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The Metric of Binocular Visual Space*

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The aim of this paper is to show that the so-called visual space has a uniquely determined non-Euclidean metric, or psychometric distance function, the numerical parameters of which depend on the individual observer. Certain well-known phenomena of space perception, such as the horopter, the alley experiment and size constancy, are explained on the basis of the distance function. Methods of measuring the personal parameters of the metric are developed, and applications of the theory to the field of binocular instruments and pictorial representation of space are suggested.

1. PURPOSE

VISUAL sensations are established under the influence of many different factors. They are the result of an activation of our mind by physical light stimuli and are distinguished by a remarkable degree of certainty and definiteness which tempts us to believe that the external world itself is revealed to us, and not merely an image of our own making. Through vision, external objects are presented to us in a three-dimensional continuum, endowed with certain sensed qualities of color, brightness, form, and localization.

Yet it is well known that the final form of visual sensations actually depends on the observer's psychological condition which in turn is the result of his entire personal development, his former experience, his prejudices, even his instantaneous wishes and intentions.

Recognition of this fact does not mean that all attempts of a quantitative investigation of visual perception must be abandoned. Visual sensations are not merely the chance result of coincidental psycho-

logical conditions but are governed by certain constant factors which, though not determining the sensations completely, play an integral part in their formation. These factors must be related to the external physical situation as well as to certain constant elements in the personality of the observer. The denial of the existence of such constant elements in the relationship of a living organism to external nature would be tantamount to denying all objectivity in visual sensations and therefore would be absurd.

In the following we are concerned with a special group of constant elements in visual sensations, namely, those which influence the sensed qualities of *form* and *localization* in the three-dimensional continuum of visual perception. Our aim is to show that this continuum, the so-called *Visual Space*, is a metric mathematical space. It possesses a uniquely determined metric, which varies in its numerical parameters from observer to observer. But its general form is invariant; it is the metric of the three-dimensional hyperbolic geometry.

2. CONSIDERATION LIMITED TO BINOCULAR FACTORS OF VISUAL PERCEPTION

This surprising conclusion is arrived at by an inductive analysis of certain experimental facts which are obtained under simplified laboratory conditions. The observations are made in a dark room with the observer's head, and therefore approximately also the rotational centers of his eyes, fixed by means of a

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† This is the last paper Dr. Luneburg wrote before his sudden, unexpected death in August, 1949. It was not written for publication and has been edited by the Knapp Laboratory staff working on this project and chiefly by Paul Boeder, consultant to the project. The paper is believed to be of great significance because here for the first time, Luneburg removes the restriction on freedom of observation and introduces the probable mathematical function involved in motion.

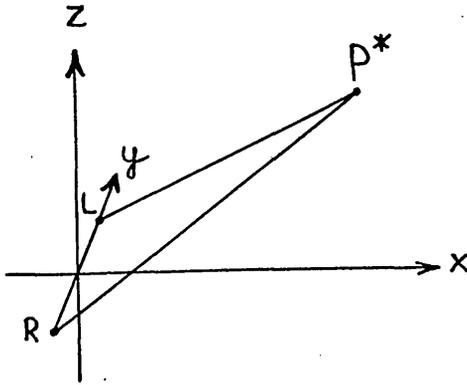


FIG. 1. Cartesian coordinate system for physical space; L and R represent centers of rotation of left and right eyes.

headrest. Let us assume that the centers of rotation, R and L , are the points $y = \pm 1$ of the y axis of a Cartesian coordinate system whose x, y -plane coincides with the physical horizontal plane (Fig. 1).

Simple geometrical configurations are presented to the observer; for instance, isolated light points, lighted lines or surfaces. In judging such configurations, the observer has to rely almost entirely on his primitive ability to see; intellectual clues of interpretation are practically excluded. We may assume that visual sensations such as these reveal more clearly the underlying constant factors which we wish to investigate.

There is a great difference between monocular and binocular vision under the artificial conditions just described. With one eye our judgment of spatial form and localization is erratic and inconsistent. Binocular observation, on the other hand, retains a high degree of certainty. True, in ordinary situations monocular observation is not necessarily inferior to binocular vision. But from experiments in the dark room we conclude that binocular vision adds certain elements to visual perception which monocular vision lacks. This is also demonstrated in a stereoscope which provides a situation of clueless vision comparable to that of the dark room. With a stereoscope, the monocular observer is dependent on intellectual clues of perspective and interpretation, whereas the binocular observer obtains a striking sensation of depth.

The following investigation is concerned with these special binocular or stereoscopic factors of visual perception. No claim is made of a general theory of space perception, nor even of a theory of visual perception integrating binocular as well as other psychological factors. In spite of this limitation, the end result is not without significance for the broader aim inasmuch as it will make possible a precise characterization of the variations which may be caused by the instantaneous psychological condition of the observer.

3. FACTS REQUIRING RECOGNITION

A theory of binocular vision must account for the following experimental facts.

(a) A single isolated light point P^* is given in the dark room. Let x, y, z be its physical Cartesian coordinates. The observer sees the point P^* as a "sensed" point P in his three-dimensional visual space. It has, at any instant, a definite localization in this space, relative to certain subjective planes of reference of the observer, and relative to an apparent center of observation. The fact that two eyes are involved in creating the sensation of the point P^* is not represented in the visual space. We may say that binocular observation in the physical space corresponds to an apparent monocular observation in the visual space. Besides an apparent center of observation, there are three planes of reference of which we are conscious, namely, the subjective horizontal plane, the subjective median plane, and the subjective frontal plane. Mathematically we represent this in a Euclidean space by means of a Cartesian coordinate system, ξ, η, ζ . The origin $\xi = \eta = \zeta = 0$ represents the apparent center of observation, the ξ, η -plane the subjective horizontal plane, the ξ, ζ -plane the median plane, and the η, ζ -plane the frontal plane of the visual space. In general, these planes are correlated approximately to the corresponding physical planes, i.e. a point P^* in the x, y -plane is seen as a point P near the ξ, η -plane, a point P^* in the x, z -plane as a point near to the median plane (Fig. 2).

Since the point P^* is seen, at any instant, as a point P in some definite localization relative to the three subjective planes of reference, we might be tempted to search for a constant and necessary one-to-one correspondence between physical points and sensed points in the visual space. However such an attempt would fail, since it would mean that visual sensations are uniquely determined by the physical stimuli, independent of psychological factors. It would also be in disagreement with actual observations. Visual localization of an isolated, repeatedly shown, physical point P^* is found to be highly erratic. The point P seems to be floating in space indeterminately, now it seems near, now far. This uncertainty of correlating visual and physical localizations becomes obvious if a similar experiment is made with a stereoscope. It is even possible to disturb the observer's judgment of localization with respect to the subjective reference planes, for example, by placing him on a slowly rotating chair.

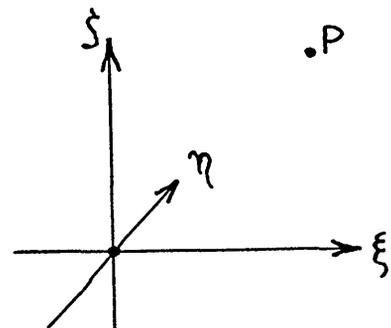


FIG. 2. Subjective coordinate system for visual space.

A point P seems to remain in the median plane, though the physical x, y -plane has been rotated into a different position.

The net result of experiments with an isolated point is purely negative: *Binocular observation of a single point does not differ from monocular observation.* Both are equally uncertain as to correlating a sensed point P to the physical coordinates of the stimulating point P^* .

(b) The situation is quite different when the object consists of several isolated points $P_1^*, P_2^*, P_3^*, \dots$. Aside from sensing these points as points P_1, P_2, P_3, \dots , in a three-dimensional continuum, we have a rather definite sensation of distance of the points from one another. We also find that the sizes of these sensed distances can be compared. Thus, denoting the sensed distance between two points P_1 and P_2 by (P_1P_2) , our visual sensation of three points P_1, P_2, P_3 contains an immediate judgment of inequalities $(P_1P_2) > (P_2P_3)$ or $(P_1P_2) < (P_2P_3)$. The same is true for two pairs of points P_1, P_2 and P_3, P_4 . There is an immediate perception of inequalities $(P_1P_2) > (P_3P_4)$ or $(P_1P_2) < (P_3P_4)$.

Moreover it is found, for binocular observation, that the inequality signs are determined by the physical coordinates of the stimulating points $P_1^*, P_2^*, P_3^*, P_4^*$. This means that the scores of repeated tests with the same two pairs of stimulating points are highly correlated, regardless of other physical qualities of the points such as brightness and color. A corresponding correlation is not found in monocular vision where such additional qualities may easily change our judgment. We note here that in general the sensed inequalities do not correspond to those between the physical distances $(P_1^*P_2^*)$ and $(P_3^*P_4^*)$. Moreover, there is no sensation of absolute localization of the whole configuration, that is, there is no certainty in judging the physical coordinates of the points P^* from the sensed coordinates of the points P .

(c) The following extremely interesting observation is due to Helmholtz.¹ An observer, placed into the position described previously, is given the task of arranging a number of light points so that they appear to lie on a horizontal straight line, symmetric to the median plane. It is found that the points are set

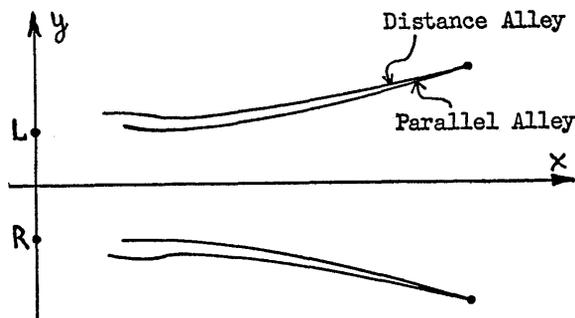


FIG. 4. Distance and parallel alleys.

consistently on certain physical curves which are, in general, not straight in the physical sense. The form of these so-called *horopter* curves depends on the distance x_0 , of the center point. At a certain distance $x_0 = a$ the horopter is practically straight. At nearer distances, $x_0 < a$, the horopters are concave to the observer, at greater distances, $x_0 > a$, they are convex (Fig. 3). The distance, a , of the straight horopter depends on the individual observer. The Helmholtz horopter experiment reveals a new element in visual sensations, namely, that there is an *immediate feeling for straightness* which allows us to judge whether *three points PRQ lie on a subjective straight line.*

(d) A related experiment was made by Hillebrand; it is the so-called *Alley experiment*.² It is well known that physically parallel lines like railroad tracks, or the curbs of a straight road, seem to converge in the distance. The same is true for two physically parallel rows of light points observed in the dark room. In the Hillebrand experiment the rows of light points are arranged so that they no longer converge but seem to form a parallel "alley". The resulting divergent physical curves have a characteristic form comparable to that of hyperbolas.

(e) Hillebrand's experiment was repeated by Blumenfeld³ under improved technical conditions. The general result was the same but an additional observation was made. When the observer was instructed to pay predominant attention to the distance between opposite points, and, by making them equal, to set up alleys of apparently equidistant walls, the resulting curves had the general form found by Hillebrand, but did not appear parallel and straight. The "distance alleys" differed from the "parallel alleys" and were found to lie consistently outside the parallel alleys (Fig. 4).

4. THE PSYCHOMETRIC DISTANCE FUNCTION

The preceding experiments indicate strongly that the visual space, as revealed in dark room experiments, can be characterized mathematically as a metric space. This means that we may assume the existence of a dis-

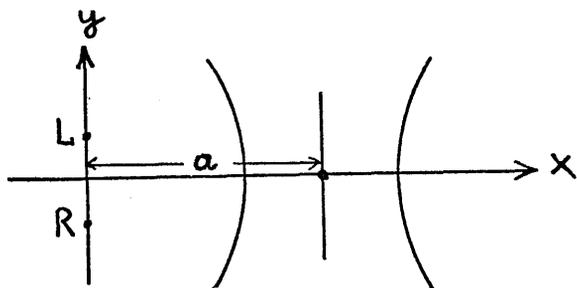


FIG. 3. Horopter curves for different fixations.

¹ H. V. Helmholtz, *Treatise on Psychological Optics*, J. P. C. Southall, Editor (Optical Society of America, 1925), Vol. 3, pp. 482 f.

² F. Hillebrand, *Denkschr. Akad. Wiss. Wien, math.-nat. Kl.* 72, 255 (1902).

³ W. Blumenfeld, *Zeits. f. Physiol. d. Sinnesorgane* 65, 241 (1913).

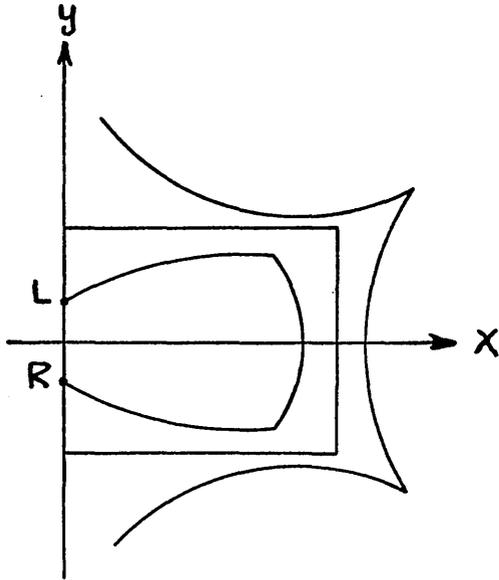


FIG. 5. Equivalent configurations.

tance function, $D(P_1, P_2)$, which accounts quantitatively for the observations just described. In other words, we assume that to any two points P_1, P_2 of the visual space, a non-negative number can be assigned which measures the sensed distance, (P_1P_2) , between these two points.

Any such distance function must satisfy the following conditions:

- (1) $D(P_1, P_2) = D(P_2, P_1) > 0$, if P_1, P_2 are sensed as different.
- (2) $D(P_1, P_2) = 0$ if $P_1 = P_2$.
- (3) $D(P_1, P_2) + D(P_2, P_3) \cong D(P_1, P_3)$ for any 3 points P_1, P_2, P_3 .

The physical distance $D^*(P_1^*, P_2^*)$ though it obviously satisfies the condition (4.1) cannot be considered as a distance function of the corresponding sensations because it would not measure the sensed distance correctly. In fact, we have already seen that it would not satisfy the following further condition which must be fulfilled by the distance function:

- (1) If (P_1, P_2) and (P_3, P_4) are any two pairs of sensed points and $(P_1, P_2) > (P_3, P_4)$ then $D(P_1, P_2) > D(P_3, P_4)$.
- (2) If P_1, P_2, P_3 are sensed on an apparently straight line, then $D(P_1, P_2) + D(P_2, P_3) = D(P_1, P_3)$ and vice versa.

A distance function $D(P_1, P_2)$ which satisfies both conditions (4.1) and (4.2) is called a *psychometric distance function*. Our problem is to determine this function for the visual space, and, if possible, to express it in terms of the physical coordinates of the stimulating points P_1^*, P_2^* .

5. THE INDETERMINACY OF THE DISTANCE FUNCTION

First we ask whether such a function is uniquely determined by the preceding conditions. For if a multiplicity of possible distance functions existed, any one of them would give us merely an empirical description of certain experimental findings. But if the function is uniquely determined, the establishment of its explicit form must represent an interesting psychological result concerning the nature of visual sensations.

Obviously, D is not uniquely determined since along with $D(P_1, P_2)$ any other function $\bar{D}(P_1, P_2) = C \cdot D(P_1, P_2)$, C being a positive constant, satisfies the conditions. It can be shown, however, that this indeterminacy is the only one possible.

In order to show this mathematically, we formulate certain basic assumptions which are concerned with the compactness and the convexity of the visual space.

(a) *The visual space is finitely compact:* To any infinite sequence P_n such that $D(P_1, P_n)$ is bounded, there exists a subsequence P_{n_r} and a point P of the visual space such that $D(P_{n_r}, P) \rightarrow 0$.

(b) *The visual space is internally convex:* To any pair of points $P_1, P_2 (P_1 \neq P_2)$ there exists a third point Q such that $D(P_1, Q) + D(Q, P_2) = D(P_1, P_2)$.

General postulates of this nature, of course, cannot be tested by experiments since an infinity of points is involved. Yet, they are not arbitrary assumptions but formulations of certain observations which we make with regard to our own visual sensations.

The uniqueness of the psychometric distance function can now be readily deduced. Suppose there are two such functions $D(P_1, P_2)$ and $\bar{D}(P_1, P_2)$, both satisfying all requirements. Then, whenever $(P_1P_2) > (P_3P_4)$ then $D(P_1, P_2) > D(P_3, P_4)$ and $\bar{D}(P_1, P_2) > \bar{D}(P_3, P_4)$. It follows that $\bar{D}(P_1, P_2) > \bar{D}(P_3, P_4)$ whenever $D(P_1, P_2) > D(P_3, P_4)$. This is possible only if $\bar{D} = f(D)$, for all points where D and \bar{D} are defined. Consider next any two points P_1, P_2 of the visual space and assume that $D(P_1, P_2) = a > 0$. As a consequence of compactness and internal convexity, it can be shown that to any number t of the interval $0 \leq t \leq a$ there exists a point Q such that $D(P_1, Q) = t$ and $D(Q, P_2) = a - t$. On account of the straight line condition (4.2) it follows that $\bar{D}(P_1, Q) + \bar{D}(Q, P_2) = \bar{D}(P_1, P_2)$, or $f(t) + f(a - t) = f(a)$ for any t in $0 \leq t \leq a$ and for any a determined by a pair of points P_1, P_2 . This leads to the conclusion $f(t) = Ct$ and hence $\bar{D}(P_1, P_2) = C \cdot D(P_1, P_2)$, where C is a constant.

Thus, the only possible indeterminacy of the psychometric distance function is a multiplicative constant. This means that the ratio $D(P_1, P_2)/D(P_3, P_4)$ of two distances is a uniquely determined function of the four points in question. This result is in agreement with the observation made previously that the inequality sign in the sensation $(P_1P_2) > (P_3P_4)$ is determined by the physical coordinates of the points $P_1^*, P_2^*, P_3^*, P_4^*$. We conclude that the ratio $D(P_1, P_2)/D(P_3, P_4)$ will be a function of the physical coordinates of the four stimu-

lating points which is invariant for an individual observer. It establishes a constant relationship between the objective physical space and the subjective visual space of the observer which is unaffected by other psychological factors. Any parameters in this relation, which are not the physical coordinates of the four points, must be constant factors of the personality of the observer, characterizing his visual reaction to external nature.

6. THE FORM OF THE DISTANCE FUNCTION

In order to find an explicit expression for the psychometric function $D(P_1, P_2)$, we make use of certain general psychological observations and of certain instructive experiments. These observations are concerned with the thesis that there is no absolute localization even in binocular vision. A given physical configuration provides no immediate visual clues which could determine localization and absolute phenomenological size of the sensed configuration. Localization and size are determined by other non-visual psychological factors, in particular, by the intellectual experience of the observer.

Depending upon such factors, the same physical configuration may be seen in different localizations and different phenomenological sizes. Vice versa, entirely different physical configurations may give rise to the same visual sensation.

This principle is readily seen to apply to monocular vision, in fact, this is the tacit assumption in reproducing a visual sensation by painting. Its validity in binocular vision is indicated by observations of an isolated point. But there are more convincing demonstrations.

Ames of the Dartmouth Eye Institute has shown that to a given ordinary rectangular room, a set of distorted rooms can be constructed, which, if viewed from a fixed position, are indistinguishable from the original room (Fig. 5). In fact, the illusion of seeing the original room in all cases is inescapable if suitable suggestive details, for example, window frames, are included. Obviously, it is our experience of living in rectangular rooms which causes us to see the rectangular form even though the physical reality is quite different. The distorted rooms, of course, cannot be constructed arbitrarily. Certain metric clues must be identical in all of them.

In order to combine this principle of the lack of absolute localization with our previous result concerning the existence of a psychometric distance function we are forced to the conclusion that $D(P_1, P_2)$ must be a distance function of a homogeneous metric space, that is, a Riemannian space of constant curvature. For only in these spaces form and localization are completely uncorrelated, which means that the form of a configuration provides no clues for its localization.

The preceding considerations warrant the introduction of the hypothesis of constant curvature, but there

is another observation which adds strong support to the hypothesis. Observations in the dark room reveal that we not only have an immediate sensation of straightness but also of plane surfaces. If light points or pliable wires are arranged in apparently plane surfaces, the resulting physical surfaces are curved. If any two points of a "visual plane" are connected by a visually straight line, it would not leave the visual plane. It is well known that two arbitrarily located dimensional submanifolds of this type exist only in the homogeneous spaces. Thus, the very existence of an immediate sensation of "planeness" can be considered as additional evidence in favor of the hypothesis of constant curvature.

The mathematical consequence of this hypothesis is that there exists in the visual space a subjective coordinate system, such that the psychometric distance function $D(P_1, P_2)$ is given by the formula:

$$\frac{2}{(-K)^{\frac{1}{2}}} \sin \left[\frac{1}{2} (-K)^{\frac{1}{2}} \cdot \frac{D}{C} \right] = \frac{[(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2]^{\frac{1}{2}}}{\left[\left(1 + \frac{K}{4} \rho_1^2 \right) \left(1 + \frac{K}{4} \rho_2^2 \right) \right]^{\frac{1}{2}}} \tag{6.1}$$

where $\rho_n^2 = \xi_n^2 + \eta_n^2 + \zeta_n^2$, and K and C are real valued constants which are as yet undetermined. We interpret ξ, η, ζ as cartesian coordinates in a Euclidean space. Thus we obtain a Euclidean map of the visual space, which, in general, is not isometric, since the distance of two points is given by a distance function (6.1) which is non Euclidean if $K \neq 0$.

The question whether the geometry of the visual space is elliptic, hyperbolic or Euclidean, i.e. whether in (6.1) $K > 0, K < 0$ or $K = 0$ can only be answered by

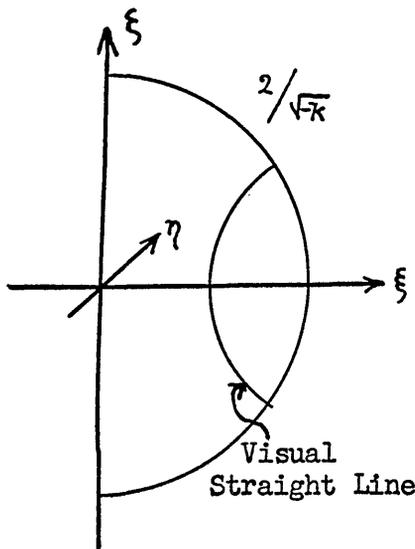


FIG. 6. A visually straight line represented in hyperbolic space.

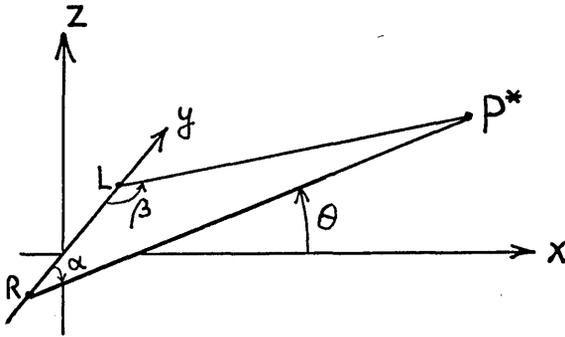


FIG. 7. Bipolar coordinates of a physical point.

experiments. All experiments carried out thus far indicate that K is negative, which means that the visual space must be a part of a hyperbolic or Lobachevski space.

The total hyperbolic space is represented in our map by the interior of the sphere $\xi^2 + \eta^2 + \zeta^2 = -4/K$. The visual straight lines are given by circles which intersect the basic sphere at right angles; all of these geodesics are thus represented by plane curves which lie in the planes through the origin (Fig. 6).

The visual planes are represented by spheres which intersect the basic sphere at right angles.

Because of the symmetry of the metric (6.1) with regard to the origin, we identify the origin $\xi = \eta = \zeta = 0$ with the apparent center of observation in the visual space. Furthermore we coordinate the ξ, η -plane to the subjective horizontal plane, the ξ, ζ -plane to the median plane, and the η, ζ -plane to the frontal plane.

7. RELATION OF VISUAL TO PHYSICAL SPACE

Next we establish a relation of the subjective coordinates ξ, η, ζ of a sensed point P to the physical coordinates of its stimulating point P^* . As stated before, no such relation can be constant or necessary since there is no absolute localization in the visual space. Hence, if we found one possible relation $(x, y, z) = f(\xi, \eta, \zeta)$ between the coordinates, any other relation $(x, y, z) = f(\xi', \eta', \zeta')$ would be visually equivalent provided that $(\xi', \eta', \zeta') = g(\xi, \eta, \zeta)$ represents a motion, i.e. a congruence transformation of the visual space.

The problem of finding a coordination of the two spaces can actually be solved only individually since the physiological condition of the eyes is a determining factor. In case of defective vision, the results are different for observations with or without glasses. The visual space and its metric, of course, are independent of these conditions; only the correlation of visual sensations to the physical qualities of external objects is affected. Since such differences, therefore, can be removed to a certain degree by corrective glasses, we are justified in choosing among the possible coordinations one which applies to a normal observer under usual conditions.

For an observer whose eyes are functioning equally well, we may assume that objects in the physical x, y -

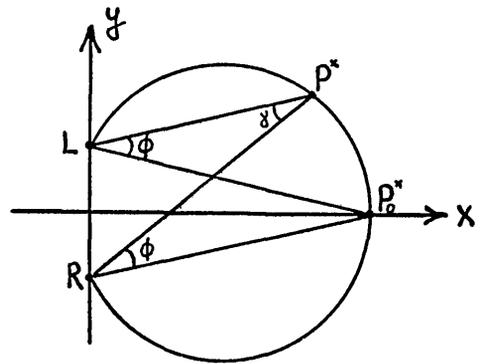


FIG. 8. Bipolar parallax and bipolar latitude of a physical point.

plane are seen in the subjective horizontal ξ, η -plane, and also, that points in the x, z -plane are placed into the median ξ, ζ -plane. For the further description of the required relationship we introduce, in the physical space, a special bipolar system of angular coordinates which is better adapted to the physiological mechanism of binocular vision. Consider the plane through a point P^* and through the eyes R and L . Its angle θ with the x, y -plane is called the *angle of elevation*. In this plane of elevation, two lines are drawn from R and L toward P^* . These lines determine the positions of the optical axes of the eyes if the point P^* is fixated. We characterize these lines by their angles α and β with the y axis as shown in Fig. 7. The localization of the point P^* is then determined by the three angles α, β, θ . Instead of α and β we shall use the linear combinations:

$$\begin{aligned} \gamma &= \pi - \alpha - \beta \\ \phi &= \frac{1}{2}(\beta - \alpha) \end{aligned} \tag{7.1}$$

which are called the *bipolar parallax* and the *bipolar latitude* of a point P^* (Fig. 8). In any plane of elevation, the curves $\gamma = \text{const.}$ are circles through the eyes, the so-called Vieth-Mueller circles, whereas the curves $\phi = \text{const.}$ are hyperbolas, the Hillebrand hyperbolas, whose asymptotes go through the point $x = y = z = 0$ (Fig. 9).

If a Vieth-Mueller circle is rotated about the y axis, a torus surface $\gamma = \text{const.}$ is obtained which shall be called a *Vieth-Mueller torus*.

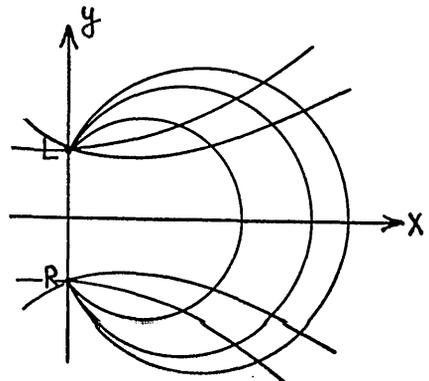


FIG. 9. Curves of constant bipolar parallax and latitude.

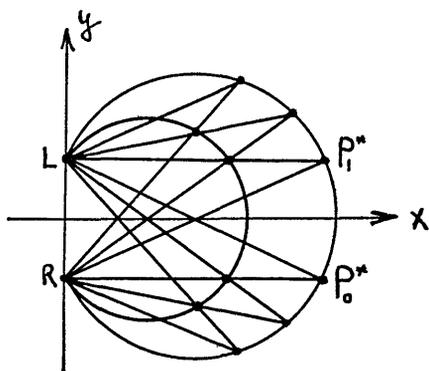


FIG. 10. Testing correlation between physical and visual spaces on Vieth-Mueller circle by changing convergence.

For infinitely distant points of the physical space, we have $\gamma=0$. Since these points give the visual impression of forming an apparently spherical dome of finite radius around the observer,[‡] we represent them, in the ξ, η, ζ -space, as points on a sphere concentric to the origin. This sphere will include the total visual part of the visual space. We choose arbitrarily a sphere of radius 2 to represent physical infinity.

This normation makes the constant K in the expression (6.1) for $D(P_1, P_2)$ a dimensionless parameter, with individual differences in K indicating differences in visual perception.

Since we have no visual sensations of infinite size, we conclude that K can never be smaller than -1 , so that the basic sphere, $2/(-K)^{1/2}$, in case $K \leq 0$, lies always outside the sphere of radius 2 which represents physical infinity. The two spheres coincide if $K = -1$; in which case astronomical objects would give an immediate sensation of immense size and distance. Though this limiting case is never reached, we conclude that visual sensation of size and distance relations agrees with physical reality better the nearer the value of K is to -1 . Thus, individual differences in the parameter K may correlate with individual differences in certain mechanical skills.

Further indication of the correlation of the ξ, η, ζ -space to the physical space is obtained by the following observation. We arrange a number of equidistant light points on a Vieth-Mueller circle, as shown in Fig. 10. These points are seen in the visual space as equidistant points on a circle about the center of observation. If the distance between the points P_0^*, P_1^* is made equal to the interpupillary distance of the eyes, the configuration can be "projected" toward infinity ($\gamma=0$) by simply diverging the eyes to a position of parallel optical axes. By overconverging, the configuration may be projected on smaller Vieth-Mueller circles (Fig. 10). In this way we can conveniently verify the

[‡] In a more careful investigation, the fact that the sky appears to be a flattened dome could be considered. Thus, the following arguments are to be taken as approximations referring to visual sensation in the neighborhood of the horizontal plane.

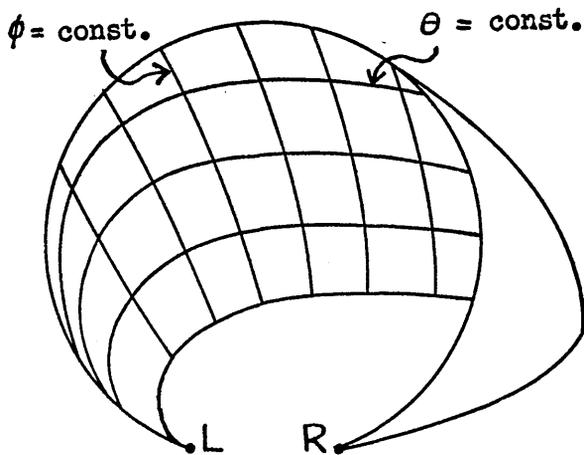


FIG. 11. Torus of constant binocular parallax, showing curves of constant elevation and constant binocular latitude.

original observation for a set of such circles. If the point configuration is rotated about the y axis, and arrested at a number of equidistant θ positions, a set of points is obtained which lie on certain curves $\phi = \text{const.}$ and $\theta = \text{const.}$ of the Vieth-Mueller torus $\gamma = \text{const.}$ Wires placed along these curves would form a network like that shown in Fig. 11. If this is observed from the points R and L , the resulting sensation is approximately that of meridians and circles of latitude on a sphere around the observer. This means that the torus surfaces $\gamma = \text{const.}$ appear as concentric spheres around the center of observation, and the bipolar angles ϕ and θ as polar coordinates in the ξ, η, ζ -space. The observation may be verified for a number of such torus configurations by simply overconverging or diverging the eyes as before. Mathematically, these observations are expressed by the equations:

$$\begin{aligned} \xi &= f(\gamma) \cos \phi \cos \theta \\ \eta &= f(\gamma) \sin \phi \\ \zeta &= f(\gamma) \cos \phi \sin \theta \end{aligned} \tag{7.2}$$

which relate physical and subjective coordinates. The function $f(\gamma)$ is a non-negative function of γ which increases monotonically if γ decreases from positive values to $\gamma=0$. From the preceding convention it follows that $f(0) = 2$.[§]

Even for the neighborhood of the horizontal plane, this formulation should be considered no more than a first approximation. There is experimental evidence of small Vieth-Mueller circles giving an elliptical rather than a circular impression. Therefore, it is possible that a more precise formulation may require the replacement of the relations (7.2) by the following:

[§] A more careful experimentation may lead to the result that, at greater distances, the network configuration is not seen as a sphere but as an oval, flattened at the top. The previously mentioned impression of the sky, and also the so-called moon phenomenon, seem to indicate this possibility. In this case, the relation $\zeta = f(\gamma) \cos \phi \sin \theta$ would have to be replaced by a relation $\zeta = f_1(\gamma) \cos \phi \sin \theta$ with $f_1(\gamma) < f(\gamma)$.

$\xi = f_1(\gamma) \cos \phi \cos \theta, \quad \eta = f_2(\gamma) \sin \phi, \quad \zeta = f_3(\gamma) \cos \phi \sin \theta,$
with three different functions $f_1(\gamma), f_2(\gamma), f_3(\gamma)$.

8. BOUNDARIES OF VISUAL AND PHYSICAL SPACE IN THE HORIZONTAL PLANE

It is interesting to note that the experiments of Helmholtz, Hillebrand, and Blumenfeld can be fully explained on the basis of these results, that is, without any further information about the function $f(\gamma)$. Consider the transformations (7.2) in the horizontal plane:

$$\begin{aligned} \xi &= f(\gamma) \cos \phi \\ \eta &= f(\gamma) \sin \phi. \end{aligned} \tag{8.1}$$

This transformation is continuous and one-to-one in the interior of the half-plane $x > 0$; it is, however, quite irregular on the boundary (Fig. 12). In fact, the boundary of this half-plane is characterized by the following conditions for the bipolar coordinates γ and ϕ :

- A: $\gamma = 0, \phi = \pi/2, \quad \text{or } x = 0, y > 1$
- L: $\gamma = \pi - 2\phi, \quad \text{or } x = 0, y = +1 \text{ (left eye)}$
- B: $\gamma = \pi, \phi = 0, \quad \text{or } x = 0, -1 < y < 1$
- R: $\gamma = \pi + 2\phi, \quad \text{or } x = 0, y = -1 \text{ (right eye)}$
- C: $\gamma = 0, \phi = -\pi/2, \quad \text{or } x = 0, y < -1$

The Sections A, B, C of the y axis are contracted into three isolated points A: $\xi = 0, \eta = 2$; B: $\xi = f(\pi), \eta = 0$, C: $\xi = 0, \eta = -2$ whereas the two eyes are stretched out into two curves $\rho_1 = f(\pi - 2\phi), \rho_2 = f(\pi + 2\phi)$ which connect the point B with A and C respectively. These two curves join at B at an acute angle, as follows from the fact that $f(\gamma)$ increases with decreasing γ , i.e. $f'(\gamma) < 0$, and hence $d\rho_1/d\phi > 0, d\rho_2/d\phi < 0$.

The half-plane $x > 0$ is thus imaged into a scythe-shaped domain of the ξ, η -plane as shown in Fig. 13. The image of the half-space $x > 0$ is obtained by simply rotating this domain around the η -axis.

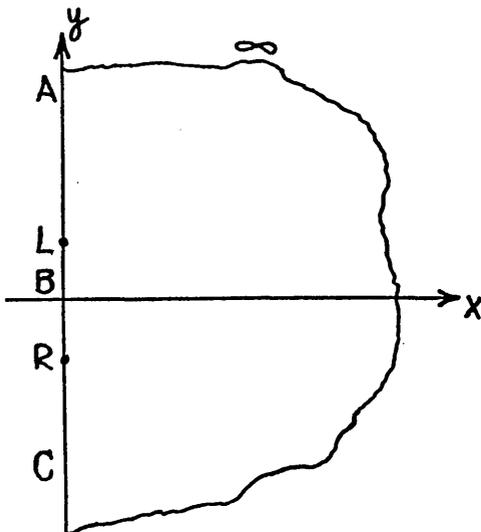


FIG. 12. Boundaries of the horizontal half-plane in physical space.

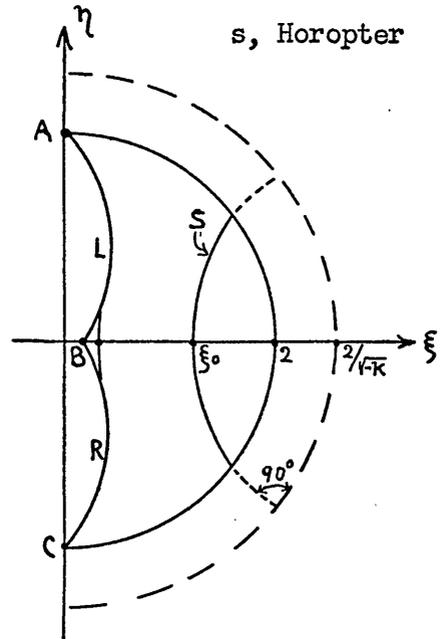


FIG. 13. Boundaries of the horizontal half-plane of visual space and a frontal plane horopter, S, mapped on a ξ, η -diagram.

9. HOROPTERS AND ALLEYS

We interpret Helmholtz' horopter curves as those geodesics of the ξ, η -plane which are symmetric to the ξ axis (Fig. 13). Their equation is:

$$\frac{K}{4} - (\xi^2 + \eta^2) - 1 = C\xi. \tag{9.1}$$

If $K < 0$, they are circles normal to the basic circle $2/(-K)^{1/2}$ and normal to the ξ axis. Two different groups of circles can be distinguished. Starting from a point ξ_0 of the ξ axis, they either reach directly the circle $\xi^2 + \eta^2 = 4$ representing infinity, or they are intercepted by the curves representing the eyes.

Their counterparts in the physical x, y -plane, therefore, must be curves which, starting from x_0 , either go

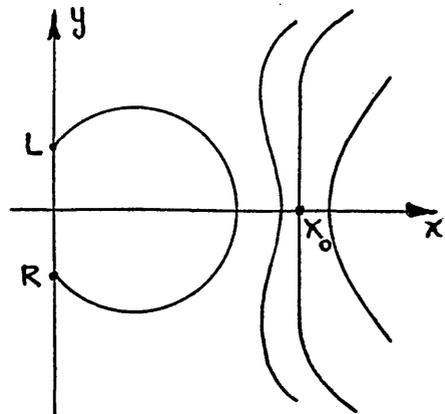


FIG. 14. Frontal plane horopters.

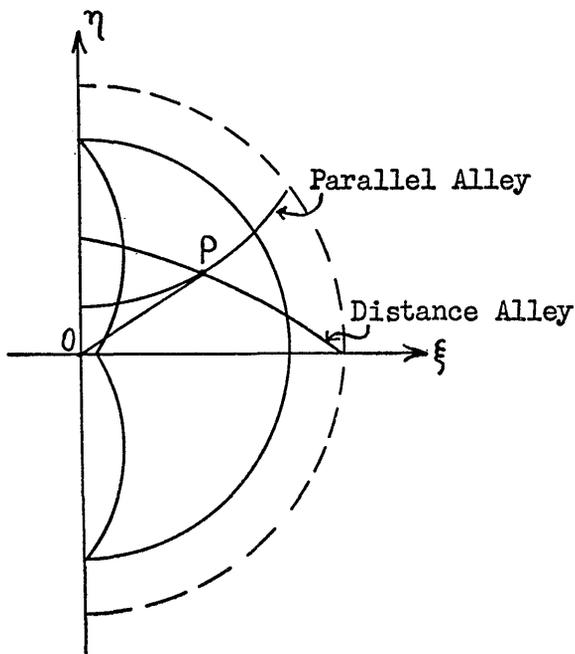


FIG. 15. Distance and parallel alleys in map of subjective horizontal plane.

asymptotically toward infinity or approach the eyes R and L . Thus they are curves of precisely the type observed in the horopter experiment. In particular, there must be one curve, which, in the neighborhood of the x axis, is practically straight (Fig. 14).

This situation not only exists for the case $K < 0$, but also for $K = 0$ or $K > 0$. If $K = 0$, the horopter curves (9.1) are straight lines, if $K > 0$, they are circles through the points $\eta = \pm 2/(K)^{1/2}$ of the η axis. In either case, as before, the two different groups can be distinguished.

The distance alleys of Blumenfeld are given by the equation:

$$\frac{K}{4}(\xi^2 + \eta^2) + 1 = C\eta. \tag{9.2}$$

If $K < 0$, they are circles through the points $\xi = \pm 2/(-K)^{1/2}$ of the ξ axis. Since these curves lie to the left of the radius vector OP to any of their points, it follows that their physical counterparts must lie above the corresponding sections of the hyperbolas $\phi = \text{const.}$ This means that they must have the tendency of bending away from the curves $\phi = \text{const.}$ which is characteristic for the alley curves in the Blumenfeld experiment.

The parallel alleys of Hillebrand, given by the geodesics:

$$\frac{K}{4}(\xi^2 + \eta^2) - 1 = -C\eta \tag{9.3}$$

are symmetric to the η axis. Hence, for $K < 0$ they are circles normal to the basic circle $2/(-K)^{1/2}$ and normal to the η axis. Also these curves lie above any radius vector OP and thus have the same characteristic form as the distance alleys.

If a parallel alley and a distance alley are started from the same point P , the parallel alley lies inside the distance alley, in agreement with actual observations (Figs. 15 and 16).

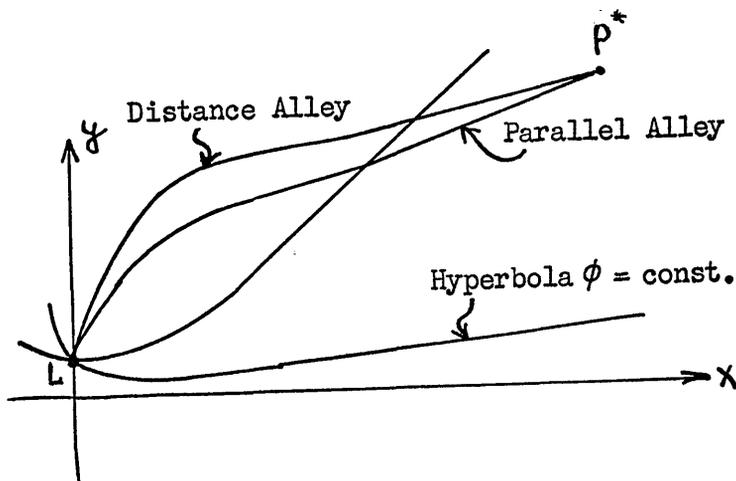
If $K = 0$, parallel alleys and distance alleys are identical, namely, straight lines $\eta = \text{const.}$ in the ξ, η -plane. For $K > 0$, distance alleys are represented by circles normal to the sphere $\rho = 2/(K)^{1/2}$, parallel alleys by circles through the points $2/(K)^{1/2}$ of the ξ axis. In this case, therefore, the distance alley should lie inside the parallel alley. The Blumenfeld experiment contradicts this, thus giving a direct demonstration that the space of binocular vision is a Lobachevski space.

10. USE OF ALLEY DATA TO DETERMINE $f(\gamma)$ AND K

The Hillebrand-Blumenfeld alleys can be used to determine the function $f(\gamma)$ and the constant K . Consider a distance and a parallel curve which starts from the same point:

$$\begin{aligned} \xi_0 &= f(\gamma_0) \cos \phi_0 \\ \eta_0 &= f(\gamma_0) \sin \phi_0. \end{aligned} \tag{10.1}$$

FIG. 16. Distance and parallel alleys in physical horizontal plane.



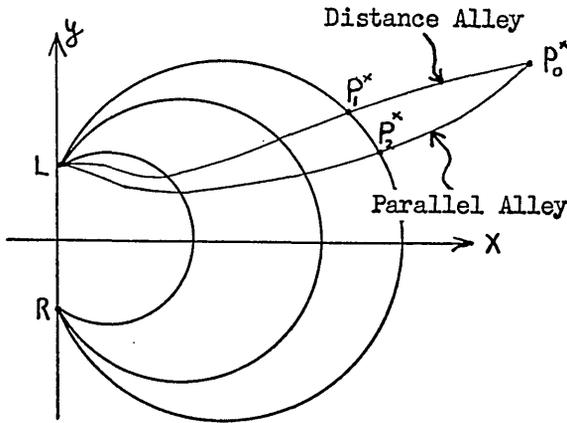


FIG. 17. Distance and parallel alleys with a common point in a system of Vieth-Mueller circles.

From (9.2) and (9.3) it follows that

$$\frac{\frac{K}{4}f^2(\gamma)+1}{f(\gamma)\sin\phi_1} = \frac{\frac{K}{4}f^2(\gamma_0)+1}{f(\gamma_0)\sin\phi_0} \quad (\text{distance curve})$$

and

$$\frac{\frac{K}{4}f^2(\gamma)-1}{f(\gamma)\sin\phi_2} = \frac{\frac{K}{4}f^2(\gamma_0)-1}{f(\gamma_0)\sin\phi_0} \quad (\text{parallel curve}),$$

or

$$\frac{K}{4}f^2(\gamma_0)\left(\frac{f(\gamma)}{f(\gamma_0)} - \frac{\sin\phi_1}{\sin\phi_0}\right) = -\frac{f(\gamma_0)}{f(\gamma)} + \frac{\sin\phi_1}{\sin\phi_0} \quad (10.2)$$

$$\frac{K}{4}f^2(\gamma_0)\left(\frac{f(\gamma)}{f(\gamma_0)} - \frac{\sin\phi_2}{\sin\phi_0}\right) = -\frac{\sin\phi_2}{\sin\phi_0} + \frac{f(\gamma_0)}{f(\gamma)}$$

A Vieth-Mueller circle of parallax γ is intersected by the two curves at P_1^* and P_2^* of different bipolar lati-

tudes, ϕ_1 and ϕ_2 (Fig. 17). We determine the ratios

$$S_1 = \frac{\sin\phi_1}{\sin\phi_0} \quad \text{and} \quad S_2 = \frac{\sin\phi_2}{\sin\phi_0}$$

by measurements. From (10.2) it follows that

$$\frac{\frac{f(\gamma)}{f(\gamma_0)} - S_1}{\frac{f(\gamma)}{f(\gamma_0)} - S_2} = -\frac{\frac{f(\gamma_0)}{f(\gamma)} - S_1}{\frac{f(\gamma_0)}{f(\gamma)} - S_2},$$

or

$$\frac{1}{2}\left(\frac{f(\gamma)}{f(\gamma_0)} + \frac{f(\gamma_0)}{f(\gamma)}\right) = \frac{1+S_1S_2}{S_1+S_2},$$

and hence

$$\frac{f(\gamma_0)}{f(\gamma)} = \frac{1+S_1S_2 + [(S_1^2-1)(S_2^2-1)]^{\frac{1}{2}}}{S_1+S_2} \quad (10.3)$$

Since parallel and distance curves are in general not far apart, we may replace S_1 and S_2 by an intermediate value $\sin\phi/\sin\phi_0 = S$. Then it follows from (10.3) that

$$\frac{f(\gamma_0)}{f(\gamma)} \approx S = \frac{\sin\phi}{\sin\phi_0} \quad (10.4)$$

i.e. the function $\log f(\gamma)$ is obtained directly if a distance or a parallel alley is plotted in the coordinates γ and $\log \sin\phi$.

The experiments seem to indicate that the function $\log f(\gamma)$ in first approximation can be considered as a linear function of γ . Then $\log f(\gamma) = -\sigma\gamma + C$ or $f(\gamma) = 2e^{-\sigma\gamma}$, σ being a new parameter of the observer.

The constant $(K/4)f^2(\gamma_0)$ can be found from (10.2) by introducing the expression (10.3) in one of the Eqs.

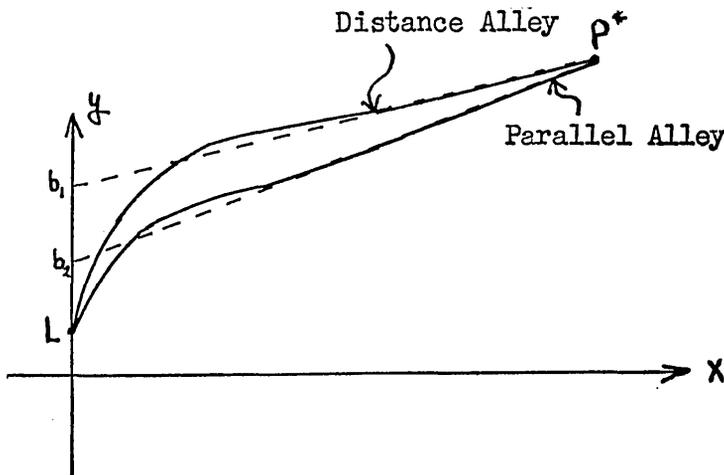


FIG. 18. Tangents to distance and parallel alleys at the common point.

(10.2). The result may be written as follows:

$$\frac{-\frac{K}{4}f^2(\gamma_0)+1}{\frac{K}{4}f^2(\gamma_0)+1} = \frac{(S_1^2-1)^{\frac{1}{2}} (\sin^2\phi_1-\sin^2\phi_0)^{\frac{1}{2}}}{(S_2^2-1)^{\frac{1}{2}} (\sin^2\phi_2-\sin^2\phi_0)^{\frac{1}{2}}}$$

This yields at the end point of the alleys

$$\left[\frac{1-\frac{K}{4}f^2(\gamma_0)}{1+\frac{K}{4}f^2(\gamma_0)} \right]^2 \frac{d\phi_1}{d\gamma} = \frac{b_1}{b_2}$$

It is readily shown that $2(d\phi_1/d\gamma)=b_1$ and $2(d\phi_2/d\gamma)=b_2$ determine the intersection of the tangents of the two curves at P^* with the y axis (Fig. 18). Thus, the ratio b_1/b_2 can be used directly to find the constant $(K/4)f^2(\gamma_0)$. If the two alleys start at the same infinitely distant point, we have $f(\gamma_0)=2$, and hence the simple relation

$$\left(\frac{1-K}{1+K} \right)^2 = \frac{b_1}{b_2}$$

11. METHODS OF MEASURING K AND $f(\gamma)$

The alley experiments provide a convincing demonstration of the non-Euclidean hyperbolic character of the binocular visual space. But they are not too well suited for the problem of measuring the parameters of the metric. The mechanical construction has to be quite elaborate, if undesirable clues are to be eliminated that would invalidate the measurements. The setting up of a pair of alleys requires time, especially with new observers. Consistency is generally lacking at the beginning so that each new observer must go through a series of preparatory experiments.

In order to remove these difficulties, other methods

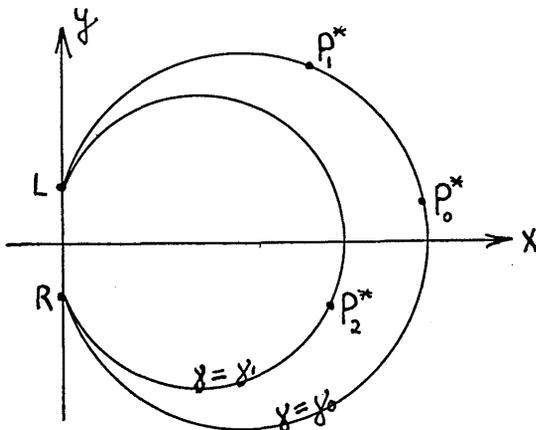


FIG. 19. Points on two Vieth-Mueller circles.

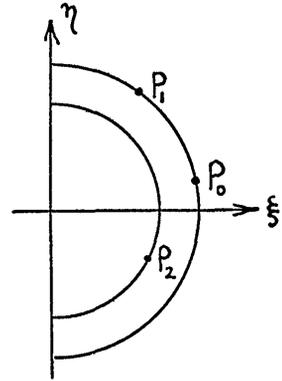


FIG. 20. The subjective correlation of Fig. 19.

of measurement had to be investigated. The following method so far seems to promise the best results.

Three isolated light points P_0^*, P_1^*, P_2^* are given in the horizontal plane. P_0^* and P_1^* are movable on a fixed Vieth-Mueller circle of parallax γ_0 , P_2^* on a second fixed Vieth-Mueller circle of parallax γ_1 (Fig. 19).

In the ξ, η -plane, these points are represented by the two points P_0 and P_1 on a circle of radius $\rho_0=f(\gamma_0)$, and by a third point P_2 on a circle $\rho_1=f(\gamma_1)$ (Fig. 20). Both circles are concentric to the origin. The task of the observer is to set the point P_2^* in such a position on its circle that the visual distance $D(P_0P_2)$ is equal to the visual distance $D(P_0P_1)$. This is carried out for a number of settings P_0^*, P_1^* on the Vieth-Mueller circle γ_0 . From (6.1) it follows that the distances are equal if

$$\frac{\{2\rho_0^2[1-\cos(\phi_1-\phi_0)]\}^{\frac{1}{2}}}{1+\frac{K}{4}\rho_0^2} = \frac{[\rho_0^2+\rho_1^2-2\rho_0\rho_1\cos(\phi_2-\phi_1)]^{\frac{1}{2}}}{\left(1+\frac{K}{4}\rho_0^2\right)^{\frac{1}{2}}\left(1+\frac{K}{4}\rho_1^2\right)^{\frac{1}{2}}}, \quad (11.1)$$

i.e. if

$$4 \sin^2\frac{1}{2}(\phi_2-\phi_0) = \frac{\rho_0}{\rho_1} \frac{1+\frac{K}{4}\rho_1^2}{1+\frac{K}{4}\rho_0^2} \times 4 \sin^2\frac{1}{2}(\phi_1-\phi_0) - \left[\left(\frac{\rho_0}{\rho_1}\right)^{\frac{1}{2}} - \left(\frac{\rho_1}{\rho_0}\right)^{\frac{1}{2}} \right]^2. \quad (11.2)$$

The quantities

$$A = \frac{\rho_0}{\rho_1} \frac{1+\frac{K}{4}\rho_1^2}{1+\frac{K}{4}\rho_0^2}$$

and

$$B = \left[\left(\frac{\rho_0}{\rho_1} \right)^{\frac{1}{2}} - \left(\frac{\rho_1}{\rho_0} \right)^{\frac{1}{2}} \right]^2 = \frac{\rho_0}{\rho_1} + \frac{\rho_1}{\rho_0} - 2 \quad (11.3)$$

are constants of the apparatus and of the observer.

The quantities

$$X = 4 \sin^2 \frac{1}{2} (\phi_1 - \phi_0); \quad Y = 4 \sin^2 \frac{1}{2} (\phi_2 - \phi_0) \quad (11.4)$$

are measured after each setting. It follows that

$$Y = AX - B, \quad (11.5)$$

so that a graphical plot of Y against X should yield a straight line. The experimental points are actually found scattered about a certain average straight line. The scattering may be considerable for a new observer, but it decreases rapidly with some training. It is interesting to note that young children display a remarkable degree of certainty and consistency in such observations.

From a suitable number of observations we obtain by the method of least squares the best straight line representation of the observation, and from the constant B the ratio

$$\frac{\rho_1}{\rho_0} = \frac{f(\gamma_1)}{f(\gamma_0)},$$

and from A the quantity $K \cdot f^2(\gamma_0)$.

Repetition of these experiments with other pairs of Vieth-Mueller circles yields further ratios $f(\gamma_{i+1})/f(\gamma_i)$, and thus, with $f(0)=2$, a numerical tabulation of $f(\gamma)$ is obtained. Measurements such as these, as well as other observations, seem to indicate that a first approximation of the ratio $f(\gamma_1)/f(\gamma_0)$ depends only on the difference $\gamma_1 - \gamma_0$. This implies that $\log f(\gamma)$ is a linear function of γ , and hence

$$f(\gamma) = 2e^{-\sigma\gamma}, \quad (11.6)$$

where σ is an individual constant of the observer.

The numerical value of σ characterizes the degree of depth perception of a person. There is no doubt that considerable differences exist in the quality of depth perception; σ -values ranging from 4 to 15 have been found.

The parameter σ can be determined from observations on a single pair of Vieth-Mueller circles. We have $2 \sinh(\sigma/2)(\gamma_1 - \gamma_0) = (B)^{\frac{1}{2}}$, and hence, in sufficient approximation

$$\sigma = \frac{(B)^{\frac{1}{2}}}{\gamma_1 - \gamma_0}. \quad (11.7)$$

If σ is known, the constant A can be utilized to calculate K . However, if the two Vieth-Mueller circles are not far apart, the variation of A with respect to K is not sufficient to provide a good measurement of this constant. A more effective arrangement is the following:

Consider four points $P_0^*, P_1^*, P_2^*, P_3^*$. The first pair is movable on a Vieth-Mueller circle γ_0 , the second pair on a smaller circle of parallax $\gamma_1 > \gamma_0$. The task of the observer is to place the point P_3^* , for a given setting of P_0^*, P_1^*, P_2^* so that

$$D(P_0, P_1) = D(P_2, P_3).$$

This leads to the condition

$$2 \sin^2 \frac{1}{2} (\phi_3 - \phi_2) = \frac{\rho_0}{\rho_1} \frac{1 + \frac{K}{4} \rho_1^2}{1 + \frac{K}{4} \rho_0^2} 2 \sin^2 \frac{1}{2} (\phi_1 - \phi_0) \quad (11.8)$$

or

$$Y = AX$$

where

$$X = 2 \sin^2 \frac{1}{2} (\phi_1 - \phi_0) \quad \text{and} \quad Y = 2 \sin^2 \frac{1}{2} (\phi_3 - \phi_2).$$

A graphical plot of the observed quantities Y against the settings X should yield a straight line through the origin. If the two Vieth-Mueller circles are chosen far enough apart, the direction A of this straight line varies considerably with K . Hence, if σ is known, we find K from the relation*

$$A = e^{\sigma(\gamma_1 - \gamma_0)} \cdot \frac{1 + Ke^{-2\sigma\gamma_1}}{1 + Ke^{-2\sigma\gamma_0}}. \quad (11.9)$$

12. THE HOMOGENEITY OF VISUAL SPACE

The essential feature of the preceding experiments is that distance observations are made on sets of points that are confined to two fixed Vieth-Mueller circles, that is, to curves which are sensed as two subjective circles around the observer. A consequence of this restriction is that certain simple trigonometric functions of the observed angular differences become linear functions of each other, thus providing an elegant method of statistical evaluation. This applies even if the points do not lie in the horizontal plane or the same plane of elevation, provided they are confined to two different Vieth-Mueller torus surfaces, that is, to two surfaces sensed as two subjective spheres around the observer. In this more general case, spherical arcs Ω_{ik} , for instance the arc Ω_{12} between two points P_1^*, P_2^* given by

$$\cos \Omega_{12} = \cos \phi_1 \cos \phi_2 \cos(\theta_2 - \theta_1) + \sin \phi_1 \sin \phi_2,$$

replace the differences $\phi_k - \phi_i$ in the horizontal plane.

Hence, with $X = 4 \sin^2 \frac{1}{2} \Omega_{01}$; $Y = 4 \sin^2 \frac{1}{2} \Omega_{02}$, in the

* This formula is based on the approximation $f(\gamma) = 2e^{-\sigma\gamma}$. If this approximation is not accepted, the quantity K may be found from

$$A = \frac{f(\gamma_0)}{f(\gamma_1)} \frac{1 + \frac{K}{4} f^2(\gamma_1)}{1 + \frac{K}{4} f^2(\gamma_0)},$$

if $f(\gamma)$ is known from observations of the first type.

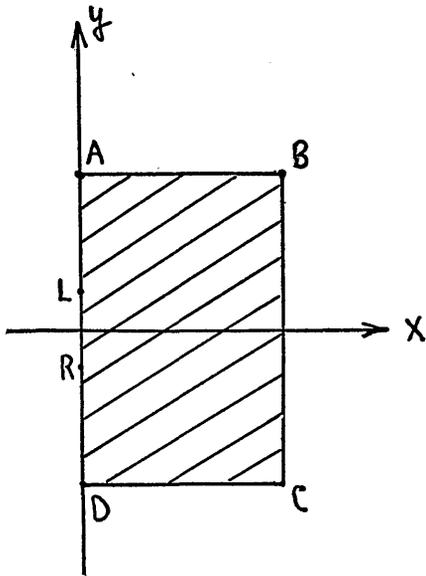


FIG. 21. Horizontal cross-section of Ames' experimental room.

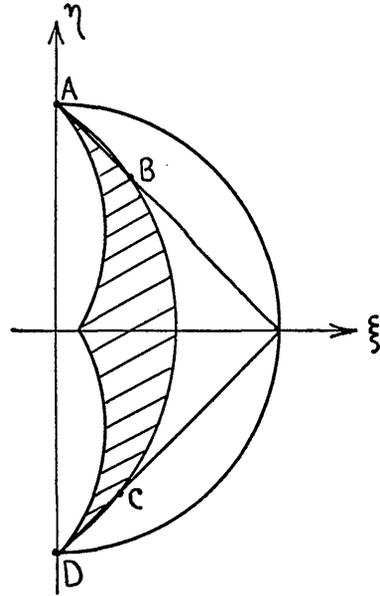


FIG. 22. Rectangle of Fig. 21 mapped in the subjective plane.

experiment of the first type, we should still get points X, Y scattered about the same straight line $Y = AX - B$.

And with $X = 2 \sin \frac{1}{2} \Omega_{01}$; $Y = 2 \sin \frac{1}{2} \Omega_{23}$, we should find points about $Y = AX$ in the second experiment. This follows from the special form of the distance function $D(P_0, P_1)$:

$$\frac{2}{(-K)^{\frac{1}{2}}} \sin \left[\frac{1}{2} (-K)^{\frac{1}{2}} \cdot \frac{D}{C} \right] = \frac{(\rho_0^2 + \rho_1^2 - 2\rho_0\rho_1 \cos \Omega_{01})^{\frac{1}{2}}}{\left(1 + \frac{K}{4}\rho_0^2\right)^{\frac{1}{2}} \left(1 + \frac{K}{4}\rho_1^2\right)^{\frac{1}{2}}} \quad (12.1)$$

More generally, this is true for any metric space whose distance function $D(P_0, P_1)$ has the form

$$D(P_0, P_1) = F[M(\rho_0, \rho_1) + N(\rho_0, \rho_1) \cos \Omega_{01}] \quad (12.2)$$

where $M(\rho_0, \rho_1)$ and $N(\rho_0, \rho_1)$ are symmetric functions of ρ_0 and ρ_1 .

Since the linearity of the graphical representation is one of the most consistent observations in these experiments, the question may be considered whether this fact should be interpreted as an experimental verification of the hypothesis of constant curvature. In other words: Are the homogeneous spaces the only Riemannian spaces where the distance function $D(P, Q)$ can be written in the form (12.2)?

Since the metric, defined by (12.2), is symmetric to the origin, the associated line element ds^2 can be assumed to have the form

$$ds^2 = n^2(\rho)(d\xi^2 + d\eta^2 + d\zeta^2) \quad (12.3)$$

so that the distance function $D(\xi_0, \eta_0, \zeta_0; \xi, \eta, \zeta)$ must

be a solution of the partial differential equations

$$\begin{aligned} D_{\xi}^2 + D_{\eta}^2 + D_{\zeta}^2 &= n^2(\rho) \\ D_{\xi_0}^2 + D_{\eta_0}^2 + D_{\zeta_0}^2 &= n^2(\rho_0). \end{aligned} \quad (12.4)$$

The problem is therefore to determine those functions $n^2(\rho)$ for which the differential equations admit solutions of the special form (12.2).

The answer is that this is possible then, and only then, if $n(\rho)$ has the form

$$n(\rho) = \frac{C}{1 + \frac{K}{4}\rho^2} \quad (12.5)$$

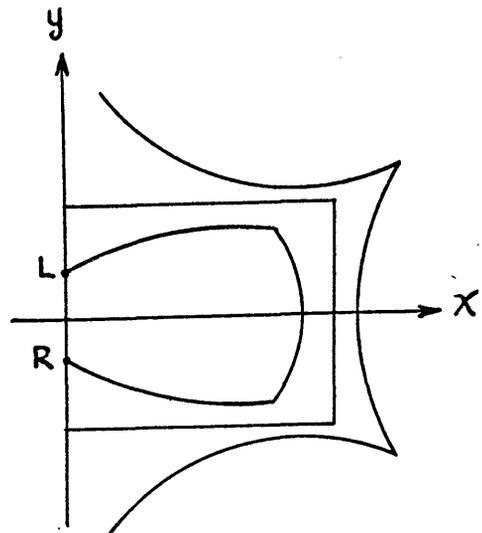


FIG. 23. Horizontal cross sections of visually congruent rooms.

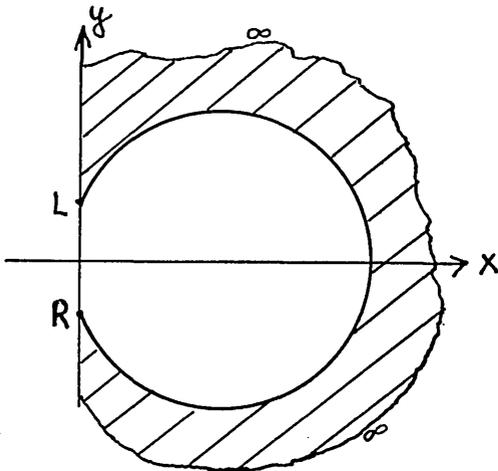
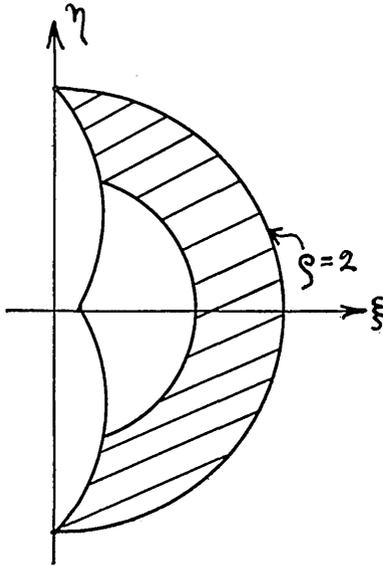


FIG. 24. A domain in the physical plane and the correlated domain in the subjective plane.

where C and K are constants. In other words, the space must be Riemannian space of constant curvature.

Therefore, the linearity of the relation between X and Y is indeed a direct indication of the homogeneous character of the visual space.

13. CONGRUENT CONFIGURATIONS⁴

There are a number of practical applications of the theory of binocular vision which depend on the knowledge of the numerical values of the parameters K and the function $f(\gamma)$. We consider first the possibility of psychometric reproduction of visual sensations. Since in homogeneous spaces a given configuration can be moved freely without metric distortion, we can con-

⁴R. K. Luneburg, *Mathematical Analysis of Binocular Vision* (Princeton University Press, Princeton, 1947); *Metric Methods in Binocular Visual Perception, Studies and Essays*, Courant Anniversary Volume (Interscience Publishers, Inc., New York, 1948).

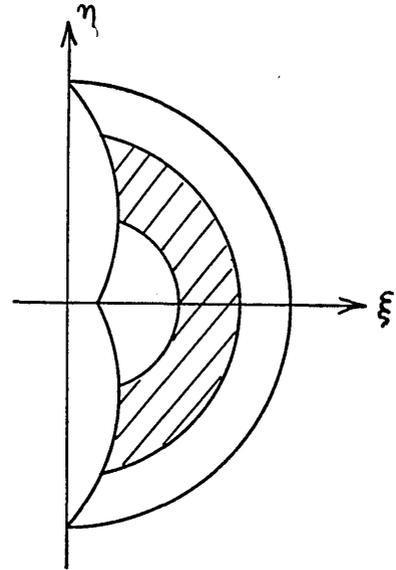


FIG. 25. The domain of Fig. 24 after a congruency-transformation.

struct, to any given configuration, a metrically congruent one in any localization or orientation.

The original rectangular room in the Ames experiment may serve as an example. The cross section made with the horizontal plane is shown in Fig. 21, its counterpart in the ξ, η -plane in Fig. 22.

Consider the following hyperbolic motion:

$$\begin{aligned} \xi' &= -a - \left(a^2 + \frac{4}{K}\right) \frac{\xi - a}{(\xi - a)^2 + \eta^2 + \zeta^2} \\ \eta' &= \left(a^2 + \frac{4}{K}\right) \frac{\eta}{(\xi - a)^2 + \eta^2 + \zeta^2} \\ \zeta' &= \left(a^2 + \frac{4}{K}\right) \frac{\zeta}{(\xi - a)^2 + \eta^2 + \zeta^2} \end{aligned} \tag{13.1}$$

which transforms median and horizontal planes into

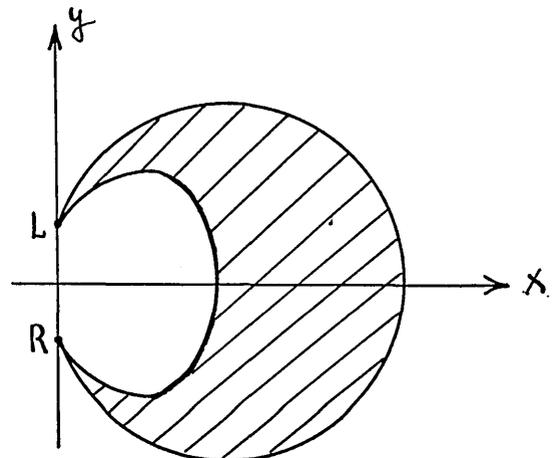
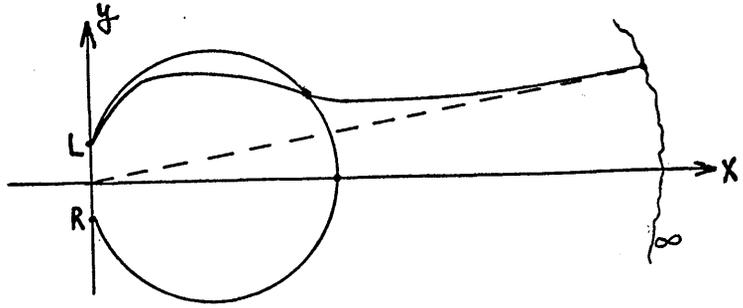


FIG. 26. The domain of Fig. 25 in the physical plane.

FIG. 27. Illustrating difference between psychometric representation and central projection.



themselves. Submitting the "room" *A, B, C, D* to this transformation, we find that all the points of the boundary move either to the right or to the left, depending on the parameter α . The physical counterparts of these visually congruent rooms are no longer rectangular but have the form shown in Fig. 23. They conform with Ames' experimental result, and can be used as a quantitative basis for the design of such rooms.

Next, we consider the problem of psychometric reproduction of landscapes, for example, a scenery which lies between a certain Vieth-Mueller circle γ and physical infinity $\gamma=0$. The corresponding domain in the ξ, η, ζ -space lies between two concentric spheres, $\rho=\rho(\gamma)$ and $\rho=2$ (Fig. 24). Submitting this domain to a transformation (13.1) [$\alpha > 2/(-K)^{\frac{1}{2}}$], we obtain a domain bounded by two non-concentric spheres as shown in Fig. 25.

The corresponding domain in the physical space is obtained by rotating the shaded area in Fig. 26 about the y axis. The entire original part of the space is condensed into this narrow shell. With the aid of (13.1) a relief reproduction of the original scenery could be constructed which is visually indistinguishable from the original.||

If the inner boundary $\rho = \rho(\gamma)$ is sufficiently far away, the relief becomes so flat that it may be replaced by a painting or projected image on the outer screen surface, which is metrically congruent to infinity. In order to find the coordination of a point on this screen, we can use the Blumenfeld alleys since points on these alleys have a constant visual distance from the x axis. If the screen is chosen so small that it lies in the domain where the distance alleys bend away from their asymptotes, the image points depart considerably from the position of central projection (Fig. 27). This explains why the main feature of a landscape appears often disappointingly small compared with the original impression. Therefore in order to obtain a true psychometric projection of a scenery, we should project upon a curved screen designed in accordance with explicit mathematical principles. Furthermore, we should use a projection lens which possesses a strong amount of specified pincushion distortion.

|| This illusion can be accomplished to a remarkable degree, as for instance the so-called Habitual groups of animal life in the Los Angeles County Museum demonstrate. As a matter of fact, it is nearly impossible to judge the actual dimensions of the scenery.

14. PHENOMENOLOGICAL SIZE AND DISTANCE

Our last result is related to a well-known fact. The phenomenological size of physical objects which are moving away from us, does not decrease in proportion to the angle of view. The size seems to remain constant for a while until, at greater distances, it decreases rapidly.

Consider two points P_0^* and P_1^* in the median plane and symmetric to the x axis (Fig. 28). We judge their apparent distance D according to the formula:

$$\frac{2}{(-K)^{\frac{1}{2}}} \sin \left[\frac{1}{2} (-K)^{\frac{1}{2}} \cdot \frac{D}{C} \right] = \frac{[2\rho^2(1 - \cos 2\theta)]^{\frac{1}{2}}}{\frac{K}{1 + \frac{\rho^2}{4}}} = \frac{2\rho \sin \theta}{1 + \frac{\rho^2}{4}} \quad (14.1)$$

Since

$$\sin \theta = \frac{z}{(x^2 + z^2)^{\frac{1}{2}}} = z \tan \frac{\gamma}{2}$$

and $\rho = 2e^{-\sigma\gamma}$, it follows that

$$\frac{2}{(-K)^{\frac{1}{2}}} \sin \left[\frac{1}{2} (-K)^{\frac{1}{2}} \cdot \frac{D}{C} \right] = z \frac{4 \tan \frac{\gamma}{2}}{e^{\sigma\gamma} + Ke^{-\sigma\gamma}} \quad (14.2)$$

If z is kept unchanged, but the position x of the two points is altered, D/C becomes a well-defined function

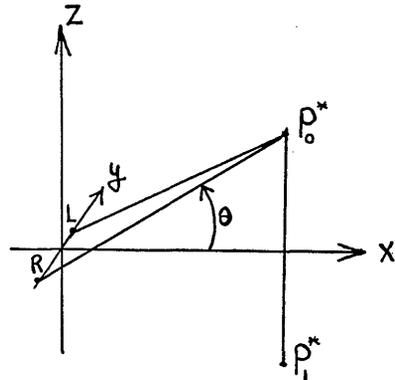


FIG. 28. Showing a line segment in the median plane.

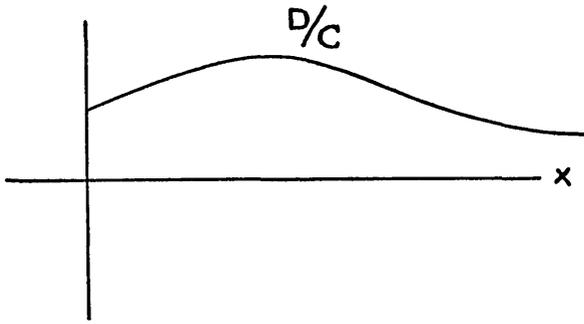


FIG. 29. Showing relative apparent size of line segment of Fig. 28 as a function of its position along the x axis.

of γ , that is, of the distance x of the two points from the observer. For small values of z we may write approximately

$$\frac{D}{C} = z \frac{4 \tan \frac{\gamma}{2}}{e^{\sigma\gamma} + K e^{-\sigma\gamma}} \tag{14.3}$$

Thus D approaches zero if $x \rightarrow \infty$, unless $K = -1$, in which case a finite limit is reached asymptotically (Fig. 29). We conclude that constancy of size is maintained longer, the nearer to -1 the value of the observer's constant K is found to be.*

15. APPLICATIONS TO BINOCULAR OPTICAL INSTRUMENTS

The interpretation of the visual space as a metric space provides a theoretical basis for an investigation of binocular optical instruments, such as range finders, binocular field glasses, binocular microscopes, stereoscopic projectors, or even ordinary spectacles. Such instruments do not change the geometric character of

* The same results are obtained qualitatively without the assumption $f(\gamma) = 2e^{-\sigma\gamma}$. Instead of (14.3) we have

$$\frac{D}{C} = z \frac{2f(\gamma)}{1 + \frac{K}{4} f^2(\gamma)}$$

the visual space but merely its coordination to the physical space. This fact may be expressed by stating that any optical instrument represents a certain point transformation of the visual space. Except for trivial cases, these transformations are not isometric, and therefore a metrically distorted view of external objects is introduced. A surface which originally gave the impression of a physically plane surface, may become apparently convex or concave. In a theoretical investigation of this situation the theory of curved surface embedded into a hyperbolic space thus finds a significant application.

16. INCORPORATION OF MOTION IN THEORY

The artificial restriction of the freedom of observation which has been introduced at the beginning is not maintained in usual visual sensations. We move our head, even our body, quite freely. This leads us to the problem of incorporating such motions into the theory of visual perception. Obviously, this problem is intimately related to the general problem of visual sensations of moving objects or of a moving observer. In order to approach this subject, a psychometric function should be established which measures the sensation of velocity of visual objects. Certain simple observations lead to the conclusion that this function is most probably given by a function of the type

$$w(v) = C \log \frac{c+v}{c-v}, \tag{16.1}$$

where c is a certain constant representing the extreme velocity of eye motions, whereas v is the image of the physical velocity in the ξ, η, ζ -space. This psychometric function is real valued only if $v < c$, which means that physical velocities $v > c$ are no longer recognized as velocity sensations. This corresponds to the fact that a moving light point produces the sensation of a streak of light if a certain velocity is exceeded. A space-time metric introduced to describe this situation mathematically will then be closely related to the space-time metric of special relativity.