf{r}}_S^B\right)\end{aligned} \tag{B-71}$$

since $\mathbf{r}_B^E = \mathbf{r}_M^E/(1 + \mu)$. Equation (B-71) will be evaluated assuming that both orbits are circular. The two inertial velocity vectors, with rectangular components referred to the mean ascending node of the lunar orbit on the ecliptic and the ecliptic of date, are given by

$$\mathring{\mathbf{r}}_M^E = \begin{bmatrix} -\sin(\text{☾} - \text{☊}) \\ \cos(\text{☾} - \text{☊})\cos i \\ \cos(\text{☾} - \text{☊})\sin i \end{bmatrix} \mathring{s}_M \tag{B-72}$$

$$\mathring{\mathbf{r}}_S^B = \begin{bmatrix} -\sin(L - \text{☊}) \\ \cos(L - \text{☊}) \\ 0 \end{bmatrix} \mathring{s}_c \tag{B-73}$$

Substituting Eqs. (B-72) and (B-73) into Eq. (B-71) gives

$$\begin{aligned}\left(\frac{dA1}{dET}\right)_{MA} = & -\frac{\mathring{s}_M \mathring{s}_c}{c^2(1+\mu)}[\sin(\text{☾} - \text{☊})\sin(L - \text{☊}) \\ & + \cos(\text{☾} - \text{☊})\cos(L - \text{☊})\cos i]\end{aligned} \tag{B-74}$$

Using trigonometric identities gives

$$\begin{aligned}\left(\frac{dA1}{dET}\right)_{MA} = & -\frac{\mathring{s}_M \mathring{s}_c}{2c^2(1+\mu)}[(1 + \cos i)\cos(\text{☾} - L) \\ & - (1 - \cos i)\cos(\text{☾} + L - 2\text{☊})]\end{aligned} \tag{B-75}$$

The magnitude of the second term is about 0.8×10^{-14}, which is smaller than the retention criterion of 5.4×10^{-13} for a monthly term. Ignoring this term and denoting $\text{☾} - L$ by D (see Section II) gives

$$\left(\frac{dA1}{dET}\right)_{MA} = -\frac{\mathring{s}_M \mathring{s}_c}{2c^2(1+\mu)}(1 + \cos i)\cos D \tag{B-76}$$

Inserting numerical values gives

$$\left(\frac{dA1}{dET}\right)_{MA} = -4.1172 \times 10^{-12} \cos D \tag{B-77}$$

The first-order monthly eccentricity terms would have a maximum coefficient of about 3×10^{-13}. Since this value is less than the retention criterion of 5.4×10^{-13}, the assumption of circular orbits is valid.

E. Term *DM*: Product of Daily and Monthly Velocity Components in s^2

Repeating Eq. (B-20),

$$\left(\frac{dA1}{dET}\right)_{DM} = -\frac{1}{c^2}\left(\mathring{\mathbf{r}}^{E}_{STN} \cdot \mathring{\mathbf{r}}^{B}_{E}\right)$$

$$= \frac{1}{c^2\,(1+\mu)}\left(\mathring{\mathbf{r}}^{E}_{STN} \cdot \mathring{\mathbf{r}}^{E}_{M}\right) \qquad \text{(B-78)}$$

For the purpose of deriving this small term, the 5-deg inclination of the lunar orbit to the ecliptic and the eccentricity of the orbit are ignored. It will be seen that these assumptions are justified. The two inertial velocity vectors, with rectangular components referred to the mean equinox and ecliptic of date, are

$$\mathring{\mathbf{r}}^{E}_{STN} = \begin{bmatrix} -\sin(\theta_M + \lambda) \\ \cos(\theta_M + \lambda)\cos\epsilon \\ -\cos(\theta_M + \lambda)\sin\epsilon \end{bmatrix} \mathring{s}_{STN} \qquad \text{(B-79)}$$

$$\mathring{\mathbf{r}}^{E}_{M} = \begin{bmatrix} -\sin \text{☾} \\ \cos \text{☾} \\ 0 \end{bmatrix} \mathring{s}_{M} \qquad \text{(B-80)}$$

Substituting Eqs. (B-79) and (B-80) into Eq. (B-78) gives

$$\left(\frac{dA1}{dET}\right)_{DM} = \frac{\mathring{s}_{STN}\,\mathring{s}_M}{c^2\,(1+\mu)}\left[\sin(\theta_M + \lambda)\sin \text{☾}\right.$$

$$\left. + \cos(\theta_M + \lambda)\cos \text{☾} \cos\epsilon\right] \qquad \text{(B-81)}$$

Using trigonometric identities and Eq. (B-67) gives

$$\left(\frac{dA1}{dET}\right)_{DM} \doteq \frac{\mathring{\theta}_M\,\mathring{s}_M}{2c^2\,(1+\mu)}\,u\left[(1+\cos\epsilon)\cos(\theta_M + \lambda - \text{☾})\right.$$

$$\left. - (1-\cos\epsilon)\cos(\theta_M + \lambda + \text{☾})\right] \qquad \text{(B-82)}$$

The numerical coefficient of the smaller daily term (not including the value of u) is 0.4×10^{-18}, which is less than the retention criterion of 2.8×10^{-18}. Ignoring this term and using Eq. (B-59) and $D = \text{☾} - L$ to eliminate θ_M and ☾ in favor of UT and D gives

$$\left(\frac{dA1}{dET}\right)_{DM} = -\frac{\mathring{\theta}_M\,\mathring{s}_M}{2c^2\,(1+\mu)}(1+\cos\epsilon)\,u\cos(\text{UT} + \lambda - D) \qquad \text{(B-83)}$$

Inserting numerical values gives

$$\left(\frac{dA1}{dET}\right)_{DM} = -0.9684 \times 10^{-17}\,u\cos(\text{UT} + \lambda - D) \qquad \text{(B-84)}$$

If the moon moved in the earth's equatorial plane, the factor $(1 + \cos\epsilon)/2$ would not be present in Eq. (B-83), and the numerical coefficient of Eq. (B-84) would be changed to 1.0100×10^{-17}. The change of 4.2×10^{-19} is less than the retention criterion of 2.8×10^{-18}.

Thus, in the derivation of $(dA1/dET)_{DM}$, the average inclination ϵ ($\approx 23°5$) of the lunar orbit plane to the earth's equatorial plane is not significant. Hence, neglecting the periodic variation in this inclination of $\pm i \approx 5$ deg with a period of 18.6 years is certainly justified. Also, the first-order eccentricity terms of Eq. (B-80) would produce terms similar to Eq. (B-84) but with a numerical coefficient of about 5×10^{-19}, which is less than the retention criterion of 2.8×10^{-18}.

VII. Integration of *d*A1/*d*ET To Give Expression for ET − A1

The integrable expression for $dA1/dET$ is obtained by substituting Eqs. (B-46), (B-61), (B-70) (except the fourth term, which is not significant), (B-77), and (B-84) into Eq. (B-15):

$$\begin{aligned}\frac{dA1}{dET} = 1 &+ \frac{\Delta f_{\text{cesium}}}{f_{\text{cesium}}} - (1.658 \times 10^{-3}\,\text{s})(\cos E)(\mathring{E}) \\ &- 2.310224 \times 10^{-14}\,u\cos(\text{UT} + \lambda) \\ &- 3.8738 \times 10^{-16}\,u\cos(\text{UT} + \lambda - M) \\ &- 7.287 \times 10^{-18}\,u\cos(\text{UT} + \lambda - 2M) \\ &+ 0.997334 \times 10^{-15}\,u\cos(\text{UT} + \lambda + 2L) \\ &+ 1.6675 \times 10^{-17}\,u\cos(\text{UT} + \lambda + 2L + M) \\ &- 4.1172 \times 10^{-12}\cos D \\ &- 0.9684 \times 10^{-17}\,u\cos(\text{UT} + \lambda - D)\end{aligned} \qquad \text{(B-85)}$$

where the fourth term is the sum of Eq. (B-61) and the first term of Eq. (B-70). In order to integrate this expression, it is multiplied by dET, and each of the last seven terms is multiplied by the analytical expression for the constant derivative of the argument of the cosine function with respect to ET and divided by the corresponding numerical value. The result (in units of seconds) is

$$
\begin{aligned}
d\mathrm{A1} = d\mathrm{ET}\left(1+\frac{\Delta f_{\mathrm{cesium}}}{f_{\mathrm{cesium}}}\right) &- 1.658\times10^{-3}(\cos E)\,dE \\
&-0.317679\times10^{-9}u\,[\cos(\mathrm{UT}+\lambda)]\,d\mathrm{UT} \\
&-5.341\times10^{-12}u\,[\cos(\mathrm{UT}+\lambda-M)]\,(d\mathrm{UT}-dM) \\
&-1.01\times10^{-13}u\,[\cos(\mathrm{UT}+\lambda-2M)]\,(d\mathrm{UT}-2dM) \\
&+1.3640\times10^{-11}u\,[\cos(\mathrm{UT}+\lambda+2L)]\,(d\mathrm{UT}+2dL) \\
&+2.27\times10^{-13}u\,[\cos(\mathrm{UT}+\lambda+2L+M)]\,(d\mathrm{UT}+2dL+dM) \\
&-1.672\times10^{-6}\,[\cos D]\,dD \\
&-1.38\times10^{-13}u\,[\cos(\mathrm{UT}+\lambda-D)]\,(d\mathrm{UT}-dD)
\end{aligned}
\qquad \text{(B-86)}
$$

As indicated after Eq. (64), the master A1 clock was set up on January 1, 1958, 0^{h} UT2. Integrating Eq. (B-86) from this initial epoch (denoted by subscript 0) to the current epoch (denoted as A1 or ET) gives, in seconds,

$$
\begin{aligned}
\mathrm{ET}-\mathrm{A1} = (\mathrm{ET}-\mathrm{A1})_0 - (\mathrm{ET}-\mathrm{ET}_0)&\frac{\Delta f_{\mathrm{cesium}}}{f_{\mathrm{cesium}}} \\
&+1.658\times10^{-3}\sin E\,\Big|_{0}^{\mathrm{ET}} \\
&+0.317679\times10^{-9}u\sin(\mathrm{UT}+\lambda)\,\Big|_{0}^{\mathrm{ET}} \\
&+5.341\times10^{-12}u\sin(\mathrm{UT}+\lambda-M)\,\Big|_{0}^{\mathrm{ET}} \\
&+1.01\times10^{-13}u\sin(\mathrm{UT}+\lambda-2M)\,\Big|_{0}^{\mathrm{ET}} \\
&-1.3640\times10^{-11}u\sin(\mathrm{UT}+\lambda+2L)\,\Big|_{0}^{\mathrm{ET}} \\
&-2.27\times10^{-13}u\sin(\mathrm{UT}+\lambda+2L+M)\,\Big|_{0}^{\mathrm{ET}} \\
&+1.672\times10^{-6}\sin D\,\Big|_{0}^{\mathrm{ET}} \\
&+1.38\times10^{-13}u\sin(\mathrm{UT}+\lambda-D)\,\Big|_{0}^{\mathrm{ET}}
\end{aligned}
\qquad \text{(B-87)}
$$

The initial condition $(\mathrm{ET}-\mathrm{A1})_0$ equals ET − UT2 on January 1, 1958, 0^{h} UT2, since the master A1 clock was set equal to UT2 at this epoch. Denoting this quantity minus the initial values of the periodic relativity terms of Eq. (B-87) evaluated with u and λ of the master A1 clock as ΔT_{1958} gives Eq. (65) for ET − A1.

Appendix C

Derivation of Light Time Equation

The light time equation is derived in Section I without making the usual assumption that light moves along a straight line from the transmitter to the receiver. In Section II, this same equation is obtained by assuming straight line motion between two points at the coordinate speed v_c given by Eq. (86). The results are the same because terms are retained to order $1/c^3$ only and the bending affects terms of order $1/c^5$ and greater.

I. Derivation Without Assumption of Straight Line Motion

Substitution of $d\phi$ from Eq. (79) into Eq. (80), setting $dr/dt = 0$ when $r = R$ (the minimum value of r on the light path), and ignoring $1/c^4$ terms gives

$$dt = \pm \frac{1}{c} \frac{r\left(1 + \frac{(1+\gamma)\mu}{c^2 r}\right)^2 dr}{\left\{r^2\left[1 + \frac{(1+\gamma)\mu}{c^2 r}\right]^2 - R^2\left[1 + \frac{(1+\gamma)\mu}{c^2 R}\right]^2\right\}^{1/2}} \tag{C-1}$$

Making the following change of variable:

$$\rho = r + \frac{(1+\gamma)\mu}{c^2} \tag{C-2}$$

$$\rho_0 = R + \frac{(1+\gamma)\mu}{c^2} \tag{C-3}$$

gives, ignoring $1/c^4$ terms,

$$dt = \pm \frac{1}{c} \frac{\rho\left[1 + \frac{(1+\gamma)\mu}{c^2 \rho}\right] d\rho}{(\rho^2 - \rho_0^2)^{1/2}} \tag{C-4}$$

Writing the right-hand side of Eq. (C-4) as two terms and replacing ρ and ρ_0 by r and R in the $1/c^3$ term gives

$$dt = \pm \frac{1}{c} \frac{\rho d\rho}{(\rho^2 - \rho_0^2)^{1/2}} \pm \frac{(1+\gamma)\mu}{c^3} \frac{dr}{(r^2 - R^2)^{1/2}} \tag{C-5}$$

Integrating from point 1 (r_1 or ρ_1, t_1) to point 2 (r_2 or ρ_2, t_2) gives

$$t_2 - t_1 = \pm \frac{1}{c} \left[(\rho_2^2 - \rho_0^2)^{1/2} - (\rho_1^2 - \rho_0^2)^{1/2}\right] \pm \frac{(1+\gamma)\mu}{c^3} \ln\left[\frac{r_2 + (r_2^2 - R^2)^{1/2}}{r_1 + (r_1^2 - R^2)^{1/2}}\right] \tag{C-6}$$

where the plus sign applies when r is strictly increasing from point 1 to point 2, and the minus sign applies when r is strictly decreasing from point 1 to point 2. From the second form of Eq. (82), and referring to Fig. 2,

$$x = r\cos\phi = R + \frac{(1+\gamma)\mu}{c^2} - \frac{(1+\gamma)\mu r}{c^2 R} \tag{C-7}$$

Substituting Eq. (C-3) into Eq. (C-7) gives

$$\rho_0 = x + \frac{(1+\gamma)\mu r}{c^2 R} \tag{C-8}$$

Substituting Eqs. (C-2) and (C-8) into $(\rho^2 - \rho_0^2)^{1/2}$, evaluating x in a $1/c^2$ term with Eq. (C-7), and ignoring $1/c^4$ terms gives the result that

$$(\rho^2 - \rho_0^2)^{1/2} = (r^2 - x^2)^{1/2} = \pm y \tag{C-9}$$

where the minus sign applies for negative y (decreasing r). Substituting this result into Eq. (C-6) gives

$$t_2 - t_1 = \frac{y_2 - y_1}{c} \pm \frac{(1+\gamma)\mu}{c^3} \ln\left[\frac{r_2 + (r_2^2 - R^2)^{1/2}}{r_1 + (r_1^2 - R^2)^{1/2}}\right] \tag{C-10}$$

where the minus sign applies for decreasing r. The argument of the logarithm in Eq. (C-10) may be expressed as (as explained below)

$$\frac{r_2 + (r_2^2 - R^2)^{1/2}}{r_1 + (r_1^2 - R^2)^{1/2}} = \frac{r_1 - (r_1^2 - R^2)^{1/2}}{r_2 - (r_2^2 - R^2)^{1/2}} = \frac{r_1 + r_2 + \left[(r_2^2 - R^2)^{1/2} - (r_1^2 - R^2)^{1/2}\right]}{r_1 + r_2 - \left[(r_2^2 - R^2)^{1/2} - (r_1^2 - R^2)^{1/2}\right]}$$

$$\approx \frac{r_1 + r_2 + \left[(r_2^2 - x^2)^{1/2} - (r_1^2 - x^2)^{1/2}\right]}{r_1 + r_2 - \left[(r_2^2 - x^2)^{1/2} - (r_1^2 - x^2)^{1/2}\right]} = \frac{r_1 + r_2 \pm (y_2 - y_1)}{r_1 + r_2 \mp (y_2 - y_1)} \tag{C-11}$$

The second form is obtained from the first by multiplying and dividing by

$$[r_1 - (r_1^2 - R^2)^{1/2}]\,[r_2 - (r_2^2 - R^2)^{1/2}]$$

The third form is obtained from the first two forms by adding the numerators and denominators. The fourth form is obtained by replacing R by x and ignoring the $1/c^2$ terms of Eq. (C-7) since they produce $1/c^5$ terms in the light time equation. The fifth form follows from Eq. (C-9); hence, the lower sign in the numerator and denominator applies for decreasing r (negative y). Substituting the final form of Eq. (C-11) into Eq. (C-10) gives

$$t_2 - t_1 = \frac{y_2 - y_1}{c} + \frac{(1+\gamma)\mu}{c^3} \ln\left[\frac{r_1 + r_2 + (y_2 - y_1)}{r_1 + r_2 - (y_2 - y_1)}\right] \tag{C-12}$$

which applies when r is strictly increasing or strictly decreasing.

The quantity $y_2 - y_1$ is the y component of the straight line distance r_{12} between the transmitter (point 1) and the receiver (point 2). From Fig. 2, the maximum angle of r_{12} to the y axis is $(1+\gamma)\,\mu/c^2 R$. Hence,

$$r_{12} - (y_2 - y_1) < \frac{(y_2 - y_1)}{\cos\left[\dfrac{(1+\gamma)\mu}{c^2 R}\right]} - (y_2 - y_1) \approx \frac{(y_2 - y_1)}{2}\left[\frac{(1+\gamma)\mu}{c^2 R}\right]^2 \tag{C-13}$$

which is of order $1/c^4$. Since $1/c^5$ terms are ignored in the light time equation, $y_2 - y_1$ may be replaced by r_{12} in Eq. (C-12), giving

$$t_2 - t_1 = \frac{r_{12}}{c} + \frac{(1+\gamma)\mu}{c^3} \ln\left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}}\right) \tag{C-14}$$

which applies when the sign of $\dot{r}$ does not change between t_1 and t_2. For the case where r passes through a minimum between r_1 and r_2 (see Fig. 2), the total light time is the time for light to travel from r_1 to the minimum radius R plus the time for light to travel from R to r_2. Let light reach the radius $r = R$ at ephemeris time t_R. Then,

$$t_2 - t_1 = (t_2 - t_R) + (t_R - t_1) \tag{C-15}$$

From Eq. (C-10) with the positive sign,

$$t_2 - t_R = \frac{y_2}{c} + \frac{(1+\gamma)\mu}{c^3}\{\ln[r_2 + (r_2^2 - R^2)^{1/2}] - \ln R\} \tag{C-16}$$

From Eq. (C-10) with the negative sign and the argument of the logarithm replaced by the second form of Eq. (C-11),

$$t_R - t_1 = -\frac{y_1}{c} + \frac{(1+\gamma)\mu}{c^3}\{\ln R - \ln[r_1 - (r_1^2 - R^2)^{1/2}]\} \tag{C-17}$$

Substituting Eqs. (C-16) and (C-17) into Eq. (C-15) and replacing $(y_2 - y_1)$ with r_{12} gives

$$t_2 - t_1 = \frac{r_{12}}{c} + \frac{(1+\gamma)\mu}{c^3} \ln\left[\frac{r_2 + (r_2^2 - R^2)^{1/2}}{r_1 - (r_1^2 - R^2)^{1/2}}\right] \tag{C-18}$$

The argument of the logarithm may be expressed as

$$\frac{r_2 + (r_2^2 - R^2)^{1/2}}{r_1 - (r_1^2 - R^2)^{1/2}} = \frac{r_1 + (r_1^2 - R^2)^{1/2}}{r_2 - (r_2^2 - R^2)^{1/2}} = \frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \tag{C-19}$$

This result was obtained by using the procedures used in the derivation of Eq. (C-11) and by setting $y_2 - y_1 = r_{12}$. Substituting Eq. (C-19) into Eq. (C-18) gives Eq. (C-14). Hence Eq. (C-14) is the general result which is valid regardless of whether r increases, decreases, or passes through a minimum between r_1 and r_2. Equation (C-14) is Eq. (88) of the text.

II. Derivation Assuming Straight Line Motion

The geometry for straight line motion is shown in Fig. C-1, where S indicates the position of the sun. Light is emitted at point 1 at ephemeris time t_1, moves along the straight line path at the coordinate speed v_c given by Eq. (86), and arrives at point 2 at time t_2. Let

$\mathbf{r}_1, \mathbf{r}_2 =$ heliocentric position vectors of points 1 and 2 at ephemeris times t_1 and t_2, respectively, with rectangular components referred to a nonrotating coordinate system

$\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$

$r_1, r_2, r_{12} =$ magnitudes of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_{12}$, respectively

Then, the angles β_1 and β_2 are computed from

$$\cos\beta_1 = \frac{\mathbf{r}_1}{r_1} \cdot \frac{\mathbf{r}_{12}}{r_{12}} \qquad 0 < \beta_1 < \pi \qquad \text{(C-20)}$$

$$\cos\beta_2 = \frac{\mathbf{r}_2}{r_2} \cdot \frac{\mathbf{r}_{12}}{r_{12}} \qquad 0 < \beta_2 < \pi \qquad \text{(C-21)}$$

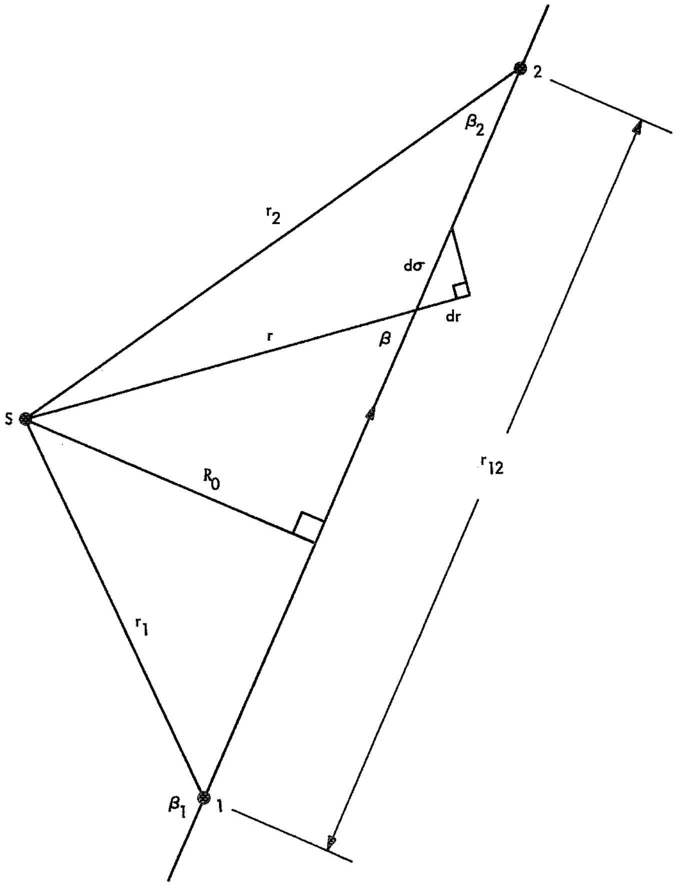

Fig. C-1. Geometry for straight line motion

For a photon passing the sun S on an infinitely long track, the angle β decreases from π to zero, passing through $\pi/2$ at the point of closest approach, where $r = R_0$.

The time for light to travel from point 1 to point 2 is the integral of the differential of coordinate distance $d\sigma$ divided by the coordinate speed of light v_c along the straight line path joining the two points:

$$t_2 - t_1 = \int_0^{r_{12}} \frac{d\sigma}{v_c} \qquad \text{(C-22)}$$

From Eq. (86),

$$v_c \approx c\left[1 - \frac{(1+\gamma)\mu}{c^2 r}\right] \qquad \text{(C-23)}$$

Ignoring terms of order $1/c^5$,

$$t_2 - t_1 = \frac{1}{c}\int_0^{r_{12}} \left[1 + \frac{(1+\gamma)\mu}{c^2 r}\right] d\sigma$$

$$= \frac{r_{12}}{c} + \frac{(1+\gamma)\mu}{c^3}\int_0^{r_{12}} \frac{d\sigma}{r} \qquad \text{(C-24)}$$

From Fig. C-1,

$$d\sigma \cos\beta = dr \qquad \text{(C-25)}$$

Note that $\cos\beta$ and dr have the same sign and thus $d\sigma$ is always positive. Also, from Fig. C-1,

$$r \sin\beta = R_0 \qquad \text{(C-26)}$$

and

$$\frac{dr}{\cos\beta} = -\frac{r\,d\beta}{\sin\beta} \qquad \text{(C-27)}$$

Substituting Eq. (C-27) into Eq. (C-25) gives

$$d\sigma = -\frac{r\,d\beta}{\sin\beta} \qquad \text{(C-28)}$$

Substituting Eq. (C-28) into Eq. (C-24) gives

$$t_2 - t_1 = \frac{r_{12}}{c} - \frac{(1+\gamma)\mu}{c^3}\int_{\beta_1}^{\beta_2} \frac{d\beta}{\sin\beta} \qquad \text{(C-29)}$$

This may be integrated directly, giving

$$t_2 - t_1 = \frac{r_{12}}{c} + \frac{(1+\gamma)\mu}{c^3}\ln\left(\frac{\tan\frac{1}{2}\beta_1}{\tan\frac{1}{2}\beta_2}\right) \tag{C-30}$$

For light moving radially to or from the sun, $\beta_1 = \beta_2 = \pi$ or zero, respectively, and Eq. (C-30) is indeterminate. For this case, the time for light to travel radially from r_1 to r_2 (to or from the sun, not through the sun), denoting the larger and smaller values of r_1 and r_2 as r_{larger} and r_{smaller}, is

$$t_2 - t_1 = \int_{r_{\text{smaller}}}^{r_{\text{larger}}} \frac{dr}{v_c} = \frac{1}{c}\int_{r_{\text{smaller}}}^{r_{\text{larger}}}\left[1 + \frac{(1+\gamma)\mu}{c^2 r}\right] dr \tag{C-31}$$

which integrates to

$$t_2 - t_1 = \frac{r_{12}}{c} + \frac{(1+\gamma)\mu}{c^3}\ln\left(\frac{r_{\text{larger}}}{r_{\text{smaller}}}\right) \tag{C-32}$$

It will be shown that Eq. (C-32) for the radial case and Eq. (C-30) for all other cases are equivalent to Eq. (C-14) derived without the assumption of straight line motion. The argument of the logarithm of Eq. (C-30) may be written as

$$\frac{\tan\frac{1}{2}\beta_1}{\tan\frac{1}{2}\beta_2} = \frac{\sin\beta_1}{1+\cos\beta_1}\cdot\frac{1+\cos\beta_2}{\sin\beta_2} \tag{C-33}$$

However, from Fig. C-1,

$$r_2 \sin\beta_2 = r_1 \sin\beta_1 \tag{C-34}$$

Thus,

$$\frac{\tan\frac{1}{2}\beta_1}{\tan\frac{1}{2}\beta_2} = \frac{r_2(1+\cos\beta_2)}{r_1(1+\cos\beta_1)} \tag{C-35}$$

The argument of the logarithm may also be written as

$$\frac{\tan\frac{1}{2}\beta_1}{\tan\frac{1}{2}\beta_2} = \frac{1-\cos\beta_1}{\sin\beta_1}\cdot\frac{\sin\beta_2}{1-\cos\beta_2} = \frac{r_1(1-\cos\beta_1)}{r_2(1-\cos\beta_2)} \tag{C-36}$$

Adding the numerators and denominators of Eqs. (C-35) and (C-36) gives

$$\frac{\tan\frac{1}{2}\beta_1}{\tan\frac{1}{2}\beta_2} = \frac{r_1 + r_2 + (r_2\cos\beta_2 - r_1\cos\beta_1)}{r_1 + r_2 - (r_2\cos\beta_2 - r_1\cos\beta_1)} \tag{C-37}$$

Regardless of whether r increases, decreases, or passes through a minimum between r_1 and r_2,

$$r_2\cos\beta_2 - r_1\cos\beta_1 = r_{12} \tag{C-38}$$

Thus,

$$\frac{\tan\frac{1}{2}\beta_1}{\tan\frac{1}{2}\beta_2} = \frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}} \tag{C-39}$$

and Eq. (C-30) is equivalent to Eq. (C-14). For the radial case, Eq. (C-32) is equivalent to Eq. (C-14) since

$$\left(\frac{r_1 + r_2 + r_{12}}{r_1 + r_2 - r_{12}}\right)_{\text{radial}} = \frac{2r_{\text{larger}}}{2r_{\text{smaller}}} = \frac{r_{\text{larger}}}{r_{\text{smaller}}} \tag{C-40}$$

Thus, if terms to order $1/c^3$ only are retained in the light time equation, it is valid to neglect the bending of light.

In the original version of the DPODP, the light time is evaluated with Eq. (C-30) or Eq. (C-32), using β_1 and β_2 from Eqs. (C-20) and (C-21). However, it is planned to replace these equations with Eq. (C-14).

As previously mentioned, the form of Eq. (C-14) has been derived by Holdridge (Ref. 23) and by Tausner (Ref. 24). The form of Eq. (C-18), evaluated along a straight line path, has been derived by Ross and Schiff (Ref. 64).

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