EXPLANATORY SUPPLEMENT TO THE EPHEMERIS

EXPLANATORY SUPPLEMENT

TO

THE ASTRONOMICAL EPHEMERIS

AND

THE AMERICAN EPHEMERIS AND NAUTICAL ALMANAC

PREPARED JOINTLY BY THE NAUTICAL ALMANAC OFFICES
OF THE UNITED KINGDOM AND
THE UNITED STATES OF AMERICA

Issued by H.M. Nautical Almanac Office
by Order of the
Science Research Council

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PREFACE

The purpose of this *Explanatory Supplement* is to provide the users of *The Astronomical Ephemeris* (prior to 1960 entitled *The Nautical Almanac and Astronomical Ephemeris*) and *The American Ephemeris and Nautical Almanac* with fuller explanations of their content, derivation, and use than can conveniently be included in the publications themselves. A rigorous treatment is given of the fundamental basis of the tabulations; this is supplemented by a detailed derivation, showing how each tabulated quantity is obtained from basic data. The use of the ephemerides is also explained and illustrated, but completeness is not attempted. Auxiliary tables, lists of constants, and miscellaneous data are added, partly for convenience of use with the Ephemeris and partly for reference.

By its nature this Supplement must primarily be a reference book. However, it is hoped that certain sections will come to be regarded as full, connected, and authoritative treatments of the subjects with which they deal, and that the tables and other data will prove of general use in astronomical computing. An account of its origins and much information of a general nature about the purpose and scope of the unified Ephemeris is given in section 1, " Introduction ".

Although published in the United Kingdom, the *Explanatory Supplement* has been prepared jointly by the Nautical Almanac Office, United States Naval Observatory, under the immediate supervision of its Director, Edgar W. Woolard, and by H.M. Nautical Almanac Office, Royal Greenwich Observatory, under the immediate supervision of its Superintendent, D. H. Sadler. It has been edited by G. A. Wilkins, assisted by Miss A. W. Springett.

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January, 1960

NOTE ON 1974 REPRINT

It is regretted that it has not been possible to revise this *Explanatory Supplement* to take account of the many changes that have been made in *The Astronomical Ephemeris* and *The American Ephemeris and Nautical Almanac* since the editions for 1960. *The Supplement to the A.E. 1968* has, however, been reprinted, with change of pagination, before the Index of this volume; it includes a specification of the IAU system of astronomical constants, an account of its introduction into the almanacs for 1968, a list of the principal consequential changes in this *Explanatory Supplement*, and a list of the known errors in the original edition. The errata and corrections listed on pages 520 to 521 have all been carried through, or otherwise noted, on the relevant pages of this edition. Some other amendments have also been made; in particular, some of the reference data given in section 18 have been brought up to date. The changes described on pages 514 to 519 have not been made, although attention has normally been drawn in footnotes to the changes that would be appropriate to the new system of constants.

All changes in the bases of the ephemerides have been mentioned in the Prefaces to the editions in which they were first made, and corresponding changes have been made in the Explanations at the ends of the volumes. Even apart from these changes, this *Explanatory Supplement* is now out of date in a number of respects, and so should be used with care. In particular, the following points should be noted:

(a) Even where the basis of an ephemeris has not been changed, an improved method of computation may have been used, so that the numerical example may not define precisely the technique used.

(b) For certain purposes the printed fundamental ephemerides are of inadequate precision, but improved ephemerides are now available. Further details can be obtained from the Bureau International d'Information sur les Ephémérides Astronomiques, 3 Rue Mazarine, Paris (6^{e}), France.

(c) The second of the international system (SI) of units is now defined in terms of the frequency of a particular caesium resonance, and a scale of "international atomic time" is currently available for reference purposes. As from 1972 January 1 the principal time signals are based on a scale (UTC) that differs from IAT by an exact number of seconds and from UT1 by an amount that does not normally exceed 0·7 seconds.

It is hoped that, in spite of these deficiencies, this *Explanatory Supplement* will continue to be of value to all who require information in the fields that it aims to cover until such time as a completely revised edition can be prepared. General suggestions concerning the nature of such a revision, as well as notes on specific amendments, should be sent to the Superintendent of H.M. Nautical Almanac Office (G. A. Wilkins), Royal Greenwich Observatory, or to the Director of the Nautical Almanac Office (R. L. Duncombe), U.S. Naval Observatory.

January, 1973

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*See also reprint of the *Supplement to the A.E. 1968* at the end of this volume.

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1. INTRODUCTION

A. ORIGIN OF THE SUPPLEMENT

The Nautical Almanac for 1931 was completely redesigned; for the first time it included a comprehensive *Explanation* and a *Derivation* illustrating the calculation of every quantity tabulated in the Almanac. Although the *Derivation* was discontinued after that year, the *Explanation* was continued in full and was gradually expanded. This was the consequence partly of newly-added matter, requiring detailed explanation, and partly of more comprehensive illustrations of the use of the tabulated data, such as, for example, in the case of eclipses. The Almanacs for the years 1937, 1938, 1939, and 1940 contained (with appendices) 951, 940, 912, and 920 pages respectively. All of this added material was (and much still is) of considerable value, but much was inappropriate for the day-to-day use of the Almanac as an astronomical ephemeris; and much was of permanent rather than ephemeral interest. Many practical astronomers complained of the unwieldy volume and more than one suggested the separation of the permanent tables and explanations from the purely ephemeral data. The omission of most of the apparent places of stars in the edition for 1941, consequent on the introduction of the international volume of *Apparent Places of Fundamental Stars*, reduced the number of pages to 759. At this juncture a drastic cut was imposed on the overall size of subsequent editions by the exigencies of war. The opportunity was taken of inaugurating a policy that had been under consideration on its own merits. To quote from the *Preface* to the edition for 1942:

" It is intended that in future, starting with this edition, the *Nautical Almanac* should in general contain, in addition to the ephemeral data which will continue to be printed in the established form, only such auxiliary tables and explanations as are necessary for the user to extract the ephemeral data in the form he requires. In previous editions considerably more auxiliary tables and more detailed explanations than have been required for this purpose have been given and, although these have been of considerable benefit to some users, they have detracted from the convenience of the *Almanac* for the majority of routine observers and computers. As it has not been possible to include all the auxiliary tables, illustrations and explanations required in the application of the tabulated ephemeral data, the *Almanac* has never been completely self-contained; with this in mind, it is further intended to publish a separate supplement, which will be of a permanent character and which will contain all the permanent tables and explanations previously given, together with such added information as can be included in the rather wider scope provided by a separate publication. It is considered that this separation of the ephemeral data from the permanent tables and explanation wili not only lead to a desirable reduction in the size of the *Almanac*, but will also add to the convenience of the user requiring both books; it is easier to refer to two books at once than to two different places in the same book.

" It is possible that publication of the Supplement will be delayed for some time; in the meantime reference should be made to the relevant portions of previous editions."

It was, unfortunately, not possible to take any active steps towards the preparation of the promised " Supplement " until several years after the end of the war. At one time it was hoped that it would be possible to issue the Supplement to relate to *The Nautical Almanac* for 1952, and much work was actually done, particularly in the preparation of detailed examples of eclipse calculation; but this hope could not be fulfilled.

With the introduction of the concept of ephemeris time at the Paris (1950) Conference on the Fundamental Constants of Astronomy, it became clear that substantial changes in the Almanac could not be long delayed. This view was confirmed at the Rome (1952) General Assembly of the International Astronomical Union, when a series of recommendations involving fundamental changes in the ephemerides was agreed, to become effective as from 1960. The advantages of still further delaying the Supplement were evident; by relating it to the edition of 1960 it could present the new system as a unified whole, without the complication of a detailed explanation of the old. And it was accordingly agreed to introduce the Supplement as from 1960.

In 1954 the first steps were taken to achieve the " conformity " of *The Nautical Almanac* and *The American Ephemeris and Nautical Almanac;* and this has eventually led to their complete unification as from 1960. The plans for the publication of the Supplement naturally affected the contents of the unified Ephemeris, particularly in regard to the explanation and auxiliary tables; and as the Supplement would apply equally to *The American Ephemeris* it was natural that it should become a joint production.

This Supplement has accordingly been prepared jointly by the Nautical Almanac Office, U.S. Naval Observatory, and by H.M. Nautical Almanac Office. Although the latter has perforce accepted editorial responsibility, and the general work of compilation has been shared, the principal authors have been as follows:

In H.M. Nautical Almanac Office:

D. H. Sadler, Flora M. McBain Sadler, J. G. Porter, G. A. Wilkins, and H. W. P. Richards. H. M. Smith (Time Department) prepared section 15.

In Nautical Almanac Office, U.S. Naval Observatory:

G. M. Clemence, E. W. Woolard, Simone D. Gossner, and A. Thomas.

In both Offices other members of the staff, not named individually, have shared in the work of compilation and proof reading.

The note on page vi indicates the policy that has been adopted in the editing of this reprint of the original edition, and draws attention to its current deficiencies.

B. HISTORY OF THE EPHEMERIDES

The brief histories that follow are concerned solely with the major changes of form and content, and are intended as a general introduction to the detailed analyses given in section 7.

1. The Astronomical Ephemeris

"The Commissioners of Longitude, in pursuance of the Powers vested in them by a late Act of Parliament, present the Publick with the NAUTICAL ALMANAC and ASTRONOMICAL EPHEMERIS for the Year 1767, to be continued annually; a Work which must greatly contribute to the Improvement of Astronomy, Geography, and Navigation. This EPHEMERIS contains every Thing essential to general Use that is to be found in any Ephemeris hitherto published, with many other useful and interesting Particulars never yet offered to the Publick in any Work of this Kind. The Tables of the Moon had been brought by the late Professor MAYER of Gottingen to a sufficient Exactness to determine the Longitude at Sea, within a Degree, as appeared by the Trials of several Persons who made Use of them. The Difficulty and Length of the necessary Calculations seemed the only Obstacles to hinder them from becoming of general Use: To remove which this EPHEMERIS was made; the Mariner being hereby relieved from the Necessity of calculating the Moon's Place from the Tables, and afterwards computing the Distance to Seconds by Logarithms, which are the principal and only very delicate Part of the Calculus; so that the finding the Longitude by the Help of the EPHEMERIS is now in a Manner reduced to the Computation of the Time, an Operation . . . ."

"All the Calculations of the EPHEMERIS relating to the Sun and Moon were made from Mr. MAYER's last manuscript Tables, received by the Board of Longitude after his Decease, which have been printed under my Inspection, and will be published shortly. The Calculations of the Planets were made from Dr. HALLEY's Tables; and those of . . . ."

The above extracts from the *Preface* to the first edition, for 1767, of *The Nautical Almanac and Astronomical Ephemeris* were written by Nevil Maskelyne, then Astronomer Royal. The main incentive for, and the main emphasis of, the publication was the determination of longitude at sea using the method of lunar distances. The ephemerides were all given in terms of apparent solar time, for the reasons given in the *Explanation.*

"It may be proper first to premise, that all the Calculations are made according to apparent Time by the Meridian of the Royal Observatory at Greenwich."

"What has been shewn concerning the Equation of Time chiefly respects the Astronomer, the Mariner having little to do with it in computing his Longitude from the Moon's Distances from the Sun and Stars observed at Sea with the Help of the Ephemeris, all the Calculations thereof being adapted to apparent Time, the same which he will obtain by the Altitudes of the Sun or Stars in the Manner hereafter prescribed.

"But if Watches made upon Mr. John Harrison's or other equivalent Principles should be brought into Use at Sea, the apparent Time deduced from an Altitude of the Sun must be corrected by the Equation of Time, and the mean Time found compared with that shewn by the Watch, the Difference will be the Longitude in Time from the Meridian by which the Watch was set; as near as the Going of the Watch can be depended upon."

Apart from many changes in the sources of the data, and in particular the tables from which the Moon's position was calculated, the main pages of the Almanac remained essentially unchanged until 1834. For that year, to quote from the *Preface:*

"The NAUTICAL ALMANAC and ASTRONOMICAL EPHEMERIS for the Year 1834, has been constructed in strict conformity with the recommendations of the ASTRONOMICAL SOCIETY of LONDON, as contained in their Report . . . ; and will, it is believed, be found to contain almost every aid that the Navigator and Astronomer can require."

The changes were both fundamental and substantial, and involved almost doubling the size. The most fundamental change was to replace apparent time by mean time as the argument of the ephemerides. In the words of the *Report:*

"The attention of the Committee was, in the first instance, directed to a subject of general importance, as affecting almost *all* the results in the Nautical Almanac ; viz., whether the quantities therein inserted should in future be given for *apparent* time (as heretofore), or for *mean solar* time. Considering that the latter is the most convenient, not only for every purpose of Astronomy, but also (from the best information they have been able to obtain) for all the purposes of Navigation; at the same time that it is less laborious to the computer, and has already been introduced with good effect into the national Ephemerides of Coimbra and Berlin, the Committee recommend the abolition of the use of *apparent* time in all the computations of the Nautical Almanac; excepting . . . . "

The direction of at least some of the other changes was influenced by the view that was expressed in the *Report* as:

"And here perhaps it may be proper to remark, that, although in these discussions the Committee have constantly kept in view the principal object for which the Nautical Almanac was originally formed, viz., the promotion and advancement of *nautical* astronomy, they have not been unmindful that, by a very slight extension of the computations, and by a few additional articles (of no great expense or labour), the work might be rendered equally useful for all the purposes of *practical* astronomy."

The requirements of the navigator were by no means overlooked; in particular the number and presentation of "lunar distances", including distances from the planets, was greatly improved. However, the *Explanation and Use* of the previous editions, which had still been based on Maskelyne's, was replaced by a completely new *Explanation* in which little reference was made to the use of the ephemerides for navigation; tables of refraction were excluded and no example was given of clearing an observed lunar distance for the effects of semi-diameter, parallax, and refraction.

Apart from the omission of lunar distances in 1907 the first part of the Almanac, containing the ephemerides of the Sun and Moon, remained unchanged in form, though of course based from time to time on different data and tables, until 1931. At various dates other matter was added, particularly ephemerides of the Moon and planets at transit on the Greenwich meridian and the apparent places of many more stars; later, ephemerides for physical observations were added and in 1929, anticipating the redesign in 1931, ephemerides of the Sun referred to the standard equinox of 1950·0 were given for the years 1928 and 1929.

Much of the added matter was of no interest to the practical navigator, and in 1896 "Part I (containing such data as are more particularly required for navigational purposes)" was "also published separately for the convenience of sailors". This consisted of a straight reprint of the monthly pages comprising the first part of the Almanac, with selections from the other data and a few pages specially prepared.

In the *Preface* to the edition for 1914 it was announced briefly that " Part I has been remodelled for the convenience of sailors "; thus was introduced *The Nautical Almanac, Abridged for the Use of Seamen*, which was specially designed for its purpose. This Almanac was redesigned in 1929 and again in 1952, when it was renamed *The Abridged Nautical Almanac;* it was rearranged in a different form in 1958 and, as from 1960, it takes on the appropriate portion of the original title, namely *The Nautical Almanac.*

Prior to the revision in 1931 a fundamental change, requiring consequential changes in the Almanac, had taken place in the measure of mean solar time. Before 1925 the astronomical day was considered to start at noon, and the principal ephemerides had been given for 0^h, i.e. noon, on each day. As from 1925 January 1 the tabular day was brought into coincidence with the civil day and was considered to start at midnight; the ephemerides were still given for 0^h, now indicating midnight.

The revision of 1931 was much more than a rearrangement of the same data in a different form; the changes of page size, of presentation, of provision for interpolation, and of content were less important than the complete break with the century-old lay-out designed primarily for navigation for which the Almanac had ceased to provide. The new form could be, and was, designed for the astronomer without the necessity for considering the requirements of navigation. Its arrangement has remained basically unchanged, though there has been frequent change in content of the less fundamental matter.

Major changes were introduced in the edition for 1960 when the Almanac was unified with *The American Ephemeris*, the principal one being the use of ephemeris time, instead of universal time (mean solar time on the meridian of Greenwich), as the argument for the fundamental ephemerides. This change still further emphasized the unsuitability of the volume for navigation, and led to the adoption for its new title of the appropriate part of the original full title, namely *The Astronomical Ephemeris.* The changes are fully described in the *Preface* to the edition for 1960.*

2. The American Ephemeris and Nautical Almanac

During the first half of the nineteenth century, *The Nautical Almanac* remained in general use on American ships and among astronomers and surveyors in the United States. However, with the continued development of the country, and its growth as a maritime nation, an increasing need for a national almanac was felt and eventually led to the establishment of a Nautical Almanac Office in the Navy Department by an Act of Congress approved in 1849. The Office was set up in Cambridge, Massachusetts, where library and printing facilities were available, and began work during the latter part of 1849. The first volume of *The American Ephemeris and Nautical Almanac* was for the year 1855, and was published in 1852. The Office was moved to Washington in 1866, but was not located at the Naval Observatory until 1893.

*Some additional notes on the history of *The Astronomical Ephemeris* are given on pages ix–xviii of the volume for 1967, the two-hundredth anniversary edition.

For the years 1855–1915 inclusive, the volume was divided at first into two parts, then, beginning with 1882, into three. The first part during this entire period was an ephemeris for the use of navigators that was also reprinted separately, with the inclusion of a few pages from the remainder of the volume, as *The American Nautical Almanac*. It comprised 12 monthly sections, for the meridian of Greenwich, each containing ephemerides of the Sun, Moon, and lunar distances for the month; following the monthly sections were ephemerides of Venus, Mars, Jupiter, and Saturn for the year, and, beginning with 1882, of Mercury, Uranus, and Neptune.

The second part of the volume contained ephemerides of the Sun, Moon, planets, and principal stars, for meridian transit at Washington; and data on eclipses, occultations, and a few other phenomena, which in 1882 were formally grouped as a third part with the title " Phenomena ". The explanatory sections and a few miscellaneous tables completed the volume.

During the period 1855–1915, few changes were made in the form or content. The nautical part remained virtually unaltered; lunar distances were omitted, beginning with 1912, but a page explaining how to calculate them continued to be included. The principal revisions in the other parts of the volume were in 1882 and 1912–1913. The rearrangement of the 1882 volume was accompanied by some additions and omissions. The principal omission was the ephemeris of Moon-culminating stars for determining longitude. The principal additions were: the physical ephemerides of Mercury and Venus, in place of the former meagre data for the apparent disks, for the reduction of meridian and photometric observations; daily diagrams of the configurations of the four great satellites of Jupiter; and ephemerides for the identification of the satellites of Mars, Saturn, Uranus, and Neptune. In the volume for 1912, the ephemerides of the satellites were extended to include tables for determining the approximate position angle and apparent distance; in 1913, physical ephemerides were added for the Sun, Moon, Mars, and Jupiter. These revisions, and minor additions, omissions, and rearrangements, are described in more detail in section 7.

In the volume for 1916, the first to be issued under the international agreements resulting from the Paris Conference of 1911, extensive revisions were made in the form and arrangement that had been retained essentially unchanged since 1882; but the content remained substantially the same. The arrangement of the Greenwich ephemerides of the Sun and Moon by monthly sections was discontinued, and replaced by annual ephemerides. At the same time, *The American Nautical Almanac* was no longer a reprint of part of *The American Ephemeris*, but a separately prepared volume especially designed for the navigator.

In 1925, the astronomical reckoning of time from 0^h at noon was replaced by the civil reckoning from midnight.

During the interval from 1916 until the fundamental revisions in 1960 when *The American Ephemeris* was unified with *The Astronomical Ephemeris*, the revisions of form and content were mostly only in details; but a few major changes occurred, and in the volumes for 1934–1937 a number of further subdivisions and rearrangements of the contents were made. In 1937, the volume had become formally divided into seven parts; the part constituting the ephemeris for Washington had been reduced to only ephemerides of the Sun, Moon, and planets for meridian transit at Washington, all the other material having been transferred to other parts and referred to the Greenwich meridian.

Because of the limited usefulness of the Washington-transit ephemerides except to observers on the Washington meridian, the publication of this part was discontinued beginning with the 1951 volume. Otherwise, the general form and arrangement adopted in 1937 were retained until 1960. The other principal changes in content during 1916–1959 were the following: In 1919, tables of the rising and setting of the Sun and the Moon were added. In 1941, the number of stars for which apparent places were given, after having reached 887, was decreased to 212 when *Apparent Places of Fundamental Stars* was first published; in 1957, apparent places were omitted entirely, but precise mean places of 1551 stars which had been given beginning with 1951 were continued. The elements and predictions of occultations were successively extended to more and fainter stars, and to additional standard stations, because of their importance for determining the departures of the Moon from gravitational theory that are due to variations in the rotation of the Earth. An ephemeris of Pluto was added to the planetary ephemerides in 1950; and ephemerides of Ceres, Pallas, Juno, and Vesta in 1952.

C. HISTORY OF INTERNATIONAL CO-OPERATION

Formal co-operation may be regarded as dating from the International Meridian Conference held in Washington in October 1884 at the invitation of the Government of the United States. The resolutions of that conference included:

"... the adoption of the meridian passing through the centre of the transit instrument at the Observatory of Greenwich as the initial meridian for longitude."

"That from this meridian longitude shall be counted in two directions up to 180 degrees, east longitude being plus and west longitude minus."

"... the adoption of a universal day for all purposes for which it may be found convenient ..."

"That this universal day is to be a mean solar day; is to begin for all the world at the moment of mean midnight of the initial meridian, coinciding with the beginning of the civil day and date of that meridian; and is to be counted from zero up to twenty-four hours."

"That the Conference expresses the hope that as soon as may be practicable the astronomical and nautical days will be arranged everywhere to begin at mean midnight."

Although the other resolutions are now in use, it has been customary for many years in astronomy, but not in all other related sciences, to treat west longitude as positive, and east longitude as negative. This is the convention adopted in the Ephemeris.

At the invitation of the Bureau des Longitudes the directors of the national ephemerides, and other astronomers, met in Paris in May 1896 for the Conférence

Internationale des Étoiles Fondamentales. In addition to adopting resolutions concerning the fundamental catalogue, and the calculation and publication of apparent places of stars, the Conference adopted the following fundamental constants:

Nutation	$9''\cdot 21$
Aberration	$20''\cdot 47$
Solar parallax	$8''\cdot 80$

which are still in operation. It also agreed to adopt Newcomb's definitive values (which were not then in final form) of luni-solar and planetary precession.

Active co-operation between the offices of the national ephemerides dates from the Congrès International des Éphémérides Astronomiques held at the Paris Observatory in October 1911. This conference was called, on the initiative of the Bureau des Longitudes, by B. Baillaud, Director of the Observatory and President of the Comité International Permanent de la Carte Photographique du Ciel. Its purpose was " d'établir une entente permettant d'augmenter, sans nouveaux frais, la masse des données numériques fournies annuellement aux observateurs et aux calculateurs ". Although the Conference was primarily concerned in obtaining a greatly increased list of apparent places of stars, it extended its attention to all the ephemerides of bodies in the solar system. Its comprehensive recommendations covered the distribution of calculations between the five principal ephemeris offices (France, Germany, Great Britain, Spain, and the United States), specified standards of calculation and presentation, arranged for publication of additional data, and fixed the values of two further constants to be used in the ephemerides: the flattening of the Earth (1/297) and the semi-diameter of the Sun at unit distance for eclipse calculations ($15'\ 59''\cdot 63$). Most of these recommendations are still in force.

Official approval was in some cases necessary for the adoption of these recommendations, as illustrated by the following extract from the Act of Congress of August 22, 1912 (37 Stat. L., 328, 342):

" The Secretary of the Navy is hereby authorized to arrange for the exchange of data with such foreign almanac offices as he may from time to time deem desirable, with a view to reducing the amount of duplication of work in preparing the different national nautical and astronomical almanacs and increasing the total data which may be of use to navigators and astronomers available for publication in the American Ephemeris and Nautical Almanac: *Provided* . . ."

Here follows a number of provisions, the most important astronomically being the repeal of the proviso in the appropriation Act of September 28, 1850 (9 Stat. L., 513, 515) that " hereafter the meridian of the observatory at Washington shall be adopted and used as the American meridian for all astronomical purposes, and that the meridian of Greenwich shall be adopted for all nautical purposes ".

Such exchange agreements have been carried out in spite of international difficulties.

In 1919 the International Astronomical Union was founded; Commission 4 (Ephemerides), which numbers among its members the directors of the national ephemerides, thereafter provided the formal contacts by which the previous agreements could be continued and extended.

Flattening the Earth?

The 1911 agreements had been directed almost entirely to the reduction of the total amount of work by the avoidance of duplicate calculation. In 1938 Commission 4 recommended that the principle should be extended to the avoidance of duplicate publication by the collection in a single volume of the apparent places of stars then printed in each of the principal ephemerides. This recommendation, coupled with the adoption of the *Dritter Fundamentalkatalog des Berliner Astronomischen Jahrbuchs* (FK3), was implemented for 1941 by the publication, under the auspices of the International Astronomical Union, of the international volume *Apparent Places of Fundamental Stars*. By this means astronomers gained access to the apparent places of stars in one volume, and the individual ephemeris offices were saved the work of the compilation and proof reading, as well as the cost of type setting, of most of the stars which they previously published.

Continuing the precedents of the 1896 and 1911 conferences, the Director of the Paris Observatory (Professor A. Danjon) convened a further conference that was held in Paris in March 1950 to discuss the fundamental constants of astronomy. The leading recommendation was " . . . . that no change be made in the conventionally adopted value of any constant ". But the recommendations with the most far-reaching consequences were those which defined ephemeris time and brought the lunar ephemeris into accordance with the solar ephemeris in terms of ephemeris time. These recommendations were addressed to the International Astronomical Union and were formally adopted by Commission 4 and the General Assembly of the Union in Rome in September 1952.

Commission 4 had, at various times, made arrangements for the redistribution of calculations between the ephemeris offices; for example, the Institute for Theoretical Astronomy in Leningrad contributed apparent places of stars to the international volume for the years 1951–1959. With the availability of fast automatic calculating machines it is now both practicable and efficient for large blocks of work, such as the calculation of apparent places of stars, to be done in one office; and at the 1955 General Assembly of the Union in Dublin, a general redistribution of calculations on these lines was agreed by the directors of the national ephemerides and confirmed by Commission 4. Full details of these agreements, of changes in the bases of the ephemerides, and of the discussions leading to the introduction of *Apparent Places of Fundamental Stars* are given in the reports of Commission 4 in *Transactions of the International Astronomical Union*.

The logical development of this co-operation would appear to be a single international ephemeris; this is not yet practicable. Following the successful unification of the navigational almanacs, and greatly assisted by the common language, it was however agreed in 1954 to unify the British and American ephemerides as from the year 1960; and this has now been done. In reporting this agreement to Commission 4, it was announced that reproducible material for the whole Ephemeris, with the exception of the short introductory section, would be made available to other ephemeris offices through H.M. Stationery Office at a small fee. And the hope was expressed that use would be made of this facility to effect a considerable saving of type setting and proof reading, while still preserving for each country its own ephemeris with its own language headings and explanations and its own selection of material.

The *Berliner Astronomisches Jahrbuch* (published annually since 1776) and the *Astronomisch-Geodätisches Jahrbuch* (introduced for the year 1947) ceased publication with the years 1959 and 1957 respectively; in Germany either the British or American editions of the unified Ephemeris will be used and there will be no separate German edition.

References

The proceedings, recommendations, and resolutions of the international conferences referred to above have been published as follows:—

Protocols of the Proceedings of the *International Conference held at Washington for the purpose of fixing a Prime Meridian and a Universal Day. October* 1884. Washington, D.C., 1884.

Procès-Verbaux of the *Conférence Internationale des Étoiles Fondamentales de* 1896. Paris, Bureau des Longitudes, 1896.

Congrès International des Éphémérides Astronomiques tenu à l'Observatoire de Paris du 23 *au* 26 *Octobre* 1911. Paris, Bureau des Longitudes, 1912. A full account, with English translations of the resolutions, is given in *M.N.R.A.S.*, **72**, 342-345, 1912.

Colloque International sur les Constantes Fondamentales de l'Astronomie. Observatoire de Paris, 27 Mars—1er Avril 1950. *Colloques Internationaux du Centre National de la Recherche Scientifique*, **25**, 1-131, Paris, 1950. The proceedings and recommendations are also available in *Bull. Astr.*, **15**, parts 3-4, 163-292, 1950. *

The reports and recommendations of Commission 4 of the International Astronomical Union have been published as follows:

Trans. I.A.U.,	Assembly	
1, 159, 207; 1923.	Rome	1922
2, 18–19, 178, 229; 1926.	Cambridge, England	1925
3, 18, 224, 300; 1929.	Leiden	1928
4, 20, 222, 282; 1933.	Cambridge, Mass.	1932
5, 29–33, 281–288, 369–371; 1936.	Paris	1935
6, 20–25, 336, 355–363; 1939.	Stockholm	1938
7, 61, 75–83; 1950.	Zürich	1948
8, 66–68, 80–102; 1954.	Rome	1952
9, 80–91; 1957.	Dublin	1955
10, 72, 85–99; 1960.	Moscow	1958
11, **A**, 1–8; 1962. **B**, 164–167, 441–462; 1962.	Berkeley	1961
12, **A**, 1–10; 1965. **B**, 101–105, 593–625; 1966.	Hamburg	1964
13, **A**, 1–9; 1967. **B**, 47–53, 178–182; 1968.	Prague	1967
14, **A**, 1–9; 1970. **B**, 79–85, 198–199; 1971.	Brighton	1970

D. SCOPE AND PURPOSE OF THE EPHEMERIS

The Astronomical Ephemeris and *The American Ephemeris and Nautical Almanac* are identical in content and presentation, apart from a few preliminary pages. Except in the few cases where distinction is desirable they will be referred to collectively as " the Ephemeris " or by the initials *A.E.*

Scope. Now that other publications provide for the practical requirements of navigators and surveyors, the Ephemeris need no longer do so. Its content is

*See page 174 for references to proceedings of later conferences on the system of astronomical constants.

accordingly restricted to providing fundamental ephemerides of the Sun, Moon, and planets to the highest precision, and ephemerides derived from them for the requirements of the practical astronomer.

Fundamental ephemerides. The main purpose of the fundamental ephemerides * of the Sun, Moon, and planets is to provide a rigorous continuous reference system, to which observations, if necessary spread over many years, can be referred. In order to achieve this the ephemerides should be calculated strictly in accordance with a self-consistent theory, which can be specified precisely in regard to both form and numerical constants. It will suffice that the adopted constants be close enough to their true values for any possible variations to lead to linear changes in the ephemerides; but it is important that all known physical forces and effects be fully incorporated.

The ephemerides are calculated in accordance with the Newtonian law of gravitation, modified by the theory of general relativity. The values of the adopted constants are given partly in section 6 and partly in section 4 under the individual body concerned; most are collected together in section 18. The independent variable of the ephemerides is ephemeris time, which is independent of the unpredictable variations in the speed of rotation of the Earth. The highest standard of precision in the calculations is achieved for the five outer planets Jupiter, Saturn, Uranus, Neptune, and Pluto; the calculations for the Sun, Mercury, Venus, and Mars do not at present reach the same standard. For the Moon other requirements are very severe; extremely accurate values of the Moon's motion, over short intervals of time, are required for the determination of relative positions on the Earth through observations of eclipses and occultations; consistent positions of the Moon over long intervals (say 10 years) of time are needed for the practical determination of the length of the fundamental unit of time, the ephemeris second. But the precision of the ephemeris is reasonably adequate for the present.

It is convenient, but not necessary, that the fundamental ephemerides should give positions sufficiently close to the actual positions to provide for the observational astronomer and as a basis for further predictions. The ephemerides are, in fact, amply close enough for this secondary purpose in terms of ephemeris time; but the correction to universal time is large enough to make its application necessary for the ephemeris of the Moon.

Other data. The only other data of a fundamental character given in the Ephemeris are those required for the calculation of apparent places of stars; these include the values of precession and nutation required to specify the observational frame of reference. Apparent places themselves are not included as they are given in *Apparent Places of Fundamental Stars.* Some deduced data, such as the Besselian elements of eclipses, are of the highest precision; but generally all other ephemeral data are intended to assist observation and are not of adequate accuracy for precise comparison with observations. In particular the theories on which the orbits of the satellites are based are too imperfect to provide ephemerides of a fundamental character.

*In most cases new ephemerides of higher precision and accuracy are now available, but the ephemerides in the *A.E.* provide a useful common standard of reference.

E. SCOPE AND PURPOSE OF THE SUPPLEMENT*

As stated in the *Preface* the purpose of this *Explanatory Supplement* is to provide users of the Ephemeris with fuller explanations of its content, derivation, and use than can conveniently be included in the Ephemeris itself. To a limited extent it also provides the auxiliary tables and reference data required in the application of the data tabulated in the Ephemeris; but, because of the availability of other publications and of changing methods of calculation, these requirements are much less than when the Supplement was first proposed in 1940.

In particular it has been decided not to include a section on observatories, as originally planned. The list of observatories in the editions of *The Nautical Almanac* prior to 1942 changed little from year to year, and formed one of the motivations for a separate supplement; in recent years, however, the rapid increase in the number of observatories, both optical and radio, together with more frequent changes of position, make any list incomplete and out-of-date in one or two years. The list of observatories in *A.E.*, pages 434–452 in 1960, contains full details of place, description, positions, and certain derived constants for use in the reduction of observations, for some 320 optical and 27 radio observatories; the list includes only major observatories and those specifically engaged on observations requiring an accurate knowledge of position for their reduction. It is necessarily prepared some two years before the year of the Ephemeris in which it is printed, and is out-of-date to that extent. A full description of the list and an explanation of the quantities tabulated are given in the Ephemeris itself.

The data in the Ephemeris will suffice for most requirements for the reduction of current observations. Much more detailed information about the equipment, programmes of observation, and staff of observatories is given in the publication *Les Observatoires Astronomiques et les Astronomes* by F. Rigaux, published in 1959 by l'Observatoire Royal de Belgique under the auspices of the International Astronomical Union. As with all such lists the data, particularly as regards individual astronomers, are rapidly becoming out of date. No derived constants are given, and the positional data are not always complete or specific. In any case, users who require precise positions for the reduction of observations should obtain positions for the particular telescope used.

It was originally planned to include a comprehensive list of former observatories, on the lines of the lists published in the editions of *The Nautical Almanac* for the years 1929 to 1938 inclusive. Changes of position of several observatories have added to this list in recent years. But the small amount of additional data hardly justifies the re-publication of data that must now be rarely, if ever, used.

The Ephemeris does not contain all ephemerides of position. Ephemerides of the stars, of minor planets, of comets, and of other bodies are tabulated in other publications, mainly for the general convenience of users; it is proper to regard these as forming an integral part of the totality of astronomical ephemerides. The

*See also note on page vi regarding the 1974 reprint.

scope of the Supplement is accordingly extended to include reference to such ephemerides; but, generally, less detailed explanations and derivations are given for these.

It is a necessary preliminary to the main purpose to define frames of reference and systems of coordinates with some care. In doing so the text-book approach has been deliberately avoided: all elementary definitions and proofs have in general been omitted. An attempt has been made to combine complete rigour of treatment with practical requirements, giving the errors of all approximate procedures; but no attempt has been made to be comprehensive.

The treatment of the main sections varies according to the nature of their content, particularly as to whether they refer to fundamental data, or derived quantities, or to ephemerides in the Ephemeris or elsewhere.

In one section only, that on *Systems of Time Measurement*, has an attempt been made to give a completely exhaustive, and authoritative, treatment of the subject. This subject is fundamental to the whole purpose of the Ephemeris and is one of extreme difficulty, especially in view of the many recent changes in both conception and practical determination. It is hoped that this section will be regarded as providing authoritative and precise statements as to the definitions of *Universal Time*, *Sidereal Time*, and *Ephemeris Time* and of the relationships between them.

The most important specific function of the Supplement is to define precisely, for each individual ephemeris: the quantity tabulated; the fundamental data on which it is based; and how it is derived from those data. No such definition can be regarded as complete, or as free from possible misunderstanding, until it is illustrated by a numerical example in which every figure is derived from the stated fundamental data by means of the stated procedure and formulae; only by such means can ambiguities of wording be clarified, and procedures and formulae verified. To achieve this purpose fully, numerical examples should be chosen so as together to cover all cases and to avoid accidentally-small contributions in which significant errors of principle might lead to negligible numerical differences. In principle the tabulated values should be reproduced exactly; but in practice there must always be a small, and almost always negligible, area of uncertainty in which a real difference of principle may be masked by legitimate variations of procedure and by accumulation of rounding-off errors owing to differences in computing methods.

Although the "derivation", as understood above, of every ephemeris is illustrated numerically in the Supplement no claim is made to have achieved complete coverage. The single examples given cannot cover every case and may sometimes leave uncertainties due to unsuitable choice of date and time; this is especially so as a fixed epoch (1960 March 7 at 0^h E.T.) has been adopted for most of the examples. Moreover, the examples have been calculated on a desk calculating machine one stage at a time, recording intermediate results where necessary; the final results may therefore differ both from the values printed in the Ephemeris, which are calculated systematically on punched-card machines, and from those obtained by adopting different stages in the calculation. None of these deficiencies is likely to be serious, or to result in difficulties of interpretation, provided the limitations are understood. It is intended that every printed figure should be obtainable directly, correctly rounded off, from the stated formula using the actual printed values of the basic data and intermediate results quoted; however, with a calculating machine, there are different methods of accumulating products and of doing continuous multiplications, and in a few cases, by oversight, the rounding-off of an intermediate or final result may differ from that formally obtained. Similarly, values of trigonometric functions may differ according to the interval and number of figures in the tables used.

The numerical examples are designed primarily to illustrate unambiguously the formulae quoted, and they do not necessarily indicate either the best method of calculation or the method actually used. It is not possible to illustrate numerically many of the actual methods used for systematic calculation on punched-card and electronic computing machines.

Details of methods of calculation are omitted from the numerical examples; a short note on computing techniques, particularly in regard to the solution of spherical triangles, is given in section 16A.

F. OTHER PUBLICATIONS OF RELEVANCE

For convenience of reference, there are listed below the full titles, descriptions, and adopted abbreviations of British, American, and other publications which are likely to be of interest to astronomers; the British publications may be obtained through H.M. Stationery Office and the American publications through the Superintendent of Documents, U.S. Government Printing Office.

1. Unified publications; British and American editions

The Nautical Almanac (*N.A.*) (about 276 + xxxv pages) contains data for astronomical navigation at sea. Of astronomical interest are: the Greenwich Hour Angle (G.H.A.) and Declination (Dec.) to 0'·1 for each hour for the Sun, Moon, Venus, Mars, Jupiter, and Saturn; times of sunrise, sunset, and beginning and end of civil and nautical twilights for latitudes N. 72° to S. 60° for every third day; times of moonrise and moonset for latitudes N. 72° to S. 60° for every day.

The Air Almanac (*A.A.*) (four-monthly edition, about 242 + 90 pages) contains data for astronomical navigation in the air. Chief astronomical interest lies in the tabulations of G.H.A. and Dec. of the Sun (to 0'·1), and of the Moon and three planets (to 1'), for each 10^m.

Sight Reduction Tables for Marine Navigation, U.S. Naval Oceanographic Office, H.O. Pub. No. 229, six volumes each covering 15° of latitude, 1970 onwards. Reproduced as (British) Hydrographic Department, N.P. 401, 1971 onwards. These tables give altitude to 0'·1, with variations for declination, and azimuth to 0°·1, with arguments latitude, hour angle, and declination, all at 1° interval. They provide all solutions of the spherical triangle, given two sides and the included angle, to find a third side and adjacent angle.

Sight Reduction Tables for Air Navigation, U.S. Naval Oceanographic Office H.O. Pub. No. 249, reproduced as (British) Air Publication, A.P. 3270; vol. 1, Selected Stars (epoch 1975·0), 1973; vols. 2 and 3, Declinations 0°–29°, 1953. Volume 1 contains the altitude to 1′ and the azimuth to 1° for the seven most suitable stars for navigation, for each degree of latitude and for each degree of local sidereal time. Volumes 2 and 3 give similar data for each degree of declination to 29° and for each degree of hour angle; tabulations extend to depressions of at least 5° below the horizon.

2. British publications

The Star Almanac for Land Surveyors (*S.A.*) (about 90 pages) is designed for topographical surveyors. Its principal interest lies in the apparent places (to $0^s{\cdot}1$ and 1″) of 685 stars, including all stars not fainter than magnitude 4·0.

Planetary Co-ordinates for the years 1960–1980 *referred to the equinox of* 1950·0 (*Planetary Co-ordinates*) 180 pages, 1958.* (Earlier volumes covering the years 1900–1940 and 1940–1960 were published in 1933 and 1939, respectively.) These volumes are intended mainly for the calculation of perturbations of comets and minor planets. They give heliocentric, spherical and rectangular coordinates, referred to the standard equinox of 1950·0, of the planets, together with auxiliary tables, explanations, and illustrations; the volume for the years 1960–1980 also contains a comprehensive collection of formulae.

Interpolation and Allied Tables (*I.A.T.*) 80 pages, 1956,† is a collection of interpolation tables and formulae of numerical analysis, with explanations and illustrations, designed as a working handbook for the computer.

Subtabulation, 54 pages, 1958, contains descriptions and tables for various methods of subtabulation, many of which are used in the compilation of the Ephemeris.

Seven-figure Trigonometrical Tables for every Second of Time, 101 pages, 1939, reprinted 1961.

Five-figure Tables of Natural Trigonometrical Functions (for every 10″), 123 pages, 1947, reprinted 1969.

Greenwich Observations. A complete list of the appendices and special investigations included in the annual volumes of *Observations made at the Royal Observatory, Greenwich*, and a list of the separate publications of the Observatory are given in the volume for 1946, published in 1955. In particular:

"Reduced observations of lunar occultations for the years 1943–1947", published in 1952, as an appendix to the *Observations* for 1939.

Royal Observatory Annals (*R.O. Ann.*). This series of publications includes: Number 1, "Nutation 1900–1959", 1961; values based on E. W. Woolard's series, see section 2C. There are also *Royal Observatory Bulletins* (*R.O. Bull.*).

Annals of Cape Observatory. This series includes many papers and much observational data that are also of relevance to the ephemerides.

3. American publications

The Ephemeris, U.S. Department of the Interior, Bureau of Land Management, 30 pages. For surveyors.

Improved Lunar Ephemeris, 1952–1959. A Joint Supplement to *The American Ephemeris* and *The* (British) *Nautical Almanac* (*I.L.E.*), xiv + 422 pages, 1954. Extends the lunar ephemeris in *A.E.* backwards to 1952, and includes a detailed account of the basic computation from Brown's theory. It also gives revised values of nutation and aberration for 1952–1959 and an account of their calculation.

* Reprinted 1962.

† Reprinted 1972.

Tables of Sunrise, Sunset, and Twilight, Supplement to *The American Ephemeris*, 1946 (*S.S.T.*), 196 pages, 1945. Contains permanent and comprehensive tables of the times of sunrise, sunset and twilight for each degree of latitude to 75°; variations are given by which times can be calculated simply for any year and any place.

Astronomical Papers prepared for the use of The American Ephemeris and Nautical Almanac (*A.P.A.E.*). Introduced in 1882, there are now sixteen volumes, almost every part of which is of direct interest to users of the Ephemeris. A full list of the contents follows:

Volume I.

I. Simon Newcomb. " On the recurrence of solar eclipses, with tables of eclipses from B.C. 700 to A.D. 2300 ". 1879.

II. Simon Newcomb, aided by John Meier. " A transformation of Hansen's lunar theory, compared with the theory of Delaunay ". 1880.

III. Albert A. Michelson. " Experimental determination of the velocity of light made at the United States Naval Academy, Annapolis ". 1880.

IV. Simon Newcomb. " Catalogue of 1098 standard clock and zodiacal stars". 1882.

V. George W. Hill. " On Gauss's method of computing secular perturbations, with an application to the action of Venus on Mercury ". 1881.

VI. Simon Newcomb. " Discussion of observed transits of Mercury, 1677–1881 ". 1882.

Volume II.

I. Simon Newcomb and John Meier. " Formulae and tables for expressing corrections to the geocentric place of a planet in terms of symbolic corrections to the elements of the orbits of the Earth and planet ". 1883.

II. Truman Henry Safford. " Investigation of corrections to the Greenwich planetary observations, from 1762 to 1830 ". 1883.

III. Simon Newcomb. " Measures of the velocity of light made under the direction of the Secretary of the Navy during the years 1880–1882 ". 1885.

IV. Albert A. Michelson. " Supplementary measures of the velocities of white and colored light in air, water, and carbon disulphide, made with the aid of the Bache fund of the National Academy of Sciences ". 1885.

V. Simon Newcomb. " Discussion of observations of the transits of Venus in 1761 and 1769 ". 1890.

VI. Simon Newcomb. " Discussion of the north polar distances observed with the Greenwich and Washington transit circles, with a determination of the constant of nutation ". 1891.

Volume III.

I. Simon Newcomb. " Development of the perturbative function and its derivatives, in sines and cosines of multiples of the eccentric anomalies, and in powers of the eccentricities and inclinations ". 1884.

II. George W. Hill. " Determination of the inequalities of the Moon's motion which are produced by the figure of the Earth ". 1884.

III. Simon Newcomb. " On the motion of Hyperion ". 1884.

IV. George W. Hill. " On certain lunar inequalities due to the action of Jupiter and discovered by Mr. E. Neison ". 1885.

V. Simon Newcomb. " Periodic perturbations of the longitudes and radii vectores of the four inner planets of the first order as to the masses ". 1891.

Volume IV.

G. W. Hill. " A new theory of Jupiter and Saturn ". 1890.

Volume V.

I. Simon Newcomb. " Development of the perturbative function in cosines of multiples of the mean anomalies and of angles between the perihelia and common node and in powers of the eccentricities and mutual inclination ". 1895.

II. Simon Newcomb. " Inequalities of long period, and of the second order as to the masses, in the mean longitudes of the four inner planets ". 1895.

III. Simon Newcomb. " Theory of the inequalities in the motion of the Moon produced by the action of the planets ". 1895.

IV. Simon Newcomb. " Secular variations of the orbits of the four inner planets ". 1895.

V. Simon Newcomb. " On the mass of Jupiter and the orbit of Polyhymnia ". 1895.

Volume VI. Tables of the four inner planets.

I. Simon Newcomb. " Tables of the motion of the Earth on its axis and around the Sun ". 1895.

II. Simon Newcomb. " Tables of the heliocentric motion of Mercury ". 1895.

III. Simon Newcomb. " Tables of the heliocentric motion of Venus ". 1895.

IV. Simon Newcomb. " Tables of the heliocentric motion of Mars ". 1898.

Volume VII.

I. George William Hill. " Tables of Jupiter, constructed in accordance with the methods of Hansen ". 1895.

II. George William Hill. " Tables of Saturn, constructed in accordance with the methods of Hansen ". 1895.

III. Simon Newcomb. " Tables of the heliocentric motion of Uranus ". 1898.

IV. Simon Newcomb. " Tables of the heliocentric motion of Neptune ". 1898.

Volume VIII.

I. Simon Newcomb. " A new determination of the precessional constant with the resulting precessional motions ". 1897.

II. Simon Newcomb. " Catalogue of fundamental stars for the epochs 1875 and 1900 reduced to an absolute system ". 1899.

III. Henry B. Hedrick. " Catalogue of zodiacal stars for the epochs 1900 and 1920 reduced to an absolute system ". 1905.

Volume IX.

I. Simon Newcomb. " Researches on the motion of the Moon. Part II ". 1912.

II. Frank E. Ross. " New elements of Mars and tables for correcting the heliocentric positions derived from Astronomical Papers, Vol. VI, Part IV ". 1917.

III. W. S. Eichelberger and Arthur Newton. " The orbit of Neptune's satellite and the pole of Neptune's equator ". 1926.

Volume X.

I. W. S. Eichelberger. " Positions and proper motions of 1504 standard stars for the equinox 1925·0 ". 1925.

II. James Robertson. " Catalog of 3539 zodiacal stars for the equinox 1950·0 ". 1940.

Volume XI.

I. G. M. Clemence. " The motion of Mercury, 1765–1937 ". 1943.

II. G. M. Clemence. " First-order theory of Mars ". 1949.

III. H. R. Morgan. " Definitive positions and proper motions of primary reference stars for Pluto ". 1950.

IV. Paul Herget, G. M. Clemence, and Hans G. Hertz. " Rectangular coordinates of Ceres, Pallas, Juno, Vesta, 1920–1960 ". 1950.

Volume XII.

W. J. Eckert, Dirk Brouwer, and G. M. Clemence. " Coordinates of the five outer planets, 1653–2060 ". 1951.

Volume XIII.

I. A. J. J. van Woerkom. " The motion of Jupiter's fifth satellite, 1892–1949 ". 1950.

II. Dirk Brouwer and A. J. J. van Woerkom. " The secular variations of the orbital elements of the principal planets ". 1950.

III. H. R. Morgan. " Catalog of 5268 standard stars, 1950·0, based on the normal system N30 ". 1952.

IV. G. M. Clemence. " Coordinates of the center of mass of the Sun and the five outer planets, 1800–2060 ". 1953.

V. G. M. Clemence. " Perturbations of the five outer planets by the four inner ones ". 1954.

Volume XIV.

Paul Herget. " Solar coordinates 1800–2000 ". 1953.

Volume XV.

I. Edgar W. Woolard. " Theory of the rotation of the Earth around its center of mass ". 1953.

II. Hans G. Hertz. " The mass of Saturn and the motion of Jupiter 1884–1948 ". 1953.

III. Paul Herget. " Coordinates of Venus 1800–2000 ". 1955.

Volume XVI.

I. Raynor L. Duncombe. " The motion of Venus, 1750–1949 ". 1958.

This list is continued on page 522.

4. Other publications

The following are international volumes published under the auspices of the International Astronomical Union.

Apparent Places of Fundamental Stars (*A.P.F.S.*), about xl + 500 pages, contains the apparent places of the 1535 stars in FK3.* It contains explanations in English, French, German, Russian, and Spanish. From its inception in 1941 until 1959 it was compiled by H.M. Nautical Almanac Office and published by H.M. Stationery Office, London. It is now compiled and issued by the Astronomisches Rechen-Institut in Heidelberg, and is published by Verlag G. Braun, Karl-Friedrich-Strasse 14, Karlsruhe, Germany.

Ephemerides of the Minor Planets (*E.M.P.*), about 170 pages, contains elements and search ephemerides of all known minor planets. A brief introduction in English is given, and a full translation of the Russian text is also available. It is now compiled by the Institute of Theoretical Astronomy, Leningrad, and is published by the Academy of Sciences of U.S.S.R. From 1898 to 1946 it was prepared by the Astronomisches Rechen-Institut, Berlin, and from 1947 to 1951 both by the Minor Planet Center, Cincinnati, and by the Institute of Theoretical Astronomy.

Notes on other publications and circulars giving current ephemerides of minor

* FK4 in *A.P.F.S.* 1964 onwards.

planets, comets, and satellites are given in the relevant sections of this Supplement.

The following tables may be used for approximate calculation of astronomical phenomena for dates in the past or future for which no fundamental ephemerides are available.

Schoch, K. *Planeten-Tafeln für Jedermann*, Berlin-Pankow, Linser-Verlag G.m.b.H., 1927.

Ahnert, P. *Astronomisch-chronologische Tafeln für Sonne, Mond and Planeten*, Leipzig, Barth, 1960.

Neugebauer, P. V. *Astronomische Chronologie*, 2 volumes, Berlin and Leipzig, Walter de Gruyter, 1929.

Baehr, U. *Tafeln zur Behandlung chronologischer Probleme*, *Veröff. Astr. Rechen-Inst. zu Heidelberg*, no. 3, 1955.

Neugebauer, P. V. *Tafeln zur astronomischen Chronologie*, 3 volumes, Leipzig, 1912–1925. Some of the tables in these volumes have been superseded or corrected by tables in the preceding two references.

5. A note on references

In addition to the abbreviations given above the following are used in this Supplement in references to astronomical journals and publications.

A.J.	*The Astronomical Journal*
Ast. Nach.	*Astronomische Nachrichten*
Bull. Astr.	*Bulletin Astronomique*, Paris
J.B.A.A.	*Journal of the British Astronomical Association*
M.N.R.A.S.	*Monthly Notices of the Royal Astronomical Society*
Mem. R.A.S.	*Memoirs of the Royal Astronomical Society*
P.A.S.P.	*Publications of the Astronomical Society of the Pacific*
Trans. I.A.U.	*Transactions of the International Astronomical Union*

G. SUMMARY OF NOTATIONS

In general, notations are defined and explained as they occur, and no attempt is made to adopt a consistent system throughout the Supplement. The adopted symbols may differ from those recommended by the International Astronomical Union (*Trans. I.A.U.*, **6**, 345, 1939), and may also differ in different sections.

Symbols are generally used to denote the physical quantities which they represent rather than the numerical expression of those quantities in some particular units. Thus the day numbers C, D are angular displacements which may be expressed in seconds of arc, in seconds of time, or in radians. Where it is desired to use a symbol for the numerical value, this is either specifically stated or the unit used is indicated after the symbol: for example, n^s and n'' are the numbers of seconds of time and arc in the annual general precession in declination n. Angles are otherwise expressed in radians, so that powers of small angles occurring in expansions do not require to be modified by powers of sin 1″, as is often done;

occasionally the square of a small angle, say θ^2, may be written as $\theta \sin \theta$ to emphasise this point.

The following summary refers to those symbols and notations that are used consistently throughout the Supplement.

1. Subscripts for reference systems

The reference system for equatorial or ecliptic coordinates is defined by the equinox and either the equator or the ecliptic; there are four such systems in general use. In many applications it suffices to specify the reference system in precise terms such as:

" referred to the mean equinox and equator (or ecliptic) of date "

and thereafter to use appropriate symbols without subscripts to denote the reference system; this specification may be abbreviated in later references in the same application to:

" for mean equinox of date ".

Where necessary to avoid confusion or circumlocution, or merely to assist interpretation, the following subscripts are used consistently to indicate the reference system to which the coordinates are referred. Positions may be geometric, apparent, or astrometric according to the corrections applied for aberration, and subscripts are adopted for all combinations of reference systems and positions that are in use:

Reference system	Position: geometric	Position: apparent	Position: astrometric
Mean equinox of 1950·0	S		R
Mean equinox of beginning of year	B	*	
Mean equinox of date	M, C		
True equinox of date	T	A	

C is used as an alternative to M for ecliptic coordinates.

* No symbol is used for this combination, although it is implicitly used as an intermediate step in the calculation of apparent places of stars.

2. Symbols for heliocentric and geocentric coordinates

Heliocentric:

spherical ecliptic	l, b, r	with appropriate subscripts
rectangular equatorial	x, y, z	with appropriate subscripts
rectangular ecliptic, geometric for mean equinox of date	x_C, y_C, z_C	

Geocentric:

spherical ecliptic	λ, β, Δ	with appropriate subscripts
spherical equatorial	α, δ, Δ	with appropriate subscripts
rectangular equatorial	ξ, η, ζ	with appropriate subscripts
rectangular equatorial (Sun)	X, Y, Z	

3. Precession and nutation

ψ = annual luni-solar precession in longitude
p = annual general precession in longitude
m = annual general precession in right ascension
n = annual general precession in declination
ϵ = obliquity of the ecliptic
$\Delta\psi$ = (total) nutation in longitude
$d\psi$ = short-period terms of nutation in longitude
$\Delta\epsilon$ = (total) nutation in obliquity
$d\epsilon$ = short-period terms of nutation in obliquity

4. Fundamental epochs and measures of time

Ephemeris time. The fundamental epoch to which the elements of the Sun, Moon, and planets are referred is:

1900 January 0 at 12^h ephemeris time
= 1900 January 0·5 E.T. = J.E.D. 241 5020·0 E.T.

Ephemeris time is measured conventionally in years, months, days, and subdivisions of a day. The interval T of ephemeris time from the fundamental epoch contains:

T Julian centuries of 36525 days, each of 86400 ephemeris seconds;
d, or 10000 D, ephemeris days ($d = 36525\ T$; $D = 3{\cdot}6525\ T$).

When desirable to emphasise that these relate to an interval of ephemeris time, a subscript E is added thus: T_E, d_E, D_E.

Universal time. The fundamental epoch which is used in the definition and derivation of universal time is:

1900 January 0 at 12^h universal time
= 1900 January 0·5 U.T. = J.D. 241 5020·0 U.T.

The interval T_U of universal time from this epoch contains:

T_U Julian centuries of 36525 days, each of 86400 seconds of U.T.;
d_U, or 10000 D_U, days of U.T. ($d_U = 36525\ T_U$; $D_U = 3{\cdot}6525\ T_U$).

The subscript U is always used, unless the context makes it superfluous.

E.T. – U.T. At any instant the measure of ephemeris time (epoch + T_E) is equal to the measure of universal time (epoch + T_U) + ΔT; thus:

$$\Delta T = \text{E.T.} - \text{U.T.} = T_E - T_U$$

ΔT is most conveniently expressed in seconds of time.

It must be emphasised that the fundamental epochs used for ephemeris time and universal time, although denoted by the same measure, do not correspond to the same instant of time; in fact at each epoch ΔT_0 is about -4^s, i.e. the epoch of E.T. is 4^s later than that of U.T. The interval of time between two instants, the later one being indicated by a prime, can be expressed as:

$$T'_E - T_E \text{ of ephemeris time}$$

or as: $$T'_U - T_U = (T'_E - T_E) - (\Delta T' - \Delta T) \text{ of universal time}$$

The difference in the two measures involves the values of ΔT at both instants; it is only because the two fundamental epochs have the same measure that it is possible to write:

$$T_E = T_U + \Delta T$$

The Besselian solar year. For certain applications it is more convenient to measure time in units of tropical centuries of 36524·21988 ephemeris days, the fundamental epoch being the beginning of the Besselian (fictitious) solar year 1900·0, or 1900 January $0^d.813$ E.T. In the great majority of such cases the difference in length of the century is not significant: the same symbol T is accordingly used, though always with a specific explanation. The difference between the lengths of the Besselian solar year and the tropical year ($0^s.148\ T$) can always be neglected and multiples of 0·01 in T thus relate to the beginning of the corresponding Besselian year (see section 2B).

The fraction of the tropical year is denoted by τ, measured backwards or forwards from the beginning of the Besselian year; a unit difference in τ corresponds to a difference of 0·01 in T.

An interval of time measured in tropical years is denoted by t. Initial and general epochs are denoted by t_0 and t respectively. The context will indicate the meaning to be attached to t_0 and t:

$t - 1950{\cdot}0$ clearly implies that t is an epoch, e.g. 1960·0

$1950{\cdot}0 + t$ clearly indicates that t is an interval, e.g. 10·0

In some contexts the epoch t_0 is used for that of $1900{\cdot}0 + T_0$, and the epoch t for that of $1900{\cdot}0 + T_0 + T$; $t_0 = 100\ T_0$ and $t = 100\ T$ are both intervals, but are used conventionally to describe epochs.

Other notations for time. τ is also used to denote light-time in the application of corrections for aberration.

Special notations for time, defined as they occur, are used in the sections on eclipses and occultations, and in respect of some of the satellites. No attempt has been made to adhere to a single uniform notation throughout.

5. Day numbers and star-constants

A, B, C, D, E	Besselian day numbers
f, g, G, h, H, i	Independent day numbers
f', g', G'	Independent day numbers (short-period terms)
a, b, c, d	Star constants in right ascension
a', b', c', d'	Star constants in declination
J	Second-order day number in right ascension
J'	Second-order day number in declination

For derivation and formulae see section 5C.

6. Figure of the Earth

ϕ = geographic, or geodetic, latitude—see special note in section 2F

ϕ' = geocentric latitude $\qquad \tan \phi' = (1 - e^2) \tan \phi$

ϕ_1 = parametric latitude $\qquad \tan \phi_1 = (1 - f) \tan \phi$

e = ellipticity, or eccentricity, of the Earth's meridian

f = flattening $\qquad 1 - f = (1 - e^2)^{\frac{1}{2}}$

ρ = geocentric distance in units of the Earth's equatorial radius

S, C = auxiliary functions such that $\rho \sin \phi' = S \sin \phi$

$\rho \cos \phi' = C \cos \phi = \cos \phi_1$

For other relations and formulae see sections 2F and 9B.

2. COORDINATE AND REFERENCE SYSTEMS

A. COORDINATE SYSTEMS

The fundamental astronomical reference systems are based on the celestial equator, coplanar with the Earth's equator, and the ecliptic, the plane* of the Earth's orbit round the Sun. The angular coordinates in these planes are measured from the ascending node of the ecliptic on the equator, or the point at which the Sun in its annual apparent path round the Earth crosses the equator from south to north; and they are measured positively to the east, that is in the direction of the Sun's motion with respect to the stars. The ascending node of the ecliptic on the equator is referred to as " the vernal equinox ", " the first point of Aries ", or simply as " the equinox ". The axes of the corresponding rectangular coordinate systems are right-handed, i.e. the x-axis is directed towards the equinox, the y-axis to a point 90° to the east, while the z-axis is positive to the north.

The position of a point in space may be specified astronomically by reference to a wide variety of coordinate systems; and it may be given by means of (among other less usual systems) either spherical coordinates, consisting of a direction and a distance, or rectangular coordinates, consisting of the projections of the distance on three rectangular axes. The systems are determined by the two following characteristics:

(a) *Origin of coordinates*—and designation.

- (i) The observer—topocentric.
- (ii) The centre of the Earth—geocentric.
- (iii) The centre of the Sun—heliocentric.
- (iv) The centre of mass of the solar system—barycentric.

(b) *Reference planes and directions*—and designation of spherical coordinates.

- (i) The horizon and the local meridian—azimuth and altitude.
- (ii) The equator and the local meridian—hour angle and declination.
- (iii) The equator and the equinox—(equatorial) right ascension and declination.
- (iv) The ecliptic and the equinox—(ecliptic or celestial) longitude and latitude.
- (v) The plane of an orbit and its equatorial or ecliptic node—orbital longitude and latitude.

*More strictly, the mean plane of the orbital motion, ignoring periodic perturbations.

Barycentric coordinates are often referred to the centre of mass of the Sun and the inner planets, and less often to other combinations. The equator, the ecliptic, and the equinox are constantly in motion due to the effects of precession and nutation, and must be further specified; this is done in sub-sections B and C. A notation to distinguish the various systems in current use is introduced in section 1G.

The reduction from geocentric to topocentric coordinates depends on the figure of the Earth, and is considered in detail in sub-section F. In most cases of astronomical interest, the differences are so small that they can be applied as first-order differential corrections.

Positions may be of several kinds, including: the *geometric* position derived from the actual position at the time of observation; the *apparent* position in which an observer, situated at the origin of coordinates, would theoretically see the object; and the *astrometric* position, in which corrections have been made for some small terms of aberration in order that it may be directly comparable with the tabulated catalogue positions of stars. The apparent position is derived from the geometric position by the application of corrections for aberration, and where relevant for refraction. However, refraction is dependent on the observer's local reference system and is invariably treated as a correction to the observation rather than to the ephemeris position; exceptions only occur for phenomena that are essentially topocentric, such as rising and setting and (in principle, though the correction is neglected in practice) for eclipses and occultations. For geocentric coordinates the apparent position is the direction in which an observer at the centre of the Earth would see the object, and refraction does not enter. Aberration is dealt with in sub-section D and refraction briefly in sub-section E.

In the present sub-section the effects of precession, nutation, aberration, refraction, and parallax are ignored in order to present the relationships between the coordinate systems. The general notation used is restricted to this purpose and should not be confused with the more detailed notation in section 1G necessary to distinguish between the different kinds of position.

Not all combinations of (a) and (b) occur and many are not used in the Ephemeris; (a) (iv), in particular, is therefore not referred to again. Moreover, if corrections for parallax be deferred, there is no difference between (a) (i) and (a) (ii), which can be treated together.

For *geocentric* spherical coordinates there are thus the four practical reference systems of:

(i) azimuth (A) measured from the north through east in the plane of the horizon, and altitude (a) measured perpendicular to the horizon; in astronomy the zenith distance ($z = 90° - a$) is more generally used, but the altitude is retained in the formulae for reasons of symmetry;

(ii) hour angle (h) measured westwards in the plane of the equator from the meridian, and declination (δ) measured perpendicular to the equator, positive to the north;

(iii) right ascension (α) measured from the equinox eastwards in the plane of the equator, and declination (δ);

(iv) longitude (λ) measured from the equinox eastwards in the plane of the ecliptic, and latitude (β) measured perpendicular to the ecliptic, positive to the north.

The formulae connecting these coordinates are:

Azimuth/altitude : Hour angle/declination

$$\begin{aligned} \cos a \sin A &= -\cos \delta \sin h \\ \cos a \cos A &= \sin \delta \cos \phi - \cos \delta \cos h \sin \phi \\ \sin a &= \sin \delta \sin \phi + \cos \delta \cos h \cos \phi \end{aligned}$$

$$\begin{aligned} \cos \delta \sin h &= -\cos a \sin A \\ \cos \delta \cos h &= \sin a \cos \phi - \cos a \cos A \sin \phi \\ \sin \delta &= \sin a \sin \phi + \cos a \cos A \cos \phi \end{aligned}$$

where ϕ is the latitude of the observer. Note that the conversion corresponds to a simple rotation of the frame of reference through an angle $90° - \phi$ in the plane of the meridian.

Hour angle/declination : Right ascension/declination

The two systems are identical apart from the origin, and direction, of measurement of hour angle and right ascension, which are connected by the relation:

$$h = \text{local sidereal time} - \alpha$$

since local sidereal time is the hour angle of the equinox.

Right ascension/declination : Longitude/latitude

$$\begin{aligned} \cos \delta \cos \alpha &= \cos \beta \cos \lambda \\ \cos \delta \sin \alpha &= \cos \beta \sin \lambda \cos \epsilon - \sin \beta \sin \epsilon \\ \sin \delta &= \cos \beta \sin \lambda \sin \epsilon + \sin \beta \cos \epsilon \end{aligned}$$

$$\begin{aligned} \cos \beta \cos \lambda &= \cos \delta \cos \alpha \\ \cos \beta \sin \lambda &= \cos \delta \sin \alpha \cos \epsilon + \sin \delta \sin \epsilon \\ \sin \beta &= -\cos \delta \sin \alpha \sin \epsilon + \sin \delta \cos \epsilon \end{aligned}$$

where ϵ is the obliquity of the ecliptic (corresponding to the particular equator and ecliptic used). Geocentric longitude and latitude are used now only for the Sun and Moon. Note that the conversions correspond to a simple rotation round the x-axis through an angle ϵ.

The corresponding equatorial rectangular coordinates and distance are denoted by X, Y, Z, and R for the Sun and by ξ, η, ζ, and Δ for the planets; they are derived from the spherical coordinates by the formulae:

$$\begin{aligned} X/R \text{ or } \xi/\Delta &= \cos \delta \cos \alpha \\ Y/R \text{ or } \eta/\Delta &= \cos \delta \sin \alpha \\ Z/R \text{ or } \zeta/\Delta &= \sin \delta \end{aligned}$$

Geocentric ecliptic rectangular coordinates are rarely (if ever) used.

For *heliocentric* coordinates there are only the two practical reference systems—the equatorial and the ecliptic; and in the equatorial system only rectangular coordinates are used. The relationships between the ecliptic rectangular coordinates (x_C, y_C, z_C), the ecliptic longitude, latitude, and distance (l, b, r), and the equatorial rectangular coordinates (x, y, z) are:

$$\begin{aligned} x_C &= r \cos b \cos l = x \\ y_C &= r \cos b \sin l = +y \cos \epsilon + z \sin \epsilon \\ z_C &= r \sin b = -y \sin \epsilon + z \cos \epsilon \\ x &= x_C = r(\cos b \cos l) \\ y &= y_C \cos \epsilon - z_C \sin \epsilon = r(\cos b \sin l \cos \epsilon - \sin b \sin \epsilon) \\ z &= y_C \sin \epsilon + z_C \cos \epsilon = r(\cos b \sin l \sin \epsilon + \sin b \cos \epsilon) \end{aligned}$$

The conversion from *heliocentric* to *geocentric* coordinates is performed in terms of equatorial rectangular coordinates through:

$$\xi = x + X \qquad \eta = y + Y \qquad \zeta = z + Z$$

where X, Y, Z are the geocentric coordinates of the Sun.

The calculation of the spherical coordinates from the rectangular coordinates, or from the known direction cosines, typified by:

$$\Delta \cos \delta \cos \alpha = \xi \qquad \Delta \cos \delta \sin \alpha = \eta \qquad \Delta \sin \delta = \zeta$$

is performed by:

$$\tan \alpha = \eta/\xi \qquad \cot \alpha = \xi/\eta$$
$$\Delta = (\xi^2 + \eta^2 + \zeta^2)^{\frac{1}{2}} \qquad \sin \delta = \zeta/\Delta$$

The quadrant of α is determined by the signs of ξ and η, and that of δ by the sign of ζ; Δ and $\Delta \cos \delta$ are always positive. The formulae for α and δ may be written:

$$\begin{aligned} \alpha &= \tan^{-1}\eta/\xi \quad \text{or arctan } \eta/\xi \\ &= \cot^{-1}\xi/\eta \quad \text{or arccot } \xi/\eta \\ \delta &= \sin^{-1}\zeta/\Delta \quad \text{or arcsin } \zeta/\Delta \end{aligned}$$

provided that the appropriate values, and not necessarily the principal values, of the multi-valued functions are taken.

Notes on the technique of practical calculation using these formulae, and on the most suitable trigonometric tables to use, are given in section 16A.

Many of the conversions above correspond to a simple rotation of the frame of reference about one of its axes. These are special cases of the general conversion from a set of axes designated by x, y, z to a set designated by x', y', z'; the two systems are connected by the formulae:

$$\begin{aligned} x &= l_1 x' + l_2 y' + l_3 z' & x' &= l_1 x + m_1 y + n_1 z \\ y &= m_1 x' + m_2 y' + m_3 z' & y' &= l_2 x + m_2 y + n_2 z \\ z &= n_1 x' + n_2 y' + n_3 z' & z' &= l_3 x + m_3 y + n_3 z \end{aligned}$$

where l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of x', y', z' referred to the system x, y, z. The direction cosines satisfy the relations typified by:

$$l_1^2 + m_1^2 + n_1^2 = 1 \qquad l_1^2 + l_2^2 + l_3^2 = 1$$
$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \qquad m_1 n_1 + m_2 n_2 + m_3 n_3 = 0$$

These nine quantities can be expressed in terms of the Eulerian angles θ, ϕ, ψ by:

$$
\begin{aligned}
l_1 &= +\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi \\
l_2 &= -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi \\
l_3 &= +\cos\phi\sin\theta \\
\\
m_1 &= +\sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi \\
m_2 &= -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi \\
m_3 &= +\sin\phi\sin\theta \\
\\
n_1 &= -\sin\theta\cos\psi \\
n_2 &= +\sin\theta\sin\psi \\
n_3 &= +\cos\theta
\end{aligned}
$$

In this case the conversion corresponds to a rotation ϕ about the z-axis, θ about the new position of the y-axis, and ψ about the new (and final) position of the z-axis. The transformation is equivalent to a single rotation about some line not in general coincident with one of the axes; but such single rotations are not frequently encountered in astronomical practice.

B. PRECESSION

The equator and the ecliptic, and hence the equinox, are continuously in motion. The motion of the equator, or of the celestial pole, is due to the gravitational action of the Sun and Moon on the equatorial bulge of the Earth: it consists of two components, one *luni-solar precession* being the smooth long-period motion of the *mean pole* of the equator round the pole of the ecliptic in a period of about 26,000 years, and the other *nutation* being a relatively short-period motion that carries the actual (or *true*) pole round the mean pole in a somewhat irregular curve, of amplitude about 9″ and main period 18·6 years. The motion of the ecliptic, that is of the mean plane of the Earth's orbit, is due to the gravitational action of the planets on the Earth as a whole and consists of a slow rotation of the ecliptic about a slowly-moving diameter, the ascending node of the instantaneous position of the ecliptic on the immediately preceding position being in longitude about 174°; this motion is known as *planetary precession* and gives a precession of the equinox of about 12″ a century and a decrease of the obliquity of the ecliptic of about 47″ a century.

In this sub-section the effects of the motions of only the *mean* poles of the equator and ecliptic, known as *general precession*, are considered; the effect of *nutation* is dealt with separately in sub-section C. The treatment is restricted to the development of formulae for the practical application of corrections to coordinates and orbital elements.

Rigorous formulae

The effect of precession on the coordinates of a fixed object is illustrated in figure 2.1, in which the position of a star S is referred at an initial time t_0 to a system of equatorial axes defined by the mean pole of the equator P_0 and the mean equinox X_0; at this initial epoch the pole of the ecliptic is at C_0. P, X, and C are the respective positions of these points at a subsequent time t. Although at any instant P moves, owing to luni-solar precession, in a direction perpendicular to the colure CP, i.e. towards X, the arc P_0P is not perpendicular either to C_0P_0 or to CP; owing to planetary precession C is itself in motion along a curve which is always convex to CP. This complex motion is specified by means of the three angles ζ_0, z, θ (where $90° - \zeta_0$ is the right ascension of the ascending node of the equator of epoch t on the equator of t_0 reckoned from the equinox of t_0, $90° + z$ is the right ascension of the node reckoned from the equinox of t, and θ is the inclination of the equator of t to the equator of t_0) together with the corresponding values of the obliquity of the ecliptic ϵ.

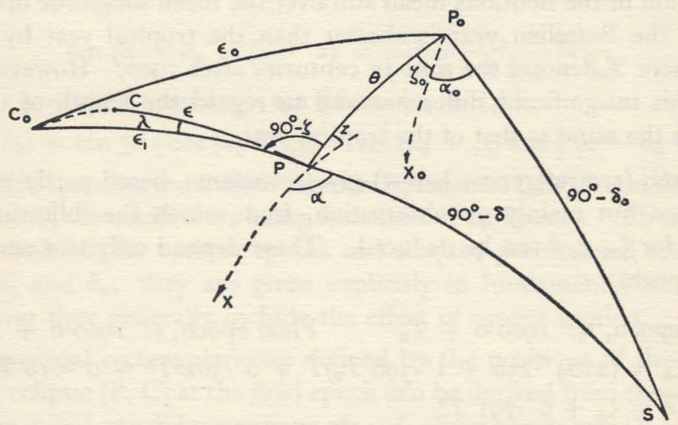

Figure 2.1. Precession—polar diagram

In the figure:

$\zeta_0 = X_0P_0P$ $\qquad \zeta = 90° - P_0PC_0$

$z = 90° - P_0PC = \zeta - C_0PC = \zeta - \lambda$

$\theta = P_0P$ The great circle P_0P is not the actual path taken by the moving pole.

P_0C_0P = luni-solar precession in longitude in the interval $t - t_0$

C_0C = planetary precession in longitude in the interval $t - t_0$

$\lambda = C_0PC$ = planetary precession on the equator in the interval $t - t_0$

$\epsilon_0 = C_0P_0$ = obliquity of the ecliptic at t_0

$\epsilon =$ CP = obliquity of the ecliptic at t

$\epsilon_1 = C_0P$

Figure 2.1 has been drawn for an epoch t for which λ is negative, i.e. for which z is greater than ζ.

Positions referred to the reference system specified by the mean pole P, the mean equinox X, and the pole of the ecliptic C at time t are designated formally as being referred to "the mean equinox and equator (or ecliptic) of epoch t". Where no confusion can be caused this is abbreviated to "mean equinox of epoch t". In practice three reference systems are used: the mean equinox of 1950·0 (occasionally referred to as the "standard" equinox), the mean equinox of the beginning of the Besselian year, and the mean equinox of date (i.e. the epoch of the reference system is the same as the date and time for which the position is given). Where necessary quantities referred to these systems are distinguished by subscripts S or R, B, M or C respectively. (See section 1G).

The beginning of the Besselian (fictitious) solar year is the instant when the right ascension of the fictitious mean sun, affected by aberration and measured from the mean equinox, is $18^h 40^m$. This instant always occurs near the beginning of the calendar year and is denoted by the notation ·0 after the year; for example, as given in *A.E.*, page 2, the beginning of the Besselian solar year 1960 is January $1^d{\cdot}345$ E.T. = 1960·0. Because of the excess of the secular acceleration of the right ascension of the fictitious mean sun over the mean longitude of the Sun (see section 3B) the Besselian year is shorter than the tropical year by the amount $0^s{\cdot}148T$, where T denotes the time in centuries after 1900. However, it is usual to ignore this insignificant difference and to regard the length of the Besselian solar year as the same as that of the tropical year.

Newcomb (see references below) gives constants, based partly on theoretical considerations but mainly on observation, from which the following numerical expressions for ζ_0, z, θ can be deduced. These depend only to a small extent on the initial epoch.

Initial epoch, t_0: $1900{\cdot}0 + T_0$ Final epoch, t: $1900{\cdot}0 + T_0 + T$

$$\zeta_0 = (2304''{\cdot}250 + 1''{\cdot}396\, T_0)T + 0''{\cdot}302\, T^2 + 0''{\cdot}018\, T^3$$
$$z = \zeta_0 + 0''{\cdot}791\, T^2$$
$$\theta = (2004''{\cdot}682 - 0''{\cdot}853\, T_0)T - 0''{\cdot}426\, T^2 - 0''{\cdot}042\, T^3$$

where T_0, T are measured in tropical centuries; the small secular changes in the coefficients of T^2 are here ignored.

The series given are for the conversion from the mean equinox of the initial epoch to an epoch T centuries later; it can be verified that ζ_0, z, θ, for initial epoch t_0 and interval T, are identically equal to $-z$, $-\zeta_0$, $-\theta$, respectively, for epoch $t_0 + T$ and interval $-T$. When values are tabulated for reduction from the mean equinox of t_0 to that of t, the same values can therefore be used for reduction from the mean equinox of t to that of t_0 by replacing ζ_0, z, θ (for t_0) by $-z$, $-\zeta_0$, $-\theta$.

Values for the reduction from the mean equinox of the beginning of the current year to the standard equinox of 1950·0 are given in *A.E.*, page 50, and for a selection of years in the three volumes of *Planetary Co-ordinates*; the reduction from the standard equinox of 1950·0 can be obtained by the simple substitution mentioned above.

Values of ζ_0, z, θ and related precessional elements M, N are given in table 2.1 for years 1900 to 1980 at intervals of one year. The main tabulation is given for reduction to the initial epoch 1950·0 but appropriate formulae are given with the table so that it may be used for reductions between any two epochs. Values for reduction from the mean equinox of selected years back to 1755 to that of the current year are given in *A.E.*, Table III.

Rigorous formulae for the reduction of positions from one epoch to another are easily deduced from figure 2.1; in triangle P_0PS:

$$P_0S = 90^\circ - \delta_0 \qquad PP_0S = \alpha_0 + \zeta_0 \qquad PP_0 = \theta$$
$$PS = 90^\circ - \delta \qquad P_0PS = 180^\circ - (\alpha - z)$$

where α_0, δ_0 and α, δ are right ascension and declination for the initial and final epochs respectively. Then α, δ are given by:

$$\cos\delta\sin(\alpha - z) = \cos\delta_0\sin(\alpha_0 + \zeta_0)$$
$$\cos\delta\cos(\alpha - z) = \cos\theta\cos\delta_0\cos(\alpha_0 + \zeta_0) - \sin\theta\sin\delta_0$$
$$\sin\delta = \cos\theta\sin\delta_0 + \sin\theta\cos\delta_0\cos(\alpha_0 + \zeta_0)$$

The rigorous formulae for α, δ may be written in the form:

$$\tan(\alpha - \alpha_0 - \zeta_0 - z) = \frac{q\sin(\alpha_0 + \zeta_0)}{1 - q\cos(\alpha_0 + \zeta_0)}$$

where $q = \sin\theta\,\{\tan\delta_0 + \tan\tfrac{1}{2}\theta\cos(\alpha_0 + \zeta_0)\}$

$$\tan\tfrac{1}{2}(\delta - \delta_0) = \tan\tfrac{1}{2}\theta\,\{\cos(\alpha_0 + \zeta_0) - \sin(\alpha_0 + \zeta_0)\tan\tfrac{1}{2}(\alpha - \alpha_0 - \zeta_0 - z)\}$$

which permits expansion in terms of the small quantities ζ_0, z, θ, and thus in a series in the interval T, the coefficients being functions of α_0, δ_0 only. These coefficients have been tabulated, for various epochs and adopted precessional constants, with arguments α_0 and δ_0; they are given explicitly in fundamental star catalogues, where however they generally include the effect of proper motion.

The equatorial rectangular axes defined by the positions of the poles of the equator and ecliptic (P, C) at the final epoch can be derived from those (defined by P_0, C_0) at the initial epoch by rotations of: $-\zeta_0$ about the z_0-axis (P_0); θ about the y-axis; and $-z$ about the z-axis (P). The direction cosines of one set of axes referred to the other may be expressed in terms of ζ_0, z, θ (see sub-section A); in particular the direction cosines of the initial axes referred to the final axes are:

$$\begin{aligned}
X_x &= \cos\zeta_0\cos\theta\cos z - \sin\zeta_0\sin z\\
Y_x &= -\sin\zeta_0\cos\theta\cos z - \cos\zeta_0\sin z\\
Z_x &= -\sin\theta\cos z\\
X_y &= \cos\zeta_0\cos\theta\sin z + \sin\zeta_0\cos z\\
Y_y &= -\sin\zeta_0\cos\theta\sin z + \cos\zeta_0\cos z\\
Z_y &= -\sin\theta\sin z\\
X_z &= \cos\zeta_0\sin\theta\\
Y_z &= -\sin\zeta_0\sin\theta\\
Z_z &= \cos\theta
\end{aligned}$$

where, for example, Y_x is the direction cosine of the initial y-axis referred to the final x-axis.

2.1—EQUATORIAL PRECESSIONAL ELEMENTS

FOR REDUCTION TO 1950·0 OR OTHER EPOCHS

Date t_0	ζ_0	z	$\sin\theta$	$\cos\theta - 1$	M	$N = \theta$	$N = \theta$
	s	s	Unit = 10^{-8}	Unit = 10^{-8}	s	s	″
1900·0	+76·814	+76·827	+48 5892	−1180	+153·640	+66·815	+1002·23
1901	75·278	75·291	47 6174	1133	150·568	65·479	982·18
1902	73·742	73·754	46 6455	1088	147·495	64·142	962·14
1903	72·206	72·218	45 6737	1043	144·423	62·806	942·09
1904	70·670	70·681	44 7018	999	141·350	61·469	922·04
1905·0	+69·134	+69·145	+43 7300	− 956	+138·278	+60·133	+ 902·00
1906	67·598	67·608	42 7581	914	135·206	58·797	881·95
1907	66·062	66·072	41 7862	873	132·133	57·460	861·91
1908	64·526	64·535	40 8144	833	129·061	56·124	841·86
1909	62·990	62·999	39 8426	794	125·988	54·787	821·81
1910·0	+61·454	+61·462	+38 8707	− 756	+122·916	+53·451	+ 801·77
1911	59·918	59·926	37 8989	718	119·843	52·115	781·72
1912	58·382	58·389	36 9270	682	116·771	50·778	761·68
1913	56·845	56·853	35 9552	646	113·698	49·442	741·63
1914	55·309	55·316	34 9834	612	110·626	48·105	721·59
1915·0	+53·773	+53·780	+34 0116	− 579	+107·553	+46·769	+ 701·54
1916	52·237	52·243	33 0397	545	104·480	45·433	681·49
1917	50·701	50·707	32 0679	513	101·408	44·097	661·45
1918	49·164	49·170	31 0961	483	98·335	42·760	641·40
1919	47·628	47·634	30 1243	453	95·263	41·424	621·36
1920·0	+46·092	+46·097	+29 1525	− 424	+ 92·190	+40·088	+ 601·31
1921	44·556	44·561	28 1807	397	89·117	38·751	581·27
1922	43·020	43·024	27 2089	370	86·044	37·415	561·22
1923	41·484	41·488	26 2371	344	82·971	36·079	541·18
1924	39·948	39·951	25 2653	319	79·899	34·742	521·13
1925·0	+38·411	+38·415	+24 2935	− 295	+ 76·826	+33·406	+ 501·09
1926	36·875	36·878	23 3217	272	73·753	32·070	481·05
1927	35·339	35·341	22 3499	250	70·680	30·733	461·00
1928	33·802	33·805	21 3781	229	67·607	29·397	440·96
1929	32·266	32·268	20 4064	208	64·534	28·061	420·91
1930·0	+30·730	+30·732	+19 4346	− 189	+ 61·462	+26·725	+ 400·87
1931	29·193	29·195	18 4628	171	58·389	25·388	380·82
1932	27·657	27·659	17 4911	153	55·316	24·052	360·78
1933	26·121	26·122	16 5193	137	52·243	22·716	340·74
1934	24·584	24·586	15 5476	121	49·170	21·379	320·69
1935·0	+23·048	+23·049	+14 5758	− 106	+ 46·097	+20·043	+ 300·65
1936	21·511	21·512	13 6041	93	43·024	18·707	280·60
1937	19·975	19·976	12 6323	80	39·951	17·371	260·56
1938	18·439	18·439	11 6606	68	36·878	16·034	240·52
1939	16·902	16·903	10 6888	57	33·805	14·698	220·47
1940·0	+15·366	+15·366	+ 9 7171	− 47	+ 30·732	+13·362	+ 200·43
1941	13·829	13·830	8 7454	38	27·659	12·026	180·39
1942	12·293	12·293	7 7737	30	24·586	10·690	160·34
1943	10·756	10·756	6 8019	23	21·512	9·353	140·30
1944	9·220	9·220	5 8302	17	18·439	8·017	120·26
1945·0	+ 7·683	+ 7·683	+ 4 8585	− 12	+ 15·366	+ 6·681	+ 100·21
1946	6·146	6·146	3 8868	8	12·293	5·345	80·17
1947	4·610	4·610	2 9151	4	9·220	4·009	60·13
1948	3·073	3·073	1 9434	− 2	6·146	2·672	40·09
1949·0	+ 1·537	+ 1·537	+ 0 9717	0	+ 3·073	+ 1·336	+ 20·04

2.1—EQUATORIAL PRECESSIONAL ELEMENTS

FOR REDUCTION TO 1950·0 OR OTHER EPOCHS

Date t_0	ζ_0	z	$\sin\theta$	$\cos\theta - 1$	M	$N = \theta$	$N = \theta$
	s	s	Unit = 10^{-8}		s	s	″
1950·0	0·000	0·000	0	0	0·000	0·000	0·00
1951	− 1·537	− 1·537	− 9717	0	− 3·073	− 1·336	− 20·04
1952	3·073	3·073	1 9434	− 2	6·147	2·672	40·08
1953	4·610	4·610	2 9151	4	9·220	4·008	60·13
1954	6·147	6·147	3 8867	8	12·293	5·345	80·17
1955·0	− 7·683	− 7·683	− 4 8584	− 12	− 15·367	− 6·681	− 100·21
1956	9·220	9·220	5 8301	17	18·440	8·017	120·25
1957	10·757	10·757	6 8017	23	21·513	9·353	140·30
1958	12·294	12·293	7 7734	30	24·587	10·689	160·34
1959	13·830	13·830	8 7450	38	27·660	12·025	180·38
1960·0	−15·367	−15·367	− 9 7167	− 47	− 30·734	−13·361	− 200·42
1961	16·904	16·903	10 6883	57	33·807	14·698	220·46
1962	18·441	18·440	11 6600	68	36·881	16·034	240·50
1963	19·977	19·977	12 6316	80	39·954	17·370	260·55
1964	21·514	21·513	13 6032	93	43·028	18·706	280·59
1965·0	−23·051	−23·050	−14 5749	− 106	− 46·101	−20·042	− 300·63
1966	24·588	24·587	15 5465	121	49·175	21·378	320·67
1967	26·125	26·123	16 5181	137	52·248	22·714	340·71
1968	27·662	27·660	17 4897	153	55·322	24·050	360·75
1969	29·199	29·197	18 4613	171	58·395	25·386	380·79
1970·0	−30·736	−30·733	−19 4330	− 189	− 61·469	−26·722	− 400·83
1971	32·273	32·270	20 4046	208	64·543	28·058	420·87
1972	33·809	33·807	21 3762	229	67·616	29·394	440·92
1973	35·346	35·344	22 3477	250	70·690	30·730	460·96
1974	36·883	36·880	23 3193	272	73·764	32·066	481·00
1975·0	−38·420	−38·417	−24 2909	− 295	− 76·837	−33·402	− 501·04
1976	39·957	39·954	25 2625	319	79·911	34·738	521·08
1977	41·494	41·491	26 2341	344	82·985	36·074	541·12
1978	43·031	43·027	27 2056	370	86·059	37·410	561·16
1979	44·568	44·564	28 1772	397	89·133	38·746	581·20
1980·0	−46·106	−46·101	−29 1488	− 424	− 92·206	−40·082	− 601·24

These values are for the reduction from the epoch t_0, in the left-hand argument column, to the epoch 1950·0. For reduction from 1950·0 to t_0 enter the table with t_0 as argument, reverse the signs of all respondents except $\cos\theta - 1$, and interchange ζ_0 and z.

For reduction from the epoch $t_0 + \Delta t$ to 1950·0 + Δt, and vice versa, take out values from the table using argument t_0, and multiply:

$$\zeta_0,\ z,\ M \text{ by } (1 + 0{\cdot}0000\ 06\ \Delta t)$$

and

$$N,\ \theta,\ \sin\theta \text{ by } (1 - 0{\cdot}0000\ 04\ \Delta t).$$

Over the range of the table $\tan\frac{1}{2}\theta$ can be taken as $\frac{1}{2}\sin\theta$.

Formulae for the reduction of equatorial spherical coordinates include:

$$\alpha - \alpha_0 = M + N \sin\tfrac{1}{2}(\alpha + \alpha_0) \tan\tfrac{1}{2}(\delta + \delta_0)$$
$$\delta - \delta_0 = N \cos\tfrac{1}{2}(\alpha + \alpha_0)$$

where α_0, δ_0 are for epoch t_0, and α, δ are for epoch 1950·0.

Rectangular coordinates x, y, z referred to the final epoch t can thus be expressed in terms of rectangular coordinates x_0, y_0, z_0 referred to the initial epoch t_0 by:

$$\begin{aligned} x &= X_x x_0 + Y_x y_0 + Z_x z_0 \\ y &= X_y x_0 + Y_y y_0 + Z_y z_0 \\ z &= X_z x_0 + Y_z y_0 + Z_z z_0 \end{aligned}$$

These formulae are precisely equivalent to those connecting the spherical coordinates above.

In systematic computation these and similar formulae are often modified so as to give the reductions $(x - x_0)$, $(y - y_0)$, $(z - z_0)$ to be applied to the known x_0, y_0, z_0 to give x, y, z; e.g. the first formula is written as:

$$x = x_0 + (X_x - 1)\, x_0 + Y_x y_0 + Z_x z_0$$

For reduction from the epoch t to the epoch t_0:

$$\begin{aligned} x_0 &= X_x x + X_y y + X_z z = + X_x x - Y_x y - Z_x z \\ y_0 &= Y_x x + Y_y y + Y_z z = - X_y x + Y_y y + Z_y z \\ z_0 &= Z_x x + Z_y y + Z_z z = - X_z x + Y_z y + Z_z z \end{aligned}$$

in which the first set of formulae is rigorous, but the second (which has been largely used) depends on the approximate equality of X_y and $-Y_x$; X_z and $-Z_x$; and Y_z and Z_y; the approximation is so good that the numerical values are identical. For reduction from the mean equinox of 1950·0 + T_0 to that of 1950·0 + T_0 + T:

$$\begin{aligned} X_x - 1 &= -(2\,9696 + 26\, T_0)\, T^2 - 13\, T^3 \\ Y_x = -X_y &= -(223\,4941 + 1355\, T_0)\, T - 676\, T^2 + 221\, T^3 \\ Z_x = -X_z &= -(97\,1690 - 414\, T_0)\, T + 207\, T^2 + 96\, T^3 \\ Y_y - 1 &= -(2\,4975 + 30\, T_0)\, T^2 - 15\, T^3 \\ Y_z = Z_y &= -(1\,0858 + 2\, T_0)\, T^2 \\ Z_z - 1 &= -(4721 - 4\, T_0)\, T^2 \end{aligned}$$

where T_0, T are measured in tropical centuries, and the coefficients on the right-hand side are in units of the eighth decimal. Numerical values for reduction from (and to) the equinox of 1950·0 to (and from) the mean equinox of the beginning of the year are given at various intervals for the years 1800 to 1980 in the three volumes of *Planetary Co-ordinates*.

For reduction from 1950·0 to the mean equinox of date it is more convenient to have these expressions in terms of days (d) measured from some convenient zero near the epoch of the standard mean equinox of 1950·0; this is chosen to be J.D. 243 3000·5 (1949 March 25·0), so that 1950·0 corresponds to d = 281·923.

$$\begin{aligned} X_x - 1 &= - 2 + 125{\cdot}5\, D - 2226{\cdot}0\, D^2 - 0{\cdot}3\, D^3 \\ Y_x = -X_y &= +1\,7251 - 61\,1903{\cdot}6\, D - 51{\cdot}1\, D^2 + 4{\cdot}5\, D^3 \\ Z_x = -X_z &= +7500 - 26\,6040{\cdot}8\, D + 15{\cdot}3\, D^2 + 2{\cdot}0\, D^3 \\ Y_y - 1 &= - 1 + 105{\cdot}6\, D - 1872{\cdot}1\, D^2 - 0{\cdot}3\, D^3 \\ Y_z = Z_y &= - 1 + 45{\cdot}9\, D - 814{\cdot}0\, D^2 \\ Z_z - 1 &= + 20{\cdot}0\, D - 353{\cdot}9\, D^2 \end{aligned}$$

where $D = d/10000$, and the coefficients on the right-hand side are in units of the eighth decimal. Numerical values, calculated from the original expressions, are given in table 2.2 with argument Julian date at intervals of 1000 days.

Annual motions and approximate formulae

The precessional motions during a short interval of time (of the order of a year) are small, and in many cases it is adequate to use first-order corrections equal to the rates of change multiplied by the interval. In figure 2.2 the effect of precession on the mean equator, ecliptic, and mean equinox is illustrated diagrammatically. Q_0, E_0, X_0 are the equator, ecliptic, and equinox respectively at the initial epoch t_0, and Q, E, X at an epoch t taken to be one tropical year later; the interval is taken as being sufficiently small for the actual displacements to be regarded as annual rates of change.

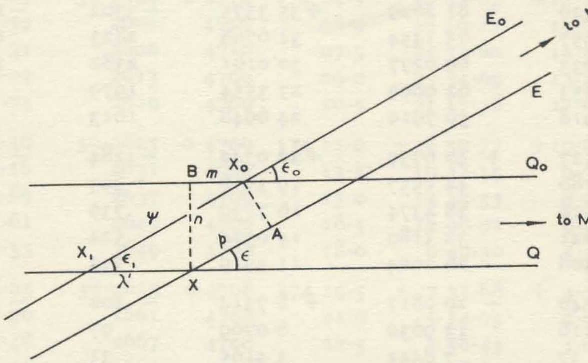

Figure 2.2. Precession—equatorial diagram

The two equators intersect at M, about 90° from X_0, and the two ecliptics intersect at N, about 174° from X_0; M, N are the axes about which the equator and ecliptic rotate. X_1 is the intersection of the equator Q of epoch t with the ecliptic E_0 of epoch t_0. Then:

$\Pi = X_0N$ = longitude of the axis of rotation of the ecliptic, i.e. of the ascending node of the instantaneous position of the ecliptic on the immediately preceding position; it is referred to the mean equinox of date

$\pi = X_0NX$ = annual rate of rotation of the ecliptic

$\zeta_0 = 90° - X_0M$ $\quad z = XM - 90°$

$\epsilon_0 = E_0X_0Q_0$ = obliquity of the ecliptic at epoch t_0

$\epsilon = EXQ$ = obliquity of the ecliptic at epoch t

$\epsilon_1 = E_0X_1Q$

$\psi' = X_0X_1$ = annual luni-solar precession

$\lambda' = X_1X$ = annual planetary precession on the equator

$p = XN - X_0N$ (or XA) = annual general precession in longitude $= \psi' - \lambda' \cos \epsilon_1$

$m = XM - X_0M$ (or BX_0) = annual general precession in right ascension $= \psi' \cos \epsilon_1 - \lambda'$

$n = X_0MX$ (or BX) = annual general precession in declination = rate of change of $\theta = \psi' \sin \epsilon_1$

The quantity here denoted by ψ' is denoted by ψ in the Ephemeris, and in figure 2.2.

FOR REDUCTION OF EQUATORIAL RECTANGULAR COORDINATES FROM (AND TO) THE MEAN EQUINOX OF 1950·0

Julian Date t	$X_x - 1$	$Y_x = -X_y$	$Z_x = -X_z$	$Y_y - 1$	$Z_y = Y_z$	$Z_z - 1$
	In units of the eighth decimal					
241 5000·5	−7438	+111 8485	+48 6412	−6255	−2720	−1183
6000	6647	105 7317	45 9804	5590	2431	1057
7000	5900	99 6147	43 3197	4962	2158	938
8000	5198	93 4976	40 6589	4371	1901	827
241 9000	4540	87 3803	37 9982	3818	1660	722
242 0000·5	−3926	+ 81 2629	+35 3375	−3302	−1436	− 624
1000	3357	75 1454	32 6768	2823	1228	534
2000	2833	69 0277	30 0161	2382	1036	451
3000	2353	62 9099	27 3554	1979	860	374
4000	1918	56 7919	24 6948	1613	701	305
242 5000·5	−1527	+ 50 6739	+22 0342	−1284	− 558	− 243
6000	1180	44 5557	19 3736	993	432	188
7000	878	38 4374	16 7130	739	321	140
8000	621	32 3189	14 0524	522	227	99
242 9000	408	26 2004	11 3919	343	149	65
243 0000·5	− 240	+ 20 0817	+ 8 7314	− 202	− 88	− 38
1000	116	13 9630	6 0709	97	42	18
2000	37	7 8441	3 4105	31	13	− 6
3000	2	+ 1 7251	+ 7500	1	1	0
4000	11	− 4 3940	− 1 9104	10	4	− 2
243 5000·5	− 66	− 10 5132	− 4 5707	− 55	− 24	− 10
6000	164	16 6324	7 2310	138	60	26
7000	308	22 7518	9 8913	259	112	49
8000	495	28 8713	12 5516	417	181	79
243 9000	728	34 9908	15 2118	612	266	116
244 0000·5	−1005	− 41 1105	−17 8720	− 845	− 367	− 160
1000	1326	47 2302	20 5322	1115	485	211
2000	1692	53 3500	23 1923	1423	619	269
3000	2102	59 4699	25 8523	1768	769	334
4000	2557	65 5899	28 5123	2151	935	407
244 5000·5	−3057	− 71 7099	−31 1723	−2571	−1118	− 486
Mean $\delta_0^2 + \delta_1^2$	−89	−2	+1	−75	−32	−14

For interpolation to full eight-decimal precision in this table, second differences must be taken into account; they are sensibly constant over the range of tabulation, and mean values of the double second difference are given at the foot of each column for use with the interpolation formula:

$$f_p = (1 - p) f_0 + p f_1 + B_2 (\delta_0^2 + \delta_1^2)$$

If x_0, y_0, z_0 are the equatorial coordinates for the mean equinox of 1950·0 and x, y, z are for the mean equinox of date (t), then the formulae for reduction are:

From 1950·0 to t

$$x = x_0 + (X_x - 1)x_0 + Y_x y_0 + Z_x z_0$$
$$y = y_0 + X_y x_0 + (Y_y - 1)y_0 + Z_y z_0$$
$$z = z_0 + X_z x_0 + Y_z y_0 + (Z_z - 1)z_0$$

From t to 1950·0

$$x_0 = x + (X_x - 1)x + X_y y + X_z z$$
$$y_0 = y + Y_x x + (Y_y - 1)y + Y_z z$$
$$z_0 = z + Z_x x + Z_y y + (Z_z - 1)z$$

where $(X_x - 1)$, Y_x, . . . are obtained by interpolation to the argument t.

FOR REDUCTION TO (AND FROM) THE MEAN EQUINOX OF 1950·0

Julian Date t_0	ϵ 23° 26′ +	p	π	Π	a	b	c	c'
	″	″	″	° ′	′ ″	″	° ′	° ′
241 5000·5	68·28	50·2564	0·4711	173 57·0	+41 55·82	+23·57	6 10·2	5 28·3
6000	67·00	2570	4711	58·5	39 38·23	22·28	08·3	28·7
7000	65·72	2576	4711	174 00·0	37 20·63	20·99	06·4	29·1
8000	64·44	2582	4710	01·5	35 03·02	19·70	04·5	29·5
241 9000	63·15	2588	4710	03·0	32 45·42	18·41	02·6	29·9
242 0000·5	61·87	50·2594	0·4710	174 04·5	+30 27·82	+17·12	6 00·7	5 30·3
1000	60·59	2600	4710	06·0	28 10·21	15·83	5 58·8	30·7
2000	59·31	2606	4710	07·5	25 52·60	14·54	57·0	31·1
3000	58·02	2613	4709	09·0	23 34·99	13·25	55·1	31·5
4000	56·74	2619	4709	10·5	21 17·38	11·97	53·2	31·9
242 5000·5	55·46	50·2625	0·4709	174 12·0	+18 59·77	+10·68	5 51·3	5 32·3
6000	54·18	2631	4709	13·5	16 42·15	9·39	49·4	32·7
7000	52·89	2637	4709	15·0	14 24·53	8·10	47·5	33·1
8000	51·61	2643	4709	16·5	12 06·92	6·81	45·6	33·5
242 9000	50·33	2649	4708	18·0	9 49·30	5·52	43·7	33·9
243 0000·5	49·05	50·2655	0·4708	174 19·5	+ 7 31·68	+ 4·23	5 41·8	5 34·3
1000	47·76	2661	4708	21·0	5 14·05	2·94	39·9	34·7
2000	46·48	2667	4708	22·5	2 56·43	1·65	38·0	35·0
3000	45·20	2673	4708	24·0	+ 0 38·80	+ 0·36	36·1	35·4
4000	43·92	2679	4707	25·5	− 1 38·83	− 0·93	34·2	35·8
243 5000·5	42·63	50·2685	0·4707	174 27·0	− 3 56·46	− 2·21	5 32·3	5 36·2
6000	41·35	2692	4707	28·5	6 14·09	3·50	30·4	36·6
7000	40·07	2698	4707	30·0	8 31·72	4·79	28·5	37·0
8000	38·78	2704	4707	31·5	10 49·36	6·08	26·6	37·4
243 9000	37·50	2710	4706	33·0	13 06·99	7·37	24·7	37·8
244 0000·5	36·22	50·2716	0·4706	174 34·5	−15 24·63	− 8·66	5 22·8	5 38·2
1000	34·94	2722	4706	36·0	17 42·27	9·95	20·9	38·6
2000	33·65	2728	4706	37·5	19 59·91	11·23	19·0	39·0
3000	32·37	2734	4706	39·0	22 17·56	12·52	17·1	39·4
4000	31·09	2740	4705	40·5	24 35·20	13·81	15·2	39·8
244 5000·5	29·81	50·2746	0·4705	174 42·0	−26 52·85	−15·10	5 13·3	5 40·2

If λ_0, β_0 and Ω_0, ω_0, i_0 are ecliptic coordinates and elements for the mean equinox of date (t_0), and if λ, β and Ω, ω, i are for the mean equinox of 1950·0, then over the range of the above table the formulae for reduction are:

From t_0 to 1950·0

$$\lambda = \lambda_0 + a - b\cos(\lambda_0 + c)\tan\beta_0$$
$$\beta = \beta_0 + b\sin(\lambda_0 + c)$$
$$\Omega = \Omega_0 + a - b\sin(\Omega_0 + c)\cot i_0$$
$$\omega = \omega_0 + b\sin(\Omega_0 + c)\operatorname{cosec} i_0$$
$$i = i_0 + b\cos(\Omega_0 + c)$$

From 1950·0 to t_0

$$\lambda_0 = \lambda - a + b\cos(\lambda + c')\tan\beta$$
$$\beta_0 = \beta - b\sin(\lambda + c')$$
$$\Omega_0 = \Omega - a + b\sin(\Omega + c')\cot i$$
$$\omega_0 = \omega - b\sin(\Omega + c')\operatorname{cosec} i$$
$$i_0 = i - b\cos(\Omega + c')$$

where a, b, c, c' are obtained by linear interpolation to the argument t_0.

The following numerical values are deduced from Newcomb's discussion; they are derived† from fundamental values of the precessional constant, the obliquity, the speed of rotation of the ecliptic, and the longitude of the axis of rotation.

$$\begin{array}{llll}
\psi' = 50''\cdot3708 + 0''\cdot0050\,T & & m = & 3^{s}\cdot07234 + 0^{s}\cdot00186\,T \\
\lambda' = 0''\cdot1247 - 0''\cdot0188\,T & & n = & 1^{s}\cdot33646 - 0^{s}\cdot00057\,T \\
p = 50''\cdot2564 + 0''\cdot0222\,T & & = & 20''\cdot0468 - 0''\cdot0085\,T \\
\Pi = 173^{\circ}\,57'\cdot06 + 54'\cdot77\,T & & \pi = & 0''\cdot4711 - 0''\cdot0007\,T
\end{array}$$

$$\epsilon = 23^{\circ}\,27'\,08''\cdot26 - 46''\cdot845\,T - 0''\cdot0059\,T^2 + 0''\cdot00181\,T^3$$

where T is measured in tropical centuries from 1900·0.

Values of these quantities (except ψ', λ') for the current year are given in *A.E.*, page 50. Values of ϵ, p, π, Π are given in table 2.3, which has argument Julian date and an interval of 1000 days.

If p_{m}, m_{m}, n_{m}, Π_{m}, π_{m} are the values of the above quantities at an epoch midway between the initial epoch t_0 and a subsequent epoch t, then:

M = general precession in right ascension $= m_{\text{m}}(t - t_0) = \zeta_0 + z$

N = precession in declination $= n_{\text{m}}(t - t_0) = \theta$

a = general precession in longitude $= p_{\text{m}}(t - t_0)$

b = inclination of ecliptic of epoch t to that of epoch $t_0 = \pi_{\text{m}}(t - t_0)$

$c = 180^{\circ} - \Pi_{\text{m}} + \tfrac{1}{2}a$ $\qquad c' = 180^{\circ} - \Pi_{\text{m}} - \tfrac{1}{2}a$

Values of the above quantities, for reduction to and from 1950·0, are given for
* the current year in *A.E.*, page 50, and for 1800 to 1980 in the three volumes of *Planetary Co-ordinates*. Values of M, N are given in table 2.1, and values of a, b, c, c' are given with the values of ϵ, p, π, Π in table 2.3. They may be used for the reduction of positions from one mean equinox to another, provided the time interval is not too long nor the position too close to the pole.

The formulae for the reduction of equatorial coordinates from the mean equinox of t_0 to the mean equinox of t, or from t to t_0 are:

$$\alpha - \alpha_0 = M + N \sin \tfrac{1}{2}(\alpha + \alpha_0) \tan \tfrac{1}{2}(\delta + \delta_0)$$
$$\delta - \delta_0 = N \cos \tfrac{1}{2}(\alpha + \alpha_0)$$

where the right-hand sides are evaluated by successive approximation, if necessary.

The formulae for the reduction of ecliptic coordinates and ecliptic elements are:

From mean equinox of t_0 to mean equinox of t	From mean equinox of t to mean equinox of t_0
$\lambda = \lambda_0 + a - b\cos(\lambda_0 + c)\tan\beta$	$\lambda_0 = \lambda - a + b\cos(\lambda + c')\tan\beta_0$
$\beta = \beta_0 + b\sin(\lambda_0 + c)$	$\beta_0 = \beta - b\sin(\lambda + c')$
$\Omega = \Omega_0 + a - b\sin(\Omega_0 + c)\cot i$	$\Omega_0 = \Omega - a + b\sin(\Omega + c')\cot i_0$
$\omega = \omega_0 + b\sin(\Omega_0 + c)\operatorname{cosec} i$	$\omega_0 = \omega - b\sin(\Omega + c')\operatorname{cosec} i_0$
$i = i_0 + b\cos(\Omega_0 + c)$	$i_0 = i - b\cos(\Omega + c')$

where the final coordinates and elements on the right-hand sides are evaluated by successive approximation, if necessary, although the initial values are usually sufficiently accurate. Note that when i is small:

$$\Omega + \omega = \Omega_0 + \omega_0 + a$$

*Page 11 in *A.E.* 1972–3, page 9 from 1974.

†See also section 6.

For the transformation of equatorial or ecliptic rectangular coordinates the rigorous formulae obtained above are still the most suitable, though some simplifications may be made in the actual calculations.

Over short intervals of time, of less than a year, or to lower precision the formulae may be simplified by using constants for the coefficients and by ignoring second-order terms. Thus the reduction from the mean equinox of the beginning of one year to that of the next following year may be made through:

$$\alpha = \alpha_0 + 3^s{\cdot}073 + 1^s{\cdot}336 \sin \alpha \tan \delta$$
$$\delta = \delta_0 + 20''{\cdot}04 \cos \alpha$$
$$\lambda = \lambda_0 + 50''{\cdot}27 - 0''{\cdot}47 \cos (\lambda + 6^\circ) \tan \beta$$
$$\beta = \beta_0 + 0''{\cdot}47 \sin (\lambda + 6^\circ)$$

$$\Omega = \Omega_0 + 0^\circ{\cdot}01396 - 0^\circ{\cdot}00013 \sin (\Omega + 6^\circ) \cot i$$
$$\omega = \omega_0 + 0^\circ{\cdot}00013 \sin (\Omega + 6^\circ) \operatorname{cosec} i$$
$$i = i_0 + 0^\circ{\cdot}00013 \cos (\Omega + 6^\circ)$$

where it does not matter to which equinox the coordinates are referred.

Approximate annual corrections to right ascension and declination may be taken directly from table 2.4, which has been calculated from the formulae:

in right ascension	$3^s{\cdot}0730 + 1^s{\cdot}3362 \sin \alpha \tan \delta$
in declination	$20''{\cdot}043 \cos \alpha$

Coefficients for the approximate reduction from the standard equinox of 1950·0 to the true equinox during the current year are given in the *A.E.*, Table IV.

Differential precession and nutation

The rotation of the frame of reference due to precession (and nutation) causes small changes in the relative coordinates of two adjacent points and, in particular, changes the position angles of a star with respect to others; if t_0 is the epoch to which the angle is to be referred, and t is the epoch of the observation, then the observed position angle must be corrected by applying the angle PSP_0 (see figure 2.1) or $-(n \sec \delta \sin \alpha)(t - t_0)$, where n may be taken as $0^\circ{\cdot}0056$ and the time interval is in tropical years.

Over a small area in the sky the effect of precession (and nutation) varies slowly; thus the corrections for precession and nutation for moving objects will differ little from those for neighbouring stars, to which the positions of the moving object may be referred. Since the positions of the stars for equinox 1950·0, or for the beginning of the year, will be known it is only necessary to apply corrections for *differential precession and nutation* (and similarly for aberration and refraction) to yield the positions of the moving objects referred to the same equinox.

The effect of differential nutation is always small, and it is convenient to combine precession and nutation in one correction.

If $\Delta\alpha$ and $\Delta\delta$ are the observed differences of coordinates in the sense moving object minus star, then the corrections for differential precession and nutation for reduction to the nearest yearly equinox are, in the notation of section 5:

in right ascension	$-\{g \cos (G + \alpha) \tan \delta\} \Delta\alpha - \{g \sin (G + \alpha) \sec^2 \delta\} \Delta\delta$
in declination	$+\{g \sin (G + \alpha)\} \Delta\alpha$

Annual precession in right ascension, varies with declination

α	+80°	+70°	+60°	+50°	+40°	+30°	+15°	0°	−15°	−30°	−40°	−50°	−60°	−70°	−80°	Ann. Prec. in Dec.
h	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	′
0·0	+3·1	+3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	+3·1	+ 3·1	+20
0·5	4·1	3·6	3·4	3·3	3·2	3·2	3·1	3·1	3·0	3·0	2·9	2·9	2·8	2·6	2·1	20
1·0	5·0	4·0	3·7	3·5	3·4	3·3	3·2	3·1	3·0	2·9	2·8	2·7	2·5	2·1	1·1	19
1·5	6·0	4·5	4·0	3·7	3·5	3·4	3·2	3·1	2·9	2·8	2·6	2·5	2·2	1·7	+ 0·2	19
2·0	6·9	4·9	4·2	3·9	3·6	3·5	3·3	3·1	2·9	2·7	2·5	2·3	1·9	1·2	− 0·7	17
2·5	7·7	5·3	4·5	4·0	3·8	3·5	3·3	3·1	2·9	2·6	2·4	2·1	1·7	0·8	− 1·5	16
3·0	8·4	5·7	4·7	4·2	3·9	3·6	3·3	3·1	2·8	2·5	2·3	1·9	1·4	0·5	− 2·3	+14
3·5	9·1	6·0	4·9	4·3	4·0	3·7	3·4	3·1	2·8	2·5	2·2	1·8	1·2	+0·2	− 2·9	12
4·0	9·6	6·3	5·1	4·5	4·0	3·7	3·4	3·1	2·8	2·4	2·1	1·7	1·1	−0·1	− 3·5	10
4·5	10·1	6·5	5·2	4·5	4·1	3·8	3·4	3·1	2·7	2·4	2·0	1·6	0·9	−0·3	− 3·9	8
5·0	10·4	6·6	5·3	4·6	4·2	3·8	3·4	3·1	2·7	2·3	2·0	1·5	0·8	−0·5	− 4·2	5
5·5	10·6	6·7	5·4	4·7	4·2	3·8	3·4	3·1	2·7	2·3	2·0	1·5	0·8	−0·6	− 4·4	+ 3
6·0	10·7	6·7	5·4	4·7	4·2	3·8	3·4	3·1	2·7	2·3	2·0	1·5	0·8	−0·6	− 4·5	0
6·5	10·6	6·7	5·4	4·7	4·2	3·8	3·4	3·1	2·7	2·3	2·0	1·5	0·8	−0·6	− 4·4	− 3
7·0	10·4	6·6	5·3	4·6	4·2	3·8	3·4	3·1	2·7	2·3	2·0	1·5	0·8	−0·5	− 4·2	5
7·5	10·1	6·5	5·2	4·5	4·1	3·8	3·4	3·1	2·7	2·4	2·0	1·6	0·9	−0·3	− 3·9	8
8·0	9·6	6·3	5·1	4·5	4·0	3·7	3·4	3·1	2·8	2·4	2·1	1·7	1·1	−0·1	− 3·5	10
8·5	9·1	6·0	4·9	4·3	4·0	3·7	3·4	3·1	2·8	2·5	2·2	1·8	1·2	+0·2	− 2·9	12
9·0	8·4	5·7	4·7	4·2	3·9	3·6	3·3	3·1	2·8	2·5	2·3	1·9	1·4	0·5	− 2·3	−14
9·5	7·7	5·3	4·5	4·0	3·8	3·5	3·3	3·1	2·9	2·6	2·4	2·1	1·7	0·8	− 1·5	16
10·0	6·9	4·9	4·2	3·9	3·6	3·5	3·3	3·1	2·9	2·7	2·5	2·3	1·9	1·2	− 0·7	17
10·5	6·0	4·5	4·0	3·7	3·5	3·4	3·2	3·1	2·9	2·8	2·6	2·5	2·2	1·7	+ 0·2	19
11·0	5·0	4·0	3·7	3·5	3·4	3·3	3·2	3·1	3·0	2·9	2·8	2·7	2·5	2·1	1·1	19
11·5	4·1	3·6	3·4	3·3	3·2	3·2	3·1	3·1	3·0	3·0	2·9	2·9	2·8	2·6	2·1	20
12·0	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	−20
12·5	2·1	2·6	2·8	2·9	2·9	3·0	3·0	3·1	3·1	3·2	3·2	3·3	3·4	3·6	4·1	20
13·0	1·1	2·1	2·5	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·7	4·0	5·0	19
13·5	+0·2	1·7	2·2	2·5	2·6	2·8	2·9	3·1	3·2	3·4	3·5	3·7	4·0	4·5	6·0	19
14·0	−0·7	1·2	1·9	2·3	2·5	2·7	2·9	3·1	3·3	3·5	3·6	3·9	4·2	4·9	6·9	17
14·5	−1·5	0·8	1·7	2·1	2·4	2·6	2·9	3·1	3·3	3·5	3·8	4·0	4·5	5·3	7·7	16
15·0	−2·3	0·5	1·4	1·9	2·3	2·5	2·8	3·1	3·3	3·6	3·9	4·2	4·7	5·7	8·4	−14
15·5	−2·9	+0·2	1·2	1·8	2·2	2·5	2·8	3·1	3·4	3·7	4·0	4·3	4·9	6·0	9·1	12
16·0	−3·5	−0·1	1·1	1·7	2·1	2·4	2·8	3·1	3·4	3·7	4·0	4·5	5·1	6·3	9·6	10
16·5	−3·9	−0·3	0·9	1·6	2·0	2·4	2·7	3·1	3·4	3·8	4·1	4·5	5·2	6·5	10·1	8
17·0	−4·2	−0·5	0·8	1·5	2·0	2·3	2·7	3·1	3·4	3·8	4·2	4·6	5·3	6·6	10·4	5
17·5	−4·4	−0·6	0·8	1·5	2·0	2·3	2·7	3·1	3·4	3·8	4·2	4·7	5·4	6·7	10·6	− 3
18·0	−4·5	−0·6	0·8	1·5	2·0	2·3	2·7	3·1	3·4	3·8	4·2	4·7	5·4	6·7	10·7	0
18·5	−4·4	−0·6	0·8	1·5	2·0	2·3	2·7	3·1	3·4	3·8	4·2	4·7	5·4	6·7	10·6	+ 3
19·0	−4·2	−0·5	0·8	1·5	2·0	2·3	2·7	3·1	3·4	3·8	4·2	4·6	5·3	6·6	10·4	5
19·5	−3·9	−0·3	0·9	1·6	2·0	2·4	2·7	3·1	3·4	3·8	4·1	4·5	5·2	6·5	10·1	8
20·0	−3·5	−0·1	1·1	1·7	2·1	2·4	2·8	3·1	3·4	3·7	4·0	4·5	5·1	6·3	9·6	10
20·5	−2·9	+0·2	1·2	1·8	2·2	2·5	2·8	3·1	3·4	3·7	4·0	4·3	4·9	6·0	9·1	12
21·0	−2·3	0·5	1·4	1·9	2·3	2·5	2·8	3·1	3·3	3·6	3·9	4·2	4·7	5·7	8·4	+14
21·5	−1·5	0·8	1·7	2·1	2·4	2·6	2·9	3·1	3·3	3·5	3·8	4·0	4·5	5·3	7·7	16
22·0	−0·7	1·2	1·9	2·3	2·5	2·7	2·9	3·1	3·3	3·5	3·6	3·9	4·2	4·9	6·9	17
22·5	+0·2	1·7	2·2	2·5	2·6	2·8	2·9	3·1	3·2	3·4	3·5	3·7	4·0	4·5	6·0	19
23·0	1·1	2·1	2·5	2·7	2·8	2·9	3·0	3·1	3·2	3·3	3·4	3·5	3·7	4·0	5·0	19
23·5	2·1	2·6	2·8	2·9	2·9	3·0	3·0	3·1	3·1	3·2	3·2	3·3	3·4	3·6	4·1	20
24·0	+3·1	+3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	3·1	+3·1	+ 3·1	+20

The precession in right ascension is positive, except where indicated in high declinations.

Tables based on similar formulae are given in *A.E.*, Table VI, for reduction to the nearest equinox of the beginning of a Besselian year, and also to that of 1950·0; the corrections are given in the form:

in right ascension $e \tan \delta \, \Delta\alpha \quad - f \sec^2\delta \, \Delta\delta$

in declination $f \, \Delta\alpha$

where $e = 10^2 j \sin (J + \alpha) \qquad j \sin J_0 = n\,(t - A) \sin 1' \qquad J = J_0 - 1^{s}\!\cdot\!5\, t$

$f = 10^2 j \cos (J + \alpha) \qquad j \cos J_0 = B \sin 1'$

and t is the number of years from date to the equinox desired. The small correction to J_0 allows for the fact that it is more accurate to use the right ascension of the star for the mid-epoch. With j, e, f measured in the units indicated (chosen for convenience of tabulation), and with $\Delta\alpha$, $\Delta\delta$ in units of 1′, the corrections are in units of 0″·01.

References

The numerical expressions for the precessional motions are those of S. Newcomb; fundamental and derived values are given in:

Newcomb, S. *The elements of the four inner planets and the fundamental constants of astronomy.* Supplement to *The American Ephemeris* for 1897, Washington, 1895.

Newcomb, S. *A new determination of the precessional constant with the resulting precessional motions, A.P.A.E.*, **8**, part 1, 1897.

Newcomb, S. *A compendium of spherical astronomy.* New York, Macmillan, 1906; reprinted, New York, Dover Publications, 1960.

Peters, J. *Präzessionstafeln für das Äquinoktium* 1950·0. *Veröff. Astr. Rechen-Inst. zu Berlin-Dahlem*, no. 50, 1934.

C. NUTATION

Nutation is essentially that part of the precessional motion of the pole of the Earth's equator which depends on the periodic motions of the Sun and Moon in their orbits round the Earth. The progressive long-period motion of the *mean* pole has been considered as luni-solar precession in sub-section B; nutation is the somewhat irregular elliptical motion of the *true* pole about the *mean* pole in a period of about 19 years with an amplitude of about 9″. The principal term depends on the longitude of the node of the Moon's orbit and has a period of 6798 days or 18·6 years; the amplitude of this term, 9″·210, is known as the *constant of nutation.* In the complicated theory of the gravitational action of the Sun and Moon on the rotating non-spherical Earth, other terms arise which depend on the mean longitudes and mean anomalies of the Sun and Moon and on their combinations with the longitude of the Moon's node. The resulting shift of the mean to the true pole can be resolved into corrections to the longitude ($\Delta\psi$, nutation in longitude) and to the mean obliquity ($\Delta\epsilon$, nutation in obliquity), and expressions for these in series constitute the formal specification of the nutation. The theory and the numerical series upon which the nutation is now based are developed in full detail by E. W. Woolard in *A.P.A.E.*, 15, part I, 1953, to which reference should be made for further information.

The terms divide naturally into those not depending on the Moon's longitude, which can be interpolated at intervals of 10 days, and those depending on the Moon's longitude, with periods of less than about 60 days, which cannot be so interpolated. Nutation is therefore conventionally divided into long-period and short-period terms; the latter, consisting of terms with periods less than 35 days, are summed separately as $d\psi$ and $d\epsilon$, being the short-period terms of nutation in longitude and obliquity respectively. In certain special applications, such as the tabulation of the apparent positions of stars at intervals of 10 days, only the long-period terms of nutation are included; and data are provided for the individual application of corrections for the much smaller short-period terms after interpolation.

The terms included in the nutation are given in table 2.5. There are 69 terms in $\Delta\psi$, of which 46 are of short period and are summed separately as $d\psi$; for the obliquity there are 40 terms in $\Delta\epsilon$, including 24 terms in $d\epsilon$. The series include all terms with coefficients of $0''\cdot0002$ or greater. In the table the terms are grouped according to their periods, and are arranged in order of magnitude of the coefficient of the nutation in longitude within each group.

These series may be compared with those used prior to 1960 (with a maximum of 22 terms in longitude and 15 in obliquity), which are given in section 7; values were then tabulated only to $0''\cdot01$.

Values of the nutation have been calculated from the series given above for 0^h E.T. on each day from 1900 to 2000. Those for 1900 to 1959 are published in *Royal Observatory Annals*, Number 1. The values for 1952 to 1959 have also been included in the *Improved Lunar Ephemeris* 1952–1959. In each publication there is given a description of the method used for the calculation on punched-card machines.

The nutation in longitude ($\Delta\psi$), to be added to longitudes measured from the mean equinox of date, is tabulated to $0''\cdot001$ for 0^h E.T. on each day of the year in *A.E.*, pages 18 to 32. The nutation in obliquity ($\Delta\epsilon$) is not tabulated directly as such, but enters into the obliquity of the ecliptic on pages 18 to 32 and is obtainable immediately as $-B$, the Besselian day number given in *A.E.*, pages 266 to 285; the short-period terms in both longitude and obliquity, $d\psi$ and $d\epsilon$, are also tabulated on the latter pages, all to $0''\cdot001$. The long-period terms $\Delta\psi - d\psi$ and $\Delta\epsilon - d\epsilon$ are not separately tabulated in the Ephemeris, though special values at intervals of 10 sidereal days are incorporated into the day numbers used for the calculation of the apparent places of 10-day stars, published in *Apparent Places of Fundamental Stars*.

The intersection of the true equator (as affected by both precession and nutation) with the true ecliptic is known as the true equinox of date; and, where distinction is desirable, all coordinates referred to this reference system of the true equinox, true equator, and true ecliptic of date are prefixed by the words "true" or "apparent", the latter being used when the direction is affected by aberration.

The *equation of the equinoxes*, which in editions prior to 1960 was called "nutation in right ascension", is the right ascension of the mean equinox referred to the true equator and equinox. It is equal to $\Delta\psi \cos \epsilon$ and represents the difference between the mean and true right ascensions for a body on the equator; it is thus the difference between mean and apparent sidereal time. The equation of the equinoxes is tabulated to $0^s.001$ in *A.E.*, pages 10 to 17, and is incorporated into the apparent sidereal time on the same pages.

The simplest and most direct method of converting positions from the mean equinox and the mean equator to the true equinox and the true equator is to add $\Delta\psi$ to longitude, since the ecliptic and therefore the latitude is unchanged by nutation. In converting the ecliptic coordinates to equatorial coordinates the true obliquity ($\epsilon = \epsilon_0 + \Delta\epsilon$) must be used. It should, however, be remembered that coordinates referred to the true equinox cannot be interpolated at intervals longer than one day. First-order corrections $\Delta\alpha$, $\Delta\delta$ to right ascension and declination may be calculated directly from:

$$\begin{aligned}\Delta\alpha &= (\cos\epsilon + \sin\epsilon \sin\alpha \tan\delta)\, \Delta\psi - \cos\alpha \tan\delta\, \Delta\epsilon \\ \Delta\delta &= \sin\epsilon \cos\alpha\, \Delta\psi + \sin\alpha\, \Delta\epsilon\end{aligned}$$

but these are invariably combined with the reduction for precession from the mean equinox of the beginning of the year by means of day numbers. The method, as applied to stars, is described in detail in section 5.

Equatorial rectangular coordinates referred to the mean equinox can be converted to the true equinox by the application of the corrections:

$$\begin{aligned}\Delta x &= -\,(y\cos\epsilon + z\sin\epsilon)\,\Delta\psi \\ \Delta y &= +\,x\cos\epsilon\,\Delta\psi - z\,\Delta\epsilon \\ \Delta z &= +\,x\sin\epsilon\,\Delta\psi + y\,\Delta\epsilon\end{aligned}$$

Second-order terms, which are neglected, can only reach one unit in the eighth figure. These formulae are used for the Sun and planets (see sections 4B and 4D). The reduction for nutation can also be combined with that for precession by pre-multiplying the matrix of coefficients X_x, Y_x, ... by the matrix whose corresponding elements are:

$$\begin{matrix} 1 & -\Delta\psi\cos\epsilon & -\Delta\psi\sin\epsilon \\ +\Delta\psi\cos\epsilon & 1 & -\Delta\epsilon \\ +\Delta\psi\sin\epsilon & +\Delta\epsilon & 1 \end{matrix}$$

It is not sufficient merely to add $-\Delta\psi\cos\epsilon$ to Y_x, $-\Delta\psi\sin\epsilon$ to Z_x, . . . as second-order terms may be significant.

Differential nutation. For objects within a small area of the sky differential nutation is always combined with differential precession; see sub-section B and *A.E.*, Table VI.

Short-period nutation. Corrections for the short-period terms of nutation may be obtained directly from table 5.2, which is described in section 5D.

The fundamental arguments, corrected for amendment to Brown's tables,* are:

$$l = 296^\circ\ 06'\ 16''\cdot 59 + 1325^r\ 198^\circ\ 50'\ 56''\cdot 79T + 33''\cdot 09T^2 + 0''\cdot 0518T^3$$
$$= 296^\circ\cdot 10460\ 8 + 13^\circ\cdot 06499\ 24465d + 0^\circ\cdot 00068\ 90D^2 + 0^\circ\cdot 00000\ 0295D^3$$

$$l' = 358^\circ\ 28'\ 33''\cdot 00 + 99^r\ 359^\circ\ 02'\ 59''\cdot 10T - 0''\cdot 54T^2 - 0''\cdot 0120T^3$$
$$= 358^\circ\cdot 47583\ 3 + 0^\circ\cdot 98560\ 02669d - 0^\circ\cdot 00001\ 12D^2 - 0\cdot 00000\ 0068D^3$$

$$F = 11^\circ\ 15'\ 03''\cdot 20 + 1342^r\ 82^\circ\ 01'\ 30''\cdot 54T - 11''\cdot 56T^2 - 0''\cdot 0012T^3$$
$$= 11^\circ\cdot 25088\ 9 + 13^\circ\cdot 22935\ 04490d - 0^\circ\cdot 00024\ 07D^2 - 0^\circ\cdot 00000\ 0007D^3$$

$$D = 350^\circ\ 44'\ 14''\cdot 95 + 1236^r\ 307^\circ\ 06'\ 51''\cdot 18T - 5''\cdot 17T^2 + 0''\cdot 0068T^3$$
$$= 350^\circ\cdot 73748\ 6 + 12^\circ\cdot 19074\ 91914d - 0^\circ\cdot 00010\ 76D^2 + 0^\circ\cdot 00000\ 0039D^3$$

$$\Omega = 259^\circ\ 10'\ 59''\cdot 79 - 5^r\ 134^\circ\ 08'\ 31''\cdot 23T + 7''\cdot 48T^2 + 0''\cdot 0080T^3$$
$$= 259^\circ\cdot 18327\ 5 - 0^\circ\cdot 05295\ 39222d + 0^\circ\cdot 00015\ 57D^2 + 0^\circ\cdot 00000\ 0046D^3$$

where the fundamental epoch is 1900 January $0^d\cdot 5$ E.T. = J.E.D. 241 5020·0, and

T is measured in Julian centuries of 36525 days,

d is measured in days,

D is measured in units of 10 000 days.

Period (days)	ARGUMENT Multiple of l	l'	F	D	Ω	LONGITUDE Coefficient of sine argument		OBLIQUITY Coefficient of cosine argument	
						Unit = 0″·0001			
6798					+1	−172327	−173·7 T	+92100	+9·1 T
3399					+2	+ 2088	+ 0·2 T	− 904	+0·4 T
1305	−2		+2		+1	+45		−24	
1095	+2		−2			+10			
6786		−2	+2	−2	+1	− 4		+ 2	
1616	−2		+2		+2	− 3		+ 2	
3233	+1	−1		−1		− 2			
183			+2	−2	+2	−12729	−1·3 T	+5522	−2·9 T
365		+1				+ 1261	−3·1 T		
122		+1	+2	−2	+2	− 497	+1·2 T	+ 216	−0·6 T
365		−1	+2	−2	+2	+ 214	−0·5 T	− 93	+0·3 T
178			+2	−2	+1	+ 124	+0·1 T	− 66	
206	+2			−2		+45			
173			+2	−2		−21			
183		+2				+16	−0·1 T		
386		+1			+1	−15		+ 8	
91		+2	+2	−2	+2	−15	+0·1 T	+ 7	
347		−1			+1	−10		+ 5	
200	−2			+2	+1	− 5		+ 3	
347		−1	+2	−2	+1	− 5		+ 3	
212	+2			−2	+1	+ 4		− 2	
120		+1	+2	−2	+1	+ 3		− 2	
412	+1			−1		− 3			

*See note on page 523.

Unit = 0″·0001

Period (days)	ARGUMENT Multiple of l	l'	F	D	Ω	LONGITUDE Coefficient of sine argument		OBLIQUITY Coefficient of cosine argument	
13·7			+2		+2	−2037	−0·2T	+884	−0·5T
27·6	+1					+675	+0·1T		
13·6			+2		+1	−342	−0·4T	+183	
9·1	+1		+2		+2	−261		+113	−0·1T
31·8	+1			−2		−149			
27·1	−1		+2		+2	+114		−50	
14·8				+2		+60			
27·7	+1				+1	+58		−31	
27·4	−1				+1	−57		+30	
9·6	−1		+2	+2	+2	−52		+22	
9·1	+1		+2		+1	−44		+23	
7·1			+2	+2	+2	−32		+14	
13·8	+2					+28			
23·9	+1		+2	−2	+2	+26		−11	
6·9	+2		+2		+2	−26		+11	
13·6			+2			+25			
27·0	−1		+2		+1	+19		−10	
32·0	−1			+2	+1	+14		−7	
31·7	+1			−2	+1	−13		+7	
9·5	−1		+2	+2	+1	−9		+5	
34·8	+1	+1		−2		−7			
13·2		+1	+2		+2	+7		−3	
9·6	+1			+2		+6			
14·8				+2	+1	−6		+3	
14·2		−1	+2		+2	−6		+3	
5·6	+1		+2	+2	+2	−6		+3	
12·8	+2		+2	−2	+2	+6		−2	
14·7				−2	+1	−5		+3	
7·1			+2	+2	+1	−5		+3	
23·9	+1		+2	−2	+1	+5		−3	
29·5				+1		+4			
15·4		+1		−2		−4			
29·8	+1	−1				+4			
26·9	+1		−2			+4			
6·9	+2		+2		+1	−4		+2	
9·1	+1		+2			+3			
25·6	+1	+1				−3			
9·4	+1	−1	+2		+2	−3			
13·7	−2				+1	−2			
32·6	−1		+2	−2	+1	−2			
13·8	+2				+1	+2			
9·8	−1	−1	+2	+2	+2	−2			
7·2		−1	+2	+2	+2	−2			
27·8	+1				+2	−2			
8·9	+1	+1	+2		+2	+2			
5·5	+3		+2		+2	−2			

D. ABERRATION

Because the velocity of light is finite, the apparent direction of a moving celestial object from a moving observer is not the same as the geometric direction of the object from the observer at the same instant. This displacement of the apparent position from the geometric position may be attributed in part to the motion of the object, and in part to the motion of the observer, these motions being referred to an inertial frame of reference. The former part, independent of the motion of the observer, may be considered to be a *correction for light-time*; the latter part, independent of the motion or distance of the object, is referred to as *stellar aberration*, since for the stars the correction for light-time is, of necessity, ignored. The sum of the two parts is called *planetary aberration* since it is applicable to planets and other members of the solar system.

Correction for light-time. Let P and E (see figure 2.3) be the geometric positions of an object and a stationary observer at time t, and let P′ be the geometric position of the object at time $t - \tau$, where τ is the light-time, i.e. the time taken for the light to travel from the point of emission, in this case P′, to the point of observation, E. Then, since E is regarded as stationary, the direction EP′ is the apparent direction of the object at time t, i.e. the apparent direction at time t is the same as the geometric direction of the object at time $t - \tau$.

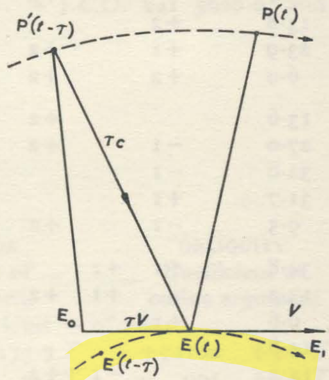

Figure 2.3. *Planetary aberration*

Stellar aberration. The light which is received at the instant of observation t was emitted, at a previous instant, from the position which the object occupied at time $t - \tau$ towards the position which the observer was later to occupy at time t; but when the light reaches the observer it appears to be coming, not from this actual direction but from its direction relative to the moving observer. Let the object be considered stationary at P′, the position it occupies at time $t - \tau$, and let E be moving in the direction EE_1 with an instantaneous velocity V. Then, according to classical theory, the apparent direction of the object is that of the vector difference of the velocity of light c in the direction P′E and the velocity V in the direction EE_1. The apparent angular displacement is independent of the distance, but, by definition of the light-time τ, $P'E = \tau c$ so that if E_0E is drawn in the direction of motion and of magnitude τV the apparent direction of the object is E_0P'. Thus the apparent direction at time t would be the same as the geometric direction at time $t - \tau$ were E moving with a constant rectilinear velocity V, i.e. if E_0 were identical with E′, the position of the observer at time $t - \tau$.

The displacement is toward the apex of the motion of the observer; its magnitude ($\Delta\theta$) depends upon the ratio of the velocity of the observer (V) to the velocity of light (c), and is given by the solution of the triangle $EP'E_0$, where θ is the angle $P'EE_1$ between the direction of P' and the direction of motion.

$$\sin \Delta\theta = \frac{V}{c} \sin (\theta - \Delta\theta), \text{ or } \tan \Delta\theta = \frac{V \sin \theta}{c + V \cos \theta}$$

Expanding in powers of V/c:

$$\Delta\theta = \frac{V}{c} \sin \theta - \tfrac{1}{2} \left(\frac{V}{c}\right)^2 \sin 2\theta + \ldots$$

Since V/c is about 0·0001 or 20″, the second-order term has a maximum of about 0″·001.

The rigorous relativistic theory for stellar aberration gives a different coefficient for the second-order term, but this effect is too small to be observed and is generally ignored.

The motion of an observer on the Earth is the resultant of the diurnal rotation of the Earth, the orbital motion of the Earth about the centre of mass of the solar system, and the motion of this centre of mass in space. The stellar aberration is therefore made up of three components which are referred to as the *diurnal aberration*, the *annual aberration*, and the *secular aberration*. The stars and the centre of mass of the solar system may each be considered to be in uniform rectilinear motion; in this case the correction for light-time and the secular aberration are indistinguishable and the aberrational displacement due to the relative motion is merely equal to the proper motion of the star multiplied by the light-time; it is constant for each star, is in general not known, and is ignored.

The term " stellar aberration " is sometimes loosely used in this Supplement in contexts where " annual aberration " should strictly be used.

Annual aberration. In accordance with recommendations of the International Astronomical Union (*Trans. I.A.U.*, 7, 75, 1950; **8**, 67 and 90, 1954) the annual aberration is calculated as from 1960 from the actual motion of the Earth, referred to an inertial frame of reference and to the centre of mass of the solar system. The resulting aberrational displacement $\Delta\theta$ may be resolved into corrections to the directional coordinates by the standard methods. If, for example, $-X'$, $-Y'$, $-Z'$ are the components of the Earth's velocity parallel to equatorial rectangular axes, the corrections to right ascension and declination, referred to the same equator and equinox, in the sense " apparent place *minus* mean place " are, to the first order in V/c:

$$\cos \delta \, \Delta\alpha = \frac{X'}{c} \sin \alpha - \frac{Y'}{c} \cos \alpha$$

$$\Delta\delta = \frac{X'}{c} \cos \alpha \sin \delta + \frac{Y'}{c} \sin \alpha \sin \delta - \frac{Z'}{c} \cos \delta$$

These formulae are usually simplified by the use of the aberrational day numbers C and D (or h, H, i), discussed in detail in section 5; in this simplification the assumption is implicitly made that the direction of motion of the Earth lies in the ecliptic, but the resulting error is negligible. The effect of second-order terms is included in the expressions for the second-order day numbers J and J'.

Prior to 1960 it was customary, for computational convenience, to approximate to the motion of the Earth by taking:

$$X' = -kc \sin \lambda \qquad Y' = +kc \cos \lambda \cos \epsilon \qquad Z' = Y' \tan \epsilon$$

where k is the constant of aberration and λ is the Sun's true longitude. In addition to small periodic terms due to the action of the Moon and planets, this procedure neglects terms depending on the eccentricity and longitude of perihelion of the Earth's orbit (*M.N.R.A.S.*, **110,** 467, 1950). These terms, of order about $0''\cdot34$, are constant throughout the year for any particular star, and change very slowly during the centuries; they are here represented symbolically by E, in the sense of apparent place *minus* mean place. It is assumed that the observed apparent places of stars have in the past been reduced to give catalogue mean places of stars that already contain the constant part E of the aberrational reduction to apparent place, and so it is desirable to subtract the effect of E from the aberration calculated from the actual motion of Earth; this procedure is recommended by the International Astronomical Union (*Trans. I.A.U.*, **7,** 75, 1950; **8,** 90, 1954). It can be accomplished by applying constant corrections to the components of the Earth's motion.

The sense in which the E-terms are measured can best be appreciated by using symbolic notation; let:

$A \equiv$ apparent place; $M \equiv$ true mean place; $M_0 \equiv$ catalogue mean place;
$R \equiv$ the complete star reduction, including the correction for aberration calculated from the true motion of the Earth, in the sense apparent $-$ mean;
$E \equiv$ the E-terms of aberration, in the sense here used.

Then: the true mean place $M = A - R$
the catalogue mean place $M_0 = A - (R - E) = M + E$

Thus the modified star reduction, to be applied to the catalogue mean place to give the apparent place, is $R - E$ since:

$$M_0 + (R - E) = A$$

In this sense the E-terms are:

in longitude (λ) $+ke \sec \beta \cos (\varpi - \lambda)$
in latitude (β) $+ke \sin \beta \sin (\varpi - \lambda)$

*where $k = 20''\cdot47$ is the constant of aberration, $e = 0\cdot01675 - 0\cdot00004\,T$ is the eccentricity of the Earth's orbit, and $\varpi = \Gamma - 180° = 101°\cdot22 + 1°\cdot72\,T$ is the longitude of perihelion of the Earth's orbit (see section 4B).

For systematic application to right ascension and declination the E-terms are best expressed in terms of corrections $\Delta C = +ke \cos \varpi \cos \epsilon$, $\Delta D = +ke \sin \varpi$ to the day numbers C, D, in such a way that the E-terms are:

in right ascension $c\Delta C + d\Delta D$
in declination $c'\Delta C + d'\Delta D$

Full details of the practical application to the calculation of the day numbers C and D are given in section 5D, and numerical expressions are given in section 4G. For bodies in the solar system the E-terms vary, and so the annual aberration is calculated, implicitly, without modification from the actual motion of the Earth.

The value that is used for c corresponds to the adopted value of $20''\cdot47$ for the

*$20''\cdot496$ from 1968, corresponding to $c = 2\cdot997\,925 \times 10^8$ m/s.

constant of aberration, and not to the actual velocity of light. It is equivalent to a light-time of $0^d.0057683 = 498^s.38$ for unit distance.*

Diurnal aberration. The rotation of the Earth on its axis carries the observer towards the east with a velocity $v_0 \rho \cos \phi'$, where v_0 (0·46 km/sec) is the equatorial rotational velocity of the surface of the Earth. The corresponding constant of diurnal aberration is:

$$\frac{v_0}{c} \rho \cos \phi' = 0''.320 \rho \cos \phi' = 0^s.0213 \rho \cos \phi'$$

The aberrational displacement may be resolved into corrections (apparent minus mean) in right ascension and declination:

$$\Delta\alpha = 0^s.0213 \rho \cos \phi' \cos h \sec \delta$$
$$\Delta\delta = 0''.320 \rho \cos \phi' \sin h \sin \delta$$

where h is the hour angle. The effect is small but is of importance in meridian observations; for a star at transit $\Delta\delta$ is zero, but

$$\Delta\alpha = \pm 0^s.0213 \rho \cos \phi' \sec \delta$$

according as the star is above or below the pole. A correction of this amount is usually subtracted from the observed time of transit instead of being added to the right ascension. Values of the correction are tabulated in table 2.6.

Correction for light-time. When a correction for light-time is required, it is usually combined with that for annual aberration; the combined correction for planetary aberration is described in the following paragraph. The correction for light-time alone could be obtained, if desired, by a comparison of the geometric ephemeris at time t with that derived by combining the geometric position of the Earth at time t with the geometric position of the object at time $t - \tau$.

Planetary Aberration. The displacement of the apparent position from the geometric position at the same instant by planetary aberration may be obtained from the two independent components due to the instantaneous motion of the Earth and the motion of the body during the light-time; but the practical methods that are used give the planetary aberration directly from the geocentric ephemeris, without explicit separation of the two components. The errors of these methods may be deduced by comparison with the results of using the heliocentric motions (strictly barycentric motions, although the maximum errors due to the motion of the Sun with respect to the centre of mass of the solar system are quite negligible) of the Earth and the body. Such methods are not generally practicable as the geocentric distances must be calculated to give the light-time.

Since the E-terms vary for a moving object, such as a planet, annual aberration must in this case be calculated, without modification, from the actual motion of the Earth. It may be allowed for *exactly* by displacing the Earth a distance τV in the direction opposite to the Earth's *instantaneous* velocity V. If the planet's motion in the light-time τ can be regarded as rectilinear and uniform, the position of the planet at time $t - \tau$ may be obtained by a displacement of distance τv in the direction opposite to the planet's instantaneous velocity v.

*$0^d.005\,7756 = 499^s.012$ from 1968.

Lat.	0°	10°	20°	30°	35°	40°	45°	50°	52°	54°	56°	58°	60°
Dec. °						Unit $0^s.001$							
0	21	21	20	18	17	16	15	14	13	13	12	11	11
5	21	21	20	19	18	16	15	14	13	13	12	11	11
10	22	21	20	19	18	17	15	14	13	13	12	11	11
15	22	22	21	19	18	17	16	14	14	13	12	12	11
20	23	22	21	20	19	17	16	15	14	13	13	12	11
25	24	23	22	20	19	18	17	15	14	14	13	12	12
30	25	24	23	21	20	19	17	16	15	14	14	13	12
35	26	26	24	23	21	20	18	17	16	15	15	14	13
40	28	27	26	24	23	21	20	18	17	16	16	15	14
45	30	30	28	26	25	23	21	19	19	18	17	16	15
50	33	33	31	29	27	25	23	21	20	20	19	18	17
52	35	34	33	30	28	27	25	22	21	20	19	18	17
54	36	36	34	31	30	28	26	23	22	21	20	19	18
56	38	38	36	33	31	29	27	24	23	22	21	20	19
58	40	40	38	35	33	31	28	26	25	24	23	21	20
60	43	42	40	37	35	33	30	27	26	25	24	23	21
62	45	45	43	39	37	35	32	29	28	27	25	24	23
64	49	48	46	42	40	37	34	31	30	29	27	26	24
66	52	52	49	45	43	40	37	34	32	31	29	28	26
68	57	56	54	49	47	44	40	37	35	33	32	30	28
70	62	61	59	54	51	48	44	40	38	37	35	33	31
71	66	65	62	57	54	50	46	42	40	39	37	35	33
72	69	68	65	60	57	53	49	44	43	41	39	37	35
73	73	72	69	63	60	56	52	47	45	43	41	39	36
74	77	76	73	67	63	59	55	50	48	45	43	41	39
75	82	81	77	71	68	63	58	53	51	48	46	44	41
76	88	87	83	76	72	68	62	57	54	52	49	47	44
77	95	93	89	82	78	73	67	61	58	56	53	50	47
78	103	101	96	89	84	79	73	66	63	60	57	54	51
79	112	110	105	97	92	86	79	72	69	66	63	59	56
° ′						Unit $0^s.01$							
80 00	12	12	12	11	10	9	9	8	8	7	7	7	6
81 00	14	13	13	12	11	10	10	9	8	8	8	7	7
82 00	15	15	14	13	13	12	11	10	9	9	9	8	8
83 00	18	17	16	15	14	13	12	11	11	10	10	9	9
84 00	20	20	19	18	17	16	14	13	13	12	11	11	10
85 00	24	24	23	21	20	19	17	16	15	14	14	13	12
85 30	27	27	26	24	22	21	19	17	17	16	15	14	14
86 00	31	30	29	26	25	23	22	20	19	18	17	16	15
86 30	35	34	33	30	29	27	25	22	22	21	20	19	17
87 00	41	40	38	35	33	31	29	26	25	24	23	22	20
87 30	49	48	46	42	40	37	35	31	30	29	27	26	24
88 00	61	60	57	53	50	47	43	39	38	36	34	32	31
88 10	67	66	63	58	55	51	47	43	41	39	37	35	33
88 20	73	72	69	64	60	56	52	47	45	43	41	39	37
88 30	82	80	77	71	67	62	58	52	50	48	46	43	41
88 40	92	90	86	79	75	70	65	59	56	54	51	49	46
88 50	105	103	98	91	86	80	74	67	65	62	59	56	52
89 00	122	120	115	106	100	94	86	79	75	72	68	65	61

This correction is to be *subtracted* from the observed time of transit for transits above pole, and *added* to the time of transit for transits below pole.

Thus the practical determination of planetary aberration may be based on either of the two principles:

(1) To the order of accuracy that the motion of the *object* during the light-time is rectilinear and uniform, the planetary aberration depends upon the instantaneous velocity of the observer *relative to the object* at the time of observation in exactly the same way as stellar aberration depends upon the instantaneous total velocity of the observer.

(2) To the order of accuracy that the motion of the *Earth* during the light-time is rectilinear and uniform, the directly observed apparent position at time t is the same as the geometric position that the object occupied at time $t - \tau$ relative to the position that the Earth occupied at time $t - \tau$.

From (1) it follows that the apparent position at time t may be determined by applying to the geometric position at t a correction consisting of the light-time multiplied by the instantaneous velocity of the Earth relative to the object; since this relative motion is the negative of the geocentric motion of the object, the correction to the geometric value of any geocentric coordinate q is $-\tau\, dq/dt$ where the instantaneous rate of change dq/dt is obtained by numerical differentiation of the geometric ephemeris. Thus:

$$\begin{aligned}\text{Apparent position} &= \text{geometric position} - \tau \text{ (rate of change)}\\ &= \text{geometric position} - 0{\cdot}0057683\, \Delta \text{ (daily motion)*}\end{aligned}$$

The departure from rectilinear and uniform motion gives rise to errors of order $0''{\cdot}001\ \Delta/a^2$ where a is the mean distance of the planet; this may reach $0''{\cdot}01$ for Mercury but does not exceed $0''{\cdot}001$ for the outer planets; second-order terms may reach $0''{\cdot}001$ or $0''{\cdot}002$ and are neglected.

Alternatively from (2) it follows that, if the light-time is not too great, the apparent position at time t may be obtained by interpolating the geometric ephemeris to time $t - \tau$; or, conversely, from an observed position, the geometric position at the preceding instant when light left the object is immediately obtained by ante-dating the time t of observation to $t - \tau$. The error is generally larger than in using (1) and, for the outer planets, may reach $0''{\cdot}001\ \Delta$. Corrections for the effect of curvature of the Earth's orbit may be applied if high precision is required; but no formulae for these are given here.

Strictly, the light-time corresponds to the distance from the position of the Earth at time t to the position of the body at time $t - \tau$; but, as far as the planets are concerned, the maximum error arising from using the geocentric distance at time t for calculating the light-time is only $0''{\cdot}0005$.

Illustrations of the application of corrections for planetary aberration are given in sections 4B and 4D.

Differential aberration. The differential coordinates of a moving object with respect to a fixed star will be affected by differential aberration; if $\Delta\alpha$, $\Delta\delta$ are the observed differences of the coordinates in the sense

*0·005 7756 from 1968.

The corrections for differential aberration to be added to the observed differences (in the sense moving object minus star) of right ascension and declination to give the true differences are:

$$\text{in right ascension} \quad a\,\Delta\alpha + b\frac{\Delta\delta}{10} \quad \text{in units of } 0^{s}\!.001$$

$$\text{in declination} \quad c\,\Delta\alpha + d\frac{\Delta\delta}{10} \quad \text{in units of } 0''\!.01$$

where $\Delta\alpha$, $\Delta\delta$ are the observed differences in units of 1^{m} and $1'$ respectively and where a, b, c, d are the coefficients defined by:

$$a = -\tfrac{1}{15}\,10^{3} \sin 1^{m}\, h \cos (H + \alpha) \sec \delta \qquad b = -\tfrac{1}{15}\,10^{4} \sin 1'\, h \sin (H + \alpha) \sec \delta \tan \delta$$

$$c = +\,10^{2} \sin 1^{m}\, h \sin (H + \alpha) \sin \delta \qquad d = -\,10^{3} \sin 1'\, h \cos (H + \alpha) \cos \delta$$

in which a constant value of $19''\!.6$ is used for h. The values of these coefficients are tabulated without signs on the opposite page with main arguments $H + \alpha$, for the first quadrant only, and δ, also without a sign; an auxiliary argument for the second quadrant is also given on the extreme right-hand side. Interpolation in this table is, in general, unnecessary. The day number H is obtained from the critical table below; this table may be used unchanged for all years. The signs of a, b, c, d, which depend on the quadrant of $H + \alpha$ and the sign of the declination, and the argument in the first quadrant corresponding to the actual value of $H + \alpha$ are taken from the second table below; the auxiliary argument is also indicated in the second and fourth quadrants.

Date	H	Date	H	Date	H	Date	H	Date	H
Dec. 26	h	Mar. 11	h	May 23	h	Aug. 14	h	Oct. 25	h
	23·5		18·5		13·5		8·5		3·5
Jan. 3		17		31		22		Nov. 2	
	23·0		18·0		13·0		8·0		3·0
11		24		June 9		29		9	
	22·5		17·5		12·5		7·5		2·5
19		31		17		Sept. 6		17	
	22·0		17·0		12·0		7·0		2·0
26		Apr. 7		26		13		24	
	21·5		16·5		11·5		6·5		1·5
Feb. 3		15		July 5		20		Dec. 2	
	21·0		16·0		11·0		6·0		1·0
10		22		13		27		10	
	20·5		15·5		10·5		5·5		0·5
17		29		21		Oct. 4		18	
	20·0		15·0		10·0		5·0		0·0
25		May 7		29		11		26	
	19·5		14·5		9·5		4·5		23·5
Mar. 4		15		Aug. 6		18		Jan. 3	
	19·0		14·0		9·0		4·0		
11		23		14		25			

In critical cases ascend

Signs of a, b, c, d — Tabular arguments to be used

$H + \alpha$	Positive δ: a b c d	$H + \alpha$	Negative δ: a b c d	$H + \alpha$	Argument on the left	Argument on the right
h 0		h 0		h 0		
	− − + −		− + − −		$H + \alpha$	
6		6		6		
	+ − + +		+ + − +		$12^{h} - (H + \alpha)$	$H + \alpha$
12		12		12		
	+ + − +		+ − + +		$(H + \alpha) - 12^{h}$	
18		18		18		
	− + − −		− − + −		$24^{h} - (H + \alpha)$	$(H + \alpha) - 12^{h}$
24		24		24		

δ	0°				10°				20°				30°				40°				
$H+\alpha$	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	$H+\alpha$
h																					h
0	6	0	0	6	6	0	0	6	6	0	0	5	7	0	0	5	7	0	0	4	12
1	6	0	0	6	6	0	0	5	6	0	1	5	6	1	1	5	7	1	1	4	11
2	5	0	0	5	5	0	1	5	5	1	1	5	6	1	2	4	6	2	3	4	10
3	4	0	0	4	4	0	1	4	4	1	2	4	5	2	3	3	5	3	4	3	9
4	3	0	0	3	3	1	1	3	3	1	3	3	3	2	4	2	4	4	5	2	8
5	1	0	0	1	1	1	1	1	2	1	3	1	2	2	4	1	2	4	5	1	7
6	0	0	0	0	0	1	1	0	0	1	3	0	0	3	4	0	0	4	5	0	6

	45°				50°				55°				60°				65°				
	a	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	
h																					h
0	8	0	0	4	9	0	0	4	10	0	0	3	11	0	0	3	13	0	0	2	12
1	8	1	2	4	9	2	2	4	10	2	2	3	11	3	2	3	13	5	2	2	11
2	7	3	3	3	8	4	3	3	9	5	4	3	10	7	4	2	12	10	4	2	10
3	6	4	4	3	6	5	5	3	7	7	5	2	8	9	5	2	10	14	5	2	9
4	4	5	5	2	4	6	6	2	5	8	6	2	6	11	6	1	7	17	7	1	8
5	2	5	6	1	2	7	6	1	3	9	7	1	3	13	7	1	3	19	7	1	7
6	0	5	6	0	0	7	7	0	0	9	7	0	0	13	7	0	0	19	8	0	6

	62°				64°				66°				68°				70°				
	a	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	
h																					h
0·0	12	0	0	3	13	0	0	2	14	0	0	2	15	0	0	2	17	0	0	2	12·0
0·5	12	2	1	3	13	2	1	2	14	3	1	2	15	3	1	2	17	4	1	2	11·5
1·0	12	4	2	3	13	5	2	2	14	5	2	2	15	6	2	2	16	8	2	2	11·0
1·5	11	6	3	2	12	7	3	2	13	8	3	2	14	10	3	2	15	12	3	2	10·5
2·0	11	8	4	2	11	9	4	2	12	10	4	2	13	13	4	2	14	15	4	2	10·0
2·5	10	9	5	2	10	11	5	2	11	13	5	2	12	15	5	2	13	19	5	2	9·5
3·0	9	11	5	2	9	13	5	2	10	15	6	2	11	18	6	2	12	22	6	1	9·0
3·5	7	12	6	2	8	14	6	2	9	17	6	1	9	20	6	1	10	24	6	1	8·5
4·0	6	13	7	1	7	15	7	1	7	18	7	1	8	22	7	1	8	26	7	1	8·0
4·5	5	14	7	1	5	16	7	1	5	19	7	1	6	23	7	1	6	28	7	1	7·5
5·0	3	15	7	1	3	17	7	1	4	20	8	1	4	24	8	1	4	29	8	1	7·0
5·5	2	15	7	0	2	18	8	0	2	21	8	0	2	25	8	0	2	30	8	0	6·5
6·0	0	15	8	0	0	18	8	0	0	21	8	0	0	25	8	0	0	31	8	0	6·0

	71°				72°				73°				74°				75°				
	a	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	
h																					h
0·0	18	0	0	2	18	0	0	2	20	0	0	2	21	0	0	2	22	0	0	1	12·0
0·5	17	4	1	2	18	5	1	2	19	6	1	2	21	6	1	2	22	7	1	1	11·5
1·0	17	9	2	2	18	10	2	2	19	11	2	2	20	12	2	2	21	14	2	1	11·0
1·5	16	13	3	2	17	15	3	2	18	16	3	2	19	18	3	1	20	21	3	1	10·5
2·0	15	17	4	2	16	19	4	2	17	21	4	1	18	24	4	1	19	27	4	1	10·0
2·5	14	21	5	1	15	23	5	1	15	26	5	1	16	29	5	1	17	33	5	1	9·5
3·0	12	24	6	1	13	27	6	1	14	30	6	1	15	34	6	1	16	39	6	1	9·0
3·5	11	27	6	1	11	30	6	1	12	34	6	1	13	38	7	1	13	43	7	1	8·5
4·0	9	29	7	1	9	33	7	1	10	37	7	1	10	42	7	1	11	48	7	1	8·0
4·5	7	31	7	1	7	35	8	1	8	39	8	1	8	44	8	1	8	51	8	1	7·5
5·0	5	33	8	0	5	37	8	0	5	41	8	0	5	46	8	0	6	53	8	0	7·0
5·5	2	34	8	0	2	38	8	0	3	42	8	0	3	48	8	0	3	54	8	0	6·5
6·0	0	34	8	0	0	38	8	0	0	43	8	0	0	48	8	0	0	55	8	0	6·0

moving object minus star, then the corrections for differential aberration are: in right ascension

$$-h \cos (H + \alpha) \sec \delta \, \Delta\alpha - h \sin (H + \alpha) \sec \delta \tan \delta \, \Delta\delta$$

in declination

$$+h \sin (H + \alpha) \sin \delta \, \Delta\alpha - h \cos (H + \alpha) \cos \delta \, \Delta\delta$$

in which a small term $i \sin \delta \, \Delta\delta$ has been omitted from the correction in declination; this may reach $0''.02$ near the pole for $\Delta\delta = 10'$; h, H, i are the independent day numbers defined in section 5D.

To the precision required, these corrections may be regarded as independent of the year or of the equinox required, and may be tabulated as functions of position and date. Permanent tables of this kind are given in *A.E.*, Table V, and in table 2.7 of this Supplement; in these tables the coefficients of $\Delta\alpha$, $\Delta\delta$ in the above equations are tabulated using a mean value of $19''.6$ for h.

The corrections should be applied with those for differential precession and nutation (see sub-section B) to give mean positions referred to the same equinox as those of the stars.

Astrometric positions. An astrometric position is obtained by adding the planetary aberration to the geometric ephemeris and then subtracting stellar aberration from which the E-terms have been omitted. The astrometric ephemeris is therefore rigorously comparable with observations that are referred to catalogue mean places of comparison stars, it being only necessary to correct the observations for geocentric parallax. Such positions are discussed in more detail in section 4D.

E. REFRACTION

In the Ephemeris atmospheric refraction enters into only a very few topocentric phenomena, such as the times of rising and setting of the Sun and Moon, and, in theory though neglected in practice, the local predictions of eclipses. Consequently all explanations of the theory, and of the practical calculation and application of numerical tables, are omitted; they are adequately covered in the references given at the end of this sub-section.

Rising and setting phenomena. As described in section 13 a constant of $34'$ is used for the horizontal refraction in the calculation of the times of rising and setting: that is, the zenith distance of the object (upper limb of the Sun or Moon) is $90° \, 34'$.

Eclipses and occultations. Owing to refraction and parallax the geometric direction from an object M outside the Earth's atmosphere to an observer at P is not the same as the initial direction of the ray of light from M to P; the difference is only significant for an object as close as the Moon, and then only at low altitudes. Thus the condition that two objects M and S shall appear to be coincident to an

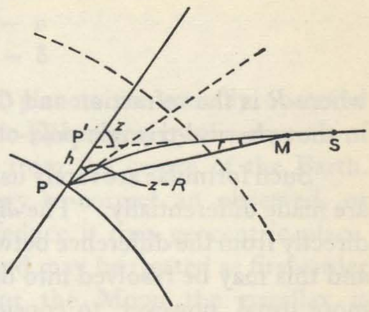

Figure 2.4. Refraction

observer at P on the Earth's surface is not precisely the condition that the geometrical direction PMS is a straight line. Let $z - R$ be the apparent zenith distance at P, R being the refraction; then the precise condition is that the geometric direction P′MS is a straight line, where P′ is the point vertically above P at which the true zenith distance is z (see figure 2.4). The height (h) of P′ above P can be calculated from the formula given below; it is independent of the distances of M and S. Full allowance can therefore be made for refraction by treating the observer as though he were at P′, instead of at P, that is by increasing the height of the observer above the spheroid by h.

It can be shown (Chauvenet, Vol. 1, page 516) that, for a spherical atmosphere:

$$1 + \frac{h}{\rho} = \frac{\mu_0 \sin (z - R)}{\sin z}$$

where ρ is the (geocentric) radius and μ_0 the index of refraction of the atmosphere at P. To the order of accuracy required the correction can be made either by increasing the height above sea level by h, or by multiplying the geocentric rectangular coordinates ξ, η, ζ of the observer by $1 + h/\rho$. Values of h in metres and of $1 + h/\rho$, based on mean values of the quantities concerned, are given for a point at sea level in the following table. It will be seen that the corrections are only significant for altitudes less than about 10°, when, however, the refraction may differ considerably from its mean value.

z	h	$1 + \frac{h}{\rho}$	z	h	$1 + \frac{h}{\rho}$
°	m		°	m	
0	0	1·000000	82	100	1·000016
30	0	1·000000	84	170	1·000026
60	5	1·000001	86	290	1·000046
70	20	1·000003	88	610	1·000095
75	30	1·000005	89	940	1·000148
80	70	1·000011	90	1540	1·000242

Artificial satellites. Referring to figure 2.4 it will be seen that the refraction correction applicable to the observation of a close object is not R but $R - r$ where r is the angle P′MP. For objects only a few hundred kilometres above the Earth's surface r can be of the order of a minute of arc, and must be allowed for in the reduction of precise observations.

Corrections to right ascension and declination. Refraction affects the observed zenith distance. If an object is observed on the meridian the refraction is a direct correction to the observed declination, and the deduced right ascension is unaffected.

If α', δ' are the observed right ascension and declination of an object not on the meridian, the corrections required to give the true values α, δ are, to first order:

$$\alpha - \alpha' = - R \sec \delta' \sin C$$
$$\delta - \delta' = - R \cos C$$

where R is the refraction and C is the parallactic angle, i.e. the angle at the object in the spherical triangle pole-object-zenith.

Such formulae are rarely used since most observations made out of the meridian are made differentially. The *differential refraction* of two objects may be calculated directly from the difference between the refractions appropriate to the two altitudes; and this may be resolved into differences of right ascension and declination. It is more usual, however, to consider such corrections as linear over the small area covered by a photographic plate and to allow for them by means of the plate constants determined from the coordinates of the standard stars.

References

Text-books that give the basic theory of refraction and of the corrections for it include:

Chauvenet, W. *A manual of spherical and practical astronomy*, 2 volumes, Philadelphia, 5th ed. 1892; reprinted, New York, Dover Publications, 1960. (Refraction in the local prediction of eclipses is discussed in Vol. 1, pages 515-517.)

Smart, W. M. *Text-book on spherical astronomy*, Cambridge University Press, 4th ed. 1944.

Theoretical discussions of the calculation of refraction include:

Harzer, P. Berechnung der Ablenkung der Lichtstrahlen in der Atmosphäre der Erde auf rein meteorologisch-physikalischer Grundlage. *Publikation der Sternwarte in Kiel*, no. 13, 1924.

Willis, J. E. A determination of astronomical refraction from physical data. *Transactions of the American Geophysical Union*, 1941, part II, 324-336.

Garfinkel, B. An investigation in the theory of astronomical refraction. *A.J.*, **50**, 169-179, 1944. Astronomical refraction in a polytropic atmosphere. *A.J.*, **72**, 235-254, 1967.

Tables used for the calculation of refraction corrections include:

Greenwich. Refraction tables arranged for use at the Royal Observatory, Greenwich, by P. H. Cowell, M.A., Chief Assistant. *Greenwich Observations for* 1898, appendix I, London, 1900.

Washington. Tables XXVI-XXXIV of " Reduction tables for transit circle observations ", *Publications of the United States Naval Observatory*, 2nd series, **4**, appendix II, Washington, 1904. (The tables " follow the form used in the Refraction Tables published by the Observatory in 1887, but are based upon the Pulkowa Refraction Tables instead of upon those of Bessel ".)

Harzer, P. Gebrauchstabellen zur Berechnung der Ablenkungen der Lichtstrahlen in der Atmosphäre der Erde für die Beobachtungen am grossen Kieler Meridiankreise. *Publikation der Sternwarte in Kiel*, no. 14, 1924.

Pulkovo. *Refraction tables of Pulkova Observatory*. 3rd ed. (including some supplementary tables taken from Harzer), 1930; 4th ed. (B. A. Orlov) 1956.

F. PARALLAX

Introduction

The positions in which the Sun, Moon, and planets are actually observed differ from the geocentric positions tabulated in the Ephemeris by the amount of parallax due to the displacement of the observer from the centre of the Earth. Before comparison with theory, it is thus necessary to correct an observed, or topocentric place, by applying parallax so as to reduce it to a geocentric place. For the Sun and planets the corrections are small and may be treated as first-order quantities whose squares can be neglected. For the Moon the parallax is sufficiently large to require third-order terms in the general expression for the corrections; and it is better to use exact formulae. For artificial satellites of the Earth the parallax may be so large that exact formulae, based on the actual position of the observer relative to the centre of the Earth, must be used.

The geocentric positions of stars are similarly affected by the annual parallax due to the displacement of the Earth from the centre of the Sun. In this case it is usual to include the parallax in the ephemeris position, so that it is directly comparable with the observed position.

Details of the corrections and the method of calculation follow.

The figure of the Earth

In calculating parallax corrections, the dimensions of the Earth are usually taken to be those of Hayford's spheroid (see section 6).* This is defined by the equatorial radius (a) and the flattening (f), for which an exact value of 1/297 is adopted. The adopted value for a is 6378·388 km, from which the polar radius $b = a(1 - f)$ is derived as 6356·912 km. Otherwise the notation used is:

ϕ = geographic (or geodetic) latitude ϕ' = geocentric latitude

ρ = geocentric distance, i.e. the distance of the observer from the centre of the Earth, in units of the Earth's equatorial radius.

The latitude ϕ is variously referred to as the geographic latitude, or the geodetic latitude; on the spheroid the two are identical, but on the actual Earth they differ on account of gravity anomalies. No significance is to be attached in this Supplement to the use of one term or the other. †

The position of an observer relative to the centre of the Earth is most readily expressed in rectangular coordinates; in the meridian section of the Earth through the observer these are:

$$\rho \sin \phi' = S \sin \phi \qquad \rho \cos \phi' = C \cos \phi$$

which serve to define the auxiliary functions S and C. It should be noted that:

$$\tan \phi' = \frac{b^2}{a^2} \tan \phi = (1 - f)^2 \tan \phi$$

whence $S = (1 - f)^2 C \qquad C = \{\cos^2\phi + (1 - f)^2 \sin^2 \phi\}^{-\frac{1}{2}}$

and $\rho^2 = \frac{1}{2}(S^2 + C^2) + \frac{1}{2}(C^2 - S^2) \cos 2\phi$

$$= C^2 \{ \cos^2 \phi + (1 - f)^4 \sin^2 \phi \}$$

*From 1968: $f = 1/298{\cdot}25$, $a = 6\,378\,160$ m, $b = 6\,356\,775$ m

†The difference is however significant for some applications.

The following expressions, which contain terms up to f^3, may then be derived for S, C, ρ, and $\phi - \phi'$ in terms of the geographic latitude (ϕ) and the flattening (f); they assume that the observer is at sea level.

$$S = 1 - \tfrac{3}{2}f + \tfrac{5}{16}f^2 + \tfrac{3}{32}f^3 - (\tfrac{1}{2}f - \tfrac{1}{2}f^2 - \tfrac{5}{64}f^3)\cos 2\phi + (\tfrac{3}{16}f^2 - \tfrac{3}{32}f^3)\cos 4\phi - \tfrac{5}{64}f^3 \cos 6\phi$$

$$C = 1 + \tfrac{1}{2}f + \tfrac{5}{16}f^2 + \tfrac{7}{32}f^3 - (\tfrac{1}{2}f + \tfrac{1}{2}f^2 + \tfrac{27}{64}f^3)\cos 2\phi + (\tfrac{3}{16}f^2 + \tfrac{9}{32}f^3)\cos 4\phi - \tfrac{5}{64}f^3 \cos 6\phi$$

$$\rho = 1 - \tfrac{1}{2}f + \tfrac{5}{16}f^2 + \tfrac{5}{32}f^3 + (\tfrac{1}{2}f - \tfrac{13}{64}f^3)\cos 2\phi - (\tfrac{5}{16}f^2 + \tfrac{5}{32}f^3)\cos 4\phi + \tfrac{13}{64}f^3 \cos 6\phi$$

$$\phi - \phi' = (f + \tfrac{1}{2}f^2)\sin 2\phi - (\tfrac{1}{2}f^2 + \tfrac{1}{2}f^3)\sin 4\phi + \tfrac{1}{3}f^3 \sin 6\phi$$

Inserting the adopted numerical value of $f = 1/297$ leads to the following series:

$$S = 0{\cdot}99495\,304 - 0{\cdot}00167\,783 \cos 2\phi + 0{\cdot}00000\,212 \cos 4\phi$$

$$C = 1{\cdot}00168\,705 - 0{\cdot}00168\,919 \cos 2\phi + 0{\cdot}00000\,214 \cos 4\phi$$

$$\rho = 0{\cdot}99832\,005 + 0{\cdot}00168\,349 \cos 2\phi - 0{\cdot}00000\,355 \cos 4\phi + 0{\cdot}00000\,001 \cos 6\phi$$

$$\phi - \phi' = 695''{\cdot}66 \sin 2\phi - 1''{\cdot}17 \sin 4\phi$$

Values of S and C, calculated from the first two of these series, are tabulated in *A.E.*, Table VII; they may also be found together with ρ and $\phi - \phi'$ in table 2.8 of this Supplement.* A correction for the height of the observer above sea level is necessary for the calculation of his actual coordinates $\rho \sin \phi'$ and $\rho \cos \phi'$. To an adequate approximation the geocentric radius is increased by $0{\cdot}1568\,h$ or $0{\cdot}0478\,H \times 10^{-6}$ and the angle of the vertical $\phi - \phi'$ is unchanged, where h is the height above sea level in metres and H is the height in feet. The addition of this correction leads to the expressions:

$$\rho \sin \phi' = (S + 0{\cdot}1568\,h \times 10^{-6}) \sin \phi = (S + 0{\cdot}0478\,H \times 10^{-6}) \sin \phi$$

$$\rho \cos \phi' = (C + 0{\cdot}1568\,h \times 10^{-6}) \cos \phi = (C + 0{\cdot}0478\,H \times 10^{-6}) \cos \phi$$

$$\tan \phi' = (0{\cdot}9932773 + 0{\cdot}0011\,h \times 10^{-6}) \tan \phi = (0{\cdot}9932773 + 0{\cdot}0003\,H \times 10^{-6}) \tan \phi$$

Values of these three quantities are given for the principal observatories in the list of observatories in the Ephemeris.

Example 2.1. *Geocentric coordinates of Washington*

The geographic coordinates of a point at U.S. Naval Observatory, Washington, D.C., are $\lambda = +5^h\, 08^m\, 15^s{\cdot}75$, $\phi = +38°\, 55'\, 12''{\cdot}3$, and height = 85m.

	° ′ ″	sin	cos	tan
ϕ	+ 38 55 12·3	+0·62823 58	+0·77802 30	+0·80747 72
2ϕ	77 50 25	+0·97756	+0·21064	
4ϕ	155 41	+0·412	−0·911	

From the series:

S	0·99459 77	ρ	0·99867 8
C	1·00132 93	$\phi - \phi'$	11′ 19″·6

It may be confirmed that (to the accuracy of the table) the same values are obtained by interpolation in table 2.8 to $\phi = 38°{\cdot}920$.

For a height of 85m a correction of $13{\cdot}3 \times 10^{-6}$ must be applied to S and C, before forming $\rho \sin \phi'$ and $\rho \cos \phi'$, and to ρ. Thus:

$\rho \sin \phi'$	+0·62485 0	ρ	0·99869 1
$\rho \cos \phi'$	+0·77906 8	ϕ'	+38° 43′ 52″·7

The correction for height to $\phi - \phi'$ is here negligible.

*The coefficients and table given here are for the Hayford spheroid in use before 1968. See note on page 515.

ϕ	S	Δ	C	Δ	ρ	Δ	$\phi-\phi'$	Δ
°	0·99		1·00		0·99		"	
0	3277	1	0000	1		1	0	24
1	3278	3	0001	3	9999	3	24	24
2	3281	5	0004	5	9996	5	48	24
3	3286	8	0009	7	9991	7	72	24
4	3294	9	0016	10	9984	9	96	24
5	3303	11	0026	11	9975	11	120	24
6	3314	13	0037	13	9964	14	144	24
7	3327	15	0050	15	9950	15	168	23
8	3342	17	0065	17	9935	17	191	23
9	3359	19	0082	19	9918	19	214	23
10	3378	21	0101	21	9899	21	237	23
11	3399	23	0122	23	9878	22	260	22
12	3422	24	0145	25	9856	25	282	22
13	3446	27	0170	27	9831	26	304	22
14	3473	28	0197	28	9805	29	326	21
15	3501	30	0225	30	9776	30	347	21
16	3531	32	0255	32	9746	32	368	20
17	3563	33	0287	34	9714	33	388	20
18	3596	35	0321	35	9681	35	408	19
19	3631	37	0356	37	9646	37	427	19
20	3668	38	0393	39	9609	38	446	18
21	3706	40	0432	40	9571	40	464	18
22	3746	41	0472	42	9531	41	482	17
23	3787	43	0514	43	9490	43	499	17
24	3830	44	0557	44	9447	44	516	16
25	3874	46	0601	46	9403	46	532	15
26	3920	46	0647	47	9357	46	547	15
27	3966	48	0694	48	9311	48	562	14
28	4014	49	0742	49	9263	49	576	13
29	4063	50	0791	50	9214	50	589	12
30	4113	51	0841	52	9164	52	601	12
31	4164	52	0893	52	9112	52	613	11
32	4216	53	0945	54	9060	53	624	11
33	4269	54	0999	54	9007	54	635	9
34	4323	55	1053	55	8953	54	644	9
35	4378	55	1108	55	8899	56	653	8
36	4433	56	1163	57	8843	56	661	7
37	4489	56	1220	57	8787	57	668	6
38	4545	57	1277	57	8730	57	674	6
39	4602	58	1334	58	8673	57	680	5
40	4660	57	1392	58	8616	58	685	4
41	4717	59	1450	58	8558	59	689	3
42	4776	58	1508	59	8499	58	692	2
43	4834	58	1567	59	8441	59	694	1
44	4892	59	1626	59	8382	58	695	1
45	4951		1685		8324		696	

ϕ	S	Δ	C	Δ	ρ	Δ	$\phi-\phi'$	Δ
°	0·99		1·00		0·99		"	
45	4951	58	1685	59	8324	59	696	1
46	5009	59	1744	59	8265	59	695	1
47	5068	58	1803	59	8206	58	694	2
48	5126	59	1862	58	8148	59	692	3
49	5185	57	1920	58	8089	58	689	4
50	5242	58	1978	58	8031	58	685	4
51	5300	57	2036	58	7973	57	681	5
52	5357	57	2094	57	7916	57	676	7
53	5414	56	2151	56	7859	56	669	7
54	5470	55	2207	56	7803	56	662	8
55	5525	55	2263	55	7747	55	654	8
56	5580	54	2318	55	7692	54	646	10
57	5634	53	2373	53	7638	54	636	10
58	5687	53	2426	53	7584	52	626	11
59	5740	51	2479	52	7532	52	615	12
60	5791	50	2531	50	7480	50	603	12
61	5841	49	2581	50	7430	50	591	13
62	5890	49	2631	48	7380	48	578	14
63	5939	46	2679	47	7332	48	564	15
64	5985	46	2726	46	7284	45	549	15
65	6031	45	2772	45	7239	45	534	16
66	6076	42	2817	43	7194	43	518	16
67	6118	42	2860	42	7151	42	502	18
68	6160	40	2902	41	7109	40	484	17
69	6200	39	2943	38	7069	39	467	19
70	6239	37	2981	38	7030	37	448	19
71	6276	35	3019	35	6993	36	429	19
72	6311	34	3054	34	6957	34	410	20
73	6345	32	3088	32	6923	32	390	20
74	6377	30	3120	31	6891	31	370	21
75	6407	29	3151	29	6860	28	349	21
76	6436	26	3180	27	6832	27	328	22
77	6462	25	3207	25	6805	25	306	22
78	6487	23	3232	23	6780	23	284	23
79	6510	21	3255	21	6757	22	261	22
80	6531	19	3276	19	6735	19	239	23
81	6550	18	3295	18	6716	17	216	24
82	6568	15	3313	15	6699	16	192	23
83	6583	13	3328	13	6683	13	169	24
84	6596	11	3341	12	6670	11	145	24
85	6607	10	3353	9	6659	9	121	24
86	6617	7	3362	7	6650	8	97	24
87	6624	5	3369	5	6642	5	73	24
88	6629	3	3374	3	6637	3	49	25
89	6632	1	3377	1	6634	1	24	24
90	6633		3378		6633		0	

The above table enables rectangular and polar geocentric coordinates to be calculated for an observer in geographic (geodetic) latitude ϕ, from the formulae:

$$\rho \sin \phi' = (S + 0{\cdot}1568h \times 10^{-6}) \sin \phi = (S + 0{\cdot}0478\, H \times 10^{-6}) \sin \phi$$

$$\rho \cos \phi' = (C + 0{\cdot}1568h \times 10^{-6}) \cos \phi = (C + 0{\cdot}0478\, H \times 10^{-6}) \cos \phi$$

where, h (H) is the height of the observer above the surface of the Earth in metres (feet). For reasonable heights, $0{\cdot}1568h \times 10^{-6}$ or $0{\cdot}0478H \times 10^{-6}$ can be added to ρ, and $\phi - \phi'$ can be used unchanged.

Parallax of the Moon

The topocentric hour angle (h), declination (δ), and distance (r) of the centre of the Moon are related to the geocentric coordinates (h_0, δ_0, r_0) by the exact equations:

$$
\begin{aligned}
F \cos \delta \sin h &= \cos \delta_0 \sin h_0 &&\equiv A \\
F \cos \delta \cos h &= \cos \delta_0 \cos h_0 - \rho \cos \phi' \sin \pi &&\equiv B \\
F \sin \delta &= \sin \delta_0 - \rho \sin \phi' \sin \pi &&\equiv C
\end{aligned}
$$

where $F = r/r_0$, π is the geocentric equatorial horizontal parallax of the centre of the Moon, and ρ, ϕ' are the geocentric coordinates of the observer.

If the ephemeris position is known A, B, C can be calculated numerically; then:

$$F^2 = A^2 + B^2 + C^2$$

$$\tan h = A/B \qquad \sin h = A/(A^2 + B^2)^{\frac{1}{2}} \qquad \cos h = B/(A^2 + B^2)^{\frac{1}{2}}$$

$$\tan \delta = C/(A^2 + B^2)^{\frac{1}{2}} \qquad \sin \delta = C/F \qquad \cos \delta = (A^2 + B^2)^{\frac{1}{2}}/F$$

Some simplification of this calculation can be achieved by rounding δ_0 and h_0 to avoid interpolation in the trigonometric tables; if the values actually used are δ_0' and h_0', giving rise to δ' and h', then:

$$\delta = \delta' + (\delta_0 - \delta_0') \qquad h = h' + (h_0 - h_0') \qquad F = F'$$

with errors not exceeding 1/60 of the " roundings ".

The reverse problem of deducing the geocentric position from the observed coordinates is less readily solved. The equations take the form:

$$
\begin{aligned}
G \cos \delta_0 \sin h_0 &= \cos \delta \sin h &&\equiv A_0 \\
G \cos \delta_0 \cos h_0 &= \cos \delta \cos h + G \rho \cos \phi' \sin \pi &&\equiv B_0 \\
G \sin \delta_0 &= \sin \delta + G \rho \sin \phi' \sin \pi &&\equiv C_0
\end{aligned}
$$

where $G = r_0/r = 1/F$.

These equations may be solved in precisely the same manner if r is known, possibly by observation, and r_0 is deduced from the ephemeris, to provide a sufficiently accurate value of G to substitute on the right-hand side of the equations.

The general solution, in which h, δ are observed and π is known (from the ephemeris), is as follows. Let:

$$g = \rho \cos \phi' \cos \delta \cos h + \rho \sin \phi' \sin \delta$$

and substitute:

$$G_0 = 1 + g \sin \pi + \tfrac{1}{2} (1 + g^2) \sin^2 \pi$$

for G on the right-hand side of the equations. The adequacy of this approximation for G, and the accuracy of the calculation, is checked by comparison with the value of G determined from $(A_0^2 + B_0^2 + C_0^2)^{\frac{1}{2}}$. Alternatively G_0 is taken as unity, G_1 is formed from $(A_0^2 + B_0^2 + C_0^2)^{\frac{1}{2}}$ and used instead of G_0 to form G_2, and the process continued until G is known with sufficient accuracy. Two such approximations generally suffice.

The topocentric distance of the Moon is F times the geocentric distance, so that the apparent semi-diameter is greater than the tabulated value, the augmentation being $1/F = G$.

The formulae may be expressed in alternative forms to give directly the effect of parallax on the coordinates. If $\Delta\alpha = \alpha_0 - \alpha$, $\Delta\delta = \delta_0 - \delta$ are the corrections to be applied to the topocentric position to give the geocentric position, then, in terms of the geocentric position:

$$\tan (h - h_0) = \tan \Delta\alpha = \frac{\rho \cos \phi' \sin \pi \sin h_0}{\cos \delta_0 - \rho \cos \phi' \sin \pi \cos h_0}$$

$$\tan \Delta\delta = \frac{\rho \sin \phi' \sin \pi (\cos \delta_0 - m \sin \delta_0)}{1 - \rho \sin \phi' \sin \pi (m \cos \delta_0 + \sin \delta_0)}$$

where

$$m = \cot \phi' \frac{\cos \frac{1}{2} (h + h_0)}{\cos \frac{1}{2} (h - h_0)}$$

$$= \cot \phi' \{ \cos h_0 - \sin h_0 \tan \tfrac{1}{2} (h - h_0) \}$$

There are no simple corresponding expressions in terms of the topocentric position.

Example 2.2. Parallax of the Moon at Washington

1960 March 13 at $3^h\ 17^m\ 48^s.0$ U.T.

For the purpose of this example, the time of observation is taken to be exactly equivalent to $3^h\ 18^m\ 24^s.0$ E.T.; the coordinates of Washington are taken from example 2.1.

The geocentric coordinates of the Moon are obtained by interpolation in the Ephemeris as:

$\alpha_0\ 11^h\ 22^m\ 16^s.16$ $\qquad \delta_0\ +3^\circ\ 35'\ 24''.4$ $\qquad \pi\ 57'\ 21''.71$

The geocentric hour angle is then calculated as follows:

	h m s
S.T. at 0^h (*A.E.*, page 11)	11 22 30·237
U.T. of observation	3 17 48·000
Increment (*A.E.*, Table IX)	+ 32·493
$-\lambda$	− 5 08 15·75
$-\alpha_0$	− 11 22 16·16
Sum = h_0	22 10 18·82

Then

$\sin \delta_0$ +0·06261 83	$\sin h_0$ −0·46053 42	$\sin \pi$ +0·01668 51
$\cos \delta_0$ +0·99803 76	$\cos h_0$ +0·88764 20	$\rho \sin \phi' \sin \pi$ +0·01042 57
		$\rho \cos \phi' \sin \pi$ +0·01299 88
A −0·45963 04	F 0·98789 72	
B +0·87290 13	$\tan h$ −0·52655 48	h $22^h\ 08^m\ 55^s.38$
C +0·05219 26	$\sin \delta$ +0·05283 20	δ $+ 3^\circ\ 01'\ 42''.5$

If $h_0' = 22^h\ 10^m\ 19^s$ and $\delta_0' = +3^\circ\ 35'\ 20''$ had been used, no interpolation would have been required in the trigonometric tables, and h', δ' would have been obtained as:

$h' = \ 22^h\ 08^m\ 55^s.56$ giving $h - h_0 = \alpha_0 - \alpha = -1^m\ 23^s.44$

$\delta' = +\ 3^\circ\ 01'\ 38''.0$ giving $\delta - \delta_0 = -33'\ 42''.0$

For the alternative method five-figure tables and working suffice for a precision of $0''.1$; using h_0', δ_0' as above, to avoid interpolation:

$\rho \sin \phi' \sin \pi$ +0·01042 57	$\sin \delta_0'$ +0·06260	$\sin h_0'$ −0·46052
$\rho \cos \phi' \sin \pi$ +0·01299 88	$\cos \delta_0'$ +0·99804	$\cos h_0'$ +0·88765

$$\tan (h - h_0) = \frac{-0.00598\ 62}{+0.98650} = -0.00606\ 81$$

$$m = 0.88625 \cot \phi' = 1.10499$$

where $\tan \frac{1}{2} (h - h_0)$ is taken as one-half of $\tan (h - h_0)$

$$\tan (\delta_0 - \delta) = \frac{+0.00968\ 41}{+0.98785} = +0.00980\ 32$$

whence

$$h - h_0 = -0^h\ 01^m\ 23^s.44$$

$$\delta - \delta_0 = -0^\circ\ 33'\ 42''.0$$

The reverse process, assuming the observed position and horizontal parallax to be:

$\alpha\ 11^h\ 23^m\ 39^s{\cdot}60 \qquad \delta\ +3^\circ\ 01'\ 42''{\cdot}5 \qquad \pi\ 57'\ 21''{\cdot}71$

is as follows:

h	$-1^h\ 51^m\ 04^s{\cdot}62$	δ	$+3^\circ\ 01'\ 42''{\cdot}5$
$\sin h$	$-0{\cdot}46591\ 18$	$\sin\delta$	$+0{\cdot}05283\ 22$
$\cos h$	$+0{\cdot}88483\ 12$	$\cos\delta$	$+0{\cdot}99860\ 34$
$\sin\pi$	$0{\cdot}01668\ 51$	$\rho\sin\phi'\sin\pi$	$+0{\cdot}01042\ 57$
$\sin^2\pi$	$0{\cdot}00027\ 84$	$\rho\cos\phi'\sin\pi$	$+0{\cdot}01299\ 88$

The solution of the equations is commenced with:

$g = +0{\cdot}7214 \qquad G_0 = 1{\cdot}012248$

Then $A_0\ -0{\cdot}46526\ 11 \qquad B_0\ +0{\cdot}89675\ 35 \qquad C_0\ +0{\cdot}06338\ 56$

from which $G = 1{\cdot}01225\ 12$. With this value of G, no change occurs in A_0, B_0, or C_0. Hence:

$\tan h_0$	$-0{\cdot}51882\ 83$	h_0	$-1^h\ 49^m\ 41^s{\cdot}17$
$\sin\delta_0$	$+0{\cdot}06261\ 84$	δ_0	$+3^\circ\ 35'\ 24''{\cdot}4$

When the Moon is at transit $h = h_0 = 0$, and the parallax applies to the declination only. The correction $\Delta\delta$ to be applied to the topocentric declination at transit to give the geocentric value is given by:

$$\tan\Delta\delta = \frac{\rho\sin\pi\sin(\phi' - \delta_0)}{1 - \rho\sin\pi\cos(\phi' - \delta_0)}$$

or, by using the observed declination and geocentric latitude of the observatory, through the equivalent expression:

$$\sin\Delta\delta = \rho\sin\pi\sin(\phi' - \delta)$$

This may be put in the approximate form:

$$\Delta\delta = 0{\cdot}999988\ \rho\ \pi\sin(\phi' - \delta)$$

with an error not exceeding $0''{\cdot}04$.

Example 2.3. Parallax of the Moon at transit at Washington

On 1960 August 7 the observed declination at transit is assumed to be $-14^\circ\ 19'\ 57''{\cdot}6$. The U.T. of the observation is $5^h\ 16^m\ 48^s{\cdot}88$ and this is assumed to be equivalent to $5^h\ 17^m\ 25^s$ E.T. The coordinates of Washington are assumed to be those of example 2.1. From the Ephemeris:

δ_0	$-13^\circ\ 31'\ 32''$	π	$60'\ 40''{\cdot}85$
$\phi' - \delta_0$	$+52^\circ\ 15'\ 25''$	ρ	$0{\cdot}99869$
$\sin(\phi' - \delta_0)$	$+0{\cdot}79076$	$\sin\pi$	$0{\cdot}01765\ 04$
$\cos(\phi' - \delta_0)$	$+0{\cdot}61212$	$\rho\sin\pi$	$0{\cdot}01762\ 73$

$$\tan\Delta\delta = \frac{+0{\cdot}01393\ 90}{+0{\cdot}98921\ 0} = +0{\cdot}01409\ 10 \qquad \Delta\delta\ +0^\circ\ 48'\ 26''{\cdot}3$$

From the observation:

$\phi' - \delta$	$53^\circ\ 03'\ 50''{\cdot}3$	$\sin\Delta\delta$	$+0{\cdot}01408\ 97$
$\sin(\phi' - \delta)$	$+0{\cdot}79931$	$\Delta\delta$	$+0^\circ\ 48'\ 26''{\cdot}3$

From the approximate formula:

$$\Delta\delta = 0{\cdot}99998\ 8 \times 0{\cdot}99869 \times 0{\cdot}79931 \times 3640''{\cdot}85$$
$$= 2906''{\cdot}3 = 0^\circ\ 48'\ 26''{\cdot}3$$

The most important use of parallax corrections in the above form, when the Moon is not on the meridian, is in the reduction of observations made with the Markowitz dual-rate Moon camera. In the calculation of eclipses and the reduction of occultations the methods used, involving Besselian elements, do not require parallax corrections in the above form.

Parallax of the Sun and planets

For the more distant bodies such as the Sun, the planets, or comets, whose parallax amounts to only a few seconds of arc, the above formulae may be greatly simplified. It is sufficient to restrict the expressions to terms of the first order; expressed as corrections to be applied to the observed positions they are:

$$\Delta\alpha = \pi\,\{\,\rho\cos\phi'\sin h\sec\delta\,\}$$
$$\Delta\delta = \pi\,\{\,\rho\sin\phi'\cos\delta - \rho\cos\phi'\cos h\sin\delta\,\}$$

where h, δ may be replaced by h_0, δ_0.

When the horizontal parallax of the object is not available, it may be calculated * from $\pi = 8''{\cdot}80/\Delta$ where Δ is the geocentric distance. In preliminary work on comets and minor planets, where the geocentric distance is unknown, it is convenient to calculate parallax factors p_α and p_δ for each observation; these may then be used, once the geocentric distances are determined, to give the parallax corrections in the form:

$$\Delta\alpha = p_\alpha/\Delta \qquad \Delta\delta = p_\delta/\Delta$$

The parallax factors are calculated from:

$$p_\alpha = 8''{\cdot}80\,\rho\cos\phi'\sin h\sec\delta = 0^s{\cdot}587\,\rho\cos\phi'\sin h\sec\delta$$
$$p_\delta = 8''{\cdot}80\,(\rho\sin\phi'\cos\delta - \rho\cos\phi'\cos h\sin\delta)$$

The hour angle h is found from $h = \theta - \alpha$ where θ is the local sidereal time at universal time t, given by:

$$\theta = \text{S.T. at } 0^h \text{ U.T.} + t^\star - \lambda$$

where S.T. at 0^h U.T. is obtained from *A.E.*, pages 10–17†, $t^\star$ is the sidereal equivalent of t, and λ is the longitude, measured positively to the west. Since t is usually given in decimals of a day, $t^\star$ is most readily determined from table 17.3.

Example 2.4. Parallax factors for a minor planet

Observation of Vesta 1960 March $7^d\ 02^h\ 34^m\ 21^s$ U.T. at Johannesburg

$t = 0^d{\cdot}10719$ $\quad\alpha\ 17^h\ 57^m\ 21^s{\cdot}50$ $\quad\delta\ -18^\circ\ 43'\ 31''{\cdot}3$

	h m
Sidereal time at 0^h U.T. (*A.E.*, p. 11)	10 58·8
$t^\star$ (from table 17.3)	2 34·8
Correction for longitude, $-\lambda$	+ 1 52·3
Local sidereal time, θ	15 25·9
Right ascension, α	17 57·4
Local hour angle, $h = \theta - \alpha$	21 28·5

$\sin\delta$ −0·3210	$\sin h$ −0·6139	$\rho\sin\phi'$ −0·43867
$\cos\delta$ +0·9471	$\cos h$ +0·7894	$\rho\cos\phi'$ +0·89824
$\sec\delta$ +1·0559		

$p_\alpha\ -0^s{\cdot}342 \qquad p_\delta\ -1''{\cdot}65$

For a fixed observatory parallax corrections may be further simplified by forming two permanent tables. The first of these is similar to table 17.3 but gives $t^\star - \lambda$ directly in the first part of the table. The second table gives the coefficients A, B, C in the expressions:

$$p_\alpha = A\sin h \qquad p_\delta = B - C\cos h$$

where* $A = 8''{\cdot}80\,\rho\cos\phi'\sec\delta = 0^s{\cdot}587\,\rho\cos\phi'\sec\delta$

$B = 8''{\cdot}80\,\rho\sin\phi'\cos\delta$

$C = 8''{\cdot}80\,\rho\cos\phi'\sin\delta$

*$8''{\cdot}794 = 0^s{\cdot}5863$ from 1968.

†Pages 12 to 19 in *A.E.* 1972 onwards.

Thus for Copenhagen ($\lambda = -0^h 50^m.3$, $\rho \sin \phi' = +0.82231$, $\rho \cos \phi' = +0.56501$) the two tables start as follows:

t	$t^* - \lambda$	δ	A	B	C
d	h m	°	s	″	″
0·00	0 50·3	+0	+0·331	+7·24	+0·00
·01	1 04·7	1	·332	7·24	·09
·02	1 19·2	2	·332	7·23	·17
·03	1 33·6	3	·332	7·23	·26
·04	1 48·1	4	·332	7·22	·35
0·05	2 02·5	5	+0·333	+7·21	+0·43
·06	2 16·9	6	·333	7·20	·52
·07	2 31·4	7	·334	7·18	·61
·08	2 45·8	8	·335	7·17	·69
·09	3 00·3	9	·336	7·15	·78

Then h = S.T. at 0^h U.T. + $(t^* - \lambda) - \alpha$

An alternative method of correction when the geocentric distance is not accurately known, is to modify the solar coordinates X, Y, Z so as to refer them to the point of observation. These topocentric corrections are:

$$\Delta X = -a\rho \cos \phi' \cos \theta = \Delta_{xr} \cos \theta$$
$$\Delta Y = -a\rho \cos \phi' \sin \theta = \Delta_{xr} \sin \theta$$
$$\Delta Z = -a\rho \sin \phi' = \Delta_z$$

* where a is the Earth's equatorial radius in astronomical units $= 426.64 \times 10^{-7}$, and θ is the local sidereal time. The factors Δ_{xr} and Δ_z are given for each observatory in the last two columns of the list of observatories in the Ephemeris.

Example 2.5. Parallax factors for a minor planet (continued)

Using the data of example 2.4:

$\Delta_{xr} = -383 \times 10^{-7}$ $\quad \sin\theta = -0.7823 \quad \Delta X \; +239 \times 10^{-7}$

$\Delta_z = +187 \times 10^{-7}$ $\quad \cos\theta = -0.6229 \quad \Delta Y \; +300 \times 10^{-7}$

$\Delta Z \; +187 \times 10^{-7}$

When the horizontal parallax is small, S and C may often be taken as unity leading to a simplification of the formulae; this will not be applied to a fixed observatory.

Annual parallax

If π is the annual parallax of a star, and X, Y, Z are the solar coordinates, the star is displaced from its mean place α, δ by amounts $\Delta\alpha$, $\Delta\delta$ given by:

$$\cos\delta\,\Delta\alpha = \pi\,(Y\cos\alpha - X\sin\alpha)$$
$$\Delta\delta = \pi\,(Z\cos\delta - X\cos\alpha\sin\delta - Y\sin\alpha\sin\delta)$$

These expressions may be simplified by using the star constants c, d, c', d' (see section 5):

$$\Delta\alpha = \pi\,(Yc - Xd)$$
$$\Delta\delta = \pi\,(Yc' - Xd')$$

Thus, corrections for annual parallax may be included with the aberration terms of the reduction from mean to apparent place, as follows:

$$\Delta\alpha = (C + \pi Y)c + (D - \pi X)d$$
$$\Delta\delta = (C + \pi Y)c' + (D - \pi X)d'$$

*$a = 426.35 \times 10^{-7}$ a.u. from 1968.

When the annual parallax is small enough, a further simplification can be made by writing the expressions in the form:

$$\Delta\alpha = C\,(c + d\,\pi\,k_1) + D\,(d - c\,\pi\,k_2)$$
$$\Delta\delta = C\,(c' + d'\,\pi\,k_1) + D\,(d' - c'\,\pi\,k_2)$$

where $k_1 = R \sec \epsilon / k$, $k_2 = R \cos \epsilon / k$, in which R is the Sun's radius vector and k is * the aberration constant $= 20''\cdot47$. The variation in R throughout the year amounts to 1/60 of the mean value, so for a small parallax R may be taken as unity. The method uses, in effect, modified values of the star constants, and is particularly valuable in the routine calculation of an ephemeris. The maximum error in the case of α Centauri is $0''\cdot013$.

Example 5.4 illustrates the application of this correction.

It should be noted that, in the above formulae, the corrections for annual parallax are, contrary to normal practice, given in the sense of (observed − tabulated); they are included in the apparent places of the stars which are directly comparable with observation.

*$20''\cdot496$ from 1968.

3. SYSTEMS OF TIME MEASUREMENT

A. INTRODUCTION

The systems of time measurement in use in present-day (post 1960) astronomy are a development of those in use before the variable rotation of the Earth was recognised. A complete appreciation thus requires a full understanding of the earlier concepts. However, consideration of both systems together is necessarily complicated, and it is desirable to have a general understanding of present-day systems before considering how they have been developed.

In this introductory sub-section there is given a general description of the systems of time measurement in use in astronomy from 1960 onwards. Detailed developments are given in subsequent sub-sections.

A fundamental necessity of any system of time measurement is a one-to-one relationship between the adopted numerical expression, or *measure*, of the time (usually in the conventional form of years, months, days, hours, minutes, seconds, and decimals of seconds) and some observable physical phenomenon that is either repetitive and countable, or continuous and measurable, or both. The phenomenon and the precise form of relationship are chosen so that the resulting time-system satisfies some particular requirement; the relationship may be simple and regular, as in a direct count of oscillations, or complex and irregular, as in the motion of the Moon. In all systems there is an additional practical requirement that the time should be free from short-period irregularities to permit interpolation and extrapolation by man-made clocks.

In astronomy there are three such particular requirements, each closely related to some natural observable motion and each leading to a different system of time measurement. The natural motions and the resulting time-systems are:

(i) The alternation of day and night, or the diurnal motion of the Sun: *Universal Time.*

(ii) The period of rotation of the Earth, or the diurnal motions of the stars: *Sidereal Time.*

(iii) The orbital motions of the Earth, Moon, and planets in the solar system: *Ephemeris Time.*

Both here and in the following sub-sections, to allow a more logical development, the three time-systems are considered in the reverse order to that above. It is emphasised that many complicated details, that do not affect the broad principles, are omitted in this sub-section.

Ephemeris time is theoretically uniform, since the length of the ephemeris second is fixed by definition. The relationship through which ephemeris time is determined in practice is that it is the independent time-argument of the ephemerides of the Sun, Moon, and planets. These ephemerides are thus to be computed in such a way that the ephemeris time determined from them is in accord with its theoretical definition. But, in particular, an observationally-determined value is used for the coefficient of the secular term in the mean longitude of the Moon. Ephemeris time determined from this relationship will depart from the theoretical uniform time in so far as the theory of the motions is inadequate, and the observationally-determined values are erroneous; the possible departure through these causes is small, of the order of two or three seconds in a century.

Sidereal time is directly related to the rotation of the Earth; equal intervals of angular motion correspond to equal intervals of sidereal time.

There is a fundamental difference between the two systems of time measurement: sidereal time reflects the actual rotation of the Earth; ephemeris time is defined to be uniform and is, in practice, determined through the motion of the Moon in its orbit round the Earth. It is thus not possible to express one system in terms of the other; the relation between them must be determined empirically. In fact, the speed of rotation of the Earth is known to be subject to unpredictable variations in terms of ephemeris time.

The diurnal motion of the Sun involves both the diurnal rotation of the Earth, related to sidereal time, and the motion of the Earth in its orbit round the Sun, related to ephemeris time. Although it would be possible to define a system of time measurement by means of a relationship to the hour angle of the Sun, this system could never be related precisely to sidereal time and could not, therefore, be determined by observations of star transits.

Universal time, for this reason, is directly related to sidereal time by means of a numerical formula; it contains no reference to ephemeris time and is not precisely related to the hour angle of the Sun. Although it is continuous with universal time as practically determined in the past, it is only since the variable rate of rotation of the Earth was recognised that it has been realised that universal time is not a precise measure of mean solar time as generally understood; it is related to the hour angle of a point moving with the mean speed of the Sun in its orbit by means of an empirical correction, which must be determined by observation.

Universal time and sidereal time are rigorously related in such a way that an expression of time in one system can be converted, by means of the numerical formula, to an equivalent expression of time in the other. A knowledge of one is equivalent to a knowledge of the other. The two systems of time measurement are not independent and the use of one instead of the other is purely a matter of convenience: sidereal time is the more convenient for observations of star transits; universal time is the more convenient for many other purposes.

Uniform time and the laws of motion

The astronomical reference systems of position and time are established empirically, by observations of the apparent motions that define them; but these apparent motions reflect actual motions of the Earth and the other celestial bodies, and consequently the reference systems can be constructed on an exact dynamical foundation by means of the gravitational theories of the motions in the solar system. In particular, the measurement of time may be based upon the primary standard that is implicitly defined by the dynamical laws of motion. A clear understanding of the astronomical measurement of time on this basis requires two cardinal principles to be kept in mind:

(a) In astronomy, we are concerned, not with defining time, but only with measuring it. To define a measure of time, it is not necessary to know the ultimate nature of time; we need only devise practicable means for realising a unit of time and for comparing any interval of time with this unit.

(b) A measure of time, like any physical measure, is entirely conventional. Any particular measure may be adopted on the basis of its relative advantages for the specific purposes at hand; no restriction to a unique measure is imposed by physical principles, and no ultimate standard of reference is physically attainable.

For astronomical purposes, the most advantageous fundamental standard for the effective correlation and systematic representation of observed phenomena in terms of the measure of time is the independent variable of the accepted dynamical equations of motion. This measure of time may be characterized as the measure in which observed motions agree with the dynamical theories constructed from the laws of motion; in effect, it is therefore defined by these laws. In the terminology of the traditional formulation of the foundations of dynamics in terms of intuitive concepts, this independent variable is " uniform time ", measured in the invariable unit which, by the law of inertia, would be determined by successive equal rectilinear displacements of a particle moving under no forces. From the preceding principles, however, it follows that a uniform measure necessarily is uniform only by definition. No absolute standard of comparison is accessible, but this is immaterial; an accessible standard that does not lead to any contradiction between theory and observation is all that is required.

The measure of time defined by the laws of motion is not immediately accessible, but the dynamical theory of an observable motion provides a means of obtaining it from the empirical measure determined directly by this motion. Abstractly, uniform time is by definition the independent variable of the equations of motion, inclusive of effects required by relativity; operationally, a uniform measure of time is a measure in terms of which the observed motions of celestial bodies are in agreement with rigorous dynamical theories of these motions.

For designating a measure of time that is defined by the laws of dynamics, ephemeris time has been introduced. It is uniform in the sense that the length of the ephemeris second is defined to be a constant. The dynamical theories of the motions of celestial bodies are developed, in accordance with the fundamental laws of motion, so that the independent variable is ephemeris time as so defined. Beginning with 1960 the designation " Ephemeris Time " is used for the tabular argument in the fundamental ephemerides of the Sun, Moon, and planets.

B. ASTRONOMICAL MEASURES OF TIME AND RELATED CONCEPTS

1. Ephemeris time

Ephemeris time is a uniform measure of time depending for its determination on the laws of dynamics. It is the independent variable in the gravitational theories of the Sun, Moon, and planets, and the argument for the fundamental ephemerides in the Ephemeris.

The measure of ephemeris time has been chosen to agree as nearly as possible with that of universal time during the nineteenth century and it is unlikely that the two measures will differ by more than a few minutes in the twentieth century. Ephemeris time is accordingly expressed in the conventional units of centuries, years, months, days, hours, minutes, and seconds. The numerical values of the ephemeris time and the universal time at the same instant differ only slightly; to avoid possible confusion it is essential to indicate unambiguously which measure of time is being used.

The fundamental epoch from which ephemeris time is measured is the epoch that Newcomb designated as 1900 January 0, Greenwich Mean Noon, but which is now properly designated as 1900 January 0, 12^h E.T. The instant to which this designation is assigned is the instant near the beginning of the calendar year A.D. 1900 when the geometric mean longitude of the Sun, referred to the mean equinox of date, was $279° 41' 48''\cdot04$.

This instant is definitive, but the determination of it depends on observations of the Sun, which are compared with an apparent ephemeris. The observations are themselves definitive, but the apparent ephemeris as deduced from the geometric mean longitude depends on the value adopted for the constant of aberration. All relevant observations and determinations have been made using $20''\cdot47$ for the constant of aberration; a change in this value will lead to a change in our determination of the instant of the fundamental epoch and thus to a corresponding change in the measures of ephemeris time assigned to all other instants. This particular difficulty could have been avoided by specifying the epoch as the instant when the geometric mean longitude of the Sun, reduced by the constant of aberration and referred to the mean equinox of date, was $279° 41' 27''\cdot57$; but there are objections to the implied use of the " apparent mean longitude ". *

The primary unit of ephemeris time is the tropical year at the fundamental epoch of 1900 January 0, 12^h E.T.; the tropical year is defined as the interval during which the Sun's mean longitude, referred to the mean equinox of date, increases by 360°. The adopted measure of this unit is determined by the coefficient of T, measured in centuries of 36525 ephemeris days, in Newcomb's expression for the geometric mean longitude of the Sun, referred to the mean equinox of date, namely:

$$L = 279° 41' 48''\cdot04 + 1296\,02768''\cdot13\, T + 1''\cdot089\, T^2$$

The tropical year at 1900 January 0, 12^h E.T. will accordingly contain:

$$\frac{360 \times 60 \times 60}{1296\,02768\cdot13} \times 36525 \times 86400 = 315\,56925\cdot9747 \text{ ephemeris seconds}$$

*See note in paragraph 4 on page 502.

The following definition of ephemeris time, in accord with the above concepts, was adopted by the tenth General Assembly of the International Astronomical Union (Moscow, 1958; *Trans. I.A.U.*, **10**, 72, 1960) in the following terms (English translation, loc. cit. page 500):

" Ephemeris time is reckoned from the instant, near the beginning of the calendar year A.D. 1900, when the geometric mean longitude of the Sun was 279° 41′ 48″·04, at which instant the measure of ephemeris time was 1900 January 0^d 12^h precisely."

The ephemeris second had already been adopted as the fundamental invariable unit of time by the Comité International des Poids et Mesures (*Procès Verbaux des Séances*, deuxième série, **25**, 77, 1957) in the words:

" La seconde est la fraction 1/31 556 925,9747 de l'année tropique pour 1900 janvier 0 à 12 heures de temps des éphémérides ".*

As explained in sub-section D, on the historical development of ephemeris time, this definition of ephemeris time makes it equivalent to the system of time measurement used by Newcomb in his theories of the motion of bodies in the solar system. Newcomb considered it to be mean solar time and to be uniform in the sense of sub-section A; but it can be identified directly with ephemeris time so that the ephemerides derived from Newcomb's tables of the Sun and planets can be regarded as having ephemeris time as the independent time argument. The origin and rate of ephemeris time are defined to make the Sun's geometric mean longitude agree with Newcomb's expression; the symbol T in that expression therefore represents the measure of ephemeris time, not only in the theory of the motion of the Earth round the Sun but also in those of the heliocentric motions of the other planets. The first two terms of the Sun's geometric mean longitude are now thus defined to be absolute constants; the corresponding values for the Moon and other planets are, however, subject to possible revision to bring them into accord with observation. The mean longitude of any other planet, or even that of the Moon, could have been so used to define the origin and rate of a uniform time system; and ephemerides of the Sun, Moon, and planets could have been constructed with this time system as independent argument.

The measure of ephemeris time at the instant at which an observation of the Sun, Moon, or planet is made can be obtained by comparing the observed position with the gravitational ephemeris of the body; the ephemeris time will be the value of the argument for which the ephemeris position is the same as the observed position. In practice ephemeris time is obtained by the comparison of observed positions of the Sun, Moon, and planets with their corresponding ephemerides. Observations of the Moon, whose geocentric motion is much greater than those of other bodies, are the most effective and expeditious; but, even so, an accurate determination requires observations over an extended period. In practice universal time, which may be determined very accurately, with little delay, from observations of the diurnal motions of the stars, is used as an intermediary measure of time; the difference in the two measures of time, ΔT = E.T. − U.T., which can be readily formed for each observation, is a suitable quantity for combination

*See additional note on page 95.

over an extended period. The practical determination of ephemeris time is discussed more fully in sub-section C.

Ephemeris time was originally defined (1950 Paris Conference on the Fundamental Constants of Astronomy, *Colloques Internationaux du Centre National de la Recherche Scientifique*, **25**, 1–131, Paris, 1950; reprinted from *Bull. Astr.*, **15**, 163–292, 1950) by means of a formula, depending on the observed correction to the lunar ephemeris, for the correction ΔT to be applied to the measure of universal time to give ephemeris time. The use of this formula is precisely equivalent to determining ephemeris time by comparison of observations with the gravitational ephemeris of the Moon. This operational definition has now been superseded by the fundamental definition given above. The latter is independent of possible amendments of either theory or observation; but the former represents, to the best of our present theoretical and observational knowledge, the only practical way of realising the fundamental definition. If in the future a more precise lunar ephemeris is constructed, it will not affect either the definition or the measure of ephemeris time; but it will affect both the operational definition and our determination of ephemeris time.

Julian date

To facilitate chronological reckoning astronomical days, beginning at Greenwich noon, are numbered consecutively from an epoch sufficiently far in the past to precede the historical period. The number assigned to a day in this continuous count is the *Julian Day Number* which is defined to be 0 for the day starting at Greenwich mean noon on B.C. 4713 January 1, Julian proleptic calendar. The Julian day number therefore denotes the number of days that has elapsed, at Greenwich noon on the day designated, since the above epoch. The *Julian Date* (J.D.) corresponding to any instant is, by a simple extension of the above concept, the Julian day number followed by the fraction of the day elapsed since the preceding noon.

Although introduced as a continuous count, and measure, of mean solar days the Julian day number and the Julian date can conveniently be applied to ephemeris time, in which case the Julian date will differ from the conventional one by ΔT; the Julian day number will represent the number of ephemeris days that have elapsed, at the preceding 12^h E.T., since 12^h E.T. on B.C. 4713 January 1. It is not necessary in this definition to know to what universal time this epoch corresponds, i.e. to know ΔT at the epoch; in fact the measure may be regarded as conventional, applicable to both systems of time measurement, as in the case of calendar dates. The terminology *Julian Ephemeris Date* (J.E.D.) may be used when necessary to distinguish the Julian date in ephemeris time with the day beginning at 12^h E.T. from the Julian date in universal time with the day beginning at 12^h U.T.; such a distinction may be essential in dating orbital elements, or in formulae for light-curves of variable stars, where the time must be given to a large number of decimal places. The fundamental epoch 1900 January 0^d 12^h E.T. is J.E.D. 241 5020·0.

The value of J.D. − 240 0000·5 is sometimes used to specify current dates and is known as the Modified Julian Date. It is recommended that the numerical definition be given whenever truncated values are used.

2. Sidereal time

In general terms, sidereal time is the hour angle of the (vernal) equinox, or the first point of Aries. Apart from the motion of the equinox itself, due to precession and nutation, sidereal time is thus a direct measure of the diurnal rotation of the Earth. To each local meridian on the Earth there corresponds a local sidereal time, connected with the sidereal time of the Greenwich meridian by means of the relation:

$$\text{local sidereal time} = \text{Greenwich sidereal time} - \text{longitude}$$

Sidereal time is conventionally measured in hours, minutes, and seconds, so that longitude in the above equation is measured (positively to the west) in time at the rate of one hour to 15°. An object transits over the local meridian when the (local) sidereal time is equal to its right ascension.

The sidereal time measured by the hour angle of the true equinox, i.e., the intersection of the true equator of date with the ecliptic of date, is *apparent sidereal time*; the position of the true equinox is affected by the nutation of the axis of the Earth, which consequently introduces periodic inequalities into the apparent sidereal time. The time measured by the diurnal motion of the mean equinox of date, which is affected by only the secular inequalities due to the precession of the axis, is *mean sidereal time*. Apparent sidereal time *minus* mean sidereal time is the *equation of the equinoxes* due to the nutation; in the ephemerides immediately preceding 1960, it was called the " nutation in right ascension ". The period of one diurnal circuit of the equinox in hour angle, between two consecutive upper meridian transits, is a sidereal day; it is reckoned from 0^h at upper transit which is known as sidereal noon.

In the practical determination of time (see sub-section C) allowance must be made for the variation in the position of the meridian due to the motion of the geographic poles, and may also be made for short-period irregularities in the rate of rotation of the Earth. With this understanding Greenwich sidereal time may formally be defined as the Greenwich hour angle of the first point of Aries.

Sidereal time is determined in practice from observations of the transits of stars, either over the local meridian or, with a prismatic astrolabe, over the small circle corresponding to a constant altitude.

Owing to precession the mean sidereal day, of 24 hours of mean sidereal time, is about $0^s{\cdot}0084$ shorter than the actual period of rotation of the Earth; the apparent sidereal day, nominally of 24 hours of apparent sidereal time, differs from the period of rotation by a variable amount depending on the nutation.

Apparent sidereal time, because of its variable rate, is used only as a measure of epoch; it is not used as a measure of time-interval. Observations of the diurnal motions of the stars provide a direct measure of apparent sidereal time; as their right ascensions are measured from the true equinox. But in many practical methods of determining time the right ascensions are diminished by the equation of the equinoxes, so that mean sidereal time is deduced directly from the observations.

Greenwich sidereal date

In order to facilitate the enumeration of successive sidereal days the concepts of *Greenwich Sidereal Date* (G.S.D.) and *Greenwich Sidereal Day Number*, analogous to those of Julian date and Julian day number, have been introduced. The Greenwich sidereal date is defined as the interval in sidereal days, determined by the equinox of date, that has elapsed on the Greenwich meridian since the beginning of the sidereal day which was in progress at J.D. 0·0. The integral part of the Greenwich sidereal date is the Greenwich sidereal day number; it is a means of consecutively numbering the successive sidereal days beginning at the instants of transit of the equinox over the Greenwich meridian. The zero day is the sidereal day that was in progress at the beginning of the Julian era. The non-integral part of the Greenwich sidereal date is simply the Greenwich sidereal time expressed either in hours, minutes, and seconds, or in fractions of a sidereal day. These concepts can be applied equally well to mean or apparent sidereal time.

There is no direct relationship between Greenwich sidereal date and Julian ephemeris date, as the latter differs from the Julian date (in U.T.) by the unknown difference E.T. − U.T.

The relationships between Greenwich sidereal date, Julian date, and calendar date are considered in section 14H.

The ratio of the length of the mean sidereal day to the period of rotation of the Earth is $0{\cdot}99999\ 99029\ 07 - 59 \times 10^{-12}\ T$; the period of rotation is $1{\cdot}0 + (970\ 93 + 59\ T) \times 10^{-12}$ mean sidereal days. These numbers are not rigorously constant because the sidereal motion of the equinox due to precession is proportional to the length of the day, that is to the period of the rotation of the Earth, whereas the angular measure of the complete rotation is, of course, constant. However, the conceivable change in the period of rotation is such that the effect of a variation in the daily precessional motion is inappreciable. The secular variations are almost inappreciable (see sub-section B.3).

3. Universal time

Universal time is the precise measure of time used as the basis for all civil time-keeping; it conforms with a very close approximation to the mean diurnal motion of the Sun.*

It is, and since the introduction of Newcomb's *Tables of the Sun* has been, defined as 12 hours + the Greenwich hour angle of a point on the equator whose right ascension, measured from the mean equinox of date, is:

$$R_U = 18^h\ 38^m\ 45^s{\cdot}836 + 86\ 40184^s{\cdot}542\ T_U + 0^s{\cdot}0929\ T_U^2$$

where T_U is the number of Julian centuries of 36525 days of universal time elapsed since the epoch of Greenwich mean noon (regarded as 12^h U.T.) on 1900 January 0. The expression for R_U is identical with that given by Newcomb (*Tables of the Sun, A.P.A.E.*, **6**, part 1, page 9, 1895) for the right ascension of the fictitious mean sun, with the exception that Newcomb used T instead of T_U and did not specify in

*See note on page vi regarding the current basis of civil time scales. In general the term "universal time" (U.T.) may be identified throughout this Supplement with the system of U.T.1 defined on page 86.

what measure of time T was to be reckoned. Newcomb, not recognising the variable rotation of the Earth, considered that T was measured in mean solar time applicable alike to orbital motions and to hour angles; as explained in sub-section B.1, Newcomb's T may now be identified with ephemeris time. The point on the equator whose right ascension is R_U is not identical with the " fictitious mean sun " as defined by Newcomb; the right ascension of the fictitious mean sun is:

$$R_E = 18^h\,38^m\,45^s{\cdot}836 + 86\,40184^s{\cdot}542\,T_E + 0^s{\cdot}0929\,T_E^2$$

where T_E is the number of Julian centuries of 36525 days of ephemeris time elapsed since the epoch of 12^h E.T. on 1900 January 0. R_E differs from R_U by $0{\cdot}002738\,\Delta T$ where ΔT is the difference E.T. − U.T.

The implications of this distinction are considered in sub-section B.4.

The measure of universal time at time T_U, expressed in hours, minutes, and seconds, is thus:

$$12^h + \text{the Greenwich hour angle of the mean equinox of date} - R_U$$

The date expressed in the form either of a calendar date or of a Julian date (see sub-section B.1), is that corresponding to the time T_U.

The Greenwich hour angle of the mean equinox of date is Greenwich mean sidereal time, by definition. At 12^h U.T. the Greenwich mean sidereal time will therefore be R_U, which may now be described as " the mean sidereal time of 12^h U.T. "; it may thus be distinguished from the right ascension of the fictitious mean sun.

Although universal time is no longer definable as " 12^h + the Greenwich hour angle of the fictitious mean sun " it is sufficiently close, compared with the deviation between the mean sun and the true Sun, to justify the retention of the terms " mean solar time " and " mean solar day " in the sense in which they have been used in the past. The continued use of these descriptive terms is not to be regarded as identifying universal time with a precise measure of mean solar time; with this understanding, the danger of confusion is small. In this sense, universal time may be identified with Greenwich mean time.

As with sidereal time, there are local mean solar times corresponding to 12^h + the local hour angle of the point whose right ascension is R_U. These times are connected with universal time (Greenwich mean time) by means of the relation:

$$\text{local mean time} = \text{universal time} - \text{longitude}$$

The point whose right ascension is R_U is not observable and practical determinations of universal time are made, through the intermediary of sidereal time, by the observations of the diurnal motion of the stars. For the practical calculation of universal time, an ephemeris of sidereal time with argument universal time is calculated from the relation:

$$\text{Greenwich mean sidereal time} = \text{U.T.} + R_U + 12^h$$

for 0^h U.T. of every day; at U.T. = 0^h the value of the right-hand side is obtained by adding 12^h to the expression R_U for the mean sidereal time of 12^h U.T., and the relation becomes:

$$\text{G.M.S.T. of } 0^h \text{ U.T.} = 6^h\, 38^m\, 45^s{\cdot}836 + 86\,40184^s{\cdot}542 T_U + 0^s{\cdot}0929 T_U^2$$

where T_U takes on successive values at a uniform interval of 1/36525. The apparent sidereal time is obtained by adding the equation of the equinoxes to the mean sidereal time. The sidereal time at 0^h U.T. on successive dates, calculated from this expression, is tabulated in the ephemeris of *Universal and Sidereal Times* in *A.E.*, pages 10–17. These tabular times are the Greenwich hour angles of the equinox that conventionally define the instants of successive midnights of universal time; they are the means of observationally identifying these instants, and of determining the universal time at any other instant. The instant that is designated as 0^h U.T. each day is the moment at which the equinox during its apparent diurnal motion reaches a Greenwich hour angle equal to the value tabulated. At the instant of any observed Greenwich sidereal time, the interval which has elapsed since 0^h U.T., expressed in sidereal time, is immediately obtained by subtracting the tabular sidereal time at 0^h U.T. from the observed sidereal time at the instant; and the universal time at this instant is the equivalent measure of this interval in mean solar time.

Alternatively, use can be made of the tabulations, also given in *A.E.*, pages 10–17, of the universal times corresponding to the instants of 0^h Greenwich (mean and apparent) sidereal times, that is to the instants at which the mean and true equinoxes transit over the Greenwich meridian. An observed sidereal time may be converted to the equivalent interval of mean solar time, which is then added to the tabular universal time to give the universal time at the instant of observation. Examples of the methods of calculation and use of these tables are given in sub-section C.

The mean solar measure of an interval is obtained by multiplying the sidereal measure by the ratio of the sidereal day to the mean solar day. The mean solar day, of 24 mean solar hours, is the interval of time between the two instants at which the equinox reaches the tabular hour angles for two consecutive dates, corrected for the variations of the meridian due to the motion of the geographic poles and to variations of the vertical. From this formal definition and the conventional method of calculating the tabular hour angles of the equinox that determine 0^h U.T. on successive dates, it follows that the hour angle which the equinox describes during one mean solar day consists of a complete circuit of 24^h plus a further angle equal to the tabular increase in the mean sidereal time of 12^h U.T. for a numerical increase in T of one day. The interval of mean sidereal time in a mean solar day is therefore:

$$24^h + \frac{86\,40184^s{\cdot}542 + 0^s{\cdot}1858\, T_U}{36525} = 86636^s{\cdot}55536\,05 + 0^s{\cdot}00000\,5087\, T_U$$

and the ratio of a sidereal day of 86400 mean sidereal seconds to this interval is:

$$\frac{\text{mean sidereal day}}{\text{mean solar day}} = 0{\cdot}99726\,95664\,14 - 0{\cdot}586\, T_U \times 10^{-10}$$

they like to confuse

Inversely, the ratio of the mean solar day to the mean sidereal day is:

$$\frac{86636^{s}{\cdot}55536\ 05 + 0^{s}{\cdot}00000\ 5087\ T_U}{86400^{s}} = 1{\cdot}00273\ 79092\ 65 + 0{\cdot}589\ T_U \times 10^{-10}$$

Disregarding the inappreciable secular variations, the equivalent measures of the lengths of the days are:

mean sidereal day	$23^h\ 56^m\ 04^s{\cdot}09054$ of mean solar time
mean solar day	$24^h\ 03^m\ 56^s{\cdot}55536$ of mean sidereal time

The conversion tables 17.1 and 17.2 are based on these values.

The determination of mean solar time by the established method of converting the elapsed interval since 0^h U.T. from sidereal measure to mean solar measure with a fixed conversion factor keeps the ratio of the mean solar day to the sidereal day constant, irrespective of variations in the rate of rotation of the Earth. These variations cause inequalities in mean solar time as conventionally determined from the tabular hour angles of the equinox that formally define 0^h U.T., and the length of the mean solar day is slightly variable; but the ratio of the sidereal and the mean solar measures is not altered by variations in the rotation of the Earth. The effect on the length of the mean solar day of the variations in the daily motion of precession is entirely inappreciable, as precession affects the hour angle of the equinox and the right ascension of the mean sun alike. The measure of mean solar time depends only upon the motion of the equinox in hour angle that is due to the rotation of the Earth; the ratio of the mean solar day to the period of rotation is constant to 12 decimals or more.

The numerical value of this ratio is 1·00273 78119 06; the period of rotation of the Earth in mean solar time is:

$$0^d{\cdot}99726\ 96632\ 42 = 23^h\ 56^m\ 04^s{\cdot}09890\ 4$$

and the rate of rotation is $15''{\cdot}04106\ 7$ per mean solar second.

Universal time is obtained, through the intermediary of sidereal time, from observations of the transits of stars. It is thus subject to the same irregularities (divided by the factor 1·002738) as those affecting the determination of sidereal time (see sub-section B.2), namely the variation in the local meridian due to the motion of the geographic poles and the short-period variations in the rate of rotation. These irregularities are removed to provide a measure of time which is free of short-period variations (see sub-section C).

4. The ephemeris meridian

Ephemeris time is independent of the rotation of the Earth and is consequently unsuitable for the calculation of hour angles, which do depend on that rotation. For facilitating practical calculations of phenomena that depend upon hour angle and geographic location, the concept of an auxiliary reference meridian, known as the *ephemeris meridian*, has been introduced. The position of the ephemeris meridian in space is conceived as being where the Greenwich meridian would have been if the Earth had rotated uniformly at the rate implicit in the definition of ephemeris time; it is $1{\cdot}002738\ \Delta T$ east of the actual meridian of Greenwich on the surface of the Earth, where ΔT is the difference E.T. − U.T.

When referred to the ephemeris meridian, phenomena depending on the rotation of the Earth may be calculated in terms of ephemeris time by methods formally the same as those by which calculations referred to the Greenwich meridian are made in terms of universal time. The hour angle and the meridian transit of the equinox, which determine the tabular sidereal time at 0^h universal time and the universal time at 0^h sidereal time, are referred to the actual geographic meridian of Greenwich. The numerical value formally obtained from the same numerical relation as that used to compute the sidereal time at 0^h universal time, but with T reckoned expressly in ephemeris time, is the hour angle of the equinox referred to the ephemeris meridian at 0^h ephemeris time, and is called *Ephemeris Sidereal Time* (E.S.T.). Numerically, therefore, the tabular values of sidereal time at 0^h universal time are equally the values of ephemeris sidereal time at 0^h ephemeris time. Ephemeris transit occurs at the instant when the ephemeris sidereal time is equal to the right ascension.

The hour angle of an object referred to the ephemeris meridian is known as the *Ephemeris Hour Angle* (E.H.A.) of that object; it may be calculated from the relation:

ephemeris hour angle = ephemeris sidereal time − right ascension

Longitude measured from the ephemeris meridian is distinguished by the term *ephemeris longitude*; the ephemeris longitude of a place at which the local hour angle has a particular value may be obtained by taking the difference between the local and ephemeris hour angles.

All calculations into which the rotation of the Earth enters may be carried out in terms of ephemeris time, referred to the ephemeris meridian, in precisely the same way as in universal time referred to the Greenwich meridian. In the former case, the precise positions of the meridians on the Earth's surface, specified by their ephemeris longitudes, will not be known until ΔT is known; in the latter case a value of ΔT is necessary before the tabulated ephemerides can be interpolated to universal time. The use of the ephemeris meridian enables such calculations to be carried out precisely as far in advance as required; as soon as a sufficiently accurate value of ΔT can be extrapolated, or determined, the longitudes and hour angles can be referred to the Greenwich meridian and the times, in E.T., expressed in terms of U.T. This procedure is followed in predictions of the general circumstances of eclipses.

Apart from its practical advantages the concept of the ephemeris meridian is valuable in providing a clear picture of the relation between ephemeris time and universal time. At any instant:

E.T. + 12^h = the ephemeris hour angle of the fictitious mean sun, whose right ascension is R_E
= ephemeris sidereal time − R_E

U.T. + 12^h = the Greenwich hour angle of the point whose right ascension is R_U
= Greenwich sidereal time − R_U

If ΔT is the excess of the measure of E.T. over that of U.T., i.e. E.T. − U.T. $= \Delta T$, then:

$$R_E = R_U + 0{\cdot}002738\ \Delta T$$

$$\begin{aligned}\text{E.T.} &= \text{U.T.} + \text{ephemeris sidereal time} - R_E \\ &\quad - \text{Greenwich sidereal time} + R_U \\ &= \text{U.T.} + 1{\cdot}002738\ \Delta T - 0{\cdot}002738\ \Delta T = \text{U.T.} + \Delta T\end{aligned}$$

At a time ΔT *later* the Greenwich meridian will have moved through an angle $1{\cdot}002738\ \Delta T$ and thus will be in the same position as the ephemeris meridian at the earlier time; and the right ascension R_U will have increased by $0{\cdot}002738\ \Delta T$ and thus will be R_E, the same as the right ascension of the fictitious mean sun at the earlier time. The relationship between the ephemeris meridian and the fictitious mean sun at any instant is precisely the same as that between the Greenwich meridian and the point whose right ascension is R_U at a time ΔT later; the two systems of time measurement are identical except that the system of universal time relates to a time ΔT later than that of ephemeris time.

The speed of rotation of the ephemeris meridian is such that it makes one complete revolution of 360°, relative to the mean equinox, in $23^h\ 56^m\ 04^s{\cdot}09890\,4$ of ephemeris time; the ephemeris meridian coincided with the Greenwich meridian at some date between 1900 and 1905.

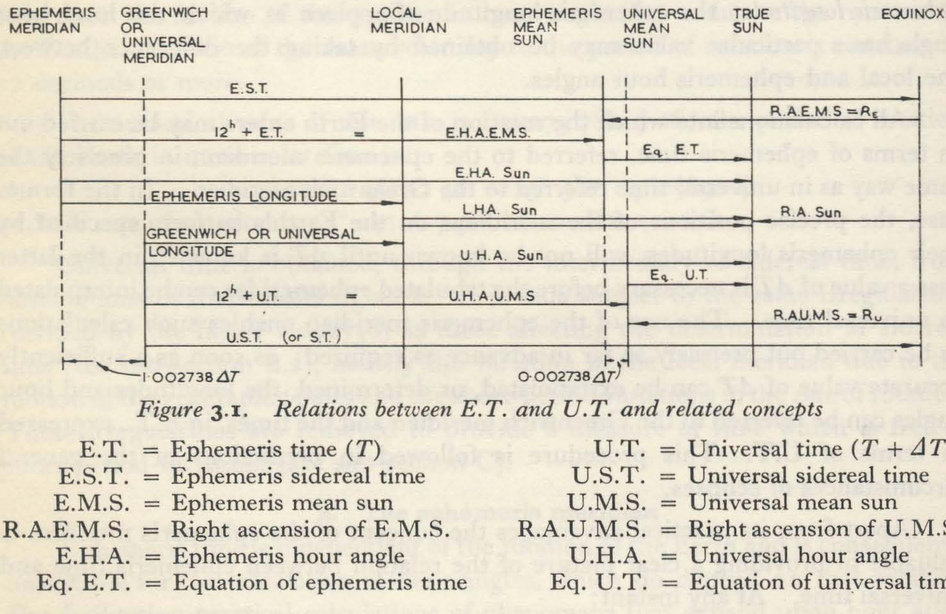

Figure 3.1. *Relations between E.T. and U.T. and related concepts*

E.T.	= Ephemeris time (T)	U.T.	= Universal time ($T + \Delta T$)
E.S.T.	= Ephemeris sidereal time	U.S.T.	= Universal sidereal time
E.M.S.	= Ephemeris mean sun	U.M.S.	= Universal mean sun
R.A.E.M.S.	= Right ascension of E.M.S.	R.A.U.M.S.	= Right ascension of U.M.S.
E.H.A.	= Ephemeris hour angle	U.H.A.	= Universal hour angle
Eq. E.T.	= Equation of ephemeris time	Eq. U.T.	= Equation of universal time

U.S.T. and U.H.A. are identical with G.S.T. (Greenwich sidereal time) and G.H.A. (Greenwich hour angle)

The accompanying diagram (figure 3.1), which is intended solely for illustration, shows clearly the relationship between the two systems: E.T. and the ephemeris meridian in the upper part of the diagram, and U.T. and the Greenwich meridian in the lower. In the diagram certain unconventional terminologies and notations have been introduced to facilitate comparison; these are only used in this limited context. A clear distinction is drawn between the fictitious mean sun (termed the ephemeris mean sun) with right ascension R_E, and the point (termed the universal mean sun) with right ascension R_U.

Two distinct concepts, termed respectively the equation of ephemeris time and the equation of universal time, have been introduced to replace the single concept " equation of time ". On the one hand, the equation of ephemeris time is the logical successor to the equation of time regarded as the excess of the hour angle (or defect of right ascension) of the true Sun over that of the fictitious mean sun; and this is the quantity tabulated in the Ephemeris for 0^h E.T. under the heading "Equation of Time". On the other hand, the equation of universal time is the logical successor to the equation of time regarded as the excess of apparent solar time over mean solar time; this is the quantity required to convert 12^h + U.T. into the G.H.A. of the Sun, but it cannot be tabulated without a knowledge of ΔT.

As from 1965 the tabulation in the Ephemeris of the equation of time will be replaced by the tabulation of the E.T. of ephemeris transit of the Sun. The term " equation of time " will thenceforward be used exclusively for the concept termed here " the equation of universal time ". The equation of time will then be defined as the correction to be applied to 12^h + U.T. to obtain G.H.A. Sun, or more generally the correction to be applied to 12^h + L.M.T. to obtain L.H.A. Sun; it is now so tabulated in the almanacs for navigators and surveyors. The concept of the equation of ephemeris time will no longer be used.

5. Mean solar time

The purpose of this sub-section is to describe the historical development of the concept of mean solar time, prior to the realization of the variability of the rotation of the Earth, and to discuss the consequences of that variability upon the definition of universal time.

A reckoning of time which conforms more or less closely to the recurrence of daylight and darkness determined by the diurnal motion of the Sun, and which is quickly obtainable with high precision from observation, is a practical necessity. Because of the variations in the rate of motion of the Sun in hour angle, due to the inequalities in the annual motion along the ecliptic and to the inclination of the ecliptic to the equator, the measure of time that is directly defined by the actual diurnal motion of the Sun, known as apparent solar time, is impracticable for the purpose of precise timekeeping. Instead, mean solar time was introduced, determined by the apparent diurnal motion of an abstract fiducial point at nearly the same hour angle as the Sun, but located on the mean celestial equator of date and characterized by a uniform sidereal motion along the equator at a rate virtually equal to the mean rate of the annual motion of the Sun along the ecliptic. Relative to any meridian of longitude, this point has a diurnal motion in hour angle virtually the same as the average diurnal motion of the Sun, and uniform except for variations of the local meridian; the position in hour angle is never more than 16^m from the Sun.

The precise position of this moving point was abstractly defined by an expression for its right ascension, which fixes its position among the stars at every instant and is a means of determining its diurnal motion from the observable diurnal motions of the stars. The practice in the past has been to adopt for the right ascension, measured from the mean equinox of date, an expression as nearly identical with the expression for the mean longitude of the Sun as is possible, consistent with a sidereal motion at a constant rate. This expression for the right ascension differs from that for the mean longitude of the Sun by only a slight, progressively increasing, excess of $0^{s}{\cdot}0203\ T^2$ where T is the number of centuries from 1900, due to the secular acceleration of the Sun and to the different rates of the general precession on the ecliptic and the equator. This abstract fiducial point has therefore traditionally been known as the fictitious mean sun; but it has no physical counterpart, and the term is essentially only a name for a mathematical expression.

The system of measuring and determining mean solar time was expressly devised to obtain a measure in agreement with the rotation of the Earth, because, prior to the realization that the rate of rotation is variable, the measure of time that it defines was considered to be uniform. It was for the purpose of obtaining a uniform measure in this way that mean solar time was defined in terms of the diurnal motion of a fictitious mean sun, not by supposing the actual mean sun transferred to the equator, since the mean motion of the Sun in longitude has a secular acceleration.

The definition of the measure of mean solar time was obtained, in the form of the relation to sidereal time, from the formula for the right ascension of the fictitious mean sun. On the Greenwich meridian, in terms of the position of the mean equinox and the position of the fictitious mean sun relative to the mean equinox, mean solar time was defined as:

$$\text{G.H.A. mean equinox of date} - \text{R.A. fictitious mean sun} + 12^{h}$$

For the right ascension of the fictitious mean sun, the numerical formula from whatever tables of the Sun were in current use was adopted. The measure represented by this expression is universal time; it is the mean solar time on the Greenwich meridian reckoned in days of 24 mean solar hours beginning with 0^{h} at midnight, and is the conventional standard measure of mean solar time.

However, because of the variations in the rate of rotation of the Earth, universal time, so defined, does not rigorously conform to the traditional geometric interpretation that originally motivated this method before these variations had been recognized. The right ascension of the fictitious mean sun increased by 12^{h} was taken as the value of the hour angle of the mean equinox to define 0^{h} U.T. in order that mean midnight would be the instant of lower meridian transit of the fictitious mean sun, and the measure of mean solar time at any other instant, reckoned from midnight, would be the hour angle of the fictitious mean sun increased by 12^{h}. In practice, to obtain the tabular values of the hour angle of the mean equinox that determine successive intervals of a mean solar day, the right ascension of the mean sun was calculated from successive values of T at uniform numerical intervals of 1/36525. The instants at which the equinox reaches these tabular hour angles during its diurnal motion depend on the variable rotation of the Earth, and are at slightly unequal intervals of uniform time; consequently, the actual amount of the sidereal motion of the fictitious mean sun during successive mean solar days is not invariable, and the hour angle of the mean sun at midnight depends on the accumulated departures of the sidereal motion from the tabular amounts.

In contrast, Newcomb's expression was intended to represent a variation of right ascension entirely independent of the rotation of the Earth and due to a rigorously uniform sidereal motion of the mean sun that increases its right ascension by a constant amount per unit increase in the numerical value of T, and to the motion of the equinox that is caused by the general precession in right ascension.

The hour angle of the mean equinox and the actual right ascension of the mean sun increased by 12^h do not both reach the tabular value of the mean sidereal time of 0^h U.T. at identically the same instant. The tabular value is, by definition, the hour angle which the equinox reaches at mean midnight, but is not precisely equal to the right ascension of the fictitious mean sun increased by 12^h at this instant. At this hour angle of the equinox, the fictitious mean sun is not exactly on the lower meridian; the designation " Right Ascension of Mean Sun + 12^h ", sometimes applied to the sidereal time of 0^h U.T. prior to 1960, is inexact when the departure of mean solar time from a uniform measure is explicitly recognized, and was therefore eliminated from the Ephemeris when a formal distinction was made between universal time and ephemeris time. In the expression for the right ascension of the fictitious mean sun, the inequalities are entirely due to the motion of the equinox, and strictly T should be interpreted as denoting a uniform measure of time; but the practical procedure is equivalent to reckoning T in mean solar days. This is immaterial for the purpose of defining a formal measure of time; but it has the consequence that, geometrically, mean solar time is not exactly the hour angle of the fictitious mean sun increased by 12^h as it ordinarily has been described, and likewise the mean solar day is not exactly the period of one diurnal circuit of the fictitious mean sun in hour angle as it would be were there no variations in the rate of rotation of the Earth.

The operational procedure used in practice for determining universal time constitutes the actual definition, and supersedes the traditional descriptive characterization. Geometrically, mean solar time and the mean solar day are determined, not by the meridian transit and the hour angle of the fictitious mean sun, but entirely by the diurnal motion of the vernal equinox, in accordance with a conventional formula that specifies a prescribed relation that mean solar time shall have to the observed sidereal time measured by the hour angle of the equinox. The instant of 0^h U.T. is precisely defined by the numerical expression from which the tabular sidereal times of 0^h U.T. are calculated; universal time as obtained in accordance with the established practical method, from the observed sidereal time at the instant and the tabular sidereal time at 0^h U.T., is essentially a formal measure defined by this abstract expression.

They are trying to say that the addition of the new time is not a direct representation of reality?

Although this conventional formal measure of time is not the exact equivalent of the traditional geometric representation of mean solar time, it is numerically identical with the measure of mean solar time that always was actually obtained in practice. Likewise, it is characterized by being strictly in accordance with the measure of time defined by the rotation of the Earth; the mean solar day, when determined from observations of stars and corrected for variations of the meridian, is rigorously proportional to the period of the rotation.

C. THE PRACTICAL DETERMINATION OF TIME

Accurate timekeeping depends upon determining the error of a clock on successive nights by means of determinations of time from astronomical observations. The observed measure of time compared with the reading of the clock at the instant of observation gives the error of the clock; from the successive clock errors, the rate of gain or loss is found, with which the clock error at any intermediate instant may be obtained by interpolation, and over limited periods in advance by extrapolation.

For timekeeping of the highest precision, quartz-crystal clocks have entirely superseded the pendulum clock. A perfect clock, which would run uniformly and have an absolutely constant rate, has not been realized; but the best clocks now available have rates more uniform than the rotation of the Earth. Atomic oscillators are also becoming an important aid in timekeeping, although it is not yet known whether the gravitational and atomic time scales are identical.

Crystal-controlled clocks are more accurate than the individual nightly determinations of time by observation. The clocks are used in practice to smooth out the random errors of observation from night to night, as well as to interpolate between observations; the crystal oscillators that constitute the primary time standards vary in frequency from day to day by only about 2 parts in 10^{10}. However, the length of time over which the clock rate may be extrapolated with confidence is inevitably limited. To maintain a precise standard of time, and to make exact measurements of long intervals, continual direct determinations from astronomical observations are essential.

The determination of sidereal time by observation. To determine the hour angle of the equinox by observations of stars, the location of the equinox among the selected stars is found from ephemerides of their apparent positions. The diurnal motions depend upon the instantaneous rotational motion of the Earth determined by the position of the axis in space and within the Earth, and by the rate of rotation. The instruments are necessarily oriented with reference to local gravity. Consequently, the measure of time obtained directly from the immediately observed positions of the stars in their diurnal circuits is the apparent sidereal time referred to the instantaneous local meridian. In principle, the time may be found from observations of stars at any point of their diurnal arcs, and many different methods have been used, depending on circumstances and on the precision needed.

For meridian observations, the most precise instrument is the photographic zenith tube, for which no corrections are required for level, azimuth, collimation, or flexure. Each observation gives a measure of both the time and the latitude. Determinations of time by extra-meridian observations, comparable in precision to determinations with the photographic zenith tube, may be made with the Danjon impersonal prismatic astrolabe. With this instrument, the stars are observed when at an altitude of 60°. Each observation of one star gives a linear relation between time, latitude, and declination; two groups of stars are observed, one before midnight and one after midnight. Brief descriptions of these instruments are given in section 15B.

The external probable error of the time determined from the observations on one night by these methods is of the order of ±4 milliseconds.

The relative positions of the stars observed with these instruments are determined from the observations themselves, and thus are independent of errors in star catalogues. But even though the star places are mutually consistent, they are still dependent on the particular coordinate system (or "equinox") to which they are referred; different systems would give rise to differing determinations of time. The International Astronomical Union recommended in Stockholm in 1938 (*Trans. I.A.U.*, **6**, 342, 1939) that the system of the FK3 be used; and the adopted practice is equivalent to using a zero determined by the average of the FK3 stars in the corresponding declination belt. The FK3 system will be replaced by that of FK4 as soon as it becomes available (*Trans. I.A.U.*, **10**, 79, 1960).

The varying rate of gain or loss of the clock on apparent sidereal time, and the accumulated error at the times of observation, depend both upon the irregularities of the clock and upon the inequalities in sidereal time. To facilitate the separation of the clock irregularities from the variations in the measure of time, in order to determine accurate clock errors and rates, the transit ephemerides of the stars are often expressed in terms of a more uniform argument than apparent sidereal time, by calculating the mean sidereal time of transit and, for convenience, further converting it to mean solar time.

The mean sidereal time at transit is obtained by omitting from the apparent right ascension the terms of the reduction for nutation that are independent of the coordinates of the star; these terms, common alike to all stars, represent the equation of the equinoxes, which causes the inequality in sidereal time that is due to the nutation of the axis of the Earth. The remaining terms of the reduction for nutation, peculiar to each star, represent the irregularities in the diurnal motion of the star that are produced by the nutation of the axis.

As long as a particular inequality in sidereal time is negligibly small compared to the irregularities of the clock and the inevitable errors of the observations, it may be disregarded in calculating the right ascensions of the stars and in reducing the observations. With the continual increase in the accuracy of observations and the development of more precise clocks, an increasing number of the inequalities have successively become distinguishable from the irregularities of the clock. To obtain a standard of comparison that is as nearly uniform as the running of the clock, successively greater refinements in computation have been necessary, by the inclusion of additional terms of the nutation and, more recently, the application of corrections for the variations of the meridian due to the polar motion. Moreover, the rates of the crystal oscillators now available are so nearly uniform, and the accuracy of the observational comparisons with the stars is so great, that it has also become the practice to include corrections to the observed time for the periodic seasonal variations in the rate of rotation of the Earth.

The calculation of mean solar time. The definition of universal time was left unchanged when ephemeris time was formally introduced into astronomical practice. The practical method of determining universal time that was in established use before 1960 was retained, and the numerical reckoning of universal time was continued without discontinuity except for increased precision resulting from the use of improved values of the nutation.

The sidereal time (hour angle of first point of Aries) at 0^h universal time, and the universal time at 0^h sidereal time (transit of first point of Aries), which formerly were included in the ephemeris of the Sun, are tabulated in the separate ephemeris of *Universal and Sidereal Times* in *A.E.*, pages 10–17,* both for the mean equinox of date and for the true equinox with the short-period terms of nutation included. This ephemeris also contains the equation of the equinoxes, which in the volumes immediately preceding 1960 was designated as the nutation in right ascension and was included with the ephemeris of the Sun.

In the tabulations for 0^h U.T., the argument is the calendar date and the equivalent Julian date. In the tabulations for 0^h S.T. the argument is the Greenwich sidereal date (G.S.D.), defined as the number of sidereal days determined by the equinox of date that have elapsed at Greenwich since the beginning of the sidereal day which was in progress at J.D. 0·0. The integral part of the G.S.D., the Greenwich sidereal day number, is a means of consecutively numbering successive sidereal days. (See sub-section B.2.).

Example 3.1. *Universal and sidereal times*
1960 March 7 at 0^h U.T.

Julian date at 0^h on 1960 March 7 (*A.E.*, p. 2)	243 7000·5
Julian date at epoch from which T_U is measured	241 5020·0
Interval in days, d	2 1980·5
Fraction of Julian century, $T_U = d/36525$	0·60179 32922 7

	h	m	s
$R_U + 12^h = 6^h\ 38^m\ 45^s{\cdot}836$	6	38	45·836
$+86\ 40184^s{\cdot}542\ T_U = 236^s{\cdot}55536\ 049\,d$	4	20	05·1013
$+ \quad 0^s{\cdot}0929\ T_U^2 = 0^s{\cdot}00696(d/10000)^2$			0·0336
Sum = Mean sidereal time at 0^h	10	58	50·971
Equation of the equinoxes ($\Delta\psi \cos \epsilon$) = $-0''{\cdot}744 \times 0{\cdot}9174$			− 0·046
Sum = Apparent sidereal time at 0^h	10	58	50·925

*On pages 12 to 19 in *A.E.* 1972 onwards.

The universal time of transit of the mean equinox is obtained by:

U.T. of transit $= 0{\cdot}99726\ 95664\ (24^h - \text{mean S.T. at } 0^h)$
$= (24^h - \text{mean S.T. at } 0^h)\ (1 - 0{\cdot}00273\ 04336)$

	h	m	s
24^h − mean S.T. at 0^h on 1960 March 7	13	01	09·029
$-0{\cdot}00273\ 04336\ (24^h - \text{mean S.T. at } 0^h)$ (*A.E.*, Table VIII)		− 2	07·973
Sum = U.T. of transit of mean equinox	12	59	01·056
Correction to true equinox $(-0{\cdot}9973\ \Delta\psi \cos\epsilon)$			
$= -0{\cdot}9973 \times -0''{\cdot}746 \times 0{\cdot}9174$			+ 0·046
Sum = U.T. of transit of true equinox	12	59	01·102

The nutation in longitude ($\Delta\psi$) is obtained from the series, and must be interpolated to the U.T. required; the obliquity (ϵ) is a constant to the precision here required. The U.T. of transit of the mean equinox can be obtained directly from the series:

$17^h\ 16^m\ 25^s{\cdot}628 - 235^s{\cdot}90946\ 18\ (\text{G.S.D.} - 242\ 1634) - 0^s{\cdot}0926\ T_U^2$

The practical calculation of universal time from the observed sidereal time with the aid of these tabulations is illustrated by the following example. For full precision it is necessary to use the quantities relating to the mean equinox (e.g. mean sidereal time), interpolating the equation of the equinoxes to the actual universal time concerned.

Example 3.2. *Derivation of universal time from observed sidereal time*

On 1960 March 7, in longitude $5^h\ 08^m\ 15^s{\cdot}75$ west at approximately 2^h local mean time, the observed apparent sidereal time was $13^h\ 05^m\ 37^s{\cdot}249$; the corresponding U.T. (about 7^h on March 7) is obtained as follows:

		h	m	s
Observed local apparent sidereal time		13	05	37·249
Equation of the equinoxes (interpolated to 7^h U.T.)				− 0·046
Observed local mean sidereal time		13	05	37·295
Longitude (add if west)		+ 5	08	15·75
Greenwich mean sidereal time		18	13	53·045
Reduction to mean solar time (*A.E.*, Table VIII)			− 2	59·207
Equivalent interval of mean solar time		18	10	53·838
U.T. of preceding transit of mean equinox (*A.E.*, p. 11)	March 6^d	13	02	56·966
U.T. of observation	March 7^d	7	13	50·804

Alternatively use can be made of the tabulated sidereal time as follows:

	h	m	s
Greenwich mean sidereal time (as above)	18	13	53·045
Greenwich mean sidereal time at 0^h U.T. on March 7 (*A.E.*, p. 11)	10	58	50·971
Difference = mean sidereal time interval	7	15	02·074
Reduction to mean solar time (*A.E.*, Table VIII)		− 1	11·269
U.T. of observation	7	13	50·805

The apparent sidereal time corresponding to a given U.T. may be calculated directly. In this case the figures are the same as above; but the reduction from mean solar time to mean sidereal time ($1^m\ 11^s{\cdot}270$) is taken from *A.E.*, Table IX, with the U.T. argument $7^h\ 13^m\ 50^s{\cdot}805$.

The universal time calculated directly from the immediately observed sidereal time referred to the instantaneous meridian is denoted by U.T.0. This measure of universal time contains inequalities due not only to the variations in the rate of rotation of the Earth but also to the variations of the meridian. In practice, the variations of the meridian due to variations of the vertical may be neglected, as they are too small in comparison with errors of observation to be significant except in an analysis of a long series of observations; but, because of the high accuracy that has been reached in timekeeping, the inequalities due to the polar motion have become of practical importance. The variations in the rate of rotation of the Earth comprise secular, irregular, and periodic seasonal and tidal inequalities. The tidal variations are almost inappreciable, and the secular variation becomes appreciable only after very long intervals; the irregular variations may reach relatively large magnitudes, but are highly erratic. The seasonal inequality is large enough to be of practical significance; and, as far as observations have yet shown, it appears to be remarkably stable from year to year. Accordingly, beginning with 1956, in conformity with resolutions of the International Astronomical Union, determinations of universal time by the national time services have been corrected for the annual variation in the rate of rotation, and for the variation in the position of the meridian due to the motion of the geographic poles.

Corrections for the polar motion were first applied in daily practice at the Royal Greenwich Observatory, beginning with 1947. Previously, these corrections had been applied only in the annual analyses of time signals by the Bureau International de l'Heure. In 1955, a special Rapid Latitude Service was established by the International Astronomical Union, for determining the motion of the pole on a nearly current basis in order that accurate corrections to time determinations may be derived. Universal time reduced to an invariable mean Greenwich meridian by correcting U.T.0 for the observed polar motion is denoted by the notation U.T.1. The corrections for each time station are issued periodically by the Bureau International de l'Heure; time signals are based on extrapolated values, and definitive time signal corrections on interpolated values.

The correction for seasonal variation is extrapolated a year in advance, and published by the Bureau International de l'Heure for use by all observatories engaged in the determination of time. The measure of universal time obtained by correcting U.T.0 for the observed polar motion and for the extrapolated seasonal variation in the rate of rotation of the Earth is denoted by the notation U.T.2. The correction for the annual variation does not wholly eliminate the variability in the length of the mean solar day, but U.T.2 is virtually free of periodic variations. (See section 15A for further details).

The determination of ephemeris time. To determine the correction ΔT for reducing universal time to ephemeris time, an observed position of a celestial body recorded in universal time is compared with a gravitational ephemeris in which the argument is the measure of time defined by Newcomb's *Tables of the Sun*; by inverse interpolation in the ephemeris, to the value of the argument for which the tabular position is the same as the observed position, the difference of the two measures of time is immediately obtained.

Observations of the Moon are the most effective means for the practical determination of ΔT. However, a direct comparison, in the way just described, with the lunar ephemeris calculated from Brown's *Tables of the motion of the Moon* does not give ΔT immediately, because Brown's theory is not strictly gravitational and his tables are not in complete accord with Newcomb's *Tables of the Sun.* In terms of the departure of the Moon from Brown's tables, the relation of ephemeris time to universal time, found from discussions of observations of the Sun, Moon, and planets over periods extending back to ancient times, is represented by:

$$\Delta T = +24^s{\cdot}349 + 72^s{\cdot}318\, T + 29^s{\cdot}950\, T^2 + 1{\cdot}82144\, B$$

where T is reckoned in Julian centuries from 1900 January 0 Greenwich mean noon, and where:

$$B = (L_O - L_C) + 10''{\cdot}71 \sin (140^\circ{\cdot}0\, T + 240^\circ{\cdot}7) - 4''{\cdot}65 - 12''{\cdot}96\, T - 5''{\cdot}22\, T^2$$

in which L_C is the tabular mean longitude of the Moon, and L_O is the observed mean longitude, referred to Newcomb's equinox, at the observed universal time.

Brown's theory is reduced to a gravitational theory in which the measure of time is the same as defined by Newcomb's *Tables of the Sun* by eliminating the empirical term from the mean longitude of the Moon, and applying to the tabular mean longitude the further correction:

$$\Delta L = -8''{\cdot}72 - 26''{\cdot}74\, T - 11''{\cdot}22\, T^2$$

Consequential corrections are required to some of the periodic terms in longitude, latitude, and parallax. Beginning with 1960, the lunar ephemeris is calculated from this amended theory, directly from the theoretical expressions for the longitude, latitude, and parallax, instead of from Brown's tables as formerly. This improved ephemeris has also been made available for 1952–1959 in the *Improved Lunar Ephemeris.*

The development of means for photographic determinations of the position of the Moon among the stars, and the introduction of the improved ephemeris of the Moon with which the observed position may be directly compared, enable ΔT to be obtained more expeditiously than by the methods previously available. Formerly, ΔT was determined principally by means of meridian observations of the Moon and observations of occultations of stars, compared with the tabular positions in the lunar ephemeris calculated from Brown's tables; the determination of a definitive value by these methods requires several years. From photographic positions of the Moon obtained with the dual-rate camera devised by Markowitz, accurate values of ΔT should be determined within a relatively brief period.

Strictly a distinction should be drawn between U.T. + ΔT and E.T., when ΔT is determined as above from observations of the Moon. U.T. + ΔT differs from E.T. in two main respects:

(a) by a quadratic expression in T of the form $a + bT + cT^2$, the coefficients of which have been observationally determined to be zero, but which almost certainly differ from zero by significant amounts (it should be noted that the term cT^2 is of a more fundamental physical character than $a + bT$);

(b) by any deficiencies that may be present in Brown's theory of the motion of the Moon, including revision of any constants involved; in particular Brown uses 1/294 for the flattening of the Earth.

Thus U.T. + ΔT may differ systematically from ephemeris time as defined by reference to the Sun's mean longitude. This is of little consequence to astronomy since the values of ΔT are the best that can be obtained and their significance is fully understood; but it could assume importance in relation to the precise determination of the unit of time. In so far as the use of the Ephemeris is concerned no formal distinction is necessary, and none is made; thus the same symbol ΔT is used to denote the actual difference E.T. − U.T., although it is realised that the observations do not relate directly to this quantity.

Only for comparatively recent years can reasonably accurate values of ΔT be obtained from the available observations; but fairly reliable values may be determined back to the beginning of the nineteenth century, and approximate estimates may be made back into the seventeenth century. Table 3.1 gives the values that were derived in a comprehensive investigation by Brouwer (*A.J.*, **57**, 125, 1952), supplemented by other determinations for more recent years.

The results of a recent estimation of the variations of ΔT during the past three centuries are illustrated in figure 3.2. The large differences from the general trend of Brouwer's values are due to the use of a different value for the tidal deceleration in the Moon's mean longitude.

The annual values of ΔT are tabulated for a limited interval ending with the current year in *A.E.*, page vii or viii. For years up to 1948 inclusive, they are taken from Brouwer's smoothed values; for the later years, definitive values available at the time the Ephemeris is prepared are supplemented by provisional and extrapolated values to extend the table to the current year.*

D. HISTORICAL DEVELOPMENT OF SYSTEMS OF TIME MEASUREMENT

Until the introduction of the pendulum clock in the latter half of the seventeenth century, no means of reasonably accurate timekeeping was available. Besides the sundial, methods had been known since ancient times for determining local time by observations of the Sun or stars, within the limits of accuracy of the existing instruments, and the concept of mean solar time together with the principles for determining the equation of time extends back to ancient Greek astronomy; but with the crude mechanical timekeeping devices that were available, satisfactory measurements of intervals of time for interpolating between astronomical observations could not be made. The earliest mechanical clocks introduced during medieval times were not much improvement, and the early pendulum clocks were not highly reliable; not until the late eighteenth century had clocks become sufficiently improved, and watches and chronometers sufficiently perfected, for accurate time to be generally available, especially at sea.

*Current years of the *A.E.* now show on page vii the relationships between I.A.T., E.T., U.T.1 and U.T.C. from 1956 onwards.

As long as the only means of obtaining accurate time was by direct astronomical observation, apparent solar time was in general use for practical purposes, and it was the argument in *The Nautical Almanac* and other national ephemerides until the early nineteenth century. Determinations of local apparent time were commonly made by observing altitudes of the Sun or stars; this is still one of the most generally useful methods, especially at sea. Mean time when needed for any purpose was obtained by applying the equation of time to the apparent time.

The equation of time, in the sense of the correction to be applied to apparent time in order to obtain mean time, had been tabulated in the national ephemerides from their earliest inception, for the express purpose of regulating clocks and of determining the argument for entering astronomical tables. As clocks were improved, and chronometers were perfected and came into extensive use at sea, apparent time was gradually superseded during the late eighteenth and early nineteenth centuries by local mean solar time for general civil use. When apparent time was replaced by mean time as the argument in the national ephemerides, the equation of time was supplemented by the addition of an ephemeris of sidereal time at mean noon to facilitate the determination of mean solar time, independently of the equation of time, by the alternative method of calculating the mean time from sidereal time.

The equation of time has since come to signify the opposite of the original concept. It now denotes the correction for obtaining apparent time from the mean time kept by clocks and chronometers, which are regulated by determinations of mean time from observations of sidereal time.

Previous to 1925, mean solar time was reckoned from noon in astronomical practice. The mean solar day beginning at noon, 12^h *after* the midnight at the beginning of the same civil date, was known as the *astronomical day*. Mean solar time reckoned from mean noon on the meridian of Greenwich was designated *Greenwich Mean Time* (G.M.T.); reckoned from mean noon on a local meridian, *Local Mean Time* (L.M.T.). Beginning with the volumes for 1925, universal time was introduced in the national ephemerides under various names, a discontinuity of 12^h being made in the arguments, so that December 31·5 in the volumes for 1924 designated the same instant as January 1·0 in the volumes for 1925. In *The Nautical Almanac* the designation *Greenwich Mean Time* (G.M.T.) was still used for the new reckoning, together with *Local Mean Time* (L.M.T.) where appropriate, whereas in *The American Ephemeris* the designation *Greenwich Civil Time* (G.C.T.) was adopted, together with *Local Civil Time* (L.C.T.). This confusion in terminology was finally removed by dropping both designations and substituting *Universal Time* (U.T.); it is, however, now called *Greenwich Mean Time* (G.M.T.) in the navigational publications of English-speaking countries.* Care is necessary to avoid confusion; to distinguish the two reckonings that have both been called *Greenwich Mean Time*, the designation *Greenwich Mean Astronomical Time* (G.M.A.T.) should be used for the reckoning from noon. The designation U.T. always refers to time reckoned from Greenwich midnight, even for epochs before 1925.

*In astronavigation the argument G.M.T. implies U.T.1, but in general communications G.M.T. usually means U.T.C. For astronomical purposes the term U.T. is preferable.

ΔT = E.T. − U.T.

Year	ΔT	Year	ΔT	Year	ΔT	Year	ΔT	Year	ΔT
	s		s		s		s		s
1621	+98	1820·5	+5·15	1860·5	+4·27	1900·5	− 3·90	1940·5	+24·20
1635	+38	1821	4·64	1861	2·68	1901	− 2·87	1941	24·99
1639	−13	1822	5·36	1862	2·75	1902	− 0·58	1942	24·97
1645	+13	1823	3·49	1863	2·67	1903	+ 0·71	1943	25·72
1653	−10	1824	3·27	1864	1·94	1904	+ 1·80	1944	26·21
1662	− 5	1825·5	+2·45	1865·5	+1·39	1905·5	+ 3·08	1945·5	+26·37
		1826	4·03	1866	1·66	1906	4·63	1946	26·89
		1827	1·76	1867	0·88	1907	5·86	1947	27·68
		1828	3·30	1868	+0·33	1908	7·21	1948	28·13
		1829	1·00	1869	−0·17	1909	8·58	1949	28·94
1681	−13·5	1830·5	+2·42	1870·5	−1·88	1910·5	+10·50	1950·5	+29·42
1710	12·0	1831	0·94	1871	3·43	1911	12·10	1951	29·66
1727	7·6	1832	2·31	1872	4·05	1912	12·49	1952	30·29
1738	2·9	1833	+2·27	1873	5·77	1913	14·41	1953	30·96
1747	− 0·4	1834	−0·22	1874	7·06	1914	15·59	1954	31·09
		1835·5	+0·03	1875·5	−7·36	1915·5	+15·81	1955·5	+31·59
		1836	−0·05	1876	7·67	1916	17·52	1956	31·52
		1837	−0·06	1877	7·64	1917	19·01	1957	31·92
		1838	−0·57	1878	7·93	1918	18·39	1958	32·45
		1839	+0·03	1879	7·82	1919	19·55	1959	32·91
1760·9	+ 2·1	1840·5	−0·47	1880·5	−8·35	1920·5	+20·36	1960·5	+33·39
1774·1	6·6	1841	+0·98	1881	7·91	1921	21·01	1961	33·80
1785·1	8·3	1842	−0·86	1882	8·03	1922	21·81	1962	34·23
1792·6	7·4	1843	+2·45	1883	9·14	1923	21·76	1963	34·73
1801·8	+ 5·7	1844	+0·22	1884	8·18	1924	22·35	1964	35·40
1811·9	+ 4·7	1845·5	+0·37	1885·5	−7·88	1925·5	+22·68	1965·5	+36·14
		1846	2·79	1886	7·62	1926	22·94	1966	36.99
		1847	1·20	1887	7·17	1927	22·93	1967	37·87
		1848	3·52	1888	8·14	1928	22·69	1968	38·75
		1849	1·17	1889	7·59	1929	22·94	1969	39·70
		1850·5	+2·67	1890·5	−7·17	1930·5	+23·20	1970·5	+40·70
		1851	3·06	1891	7·94	1931	23·31	1971	41·68
		1852	2·66	1892	8·23	1932	23·63	1972	42·82
		1853	2·97	1893	7·88	1933	23·47		
		1854	3·28	1894	7·68	1934	23·68		
		1855·5	+3·31	1895·5	−6·94	1935·5	+23·62		
		1856	3·33	1896	6·89	1936	23·53		
		1857	3·23	1897	7·11	1937	23·59		
		1858	3·60	1898	5·87	1938	23·99		
		1859·5	+3·52	1899·5	−5·04	1939·5	+23·80		

For the years 1621 to 1948·5 the values of ΔT are the unsmoothed values given by Brouwer in *A.J.*, **57,** 125–146, 1952 under the heading Δt in Table VIII; Brouwer also gives smoothed values and certain derived data. For 1949·5 to 1955·5 the values have been derived at the U.S. Naval Observatory, generally from a straight mean of the meridian and occultation results. From 1956·5 onwards the values have been derived from an atomic time scale that has been fitted to the observed values of ephemeris time from lunar observations. See page vii of the current Ephemeris for later values.

See also note on page 523.

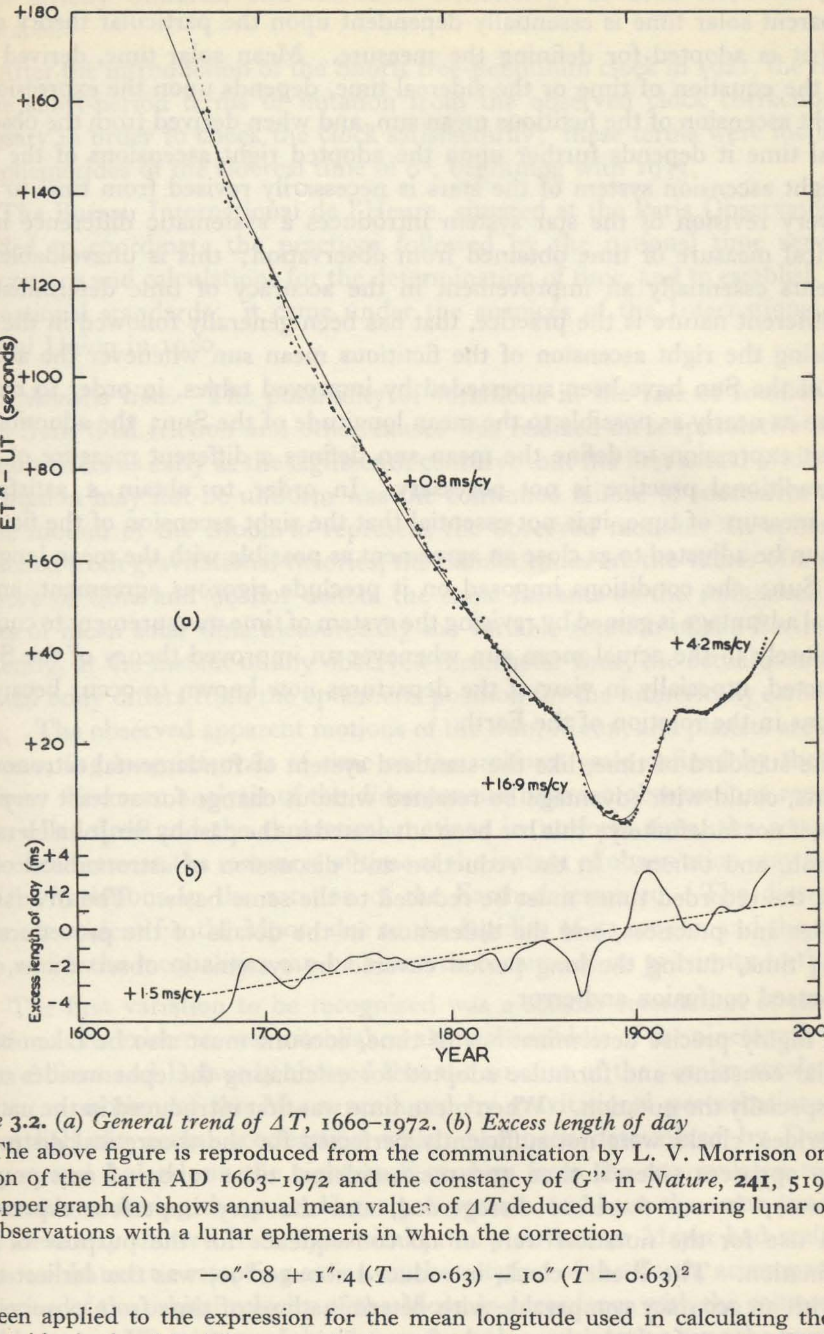

Figure 3.2. *(a) General trend of* ΔT, *1660–1972. (b) Excess length of day*

The above figure is reproduced from the communication by L. V. Morrison on "The rotation of the Earth AD 1663–1972 and the constancy of G" in *Nature*, **241**, 519, 1973. The upper graph (a) shows annual mean values of ΔT deduced by comparing lunar occultation observations with a lunar ephemeris in which the correction

$$-0''.08 + 1''.4\ (T - 0.63) - 10''\ (T - 0.63)^2$$

has been applied to the expression for the mean longitude used in calculating the lunar ephemerides in the *A.E.* The correction is such that the time-scale (denoted by ET*) of the amended ephemeris corresponds as closely as possible to the international atomic time scale over the period 1955–1972. The three solid curves correspond to constant rates of increase in the length of the day. The lower graph (b) shows the excess length (in milliseconds) of the mean solar day compared with an ephemeris day. The dashed line shows the average rate of increase in the length of the day over the period in milliseconds per century.

Systems of mean time. A measurement of time expressly based on an average of apparent solar time is essentially dependent upon the particular theory of the Sun that is adopted for defining the measure. Mean solar time, derived from either the equation of time or the sidereal time, depends upon the expression for the right ascension of the fictitious mean sun, and when derived from the observed sidereal time it depends further upon the adopted right ascensions of the stars. The right ascension system of the stars is necessarily revised from time to time, and every revision of the star system introduces a systematic difference in the numerical measure of time obtained from observation; this is unavoidable, and represents essentially an improvement in the accuracy of time determinations. Of a different nature is the practice, that has been generally followed in the past, of revising the right ascension of the fictitious mean sun whenever the adopted tables of the Sun have been superseded by improved tables, in order to make it conform as nearly as possible to the mean longitude of the Sun; the adoption of a different expression to define the mean sun defines a different measure of time. This traditional practice is not necessary. In order to obtain a satisfactory formal measure of time, it is not essential that the right ascension of the fictitious mean sun be adjusted to as close an agreement as possible with the mean longitude of the Sun; the conditions imposed on it preclude rigorous agreement, and no practical advantage is gained by revising the system of time measurement to conform more closely to the actual mean sun whenever an improved theory of the Sun is constructed, especially in view of the departures now known to occur because of variations in the rotation of the Earth.

The standard of time, like the standard system of fundamental astronomical constants, could with advantage be retained without change for at least very long periods, if not indefinitely; this has been advocated in the past by Sir John Herschel, Newcomb, and others. In the reduction and discussion of astronomical observations, the recorded times must be reduced to the same basis. The diversity of the tables and practices, and the differences in the details of the procedures for deriving time, during the long period covered by systematic observations, have often caused confusion and error.

In highly precise determinations of time, account must also be taken of the particular constants and formulae adopted for calculating the ephemerides of the stars, especially the nutation. When mean time was first introduced in the national ephemerides, clocks were not sufficiently perfected for the theoretical distinction between apparent sidereal time and mean sidereal time to be of any practical importance; it was therefore disregarded, and the very imperfect expressions then in use for the nutation were of no consequence for the purpose of time determination. The Riefler clock, introduced about 1890, was the earliest timepiece with an accuracy comparable with determinations of time from observation. As the accuracy of clocks increased, the explicit recognition of mean sidereal time as distinguished from apparent sidereal time became necessary, just as mean solar time had become necessary at an earlier stage in the development of clocks. The term uniform sidereal time was often used at first; but this measure is not strictly uniform, and the same terminology as used for solar time is preferable.

After the introduction of the Shortt free-pendulum clock in 1921, the removal of the short-period terms of nutation from the observed clock corrections was necessary in order to check the clock satisfactorily; these terms were included in the ephemerides of the sidereal time of 0^h, beginning with 1933.

The Bureau International de l'Heure, situated at the Paris Observatory, was founded to coördinate the practices followed by the national time services in observations and calculations for the determination of time, and to establish precise international standards; it came under the auspices of the International Astronomical Union in 1920.

Ephemeris time. The possibility of variations in the rate of rotation of the Earth from tidal friction and other causes was realized on a speculative basis by several writers as early as the eighteenth century; but the first actual evidence that the rotation may not be uniform was the continued failure of successive theories of the motion of the Moon to represent the observed motion. In ephemerides calculated from gravitational theories, the tabular times are the values of a uniform measure of time and do not denote the same instants as the numerically equal values of mean solar time measured by the variable rotation of the Earth. Consequently, at the instant of any observed mean solar time, the actual position of a celestial body differs from the ephemeris position for the numerically same tabular time. The observed apparent motions of the Sun, Moon, and planets are a means of measuring long intervals of time on the uniform scale defined by the laws of motion. From an analysis of the discrepancies between observations recorded in mean solar time and the theoretical motions in uniform time, the accumulated difference between the measures of time at the instant of observation may be found, and the variations in the rotation of the Earth determined. The discrepancies are most evident for the Moon, due to the rapidity of its motion and the accuracy with which the inequalities can be observed because of its proximity to the Earth.

The first variation to be recognized was a secular retardation of the rate of rotation. Its existence was established about the middle of the nineteenth century, when Adams and Delaunay showed that the amount of the secular acceleration of the mean motion of the Moon produced by gravitational perturbations is only about half the actual acceleration which had been determined by Dunthorne, Mayer, and Lalande in the eighteenth century from the accumulated records of observations during the preceding 2,500 years. At about the same time, Ferrel and Delaunay showed from dynamical principles that, as Mayer had realized, the tides would exert a retarding action on the rotation of the Earth, accompanied by a variation of the orbital velocity of the Moon in accordance with the conservation of momentum. The excess of the observed secular acceleration over the gravitational value is therefore ascribed to the tidal retardation of the rotation.

In addition to the secular departure of the Moon from theory, further variations that are irregular in character occur. In the construction of lunar tables, the principal part of this additional departure has been represented by a long-period empirical term in the longitude, based on past observations; but the different empirical terms adopted by successive investigators have invariably failed to represent subsequent observations. Moreover, further small irregular fluctuations of shorter duration still remained; Newcomb suggested that these may be due to irregular variations in the rate of rotation of the Earth, but at that time conclusive evidence could not be obtained.

Not until after Brown's lunar theory had become available for comparison with observation could confidence be felt that the gravitational theory of the motion of the Moon was sufficiently free from imperfections to enable the discrepancies with observation to be ascribed with assurance to variations in the rotation of the Earth. Furthermore, any apparent fluctuations in the motion of the Moon that are due to variations of the rotation must be accompanied by exactly similar fluctuations in the motions of the other bodies in the solar system, proportional in magnitude to the respective mean motions; these deviations are difficult to detect with certainty, but their existence was finally established by the virtually conclusive investigation by Spencer Jones in 1939 (*M.N.R.A.S.*, **99**, 541, 1939) and later confirmation of his results by others.

Meanwhile, the accuracy of crystal-controlled clocks was becoming comparable with that of the rotation of the Earth. By intercomparisons of the observed rates of the clocks of different national time services, Stoyko in 1937 detected a periodic seasonal variation in the rate of rotation. It has since been confirmed, and accordant results obtained at different observatories, as the clocks and the astronomical observations have been further improved. Even the minute variations due to Earth tides can now be detected.

Because of the secular, irregular, and periodic variations in the rate of rotation of the Earth and in the measure of mean solar time determined by this rotation, a proposal to establish a more uniform fundamental standard of time was referred to the International Astronomical Union in 1948 by the Comité International des Poids et Mesures, and was considered at the Conference on the Fundamental Constants of Astronomy held at Paris in 1950. At this Conference, the measure defined by Newcomb's *Tables of the Sun* was proposed by Clemence. The correction ΔT that reduces universal time to the measure defined by Newcomb's tables, and the correction to the mean longitude of the Moon that enables this measure to be determined from observations of the Moon, had previously been derived by Clemence (*A.J.*, **53**, 169, 1948) from the results found by Spencer Jones for the departures of the Sun and Moon from their tabular positions. The Conference adopted a resolution recommending that this measure of time be adopted, be expressed in units of the sidereal year at 1900·0, and be designated by the name *Ephemeris Time* which had been suggested by Brouwer. This recommendation was adopted in 1952 by the International Astronomical Union at its General Assembly in Rome.

Further consideration indicated that the tropical year would be preferable as the unit, since it is directly accessible to observation and somewhat more fundamental; the sidereal year cannot be determined without a knowledge of the value of the precession. Accordingly, the Comité International des Poids et Mesures at its session in September, 1954, in Paris, proposed to the Tenth General Conference on Weights and Measures which met in Paris during the following month that the fundamental unit of time be the second, redefined as 1/315 56925·975 of the length of the tropical year for 1900·0. The Conference authorized the Comité to adopt a unit after formal action on the definition had been taken by the International Astronomical Union. The Union, at its General Assembly in Dublin in September, 1955, approved the definition proposed by the Comité. However, the tropical year is understood to be the mean tropical year defined by Newcomb's expression for the geometric mean longitude of the Sun; the value of the second required for exact agreement with Newcomb's tables is 1/315 56925·97474 of the tropical year. Consequently, the Comité at its session in Paris in October, 1956, under the authority given by the Tenth General Conference, adopted in place of the value formerly recommended the slightly more precise value 1/315 56925·9747 of the tropical year at 1900 January 0, 12^h E.T. At this session, a Comité Consultatif pour la Définition de la Seconde was established, to coordinate the work of physicists on atomic standards and of astronomers on the astronomical standard of ephemeris time.

The fundamental epoch of ephemeris time was defined in 1958 by the International Astronomical Union at its General Assembly in Moscow (see sub-section B.1.).

References

The following may be consulted for further details on the matters discussed above:

Clemence, G. M. On the system of astronomical constants. *A.J.*, **53**, 169, 1948.

Jones, H. S. The determination of precise time. *Annual report of the Smithsonian Institution* for 1949, 189.

Brouwer, D. A study of the changes in the rate of rotation of the Earth. *A.J.*, **57**, 125, 1952.

Sadler, D. H. Ephemeris time. *Occ. Notes R.A.S.*, **3**, 103, 1954.

Sadler, D. H., and Clemence, G. M. *The improved lunar ephemeris* 1952–1959. A joint supplement to *The American Ephemeris* and *The* (British) *Nautical Almanac*. Washington, 1954.

Markowitz, W., Hall, R. G., Essen, L., and Parry, J. V. L. Frequency of cesium in terms of ephemeris time. *Physical Review Letters*, **1**, 105, 1958.

Reports of Commission 4 of the International Astronomical Union. *Trans. I.A.U.*, **7**, 75, 1950; **8**, 80, 1954; **9**, 80, 1957; **10**, 85, 1960.

Additional Note (1973)

As a result of the rapid development of atomic time standards, the unit and scale of time for general use are no longer based on the ephemeris second. The SI second is now defined so that the frequency corresponding to a certain resonance of the caesium atom is 9 192 631 770 cycles per second. This numerical value is such that the SI second is equal to the observationally determined value of the ephemeris second over the period 1956 to 1965. The scale of international atomic time (I.A.T.) is such that ephemeris time, as defined by the current lunar ephemeris, is equal to I.A.T. + $32^s{\cdot}2$ with an accuracy of about $0^s{\cdot}1$.

4. FUNDAMENTAL EPHEMERIDES

A. INTRODUCTION

The first part of the Ephemeris (pages 10 to 235 in 1960) is devoted to the fundamental ephemerides of the Sun, Moon, and planets, which are designed to provide a rigorous reference system to which observations can be referred (see section 1D). The purpose of this section is primarily to specify in precise detail the bases of these ephemerides; and this is done for the Sun, Moon, and planets in sub-sections B, C, and D respectively. Sub-section E contains brief references to the ephemerides of other members of the solar system, for which no data are given in the Ephemeris.

Sub-sections F and G deal with the more practical uses of the ephemerides: the formation of transit ephemerides to facilitate observation and the reduction of the observations, and the comparison of observations with theory.

The tabular argument for the ephemerides is expressed in ephemeris time; see section 3 for the definition of ephemeris time and a discussion on its relation to universal time. In the astronomical system of measures, the usual unit is the ephemeris day of 86400 ephemeris seconds. The fundamental unit of mass is the mass of the Sun. The unit of length is the astronomical unit (a.u.), defined as the unit of distance in terms of which, in Kepler's Third Law $n^2 a^3 = k^2 (1 + m)$, the semi-major axis (a) of an elliptic orbit must be expressed in order that the Gaussian constant k may be exactly 0·01720 20989 5, when the unit of time is the ephemeris day (*Trans. I.A.U.*, **6**, 20, 336, 357, 1939); in astronomical units, the mean distance of the Earth from the Sun, calculated by Kepler's law from the observed sidereal mean motion n and adopted mass m, is 1·00000 003.

A full discussion of the system of astronomical constants is given in section 6; no change has been made in the conventionally adopted value of any fundamental constant in recent years.*

The notation used in this section is summarized in section 1G. In particular the symbol T is used to denote time measured from the fundamental epoch of 1900 January 0 at 12^h E.T. in Julian centuries of 36525 days each of 86400 ephemeris seconds.

Except where otherwise stated, the tabular positions are *apparent* positions, that is the positions in which the Sun, Moon, and planets would actually be seen from the centre of the Earth, displaced by planetary aberration (section 2D) and

*New values were introduced into the ephemerides for 1968 onwards. See pages 497 to 521 and the Explanations of current years of the *A.E.*

referred to the coordinate system determined by the instantaneous equator, ecliptic, and equinox. The value used for the light-time at unit distance is * $498^s.38$ corresponding to the adopted constant of aberration. For comparison with photographic observations, *astrometric* positions are given for Pluto and the minor planets, for the latter in addition to the apparent positions. Full details of the methods to be used for the comparison of observations with these ephemerides are given in sub-section G. Ephemerides that are intended for theoretical purposes, where a fixed reference system is needed, are referred to the mean equinox at a convenient epoch, usually 1950·0. The methods of passing from one reference system to another, and in fact from one coordinate system to another, are dealt with in section 2.

In order to standardize the dates for which osculating elements of planets, minor planets, and comets are given, the International Astronomical Union (*Trans. I.A.U.*, **5**, 315, 1936) recommended fixed epochs of osculation at the midnights following integral Julian dates that are exactly divisible by 400. This followed an earlier resolution (*Trans. I.A.U.*, **3**, 226, 301, 1929) that the dates used in giving the osculation epochs of elements of comets and minor planets should be the midnight following an integral Julian date that is exactly divisible by 40. More recently the Union (*Trans. I.A.U.*, **7**, 65, 1950) recommended the use of standard 10-day dates (midnights following integral Julian dates exactly divisible by 10) for ephemerides of minor planets and comets, thus superseding an earlier recommendation to use 8- (4- or 2-) day dates. The tabular dates for which elements are given in the Ephemeris conform to this system of 400-, (80-), 40-, 10-day dates. The 400-day dates from 1960 onwards are:

243 7200·5	1960 Sept. 23	244 0000·5	1968 May 24	244 2800·5	1976 Jan. 23
243 7600·5	1961 Oct. 28	244 0400·5	1969 June 28	244 3200·5	1977 Feb. 26
243 8000·5	1962 Dec. 2	244 0800·5	1970 Aug. 2	244 3600·5	1978 Apr. 2
243 8400·5	1964 Jan. 6	244 1200·5	1971 Sept. 6	244 4000·5	1979 May 7
243 8800·5	1965 Feb. 9	244 1600·5	1972 Oct. 10	244 4400·5	1980 June 10
243 9200·5	1966 Mar. 16	244 2000·5	1973 Nov. 14	244 4800·5	1981 July 15
243 9600·5	1967 Apr. 20	244 2400·5	1974 Dec. 19	244 5200·5	1982 Aug. 19

The tabular quantities at times other than those for which they are tabulated may be obtained by interpolation (see section 16); for this purpose first differences are included in many of the ephemerides.

With the exception of the E.T. of ephemeris transit, none of the ephemerides considered in this section involves hour angles or is concerned with the rotation of the Earth. For the purpose of constructing almanacs for navigational or surveying purposes, the ephemerides in terms of E.T. may be converted to ephemerides in terms of U.T. by interpolating the tabular values to a time ΔT *later* than those for which they are tabulated. An ephemeris for 0^h U.T. can be obtained by interpolating the tabulated ephemeris to an E.T. of $0^h + \Delta T$. If (as is almost always the case) second differences are negligible, the interpolated values are obtained by adding algebraically to each tabular value the correction $(\Delta T/h) \times$ the first difference, where h is the tabular interval in the same units as ΔT. The derivation of U.T. of Greenwich transit from the tabulated E.T. of ephemeris transit is discussed in sub-section F.

*$499^s.012$ from 1968.

B. THE SUN *

The ephemerides of the Sun are derived from the geometric longitude referred to the mean equinox of date, the latitude referred to the ecliptic of date, the logarithm of the radius vector, and the mean obliquity of date, that are taken from Newcomb's *Tables of the Sun* (*A.P.A.E.*, **6**, part 1, 1895), afterwards referred to as " the Tables ". The mean orbital elements and constants on which these tables are based are as follows, where the time interval from the epoch is denoted by T when measured in Julian centuries of 36525 ephemeris days, by $D = 3{\cdot}6525T$ when measured in units of 10000 ephemeris days, and by $d = 10000D = 36525T$ when measured in ephemeris days.

Epoch 1900 January 0·5 E.T. = J.D. 241 5020·0

Geometric mean longitude, mean equinox of date

$$L = 279^\circ\ 41'\ 48''{\cdot}04 + 1296\ 02768''{\cdot}13T + 1''{\cdot}089T^2$$
$$= 279^\circ{\cdot}69667\ 8 + 0^\circ{\cdot}98564\ 73354d + 0^\circ{\cdot}00002\ 267D^2$$

Mean longitude of perigee, mean equinox of date

$$\Gamma = 281^\circ\ 13'\ 15''{\cdot}04 + 6189''{\cdot}03T + 1''{\cdot}63T^2 + 0''{\cdot}012T^3$$
$$= 281^\circ{\cdot}22084\ 4 + 0^\circ{\cdot}00004\ 70684d + 0^\circ{\cdot}00003\ 39D^2 + 0^\circ{\cdot}00000\ 007D^3$$

Mean anomaly, $L - \Gamma$

$$g = 358^\circ\ 28'\ 33''{\cdot}00 + 1295\ 96579''{\cdot}10T - 0''{\cdot}54T^2 - 0''{\cdot}012T^3$$
$$= 358^\circ{\cdot}47583\ 3 + 0^\circ{\cdot}98560\ 02670d - 0^\circ{\cdot}00001\ 12D^2 - 0^\circ{\cdot}00000\ 007D^3$$

Eccentricity

$$e = 0{\cdot}01675\ 104 - 0{\cdot}00004\ 180T - 0{\cdot}00000\ 0126T^2$$
$$= 0{\cdot}01675\ 104 - 0{\cdot}00001\ 1444D - 0{\cdot}00000\ 00094D^2$$

Mean obliquity of the ecliptic

$$\epsilon = 23^\circ\ 27'\ 08''{\cdot}26 - 46''{\cdot}845T - 0''{\cdot}0059T^2 + 0''{\cdot}00181T^3$$
$$= 23^\circ{\cdot}45229\ 4 - 0^\circ{\cdot}01301\ 25T - 0^\circ{\cdot}00000\ 164T^2 + 0^\circ{\cdot}00000\ 0503T^3$$
$$= 23^\circ{\cdot}45229\ 4 - 0^\circ{\cdot}00356\ 26D - 0^\circ{\cdot}00000\ 0123D^2 + 0^\circ{\cdot}00000\ 00103D^3$$

Annual rate of rotation of the ecliptic

$$\pi = 0''{\cdot}4711 - 0''{\cdot}0007T = 0^\circ{\cdot}00013\ 086 - 0^\circ{\cdot}00000\ 0053D$$

Longitude of axis of rotation

$$\Pi = 173^\circ\ 57'{\cdot}06 + 54'{\cdot}77T = 173^\circ{\cdot}9510 + 0^\circ{\cdot}2499D$$

The expression for the obliquity of the ecliptic was originally given by Newcomb with T measured in tropical centuries (see section 2B), but was later given without change in the coefficients in terms of Julian centuries; the difference is so small that either form may be used (and is so used) according to whether values are required for the beginning of a Besselian year, or for the epoch of date. The latter (T in Julian centuries) should be regarded as the definitive one if distinction is ever required.

The mean distance a of the Sun (strictly, the constant part of the radius vector), as adopted by Newcomb, is derived from his expressions for the mean motion with the addition of corrections for the action of the planets. Newcomb's value of $\log a = 0{\cdot}00000\ 010$, from which $a = 1{\cdot}00000\ 023$, thus differing from that derived simply from Kepler's law. The expressions for the Sun's mean motion lead also to the following lengths of the principal years.

*Formulae for the corrections to reduce the tabulated values to the IAU system of astronomical constants are given in the *A.E.* for 1968 onwards.

Tropical year (equinox to equinox)

$365^{d}\cdot24219\ 878 - 0^{d}\cdot00000\ 614T = 365^{d}\ 05^{h}\ 48^{m}\ 46^{s}\cdot0 - 0^{s}\cdot530T$

Sidereal year (fixed star to fixed star)

$365^{d}\cdot25636\ 042 + 0^{d}\cdot00000\ 011T = 365^{d}\ 06^{h}\ 09^{m}\ 09^{s}\cdot5 + 0^{s}\cdot01T$

Anomalistic year (perigee to perigee)

$365^{d}\cdot25964\ 134 + 0^{d}\cdot00000\ 304T = 365^{d}\ 06^{h}\ 13^{m}\ 53^{s}\cdot0 + 0^{s}\cdot26T$

Eclipse year (Moon's node to Moon's node)

$346^{d}\cdot62003\ 1 + 0^{d}\cdot00003\ 2T = 346^{d}\ 14^{h}\ 52^{m}\ 50^{s}\cdot7 + 2^{s}\cdot8T$

The values of L and g for every tenth day, the values of Γ and e at the beginning of the calendar year, and of π, Π, and the trigonometric functions of ϵ for the beginning of the Besselian year, are tabulated in *A.E.*, page 50.* They are derived simply by substituting the appropriate value of T in the expressions quoted above. The mean obliquity is not tabulated, though it is required for the calculation of the true obliquity.

Example 4.1. *Sun's mean elements and precessional constants*

1960 March 7 at 0^{h} E.T.

Julian date at 0^{h} on 1960 March 7 (*A.E.*, p. 2)	243 7000·5
Julian date at epoch of the Tables	241 5020·0
Interval in days, d	2 1980·5
Fraction of Julian century, $T = d/36525$	0·60179 33

	L (°)	g (°)
Constant term	279·69668	358·47583
term in d	+65·02126	+63·98667
term in T^2 or D^2	+ 11	− 5
term in T^3 or D^3		0
Sum	344·71805	62·46245

Epoch 1960·0

For the mean obliquity of the ecliptic ϵ, and the precessional constants, T in section 2B is measured in tropical centuries and its value for 1960·0 is 0·60; for example:

$$\epsilon = 23^{\circ}\ 27'\ 08''\cdot26 - 28''\cdot107 - 0''\cdot002 + 0''\cdot000 = 23^{\circ}\ 26'\ 40''\cdot15$$
$$p = 50''\cdot2564 + 0''\cdot0133 = 50''\cdot2697$$

In the Ephemeris the geocentric spherical coordinates are presented on facing pages (18 to 33):† ecliptic coordinates on the left-hand pages and equatorial coordinates on the right-hand pages. The tabulated quantities are described below. No illustration is given of the derivation of the geocentric ecliptic coordinates, since detailed precepts and illustrations are given in the Tables. These precepts are not followed precisely, though no significant departure is made; errors and misprints in the Tables have naturally been corrected before use; the entries in some tables have been replaced by direct calculation of the terms in the formulae from which they were constructed; and, for computing convenience, modifications have been introduced in the intervals of calculation and methods of subtabulation. The resulting coordinates, which are calculated to at least one more decimal than is printed, are smoother than those that would have been derived by the rigid application of Newcomb's precepts but differ from them by negligible amounts.

*See pages 11 and 216 in *A.E.* 1972–3, pages 9 and 216 in *A.E.* 1974 onwards.

†See pages 20 to 35 in *A.E.* 1972 onwards.

The longitude is the geometric longitude referred to the mean equinox of the beginning of the year; it is derived from the quantity obtained from the Tables by subtracting the precession in longitude, which is the precessional displacement of the equinox along the ecliptic since the beginning of the Besselian year. It may be reduced to the (standard) mean equinox of 1950·0 by applying the reduction, which is constant during any one year, given in the footnote.

The apparent longitude is not tabulated directly, since it is less likely to be used; it can be obtained by applying to the tabulated longitude the reduction to apparent longitude, given in the adjacent column. This reduction is the sum of: the precession in longitude from the beginning of the year to date $= p\tau$; the nutation in longitude, including short-period terms, $= \Delta\psi$; and the correction for aberration, taken as $-20''\cdot47/R$, where R is the Sun's radius vector. The precession in longitude and the nutation in longitude are tabulated to $0''\cdot001$, a precision considerably in excess of that possible for the Sun's longitude. The correction for aberration is not tabulated: the aberration is the change of the geometric longitude in the time taken by light to travel the distance R from the Sun to the Earth; since, by the laws of celestial mechanics, the motion in longitude must be proportional to $1/R^2$, the correction is proportional to $1/R$. The constant of proportionality is by definition the constant of aberration, for which the value of $20''\cdot47$ is adopted.

Example 4.2. *Longitude of the Sun*

1960 March 7 at 0^h E.T.

Beginning of Besselian year 1960·0 is 1960 January $1^d\cdot345$ = J.D. 243 6934·845				
Interval in days to 0^h on 1960 March 7				65·655
Fraction (τ) of the tropical year				0·1797 57
Precession in longitude on 1960 March 7, $p\tau$	$+\ 9''\cdot036$			
Nutation in longitude, $\Delta\psi$ (from series)	$-\ 0\cdot744$			
Aberration (R = 0·9924 841)	$-20\cdot625$			
Reduction to apparent longitude	$-12\cdot333$	°	′	″
Longitude referred to mean equinox of date (Tables)		346	26	23·47
Precession in longitude				9·04
Longitude referred to mean equinox of 1960·0		346	26	14·43

The latitude is tabulated for the ecliptics of date, of the beginning of the Besselian year, and of 1950·0. The latitude for the mean ecliptic of date is given directly by the Tables; and, since the latitude is unaffected by nutation and the correction for aberration is negligible, this may be regarded as the apparent latitude. It is reduced to the ecliptic of the beginning of the year by applying the correction $-0''\cdot471\ \tau \sin(\lambda_B + 5^\circ\cdot5)$, and to the ecliptic of 1950·0 by the further addition of $b \sin(\lambda_B + c)$, where b and c (for 1950·0) are the values tabulated in *A.E.*, page 50,* and λ_B is the tabulated longitude referred to the equinox of the beginning of the year. Since there are only a small number of terms which have to be combined to form the latitude, it is tabulated to the same precision, $0''\cdot01$, as the individual contributions in the Tables.

*See page 11 in *A.E.* 1972–3, and page 9 in *A.E.* 1974 onwards.

Example 4.3. *Latitudes of the Sun*

1960 March 7 at 0^h E.T.

Latitude, ecliptic of date (Tables)	$-0''.65$
$-0''.471\ \tau \sin(346°.5 + 5°.5) = -''.471 \times .18 \times -.14$	$+ .01$
Latitude, ecliptic of 1960·0	-0.64
$+ b \sin(346° 26' + c)$; $b = -4''.71$, $c = 5° 29'$ (*A.E.*, p. 50)	$+ .66$
Latitude, ecliptic of 1950·0	$+0.02$

The horizontal parallax is the angle subtended at the Sun by the equatorial radius of the Earth; the tabulated values are calculated by dividing the adopted constant, 8″·80, by the radius vector*. The latter, given on the right-hand pages, is obtained directly from the values of its logarithm from the Tables; it is the actual geometric distance from the centre of the Sun to that of the Earth, measured in astronomical units, at the time stated; no correction for aberration is applied.

The semi-diameter is the apparent value as seen from the centre of the Earth, and is obtained by dividing an adopted value (16′ 01″·18) at unit distance by the radius vector. The adopted semi-diameter at unit distance is an enhanced value, which includes an allowance for irradiation, although this should strictly be independent of the distance; the variation of the correction from its mean value can be about 0″·02, i.e. 0·017 of the mean value. The adopted value is that used in *The Nautical Almanac* for years immediately preceding 1960 and differs from that (16′ 01″·50) used before 1960 in *The American Ephemeris* because of a different allowance for irradiation. A smaller value (15′ 59″·63) is used in the calculation of eclipses (see section 9A). The values adopted for the semi-diameter at unit distance are not necessarily the best possible; but they are sufficiently close to the true values for any variation to be treated, in the analysis of observations, as a small quantity.

Example 4.4. *Radius vector, H.P., and S.D. of the Sun*

1960 March 7 at 0^h E.T.

Log radius vector (Tables)	9·9967 2355
Radius vector (R)	0·9924 841
Horizontal parallax $= 8''.80/R$	$= 8''.87$
Semi-diameter $= (16'\ 01''.18)/R$	$= 16'\ 08''.46$

The Sun's apparent right ascension and declination are referred to the true equinox and equator of date, and are corrected for aberration. In principle, they are derived from the corresponding apparent longitude, latitude, and obliquity of the ecliptic by the standard conversion formulae; in practice, however, a different procedure is followed for reasons given in detail below. In the direct conversion the true obliquity of the ecliptic, as tabulated on the left-hand pages, is used; it is obtained by applying the nutation in obliquity ($\Delta\epsilon$) to the mean obliquity (ϵ_M). Two minor modifications may be introduced: the first is due to the small range in the values of the obliquity of the ecliptic which enables its trigonometric functions to be expressed as linear series of the obliquity itself, or of its difference from some adopted mean; the second arises from the small range

*The name "True Distance" is used in the *A.E.* for 1972 onwards.

of latitude which enables the conversion formulae to be written in the form:

$$\cos\alpha\cos\delta = X/R = \cos\lambda$$
$$\sin\alpha\cos\delta = Y/R = \sin\lambda\cos\epsilon - 19{\cdot}29\,\beta'' \times 10^{-7}$$
$$\sin\delta = Z/R = \sin\lambda\sin\epsilon + 44{\cdot}48\,\beta'' \times 10^{-7}$$

where β'' is the number of seconds of arc in the latitude and where all coordinates, and the obliquity of the ecliptic, are referred to the same equinox, equator, and ecliptic. The approximation is adequate even for 50 years from the adopted epoch when β'' may reach 24.

Example 4.5. *Equatorial coordinates of the Sun — direct method*

1960 March 7 at 0^h E.T.

Mean obliquity (T = 0·60 + 0·00179), ϵ_M		23° 26′ 40″·067	
Nutation in obliquity, $\Delta\epsilon$		−8·836	
True obliquity of the ecliptic, $\epsilon_T = \epsilon_A$		23 26 31·231	
Apparent longitude, λ_A = 346° 26′ 14″·43 − 12″·33 = 346° 26′ 02″·10			
$\sin\lambda_A$ −0·2345 6671	$\sin\epsilon_T$ +0·3978 2067		β''_A −0·65
$\cos\lambda_A$ +0·9721 0002	$\cos\epsilon_T$ +0·9174 6319		
X_A/R +0·9721 0002	$\tan\alpha_A$ −0·2213 8161		$\alpha_A = 23^h\ 10^m\ 04^s{\cdot}103$
Y_A/R −0·2152 0507			
Z_A/R −0·0933 1838	= $\sin\delta_A$		δ_A = −5° 21′ 16″·34
Sum of squares $1{\cdot}0 - 1 \times 10^{-8}$			

The quadrant in which α lies is determined by the signs of X_A/R and Y_A/R, since $\cos\delta_A$ is positive. The small difference from the tabulated value of α_A is partly due to the approximation used in calculating the aberration.

In practice the above conversion is done with geometric values of the coordinates in three steps: firstly, from λ_M, β_M to X_M, Y_M, Z_M, referred to the mean equinox, ecliptic, and equator of date; then to X_T, Y_T, Z_T, referred to the true equinox and equator; and thirdly to α_T, δ_T, referred to the true equinox. This is done to provide unpublished geometric values of the Sun's equatorial rectangular coordinates X_M, Y_M, Z_M referred to the mean equinox of date. These are used as a first step towards the geometric values of the equatorial rectangular coordinates referred to the mean equinoxes and equators of 1950·0 and of the beginning of the year, as published, and to the true equinox and equator of date, as required in the systematic conversion of heliocentric to geocentric coordinates for the planets. For the sole purpose of calculating apparent right ascension and declination it would be much easier to start from the apparent longitude and latitude, as above, or even from the geometric longitude and latitude referred to the true equinox and ecliptic. The method actually used entails an additional step in the conversion and requires that the right ascension and declination must each subsequently be corrected for planetary aberration; the advantages in being able to use a standard systematic procedure for the geometric ephemerides of both Sun and planets outweigh these disadvantages.

The corrections to equatorial rectangular coordinates to allow for nutation, that is to convert from mean to true equinox and equator, are given in section 2C; in the case of the Sun they may be reduced to:

$$10^7 (X_T - X_M) = -52{\cdot}84\, \Delta\psi''\, Y_M$$
$$10^7 (Y_T - Y_M) = +44{\cdot}48\, \Delta\psi''\, X_M - 21{\cdot}03\, \Delta\epsilon''\, Y_M$$
$$10^7 (Z_T - Z_M) = +19{\cdot}29\, \Delta\psi''\, X_M + 48{\cdot}48\, \Delta\epsilon''\, Y_M$$

where $\Delta\psi''$ and $\Delta\epsilon''$ are the number of seconds of arc in the nutation in longitude and obliquity respectively.

Example 4.6. *Equatorial coordinates of the Sun—indirect method*
1960 March 7 at 0^h E.T.

Longitude, mean equinox of date, λ_M		346° 26′ 23″·47
Obliquity, mean equinox of date, ϵ_M		23 26 40 ·067
sin λ_M −0·2344 6600	sin ϵ_M +0·3978 5998	β'' −0·65
cos λ_M +0·9721 2432	cos ϵ_M +0·9174 4615	R 0·9924 841
X_M +0·9648 1793	Y_M −0·2134 9194	Z_M −0·0925 8639

$$(X_M^2 + Y_M^2 + Z_M^2 - R^2 = 0 \times 10^{-8})$$
$$\Delta\psi = -0''{\cdot}744 \qquad \Delta\epsilon = -8''{\cdot}836$$

$X_T - X_M$ −0·0000 0084	$Y_T - Y_M$ −0·0000 0716	$Z_T - Z_M$ +0·0000 0776
X_T +0·9648 1709	Y_T −0·2134 9910	Z_T −0·0925 7863

$$(X_T^2 + Y_T^2 + Z_T^2 - R^2 = 0 \times 10^{-8})$$

tan $\alpha_T = Y_T/X_T$ −0·2212 8453	α_T (geometric)	$23^h\ 10^m\ 05^s{\cdot}376$
Correction for aberration, −0·0028 841 × R × $444^s{\cdot}4$		−1 ·272
Apparent right ascension, α_A		23 10 04 ·104
sin $\delta_T = Z_T/R$ −0·0932 7971	δ_T (geometric)	−5° 21′ 08″·33
Correction for aberration, −0·0028 841 × R × +2797″		−8 ·01
Apparent declination, δ_A		−5 21 16 ·34

The above corrections for aberration are calculated from the formula:
−0·0057 683 × distance × daily motion = −0·0028 841 × R × double first difference since the effects of third differences are negligible.

Although the precision of the initial data does not justify the retention of eight decimals, they are used to illustrate the consistency of different methods of calculation.

The equation of time, which is tabulated in the sense apparent minus mean, is the excess of the right ascension of the fictitious mean sun over that of the true Sun. The tabular value at 0^h E.T. is obtained by subtracting the apparent right ascension of the Sun at 0^h E.T. from the apparent sidereal time of 0^h U.T. increased by 12^h, which is the same numerically as the right ascension of the fictitious mean sun at 0^h E.T. The values for 1960 January 0 and January 1 at 0^h E.T. are therefore numerically the same as the tabular values for 1959 December 31 and December 32 at 0^h U.T. given in the ephemerides before the introduction of ephemeris time.

Example 4.7. *Equation of time*
1960 March 7 at 0^h E.T.

12^h + apparent sidereal time at 0^h U.T.	$22^h\ 58^m\ 50^s{\cdot}925$
Apparent right ascension of the Sun at 0^h E.T.	23 10 04 ·104
Equation of time at 0^h E.T. (apparent − mean)	−11 13 ·179

As explained in sub-section 3B.4 the equation of time (there termed, for purposes of explanation, the equation of ephemeris time) as tabulated in the Ephemeris differs from the excess of the Greenwich hour angle of the Sun over 12^h + U.T., or from the excess of apparent solar time (considered as 12^h + the hour angle of the Sun) over mean solar time. By reference to the diagram (figure 3.1) there given, it is seen that the difference is $-0{\cdot}002738\,\Delta T$; if the equation of time is denoted by E then, at any instant:

$$\text{G.H.A. Sun} = 12^{\text{h}} + \text{U.T.} + E - 0{\cdot}002738\,\Delta T$$

Since E is tabulated for 0^{h} E.T. a knowledge of ΔT is strictly required before E can be interpolated to a given U.T.; but the variation of E is so small that a very approximate value of ΔT will suffice. From this relation it follows that:

at Greenwich mean noon, 12^{h} U.T., the hour angle of the Sun is:

$$E(12^{\text{h}}) - 0{\cdot}002738\,\Delta T$$

at Greenwich transit of the Sun, the U.T. is:

$$12^{\text{h}} - \{E(12^{\text{h}} - E) - 0{\cdot}002738\,\Delta T\}$$

where the time in parentheses () after E indicates the U.T. to which it must be interpolated; the variation of E in time $0{\cdot}002738\,\Delta T$ can be ignored.

For practical use (for example, for navigation or surveying) the quantity required is therefore $E - 0{\cdot}002738\,\Delta T$ tabulated in terms of U.T.; strictly this can only be done when ΔT is known, but an approximate value of ΔT suffices. The maximum daily difference of E is 30^{s}, so that the maximum errors arising through an error of, say, 3^{s} in the extrapolated value of ΔT are: $0^{\text{s}}{\cdot}001$ in the interpolation of E and $0^{\text{s}}{\cdot}008$ through the correction $0{\cdot}002738\,\Delta T$. To navigational accuracy, it suffices at the present time to subtract a mean value of $0^{\text{s}}{\cdot}10$ from the tabulated values.

From 1965 onwards the equation of time in the Ephemeris will be replaced by the ephemeris time of ephemeris transit, as given for the Moon and planets. Reference to section 3B.4 shows that:

at 12^{h} E.T. the ephemeris hour angle of the Sun is $E(12^{\text{h}})$

at ephemeris transit of the Sun the E.T. is $12^{\text{h}} - E(12^{\text{h}} - E)$

where the time in parentheses () after E indicates the E.T. to which it must be interpolated. Thus the tabulated quantity will be $12^{\text{h}} - E$ interpolated to an E.T. of $12^{\text{h}} - E$; it will actually be calculated in a way similar to that used for the Moon and planets, namely by finding the E.T. at which the right ascension equals the ephemeris sidereal time. The ephemeris sidereal time at 0^{h} E.T. is the same as apparent sidereal time at 0^{h} U.T. The fraction of the day p is thus obtained from the equation:

(right ascension − ephemeris sidereal time) at 0^{h} E.T.
$-\,p \times \{\,24^{\text{h}}$ − the following daily difference of (R.A. − E.S.T.)}
$-0{\cdot}0625$ (double second difference of R.A.) = 0

The actual difference of the ephemeris sidereal time is used to allow for the variation of the equation of the equinoxes. The equation is solved to give p directly in hours, minutes, and seconds. Thus:

$$24^{\text{h}}p = 24^{\text{h}}p_0 + (p_0 + \tfrac{1}{2}\delta p_0)\,24^{\text{h}}\,\delta p_0 - 0{\cdot}0625\text{ (double second difference of R.A.)}$$

where

$24^{\text{h}}p_0 =$ (R.A. − E.S.T.) at 0^{h} E.T.

$24^{\text{h}}\,\delta p_0 =$ following first daily difference of R.A. − E.S.T.

Example 4.8. *E.T. of ephemeris transit of the Sun*
1960 March 7

The relevant quantities and their differences are:

R.A. Sun at 0^h E.T.			App. S.T. at 0^h U.T.	R.A. − S.T.	
	h m s	s	h m s	h m s	
7	23 10 04·104	−0·420	10 58 50·925	12 11 13·179	
	221ˢ·998				−14ˢ·558
8	23 13 46·102	−0·390	11 02 47·481	12 10 58·621	

where App. S.T. at 0^h U.T. = E.S.T. at 0^h E.T. Then:

	h m s		
$24^h\, p_0$ = R.A. − S.T.	12 11 13·179	p_0 =	0·50779
$24^h\, \delta p_0 = -14^s{\cdot}558$		$\frac{1}{2}\delta p_0$ =	− ·00008
$(p_0 + \frac{1}{2}\,\delta p_0)\ 24^h\,\delta p_0$	−7·391	$p_0 + \frac{1}{2}\delta p_0$ =	·50771
Double second difference of R.A. = $-0^s{\cdot}810$			
$-0{\cdot}0625 \times -0^s{\cdot}810$	+0·051		
Sum = E.T. of ephemeris transit	12 11 05·839		

The conversion to fractions of a day is conveniently done by means of table 17.5.

As from 1965 the term " equation of time " will be used exclusively for the quantity $E - 0{\cdot}002738\ \Delta T$, which in figure 3.1 was termed "the equation of universal time "; the equation of time will be defined as the correction to be applied to 12^h + U.T. to obtain G.H.A. Sun, or more generally the correction to be applied to 12^h + L.M.T. to obtain L.H.A. Sun. The concept of the equation of ephemeris time (that is E) will no longer be used.

The geocentric equatorial rectangular coordinates of the Sun tabulated in the Ephemeris are derived, as indicated above, from the longitude, latitude, radius vector, and obliquity. The geometric values referred to the true equator and equinox of date as fundamental plane and point of reference, as obtained in the course of conversion to right ascension and declination, are not tabulated; instead geometric values referred to the mean equator and equinox of (a) the beginning of the year (*A.E.*, pages 34–41) and (b) 1950·0 (*A.E.*, pages 42–49) are given.*

These rectangular coordinates may be converted from one reference system to another by means of the formulae of sections 2B and 2C. Both precession and nutation may be incorporated in the same conversion formulae, but, because it is not possible to tabulate or subtabulate coordinates referred to the true equinox at an interval of ten days, the conversion from mean to true equinox is always done, as described above, as a separate calculation.

As applied to the Sun the formulae take the forms:
(a) from mean equinox and equator of date (X_M, Y_M, Z_M) to the beginning of the year (X_B, Y_B, Z_B)

$$X_B = X_M + 2656{\cdot}5\ \tau\ Y_M \times 10^{-7}$$
$$Y_B = Y_M - 2234{\cdot}9\ \tau\ X_M \times 10^{-7}$$
$$Z_B = Z_M - 971{\cdot}7\ \tau\ X_M \times 10^{-7}$$

where τ is the fraction of the year. The approximations used may give rise to maximum errors of $0{\cdot}2 \times 10^{-7}$ in X_B and Y_B; these are ignored in practice. Note that $(Z_B - Z_M) = 0{\cdot}4348\ (Y_B - Y_M)$.

*For *A.E.* 1972 onwards: the values for the *nearest* beginning of year are given on pages 36 to 43; the values for 1950·0 are given on pages 44 to 51.

(b) from mean equinox and equator of date (X_M, Y_M, Z_M) to 1950·0 (X_S, Y_S, Z_S)

$$X_S = X_x X_M + X_y Y_M + X_z Z_M$$
$$Y_S = Y_x X_M + Y_y Y_M + Y_z Z_M$$
$$Z_S = Z_x X_M + Z_y Y_M + Z_z Z_M$$

where X_x, Y_x, ... are functions of the precessional elements and can be calculated from simple series for any date (see section 2B); they are systematically tabulated at intervals of 1000 days in table 2.2.

Example 4.9. *Conversion of equatorial rectangular coordinates of the Sun*

1960 March 7 at 0^h E.T.

Fraction of tropical year, τ = 0·179757

X_M	+0·9648 1793	Y_M	−0·2134 9194	Z_M	−0·0925 8639
$X_B - X_M$	− 1019	$Y_B - Y_M$	− 3876	$Z_B - Z_M$	− 1685
X_B	+0·9648 0774	Y_B	−0·2135 3070	Z_B	−0·0926 0324

For conversion from 1950·0 to the mean equinox of date (by direct calculation from series in section 2B, rounded differently from some values in table 2.2):

X_x	+0·9999 9692	Y_x	−0·0022 7519	Z_x	−0·0009 8914
X_y	+0·0022 7519	Y_y	+0·9999 9741	Z_y	−0·0000 0113
X_z	+0·0009 8914	Y_z	−0·0000 0113	Z_z	+0·9999 9951

With appropriate changes of formula to convert from date to 1950·0:

X_S	+0·9642 3764	Y_S	−0·2156 8643	Z_S	−0·0935 4044

$$(X_S^2 + Y_S^2 + Z_S^2 - R^2 = -1 \times 10^{-8})$$

C. THE MOON*

Beginning with the volume for 1960, the lunar ephemeris is calculated directly from Brown's theory instead of from his *Tables of the Motion of the Moon* (New Haven, Yale University Press, 1919); but in order to obtain a strictly gravitational ephemeris expressed in the same measure of time as defined by Newcomb's *Tables of the Sun*, the orbital elements upon which Brown's tables are based have been amended by removing the empirical term and by applying to the mean longitude the correction:

$$-8''\cdot72 - 26''\cdot74\ T - 11''\cdot22\ T^2$$

where T is measured in Julian centuries from 1900 January 0·5 E.T. A description of the method of calculating the ephemeris, and a comparison of the positions with tabular positions from Brown's tables, are included in the *Improved Lunar Ephemeris* 1952–1959, which was issued in 1954 to make the amended ephemeris available before 1960. The complete description there given, with its detailed list of all terms included, constitutes the formal specification of the present lunar ephemeris. Notes on the history of the introduction of the improved ephemeris are given in section 3.

In the following expressions for the fundamental orbital elements and related quantities the time interval from the epoch is denoted by T when measured in Julian centuries of 36525 ephemeris days, by D = 3·6525T when measured in units of 10000 ephemeris days, and by d = 10000D = 36525T when measured in

*Important changes in the basis of the lunar ephemeris were introduced in 1968 and 1972–3; details are given on pages 497 to 513 and in the relevant volumes of the *A.E.*

ephemeris days. The symbols for the geometric mean longitudes and mean longitudes of perigee of the Sun and Moon differ from those used by Brown, whose notation is also used in the *Improved Lunar Ephemeris*. Thus L, Γ, ☾, Γ' in the Ephemeris and this Supplement correspond to L', ϖ', L, ϖ in Brown's tables and the *Improved Lunar Ephemeris*; the symbols ☊ and D are used in common. Brown's symbols $l = L - \varpi$, $l' = L' - \varpi'$, $F = L -$ ☊ have been retained in the arguments of the series for the nutation in table 2.5, although g, and not l', has been used in the Ephemeris and sub-section B for the mean anomaly of the Sun.

Epoch 1900 January 0·5 E.T. = J.D. 241 5020·0

$$☾ = 270° \; 26' \; 02''\!\cdot\!99 + 1336^r \; 307° \; 52' \; 59''\!\cdot\!31T - 4''\!\cdot\!08T^2 + 0''\!\cdot\!0068T^3$$
$$= 270°\!\cdot\!43416\;4 + 13°\!\cdot\!17639\;65268d - 0°\!\cdot\!00008\;50D^2 + 0°\!\cdot\!00000\;0039D^3$$

$$\Gamma' = 334° \; 19' \; 46''\!\cdot\!40 + 11^r \; 109° \; 02' \; 02''\!\cdot\!52T - 37''\!\cdot\!17T^2 - 0''\!\cdot\!045T^3$$
$$= 334°\!\cdot\!32955\;6 + 0°\!\cdot\!11140\;40803d - 0°\!\cdot\!00077\;39D^2 - 0°\!\cdot\!00000\;026D^3$$

$$☊ = 259° \; 10' \; 59''\!\cdot\!79 - 5^r \; 134° \; 08' \; 31''\!\cdot\!23T + 7''\!\cdot\!48T^2 + 0''\!\cdot\!008T^3$$
$$= 259°\!\cdot\!18327\;5 - 0°\!\cdot\!05295\;39222d + 0°\!\cdot\!00015\;57D^2 + 0°\!\cdot\!00000\;005D^3$$

$$D = 350° \; 44' \; 14''\!\cdot\!95 + 1236^r \; 307° \; 06' \; 51''\!\cdot\!18T - 5''\!\cdot\!17T^2 + 0''\!\cdot\!0068T^3$$
$$= 350°\!\cdot\!73748\;6 + 12°\!\cdot\!19074\;91914d - 0°\!\cdot\!00010\;76D^2 + 0°\!\cdot\!00000\;0039D^3$$

where

☾ = the mean longitude of the Moon, measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit, and then along the orbit;

Γ' = the mean longitude of the lunar perigee, measured in the ecliptic from the mean equinox of date to the mean ascending node of the lunar orbit, and then along the orbit;

☊ = the longitude of the mean ascending node of the lunar orbit on the ecliptic, measured from the mean equinox of date;

D = ☾ $- L$ = the mean elongation of the Moon from the Sun.

The expressions for the mean longitudes of the Moon and of the lunar perigee, and hence of D, include implicit partial corrections for aberration (*A.J.*, **57**, 46, 1952).*

The constant of eccentricity (e) is 0·05490 0489.

The constant of inclination (γ) is 0·04488 6967; it is the sine of half the inclination to the ecliptic.

The constant of sine parallax (α) is 3422″·5400; it corresponds to an equatorial horizontal parallax of 57′02″·70 and to a perturbed mean distance of 60·2665 equatorial radii of the Earth.

The adopted ratio of the mass of the Earth to the mass of the Moon is 81·53, in the lunar theory.

The lengths of the mean months at the epoch are:

	d	d h m s
Synodic month (new moon to new moon)	29·530 589	29 12 44 02·9
Tropical month (equinox to equinox)	27·321 582	27 07 43 04·7
Sidereal month (fixed star to fixed star)	27·321 661	27 07 43 11·5
Anomalistic month (perigee to perigee)	27·554 551	27 13 18 33·2
Draconic month (node to node)	27·212 220	27 05 05 35·8

The secular variations do not exceed a few hundredths of a second per century, and depend partly upon the variations in the rate of rotation of the Earth.

The values of Γ', ☊, ☾, and D for every tenth day at 0^h E.T. are tabulated in *A.E.*, page 51†. This page also contains, for every tenth day, the values of:

i = the inclination of the mean equator of the Moon to the true equator of the Earth,

*See note on page 523.

†Page 215 in *A.E.* 1972 onwards.

Δ = the arc of the mean equator of the Moon from its ascending node on the true equator of the Earth to its ascending node on the ecliptic of date,

Ω' = the arc of the true equator of the Earth from the true equinox of date to the ascending node of the mean equator of the Moon,

calculated with Hayn's value of 1° 32′·1 for the inclination (I) of the mean lunar equator to the ecliptic; the ascending node of the mean lunar equator on the ecliptic is at the descending node of the mean lunar orbit, $\Omega \pm 180°$. They are calculated from the following formulae, in which ϵ is the true obliquity and the node is referred to the true equinox by increasing Ω by the nutation in longitude $\Delta\psi$.

$$\sin \Delta \;\sin i = -\sin \epsilon \sin (\Omega + \Delta\psi)$$
$$\cos \Delta \;\sin i = \sin I \cos \epsilon - \cos I \sin \epsilon \cos (\Omega + \Delta\psi)$$
$$\cos i = \cos I \cos \epsilon + \sin I \sin \epsilon \cos (\Omega + \Delta\psi)$$
$$\sin \Omega' \sin i = -\sin I \sin (\Omega + \Delta\psi)$$
$$\cos \Omega' \sin i = \cos I \sin \epsilon - \sin I \cos \epsilon \cos (\Omega + \Delta\psi)$$

These formulae, which are derived from figure 4.1, give Δ, Ω', i without ambiguity of quadrant.

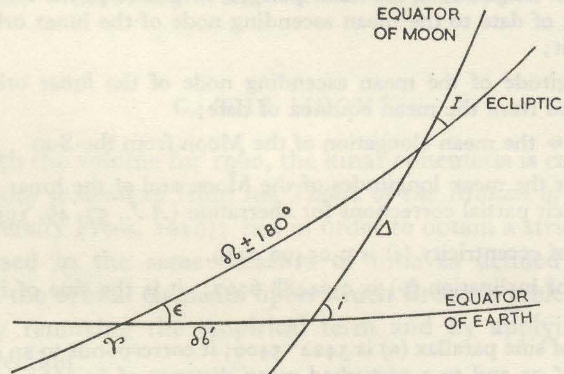

Figure 4.1. *Notation for elements of Moon's equator*

Example 4.10. *Mean elements of the Moon and auxiliary quantities*

1960 March 7 at 0^h E.T.

From example 4.1. $d = 21980{\cdot}5$ $T = 0{\cdot}60179\ 33$

The Moon's mean longitude, and mean longitudes of perigee and node are obtained as:

	☾	Γ'	☊
	°	°	°
Constant term	270·43416 4	334·32955 6	259·18327 5
term in d	+183·78385 7	+288·71738 7	− 83·95368 7
term in T^2 or D^2	− 41 0	− 373 9	+ 75 3
term in T^3 or D^3	+ 0	− 3	+ 0
Sum	94·21761 1	263·04320 1	175·23034 1

$\Delta\psi = -0''{\cdot}744 = -0°{\cdot}00020\ 7$ $\qquad \Omega + \Delta\psi = 175{\cdot}23013\ 4$

$\sin \epsilon_T$, $\cos \epsilon_T$ are taken from example 4.5 and $I = 1^\circ\ 32'{\cdot}1$

$\sin \epsilon_T$ +0·3978 207	$\sin(\Omega + \Delta\psi)$ +0·0831 537	$\sin I$ +0·0267 876
$\cos \epsilon_T$ +0·9174 632	$\cos(\Omega + \Delta\psi)$ −0·9965 367	$\cos I$ +0·9996 412
$\sin \Delta \sin i$ −0·0330 803	$\cos i$ +0·9065 142	$\sin \Omega' \sin i$ −0·0022 275
$\cos \Delta \sin i$ +0·4208 773		$\cos \Omega' \sin i$ +0·4221 695
Sum of squares = 1·0 + 0 × 10^{-7}		Sum of squares = 1·0 + 0 × 10^{-7}
$\sin i$ +0·4221 753	i +24°·97200	$\sin i$ +0·4221 754
$\tan \Delta$ −0·0785 984		$\tan \Omega'$ −0·0052 763
Δ 355°·50588		Ω' −0°·30231

The longitude referred to the mean equinox of date, the latitude referred to the ecliptic of date, and the horizontal parallax are calculated for every half-day from Brown's theoretical expressions, with the corrections required for the amendment to the mean longitude, as specified in the *Improved Lunar Ephemeris*. The apparent longitude, as tabulated, is obtained by adding nutation in longitude and applying the following correction for the residual terms in aberration not included in Brown's theory (see *A.J.*, **57**, 46–47, 1952):

$$+0''{\cdot}018 \cos(\text{☾} - \Gamma' - 2D) + 0''{\cdot}007 \cos 2D$$

The latitude and horizontal parallax are printed without amendment.

Example 4.11. *Apparent longitude, latitude, and parallax of the Moon*
1960 March 7 at 0^h E.T.

Moon's longitude, mean equinox of date	93° 09′ 52″·762
Correction for neglected aberration (☾ − Γ' = 191°·2, D = 109°·5)	+ 0·011
Nutation in longitude	− 0·744
Moon's apparent longitude, λ	93 09 52·029
Moon's apparent latitude, β	− 5 13 19·726
Moon's horizontal parallax, π	54 17·5745

The fundamental data, as above, are quoted without derivation as this is described in the *Improved Lunar Ephemeris*.

The semi-diameter (s) is derived from the horizontal parallax (π) by means of the following accurate relation, in which the adopted semi-diameter at mean distance is due to Newcomb.

$$\frac{\sin s}{\sin \pi} = \frac{\sin\{\text{semi-diameter at mean distance } (15'\ 32''{\cdot}58)\}}{\sin\{\text{equatorial horizontal parallax at mean distance } (57'\ 02''{\cdot}70)\}}$$

This leads to:*

$$\sin s = 0{\cdot}272481 \sin \pi$$

or, with an error not exceeding 0″·001:

$$s'' = 0{\cdot}0796 + 0{\cdot}272446\ \pi''$$

where s'' and π'' are respectively the number of seconds in s and π.

Example 4.12. *Semi-diameter of the Moon*
1960 March 7 at 0^h E.T.

Moon's H.P., π	54′ 17″·5745	$\sin \pi$	0·01579 25103
Moon's S.D., s	14 47 ·593	0·272481 $\sin \pi$	0·00430 31590

Note that s = 0″·0796 + 0·272446 π = 887″·593

The latter is used in practice; ten decimals are retained in the direct calculation to illustrate the accuracy of the approximation.

*For 1968 onwards: $\sin s = 0{\cdot}2724880 \sin \pi$
$s'' = 0{\cdot}0799 + 0{\cdot}272453\ \pi''$

The apparent right ascension and declination, which are tabulated to $0^s.001$ and $0''.01$ respectively for each hour in *A.E.*, pages 68–159, are referred to the true equinox and equator of date and are fully corrected for aberration. For 0^h and 12^h they are derived directly from the half-daily values of the apparent longitude and latitude, using the true obliquity of the ecliptic, by means of the formulae:

$$\cos\delta \cos\alpha = \cos\beta \cos\lambda$$
$$\cos\delta \sin\alpha = \cos\beta \sin\lambda \cos\epsilon - \sin\beta \sin\epsilon$$
$$\sin\delta = \cos\beta \sin\lambda \sin\epsilon + \sin\beta \cos\epsilon$$

These values are then subtabulated to twelfths to give the hourly ephemeris; the method of subtabulation is fully described in the *Improved Lunar Ephemeris* and is not illustrated here.*

Example 4.13. *Right ascension and declination of the Moon*
1960 March 7 at 0^h E.T.

$\sin\lambda$	+0·9984 7520	$\sin\beta$	−0·0910 1751	$\sin\epsilon$	+0·3978 2067
$\cos\lambda$	−0·0552 0204	$\cos\beta$	+0·9958 4929	$\cos\epsilon$	+0·9174 6319
$\cos\delta\cos\alpha$	−0·0549 7291				
$\cos\delta\sin\alpha$	+0·9484 7057	$\cot\alpha$	−0·0579 5953	α	$6^h\ 13^m\ 16^s.110$
$\sin\delta$	+0·3120 6014			δ	+18° 11′ 00″·34
Sum of squares $1{\cdot}0 - 3 \times 10^{-8}$					

The discrepancy in the sum of the squares is about as large as one can expect, though even larger values are possible.

The ephemeris transit of the Moon across the ephemeris meridian occurs when the ephemeris hour angle is either 0^h, for upper (U) transits, or 12^h, for lower (L) transits. The ephemeris hour angle is ephemeris sidereal time minus right ascension, so that the time of transit is the ephemeris time at which:

$$\text{ephemeris sidereal time} - \text{right ascension} = 0^h\ (U) \text{ or } 12^h\ (L)$$

If this time is $(P + p)^h$, where P is integral and p lies between 0 and 1, this equation may be written:

$$\begin{aligned}\text{ephemeris sidereal time at } P^h &- \text{right ascension at } P^h \\ &+ p\,(3609^s.856 - \text{hourly difference of R.A.}) \\ &- \text{second-difference correction} = 0^h \text{ or } 12^h\end{aligned}$$

The second-difference correction is small and cannot affect p by more than $0^h.00002$; it may therefore be neglected, so that p may be determined directly once the correct hour P has been selected. The ephemeris sidereal time for 0^h E.T. is the same numerically as the apparent sidereal time (hour angle of the first point of Aries) at 0^h U.T.

The times are tabulated to four decimals of an hour; this is adequate for the planning and reduction of meridian observations of the Moon but is inadequate for the derivation of a precise ephemeris at transit. As explained in sub-section F, it is preferable that the comparison of observation with theory be made by means of inverse interpolation in the hourly ephemeris.

*For *A.E.* 1972 onwards hourly values of the right ascension, declination *and* horizontal parallax are tabulated on pages 68–189. They are computed from Chebyshev series which have been derived from the half-daily values.

Example 4.14. *Lower transit of the Moon*
1960 March 7

Inspection of the ephemerides of the apparent right ascension of the Moon and of apparent sidereal time indicates that the Moon will transit the lower meridian between 7^h and 8^h.

	h m s
Apparent sidereal time at 0^h U.T.	10 58 50·9
12^h + apparent right ascension of the Moon at 0^h	18 13
Difference, indicating transit between 7^h and 8^h	7 14
Increment of sidereal time in 7^h	7 01 09·0
Apparent sidereal time at 7^h	17 59 59·9
12^h + apparent right ascension at 7^h	18 27 50·9
Difference (12^h + R.A. − S.T.)	27 51·0
Hourly difference of R.A. at $7^h = 125^s{\cdot}08$	

$$\text{Fraction of hour to ephemeris transit} = \frac{1671{\cdot}0}{3609{\cdot}86 - 125{\cdot}08} = 0^h{\cdot}4795$$

Note that only low precision is required, so that the change in nutation between 0^h and 7^h can be ignored.

Usually there are both an upper and a lower transit each day; but on one day in each month near full moon only one transit (lower) occurs and similarly on one day near new moon there is only one transit (upper).

For *A.E.* 1972 onwards, half-daily sets of polynomial coefficients for the calculation of the true geocentric distance of the Moon in units of the Earth's equatorial radius are tabulated on pages 190 to 197.

D. THE PLANETS *

Authorities

The elements and ephemerides of the *inner planets* Mercury, Venus, and Mars are obtained from the same tables as were used for the years immediately preceding 1960. The orbital longitudes and the heliocentric ecliptic longitudes referred to the mean equinox of date, the heliocentric latitudes referred to the ecliptic of date, and the radii vectores are taken from Newcomb's tables of these planets (*A.P.A.E.*, **6**, parts II, III, IV, 1895–1898); for Mars, the corrections derived by Ross (*A.P.A.E.*, **9**, part II, 1917) are applied.

The elements and ephemerides of the *outer planets* Jupiter, Saturn, Uranus, Neptune, and Pluto, beginning with 1960, are derived from the heliocentric rectangular coordinates obtained by numerical integration in *A.P.A.E.*, **12**, 1951, afterwards referred to as " Vol. XII ". Perturbations by the inner planets, taken from *A.P.A.E.*, **13**, part V, 1954, are included in the geocentric ephemerides, but are omitted from the heliocentric ephemerides, and from the heliocentric orbital elements tabulated in *A.E.*, page 177.† The geocentric right ascensions and declinations are tabulated to one more decimal, namely $0^s{\cdot}001$ and $0''{\cdot}01$ respectively, than for years preceding 1960 and for the inner planets.

*Formulae to reduce the tabulated ephemerides to the IAU system of astronomical constants are given in the *A.E.* for 1968 onwards. For 1972 onwards the ephemerides of the outer planets and minor planets are based on the IAU system.

†Page 217 in *A.E.* 1972 onwards.

In these tables and ephemerides, the values adopted for the masses of the planets, including atmospheres and satellites, are:

	Reciprocal Mass		Reciprocal Mass
Mercury	6 000 000	Uranus	22 869
Venus	408 000	Neptune	
Earth	329 390	For four inner planets ...	19 700
Mars	3 093 500	For five outer planets ...	19 314
Jupiter	1 047·355	Pluto	360 000
Saturn	3 501·6		

In the planetary theory the adopted ratio of the mass of the Earth to the mass of the Moon is 81·45; and the ratio of the mass of the Sun to the mass of the Earth alone is 333 432.

The ephemerides of the *minor planets* Ceres, Pallas, Juno, and Vesta are derived from unpublished heliocentric equatorial rectangular coordinates calculated by Herget by means of numerical integration using the Naval Ordnance Research Calculator (NORC). An adaptation of Hansen's method was used, with an interval of ten days. The integrations were adjusted along the entire orbits to the previous integrations (*A.P.A.E.*, **11**, part IV, 1950) that were used for the ephemerides before 1960; a smooth join-on at 1960 was obtained by taking most of the equations of condition near this epoch. Differences from the previous orbits are attributable to accumulation of rounding errors in the former integrations. The largest discontinuity at 1960 is 0″·07 for Vesta, which is smaller than the amounts that may be reached by the non-gravitational parts of the previous coordinates.*

Elements

An unperturbed orbit of a planet about the Sun is completely defined by six *elements*, which may be chosen in various ways and which may be referred, as desired, to any reference system. The adopted elements for the planets, as tabulated in *A.E.*, pages 176–177,† are referred to the mean equinox and ecliptic of date, and are:

i = the inclination of the orbit to the ecliptic;

Ω = the longitude of the ascending node of the orbit on the ecliptic, measured from the equinox;

ϖ = the longitude of perihelion, measured from the equinox along the ecliptic to the node, and then along the orbit from node to perihelion, i.e., $\varpi = \Omega + \omega$, where ω is the argument of perihelion;

a = the semi-major axis of the orbit; n, the mean daily motion, and a are related by $n^2a^3 = k^2(1 + m)$ where k is the Gaussian gravitational constant and m is the mass of the planet expressed in terms of the Sun's mass;

e = the eccentricity of the orbit;

M = the mean anomaly, defined by the relation $M = n$ (time in days since perihelion passage); this is related to L, the mean longitude, by the relation $L = M + \varpi$.

For the inner planets Mercury, Venus, and Mars the elements given in *A.E.*,

*For *A.E.* 1972 onwards the ephemerides of the minor planets are based on heliocentric rectangular coordinates calculated by R. L. Duncombe, *A.P.A.E.*, **20**, part II, 1969.

†Pages 216–217 in *A.E.* 1972 onwards.

page 176,* are *mean elements*. They represent, for each planet, the elements of a mean reference orbit which is used as a basis from which to derive the actual motion of the planet through the theory of general perturbations. The numerical values are determined to provide the best agreement with observation over the period on which Newcomb based his tables; and they may be used to represent the actual orbits very approximately, say to within 1′ in position. The reference orbits themselves contain small secular changes, due to the action of the other planets, but the elements i, ☊, ϖ vary more rapidly owing to the constantly changing reference system of mean equinox and ecliptic. With the exception of the mean anomalies, which are tabulated at intervals of 10 days, the elements are given in each edition of the Ephemeris for a single epoch only; this is always one of the standard 400-day dates. The variations in 100 days, almost entirely due to precession, are given for i, ☊, ϖ. The numerical values are derived from the following mean elements given in the Tables. These elements are referred to the mean equinox and ecliptic of date. The time interval from the epoch is denoted by T when measured in Julian centuries of 36525 ephemeris days, by $D = 3{\cdot}6525T$ when measured in units of 10000 ephemeris days, and by $d = 10000D = 36525T$ when measured in ephemeris days. n^* is the sidereal mean motion in a Julian year.

Epoch 1900 January 0·5 E.T. = J.D. 241 5020·0

Mean elements of Mercury

$i = 7^\circ\ 00'\ 10''{\cdot}37 + 6''{\cdot}699T - 0''{\cdot}066T^2$

$☊ = 47^\circ\ 08'\ 45''{\cdot}40 + 4266''{\cdot}75T + 0''{\cdot}626T^2$

$\varpi = 75^\circ\ 53'\ 58''{\cdot}91 + 5599''{\cdot}76T + 1''{\cdot}061T^2$

$n^* = 53\ 81016''{\cdot}3093 - 0''{\cdot}00049\ 5T \qquad a = 0{\cdot}38709\ 86$

$e = 0{\cdot}20561\ 421 + 0{\cdot}00002\ 046T - 0{\cdot}00000\ 0030T^2$

$M = 102^\circ\ 16'\ 45''{\cdot}77 + (415^r + 2\ 61055''{\cdot}04)T + 0''{\cdot}024T^2$

$ = 102^\circ{\cdot}27938\ 1 + 4^\circ{\cdot}09233\ 44364d + 0^\circ{\cdot}00000\ 050D^2$

$L = 178^\circ\ 10'\ 44''{\cdot}68 + (415^r + 2\ 66654''{\cdot}80)T + 1''{\cdot}084T^2$

$ = 178^\circ{\cdot}17907\ 8 + 4^\circ{\cdot}09237\ 70233d + 0^\circ{\cdot}00002\ 26D^2$

Mean elements of Venus

$i = 3^\circ\ 23'\ 37''{\cdot}07 + 3''{\cdot}621T - 0''{\cdot}0035T^2$

$☊ = 75^\circ\ 46'\ 46''{\cdot}73 + 3239''{\cdot}46T + 1''{\cdot}476T^2$

$\varpi = 130^\circ\ 09'\ 49''{\cdot}8 + 5068''{\cdot}93T - 3''{\cdot}515T^2$

$n^* = 21\ 06641''{\cdot}3832 + 0''{\cdot}00009\ 6T \qquad a = 0{\cdot}72333\ 16$

$e = 0{\cdot}00682\ 069 - 0{\cdot}00004\ 774T + 0{\cdot}00000\ 0091T^2$

$M = 212^\circ\ 36'\ 11''{\cdot}59 + (162^r + 7\ 12093''{\cdot}95)T + 4''{\cdot}6298T^2$

$ = 212^\circ{\cdot}60321\ 9 + 1^\circ{\cdot}60213\ 01540d + 0^\circ{\cdot}00009\ 6400D^2$

$L = 342^\circ\ 46'\ 01''{\cdot}39 + (162^r + 7\ 17162''{\cdot}88)T + 1''{\cdot}1148T^2$

$ = 342^\circ{\cdot}76705\ 3 + 1^\circ{\cdot}60216\ 87039d + 0^\circ{\cdot}00002\ 3212D^2$

Mean elements of Mars

$i = 1^\circ\ 51'\ 01''{\cdot}20 - 2''{\cdot}430T + 0''{\cdot}0454T^2$

$☊ = 48^\circ\ 47'\ 11''{\cdot}19 + 2775''{\cdot}57T - 0''{\cdot}005T^2 - 0''{\cdot}0192T^3$

$\varpi = 334^\circ\ 13'\ 05''{\cdot}53 + 6626''{\cdot}73T + 0''{\cdot}4675T^2 - 0''{\cdot}0043T^3$

$n^* = 6\ 89050''{\cdot}9262 + 0''{\cdot}00016\ 9T \qquad a = 1{\cdot}52369\ 15$

$e = 0{\cdot}09331\ 290 + 0{\cdot}00009\ 2064T - 0{\cdot}00000\ 0077T^2$

$M = 319^\circ\ 31'\ 45''{\cdot}93 + (53^r + 2\ 15490''{\cdot}60)T + 0''{\cdot}6509T^2 + 0''{\cdot}0043T^3$

$ = 319^\circ{\cdot}52942\ 5 + 0^\circ{\cdot}52402\ 07666d + 0^\circ{\cdot}00001\ 3553D^2 + 0^\circ{\cdot}00000\ 0025D^3$

$L = 293^\circ\ 44'\ 51''{\cdot}46 + (53^r + 2\ 22117''{\cdot}33)T + 1''{\cdot}1184T^2$

$ = 293^\circ{\cdot}74762\ 8 + 0^\circ{\cdot}52407\ 11638d + 0^\circ{\cdot}00002\ 3287D^2$

*Page 216 in *A.E.* 1972 onwards.

Example 4.15. *Mean elements of Venus*
1960 September 23 at 0^h E.T.

1960 Sept. 23^d 0^h E.T. J.D. 243 7200·5
Epoch of T J.D. 241 5020·0
d = 22180·5 T = 0·607269 T^2 = 0·3688

	Inclination,	Node, ☊	Perihelion, ϖ	Eccentricity, e
	° ′ ″	° ′ ″	° ′ ″	
Constant	3 23 37·07	75 46 46·73	130 09 49·8	0·00682 069
T	+ 2·199	+32 47·224	+51 18·20	−2 899
T^2	− 0·001	+ 0·544	− 1·30	+ 3
Sum	3 23 39·27	76 19 34·50	131 01 06·7	0·00679 173
=	3°·39424 2	76°·32625 0	131°·01853	

The mean distance and mean motion are practically constants.

	Mean longitude, L	Mean anomaly, M
	° ′ ″	° ′ ″
Constant	342 46 01·39	212 36 11·6
d	256 54 10·575	256 02 52·37
T^2 or D^2	+0·411	+1·71
Sum	239 40 12·38	108 39 05·7
ϖ	131 01 06·7	= 108·65158
$M = L - \varpi$	108 39 05·7	

Mean elements for the outer planets Jupiter, Saturn, Uranus, Neptune, and Pluto are not available and are not easily derivable from the numerical integrations, which do not require any reference orbits as a basis. Accordingly, *osculating elements* are given instead. Osculating elements at a particular epoch are defined as the elements of an unperturbed elliptical orbit, referred to as the osculating orbit, in which the position and velocity of the planet at the epoch are identical with the actual position and velocity of the planet in its perturbed orbit at the same instant. The osculating elements therefore contain the effects of the perturbations due to the other planets, so that, unlike the mean elements, they are subject to periodic variations. Whereas the elements of one of the inner planets in *A.E.*, * page 176, refer to a slowly varying orbit, those of one of the outer planets on page * 177 refer to a different orbit on each date, and the changes shown do not reflect the real changes of a mean orbit. Osculating elements have the advantage, however, that they may be used to give the actual position and motion of the planet at the epoch of osculation, and a good approximation to its actual orbit over short periods.

The osculating elements are tabulated in the Ephemeris, at intervals of 40 days for Jupiter, Saturn, Uranus, and Neptune and of 80 days for Pluto. There is no simple relation between the osculating and mean elements; but for comparison mean elements of Jupiter, Saturn, Uranus, and Neptune are given below. The elements of Jupiter and Saturn are taken from Hill's tables of these planets (*A.P.A.E.*, **7**, parts I and II, 1898) for the epoch of the tables and reduced to 1960 and 1970 by applying variations that approximate to those of Leverrier and Gaillot. The elements of Uranus and Neptune are taken from Newcomb's tables of the planets (*A.P.A.E.*, **7**, parts III and IV, 1898) and are affected by long-period variations. The elements are referred to the mean equinox and ecliptic of the epoch.

*Pages 216 and 217 in *A.E.* 1972 onwards.

Mean elements of the outer planets *

	Jupiter		Saturn	
Epoch	1960 Jan. 1·5	1970 Jan. 0·5	1960 Jan. 1·5	1970 Jan. 0·5
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
i	1 18 19·3	1 18 17·3	2 29 23·7	2 29 22·1
☊	100 02 40·0	100 08 43·9	113 18 26·9	113 23 41·2
ϖ	13 40 41·6	13 50 21·6	92 15 52·1	92 27 37·4
L	259 49 52·05	203 25 11·28	280 40 16·88	43 00 20·29
a	5·20280 3	5·20280 3	9·53884 3	9·53884 3
n	299″·1284	299″·1284	120″·4550	120″·4550
e	0·04843 54	0·04845 17	0·05568 18	0·05564 71
	Uranus		Neptune	
Epoch	1960 Jan. 1·5	1970 Jan. 0·5	1960 Jan. 1·5	1970 Jan. 0·5
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
i	0 46 23·0	0 46 23·2	1 46 25·5	1 46 22·2
☊	73 47 46·6	73 50 50·4	131 20 23·2	131 26 59·8
ϖ	170 00 39·3	170 10 23·9	44 16 26·1	44 21 42·2
L	141 18 17·87	184 17 24·64	216 56 27·22	238 55 24·26
a	19·18193 9	19·18188 2	30·05777 9	30·05790 0
n	42″·235	42″·235	21″·532	21″·532
e	0·04720 95	0·04723 67	0·00857 47	0·00858 24

The osculating elements are derived directly from the heliocentric equatorial rectangular coordinates as published in Vol. XII; as with the heliocentric longitudes and latitudes, the corrections due to the action of the inner planets have not been applied. The following procedure is used in principle, though modifications in detail are introduced for computational convenience.

The tabulated equatorial rectangular coordinates x_S, y_S, z_S, and their instantaneous rates of change x_S', y_S', z_S', referred to the mean equinox and equator of 1950·0, are converted directly to ecliptic rectangular coordinates and rates of change, referred to the mean equinox and ecliptic of date, by the formulae:

$$x_C = X_1\,x_S + Y_1\,y_S + Z_1\,z_S$$
$$y_C = X_2\,x_S + Y_2\,y_S + Z_2\,z_S$$
$$z_C = X_3\,x_S + Y_3\,y_S + Z_3\,z_S$$

in which the direction cosines $X_1, X_2, \ldots Y_1, \ldots Z_3$ of the ecliptic axes are calculated from:

$$X_1 = X_x \quad X_2 = X_y \cos\epsilon_M + X_z \sin\epsilon_M \quad X_3 = X_z \cos\epsilon_M - X_y \sin\epsilon_M$$

with similar equations in Y and Z, where ϵ_M is the mean obliquity of the ecliptic. Similar formulae apply for the rates of change x_C', y_C', z_C'. X_x, X_y, .. are the direction cosines of the equatorial axes for mean equinox and equator of date, referred to those for 1950·0, for which expansions are given in section 2B and which are used, though in the reverse direction, for the conversion of the Sun's coordinates in sub-section B.

The direction cosines X_1, X_2, .. can also be expressed directly as series expansions; thus use of the expressions for X_x, Y_y, .. from section 2B and for ϵ from sub-section B leads to:

*For osculating elements for Pluto see page 491.

$$
\begin{aligned}
X_1 &= +\ 9999\ 9998 + \quad 125{\cdot}5D - 2226{\cdot}0D^2 - 0{\cdot}3D^3\\
Y_1 &= + \quad 1\ 7251 - 61\ 1903{\cdot}6D - \quad 51{\cdot}1D^2 + 4{\cdot}5D^3\\
Z_1 &= + \quad 7500 - 26\ 6040{\cdot}8D + \quad 15{\cdot}3D^2 + 2{\cdot}0D^3\\
X_2 &= - \quad 1\ 8811 + 66\ 7235{\cdot}6D + \quad 40{\cdot}8D^2 - 4{\cdot}9D^3\\
Y_2 &= +\ 9174\ 3624 + \quad 2589{\cdot}4D - 2041{\cdot}5D^2 - 0{\cdot}3D^3\\
Z_2 &= +\ 3978\ 8279 - \quad 5655{\cdot}1D - \quad 887{\cdot}8D^2\\
X_3 &= - \quad 17 + \quad 608{\cdot}4D + \quad 7{\cdot}1D^2\\
Y_3 &= -\ 3978\ 8279 + \quad 5705{\cdot}2D - \quad 1{\cdot}7D^2\\
Z_3 &= +\ 9174\ 3625 + \quad 2474{\cdot}4D - \quad 0{\cdot}9D^2 - 0{\cdot}1D^3
\end{aligned}
$$

where D is measured in units of 10000 days from J.D. 243 3000·5 and where the coefficients on the right-hand side are in units of the eighth decimal.

The rates of change are to be formed with $1/K$ days as the unit of time, where $K = k(1 + m)^{\frac{1}{2}}$ in which k is the Gaussian gravitational constant and m is the mass of the planet in terms of that of the Sun as unity. Moreover, it is convenient, both for the simplification of the formulae and for numerical computation, to normalise the coordinates and rates of change by putting:

$$x = \frac{1}{r}x_0, \text{ etc. leading to } x^2 + y^2 + z^2 = 1$$

$$x' = r^{\frac{1}{2}}x'_0, \text{ etc. leading to } x'^2 + y'^2 + z'^2 = 2 - \frac{r}{a}$$

where, for simplicity, subscripts are omitted in the normalised coordinates. Since $r^2 = x_s^2 + y_s^2 + z_s^2$ is independent of the reference system it is advantageous to normalise the coordinates and rates of change before conversion. In this case the rate of change, used instead of x'_s, will be:

$$(r^{\frac{1}{2}}/K) \times (\text{the instantaneous daily rate of change of } x_s)$$

With the six values of x, y, z, x', y', z' the formulae for the elements are as follows:

$$
\begin{aligned}
x^2 + y^2 + z^2 - 1 &= 0 \text{ (check)}\\
x'^2 + y'^2 + z'^2 - 1 &= m\\
xx' + yy' + zz' &= l
\end{aligned}
$$

$$a = \frac{r}{1 - m} \qquad e^2 = l^2(1 - m) + m^2$$

$$n = Ka^{-\frac{3}{2}} \text{ radians/day} \qquad \rho^2 = 1 + m - l^2 = \frac{1 - e^2}{1 - m}$$

Take e_0^{-1} to be an integer (or, if e is large, with one decimal), constant for each planet, such that $e_0^{-1}e < 1$.

Let:

$$l_0 = e_0^{-1}l \qquad m_0 = e_0^{-1}m$$

$$\rho_1^2 = e_0^{-2}e^2\rho^2 = \{l_0^2(1 - m) + m_0^2\}\rho^2$$

Then:

$$
\begin{aligned}
p_x &= \rho(m_0 x - l_0 x') & q_x &= l_0 x + (m_0 - ll_0)x'\\
p_y &= \rho(m_0 y - l_0 y') & q_y &= l_0 y + (m_0 - ll_0)y'\\
p_z &= \rho(m_0 z - l_0 z') & q_z &= l_0 z + (m_0 - ll_0)z'
\end{aligned}
$$

Checks:

$$p_x q_x + p_y q_y + p_z q_z = 0$$

$$p_x^2 + p_y^2 + p_z^2 = q_x^2 + q_y^2 + q_z^2 = e_0^{-2}e^2\rho^2 = \rho_1^2$$

Then:

$$\tan E = \frac{l_0(1-m)^{\frac{1}{2}}}{m_0} \qquad M = E - e_0\, l_0\,(1-m)^{\frac{1}{2}} \text{ in radians}$$

$$= E - (57^\circ{\cdot}2957\,80\, e_0)\, l_0\,(1-m)^{\frac{1}{2}}$$

$$\tan\omega = \frac{p_z}{q_z}$$

$$\tan\varpi = \frac{p_y - q_x}{p_x + q_y} \qquad \Omega = \varpi - \omega$$

$$\tan v = \frac{\rho l_0}{m_0 - l l_0}$$

$v + \varpi$ is the orbital longitude, which must check with the value obtained by direct calculation.

$$\sin^2 i = \frac{p_z^2 + q_z^2}{\rho_1^2}$$

In all cases the quadrants of E, ω, ϖ, v are determined by the signs of numerator (corresponding to sine) and denominator (corresponding to cosine); i is always positive. If ρ_1 is calculated as $e_0^{-1}e\rho$ the value of e used must be taken to eight significant figures.

The formulae for e, E, v are derived from the relations:

$$e\sin E = l(1-m)^{\frac{1}{2}} \qquad e\sin v = \rho l$$
$$e\cos E = m \qquad e\cos v = m - l^2$$

There are many formulae for the three spherical elements i, Ω, ω; those given above appear to be the most suitable for small eccentricities and inclinations.

Let:

$$r_x = yz' - zy' \qquad r_y = zx' - xz' \qquad r_z = xy' - yx'$$

Checks:

$$r_x^2 + r_y^2 + r_z^2 = \rho^2$$

$$q_x r_x + q_y r_y + q_z r_z = 0 \qquad r_x p_x + r_y p_y + r_z p_z = 0$$

Then the following are the more important relations (for brevity ρ_1 is written for $e_0^{-1}e\rho$):

$$\begin{aligned} \rho_1 \sin i \sin\omega &= p_z \\ \rho_1 \sin i \cos\omega &= q_z \qquad \text{(A)} \\ \rho \cos i &= r_z \end{aligned} \qquad \begin{aligned} \rho \sin i \sin\Omega &= r_x \\ \rho \sin i \cos\Omega &= -r_y \qquad \text{(B)} \\ \rho \cos i &= r_z \end{aligned}$$

$$\begin{aligned} \rho_1 \cos i \sin\omega &= -p_x \sin\Omega + p_y \cos\Omega \\ \rho_1 \cos i \cos\omega &= -q_x \sin\Omega + q_y \cos\Omega \qquad \text{(C)} \\ \rho \sin i &= +r_x \sin\Omega - r_y \cos\Omega \end{aligned}$$

$$\begin{aligned} \rho_1 \cos i \sin\Omega &= -p_x \sin\omega - q_x \cos\omega \\ \rho_1 \cos i \cos\Omega &= +p_y \sin\omega + q_y \cos\omega \qquad \text{(D)} \\ \rho_1 \sin i &= +p_z \sin\omega + q_z \cos\omega \end{aligned}$$

$$\begin{aligned} \rho_1(1 \pm \cos i)\sin(\omega \pm \Omega) &= \pm p_y - q_x \\ \rho_1(1 \pm \cos i)\cos(\omega \pm \Omega) &= \pm q_y + p_x \qquad \text{(E)} \end{aligned}$$

in which the upper sign is normally taken.

It is clear that either (A) and (E) or (B) and (E) form a complete set of equations; (A) and (B) should not be used together if i is small since the resulting uncertainties in ω and Ω will be independent. The use of (B) to determine i and Ω is rather more logical than the use of (A) to determine i and ω, but the latter is adopted as thereby the calculation of r_x, r_y, r_z is avoided.

The precision obtainable in the angular elements Ω and ϖ is dependent on the magnitudes of i and e respectively. If the standard precision is of order 1×10^{-8}, as is the case with eight decimals in x, x', ... , and as is actually used in practice, Ω cannot be found more precisely than 10^{-8} cosec i, or $6^\circ \times 10^{-7}$ cosec i; in the worst case, for Uranus, this limit is about $0^\circ{\cdot}00005$ or $0''{\cdot}2$. Similarly, ϖ cannot be found more accurately than $10^{-8}/e$ or $6^\circ \times 10^{-7}/e$; in the worst case, for Neptune, the osculating value of e can be as small as 0·003 corresponding to a limit of precision of $0^\circ{\cdot}0002$ or $1''$. However such uncertainties only reflect the relative unimportance of precision in these quantities, and *any* values of Ω and ϖ, within the range of permitted values, will lead to reproduction of the eight-figure coordinates and velocities, *provided* that the other elements are consistent with the adopted values. For example, the orbital longitude $(v + \varpi)$ is clearly independent of the precise position of either node or perihelion, and consequently v must contain the negative of any uncertainty introduced into ϖ. One method of assuring this is to adopt the values of e and E, or of e and v, obtained directly from the coordinates and velocities, as exact; then all quantities are well determined with the exception of either ω or Ω. The longitude of perihelion $(\varpi = \omega + \Omega)$ should be consistent with the adopted values of E or v and must be evaluated from a formula that ensures this. The procedure adopted above is the same in principle, but is rather simpler in practice. The argument of perihelion (ω) is poorly determined if i is small, but the adopted value is treated as exact to give a value of Ω as $\varpi - \omega$ consistent with the well-determined value of ϖ. It is equally correct to determine Ω directly, to treat the adopted value as exact, and thus to deduce ω as $\varpi - \Omega$. The values obtained will differ in the two calculations, but the difference will not be significant. There are many other arrangements of these formulae, but with small i and e careful selection is required to ensure a consistency greater than the apparent precision.

Example 4.16. *Osculating elements of Uranus*
1960 March 7 at 0^h E.T.

Coordinates	$\delta_{-\frac{1}{2}} + \delta_{+\frac{1}{2}}$	$\delta^3_{-\frac{1}{2}} + \delta^3_{+\frac{1}{2}}$	Velocity components
x_S −13·9325 597	−0·2074 45985	+15368	$80\,Kx'_S$ −0·2074 48546
y_S +10·9322 454	−0·2327 50932	+17057	$80\,Ky'_S$ −0·2327 53775
z_S + 4·9871 943	−0·0990 61430	+ 7272	$80\,Kz'_S$ −0·0990 62642

The factor 80 arises from the use of double differences at an interval of 40 days.

r^2	338·5023 16	k	0·01720 20989 5
r	18·3984 325	m	1/22869
$r^{\frac{1}{2}}$	4·2893 394	$(1 + m)^{\frac{1}{2}}$	1·0000 21864
$\frac{1}{r}$	0·0543 52456	$\frac{r^{\frac{1}{2}}}{80\,K}$	3·1168 040

Normalised equatorial coordinates and velocities for 1950·0:

x_S/r	−0·7572 6884	$r^{\frac{1}{2}}\,x'_S$	−0·6465 7646
y_S/r	+0·5941 9439	$r^{\frac{1}{2}}\,y'_S$	−0·7254 4790
z_S/r	+0·2710 6626	$r^{\frac{1}{2}}\,z'_S$	−0·3087 5884

From example 4.9, for conversion from 1950·0 to mean equinox of date:

X_x +0·9999 9692	Y_x −0·0022 7519	Z_x −0·0009 8914
X_y +0·0022 7519	Y_y +0·9999 9741	Z_y −0·0000 0113
X_z +0·0009 8914	Y_z −0·0000 0113	Z_z +0·9999 9951

For 1960 March 7 (from *Planetary Co-ordinates*, Table I):

$\sin \epsilon_M$ 0·3978 5998 $\cos \epsilon_M$ 0·9174 4615

The coefficients for transfer of equatorial coordinates referred to 1950·0 to ecliptic coordinates referred to mean equinox of date:

X_1 +0·9999 9692	Y_1 −0·0022 7519	Z_1 −0·0009 8914
X_2 +0·0024 8090	Y_2 +0·9174 4332	Z_2 +0·3978 5875
X_3 +0·0000 0228	Y_3 −0·3978 5999	Z_3 +0·9174 4615

From these, using no subscripts for normalised ecliptic coordinates and velocities referred to mean equinox of date:

x −0·7588 8654	x' −0·6446 1853
y +0·6511 0705	y' −0·7900 0383
z +0·0122 8080	z' +0·0053 5561

It is verified that $x^2 + y^2 + z^2 - 1 = -1 \times 10^{-8}$

$$x'^2 + y'^2 + z'^2 = (r^{\frac{1}{2}} x'_S)^2 + (r^{\frac{1}{2}} y'_S)^2 + (r^{\frac{1}{2}} z'_S)^2$$

Calculation of elements from x, y, z; $x', y' z'$

x −0·7588 8654	x' −0·6446 1853	$\Sigma x^2 - 1$ — 1
y +0·6511 0705	y' −0·7900 0383	$m = \Sigma x'^2 - 1$ +0·0396 6778
z +0·0122 8080	z' +0·0053 5561	$l = \Sigma xx'$ −0·0251 1897
r 18·3984 325	e^2 0·0021 794665	ρ^2 1·0390 3682
a 19·1584 038	e 0·0466 8476	(Check) ρ^2 1·0390 3682
n 0·0002 0514 076	e_0^{-1} 20	ρ 1·0193 3156
= 0°·0117 5369 9	$e_0^{-2}\, e^2$ 0·8717 8660	$(1 - m)^{\frac{1}{2}}$ 0·9799 6542

Note: $e_0^{-2}\, e^2$ is not $(e_0^{-1}\, e)^2$ but is, in this case, $400e^2$.

m_0 +0·7933 5560	l_0 −0·5023 7940	$m_0 - ll_0$ +0·7807 3635
ρm_0 +0·8086 9240	ρl_0 −0·5120 9118	
p_x −0·9438 0924	q_x −0·1220 2815	$p_x + q_y$ −1·8876 9672
p_y +0·1219 9133	q_y −0·9438 8748	$p_y - q_x$ +0·2440 1948
p_z +0·0126 7395	q_z −0·0019 8830	
Σp^2 0·9058 1840	Σq^2 0·9058 1840	$\rho_1^2 = e_0^{-2}\, e^2 \rho^2$ 0·9058 1839

$\cot \omega$ −0·1568 81	ω 98°·9159	
$\tan \varpi$ −0·1292 6837	ϖ 172·6343 15	$\Omega = \varpi - \omega$ 73°·7184 15
$\tan v$ −0·6559 0795	v 326·7388 13	$v + \varpi$ 139 ·3731 28
$\sin^2 i$ 0·0001 8169 46	i 0·7723 37	
$\tan E$ −0·6205 4700	E 328·1784 54	
−57°·2957 80 $e_0\, l_0\, (1 - m)^{\frac{1}{2}}$	+1·4103 77	
	M 329·5888 31	L 142°·2231 46

Using $\tan \Omega = r_x / -r_y =$ +0·0131 8895/+0·0038 5213

$\cot \Omega$ +0·2920 73 Ω 73°·7183 ω 98°·9160 15

In this case the agreement between the two sets of values of Ω and ω is just about what might be expected; a discrepancy of 1×10^{-8} in q_z or r_y (which can easily arise through an accumulation of roundings) gives rise to differences of 0°·00006. The values in the Ephemeris were calculated using more than eight significant figures in the fundamental data. For the check on the orbital longitude, $v + \varpi$, see example 4.19.

(For *A.E.* 1960 the values printed on page 177 are erroneous; the correct values are given on page xii.)

The elements are given in the Ephemeris to varying numbers of decimals. For the inner planets a, n, e are given to six decimals and i, ☊, ϖ to 0°·00001; but the mean anomaly is only given to 0°·001. For the outer planets a is given to five or six decimals; n and e to seven decimals; i to 0°·00001; ☊, ϖ, and the mean anomaly are given to 0°·0001; to which precision they are determined absolutely, except possibly for Neptune, from the eight-figure calculations. The elements cannot be used, as they stand, to reproduce the planetary positions to full eight-figure accuracy.

No elements are given in the Ephemeris for the minor planets. The following are approximate elements for the epoch 1957 June 11 at 0^h E.T., referred to the mean equinox and ecliptic of 1950·0.

	Ceres	Pallas	Juno	Vesta
i	10·607°	34·798°	12·993°	7·132°
☊	80·514	172·975	170·438	104·102
ϖ	152·367	122·734	56·571	253·236
M	279·880	271·815	329·336	79·667
a	2·7675	2·7718	2·6683	2·3617
n	0°·21408	0°·21358	0°·22612	0°·27157
e	0·07590	0·23402	0·25848	0·08888

Heliocentric positions

The heliocentric ecliptic longitudes, latitudes, and radii vectores of Mercury, Venus, and Mars that are given in *A.E.*, pages 160 to 173,* are obtained directly from Newcomb's tables, with the application of Ross's corrections in the case of Mars. The longitude and latitude are rounded off to 0″·1 and are referred, as in the Tables, to the mean equinox and ecliptic of date; the radii vectores are deduced from the logarithmic values given in the Tables. They are given at intervals of one day for Mercury, two days for Venus, and four days for Mars.

The heliocentric orbital longitude, tabulated on the same pages, is the longitude of the planet in its orbit measured from the mean equinox of date along the ecliptic to the node and then along the orbit; it is derived from the Tables in the course of finding the ecliptic longitude. The difference between the orbital and ecliptic longitudes is a small quantity R, known as the reduction to the ecliptic; it is a simple function of u, the arc from node to planet, and of i, the inclination, and is given by:

$$\sin R = -\tan^2 \tfrac{1}{2}i \sin (2u + R)$$
$$R = -\tan^2 \tfrac{1}{2}i \sin 2u + \tfrac{1}{2} \tan^4 \tfrac{1}{2}i \sin 4u + \dots$$

u is obtained directly from the Tables and R is tabulated; thus:

$$\text{orbital longitude} = u + ☊$$
$$\text{ecliptic longitude} = u + ☊ + R$$

The daily motion of the orbital longitude is tabulated as a concept in its own right, and as an aid to interpolation; it is the first derivative at the instant of tabulation and in an unperturbed orbit it is inversely proportional to the square of the radius vector. The orbital latitude is the displacement in latitude of the planet from its mean (reference) orbit and comprises only the perturbations due to the other

*Pages 198 to 211 in *A.E.* 1972 onwards.

planets; it is negligibly small for Mercury, and is given to $0''{\cdot}01$ for Venus and Mars.

For computational convenience several modifications are in practice introduced in the use of the Tables, but none affects their standing as the authority; and the calculated values will differ from those obtained by formal use of the Tables by quantities which are small compared with the possible error introduced by adding a number of rounded-off contributions. No detailed derivation is given since numerical examples are included in the Tables.

Example 4.17. *Angular momentum of the inner planets*

Values of (daily motion) × (radius vector)^2 in degrees/day, showing the approximate constancy of the angular momentum.

1960	Mercury	Venus	Mars
Mar. 7	0·6001 18	0·8382 51	1·2113 93
June 15	·6001 19	·8382 49	1·2113 54
Sept. 23	·6001 24	·8382 51	1·2113 82
Dec. 32	·6001 21	·8382 42	1·2114 32

For the outer planets the similar data given in *A.E.*, pages 174–176,* at intervals of 10 days for Jupiter and Saturn, 40 days for Uranus and Neptune, and 80 days for Pluto, are derived from the heliocentric equatorial rectangular coordinates for equinox 1950·0 that are given in Vol. XII. As in the determination of the orbital elements, the tabulated coordinates (x_S, y_S, z_S) are first converted to ecliptic rectangular coordinates (x_C, y_C, z_C) referred to the mean equinox and ecliptic of date by:

$$x_C = X_1\, x_S + Y_1\, y_S + Z_1\, z_S \quad \text{etc.}$$

(Normalised coordinates are now used in practice, since they are required for the calculation of the osculating elements.) The heliocentric longitude (l_M), latitude (b_M), and radius vector (r) are then obtained from:

$$r \cos b_M \cos l_M = x_C$$
$$r \cos b_M \sin l_M = y_C$$
$$r \sin b_M = z_C$$

The intermediate values for Jupiter and Saturn are obtained by subtabulation.

Example 4.18. *Heliocentric longitude, latitude, and radius vector of Uranus*
1960 March 7 at 0^h E.T.

From example 4.16 the normalised ecliptic rectangular coordinates for mean equinox of date are: (r = 18·3984 325)

x_C −0·7588 8654	y_C +0·6511 0705	z_C +0·0122 8080
whence tan l_M −0·8579 7681		sin b_M +0·0122 8080
l_M 139° 22′ 16″·22		b_M +0° 42′ 13″·16

The orbital longitude (u + ☊) is derived from the ecliptic longitude by *subtracting* the reduction to the ecliptic (R), which is expressed in terms of the ecliptic longitude as:

$$\sin R = -\tan^2 \tfrac{1}{2}i \sin \{ 2\,(l - \Omega) - R \}$$
$$R = -\tan^2 \tfrac{1}{2}i \sin 2\,(l - \Omega) - \tfrac{1}{2} \tan^4 \tfrac{1}{2}i \sin 4\,(l - \Omega) + \dots$$

or, for Pluto for which i is large, by determining u directly from:

$$\tan u = \tan\,(l - \Omega) \sec i$$

*Pages 212 to 214 in *A.E.* 1972 onwards.

It may alternatively be derived, or checked, from the osculating elements as the sum of the true anomaly (v) and the longitude of perihelion (ϖ); and it is for this reason that the calculation of the true anomaly, which is not otherwise required, is included in that of the elements. There are other methods of linking the two calculations, and it is possible to derive the orbital longitude ($u + \Omega$) almost directly from the rectangular coordinates by:

$$r \cos (u + \Omega) = x_C + (\sec i - 1) \sin \Omega \, (x_C \sin \Omega - y_C \cos \Omega)$$
$$r \sin (u + \Omega) = y_C - (\sec i - 1) \cos \Omega \, (x_C \sin \Omega - y_C \cos \Omega)$$

Example 4.19. *Orbital longitude of Uranus*
1960 March 7 at 0^h E.T.

	°		
i	0·7723 37	$\tan^2 \frac{1}{2}i$	0·0000 45425
l	139·3711 72		
Ω	73·718		
$l - \Omega$	65·653	$\sin 2 (l - \Omega)$	+0·75119

R (in degrees) $= -0°{\cdot}0019\ 55$ (the second term is negligible.)

$l - R$ = orbital longitude $= 139°{\cdot}3731\ 27$ which may be compared with the value of $v + \varpi$ in example 4.16.

Since the orbital elements are osculating elements, the orbital latitudes are zero by definition.

Heliocentric positions of the minor planets are not tabulated.

Equatorial rectangular coordinates (x_T, y_T, z_T) referred to the true equinox and equator of date are required at daily intervals for all planets as an essential step in the calculation of the geocentric ephemerides.

For the inner planets heliocentric equatorial rectangular coordinates (x_M, y_M, z_M) referred to the mean equinox and equator of date, are first derived from the heliocentric positions given by the Tables by:

$$x_M = r \cos l_M \cos b_M = x_C$$
$$y_M = r (\sin l_M \cos b_M \cos \epsilon_M - \sin \epsilon_M \sin b_M) = y_C \cos \epsilon_M - z_C \sin \epsilon_M$$
$$z_M = r (\sin l_M \cos b_M \sin \epsilon_M + \cos \epsilon_M \sin b_M) = y_C \sin \epsilon_M + z_C \cos \epsilon_M$$

where ϵ_M is the mean obliquity of the ecliptic. These are calculated at intervals of one day.

For the outer planets the heliocentric equatorial rectangular coordinates are derived from the values given in Vol. XII, as corrected for the action of the inner planets. These coordinates (x_S, y_S, z_S), which are referred to the mean equinox and equator of 1950·0, are converted to the mean equinox and equator of date by the formulae:

$$x_M = X_x x_S + Y_x y_S + Z_x z_S$$
$$y_M = X_y x_S + Y_y y_S + Z_y z_S$$
$$z_M = X_z x_S + Y_z y_S + Z_z z_S$$

in which X_x, X_y, .. are defined in section 2B. This conversion is done at intervals of 10 days and the coordinates are then subtabulated to single days, except for Pluto. A similar procedure is followed for the minor planets, for which apparent places are calculated although not directly tabulated in the Ephemeris.

Example 4.20. *Heliocentric equatorial rectangular coordinates referred to the mean equinox of date*

Venus, 1960 March 7 at 0^h E.T.

From Newcomb's tables (see *A.E.*, page 168):

l_M	279° 36′ 23″·89	b_M	−1° 20′ 35″·23	r	0·7274 9725
$\sin l_M$	−0·9859 7672	$\sin b_M$	−0·0234 3971	$\sin \epsilon_M$	+0·3978 5998
$\cos l_M$	+0·1668 8294	$\cos b_M$	+0·9997 2525	$\cos \epsilon_M$	+0·9174 4615
x_C	+0·1213 7352	y_C	−0·7170 9828	z_C	−0·0170 5232
x_M	+0·1213 7352	y_M	−0·6511 1462	z_M	−0·3009 4929

(Check: $x_M^2 + y_M^2 + z_M^2 - r^2 = +1 \times 10^{-8}$)

Jupiter, 1960 March 7 at 0^h E.T.

Vol. XII.	x_S	−0·9094 217	y_S	−4·7990 702	z_S	−2·0365 839
Corrections		− 19		− 27		− 13
	x_S	−0·9094 236	y_S	−4·7990 729	z_S	−2·0365 852

Using values of X_x, Y_x, ... given in example 4.16:

x_M	−0·8964 875	y_M	−4·8011 273	z_M	−2·0374 783

Vesta, 1960 March 7 at 0^h E.T.

Unpublished	x_S	−0·9891 498	y_S	−1·8175 824	z_S	−0·5957 276

Using values of X_x, Y_x, ... given in example 4.16:

x_M	−0·9844 222	y_M	−1·8198 275	z_M	−0·5967 037

The coordinates (x_M, y_M, z_M) for mean equinox and equator of date, now available for 0^h E.T. for every day for all planets except Pluto, are converted to the true equinox and equator of date by means of the formulae:

$$x_T = x_M - (y_M \cos \epsilon_T + z_M \sin \epsilon_T)\, \Delta\psi$$
$$y_T = y_M + x_M \cos \epsilon_T\, \Delta\psi - z_M\, \Delta\epsilon$$
$$z_T = z_M + x_M \sin \epsilon_T\, \Delta\psi + y_M\, \Delta\epsilon$$

in which (see section 2C) ϵ_T is the true obliquity and the second-order terms are negligible. These coordinates are then combined with the similar coordinates for the Sun (X_T, Y_T, Z_T), either at the same or a separate operation, to give the geocentric coordinates (ξ_T, η_T, ζ_T). The conversion from the mean to true reference system could, of course, be done after the formation of the geocentric coordinates, and (for technical reasons) this is actually adopted for the outer planets.

Example 4.21. *Conversion of heliocentric equatorial rectangular coordinates from mean to true equinox of date*

1960 March 7 at 0^h E.T.

$\Delta\psi = -0''{\cdot}744 = -0{\cdot}0000\ 0361 \qquad \Delta\psi \sin \epsilon_T = -0{\cdot}0000\ 0144$

$\Delta\epsilon = -8''{\cdot}836 = -0{\cdot}0000\ 4284 \qquad \Delta\psi \cos \epsilon_T = -0{\cdot}0000\ 0331$

whence:

$$x_T - x_M = (+331 y_M + 144 z_M) \times 10^{-8}$$
$$y_T - y_M = (-331 x_M + 4284 z_M) \times 10^{-8}$$
$$z_T - z_M = (-144 x_M - 4284 y_M) \times 10^{-8}$$

Unit: 10^{-8}	$x_T - x_M$	$y_T - y_M$	$z_T - z_M$
Venus	− 259	−1329	+ 2772
Jupiter	−1883	−8432	+2 0697
Vesta	− 688	−2230	+ 7938

From example 4.6: X_T +0·9648 1709 Y_T −0·2134 9910 Z_T −0·0925 7863

Hence, using x_M etc. from example 4.20:

	ξ_T	η_T	ζ_T
Venus	+1·0861 8802	−0·8646 2701	−0·3935 0020
Jupiter	+0·0683 108	−5·0147 107	−2·1298 500
Vesta	−0·0196 120	−2·0333 489	−0·6892 030

As a check the values of ξ_T, η_T, ζ_T for Venus are calculated by first forming ξ_M, η_M, ζ_M and then correcting for nutation:

X_M	+0·9648 1793	Y_M	−0·2134 9194	Z_M	−0·0925 8639
ξ_M	+1·0861 9145	η_M	−0·8646 0656	ζ_M	−0·3935 3568
$\xi_T - \xi_M$	− 343	$\eta_T - \eta_M$	− 2045	$\zeta_T - \zeta_M$	+ 3548
ξ_T	+1·0861 8802	η_T	−0·8646 2701	ζ_T	−0·3935 0020

As another check the values of x_T, y_T, z_T for Jupiter are calculated directly from the corrected values of x_S, y_S, z_S using coefficients X'_x, X'_y . . . for direct conversion from the mean equinox of 1950·0 to the true equinox of date (see section 2C).

From example 4.16 and above, the modified coefficients are:

X'_x	+0·9999 9693	Y'_x	−0·0022 7188	Z'_x	−0·0009 8770
X'_y	+0·0022 7192	Y'_y	+0·9999 9742	Z'_y	+0·0000 4171
X'_z	+0·0009 8760	Y'_z	−0·0000 4397	Z'_z	+0·9999 9951
x_T	−0·8965 064	y_T	−4·8012 116	z_T	−2·0372 713

These agree within the limits of rounding-off errors.

Equatorial rectangular coordinates (x_S, y_S, z_S) referred to the mean equinox and equator of 1950·0 are also required for every tenth day for the inner planets for inclusion in the volumes of *Planetary Co-ordinates.** They are formed by the standard conversion formulae using values of l_S, b_S for the mean equinox and ecliptic of 1950·0, obtained by:

$$l_S = l_M + a - b \cos (l_M + c) \tan b_M$$
$$b_S = b_M + b \sin (l_M + c)$$

where a, b, c are the precessional constants for reduction to 1950·0.

The same formulae are used to obtain l_S, b_S for the outer planets.

Values of x_S, y_S, z_S for the inner planets could alternatively be obtained by the standard transformation from the values of x_M, y_M, z_M. The ecliptic longitude and latitude (l_S, b_S) could be derived from the usual formulae:

$$r \cos b_S \cos l_S = x_S$$
$$r \cos b_S \sin l_S = y_S \cos \epsilon_S + z_S \sin \epsilon_S$$
$$r \sin b_S = -y_S \sin \epsilon_S + z_S \cos \epsilon_S$$

where ϵ_S is the obliquity of the ecliptic at 1950·0. These formulae are used as a check on the calculations.

Geocentric positions

The apparent right ascensions and declinations of the planets, except for Pluto and the minor planets, are tabulated for 0^h E.T. on each day in *A.E.*, pages 178–233.† They are given to $0^s·01$ in right ascension and $0''·1$ in declination for the inner planets, but to an increased precision of $0^s·001$ and $0''·01$ for the outer planets. They are referred to the true equinox of date and are affected by planetary aberration.

*See page 15.

†Pages 218 to 273 in *A.E.* 1972 onwards.

The geometric values are derived from the geocentric equatorial rectangular coordinates (ξ_T, η_T, ζ_T), referred to the true equinox of date, by the formulae:

$$\Delta \cos \delta_T \cos \alpha_T = \xi_T = x_T + X_T$$
$$\Delta \cos \delta_T \sin \alpha_T = \eta_T = y_T + Y_T$$
$$\Delta \sin \delta_T = \zeta_T = z_T + Z_T$$
$$\Delta^2 = \xi_T^2 + \eta_T^2 + \zeta_T^2 \qquad \tan \alpha_T = \eta_T/\xi_T \qquad \sin \delta_T = \zeta_T/\Delta$$

where x_T, y_T, z_T are the heliocentric geometric equatorial rectangular coordinates of the planet, referred to the true equinox and equator of date, and X_T, Y_T, Z_T are the similar geocentric coordinates of the Sun, obtained as described in sub-section B. These are subsequently converted to apparent positions α_A, δ_A by the application of the correction for planetary aberration in the form:

$$\alpha_A = \alpha_T - 0{\cdot}0057683\,\Delta \times \text{instantaneous daily motion of } \alpha_T$$
$$\delta_A = \delta_T - 0{\cdot}0057683\,\Delta \times \text{instantaneous daily motion of } \delta_T \quad *$$

where Δ is the (uncorrected) geometric distance of the planet from the Earth, as tabulated.

Example 4.22. *Apparent right ascension and declination of the planets*
1960 March 7 at 0^h E.T.

From the values of ξ_T, η_T, ζ_T in the previous example we deduce:

	Venus	Jupiter	Vesta
Δ^2	2·0822 2669	29·6882 51	4·6098 932
Δ	1·4429 9227	5·4486 926	2·1470 662
0·0057 683Δ	0·0083 23	0·0314 3	0·0123 8
tan or cot α_T	t −0·7960 1965	c − 0·0136 2208	c +0·0096 4517
α_T	$21^h\ 25^m\ 55^s{\cdot}085$	$18^h\ 03^m\ 07^s{\cdot}305$	$17^h\ 57^m\ 47^s{\cdot}373$
correction*	−2 ·455	−1 ·023	−1 ·261
α_A	21 25 52 ·63	18 03 06 ·282	17 57 46 ·112
sin δ_T	− 0·2726 9737	− 0·3908 9194	− 0·3209 9755
δ_T	−15° 49′ 29″·422	−23° 00′ 36″·036	−18° 43′ 23″·747
correction*	−10 ·304	−0 ·025	+0 ·322
δ_A	−15 49 39 ·73	−23 00 36 ·06	−18 43 23 ·42

These agree with the values tabulated in the Ephemeris within legitimate variations due to small accumulations of error.

*To illustrate the application of corrections for aberration the following daily motions are taken from the ephemerides:

	Venus	Jupiter	Vesta
α	$+295^s{\cdot}0$	$+32^s{\cdot}54$	$+101^s{\cdot}88$
δ	+1238″	+0″·8	−26″·0

The tabulated values of the semi-diameters are obtained by dividing the semi-diameters at unit distance by the geocentric distance Δ; the adopted semi-diameters at unit distance and the authorities for each are as given in section 7C for the years immediately preceding 1960. The semi-diameters of the minor planets are not known with great accuracy and are not tabulated.

The (equatorial) horizontal parallaxes are the values of the solar parallax 8″·80 divided by the geocentric distance Δ. *

*0·0057756 and 8″·794 for outer and minor planets for 1972 onwards.

Example 4.23. *Semi-diameter and horizontal parallax of Venus and Jupiter*
1960 March 7 at 0^h E.T.

	Venus		Jupiter	
Distance, Δ		1·4430		5·4487
H.P. $= 8''\cdot80/\Delta$		$6''\cdot10$		$1''\cdot62$
S.D.	$8''\cdot41/\Delta$	$5''\cdot83$	$98''\cdot47/\Delta$	$18''\cdot07$
polar S.D.			$91''\cdot91/\Delta$	$16''\cdot87$

The ephemeris time of ephemeris transit is the time at which the centre of the planet is on the ephemeris meridian; it is tabulated to the nearest second, which is always adequate for the interpolation of the right ascension and declination to the time of transit. As for the Moon (sub-section C) the time of transit is the ephemeris time at which:

$$\text{ephemeris sidereal time} - \text{right ascension} = 0$$

If p is the corresponding fraction of the day, then:

$$\text{ephemeris sidereal time at } 0^h - \text{right ascension at } 0^h$$
$$+ p\,(86636^s\cdot56 - \text{following daily difference of R.A.})$$
$$- \text{second-difference correction} = 0$$

The small daily change in the equation of the equinoxes may be ignored and the second-difference correction is only appreciable for the inner planets; the correction can be expressed simply in the form of a correction to the time of ephemeris transit as:

$$+B_2(p) \times \text{double second difference of R.A. in seconds}$$

where B_2 is the Bessel second-difference interpolation coefficient. The ephemeris sidereal time at 0^h E.T. is numerically the same as the apparent sidereal time at 0^h U.T.

Example 4.24. *Ephemeris transit of Mercury*
1960 March 7

	h m s
Apparent sidereal time at 0^h U.T. (*A.E.*, page 11)	10 58 50·925
Apparent right ascension at 0^h E.T. (*A.E.*, page 179)	23 31 19·32
Difference	12 32 28·4

i.e. right ascension − ephemeris sidereal time, at 0^h E.T. $= 45148^s\cdot4$
Following daily (first) difference of R.A. $= -176^s\cdot20$
Double second difference of R.A. $= -34^s$
First approximation to $p = 45148\cdot4/(86636\cdot56 + 176\cdot20)$
$= 0\cdot520066$ $\quad B_2(p) = -0\cdot0624$

First approximation to 24^hp	12 28 53·7
Second-difference correction $= -0\cdot0624 \times -34^s$	+ 2·1
Ephemeris transit	12 28 55·8

Astrometric positions

For Pluto and the minor planets Ceres, Pallas, Juno, and Vesta, for which observations are generally made photographically, *astrometric positions* are given in * *A.E.*, pages 234–265, instead of apparent positions; the differences "apparent minus astrometric" are also given for the minor planets to enable apparent positions to be derived for comparison with meridian observations.

*Pages 274 to 307 in *A.E.* 1972 onwards.

The astrometric position of a planet is directly comparable with the mean places of stars as given in star catalogues*; it is conventionally referred at present to the mean equinox and equator of 1950·0. Since a planet is a moving object, the purely geometric position must be corrected for the effect of light-time; and this corrected position must then be adjusted by the constant part E (see section 2D) of the aberrational reduction to apparent place, which is already included in the mean places of the stars. There are several methods of calculating such an astrometric ephemeris; the simplest in principle, and the one actually adopted, is to form a geometric ephemeris for the mean equinox and equator of 1950·0, then to apply the full correction for planetary aberration, and finally to remove annual aberration calculated from the same day numbers (C, D) as used for the stars. An alternative method, to be used as a check, is to apply the correction for light-time (based on the geocentric distance) to the heliocentric rectangular coordinates before combination with those of the Sun, and to adjust the resulting geocentric position for the E-terms of annual aberration. A third method, applicable particularly to the minor planets, is to calculate apparent positions in the normal way and to reduce these to " mean places referred to mean equinox and equator of 1950·0" by the standard formulae applicable to the stars (see section 5).

For both Pluto and the minor planets heliocentric equatorial rectangular coordinates for mean equinox and equator of 1950·0 are available from the basic numerical integrations; these are combined directly with the similar coordinates of the Sun (X_S, Y_S, Z_S; see section B) to give geocentric rectangular coordinates (ξ_S, η_S, ζ_S) from which the geometric spherical coordinates (α_S, δ_S and Δ) are derived by the standard formulae. The correction for planetary aberration is applied to α_S, δ_S in the usual way by:

$$-0{\cdot}0057683\ \Delta \times \text{instantaneous daily motion}\ ^{\dagger}$$

For Pluto, for which the geometric ephemeris is calculated at intervals of four days, third differences must be taken into account in deriving the daily motions. The effect of annual aberration is then removed by applying the corrections $-(Cc + Dd)$ to right ascension, and $-(Cc' + Dd')$ to declination, where C, D are the aberrational day numbers referred to the mean equinox and equator of 1950·0, and c, d, c', d' are the " star constants " appropriate to the geometric positions.

For the minor planets columns headed " App. − Astr." are included in the Ephemeris to provide a ready means of deriving the apparent from the astrometric position; these are simply the star reductions from mean places for 1950·0 to apparent places. They are actually calculated from the differences of the apparent (α_A, δ_A) and astrometric (α_R, δ_R) positions, each taken to one more decimal than printed.

The ephemerides are given only during the period when these planets can conveniently be observed, and are therefore omitted when the planets are within about 40° of the Sun. ‡

*The star places must be corrected for proper motion and parallax.

†0.0057756 for 1972 onwards.

‡There are no omissions in *A.E.* 1972 onwards.

Example 4.25. Astrometric position of Vesta
1960 March 7 at 0^h E.T.

From example 4.20	x_S −0·9891 498	y_S −1·8175 824	z_S −0·5957 276
From example 4.9	X_S +0·9642 376	Y_S −0·2156 864	Z_S −0·0935 404
	ξ_S −0·0249 122	η_S −2·0332 688	ζ_S −0·6892 680

whence $\Delta^2 =$ 4·6098 930 $\qquad \Delta =$ 2·1470 661

$\cot \alpha_S =$ +0·0122 5229	α_S $17^h\ 57^m\ 11^s{\cdot}527$
Correction for planetary aberration (see below)	−1 ·261
− annual aberration (see below)	+0 ·326
	α_R 17 57 10 ·592
$\sin \delta_S =$ −0·3210 2784	δ_S $-18^\circ\ 43'\ 30''{\cdot}34$
Correction for planetary aberration (see below)	+0 ·34
− annual aberration (see below)	+1 ·62
	δ_R −18 43 28 ·38

For the purpose of calculating the corrections for planetary aberration the following are the daily motions in right ascension and declination, taken from *A.E.*, page 259:

$$\alpha\ +101^s{\cdot}86 \qquad \delta\ -27''{\cdot}5$$

These are to be multiplied by $-0{\cdot}0057\ 683\ \Delta = -0{\cdot}0123\ 8$

For the calculation of annual aberration, the star constants c, d, c', d' are obtained as (see section 5):

$\sin \alpha_S = -0{\cdot}9999$	$\sin \delta_S = -0{\cdot}3210$	c −0·0130	d −1·0558
$\cos \alpha_S = -0{\cdot}0123$	$\cos \delta_S = +0{\cdot}9471$	c' +0·0898	d' +0·0039
$\tan \epsilon_S = +0{\cdot}4337$	$\sec \delta_S = +1{\cdot}0559$		

From basic calculations, referred to equinox 1950·0: $C\ -18''{\cdot}244 \qquad D\ +4''{\cdot}852$

As a check, the equatorial rectangular coordinates x_S, y_S, z_S are antedated by the light-time $0^d{\cdot}01238$ (the corrections to be applied are −1342, +570, +403 × 10^{-7}) before combination with X_S, Y_S, Z_S. The resulting values of ξ_S, η_S, ζ_S (distinguished by primes) are:

$$\xi'_S\ -0{\cdot}0250\ 464 \qquad \eta'_S\ -2{\cdot}0332\ 118 \qquad \zeta'_S\ -0{\cdot}6892\ 277$$

$\Delta'^2 =$ 4·6096 124	$\Delta' =$ 2·1470 008
$\cot \alpha'_S =$ +0·0123 1864	α'_S $17^h\ 57^m\ 10^s{\cdot}615$
E-terms of stellar aberration (see below)	−0 ·024
	α_R 17 57 10 ·591
$\sin \delta'_S =$ −0·3210 1884	δ'_S $-18^\circ\ 43'\ 28''{\cdot}38$
E-terms of aberration (see below)	−0 ·00
	δ_R −18 43 28 ·38

The E-terms are calculated from the formulae:

$$\text{correction to } \alpha'_S = c\Delta C + d\Delta D \text{ where } \Delta C = -0''{\cdot}066$$
$$\text{correction to } \delta'_S = c'\Delta C + d'\Delta D \qquad \Delta D = +0{\cdot}335$$

For a further check the reduction from the apparent position to the astrometric position is calculated directly. Since the " star constants " so far available are for the mean equinox of 1950·0 they cannot be used for this calculation; independent day numbers are therefore used for the reduction from apparent place to mean equinox of 1960·0. From example 4.22:

α_A 17 57 46·112 (h m s)		δ_A $-18^\circ\ 43'\ 23''{\cdot}42$	
G 4 37 56	f $+0^s{\cdot}5070$	$\cos \delta$ +0·9471	$i - 7''{\cdot}916$
$(G + \alpha)$ 22 35 42	$\sin (G + \alpha)$ −0·3596	$\cos (G + \alpha)$ +0·9331	$g + 9''{\cdot}435$
	$\tan \delta$ −0·3389		
H 18 58 58			
$(H + \alpha)$ 12 56 44	$\sin (H + \alpha)$ −0·2450	$\cos (H + \alpha)$ −0·9695	$h + 18''{\cdot}876$
	$\sec \delta$ +1·0559	$\sin \delta$ −0·3210	

Reductions from mean to apparent place are $+0^s\cdot258$ and $+7''\cdot18$, giving the position referred to the mean equinox of 1960·0, and hence the astrometric position:

	h m s		° ′ ″
α_B	17 57 45·854	δ_B	−18 43 30·60
M −	30·734		
$N \sin \alpha_m \tan \delta_m$ −	4·529	$N \cos \alpha_m$ +	2·21
α_R	17 57 10·591	δ_R	−18 43 28·39

which are in good agreement. For the reduction to 1950·0 the formulae and constants of *A.E.*, page 50, are used:

$$M = -30^s\cdot734 \qquad N = -13^s\cdot361 = -200''\cdot42$$

Table 2.4 gives 35^s and $0''$ for the approximate precession over 10 years so that to sufficient precision:

α_m	$17^h\ 57^m\ 28^s\cdot3$	$\sin \alpha_m$	$-0\cdot99994$	$\cos \alpha_m$	$-0\cdot01103$
δ_m	$-18°\ 43'\ 31''$	$\tan \delta_m$	$-0\cdot33897$		

The reductions " apparent − astrometric " are:

$$+35^s\cdot521 \quad \text{and} \quad +4''\cdot97$$

which agree with the values printed in *A.E.*, page 259, and derived from the differences of independently calculated apparent and astrometric places.

In the ephemerides of the minor planets in the Ephemeris the dates on which the lunar inequality in right ascension attains its numerical maxima are specifically indicated.* The lunar inequality arises from the perturbations in the Earth's orbit due to the presence of the Moon; observations of right ascension at the times indicated consequently contribute the greatest weight, by this method, to the determination of the mass of the Moon.

E. EPHEMERIDES OF OTHER MEMBERS OF THE SOLAR SYSTEM

Introduction

The ephemerides in the Ephemeris are restricted to those of the Sun and Moon, the major planets and their principal satellites, and the four large minor planets; limitations of space preclude the publication in this volume of data for the other satellites, the other very numerous minor planets, the periodic comets, and the recurrent meteor showers. Ephemerides of these other members of the solar system are, however, provided annually in a number of publications, which, in conjunction with the Ephemeris, provide for the whole solar system as completely as the present state of our knowledge will allow.

Most of the minor planets are faint bodies which are observed mainly to avoid mistaken identity; only a few have orbits of unusual theoretical interest or of practical importance. Of these, (1) Ceres, (2) Pallas, (3) Juno, (4) Vesta are widely observed for their value in correcting the fundamental systems of right ascension and declination; the observations of these planets also lead to values of the elements of the Earth's orbit, and of related quantities such as the mass of Venus and the lunar inequality. A list of minor planets to be observed has been proposed in the U.S.S.R. in connection with Zverev's *Catalogue of Faint Stars*

*These dates are not specifically indicated in *A.E.* 1972 onwards.

(see section 5 and *Trans. I.A.U.*, **9**, 285, 1957); in addition to the four minor planets already mentioned, the list contains (6) Hebe, (7) Iris, (11) Parthenope, (18) Melpomene, (39) Laetitia, and (40) Harmonia. A further group of 13 minor planets, which have been shown to be useful for the determination of the mass of Jupiter, is under observation at Washington. Certain individual planets are used for special investigations; thus, (433) Eros was used for the measurement of the solar parallax, (51) Nemausa, which is always near the equator at opposition, is useful in correcting the fundamental system of declinations, and (1566) Icarus, whose perihelion is only 0·19 astronomical units from the Sun, may afford a correction to the mass of Mercury as well as an independent verification of the general theory of relativity.

Although comets are also members of the solar system, most have orbits that are so large that an assumption of parabolic motion enables observations to be represented with sufficient precision. The term " periodic comets " is therefore restricted to those comets whose periods are less than about two hundred years. A few of these have orbits of small eccentricity, so that these comets may be observed at every opposition; but most travel in highly eccentric orbits and are visible only when they approach perihelion under favourable conditions. So little is known about comets that observations of structure and brightness have an intrinsic interest, and approximate ephemerides are published to facilitate observation. Accurate measurements of position at more than one perihelion passage are, however, essential for the precise determination of an orbit; this in turn leads to a study of the past history of the comet, and to a prediction of its return after one or more revolutions.

The ephemerides of all comets, and of most of the minor planets, are calculated by applying special perturbations to an osculating orbit, the coordinates of the body being obtained at intervals of 20 or 40 days by numerical integration of the equations of motion. Since the initial elements of the orbits are often poorly determined, approximate methods are frequently used, and the resulting ephemerides are intended solely as a guide to the planning of observations. The positions are tabulated to $0^m\cdot1$ in right ascension and to $1'$ in declination, and are generally referred to the equinox of 1950·0. An essential feature of a search ephemeris is an indication of the possible error in position that may arise from an error in the mean motion; the ephemeris therefore contains the " variations " in right ascension and declination. These are the approximate changes caused by a change of one day in the time of perihelion passage, or, in the case of a minor planet, of 1° of mean anomaly.

Minor planets

Orbital elements of some precision are known for more than 1600 minor planets, but this represents less than five per cent. of the total number that a statistical estimate suggests as being within reach of modern instruments. The majority of the orbits have a small inclination to the ecliptic and a small eccentricity, and their semi-major axes lie between the limits 2·1 and 3·1 astronomical units.

These orbital elements do not exhibit a random distribution and there are correlations between them; the longitudes of perihelion, for example, show a tendency to cluster about a mean value close to that of the perihelion of Jupiter's orbit. There are a small number of exceptional orbits which do not lie within the main asteroidal belt, and some of these, e.g., (1566) Icarus, have perihelia lying inside the orbits of the Earth or even of Mercury; at the other extreme, the orbit of (944) Hidalgo is of exceptional interest because of its high inclination of 42° and large aphelion distance of 9·6 astronomical units.

The astrometric ephemerides of Ceres, Pallas, Juno, and Vesta (see subsection D) are immediately available for the comparison of observation with theory. Ephemerides of comparable precision, but usually of the geometric form, have been published at different times for (433) Eros, (173) Ino, (1566) Icarus, and (51) Nemausa. Search ephemerides for all the known minor planets are given annually in *Ephemerides of the Minor Planets* published as a co-operative effort by the Institute of Theoretical Astronomy at Leningrad; a brief introduction in English is given, but a full translation of the Russian text is also available. The volume also contains the number, name (in Roman characters), and elements of the orbits (equinox 1950·0), together with the magnitude at mean opposition distance. The approximate ephemerides of all planets which are at opposition during the year are tabulated in order of date of opposition. For economy of presentation, the data are given in a coded form, as in the following example for the year 1960:

4 Vesta		$6^{m}\cdot 2$		21°		1958
		h m		° ′		
June 5		19 08·5		−19 53		2·168
			6·4		−47	
15		19 02·1		−20 40		−8′
			8·9		−55	
25		18 53·2		−21 35		$+9^{m}\cdot 5$
	2		10·1		−56	
July 5		18 43·1		−22 31		−0′·8
			9·7		−53	
15		18 33·4		−23 24		
			8·0		−48	
25		18 25·4		−24 12		1·158

The top line gives number, name, magnitude, mean anomaly on the third date, and the year of the last observation available at the time of calculation; it is followed by geocentric positions for six standard 10-day dates for 0^{h} U.T., with first differences.. The date of opposition (day of month) is given to the left of the right ascension column; and the last column gives, in order, and for the third date: r, the variations in declination (in minutes of arc) and in right ascension (in minutes of time) for 1° change in mean anomaly, the variation in declination for a variation in right ascension of 1^{m}, and finally Δ. The volume also gives more extended ephemerides, usually for about 200 days, for those planets brighter than magnitude 11·5, and these include magnitudes and phase angles for each date. The magnitudes are calculated from a formula of the type $m = g + 5 \log r \Delta$, the constant g being given in the table of elements.

Corrections to the orbital elements are continually being made, and much of this work is now done on electronic computers, particularly those of the Minor Planet Center at Cincinnati. This centre is responsible for a large share of the minor planet calculations, and for the publication of the *Minor Planet Circulars*, which contain the latest reports of observations, discoveries, orbits, and ephemerides. The Minor Planet Center also provides a service whereby differential corrections and perturbations are quickly and efficiently calculated; the resulting values are supplied to the Astronomisches Rechen-Institut at Heidelberg and other collaborators, who calculate new elements and ephemerides for inclusion in the Russian volume. Details of the methods used at Cincinnati are given in *Minor Planet Circulars* 1504 to 1508. For results of similar work carried out in Leningrad and other centres in the U.S.S.R., reference may be made to *Astronomical Circulars of the U.S.S.R.*, and to the *Publications* of the Pulkovo and Crimean Astrophysical Observatories.

A list of mean photographic magnitudes of all numbered minor planets has been prepared by T. Gehrels and has been published in *Trans. I.A.U.*, **10**, 305-316, 1960.

Comets

Most of the known comets, travelling in large orbits of high eccentricity, are visible only in the neighbourhood of perihelion, and only a small number, with moderately eccentric orbits, can be seen at each opposition. Ephemerides of comets are therefore restricted to the period of visibility of the short-period comets. The early recovery of a comet is important in order to provide observations over as long an arc as possible, and modern instruments can detect comets of magnitude 19 or 20; this sets a limit to the period of the ephemeris, although positions are not given if the comet is within 30° of the Sun. About fifty comets, having periods less than 100 years, are kept under review, and the observations made at each return to perihelion are used to correct the orbits. The improved elements may then be used for the calculation of special perturbations and of a geometric ephemeris for the next return. Details of perturbation methods, and all necessary data, are given in the volume of *Planetary Co-ordinates*, 1960–1980.

Search ephemerides of this nature are published annually in *The Handbook of the British Astronomical Association.* The positions, referred to equinox 1950·0, are tabulated at intervals of 10 days, together with values of the heliocentric distance r, the geocentric distance Δ, the variations in right ascension and declination, and the estimated magnitude m. The law of brightness of a comet differs from that used for a minor planet, since a comet has some inherent light of its own which varies with the heliocentric distance. Magnitude formulae are therefore expressed in terms of a fourth or even sixth power of r:

$$\text{fourth power} \quad m = g + 10 \log r + 5 \log \Delta$$
$$\text{sixth power} \quad m = g + 15 \log r + 5 \log \Delta$$

Ephemerides for individual comets are also to be found in the publications of various observatories. A more general distribution of cometary information, including ephemerides, positions, orbits, and announcements of new discoveries, is made in the *I.A.U. Circulars*, the *Harvard Announcement Cards*, the *Astronomical*

Circulars of the U.S.S.R., the *Nachrichtenblatt der Astronomischen Zentralstelle* (not issued after 1959), and the *British Astronomical Association Circulars.*

Meteors

The modern study of meteors has had considerable influence on upper-atmosphere research as well as on the study of interplanetary matter and its relationship with the solar corona and the zodiacal light. The number of meteors falling on the Earth every 24 hours is estimated to be of the order of 10^{10}, and these are mainly sporadic meteors whose individual orbits and origin are unknown. Statistical methods show, however, that meteoric dust has its origin in the solar system, each individual particle travelling in an orbit about the Sun. The meteor showers which are observed annually on fixed dates occur when the Earth intersects a stream of meteoric dust. If the position of the apparent radiant of the meteors is measured and if their velocity can be determined, then it is possible to calculate the elements of the orbit of the stream; this has been achieved in a few cases by photographic or radar methods. Some of these orbits resemble those of known comets, while others, particularly those of the sporadic meteors, and the daylight showers detected by radar methods, are very much smaller and show analogies with the orbits of certain minor planets. In the absence of any precise knowledge of the structure of meteor streams or of the elements of their orbits, the published times and positions of the radiants of meteor showers are only approximate and are based entirely on observational experience. Annual predictions of this kind, with estimated hourly rates of visual meteors, are given in *The Handbook of the British Astronomical Association* and in *The Observer's Handbook* of the Royal Astronomical Society of Canada.

Artificial planets and satellites

The successful launching of artificial satellites and planets foreshadows the addition to the solar system of essentially permanent members for which orbits and ephemerides will be required. Before it is possible for such ephemerides to be included in the Ephemeris, the objects must necessarily be in orbits entirely outside the retarding influence of the Earth's atmosphere and must also have been observed accurately over a long interval. The short periods and rapid motions will make prediction uncertain and tabulation difficult. Ephemerides are first required in advance to facilitate observation and then, to a higher precision, in arrear for the interpretation and analysis of the observations. Such preliminary ephemerides can best be provided by the special organizations set up in the launching countries and elsewhere. It is only after several years, and the accumulation of a large number of accurate observations of position, that sufficiently precise values of the elements and of the gravitational field of the Earth will enable ephemerides to be constructed that can be used for a strict comparison of observation and theory.

Positional data may be required to relatively low precision for the interpretation of radio observations, even though the objects are not readily observable optically. Similar data will be required for navigational purposes if the objects are readily observable; in principle a single observation of position of a close Earth satellite, relative to the star background, will then suffice to determine the observer's position on the Earth's surface. Precise ephemerides will be necessary for use in the determination of time; the rapid motions provide in principle efficient means of determining the measure of the adopted time scale, to an observational precision many times that obtainable by observing the Moon. The adopted time scale will theoretically be connected to ephemeris time by means of a linear relation; but the coefficients of this relationship can only be determined by the analysis of a series of observations of both Sun and object over a long interval—of the order of one hundred years.

Because of the rapid development, no details are given here of the many organizations concerned with the calculation of orbits and ephemerides of the satellites and probes, none of which so far (January, 1960) satisfies the rigid conditions of long life, freedom from atmospheric perturbations, and ready observability, required to permit the calculation of a precise ephemeris. Similarly, no references are given to the published orbits and ephemerides of existing objects.

F. EPHEMERIDES AT TRANSIT

Introduction

The Sun, Moon, and planets are most generally observed at transit with meridian instruments, and it has been customary to include ephemerides at transit to allow for the requirements of setting the instruments, reducing the observations, and comparing observation with theory. Ephemerides at transit over the Greenwich meridian were given in *The Nautical Almanac* up to and including 1959 and over the Washington meridian in *The American Ephemeris* up to and including 1950. As from 1960, however, the fundamental ephemerides are tabulated in terms of ephemeris time and it is not theoretically possible to give precise transit ephemerides over particular meridians on the Earth. Moreover, it is more satisfactory and little more difficult to compare observations directly with the ephemeris for 0^h E.T. rather than through the intermediary of a transit ephemeris. For these reasons ephemerides at transit have been omitted from the Ephemeris as from 1960.

Low-precision data at transit are required for planning observations, for setting the instrument, and for reducing the observation to the geocentric apparent place of the centre of the object. These data can be obtained fairly simply, as shown below, for any individual observatory from the ephemerides for 0^h E.T.; and it is more satisfactory for the few observatories that observe on the meridian to do this rather than that a single transit ephemeris over the ephemeris meridian, which would itself need to be interpolated, be published.

U.T. of Greenwich transit

The E.T. of ephemeris transit is given in the 0^h ephemerides for the Moon and planets, and can easily be deduced for the Sun from the tabulated equation of time; as from 1965 the E.T. of ephemeris transit of the Sun will be tabulated in the Ephemeris (see section 3B.4). The longitude of the ephemeris meridian, expressed in time measure, is $1{\cdot}002738\ \Delta T$ east of the Greenwich meridian, so that the E.T. of transit over the Greenwich meridian is later than that of ephemeris transit by:

$$\Delta T/(1 - 0{\cdot}99727 \times \text{rate of change of R.A.})$$

or, approximately:

$$\Delta T\ (1 + \text{rate of change of R.A.})$$

if the right ascension is expressed in the same unit as the time interval. The second term in the bracket has maxima of about 0·05 for the Moon, and about 0·01 for the planets (Mercury). The U.T. of Greenwich transit is therefore later than the tabulated E.T. of ephemeris transit by the small quantity:

$$\Delta T \times \text{rate of change of R.A.}$$

which may at present reach about 2^s, or $0^h{\cdot}0005$, for the Moon, but which is less than $0^s{\cdot}3$ for the planets.

For the Sun the E.T. of ephemeris transit is simply 12^h − the equation of time interpolated to the time of transit; the U.T. of Greenwich transit differs only by $+0{\cdot}002738\ \Delta T$.

E.T. of transit over observer's meridian

The E.T. of transit over a meridian in west longitude λ is obtained by interpolating the E.T. of ephemeris transit to ephemeris longitude $\lambda^* = \lambda + 1{\cdot}002738\ \Delta T$, that is to the fraction λ^{*r} towards the next following E.T. of transit, where λ^{*r} is λ^* expressed as a fraction of a revolution. Since the interval between consecutive transits is always about 24^h (25^h for the Moon), an error in the estimate of ΔT affects directly the E.T. of transit. It is therefore interesting to examine the precision required in E.T. in order to interpolate the ephemerides to full tabular precision; this is given in the table below.

Body	Tabular precision α	δ	transit	Precision of E.T. required
	s	″		
Sun	0·01	0·1	—	3^s
Moon	0·001	0·01	$0^h{\cdot}0001$	$0^h{\cdot}000003$
,,	(0·01	0·1		·00003)
Inner planets	0·01	0·1	1^s	1^s
Outer planets	0·001	0·01	1^s	1^s
,, ,,	(0·01	0·1	1^s	7^s) (Jupiter)
Minor planets	0·01	0·1	1^s	3^s

ΔT is known several years in advance to barely sufficient accuracy, but (except for the Moon) the E.T. of ephemeris transit is given to sufficient precision to enable precise transit ephemerides to be formed when a reliable estimate of ΔT is available.

U.T. of transit over observer's meridian

The U.T. of transit over a meridian in west longitude λ is:

$$\text{E.T. of ephemeris transit} - \Delta T + \lambda^{\star r}(24^h + d) + \text{second-difference correction}$$

where d is the excess over 24^h of the difference to the next following (for west longitudes) transit. This may be written:

$$\text{E.T. of ephemeris transit} + 0{\cdot}002738\, \Delta T + \lambda + \lambda^{\star r} d + \text{second-difference correction}$$

It is to be noted that this time depends only slightly on ΔT. This is equivalent to interpolating E.T. of ephemeris transit to longitude λ (not $\lambda^\star$) and applying the small correction:

$$(1{\cdot}002738\, \Delta T^d)\, d + 0{\cdot}002738\, \Delta T$$

in which ΔT^d is ΔT expressed as a fraction of a day; this correction can generally be ignored, except possibly for the Moon.

The local mean time is U.T. $- \lambda$ and is obtained by omitting the term " $+\lambda$ " in the above expression for U.T.; this is equivalent to interpolating the tabular E.T. of ephemeris transit, ignoring the change of day or the increment of 24^h, to longitude $\lambda^\star$ (not λ), and applying the small correction $+0{\cdot}002738\, \Delta T$. The local mean time can be converted to standard time by the application of the difference of longitude from the standard meridian.

Approximate transit ephemerides

The errors introduced by linear interpolation, ignoring second and higher differences, are largest for Mercury, for which they may reach 6^s in right ascension and $1'$ in declination; but for Venus they are only about 1^s and $0'{\cdot}1$ respectively, and are less for the Sun and the other planets. It is therefore suggested that for the purpose of planning observations and setting instruments linear interpolation to the E.T. of local transit will suffice for the Sun and planets, with perhaps an approximate allowance for second differences for Mercury (when observable). For a precision of about 1^s in right ascension and $0'{\cdot}1$ in declination (say 3^s and $0'{\cdot}5$ for Mercury) E.T. need be known only to the nearest 5^m, or $0^d{\cdot}003$. The interpolating factor in the 0^h ephemerides is simply the fraction of the day, as given by the E.T. of ephemeris transit; this can be formed, to $0^d{\cdot}001$ for convenience, at a wide interval and the first-difference corrections to the 0^h values of right ascension and declination calculated at the same interval. The corrections can then be subtabulated and applied to the 0^h values to provide the approximate ephemerides required. These calculations are best done systematically for each observatory.

For the Moon the corresponding precision of E.T. is about $0^h{\cdot}003$ or 15^s, but $0^h{\cdot}01$ will probably suffice; an allowance for ΔT is necessary but no great precision is required, and for any observatory a fixed value of $\lambda^\star$ can be used for many years.

If the difference between the E.T. of upper (U) transit and that of the preceding or following lower (L) transit is $12^h + d$, then the increment to be applied to the tabular E.T. of the upper transit, to give (to this precision) the E.T. of upper transit over the local meridian, is $\lambda^* + \frac{\lambda^{*h}}{12}d$, where λ^{*h} is expressed in hours. The day and hour is then used to select the appropriate entry in the hourly ephemeris, and the fraction of the hour used for linear interpolation. Such an approximate ephemeris would require correction for parallax in declination (as well as for refraction) before being used for the setting of instruments.

If only the sidereal time of meridian passage is required it can be obtained from the U.T. of transit by means of the tabular relationship between S.T. and U.T. given in *A.E.*, pages 10–17. The precision will be the same as that to which the U.T. of transit is known, namely 1^s for the planets and $0^h{\cdot}0001$ or $0^s{\cdot}4$ for the Moon; to this precision second differences are required in interpolation for the Moon and Mercury. The local mean time of transit (t) can be converted directly to local apparent sidereal time as:

$$\text{Greenwich apparent sidereal time at } t + 0{\cdot}002738\,\lambda$$

where the variation in the equation of the equinoxes is neglected and the term $0{\cdot}002738\,\lambda$ is the reduction for longitude tabulated in *A.E.*, Table IX. It should be noticed that the small term $+0{\cdot}002738\,\Delta T$ in the expression for the local mean time of transit can be incorporated by replacing λ by λ^*. To the precision of about 1^s considered here the local apparent sidereal time can be calculated directly from the local mean time of transit, or from the U.T. of transit, by means of a linear relation of the form $a + bt$.

For observations of the limbs of the Sun and Moon approximate values of the semi-diameters in arc and in time (sidereal time of semi-diameter passing the meridian) are also required.

Reduction of observations

Apart from instrumental factors, corrections for the following may be necessary before an observation of the limb of the Sun, Moon, or planet at transit can be compared with the apparent geocentric position of its centre: refraction, parallax, vertical semi-diameter, sidereal time of semi-diameter passing the meridian, and phase or defective illumination of the limb. Refraction and parallax, considered in sections 2E and 2F, affect declination only and are not further dealt with here, except in so far as the equatorial horizontal parallax at transit is required. Corrections for semi-diameter must be made for both right ascension (or time of transit) and declination, unless both limbs are observed and the mean taken; in any case values of these semi-diameters at transit are required.

The following table gives approximately the precision required in E.T. of transit in order to obtain the horizontal parallax (H.P.) and semi-diameters (S.D.) sufficiently precisely for the reduction of observations.

Body	Precision required S.D.		Precision required H.P.	Precision required in E.T.		
	$''$	^{s}	$''$			
Sun	0·1	or 0·01	0·1	1^h		
Moon	0·01	0·001	0·01	5^s	or	$0^h{\cdot}002$
,,	0·1	0·01	0·1	1^m	or	$0^h{\cdot}02$
Inner planets	0·1	0·01	0·1	5^h		
Outer planets	0·01	0·001	0·01	5^h		

There is thus never any difficulty in determining the appropriate times and interpolating factors, though an estimate of ΔT is necessary for the Moon. For the Sun and planets the interpolations of the horizontal parallax and the semi-diameters are trivial, but second differences must be used for the Moon. Ephemerides can be systematically prepared by observatories engaged in meridian observations.

The semi-diameter and horizontal parallax are merely the values at 0^h interpolated to the time of transit. The sidereal time of the semi-diameter passing the meridian is to be calculated from the formula:

$$\text{equatorial S.D.} \times \sec\,(\text{declination at transit}) \times S$$

where
$$S = \frac{1}{1 - \Delta\alpha^s/3609{\cdot}86}$$

and $\Delta\alpha^s$ is the hourly rate of change of the right ascension in seconds at the time of transit. If the equatorial semi-diameter is expressed in seconds of arc it may be converted to seconds of time by dividing by 15. For all bodies except the Moon S may be expanded as:

$$S = 1 + \Delta\alpha^s/3609{\cdot}86$$

and an approximate value used for $\Delta\alpha^s$. But for the Moon it is necessary to use both the exact formula and the value of $\Delta\alpha^s$ interpolated to the time of transit.

The parallax correction $\Delta\delta$ to be applied to the observed declination δ is given (section 2F) by:

$$\tan \Delta\delta = \frac{\rho \sin \pi \sin (\phi' - \delta_0)}{1 - \rho \sin \pi \cos (\phi' - \delta_0)}$$

where π, δ_0 are the geocentric horizontal parallax and declination, and ρ, ϕ' are the geocentric radius and latitude of the observer.

If δ is known, then $\Delta\delta$ is given by:

$$\sin \Delta\delta = \rho \sin \pi \sin (\phi' - \delta)$$

or, with an error not exceeding $0''{\cdot}04$,

$$\Delta\delta = 0{\cdot}999988\, \rho\, \pi \sin (\phi' - \delta)$$

The apparent topocentric semi-diameter is always greater than the geocentric value since the observer must be closer to the body observed; it is, however, only for the Moon that this augmentation is significant. Clearly it does not affect the sidereal time of the semi-diameter passing the meridian, but it must be taken into account for the vertical semi-diameter. It may most easily be allowed for by combining it with the correction for parallax (see section 2F) by applying the parallax correction directly to the observed northern or southern limb. The augmented semi-diameter at transit may however be calculated directly from the rigorous equations of section 2F, putting $h = h_0 = 0$. The resulting expression for G, by which the geocentric semi-diameter is to be multiplied to give the topocentric value, may be used in different forms according to requirements:

$$G = \frac{\sin(\phi' - \delta)}{\sin(\phi' - \delta_0)} \quad \text{or} \quad \frac{\cos(\delta - \delta_0)}{1 - \rho \sin\pi \cos(\phi' - \delta_0)}$$

or, with adequate accuracy for the Moon:

$$G = 1 + \rho \sin\pi \cos(\phi' - \delta_0) + (\rho \sin\pi)^2 \{ 1 - \tfrac{3}{2} \sin^2(\phi' - \delta_0) \} + \dots$$

It may be verified that the difference between the parallax corrections for the centre and limb is the augmentation.

For the planets augmentation is negligible and the parallax correction may be simplified to:

$$\Delta\delta = \rho\, \pi \sin(\phi' - \delta)$$

where $\phi' - \delta$ is the zenith distance. However, for Jupiter and Saturn the disk of the planet is not circular, and allowance must be made both for the ellipticity of the planet and the orientation of the axis of rotation. If P is the position angle of the axis of rotation (see section 11) the apparent vertical and horizontal semi-diameters at transit are given approximately by:

$$\begin{aligned} \text{vertical S.D.} &= (\text{polar S.D.}) \times (1 - e^2 \sin^2 P)^{-\frac{1}{2}} \\ \text{horizontal S.D.} &= (\text{polar S.D.}) \times (1 - e^2 \cos^2 P)^{-\frac{1}{2}} \\ &= (\text{equatorial S.D.}) \times (1 - e^2 \sin^2 P)^{+\frac{1}{2}} \end{aligned}$$

where e is the eccentricity and the inclination of the axis towards the observer is neglected. For Jupiter and Saturn $e^2 = 0{\cdot}129$ and $0{\cdot}199$, and P may reach 25° and 7°. The correcting factor $(1 - e^2 \sin^2 P)^{-\frac{1}{2}}$ can be taken as unity for Saturn but may reach an extreme of 1·01 for Jupiter and should strictly be taken into account. The following are the factors by which the polar semi-diameter, tabulated in the Ephemeris, should be multiplied to give the apparent vertical and horizontal semi-diameters; the last line of the table gives the factors by which the sidereal time of semi-diameter crossing the meridian, calculated as above, should be multiplied:

	Jupiter						Saturn
± P	0°	5°	10°	15°	20°	25°	—
Vertical	1·000	1·001	1·002	1·004	1·008	1·012	1·000
Horizontal	1·071	1·071	1·069	1·066	1·062	1·058	1·117
Time of crossing	1·000	0·999	0·998	0·996	0·992	0·988	1·000

A correcting factor should also be applied to measurements of Saturn's rings; this is generally necessary only when the rings are open and the north or south limb of the planet is hidden from view. The eccentricity of the rings is given by:

$$e^2 = (a^2 - b^2)/a^2$$

where a, b are the axes tabulated in *A.E.*, pages 374 and 375.* The vertical semi-diameter then becomes:

$$\tfrac{1}{2}\, b\, (1 - e^2 \sin^2 P)^{-\frac{1}{2}}$$

and may reach 1·011 times the tabulated value.

*Pages 410 and 411 in *A.E.* 1972.

If the limb of the Moon or of a planet is not fully illuminated, a correction for phase can in some cases be applied. For the planets (see section 11) the corrections may be incorporated with those for semi-diameter by using, in place of the full values:

$$\text{sidereal time of S.D. passing meridian} \times (1 - \tfrac{1}{2}\sin^2 i \sin^2 \Theta)$$
$$\text{vertical S.D.} \times (1 - \tfrac{1}{2}\sin^2 i \cos^2 \Theta)$$

in which i, the angle between the Earth and Sun as seen from the planet, is less than 90° (i.e. the planet is gibbous) and Θ is the position angle of the mid-point of the bright limb tabulated in the Ephemeris for the inner planets. For Mars, Jupiter, and Saturn the position angle of the greatest defect of illumination is tabulated in the Ephemeris, and differs from Θ by 180°; it may therefore be used in place of Θ in the above expressions.

For Mercury and Venus a correction in declination may also be applied when the planets are in the crescent phase. This occurs when i is greater than 90°, and the defective illumination of the north or south limb reduces the tabulated semi-diameter to:

$$\text{semi-diameter} \times \sin \Theta$$

where $\sin \Theta$ is taken as positive in all cases.

The corrections for phase may also be applied directly, especially if both limbs are observed and the mean taken; they may be deduced from the formulae above.

Similar formulae apply to the Moon, but the more rapidly changing phases make it necessary to specify the form of correction more exactly. Limb I (preceding or west) is illuminated between new moon and full moon, and limb II (following or east) is illuminated between full moon and new moon. A correction for defective illumination of the non-illuminated limb may be applied near full moon, the difference of right ascensions of the Sun and Moon at transit being usually limited to the range $11^h 40^m$ to $12^h 20^m$. The appropriate reduction to the right ascension or time of transit of the centre of the Moon is then:

$$\text{sidereal time of S.D. passing the meridian} \times \{ 1 - \tfrac{1}{2}\sin^2 (\alpha - \alpha_{\odot}) \cos^2 \delta_{\odot} \}$$

where $\delta_{\odot}$ is the Sun's declination and $\alpha - \alpha_{\odot}$ is the difference between the right ascensions of the Moon and the Sun; the defective illumination is:

$$\text{semi-diameter} \times \tfrac{1}{2}\sin^2 (\alpha - \alpha_{\odot}) \cos^2 \delta_{\odot}$$

The corresponding modified vertical semi-diameter for a non-illuminated north or south limb, for which a correction for defective illumination can be made in certain circumstances, is

$$\text{semi-diameter} \times (1 - \tfrac{1}{2}\sin^2 \psi)$$

where ψ is the altitude of the Sun above the horizon as seen from the north point of the Moon's disk and is given by:

$$\sin \psi = \sin \delta_{\odot} \cos \delta - \cos \delta_{\odot} \sin \delta \cos (\alpha - \alpha_{\odot})$$

or, with the restricted range of $\alpha - \alpha_{\odot}$ noted above:

$$\psi = \delta_{\odot} + \delta = \delta_{\odot} + \delta_0 - \Delta\delta$$

where $\Delta\delta$ is the parallax correction. If ψ is positive the north limb is illuminated and the correction for defective illumination is applied to the south limb, and vice versa. This procedure should not be used if ψ exceeds about 3°. The defective illumination in declination is:

$$\text{semi-diameter} \times \tfrac{1}{2} \sin^2 \psi$$

Example 4.26. *The transit of the Moon at Washington*
1960 August 7

The coordinates of Washington are taken from example 2.1 and ΔT is assumed to be $+36^s$.

Geographic longitude	$\lambda = +5^h\, 08^m\, 15^s{\cdot}75$	$= 5^h{\cdot}13771$	$\frac{1}{12}\lambda^h = 0{\cdot}42814$
Ephemeris longitude	$\lambda^* = \lambda + 1{\cdot}002738\, \Delta T$	$= 5^h{\cdot}14774$	$\frac{1}{12}\lambda^{*h} = 0{\cdot}42898$

E.T. of local transit. From *A.E.*, page 61:
Ephemeris upper transit occurs at August $6^d\, 23^h{\cdot}9333$
Interpolation for λ^* gives August $7^d\, 05^h{\cdot}2903 = 7^d{\cdot}2204$

U.T. of local transit may be obtained from U.T. = E.T. $- \Delta T = 5^h{\cdot}2803$
or from interpolation of ephemeris transit for λ, giving: $5^h{\cdot}2798$
The correction $(1{\cdot}002738\, \Delta T$ in days$)\, d + 0{\cdot}002738\, \Delta T$
$= 0{\cdot}00042 \times 0^h{\cdot}96 + 0^h{\cdot}00003 = 0^h{\cdot}00043$

Interpolation of the ephemeris to E.T. of local transit gives:

A.E., page 122.	Approximate right ascension	$21^h\, 11^m\, 29^s{\cdot}1$
	Approximate declination	$-13°\, 31'\, 32''$
A.E., page 61.	Semi-diameter	$16'\, 32''{\cdot}01 = 992''{\cdot}01$
	Equatorial horizontal parallax	$60'\, 40''{\cdot}85$

Local sidereal time of transit.

Local mean time of transit = 1960 August	$7^d\, 00^h{\cdot}1426$
Greenwich apparent sidereal time at 0^h	$21^h\, 02^m\, 03^s{\cdot}8$
Interval of $0^h{\cdot}1426$	+ 8 33 ·4
Increment (*A.E.*, Table IX)	+ 1 ·4
Reduction for longitude λ (*A.E.*, Table IX)	+50 ·6
	21 11 29 ·2

Sidereal time of semi-diameter passing the meridian.

A.E., page 122.	$\Delta\alpha$ interpolated to time of transit	$= 149^s{\cdot}815$
	$S = 1/(1 - \Delta\alpha^s/3609{\cdot}86)$	$= 1{\cdot}0432\ 99$
	$\frac{1}{15} \sec \delta$	$= 0{\cdot}0685\ 67$
	Semi-diameter, S.D.	$= 992''{\cdot}01$
	Sidereal time of S.D. passing the meridian	$= 70^s{\cdot}964$

The error introduced by using for $\Delta\alpha$ the tabulated first difference $(= 149^s{\cdot}761)$ would amount in this example to $0^s{\cdot}001$, but might reach $0^s{\cdot}005$ in extreme cases.

Parallax.

ϕ'	$+38°\, 43'\, 53''$	ρ	$0{\cdot}9986\ 91$
δ_0	$-13\ 31\ 32$	π	$60'\, 40''{\cdot}85$
$\phi' - \delta_0$	$+52\ 15\ 25$		
		$\sin \pi$	$0{\cdot}0176\ 504$
$\sin(\phi' - \delta_0)$	$+0{\cdot}7907\ 64$	$\rho \sin \pi$	$0{\cdot}0176\ 273$
$\cos(\phi' - \delta_0)$	$+0{\cdot}6121\ 21$	$(\rho \sin \pi)^2$	$0{\cdot}0003\ 107$
$A \equiv \rho \sin \pi \sin(\phi' - \delta_0)$	$+0{\cdot}0139\ 390$	$\tan \Delta\delta = A/(1 - B)$	$+0{\cdot}0140\ 910$
$B \equiv \rho \sin \pi \cos(\phi' - \delta_0)$	$+0{\cdot}0107\ 900$	$\Delta\delta$	$+\ 48'\, 26''{\cdot}3$
		$\delta = \delta_0 - \Delta\delta$	$-14°\, 19'\, 58''$

Augmentation.

ϕ'	$+38° \; 43' \; 53''$	$\sin(\phi' - \delta)$	$+0{\cdot}7993 \; 09$
δ	$-14 \;\; 19 \;\; 58$	$\cos(\phi' - \delta)$	$+0{\cdot}6009 \; 20$
$\phi' - \delta$	$+53 \;\; 03 \;\; 51$	$G = \sin(\phi' - \delta)/\sin(\phi' - \delta_0)$	$1{\cdot}0108 \; 06$

Geocentric semi-diameter $992''{\cdot}01 = 16' \; 32''{\cdot}01$

Topocentric semi-diameter $1002''{\cdot}73 = 16' \; 42''{\cdot}73$

The two alternative methods give similar results:

			G
$C \equiv \cos \Delta\delta$	$0{\cdot}9999 \; 01$	$C/(1 - B)$	$1{\cdot}0108 \; 08$
$D \equiv 1 - \frac{3}{2}\sin^2(\phi' - \delta_0)$	$+0{\cdot}0620 \; 38$	$1 + B + D(\rho \sin \pi)^2$	$1{\cdot}0108 \; 09$

Defective illumination.

Transit occurs 3^h after full moon, therefore the east limb (limb II) is illuminated.

A.E., page 27. At transit, Sun's right ascension $= 9^h \; 08^m \; 38^s$

Sun's declination $= +16° \; 25' \; 41''$

$\alpha - \alpha_\odot$ $\;12^h \; 02^m \; 51^s$		
$\sin(\alpha - \alpha_\odot)$ $-0{\cdot}0124$	$\sin \delta_\odot$ $+0{\cdot}2828$	$\sin \delta$ $-0{\cdot}2476$
$\cos(\alpha - \alpha_\odot)$ $-0{\cdot}9999$	$\cos \delta_\odot$ $+0{\cdot}9592$	$\cos \delta$ $+0{\cdot}9689$

Defective illumination of west limb $= \frac{1}{2}\sin^2(\alpha - \alpha_\odot)\cos^2\delta_\odot \times$ S.D. $= 0^s{\cdot}005$

$\sin\psi = +0{\cdot}0365$ (alternatively, $\psi = 16° \; 26' - 14° \; 20' = 2° \; 06'$, and $\sin\psi = +0{\cdot}0366$)

Since ψ is positive, the north limb is illuminated, and the correction is to the south limb.

Defective illumination of south limb $= \frac{1}{2}\sin^2\psi \times$ S.D. $= 0''{\cdot}66$

G. COMPARISON OF OBSERVATION WITH THEORY

The ephemerides of the Sun, Moon, and planets in the Ephemeris are intended to form a consistent basis for the comparison of observation with theory. Within the limitations of the theories on which they are based, they only depart from the actual positions *systematically* through errors in the constants adopted for their orbital elements and masses; they are calculated to a precision such that the inevitable random errors due to roundings at various stages are quite negligible compared to the random errors of observation. However, the departures from the actual positions may in some cases be quite large: this is particularly so for the minor planets, which may be as much as $5''$ away from the ephemeris positions, and for Pluto, whose right ascension is now (1960) about $0^s{\cdot}5$ greater than the tabular values. These departures are systematic, and do not affect the comparison; but corrections to the ephemerides may be necessary when extreme precision, such as for planetary occultations, is required.

In general the ephemerides consist of apparent positions referred to the true equinox of date, so that they are as directly comparable with observation as is practicable. The adopted procedure is to apply such corrections as are necessary to the observed position, which is then compared directly with the ephemeris position, interpolated to the time (E.T., or, more strictly, U.T. with an approximate correction to E.T.) of observation. In the case of meridian observations, corrections may have to be applied for instrumental errors, refraction (section 2E), parallax (section 2F), semi-diameter and defective illumination (sub-section F), diurnal aberration (section 2D), and latitude variation, if not included with parallax; when thus corrected they are directly comparable with the ephemerides.

The observed positions of faint objects, particularly of those that cannot readily be observed on the meridian, are found by differential methods in which the differences between the coordinates of the moving object and those of nearby stars are measured. In the reduction of a photographic plate the effects of differential refraction and aberration are allowed for in the plate constants, and the coordinates of the moving object are obtained directly in the same form as those of the reference stars and referred to the same equinox and equator. Since the positions of the reference stars are usually mean places taken from a fundamental catalogue, the observed position is an astrometric position; when reduced to the standard equinox of 1950·0 (if not already for that equinox) it is directly comparable with the astrometric ephemeris. Differential precession and nutation do not enter in the reduction of a photographic plate, but a correction for parallax must be applied to the observed position.

A micrometer measure of the position of a moving object with respect to a neighbouring star is sometimes made visually, and this also leads to a position that is comparable with an astrometric ephemeris. Corrections are strictly necessary, though often negligible in practice, for the differential variations in refraction, aberration, precession, and nutation between the positions of the star and the moving object; they should be applied to the position of the star together with the differences of the coordinates.

In all cases it is desirable to apply corrections to the observed position so that it is directly comparable with the ephemeris; but in general the residuals are so small that corrections can be applied, with reversed signs, to the ephemeris position if for some reason this is more convenient.

The ephemerides are tabulated in terms of E.T. and the observations are made in terms of U.T. Generally ΔT (= E.T. − U.T.) is one of the unknowns to be determined by analysis of the residuals. For relatively slow-moving objects ΔT may be sufficiently well known for the U.T. of observation to be corrected to E.T. For fast-moving objects, such as the Moon, for which ΔT is the principal unknown, the residuals themselves can be expressed in time by finding, by inverse interpolation in the ephemeris, the E.T. corresponding to the observed coordinate; ΔT then enters the equations of condition with coefficient unity.

The differences between the observed and calculated positions are normally expressed in the form of O − C (observed − computed) residuals in $\Delta\alpha \cos \delta$ and $\Delta\delta$. Over an interval in which the effect of errors in the adopted constants varies linearly with the time, a number of such residuals can be combined by taking their weighted mean to give a mean residual applicable to the average time of observation. This is then equated to the linear combination of the errors in the unknowns; from these equations of condition, each with its appropriate weight, normal equations are formed and solved for the unknowns.

When neither an apparent nor an astrometric ephemeris is available, as for minor planets other than those given in the Ephemeris, the comparison may have to be made with a geometric ephemeris or even with a heliocentric geometric ephemeris. The geometric ephemeris has to be corrected for the effect of light-time (τ) by interpolating it to time $t_U - \tau + \Delta T$ where t_U is the observed U.T. of observation; in most cases this will be an adequately precise correction for aberration, but a more accurate procedure may be used if necessary (see section 2D).

For geocentric ephemerides O − C residuals can be formed as in the following table; ΔT may be either a definitive value or an approximation to the actual value to be determined from the observations.

Observed position at U.T. = t_U	Corrections to be applied	Ephemeris position	Interpolated to E.T.
apparent	—	apparent	$t_U + \Delta T$
,,	− (precession and nutation; annual aberration)	astrometric	$t_U + \Delta T$
,,	− (precession and nutation)	geometric	$t_U - \tau + \Delta T$
astrometric	+ (precession and nutation; annual aberration)	apparent	$t_U + \Delta T$
,,	—	astrometric	$t_U + \Delta T$
,,	+ (annual aberration)	geometric	$t_U - \tau + \Delta T$

The positive sense of the corrections is that adopted in reducing star positions from mean to apparent place; the positive correction for precession and nutation is that from the mean equinox of 1950·0 (or the mean equinox to which the astrometric or geometric ephemerides are referred) to the true equinox of date.

When a geocentric ephemeris is not available the following procedure may be used. The Sun's equatorial rectangular coordinates X, Y, Z for E.T. $= t_U + \Delta T$ are combined geometrically with the (ephemeris) heliocentric rectangular co-ordinates x, y, z of the moving object, referred to the same equinox, but for E.T. $= t_U - \tau + \Delta T$. The resulting right ascension and declination differ from those in an astrometric ephemeris by the small E-terms of aberration (see section 2D), and are thus not directly comparable with an observed astrometric position; even when referred to the true equinox of date they are not directly comparable with an observed apparent position. The O − C residuals are thus formed as:

$$\left\{\begin{array}{l}\text{observed astrometric}\\ \text{position at U.T.} = t_U\end{array}\right\} - \left\{\begin{array}{ll}\text{ephemeris position derived from} & \text{E-terms}\\ X, Y, Z \text{ at E.T.} = t_U + \Delta T & + \quad \text{of}\\ \text{and } x, y, z \text{ at E.T.} = t_U - \tau + \Delta T & \text{aberration}\end{array}\right\}$$

The E-terms of aberration, in the sense implied above, are approximately:

in right ascension $-0''\!\cdot\!341 \sin(11^h\,15^m + \alpha) \sec\delta$

in declination $-0''\!\cdot\!341 \cos(11^h\,15^m + \alpha) \sin\delta - 0''\!\cdot\!029 \cos\delta$

The small errors in this procedure are usually negligible.

The form of the equations of condition and their solution are not discussed here.

5. MEAN AND APPARENT PLACES OF STARS

A. MEAN PLACES AND STAR CATALOGUES

The mean place of a star is its heliocentric position referred to a specified mean equinox and equator, generally that of the beginning of a Besselian solar year. At the stated epoch of observation (which is almost always reduced in star catalogues to that of the reference system) it represents the geometric direction of the star, modified conventionally by:

(a) the effect of secular aberration, which is unknown;
(b) the E-terms of aberration (see section 2D).

In all star catalogues, both observed and fundamental, the E-terms of aberration are included in the mean places. The rigorous calculation of a mean place for any other equinox would therefore necessitate the removal of the E-terms before applying precession (and proper motion); and the correct E-terms, calculated afresh for the new position and epoch, would then be put back in the mean place. Such rigorous methods need only be used in the case of a close circumpolar star for which precession is applied over a long period of time, and particularly where the resulting mean place is to be used in the formation of a fundamental catalogue, or in the study of proper motions.

In normal cases, where the mean place is to be derived from a fundamental catalogue, it is sufficiently accurate to apply precession to the catalogue mean place, the E-terms being regarded as constant. The principal errors introduced by this procedure are: (1) errors in the E-terms caused by the variation of the elements of the Earth's orbit, and (2) cross-terms involving precession and the E-terms. The errors are similar in form, but opposite in sign, and the total error is therefore small. In the region of the pole, the maximum centennial errors arising in this way are:

in $\cos\delta\, \Delta\alpha$	(1) $0^s{\cdot}0006$	(2) $0^s{\cdot}0005$	Total $0^s{\cdot}0001$
in $\Delta\delta$	(1) $0''{\cdot}009$	(2) $0''{\cdot}008$	Total $0''{\cdot}002$

Thus the simple procedure is adequate in all cases where the time interval is not too long.

Mean places of stars are in fact deduced from observed apparent positions by removing the effects of annual parallax and stellar aberration, allowing where necessary for proper motion and orbital motion, and transforming to the adopted mean equinox by removing precession and nutation from that mean equinox to the true equinox of date. They are therefore to be regarded as fundamental reference data, with no simple geometric significance, in which observations at different times and from different places may be combined and from which the apparent positions of stars may conveniently be derived. No attempt is made to correct for the proper motion or any orbital motion of the star during the time taken by light to travel from it; since the light-times differ from star to star, the directions of stars represented by their mean places for a particular epoch do not form a consistent pattern at that or at any other epoch.

For double and multiple systems mean places generally refer to the centre of gravity of the system, but are sometimes given for individual stars of a system. The mean places of a star for different epochs but the same mean equinox differ only in respect of the proper motion of the star (and exceptionally for orbital motion) during the interval. In reducing the mean place of a star from the mean equinox and epoch (t_0) of one date to another (t), proper motion (referred to the mean equinox of t_0) during the interval ($t - t_0$) should first be applied, to be followed by the reduction for precession. Rigorous formulae are given in section 2B for this reduction, but as there shown the corrections may be expanded in a power series in the time interval ($t - t_0$); the coefficients in such series may be calculated from the formulae in section 2B or taken, for certain specified epochs, from special tables such as Peters' 'Präzessionstafeln für das Äquinoktium 1950·0' (*Veröffentlichungen des Astronomischen Rechen-Instituts zu Berlin-Dahlem*, no. 50, 1934). The coefficients of ($t - t_0$) and $(t - t_0)^2$ are known as the annual and secular terms of precession. Similar quantities are given in star catalogues, but these generally include the effect of proper motion, and are then known as annual variation and secular variation respectively; in some cases the coefficient of the third term is also published. It is to be noted that there has been in the past some confusion of practice in that some catalogues tabulate the coefficient of $(t - t_0)^2$, while others give double this value; in order to avoid this confusion the International Astronomical Union has recommended (*Trans. I.A.U.*, **6**, 336, 1939) that the exact form of precession terms should be made clear by a formula printed at the foot of each page.

Most star catalogues are observational catalogues published by individual observatories. The mean places which they contain are derived from observations made at the one observatory and may contain systematic errors peculiar to the instruments and observing methods used; the observations are reduced to a common mean equinox and combined to give a mean place for the mean epoch of observation; available proper motions may be used to reduce the positions to a common epoch. In contrast, fundamental catalogues are derived by the combination of all available observational catalogues, which are collated to provide indications of systematic, as well as accidental, error; positions and derived proper motions are usually given for the epoch of a standard mean equinox such as that of 1950·0.

Example 5.1. *Mean places of α Centauri for* 1960·0 *and* 1961·0

The mean place and centennial proper motion of α Centauri for 1950·0 are given in FK3 (no. 538) together with the centennial variations as follows:

α_0	$14^h\ 36^m\ 11^s{\cdot}309$	δ_0	$-60°\ 37'\ 49''{\cdot}26$
$\dfrac{d\alpha}{dT}$	$+\ 408{\cdot}006$	$\dfrac{d\delta}{dT}$	$-\ 1485{\cdot}80$
$\dfrac{d^2\alpha}{dT^2}$	$+\ 7{\cdot}398$	$\dfrac{d^2\delta}{dT^2}$	$+\ 34{\cdot}75$
μ_α	$-\ 48{\cdot}890$	μ_δ	$+\ 70{\cdot}76$
$\dfrac{d\mu_\alpha}{dT}$	$-\ 0{\cdot}716$	$\dfrac{d\mu_\delta}{dT}$	$-\ 3{\cdot}37$

The transformation to an equinox t years after 1950·0 is made through:

$$\alpha_t = \alpha_0 + \frac{t}{100}\frac{d\alpha}{dT} + \tfrac{1}{2}\left(\frac{t}{100}\right)^2 \frac{d^2\alpha}{dT^2}$$

$$\mu_t = \mu_0 + \frac{t}{100}\frac{d\mu}{dT}$$

where t is in tropical years and T in tropical centuries of 100 tropical years, with similar expressions for the declination. The value of $d\mu/dT$ used in the last equation should strictly be that for the middle of the interval, and this can be derived, if necessary, from the values given in FK3.

	1960·0	1961·0		1960·0	1961·0
α_0	$14^h\ 36^m\ 11^s{\cdot}309$	$11^s{\cdot}309$	δ_0	$-60°\ 37'\ 49''{\cdot}26$	$37'\ 49''{\cdot}26$
$\left(\dfrac{t}{100}\right)\dfrac{d\alpha}{dT}$	$+\ 40{\cdot}8006$	$44{\cdot}8807$	$\left(\dfrac{t}{100}\right)\dfrac{d\delta}{dT}$	$-\ 2\ 28{\cdot}580$	$2\ 43{\cdot}438$
$\tfrac{1}{2}\left(\dfrac{t}{100}\right)^2\dfrac{d^2\alpha}{dT^2}$	$+\ 0{\cdot}0370$	$0{\cdot}0448$	$\tfrac{1}{2}\left(\dfrac{t}{100}\right)^2\dfrac{d^2\delta}{dT^2}$	$+\ 0{\cdot}174$	$0{\cdot}210$
α	$14\ 36\ 52{\cdot}1466$	$56{\cdot}2345$	δ	$-60\ 40\ 17{\cdot}666$	$40\ 32{\cdot}488$
$\mu_{\alpha 0}$	$-\ 0^s{\cdot}48890$	$-\ 0^s{\cdot}48890$	$\mu_{\delta 0}$	$+\ 0''{\cdot}7076$	$+\ 0''{\cdot}7076$
$\left(\dfrac{t}{100}\right)\dfrac{d\mu_\alpha}{dT}$	$-\ 0{\cdot}00072$	$-\ 0{\cdot}00079$	$\left(\dfrac{t}{100}\right)\dfrac{d\mu_\delta}{dT}$	$-\ 0{\cdot}0034$	$-\ 0{\cdot}0037$
μ_α	$-\ 0{\cdot}48962$	$-\ 0{\cdot}48969$	μ_δ	$+\ 0{\cdot}7042$	$+\ 0{\cdot}7039$

Lists of star catalogues for the eighteenth and nineteenth centuries are given in the volumes of *Geschichte des Fixsternhimmels* (Karlsruhe; 1922-1957; Berlin 1952-1959). A further list for the period 1900-1925 is given in *Index der Sternörter* 1900-1925 (Bergedorf, 1928). Foremost among the observational catalogues in these lists are those compiled under the auspices of the Astronomische Gesellschaft (A. G.) as a co-operative effort by a number of observatories. This series of volumes was begun in 1863, and gives the positions of all stars shown in the Bonn Durchmusterung (B. D.) from declination +80° to −18° to magnitude 9·0. The observations made by each observatory were confined to a narrow zone of declinations best suited to the latitude of the observatory. In more recent times a re-observation of the A.G. zones has been undertaken by photography, and the new positions are given in:

AGK2 'Zweiter Katalog der Astronomischen Gesellschaft für das Äquinoktium 1950'; there are ten volumes covering declinations +90° to +20° (Hamberg-Bergedorf, 1951-1954), and five volumes covering declinations +20° to −2° (Bonn, 1957-1958).

Yale — Catalogues of the zones +50° to +60° and −30° to +30° are published in *Transactions of the Astronomical Observatory of Yale University* (New Haven, 1925 onwards); the positions are also for equinox 1950·0.

Among modern fundamental catalogues, the following are representative:

GC — 'General catalogue of 33342 stars for the epoch 1950', in 5 volumes, Washington, 1937. This gives positions and proper motions of all stars brighter than magnitude 7, with some thousands of fainter stars.

FK3 — 'Dritter Fundamentalkatalog des Berliner Astronomischen Jahrbuchs' I Teil: *Veröffentlichungen des Astronomischen Rechen-Instituts*, no. 54, 1937. This gives positions for 1925·0 and 1950·0 of the 925 stars of Auwers, 'Neue Fundamentalkatalog' (1910).
II Teil: *Abhandlungen der Preussischen Akademie der Wissenschaften, Phys.-Math. Klasse*, no. 3, 1938. This gives the positions of 666 additional stars for equinox 1950·0.

FK4 — 'Fourth Fundamental Catalogue (FK4)', *Veröffentlichungen des Astronomischen Rechen-Instituts, Heidelberg*, no. 10, 1963. This resulted from a revision of FK3; the re-examination of the available observations showed that no change in the equinox was justified.

N30 — 'Catalog of 5268 standard stars, 1950·0, based on the normal system N30', *Astronomical Papers of the American Ephemeris*, **13**, part III, 1952. The positions are derived from more than 70 catalogues with epochs of observation between 1917 and 1949.

ZC — 'Catalog of 3539 zodiacal stars for the equinox 1950·0', *Astronomical Papers of the American Ephemeris*, **10**, part II, 1940. This gives positions of all stars to magnitude 7 in the zodiacal zone with many fainter ones, and is intended for use in occultation work; it is based on 90 catalogues, and positions are reduced to the FK3 system.

PFKSZ — 'A preliminary general catalogue of fundamental faint stars between declinations +90° and −20°', M. S. Zverev and D. D. Polozhentsev, *Publications of the Main Astronomical Observatory of Pulkovo*, **72**, 1958.

In the Ephemeris, pages 288 to 298 in 1960,* the mean places of 1078 stars are tabulated for the beginning of the Besselian year, to a precision of $0^s\!.1$ in right ascension and 1″ in declination; this is adequate for purposes of identification and setting of telescopes. The list includes all bright stars to a limiting magnitude of 4·75, excepting 8 stars within 30″ of a brighter tabulated star; variable stars are included if their maxima are brighter than magnitude 4·7. The positions are derived from GC, and are tabulated in order of mean right ascension for the equinox and epoch of 1950·0. Approximate values of the changes due to precession in one year are given in table 2.4. The three-letter abbreviations that are used for the constellation names are as recommended by the International Astronomical Union and are given in section 18. Some stars are identified by their numbers in the following catalogues: Bonn Durchmusterung (north of −23°), Cordoba Durchmusterung (south of −23°), Bradley (Br.), Piazzi (Pi.), Gould (G.), and Hevelius (H.). The magnitude (to the nearest tenth except for variable stars) and spectral type of each star are given in the list.

*Pages 332 to 342 in *A.E.* 1972 onwards.

B. APPARENT PLACES

The apparent place of a star is the geocentric position, referred to the true equinox and equator of date, in which the star would be observed. It differs from the position in which the star is actually observed by the effects of refraction and diurnal aberration (see section 2D); geocentric parallax is, of course, negligible. An observed position is therefore directly comparable with an apparent place after correction for instrumental errors, and for refraction (section 2E) and diurnal aberration (section 2D) where appropriate.

No apparent places of stars are provided in the Ephemeris since they are available in *Apparent Places of Fundamental Stars*. This contains the positions of 1535 stars taken from FK3,* for each upper transit at Greenwich for the 52 circumpolar stars with declinations greater than $\pm 81°$; and for every tenth upper transit for the remaining 1483 stars. The positions are given to $0^s{\cdot}001$ in right ascension ($0^s{\cdot}01$ for the circumpolar stars) and $0''{\cdot}01$ in declination, and in the tabulation of the 10-day stars, first differences are provided. The positions on intermediate dates may be obtained by interpolation, using second differences, this being possible because in calculating these positions the effect of nutation on the frame of reference is restricted to long-period terms only. Special provision is made for the calculation of the effect of short-period terms of nutation.

The apparent places are given in order of right ascension from 0^h to 24^h, and are tabulated for every tenth Greenwich transit. In the volumes for 1941 to 1959, the first tabulated transit is that transit that occurred after the transit of the first point of Aries on January 0^d (i.e. at about 17^h); the first entry therefore varied from January $0^d{\cdot}7$ U.T. for the first star to January $1^d{\cdot}7$ for the last star. As from the volume for 1960, a continuous form of tabulation has been adopted (*Trans. I.A.U.*, **9**, 90, 1957) and positions are given for Greenwich transits occurring on Greenwich sidereal dates whose integral part is divisible by 10. The U.T. of transit is indicated as before, with the months given by roman numerals.

Similar but shorter lists of apparent places of FK3 stars are also published in *Connaissance des Temps* (Paris), *Almanaque Náutico* (San Fernando), *Japanese Ephemeris* (Tokyo), *Astronomical Yearbook U.S.S.R.* (which also includes a number of stars not given in FK3) and, in years prior to 1960, in *The American Ephemeris* and in *Berliner Astronomisches Jahrbuch*.

To meet the requirements of observers with photographic zenith telescopes or prismatic astrolabes, the apparent places or times of transit of the selected stars are often calculated by the national ephemeris offices and supplied to the individual observatories. Reductions from apparent to mean place are increasingly being calculated by electronic computers for each individual observation in preference to the systematic pre-calculation previously adopted.

*From FK4 in *A.P.F.S.* 1964 onwards.

C. REDUCTION FROM MEAN TO APPARENT PLACE

The calculation of the apparent place of a star for date $t + \tau$ (where t represents the beginning of a Besselian solar year and τ a fraction of a tropical year) first necessitates the calculation of the mean place for mean equinox and epoch of t. The reduction then involves the application of corrections for precession from the beginning of year to date (i.e. for the interval τ), nutation, stellar aberration, annual parallax, proper motion, and orbital motion.

Proper motion, orbital motion, and stellar aberration do not affect the frame of reference, but cause changes in the actual direction in which the star is observed; the corresponding corrections must therefore be calculated with respect to a particular reference system and applied to the position of the star in the same system. Precession and nutation, however, are changes in the frame of reference and do not affect the actual direction in which the star is observed: these two corrections, and that for annual aberration, are sufficiently large to make their order of application of significance if cross-product terms are neglected. The corrections for parallax, proper motion, and orbital motion are generally very small and can be applied at any convenient stage.

Since nutation is calculated from the longitudes of the Sun and Moon referred to mean equinox of date, it is (theoretically) necessary to apply precession before nutation. There are then two methods for calculating the reduction to apparent place: if precession and nutation are applied first (method i) then the aberration correction should strictly be applied to a fixed star whose coordinates (referred to the moving frame of reference) are continuously changing; if aberration is applied first (method ii) then the corrections for precession and nutation should strictly be applied to the changing positions of the star. The two methods give, of course, identical results, but for systematic calculation it is practically essential to apply corrections for precession, nutation, and aberration to a fixed star, any residual corrections (if appreciable) being applied separately. As might be expected, the largest correcting term is the same for both methods, but the other terms differ. An analysis of the magnitude of the residual terms, taken in conjunction with the second-order terms of precession, nutation, and aberration themselves, shows conclusively that the second method leads to smaller residual errors (Porter, J. G., and Sadler, D. H. The accurate calculation of apparent places of stars. *M.N.R.A.S.*, **113**, 455-467, 1953). (Although not really relevant, it is also to be noted that it is more logical to apply aberration with respect to a fixed frame of reference.)

The present availability of fast computing machines has now made possible the transformation from mean to apparent place by rigorous formulae that do not involve expansion in series (see section 2B). Such methods are also used for the individual calculation of the mean place from the observed apparent place.

Reduction for precession and nutation

From sections 2B and 2C it is seen that, to the first order, the combined reduction for precession and nutation from the mean equinox of t to the true equinox of $t + \tau$ is:

in right ascension $\quad (m + n \sin \alpha \tan \delta)\left(\tau + \frac{\Delta\psi}{\psi'}\right) - \cos \alpha \tan \delta \, \Delta\epsilon + \lambda' \frac{\Delta\psi}{\psi'}$

in declination $\quad n \cos \alpha \left(\tau + \frac{\Delta\psi}{\psi'}\right) + \sin \alpha \, \Delta\epsilon$

where α, δ is the mean place for epoch t; m, n, λ', ψ' are the precessional constants defined in section 2B, and $\Delta\psi$, $\Delta\epsilon$ are the (total) nutation in longitude and obliquity respectively.

These expressions may be written in the forms:

in right ascension $\quad Aa + Bb + E \quad = f + g \sin (G + \alpha) \tan \delta$

in declination $\quad Aa' + Bb' \quad = \quad g \cos (G + \alpha)$

where A, B, E are known as Besselian day numbers, defined by:

$$A = n\tau + n \frac{\Delta\psi}{\psi'} = n\tau + \sin \epsilon \, \Delta\psi$$

$$B = -\Delta\epsilon$$

$$E = \lambda' \frac{\Delta\psi}{\psi'}$$

and f, g, G are independent day numbers, derived from:

$$f = \frac{m}{n} A + E = m\tau + \cos \epsilon \, \Delta\psi$$

$$g \sin G = B \qquad\qquad g \cos G = A$$

and a, a', b, b' are star constants, defined by:

$$a = \frac{m}{n} + \sin \alpha \tan \delta \qquad\qquad a' = +\cos \alpha$$

$$b = \cos \alpha \tan \delta \qquad\qquad b' = -\sin \alpha$$

Note that they are constant only in so far as they are calculated for the mean equinox of a fixed epoch.

An approximate correction for precession and nutation from the mean equinox of 1950·0 to the true equinox of date may be made by means of Table IV in the Ephemeris.

A more rigorous reduction for the combined effect of precession and nutation gives rise to the second-order terms numbered 1 to 4 in table 5.1.

Reduction for annual aberration

The first-order expressions for annual aberration given in section 2D are functions of the components of the Earth's velocity (x', y', z') or of the corresponding solar components ($-X'$, $-Y'$, $-Z'$). With the approximation $z' = y' \tan \epsilon$, the reduction for annual aberration may be written in the form:

in right ascension $\quad Cc + Dd = h \sin (H + \alpha) \sec \delta$

in declination $\quad Cc' + Dd' = h \cos (H + \alpha) \sin \delta + i \cos \delta$

where C and D are the Besselian day numbers, calculated from the ratios of $+y'$ (or $-Y'$) and $-x'$ (or $+X'$) to the adopted value of the velocity of light; and c, c', d, d' are star constants, defined by:

$$c = \cos \alpha \sec \delta \qquad c' = \tan \epsilon \cos \delta - \sin \alpha \sin \delta$$
$$d = \sin \alpha \sec \delta \qquad d' = \cos \alpha \sin \delta$$

In the alternative form of reduction, the independent day numbers h, H, i are defined by:

$$h \sin H = C \qquad h \cos H = D \qquad i = C \tan \epsilon$$

The aberrational day numbers C and D tabulated in the Ephemeris are calculated from the true motion of the Earth referred to a fixed equinox, and to the centre of gravity of the solar system. In all ephemerides prior to 1960 it was customary to calculate them from expressions involving the Sun's longitude, and this process involved assumptions which are discussed in section 2D.

The rigorous reduction of the formulae for aberration gives the additional second-order terms numbered 5 to 8 in table 5.1.

Combined reduction

The total effect of precession, nutation, and annual aberration is given by the sum of the terms already discussed; to the first order the combined reduction is:

in right ascension $Aa + Bb + Cc + Dd + E$
$$= f + g \sin (G + \alpha) \tan \delta + h \sin (H + \alpha) \sec \delta$$

in declination $Aa' + Bb' + Cc' + Dd'$
$$= g \cos (G + \alpha) + h \cos (H + \alpha) \sin \delta + i \cos \delta$$

For full precision, the second-order terms should be included; for practical reasons the star constants must strictly be regarded as constants for a sequence of dates (that is, regarded as being calculated for a fixed position referred to a fixed equinox). Thus there arise additional second-order terms when the two corrections are combined; these additional terms differ according to the method employed, there being fewer terms of this kind when method (ii) is used. The additional terms in $\Delta\alpha \cos \delta$ are:

method (i) $fh \cos (H + \alpha) + gh \sin (G + H + 2\alpha) \tan \delta$

method (ii) $+ gh \sin (G + H + 2\alpha) \tan \delta + gi \sin (G + \alpha)$

and the additional terms in $\Delta\delta$ are:

method (i) $-fh \sin (H + \alpha) \sin \delta - gh \sin (G + \alpha) \sin (H + \alpha) \sec \delta$
$+ gh \cos (G - H) \cos \delta - gi \cos (G + \alpha) \sin \delta$

method (ii) $-gh \sin (G + \alpha) \sin (H + \alpha) \sec \delta$

The terms arising in method (ii) are included in table 5.1 (terms 9 and 10) which therefore gives all the second-order terms that can arise when this method is used.

It will be seen from the table that terms 3, 5, and 9 are the most significant terms; these involve tan δ or sec δ, and some form of correction for them is essential

Table 5.1. Second-order terms in star reductions

Term No.	$\Delta\alpha \cos\delta$	$\Delta\delta$
1	$+fg \cos(G+\alpha) \sin\delta$	$-fg \sin(G+\alpha)$
2	$-\frac{1}{2} fg \cos G \cos\alpha \sin\delta$	$+\frac{1}{2} fg \cos G \sin\alpha$
3	$+\frac{1}{2}g^2 \sin 2(G+\alpha) \tan\delta \sin\delta$	$-\frac{1}{2}g^2 \sin^2(G+\alpha) \tan\delta$
4	$+\frac{1}{2}g^2 \cos\alpha \sin(2G+\alpha) \cos\delta$	—
5	$+\frac{1}{2}h^2 \sin 2(H+\alpha) \sec\delta$	$-\frac{1}{2}h^2 \sin^2(H+\alpha) \tan\delta$
6	—	$+\frac{1}{2}h^2 \cos^2(H+\alpha) \sin 2\delta$
7	—	$+hi \cos(H+\alpha) \cos 2\delta$
8	—	$-\frac{1}{2}i^2 \sin 2\delta$
9	$+gh \sin(G+H+2\alpha) \tan\delta$	$-gh \sin(G+\alpha) \sin(H+\alpha) \sec\delta$
10	$+gi \sin(G+\alpha)$	—

for stars of high declination. The form of correction discussed by Fabritius (*Ast. Nach.*, **87**, 113 and 129, 1876) may be used; if $\Delta\alpha$ and $\Delta\delta$ are the first-order corrections, then the complete reductions are:

in right ascension $\quad \Delta\alpha + \Delta\alpha\, \Delta\delta \tan\delta$

in declination $\quad \Delta\delta - \frac{1}{2}(\Delta\alpha)^2 \sin\delta \cos\delta$

This method is inadequate in principle, since the formulae apply strictly to the solution of a single spherical triangle, and not to the complicated geometry of precession and nutation. The Fabritius method removes the terms dependent on $\tan\delta$ and $\sec\delta$ but introduces several other terms which are, however, independent of $\sec\delta$.

The most advantageous method of correcting the first-order reduction is to introduce additional day numbers J and J', which can be tabulated in the ephemerides, to give additional reductions in the form:

in right ascension $\quad +J \tan^2\delta$

in declination $\quad +J' \tan\delta$

The full expressions for J and J' are derived from terms 3, 5, and 9, replacing $\sec\delta$ by $\pm\tan\delta$, with an error that vanishes at the poles. This gives:

$$J = +\{g \sin(G+\alpha) \pm h \sin(H+\alpha)\}\{g \cos(G+\alpha) \pm h \cos(H+\alpha)\}$$
$$= +\{(A \pm D) \sin\alpha + (B \pm C) \cos\alpha\}\{(A \pm D) \cos\alpha - (B \pm C) \sin\alpha\}$$
$$J' = -\tfrac{1}{2}\{g \sin(G+\alpha) \pm h \sin(H+\alpha)\}^2$$
$$= -\tfrac{1}{2}\{(A \pm D) \sin\alpha + (B \pm C) \cos\alpha\}^2$$

the upper sign being taken for positive declinations, and the lower sign for negative declinations.* These day numbers may be tabulated as simple functions of right ascension and date, so that the complete reduction may be made in one operation. The method is therefore of considerable advantage in the routine calculation of a number of star places; in the Fabritius method, the second-order corrections cannot be made until the first-order terms have been calculated.

A full discussion of these corrections is given in 'The accurate calculation of apparent places of stars' (*M.N.R.A.S.*, **113**, 455-467, 1953). In this paper the expressions for J and J' (equations 14 and 15 on page 460) were given for

*See note on page 523.

northern declinations only; for southern declinations the sign of the term in $2gh$ in both equations should be reversed. Also, in the second-order term no. 12 of Table I, $(1 - \sin \delta)$ should be replaced for southern stars by $(1 + \sin \delta)$. The substance and conclusions of the paper are unaffected by these omissions. In particular, the discussion clearly shows the advantages of restricting the range of τ to $\pm\frac{1}{2}$; this has the effect of reducing all the second-order terms in f and g, which are functions of time. A more detailed analysis of the magnitude of the neglected terms in different methods has confirmed the conclusion that if τ is allowed to reach $+1$, and no second-order corrections are applied, there are unavoidable errors of 0″·010, even at declinations of 45°. A maximum error of this magnitude may also be reached (at the poles) when the Fabritius method of correction is used, but this maximum is reduced to 0″·007 when the tabulated values of J and J' are used. If τ is restricted to $\pm\frac{1}{2}$ these maxima are reduced to 0″·005 using the Fabritius method and 0″·003 using J and J', while the range of declinations over which second-order corrections may be neglected is correspondingly increased. This is shown in the following table, which gives the upper limit of declination for a given error, when no second-order correction is applied.

Limiting error	0″·007	0″·008	0″·009	0″·010	0″·012	0″·014	0″·016	0″·018	0″·020
$0 < \tau < +1$	5°	25°	35°	43°	49°	54°	57°	60°	63°
$-\frac{1}{2} < \tau < +\frac{1}{2}$	57	62	67	70	72	74	76	78	81

As a result of this analysis, it is considered that the best choice is method (ii) with τ restricted to $\pm\frac{1}{2}$ and the provision of tabulated values of J and J' for the correction of second-order terms; this has been adopted in the Ephemeris.

Proper motion, orbital motion, and parallax

In all cases where the proper motion of a star is known and is appreciable, a correction for the change of position should be included both in the mean place and in the star reductions. The components of proper motion in right ascension μ_α and in declination μ_δ are given in the catalogues, and are incorporated wholly or partially in the positions. In fundamental catalogues the epoch to which proper motion is included is identical with that of the equinox to which the catalogue refers, but in observational catalogues the epoch may differ from that of the equinox to which the positions are referred. In all cases, the correction to the position consists of the product of the proper motion and the number of years from the epoch to the required date. In some catalogues the secular variations of the proper motions are also tabulated and in such cases the mean value of the proper motion during the interval is to be used.

The remaining correction for the fraction of the year τ is incorporated in the star reductions:

in right ascension $+\tau\mu_\alpha$ in declination $+\tau\mu_\delta$

in which the value of the proper motion is that for the year and not for the original epoch.

In a few cases a correction for orbital motion is necessary, but this can generally be considered to vary linearly with the time during the course of a year. The corrections in right ascension and declination for the beginning of each year from 1925 to 1950 are tabulated, for the four stars so affected in FK3, in an appendix to that catalogue, and these have been extended for later years by values supplied by the Astronomisches Rechen-Institut.

The correction for annual parallax may be included with that for annual aberration, as shown in section 2F. If π is the annual parallax of the star, the combined aberration-parallax corrections become:

$$\text{in right ascension} \quad (C + \pi Y)\, c + (D - \pi X)\, d$$
$$\text{in declination} \quad (C + \pi Y)\, c' + (D - \pi X)\, d'$$

A more convenient form of these expressions may be used in cases where the parallax is small; the corrections then become:

$$\text{in right ascension} \quad C\,(c + d\pi \sec \epsilon/k) + D\,(d - c\pi \cos \epsilon/k)$$
$$= C\,(c + 0{\cdot}0532\, d\pi) + D\,(d - 0{\cdot}0448\, c\pi)$$
$$\text{in declination} \quad C\,(c' + d'\pi \sec \epsilon/k) + D\,(d' - c'\pi \cos \epsilon/k)$$
$$= C\,(c' + 0{\cdot}0532\, d'\pi) + D\,(d' - 0{\cdot}0448\, c'\pi)$$

where k is the constant of aberration = 20″·47, and π is expressed in seconds of arc. These formulae assume a mean value of unity for the Sun's radius vector (section 2F); the error is negligible if the parallax of the star is less than about 0″·2. *

D. DAY NUMBERS

As a result of resolutions adopted at the 1952 meeting of the International Astronomical Union (*Trans. I.A.U.*, 7, 75-76, 1950; **8**, 67, 1954) changes have been made as from 1960 in the definitions of the day numbers and star constants. These changes, together with the considerations of the previous sub-section, lead to the definitions and methods of derivation that follow. Two complete examples of the use of these day numbers in the calculation of the apparent places of stars are given at the end of this sub-section.

Besselian day numbers

The Besselian day numbers are now defined by:

$$A = n\tau + n\,\frac{\Delta\psi}{\psi'} = n\tau + \sin\epsilon\, \Delta\psi$$

$$B = -\Delta\epsilon$$

C, D = aberrational day numbers, calculated from the components of the Earth's true velocity referred to the centre of mass of the solar system, corrected for the effect of the E-terms, and referred to the equinox of the epoch from which τ is measured;

$$E = \lambda'\,\frac{\Delta\psi}{\psi'}$$

J, J' = second-order day numbers

*20″·496 for 1968 onwards.

where: τ is measured from the nearest beginning of a Besselian year; n, λ', ψ' are the precessional constants defined in section 2B; and $\Delta\psi$, $\Delta\epsilon$ are the (total) nutation in longitude and obliquity respectively (section 2C).

These day numbers are used in conjunction with star constants defined as follows:

$$a = m/n + \sin\alpha_0 \tan\delta_0 \qquad a' = \cos\alpha_0$$
$$b = \cos\alpha_0 \tan\delta_0 \qquad b' = -\sin\alpha_0$$
$$c = \cos\alpha_0 \sec\delta_0 \qquad c' = \tan\epsilon\cos\delta_0 - \sin\alpha_0\sin\delta_0$$
$$d = \sin\alpha_0 \sec\delta_0 \qquad d' = \cos\alpha_0 \sin\delta_0$$

where $m/n = 2{\cdot}29887 + 0{\cdot}00237\,T$, T being measured in centuries from 1900·0, and α_0, δ_0 is the mean place of the star for the beginning of a Besselian year, i.e. it is corrected for precession and proper motion to the equinox of the epoch from which τ is measured.

The apparent places are to be calculated from:

$$\alpha = \alpha_0 + \tau\mu_\alpha + Aa + Bb + Cc + Dd + E + J\tan^2\delta_0$$
$$\delta = \delta_0 + \tau\mu_\delta + Aa' + Bb' + Cc' + Dd' + J'\tan\delta_0$$

The day numbers A, B, C, D are tabulated in the Ephemeris in seconds of arc; when used for calculating the star reduction in right ascension, either they or the star constants by which they are multiplied should be divided by 15 to express the reduction in seconds of time.

Independent day numbers

Formulae involving Besselian day numbers are best employed in systematic calculation of a number of star places, but for an occasional reduction the independent day numbers are more suitable. They are defined as follows:

$$f = (m/n)\,A + E \qquad h\sin H = C$$
$$g\sin G = B \qquad h\cos H = D$$
$$g\cos G = A \qquad i = C\tan\epsilon$$

The apparent place is formed from:

$$\alpha = \alpha_0 + \tau\mu_\alpha + f + g\sin(G+\alpha_0)\tan\delta_0 + h\sin(H+\alpha_0)\sec\delta_0 + J\tan^2\delta_0$$
$$\delta = \delta_0 + \tau\mu_\delta + g\cos(G+\alpha_0) + h\cos(H+\alpha_0)\sin\delta_0 + i\cos\delta_0 + J'\tan\delta_0$$

The day numbers g, h are tabulated in the Ephemeris in seconds of arc; when used for calculating the star reduction in right ascension, they should be divided by 15 to express the reduction in seconds of time.

Values of f, g, G for the approximate reduction from the standard equinox of 1950·0 to a true equinox during the current year are given in *A.E.*, Table IV.

Short-period terms

The day numbers A, B that are tabulated in the Ephemeris as from 1960 contain both long-period and short-period terms of nutation. In certain cases, such as the 10-day ephemerides in *Apparent Places of Fundamental Stars*, the effect of the short-period terms is omitted from the apparent places because of the difficulty of interpolation at such an interval. For such stars, therefore, the short-period terms must be calculated separately, and applied to the tabulated places after interpolation. For single reductions, the use of independent day numbers is suitable, and the appropriate quantities are defined as:

$$f' = +\mathrm{d}\psi \cos \epsilon$$
$$g' \sin G' = -\mathrm{d}\epsilon$$
$$g' \cos G' = +\mathrm{d}\psi \sin \epsilon$$

where $\mathrm{d}\psi$ and $\mathrm{d}\epsilon$ are the short-period terms of nutation in longitude and obliquity respectively. The corrections, to be added to the apparent places including long-period terms only, are made through the expressions:

$$\Delta\alpha = f' + g' \sin (G' + \alpha) \tan \delta$$
$$\Delta\delta = g' \cos (G' + \alpha)$$

For systematic work where a number of reductions are to be made, these formulae may be written:

$$\Delta\alpha = \mathrm{d}\alpha(\psi) \,.\, \mathrm{d}\psi + \mathrm{d}\alpha(\epsilon) \,.\, \mathrm{d}\epsilon$$
$$\Delta\delta = \mathrm{d}\delta(\psi) \,.\, \mathrm{d}\psi + \mathrm{d}\delta(\epsilon) \,.\, \mathrm{d}\epsilon$$

where $\mathrm{d}\alpha(\psi) = \cos \epsilon + \sin \alpha \tan \delta \sin \epsilon \qquad \mathrm{d}\delta(\psi) = \cos \alpha \sin \epsilon$

$\mathrm{d}\alpha(\epsilon) = -\cos \alpha \tan \delta \qquad \mathrm{d}\delta(\epsilon) = \sin \alpha$

These coefficients are given (divided by 15 for α) in *Apparent Places of Fundamental Stars* for each of the 10-day stars, and the short-period terms of nutation are tabulated in Table I of the same volume, as well as in *A. E.*, pages 266 to 280. *

Corrections for short-period terms may also be obtained, without multiplication, with the aid of table 5.2; this is a triple-entry permanent table which is entered with arguments g', $(G' + \alpha)$, and δ. The complete correction in right ascension is $f' + (\Delta\alpha - f')$, but the correction in declination is given directly. The table provides a simple means of calculating the corrections for short-period terms for a moving object, for example for the Sun, Moon, and planets as given in the national ephemerides before 1960.

Before 1960, the ephemerides gave short-period day numbers in the Besselian form. The relations between these, the star constants, and the quantities defined above, are as follows:

$$A' = \mathrm{d}\psi/\psi' \qquad f' = mA' \text{ approximately}$$
$$B' = -\mathrm{d}\epsilon \qquad g' \sin G' = B'$$
$$g' \cos G' = nA'$$
$$\mathrm{d}\alpha(\psi) = a \sin \epsilon \text{ approximately} \qquad \mathrm{d}\delta(\psi) = a' \sin \epsilon$$
$$\mathrm{d}\alpha(\epsilon) = -b \qquad \mathrm{d}\delta(\epsilon) = -b'$$

Derivation

The numerical values of the day numbers *A*, *B*, *E* that are tabulated in the Ephemeris are derived for each day from the expressions given above, using the precessional constants and values of nutation previously defined in sections 2B and 2C. The value of τ used in calculating *A* is obtained by dividing the

*Pages 308 to 322 in *A.E.* 1972 onwards.

number of days from the nearest beginning of a Besselian solar year by 365·2422, the approximate number of days in a tropical year.

The aberrational day numbers C, D are derived, as from 1960, from the true velocity of the Earth in its orbit, referred to the centre of mass of the solar system and to an inertial frame of reference. If x', y', z' are the components of this velocity, then the components of the aberrational vector corresponding to C and D are given by the ratios $+y'/c$ and $-x'/c$ where c is the velocity of light. These expressions would give a complete correction for annual aberration, including the small E-terms due to the eccentricity of the Earth's orbit. Since these terms are constant for each star, it has been customary, by convention (see section 2D), to allow them to remain in the mean places; they must therefore be removed, thus leading to the expressions:

$$C = +y'/c - ke \cos \varpi \cos \epsilon$$
$$D = -x'/c - ke \sin \varpi$$

* where k is the constant of aberration $= 20''\cdot 47$, e is the eccentricity of the Earth's orbit, ϵ is the obliquity of the ecliptic, and ϖ is the longitude of the perihelion of the Earth's orbit. These expressions assume, as explained in section 2D, that the motion of the Earth lies entirely in the ecliptic, so that $z' = y' \tan \epsilon$. It can be shown that the error involved is negligible. (x', y', z' are equatorial components.)

In these expressions the value of c must be consistent with that of k; inserting * numerical values for the equinox 1950·0, and expressing C_0, D_0 for this equinox in seconds of arc:

$$C_0 = +1189''\cdot 80\,(y' + 553)\,10^{-7}$$
$$D_0 = -1189''\cdot 80\,(x' + 2815)\,10^{-7}$$

in which x', y' are in units of 10^{-7} astronomical units per day. These velocities are derived from the differences of the solar coordinates (X, Y, Z) for equinox 1950·0, corrected to the centre of mass of the solar system. If x_n, y_n, z_n are the heliocentric coordinates of the n^{th} planet, then the coordinates of the Earth (x_G, y_G, z_G) referred to the centre of mass of the whole system are given by the expressions of the form:

$$x_G = -X - \Sigma\{m_n x_n/(1 + \Sigma m_n)\}$$

where m_n represents the planet's mass in terms of the Sun's mass. Only the planets Jupiter, Saturn, Uranus, and Neptune need be considered, and the formulae are written in the form:

$$x_G = -X - \Sigma m_n' x_n$$

where $m_n' = m_n/(1 + \Sigma m_n)$ has the following values:

Jupiter	0·000953	Uranus	0·000044
Saturn	0·000285	Neptune	0·000052

and the coordinates of the planets are taken from *Planetary Co-ordinates*.

In the routine calculation of the day numbers, the velocities are formed in two stages, using:

$$x_G' = -X' - \Sigma m_n' x_n'$$

The daily differences of the solar coordinates (X, Y) are used to form the

*20″·496 and 1191″·30 for 1968 onwards.

			Declination										
$G' + \alpha$	g'		10°	20°	30°	40°	50°	55°	60°	65°	70°	75°	80°
		$\Delta\delta$					$\Delta\alpha - f'$						
h	″	″											
	0·05	0·05											
0·0	·07	·07											
or	·09	·09	For $G' + \alpha = 0^h$ or 12^h, $\Delta\alpha = f'$ for all declinations										
12·0	·11	·11											
	·13	·13											
			s	s	s	s	s	s	s	s	s	s	s
	0·05	0·05	0·000	0·000	0·000	0·001	0·001	0·001	0·001	0·00	0·00	0·00	0·00
1·0	·07	·07	0	0	1	1	1	2	2	0	0	0	1
11·0	·09	·09	0	1	1	1	2	2	3	0	0	1	1
13·0	·11	·11	0	1	1	2	2	3	3	0	1	1	1
23·0	·13	·13	0	1	1	2	3	3	4	0	1	1	1
	0·05	0·04	0·000	0·001	0·001	0·001	0·002	0·002	0·003	0·00	0·00	0·01	0·01
2·0	·07	·06	0	1	1	2	3	3	4	1	1	1	1
10·0	·09	·08	1	1	2	3	4	4	5	1	1	1	2
14·0	·11	·10	1	1	2	3	4	5	6	1	1	1	2
22·0	·13	·11	1	2	3	4	5	6	8	1	1	2	2
	0·05	0·04	0·000	0·001	0·001	0·002	0·003	0·003	0·004	0·01	0·01	0·01	0·01
3·0	·07	·05	1	1	2	3	4	5	06	1	1	1	2
9·0	·09	·06	1	2	2	4	5	6	07	1	1	2	2
15·0	·11	·08	1	2	3	4	6	7	09	1	1	2	3
21·0	·13	·09	1	2	4	5	7	9	11	1	2	2	3
	0·05	0·03	0·001	0·001	0·002	0·002	0·003	0·004	0·005	0·01	0·01	0·01	0·02
4·0	·07	·04	1	1	2	3	5	06	07	1	1	2	2
8·0	·09	·05	1	2	3	4	6	07	09	1	1	2	3
16·0	·11	·06	1	2	4	5	8	09	11	1	2	2	4
20·0	·13	·07	1	3	4	6	9	11	13	2	2	3	4
	0·05	0·01	0·001	0·001	0·002	0·003	0·004	0·005	0·006	0·01	0·01	0·01	0·02
5·0	·07	·02	1	2	3	4	05	06	08	1	1	2	3
7·0	·09	·02	1	2	3	5	07	08	10	1	2	2	3
17·0	·11	·03	1	3	4	6	08	10	12	2	2	3	4
19·0	·13	·03	1	3	5	7	10	12	14	2	2	3	5
	0·05	0·00	0·001	0·001	0·002	0·003	0·004	0·005	0·006	0·01	0·01	0·01	0·02
6·0	·07	·00	1	2	3	4	06	07	08	1	1	2	3
or	·09	·00	1	2	3	5	07	09	10	1	2	2	3
18·0	·11	·00	1	3	4	6	09	10	13	2	2	3	4
	0·13	0·00	0·002	0·003	0·005	0·007	0·010	0·012	0·015	0·02	0·02	0·03	0·05

Correction in right ascension $= (\Delta\alpha - f') + f'$

Correction in declination $= \Delta\delta$

where $(\Delta\alpha - f')$ and $\Delta\delta$ are applied with the signs given by the following table.

$G' + \alpha$	All declinations	Northern declinations ($\delta > 0$)	Southern declinations ($\delta < 0$)
$0^h - 6^h$	$\Delta\delta$ is positive	$\Delta\alpha - f'$ is positive	$\Delta\alpha - f'$ is negative
$6^h - 12^h$	$\Delta\delta$ is negative	$\Delta\alpha - f'$ is positive	$\Delta\alpha - f'$ is negative
$12^h - 18^h$	$\Delta\delta$ is negative	$\Delta\alpha - f'$ is negative	$\Delta\alpha - f'$ is positive
$18^h - 24^h$	$\Delta\delta$ is positive	$\Delta\alpha - f'$ is negative	$\Delta\alpha - f'$ is positive

f', g', G' are tabulated in *A.E.*, pages 267 to 281 (1960).

derivatives $-X'$, $-Y'$ for every fifth day. The sum $\Sigma\, m_n'\, x_n$ is evaluated and differenced at intervals of 100 days, and the value of the derivative $-\Sigma\, m_n'\, x_n'$ is then calculated from the differences (see section 16C).

The final subtabulations, additions, and multiplications are performed in one operation to give daily values of C_0 and D_0.

The conversion of the day numbers C_0 and D_0 from equinox 1950·0 to any other equinox t is given by the first-order expressions:

$$C_t = C_0 - p \cos \epsilon\,(t - 1950)\, D_0 = C_0 - 0{\cdot}0002235\,(t - 1950)\, D_0$$
$$D_t = D_0 + p \sec \epsilon\,(t - 1950)\, C_0 = D_0 + 0{\cdot}0002656\,(t - 1950)\, C_0$$

where p is the annual general precession $= 50''{\cdot}27$. The error due to neglecting second-order terms in these expressions is less than $0''{\cdot}0005$ for time intervals up to 30 years. The tabulated values are referred to the equinox of the nearest beginning of a Besselian year.

The values of A, B, C, D for 0^h sidereal time are obtained from the daily values at 0^h E.T. by interpolation. The independent day numbers are calculated from the Besselian day numbers for every day by the formulae quoted above.

Example 5.2. *Besselian and independent day numbers*
1960 March 7

τ	$+0{\cdot}17976$	$\Delta\psi$	$-0''{\cdot}744$	$d\psi$	$+0''{\cdot}008$
n	$20''{\cdot}0417$	$\Delta\epsilon$	$-8''{\cdot}836$	$d\epsilon$	$-0''{\cdot}066$
$\frac{1}{15}(m/n)$	$0{\cdot}15335$	$\sin\epsilon$	$+0{\cdot}39786$	$\cos\epsilon$	$+0{\cdot}91745$

$\lambda'/\psi' = 0{\cdot}002476 - 0{\cdot}000373T = 0{\cdot}002$ where T is in tropical centuries from 1900·0

Besselian day numbers.

$$A = n\tau + \Delta\psi \sin\epsilon \quad +3''{\cdot}307$$
$$B = -\Delta\epsilon \quad +8''{\cdot}836$$
$$E = (\lambda'/\psi')\,\Delta\psi \quad -0^s{\cdot}0001$$

The aberrational day numbers C and D are formed from the coordinates of the Sun and planets for equinox 1950·0. The differences of the Sun's coordinates for March 7 (*A.E.*, page 43) and, hence, the velocity components are, in units of the seventh decimal:

$$\mu\delta X + 43509 \qquad \mu\delta^3 X - 8 \qquad X' = \mu\delta X - \tfrac{1}{6}\mu\delta^3 X + 43510$$
$$\mu\delta Y + 153888 \qquad \mu\delta^3 Y - 46 \qquad Y' = \mu\delta Y - \tfrac{1}{6}\mu\delta^3 Y + 153896$$

For the planets, coordinates at intervals of 100 days are taken from *Planetary Co-ordinates*:

J.D.	Jupiter	Saturn	Uranus	Neptune	unit 10^{-5}	
	x-coordinate				$\Sigma m'x$	$\Sigma m'x'$
243 6900·5	−1·634	+1·451	−13·670	−24·296	−301	
						+84
7000·5	−0·909	+1·971	−13·933	−24·109	−217	
						+85
7100·5	−0·167	+2·485	−14·189	−23·918	−132	
	y-coordinate				$\Sigma m'y$	$\Sigma m'y'$
243 6900·5	−4·672	−9·181	+11·221	−17·008	−746	
						−12
7000·5	−4·799	−9·086	+10·932	−17·236	−758	
						−2
7100·5	−4·831	−8·965	+10·639	−17·463	−760	

Since the interval is 100 days, the values for 1960 March 7 = J.D. 243 7000·5 are obtained directly in units of 10^{-7} as:

$$\Sigma m'x' = +84 \qquad \Sigma m'y' = -7$$

Then $C_0 = -1189''\cdot80\,(Y' + \Sigma m'y' - 553)\,10^{-7} = -18''\cdot244$

$D_0 = +1189''\cdot80\,(X' + \Sigma m'x' - 2815)\,10^{-7} = +\ 4''\cdot852$

The conversion to equinox 1960·0 is given by:

$C = C_0 - D_0\,(0\cdot002235) = -18''\cdot255$

$D = D_0 + C_0\,(0\cdot002656) = +\ 4''\cdot803$

Independent day numbers.

$f = \frac{m}{n}A + E$	$+0^s\cdot5070$		
$g \sin G = B$	$+\ 8''\cdot836$	g	$9''\cdot435$
$g \cos G = A$	$+\ 3''\cdot307$	G	$4^h\ 37^m\ 55^s$
$h \sin H = C$	$-18''\cdot255$	h	$18''\cdot876$
$h \cos H = D$	$+\ 4''\cdot803$	H	$18^h\ 58^m\ 58^s$
$i = C \tan \epsilon$	$-\ 7''\cdot916$		
$f' = d\psi \cos \epsilon$	$+0^s\cdot0005$		
$g' \sin G' = -d\epsilon$	$+0''\cdot066$	g'	$0''\cdot066$
$g' \cos G' = d\psi \sin \epsilon$	$+0''\cdot0032$	G'	$5^h\ 49^m$

The second-order day numbers J, J' are formed for every tenth day and for each hour of right ascension from:

$$P_1 = (A + D) \sin \alpha + (B + C) \cos \alpha \qquad P_2 = (A + D) \cos \alpha - (B + C) \sin \alpha$$

$$Q_1 = (A - D) \sin \alpha + (B - C) \cos \alpha \qquad Q_2 = (A - D) \cos \alpha - (B - C) \sin \alpha$$

If P_1, P_2, Q_1, Q_2 are in seconds of arc, then J, J', also in seconds of arc, are given by:

$$\left.\begin{aligned} J &= P_1 P_2 \sin 1'' \\ J' &= -\tfrac{1}{2} P_1^2 \sin 1'' \end{aligned}\right\} \text{for northern declinations}$$

$$\left.\begin{aligned} J &= Q_1 Q_2 \sin 1'' \\ J' &= -\tfrac{1}{2} Q_1^2 \sin 1'' \end{aligned}\right\} \text{for southern declinations}$$

Example 5.3. *Second-order day numbers J and J'*

1960 March 7

A	$+3''\cdot31$	B	$+\ 8''\cdot84$
D	$+4\cdot80$	C	$-18\cdot26$
$A + D$	$+8\cdot11$	$B + C$	$-\ 9\cdot42$
$A - D$	$-1\cdot49$	$B - C$	$+27\cdot10$

α	0^h	1^h	2^h	3^h	4^h	5^h	6^h
$\sin \alpha$	+0·00	0·26	0·50	0·71	0·87	0·97	1·00
$\cos \alpha$	+1·00	0·97	0·87	0·71	0·50	0·26	0·00
P_1	− 9·4	− 7·0	− 4·1	− 0·9	+ 2·3	+ 5·4	+ 8·1
P_2	+ 8·1	+10·3	+11·8	+12·4	+12·3	+11·2	+ 9·4
Q_1	+27·1	+25·9	+22·8	+18·2	+12·3	+ 5·6	− 1·5
Q_2	− 1·5	− 8·5	−14·8	−20·3	−24·3	−26·7	−27·1
Northern declinations							
$J = +0\cdot032\, P_1 P_2$	−2	−2	−2	0	+1	+2	+2
$J' = -0\cdot024\, P_1^2$	−2	−1	0	0	0	−1	−2
Southern declinations							
$J = +0\cdot032\, Q_1 Q_2$	− 1	− 7	−11	−12	−10	− 5	+ 1
$J' = -0\cdot024\, Q_1^2$	−18	−16	−12	− 8	− 4	− 1	0

P_1, P_2, Q_1, Q_2, are in seconds of arc, J is in units of $0^s\cdot00001$, and J' is in units of $0''\cdot0001$.

Star reductions prior to 1960

The new definitions have removed a number of inconsistencies and increased the precision of application of star reductions. The ephemerides prior to 1960 differed from present practice in the following respects:

(a) the day numbers were then referred to the equinox of the beginning of the year, so that τ could reach a value of $+1$;

(b) the day number A was then defined as:

$$A = \tau + \frac{\Delta\psi}{\psi'}$$

but it is now n times this quantity, and, like the other day numbers, is expressed in seconds of arc;

(c) the star constants a, a' were then defined by:

$$a = m + n \sin \alpha \tan \delta \qquad a' = n \cos \alpha$$

but these expressions are now divided by n;

(d) the day number B, and the nutation terms in A, were then derived from long-period nutation terms only; they now include (total) nutation, and are given to an extra decimal;

(e) the aberrational day numbers C, D were then derived from theoretical expressions for the Earth's velocity based on a mean orbit about the Sun; they did not allow for perturbations by the Moon and planets, were not referred to the centre of mass of the solar system, and were given to $0''.01$ only;

(f) no allowance was then made for the correction of second-order terms.

Summary

* The day numbers tabulated in *A.E.*, pages 266–281 in 1960, give the values of the Besselian and independent day numbers, as defined above, for each day at 0^h E.T., together with τ the fraction of the year from the nearest beginning of a Besselian year. There is a discontinuity in τ, and in all the day numbers except B and E, due to the change of equinox; the data for July 1 and 2 are given for both systems. The short-period independent day numbers f', g', G' and the equivalent short-period nutation terms $d\psi$, $d\epsilon$ are also tabulated. Short-period Besselian day numbers A', B', which were in use in the ephemerides before 1960, are not now given. For convenience, an approximate indication of the sidereal time at 0^h is also tabulated. The day numbers are given in general to a precision of $0''.001$, but f, f', E are expressed in seconds of time to $0^s.0001$; G, H are given to the nearest second, G' to the nearest minute and τ to 0.0001.

† In *A.E.*, pages 282–285 in 1960, the Besselian day numbers are tabulated for 0^h S.T. on each day, and in this form will be found convenient for systematic computation of the corrections to the positions of stars at transit; the values at the time of transit are obtained by interpolation to the right ascension of the star. It will be noted that there are two entries for the day numbers on a day near

*Pages 308 to 323 in *A.E.* 1972 onwards.

†Pages 324 to 327 in *A.E.* 1972 onwards.

September 21; care should be taken, for a station not on the Greenwich meridian, to select the correct entry.

In *A.E.*, pages 286–287 and x–xi in 1960, the second-order day numbers J, J' are tabulated for 0^h E.T. on every tenth day as a function of right ascension. Interpolation in these tables is not generally necessary. *

Numerical illustrations

Example 5.4. *Apparent place of* α *Centauri*
Greenwich upper transit on 1960 July 1

The mean places and proper motions of example 5.1 are used with day numbers containing long-period nutation only, the resulting place being comparable with that given in *Apparent Places of Fundamental Stars*; the short-period terms are calculated independently. Transit occurs on July 1·83, and for this date the reduction may be made from either equinox. Corrections for parallax and for reduction from centre of gravity to the bright star are included. $\pi = 0''\cdot756$

	1960·0	1961·0		1960·0	1961·0		1960·0	1961·0
	″	″						
A	+ 8·853	−11·188	$\frac{1}{15}a$	+0·22838	0·22842	a'	−0·77475	0·77457
B	+ 9·641	+ 9·641	$\frac{1}{15}b$	+0·09193	0·09193	b'	+0·63226	0·63249
C	+ 3·250	+ 3·254	$\frac{1}{15}c$	−0·10545	0·10544	c'	−0·33881	0·33906
D	−20·150	−20·150	$\frac{1}{15}d$	−0·08605	0·08610	d'	+0·67545	0·67532
E	− 0s·0004	−0s·0004				Reduction to bright star $\Delta\alpha$	+0s·006	+0s·064
τ	+ 0·4996	−0·5004				$\Delta\delta$	+2″·00	+2″·86

	1960·0	1961·0		1960·0	1961·0
	h m s	s		° ′ ″	″
α_0	14 36 52·147	56·235	δ_0	− 60 40 17·67	32·49
$\tau\mu_\alpha$	− 0·2446	+ 0·2450	$\tau\mu_\delta$	+ 0·352	− 0·352
Aa	+ 2·0218	− 2·5556	Aa'	− 6·859	+ 8·666
Bb	+ 0·8863	+ 0·8863	Bb'	+ 6·096	+ 6·098
Cc	− 0·3427	− 0·3431	Cc'	− 1·101	− 1·103
Dd	+ 1·7339	+ 1·7349	Dd'	− 13·610	− 13·608
E	− 0·0004	− 0·0004			
Parallax	− 0·0847	− 0·0847	Parallax	− 0·146	− 0·146
Reduction	+ 0·0350	+ 0·0350	Reduction	+ 2·430	+ 2·430
α	$14^h\ 36^m\ 56^s\cdot1516$	$56^s\cdot1524$	δ	− 60° 40′ 30″·508	30″·505

Short-period terms:

f'	− 0s·0064	G'	$15^h\ 58^m$	$\sin(G' + \alpha_0)$	+ 0·9884
g'	0″·080	$G' + \alpha_0$	6 35	$\cos(G' + \alpha_0)$	− 0·1521

	s		″
f'	− 0·0064		
$g' \sin(G' + \alpha_0) \tan\delta$	− 0·0094	$g' \cos(G' + \alpha_0)$	− 0·012
$\Delta\alpha$	− 0·0158	$\Delta\delta$	− 0·012

The alternative method gives:

$d\alpha(\psi)$	+0·091	$d\delta(\psi)$	−0·308	$d\psi$	−0″·104
$d\alpha(\epsilon)$	−0·092	$d\delta(\epsilon)$	−0·632	$d\epsilon$	+0·068

$\Delta\alpha$ −0s·0157 $\Delta\delta$ −0″·011

The apparent place, including short-period terms, is therefore:

α $14^h\ 36^m\ 56^s\cdot136$ δ −60° 40′ 30″·52

*Pages 328 to 331 in *A.E.* 1972 onwards.

For comparison, the apparent place is calculated by means of independent day numbers from the mean place for 1961·0:

f $-1^s\cdot7225$	$G + \alpha_0$ $23^h\ 55^m\ 08^s$		$\sin\delta_0$ $-0\cdot87186$
g $14''\cdot756$	$\sin(G + \alpha_0)$ $-0\cdot02123$		$\cos\delta_0$ $+0\cdot48975$
h $20\ \cdot412$	$\cos(G + \alpha_0)$ $+0\cdot99977$		$\tan\delta_0$ $-1\cdot78021$
i $+1\ \cdot411$			$\sec\delta_0$ $+2\cdot04186$
	$H + \alpha_0$ $2^h\ 00^m\ 15^s$		
G $9^h\ 18^m\ 12^s$	$\sin(H + \alpha_0)$ $+0\cdot50094$		
H $11\ \ 23\ \ 19$	$\cos(H + \alpha_0)$ $+0\cdot86548$		

	h m s		° ′ ″
α_0	14 36 56·235	δ_0	−60 40 32·49
$\tau\mu_\alpha$	+ 0·2450	$\tau\mu_\delta$	− 0·352
f	− 1·7225	$g\cos(G + \alpha_0)$	+14·753
$g\sin(G + \alpha_0)\tan\delta_0$	+ 0·0372	$h\cos(H + \alpha_0)\sin\delta_0$	−15·402
$h\sin(H + \alpha_0)\sec\delta_0$	+ 1·3919	$i\cos\delta_0$	+ 0·691
Parallax	− 0·0847	Parallax	− 0·146
Reduction to bright star	+ 0·0350	Reduction to bright star	+ 2·430
Apparent place, α	14 36 56·137	Apparent place, δ	−60 40 30·52

In the above calculations more figures have been retained than would normally be required or justified; this is done to illustrate the extent of the agreement of the calculations by different methods, and the magnitude of the differences between quantities for the two equinoxes.

The inclusion of second-order terms, which would normally be neglected in this case, reduces still further the difference between the two results:

	Right ascension			Declination	
	1960·0	1961·0		1960·0	1961·0
J	$+0^s\cdot00014$	$+0^s\cdot00001$	J'	$-0''\cdot0014$	$-0''\cdot003$
$J\tan^2\delta$	+0 ·0005	0 ·0000	$J'\tan\delta$	+0 ·003	+0 ·001
α	$14^h\ 36^m\ 56^s\cdot1521$	$56^s\cdot1524$	δ	$-60°\ 40'\ 30''\cdot505$	$30''\cdot504$

Example 5.5. *Apparent place of* 20 *G. Octantis*

Greenwich upper transit on 1960 July 1

The example illustrates the use of second-order terms in calculating the apparent place of a circumpolar star; day numbers including short-period nutation are used. Transit occurs on July 1·853.

Besselian day numbers; mean place for 1960·0. $\tau = +0\cdot4997$

α_0 $15^h\ 06^m\ 25^s\cdot56$ δ_0 $-87°\ 59'\ 12''\cdot74$

μ_α $-0\ \cdot179$ μ_δ $-0\ \cdot072$

	″		s				
A	+ 8·813	E	− 0·0005	$\frac{1}{15}a$	+1·53154	a'	−0·68701
B	+ 9·572	J	+ 0·00012	$\frac{1}{15}b$	+1·30299	b'	+0·72665
C	+ 3·257			$\frac{1}{15}c$	−1·30379	c'	−0·71097
D	−20·149	J'	$-0''\cdot0015$	$\frac{1}{15}d$	−1·37903	d'	+0·68658

	h m s		° ′ ″
α_0	15 06 25·56	δ_0	−87 59 12·74
$\tau\mu_\alpha$	− 0·089	$\tau\mu_\delta$	− 0·036
Aa	+13·497	Aa'	− 6·055
Bb	+12·472	Bb'	+ 6·955
Cc	− 4·246	Cc'	− 2·316
Dd	+27·786	Dd'	−13·834
E	0·000		
$J\tan^2\delta_0$	+ 0·097	$J'\tan\delta_0$	+ 0·043
α	$15^h\ 07^m\ 15^s\cdot08$	δ	$-87°\ 59'\ 27''\cdot98$

Independent day numbers; mean place for 1961·0. $\tau = -0\cdot5003$

α_0	$15^h\ 06^m\ 56^s\cdot13$			δ_0	$-87^\circ\ 59'\ 26''\cdot57$
μ_α	$-0\cdot179$			μ_δ	$-0\cdot072$
f	$-1^s\cdot7224$	$G + \alpha_0$	$0^h\ 25^m\ 08^s$	$\sin\delta_0$	$-\ 0\cdot999385$
g	$14''\cdot755$	$\sin(G + \alpha_0)$	$+0\cdot10944$	$\cos\delta_0$	$+\ 0\cdot035062$
h	$20''\cdot411$	$\cos(G + \alpha_0)$	$+0\cdot99399$	$\sec\delta_0$	$+28\cdot5209$
i	$+1''\cdot414$			$\tan\delta_0$	$-28\cdot5038$
		$H + \alpha_0$	$2^h\ 30^m\ 10^s$		
G	$9^h\ 18^m\ 12^s$	$\sin(H + \alpha_0)$	$+0\cdot60934$	J	$+\ 0^s\cdot00001$
H	11 23 14	$\cos(H + \alpha_0)$	$+0\cdot79291$	J'	$-0''\cdot0003$

	h m s		° ′ ″
α_0	15 06 56·13	δ_0	−87 59 26·57
$\tau\mu_\alpha$	+ 0·090	$\tau\mu_\delta$	+ 0·036
f	− 1·722	$g\cos(G + \alpha_0)$	+14·666
$g\sin(G + \alpha_0)\tan\delta_0$	− 3·069	$h\cos(H + \alpha_0)\sin\delta_0$	−16·174
$h\sin(H + \alpha_0)\sec\delta_0$	+23·648	$i\cos\delta_0$	+ 0·050
$J\tan^2\delta_0$	+ 0·008	$J'\tan\delta_0$	+ 0·009
α	$15^h\ 07^m\ 15^s\cdot08$	δ	$-87^\circ\ 59'\ 27''\cdot98$

E. POLE STAR TABLES

The proximity of the second-magnitude star α Ursae Minoris, *Polaris* or the Pole Star, to the north pole of the sky has given it a special significance for the convenient determination of direction and latitude. This is particularly so in the fields of navigation and surveying, for which its constant availability for observation (in northern latitudes) and the simple methods that can be used for the reduction of observations are invaluable. For the more precise requirements of astronomy its distance from the pole is sufficiently large for the special methods of reduction no longer to confer any advantage over standard methods. Thus " Pole Star Tables " are restricted to the precision required in navigation and surveying, and belong to the corresponding " almanacs ", rather than to the Ephemeris; however, because of the general use of the Pole Star, the principal table to navigational precision is included in the Ephemeris (*A.E.* *1960, Table II, page 456).

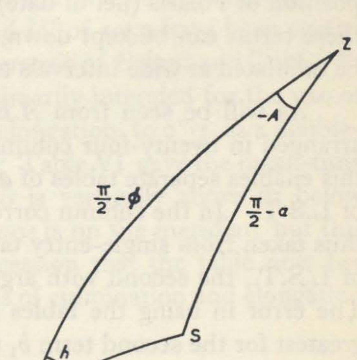

Figure 5.1. *Notation for Pole Star*

The polar distance of *Polaris* is at present (1960) about 55′·4 and is decreasing. It will reach a minimum of about 27′·5 in 2101, and it will then increase with increasing rapidity. It will reach 1° in about 2250 and 2° in about 2450.

If the polar distance of *Polaris* is denoted by p (of the order of one degree) and its local hour angle by h, then its altitude a and azimuth A, as seen from an observer in latitude ϕ, are given by solving the spherical triangle PZS (see figure 5.1) formed by the

*Pages 496 to 499 in *A.E.* 1972.

north pole, the zenith, and *Polaris*. Because p is small the solution may be expanded as:

$$a = \phi + p \cos h - \tfrac{1}{2} p \sin p \sin^2 h \tan \phi + \ldots$$
$$- A \cos \phi = p \sin h + p \sin p \sin h \cos h \tan \phi + \ldots$$

In each case the next term of the expansion is of order $p \sin^2 p \tan^2 \phi$ and cannot, for many years, exceed 0'·1 for latitudes up to 70°.

For convenience of tabulation, these expansions are rewritten in the form:

$$\begin{aligned} \phi - a = & - (p_0 \cos h_0 - \tfrac{1}{2} p_0 \sin p_0 \sin^2 h_0 \tan \phi_0) \\ & + \tfrac{1}{2} p_0 \sin p_0 \sin^2 h_0 (\tan \phi - \tan \phi_0) \\ & - (p \cos h - p_0 \cos h_0) = a_0 + a_1 + a_2 \\ A \cos \phi = & - (p_0 \sin h_0 + p_0 \sin p_0 \sin h_0 \cos h_0 \tan \phi_0) \\ & - p_0 \sin p_0 \sin h_0 \cos h_0 (\tan \phi - \tan \phi_0) \\ & - (p \sin h - p_0 \sin h_0) = b_0 + b_1 + b_2 \end{aligned}$$

in which p_0 and h_0 are the polar distance and hour angle of a convenient point close to the mean position of *Polaris* throughout the year, and ϕ_0 is a mean latitude, usually chosen to be 50°. The mean position of *Polaris* (which must not be confused with its mean place) is usually chosen to have convenient exact values for its right ascension α_0 and polar distance p_0.

The first terms (a_0, b_0) in the modified expressions are functions of the single variable, local sidereal time, since:

$$h_0 = \text{L.S.T.} - \alpha_0$$

and may be tabulated at a suitable interval of L.S.T.

The second terms (a_1, b_1) are functions both of h_0 (i.e. of L.S.T.) and of latitude and must thus be tabulated in a double-entry table with arguments L.S.T. and latitude. By incorporating a mean value (corresponding to latitude ϕ_0) in the first term, the magnitude of these terms can be kept down to about 0'·5; they may thus be tabulated at wide intervals of both latitude and L.S.T.

Similarly, the third terms (a_2, b_2) are functions both of h_0 and of the apparent position of Polaris (i.e. of date). By proper choice of p_0 and α_0, the magnitude of these terms can be kept down, during one year, to about 0'·5; and they can also be tabulated at wide intervals of both date and L.S.T.

As will be seen from *A.E.*, Table II, the single-entry table of a_0 and b_0 is arranged in twenty-four columns, each containing values for one hour of L.S.T.; this enables separate tables of a_1 and b_1, and of a_2 and b_2, to be given for each hour of L.S.T. In the column corresponding to the hour of L.S.T. all these terms are thus taken from single-entry tables—the first with argument minutes and seconds of L.S.T., the second with argument latitude, and the third with argument date. The error in using the tables for the hour, without interpolation for L.S.T., is greatest for the second term b_1 (owing to its dependence on sin $2h_0$) and may reach 0'·15 for extreme latitudes; otherwise the error is very small.

The complications of these tabulations are unnecessary for astronomical usage, but valuable for navigational use, in which simplicity of tabular entry and of interpolation are of foremost importance. Table II is essentially the same as the corresponding table used for surface navigation, apart from intervals of tabulation (1° in L.S.T. or L.H.A. of the first point of Aries) and from a further simplification for the user by adding constants (whose sum is one degree) to a_0, a_1, a_2 to make them always positive. (*The Nautical Almanac*, 1960, page 274.)

Example 5.6. *Derivation of Pole Star Table*

The adopted mean position of *Polaris* for 1960 is:

$$\alpha_0 = 1^h\,57^m;\ p_0 = 55'{\cdot}4 \text{ whence } p_0 \sin p_0 = 0'{\cdot}89.$$

$$\phi_0 \text{ is taken to be } 50°;\ \tan\phi_0 = 1{\cdot}192,\ p_0 \sin p_0 \tan\phi_0 = 1'{\cdot}06.$$

Entries will be calculated for a local sidereal time of $4^h\,30^m$, latitude 64°, and for the month of March.

$h_0 = 4^h\,30^m - 1^h\,57^m = 2^h\,33^m$

						a_0	b_0
						′	′
$-\cos h_0$	$-0{\cdot}7853$	$-\sin h_0$	$-0{\cdot}6191$	×	p_0	$-43{\cdot}51$	$-34{\cdot}30$
$+\frac{1}{2}\sin^2 h_0$	$+0{\cdot}19$	$-\sin h_0 \cos h_0$	$-0{\cdot}49$	×	$p_0 \sin p_0 \tan\phi_0$	$+\ 0{\cdot}20$	$-\ 0{\cdot}52$
						$-43{\cdot}31$	$-34{\cdot}82$

For $\phi = 64°$: $\tan\phi\ +2{\cdot}050$ $\quad \tan\phi - \tan\phi_0\ +0{\cdot}86$

$-p_0 \sin p_0 (\tan\phi - \tan\phi_0)\ -0'{\cdot}77 \qquad a_1\ +0'{\cdot}15 \qquad b_1\ -0'{\cdot}38$

Mean values for March, based on the apparent place of *Polaris* tabulated in *Apparent Places of Fundamental Stars* are:

$$\alpha = 1^h\,55^m\,17^s \qquad (\alpha - \alpha_0) = -1^m{\cdot}7 \qquad p_0 \sin(\alpha - \alpha_0) = -0'{\cdot}41$$

$$\delta = 89°\,04'\,49'' \qquad (p - p_0) = -0'{\cdot}22$$

Now:

$$a_2 = -p\cos h + p_0 \cos h_0 = -(p - p_0)\cos h_0 - p_0 \sin h_0 \sin(\alpha - \alpha_0) = +0'{\cdot}43$$

$$b_2 = -p\sin h + p_0 \sin h_0 = -(p - p_0)\sin h_0 + p_0 \cos h_0 \sin(\alpha - \alpha_0) = -0'{\cdot}19$$

It may readily be verified that a direct solution of the spherical triangle for latitude 64°, declination 89° 04′ 49″, and hour angle $2^h\,34^m\,43^s$ gives:

altitude = 64° 42′ 42″·8, corresponding to $(a_0 + a_1 + a_2) = -42'{\cdot}71$

azimuth = −1° 20′ 44″·3, corresponding to $(b_0 + b_1 + b_2) = -35'{\cdot}39$

Prior to 1960, *The American Ephemeris* contained the daily apparent place of *Polaris* and a number of tables mainly designed for the precise determination of azimuth from observations of *Polaris*. Tables I and IV of 1959 have been essentially replaced by the present Table II, but the ephemeris of *Polaris* and Tables V, VI, and VII have not been included, as they are primarily intended for the use of surveyors. Table V gave the azimuth of *Polaris* at elongation, to 0″·1, as a double-entry table with arguments latitude and declination; Table VI gave the mean-time interval which elapses from the time when *Polaris* is vertically above or below ζ^1 Ursae Majoris or δ Cassiopeiae to that when *Polaris* is on the meridian, but this interval has become so greatly lengthened by precession that the table now has little practical usefulness. Table VII gave the times of culmination and elongation of *Polaris*.

6. THE SYSTEM OF ASTRONOMICAL CONSTANTS*

The constants of importance in the dynamics of the solar system comprise the elements of the orbits of the several bodies, their masses relative to that of the Sun, the constants specifying their size, shape, orientation, rotation, and inner constitution, and the velocity of light. The constants connected with the Earth are of special importance, because all conclusions about the motions of other celestial bodies depend on them; this small group of constants is called the system of astronomical constants. The word *system* is appropriate for two distinct reasons. In the first place, the constants are not all independent of one another; once the values of some of them are known, others can be calculated without further recourse to observations. In the second place, when taken together with theory, the constants constitute a model of the Earth and its motions, which serves for the calculation of ephemerides. Analysis of the discrepancies between the ephemerides and the observations leads in turn to new knowledge of the dynamics of the solar system and to more accurate values of the constants.

This section is devoted to a list of the conventionally adopted values of the constants comprising the system, their definitions, a discussion of some of the more important relations among them, and a statement of the known inconsistencies in the system, awareness of which is necessary in some specialized investigations if misinterpretations are to be avoided. For further information on these matters see the references at the end of the section.

In table 6.1 are given the conventionally adopted values of the constants comprising the system. It should be said at once that it is not possible to set precise limits on such a list. Some of the constants, such as the polar radius of the Earth, are so easily derived from others that they might be omitted. On the other hand, the motion of the ecliptic is calculated from the adopted masses of all the planets; it would be proper to add them to the list but no useful purpose would be served, because it is more expedient to consider the calculation as a part of the theory of the motion of the Earth around the Sun.

The values of the equatorial radius (a) and the flattening (f) of the Earth have been adopted by the International Union of Geodesy and Geophysics (G. Perrier, "Comptes Rendus de la Section de Géodésie, Madrid, 1924", *Bulletin Géodésique*, no. 7, 552–6, 1925), as has also the expression for the value of gravity.

*See pages 497 to 521 for an account of the IAU system of astronomical constants that was introduced in 1968.

Table 6.1. The system of astronomical constants

Equatorial radius of the Earth	$a = 6\ 378\ 388$m
Flattening of the Earth	$f = 1/297$
Polar radius of the Earth	$a(1 - f) = 6\ 356\ 911{\cdot}946$m
Normal gravity	$g = 978{\cdot}049\ (1 + 0{\cdot}00528\ 84 \sin^2 \phi - 0{\cdot}00000\ 59 \sin^2 2\phi)$ cm/sec^2
Solar parallax	$\pi = 8''{\cdot}80$
Constant of nutation, 1900·0	$9''{\cdot}21$
Constant of aberration	$k = 20''{\cdot}47$
General precession in longitude, per tropical century ...	$p = 5025''{\cdot}64 + 2''{\cdot}22T$
Precession in right ascension, per tropical century ...	$m = 4608''{\cdot}50 + 2''{\cdot}79T$
Precession in declination, per tropical century	$n = 2004''{\cdot}68 - 0''{\cdot}85T$
Speed of rotation of the ecliptic, per tropical century	$\pi = 47''{\cdot}11 - 0''{\cdot}07T$
Longitude of axis of rotation of the ecliptic ...	$\Pi = 173° 57' 03''{\cdot}6 + 3286''{\cdot}2T$
Obliquity of the ecliptic	$\epsilon = 23° 27' 08''{\cdot}26 - 46''{\cdot}845T - 0''{\cdot}0059\ T^2 + 0''{\cdot}00181\ T^3$
Equatorial horizontal parallax of the Moon at distance 60·2665 equatorial radii of the Earth	$57' 02''{\cdot}70$
Velocity of light	$c = 299\ 860$ km/sec $= 186\ 324$ statute miles/sec
Light travels unit distance in $498^s{\cdot}580$ (from solar parallax) or $498^s{\cdot}38$ (from constant of aberration)	
Gaussian constant of gravitation ...	$k = 0{\cdot}01720\ 20989\ 50000 = 3548''{\cdot}18760\ 69651$
Mass ratio Earth: Moon	81·45 (for lunar inequality) or 81·53 (in Brown's lunar theory)
Mass ratio Sun: (Earth plus Moon)	329 390

T denotes centuries from 1900·0, no distinction being necessary between the tropical century and the Julian century; see, however, section 4B.

The values of the solar parallax, constant of nutation, and constant of aberration were adopted by the Paris Conference of 1896. The precessional constants and the motion of the ecliptic are Newcomb's. The parallax of the Moon is Brown's. The velocity of light is the value of Newcomb (1882). The mass ratio Earth: Moon of 81·45 is that used in Newcomb's tables for calculating the lunar inequality in the solar coordinates, while the value 81·53 is used in Brown's lunar theory. The mass ratio Sun: (Earth plus Moon) is that used in Newcomb's tables of the Sun and planets.

The equatorial radius and the flattening define the size and shape of the ellipsoid (known as Hayford's spheroid or the International Ellipsoid of Reference) that is substituted for the actual Earth in astronomical calculations. The polar radius is derived from the equatorial radius and the flattening as $a\ (1 - f)$; for consistency with them it is given to three extra figures.

The expression for gravity, in which ϕ is the geodetic latitude, includes the effect of centrifugal force due to the rotation of the Earth.

The solar parallax is defined as the angle subtended by the equatorial radius of the Earth at a distance of one astronomical unit.

The constant of nutation is the coefficient that is multiplied by the cosine of the longitude of the Moon's ascending node on the ecliptic, in the expression for the nutation in obliquity.

The constant of aberration (k) is the value of the ratio, expressed in seconds of arc, of the Earth's mean orbital velocity, conventionally taken to be the component perpendicular to the radius vector, to the velocity of light. The relation between it and the solar parallax is given later.

The speed of the general precession in longitude is inferred from observations combined with theory. It has often been called the constant of precession, but it is preferable to reserve the term constant of precession, as Newcomb did, for the function:

$$P = \left(A + B \frac{\mu}{1 + \mu}\right) H$$

where P is the constant of precession, properly so called, μ is the mass ratio Moon: Earth, H is the mechanical ellipticity of the Earth (to be distinguished from the flattening), and A and B are functions of the elements of the orbits of the Earth and Moon, and of the mass ratio Sun: (Earth plus Moon). The quantity P is very nearly constant; it is diminishing at the rate of $0''.004$ per century, mainly because of the secular decrease of the eccentricity of the Earth's orbit.

The precessional constant (P) is connected with the general precession in longitude (p) by the relation:

$$p = P \cos \epsilon - p_g - \lambda' \cos \epsilon$$

where ϵ is the obliquity of the ecliptic, λ' is the planetary precession, and p_g is the geodesic precession, equal to $1''.915$ per century. The geodesic precession is a relativistic motion of the equinox along the ecliptic, similar to the general precession but in the opposite sense. The amount is given by $3V^2 n/2$, where V is the r.m.s. value of the ratio of the Earth's velocity to the velocity of light, and n is the Earth's mean angular orbital motion. (See Chazy, J., *La Théorie de la Relativité et la Mécanique Céleste.* Volume II, Paris, 1930.)

Denoting the precession in right ascension by m, and the precession in declination by n, we have:

$$m = (P \cos \epsilon - p_g) \cos \epsilon - \lambda'$$
$$n = (P \cos \epsilon - p_g) \sin \epsilon$$

The plane of the ecliptic is defined as the mean plane of the Earth's orbit; thus it is affected by secular perturbations from the action of the other planets, but not by periodic perturbations, which are considered to be synonymous with the latitude of the Earth. The ecliptic is rotating about an axis which is about 6° from the equinoxes; thus the secular change of the obliquity is slightly less numerically than the speed of rotation of the ecliptic.

The mechanical ellipticity of the Earth (H) is defined by:

$$H = \frac{C - A}{C}$$

where C and A are the polar and equatorial moments of inertia. The mechanical ellipticity is thus a dynamical constant while the flattening (f) is a geometrical one. The relation between the two is not a simple one, but an idea of it may be obtained from the relation that would hold approximately if the Earth were of uniform density:

$$H = f - \tfrac{1}{2}\sigma$$

where σ is the ratio of the centrifugal acceleration to gravity, both taken at the equator.

The time required for light to travel one astronomical unit (see section 4A) may be inferred either from the solar parallax by the formula*:

$$\tau = \frac{a}{c\pi} = 498^{s}\cdot 580$$

where τ is the light-time, a is the equatorial radius of the Earth, c is the velocity of light, and π is the solar parallax expressed in radians, or from the constant of aberration by the formula:

$$\tau = \frac{k}{n}(1 + \nu)^{-1}\cos\phi = 498^{s}\cdot 38$$

where k is the constant of aberration, n is the angular mean motion of the Earth, $0''\cdot04106\ 7043$ per second, $\sin\phi$ is the eccentricity of the Earth's orbit, 0·01672 63, and $1 + \nu$ is the mean distance of the Earth from the Sun, 1·00000 023. The method of calculating the planetary aberration used in the fundamental ephemerides of the Sun and planets is equivalent to using the smaller of the two values of τ, while for the physical ephemerides and satellites the larger of the two is conventionally used.

From the disagreement in the values of τ calculated by the two formulae it is seen that the adopted values of k and π are inconsistent with each other. In fact the product:

$$kc\pi = na(1 + \nu)\sec\phi = 1\cdot27010\ 64 \text{ m/sec}$$

is known with greater accuracy than any of the constants k, c, or π. It can therefore be used to find any one of them if the other two are known. The following table gives values of k resulting from use of the constant 1·27010 64 with several combinations of values of c and π.

c km/sec	π ″	k ″
299770	8·800	20·484
299790*	8·800	20·483
299860	8·800	20·478
299770	8·790	20·508
299790*	8·790	20·506
299860	8·790	20·501

* Modern laboratory determinations give 299 792·458.

Besides being inconsistent with the constant of aberration, the adopted value of the solar parallax is inconsistent with the mass ratio Sun: (Earth plus Moon) used in the tables of the Sun and planets. The mass ratio 329 390 corresponds to $8''\cdot79$ for the solar parallax, whereas that corresponding to $8''\cdot80$ is 328 270.

It was long supposed that an inconsistency existed between the values of the mass ratio Earth: Moon as inferred from the lunar inequality in the orbital motion

*See page 517.

of the Earth, and as inferred from the observed value of the constant of nutation. Sir Harold Jeffreys has shown, in a series of papers in the *Monthly Notices of the Royal Astronomical Society* beginning about 1950, that the old theory of the nutation, in which the Earth is considered to be a rigid body, is insufficient; it is necessary to take the departures from rigidity into account, and the subject is one of considerable difficulty.

A few other inconsistencies exist in the system of constants, but they are not so important as the ones mentioned here.

Aside from the inconsistencies in the system, some of the conventional values are shown by modern observations to be in error by appreciable amounts. The most important error is in the precession, the general centennial precession in longitude requiring a correction of about $+0''\cdot8$, due principally to the neglect of the galactic rotation in the derivation of the adopted value. The centennial change in the obliquity is known to be in error by a few tenths of a second, probably as a result of the neglect of perturbations of the second order in its calculation. The last two figures in the equatorial radius and the equatorial acceleration of gravity are without physical significance, and the flattening is probably closer to 1/298 than to 1/297. In Brown's theory of the motion of the Moon the flattening used in calculating the periodic perturbations is 1/294, the value that he found would reconcile the observed motions of the perigee and node with his calculations. It is probable, however, that his calculations failed to include other significant effects, and so the value 1/294 is not to be regarded as an actual determination of the flattening.

The errors in the adopted values of the constants do not impair the usefulness of the system in the slightest degree. The inconsistencies, on the other hand, are of some importance, since the investigator who is unaware of them may occasionally be led to erroneous conclusions, but a new system in which the inconsistencies had been removed would on the whole be much less valuable than the present one with all its imperfections. These facts deserve emphasis, because they seem to be peculiarly difficult of apprehension.

Most non-astronomers, and even many astronomers not working in the dynamics of the solar system, expect to find in a national ephemeris a list of constants the values of which, if they are not absolutely accurate, are at least as up to date as possible. The importance of self-consistency in the system is little appreciated, and the even greater importance of perpetuating a value that is known to be incorrect is admitted but seldom. The guardians of the system are accused of inaction and negligence, and even prejudice, by those who for one reason or another wish to introduce a new value for some constant.

The principal reason for retaining the system unchanged is a consequence of the methods necessarily employed in dynamical astronomy. The value of a constant is never measured directly. The method of differential corrections is employed instead. Observations made at various times are compared with an ephemeris. Analysis of the discrepancies between the observations and the ephemeris yields corrections to the values of the constants used in constructing the ephemeris, which being applied give more accurate values of the constants. If, during the period covered by the observations, the value of any constant entering into the calculation of the ephemeris has been altered, then the ephemeris at times before the alteration is inconsistent with the ephemeris at later times, and an analysis that fails to take account of the change is bound to lead to an erroneous conclusion. What must be done is either to recalculate one portion of the ephemeris, or to make the analysis of the two portions separately, combining the two results at the end. But in practice the alteration of the ephemeris has often been unknown to the investigator, especially when different portions of it have been calculated at different times by different persons. Even in cases where the alterations are known, it may be very laborious to make the analysis properly, and in any case considerable care is necessary.

During the nineteenth century the advantages of continuity in an ephemeris were not appreciated even by the compilers, and during the twentieth century some changes have been made that in retrospect hardly appear to have been justified. It would, however, be going too far to conclude that the present system should be retained for ever. It should be revised eventually, not piecemeal but as a whole, and when it is done care should be taken that there is no contradiction between the revised system and the theories of the motions of the four inner planets.

At a conference on the fundamental constants of astronomy, held at Paris in 1950, it was unanimously recommended to retain the present conventional values of the constants comprising the system, and this recommendation was approved at the General Assembly of the International Astronomical Union in 1952 (*Trans. I.A.U.*, **8,** 66, 1954).

It is useful to distinguish at least three different values of any of the constants: (a) the conventional value comprising part of the model or system of reference, to which observations are referred, and which remains unchanged for long periods of time, (b) the observed value, which changes with each new determination, and (c) an adjusted value, which rigorously satisfies the theoretical relations with the adjusted values of other constants, and which agrees with the observed value within the tolerance set by the observational errors.

References

The first thorough systematic treatment of astronomical constants seems to be that of William Harkness in " The Solar Parallax and its Related Constants including the Figure and Density of the Earth " (*Washington Observations for* 1885, Appendix III, Washington, 1891). Although it is necessarily out of date in several respects it is still worthy of study by everyone interested in the subject.

Simon Newcomb's exhaustive study of the system of constants is contained in *The Elements of the Four Inner Planets and the Fundamental Constants of Astronomy* (Supplement to *The American Ephemeris and Nautical Almanac* for 1897, Washington, 1895). To a considerable extent it formed the basis of discussion at the Paris Conference of 1896 (*Procès-Verbaux* of the *Conférence Internationale des Étoiles Fondamentales de* 1896, Bureau des Longitudes, Paris, 1896). The result of this conference was to unify the basis for the construction of astronomical ephemerides throughout the world. A further conference, in 1911, (*Congrès International des Éphémérides Astronomiques*, Bureau des Longitudes, Paris, 1912) provided for the division of the work of constructing the various national ephemerides, and also adopted values for a few constants.

In a series of papers in the *Bulletin of the Astronomical Institutes of the Netherlands* W. de Sitter developed the first rigorous treatment of the system of constants in which a departure of the Earth from a concentrically homogeneous oblate spheroid was taken into account. The culmination of this work was edited and completed by Dirk Brouwer after de Sitter's death (*Bulletin of the Astronomical Institutes of the Netherlands*, **8**, 213, 1938). The subject was treated by G. M. Clemence (*The Astronomical Journal*, **53**, 169, 1948) using de Sitter's theory with different values of the constants and a different treatment of the measure of time, and with particular attention to the inconsistencies in the conventional system.

Reference should be made to the proceedings of a third international conference held in Paris in 1950, '. . . sur les Constantes Fondamentales de l'Astronomie' (*Bulletin Astronomique*, **15**, 163-292, 1950). It was the work of this conference that led to the redefinition of the second and the recognition of ephemeris time.

Additional references (1973)

Proceedings of IAU Symposium No. 21 (Paris, 1963) on "The System of Astronomical Constants", *Bull. Astron.* **25**, 1–324, 1965.

Proceedings of IAU Joint Discussion F (Hamburg, 1964) on "The IAU System of Astronomical Constants", *Trans. IAU*, **12B**, 593–625, 1966.

Proceedings of IAU Colloquium No. 9 (Heidelberg, 1970) on "The IAU System of Astronomical Constants", *Celestial Mechanics*, **4**, no. 2, 128–280, 1971.

7. HISTORICAL LIST OF AUTHORITIES

A. INTRODUCTION

When the contents of the separate supplement were first being considered, there was a demand for a comprehensive list of authorities, including numerical values of the constants used, for the major ephemerides in each edition of *The Nautical Almanac* since 1767. At this time these ephemerides represented the only readily available basis for the comparison of observation with theory; and they are only suitable for this purpose if full allowance is made for the many different authorities and constants on which the ephemerides are based. Generally, but not always, the authorities and constants used can be found from the *Preface* or *Explanation* of the edition concerned, or of some earlier edition; in some cases, ambiguities can only be resolved by recalculation, since records of the actual calculations have not been preserved. Considerable work was done, as opportunity offered, to collect and synthesize this information.

With the availability of fast computing machines it has become practicable to calculate ephemerides, over the whole period covered by accurate observation, from current theories; such are in fact already available for the Sun (*A.P.A.E.*, **14**, 1953), Venus (*A.P.A.E.*, **15**, part III, 1955) and the five outer planets (*A.P.A.E.*, **12**, 1951).

The tabulated ephemerides have therefore been largely superseded and will not again be generally used for the comparison of observation with theory.

A knowledge of the basis of the ephemerides is, however, essential for the proper interpretation of the results of past discussions. For this reason, and for purposes of historical interest and record, the list of authorities is given in the form originally intended. But a few uncertainties and ambiguities, which could only have been resolved by excessive research or recalculation, have been allowed to remain.

The material is arranged in three main sub-sections: up to 1900, *The Nautical Almanac* and *The American Ephemeris* are treated separately; but after 1900 they are combined. Each sub-section is divided according to the body or subject (e.g. Sun, Moon, Precession, Nutation, Constants); within each division the authorities, arranged chronologically, are preceded by a short narrative of the quantities tabulated. In this narrative " nD " is used to indicate that the quantity is tabulated to n decimal places, and the term " precision " is used to indicate merely the unit of the end figure. Names and dates only are usually given for the authorities; full references are given in sub-section E. In some cases, for example for the adopted semi-diameters of the planets, detailed references to the original publications have been omitted.

Some of the tabulated ephemerides are based on theories, derivations, and constants given in *Appendices* and *Supplements* to *The Nautical Almanac* and to *The American Ephemeris*. No lists of these appendices and supplements are readily available, and the opportunity is therefore taken of including, in sub-section F, complete lists of all such appendices and supplements with a brief description of their contents. Details of miscellaneous ephemerides of auxiliary quantities that were occasionally given in appendices or supplements have been omitted from this section.

B. LIST OF AUTHORITIES FOR TABULATIONS IN *THE NAUTICAL ALMANAC*, 1767-1900

1. Sun

All ephemerides from 1767 to 1833 were given with argument apparent time. Quantities tabulated for the Sun at intervals of one day were longitude and declination (each to $1''$), right ascension and equation of time (each to 1^s). Semi-diameters in arc (to $0''\cdot1$) and in time (to $0^s\cdot1$) and log distance (to 6D) were given at intervals of 6 days. From 1768 the equation of time was given to $0^s\cdot1$, and from 1772 the right ascension was also given to $0^s\cdot1$. For the years 1815–1822, the log distance was given only to 5D. In 1833 the semi-diameters and log distance were given at intervals of one day, to $0''\cdot01$, to $0^s\cdot01$, and to 7D.

The Almanac for 1834 was largely remodelled in accordance with the (Royal) Astronomical Society's report printed in that Almanac, and thereafter the argument of most ephemerides was mean time. Most quantities tabulated in time were given to $0^s\cdot01$, those in arc to $0''\cdot1$ (except the Sun's latitude, given to $0''\cdot01$), and the log radius vector to 7D. In many cases differences or variations were given. From 1848 equatorial rectangular coordinates were included, at intervals of one day and to 7D, the latitude terms being included for the first time in 1866; the values for 1845-1847 were given in the 1848 volume. No other substantial change was made before 1900.

1767–1796: Mayer's "last manuscript tables" (which assumed an annual precession of $50''\cdot3$).

1797–1804: Mayer's tables, with the mean motion corrected to the revised precession of $50''\cdot2$.

1805–1812: Delambre's tables, as given in Lalande (1792), but with certain (unspecified) coefficients determined by Maskelyne.

1813–1821: Improved tables by Delambre (1806).

1822–1832: The tables in Vince (1808, Volume III), "with the omission only of some equations which do not materially affect the results". The tables are stated by Vince

(Volume III, page 2) to have been " constructed by M. de Lambre, from the observations of Dr. Maskelyne, and the theory of M. Laplace. See Les Mémoires de l'Académie de Berlin, for 1784, 1785". [In 1832 the position of the Sun for the calculation of the transit of Mercury (and for no other purpose) was taken from Carlini's tables, (see below) corrected.]

1833: The longitude was taken from Delambre's tables, improved by Airy's corrections based on Greenwich observations.

1834–1835: Carlini's tables (1810) with Bessel's corrections (1828) and nutation as in the Astronomical Society's tables (Baily, 1825). [The elements used by Carlini are the same as those of Delambre (1806), but the arrangement is better for the construction of an ephemeris.]

1836–1863: Carlini (1832).

1864–1900: Leverrier (1858).

2. Moon

The Moon's longitude and latitude, semi-diameter, and horizontal parallax (each to 1″) and its right ascension and declination (each to 1′) were tabulated at intervals of 12^h (apparent time) for the years 1767–1833. Lunar distances (at least one star, and from 1770 one or two stars, as well as the Sun when conveniently placed) were given to a precision of 1″ for every 3^h. From 1823 the right ascension and declination were given to 1″. In 1834 the argument became mean time and, with occasional minor alterations, the tabulations were given to an extra figure until the year 1900. The right ascension (to $0^s{\cdot}01$) and the declination (to $0''{\cdot}1$) were given at intervals of one hour.

1767–1776: Mayer's last manuscript tables.

1777–1788: Mayer's tables, improved by Mason under Maskelyne's direction, based on Bradley's observations (the latter are printed in *N.A.*, 1774).

1789–1796: Mayer's tables, further improved by Mason (1780). Eight new equations were taken from Mayer's tables, the coefficients being determined from Bradley's observations. The 18th equation in longitude was omitted.

1797–1804: The same set of tables, but adjusted (as for the Sun) for the corrected value of precession.

1805–1807: Lalande (1792); the tables are the same as Mason (1780) except for the substitution of Laplace's acceleration and secular motion.

1808: Lalande's tables, with the addition of two further inequalities found by Laplace.

1809–1812: The epochs, Laplace's accelerations, and " a particular equation of his " were taken by Maskelyne from Burg's tables (see below) and hence the mean longitudes were computed. The parallax was taken from Mayer.

1813–1817: Burg (1806) on Laplace's theory, the coefficients being determined from Maskelyne's observations, and the epochs from those of Maskelyne and Bradley.

1818–1820: According to Pond's Preface, the tables of Burckhardt were used. [But a note (initialled T.Y.) at the end of the 1820 Preface states that those of Burg were used.]

1821–1833: Burckhardt (1812).

1834–1855: Burckhardt's tables, with nutation from Baily (1825).

1856: As in the previous years, but the parallax taken from Adams (1853b) and the semi-diameter taken as 0·2725 times the horizontal parallax.

1857–1861: The ratio of semi-diameter to horizontal parallax was changed to 0·273114.

1862–1882: Hansen (1857).

1883–1895: Hansen, but with Newcomb's corrections (1878b) included in the right ascension and declination.

1896: As in previous years, and with the substitution of Newcomb's Table XXXIV for Hansen's.

1897–1900: Newcomb's corrections included in horizontal parallax and semi-diameter.

3. Major planets

Ephemerides of the five " classical " planets were given at intervals of 6 days (Mercury, 3 days from 1778) until 1832. Those of Uranus (at intervals of 10 days) were introduced in 1789 and again from 1791 onwards. The adopted precision was 1′ for declination and both heliocentric and geocentric longitudes and latitudes. When the right ascension was added, in 1819, a precision of 1^m was used.

Heliocentric coordinates were omitted in 1833, while declination and geocentric longitude and latitude were given to 1″, right ascension to $0^s{\cdot}1$, and log distance to 5D, all at intervals of one day.

The intervals were changed in 1834 to one day for all planets (in 1861, when Neptune was introduced, to 4 days for Uranus and Neptune). Geocentric longitude and latitude were omitted, and the quantities were tabulated to $0^s{\cdot}01$ for right ascension, to 0″·1 for declination and heliocentric longitude and latitude, and to 7D for log distance and log radius vector. A geocentric (equatorial) ephemeris of Neptune was published between 1850 and 1860, at intervals of 5 days, usually as an appendix to later Almanacs.

Transit ephemerides were introduced in 1839 for Mercury to Uranus, and in 1861 for Neptune.

Mercury, Venus, Mars, Jupiter, Saturn

1767–1779: Halley (1749).

1780–1804: Wargentin's tables, " annexed to M. De la Lande's Astronomy ".

1805–1833: Lalande (1792). These are the tables calculated by Delambre on the theory of Laplace. The tables of Mars from 1822 were taken from " those of Lalande in the *Connaissance des Tems* [sic] for the 12th year [1803–04] "; the places of Mercury for the transit of 1832 from Lindenau's tables (see below).

Mercury

1834–1863: Lindenau (1813).
1864–1900: Leverrier (1859).

Venus

1834–1864: Lindenau (1810). For the years 1837–1848 a correction of −2′ 18″ was applied to the tabular longitude of the node.
1865–1900: Leverrier (1861a).

Mars

1834–1865: Lindenau (1811).
1866–1900: Leverrier (1861b).

Jupiter

1834–1877: Bouvard (1821).
1878–1900: Leverrier (1876a).

Saturn

1834–1879: Bouvard (1821). [For the years 1852–1879, Bouvard's Table 42 was used in the corrected form given by Adams (1849) and in *N.A.* 1851, xiv.]
1880–1900: Leverrier (1876b).

Uranus [" The Georgian " in *N.A.* 1789–1850]

1789: }
1791–1833: } As for the " classical " planets.
1834–1876: Bouvard (1821).
1877–1881: Newcomb (1873).
1882–1900: Leverrier (1877a).

Neptune

1850–1857: Computed from elements given in various issues of the *Berliner Jahrbuch* or *The Nautical Almanac*.
1858–1870: Kowalski (1855). [This is a little uncertain for the years 1859–60, as the supplements to the almanacs containing these ephemerides do not quote the authority.]
1871–1881: Newcomb (1865).
1882–1900: Leverrier (1877b).

4. Minor planets

Ephemerides of minor planets were given for the first time in the Almanac for 1834. That issue contained ephemerides, at intervals of 4 days throughout the year, of the first four planets, based on elements by Encke. The right ascension was tabulated to $0^{m}\cdot1$, declination and heliocentric longitude and latitude to $1'$, log distance and log radius vector to 4D. For one month on each side of opposition, at intervals of one day, the right ascension was given to $0^{s}\cdot01$, declination to $0''\cdot1$, log distance and log radius vector to 5D.

Similar ephemerides for the years to 1849 were based on the same elements, with variations calculated by the method given by Airy (1835).

Between 1850 and 1866 the number of planets, for which ephemerides at wider intervals were published, was increased, in some years to as many as 36, the elements used being due to a number of different authors. [The Almanac for 1856 contains a translation (by Airy) of papers by Encke (1852 a, b) on the computation of special perturbations.]

From 1867 the number of planets was decreased to five, and from 1876 to the first four, on the ground that more accurate ephemerides were to be published in the *Berliner Jahrbuch*. The elements used were:

Ceres, 1867–1881: Schubert (1854).
Ceres, 1882–1900: Godward (1878).
Pallas, 1867–1900: Farley (1856a).
Juno, 1867–1893: Hind (1855).
Juno, 1894–1900: Hind (1855), with corrections by Downing (1890).
Vesta, 1867–1900: Farley (1856b).
Astraea, 1867–1875: Farley (1856c).

5. Auxiliary quantities

Sidereal time

The sidereal time was not tabulated explicitly for the first sixty or seventy years, but from 1833 values were given at intervals of one day. The sidereal time at mean noon is stated to have been calculated from the following expressions:

1833: Sun's mean longitude $+ 6''\cdot0 - 16''\cdot5 \sin\Omega - 0''\cdot917 \sin 2\odot$

1834–1900: Sun's mean longitude + nutation, where the Sun's mean longitude at Paris mean noon of January 0 of the year 1800 + t is given by Bessel (1830a, p.xxiv) as:

$$279^\circ\ 54'\ 01''\cdot36 + 27''\cdot60584\ 4\ t + 0''\cdot00012\ 21805\ t^2 - 14'\ 47''\cdot083\ f$$

f being (for the 19th century) the number of years from the preceding leap year.

Mean obliquity of the ecliptic

The values used were the following (t being measured in years):

1767–1807: $23^\circ\ 28'\ 16'' - x(t - 1756)$ Mayer (1770, pp. 105 and [v])

x was stated by Maskelyne in several almanacs to be about half a second, but Mayer's table indicates $0''\cdot46$. The values seem to have been adjusted occasionally by Maskelyne.

1808–1833: Corrected year by year, from Greenwich observations to a current date.

1834–1863: $23^\circ\ 27'\ 54''\cdot8 - 0''\cdot457\ (t - 1800\cdot0)$ Bessel (1830a, p. xxvii)

1864–1900: $23^\circ\ 27'\ 31''\cdot83 - 0''\cdot476\ (t - 1850\cdot0)$ Leverrier (1858, p. 203)

The authorities for the values of the obliquity adopted for the conversion of the Moon's longitude and latitude to right ascension and declination were (see A. M. W. Downing, *M.N.R.A.S.*, **69**, 618, 1909):

1862–1874: Hansen (1857, p. 45; see Hansen and Olufsen, 1853, p. 5)

1875–1900: Leverrier (1858, p. 203)

Reducing all these to a common date, for comparison with Peters (1842) and Newcomb (1895a), we have:

Mayer:	$23^\circ\ 27'\ 29'' - 0''\cdot46\ (t - 1850\cdot0)$
Bessel:	$23^\circ\ 27'\ 31''\cdot95 - 0''\cdot457\ (t - 1850\cdot0)$
Leverrier:	$23^\circ\ 27'\ 31''\cdot83 - 0''\cdot476\ (t - 1850\cdot0)$
Hansen and Olufsen:	$23^\circ\ 27'\ 31''\cdot42 - 0''\cdot46784\ (t - 1850\cdot0)$
Peters:	$23^\circ\ 27'\ 30''\cdot99 - 0''\cdot4645\ (t - 1850\cdot0)$
Newcomb:	$23^\circ\ 27'\ 31''\cdot68 - 0''\cdot468\ (t - 1850\cdot0)$

Apparent obliquity of the ecliptic

Values of the apparent obliquity (to $0''\cdot1$) at intervals of three months were published from the inception of the Almanac; in 1817 and 1818, and again from 1834, the interval was changed to 10 days, and in 1834 the precision was changed to $0''\cdot01$. From 1876 to 1895 the short-period terms of nutation were included in the apparent values, which were tabulated at intervals of one day.

Precession

Mayer's value (1770, p. [52]) of $50''\cdot3$ was used in the Almanac from 1767 to 1796, but was corrected to $50''\cdot2$ from 1797 to 1833, with a corresponding adjustment of Mayer's values of the mean motions of the Sun and Moon.

Between 1834 and 1853 there is no specific statement of the values used, but from 1854 to 1895 the annual (and daily) increments were given (to 0″·0001), and the precession from the beginning of the year was tabulated (to 0″·01) at intervals of 10 days. No authority is quoted for these figures.

From 1896 to 1900, Peters' value (1842, p. 71) is stated to have been used as the authority.

The following comparison shows the various values of the annual precession that were used (T being measured in centuries from 1850·0):

1854–1856 (deduced):	$50''.2357 + 0''.025T$
1857–1895 (deduced), 1896–1900 (Peters):	$50''.2524 + 0''.0227T$
1901–1959 (Newcomb):	$50''.2453 + 0''.0222T$

Nutation

1767–1833: The "Equation of the Equinoctial Points" (nutation in longitude) was tabulated (to 0″·1) for every three months, but without any indication of the authority or of the terms included. In the years 1817 and 1818 the same "Equation ... in Sidereal Time" (or nutation in right ascension) was given (to 0^s·01) at intervals of ten days.

1834–1856: The tabulated values (to 0″·01 in longitude and 0^s·01 in right ascension) at intervals of 10 days, were based on Baily's (1825) values and included the four terms numbered 1, 2, 3, 14 in table 7.1.

1857–1880: The same terms were tabulated, and the precision and the interval of tabulation were unchanged, but the coefficients were based on Peters (1842).

1881–1892: Nutation in obliquity was also included, to a precision of 0″·01.

1893–1895: Two additional terms (nos. 5 and 7) were included in the tabulations.

1896: Long-period and short-period terms in both longitude and obliquity were included in the tabulations, the interval of which was one day. Term no. 15 was included.

1897–1900: Nine additional terms (nos. 6, 8–13, 17, 18) were included.

Constants

The following values of the principal constants have been used:

Solar parallax

1834–1869: 8″·5776	Encke (1824, p. 108)
1870–1881: 8″·95	Leverrier (1858, p. 114)
1882–1900: 8″·848	Newcomb (1867, p. 29)

Constant of aberration

1834–1849: 20″·36	Baily (1825, p. x)
1850–1856: 20″·42	Baily (1845, p. 21)
1857–1900: 20″·4451	Struve (1844, p. 275)

Constant of nutation

1834–1856: 9″·25 Baily (1825, p. xiv)

1857–1900: $9''.2231 + 0''.0009T$, where T is in centuries from 1800·0. Peters (1842, p. 75)

Semi-diameters at unit distance

Sun		
1767–1807:	962″·8	Mayer (1770, p.[56])
1808–1833:	961″·37	
1834–1852:	960″·9	Bessel (1830a, p. L)
1853–1895:	961″·82	Airy (1855, p. lxxviii)
1896–1900:	961″·18	Auwers (from observations at Greenwich, 1851–1883)
	959″·63	Auwers (1891, p. 367) [This value was used for eclipse calculations only]
Mercury		
1834–1863:	3″·23	Lindenau
1864–1900:	3″·34	Leverrier
Venus		
1834–1864:	8″·25	Delambre
1865–1895:	8″·305	Leverrier
1896–1900:	8″·40	Auwers
Mars		
1834:	4″·57	
1835–1865:	4″·435	Littrow
1866–1895:	5″·55 [sic]	Leverrier
1896–1900:	4″·68	Hartwig
Jupiter (equatorial)		
1837–1881:	99″·704	Struve
1882–1895:	98″·19	Leverrier
1896–1900:	97″·36	Schur
Jupiter (polar)		
1834:	93″·37	
1835–1856:	93″·4	Delambre
1857–1881:	92″·426	Equatorial S.D. × 0·927
1882–1895:	92″·200	Equatorial S.D. × 0·939
1896–1900:	91″·10	Schur
Saturn (equatorial)		
1834:	88″·72	
1837–1881:	81″·106	Bessel
1882–1895:	83″·31	Leverrier
1896–1900:	84″·75	Meyer
Saturn (polar)		
1835–1856:	75″·25	Bessel
1857–1881:	75″·19	Equatorial S.D. × 0·927
1882–1895:	74″·56	Equatorial S.D. × 0·895
1896–1900:	76″·88	Meyer
Uranus		
1834:	37″·20	
1835–1881:	37″·25	Delambre
1882–1895:	34″·28	Leverrier
1896–1900:	34″·28	Hind
Neptune		
1899–1900:	36″·56	Barnard; however, no values were tabulated

This table gives the values adopted, beginning with each year shown, of the coefficient of sine (Argument) in the nutation in longitude ($\Delta\psi$), and of cosine (Argument) in the nutation in obliquity ($\Delta\epsilon$); a blank indicates that the term was not included in the tabulations for that year, a leader (...) that the coefficient used was the same as in the preceding entry. The sub-heading $\Delta\epsilon$ is omitted for those terms that do not occur in the obliquity. Terms numbered 1 to 13 all have periods greater than 100 days, and are known as long-period terms; numbers 14 to 25 have periods shorter than 35 days, and are known as short-period terms.

Term No.	1		2		3		14	
Argument	☊		2☊		$2L$		2☾	
	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\epsilon$
	″	″	″	″	″	″	″	″
1834	−17·2985	+9·2500	+0·2082	−0·0903	−1·2550	+0·5447	−0·2074	+0·0900
1857	−17·2524	+9·2236	+0·2063	−0·0895	−1·2691	+0·5507	...	+0·0885
1896	...	...	...	...	...	...	−0·2041	...
1901	−17·236	+9·210	+0·209	−0·090	−1·257	+0·546	−0·204	+0·088
1903	−17·235	...	...	...	−1·270	+0·551	...	...
1937	−17·234 − 0·017T	+9·210 + 0·0009T	...	...	−1·272	...	...	...

None of the remaining coefficients is as large as 1″; the zero in the units place is consequently omitted, to save space.

Term No.	4	5		6		7		8	
Argument	$L - \Gamma$	$L + \theta$		$3L - \Gamma$		$L + \Gamma$		$2L - $ ☊	
	$\Delta\psi$	$\Delta\psi$	(θ)	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\epsilon$*	$\Delta\psi$	$\Delta\epsilon$
	″	″	° ′	″	″	″	″	″	″
1893		+·1476	82 02				+·0093		
1897		...	...	−·0058	+·0027		...	+·0125	−·0067
1901		+·110	75·3	−·049	+·021		−·009		
1903		+·107	74·4	−·050	...				
1912		...	74·3	...	+·022		...	+·012	
1918	+·126			...	...	+·021	...	...	−·007

Term No.	9		10	11	12		13	
Argument	$2\Gamma' - $ ☊		$2L - 2\Gamma'$	$2L - 2$☊	$\Gamma' + 90°$		$2\Gamma'$	
	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\psi$	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\epsilon$
	″	″	″	″	″	″	″	″
1897	+·0044	−·0024	+·0053	−·0024	+·0026	−·0023	+·0020	−·0008
1901								
1937	+·005	−·003	+·004					

Term No.	15	16	17		18		19	20	
Argument	☾ $- \Gamma'$	2☾ $- 2\Gamma'$	2☾ $-$ ☊		3☾ $- \Gamma'$		☾ $- 2L + \Gamma'$	☾ $+ \Gamma'$	
	$\Delta\psi$	$\Delta\psi$	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\psi$	$\Delta\epsilon$
	″	″	″	″	″	″	″	″	″
1896	+·0677								
1897	...		−·0339	+·0181	−·0261	+·0113			
1901	+·067		−·034	+·018	−·026	+·011	+·015	+·012	−·005
1903	+·068		...	...	...	...	...	+·011	...
1937	...	+·003	...	...	...	...	...	...	...

Term No.	21	22		23		24		25	
Argument	2☾ $- 2L$	☾ $- \Gamma' +$ ☊		☾ $- \Gamma' -$ ☊		3☾ $- 2L + \Gamma'$		3☾ $- \Gamma' -$ ☊	
	$\Delta\psi$	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\epsilon$	$\Delta\psi$	$\Delta\epsilon$
	″	″	″	″	″	″	″	″	″
1901	+·006								
1937	...	+·006	−·003	+·006	+·003	−·005	+·002	−·004	+·002

* The sign of term no. 7 in $\Delta\epsilon$ is given incorrectly in the Almanac from 1893 to 1900.

C. LIST OF AUTHORITIES FOR TABULATIONS IN *THE AMERICAN EPHEMERIS*, 1855–1900

During the period 1855–1900, the twelve monthly sections that formed the principal content of the first part of *The American Ephemeris* contained the Greenwich ephemerides of the Sun and Moon; these sections remained virtually unchanged throughout the period. Following them in the first part of the volume were Greenwich ephemerides of Venus, Mars, Jupiter, and Saturn; similar ephemerides for Mercury, Uranus, and Neptune, and heliocentric ephemerides for all seven planets were added in 1882. An ephemeris of the rectangular coordinates of the Sun was also included, although for the period 1875–1881 it was relegated to the second part.

The second part contained the ephemeris for the meridian of Washington, and included further ephemerides of the Sun, Moon, and planets, partly for Washington noon and midnight and partly for meridian transit at Washington. These ephemerides, which were revised somewhat from time to time, included a tabulation of the obliquity, precession, and nutation until 1882, when it was transferred to the first part of the volume.

1. Sun

In the Greenwich ephemerides of the Sun, the apparent right ascension (to $0^s{\cdot}01$), the declination (to $0''{\cdot}1$), and the equation of time (to $0^s{\cdot}01$) were tabulated at intervals of one day for apparent noon and (except in 1855) for mean noon; the semi-diameter (to $0''{\cdot}01$) and the sidereal time of semi-diameter passing the meridian (to $0^s{\cdot}01$) were given for apparent noon; and (except in 1855) the longitude (to $0''{\cdot}1$) referred both to the true equinox of date and to the mean equinox of the beginning of the year, the latitude (to $0''{\cdot}01$), and the log radius vector (to 7D) were given for mean noon. In the Washington ephemerides, the right ascension and declination were given for mean and apparent noon; the equation of time, the semi-diameter, and the sidereal time of semi-diameter passing the meridian were given for apparent noon; and during 1855–1881, the longitude, latitude, and log radius vector were given for Washington mean noon and midnight. The tabular precisions were the same as in the Greenwich ephemerides.

During the period 1855–1881 the equatorial rectangular coordinates of the Sun (to 7D) were tabulated for Greenwich mean noon, referred to the true equinox and equator of date, and also for Washington mean noon and midnight, referred both to the true equinox of date and the mean equinox of the beginning of the year. In 1882, these ephemerides were replaced by a tabulation (to 7D) for Greenwich mean noon and midnight, referred both to the true equinox of date and mean equinox of the beginning of the year.

The horizontal parallax (to $0''{\cdot}01$) and aberration (to $0''{\cdot}01$) of the Sun were tabulated at intervals of 10 days, for 0^h Washington sidereal time during 1855–1864, for Washington mean noon during 1865–1881, and for Greenwich mean noon thereafter.

1855–1857: Carlini (1810), with Bessel's revisions (1828).
1858–1874: Hansen and Olufsen (1853).
1875–1899: Hansen and Olufsen (1853), with aberration according to Struve (1844).
1900: Newcomb (1895a).

2. Moon

In the Greenwich ephemerides of the Moon, the right ascension (to $0^s{\cdot}01$) and declination (to $0''{\cdot}1$) were tabulated for every hour; the semi-diameter and horizontal parallax (to $0''{\cdot}1$) were given for noon and midnight; and, during 1860–1900, the longitude and latitude (to $0''{\cdot}1$) were given for noon and midnight.

In the Washington ephemerides, the right ascension (to $0^s{\cdot}01$) and declination (to $0''{\cdot}1$) were tabulated for upper and lower culmination during 1855–1864, and for upper culmination during 1882–1900; the sidereal time of semi-diameter passing the meridian (to $0^s{\cdot}01$) was given for both culminations during the period 1855–1864, but only for upper culmination after 1864. The semi-diameter and horizontal parallax (to $0''{\cdot}1$) were tabulated for Washington mean noon and midnight during 1855–1881, and for upper culmination during 1882–1900.

The times of the phases, apogee, and perigee were given both in Greenwich mean time and in Washington mean time. The mean longitude and the longitude of the ascending node were given at intervals of 10 days.

1855–1856: Peirce (1853). These tables are based on Airy (1848), with corrections by Airy (1849) and Longstreth (1853). The tables used by Airy (1848) were derived from Damoiseau (1824), and were sub^tantially a development of Plana's theory (1832), modified to include two Venus inequalities discovered by Hansen (1847).
1857–1882: Peirce (1853), with parallax from tables based on formulae of Adams (1853a) and Walker (1848).
1883–1900: Hansen (1857) with corrections by Newcomb (1878b). [In the *Introduction* to *The American Ephemeris* for 1912 and following years, attention is called to the fact that these corrections were not precisely in accordance with the statement given in the volumes for 1883–1911, and the formula actually used is given.]

3. Major planets

In the Greenwich ephemerides of the planets, the apparent right ascension (to $0^s{\cdot}01$) and declination (to $0''{\cdot}1$) were tabulated for mean noon at intervals of one day for Mercury (after 1881), Venus, Mars, Jupiter, and Saturn, and (also after 1881) at intervals of 4 days for Uranus and Neptune. The time of meridian passage (to $0^m{\cdot}1$) was also given; the semi-diameter and horizontal parallax (to $0''{\cdot}1$ in general; to $0''{\cdot}01$ for Uranus and Neptune, and in 1900 for Mercury, Venus, and Mars) were tabulated at various different intervals for the different planets.

In the Washington ephemerides, the apparent right ascension (to $0^s{\cdot}01$) and declination (to $0''{\cdot}1$) were tabulated, during 1855–1869, for the inferior planets at Washington mean noon and meridian transit, and for the superior planets at Washington sidereal noon and meridian transit; from 1870, mean noon and meridian transit were the times used for all the planets, and from 1882 the ephemerides for noon were omitted. The semi-diameter and horizontal parallax (to 0″·01) were given for 0^h Washington sidereal time during 1855–1864, for Washington mean noon during 1865–1881, and (to 0″·1) for Washington meridian transit from 1882. The sidereal time of semi-diameter passing the meridian was tabulated to 0^s·01 throughout.

Included in the second part of the volume during 1855–1881 were also heliocentric ephemerides of the planets. The quantities tabulated were the rectangular coordinates (to 4D for Mercury, Venus, Mars, and Neptune; to 5D for Jupiter, Saturn, and Uranus), the log radius vector (to 4D for Mercury, Venus, and Mars; to 5D for the outer planets), and the orbital longitude (to 0′·1 for the inner planets; to 1″ for the outer planets); the attractions on the Sun were added in 1861. During the period 1855–1860 the rectangular coordinates were referred to the equinox and equator of date; beginning with 1861, they were referred to the ecliptic and mean equinox of a selected epoch, and the coordinates of the Earth were included. A table of the adopted masses and the orbital inclinations and nodes was also given.

In 1882, these heliocentric ephemerides were replaced by different ones which followed immediately after the geocentric ephemerides in Part I and contained the longitude and latitude (to 0″·1) referred to the ecliptic and mean equinox of date, the reduction to orbit, log radius vector (to 7D; in 1900, to 8D for Venus and Mars), and log geocentric distance (to 7D), for Greenwich mean noon at intervals of 8 days for Uranus and Neptune, 2 days for Mercury (one day in 1900), and 4 days for the other planets (2 days for Venus and Mars in 1900); log geocentric distance was also given for the dates intermediate between the tabular dates.

During the years preceding the completion of the planetary tables of Newcomb and Hill, the ephemerides were calculated with tables that for the most part were constructed in the Office by applying corrections to the early tables of Lindenau and Bouvard that had been based upon Laplace's theories.

Mercury

1855–1899: Winlock (1864), based on the theory of Leverrier (1845).
1900: Newcomb (1895b).

Venus

1855–1875: Manuscript tables prepared from Lindenau (1810) by applying corrections based on investigations by Airy (1832), Breen (1848), and Leverrier (1841).
1876–1899: Hill (1872). [An ephemeris for 1874–1875 calculated from these tables is given in the Appendix to the 1876 volume.]
1900: Newcomb (1895c).

Mars

1855–1899: Manuscript tables, based on Lindenau (1811), with corrections from Breen (1851) and Leverrier (1841), and various other corrections from time to time.
1900: Manuscript tables, based on the elements derived by Newcomb (1895d).

Jupiter

1855–1897: Manuscript tables, prepared from Bouvard (1821), with corrections to make them agree with observation.
1898–1900: Hill (1895a). [In the 1898 volume, it is incorrectly stated that Bouvard's tables were used for that year.]

Saturn

1855–1882: Manuscript tables prepared from Bouvard (1821), with various corrections from time to time.

1883–1899: Manuscript tables prepared from a provisional theory by Hill (1890).

1900: Hill (1895b).

Uranus

1855–1875: Bouvard (1821), with revisions by Leverrier (1846) and Peirce (1848a), and, beginning with 1859, further corrections by Runkle (1855).

1876: Manuscript tables constructed by Newcomb.

1877–1900: Newcomb (1873). [An ephemeris for 1873–1876 calculated with these tables is given in the 1877 volume.]

Neptune

1855–1869: Tables based on Peirce's theory (1848b) and Walker's elements (1848). [Ephemerides for 1853 and 1854 were given in the volumes for 1855 and 1856, respectively.]

1870–1900: Newcomb (1865). [An ephemeris for 1866–1869 calculated from these tables is given in the Appendix to the 1869 volume.]

4. Minor planets

Among the early tables constructed and printed for *The American Ephemeris* were tables of the minor planets (15) Eunomia, (40) Harmonia, (18) Melpomene, and (11) Parthenope. Ephemerides of these minor planets were not included in *The American Ephemeris*, but an " Asteroid Supplement " in the volume for 1861 contained opposition ephemerides for 33 minor planets for 1859, and the orbital elements of (1) — (55).

5. Auxiliary quantities

Sidereal time

The sidereal time was tabulated (to $0^s\cdot01$) for every Washington mean noon, and (except in 1855) for every Greenwich mean noon. In the Greenwich ephemeris, except in 1855, the mean time of 0^h sidereal time (to $0^s\cdot01$) was also given for every day.

Obliquity of the ecliptic

The authorities and adopted expressions for the mean obliquity were:

1855–1881: Peters (1842).

$$23^\circ\ 27'\ 54''\cdot22 - 0''\cdot4645t - 0''\cdot00000\ 14t^2$$

where t is reckoned in years from 1800.

1882–1899: Hansen and Olufsen (1853).

$$23^\circ\ 27'\ 31''\cdot42 - 0''\cdot46784t$$

where t is reckoned in years from 1850.

1900: Newcomb (1895d).

$$23^\circ\ 27'\ 08''\cdot26 - 0''\cdot468t$$

where t is reckoned in years from 1900.

The apparent obliquity was tabulated (to $0''\cdot01$) at intervals of 10 days (5 days in 1900), for 0^h Washington sidereal time during 1855–1864, for Washington mean noon during 1865–1881, and for Greenwich mean noon thereafter.

The obliquity actually used in calculating the ephemerides of the Sun, Moon, and planets in the volumes for 1865–1899 was taken from Hansen and Olufsen (1853). In 1900, it was taken from Newcomb (1895a).

Precession

For the general precession in longitude, the expression given by Peters (1842):

$$50''\cdot 2411 + 0''\cdot 00022\ 68\ t$$

where t is reckoned in years from 1800, was used until in 1900 the value given by Newcomb (1895d) was adopted:

$$50''\cdot 2482 + 0''\cdot 00022\ t$$

where t is reckoned in years from 1900.

The amount of precession since the beginning of the year was tabulated (to 0″·01) at intervals of 10 days (5 days in 1900) for 0^h Washington sidereal time during 1855–1864, for Washington mean noon during 1865–1881, and for Greenwich mean noon thereafter.

Nutation

The nutation in longitude, for which the term " equation of the equinoxes in longitude " was used during 1855–1899, and the nutation in obliquity that were used for the computations relating to the stars were calculated from the formulae given by Peters (1842) until, in the volume for 1900, they were taken from Newcomb (1895a). The formulae of Peters are given in the Appendix to the volume for 1855; during later years, provision was made for including additional small terms when required.

Beginning with 1865, the apparent obliquity and nutation that were used in the ephemerides of the Sun, Moon, and planets were taken from Hansen and Olufsen (1853), until in 1900 the obliquity and the long-period nutation were taken from Newcomb (1895a).

The equation of the equinoxes in longitude (to 0″·01) and in right ascension (to $0^s\cdot 01$ for 1855–1877, and to $0^s\cdot 001$ for 1878–1900) were tabulated at intervals of 10 days (5 days in 1900) for 0^h Washington sidereal time in 1855–1864, for Washington mean noon in 1865–1881, and for Greenwich mean noon thereafter. In 1900, the nutation in obliquity (to 0″·01) was also explicitly tabulated at intervals of 5 days.

Constants

Solar parallax

1855–1869:	8″·5776	Encke (1824)
1870–1899:	8″·848	Newcomb (1867)
1900:	8″·80	Paris (1896) [This value was used for eclipse calculations from 1896 onwards.]

Constant of aberration

1855–1899:	20″·4451	Struve (1844). [In the ephemeris of the Sun for 1869–1874 the value 20″·255 from Hansen and Olufsen (1853) was used.]
1900:	20″·47	Paris (1896)

Constant of nutation

1855–1899: 9″·2231 + 0″·0009*T*, where *T* is in centuries from 1800·0. Peters (1842)

1900: 9″·21 at 1900·0 Paris (1896)

Semi-diameters

Sun

1855–1899: 16′ 02″ at mean distance (Greenwich observations).

1883–1899: In the calculation of eclipses, 15′ 59″·788 (Bessel).

1900: 15′ 59″·63 at mean distance (Auwers); this value was used in calculating eclipses, but in the ephemeris of the Sun 1″·15 was added to it for irradiation.

Moon

1855–1868: Burckhardt's value increased by 1/500 part.

1869–1900: 0·272274π + 2″·5 for irradiation; the irradiation was omitted in the calculation of eclipses and occultations.

Planets

1855–1900:

		log distance
Mercury	3″·34	0·00
Venus	8″·546	0·00
Mars	2″·842	0·25
Jupiter	18″·78	0·70
Saturn	8″·77	0·95
Uranus	1″·68	1·30

The value for Mercury was taken from Leverrier; the others were determined by Peirce from observations in 1845 and 1846 with the mural circle at Washington. The values for Jupiter and Saturn are the polar semi-diameters; it is stated in the 1869 volume that 19″·19 was erroneously used in 1858–1869 for Jupiter in the Washington ephemeris. In the volumes for 1869–1900, the equatorial semi-diameter of Jupiter is given as 20″·00 at log distance 0·70, and that of Saturn as 9″·38 at log distance 0·95, without authority.

1882–1900:

Neptune 1″·28 at log distance 1·48; no authority given.

D. LIST OF AUTHORITIES FOR TABULATIONS IN *THE NAUTICAL ALMANAC* AND IN *THE AMERICAN EPHEMERIS*, 1901–1959

The details given in this sub-section refer specifically to *The Nautical Almanac*; any differences in the corresponding editions of *The American Ephemeris* are specially noted in brackets—in which *A.E.* refers only to *The American Ephemeris*.

1. Sun

The quantities tabulated, generally with argument mean time, are substantially the same as before—longitude, latitude, right ascension, declination, and log radius vector—the precision normally 0″·1, 0″·01, 0^s·01, 0″·1, and 7D, and (except for the mean longitude and anomaly and the equatorial rectangular coordinates) the interval one day. The mean longitude has been given at intervals of 10 days since 1906, to 0°·00001 for 1906–1911, and to 0°·0001 since 1912. [In *A.E.*, the mean longitude, at intervals of 5 days and to 0°·0001, has been given since 1934.] Equatorial rectangular coordinates referred to the true equator and equinox of date, with reductions to those of the beginning of the year, were given at intervals of 12^h and to 7D until 1930, while similar coordinates at intervals of one day for the equinox of the beginning of the year have been tabulated since 1931, and those for the standard equinox of 1950·0 (Comrie, 1926) since 1928.* [In *A.E.*, the interval of 12^h was retained until 1950; from 1931 to 1950 the coordinates were referred to the beginning of the year, and reductions to the true equinox of date were also tabulated. Coordinates for 1950·0 have been tabulated since 1938, at intervals of 12^h until 1950, of one day from 1951.] Since 1954 [in *A.E.*, since 1953] the coordinates on the "standard" 10-day dates (I.A.U., 1950) have been emphasized by the use of **bold** type.

Longitude referred to the mean equinox of the beginning of the year was included from 1931 to 1937 [in *A.E.*, from 1901 to 1915 and from 1931 to 1951]. Longitude and latitude referred to the mean equinox of 1950·0 have also been tabulated at intervals of one day to 0°·00001 from 1928 to 1937 [not in *A.E.*]; from 1938 to 1959 [also in *A.E.*] longitude given to 0″·1, latitude to 0″·01. [In *A.E.*, latitude referred to ecliptic of date given during 1901–1959.] Natural values of the radius vector (to 7D) were introduced in 1928 [in *A.E.*, in 1938], the logarithmic value being omitted from both almanacs in 1938.

1901–1959: Newcomb (1895a).

2. Moon

Values of the longitude and latitude to 0″·1, of the parallax to 0″·01 [in *A.E.*, 0″·1 until 1912] and of the semi-diameter to 0″·01 [in *A.E.*, 0″·1 until 1939] were tabulated at intervals of 12^h throughout the period. Values of the right ascension and declination to 0^s·01 and 0″·1 were given at intervals of one hour, while lunar distances continued to appear until 1906 [in *A.E.*, until 1911], and examples of their calculation were given in the issues for 1907–1919 [in *A.E.*, for 1912–1935]. Elements of the mean equator and orbit have been included throughout. [The angular distance from the Sun has been given in *A.E.* since 1937, to 0′·1 until 1941, to 1′ thereafter.]

1901–1922: Hansen (1857) with Newcomb's corrections (1878b) to right ascension, declination, parallax, and semi-diameter from 1901 to 1914, and to longitude and latitude (before conversion to equatorial coordinates) from 1915 to 1922. [*A.E.* for 1912 states that Newcomb's formula for the correction to Hansen's mean longitude was not, in fact, used for the years 1883 to 1911. The formula actually used is there quoted.]

1923–1959: Brown (1919). It is to be noted that the values of the longitude (and correspondingly of the right ascension and declination) for the year 1923 require a small correction $+0''.08 \cos (☾ - \Gamma')$.

*The coordinates for 1928 were given in the volume for 1929.

3. Major planets

Values of the apparent right ascension and declination to $0^s{\cdot}01$ and $0''{\cdot}1$ have been tabulated at intervals of one day throughout the period. [In *A.E.*, the interval for Uranus and Neptune was 4 days until 1935.] Log distance at the same interval and to 7D was given until 1934 for Mercury and until 1940 for the other planets. [In *A.E.*, the intervals until 1915 were 12^h for Mercury, one day for Venus and Mars, 2 days for Jupiter and Saturn; from 1916, one day for each of these five; for Uranus and Neptune, 4 days from 1901 to 1935.] Natural values of the distance have been given since 1935 for Mercury and since 1941 for the other six planets at intervals of one day, to 6D for Mercury to Jupiter, and to 5D for Saturn to Neptune.

Heliocentric longitude and latitude (to $0''{\cdot}1$), referred to the mean equinox of date, and log radius vector (to 7D) were tabulated at intervals of from one to four days until about 1915 and thereafter they were given, in the appendices to various almanacs between 1915 and 1920, at wider intervals (up to 40 days) for the period up to 1940. [In *A.E.*, at intervals of from one to ten days throughout the period 1901–1959.] The first two volumes of *Planetary Co-ordinates* (H.M.N.A.O., 1933 and 1939) contain similar coordinates (to $0°{\cdot}001$ or $0°{\cdot}0001$), but referred to the mean equinox of 1950·0, and natural values of the radius vector (to 4D or 5D) for the period 1920 to 1960, except that no tabulations are given for Mercury and Pluto. The volumes also contain coordinates of the four outer planets Jupiter, Saturn, Uranus, and Neptune for 1800 to 1920.

Astrometric right ascensions and declinations, and distances, of Pluto have been included since 1950, but no heliocentric coordinates of this planet have been given. [In *A.E.*, heliocentric longitude and latitude (to $0''{\cdot}1$) and log radius vector (to 7D) at intervals of 100 days have also been given since 1950.]

Mercury, Venus

1901–1959: Newcomb (1895b and 1895c).

Mars

1901: Leverrier (1861b). [In *A.E.*, tables in manuscript constructed from elements by Newcomb (1895d).]
1902: Leverrier (1861b). [In *A.E.*, Newcomb (1898a).]
1903–1921: Newcomb (1898a).
1922–1959: Newcomb (1898a) with Ross's corrections (1917).

Jupiter, Saturn

1901–1959: Hill (1895a and 1895b).

Uranus

1901–1903: Leverrier (1877a). [In *A.E.*, Newcomb (1873).]
1904–1959: Newcomb (1898b).

Neptune

1901–1902: Leverrier (1877b). [In *A.E.*, Newcomb (1865).]
1903: Leverrier (1877b). [In *A.E.*, Newcomb (1898c).]
1904–1959: Newcomb (1898c).

Pluto

1950–1959: Bower (1931).

4. Minor planets

Elements and ephemerides of the first four minor planets were published as in previous years [not in *A.E.*] from 1901 to 1913; ephemerides alone were given in 1914 and 1915. Thereafter, no tabulations were given until 1952; since that year ephemerides have been included to cover, at intervals of one day, the periods during which transit occurs between sunset and sunrise at (most) fixed observatories, although in 1958 the period of tabulation was altered to that during which the planet is " not within about 40° of the Sun ". The quantities given were apparent right ascension and declination (to $0^{s}\cdot01$ and $0''\cdot1$) with corrections " astrometric minus apparent ", and distance (to 6D).

The basis of the tabulations has been:

Ceres,	1901–1915:	Godward (1878).
Pallas,	1901–1915:	Farley (1856a).
Juno,	1901–1915:	Hind (1855), with corrections by Downing (1890).
Vesta,	1901–1906:	Farley (1856b).
Vesta,	1907–1915:	Leveau (1896).
All,	1952–1959:	Herget, Clemence, and Hertz (1950).

5. Auxiliary quantities

Sidereal time

1901–1932: The values tabulated, to $0^{s}\cdot01$ and at intervals of one day, were based on Newcomb's value (1895a) for the right ascension of the mean sun affected by aberration:

$$18^{h}\ 38^{m}\ 45^{s}\cdot836 + 8640184^{s}\cdot542\ T + 0^{s}\cdot0929\ T^2$$

where T is measured in Julian centuries from 1900 January 0 at 12^{h} U.T., and included the effect of long-period terms only of nutation.

1933–1959: The precision was changed to $0^{s}\cdot001$, and the effect of short-period terms of nutation was also included.

Obliquity of the ecliptic

Newcomb's value (1895a) for the mean obliquity was used throughout:

$$23^\circ\ 27'\ 08''\cdot26 - 46''\cdot845\ T - 0''\cdot0059\ T^2 + 0''\cdot00181\ T^3$$

[In addition, *A.E.* includes values attributed to Hansen and to Peters for the years 1901 to 1915, also those due to Leverrier for the years 1902 to 1915.]

Values of the true obliquity excluding the effect of short-period terms of nutation were given (to $0''\cdot01$) at intervals of one day until 1930 [in *A.E.*, until 1933; and at intervals of 5 days from 1934 to 1959]. Since then, only the mean obliquity for the beginning of the year, and the (daily) nutation in obliquity have been given; the calculation of the true obliquity, if required, is a simple process.

Precession

The values given were based, throughout the period, on Newcomb's determination (1897, p. 73):

$$50''\cdot2564 + 0''\cdot0222\ T$$

[*A.E.* for the years 1901 to 1911 gave also values based on Peters (1842).]

Values of the precession from the beginning of the year were tabulated to 0″·01 at intervals of one day [in *A.E.*, 5 days until 1915], while the daily and annual increments to 0″·0001 were given until 1930 [in *A.E.*, until 1915].

From 1931 [not in *A.E.*] additional precessional constants, and tables for reduction of star positions, were also included.

Nutation—The Nautical Almanac

1901–1902: The tabulations of long-period and short-period terms (separately) in both longitude and obliquity were based on the values given by Newcomb (1895a), as modified by the Paris Conference (1896). The precision was 0″·01, while six terms of long period, and seven of short period, were included, as in table 7.1.

1903–1911: The coefficients of a few of the terms were modified in accordance with the revised values given by Newcomb (1898d).

1912–1930: An additional term, no. 8, was included, in the day numbers throughout this period, but only in the tabulation of the nutation from 1918.

1931–1936: Nutation in obliquity, and the short-period nutation in longitude, were no longer tabulated explicitly, though the former was available as $-B$ (the day number). Nutation in right ascension, including short-period terms, was tabulated to 0^s·001.

1937–1959: Tabulations were the same as in former years, but 21 terms (nos. 1–4, 6–10, 14–25) were included. The coefficients used were those given by Newcomb (1898d) and are quoted in table 7.1.

Nutation—The American Ephemeris

1901–1911: The tabulations of long-period terms (at intervals of 5 days) and of short-period terms (at intervals of one day) were based, for these years, on both the values of Peters (1842) and those of the Paris Conference (1896); the terms included were (see table 7.1) nos. 1–3, 5, 6, 8–15. The argument Sun's true longitude (☉) was used, both alone and in combination, instead of Sun's mean longitude (L), from 1901 to 1911 in the Peters calculations for terms nos. 3, 5, 6, 8, 10, and 11; from 1901 to 1907 in the Paris calculations for terms nos. 8, 10, and 11. The coefficients only differ in a very few cases from those in table 7.1, and the individual discrepancies (usually of a single unit) are not listed.

1912–1936: The tabulations were based solely on the values of the Paris Conference, and after 1915 were all at intervals of one day; the terms included were nos. 1–4, 6–8, 14, 15, and 17–21.

1937–1959: Tabulations of long-period nutation in longitude, and short-period nutation in both longitude and obliquity, were given at intervals of one day, and included the terms nos. 1–4, 6–10, and 14–25. The coefficients are those listed in table 7.1.

Constants

Solar parallax	1901–1959	8″·80	Paris (1896, p. 54)
Constant of aberration	1901–1959	20″·47	Paris (1896, p. 54)
Constant of nutation	1901–1936	9″·21	Paris (1896, p. 54)
	1937–1959	9″·210 + 0″·0009 T	Newcomb (1898d, p. 241)

[*A.E.* for the years 1901–1911 included tables for the Struve and Peters constants of aberration and nutation as used formerly (sub-section C), as well as those for the above Paris values.]

Semi-diameters at unit distance—The Nautical Almanac

Sun (see sub-section B.5)	1901–1959	961″·18	Auwers
	1901–1959	959″·63	Auwers (1891, p. 367); for eclipses only.
Mercury	1901–1959	3″·34	Leverrier (1843)
Venus	1901–1920	8″·40	Auwers (1891)
	1921–1959	8″·41	Auwers (1894)
Mars	1901–1959	4″·68	Hartwig (1879)
Jupiter (equatorial)	1901–1920	97″·36	Schur (1896)
	1921–1959	98″·47	Sampson (1910)
(polar)	1901–1920	91″·10	Schur (1896)
	1921–1959	91″·91	Sampson (1910)
Saturn (equatorial)	1901–1920	84″·75	Meyer (1883)
	1921–1959	83″·33	Struve (1898)
(polar)	1901–1920	76″·88	Meyer (1883)
	1921–1959	74″·57	Struve (1898)
Uranus	1901–1930	34″·28	Hind
	1931–1959	34″·28	Barnard (1896); See (1902); Wirtz (1912)
Neptune	1901–1959	36″·56	Barnard (1902)

Semi-diameters at unit distance—The American Ephemeris

Sun	1901–1902	960″·78	Auwers (1891) + 1″·15 for irradiation
	1903–1959	961″·50	Harkness (1899); includes 1″·15 for irradiation.
	1901–1959	959″·63	Auwers (1891, p. 367); for eclipses only.
Mercury	1901–1959	3″·34	Leverrier (1843)
Venus	1901–1919	8″·55	Peirce
	1920–1959	8″·41	Auwers (1894)
Mars	1901–1919	5″·05	Peirce
	1920–1959	4″·68	Hartwig (1879)
Jupiter (equatorial)	1901–1919	100″·24	
	1920–1959	98″·47	Sampson (1910)
(polar)	1901–1919	94″·12	Peirce
	1920–1959	91″·91	Sampson (1910)
Saturn (equatorial)	1901–1911	83″·60	
	1912–1919	84″·88	Barnard (1902)
	1920–1959	83″·33	Struve (1898)
(polar)	1901–1911	78″·16	Peirce
	1912–1919	77″·47	Barnard (1902)
	1920–1959	74″·57	Struve (1898)
Uranus	1901–1919	33″·52	Peirce
	1920–1959	34″·28	Barnard (1896); See (1902); Wirtz (1912)
Neptune	1901–1919	38″·66	
	1920–1959	36″·56	Barnard (1902)

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Newcomb, S. 1873. An investigation of the orbit of Uranus, with general tables of its motion. (*Smithson. Contr.*, 262). 4to, Washington.

Newcomb, S. 1878a. Researches on the motion of the Moon. *Washington Observations* for 1875, Appendix II, 1–280.

Newcomb, S. 1878b. *Corrections to Hansen's tables of the Moon, prepared and printed for the use of the The American Ephemeris and Nautical Almanac.* Washington.

Newcomb, S. 1895a. Tables of the motion of the earth on its axis and around the Sun. *A.P.A.E.*, **6**, 1–169. [Usually referred to as *Tables of the Sun.*]

Newcomb, S. 1895b. Tables of the heliocentric motion of Mercury. *A.P.A.E.*, **6**, 171–270.

Newcomb, S. 1895c. Tables of the heliocentric motion of Venus. *A.P.A.E.*, **6**, 271–382.

Newcomb, S. 1895d. *The Elements of the Four Inner Planets and the Fundamental Constants of Astronomy.* Supplement to *The American Ephemeris* for 1897, 1–202.

Newcomb, S. 1897. A New Determination of the Precessional Constant. *A.P.A.E.*, **8**, 1–76.

Newcomb, S. 1898a. Tables of the heliocentric motion of Mars. *A.P.A.E.*, **6**, 383–586.

Newcomb, S. 1898b. Tables of the heliocentric motion of Uranus. *A.P.A.E.*, **7**, 287–416.

Newcomb, S. 1898c. Tables of the heliocentric motion of Neptune. *A.P.A.E.*, **7**, 417–471.

Newcomb, S. 1898d. Sur les formules de nutation basées sur les décisions de la Conférence de 1896. *Bull. Astr.*, **15**, 241–246.

Paris, 1896. Conférence Internationale des Étoiles Fondamentales: Procès-Verbaux. 8vo, Paris.

Peirce, B. 1848a. Perturbations of Uranus; investigations into the action of Neptune upon Uranus. *Proc. Amer. Acad. Sci.* (Boston), **1**, 144–149 and 332–342.

Peirce, B. 1848b. Ueber die Störungen des Neptun's. *Ast. Nach.*, **27**, 215–218.

Peirce, B. 1853. *Tables of the Moon.* 4to, Washington. 2nd ed. 1865.

Peters, C. A. F. 1842. *Numerus Constans Nutationis ex ascensionibus rectis stellae polaris in specula dorpatensi* 1822–1838 *observatis deductus; adjecta est disquisitio theoretica de formula nutationis.* 4to, St. Petersburg.

Plana, J. B. 1832. *Théorie du mouvement de la Lune.* 3 volumes, 4to, Turin.

Ross, F. E. 1917. New elements of Mars and tables for correcting the heliocentric positions derived from *Astronomical Papers*, Volume VI, Part IV. *A.P.A.E.*, **9**, 251–274.

Runkle, J. D. 1855. *New tables for determining the values of the coefficients, in the perturbative function of planetary motion, which depend upon the ratio of the mean distances.* 4to, Washington.

Schubert, E. 1854. On the correction of the elements of (1) Ceres. *A.J.*, **3**, 153–159 and 162–165.

Struve, F. G. W. 1844. Sur le coefficient constant dans l'aberration des étoiles fixes, déduits des observations qui ont été exécutées à l'observatoire de Poulkova par l'instrument des passages de Repsold, établi dans le premier vertical. *Mém. de l'Acad. Imp. Sci. St. Petersburg*, 6th series, **5**, 229–285.

Vince, S. 1797–1808. *A complete system of astronomy.* 3 volumes, 4to, Cambridge. 2nd ed. 1814–1825, 4to, London.

Walker, S. C. 1848. *Report of the United States Coast and Geodetic Survey for* 1848.

Winlock, J. 1864. *Tables of Mercury, for the use of The American Ephemeris and Nautical Almanac.* 4to, Washington.

F. LIST OF APPENDICES AND SUPPLEMENTS

1. The Nautical Almanac

Many issues, especially the earlier years, of the Almanac contain appendices on various astronomical and navigational subjects. A list of these and of separate supplements, together with sections of a similar nature in the prefaces and explanations to certain issues, is given below. Unaltered, or nearly unaltered, reprints in later issues are ignored.

The appendices to the almanacs for 1772 to 1778, 1787, 1788 and 1794 were collected and published in 1813 with the title *Selections from the additions that have been occasionally annexed to The Nautical Almanac from its commencement to the year* 1812, while those to the almanacs for 1835 to 1854 were similarly published in 1851 with the title *Appendices to various Nautical Almanacs between the years* 1834 *and* 1854.

1766: [Maskelyne, N., et al.] *Tables requisite to be used with The Astronomical and Nautical Ephemeris.* Separate publication, 166 pages.

1769: (1) Maskelyne, N. Instructions relative to the observation of the ensuing transit of the planet Venus over the Sun's disk, on the 3d of June 1769. 9 pages.

(2) Maskelyne, N. Use of the astronomical quadrant in taking altitudes. 38 pages.

1771: (1) [Douwes, C., and Campbell, J.] Tables for . . . finding the latitude of a ship at sea. 77 pages.

(2) [Maskelyne, N.] Determination [by John Bradley] of the position of the Lizard. 6 pages.

(3) Wargentin, P. W. Tabulae novae et correctae pro supputandis eclipsibus tertii satellitis Jovis . . . 16 pages.

1772: (1) Maskelyne, N. A correct and easy method of clearing the [lunar] distance . . . 25 pages.

(2) [Eclipses of Jupiter's third satellite, and tables of the hour angle of the Sun and Jupiter.] 6 pages.

(3) Lyons, I., and Dunthorne, R. Problems in navigation. 8 pages.

1773: (1) A table of the equations to equal altitudes. 24 pages.

(2) A catalogue of the places of 387 fixed stars, . . . [for 1760·0]. 14 pages.

1774: (1) [Mason, C.] . . . Longitudes and latitudes of the Moon, deduced from Dr. Bradley's observations, made between September 13th 1750 and November 2d 1760, and compared with a set of manuscript tables. 36 pages.

(2) [Maskelyne, N.] Elements of the lunar tables. 11 pages.

(3) Maskelyne, N. Remarks on the Hadley's quadrant. 14 pages.

(4) Lyons, I. [Astronomical problems.] 10 pages.

1778: (1) [Mason, C.] Right ascensions and zenith distances of the Moon . . . 40 pages.

(2) Lyons, I. Astronomical problems. 11 pages.

1779: Wargentin, P. W. Tabulae novae et correctae pro supputandis eclipsibus secundi satellitis Jovis . . . . 30 pages.

1781: (1) Edwards, J. Astronomical problems. 27 pages.

(2) Edwards, J. Addition to the . . . tables annexed to the Nautical Almanac of 1771. 10 pages.

1787: (1) Edwards, J. Directions for making . . . reflecting telescopes and . . . polishing . . . them . . . . 48 pages.

(2) Edwards, J., and Maskelyne, N. An account of the . . . tremors of reflecting telescopes . . . . 12 pages.

1788: Blair, R. Description of a . . . method of adjusting Hadley's quadrant. 20 pages.

1791: Maskelyne, N. Advertisement of the expected . . . comet of . . . 1788, and relative to . . . Saturn's ring in 1789 and 1790. 4 pages.

1794: Brinkley, J. Tables to improve . . . the method of finding the latitude . . . . 15 pages.

1798: Brinkley, J. Tables to improve . . . latitude . . . (second edition, revised and corrected). 16 pages.

1809–1821: Each issue contains one or more catalogues of stars, some of longitude and latitude, some of right ascension and declination; the number of stars varies between 9 and about 50.

1812: Pond, J. [On the obliquity.] 3 pages.

1818: Brinkley, J. Two practical rules for reducing the observed distance of the Moon from the Sun or a . . . star . . . . 18 pages.

1822: Brinkley, J. A practical method of computing the latitude. 16 pages.

1822–1833: Tables of . . . refraction, . . . of second differences, and . . . of star places. 16–40 pages.

1824–1833: Elements of occultations. 6–17 pages.

1826: Rules for [predicting] occultations. 8 pages.

1827: Young, T.; Henderson, T. Rules for [reducing] occultations. 4 pages.

1828–1833: Separately issued supplements for each of these years contained a number of quantities that were transferred, between 1832 and 1834, to the pages of the Almanac proper. 34–54 pages.

1829: Lax, W. An easy method of finding the latitude and time at sea . . . . 23 pages.

1831: Lax, W. An easy method of correcting the lunar distance . . . . 6 pages.

1832: (1) Airy, G. B. Corrections of the longitudes and right ascensions of the Sun . . . . 4 pages.

(2) Jenkins, H. Recalculated elements of Delambre's tables . . . of Jupiter's satellites. 12 pages.

1833: (1) [Schumacher, H. C.] Ephemeris of . . . lunar distances of Venus, Mars, Jupiter, and Saturn. 44 pages.

(2) Geocentric places of the planets. 75 pages.

1834: Report of the Committee of the Astronomical Society of London. 11 pages.

1835: (1) Woolhouse, W. S. B. New tables for . . . Jupiter's satellites . . . . 39 pages.

(2) Woolhouse, W. S. B. On the computation of an ephemeris of a comet . . . . 9 pages.

(3) Comparison of . . . Burckhardt's and Damoiseau's lunar tables . . . . 4 pages.

1836: Woolhouse, W. S. B. On Eclipses. 96 pages.

1837: (1) Airy, G. B. On the calculation of . . . perturbations . . . . 23 pages.

(2) Woolhouse, W. S. B. On the determination of the longitude . . . . 12 pages.

1839: Stratford, W. S. On the elements of the orbit of Halley's Comet . . . . 79 pages.

1851: Adams, J. C. On the perturbations of Uranus. 29 pages.

1853–1914: Each contains the elements and ephemerides of a number (from 4 to 36) of minor planets. 16–58 pages.

1854: Challis, J. On the correction of a longitude . . . . 23 pages.

1856: (1) Encke, J. F. (trans. Airy, G. B.) On a new method of computing the perturbations of planets. 33 pages.

(2) Adams, J. C. On new tables of the Moon's parallax. 20 pages.

1862: Comparison of Moon's places by Burckhardt's tables with similar ones by Hansen's tables. 2 pages.

1867: Breen, H. Corrections to . . . the [tabulated] values of the Moon's . . . parallax . . . 1831–1839. 6 pages.

1874: Predictions for the transit of Venus. 6 pages.

1881: Adams, J. C. Continuation of . . . Damoiseau's tables of Jupiter's satellites. 9 pages.

1883–1922: [Newcomb, S.] Corrections . . . to Hansen's tables of the Moon. [These contain longitude and latitude corrections only from 1883 to 1895, those for right ascension and declination being included also from 1896.] 2–4 pages.

Between about 1850 and 1900, a series of "Nautical Almanac Circulars" was issued, mostly giving details and local predictions of total eclipses.

1897: Approximate places for 1900·0 of 834 zodiacal stars . . . 11 pages.

1900: Downing, A. M. W. Continuation of . . . Damoiseau's tables of Jupiter's satellites. 7 pages.

1901–1906: *Corrections to the apparent places . . . to obtain apparent places corresponding to the Struve-Peters constants.* Separate publications, 22 pages.

1907–1919: Calculation of a lunar distance. 2 pages.

1907–1914: Ephemerides for physical observations. 30 pages.

1915: (1) Some constants and formulae. 4 pages.

(2) Heliocentric [co-ordinates] of . . . [planets]. 68 pages.

1915: *Corrections to . . . 1532 stars . . .* Separate publication, by Pulkovo Observatory, 308 pages.

1916: Heliocentric [co-ordinates] . . . of Venus. 82 pages.

1917: Heliocentric [co-ordinates] . . . of Mars. 82 pages.

1918: Derivation of quantities contained in the Nautical Almanac. 23 pages.

1920: Ross's corrections to . . . places of Mars . . . . 16 pages.

1929: Coordinates of the Sun for 1950·0 for 1928 and 1929.

1931: (1) Fotheringham, J. K. The calendar. 14 pages.

(2) Derivation of quantities . . . . 26 pages.

(3) Tables for interpolation . . . by the end-figure process. 32 pages.

1935: (1) Fotheringham, J. K. The calendar. 17 pages.

(2) Interpolation tables. 14 pages.

1936: *Interpolation and allied tables.* Reprinted for separate sale, 48 pages.

1938: *The prediction and reduction of occultations.* Separate publication, 50 pages.

1938: *The total solar eclipse of* 1940 *October* 1. Typescript, 19 pages.

1940: (1) Heliocentric co-ordinates of Mercury. 4 pages.

(2) Corrections FK3 – Eichelberger. 6 pages.

1941: *Occultation reduction elements . . . .* Separate publication, by Yale University Observatory, 37 pages.

1950: Ephemeris of Pluto.

1954: *Improved lunar ephemeris* 1952–1959. Separate publication as a "Joint Supplement to *The American Ephemeris* and *The* (British) *Nautical Almanac*", Washington, 435 pages.

2. The American Ephemeris

Unlike most of the other national ephemerides, *The American Ephemeris* was never a medium for the publication of technical articles. The volumes from 1855 to 1911, inclusive, contained an *Appendix*, but ordinarily it comprised only the miscellaneous tables regularly included every year, and the list of fundamental constants and tables used in preparing the ephemerides; in 1912, this list was transferred to the beginning of the volume, leaving the section of miscellaneous tables at the end, and the *Appendix* was discontinued. Occasionally, appendices containing various ephemerides have also been added to individual volumes; but very few technical contributions were ever included.

However, separate supplements to *The American Ephemeris* have been issued from time to time; these sometimes were separate printings of material from the appendices but more often were in addition to the contents of the volumes. Most of them contain supplementary ephemeris data, especially for total solar eclipses, but among them have also been several important technical publications.

The appendices and supplements that are of interest for their technical content other than ephemeral data are listed below; there follows also a list of the supplements giving extended data and large-scale maps for total solar eclipses.

Prior to the establishment of the series of *Astronomical Papers prepared for the use of The American Ephemeris and Nautical Almanac* (see section 1F.3 for a complete list of contents) the principal tables constructed for the Office were printed, each as a single publication, and a list of these is also appended; but many tables were prepared only in manuscript.

Appendices and supplements

1855: Chauvenet's tables for correcting lunar distances, with directions for using the tables, and explanation of their construction. Pages 13–70 of the *Appendix*.

1857: (1) Chauvenet's tables for correcting lunar distances, with directions for using the tables, and explanation of their construction. Pages 11–67.

(2) Chauvenet, W. Improved method of finding the error and rate of a chronometer by equal altitudes. Pages 69–94.

(3) Walker, S. C. Logarithms of the Le Verrier coefficients of the perturbative function of planetary motion. Pages 95–117.

Pages 11–94 of this Appendix were later reprinted as a separate publication: William Chauvenet, New method of correcting lunar distances, and Improved method of finding the error and rate of a chronometer by equal altitudes, 1866.

1874: Coffin, J. H. C. *Tables for finding the latitude of a place by altitudes of Polaris.* Supplement for 1874–1877.

This article, which included the formulae from which the tables were calculated, with instructions for using the tables, and an illustrative example, was also put into the *Appendix* (pages 25–33) in *The American Ephemeris* for 1877, the first volume in which these tables were given.

A similar supplement for the years 1878–1881, inclusive, was separately printed, and was also included in the *Appendix* in *The American Ephemeris* for each of these years. In the volume for 1882, these tables were replaced by a simple one-page table which was retained until it, in turn, was replaced by the table that was given throughout the period 1912–1959.

1895: Newcomb, S. *The elements of the four inner planets and the fundamental constants of astronomy.* Supplement for 1897.

1945: *Tables of sunrise, sunset, and twilight.* Supplement for 1946.

1950: Ephemeris of Pluto.

1954: *Improved lunar ephemeris* 1952–1959. Published as a "Joint Supplement to *The American Ephemeris* and *The* (British) *Nautical Almanac*".

Supplementary publications for solar eclipses

The series of supplements to *The American Ephemeris* that were published for the occasions when total eclipses of the Sun were visible in the United States began with a supplement to the volume for 1869; but prior to that, pamphlets had been issued for the annular eclipse of 1854 May 26 and the total eclipse of 1860 July 17.

For the total eclipse of 1869 August 7, a supplement was issued containing predicted data, and also one containing suggestions for observing the eclipse. In 1885, a publication containing reports of observations of this eclipse was issued by the Nautical Almanac Office.

Since the eclipse of 1869, supplements have been published for the total eclipses that occurred on the following dates:

1878	July 29	1936	June 19
1900	May 28	1940	October 1
1918	June 8	1945	July 9
1925	January 24	1947	May 20
1932	August 31	1954	June 30

In addition, data for two other eclipses were issued as *U.S. Naval Observatory Circulars:*

Circular no. 27. Annular eclipse of September 1, 1951: path of annular phase in the United States.

Circular no. 78. Total eclipse of October 2, 1959: path of total phase in the United States.

Tables prepared for The American Ephemeris and Nautical Almanac

Peirce, B. Tables of the Moon. 1st ed. 1853; 2nd ed. 1865.
Winlock, J. Tables of Mercury. 1864.
Hill, G. W. Tables of Venus. 1872 (on title page; cover has 1873).
Schubert, E. Tables of Melpomene. 1860.
Schubert, E. Tables of Eunomia. 1866.
Schubert, E. Tables of Harmonia. 1869.
Schubert, E. Tables of Parthenope. 1871.
Todd, D. P. Continuation of de Damoiseau's tables of the satellites of Jupiter to 1900. 1876.
Tables to facilitate the reduction of places of the fixed stars. 1st ed. 1869; 2nd ed. 1873.
Almanac catalogue of zodiacal stars. 1864.

8. CONFIGURATIONS OF THE SUN, MOON, AND PLANETS*

A. INTRODUCTION

The times of the principal astronomical phenomena involving configurations of the Sun, Moon, and planets are given in *A.E.*, pages 4–9, under the headings *Phenomena* and *Diary*. In most cases the times are given to the nearest hour, but for certain heliocentric phenomena of the planets the date only is given. The times of the greatest general interest, however, are those which mark the commencement of the seasons and the phases of the Moon (*A.E.*, page 159), and these are tabulated to the nearest minute.

B. THE SUN AND MOON

Equinoxes and solstices

The times of the *equinoxes* and *solstices* are those at which the Sun's apparent longitude is a multiple of 90°; thus (in the northern hemisphere) spring equinox, summer solstice, autumn equinox, and winter solstice correspond to apparent longitudes of the Sun of 0°, 90°, 180°, and 270° respectively. They are found by inverse interpolation of the Sun's daily ephemeris, the times so obtained being corrected from E.T. to U.T. by the addition of $-\Delta T$. The times of commencement of the seasons show a progressive change through the years, because the period of revolution of the Earth about the Sun is not commensurate with the calendar year; it is only after a complete cycle of four centuries that the seasons again commence at approximately the same times. In the present century the latest dates for the seasons occurred in 1903 and the earliest will be in 2000; by the year 2096 the seasons will begin at their earliest possible times:

		Spring	Summer	Autumn	Winter
Latest	1903	March 21^d 19^h	June 22^d 15^h	Sept. 24^d 06^h	Dec. 23^d 00^h
	2000	20 07	21 02	22 17	21 13
Earliest	2096	19 14	20 07	21 23	20 21

The total range in times is about 54 hours in each case.

Perihelion and aphelion

The dates on which the Earth is at *perihelion* and *aphelion* are those on which the Sun's radius vector is a minimum and maximum respectively. On account of

*There have been several minor changes in content and arrangement in the *A.E.* since 1960; these are described in the volumes concerned and are not noted here.

perturbations they are not always the same as the dates on which the longitude of the Sun is equal to the mean longitude of perigee or apogee.

Phases of the Moon

The times of new moon, first quarter, full moon, and last quarter are tabulated to the nearest minute of E.T. in *A.E.*, page 159, and are given to the nearest hour of U.T. in the *Diary* on pages 5–7; they are the times when the excess of the Moon's apparent longitude over the Sun's apparent longitude is 0°, 90°, 180°, and 270° respectively. These times are found by inverse interpolation in a table giving the differences of the longitudes, in the sense Moon minus Sun, at 12^h intervals. The times given on the *Phenomena* pages are adjusted in critical cases for the difference between E.T. and U.T. before rounding to the nearest hour. The *lunation numbers* given on page 159 are in continuation of E. W. Brown's series (*M.N.R.A.S.*, **93**, 603, 1933), of which no. 1 commenced on 1923 January 16.

Example 8.1. *Derivation of time of first quarter*
1960 March 5

The apparent longitudes of the Moon and Sun and their difference are tabulated; the time of first quarter is obtained by inverse interpolation of the difference to the value 90° (see section 16B).

Date E.T. 1960	Moon (*A.E.*, page 54)	Sun (*A.E.*, page 20)	Moon − Sun		
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″
Mar. 4·5	63 25 31·5	343 55 52·2	79 29 39		
				5 27 59	
5·0	69 23 33·3	344 25 55·3	84 57 38		−70
				5 26 49	
5·5	75 20 24·8	344 55 57·7	90 24 27		−34
				5 26 15	
6·0	81 16 41·5	345 25 59·7	95 50 42		

Second differences are just appreciable; the interpolation formula from which the interpolating factor p is to be determined by successive approximation is re-expressed as:

$$p = \{(f_p - f_0) - B_2\,(\delta_0^2 + \delta_1^2) - \ldots\}/\delta_{\frac{1}{2}}$$

where $f_p - f_0 = 90° - 84°\ 57'\ 38'' = 18142$; $\delta_{\frac{1}{2}} = 19609$; $\delta_0^2 + \delta_1^2 = -104$; the working unit is 1″.

First approximation: $B_2\,(\delta_0^2 + \delta_1^2) = 0$ $\qquad p = 0{\cdot}925$

Second approximation: $B_2\,(\delta_0^2 + \delta_1^2) = +1{\cdot}8$ (Table 16.1) $\qquad p = 0{\cdot}92510$

There is no further change in the second-difference correction, so the required E.T. of first quarter is March $5^d{\cdot}46255$ = March $5^d\ 11^h\ 06^m{\cdot}1$.

Owing to the rapid variations in the distance and velocity of the Moon, the intervals between successive phases are not constant, nor is it possible to check these times by differencing. A check is provided, however, by examination of the higher differences of the successive times of the same phenomenon.

The phases of the Moon do not recur on exactly the same dates in any regular cycle, but the *approximate* dates of the phases in any year can be found from the dates on which the phases occurred 19 years previously. Every 19 years, the phases recur on dates that either are the same, or else differ by only one or occasionally two days, depending partly on the number of intervening leap years and partly on the perturbations of the Moon. For example, during 1960 the dates are the same as in 1941 on thirty occasions, and differ by one day for the remaining nineteen.

Moon's perigee and apogee

The times of the Moon's *perigee* and *apogee* are those at which the distance of the Moon from the Earth is a minimum and maximum respectively; they are tabulated in *A.E.*, page 159, as well as in the *Diary*. The times are obtained in practice from the values of the Moon's horizontal parallax, which is a maximum at perigee and a minimum at apogee; inverse interpolation is used to find the times when the first derivative is zero.

Example 8.2. *Derivation of time of apogee of the Moon*
1960 March 6

The time of apogee is the time at which the horizontal parallax of the Moon is a minimum, that is, when its first derivative is zero while its second derivative is positive. (See section 16C).

Date E.T.	Moon Horizontal Parallax (*A.E.*, page 54)			
1960	′ ″	″	″	
Mar. 5·5	54 15·098		+2·652	
		−1·804		−6
6·0	54 13·294		+2·646	
		+0·842		−50
6·5	54 14·136		+2·596	

Apogee occurs near March 6^d. The contributions from third and fourth differences may be ignored as the time is only required to the nearest hour. Hence the interpolating factor is:

$$p = -\tfrac{1}{2} + 1804/2646 = +0{\cdot}18, \text{ in units of } 12^h.$$

Hence apogee occurs at March $6^d\ 02^h$.

Eclipses and occultations

Attention is also drawn in the *Phenomena* pages to the times of *eclipses*, *transits of Mercury*, and some of the more important *occultations* by the Moon. Brief notes are given of the areas of visibility of solar and lunar eclipses, and of transits of Mercury when these occur; more detailed information on these phenomena is given in the main pages of the Ephemeris. A list is also given of the times of occultations of planets and bright stars; on page 5 is tabulated the time of conjunction to the nearest hour, together with brief notes on the areas of visibility. The planets Neptune and Pluto are omitted from this list, and the only stars included are *Antares*, *Aldebaran*, *Regulus*, and *Spica*. These first-magnitude stars lie close enough to the ecliptic to be occulted by the Moon, and their occultations occur in a cycle of 18·6 years, during which time the nodes of the Moon's orbit complete one circuit of the ecliptic. During this cycle the stars may be occulted in successive lunations over a period which varies with the latitude of the star; thus *Regulus* undergoes two periods of occultation, each lasting for 19 successive lunations, *Spica* two periods of 21 lunations, *Antares* one combined period of 69 lunations, and *Aldebaran* one period of 48 lunations. Of these phenomena only a few will be visible at any particular station under favourable conditions.

C. THE PLANETS

Planetary phenomena—heliocentric

Certain heliocentric phenomena of the planets are collected together in the table in *A.E.*, page 4. The dates of *perihelion* and *aphelion* are those on which the radius vector of the planet is a minimum and maximum respectively; they are thus the times when the first derivative of the radius vector is zero. Owing to the presence of perturbations in the planetary motion, these dates may differ from those that would be obtained from the angular elements of the mean orbits. The actual disturbed motion of the planets is also used to determine the dates when they pass through the *nodes* on the ecliptic, and when they reach their *greatest latitudes north or south.* At the nodes the heliocentric latitude is zero, changing from negative to positive at the ascending node, and from positive to negative at the descending node. These dates are given each year for Mercury, Venus, and Mars, but they occur less frequently for the outer planets and in these cases are given as additional notes when necessary. The dates on which a planet has its greatest latitude north or south are determined as the times at which the first derivative of the latitude is zero.

Planetary phenomena—geocentric

The times of *conjunction* and *opposition* of the planets are those at which the difference between the apparent geocentric longitudes of the planet and the Sun is 0° and 180° respectively. In order to avoid calculating the geocentric longitude of each planet, the heliocentric geometric longitudes may be used as tabulated, and a small correction applied to correct for the effect of light-time. The difference between the Sun's geometric longitude and the heliocentric longitude of the planet (strictly for mean equinox of date) is taken to be 0° at superior conjunction of Mercury and Venus and conjunction of the outer planets; and is taken to be 180° at inferior conjunction or opposition. The following table gives the corrections that should be applied to the times so obtained; they are based on mean elements of the orbits, but the errors resulting from their use cannot amount to more than 10^m for the conjunctions of Venus and Mars, and are considerably less for the other phenomena.

	Mercury	Venus	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
Inferior Conjunction	$+7^m$	$+6^m$						
Opposition			$+7^m$	$+11^m$	$+13^m$	$+14^m$	$+15^m$	$+15^m$
Superior Conjunction	$+15^m$	$+37^m$	-24^m	-5^m	-3^m	-2^m	-2^m	-1^m

Alternatively the correction for the effect of light-time can be calculated for each individual phenomenon.

Owing to the eccentricities and inclinations of the orbits and the presence of perturbations, the times so obtained may be different from those at which the geocentric distance is a minimum (at inferior conjunction of Mercury or Venus, or opposition of a superior planet) or a maximum (at superior conjunction of Mercury or Venus, or conjunction of a superior planet).

Example 8.3. *Derivation of time of superior conjunction of Venus*
1960 June 22

(a). An approximate time can be obtained by inverse interpolation of the difference between the heliocentric longitude of Venus and the geometric longitude of the Sun to the value zero. A better value is then found by applying the correction for the effect of light-time given above in the text.

Date 0^h E.T. 1960	Venus Heliocentric Longitude (*A.E.*, page 169)	Sun Geometric Longitude (*A.E.*, page 24)	Venus − Sun	
	° ′ ″	° ′ ″	′ ″	′ ″
June 21	88 31 17	89 37 13	−65 56	
				+79 26
23	91 45 14	91 31 44	+13 30	
				+79 34
25	94 59 18	93 26 14	+93 04	

The fraction of the interval of 2^d is $65' 56''/79' 26'' = 0{\cdot}8300$, so that the approximate time is June $21^d\ 0^h + 39^h{\cdot}84 =$ June $22^d\ 15^h\ 50^m$
The light-time correction (from the table) $+37^m$
Superior conjunction occurs at E.T. June $22^d\ 16^h\ 27^m$

At the time of conjunction, the actual correction for light-time can be incorporated by correcting the heliocentric longitude of the planet for a light-time equivalent to the distance Earth to Venus: this correction is:

$$-0{\cdot}002884 \times 1{\cdot}736 \times +193' 57'' = -58''$$

corresponding to a correction to the time of $+35^m$ (as compared with $+37^m$ from the table).

(b). The apparent geocentric longitude λ is formed from the equations:

$$\cos\beta \cos\lambda = \cos\delta \cos\alpha$$
$$\cos\beta \sin\lambda = \sin\delta \sin\epsilon + \cos\delta \sin\alpha \cos\epsilon$$

$\tan\lambda$ or $\cot\lambda$ (whichever is the smaller) being found by division. The time of conjunction is obtained by inverse interpolation of the difference between the geocentric longitudes of the planet and the Sun to the value zero.

Date 0^h E.T. 1960	Venus Right Ascension (*A.E.*, page 189)	Venus Declination (*A.E.*, page 189)	Apparent Longitude Venus (*A.E.*, page 24)	Apparent Longitude Sun (*A.E.*, page 24)	Venus − Sun	
	h m s	° ′ ″	° ′ ″	° ′ ″	′ ″	′ ″
June 22	6 01 39·7	+23 46 37	90 22 49	90 34 05	−11 16	
						+16 28
23	6 07 02·1	+23 48 22	91 36 33	91 31 21	+ 5 12	

Superior conjunction occurs at June $22^d\ 0^h + (11' 16''/16' 28'')\ 24^h =$ June $22^d\ 16^h\ 25^m$.

The mean interval between successive conjunctions or oppositions of a planet is the synodic period, and this is of some value in estimating future dates of these phenomena. The following table gives the synodic periods of the planets in days, based on an assumption of mean motions; owing to the eccentricities and perturbations of the orbits, the actual interval between the phenomena may differ considerably from these periods.

	d		d
Mercury	115·88	Saturn	378·09
Venus	583·92	Uranus	369·66
Mars	779·94	Neptune	367·48
Jupiter	398·88	Pluto	366·73

In the case of the slowly-moving planets Jupiter to Pluto, the error in using these values is small, but for Mercury, Venus, and Mars the mean synodic period is less useful. Much more accurate estimates may be made for these planets by using long-period cycles, which contain, with varying degrees of accuracy, integral numbers of revolutions of both the Earth and the planet.

For Mercury	54 sidereal periods	=	13 years + 2 days
	137 ,, ,,	=	33 years − 1 day
For Venus	13 sidereal periods	=	8 years − 1 day
	395 ,, ,,	=	243 years − ½ day
For Mars	8 sidereal periods	=	15 years + 17 days
	17 ,, ,,	=	32 years − 9 days
	25 ,, ,,	=	47 years + 8 days
	42 ,, ,,	=	79 years − 1 day

Any particular phenomenon of a planet repeats itself after each cycle at the same time of year and in the same part of the sky; this is not the case after a single synodic period. For Venus a useful and accurate form of the relation is:

$$5 \text{ mean synodic periods} = 8 \text{ calendar years} - 2^{d}\!\cdot\!4$$

provided the interval does not include a non-bissextile century year.

The times of *stationary points* differ from those previously described in that the motion in right ascension is considered; since the planet is stationary in right ascension, the times are those at which the first derivative of the right ascension is zero.

Tables of *magnitudes* and *elongations* of the planets from the Sun are given in *A.E.*, pages 8 and 9, for every fifth day for the inferior planets and for every tenth day for the superior planets; approximate magnitudes of the minor planets Ceres, Pallas, Juno, and Vesta are also given. The magnitudes of the major planets are calculated from the formulae given in section 11 and visual magnitudes of the minor planets from $g + 5 \log r\,\Delta$, where g has (for the years 1962 onwards) the value 3·38 for Ceres, 4·51 for Pallas, 5·58 for Juno, and 3·55 for Vesta.

The elongation E of a planet, measured eastwards or westwards from the Sun, is calculated as the angle planet-Earth-Sun in the plane triangle formed by these three bodies, and is given by:

$$\cos E = \frac{R^2 + \Delta^2 - r^2}{2R\Delta}$$

where r is the radius vector of the planet, R that of the Sun, and Δ the geocentric distance of the planet. The elongations of Mercury and Venus may also be obtained from:

$$\sin E = \frac{r \sin i}{R}$$

where i is the angle Earth-planet-Sun and is tabulated, without sign, in *A.E.*, on pages 326 and 327. The same formula can be used for Mars, Jupiter, and Saturn near opposition when the angle i is tabulated in *A.E.*, pages 328–341; it cannot be used when E is near 90°.

The elongations are measured from 0° to 180° east or west of the Sun, and are tabulated to the nearest degree. Owing to the inclinations of the planetary orbits, the elongations do not necessarily pass through 0° or 180° as they change from east to west or west to east.

Example 8.4. *Derivation of elongation of Uranus*
1960 March 7 at 0^h E.T.

$R = 0{\cdot}9925$ $\Delta = 17{\cdot}5206$ $r = 18{\cdot}3984$

Hence $\cos E = -0{\cdot}8783$ and $E = 151^\circ{\cdot}4$. Since the date is after opposition and before conjunction, the planet is east of the Sun and the elongation is 151° E.

Near opposition ($E = 180°$) more decimals have to be retained in Δ and r, and thus in $\cos E$, to give comparable precision.

The phenomena of the planets Mercury and Venus always occur in the order: inferior conjunction — elongation west — superior conjunction — elongation east — inferior conjunction; similarly the superior planets pass successively through the configurations: opposition — elongation east — conjunction — elongation west — opposition. The superior planets may attain 180° elongation from the Sun, but the inferior planets have restricted ranges of about 47° for Venus and 28° for Mercury. The times of the *greatest elongations* of these two planets are determined from the maximum values of the angular distances from the Sun, which for this purpose are calculated at close intervals about the required times.

The dates of *greatest brilliancy* (see section 11C) of Venus are those on which the expression:

$$\frac{(r + \Delta + R)(r + \Delta - R)}{r^3 \Delta^3}$$

is a maximum; the dates so derived normally occur about a month after greatest elongation east, and a month before greatest elongation west. In the case of Mercury the dates of greatest brilliancy occur after greatest elongation west, and before greatest elongation east; but Mercury is not always readily observable at the time of its greatest brilliancy and this phenomenon is not tabulated.

D. DIARY

All of these configurations, with the exception of the heliocentric phenomena, are given in diary form in *A.E.*, pages 5 to 7. The *Diary* gives, in chronological order, and with times to the nearest hour of U.T., all the geocentric phenomena of the previous pages, together with certain additional information. Eclipses of the Sun and Moon (including penumbral lunar eclipses) are mentioned at the times of new or full moon. For the minor planets Ceres, Pallas, Juno, and Vesta, the times of conjunction, opposition, and stationary points are included; these are also listed separately on page 8. The time of *closest approach of Mars* is included in those years when the planet is in opposition; owing to the eccentricity of the Martian orbit, the time at which the geocentric distance is a minimum may differ considerably from the date of opposition.

Example 8.5. *Derivation of time of closest approach of Mars to the Earth.*
1960 December 25

This is found by inverse interpolation to zero of the first difference of the true geocentric distance of Mars.

Date 0^h E.T. 1960		Mars Distance (*A.E.*, page 201)		
Dec.	24	0·606 989		
			−158	
	25	0·606 831		+214
			+ 56	
	26	0·606 887		

Closest approach occurs at December 24·5 + 158/214 = December 25^d 06^h

The *Diary* also includes the times of *conjunction of the planets* (except Pluto) with the Moon, with each other, and with the bright stars *Antares*, *Aldebaran*, *Pollux*, *Regulus*, and *Spica*, which lie near the ecliptic. In all cases times are given for conjunction in right ascension, and are obtained by a method similar to that described for the phases of the Moon. Conjunctions of planets with the Moon are not given if they occur within 15° of the Sun, and conjunctions of the planets in pairs or with the five bright stars are omitted if the elongation of either body from the Sun is less than 10°. The difference of declination of the two bodies at the time of conjunction is also given; for conjunctions of the planets in pairs the difference is given to a tenth of a degree, but in other cases it is given to this precision only if less than one degree. It should be noted that in the case of a conjunction with the Moon, the actual observed configuration of the two bodies differs from that tabulated by the amount of the lunar parallax (see section 2F); in other cases the parallax is always small.

9. ECLIPSES AND TRANSITS

A. INTRODUCTION

The data in the Ephemeris

Elements and general circumstances are given in the Ephemeris for all solar and lunar eclipses, including penumbral lunar eclipses, that occur during the year. For solar eclipses, maps are given from which approximate local circumstances may be obtained for any particular place, and the Besselian elements are tabulated at intervals of ten minutes for the calculation of accurate predictions for any point on or above the surface of the Earth. For total or annular eclipses, the latitudes and longitudes of points on the central line and on the northern and southern limits, the duration of the total or annular phase, and the altitude of the Sun on the central line are tabulated at intervals of five minutes or less throughout the eclipse. For lunar eclipses, the circumstances and their times are the same for all parts of the Earth; any particular phase is visible from any point at which the Moon is then above the horizon.

In accord with a request from the solar eclipse Commission of the International Astronomical Union, the principal circumstances of all solar eclipses are calculated and made available to astronomers several years in advance of the publication of the Ephemeris. At present, the aim is to publish the data eight years before the corresponding eclipses. This advance information, which includes the tabulation of the path of totality or annularity, is given in *U.S. Naval Observatory Circulars*; recent numbers containing eclipse data have been published as follows: *

Circ. No.	Date	Information included
59	28 June 1955	Solar eclipses, 1960–1963
85	16 April 1958	Solar eclipses, 1963–1967
88	10 March 1960	Eclipse maps for the total solar eclipse of 1962 February 4–5, and the annular solar eclipse of 1962 July 31
89	15 June 1960	Solar eclipses, 1968–1970

Material from the section of the Ephemeris on solar and lunar eclipses is also published, in advance of publication of the Ephemeris itself, in the small booklet

*See page 522 for list of eclipse circulars published since 1960.

Astronomical Phenomena, issued annually by the Nautical Almanac Office, U.S. Naval Observatory.

All eclipse computations for the Ephemeris are now carried out on an electronic computer and do not follow precisely the methods, formulae, and examples given below. Further details of methods for the computation of solar eclipses may be found in the references listed at the end of this sub-section.

The predictions given in the Ephemeris take no account of the effects of refraction, although this may be significant for the reduction of observations intended to give precise positions. Besselian elements, however, are rigorously independent of refraction.

Corrections to the ephemerides

The basic quantities from which the calculations of solar and lunar eclipses are derived are: the apparent right ascension $\alpha_{\odot}$, declination $\delta_{\odot}$, and radius vector R of the Sun, and the right ascension $\alpha_{☾}$, declination $\delta_{☾}$, and horizontal parallax $\pi_{☾}$ of the Moon, for every hour of E.T. during the eclipse; and the ephemeris sidereal time at 0^h E.T. for the day of the eclipse and for the following day. The apparent right ascensions of the Sun and Moon, and the ephemeris sidereal time, include short-period terms of nutation. The coordinates of the Sun and Moon are given to $0^s{\cdot}001$ in right ascension and $0''{\cdot}01$ in declination; those for the Sun are otherwise unpublished to this precision, rounded values being tabulated in the Ephemeris. The ephemeris sidereal time at 0^h E.T., which is the local apparent sidereal time on the ephemeris meridian, is the same numerically as the (Greenwich) apparent sidereal time at 0^h U.T., as tabulated in the Ephemeris.

Gravitational ephemerides refer to the positions of the centres of mass of the bodies concerned. The phenomena of eclipses, however, are governed by the positions of the centres of figure of the Sun and Moon. The centre of figure of the Moon does not coincide with the centre of mass; to allow for this a correction $\Delta\beta_{☾}$ of $-0''{\cdot}5$ is applied to the tabular latitude of the Moon. As from 1964, $\Delta\beta_{☾}$ will be revised to $-0''{\cdot}6$, a value which is in better accord with the latest observed determination.

The conversion of $\Delta\beta$ into corrections $\Delta\alpha$, $\Delta\delta$ to the right ascension and declination is effected by the formulae:

$$\Delta\alpha = -\sin\epsilon\cos\lambda\sec^2\delta\,\Delta\beta$$
$$\Delta\delta = +(\cos\epsilon\cos\lambda\cos\alpha + \sin\lambda\sin\alpha)\,\Delta\beta$$

in which ϵ is the obliquity of the ecliptic, α and δ are the apparent right ascension and declination, λ is the apparent longitude; all quantities are taken by convention for the approximate time of conjunction (or opposition) in longitude, as given in Oppolzer's *Canon der Finsternisse*. The corrections are treated as constant throughout each eclipse.

The semi-diameters of the Sun and Moon used in the calculation of eclipses do not include irradiation. The adopted semi-diameter $s_{\odot}$ of the Sun at unit distance is 15′ 59″·63 (Auwers, A., *Ast. Nach.*, **128**, 367, 1891), the same, except for irradiation, as in the ephemeris of the Sun; but the apparent semi-diameter of the Moon is calculated by putting its sine equal to $k \sin \pi_{☾}$, where $\pi_{☾}$ is the horizontal parallax and k, the ratio of the Moon's radius to the equatorial radius of the Earth, is a constant. Until 1962 inclusive, the value of k is taken as 0·2722 74, and the resulting value of the apparent lunar semi-diameter differs from the tabular value in the ephemeris of the Moon. As from 1963, the value of k is taken as 0·2724 807, which leads to the same value of the semi-diameter as in the lunar * ephemeris. The value 0·2722 74 for k is retained, however, in the calculation of * duration on the central line of *total* solar eclipses, by way of applying an approximate correction for the irregularities of the lunar limb.

Illustrative examples

With the exception of example 9.17, which relates to the partial solar eclipse of 1960 September 20–21, all illustrative examples in sub-sections A, B, and C relate to the total solar eclipse of 1961 February 15. The approximate time of conjunction as given in Oppolzer's *Canon der Finsternisse* is 1961 February $15^d\ 08^h\ 11^m{\cdot}0$; the other data, including the maps, give a general picture of the circumstances of the eclipse.

The examples are arranged so as to follow immediately after the derivation of the relevant equations; explanation is in general unnecessary. As far as possible the derivation of all quantities entering into the examples is illustrated, but no references are given to the origin of the basic data. Most of the quantities are taken from the Ephemeris, either directly or by interpolation; others are taken from the unpublished auxiliary elements and are not independently derived in all examples.

For purposes of illustration more figures are often retained in the examples than would otherwise be justified by the precision of the data or the requirements of the particular calculation; but the fullest precision is required for some calculations, in particular that of the Besselian elements.

Example 9.1. *Corrections to the coordinates of the Moon*

Approximate time of conjunction 1961 February $15^d\ 08^h\ 11^m$

$\lambda_{☾}$ 326°·4	$\alpha_{☾}$ $21^h\ 53^m\ 22^s$	$\delta_{☾}$ −11°·9	$\Delta\beta_{☾}$ −0″·50
$\sin\lambda_{☾}$ −0·553	$\sin\alpha_{☾}$ −0·525	$\sec\delta_{☾}$ + 1·022	$\sin\epsilon$ 0·398
$\cos\lambda_{☾}$ +0·833	$\cos\alpha_{☾}$ +0·851	$\sec^2\delta_{☾}$ + 1·044	$\cos\epsilon$ 0·917

$$\Delta\alpha_{☾} = -0{\cdot}347\,\Delta\beta_{☾} = +0''{\cdot}17$$
$$\Delta\delta_{☾} = (+0{\cdot}650 + 0{\cdot}290)\,\Delta\beta_{☾} = -0''{\cdot}47$$

References

Oppolzer, T. R. v., *Canon der Finsternisse*. Vienna, 1887.
Reprinted, Dover Publications, 1962.

This book contains elements of 8000 solar eclipses from −1207 November 10 (Julian proleptic date) to 2161 November 17 (Gregorian date) and of 5200 lunar eclipses from −1206 April 21 to 2163 October 12 together with maps showing the paths of total and annular solar eclipses in the northern hemisphere. The main use of this volume is to aid

*For 1968 onwards the corresponding values of k are 0·2724 880 and 0·2722 81.

the historian in chronological researches but it also provides a picture of future eclipses and is useful for planning.

Chauvenet, W., *A manual of spherical and practical astronomy*. Vol. 1, 436–549, 5th ed. 1892, reprinted 1960.

This chapter contains an account of the formulae necessary for the prediction of eclipses for the Earth generally, with all the bounding limits, and for a particular place. Various corrections are also considered.

The formulae and constants given are suitable for use with logarithms. Some of these formulae were adapted to more modern requirements by Comrie, L. J., in papers in *M.N.R.A.S.*, **87**, 483–496, 1927; **93**, 175–181, 1933. Also see Mikhailov, A. A., *Ast. Nach.*, **243**, no. 5812, 49–54, 1931.

Dyson, F., and Woolley, R. v. d. R., *Eclipses of the Sun and Moon*. 1937.

This book is mainly concerned with physical observations but the introductory chapters deal with the historical importance of eclipses and with predictions.

Mitchell, S. A., *Eclipses of the Sun*. 5th ed. 1951.

Describes eclipse expeditions up to 1950.

Mikhailov, A. A., *Teoriya Zatmennii*. 2nd ed. 1954, in Russian.

Contains a detailed account of the computation of solar and lunar eclipses, and related subjects, with sections on limb corrections, corpuscular eclipses, etc. The formulae are designed for use with logarithms.

The three following papers are concerned with ionospheric and corpuscular eclipses.

H.M. Nautical Almanac Office, *M.N.R.A.S.*, **98**, 664–669 and 727–733, 1938.

Contain predictions for the ionospheric and corpuscular eclipses of 1940 October 1, but also give some general formulae.

Lewis, Isabel M., *A.J.*, **49**, no. 1122, 4–7, 1940.

Gives general formulae for the track of eclipses in the ionosphere, and includes a useful table giving the obscured fraction of the solar disk as a function of the magnitude.

B. SOLAR ECLIPSES—FUNDAMENTAL EQUATIONS

Occurrence of solar eclipses

An eclipse of the Sun or Moon will occur when the centres of the Sun, Earth, and Moon are nearly in a straight line. This condition can be fulfilled only when conjunction or opposition occurs in the vicinity of the nodes of the lunar orbit. A criterion for the occurrence of a solar eclipse may be obtained as follows:

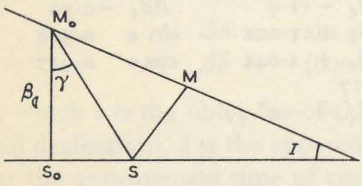

Figure 9.1. *Occurrence of solar eclipses*

Let: $\beta_☾$ be the latitude of the Moon at the time of conjunction in longitude; I the inclination of the orbit of the Moon to the ecliptic; and q the ratio of the motion of the Moon in longitude to that of the Sun. In figure 9.1 S_0, M_0 and S, M represent the positions of the centres of the Sun and Moon respectively at conjunction and at some other time at which the angle S_0M_0S is γ.

The angular distance SM is given by:

$$SM^2 = \beta_☾^2 \{ (q - 1)^2 \tan^2 \gamma + (1 - q \tan I \tan \gamma)^2 \}$$

It is convenient at this point to introduce an auxiliary angle I', such that:

$$(q - 1) \tan I' = q \tan I$$

The quantity between braces { } in the expression for SM² may thus be written:

$$(q - 1)^2 \tan^2 \gamma + \{ 1 - (q - 1) \tan I' \tan \gamma \}^2$$

The value of γ for which this expression is a minimum is obtained by setting its derivative with respect to γ equal to zero. This yields:

$$(q - 1) \tan \gamma = \sin I' \cos I'$$

and the corresponding value of the least true angular geocentric distance is:

$$SM = \beta_{☾} \cos I'$$

As viewed from a point on the surface of the Earth, this distance may be reduced by the difference of the horizontal parallaxes $\pi_{\odot}$, $\pi_{☾}$ of the Sun and Moon.

A solar eclipse will occur when the least apparent distance of the centres of the Sun and Moon is less than the sum of their apparent semi-diameters $s_{\odot}$ and $s_{☾}$, that is when:

$$\beta_{☾} \cos I' - (\pi_{☾} - \pi_{\odot}) < s_{☾} + s_{\odot}$$

$$\text{or} \quad \beta_{☾} < (\pi_{☾} - \pi_{\odot} + s_{☾} + s_{\odot}) \sec I'$$

Extreme and mean values of the quantities occurring in these and similar expressions are given in table 9.1.

Table 9.1. Extreme and mean values for solar eclipses

	Maximum	Minimum	Mean *
Inclination of Moon's orbit to ecliptic (I)	5° 18′	4° 59′	5° 08′
Ratio of motions in longitude (q)	16·2	10·9	13·5
sec I'	1·0052	1·0043	1·00472
Parallax of Moon ($\pi_{☾}$)	61′ 27″	53′ 53″	57′ 02″·70
Parallax of Sun ($\pi_{\odot}$)	8″·96	8″·65	8″·80
Semi-diameter of Moon ($s_{☾}$)	16′ 45″	14′ 41″	15′ 32″·58
Semi-diameter of Sun ($s_{\odot}$)	16′ 18″	15′ 46″	15′ 59″·63

The greatest value of $\beta_{☾}$ at time of conjunction for which a solar eclipse is possible is obtained by introducing the maximum values of $\pi_{☾}$, $s_{☾}$, and $s_{\odot}$, and the minimum value of $\pi_{\odot}$ in the above expression. Similarly, the greatest value of $\beta_{☾}$ at time of conjunction for which a solar eclipse is certain is obtained by introducing in the above expression the minimum values of $\pi_{☾}$, $s_{☾}$, and $s_{\odot}$, and the maximum value of $\pi_{\odot}$. In this calculation, it is sufficient for general purposes to use the mean† value of sec I' and the mean values of $\pi_{☾}$, $\pi_{\odot}$, $s_{☾}$, and $s_{\odot}$ in the portion 0·00472 $(\pi_{☾} - \pi_{\odot} + s_{☾} + s_{\odot})$ of the expression for $\beta_{☾}$. The criteria for the occurrence of a solar eclipse are thus:

$\beta_{☾} > 1° 34' 46''$	no eclipse
$1° 24' 36'' < \beta_{☾} < 1° 34' 46''$	eclipse possible
$\beta_{☾} < 1° 24' 36''$	eclipse certain

In all doubtful cases, the expression for $\beta_{☾}$ may be used as a precise criterion by introducing the actual values of I', $\pi_{☾}$, $\pi_{\odot}$, $s_{☾}$, and $s_{\odot}$ at the date considered.

*For 1968 onwards the mean values of the parallaxes are 57′ 02″·608 and 8″·794.

†It is more appropriate to use the maximum value of sec I' since the inclination is always close to its maximum when the Sun is near the nodes. The limits for $\beta_{☾}$ should therefore be increased by 3″.

Example 9.2. Test for occurrence of eclipse

q = ratio of daily motions (1961 February 15–16) of Moon and Sun in longitude
$= 54143''/3636''.8 = 14.9$

I (*A.E.* 1961, p. 51) $= 5^\circ.1$ $\quad \tan I = 0.089 \quad q/(q-1) = 1.072$
$\tan I' = 0.095 \quad \sec I' = 1.0045$

By interpolation in *A.E.* 1961 to 1961 February $15^d\ 08^h\ 11^m$:

$$\begin{array}{lll} & \;\;'\;\;\; '' & \\ \pi_☾ & 61\ 05.7 & \sin s_☾ = 0.2722\ 74 \sin \pi_☾ \\ s_☾ & 16\ 38.0 & s_☾ = 0''.08 + 0.2722\ 39\ \pi_☾ \\ \pi_\odot & 8.9 & \\ s_\odot & 16\ 11.4 & s_\odot = 16'\ 12''.97\ (A.E., \text{p. } 21) - 1''.55\ (\text{irradiation}) \end{array}$$

Thus $\sec I'\ (\pi_☾ - \pi_\odot + s_☾ + s_\odot) = 1.0045 \times 1^\circ\ 33'\ 46'' = 1^\circ\ 34'\ 11''$

$\beta_☾$ is $0^\circ\ 54'$ so that an eclipse is certain.

Besselian elements

The calculation of eclipses is carried out in accordance with Bessel's method. In solar eclipses the Besselian elements describe the geometric position of the shadow of the Moon relative to the Earth. The exterior tangents to the surfaces of the Sun and the Moon form the umbral cone, the interior tangents the penumbral cone. The common axis of the two cones is the axis of the shadow. The geocentric plane perpendicular to the axis of the shadow is called the fundamental plane, and is taken as the xy-plane of a system of geocentric rectangular coordinates. The x-axis is the intersection of the fundamental plane with the plane of the equator and is directed positively towards the east; the y-axis is directed positively towards the north. The z-axis is parallel to the axis of the shadow and is positive towards the Moon. See figure 9.2, which shows the projection of the observer and the shadow on the fundamental plane.

Let a and d designate the right ascension and declination of the point Z on the celestial sphere towards which the axis of the shadow is directed, and G the distance between the centres of the Sun and Moon; then:

$$G \cos d \cos a = R \cos \delta_\odot \cos \alpha_\odot - r_☾ \cos \delta_☾ \cos \alpha_☾$$
$$G \cos d \sin a = R \cos \delta_\odot \sin \alpha_\odot - r_☾ \cos \delta_☾ \sin \alpha_☾$$
$$G \sin d = R \sin \delta_\odot - r_☾ \sin \delta_☾$$

In practice, it is convenient to set:

$$g = G/R \qquad\qquad b = r_☾/R = \sin \pi_\odot/\sin \pi_☾$$

which yields:

$$g \cos d \cos a = \cos \delta_\odot \cos \alpha_\odot - b \cos \delta_☾ \cos \alpha_☾$$
$$g \cos d \sin a = \cos \delta_\odot \sin \alpha_\odot - b \cos \delta_☾ \sin \alpha_☾$$
$$g \sin d = \sin \delta_\odot - b \sin \delta_☾$$

In numerical calculations b is evaluated from:

$$b = \sin \pi_0/R \sin \pi_☾$$

* where π_0, the horizontal parallax of the Sun at mean distance, is equal to $8''.80$ and R is expressed in astronomical units as in the Ephemeris.

The equatorial coordinates a, d of the point Z, obtained above, are used to calculate the rectangular coordinates x, y, z of the Moon with respect to the

*$8''.794$ for 1968 onwards

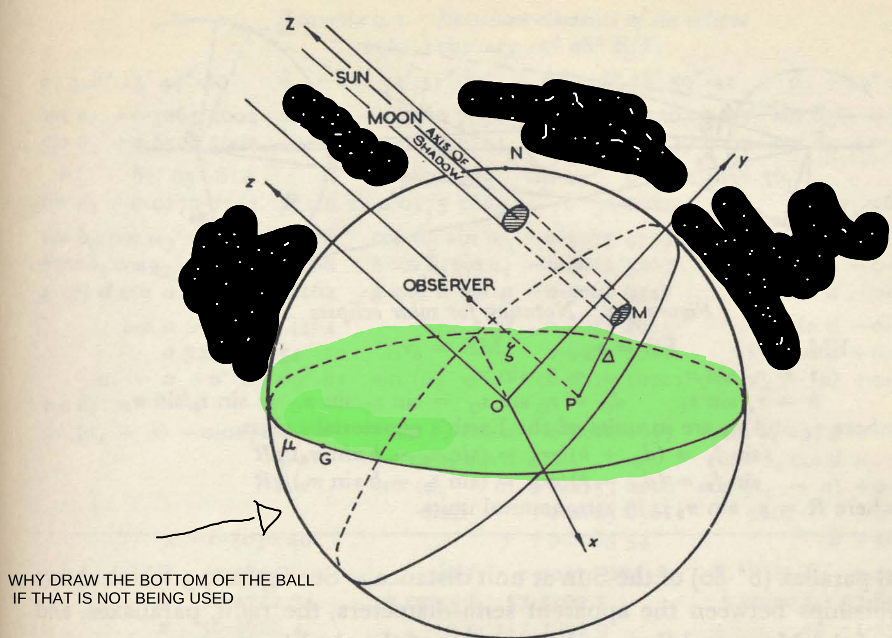

Figure 9.2. *Projection of the observer and shadow on the fundamental plane*

P Projection of observer (ξ, η)
M Projection of axis of shadow (x, y)

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fundamental plane, in units of the equatorial radius of the Earth, from:

$$x = r_{☾} \{ \cos \delta_{☾} \sin (\alpha_{☾} - a) \}$$
$$y = r_{☾} \{ \sin \delta_{☾} \cos d - \cos \delta_{☾} \sin d \cos (\alpha_{☾} - a) \}$$
$$z = r_{☾} \{ \sin \delta_{☾} \sin d + \cos \delta_{☾} \cos d \cos (\alpha_{☾} - a) \}$$

in which:

$$r_{☾} = 1/\sin \pi_{☾}$$

The coordinates x, y are also those of the intersection of the axis of shadow with the fundamental plane.

In the tabulation of Besselian elements of eclipses, the right ascension a of the point Z is conventionally replaced for practical use by the ephemeris hour angle μ of that point, given by:

$$\mu = \text{ephemeris sidereal time} - a$$

The angles f_1, f_2 which the generators of the penumbral (subscript 1) and umbral (subscript 2) cones make with the axis of the shadow are given by:

$$\sin f_1 = (\sin s_0 + k \sin \pi_0)/gR = 0{\cdot}0046\ 6401\ 6/gR$$
$$\sin f_2 = (\sin s_0 - k \sin \pi_0)/gR = 0{\cdot}0046\ 4078\ 4/gR$$

for $k = 0{\cdot}2722\ 74$. For $k = 0{\cdot}2724\ 807$, for use after 1962, the constants become * 0·0046 6402 6 (for f_1) and 0·0046 4077 6 (for f_2). In the above equations s_0, π_0 are respectively the adopted values of the semi-diameter (15′ 59″·63) and the

*For 1968 onwards:	$k = 0{\cdot}2722\ 81$	$k = 0{\cdot}2724\ 880$
Coefficients for f_1:	0·0046 6400 9	0·0046 6401 8
Coefficients for f_2:	0·0046 4079 2	0·0046 4078 3

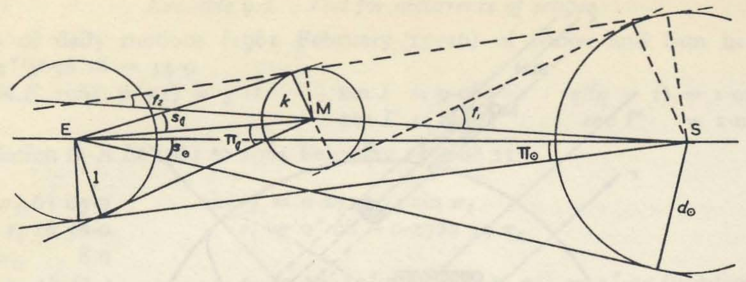

Figure 9.3. *Notation for solar eclipses*

$$EM = r_☾ \qquad ES = r_\odot \qquad MS = gr_\odot$$
$$1 = r_☾ \sin \pi_☾ \qquad 1 = r_\odot \sin \pi_\odot$$
$$k = r_☾ \sin s_☾ \qquad d_\odot = r_\odot \sin s_\odot = \sin s_\odot/\sin \pi_\odot = \sin s_0/\sin \pi_0$$

where $r_☾$ and $r_\odot$ are in units of the Earth's equatorial radius.

$$\sin f_1 = (d_\odot + k)/gr_\odot = (\sin s_0 + k \sin \pi_0)/gR$$
$$\sin f_2 = (d_\odot - k)/gr_\odot = (\sin s_0 - k \sin \pi_0)/gR$$

where $R = r_\odot \sin \pi_0$ is in astronomical units.

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* horizontal parallax (8″·80) of the Sun at unit distance. See figure 9.3 which shows the relationships between the apparent semi-diameters, the radii, parallaxes, and distances of the Moon and Sun, and the angles of the shadow cones.

The distances c_1, c_2 of the vertices of the penumbral and umbral cones above the fundamental plane are thus, in units of the equatorial radius of the Earth:

$$c_1 = z + k \operatorname{cosec} f_1$$
$$c_2 = z - k \operatorname{cosec} f_2$$

With the values of $\sin f$ and c found above, the radii l_1, l_2 of the penumbra and umbra on the fundamental plane are obtained from:

$$l_1 = c_1 \tan f_1 \qquad l_2 = c_2 \tan f_2$$

The convention of signs introduced in the formulae for c makes l_2 negative for total eclipses, positive for annular eclipses, while l_1 is always positive.

The Besselian elements $x, y, \sin d, \cos d, \mu, l_1, l_2$, tabulated in the Ephemeris, are calculated for each hour and subtabulated to an interval of 10 minutes. Although the values of $\tan f_1$ and $\tan f_2$ must be calculated for each hour for the accurate evaluation of l_1 and l_2, it is always sufficient to use the constant values of $\tan f_1$ and $\tan f_2$ for the integral hour nearest conjunction in the calculation of local circumstances and eclipse curves.

The derivatives of the Besselian elements with respect to time are calculated from the tabular values by numerical differentiation; primes are used to denote derivatives and numerical values are given as hourly variations. The hourly variations x', y', l_2' of x, y, l_2 may be obtained with sufficient precision by multiplying by six the first differences of the tabular values at intervals of 10 minutes. The hourly variations μ', d' of μ, d are constant to the precision required; they may be evaluated once only for the hourly interval which contains the time of conjunction. μ' and d' are expressed in their natural units of radians per hour.

*8″·794 for 1968 onwards.

Example 9.3. *Besselian elements of an eclipse*

1961 February 15^d 08^h E.T.

$\alpha_{☾}$	328° 13′ 44″·29	$\delta_{☾}$	−11° 53′ 31″·83	$\alpha_{\odot}$	328° 38′ 50″·42	$\delta_{\odot}$	−12° 42′ 49″·04
$\sin\alpha_{☾}$	−0·5265 2602	$\sin\delta_{☾}$	−0·2060 7054	$\sin\alpha_{\odot}$	−0·5203 0423	$\sin\delta_{\odot}$	−0·2200 7813
$\cos\alpha_{☾}$	+0·8501 5902	$\cos\delta_{☾}$	+0·9785 3714	$\cos\alpha_{\odot}$	+0·8539 8097	$\cos\delta_{\odot}$	+0·9754 8225
$\pi_{☾}$	61′ 05″·814	R	0·9878 805	$\sin\pi_0$	0·0000 4266 36		
$\sin\pi_{☾}$	0·0177 7143	$R\sin\pi_{☾}$	0·0175 5605	b	0·0024 3014	$r_{☾}$	56·2700 9

$\cos\delta_{\odot}\cos\alpha_{\odot}$	+0·8330 4328	$\cos\delta_{\odot}\sin\alpha_{\odot}$	−0·5075 4754	$\sin\delta_{\odot}$	−0·2200 7813
$b\cos\delta_{☾}\cos\alpha_{☾}$	+0·0020 2166	$b\cos\delta_{☾}\sin\alpha_{☾}$	−0·0012 5207	$b\sin\delta_{☾}$	−0·0005 0078
$g\cos d\cos a$	+0·8310 2162	$g\cos d\sin a$	−0·5062 9547	$g\sin d$	−0·2195 7735
$\tan a$	−0·6092 4464	g^2	0·9951 4625	$\sin d$	−0·2201 1219
a	328° 38′ 54″·10	g	0·9975 7017	$\cos d$	+0·9754 7456
$\alpha_{☾} - a$	−0° 25′ 09″·81	$\sin(\alpha_{☾} - a)$	−0·0073 1970	$\cos(\alpha_{☾} - a)$	+0·9999 7321
$\cos\delta_{☾}\times\sin(\alpha_{☾} - a)$	−0·0071 6260	$\sin\delta_{☾}\cos d$	−0·2010 1657	$\sin\delta_{☾}\sin d$	+0·0453 5864
		$-\cos\delta_{☾}\sin d\times\cos(\alpha_{☾} - a)$	+0·2153 8218	$+\cos\delta_{☾}\cos d\times\cos(\alpha_{☾} - a)$	+0·9545 1252
		sum	+0·0143 6561	sum	+0·9998 7116
x	−0·4030 40	y	+0·8083 54	z	+56·2628 4
gR	0·9854 801	$\sin f_1$	0·0047 3273 5	$\sin f_2$	0·0047 0916 1
k	0·2722 74	$k\operatorname{cosec} f_1$	57·5299 5	$k\operatorname{cosec} f_2$	57·8179 4
z	56·2628 4	c_1	113·7927 9	c_2	−1·5551 0
		$\tan f_1$	0·0047 3278 8	$\tan f_2$	0·0047 0921 3
		l_1	0·5385 57	l_2	−0·0073 23

" Apparent sidereal time " (*A.E.*, p. 11)	$17^h\ 40^m\ 21^s{\cdot}093$
= Ephemeris sidereal time at 8^h E.T.	265° 05′ 16″·40
a	328 38 54·10
Ephemeris hour angle, μ	296 26 22·3

From a similar calculation for February 15^d 09^h

μ =	311° 26′ 29″·2	$\sin d$ =	−0·2198 7475
hourly change =	15° 00′ 06″·9	hourly change =	+0·0002 3744
μ' =	0·2618 328	d' =	+0·0002 4341

Coordinates of the observer

For an observer located on the surface of the Earth in ephemeris longitude λ^*, geocentric latitude ϕ', at a distance ρ from the centre of the terrestrial spheroid, his geocentric rectangular coordinates ξ, η, ζ, referred to the x, y, z system of axes in units of the Earth's equatorial radius, are found in terms of the Besselian elements from:

$$\xi = \rho\cos\phi'\sin\theta$$
$$\eta = \rho\sin\phi'\cos d - \rho\cos\phi'\sin d\cos\theta$$
$$\zeta = \rho\sin\phi'\sin d + \rho\cos\phi'\cos d\cos\theta$$

in which†:

$$\theta = \mu - \lambda^*$$

Their hourly variations are found from:

$$\xi' = +\mu'\,\rho\cos\phi'\cos\theta$$
$$\eta' = +\mu'\,\rho\cos\phi'\sin d\sin\theta - d'\,(\rho\sin\phi'\sin d + \rho\cos\phi'\cos d\cos\theta)$$
$$\zeta' = -\mu'\,\rho\cos\phi'\cos d\sin\theta + d'\,(\rho\sin\phi'\cos d - \rho\cos\phi'\sin d\cos\theta)$$

†Longitude is here measured positively to the west.

or
$$\xi' = +\mu' (-\eta \sin d + \zeta \cos d)$$
$$\eta' = +\mu' \xi \sin d - d' \zeta$$
$$\zeta' = -\mu' \xi \cos d + d' \eta$$
where μ', d' are in units of radians per hour.

Precise eclipse calculations must take into account the flattening of the Earth; and, for the calculation of local circumstances for a particular observer, the above formulae present no difficulty. The geometric conditions pertaining to general eclipse phenomena, such as the track of the central line, determine the value of the coordinates ξ, η of the points on the surface of the Earth where these phenomena occur; but the evaluation of the coordinate ζ requires the value of the distance ρ of each point from the centre of the Earth. This distance is not known a priori, because it is a function of the latitude of the point. The problem of determining ζ could be solved by successive approximations, but the procedure outlined below was devised by Bessel to provide a direct computation.

If ϕ is the geodetic latitude of a point on the Earth's surface and ϕ' its geocentric latitude, then:
$$\rho \sin \phi' = (1 - e^2) \sin \phi \,(1 - e^2 \sin^2 \phi)^{-\frac{1}{2}} = S \sin \phi$$
$$\rho \cos \phi' = \cos \phi \,(1 - e^2 \sin^2 \phi)^{-\frac{1}{2}} = C \cos \phi$$
in which e is the ellipticity of the Earth's spheroid. For Hayford's spheroid, * e^2 is equal to 0·0067 2267. S and C are tabulated in table 2.8.

Let ϕ_1 be the parametric latitude, such that:
$$\sin \phi_1 = \rho \sin \phi' (1 - e^2)^{-\frac{1}{2}} \qquad \cos \phi_1 = \rho \cos \phi' \qquad \sin^2 \phi_1 + \cos^2 \phi_1 = 1$$
and set:
$$\rho_1 \sin d_1 = \sin d \qquad \rho_2 \sin d_2 = \sin d \,(1 - e^2)^{\frac{1}{2}}$$
$$\rho_1 \cos d_1 = \cos d \,(1 - e^2)^{\frac{1}{2}} \qquad \rho_2 \cos d_2 = \cos d$$
$$\eta_1 = \eta/\rho_1 \qquad \zeta_1^2 = 1 - \xi^2 - \eta_1^2$$

By means of these new quantities, the equations for ξ, η, ζ above may be transformed into:
$$\xi = \cos \phi_1 \sin \theta$$
$$\eta_1 = \sin \phi_1 \cos d_1 - \cos \phi_1 \sin d_1 \cos \theta$$
$$\zeta_1 = \sin \phi_1 \sin d_1 + \cos \phi_1 \cos d_1 \cos \theta$$
and it may be noted that:
$$\zeta = \rho_2 \sin \phi_1 \sin d_2 + \rho_2 \cos \phi_1 \cos d_2 \cos \theta$$
or
$$\zeta = \rho_2 \{ \zeta_1 \cos (d_1 - d_2) - \eta_1 \sin (d_1 - d_2)\}$$

According to the above relations, it is seen that the quantities required for the introduction of the flattening of the Earth into eclipse calculations are ρ_1, ρ_2, $\sin d_1$, $\cos d_1$, $\sin (d_1 - d_2)$, and $\cos (d_1 - d_2)$. With these quantities, ξ, η_1, ζ_1, may be calculated from given values of ξ, η to enable ϕ_1 (and thus ϕ) and θ to be deduced; and ζ can be calculated directly from values of η_1, ζ_1 and indirectly from ξ, η. They may be obtained from the following formulae:
$$\rho_1 = (1 - e^2 \cos^2 d)^{\frac{1}{2}} \qquad \rho_2 = (1 - e^2 \sin^2 d)^{\frac{1}{2}}$$
$$\rho_1 \sin d_1 = \sin d \qquad \rho_1 \cos d_1 = (1 - e^2)^{\frac{1}{2}} \cos d$$
$$\rho_1 \rho_2 \sin (d_1 - d_2) = e^2 \sin d \cos d \qquad \rho_1 \rho_2 \cos (d_1 - d_2) = (1 - e^2)^{\frac{1}{2}}$$

*e^2 = 0·0066 9454 for 1968 onwards; corresponding values of S and C are tabulated in *A.E.* Table *VII*.

The quantities ρ_1, ρ_2, and $(d_1 - d_2)$ are almost constant for the duration of the eclipse. It is always sufficient to calculate them for the integral hour nearest conjunction, while sin d_1 and cos d_1 may be tabulated with the other Besselian elements. It should be noted that the subscripts 1 and 2 used here do not refer to the penumbra and umbra.

Example 9.4. *Auxiliary elements of an eclipse*
1961 February 15^d 08^h

$\sin d$	−0·2201 12	$\cos d$	+0·9754 75	e^2	0·0067 2267
$e^2 \cos^2 d$	0·0063 97	$e^2 \sin^2 d$	0·0003 26	$e^2 \sin d \cos d$	−0·0014 43
ρ_1^2	0·9936 03	ρ_2^2	0·9996 74	$(1 - e^2)^{\frac{1}{2}}$	0·9966 33
ρ_1	0·9967 96	ρ_2	0·9998 37	$\rho_1 \rho_2$	0·9966 34
$\sin d_1$	−0·2208 20	$\cos d_1$	+0·9753 16		
$\sin (d_1 - d_2)$	−0·0014 48	$\cos (d_1 - d_2)$	0·9999 99		

Conditional and variational equations

In the plane through the observer (coordinates ξ, η, ζ) perpendicular to the axis of the shadow (coordinates x, y, ζ), and thus parallel to the fundamental plane, let Δ, Q be the distance and position angle (measured eastwards from the north, i.e. from the y-axis towards the x-axis) of the axis from the observer; then:

$$\xi = x - \Delta \sin Q$$
$$\eta = y - \Delta \cos Q$$
$$\Delta^2 = (x - \xi)^2 + (y - \eta)^2$$

At a height ζ above the fundamental plane the radius L of the shadow is given by:

$$L = l - \zeta \tan f$$

where the values of l, f appropriate to the penumbra $(L_1 = l_1 - \zeta \tan f_1)$ or umbra $(L_2 = l_2 - \zeta \tan f_2)$ are used.

At the beginning or end of an eclipse the observer is located on the surface of the cone of shadow, so that:

$$\Delta = L \qquad \Delta^2 - L^2 = 0$$

or

$$(x - \xi)^2 + (y - \eta)^2 - (l - \zeta \tan f)^2 = 0$$

When the observer is located on the northern or southern limits of the shadow, the eclipse begins and ends at the same time; the two roots of the above equation must therefore be equal, and the derivative of the left-hand side, with respect to time, must be zero. Using primes to denote derivatives with respect to time, and making the substitutions:

$$x - \xi = \Delta \sin Q \qquad y - \eta = \Delta \cos Q \qquad \Delta = L$$

the condition becomes:

$$(x' - \xi') \sin Q + (y' - \eta') \cos Q - (l' - \zeta' \tan f) = 0$$

The derivatives ξ', η', ζ' are obtained simply by substituting the expressions for ξ, η in those given above:

$$\xi' = \mu' \, (-y \sin d + \zeta \cos d + \Delta \sin d \cos Q)$$
$$\eta' = \mu' \, (+x \sin d - \Delta \sin d \sin Q) - d' \zeta$$
$$\zeta' = \mu' \, (-x \cos d + \Delta \cos d \sin Q) + d' \, (y - \Delta \cos Q)$$

If three auxiliary elements a', b', c' are introduced, with the definitions:

$$a' = -l' - \mu' x \tan f \cos d$$
$$b' = -y' + \mu' x \sin d$$
$$c' = +x' + \mu' y \sin d + \mu' l \tan f \cos d$$

then the conditional equation determining the northern and southern limits may be written, omitting terms in $d' \tan f$ and putting $\Delta = L$:

$$a' + \cos Q\,(-b' + \zeta d') + \sin Q\,(c' - \zeta \mu' \cos d \sec^2 f) = 0$$

or, setting $\sec^2 f = 1$:

$$\tan Q = (b' - \zeta d' - a' \sec Q)/(c' - \zeta \mu' \cos d)$$

At a given time, this equation determines the value of the position angle Q corresponding to positions on the northern or southern limits of the shadow, at which the eclipse both begins and ends at that time.

The above expression for Q provides a convenient method for the calculation of the limits of umbra and penumbra. It is advantageous to tabulate a'_2, b', c'_1, c'_2 at the same time as the Besselian elements, but they are not required for local predictions and are not published in the Ephemeris. In general it is unnecessary to take a'_1 into account in the calculation of the penumbral limits. Here, as elsewhere for quantities relating to the shadow, the subscripts 1 and 2 refer to the penumbra and umbra respectively.

The eclipse is a maximum (at the time of greatest phase) for a particular observer when $(L_1 - m)/(L_1 + L_2)$ is a maximum (see sub-section D). *Approximate* geometric conditions for maximum eclipse are $L - \Delta$ to be a maximum, or Δ to be a minimum; the first of these conditions leads to the same relation as for the northern and southern limits, namely:

$$(x' - \xi') \sin Q + (y' - \eta') \cos Q - (l' - \zeta' \tan f) = 0$$

whereas the second is equivalent to omitting the small contribution $(l' - \zeta' \tan f)$. Although the differences between these conditions are small, they can be significant for precise calculations; see Gossner, Simone D., A correction to the time of maximum obscuration in solar eclipses, *A.J.*, **60,** 383, 1955.

Example 9.5. *Auxiliary Besselian elements of an eclipse*

1961 February $15^d\ 08^h$

x	−0·4030 40	$\sin d$	−0·2201 12	$\tan f_1$	0·0047 33	l_1	0·5386
y	+0·8083 54	$\cos d$	+0·9754 75	$\tan f_2$	0·0047 09	l_2	−0·0073 23
μ'	0·2618 33						

x', y', l' are obtained from the differences of x, y, l (*A.E.*, p. 296)

$-l'_2$	−0·0000 57	$-y'$	−0·1430 31
$-\mu' x \tan f_2 \cos d$	+0·0004 85	$+\mu' x \sin d$	+0·0232 28
a'_2	+0·0004 28	b'	−0·1198 03

	1	2
x'	+0·5614 14	
$+\mu' y \sin d$	−0·0465 88	
$+\mu' l \tan f \cos d$	+0·0006 51	−0·0000 09
c'_1, c'_2	+0·5154 77	+0·5148 17

C. SOLAR ECLIPSES—PREDICTED DATA

General

Apart from the Besselian elements, which are given to enable local predictions to be made, the information on solar eclipses published in the Ephemeris consists of times and positions on the surface of the Earth, corresponding to the occurrence of precise phases or other conditions of the eclipse; the data are given either in tabular or graphical form. Consideration of the geometrical relationships and conditions appertaining to a particular phase of the eclipse generally enables the rectangular coordinates ξ, η of the position to be deduced. Using the auxiliary quantities ρ_1, ρ_2, $\sin d_1$, $\cos d_1$, $\sin (d_1 - d_2)$, $\cos (d_1 - d_2)$, which are tabulated at the same time as the Besselian elements, ζ and ξ, η_1, ζ_1 can then be calculated.

From these rectangular coordinates, the longitude λ and geodetic latitude ϕ of the corresponding point on the surface of the Earth are found through the following set of equations:

$$\cos \phi_1 \sin \theta = \xi$$
$$\cos \phi_1 \cos \theta = -\eta_1 \sin d_1 + \zeta_1 \cos d_1$$
$$\sin \phi_1 = +\eta_1 \cos d_1 + \zeta_1 \sin d_1$$
$$\lambda = \mu - \theta$$
$$\tan \phi = (1 - e^2)^{-\frac{1}{2}} \tan \phi_1$$

where ϕ_1 is the parametric latitude, and θ is the local hour angle of the axis of shadow.

For Hayford's spheroid (flattening 1/297) the coefficient $(1 - e^2)^{-\frac{1}{2}}$ is equal to * 1·0033 78.

Unless specifically noted otherwise, the independent variable used in delineating the tracks and curves is time; and in all calculations the time system is that of ephemeris time.

Central line of total or annular phase

The central line is the locus of points of intersection of the axis of shadow with the surface of the Earth. Thus, for each time:

$$\xi = x \qquad \eta = y \qquad \eta_1 = y/\rho_1 = y_1$$
$$\zeta_1 = (1 - \xi^2 - \eta_1^2)^{\frac{1}{2}} = (1 - x^2 - y_1^2)^{\frac{1}{2}}$$

There are two solutions. The one for which ζ_1 is negative corresponds to the phenomenon that occurs below the horizon and is usually omitted in the ephemerides.

The semi-duration s of the total, or annular, phase on the central line is given by:

$$s = L_2/n$$

where L_2 is the radius of the umbra at a height ζ above the fundamental plane, and n is the speed of the shadow relative to the observer. L_2 is found from:

$$L_2 = l_2 - \zeta \tan f_2$$

where ζ is obtained by substituting the central-line values of η_1 and ζ_1 in the

*1·0033 64 is used for 1968 onwards. See note on page 523.

equation:

$$\zeta = \rho_2 \{ \zeta_1 \cos (d_1 - d_2) - \eta_1 \sin (d_1 - d_2) \}$$

n is given by:

$$n^2 = (x' - \xi')^2 + (y' - \eta')^2$$

where:

$$\xi' = \mu' \cos \phi_1 \cos \theta = \mu' \, (-y \sin d + \zeta \cos d)$$

$$\eta' = \mu' \, x \sin d - d' \zeta$$

or, very nearly:

$$x' - \xi' = c_2' - \mu' \zeta \cos d$$

$$y' - \eta' = -b'$$

The resulting value of the semi-duration is expressed in decimals of an hour if hourly variations are used for the derivatives, and thus for n.

The positions of points on the central line, and the durations of the total or annular phase at these points, are tabulated in the Ephemeris.

Example 9.6. *Point on central line, and duration of an eclipse*

1961 February $15^d \; 08^h$

$\xi = x$	−0·4030 40	$\sin d$	−0·2201 12	$\sin d_1$	−0·2208 20	$\sin (d_1 - d_2)$	−0·0014 48
y	+0·8083 54	$\cos d$	+0·9754 75	$\cos d_1$	+0·9753 16	$\cos (d_1 - d_2)$	+0·9999 99
ρ_1	0·9967 96	μ'	+0·2618 33	$\tan f_2$	+0·0047 09	ρ_2	0·9998 37
$\eta_1 = y_1$	+0·8109 52	ζ_1^2	0·1799 16	ζ_1	0·4241 65	$(1 - e^2)^{-\frac{1}{2}}$	1·0033 78

$\cos \phi_1 \sin \theta$	−0·4030 40	ζ/ρ_2	+0·4253 39	l_2	−0·0073 23
$\cos \phi_1 \cos \theta$	+0·5927 69	ζ	+0·4252 70	$-\zeta \tan f_2$	−0·0020 03
				$L_2 = \text{sum}$	−0·0093 26
$\tan \theta$	−0·6799 28	c_2'	+0·5148 17		
θ	325° 47′·23	$-\mu' \zeta \cos d$	−0·1086 19	s	0^h·0220 21
μ	296° 26′·37	$x' - \xi' = \text{sum}$	+0·4061 98	Duration	158^s·6
λ	−29° 20′·9	$y' - \eta' = -b'$	+0·1198 03		
$\sin \phi_1$	+0·6972 70	n^2	0·1793 50	$\tan Q_0 = -(y' - \eta')/(x' - \xi')$	
$\tan \phi_1$	+0·9727 42	n	0·4234 97	=	−0·2949 37
$\tan \phi$	+0·9760 28			($\tan Q_0$ is required in example 9.7)	
ϕ	+44° 18′·3				

Northern and southern limits of umbra and penumbra

The distance from each point on the limits of the umbra to the axis of the shadow is equal to L_2, so that, at these points:

$$(x - \xi)^2 + (y - \eta)^2 = L_2^2$$

This equation may be replaced by the equivalent system:

$$\xi = x - L_2 \sin Q$$

$$\eta_1 = (y - L_2 \cos Q)/\rho_1$$

where it has been shown that, at the limits, the position angle Q must fulfil the condition:

$$\tan Q = \frac{b' - \zeta \, d' - a' \sec Q}{c_2' - \zeta \mu' \cos d}$$

The sign of $\cos Q$ is positive for the northern limit of a total eclipse and the southern limit of an annular eclipse; it is negative for the southern limit of a total eclipse and the northern limit of an annular eclipse.

The quantity ζ, required for the evaluation of L_2 and Q, is not known directly. It is therefore necessary to proceed by successive approximations. The procedure outlined below converges rapidly and rarely requires more than two approximations.

For initial values in the first approximation, assume a value for L_2 equal to that used in the computation of the duration on the central line for the corresponding time, and a value Q_0 for Q such that:

$$\tan Q_0 = \frac{b'}{c_2' - \zeta\mu' \cos d}$$

where numerator and denominator are also available from the computation of duration. If the latter is not available, as for example in the case of non-central eclipses, it is sufficient to set:

$$\zeta = 0 \qquad L_2 = l_2 \qquad \tan Q = b'/c_2'$$

From these starting values ξ and η_1 are calculated and thence:

$$\zeta_1 = (1 - \xi^2 - \eta_1^2)^{\frac{1}{2}}$$

$$\zeta = \rho_2 \{ \zeta_1 \cos (d_1 - d_2) - \eta_1 \sin (d_1 - d_2) \}$$

With this value of ζ, $\tan Q$ is calculated from the accurate formula. The term $a' \sec Q$ is small, and it is always sufficient to use the value of $\sec Q$ from the previous approximation.

The new values of ζ and Q are used to re-evaluate L_2, ξ, η_1, ζ_1, and it is then possible to proceed to the calculation of geographic positions. Near the ends of the path it may be advisable to carry through one more approximation to ζ and $\tan Q$.

The northern and southern limits of the umbra are tabulated in the Ephemeris with the central line, and are also shown on the eclipse maps.

The geometric conditions for the limits of penumbra are similar to those appertaining to the limits of umbra in an annular eclipse. The elements for the penumbra are used instead of those for the umbra. These limits need not be known very accurately, and the flattening of the Earth may be neglected. Thus, it suffices to use:

$$L_1 = l_1 - \zeta \tan f_1$$

$$\xi = x - L_1 \sin Q$$

$$\eta = y - L_1 \cos Q$$

$$\zeta = (1 - \xi^2 - \eta^2)^{\frac{1}{2}}$$

$$\tan Q = \frac{b'}{c_1' - \zeta\mu' \cos d}$$

the small terms $\zeta d'$ and $a' \sec Q$ being omitted in $\tan Q$. The sign of $\cos Q$ is positive for the southern limit and negative for the northern limit.

The initial value of ζ to be used for the first approximation is not critical and may be chosen at the discretion of the computer. It is often simplest to start by assuming ζ equal to zero, giving:

$$L_1 = l_1 \qquad \tan Q = b'/c_1'$$

which leads rapidly to an adequate value of ζ.

The values of ξ, η, ζ resulting from the third approximation may be used in the calculation of the geographic coordinates. Inasmuch as the flattening of the

Example 9.7. *Northern and southern limits of the umbra*

1961 February 15^{d} 08^{h}

ρ_1	0·9967 96	$\sin(d_1 - d_2)$	−0·0014 48	$\sin d_1$	−0·2208 20	$(1 - e^2)^{-\frac{1}{2}}$	1·0033 78
ρ_2	0·9998 37	$\cos(d_1 - d_2)$	+0·9999 99	$\cos d_1$	+0·9753 16	$\tan f_2$	0·0047 09
d'	+0·0002 43	a_2'	+0·0004 28	$\mu' \cos d$	+0·2554 12		

Initial data taken from example 9.6	First approximation Northern limit	First approximation Southern limit	Second approximation Northern limit	Second approximation Southern limit
l_2	−0·0073 23	−0·0073 23	−0·0073 23	−0·0073 23
$-\zeta \tan f_2$	−0·0020 03	−0·0020 03	−0·0019 07	−0·0020 93
L_2	−0·0093 26	−0·0093 26	−0·0092 30	−0·0094 16
$\tan Q$	−0·2949 37	−0·2949 37	−0·2925 55	−0·2976 83
$\sin Q$	−0·2828 89	+0·2828 89	−0·2807 86	+0·2853 10
$\cos Q$	+0·9591 52	−0·9591 52	+0·9597 70	−0·9584 35
$\cos Q/\rho_1$	+0·9622 35	−0·9622 35	+0·9628 55	−0·9615 16
x	−0·4030 40	−0·4030 40	−0·4030 40	−0·4030 40
$-L_2 \sin Q$	−0·0026 38	+0·0026 38	−0·0025 92	+0·0026 86
ξ = sum	−0·4056 78	−0·4004 02	−0·4056 32	−0·4003 54
y_1	+0·8109 52	+0·8109 52	+0·8109 52	+0·8109 52
$-L_2 \cos Q/\rho_1$	+0·0089 74	−0·0089 74	+0·0088 87	−0·0090 54
η_1 = sum	+0·8199 26	+0·8019 78	+0·8198 39	+0·8018 98
ζ_1^2	0·1631 47	0·1965 10	0·1633 27	0·1966 76
ζ_1	0·4039 15	0·4432 94	0·4041 37	0·4434 82
ζ	0·4050 36	0·4443 83		
b'	−0·1198 03	−0·1198 03		
$-\zeta d'$	−0·0000 98	−0·0001 08		
$-a_2' \sec Q$	−0·0004 46	+0·0004 46		
sum	−0·1203 47	−0·1194 65		
c_2'	+0·5148 17	+0·5148 17		
$-\zeta \mu' \cos d$	−0·1034 51	−0·1135 01		
sum	+0·4113 66	+0·4013 16		
$\tan Q$	−0·2925 55	−0·2976 83		

It may be verified that a further approximation changes ζ by only about 0·0002, leading to a change of less than 0·0000 01 in L_2; the corresponding change in $\tan Q$ is less than 0·0000 4 which has negligible effect on ξ, η_1.

	Northern limit	Southern limit
$\cos\phi_1 \sin\theta = \xi$	−0·4056 32	−0·4003 54
$\cos\phi_1 \cos\theta$	+0·5751 98	+0·6096 10
$\tan\theta$	−0·7052 04	−0·6567 38
θ	324° 48′·50	326° 42′·32
μ	296° 26′·37	296° 26′·37
λ	−28° 22′·1	−30° 16′·0
$\sin\phi_1$	+0·7103 61	+0·6841 74
$\tan\phi_1$	+1·0092 69	+0·9381 00
$\tan\phi$	+1·0126 78	+0·9412 69
ϕ	+45° 21′·7	+43° 16′·0

Earth is neglected, the equations are reduced to:

$$\cos\phi \sin\theta = \xi$$
$$\cos\phi \cos\theta = -\eta \sin d + \zeta \cos d$$
$$\sin\phi = +\eta \cos d + \zeta \sin d$$
$$\lambda = \mu - \theta$$

If desired, the flattening of the Earth may be introduced in the same manner as for the limits of umbra.

The northern and southern limits of the penumbra are not tabulated in the Ephemeris, but are included on the eclipse maps.

Example 9.8. *Southern limit of penumbra*

1961 February $15^d\ 08^h$

$\sin d$ −0·2201 μ' +0·2618 $\tan f_1$ +0·0047

$\cos d$ +0·9755 $\mu' \cos d$ +0·2554

	First approximation	Second approximation	Third approximation
b'	−0·1198	−0·1198	−0·1198
c_1'	+0·5155	+0·5155	+0·5155
$-\zeta\,\mu' \cos d$	0	−0·2341	−0·2371
sum	+0·5155	+0·2814	+0·2784
$\tan Q$	−0·2324	−0·4257	−0·4303
$\sin Q$	−0·2264	−0·3917	−0·3953
$\cos Q$	+0·9740	+0·9201	+0·9186
l_1	+0·5386	+0·5386	+0·5386
$-\zeta \tan f_1$	0	−0·0043	−0·0044
L_1 = sum	+0·5386	+0·5343	+0·5342
x	−0·4030	−0·4030	−0·4030
$-L_1 \sin Q$	+0·1219	+0·2093	+0·2112
ξ = sum	−0·2811	−0·1937	−0·1918
y	+0·8084	+0·8084	+0·8084
$-L_1 \cos Q$	−0·5246	−0·4916	−0·4907
η = sum	+0·2838	+0·3168	+0·3177
ζ^2	0·8404	0·8621	0·8623
ζ	0·9167	0·9285	0·9286
$\cos\phi \sin\theta = \xi$			−0·1918
$\cos\phi \cos\theta$			+0·9758
$\tan\theta$			−0·1966
θ			348°·9
μ			296°·4
λ			−52°·5
$\sin\phi$			+0·1055
ϕ			+6°·1

Outline curves

An outline curve of the shadow on the Earth's surface is the locus of all the points at which the eclipse is beginning or ending at the given time. Outline curves for the umbra are sometimes given in special eclipse publications; they are not given in the Ephemeris as the other information concerning the total or annular phase is normally adequate for most requirements. Outline curves for the penumbra, showing the places where the partial phase is beginning or ending at stated times, are usually plotted on small-scale maps and until 1960 were so presented in the Ephemeris.

To this precision the flattening of the Earth may be neglected, and l_1 may be substituted for L_1, thus avoiding the necessity for successive approximation. Thus, for a stated time:

$$\xi = x - l_1 \sin Q$$
$$\eta = y - l_1 \cos Q$$
$$\xi^2 + \eta^2 + \zeta^2 = 1$$

In this case the angle Q is used as the independent variable of the calculation. If there are points both on the northern and southern limits of penumbra for the time considered, the angle Q for the outline curve generally takes all values from 0° to 360°. If this is not so, it is necessary to find the range of Q; the extreme values of Q are those for the two points on the curve at which ζ is equal to zero. For these two points:

$$\xi^2 + \eta^2 = 1$$

If the intersection of the axis of the shadow with the fundamental plane is at distance m in position angle M from the origin, then:

$$x = m \sin M \qquad y = m \cos M$$

and, at the two extreme points:

$$(m \sin M - l_1 \sin Q)^2 + (m \cos M - l_1 \cos Q)^2 = 1$$

or

$$\cos (Q - M) = \frac{m^2 + l_1^2 - 1}{2\, l_1\, m}$$

yielding two values for the angle Q.

If the angle $(Q - M)$ is imaginary, there are no end points to the curve, and the angle Q takes all values from 0° to 360°.

Once the limits of Q have been found, it remains to determine which of the two sections of the circumference is the appropriate one. This may be found rapidly by testing whether or not an assumed value of Q, such as 0° or 90°, satisfies the equations determining the curves.

To describe the curve, discrete values of Q are taken at intervals of five or ten degrees and ξ, η, ζ and λ, ϕ are calculated as above, neglecting the flattening of the Earth.

If the curve is intended for a large-scale map, it is necessary to take into account the flattening of the Earth, and to calculate accurate values of ξ, η_1, ζ_1 from:

$$\xi = x - L_1 \sin Q$$
$$\eta_1 = (y - L_1 \cos Q)/\rho_1$$
$$\zeta_1 = (1 - \xi^2 - \eta_1^2)^{\frac{1}{2}}$$

where the value of ζ found above may be used as an initial value in obtaining L_1 from:

$$L_1 = l_1 - \zeta \tan f_1$$

The geographic coordinates λ, ϕ are then found from the accurate formulae given above.

The outline curves have been replaced in the Ephemeris by curves giving the times of middle and the semi-duration of the eclipse. These curves are actually

Example 9.9. Outline curves of an eclipse

1961 February 15^{d} 08^{h}

(a) Extreme values of Q

x	$-0·4030$	l_1	$0·5386$
y	$+0·8084$	l_1^2	$0·2901$
m	$0·9033$	m^2	$0·8159$
$\tan M$	$-0·4985$		
$2\, l_1\, m$	$0·9730$	$l_1^2 + m^2 - 1$	$+0·1060$
$\cos(Q - M)$	$+0·1089$		
	°		°
$Q - M$	$83·7$	$Q - M$	$276·3$
M	$333·5$	M	$333·5$
Q	$57·2$	Q	$249·8$

(b) Range of Q

For $Q = 0°$

x	$-0·4030$
$-l_1 \sin Q$	0
$\xi = \text{sum}$	$-0·4030$
y	$+0·8084$
$-l_1 \cos Q$	$-0·5386$
$\eta = \text{sum}$	$+0·2698$
ζ^2	$+0·76 > 0$

Thus the range of Q is $250° \leqslant Q \leqslant 360°$ and $0° \leqslant Q \leqslant 57°$

(c) Sample calculation for $Q = 20°$

$\sin Q$	$+0·3420$	$\sin d$	$-0·2201$
$\cos Q$	$+0·9397$	$\cos d$	$+0·9755$
x	$-0·4030$	$\cos\phi \sin\theta = \xi$	$-0·5872$
$-l_1 \sin Q$	$-0·1842$	$\cos\phi \cos\theta$	$+0·7990$
$\xi = \text{sum}$	$-0·5872$		
		$\tan\theta$	$-0·7349$
y	$+0·8084$	θ	$323°·7$
$-l_1 \cos Q$	$-0·5061$	μ	$296°·4$
$\eta = \text{sum}$	$+0·3023$	λ	$-\ 27°·3$
ζ^2	$0·5638$	$\sin\phi$	$+0·1296$
ζ	$0·7509$	ϕ	$+\ 7°·4$

constructed graphically from the intersections of a network of outline curves. For this purpose the outline curves are extended below the horizon by using the negative solutions for ζ for a few points at both ends of each curve.

Curves of maximum and middle eclipse,
equal semi-duration, and equal magnitude

A curve of maximum eclipse is the locus of all points at which the eclipse is at maximum at a given time. Such a curve may be substituted for the curve of middle eclipse, indicating points at which the middle of the eclipse occurs at the given time, from which it differs only slightly. In contrast to curves of middle eclipse, curves of maximum eclipse correspond to a definite geometric condition in the progress of the eclipse, and may be obtained by a simple computation.

The coordinates of the observer at the given time are used in the form:

$$\xi = x - \Delta \sin Q \qquad \eta = y - \Delta \cos Q$$

where Q fulfils the condition that the eclipse is at maximum, namely (neglecting the flattening of the Earth):

$$\tan Q = b'/(c_1' - \zeta\, \mu' \cos d)$$

Instead of Δ, the independent variable (or parameter) used to describe the curve is ζ, where again neglecting the flattening of the Earth:

$$\zeta^2 = 1 - c^2 = 1 - \xi^2 - \eta^2$$

Putting:
$$\xi = c \sin \gamma$$
$$\eta = c \cos \gamma$$

and, eliminating Δ, γ is determined from:

$$\sin(\gamma - Q) = \frac{1}{c}(x \cos Q - y \sin Q)$$

in which $\cos(\gamma - Q)$ is always taken to be positive.

For each value of ζ for which $(\gamma - Q)$ is not imaginary, there are two values, differing by 180°, of the angle Q, and thus two points on the curve. But, in most cases, only one of these points refers to an eclipsed point; the other solution may be eliminated by noting that Δ must not exceed L_1, or:

$$(x - \xi)^2 + (y - \eta)^2 \leqslant L_1^2$$

If, for the given time, there is a point on the curve of maximum eclipse at sunrise or sunset (see below), the starting value for ζ in the calculation for that time may be taken as zero; it is then increased until $(\gamma - Q)$ becomes imaginary. If there are points on both the northern and southern limits of penumbra for that time, the starting value for ζ is the smaller of the two corresponding values for these points; it is then increased until $(\gamma - Q)$ becomes imaginary.

The curves may be extended beyond the horizon by using negative values of ζ for a few points. The flattening of the Earth may be introduced, if necessary, by setting:

$$c^2 = \rho^2 - \zeta^2$$

where ρ is the geocentric radius at latitude ϕ. Then:

$$\eta_1 = \eta/\rho_1 \qquad \xi^2 + \eta_1^2 + \zeta_1^2 = 1$$

In this case, two approximations are required because ϕ is not known for the evaluation of ρ.

Once the geographic position has been calculated the semi-duration of the partial phase at each point is obtained from:

$$s = \frac{L_1 \cos \psi}{n}$$

where:

$$\sin \psi = \Delta/L_1$$
$$n^2 = (x' - \xi')^2 + (y' - \eta')^2$$
$$\xi' = \mu' \cos \phi \cos \theta$$
$$\eta' = \mu' \xi \sin d$$

The semi-duration is expressed in decimals of an hour because hourly variations are used, so that n is in units of Earth's equatorial radii per hour.

The magnitude of the eclipse is found from:

$$(L_1 - \Delta)/(L_1 + L_2)$$

where $(L_1 + L_2)$ may be replaced by $(2L_1 - 0{\cdot}5459)$* if L_2 is not available.

A direct computation of the curves of equal semi-duration is extremely laborious because both Δ and ζ are unknown, and successive approximations based on the above relations do not always converge. It is more expedient to perform

*0·5464 adopted from 1963 onwards.

Example 9.10. *Point on curve of maximum eclipse, and the semi-duration and magnitude at that point*

1961 February $15^{d}\ 08^{h}$

$\sin d$	$-0·2201$	μ'	$+0·2618$	$\tan f_1$	$0·0047$
$\cos d$	$+0·9755$	$\mu' \cos d$	$+0·2554$	$\tan f_2$	$0·0047$

$\zeta = 0·2$ (assumed)

b'	$-0·1198$	l_1	$+0·5386$	l_2	$-0·0073$	ζ^2	$+0·0400$
c_1'	$+0·5155$	$-\zeta \tan f_1$	$-0·0009$	$-\zeta \tan f_2$	$-0·0009$	c^2	$+0·9600$
$-\zeta\,\mu' \cos d$	$-0·0511$	L_1	$+0·5377$	L_2	$-0·0082$	c	$+0·9798$
$c_1' - \zeta\,\mu' \cos d$	$+0·4644$	L_1^2	$+0·2891$				
$\tan Q$	$-0·2580$			x	$-0·4030$	y	$+0·8084$

	(a)	(b)			(a)	(b)
Q	$165^\circ\ 32'$	$345^\circ\ 32'$	$x - \xi$		$-0·4608$	$+0·0196$
$\sin Q$	$+0·2498$	$-0·2498$	$y - \eta$		$+1·7865$	$-0·0756$
$\cos Q$	$-0·9683$	$+0·9683$	Δ^2		$3·4039$	$0·0061$
$x \cos Q - y \sin Q$	$+0·1883$	$-0·1883$	$\cos\phi \sin\theta = \xi$			$-0·4226$
$\sin(\gamma - Q)$	$+0·1922$	$-0·1922$	$\cos\phi \cos\theta$		Out	$+0·3897$
$\gamma - Q$	$11^\circ\ 05'$	$348^\circ\ 55'$	$\cot\theta$			$-0·9221$
γ	$176^\circ\ 37'$	$334^\circ\ 27'$	θ		because	$312^\circ·7$
$\sin\gamma$	$+0·0590$	$-0·4313$	μ			$296^\circ·4$
$\cos\gamma$	$-0·9983$	$+0·9022$	λ		$\Delta^2 > L_1^2$	$-16^\circ·3$
ξ	$+0·0578$	$-0·4226$	$\sin\phi$			$+0·8183$
η	$-0·9781$	$+0·8840$	ϕ			$+54^\circ·9$

For semi-duration and magnitude, case (b) only

Δ	$+0·0781$	$\sin\psi$	$+0·1452$	x'	$+0·5614$	y'	$+0·1430$
L_1	$+0·5377$	$\cos\psi$	$+0·9894$	$-\xi'$	$-0·1020$	$-\eta'$	$-0·0243$
L_2	$-0·0082$	$L_1 \cos\psi$	$+0·5320$	$x' - \xi'$	$+0·4594$	$y' - \eta'$	$+0·1187$
$L_1 - \Delta$	$+0·4596$			n^2	$0·2251$	n	$0·4744$
$L_1 + L_2$	$+0·5295$	Magnitude	$0·868$	Semi-duration			$1^{h}·121 = 67^{m}·3$

an inverse interpolation on the curves of maximum eclipse, provided the semi-durations at a sufficient number of points have been calculated. In the Ephemeris they are actually constructed graphically from a network of outline curves.

The curves of equal magnitude may also be obtained by inverse interpolation on the curves of maximum eclipse. They are not shown on the maps in the Ephemeris.

Curves of rising and setting and of maximum eclipse at sunrise and sunset

The rising and setting curve is the locus of the end points of the outline curves. If there are both northern and southern limits of penumbra, the rising and setting curve forms two separate loops. Otherwise, the curve assumes the shape of a distorted figure " eight ". It should be noted that the break into separate loops does not occur at the node of the figure " eight ", but at a short distance from it. At the node, the times of beginning and end of the eclipse do not coincide, whereas the duration of the eclipse is equal to zero at the point where the break occurs. Indications of the approaching break are often found in the last eclipse of a cycle to exhibit the continuous rising and setting curve. One of the loops appears somewhat strangled in the vicinity of the node. The next eclipse in the cycle will have two separate loops, and, in extreme cases, one of the loops may still contain the node. An illustration of these cases is found in the eclipses of 1894 September 28 and 1912 October 10, and again in those of 1941 September 21 and 1959 October 2. The computational procedure remains unchanged, regardless of the case involved.

If the flattening of the Earth is neglected, points on the rising and setting curve (neglecting refraction, that is assuming a zenith distance of 90°) may be calculated for each stated time by noting that ζ is equal to zero, and:

$$\xi^2 + \eta^2 = 1$$

$$\xi = x - l_1 \sin Q \qquad \eta = y - l_1 \cos Q$$

These may be written:

$$\sin \gamma = m \sin M - l_1 \sin Q$$
$$\cos \gamma = m \cos M - l_1 \cos Q$$

whence, eliminating Q:

$$\cos(\gamma - M) = \frac{m^2 + 1 - l_1^2}{2m}$$

There are two values of $(\gamma - M)$, giving two points on the curve for each given time. When the flattening of the Earth is included:

$$\xi^2 + \eta^2 = \rho^2$$

and the corresponding equations become:

$$\rho \sin \gamma = m \sin M - l_1 \sin Q$$
$$\rho \cos \gamma = m \cos M - l_1 \cos Q$$

leading to:

$$\cos(\gamma - M) = \frac{m^2 + \rho^2 - l_1^2}{2m\rho}$$

$$\eta_1 = \eta/\rho_1 \qquad \zeta_1 = 0$$

Since ϕ is not known for the initial evaluation of ρ, two approximations are necessary.

Example 9.11. ***Points on the rising and setting curves***

1961 February 15^d 08^h

x	−0·4030	l_1	+0·5386
y	+0·8084	l_1^2	+0·2901
m	+0·9033	$m^2 = x^2 + y^2$	+0·8159
$\tan M$	−0·4985		
$2m$	+1·8066	$m^2 + 1 - l_1^2$	1·5258
$\cos(\gamma - M)$	+0·8446		
	(a)		(b)
$\gamma - M$	32° 22′		327° 38′
M	333° 30′		333° 30′
γ	5° 52′		301° 08′
$\xi = \sin \gamma$	+0·1022		−0·8560
$\eta = \cos \gamma$	+0·9948		+0·5170

	(a)	(b)
$\sin d$	−0·2201	−0·2201
$\cos d$	+0·9755	+0·9755
$\cos \phi \sin \theta = \xi$	+0·1022	−0·8560
$\cos \phi \cos \theta$	+0·2190	+0·1138
$\tan \theta$ or $\cot \theta$	+0·4667	−0·1329
θ	25°·0	277°·6
μ	296°·4	296°·4
λ	− 88°·6	+ 18°·8
$\sin \phi$	+0·9704	+0·5043
ϕ	+ 76°·0	+ 30°·3

At the time given the eclipse begins at sunset in place (a) and ends at sunrise in place (b). The flattening of the Earth is neglected in the above calculation.

The curve of maximum eclipse at sunrise and sunset is the locus of the points on the curves of maximum eclipse for which ζ_1, or ζ in this case, is equal to zero. Thus, ignoring the flattening of the Earth:

$$c = 1 \qquad \tan Q = b'/c'_1$$

$$\sin(\gamma - Q) = x\cos Q - y\sin Q$$

with

$$\Delta \leqslant l_1$$

The flattening of the Earth may be introduced in the same manner as for the curves of maximum eclipse, i.e. by putting:

$$c = \rho \quad \text{and} \quad \rho\sin(\gamma - Q) = x\cos Q - y\sin Q$$

The curve of maximum eclipse in the horizon has two sections if the rising and setting curve has two separate loops; it is one continuous curve if the rising and setting curve has the shape of a distorted figure " eight ". In the latter case, the curve of maximum eclipse in the horizon passes between the node and nearer pole.

Both these curves are given on the maps in the Ephemeris.

Example 9.12. *Points on the curve of maximum eclipse at sunrise and sunset*

1961 February 15^{d} 08^{h}

b'	−0·1198	l_1	+0·5386	x	−0·4030	$\sin d$	−0·2201
c'_1	+0·5155	l_1^2	0·2901	y	+0·8084	$\cos d$	+0·9755

				(a)	(b)
$\tan Q$	−0·2324		$x - \xi$	−0·4202	+0·0225
	(a)	(b)	$y - \eta$	+1·8083	−0·0966
Q	166° 55′	346° 55′	Δ^2	3·4465	0·0098
$\sin Q$	+0·2264	−0·2264	$\cos\phi\sin\theta = \xi$	Out	−0·4255
$\cos Q$	−0·9740	+0·9740	$\cos\phi\cos\theta$		+0·1992
$\sin(\gamma - Q)$	+0·2095	−0·2095	$\cot\theta$	because	−0·4682
			θ		295°·1
$\gamma - Q$	12° 06′	347° 54′	μ	$\Delta^2 > l_1^2$	296°·4
γ	179° 01′	334° 49′	λ		+ 1°·3
$\xi = \sin\gamma$	+0·0172	−0·4255	$\sin\phi$		+0·8828
$\eta = \cos\gamma$	−0·9999	+0·9050	ϕ		+ 62°·0

First and last contacts of penumbra and umbra

The first and last contacts of the penumbra correspond to the extreme times for which there is a point on the rising and setting curve. Thus, at these points:

$$\cos(\gamma - M) = 1 \qquad \text{and} \qquad \gamma = M$$

The exact time of contact is not known and has to be obtained by successive approximation. At that time the cone of shadow is tangential to the Earth; thus:

$$x^2 + y^2 = (l_1 + \rho)^2$$

where ρ is the geocentric radius of the Earth and where the deviation of the radius from the normal at the point of tangency is neglected.

Let T_0 be an approximate time. At the time of contact $T_0 + t$:

$$(x_0 + x'\,t)^2 + (y_0 + y'\,t)^2 = (l_1 + \rho)^2$$

where the small increment in l_1 is neglected. This may be formally solved for t by using an auxiliary angle ψ_1 defined by:

$$\sin \psi_1 = \frac{x_0 y' - x' y_0}{n_1 (l_1 + \rho)} \quad \text{where } n_1^2 = x'^2 + y'^2$$

whence

$$t = \frac{l_1 + \rho}{n_1} \cos \psi_1 - \frac{x_0 x' + y_0 y'}{n_1^2}$$

where $\cos \psi_1$ is negative for first contact and positive for last contact.

In this case, although the latitude required to give ρ is not known it may be noted that:

$$\xi^2 + \eta^2 = \rho^2 \qquad \xi^2 + \eta_1^2 = 1$$

or

$$\xi = \rho \sin \gamma \qquad \xi = \sin \gamma'$$

$$\eta = \rho \cos \gamma \qquad \eta_1 = \cos \gamma'$$

But:

$$\tan \gamma = \tan M = x/y$$

and

$$\tan \gamma' = \rho_1 \tan \gamma = x/y_1 \text{ where } y_1 = y/\rho_1$$

Example 9.13. *Time and position of first contact of the penumbra*
1961 February 15

x, y, l_1 are interpolated from the Besselian elements in *A.E.*, p. 296; ρ_1 (unpublished) is taken from example 9.4.

(a) Time of contact
Initial time $T_0 = 6^h\ 10^m$

x_0	−1·4322 62	ρ_1	0·9967 96
y_0	+0·5462 65	y_{10}	+0·5480 21
m^2	2·3497 80	m_1^2	2·3517 01
ρ	0·9995 91	ρ^2	0·9991 83
x'	+0·5613 42	n_1^2	0·3355 18
y'	+0·1428 75	n_1	0·5792 39
l_1	0·5384 09		
$l_1 + \rho$	1·5380 00	$n_1 (l_1 + \rho)$	0·8908 70
$x_0 x' + y_0 y'$	−0·7259 41	$x_0 y' - x' y_0$	−0·5112 76
$-(x_0 x' + y_0 y')/n_1^2$	+2·1636 4	$\sin \psi_1$	−0·5739 06
$(l_1 + \rho) \cos \psi_1/n_1$	−2·1744 0	$\cos \psi_1$	−0·8189 21
t = sum	$-0^h{\cdot}0107\ 6$	$T = T_0 + t$	$6^h\ 09^m{\cdot}354$

(b) Point of contact
Time $T = 6^h\ 09^m{\cdot}354$

x	−1·4383 06	ρ_1	0·9967 96
y	+0·5447 27	y_1	+0·5464 78
ξ	−0·9348 01	m_1^2	2·3673 62
η_1	+0·3551 73	m_1	1·5386 23
$\cos \phi_1 \sin \theta = \xi$	−0·9348 01	$\sin d_1$	−0·2212 59
$\cos \phi_1 \cos \theta$	+0·0785 85	$\cos d_1$	+0·9752 16
$\cot \theta$	−0·0840 66		
θ	274° 48′·3		
μ	268° 46′·5		
λ	−6° 02′		
$\sin \phi_1$	+0·3463 70	$(1 - e^2)^{-\frac{1}{2}}$	1·0033 78
$\tan \phi_1$	+0·3692 26		
$\tan \phi$	+0·3704 73	ϕ	+20° 20′

Thus:

$$\rho = m/m_1$$

where

$$m^2 = x^2 + y^2 \qquad m_1^2 = x^2 + y_1^2$$

After a second approximation to the time, the geographic position of the point may be obtained from:

$$\xi = x/m_1 \qquad \eta_1 = y_1/m_1 = y/m_1\,\rho_1 \qquad \zeta_1 = 0$$

the flattening of the Earth being taken into account.

The first and last contacts of the umbra are the extreme points of the central line; thus at these points:

$$\xi = x \qquad \eta_1 = y_1 \qquad \zeta_1 = 0 \qquad x^2 + y_1^2 = 1$$

Hence, the correction t to an approximate time T_0 is:

$$t = \frac{1}{n_2}\cos\psi_2 - \frac{x_0\,x' + y_{10}\,y_1'}{n_2^2}$$

with

$$n_2^2 = x'^2 + y_1'^2$$

$$\sin\psi_2 = \frac{x_0\,y_1' - x'\,y_{10}}{n_2}$$

and where $\cos\psi_2$ is negative for first contact and positive for last contact. After a second approximation to the times, the geographic position of the point may be obtained from the final values of x and y_1.

These two points belong to the curve of maximum eclipse in the horizon.

Example 9.14. *Time and position of first contact of the umbra*

1961 February 15

(a) Time of contact

Initial time $T_0 = 7^h\ 30^m$

x_0	−0·6837 48	ρ_1	0·9967 96
y_0	+0·7368 47	y_{10}	+0·7392 15
x'	+0·5614 14	y'	+0·1429 92
		y_1'	+0·1434 52
n_2^2	0·3357 64	n_2	0·5794 51
$x_0\,x' + y_{10}\,y_1'$	−0·2778 24	$x_0\,y_1' - x'\,y_{10}$	−0·5130 91
$-(x_0\,x' + y_{10}\,y_1')/n_2^2$	+0·8274 4	$\sin\psi_2$	−0·8854 78
$\cos\psi_2/n_2$	−0·8019 3	$\cos\psi_2$	−0·4646 82
t = sum	$+0^h{\cdot}0255\ 1$	$T = T_0 + t$	$7^h\ 31^m{\cdot}531$

(b) Point of contact

Time $T = 7^h\ 31^m{\cdot}531$

$\xi = x$	−0·6694 23	ρ_1	0·9967 96
y	+0·7404 96	$\eta_1 = y_1$	+0·7428 76
$\cos\phi_1 \sin\theta = \xi$	−0·6694 23	$\sin d_1$	−0·2209 33
$\cos\phi_1 \cos\theta$	+0·1641 26	$\cos d_1$	+0·9752 90
$\cot\theta$	−0·2451 75		
θ	283° 46′·6		
μ	289° 19′·3		
λ	+5° 33′		
$\sin\phi_1$	+0·7245 20	$(1 - e^2)^{-\frac{1}{2}}$	1·0033 78
$\tan\phi_1$	+1·0511 65		
$\tan\phi$	+1·0547 16	ϕ	+46° 32′

Extreme points of limits of umbra and penumbra

For each of the extreme points on the limits of the umbra the zenith distance of the Sun is 90° and

$$\zeta_1 = 0$$

Thus:

$$\xi = x - l_2 \sin Q$$
$$\eta_1 = (y - l_2 \cos Q)/\rho_1$$
$$\xi^2 + \eta_1^2 = 1$$
$$\tan Q = (b' - a_2' \sec Q)/c_2'$$

In the evaluation of $\tan Q$, it is sufficient to use an approximate value Q_0 defined by:

$$\tan Q_0 = b'/c_2' \qquad \sec Q_0 = \pm e/c_2' \qquad e^2 = b'^2 + c_2'^2$$

in calculating the term $a_2' \sec Q$.

The sign of $\cos Q$ is positive for the northern limit of a total eclipse and the southern limit of an annular eclipse. It is negative for the southern limit of a total eclipse and the northern limit of an annular eclipse.

If T_0 is an approximate time, and the time of contact is $T_0 + t$, then:

$$(\xi_0 + \xi' t)^2 + (\eta_{10} + \eta_1' t)^2 = 1$$

where, with sufficient accuracy:

$$\xi' = x' \mp l_2 \frac{b''}{e}$$
$$\eta_1' = \frac{1}{\rho_1}\left(y' \mp l_2 \frac{c_2''}{e}\right)$$

The upper signs are used if $\cos Q$ is positive, the lower signs if $\cos Q$ is negative. The quantities b'', c_2'', are the variations of b', c_2', and may be obtained from the subtabulation of these functions. Then:

$$t = \frac{1}{n_3} \cos \psi_3 - \frac{\xi_0 \xi' + \eta_{10} \eta_1'}{n_3^2}$$

with

$$n_3^2 = \xi'^2 + \eta_1'^2 \qquad \sin \psi_3 = \frac{\xi_0 \eta_1' - \xi' \eta_{10}}{n_3}$$

where $\cos \psi_3$ is negative for the beginning and positive for the end.

After a second approximation to the times, the geographic positions of the points may be obtained from the final values of ξ and η_1. These points belong to the curve of maximum eclipse in the horizon.

The procedure for finding the extreme points on the limits of the penumbra is similar to that described for the extreme points of limits of umbra in the case of an annular eclipse. The quantities l_1 and c_1 must be substituted for l_2 and c_2.

These points are also the extremities of the curve of maximum eclipse at sunrise and sunset. They need not be known accurately, and, therefore, the flattening of the Earth may be neglected in their derivation.

Example 9.15. *Time and position of extreme point on the southern limit of the umbra*

1961 February 15 (beginning)

(a) Time of contact

Initial time $T_0 = 7^h\ 30^m$

b'	−0·1035 64	b''	−0·0324 96
c_2'	+0·5189 16	c_2''	−0·0081 84
e	0·5291 49	e^2	0·2799 99
$\sec Q_0$	−1·0197	b'	−0·1035 64
a_2'	+0·0007 53	$-a_2' \sec Q_0$	+0·0007 68
c_2'	+0·5189 16	$b' - a_2' \sec Q_0$	−0·1027 96
$\tan Q$	−0·1980 98	l_2	−0·0073 55
$\sin Q$	+0·1943 22	$\cos Q$	−0·9809 37
x_0	−0·6837 48	y_0	+0·7368 47
$-l_2 \sin Q$	+0·0014 29	$-l_2 \cos Q$	−0·0072 15
ξ_0	−0·6823 19	η_0	+0·7296 32
ρ_1	0·9967 96	η_{10}	+0·7319 77
x'	+0·5614 14	y'	+0·1429 92
$+l_2\, b''/e$	+0·0004 52	$+l_2\, c_2''/e$	+0·0001 13
ξ'	+0·5618 66	η'	+0·1431 05
		η_1'	+0·1435 65
n_3^2	0·3363 04	n_3	0·5799 17
$\xi_0\, \xi' + \eta_{10}\, \eta_1'$	−0·2782 86	$\xi_0\, \eta_1' - \xi'\, \eta_{10}$	−0·5092 30
$-(\xi_0\, \xi' + \eta_{10}\, \eta_1')/n_3^2$	+0·8274 8	$\sin \psi_3$	−0·8781 08
$\cos \psi_3 / n_3$	−0·8250 5	$\cos \psi_3$	−0·4784 62
t = sum	+0·0024 3 h	$T = T_0 + t$	7h 30m·146

(b) Position of extreme point

Time $T = 7^h\ 30^m{\cdot}146$

$\sec Q_0$ (from Q in (a))	−1·0194	b'	−0·1036 43
a_2'	+0·0007 51	$-a_2' \sec Q_0$	+0·0007 66
c_2'	+0·5188 96	$b' - a_2' \sec Q_0$	−0·1028 77
$\tan Q$	−0·1982 61	l_2	−0·0073 55
$\sin Q$	+0·1944 76	$\cos Q$	−0·9809 07
x	−0·6823 82	y	+0·7371 95
$-l_2 \sin Q$	+0·0014 30	$-l_2 \cos Q$	−0·0072 15
ξ	−0·6809 52	η	+0·7299 80
ρ_1	0·9967 96	η_1	+0·7323 26
$\cos \phi_1 \sin \theta = \xi$	−0·6809 52	$\sin d_1$	−0·2209 38
$\cos \phi_1 \cos \theta$	+0·1617 99	$\cos d_1$	+0·9752 89
$\cot \theta$	−0·2376 07		
θ	283° 21′·96		
μ	288° 58′·51		
λ	+5° 37′		
$\sin \phi_1$	+0·7142 29	$(1 - e^2)^{-\frac{1}{2}}$	1·0033 78
$\tan \phi_1$	+1·0204 55		
$\tan \phi$	+1·0239 02	ϕ	+45° 41′

Central eclipse at local apparent noon (or midnight)

The point of central eclipse at local apparent noon (or midnight) is the point on the central line at the time of conjunction of the Sun and Moon in right ascension. It is calculated as any other point on that curve, but care must be taken in the calculation of its longitude whenever the central line passes through either polar region. At the time of conjunction, x is equal to zero; therefore, the local hour angle θ is equal to 0° or 180° according as the term $\cos \phi_1 \cos \theta$ is positive or negative. In the polar regions, the eclipse may occur at lower transit of the Sun (local apparent midnight), giving:

$$\theta = 180°$$

If it is suspected that this might be the case, it is advisable to ascertain the sign of $\cos \phi_1 \cos \theta$.

Example 9.16. *Position at which central eclipse occurs at local apparent noon*

Time of conjunction of Sun and Moon in right ascension 1961 February $15^d\ 08^h\ 43^m{\cdot}075$

$x = \xi = 0$		y	$+0{\cdot}9110\ 57$	$\sin d_1$	$-0{\cdot}2206\ 49$
ρ_1	$0{\cdot}9967\ 96$	$y_1 = \eta_1$	$+0{\cdot}9139\ 85$	$\cos d_1$	$+0{\cdot}9753\ 53$
ζ_1^2	$0{\cdot}1646\ 31$	ζ_1	$0{\cdot}4057\ 47$		

$\cos \phi_1 \cos \theta >$	0	$(1 - e^2)^{-\frac{1}{2}}$	$1{\cdot}0033\ 78$
$\tan \theta$	0	$\sin \phi_1$	$+0{\cdot}8019\ 30$
θ	0° 00′	$\tan \phi_1$	$+1{\cdot}3423\ 3$
μ	307° 13′	$\tan \phi$	$+1{\cdot}3468\ 6$
λ	− 52° 47′	ϕ	+ 53° 24′

Greatest eclipse

In partial eclipses, the maximum magnitude is attained at the point on the surface of the Earth which comes closest to the axis of shadow. The magnitude varies very slowly in the vicinity of that point, and therefore its position need not be known very accurately. It is sufficient to determine the point for which the quantity m (or m^2) is a minimum since, at that point, the distance Δ of the observer from the axis of shadow is given by:

$$\Delta = m - \rho$$

The eclipse will occur in the horizon, so that:

$$\zeta = 0$$

The time of greatest eclipse must be determined by successive approximations. At an approximate time T_0:

$$\frac{d}{dt}(m^2) = 2\,(x_0\, x' + y_0\, y') \qquad \frac{d}{dt}(x_0\, x' + y_0\, y') = x'^2 + y'^2 = n_1^2$$

the terms in x'' and y'' being neglected. At the time $T_0 + t$ of greatest eclipse:

$$xx' + yy' = 0$$

since m^2 is a minimum. Thus:

$$t = -\frac{x_0\, x' + y_0\, y'}{n_1^2}$$

With:

$$\xi = \rho \sin \gamma \qquad \eta = \rho \cos \gamma$$

and

$$(x - \xi)^2 + (y - \eta)^2 = \Delta^2$$

it is found that:

$$(m \sin M - \rho \sin \gamma)^2 + (m \cos M - \rho \cos \gamma)^2 = (m - \rho)^2$$

or

$$\cos (M - \gamma) = 1$$

Thus:

$$M = \gamma$$

and, as in the case of first and last contacts of penumbra:

$$\rho = m/m_1$$

Therefore, the position of the point may be found from:

$$\xi = x/m_1 \qquad \eta_1 = y_1/m_1 \qquad \zeta_1 = 0$$

the flattening of the Earth being taken into account.

The magnitude of the greatest eclipse is given by:

$$\frac{l_1 - \Delta}{l_1 + l_2}$$

If l_2 is not available, it is sufficient to set:

$$l_1 + l_2 = 2\, l_1 - 0{\cdot}5459^*$$

where the numerical constant is a mean value of $l_1 - l_2$.

Example 9.17. *Time and place of greatest eclipse and the magnitude of eclipse*

1960 September 20–21

(a) Time of greatest eclipse

Initial time T_0 = September 20^d 23^h 00^m

x_0	+0·3717 58	x'	+0·5066 25
y_0	+1·1468 01	y'	−0·1637 67
$x_0 x' + y_0 y'$	+0·0005 34	n_1^2	0·2834 89
t	$-0^h{\cdot}0018\ 8$	$T_0 + t$	$20^d\ 22^h\ 59^m{\cdot}887$

(b) Position of place of greatest eclipse and magnitude

September $20^d\ 22^h\ 59^m{\cdot}887$

x	+0·3708 04	ρ_1	0·9966 34	$\cos\phi_1 \sin\theta$	+0·3066 43
y	+1·1471 09	y_1	+1·1509 83	$\cos\phi_1 \cos\theta$	−0·0134 72
m^2	1·4533 55	m_1^2	1·4622 57	$\cot\theta$	−0·0439 34
m	1·2055 52	m_1	1·2092 38	θ	92° 30′·9
ρ	0·9969 52	l_1	0·5557	μ	166° 40′·3
$\Delta = m - \rho$	0·2086	$l_1\ \ \Delta$	0·3471	λ	+74° 09′
		$2\, l_1 - 0{\cdot}5459$	0·5655		
Mag.	0·614			$\sin\phi_1$	+0·9517 30
				$\tan\phi_1$	+3·1007 4
ξ	+0·3066 43	$\sin d_1$	+0·0141 54	$\tan\phi$	+3·1112 1
η_1	+0·9518 25	$\cos d_1$	+0·9999 00	ϕ	+72° 11′

The data are taken from *A.E.*, p. 304, or from the unpublished elements.

*0·5464 adopted from 1963 onwards.

Note on practical computation

Many of the foregoing curves and points involve the solution of quadratic and transcendental equations; formulae have been given by which solutions can be obtained by direct calculation or by iteration, generally using auxiliary angles. It is often easier, and generally less liable to error, to tabulate a " discriminant ", which vanishes at the point required, for a few values of the independent variable; the required value of the independent variable is then obtained by the standard technique of inverse interpolation. For example, the times of first and last contacts of the penumbra require the solution of the equation:

$$D \equiv x^2 + y^2 - (l_1 + \rho)^2 = 0$$

All that is necessary is to tabulate three, or preferably four, values of D at equal intervals of time, and then use inverse interpolation to give the required times; second differences are almost constant. No theoretical approximations are necessary and the variation of l_1 can easily be taken into account. Such solutions are easier to perform than to describe, and no further details are given.

Example 9.18. *Time of first contact of the penumbra (see example* 9.13)

The time of contact occurs when

$$D \equiv x^2 + y^2 - (l_1 + \rho)^2 = 0$$

D may be calculated directly but, if terms in $(1 - \rho)^2$ are neglected, it may be replaced by the simpler:

$$D' \equiv x^2 + y^2 - (l_1 + 1)^2 + Ay^2 = 0$$

where $A = (\rho_1^{-2} - 1)/(l_1 + 1)$ should strictly be evaluated for the time of contact, but is essentially a constant. To the same precision, at the time of contact:

$$m_1 = 1 + l_1 + \{ l_1/2 \, (l_1 + 1) \} Ay^2$$

for use in the calculation of the position of the point of contact.

Six decimals are retained in the example, though this precision is not necessary. x, y, l_1 are taken from *A.E.*, p. 296.

1961 February 15

ρ_1 0·9967 96 $\rho_1^{-2} - 1$ 0·0064 39 A 0·0041 86 $\dfrac{A\, l_1}{2\,(1 + l_1)}$ 0·0007 32

E.T.	x	y	$l_1 + 1$	D'	differences	
$6^h\ 00^m$	−1·5258 18	+0·5224 54	1·5383 91	+0·2355 75		
					−2512 48	
10	1·4322 62	0·5462 65	1·5384 09	−0·0156 73		+186 45
					−2326 03	
20	1·3387 04	0·5700 79	1·5384 26	−0·2482 76		+186 48
					−2139 55	
30	1·2451 44	0·5938 96	1·5384 42	−0·4622 31		

Inverse interpolation to $D' = 0$ gives the time of first contact as:

$$6^h\ 09^m{\cdot}3538 \quad (B_2 = -0{\cdot}01511)$$

Then $m_1 = 1{\cdot}5384\ 08 + 0{\cdot}0002\ 17 = 1{\cdot}5386\ 25$ and the calculation proceeds as in example 9.13.

D. SOLAR ECLIPSES—LOCAL CIRCUMSTANCES

Introduction

In contrast with previous sub-sections the numerical examples of the calculation of the local circumstances of eclipses are collected together at the end of this sub-section. Examples are given for a partial solar eclipse (example 9.19), a total solar eclipse (example 9.20), and an annular solar eclipse (example 9.22). Not all stages are illustrated for all eclipses, but at least one illustration is given for every stage. In addition, example 9.21 illustrates the application of the alternative direct method for a total solar eclipse.

Use of eclipse maps

Approximate local circumstances for the partial phase of a solar eclipse may be obtained without calculation from the corresponding eclipse map given in the Ephemeris. On these maps the curves drawn in long dashes indicate the times midway between first and last contacts of the penumbra; these times of the middle of the eclipse should not be confused with the times of maximum eclipse, from which they may differ by several minutes. The curves drawn in short dashes give the semi-duration of the partial phase. The (ephemeris) times of first and last contacts are derived from the time of the middle of the eclipse by respectively subtracting and adding the semi-duration. The curves are extended across the rising and setting limits of the eclipse, although part of the phenomenon occurs below the horizon for observers in those regions.

Coordinates of the observer

Let ϕ be the geodetic latitude, λ the longitude of the site, and H its elevation above the spheroid. The corresponding geocentric coordinates $\rho \sin \phi'$, $\rho \cos \phi'$ are given by:

$$\rho \sin \phi' = (S + H) \sin \phi \qquad \rho \cos \phi' = (C + H) \cos \phi$$

in which H is in units of the equatorial radius of the Earth, obtained by multiplying * the height in metres by

$$0{\cdot}1567\,794 \times 10^{-6} \text{ (or the height in feet by } 0{\cdot}0477\,865 \times 10^{-6}\text{), and}$$

$$C = (1 - e^2 \sin^2 \phi)^{-\frac{1}{2}} \qquad S = (1 - e^2)\, C$$

where e is the ellipticity of the Earth's spheroid ($e^2 = 0{\cdot}0067\,2267$).*

The values of C and S may be obtained from table 2.8.*

The longitude λ must be converted to the ephemeris longitude λ^* by increasing it by $1{\cdot}0027\,38\,\Delta T$, the sidereal equivalent of ΔT.

Then the coordinates ξ, η, ζ of the observer and their hourly variations ξ', η', ζ' are found from:

$$\begin{aligned} \xi &= \rho \cos \phi' \sin \theta & \xi' &= \mu' \rho \cos \phi' \cos \theta \\ \eta &= \rho \sin \phi' \cos d - \rho \cos \phi' \sin d \cos \theta & \eta' &= \mu' \xi \sin d - \zeta d' \\ \zeta &= \rho \sin \phi' \sin d + \rho \cos \phi' \cos d \cos \theta & \zeta' &= -\mu' \xi \cos d + \eta d' \end{aligned}$$

*For 1968 onwards: H = height in metres $\times$ $0{\cdot}1567\,850 \times 10^{-6}$
= height in feet $\times$ $0{\cdot}0477\,881 \times 10^{-6}$
$e^2 = 0{\cdot}0066\,9454$. S and C may be obtained from *A.E.* Table VII.
See note on page 523.

where:

$$\theta = \mu - \lambda - 1{\cdot}0027\,38\,\Delta T$$

In most cases ζ' is not needed. If predictions to the nearest second are acceptable, the terms $\zeta d'$ and $\eta d'$ may be omitted. The Besselian elements, wherever they are used, must always be interpolated to the time assumed in the calculation.

Time of greatest phase

The approximate time of the middle of the eclipse obtained from the eclipse map differs only slightly from the time of maximum eclipse and may be used as a first approximation to the time of greatest phase. For economy of notation, the following symbols are introduced:

$$u = x - \xi \qquad u' = x' - \xi'$$
$$v = y - \eta \qquad v' = y' - \eta'$$
$$m^2 = u^2 + v^2 \qquad n^2 = u'^2 + v'^2$$

where m and n are positive quantities.

Let T_0 be an assumed time near the time T of greatest phase, and let

$$T = T_0 + t$$

The greatest phase occurs when $(L_1 - m)/(L_1 + L_2)$ is a maximum; since the variation of L with time is extremely small it suffices to determine when m, or m^2, is a minimum, that is, when:

$$uu' + vv' = 0$$

If u_0, v_0 are the values of u, v at the time T_0, the condition of greatest phase may be expressed by:

$$(u_0 + tu')\,u' + (v_0 + tv')\,v' = 0$$

or

$$t = -\,D/n^2$$

with

$$D = u_0\,u' + v_0\,v'$$

The value of t thus obtained is expressed in decimals of an hour because hourly variations have been used. Here and elsewhere, the variations of u' and v' are disregarded, because they are insignificant during the time interval t.

Beginning and end of penumbral phase

The approximate times T_0 of beginning and end are obtained from the eclipse maps. A separate calculation must be performed for each of the two phenomena.

At the time $T = T_0 + t$ of beginning or end:

$$u^2 + v^2 = L_1^2$$

or

$$(u_0 + tu')^2 + (v_0 + tv')^2 = L_1^2$$

omitting the slight variation in L_1.

Thus the value of t is obtained by solving the equation:

$$n^2\,t^2 + 2Dt + (m_0^2 - L_1^2) = 0$$

By noting that:

$$D^2 - n^2 (m_0^2 - L_1^2) = n^2 L_1^2 - (u_0 v' - u' v_0)^2$$

and setting:

$$\Delta = (u_0 v' - u' v_0)/n$$
$$\sin \psi = \Delta / L_1$$

it is found that:

$$t = \frac{L_1 \cos \psi}{n} - \frac{D}{n^2}$$

It may be noted that the term $-D/n^2$ is the correction that would be applied to T_0 to give the time of greatest phase, if T_0 were sufficiently close to this phase; the term $L_1 \cos \psi/n$ thus represents approximately the semi-duration of the partial phase which must be subtracted to obtain the time of beginning, or added to obtain the time of end. In other words, $\cos \psi$ must be taken as negative for the beginning and positive for the end, since L_1 is always positive.

In general, the times of penumbral contacts need not be known with great accuracy; thus the term $n^2 t^2$ in the equation for t may, if desired, be omitted, thereby simplifying the latter to:

$$t = (L_1^2 - m_0^2)/2D$$

Beginning and end of umbral phase

Because the central phase lasts only a few minutes, the times of umbral contacts may be calculated simultaneously with that of the greatest phase, using as T_0 the approximate time of the middle of the eclipse obtained from the map.

At the time $T = T_0 + t$ of umbral contacts:

$$u^2 + v^2 = L_2^2$$

thus, the correction t to the approximate time T_0 to obtain T is given by:

$$t = \frac{L_2 \cos \psi}{n} - \frac{D}{n^2}$$

This formula is identical to that used for the penumbral contacts, except that L_2 is used instead of L_1. Because L_2 is negative for total eclipses and positive for annular eclipses, $\cos \psi$ must be taken as positive for the beginning of the total phase and the end of the annular phase, and as negative for the end of the total phase and the beginning of the annular phase.

The semi-duration of the umbral phase is given by $\pm (L_2 \cos \psi)/n$.

For greater accuracy, the times resulting from the calculation outlined above should be taken in place of the original approximate times, and a second approximation performed.

The adopted value of ΔT must be subtracted from the final times to obtain the universal times of contacts and greatest phase.

Position angles

At the times of contact:

$$u = L \sin Q \qquad v = L \cos Q \qquad \tan Q = u/v$$

where the appropriate value of L is used. The angle Q is the position angle of the point of contact, measured eastwards from the north point of the solar limb. The quadrant of Q is determined by noting that $\sin Q$ has the sign of u, except for the contacts of the total phase for which $\sin Q$ has the opposite sign to u since L_2 is negative for total eclipses.

If only a single approximation to the times has been made it is more accurate to calculate Q by another method derived as follows.

At the time of contact, t is zero, giving:

$$\cos \psi = D/nL$$

and

$$\tan \psi = \frac{n\Delta}{D} = \frac{uv' - u'v}{D} = \frac{uv' - u'v}{uu' + vv'}$$

Let the angle N be defined by:

$$u' = n \sin N \qquad v' = n \cos N \qquad \tan N = u'/v'$$

Then:

$$\tan (N + \psi) = \frac{u' (uu' + vv') + v' (uv' - u'v)}{v' (uu' + vv') - u' (uv' - u'v)} = \frac{u}{v}$$

so that $Q = N + \psi$, where the value of ψ can be taken as that at the initial time.

The position angle V from the vertex is obtained by subtracting from Q the parallactic angle C, obtained with sufficient accuracy from:

$$\tan C = \xi/\eta$$

$\sin C$ having the same algebraic sign as ξ.

Alternative method

Times and position angles of contacts may be obtained by direct numerical solution of the equation:

$$u^2 + v^2 - L^2 = 0$$

For four, or more, times at equal intervals surrounding the phase required, a small table is made of the quantities:

$$u, v, L, u^2 + v^2 - L^2$$

The time T of contact is then found, by the standard techniques of inverse interpolation, to correspond to the zero of $u^2 + v^2 - L^2$. The position angle Q is calculated from $\sin Q = u/L$ or $\cos Q = v/L$ according to whether u or v is the smaller; u, or v, and L are interpolated to time T.

Similarly the time of greatest phase may be found numerically as the instant when the derivative of $m^2 = (u^2 + v^2)$ is zero; and, if the highest precision is required, the time of maximum of $(L_1 - m)/(L_1 + L_2)$ may be determined by similar methods.

The advantages of this method are that: there are no auxiliary formulae and angles; all numerical work is capable of simple checking; no theoretical approximations are necessary (there is no need to assume that u', v' and L are constant); and there is a direct relationship between the precision of the results and that of the data. There may be more calculation, but it is of a routine nature; quite large intervals can efficiently be used.

Magnitude

The magnitude of the eclipse is by definition the fraction of the solar diameter covered by the Moon at the time of greatest phase, expressed in units of the solar diameter. In figure 9.4 SS′ and MM′ represent the disks of the Sun and Moon, and PP′ represents the plane passing through an observer B and parallel to the fundamental plane. The point O is the intersection of the axis of shadow with the plane PP′. The exterior tangents SM, S′M′ to the Sun and Moon outline the cone of umbra and intersect PP′ respectively in A′ and A, while the interior tangents S′M, SM′ outline the cone of penumbra and intersect PP′ in P and P′. Therefore:

$$AO = L_2 \qquad PO = L_1$$

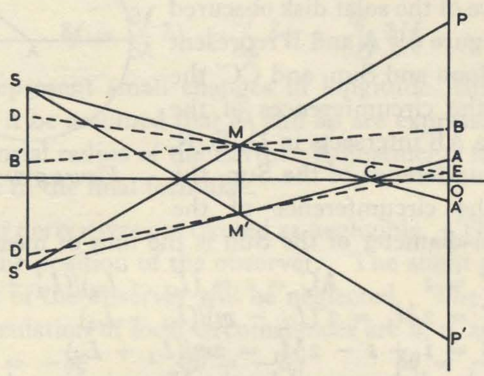

Figure 9.4. Magnitude of an eclipse

An observer B located within the penumbra sees the portion S′B′ of the solar diameter obscured by the Moon. Thus the magnitude M_1 of the partial eclipse will be given by:

$$M_1 = S'B'/SS'$$

From the figure it may be noted that:

$$S'B'/SS' = PB/PA'$$
$$PB = PO - OB = L_1 - m$$
$$PA' = PO + OA' = L_1 + L_2$$

Thus:

$$M_1 = (L_1 - m)/(L_1 + L_2)$$

An observer E located within the umbra sees the entire disk of the Moon projected in DD′ on the disk of the Sun. Therefore the magnitude M_2 of the central phase is given by:

$$M_2 = DD'/SS'$$

It is seen from the figure that:

$$DD' = S'D - S'D'$$
$$S'D/SS' = PE/PA'$$
$$S'D'/SS' = AE/PA'$$

Thus:

$$DD'/SS' = (PE - AE)/PA' = PA/PA'$$

and $$PA = PO - OA = L_1 - L_2$$
from which
$$M_2 = (L_1 - L_2)/(L_1 + L_2)$$

Although the figure illustrates the case of an annular eclipse, it may be verified that identical results are obtained for a total eclipse, provided it is noted that in the latter case:
$$OA = -L_2$$

Degree of obscuration

In the reduction of certain types of eclipse observations, it is necessary to evaluate the fraction of the surface of the solar disk obscured by the Moon. In figure 9.5 A and B represent the centres of the Moon and Sun, and CC′ the chord common to the circumferences of the two disks. The line AB intersects in D, E, F, respectively the circumference of the Sun, the chord CC′, and the circumference of the Moon. If the semi-diameter of the Sun is the unit of measurement:

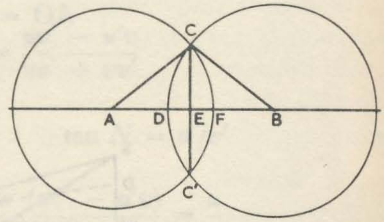

Figure 9.5. Degree of obscuration

$$BC = 1 \qquad AC = s = (L_1 - L_2)/(L_1 + L_2)$$
$$DF = 2M_1 = 2\,(L_1 - m)/(L_1 + L_2)$$
$$AB = 1 + s - 2M_1 = 2m/(L_1 + L_2)$$

The area S of the solar disk covered by the Moon is given by:
$$S = \text{segment CFC}' + \text{segment CDC}'$$
with
$$\text{segment CFC}' = 2\,(\text{sector ACF} - \text{triangle ACE})$$
$$\text{segment CDC}' = 2\,(\text{sector BCD} - \text{triangle BCE})$$
Thus:
$$S = (s^2 A + B) - (s^2 \sin A \cos A + \sin B \cos B)$$
in which the angles A and B are expressed in radians, or
$$S = s^2 A + B - s \sin C$$
because
$$CE = s \sin A = \sin B$$

Since the radius of the Sun has been taken as unity, the surface of the solar disk is equal to π, and the ratio S' of the obscured portion of the disk to the whole disk is equal to S/π.

The angles A, B, C in the triangle ABC may be evaluated from the conventional relations:
$$s^2 = 1 + (1 + s - 2M_1)^2 - 2\,(1 + s - 2M_1) \cos B$$
$$(1 + s - 2M_1)^2 = 1 + s^2 - 2s \cos C$$
Thus:
$$\cos C = (L_1^2 + L_2^2 - 2m^2)/(L_1^2 - L_2^2) \qquad 0 \leqslant C \leqslant \pi$$
$$\cos B = (L_1 L_2 + m^2)/m(L_1 + L_2) \qquad 0 \leqslant B \leqslant \pi$$
$$A = \pi - (B + C)$$

and

$$S' = (s^2 A + B - s \sin C)/\pi$$

During the annular phase, S' is equal to s^2, while it is equal to unity in the case of the total phase.

Differential corrections

The times of the phases at any point within a few miles of a point for which a precise calculation has been made may be obtained by computing differential corrections in which most of the necessary numerical quantities are already available from the original calculations.

Let the symbol δ be such that:

$$\delta f = \frac{\partial f}{\partial \lambda} \delta\lambda + \frac{\partial f}{\partial \phi} \delta\phi + \frac{\partial f}{\partial H} \delta H$$

where $\delta\lambda$, $\delta\phi$, δH represent small changes in longitude, latitude, and elevation, respectively, and let it be assumed that $\delta\lambda$ and $\delta\phi$ are expressed in radians and δH in units of the equatorial radius of the Earth. Adjustments for their expression in other units are made in the final formulae.

All second-order derivatives are treated as negligible. The Besselian elements are independent of the position of the observer. The slight change in L resulting from a displacement of the observer will be neglected. The remaining quantities entering into the calculation of local circumstances are u, v, and θ, for which:

$$\delta u = -\delta\xi \qquad \delta v = -\delta\eta \qquad \delta\theta = -\delta\lambda$$

It may be verified (from the definitions of C and S given in sub-section B) that:

$$\delta C = C^3 e^2 \sin\phi \cos\phi \, \delta\phi$$
$$\delta S = SC^2 e^2 \sin\phi \cos\phi \, \delta\phi$$

from which

$$\delta\xi = A_1 \, \delta\lambda + A_2 \, \delta\phi + A_3 \, \delta H$$

with

$$A_1 = -\rho \cos\phi' \cos\theta$$
$$A_2 = -(SC^2 + H) \sin\phi \sin\theta$$
$$A_3 = \cos\phi \sin\theta$$

and

$$\delta\eta = B_1 \, \delta\lambda + B_2 \, \delta\phi + B_3 \, \delta H$$

with

$$B_1 = -\xi \sin d$$
$$B_2 = (SC^2 + H)(\cos\phi \cos d + \sin\phi \sin d \cos\theta)$$
$$B_3 = \sin\phi \cos d - \cos\phi \sin d \cos\theta$$

In most cases it will be accurate enough to set:

$$A_2 = -C^2 \rho \sin\phi' \sin\theta$$
$$A_3 = \xi$$
$$B_2 = CS \rho \cos\phi' \cos d + C^2 \rho \sin\phi' \sin d \cos\theta$$
$$B_3 = \eta$$

The correction to an assumed time of exterior contact may be written with sufficient accuracy for the present purpose in the form:

$$t = (L_1^2 - m^2)/2D$$

Then:

$$\delta t = (u\ \delta\xi + v\ \delta\eta)/D$$

in which the small contribution of δD has been neglected.

Similarly the correction to an assumed time of greatest phase at λ, ϕ, H is given by:

$$t_m = -D/n^2$$

thus

$$\delta t_m = (u'\ \delta\xi + v'\ \delta\eta)/n^2$$

in which the small contribution of δn^2 has been neglected.

In order to calculate the value of the semi-duration (σ) of the umbral phase at a point $\lambda + \delta\lambda$, $\phi + \delta\phi$, $H + \delta H$ at which the eclipse is central, a correction $\delta\Delta$ must be applied to the value of Δ in the final approximation to the time of greatest phase at λ, ϕ, H. Since:

$$\Delta = (uv' - u'v)/n$$

its differential correction has the form:

$$\delta\Delta = (u'\ \delta\eta - v'\ \delta\xi)/n$$

in which the contribution of δn has been neglected. The new value of σ may then be calculated from:

$$\sigma = \pm L_2 \cos\psi/n$$

with

$$\sin\psi = (\Delta + \delta\Delta)/L_2$$

in which the slight change in L_2 has been neglected.

Although a direct correction to sin ψ could be evaluated in a similar manner, the procedure outlined above is preferable, because it is valid even in cases where the eclipse is not central at λ, ϕ, H.

In practice, $\delta\lambda$ and $\delta\phi$ would be expressed in minutes of arc, and δH in metres (or feet). By taking into account the expressions for $\delta\xi$ and $\delta\eta$, the corrections δt, δt_m, and $\delta\Delta$ may be written:

$$\begin{aligned}
\delta t &= p\ \delta\lambda + q\ \delta\phi + r\ \delta H \\
\delta t_m &= p_m\ \delta\lambda + q_m\ \delta\phi + r_m\ \delta H \\
\delta\Delta &= p_s\ \delta\lambda + q_s\ \delta\phi + r_s\ \delta H
\end{aligned}$$

where $\delta\lambda$, $\delta\phi$ are in minutes of arc and δH is in metres, and:

$$\begin{aligned}
p &= \sin 1'\ (uA_1 + vB_1)/D \\
q &= \sin 1'\ (uA_2 + vB_2)/D \\
r &= (uA_3 + vB_3)/Da \\
p_m &= \sin 1'\ (u'A_1 + v'B_1)/n^2 \\
q_m &= \sin 1'\ (u'A_2 + v'B_2)/n^2 \\
r_m &= (u'A_3 + v'B_3)/n^2\, a \\
p_s &= \sin 1'\ (u'B_1 - v'A_1)/n \\
q_s &= \sin 1'\ (u'B_2 - v'A_2)/n \\
r_s &= (u'B_3 - v'A_3)/na
\end{aligned}$$

in which a is the equatorial radius of the Earth in metres.

The quantities p, q, r, p_m, q_m, r_m and the resulting value of σ are naturally expressed in decimals of an hour; they may be expressed in seconds of time by multiplying by 3600.

As before, the value of ΔT must be subtracted from the final times so obtained in order to obtain the universal times.

The position angle Q for the penumbral contacts need not be known with great accuracy, and it will not be necessary to recompute it. Its value for the umbral contacts may be corrected by substituting the new value of ψ in the original computation. The parallactic angle C at the time of umbral contacts at $\lambda + \delta\lambda$, $\phi + \delta\phi$, $H + \delta H$, may be obtained with sufficient accuracy from:

$$\tan C = (\xi \mp \xi'\sigma) / (\eta \mp \eta'\sigma)$$

in which upper signs are for second contact, lower signs for third contact; ξ, ξ', η, η' are for the time of greatest phase at λ, ϕ, H. In this equation σ is to be expressed in hours; and sin C has the same algebraic sign as $(\xi \mp \xi'\sigma)$.

Final corrections for ΔT

In general, the finally adopted value of ΔT, the difference between E.T. and U.T., will differ somewhat from the value used in the predictions. It will not be necessary, however, to repeat the whole calculation for λ, ϕ, H. If the final value is equal to $\Delta T + \delta T$, it is sufficient to set:

$$\delta\lambda = 1{\cdot}0027\,38\,\delta T \qquad \delta\phi = 0 \qquad \delta H = 0$$

and to compute δt, δt_m, and σ as outlined above. It may be noted that in this case, only the quantities A_1, B_1, p, p_m, and p_s will be required. The value of $\Delta T + \delta T$ must be subtracted from the ephemeris times in order to obtain the universal times; the corrections to the previous times, expressed in U.T., will thus be $\delta t - \delta T$ and $\delta t_m - \delta T$.

Practical calculation

In view of the importance of accuracy in eclipse calculations for the precise site of the observer, it is desirable that direct calculations of the circumstances be made for at least one other site and that these calculations be used to check the coefficients of the differential corrections. These coefficients, which are particularly liable to accidental errors of calculation, may thereafter be used with confidence for the final corrections to be made in the field.

Local circumstances in the ionosphere

The preceding treatment of the calculation of local circumstances takes full account of the elevation of the observer above the spheroid. This procedure may be followed without modification for the prediction of local circumstances in the ionosphere. For convenience, the values of H required in the calculation of $\rho \sin \phi'$ and $\rho \cos \phi'$ for various heights in the ionosphere are listed below.*

Elevation	H	Elevation	H
100 km	0·015678	400 km	0·062712
200	0·031356	500	0·078390
300	0·047034	600	0·094068

*The change in the adopted radius may be ignored.

Example 9.19. *Local circumstances of the partial solar eclipse of* 1960 *September* 20–21, *at Nome, Alaska*

Station data

Latitude, ϕ	+ 64° 29′ 54″	$\sin\phi$	+0·9025 73	$\cos\phi$	+0·4305 37
		S	0·9960 08	C	1·0027 49
Longitude, λ	+165 23 36	Altitude	4 m	H	0·0000 01
1·0027 38 ΔT (ΔT = 36^s)	+ 9 01				
Ephemeris longitude, λ^*	+165 32 37	$A \equiv \rho\sin\phi'$	+0·8989 71	$\rho\cos\phi'$	+0·4317 21

Times estimated from eclipse map (*A.E.*, p. 305)

E.T. of middle of eclipse	22^h 27^m	E.T. of beginning	21^h 29^m
Semi-duration	58^m	E.T. of end	23^h 25^m

First approximation to times of contact

To simplify the notation the suffix 0, used to denote quantities calculated for an initial time, has been omitted.

	Beginning	End	Maximum
Starting time, T	21^h 29^m	23^h 25^m	22^h 27^m
μ (elements)	143° 56′ 33″	172° 57′ 07″	158° 26′ 50″
$\theta = \mu - \lambda^*$	338 23 56	7 24 30	352 54 13
$\sin\theta$	−0·3681 43	+0·1289 40	−0·1235 39
$\cos\theta$	+0·9297 69	+0·9916 52	+0·9923 40
$\rho\cos\phi'\cos\theta = B$	+0·4014 01	+0·4281 17	+0·4284 14
$\sin d$ (elements)	+0·0145 24	+0·0139 90	+0·0142 57
$\cos d$ (elements)	+0·9998 95	+0·9999 02	+0·9998 99
x (elements)	−0·3966 13	+0·5828 50	+0·0931 13
$\xi = \rho\cos\phi'\sin\theta$	−0·1589 35	+0·0556 66	−0·0533 34
$u = x - \xi$	−0·2376 78	+0·5271 84	+0·1464 47
y (elements)	+1·3950 35	+1·0785 53	+1·2368 54
$\eta = A\cos d - B\sin d$	+0·8930 47	+0·8928 94	+0·8927 72
$v = y - \eta$	+0·5019 88	+0·1856 59	+0·3440 82
$\zeta = A\sin d + B\cos d$	0·4144 16	0·4406 52	0·4411 87
$\tan f_1$ (elements)	0·0046 58	0·0046 58	0·0046 58
l_1 (elements)	0·5558 00	0·5556 08	0·5557 14
$L_1 = l_1 - \zeta\tan f_1$	0·5538 70	0·5535 55	0·5536 59
μ' (elements)	0·2618 85	0·2618 85	0·2618 85
d' (elements)	−0·0002 76	−0·0002 76	−0·0002 76
$\xi\sin d$	−0·0023 08	+0·0007 79	−0·0007 60
x' (elements)	+0·5065 91	+0·5066 18	+0·5066 28
$\xi' = B\mu'$	+0·1051 21	+0·1121 17	+0·1121 95
$u' = x' - \xi'$	+0·4014 70	+0·3945 01	+0·3944 33
y' (elements)	−0·1635 70	−0·1638 21	−0·1636 96
$\eta' = \mu'\xi\sin d - \zeta d'$	−0·0004 91	+0·0003 26	−0·0000 77
$v' = y' - \eta'$	−0·1630 79	−0·1641 47	−0·1636 19
$n^2 = u'^2 + v'^2$	0·1877 73	0·1825 75	0·1823 49
n	0·4333 28	0·4272 88	

	Beginning	End	Maximum
$D = uu' + vv'$	−0·1772 84	+0·1774 99	+0·0014 65
$\Delta = (uv' - u'v)/n$	−0·3756 35	−0·3739 35	
$\sin \psi = \Delta/L_1$	−0·6782 01	−0·6755 16	
$\cos \psi$ (from $\sin \psi$)	−0·7348 76	+0·7373 45	
	h	h	h
$-D/n^2$	+0·9441 40	−0·9721 98	−0·0080 34
$L_1 \cos \psi/n$	−0·9393 02	+0·9552 36	
sum = correction, t	+0·0048 38	−0·0169 62	−0·0080 34
= in minutes	$+0^m{\cdot}290$	$-1^m{\cdot}018$	$-0^m{\cdot}482$
Corrected E.T.	$20^d\ 21^h\ 29^m{\cdot}290$	$23^h\ 23^m{\cdot}982$	$22^h\ 26^m{\cdot}518$

Second approximation to times of contact

The details of the second approximation are omitted as they are identical in principle with those of the first approximation and are also given in *A.E.*, 1960, page 515. The effect of the second approximation is to give corrections t to the times of the first approximation as follows:

Phase	Correction, t		Corrected E.T.	U.T. ($\Delta T = 36^s$)
	h	m	h m	h m
Beginning	+0·0000 01 =	0·000	21 29·290	21 28·690
End	+0·0000 05 =	0·000	23 23·982	23 23·382
Maximum	−0·0000 33 =	−0·002	22 26·516	22 25·916

Position angles and magnitude

The values of u, v, ξ, η are normally taken unchanged from the calculations for the second approximation; they could be interpolated from those of the first approximation using u', v', ξ', η'.

	Beginning	End		Maximum
	d h m	h m		d h m
E.T.	20 21 29·290	23 23·982	E.T.	20 22 26·516
u	−0·2357	+0·5205	u	+0·1433
v	+0·5012	+0·1884	v	+0·3454
$\tan Q = u/v$ or $\cot Q$	−0·4703	+0·3620		
			$m^2 = u^2 + v^2$	0·1398
ξ	−0·1584	+0·0538	m	0·3739
η	+0·8930	+0·8929	L_1	0·5537
$\tan C = \xi/\eta$ or $\cot C$	−0·1774	+c·0603	$L_1 - m$	0·1798
	°	°		
Q	334·8	70·1	$2L_1 - 0{\cdot}5459$	0·5615
C	349·9	3·4	Mag. = $\dfrac{L_1 - m}{2L_1 - 0{\cdot}5459}$	0·320
$V = Q - C$	344·9	66·7		

Differential corrections

Illustrations of the calculation of the coefficients for the application of differential corrections to the times of contact are given in example 9.20, for the second approximation; these calculations are identical in principle for a partial eclipse, and are here omitted. They are given in *A.E.*, 1960, page 516.

Example 9.20. *Local circumstances of the total solar eclipse of* 1961 *February* 15 *at Simeis, Crimea*

Station data

Latitude, ϕ	° ′ ″ +44 24 12	$\sin\phi$	+0·6997 05	$\cos\phi$	+0·7144 32
		S	0·9949 16	C	1·0016 50
Longitude, λ	−33 59 48	Altitude	346 m	H	0·0000 54
1·0027 38 ΔT (ΔT = 36ˢ)	+ 9 01				
Ephemeris longitude, λ^*	−33 50 47	$\rho\sin\phi' = A$	+0·6961 85	$\rho\cos\phi'$	+0·7156 49

Times estimated from eclipse map (*A.E.*, 1961, p. 298)

E.T. of middle of eclipse	$8^h\ 10^m$	E.T. of beginning	$6^h\ 53^m$
Semi-duration	77^m	E.T. of end	$9^h\ 27^m$

First approximation to times of contact

Details of the first approximation are not given as they are identical in principle with those of the first approximation for a partial eclipse, illustrated in example 9.19, and follow very closely those for the second approximation below. The effect of the first approximation is to give corrections to the starting times as follows:

Phase	Correction, t		Corrected E.T.
	h	m	h m
Beginning	−0·0161 85 =	−0·971	6 52·029
End	−0·0128 73 =	−0·772	9 26·228
Maximum	−0·0504 75 =	−3·028	8 06·972

It will be seen that there is a large correction to the time of maximum eclipse, due partly to the error of estimation in the time of middle of the eclipse but more largely to the difference between the time of maximum eclipse and that of middle of the eclipse. A much closer estimate of the time of maximum eclipse for total and annular eclipses can be obtained by plotting the path of the total or annular phase in the neighbourhood of the station.

Second approximation to times of contact

	Beginning	End	Maximum
Starting time, T	$6^h\ 52^m.029$	$9^h\ 26^m.228$	$8^h\ 06^m.972$
μ (elements)	279° 26′ 41″	317° 59′ 57″	298° 10′ 58″
$\theta = \mu - \lambda^*$	313 17 28	351 50 44	332 01 45
$\sin\theta$	−0·7278 79	−0·1418 42	−0·4690 22
$\cos\theta$	+0·6857 05	+0·9898 89	+0·8831 86
$\rho\cos\phi'\cos\theta = B$	+0·4907 24	+0·7084 13	+0·6320 51
$\sin d$ (elements)	−0·2203 81	−0·2197 71	−0·2200 85
$\cos d$ (elements)	+0·9754 14	+0·9755 52	+0·9754 81
x (elements)	−1·0390 33	+0·4037 45	−0·3378 04
$\xi = \rho\cos\phi'\sin\theta$	−0·5209 06	−0·1015 09	−0·3356 55
$u = x - \xi$	−0·5181 27	+0·5052 54	−0·0021 49
y (elements)	+0·6463 71	+1·0139 76	+0·8249 75
$\eta = A\cos d - B\sin d$	+0·7872 15	+0·8348 53	+0·8182 20
$v = y - \eta$	−0·1408 44	+0·1791 23	+0·0067 55
$\zeta = A\sin d + B\cos d$	0·3252 33	0·5380 92	0·4633 34
$\tan f$ (elements)	0·0047 33	0·0047 33	0·0047 09
l (elements)	0·5384 76	0·5386 14	−0·0073 17
$L = l - \zeta\tan f$	0·5369 37	0·5360 67	−0·0094 99

	Beginning	End	Maximum	
μ' (elements)	0·2618 33	0·2618 33	0·2618 33	
d' (elements)	+0·0002 43	+0·0002 43	+0·0002 43	
$\xi \sin d$	+0·1147 98	+0·0223 09	+0·0738 73	
x' (elements)	+0·5613 92	+0·5613 40	+0·5614 10	
$\xi' = B\mu'$	+0·1284 88	+0·1854 86	+0·1654 92	
$u' = x' - \xi'$	+0·4329 04	+0·3758 54	+0·3959 18	
y' (elements)	+0·1429 40	+0·1431 18	+0·1430 39	
$\eta' = \mu'\xi \sin d - \zeta d'$	+0·0299 79	+0·0057 10	+0·0192 29	
$v' = y' - \eta'$	+0·1129 61	+0·1374 08	+0·1238 10	
$n^2 = u'^2 + v'^2$	0·2001 66	0·1601 47	0·1720 80	
n	0·4473 99	0·4001 84	0·4148 25	
$D = uu' + vv'$	−0·2402 09	+0·2145 15	−0·0000 14	
$\Delta = (uv' - u'v)/n$	+0·0054 63	+0·0052 53	−0·0070 87	
$\sin \psi = \Delta/L$	+0·0101 74	+0·0097 99	+0·7461	
$\cos \psi$ (from $\sin \psi$)	−0·9999 48	+0·9999 52	+0·6658	−0·6658
	h	h	h	h
$-D/n^2$	+1·2000 49	−1·3394 88	+0·0000 81	+0·0000 81
$L \cos \psi/n$	−1·2000 68	+1·3394 86	−0·0152 45	+0·0152 45
Sum = correction, t	−0·0000 19	−0·0000 02	−0·0151 64	+0·0153 26
= in minutes	$-0^m·001$	$0^m·000$	$-0^m·910$	$+0^m·920$
Corrected E.T.	$6^h\ 52^m·028$	$9^h\ 26^m·228$	$8^h\ 06^m·062$	$8^h\ 07^m·892$

Magnitude

At E.T. $8^h\ 06^m·977$ on February 15

L_1 (by calculation) = +0·5364 L_2 = −0·0095

The magnitude $(L_1 - L_2)/(L_1 + L_2)$ = 0·5459/0·5269 = 1·036

Position angles

The example below illustrates the procedure of calculating Q as $N + \psi$. For the beginning and end of the total phase it is necessary to interpolate ξ, η to the times of contact in order to calculate C.

	Partial Phase		Total Phase	
	Beginning	End	Beginning	End
E.T. February 15	$6^h\ 52^m·028$	$9^h\ 26^m·228$	$8^h\ 06^m·062$	$8^h\ 07^m·892$
u'	+0·4329	+0·3759	+0·3959	+0·3959
v'	+0·1130	+0·1374	+0·1238	+0·1238
$\cot N = v'/u'$	+0·2610	+0·3655	+0·3127	+0·3127
N	75°·4	69°·9	72°·6	72°·6
ψ (from $\sin \psi$)	179·4	0·6	48·3	131·7
$Q = N + \psi$	254·8	70·5	120·9	204·3
ξ	−0·5209	−0·1015	−0·3382	−0·3332
η	+0·7872	+0·8349	+0·8179	+0·8185
$\tan C = \xi/\eta$	−0·6617	−0·1216	−0·4135	−0·4071
C	326°·5	353°·1	337°·5	337°·8
$V = Q - C$	288·3	77·4	143·4	226·5

Example 9.20. Local circumstances of total solar eclipse (continued)

The values of ξ, η for the total phase are obtained from those in the calculation for the maximum by adding $\xi' t$, $\eta' t$, thus:

$$\xi = -0{\cdot}3357 + 0{\cdot}1655\ (-0^{h}{\cdot}0152 \text{ or } +0^{h}{\cdot}0153)$$
$$\eta = +0{\cdot}8182 + 0{\cdot}0192\ (-0^{h}{\cdot}0152 \text{ or } +0^{h}{\cdot}0153)$$

Differential corrections

The values of the quantities entering into the calculation of the differential corrections are all available from earlier calculations.

$a = 638 \times 10^{4}$ $\quad \sin 1' = 0{\cdot}0002\,909$ $\quad 1/a \sin 1' = 0{\cdot}0005\,39$

	Beginning	End	Maximum
$SC^2 + H$	+0·998	+0·998	+0·998
$\sin\theta \sin\phi$	−0·509	−0·099	−0·328
$\cos\theta \sin\phi$	+0·480	+0·693	+0·618
$\cos\theta \cos\phi$	+0·490	+0·707	+0·630
$M = \cos d \cos\phi + \sin d \cos\theta \sin\phi$	+0·591	+0·545	+0·561
$A_1 = -\rho \cos\phi' \cos\theta$	−0·491	−0·708	−0·632
$B_1 = -\xi \sin d$	−0·115	−0·022	−0·074
$A_2 = -(SC^2 + H) \sin\theta \sin\phi$	+0·508	+0·099	+0·327
$B_2 = (SC^2 + H)\, M$	+0·590	+0·544	+0·560
$A_3 = \cos\phi \sin\theta$	−0·520	−0·101	−0·335
$B_3 = \cos d \sin\phi - \sin d \cos\theta \cos\phi$	+0·791	+0·838	+0·822

	Beginning	End	Maximum Time		Maximum Δ	
u	−0·518	+0·505	u'	+0·396	u'	+0·396
v	−0·141	+0·179	v'	+0·124	v'	+0·124
$3600 \sin 1'/D$	−4·36	+4·88	$3600 \sin 1'/n^2$	6·09	$\sin 1'/n$	701×10^{-6}
$3600/aD$	−0·00235	+0·00263	$3600/an^2$	0·00328	$1/an$	$0{\cdot}378 \times 10^{-6}$
uA_1	+0·254	−0·358	$u'A_1$	−0·250	$u'B_1$	−0·029
vB_1	+0·016	−0·004	$v'B_1$	−0·009	$-v'A_1$	+0·078
sum	+0·270	−0·362	sum	−0·259	sum	+0·049
p	$-1^{s}{\cdot}18$	$-1^{s}{\cdot}77$	p_m	$-1^{s}{\cdot}58$	p_s	$+34{\cdot}3 \times 10^{-6}$
uA_2	−0·263	+0·050	$u'A_2$	+0·129	$u'B_2$	+0·222
vB_2	−0·083	+0·097	$v'B_2$	+0·069	$-v'A_2$	−0·041
sum	−0·346	+0·147	sum	+0·198	sum	+0·181
q	$+1^{s}{\cdot}51$	$+0^{s}{\cdot}72$	q_m	$+1^{s}{\cdot}21$	q_s	$+126{\cdot}9 \times 10^{-6}$
uA_3	+0·269	−0·051	$u'A_3$	−0·133	$u'B_3$	+0·326
vB_3	−0·112	+0·150	$v'B_3$	+0·102	$-v'A_3$	+0·042
sum	+0·157	+0·099	sum	−0·031	sum	+0·368
r	$-0^{s}{\cdot}00037$	$+0^{s}{\cdot}00026$	r_m	$-0^{s}{\cdot}00010$	r_s	$+0{\cdot}139 \times 10^{-6}$

Thus on 1961 February 15, near Simeis:

		h m s		s		s		s
U.T. of first contact	=	6 51 25·7	−	$1{\cdot}18\,\delta\lambda$	+	$1{\cdot}51\,\delta\phi$		$-0{\cdot}0003\,7\,\delta H$
U.T. of maximum eclipse	=	8 06 22·6	−	$1{\cdot}58\,\delta\lambda$	+	$1{\cdot}21\,\delta\phi$		$-0{\cdot}0001\,0\,\delta H$
U.T. of last contact	=	9 25 37·7	−	$1{\cdot}77\,\delta\lambda$	+	$0{\cdot}72\,\delta\phi$		$+0{\cdot}0002\,6\,\delta H$
Δ, at maximum eclipse	=	−0·0070 87	+	$(34{\cdot}3\,\delta\lambda$		$+126{\cdot}9\,\delta\phi$		$+0{\cdot}139\,\delta H) \times 10^{-6}$

where $\delta\lambda$, $\delta\phi$ are measured in minutes of arc and δH in metres from the adopted position of the station.

Example 9.21. The alternative direct method for the times of second and third contact for the total solar eclipse of 1961 February 15 at Simeis, Crimea

The station data are as in example 9.20; x, y, μ, $\sin d$, $\cos d$, l_2, $\tan f_2$ are copied, with simple interpolation, from the elements.

E.T.	$8^h\ 00^m$		$8^h\ 05^m$		$8^h\ 10^m$		$8^h\ 15^m$
	° ′ ″		° ′ ″		° ′ ″		° ′ ″
μ	296 26 22·3		297 41 22·8		298 56 23·4		300 11 24·0
$\theta = \mu - \lambda^*$	330 17 09·3		331 32 09·8		332 47 10·4		334 02 11·0
$\sin\theta$	−·4956 72		−·4766 06		−·4573 12		−·4378 00
$\cos\theta$	+·8685 10		+·8791 17		+·8893 06		+·8990 72
$\rho\cos\phi'\cos\theta$	+·6215 48		+·6291 39		+·6364 31		+·6434 20
$\sin d$	−·2201 12		−·2200 92		−·2200 73		−·2200 53
$\cos d$	+·9754 75		+·9754 79		+·9754 83		+·9754 87
x	−·4030 40		−·3562 56		−·3094 72		−·2626 88
ξ	−·3547 27		−·3410 83		−·3272 75		−·3133 11
$u = x - \xi$	−·0483 13		−·0151 73		+·0178 03		+·0506 23
		+331 40		+329 76		+328 20	
			−1 64		−1 56		
y	+·8083 54		+·8202 73		+·8321 93		+·8441 14
η	+·8159 21		+·8175 82		+·8191 78		+·8207 06
$v = y - \eta$	−·0075 67		+·0026 91		+·0130 15		+·0234 08
		+102 58		+103 24		+103 93	
			+66		+69		
ζ	+·4530 66		+·4604 87		+·4676 16		+·4744 50
$\tan f_2$	+·0047 09						
l_2	−·0073 23		−·0073 19		−·0073 14		−·0073 10
$L_2 = l_2 - \zeta\tan f_2$	−·0094 56		−·0094 87		−·0095 16		−·0095 44
$u^2 + v^2 - L_2^2$	+·0023 0199		+·0001 4746		+·0003 9578		+·0030 1953
		−21 5453		+2 4832		+26 2375	
			+24 0285		+23 7543		
				−2742			

The times of second and third contacts both lie in the interval from $8^h\ 05^m$ to $8^h\ 10^m$; for this interval:

$$f_0 = +1\ 4746;\ \delta_{\frac{1}{2}} = +2\ 4832;\ \delta_0^2 + \delta_1^2 = +47\ 7828;\ \delta_{\frac{1}{2}}^3 = -\ 2742$$

Inverse interpolation gives:

second contact	$p = +0{\cdot}2127\ 3$	E.T. =	$8^h\ 06^m{\cdot}064$
third contact	$p = +0{\cdot}5786\ 9$	E.T. =	8 07·893

In practice, the times of contact would be known from the path of the total phase to within one minute; an interval of one minute could be used instead of five minutes as above.

For the position angles of contact, u, v, L_2, ξ, η are interpolated, using linear interpolation, to the times of contact.

	second contact	third contact		second contact		third contact
u	−0·0081 6	+0·0039 1	$\cos Q$	−0·515	$\sin Q$	−0·412
v	+0·0048 9	+0·0086 7	Q	121°·0	Q	204°·3
L_2	−0·0094 9	−0·0095 0				
ξ	−0·3381	−0·3331	$\tan C$	−0·413	$\tan C$	−0·407
η	+0·8179	+0·8185	C	337°·6	C	337°·9

If differential corrections are required, u', v' may be obtained for the time of maximum from the differences of u, v.

Example 9.22. *Local circumstances of the annular solar eclipse of* 1961 *August* 11 *at a point in the South Atlantic Ocean, longitude W.* 1°, *latitude S.* 45°

Station data

Latitude, ϕ	$-45°\ 00'\ 00''$	$\sin\phi$	−0·7071 07	$\cos\phi$	+0·7071 07
		S	0·9949 51	C	1·0016 85
Longitude, λ	+ 1 00 00	Altitude	0	H	0·0000 00
$1{\cdot}002738\,\Delta T\ (\Delta T = 36^s)$ +	9 01				
Ephemeris longitude, λ^* +	1 09 01	$\rho\sin\phi' = A$	−0·7035 37	$\rho\cos\phi'$	+0·7082 98

First approximation to times of contact

From estimated times taken from the eclipse map, or from the plotted path of the annular phase, a first approximation yields the following times on 1961 August 11.

Beginning $8^h\ 59^m{\cdot}315$ Maximum $10^h\ 35^m{\cdot}389$ End $12^h\ 18^m{\cdot}567$

Second approximation to times of contact

The calculation follows that for a total eclipse, with the sole exception that L_2 is positive.

	Beginning	End	Maximum
Starting time, T	$8^h\ 59^m{\cdot}315$	$12^h\ 18^m{\cdot}567$	$10^h\ 35^m{\cdot}389$
μ (elements)	313° 32′ 03″	3° 21′ 24″	337° 33′ 26″
$\theta = \mu - \lambda^*$	312 23 02	2 12 23	336 24 25
$\sin\theta$	−0·7386 45	+0·0384 99	−0·4002 38
$\cos\theta$	+0·6740 95	+0·9992 59	+0·9164 11
$\rho\cos\phi'\cos\theta = B$	+0·4774 60	+0·7077 73	+0·6490 92
$\sin d$ (elements)	+0·2639 19	+0·2632 43	+0·2635 93
$\cos d$ (elements)	+0·9645 45	+0·9647 30	+0·9646 34
x (elements)	−1·0726 71	+0·5664 47	−0·2823 05
$\xi = \rho\cos\phi'\sin\theta$	−0·5231 81	+0·0272 69	−0·2834 88
$u = x - \xi$	−0·5494 90	+0·5391 78	+0·0011 83
y (elements)	−0·6730 71	−1·0305 91	−0·8453 11
$\eta = A\cos d - B\sin d$	−0·8046 04	−0·8650 40	−0·8497 52
$v = y - \eta$	+0·1315 33	−0·1655 51	+0·0044 41
$\zeta = A\sin d + B\cos d$	0·2748 55	0·4976 09	0·4406 89
$\tan f$ (elements)	0·0046 15	0·0046 15	0·0045 92
l (elements)	0·5662 86	0·5663 18	+0·0203 09
$L = l - \zeta\tan f$	0·5650 18	0·5640 22	+0·0182 85
μ' (elements)	0·2618 49	0·2618 49	0·2618 49
d' (elements)	−0·0002 11	−0·0002 11	−0·0002 11
$\xi\sin d$	−0·1380 77	+0·0071 78	−0·0747 25
x' (elements)	+0·4935 83	+0·4935 20	+0·4935 94
$\xi' = B\mu'$	+0·1250 22	+0·1853 30	+0·1699 64
$u' = x' - \xi'$	+0·3685 61	+0·3081 90	+0·3236 30
y' (elements)	−0·1074 74	−0·1078 24	−0·1076 59
$\eta' = \mu'\xi\sin d - \zeta d'$	−0·0360 97	+0·0019 85	−0·0194 74
$v' = y' - \eta'$	−0·0713 77	−0·1098 09	−0·0881 85
$n^2 = u'^2 + v'^2$	0·1409 32	0·1070 39	0·1125 13
n	0·3754 09	0·3271 68	0·3354 30

	Beginning	End	Maximum	
$D = uu' + vv'$	−0·2119 09	+0·1843 48	−0·0000 09	
$\Delta = (uv' - u'v)/n$	−0·0246 58	−0·0250 18	−0·0045 97	
$\sin\psi = \Delta/L$	−0·0436 41	−0·0443 56	−0·2514	
$\cos\psi$ (from $\sin\psi$)	−0·9990 47	+0·9990 16	−0·9679	+0·9679
$-D/n^2$	^{h} +1·5036 26	^{h} −1·7222 51	^{h} +0·0000 80	^{h} +0·0000 80
$L\cos\psi/n$	−1·5036 40	+1·7222 56	−0·0527 62	+0·0527 62
sum = correction, t	−0·0000 14	+0·0000 05	−0·0526 82	+0·0528 42
= in minutes	$-0^{m}\cdot001$	$0^{m}\cdot000$	$-3^{m}\cdot161$	$+3^{m}\cdot171$
Corrected E.T.	$8^{h}\ 59^{m}\cdot314$	$12^{h}\ 18^{m}\cdot567$	$10^{h}\ 32^{m}\cdot228$	$10^{h}\ 38^{m}\cdot560$

Magnitude

At E.T. $10^{h}\ 35^{m}\cdot394$ on August 11

L_1 (by calculation) = +0·5643 $\quad L_2$ = +0·0183

The magnitude $(L_1 - L_2)/(L_1 + L_2) = 0\cdot5460/0\cdot5826 = 0\cdot937$

Position angles and differential corrections

The calculations are the same in principle as for a total eclipse and are not illustrated.

E. LUNAR ECLIPSES

The calculation of lunar eclipses follows the same principles as that of solar eclipses. The problem is much simplified by the fact that the times and circumstances are the same for all parts of the Earth from which the Moon is visible.

Three types of lunar eclipses are distinguished:

(i) the penumbral eclipse, also called appulse, in which the Moon enters only the penumbra of the Earth;

(ii) the partial eclipse, in which the Moon enters the umbra without being entirely immersed in it;

(iii) the total eclipse, in which the Moon is entirely immersed within the umbra.

Let $\pi_{☾}$, $s_{☾}$ be the parallax and semi-diameter of the Moon, and $\pi_{\odot}$, $s_{\odot}$ the parallax and semi-diameter of the Sun. The shadow will differ somewhat from a circular cone as the Earth is not a true sphere, but it will suffice to use a mean radius for the Earth, which is equivalent to substituting for $\pi_{☾}$ a parallax π_1, reduced to latitude 45°, so that $\pi_1 = 0\cdot9983\ 33\ \pi_{☾}$. Moreover, observation has shown that * the atmosphere of the Earth has the effect of increasing the apparent radius of the shadow by about one fiftieth. Hence the apparent radii at the distance of the Moon are:

for the penumbra $\quad f_1 = 1\cdot02\ (\pi_1 + s_{\odot} + \pi_{\odot})$

and for the umbra $\quad f_2 = 1\cdot02\ (\pi_1 - s_{\odot} + \pi_{\odot})$

where f_1 and f_2 are expressed in the same units as the parallax and semi-diameter, usually in seconds of arc.

*0·9983 40 for 1968 onwards.

The angular distance (L) between the centres of the Moon and the shadow has the following value at the times of contact:

at beginning and end of the penumbral eclipse $L_1 = f_1 + s_☾$
at beginning and end of the partial eclipse $L_2 = f_2 + s_☾$
at beginning and end of the total eclipse $L_3 = f_2 - s_☾$

The condition of occurrence of a lunar eclipse is therefore:

$$\beta_☾ \cos I' < 1{\cdot}02\,(\pi_1 + s_\odot + \pi_\odot) + s_☾$$

in the case of a penumbral eclipse, and:

$$\beta_☾ \cos I' < 1{\cdot}02\,(\pi_1 - s_\odot + \pi_\odot) + s_☾$$

in the case of an umbral eclipse. Here $\beta_☾ \cos I'$ is the least true angular distance between the centres of the Moon and of the shadow; $\beta_☾$ is here used for the absolute value of the latitude of the Moon at opposition, and I' is defined in sub-section B. The introduction of numerical quantities in the manner described in sub-section B yields the following criteria:

$\beta_☾ > 1^\circ\ 36'\ 38''$	no penumbral eclipse
$1^\circ\ 26'\ 19'' < \beta_☾ < 1^\circ\ 36'\ 38''$	penumbral eclipse possible
$1^\circ\ 03'\ 46'' < \beta_☾ < 1^\circ\ 26'\ 19''$	penumbral eclipse certain; no umbral eclipse
$0^\circ\ 53'\ 26'' < \beta_☾ < 1^\circ\ 03'\ 46''$	penumbral eclipse certain; umbral eclipse possible
$\beta_☾ < 0^\circ\ 53'\ 26''$	umbral eclipse certain.

Let $\alpha_☾$, $\delta_☾$, be the right ascension and declination of the Moon, and $\alpha_\odot$, $\delta_\odot$, the right ascension and declination of the Sun. Then, the right ascension a and declination d of the point Z towards which the centre of the shadow of the Earth is directed are given by:

$$a = \alpha_\odot + 12^{\mathrm{h}} \qquad d = -\,\delta_\odot$$

and the Besselian elements x and y are obtained, as in solar eclipses, by:

$$x = \cos \delta_☾ \sin (\alpha_☾ - a)$$
$$y = \cos d \sin \delta_☾ - \sin d \cos \delta_☾ \cos (\alpha_☾ - a)$$

These equations can be replaced, with sufficient accuracy, by:

$$x = (\alpha_☾ - a) \cos \delta_☾$$
$$y = \delta_☾ - d + \epsilon$$

where

$$\epsilon = \tfrac{1}{4}\,(\alpha_☾ - a) \sin 2d \sin (\alpha_☾ - a)$$

The elements x and y are usually expressed in seconds of arc. The quantity ϵ is small and can often be ignored; it is however included in the calculations for the Ephemeris.

The angular distance (m) between the centres of the Moon and the shadow is given at all times by:

$$m = (x^2 + y^2)^{\frac{1}{2}}$$

Let x_0, y_0, be the values of x, y at some instant T_0 near opposition, and x', y', the hourly variations of x and y. Then, for any particular time of contact $T_0 + t$:

$$(x_0 + x't)^2 + (y_0 + y't)^2 = L^2$$

in which the appropriate value of L is substituted, and its slow variation ignored.

This quadratic equation is solved for t by setting:

$$n^2 = x'^2 + y'^2 \qquad \Delta = \pm \frac{1}{n}(x_0 y' - y_0 x') \qquad \Delta > 0$$

whence

$$t = -\frac{1}{n^2}(x_0 x' + y_0 y') \mp \frac{1}{n}(L^2 - \Delta^2)^{\frac{1}{2}}$$

the upper sign being used for first contact and the lower sign for last contact. If desired, the calculation may be repeated, using the times just obtained as starting times. The following cases may occur:

$L_1^2 - \Delta^2 < 0$	no eclipse
$L_1^2 - \Delta^2 > 0$ with $L_2^2 - \Delta^2 < 0$	penumbral eclipse
$L_2^2 - \Delta^2 > 0$ with $L_3^2 - \Delta^2 < 0$	partial eclipse
$L_3^2 - \Delta^2 > 0$	total eclipse

The time of greatest obscuration occurs when m is a minimum, that is, when:

$$xx' + yy' = 0$$

Thus, the correction t to a starting time T_0 to obtain the time of greatest obscuration is determined by solving:

$$(x_0 + x't)\,x' + (y_0 + y't)\,y' = 0$$

which yields:

$$t = -\frac{1}{n^2}(x_0 x' + y_0 y')$$

The greatest magnitude of the eclipse, the diameter of the Moon being unity, is:

$$\frac{1}{2s_{☾}}(L - m)$$

in which L_2 is used for partial and total eclipses, and L_1 is used for penumbral eclipses. The values of L, m, and $s_{☾}$ should be those corresponding to the time of greatest obscuration. If desired, the quantity m in the calculation of the magnitude may be replaced by Δ, since it may be easily verified that:

$$\Delta = m \text{ for } m = \{(x_0 + x't)^2 + (y_0 + y't)^2\}^{\frac{1}{2}}$$

The position angle of contact P on the limb of the Moon, measured eastwards from the north point, is given by:

$$P = M + 180°$$

where $\tan M = x/y$ and $\sin M$ has the same algebraic sign as x.

The latitudes (ϕ) and ephemeris longitudes (λ^*) of the places that have the Moon in the zenith at given times are determined by:

$$\phi = \delta_{☾} \qquad \lambda^* = \text{ephemeris sidereal time} - \alpha_{☾}$$

Example 9.23. *Circumstances of the total lunar eclipse of* 1960 *March* 13

Occurrence of total eclipse

At opposition (full moon on March 13^d 08^h 26^m) the latitude of the Moon is (*A.E.*, p. 55) about $-0° 10'$ and so is well inside the limits within which an umbral eclipse is certain.

Corrections to position of the Moon

The corrections to be applied to the lunar ephemeris to allow for the difference between the centre of mass and the centre of figure, arising from $\Delta\beta_{☾} = -0''·5$, are:

$$\Delta\alpha_{☾} = -0^s·013 = -0''·20 \qquad \Delta\delta_{☾} = -0''·46$$

Elements of the eclipse

The elements of the eclipse, given in the Ephemeris, are not used in the calculation of the circumstances; they are obtained by interpolating the ephemerides of the Sun and Moon, using the unpublished extra figures, to the time of opposition in right ascension. The latter is obtained from:

E.T.	$\alpha_{\odot}$	$\alpha_{☾}$	$\alpha_{\odot} - \alpha_{☾}$	differences	
d h	° ′ ″	° ′ ″	° ′ ″	′ ″	
13 08	353 20 59·43	173 03 51·28	180 17 08·15		
				−29 40·81	
09	353 23 16·94	173 35 49·60	179 47 27·34		−112
				29 41·93	
10	353 25 34·44	174 07 49·03	179 17 45·41		

giving p = 0·5774 26 and opposition at 1960 March 13^d 08^h 34^m 38^s·73.

Besselian elements

The calculation is illustrated for March 13^d 09^h.

	° ′ ″		° ′ ″
$\alpha_{☾}$	173 35 49·60	$\delta_{☾}$	+2 36 40·49
$\alpha_{\odot} = a + 180°$	353 23 16·94	$\delta_{\odot} = -d$	−2 51 29·90
$\alpha_{☾} - a$	+752·66	$\delta_{☾} - d$	−889·41
$\sin(\alpha_{☾} - a)$	+0·0036	$\sin 2d$	+0·100
$\cos\delta_{☾}$	+0·9989 62	ϵ	+0″·07
$x = (\alpha_{☾} - a)\cos\delta_{☾}$	+751″·88	$y = \delta_{☾} - d + \epsilon$	−889″·34
$\pi_{☾}$	3450″·24	$\sin\pi_{☾}$	0·0167 2645
$r_{\odot}$	0·9941 58	$\sin s_{☾} = 0·2722\ 74 \sin\pi_{☾}$	0·0045 5418
	″		″
$\pi_1 = 0·9983\ 33\pi_{☾}$	3444·49	$s_{☾}$	939·37
$\pi_{\odot} = 8''·80/r_{\odot}$	8·85		
$s_{\odot} = 959''·63/r_{\odot}$	965·27	$L_1 = f_1 + s_{☾}$	5446·35
		$L_2 = f_2 + s_{☾}$	3477·20
$f_1 = 1·02(\pi_1 + s_{\odot} + \pi_{\odot})$	4506·98	$L_3 = f_2 - s_{☾}$	1598·46
$f_2 = 1·02(\pi_1 - s_{\odot} + \pi_{\odot})$	2537·83		

Similar calculations for other hours lead to:

E.T.	x		y		L_1	L_2	L_3
h	″	″	″	″	″	″	″
5	−6353·54		+1359·95		5438·68	3469·45	1593·97
		+1773·88		−561·30			
6	4579·66		798·65		5440·61	3471·40	1595·10
		1775·54		562·01			
7	2804·12		+ 236·64		5442·53	3473·34	1596·22
		1777·18		562·68			
8	−1026·94		− 326·04		5444·45	3475·28	1597·34
		1778·82		563·30			
9	+ 751·88		889·34		5446·35	3477·20	1598·46
		1780·42		563·88			
10	2532·30		1453·22		5448·25	3479·12	1599·58
		1782·03		564·40			
11	4314·33		2017·62		5450·13	3481·02	1600·68
		1783·58		564·92			
12	+6097·91		−2582·54		5452·02	3482·93	1601·79

First approximation to times of contact

Starting with $T_0 = 9^h$, for time of greatest obscuration:

x	$+752''$	x'	$+1780''$	n^2	349×10^4	$xx' + yy'$	$+184 \times 10^4$	t	$-0^h.53$		
y	$-889''$	y'	$-564''$	n	$1869''$	$xy' - x'y$	$+116 \times 10^4$	Δ	$621''$		

E.T. of greatest obscuration $= T_0 + t = 8^h.47 = 8^h\,28^m$

	Penumbral eclipse	Partial eclipse	Total eclipse
L	$5446''$	$3477''$	$1598''$
$L^2 - \Delta^2$	2927×10^4	1170×10^4	217×10^4
$(L^2 - \Delta^2)^{\frac{1}{2}}$	5410	3421	1473
$(L^2 - \Delta^2)^{\frac{1}{2}}/n$	2.89	1.83	0.79
Beginning	$5^h.58 = 5^h\,35^m$	$6^h.64 = 6^h\,38^m$	$7^h.68 = 7^h\,41^m$
End	$11^h.36 = 11^h\,22^m$	$10^h.30 = 10^h\,18^m$	$9^h.26 = 9^h\,16^m$

Second approximation to times of contact

Starting with the times obtained in the first approximation, the calculation proceeds as follows. x, y, L_1, L_2, L_3 are interpolated to these times from the Besselian elements; x', y' are derived from the differences of x, y.

	Penumbral eclipse		Partial eclipse		Total eclipse	
Phase	Beginning	End	Beginning	End	Beginning	End
T_0	$5^h\,35^m$	$11^h\,22^m$	$6^h\,38^m$	$10^h\,18^m$	$7^h\,41^m$	$9^h\,16^m$
x	$-5319.0''$	$+4968.1''$	$-3455.3''$	$+3066.8''$	$-1589.9''$	$+1226.5''$
y	$+1032.6$	-2224.7	$+442.8$	-1622.5	-147.8	-1039.7
x'	$+1774.0''$	$+1783.4''$	$+1775.8''$	$+1781.7''$	$+1777.5''$	$+1780.0''$
y'	-561.4	-564.9	-562.1	-564.3	-562.8	-563.8
$n^2/100$	34622	34996	34694	34929	34763	34863
n	1860.7	1870.7	1862.6	1868.9	1864.5	1867.2
$(xy' - x'y)/100$	+ 1 1543	+ 1 1611	+ 1 1559	+ 1 1602	+ 1 1575	+ 1 1592
$(xx' + yy')/100$	−10 0156	+10 1168	+ 6 3848	+ 6 3797	− 2 7429	+ 2 7694
Δ	$620.4''$	$620.7''$	$620.6''$	$620.8''$	$620.8''$	$620.8''$
L	5439.8	5450.8	3472.6	3479.7	1597.0	1598.8
$(L^2 - \Delta^2)/100$	29 2065	29 3260	11 6738	11 7229	2 1650	2 1708
$(L^2 - \Delta^2)^{\frac{1}{2}}$	5404.3	5415.3	3416.7	3423.9	1471.4	1473.4
$(xx' + yy')/n^2$	$-2^h.8928$	$+2^h.8908$	$-1^h.8403$	$+1^h.8265$	$-0^h.7890$	$+0^h.7944$
$\mp (L^2 - \Delta^2)^{\frac{1}{2}}/n$	-2.9044	$+2.8948$	-1.8344	$+1.8320$	-0.7892	$+0.7891$
t	$-0^h.0116$	$+0^h.0040$	$+0^h.0059$	$+0^h.0055$	$-0^h.0002$	$-0^h.0053$
$T_0 + t$	$5^h\,34^m.30$	$11^h\,22^m.24$	$6^h\,38^m.35$	$10^h\,18^m.33$	$7^h\,40^m.99$	$9^h\,15^m.68$

Maximum eclipse, $T_0 = 8^h\,28^m$

x	$-197.0''$	$n^2/100$	34814	Δ	$620.8''$
y	-588.8	n	1865.9	$(xx' + yy')/n^2$	$-0^h.0054$
x'	$+1778.8''$	$(xy' - x'y)/100$	$+11583$	t	$+0^h.0054$
y'	-563.3	$(xx' + yy')/100$	-188	$T_0 + t$	$8^h\,28^m.32$

Example 9.23. *Circumstances of total lunar eclipse (continued)*

Magnitude, position angles, and sub-lunar positions

At the time of greatest obscuration, $8^h\ 28^m{\cdot}32$

$$\Delta = 621'', \quad L_2 = 3476'', \quad s_{☾} = 939''$$

so magnitude of eclipse $= (L_2 - \Delta)/2\, s_{☾} = 2855/1878 = 1{\cdot}520$

The position angles at the beginning and end of the partial phase are obtained by:

Beginning x	$-3445''$	y $+440''$	$y/x = \cot M$	$-0{\cdot}1277$	$M + 180° = P$	97°
End x	$+3077''$	y $-1626''$	$\cot M$	$-0{\cdot}5284$	P	298°

The positions at which the Moon is in the zenith are found as:

	E.T.	Sidereal time	S.T. in arc	$\alpha_{☾}$	$\lambda^* =$ S.T. $- \alpha_{☾}$	$\phi = \delta_{☾}$
Beginning	$6^h\ 38^m{\cdot}35$	$18^h\ 01^m{\cdot}94$	270° 29′·1	172° 20′·4	+ 98° 09′	+3° 01′
End	$10^h\ 18^m{\cdot}33$	$21^h\ 42^m{\cdot}53$	325° 37′·9	174° 17′·6	+151° 20′	+2° 23′

F. TRANSITS OF MERCURY

The computation of transits of Mercury for the Ephemeris is performed according to an extension of the heliocentric method described by Simon Newcomb (Discussion and results of observations on transits of Mercury, from 1677 to 1881. *A.P.A.E.*, **1**, part VI, 1882).

The heliocentric ephemeris of Mercury is taken from Newcomb's *Tables of Mercury.* The heliocentric ephemeris of the Earth is obtained by adding 180° to the longitude of the Sun, and changing the sign of its latitude, as taken from Newcomb's *Tables of the Sun.*

Occurrence of transits

Transits of Mercury and Venus across the Sun's disk can only occur when both the Earth and the planet are simultaneously very close to the same node of the planet's orbit on the ecliptic. In order that a geocentric transit should occur, the Earth must be within a range $\pm\theta$ of the node of the planet's orbit, at the instant at which the planet crosses the ecliptic. Approximately:

$$\theta = s_{\odot}\left(\frac{1}{r_1} - \frac{1}{R}\right)\left(1 - \frac{p}{p_1}\right) \operatorname{cosec} i$$

where R, r_1; p, p_1 are respectively the radii vectores and the actual daily motions of the Earth and planet, i is the inclination of the planet's orbit to the ecliptic, and $s_{\odot}$ is the Sun's semi-diameter at unit distance.

Owing to the near constancy of the longitudes of the nodes and perihelia of the planets, the Earth will be in the neighbourhood of the nodes on about the same dates each year, and the planets will be at the same points of their orbits when they pass through the ecliptic. The highly eccentric orbit of Mercury thus means that the conditions, and limits, at the November transits are very different from those at the May transits. The approximate dates at which transits can occur, the corresponding values of θ, and the deduced frequency of occurrence (assuming random distribution) are:

Mercury	Asc. node	about 10 Nov.	$\theta = 238'$	9 in 100 years
	Desc. node	,, 8 May	$= 108'$	4 ,, ,, ,,
Venus	Asc. node	,, 9 Dec.	$= 37'$	0·6 ,, ,, ,,
	Desc. node	,, 7 June	$= 41'$	0·6 ,, ,, ,,

There are, in fact, 14 transits of Mercury in the twentieth century, 10 in November, and 4 in May, including the grazing transits in May 1937 and November 1999. Future transits occur on: 1970 May 9, 1973 November 10, 1986 November 13, 1993 November 6, and 1999 November 15.

The transits of Venus are so rare that the accidental relationships (see section 8C) between the periods of revolution of the Earth and Venus are dominant. Transits have occurred at intervals of 8, $121\frac{1}{2}$, 8, $105\frac{1}{2}$, and 8 years, the last one being on 1882 December 6; the cycles of $121\frac{1}{2}$, 8, $105\frac{1}{2}$, and 8 years will continue, the next transits not being until 2004 June 8 and 2012 June 6. No further reference is here made to the transits of Venus.

Corrections to the ephemerides

G. M. Clemence has shown (*A.P.A.E.*, **11**, part 1, p. 61, 1943) that an ephemeris of Mercury derived from Newcomb's tables requires the following corrections to the elements of Mercury to represent its observed geocentric motion:

$$\Delta l = +4''{\cdot}36 + 12''{\cdot}45\,T + 5''{\cdot}100\,T^2 + 0{\cdot}310\,B$$
$$\Delta\pi = +2''{\cdot}10 + 1''{\cdot}82\,T \qquad \Delta e = -0''{\cdot}15$$
$$\Delta\theta = +0''{\cdot}2 + 1''{\cdot}57\,T \qquad \Delta i = +0''{\cdot}03$$

in which B is the fluctuation in the Moon's mean longitude, l is the mean ecliptic longitude, π is the longitude of perihelion, e is the eccentricity, θ is the longitude of the node of the orbit on the ecliptic, i is the inclination of the orbit to the ecliptic, and T is measured in centuries from 1900·0. Δi is smaller than its probable error and may be neglected. The corresponding correction to the mean longitude of the Sun, in terms of U.T., is given by (Clemence, loc. cit. p. 29):

$$\Delta\lambda_{\odot} = +1''{\cdot}00 + 2''{\cdot}97\,T + 1''{\cdot}23\,T^2 + 0{\cdot}0747\,B$$

The correction Δl to Mercury's mean ecliptic longitude, as given above, is the sum of two parts: one part reflects the difference between E.T. and U.T., due to the irregularity of the Earth's rotation, and is equal to $\Delta\lambda_{\odot}$ multiplied by the ratio of the mean motions of Mercury and the Earth (4·15209); the other part is a true correction to Mercury's mean ecliptic longitude as given by Newcomb's tables, and is the only part of Δl to be used when the transit is computed in terms of ephemeris time. The part of Δl to be retained is:

$$\Delta l - 4{\cdot}15209\,\Delta\lambda_{\odot} = +0''{\cdot}21 + 0''{\cdot}12\,T - 0''{\cdot}007\,T^2$$

in which the term in T^2 may be neglected. All the other corrections to the elements are true corrections.

The correction $\Delta\theta$ may be applied directly to θ as taken from the tables. $\Delta\pi$ and Δe affect implicitly the orbital longitude (L), but may be introduced in the form of a correction to the equation of centre (E) since:

$$\Delta L = \Delta l + \Delta E$$

The expression for E as a function of e and the mean anomaly (g) is given by Newcomb (*A.P.A.E.*, **5**, part 1, p. 21, 1895). Then:

$$\Delta E = \frac{\partial E}{\partial e}\Delta e + \frac{\partial E}{\partial g}\Delta g$$

in which Δe is given and Δg may be obtained readily from:

$$\Delta g = \Delta l - \Delta \pi$$

The following development of ΔE, omitting terms containing powers of e higher than the fourth, is obtained by partial differentiation of Newcomb's expressions for E with respect to e and g:

$$\Delta E = \{(2 - \tfrac{3}{4}e^2)\sin g + (\tfrac{5}{2}e - \tfrac{11}{6}e^3)\sin 2g + \tfrac{13}{4}e^2 \sin 3g + \tfrac{103}{24}e^3 \sin 4g\}\,\Delta e$$
$$+ \{(2e - \tfrac{1}{4}e^3)\cos g + (\tfrac{5}{2}e^2 - \tfrac{11}{12}e^4)\cos 2g + \tfrac{13}{4}e^3 \cos 3g + \tfrac{103}{24}e^4 \cos 4g\}\,\Delta g$$

Introducing $e = 0{\cdot}2056$ gives:

$$\Delta E = (1{\cdot}968 \sin g + 0{\cdot}498 \sin 2g + 0{\cdot}137 \sin 3g + 0{\cdot}037 \sin 4g)\,\Delta e$$
$$+ (0{\cdot}409 \cos g + 0{\cdot}104 \cos 2g + 0{\cdot}028 \cos 3g + 0{\cdot}008 \cos 4g)\,\Delta g$$

Δe and Δg also affect implicitly the radius vector r. The expression for $\log_e r$ as a function of e and g is given by Newcomb (loc. cit.) and, following the same procedure as for ΔE:

$$\Delta \log_e r = \{(\tfrac{1}{2}e + \tfrac{1}{8}e^3) - (1 - \tfrac{9}{8}e^2)\cos g - (\tfrac{3}{2}e - \tfrac{11}{6}e^3)\cos 2g$$
$$- \tfrac{17}{8}e^2 \cos 3g - \tfrac{71}{24}e^3 \cos 4g\}\,\Delta e$$
$$+ \{(e - \tfrac{3}{8}e^3)\sin g + (\tfrac{3}{2}e^2 - \tfrac{11}{12}e^4)\sin 2g + \tfrac{17}{8}e^3 \sin 3g$$
$$+ \tfrac{71}{24}e^4 \sin 4g\}\,\Delta g$$
$$= (0{\cdot}104 - 0{\cdot}952 \cos g - 0{\cdot}292 \cos 2g - 0{\cdot}090 \cos 3g - 0{\cdot}026 \cos 4g)\,\Delta e$$
$$+ (0{\cdot}202 \sin g + 0{\cdot}060 \sin 2g + 0{\cdot}018 \sin 3g + 0{\cdot}005 \sin 4g)\,\Delta g$$

This value of $\Delta \log_e r$ must first be expressed in radians and then multiplied by $0{\cdot}4343$ before being applied to $\log r$ as taken from Newcomb's tables. In the tables the mean anomaly g is expressed in days and is to be multiplied by $4{\cdot}0923$ to convert it to degrees for use in the above formulae; it is tabulated in *A. E.*, p. 176. The quantities ΔL, $\Delta\theta$, and $\Delta \log r$ are published in the Ephemeris.

Example 9.24. *Corrections to the ephemeris of Mercury for the transit of* 1960 *November* 7

1960 November $7^d\ 12^h$ = J.D. 243 7246·0 $T = 0{\cdot}609$ $T^2 = 0{\cdot}371$
Epoch of tables = J.D. 241 5020·0 For 1960 November $7^d\ 16^h\ 54^m$
Interval = 2 $2226^d{\cdot}0$ $g = 82^d{\cdot}921 = 339^\circ{\cdot}34$

$\Delta l = +0''{\cdot}28$ $\Delta\pi = +3''{\cdot}21$ $\Delta g = -2''{\cdot}93$ $\Delta e = -0''{\cdot}15$ $\Delta\theta = +1''{\cdot}2$

$\Delta E = -1{\cdot}18\,\Delta e + 0{\cdot}47\,\Delta g = -1''{\cdot}20$ $\Delta L = -0''{\cdot}92$

$\Delta \log_e r = -1{\cdot}05\,\Delta e - 0{\cdot}13\,\Delta g = +0{\cdot}0000\ 026$ $\Delta \log r = +0{\cdot}0000\ 011$

Details of the evaluation of the simple formulae are omitted. The last digit in ΔL has little significance, and the correction to $\log r$ does not affect the circumstances of the transit. Values of $\Delta L = -0''{\cdot}90$ and $\Delta \log r = +0{\cdot}0000\ 026$ have been used both in the Ephemeris and subsequent examples.

Effect of aberration

The contacts occur when the apparent positions of points on the limbs of Mercury and of the Sun coincide. Owing to the finite velocity of light, the ray of light from the Sun which reaches the geometric position of the observer at the instant T of contact left the Sun at some previous time $T - \tau_2$ and grazed the planet at a time $T - \tau_2 + \tau_1$. Therefore, in the calculation of transits, the geometric coordinates of Mercury at time $t - \tau_2 + \tau_1$ must be combined with the geometric coordinates of the Earth at time t.

The latitude $b_\oplus$, longitude $l_\oplus$, and radius vector R of the Earth are obtained from Newcomb's *Tables of the Sun*, by setting:

$$b_\oplus = -\beta_\odot$$
$$l_\oplus = \lambda_\odot \pm 180^\circ$$

in which $\lambda_\odot$ is the true geometric longitude of the Sun, reckoned from the equinox of date, and $\beta_\odot$ is its latitude with respect to the ecliptic of date.

The corresponding geometric coordinates l, b, r, of Mercury at time $t - \tau_2 + \tau_1$ are obtained from the tabular geometric coordinates at time t by subtracting the motion during the interval $\tau_2 - \tau_1$. If l', b', r' are the hourly variations of l, b, r, the corrections to be applied to l, b, r are:

$$-(\tau_2 - \tau_1)\, l' \qquad -(\tau_2 - \tau_1)\, b' \qquad -(\tau_2 - \tau_1)\, r'$$

in which

$$\tau_2 - \tau_1 = 0{\cdot}138439\,(R - r)$$

the factor* being the time in hours for light to travel unit distance, as derived from the constant of aberration.

Example 9.25. Coordinates corrected for effect of aberration

The hourly heliocentric ephemerides of the Earth and Mercury, taken from Newcomb's tables and corrected, for Mercury, for the corrections to the elements and for aberration, are given in the table below. The derivation is not illustrated, nor is the straightforward correction for aberration. Additional figures, obtained in the course of subtabulation from daily intervals, are retained to give smoothness in the calculations.

E.T.		Earth $l_\oplus$	Earth $b_\oplus$	Earth R	Mercury l	Mercury b	Mercury r
d	h	″	″		″	″	
Nov. 7	13	16 2465·163	0·609	0·9906 8853	15 9726·982	−1545·832	0·3159 4299
	14	2615·739	·608	7862	16 0622·464	1436·014	8 1335
	15	16 2766·318	0·607	0·9906 6871	16 1518·672	−1326·079	0·3156 8461
	16	2916·901	·606	5881	2415·604	1216·030	5 5677
	17	3067·487	·605	4892	3313·254	1105·869	4 2984
	18	3218·077	·604	3903	4211·619	995·599	3 0379
	19	3368·670	·603	2915	5110·696	885·222	1 7867
	20	16 3519·267	0·602	0·9906 1927	16 6010·479	− 774·741	0·3150 5446
	21	16 3669·867	0·601	0·9906 0940	16 6910·966	− 664·159	0·3149 3115

*0·138614 for 1968 onwards.

Geocentric contacts

The heliocentric method of computing transits of Mercury is adapted from Bessel's theory of eclipses, in which the planet is substituted for the Moon. The exterior tangents to the Sun and the planet generate the cone of umbra, which has its vertex between Mercury and the Earth. The interior tangents form the cone of penumbra, with vertex between Mercury and the Sun. The axis common to both cones is the axis of shadow. Exterior and interior contacts occur when the observer is on the surface of the penumbral and umbral cones, respectively.

As in the theory of eclipses, the fundamental plane, or xy-plane, is perpendicular to the axis of shadow and passes through the centre of the Earth. In the case of transits, however, the origin of coordinates is the intersection of the axis of shadow with the xy-plane; the x-axis is parallel to the plane of the ecliptic, positive in the direction opposite to the motion of Mercury around the Sun; the y-axis is positive towards the north.

Let m be the angular distance of the planet from the Earth as seen from the centre of the Sun, and M the position angle, reckoned from the y-axis, of the centre of the Earth with respect to the origin of coordinates, such that:

$$x = R \sin m \sin M = R \cos b_{\oplus} \sin (l - l_{\oplus})$$
$$y = R \sin m \cos M = R \{ \cos b \sin b_{\oplus} - \sin b \cos b_{\oplus} \cos (l - l_{\oplus}) \}$$

Except for M, all angles in these equations are so small that, with sufficient precision:

$$m \sin M = l - l_{\oplus} = v \qquad m \cos M = b_{\oplus} - b = u$$
$$u^2 + v^2 = m^2 \qquad m > 0$$

Using primes to denote hourly variations,

$$v' = l' - l'_{\oplus} = n \sin N \qquad u' = b'_{\oplus} - b' = n \cos N$$
$$u'^2 + v'^2 = n^2 \qquad n > 0$$

If s, $s_{\odot}$ are the semi-diameters of Mercury and the Sun at unit distance, their adopted values being $3''\cdot34$ and $959''\cdot63$ respectively, and if f_1, f_2 are the angles which the generators of the penumbral and umbral cones, respectively, make with the axis of the shadow, then:

$$f_1 = \frac{s_{\odot} + s}{r} \qquad f_2 = \frac{s_{\odot} - s}{r}$$

The corresponding radii R_1, R_2 of the shadow in the fundamental plane are found from:

$$R_1 = R \cos m \sec f_1 \{ f_1 - s_{\odot}/(R \cos m) \}$$
$$R_2 = R \cos m \sec f_2 \{ f_2 - s_{\odot}/(R \cos m) \}$$

The geocentric contacts take place at the instants at which the distance $R \sin m = Rm$ of the centre of the Earth from the origin is equal to R_1 for exterior contacts or R_2 for interior contacts, that is when:

$$m = L_1 \text{ or } L_2 \text{ with } \begin{array}{l} L_1 = R_1/R = \sec f_1 (f_1 \cos m - s_{\odot}/R) \\ L_2 = R_2/R = \sec f_2 (f_2 \cos m - s_{\odot}/R) \end{array}$$

If u_0, v_0, u'_0, v'_0, n_0 are the values of u, v, u', v', n at a time T_0 near a time of contact, *Example 9.26. Time of geocentric interior egress of Mercury from the Sun's disk on 1960 November 7*

The data are taken from example 9.25, by interpolation where necessary; the subscript o has been omitted.

Note that $s_\odot = 959''\cdot63$ and $s_\odot - s = 956''\cdot29$.

	First approximation	Second approximation	Third approximation
E.T. = T_0	17^h	$19^h\cdot176$	$19^h\cdot17799$
$l_\oplus$	16 3067″	16 3395·175″	16 3395·475″
l	16 3313	16 5269·007	16 5270·797
$v = l - l_\oplus$	+ 246	+ 1873·832	+ 1875·322
$b_\oplus$	+ 1	+ 0·603	+ 0·603
b	− 1106	− 865·785	− 865·565
$u = b_\oplus - b$	+ 1107	+ 866·388	+ 866·168
R	0·9906	0·9906 2741	0·9906 2739
r	0·3154	0·3151 5674	0·3151 5650
$l'_\oplus$	+ 151″	+ 150·596″	+ 150·596″
l'	+ 898	+ 899·554	+ 899·556
$v' = l' - l'_\oplus$	+ 747	+ 748·958	+ 748·960
$b'_\oplus$	0	− 0·001	− 0·001
b'	+ 110	+ 110·447	+ 110·448
$u' = b'_\oplus - b'$	− 110	− 110·448	− 110·449
$m^2 = u^2 + v^2$	1286×10^3	4261 875	4267 080
$n^2 = u'^2 + v'^2$	570×10^3	573 137	573 140
n	755	757·058	757·060
$u\,u' + v\,v'$	+ 62×10^3	+ 1307 731	+ 1308 874
$u\,v' - u'\,v$	+ 853×10^3	+ 855 849	+ 855 853
m	1134″	2064·431″	2065·691″
$\cos m$	1·000	0·9999 499	0·9999 499
$f_2 = 956''\cdot29/r$	3031″	3034·331″	3034·334″
$\sec f_2$	1·000	1·0001 082	1·0001 082
$959''\cdot63/R$	969″	968·709″	968·709″
$f_2 \cos m - 959''\cdot63/R$	2062	2065·470	2065·473
L_2	2062	2065·693	2065·696
$n\,L_2$	1557×10^3	1563 849	1563 856
$\sin\psi = (uv' - u'v)/n\,L_2$	+0·5478	+0·5472 709	+0·5472 710
$\cos\psi$ (from $\sin\psi$)	+0·8366	+0·8369 555	+0·8369 555
$L_2 \cos\psi/n$	$+2^h\cdot285$	$+2^h\cdot2837\ 0$	$+2^h\cdot2837\ 0$
$-(uu' + vv')/n^2$	−0·109	−2·2817 1	−2·2836 9
sum = t	+2·176	+0·0019 9	+0·0000 1
$T = T_0 + t$	19·176	19·1779 9	19·1780 0
$L_2 - m$	928″	1″·262	0″·005

The E.T. of geocentric interior egress is thus 1960 November $7^d\ 19^h\ 10^m\ 40^s.8$.

then, at the time of contact, $T = T_0 + t$:

$$(u_0 + u_0't)^2 + (v_0 + v_0't)^2 = L^2$$

in which the second-order variations of u, v are neglected. Solving for t by setting:

$$\sin \psi = (u_0 v_0' - u_0' v_0)/n_0 L$$

yields

$$t = \frac{L \cos \psi}{n_0} - \frac{u_0 u_0' + v_0 v_0'}{n_0^2}$$

where $\cos \psi$ is taken as negative for ingress and positive for egress.

The resulting times may be tested by means of the criterion:

$$m = L$$

and another approximation made if necessary.

Least angular distance of centres

The angle m is the angular separation between the centres of Mercury and the Earth as seen from the Sun. If Δ is the geometric distance of Mercury from the Earth, the angular separation S between the centres of Mercury and the Sun as seen from the Earth is given by:

$$\Delta \sin S = r \sin m$$

or, with sufficient precision:

$$S = mg \qquad \text{with} \qquad g = r/\Delta$$

Δ may be calculated from:

$$\Delta^2 = R^2 + r^2 - 2rR\,\{\cos b \cos b_\oplus \cos (l - l_\oplus) + \sin b \sin b_\oplus\}$$

in which the tabular geometric values of r, b, l (not ante-dated) must be used. The function g may then be tabulated and ante-dated to the time $t - \tau_2 + \tau_1$, as was done for r, b, l.

The minimum of S occurs when m^2g^2 is a minimum, that is, when its first derivative is equal to zero. If m_0, g_0 are the values of m, g at a time T_0, near the time T of minimum, the correction $t = T - T_0$ may be found, in hours, from:

$$t = -\frac{d}{dt}(m^2g^2) \Big/ \frac{d^2}{dt^2}(m^2g^2) \text{ evaluated at time } T_0$$

Using primes to denote first derivatives with respect to the time and neglecting second derivatives of u, v:

$$\frac{1}{2}\frac{d}{dt}(m^2g^2) = (uu' + vv')\,g^2 + gg'm^2$$

$$\frac{1}{2}\frac{d^2}{dt^2}(m^2g^2) = n^2g^2 + 4\,(uu' + vv')\,gg' + m^2g'^2$$

whence

$$t = -\frac{(u_0u_0' + v_0v_0')\,g_0^2 + m_0^2g_0g_0'}{n_0^2g_0^2 + 4\,(u_0u_0' + v_0v_0')\,g_0g_0' + m_0^2g_0'^2}$$

where g_0' may be taken as the hourly first difference of g, interpolated to the time T_0.

An alternative procedure to obtain the minimum value of S and the time at which it occurs is to tabulate this function for every ten minutes over a short interval containing the time of mid-transit, and to perform an inverse interpolation to the time for which the first difference of S is equal to zero.

Example 9.27. *Calculation of g for E.T.* 17^h *on* 1960 *November* 7

Geometric values, uncorrected for aberration, of l, b, r must be used; the values used therefore differ from those in example 9.25.

$l_\oplus$	16 3067″·5	$b_\oplus$	+0″·6			R	0·9906 489
l	16 3397·2			b	−1095″·6	r	0·3154 180
$l - l_\oplus$	+ 329·7						
		$\sin b_\oplus$	+0·0000 029	$\sin b$	−0·0053 116	$2\,Rr$	0·6249 370
$\cos(l - l_\oplus)$	+0·9999 987	$\cos b_\oplus$	+1·0000 000	$\cos b$	+0·9999 859	A	0·9999 846

$$A = \sin b_\oplus \sin b + \cos(l - l_\oplus) \cos b_\oplus \cos b$$

$$R^2 + r^2 - 2\,RrA = \Delta^2 \quad 0{\cdot}4559\ 464$$

$$\text{whence } \Delta = 0{\cdot}6752\ 380 \qquad g = 0{\cdot}4671\ 212$$

It will be noted that Δ differs very slightly from the value (0·6752 395) obtained by interpolation of the geocentric distance of Mercury on page 184 of the Ephemeris; the latter value may be corrected by applying the negative of the correction to the radius vector, namely (see example 9.24), -19×10^{-7}.

The above procedure leads to the following values of g:

E.T.	g	difference	correction for light-time	g (ante-dated)
14^h	0·4679 174			
		−2 632		
15	6 542		+0·0000 247	0·4676 789
		2 652		
16	3 890		249	·4674 139
		2 678		
17	·4671 212		251	·4671 463
		2 697		
18	·4668 515		253	·4668 768
		2 723		
19	5 792			

The light-time is 0·1384 39 $(R - r)$ and for 17^h = 0·0934 8.

Example 9.28. *Time of least geocentric distance of centres of Mercury and the Sun*

The first approximation to the time is $T_0 = 17^h$, giving $t = -0^h{\cdot}109$; most of the quantities required already occur in example 9.26. The subscript 0 is omitted, except in T_0.

Thus T_0 = November $7^d\ 16^h{\cdot}891$

$l_\oplus$	16 3051″·073	$l'_\oplus$	+150″·588	g	0·4671 756	g'	−0·0002 683
l	16 3215·375	l'	+897·930	g^2	0·2182 530	g'^2	0·0000 001
v	+ 164·302	v'	+747·342	$uu' + vv'$	− 473	gg'	−0·0001 253
$b_\oplus$	+ 0·605	$b'_\oplus$	− 0·001	n^2	57 0665	m^2	127 8008
b	− 1117·882	b'	+110·204	Num.	+263·3	Denom.	12 4549·6
u	+ 1118·487	u'	−110·205	t	$+0^h{\cdot}0021\ 1$		

$$\text{Num.} = -(uu' + vv')\,g^2 - m^2 gg'\ ; \quad \text{Denom.} = n^2 g^2 + m^2 g'^2 + 4\,(uu' + vv')\,gg'$$

Thus the time of least geocentric distance is $16^h{\cdot}89311 = 16^h\ 53^m\ 35^s{\cdot}2$, at which time $m = 1130''{\cdot}49$ and $S = 528''{\cdot}1 = 8'\ 48''{\cdot}1$.

Position angle of point of contact

The position angle P of a given point of contact, measured from the north point of the Sun's limb towards the east, may be found to within a few minutes of arc from:

$$P = M \pm 180^\circ + V$$

or

$$P = N - \psi \pm 180^\circ + V$$

in which M, N, and ψ have been defined above (see under geocentric contacts) and V is the angle between two points on the Sun's limb, one being the north point, and the other being the point nearest the north pole of the ecliptic. The angle V is calculated from:

$$\cos V = \cos \epsilon \sec \delta_\odot$$

in which ϵ is the obliquity of the ecliptic and $\delta_\odot$ is the apparent declination of the Sun as tabulated in the Ephemeris; V is in the first quadrant for November transits and in the fourth quadrant for May transits.

Example 9.29. Position angle of point of contact

The position angle of geocentric interior egress at E.T. $19^h{\cdot}178$ is formed as (see example 9.26):

u	866″	v	1875″	m	2066″		
ϵ	23°·44	$\delta_\odot$	−16°·45	$\cos M$	0·4193	M	65°·2
$\cos \epsilon$	0·9175	$\sec \delta_\odot$	1·0427	$\cos V$	0·9567	V	16 ·9

$$P = M + V \pm 180° = 262 {\cdot} 1$$

Mercury in the zenith

The latitude ϕ and ephemeris longitude λ^* of the place at which Mercury is in the zenith at a given time are found from:

$$\phi = \delta \qquad \lambda^* = \text{ephemeris sidereal time} - \alpha$$

in which α and δ are the apparent right ascension and declination of Mercury for that time, as tabulated in the Ephemeris.

Example 9.30. Mercury in the zenith

The place at which Mercury will be in the zenith at the time of geocentric interior egress at E.T. $19^h{\cdot}178$ on November 7 is found by:

Latitude, $\phi = \delta =$	−16° 29′ (*A.E.*, page 184)
Ephemeris sidereal time =	$22^h\ 18^m\ 37^s$ (*A.E.*, page 16)
Right ascension, $\alpha =$	14 50 33 (*A.E.*, page 184)
Ephemeris longitude, $\lambda^* =$	7 28 04 = +112° 01′

Local contacts

At a point on the surface of the Earth, the quantities u, v, u', v', m, n, L assume corresponding values $u_1, v_1, u_1', v_1', m_1, n_1, L_1$, which may be written in the form:

$$u_1 = u + q \qquad u_1' = u' + q' \qquad q' = dq/dt$$
$$v_1 = v + p \qquad v_1' = v' + p' \qquad p' = dp/dt$$
$$m_1^2 = u_1^2 + v_1^2 \qquad n_1^2 = u_1'^2 + v_1'^2 \qquad L_1 = L + w$$

The condition of local contact is given by:

$$m_1 = L_1$$

Thus, a time T_c of geocentric contact may be used as the first approximation to the corresponding time T_1 of local contact, and a correction t ($= T_1 - T_c$) may be calculated, in hours, as for geocentric contacts by setting:

$$\sin \psi_1 = (u_1 v_1' - u_1' v_1)/L_1 n_1$$

whence, neglecting the variation of L_1 during the interval t:

$$t = \frac{L_1 \cos \psi_1}{n_1} - \frac{u_1 u_1' + v_1 v_1'}{n_1^2}$$

in which $\cos \psi_1$ is negative for ingress and positive for egress.

Newcomb (loc. cit.) has shown that, with sufficient precision:

$$q = \rho \sin \pi_\odot \sin \beta$$
$$p = \rho \sin \pi_\odot \cos \beta \sin (l_\oplus - \lambda)$$
$$w = \rho \sin s_\odot \sin \pi_\odot \cos \beta \cos (l_\oplus - \lambda)$$

in which β, λ are the latitude and longitude of the observer's geocentric zenith, ρ is the geocentric distance, and $\pi_\odot$ is the horizontal parallax of the Sun at the time considered, obtained from:

$$\pi_\odot = 8''{\cdot}80/R \quad *$$

The coordinates β, λ are related to the observer's geocentric latitude ϕ' and the local sidereal time τ by:

$$\cos \beta \cos \lambda = \cos \phi' \cos \tau$$
$$\cos \beta \sin \lambda = \sin \phi' \sin \epsilon + \cos \phi' \cos \epsilon \sin \tau$$
$$\sin \beta = \sin \phi' \cos \epsilon - \cos \phi' \sin \epsilon \sin \tau$$

where ϵ is the obliquity of the ecliptic.

Thus the expressions for q, p, w may be written in the following form, in which $\pi_\odot$ has been substituted for its sine:

$$q = \pi_\odot (\rho \sin \phi' \cos \epsilon - \rho \cos \phi' \sin \epsilon \sin \tau)$$
$$p = \pi_\odot (-\rho \sin \phi' \sin \epsilon \cos l_\oplus - \rho \cos \phi' \cos \epsilon \cos l_\oplus \sin \tau + \rho \cos \phi' \sin l_\oplus \cos \tau)$$
$$w = \pi_\odot \sin s_\odot (\rho \sin \phi' \sin \epsilon \sin l_\oplus + \rho \cos \phi' \cos \epsilon \sin l_\oplus \sin \tau + \rho \cos \phi' \cos l_\oplus \cos \tau)$$

Therefore:

$$q' = -\tau' \pi_\odot \rho \cos \phi' \sin \epsilon \cos \tau$$
$$p' = \pi_\odot (l_\oplus' \rho \sin \phi' \sin \epsilon \sin l_\oplus + l_\oplus' \rho \cos \phi' \cos \epsilon \sin l_\oplus \sin \tau - \tau' \rho \cos \phi' \cos \epsilon \cos l_\oplus \cos \tau + l_\oplus' \rho \cos \phi' \cos l_\oplus \cos \tau - \tau' \rho \cos \phi' \sin l_\oplus \sin \tau)$$

in which the hourly variations $l_\oplus'$, τ' of $l_\oplus$, τ should be expressed in radians per hour. It is seen, therefore, that the functions u_1, v_1, u_1', v_1', and L_1 may all be expressed in the general form:

$$A + B \rho \sin \phi' + \rho \cos \phi' (C \sin \tau + D \cos \tau)$$

with

$$\tau = \mu - \lambda^*$$

λ^* being the ephemeris longitude of the observer and μ the ephemeris sidereal time of geocentric contact.

A, *B*, *C*, *D* are independent of the observer's position; they may be tabulated, together with μ, for a particular transit. Their expressions are summarised below:

	u_1	u_1'	L_1
A	u	u'	L
B	$\pi_\odot \cos \epsilon$	0	$\pi_\odot \sin s_\odot \sin \epsilon \sin l_\oplus$
C	$-\pi_\odot \sin \epsilon$	0	$\pi_\odot \sin s_\odot \cos \epsilon \sin l_\oplus$
D	0	$-\tau' \pi_\odot \sin \epsilon$	$\pi_\odot \sin s_\odot \cos l_\oplus$

*8″·794 for 1968 onwards.

	v_l	v_l'
A	v	v'
B	$-\pi_\odot \sin\epsilon \cos l_\oplus$	$l_\oplus' \pi_\odot \sin\epsilon \sin l_\oplus$
C	$-\pi_\odot \cos\epsilon \cos l_\oplus$	$\pi_\odot \sin l_\oplus (l_\oplus' \cos\epsilon - \tau')$
D	$\pi_\odot \sin l_\oplus$	$\pi_\odot \cos l_\oplus (l_\oplus' - \tau' \cos\epsilon)$

There are four sets of values, one for each of the contacts. The above formulae yield quantities expressed in seconds of arc. If preferred, they may be expressed in any other convenient unit, because the expression for t is homogeneous in u_l, v_l, u_l', v_l', and L_l; for example, in *A.E.* 1960 they have been converted into radians and multiplied by an arbitrary factor of 1000.

Example 9.31. *Local circumstances for the transit of Mercury on* 1960 *November* 7

(a) Calculation of the coefficients A, B, C, D for interior egress. The subscript l, denoting local values, is omitted. Here A, B, C, D are in seconds of arc.

Using data from the final approximation (example 9.26) and the formulae above:

$\pi_\odot$	8·883″	ϵ	23·44°	$l_\oplus$	45·39°
$\sin s_\odot$	0·0047	$\sin\epsilon$	+0·3978	$\sin l_\oplus$	+0·7119
		$\cos\epsilon$	+0·9175	$\cos l_\oplus$	+0·7023
τ'	0·2625	$l_\oplus' \cos\epsilon - \tau'$	−0·2619		
$l_\oplus'$	0·0007	$l_\oplus' - \tau' \cos\epsilon$	−0·2401		

	u	v	u'	v'	L
A	+866·168	+1875·322	−110·449	+748·960	+2065·696
B	+ 8·150	− 2·482	0	+ 0·002	+ 0·012
C	− 3·534	− 5·724	0	− 1·656	+ 0·027
D	0	+ 6·324	− 0·927	− 1·498	+ 0·029

Ephemeris sidereal time $= 22^h\ 18^m\ 36^s{\cdot}4$ $\quad \mu = 334°\ 39'{\cdot}1$

(b) Calculation of time of local interior egress at the U.S. Naval Observatory.

$\rho \sin\phi'$	0·6249	longitude, λ	+77° 03′·9	μ	334° 39′·1
$\rho \cos\phi'$	0·7791	$1{\cdot}002738\ \Delta T\ (\Delta T = 36^s)$	9·0	$\tau = \mu - \lambda^*$	257 26·2
		ephemeris longitude, λ^*	+77 12·9	$\sin\tau$	−0·9760
				$\cos\tau$	−0·2176

	u	v	u'	v'	L
$C \sin\tau + D \cos\tau$	+ 3·449	+ 4·211	+ 0·202	+ 1·942	− 0·033
A	+866·168	+1875·322	−110·449	+748·960	+2065·696
$B\ \rho \sin\phi'$	+ 5·093	− 1·551	0	+ 0·001	+ 0·007
$\rho \cos\phi'\ (C \sin\tau + D \cos\tau)$	+ 2·687	+ 3·281	+ 0·157	+ 1·513	− 0·026
sums	+873·948	+1877·052	−110·292	+750·474	+2065·677

$n^2 = u'^2 + v'^2$	57 5376	$uu' + vv'$	131 2289	$-(uu' + vv')/n^2$	$-2{\cdot}2807\ 5^h$
n	758·535	$uv' - u'v$	86 2899	$L \cos\psi/n$	+2·2730 9
nL	156 6888	$\sin\psi = (uv' - u'v)/nL$	0·5507 088	$t = \text{sum}$	−0·0076 6
		$\cos\psi$ (from $\sin\psi$)	0·8346 974		$= -27^s{\cdot}6$

$T_c = 19^h\ 10^m\ 40^s{\cdot}8 \qquad T_l = T_c + t = 19^h\ 10^m\ 13^s{\cdot}2$

U.T. of local contact $= T_l - \Delta T\ (36^s) = 19^h\ 09^m\ 37^s{\cdot}2$

The method outlined above is equivalent to Newcomb's (loc. cit.) rigorous solution of his equations (10). It is therefore suitable in all cases, including those in which the least angular distance of centres is nearly as great as the Sun's semi-diameter. When, however, this distance is small in comparison with $s_\odot$, sufficient precision may be attained in the following, simpler, manner.

At the time T_1 of local contact, the condition $L_1 = m_1$ may be written, omitting the small quantities $p't$, $q't$:

$$L_1 \sin M_1 = L \sin M + p + nt \sin N$$
$$L_1 \cos M_1 = L \cos M + q + nt \cos N$$

Eliminating M_1 from these equations and omitting terms in p^2, q^2, t^2, pt, and qt, yields:

$$L_1^2 = L^2 + 2L\,(p \sin M + q \cos M) + 2nLt \cos (M - N)$$

which may be written:

$$L_1^2 - L^2 = \frac{2L}{m}\{\,pv + qu + (uu' + vv')\,t\,\}$$

or, very nearly,

$$L_1 - L = \{\,pv + qu + (uu' + vv')\,t\,\}/m$$

As before:

$$L_1 - L = w$$

whence

$$t = \frac{mw - pv - qu}{uu' + vv'}$$

Thus t may also be written in the form:

$$t = B\,\rho \sin \phi' + \rho \cos \phi'\,(C \sin \tau + D \cos \tau)$$

with

$$B = \frac{\pi_\odot}{uu' + vv'}\,(\,v \sin \epsilon \cos l_\oplus - u \cos \epsilon + m \sin s_\odot \sin \epsilon \sin l_\oplus\,)$$

$$C = \frac{\pi_\odot}{uu' + vv'}\,(\,v \cos \epsilon \cos l_\oplus + u \sin \epsilon + m \sin s_\odot \cos \epsilon \sin l_\oplus\,)$$

$$D = \frac{\pi_\odot}{uu' + vv'}\,(\,-v \sin l_\oplus + m \sin s_\odot \cos l_\oplus\,)$$

The quantity $\sin s_\odot$ is of the same order of magnitude as others which have been neglected; terms in $s_\odot$ could therefore be omitted, but they are, however, taken into account in the values published in the Ephemeris.

Traditionally, the reduction formulae for local contacts have been published in a slightly different form, obtained by setting:

$$C = k \sin K \qquad D = k \cos K$$

Then:

$$t = B\,\rho \sin \phi' + k\,\rho \cos \phi' \cos (K - \tau)$$

or, introducing the ephemeris sidereal time μ of geocentric contact:

$$t = B\,\rho \sin \phi' + k\,\rho \cos \phi' \cos (K - \mu + \lambda^*)$$

The coefficients B and k, expressed in hours, should be converted into seconds by multiplying by 3600. Then, if λ is the geographic longitude of the observer, the universal times of local contacts will be found from:

$$\text{U.T.} = T_c + B\,\rho \sin \phi' + k\,\rho \cos \phi' \cos (K - \mu + \lambda + 1{\cdot}002738\,\Delta T) - \Delta T$$

There are four such relations, one for each contact.

Example 9.32. Reduction formulae for local circumstances for the transit of 1960 November 7

(a) Calculation of the reduction formulae for local interior egress. The data are taken from the final approximation (examples 9.26 and 9.31).

u	$+\ 866''\cdot 2$	$\pi_\odot$	$8''\cdot 883$	$\sin \epsilon$	$+0\cdot 3978$	$\sin l_\oplus$	$+0\cdot 7119$
v	$+1875\cdot 3$	$uu' + vv'$	$1308\ 874$	$\cos \epsilon$	$+0\cdot 9175$	$\cos l_\oplus$	$+0\cdot 7023$
m	$2065\cdot 7$	F	$0\cdot 0244\ 32$	$\sin s_\odot$	$0\cdot 0047$	$G = m \sin s_\odot$	$9\cdot 7$

$$F = 3600\, \pi_\odot/(uu' + vv')$$

$+v \sin \epsilon \cos l_\oplus$	$+524\cdot 0$	$+v \cos \epsilon \cos l_\oplus$	$+1208\cdot 4$	$-v \sin l_\oplus$	$-1335\cdot 0$
$-u \cos \epsilon$	$-794\cdot 7$	$+u \sin \epsilon$	$+\ 344\cdot 6$		
$+G \sin \epsilon \sin l_\oplus$	$+\ \ 2\cdot 7$	$+G \cos \epsilon \sin l_\oplus$	$+\ \ 6\cdot 3$	$+G \cos l_\oplus$	$+\ \ 6\cdot 8$
sum	$-268\cdot 0$	sum	$+1559\cdot 3$	sum	$-1328\cdot 2$
$B = \text{sum} \times F$	$-6^s\cdot 55$	$C = \text{sum} \times F$	$+38^s\cdot 097$	$D = \text{sum} \times F$	$-32^s\cdot 451$
$k^2 = C^2 + D^2$	$2504\cdot 4$	k	$+50^s\cdot 04$	μ	$334^\circ\ 39'\cdot 1$
$\cot K = D/C$	$-0\cdot 8518\ 0$	K	$130^\circ\ 25'\cdot 5$	$K - \mu$	$155\ \ 46\cdot 4$

(b) Calculation of time of local interior egress on 1960 November 7 at the U.S. Naval Observatory from the reduction formula:

λ	$+77^\circ\ 03'\cdot 9$	λ^*	$+77^\circ\ 12'\cdot 9$	T_c	$19^h\ 10^m\ 40^s\cdot 8$
$\rho \sin \phi'$	$0\cdot 6249$	$K - \mu + \lambda^* = \theta$	$232\ \ 59\cdot 3$	$B \rho \sin \phi'$	$-\ \ 4\cdot 1$
$\rho \cos \phi'$	$0\cdot 7791$	$\cos \theta$	$-0\cdot 6020$	$k \rho \cos \phi' \cos \theta$	$-23\cdot 5$
				$\text{sum} = T_1$	$19\ \ 10\ \ 13\cdot 2$

The U.T. of contact is $19^h\ 10^m\ 13^s\cdot 2 - \Delta T\,(36^s) = 19^h\ 09^m\ 37^s\cdot 2$

Map curves

Curves pertaining to the exterior contacts of the transit are shown on the map published in the Ephemeris. The ephemeris times of exterior ingress and egress at a given place may be estimated from these curves to within a second.

Curves for interior contacts are omitted for the sake of clarity; the times of interior contacts may be inferred from the fact that the elapsed times between exterior and interior ingress and between interior and exterior egress differ little from their geocentric counterparts.

The curve labelled " Transit begins at sunrise—Transit begins at sunset " is the great circle having for pole the point at which Mercury is in the zenith at the time of geocentric exterior ingress. Similarly, the curve labelled " Transit ends at sunrise—Transit ends at sunset " is the great circle having for pole the point at which Mercury is in the zenith at the time of geocentric exterior egress.

If ϕ_0, λ_0^* are the geographic coordinates of the pole of a great circle, the coordinates ϕ, λ^* of a point on this circle satisfy the condition:

$$\tan \phi = -\cot \phi_0 \cos (\lambda_0^* - \lambda^*)$$

Values of ϕ corresponding to assumed values of λ^*, at suitable intervals between 0° and $\pm 180^\circ$, may be calculated until enough points are available for plotting the curve.

Example 9.33. *Curve of exterior ingress at sunrise or sunset*

At the time of geocentric exterior ingress, Mercury is in the zenith at the point whose geographic coordinates are $\phi_0 = -16° 37'$, $\lambda_0^* = +42° 34'$. (*A.E.* 1960, page 306).

$$\text{Take } \lambda^* = 120°, (\lambda_0^* - \lambda^*) = -77° 26', \cos(\lambda_0^* - \lambda^*) = +0{\cdot}2176$$
$$\cot\phi_0 = -3{\cdot}3509, \tan\phi = +0{\cdot}7292, \phi = +36° 06'$$

The curve of ingress or egress for a selected time T_1 is calculated from the relation provided by the corresponding reduction formula for local contacts. From:

$$T_1 = T_c + t$$

and

$$t = B\,\rho\sin\phi' + k\,\rho\cos\phi'\cos(K - \mu + \lambda^*)$$

it is found that the geocentric coordinates of points on the curve at T_1 must satisfy the relation:

$$\cos(K - \mu + \lambda^*) = (t - B\,\rho\sin\phi')/k\,\rho\cos\phi'$$

In this case, successive values of ϕ are assumed and the corresponding values of λ^* are computed. For small-scale maps it is not necessary to distinguish between λ^* and λ.

Example 9.34. *Curve of exterior ingress*

The data are taken from *A.E.* 1960, page 306.

T_c	h m s 14 34 33·4			Assumed ϕ	+40° 00′		
t	+36·6	B	s +41·17	$\rho\sin\phi'$	+0·6394	$t - B\rho\sin\phi'$	s 10·28
T_1	14 35 10·0	k	−29·30	$\rho\cos\phi'$	+0·7671	$k\,\rho\cos\phi'$	−22·48
		$K - \mu$	+72° 15′			$\cos(K - \mu + \lambda^*)$	−0·4573
		λ^*	+44° 58′ or +170° 32′			$K - \mu + \lambda^*$	±117° 13′
		λ	+44 49 or +170 23				

The second solution in this case corresponds to a point for which Mercury is below the horizon.

Alternative method

As for eclipses it is often easier, and less liable to error, to solve many of the transcendental equations that arise by numerical methods; a discriminant, which vanishes at the point required, is tabulated at suitable equal intervals of the independent variable (usually time) and the required value of the independent variable is then obtained by the standard techniques of inverse interpolation. This method is very suitable for the calculation of the geocentric phases of transits of Mercury, as large intervals of time (one hour) may be used.

Example 9.35. *The geocentric phases of the transit of Mercury on* 1960 *November* 7, *by the alternative method*

By calculations similar to those in example 9.26 values of m, L_1, L_2 are obtained for each hour covering the duration of the transit, $13^h - 21^h$; u', v' and associated quantities are not required. For the time of least angular distance values of g are taken from example 9.27.

E.T.	$m^2 - L_1^2$	δ	δ^2	δ^3	$m^2 - L_2^2$	
h						
13	+556 6263				+565 3747	
		−385 7823				
14	+170 8440		+113 1966		+179 6023	
		−272 5857		+3334		−272 5763
15	−101 7417		113 5300		− 92 9740	
		−159 0557		3294		
16	−260 7974		113 8594		−252 0210	
		− 45 1963		3287		
17	−305 9937		114 1881		−297 2083	
		+ 68 9918		3252		
18	−237 0019		114 5133		−228 2077	
		+183 5051		3228		
19	− 53 4968		114 8361		− 44 6945	
		+298 3412		+3226		+298 3490
20	+244 8444		+115 1587		+253 6545	
		+413 4999				
21	+658 3443				+667 1614	

Contact	Ingress exterior	Ingress interior	Egress interior	Egress exterior
Hour	14^h	14^h	19^h	19^h
$\delta^2_{0m} + \delta^2_{1m}$	+226 7277	+226 7263	+229 9943	+229 9954
$\delta^3_{\frac{1}{2}}$	+ 3334	+ 3332	+ 3221	+ 3226
p	0·5759 64	0·6094 03	0·1779 97	0·2114 40
B_2	−0·0610 57	−0·0595 08	−0·0365 79	−0·0416 83
B_3	−0·0031	−0·0043	+0·0079	+0·0080
E.T.	$14^h\ 34^m\ 33^s{\cdot}47$	$14^h\ 36^m\ 33^s{\cdot}85$	$19^h\ 10^m\ 40^s{\cdot}79$	$19^h\ 12^m\ 41^s{\cdot}18$

E.T.	m^2g^2	δ	δ^2
15^h	72 5444		
		−34 7153	
16	37 8291		+24 9213
		− 9 7940	
17	28 0351		+24 9078
		+15 1138	
18	43 1489		

m^2g^2 is a minimum in the interval $16^h - 17^h$, and $p = 0{\cdot}8931\ 2$

$B_2 = -0{\cdot}0238\ 6$

At the minimum $m^2g^2 = 27\ 8930$

$s = mg = 528''{\cdot}14$

Least angular distance is $8'\ 48''{\cdot}14$ at E.T. $16^h\ 53^m\ 35^s{\cdot}2$

After the times of the geocentric phases have been obtained, all other data required for the calculation of local contacts may be derived by interpolation, to the times of geocentric contacts, in the hourly tabulations.

10. OCCULTATIONS*

A. INTRODUCTION

Although there is no section of the Ephemeris devoted to lunar occultations, the provision of predictions to facilitate observation and the reduction of observations form a routine commitment of H.M. Nautical Almanac Office. A world-wide coverage of predictions was undertaken starting with the year 1937, and since 1943 the Office has been responsible for the collation, reduction, and discussion of observations. More recently a start has been made in providing predictions for occultations of radio sources.

For many years previously H.M. Nautical Almanac Office had endeavoured to further the increased observation, and the reduction, of lunar occultations. Prediction elements were first published in *The Nautical Almanac* for 1824, and predictions for Greenwich were added in the edition for 1834. [*The American Ephemeris* gave prediction elements and predictions for Washington from its introduction in 1855.] Several appendices (see section 7F.1), notably those to the editions for 1826, 1827 to 1833, 1836, and 1837, were devoted to methods and rules for computing visible and observed occultations. The importance of lunar occultations in determining the fluctuation in the motion of the Moon received new emphasis with the publication in 1919 of Brown's *Tables of the Motion of the Moon*, which provided a firm basis for the comparison of observation with theory. But, until the availability in 1940 of the *Catalog of* 3539 *Zodiacal Stars for the Equinox* 1950·0, the errors in the star positions placed a severe limitation on the accuracy obtainable. Up to 1942, the reductions were done either by the observers themselves or, for amateur observers, by the American Association of Variable Star Observers and by the computing section of the British Astronomical Association. The whole work of observation and reduction was organised, at Yale University, on a world-wide scale by Professor E. W. Brown, and later by Professor D. Brouwer, who published the annual analyses of the observations for the years 1923 to 1942. H.M. Nautical Almanac Office had contributed much to this programme by the provision of forms, tables, and instructions for the use of the B.A.A. computers, in addition to an increasingly large number of predictions. The work of prediction had been simplified by J. D. McNeile who had directed much of the B.A.A. work and who, in 1928, designed and built an " occultation machine " to provide preliminary times for any station; this machine was used in the Office until replaced in 1934 by the present machine, of which a full

*Many changes in scope and technique have been made since 1960 but only a few are noted.

description is given in *The Prediction and Reduction of Occultations*, a supplement to *The Nautical Almanac* for 1938. Many data, designed to simplify the onerous calculation of the apparent places of stars, were also published in these years in *The American Ephemeris*, which also supplied the very considerable basic data of the prediction elements. The method of reduction used was essentially that developed by R. T. A. Innes in 1922, though this was adapted in 1937 to machine calculation by L. J. Comrie, who introduced the concept of occultation reduction elements, designed to facilitate the labour of reduction; these were published for the years 1937 to 1942, until in fact the Office undertook to compute the reductions themselves.

L. J. Comrie was also mainly responsible for the increased interest in, and predictions for, occultations by planets and much work was done under his direction by the B.A.A. computing section. Brief notes on such planetary occultations are given in sub-section F; sub-sections A to E are concerned only with lunar occultations.

Details are given later of stations for which predictions are made, how they are distributed and where they are published. Prediction elements, which were published in *The American Ephemeris* prior to 1960, and apparent places of occulted stars, required in the reduction of observed occultations and published in *The Nautical Almanac* until 1960, have been omitted from the Ephemeris for various reasons: in the case of predictions because it would be inappropriate to include all stations, and because other publications have agreed to carry them for North America and the British Commonwealth; in the case of prediction elements because the programme itself calls for complete coverage, and can be expanded if necessary; and in the case of apparent places because observations of predicted occultations are reduced systematically in H.M. Nautical Almanac Office on punched-card machines, and individual reductions are not in fact used.

An observation of an occultation of a star by the Moon gives a relation between the positions of the star, the limb of the Moon, and the observer, and a series of such observations can be used to determine one of these positions provided the other two are sufficiently well determined. For the visual observations, the positions of the stars and the observers are assumed known with sufficient accuracy for the statistical investigation of the results to be used to determine the values of quantities connected with the Moon and its orbit. At present (1960) these investigations are confined to determining the difference between the observed and tabular positions of the Moon. On the other hand, the position of the Moon is known more accurately than are the positions of radio sources, and so the occultations in this case are mainly used for improving the positions of the sources.

Added note (1973). Since 1960 the occultation programme has been extended in several ways. Maps of the tracks and detailed predictions of grazing occultations are issued since the observations provide useful data about the latitude of the Moon and the profile of its limb. Special predictions for the occultations of X-ray sources, etc., observed from rockets and artificial satellites are prepared. In the reduction of occultation timings allowance is made for the irregularities of the Moon's limb; the corrections are calculated automatically from a digital representation of Watts' charts (see page 303).

References

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These papers include notes on the significance of observations of occultations as well as a discussion of occultations observed up to 1908.

Chauvenet, W. *A manual of spherical and practical astronomy*. Vol. 1, 549–591 5th ed. 1892, reprinted 1960.

These pages contain the fundamental formulae for the prediction and reduction of occultations of stars and planets by the Moon.

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This book describes a convenient graphical method for predictions.

Innes, R. T. A. Reduction of occultations of stars by the Moon. *A.J.*, **35**, 155–156, 1924; errata, E. W. Brown, *A.J.*, **39**, 96, 1929.

Comrie, L. J. The reduction of lunar occultations. *A.J.*, **46**, 61–67, 1937.

These two papers give the bases of modern methods of reduction.

H.M. Nautical Almanac Office. *The prediction and reduction of occultations*. 1937.

A supplement to *The Nautical Almanac* for 1938.

This booklet contains a full account of the methods used in H.M. Nautical Almanac Office in 1937, with detailed descriptions of the occultation machine and of the calculation and use of occultation reduction elements.

H.M. Nautical Almanac Office. Reduced observations of lunar occultations for the years 1943 to 1947. Appendix to *Greenwich Observations* 1939. 1952.

This also contains a full list of the annual discussions in the *Astronomical Journal* from 1923 onwards and lists of related papers. See also note on page 303.

B. PREDICTION

To ensure that all suitable occultations will be observed whenever possible, it is considered desirable to supply the observers with predictions. The information supplied in the prediction of an occultation has been chosen to be in the most convenient form for the observer, particularly in the case of reappearances, and consists of the approximate time and position angle of the disappearance or reappearance of the star at the limb of the Moon. The time is given in U.T. to one tenth of a minute and the position angle to the nearest degree. Annual lists of these predictions are prepared for about 80 central stations, together with longitude and latitude coefficients which enable observers in the vicinity of these stations to derive times for their own positions.

Limitations imposed on predictions

The stars for which predictions are made are confined to those of magnitude 7·5 and brighter contained in the *Catalog of* 3539 *Zodiacal Stars for the Equinox* 1950·0 (Z.C.) published in *A.P.A.E.*, **10**, part II, 1940. With a view to restricting predictions to those occultations of which observations can be made in suitable circumstances, the following limitations are imposed.

At the bright limb, disappearances are given only for stars of magnitude 4·5 or brighter, and reappearances for stars of magnitude 3·5 or brighter. At the dark limb, reappearances are given only for stars of magnitude 6·5 or brighter; hence, after full moon no predictions are given for stars fainter than magnitude 6·5.

Within 24 hours of new moon no predictions are given, while within 48 hours of new moon predictions are restricted to stars of magnitude 1·9 or brighter.

Within 24 hours of full moon, except during a total lunar eclipse when the appropriate dark-limb limitations are used, the limiting magnitude is 3·0, within 48 hours 5·5, and within 72 hours 6·5.

The star must be at least ten degrees above the horizon except for stars of magnitude 1·9 and brighter when the adopted limit for altitude is two degrees.

Predictions are given during daylight hours for stars of magnitude 1·9 and brighter. For other stars the earliest and latest times for which predictions are given correspond to the following depressions of the Sun.

Magnitude	Earliest	Latest
2·0 — 4·5	Sunset	6°
4·6 — 5·5	3°	9°
5·6 — 7·5	6°	12°

For grazing occultations the criteria for rejection or for the omission of longitude and latitude coefficients, which become unreliable near grazing conditions, depend on the quantity *kn* cos ψ which occurs in the prediction calculation and is explained below. The occultations are omitted completely when the value of *kn* cos ψ at the moment of occultation is numerically less than 0·030, unless the star is brighter than 1·5 when the limiting value is 0·015; the corresponding value for the omission of latitude and longitude coefficients is 0·060.

Preparation of the list of conjunctions

If at any instant a star is within an angular distance of the Moon of less than the sum of the Moon's horizontal parallax and semi-diameter, it must be occulted as seen from some point on the Earth. During any one year the same star may be occulted more than once, as the Moon will make twelve or thirteen passages along approximately the same path in the sky. The selection of conjunctions for which occultations are possible can be made by a variety of methods, as, for example, by plotting the path of the Moon on a chart on which the star positions are already plotted. The method described here has been chosen because of its suitability for use with punched-card machines.

The positions of the Z.C. stars for the equinox of 1950·0 are available on cards as are also the hourly values of the apparent right ascension, declination ($\delta_☾$), and horizontal parallax ($\pi_☾$) of the Moon. The selection of conjunctions is made twelve months at a time from July to June, to simplify the calculation of the apparent places of the stars. For an integral hour, let:

$$A = \delta_☾ + \delta\delta_☾ \pm (Z + \pi_☾)$$
$$B = \delta_☾ \mp (Z + \pi_☾)$$

where $(Z + \pi_☾)$ is always taken with the same sign as $\delta\delta_☾$, the hourly first difference of $\delta_{☾}$, and Z is a constant for any particular selection; a sufficiently large value of Z can be chosen to ensure that A and B are the extreme limits of declination for which stars, coming to conjunction in right ascension in the following hour, can be occulted. In practice, the selection is made by calculating $(A - \delta_{\star})$ and $(B - \delta_{\star})$ and retaining those stars for which the two quantities have opposite signs.

A crude preliminary selection is made first by combining the 1950·0 positions of the stars with a mean orbit of the Moon as represented by one actual revolution from 0^h to 24^h in right ascension. In this case the constant Z is made up of the maximum values of the Moon's semi-diameter, of the reduction of the star positions from 1950·0, and of the effects of using the mean orbit. The resulting list of stars, for which $(A - \delta_{\star})$ and $(B - \delta_{\star})$ have opposite signs, includes many stars that will not lead to eventual occultations.

For the stars in this preliminary list, mean places are calculated for the beginning of the year included in the period. The selection process is then repeated using these mean places for the stars and the actual hourly positions for the Moon; the value of Z used in this case is much smaller than in the preliminary selection, as the effects of reducing the mean places from 1950·0 to the beginning of the year concerned and of using a mean orbit are removed. Apparent places of the stars are then calculated for the conjunctions so selected. These are checked by differencing and by joining on to those calculated in the preceding year.

The conjunctions that violate the restrictions depending on the magnitude of the star and the phase of the Moon are deleted from the list at this stage. They could be omitted before the calculation of the apparent places, but their retention assists the checking of the apparent places themselves and helps to ensure that no conjunction is accidentally omitted. The next stage is to calculate Besselian elements for the remaining conjunctions.

Calculation of Besselian elements

The Besselian elements for occultations are similar to those for eclipses (see section 9). In occultations, the fundamental plane passes through the centre of the Earth and is perpendicular to the line joining the star and the centre of the Moon, i.e. to the axis of the shadow. The origin of the coordinates is the centre of the Earth; the axis of x is the intersection of the Earth's equator with the fundamental plane and is taken as positive towards the east; the axis of y is perpendicular to that of x and is taken as positive towards the north. The great distance of the star implies that the fundamental plane is perpendicular to the line joining the centre of the Earth to the star, and that the Moon's shadow is essentially a cylinder whose intersection with the fundamental plane is a circle of invariable size, its diameter being equal to that of the Moon. The coordinates of the centre of this circle, i.e. of the axis of the shadow, are denoted by x and y. The adopted unit of linear measurement is the Earth's equatorial radius.

The Besselian elements are given for one instant only, namely, the time of conjunction of the star and Moon in right ascension, when x is zero. They are:

T_0 = the U.T. of conjunction in right ascension
H = the Greenwich hour angle of the star at T_0
Y = y at T_0
x', y' = the hourly variations of x and y
$\alpha_\star, \delta_\star$ = the right ascension and declination of the star.

Example 10.1. *Illustration of the selection of stars*

It is not practicable to give a numerical illustration of the general process of selection, but the following shows portions of the selection process, including Z.C. 2833, the occultation of which on 1960 September 1 is used for later examples.

For the year 1960 July to 1961 June, the mean orbit adopted for the Moon, for comparison with the 1950·0 star positions, is the revolution for 1960 December 24^d 20^h to 1961 January 21^d 03^h and the value for Z is 1° 35′. A typical run of results from the punched card machines follows; the lines with no entry under " Z.C. No." refer to the Moon and asterisks indicate the selected stars.

Z.C. No.	Mag.	Date	R.A.	Dec.	$\delta\delta_☾$	$A - \delta_\star$	$B - \delta_\star$
		d h	h m s	° ′ ″	′ ″	° ′ ″	° ′ ″
		Jan. 16 10	19 20 52·461	−18 16 52·93	+3 01·83		
2830*	6·9		19 20 55·721	−17 17 36·05		+ 1 40 10·62	−3 35 42·55
2831	5·9		19 21 23·225	−27 57 51·21		+12 20 25·78	+7 04 32·61
2832	6·1		19 21 55·187	−29 24 27·75		+13 47 02·32	+8 31 09·15
2833*	7·0		19 22 12·339	−18 39 44·43		+ 3 02 19·00	−2 13 34·17
2834	5·0		19 22 14·006	−24 36 26·82		+ 8 59 01·39	+3 43 08·22

The stars indicated by asterisks, such as Z.C. 2830 and Z.C. 2833, are retained for further examination and their mean places for 1961·0 calculated. In the selection proper, these mean places are compared with the tabular values of the Moon's position and the value of Z becomes 21′ 24″. The section of the results for September corresponding to the above preliminary selection is as follows.

Z.C. No.	Mag.	Date	R.A.	Dec.	$\delta\delta_☾$	$A - \delta_\star$	$B - \delta_\star$
		d h	h m s	° ′ ″	′ ″	° ′ ″	° ′ ″
		Sept. 01 19	19 20 49·366	−17 42 15·77	+3 13·96		
2830*	6·9		19 21 33·826	−17 16 19·87		+ 0 58 46·81	−1 47 24·65
2833*	7·0		19 22 50·796	−18 38 26·89		+ 2 20 53·83	−0 25 17·63
		Sept. 01 20	19 23 21·896	−17 39 01·81	+3 21·37		
2846*	6·9		19 25 49·133	−18 26 22·32		+ 2 12 11·16	−0 34 08·77

For the retained stars, apparent places are calculated for each conjunction with the Moon. The apparent places for Z.C. 2833, magnitude 7·0, are listed below, together with reasons why certain conjunctions are eliminated from further consideration.

Date 1960	R.A.	Dec.	Date 1961	R.A.	Dec.
d	h m s	° ′ ″	d	h m s	° ′ ″
July 09*	19 22 50·28	−18 38 18·9	Jan. 16*	19 22 49·09	−18 38 21·9
Aug. 05†	50·43	18·5	Feb. 12*	49·59	22·0
Sept. 01	50·24	18·9	Mar. 12*	50·28	21·4
Sept. 29	49·80	19·7	Apr. 08*	51·08	20·0
Oct. 26	49·31	20·4	May 05*	51·91	17·9
Nov. 22	48·94	21·0	June 02*	52·68	15·6
Dec. 19‡	19 22 48·86	−18 38 21·5	June 29*	19 22 53·26	−18 38 14·0

* after full moon and before new moon; star too faint for reappearances.
† within 48 hours of full moon.
‡ within 48 hours of new moon.

The hourly values of the Moon's position are available in terms of E.T., so the Besselian elements are first calculated in terms of E.T. and later the time and hour angle are adjusted for the extrapolated difference between U.T. and E.T.

The formulae for x and y, the coordinates of the centre of the shadow on the fundamental plane, are:

$$x \sin \pi_{☾} = \cos \delta_{☾} \sin (\alpha_{☾} - \alpha_{\star})$$
$$y \sin \pi_{☾} = \sin \delta_{☾} \cos \delta_{\star} - \cos \delta_{☾} \sin \delta_{\star} \cos (\alpha_{☾} - \alpha_{\star})$$

These may be reduced, with sufficient accuracy for prediction purposes, to:

$$x = 15 \cos \delta_{☾} \frac{\alpha_{☾} - \alpha_{\star}}{\pi_{☾}} \qquad y = \frac{\delta_{☾} - \delta_{\star}}{\pi_{☾}} + 0{\cdot}000036\, x\, (\alpha_{☾} - \alpha_{\star}) \sin \delta_{\star}$$

in which $\alpha_{☾} - \alpha_{\star}$ is in seconds of time, and $\delta_{☾} - \delta_{\star}$ and $\pi_{☾}$ are in seconds of arc. The second term in y may be replaced by $+0{\cdot}0083\, x^2 \tan \delta_{\star}$, with a further error not exceeding 0·0002.

At the time of conjunction:

$$x = 0 \qquad y = Y = \frac{\delta_{☾} - \delta_{\star}}{\pi_{☾}}$$
$$x' = 15 \cos \delta_{☾} \frac{\alpha_{☾}'}{\pi_{☾}} \qquad y' = \frac{\delta_{☾}'}{\pi_{☾}} - Y \frac{\pi_{☾}'}{\pi_{☾}}$$

where x', y', $\alpha_{☾}'$, $\delta_{☾}'$, $\pi_{☾}'$ are the hourly variations of x, y, $\alpha_{☾}$ (in seconds of time) and $\delta_{☾}$, $\pi_{☾}$ (in seconds of arc). The hourly variations at the time of conjunction of the Moon's coordinates may be obtained with sufficient accuracy by interpolating the first difference to the time of conjunction.

As x and y are linear to the precision required x' and y' are constant, and it is quite simple to calculate most of the elements by two methods so that comparison of the two results provides a complete check. In the first method, the values of x, y are calculated for the integral hour T_1 before conjunction and the integral hour T_2 immediately following. If these values are designated x_1, y_1 and x_2, y_2 respectively, then:

$$x' = x_2 - x_1 \qquad y' = y_2 - y_1$$
$$T_E \text{ (time of conjunction in E.T.)} = T_1 - x_1/x'$$
$$Y = y_1 - x_1 y'/x'$$
$$H_E = \text{ephemeris sidereal time at } 0^h \text{ E.T.} + \text{sidereal equivalent of } T_E - \alpha_{\star}$$

In the second method, the values of $\alpha_{☾}$, $\delta_{☾}$, and $\pi_{☾}$ and their variations for the time T_E, as given by the first method, are obtained by interpolation in the Moon's hourly ephemeris. Y, x', and y' are derived from the direct formulae and are compared with the values already obtained; $\alpha_{☾}$ and $\alpha_{\star}$ are compared to check the value of T_E.

H_E is calculated independently, as a check, by replacing $\alpha_{\star}$ in the formula by $\alpha_{☾}$. The adjustment from E.T. to U.T. is made by putting:

$$T_0 = T_E - \Delta T \qquad H = H_E - 1{\cdot}002738\, \Delta T$$

Example 10.2. The calculation of Besselian elements

For Z.C. 2833 on 1960 September 1, the integral hour before conjunction is 19^h, so the first part of the calculation of the Besselian elements is performed for 19^h and 20^h E.T.

E.T.	19^h		20^h
	h m s		h m s
$\alpha_☾$	19 20 49·37		19 23 21·90
$\alpha_\star$	19 22 50·24		19 22 50·24
$\alpha_☾ - \alpha_\star$	− 120·87		+ 31·66
	° ′ ″		° ′ ″
$\delta_☾$	−17 42 15·8		−17 39 01·8
$\delta_\star$	−18 38 18·9		−18 38 18·9
$\delta_☾ - \delta_\star$	+ 3363·1		+ 3557·1
$\pi_☾$	3604·8		3605·3
15 cos $\delta_☾$	14·290		14·294
sin $\delta_\star$		− 0·3196	
x	−0·4791		+0·1255
x'		+ 0·6046	
$(\delta_☾ - \delta_\star)/\pi_☾$	+0·93295		+0·98663
second term	−0·00067		−0·00005
sum = y	+0·9323		+0·9866
y'		+ 0·0543	
Y		+ 0·9753	
$T_E = 19^h - x_1/x'$		$19^h{\cdot}7924$	
E.S.T. at 0^h E.T.		$22^h{\cdot}67711$	

$$H_E = 22^h{\cdot}67711 + 1{\cdot}002738\,(19^h{\cdot}7924) - 19^h{\cdot}38062 = 23^h{\cdot}1431$$

The elements are now calculated for time T_E, i.e. for $19^h{\cdot}7924$ E.T.; the subscript ☾ is omitted, so that all quantities without a subscript refer to the Moon.

	h m s	h			
α	19 22 50·23	19·38062		$1/\pi$	0·0002 7738
α'	152·52			$x' = 15 \cos\delta\,(\alpha'/\pi)$	+0·6047
	° ′ ″				
δ	−17 39 42·7	15 cos δ	14·293		
$\delta - \delta_\star$	+ 3516·2	π	3605·2	$Y = (\delta - \delta_\star)/\pi$	+0·9753
δ'	+ 196·1	π'	+ 0·5	$y' = (\delta' - Y\pi')/\pi$	+0·0543

The calculation of H_E is the same as above, $H_E = 23^h{\cdot}1431$.

The comparison of the values derived by the two methods is satisfactory.

$$\Delta T = 36^s = 0^h{\cdot}0100$$

$$T_0 = 19^h{\cdot}7824 \qquad H = 23^h{\cdot}1331$$
$$= 19^h\,46^m{\cdot}9 \qquad = -0^h\,52^m{\cdot}0$$

An examination at this stage of the values of Y enables many of the conjunctions which will just miss the Earth to be eliminated, but it is impossible to eliminate conjunctions that will be visible from portions of the Earth from which observation is unlikely, or to apply the remaining restrictions; these must wait until the examination is made using the occultation machine and, in critical cases, until the accurate predictions are done.

The same limitations are imposed on the predictions of occultations of planets and minor planets as are used for stars. The selection of possible conjunctions and the calculation of the Besselian elements are done separately. The position of the planet is calculated for the integral hours T_1 and T_2 and allowance made for the variation of the planet's position in the calculation of the elements.

Occultation maps and limiting parallels

Maps similar to those for solar eclipses (see section 9C) could be prepared for all occultations. A full account of the calculations for such maps is given by J. T. Foxell in *Memoirs of the British Astronomical Association*, **30**, 107–148, 1934.

Because of the large numbers of occultations it is impracticable to publish a map for each occultation, and the complete coverage provided by the predictions now makes them unnecessary; they are, in effect, replaced by the occultation machine. When elements were published it was usual to include limiting parallels of latitude, outside which the star could not be occulted. The method of obtaining these parallels is developed by Chauvenet in *Spherical and Practical Astronomy*, Vol. 1, page 561. His results are summarised, in a more convenient form below.

The northern and southern limiting parallels are denoted by ϕ_1 and ϕ_2.

$\cot N = y'/x'$ y' treated as positive $0 \leqslant N \leqslant 90^\circ$
$\cos \gamma_1 = Y \sin N + 0{\cdot}2725$ $\gamma_1 < 180^\circ$
$\cos \gamma_2 = Y \sin N - 0{\cdot}2725$ $\gamma_2 < 180^\circ$
$\sin \beta = \sin N \cos \delta_\star$ $\beta < 90^\circ$

Then, for northern declinations:

(1) If $\cos \gamma_2$ is greater than $\sin \beta$ $\phi_1 = \beta + \gamma_2$
(2) If $\cos \gamma_2$ is less than $\sin \beta$ and
(a) γ_1 is imaginary $\phi_1 = +90^\circ$
(b) $\cos \gamma_1$ is greater than $\sin \beta$ $\phi_1 = +90^\circ$
(c) $\cos \gamma_1$ is less than $\sin \beta$ $\phi_1 = 180^\circ - \beta - \gamma_1$
(3) If $\cos \gamma_2$ is greater than $-\sin N$ $\sin \phi_2 = \sin (N - \gamma_2) \cos \delta_\star$
(4) If $\cos \gamma_2$ is less than $-\sin N$ $\phi_2 = -(90^\circ - \delta)$
(5) If γ_2 is imaginary and
(a) $\cos \gamma_1$ is greater than $-\sin N$ $\phi_2 = -(90^\circ - \delta)$
(b) $\cos \gamma_1$ is less than $-\sin N$ $\sin \phi_2 = \sin (N - \gamma_1) \cos \delta_\star$

For southern declinations:

(1) If $\cos \gamma_1$ is less than $-\sin \beta$ $\phi_2 = \gamma_1 - \beta - 180^\circ$
(2) If $\cos \gamma_1$ is greater than $-\sin \beta$ and
(a) γ_2 is imaginary $\phi_2 = -90^\circ$
(b) $\cos \gamma_2$ is less than $-\sin \beta$ $\phi_2 = -90^\circ$
(c) $\cos \gamma_2$ is greater than $-\sin \beta$ $\phi_2 = \beta - \gamma_2$
(3) If $\cos \gamma_1$ is less than $\sin N$ $\sin \phi_1 = \sin (N + \gamma_1) \cos \delta_\star$
(4) If $\cos \gamma_1$ is greater than $\sin N$ $\phi_1 = 90^\circ + \delta_\star$
(5) If γ_1 is imaginary and
(a) $\cos \gamma_2$ is less than $\sin N$ $\phi_1 = 90^\circ + \delta_\star$
(b) $\cos \gamma_2$ is greater than $\sin N$ $\sin \phi_1 = \sin (N + \gamma_2) \cos \delta_\star$

Examination by means of the occultation machine

The occultation machine was designed for the double purpose of eliminating conjunctions for which predictions would not be given, and of obtaining preliminary times and position angles at all stations for which predictions are necessary. It is now primarily used for determining for each selected conjunction those stations for which detailed investigation of possible occultations is to be made on an electronic computer.

Regarding the star as a source of light at an infinite distance, the Moon casts a cylindrical shadow on the Earth; at any instant the star is occulted at all places within the area formed by the intersection of this shadow and the surface of the Earth, while an occultation is either just beginning or just ending at places on the edge of the shadow. In the machine, the Earth is represented by a globe and the Moon-star-shadow system by a cylindrical beam of light of the correct radius. The initial setting of the machine is for the time of conjunction and consists only of setting the Besselian elements on appropriate scales. The driving mechanism is connected to a clock face on which a zero reading is obtained at the time of conjunction, and which is graduated in both directions to enable differences in time from that of conjunction to be read to $0^{h}\cdot 01$. Provision is made for the globe and the shadow to be moved by the operation of one handle only, so that, when once set for the time of conjunction, the machine will continue to present an accurate picture of the actual circumstances; it thus enables the path of the shadow to be visualised, and the time intervals from conjunction to the disappearances and reappearances, and the corresponding position angles, to be read off for all stations in that path. Each station is represented accurately on the globe by a black dot on a white disk.

It is thus easy for the operator to eliminate conjunctions which give rise to occultations visible only from uninhabited portions of the Earth. Of the other limitations imposed on predictions only those for sunrise and sunset are likely to cause the complete elimination of a conjunction, but they must also be borne in mind by the operator for the individual stations. An adjustable grid is used to represent the different sunrise and sunset limits, and, in recording the times and position angles, the operator adds the letter S or SS to show that a station is near the limit. The limits for altitude and graze require a judgment that is learned only by experience; low occultations and near grazes are marked L or LL and G or GG respectively. The final decision for inclusion or exclusion of the prediction can only be made when the accurate calculation has been done. Conjunctions which give rise to no predictions are examined by a second operator to ensure that no accidental omission is made.

Calculation of accurate predictions

In the numerical prediction of occultations, the only mathematical condition to be satisfied is that, at the time of disappearance or reappearance, the line joining the star and the observer must be tangential to the limb of the Moon. The problem is that of finding the time at which the projection of the observer's

Example 10.3. The use of the occultation machine

The occultation machine was set up for the time of conjunction of Z.C. 2833 on 1960 September 1. As the magnitude of this star is 7·0, preliminary times and position angles are required for disappearances at stations where the Sun is at least 6° below the horizon. The machine showed that the shadow first touches the Earth in the North Atlantic Ocean, crosses the whole of Europe and the North African coast, and finally leaves the Earth near the Caspian Sea. The first portion of the track occurs in daylight, and it is only after the leading edge of the shadow has passed stations in the British Isles, western France, Spain and the north-west coast of Africa that the 6° limit for the Sun's depression is reached. A portion of the record of time intervals from conjunction and position angles for subsequent stations follows.

Station	Interval from conjunction	Position angle	
	h	°	
Brussels	−0·82	91	S
Utrecht	−0·79	90	S
Berne	−0·78	96	
Strasbourg	−0·76	94	
Frankfurt	−0·72	91	
Munich	−0·66	95	
Prague	−0·58	92	

The entry S indicates that the Sun is very close to the 6° limit. When accurate times were obtained the predictions for this star for Brussels and Utrecht were discarded.

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position on the fundamental plane lies on a circle of constant radius (k) whose centre (x, y) is the projection of the Moon's centre. The constant radius k is the Moon's radius, here expressed in units of the Earth's equatorial radius; the value adopted for predictions is 0·2725.

If ξ, η are the coordinates on the fundamental plane of the projection of the observer's station the condition for a disappearance or reappearance is:

$$(x - \xi)^2 + (y - \eta)^2 - k^2 = 0$$

At the preliminary time t_1 this equation is not in general satisfied and the following method is used to obtain a more accurate time. The basic data available for each prediction consist of the Besselian elements at the time of conjunction (T_0, H, Y, x', y', $\delta_\star$), the geocentric coordinates (λ, $\rho \sin \phi'$, $\rho \cos \phi'$) of the station, and the preliminary values of the time interval (t_1) from the time of conjunction and of the position angle (P_0).

Then, denoting quantities at time t_1 by the subscript 1:

$$h_1 = H - \lambda + t_s$$

where $t_s = 1{\cdot}002738\, t_1$ is the sidereal equivalent of t_1

$$x_1 = x' t_1 \qquad y_1 = Y + y' t_1$$

$$\xi_1 = \rho \cos \phi' \sin h_1 \qquad \eta_1 = \rho \sin \phi' \cos \delta_\star - Q_1 \sin \delta_\star$$

$$Q_1 = \rho \cos \phi' \cos h_1$$

$$f_1 = x_1 - \xi_1 \qquad g_1 = y_1 - \eta_1$$

$$f_1' = x' - h' Q_1 \qquad g_1' = y' - h' \xi_1 \sin \delta_\star$$

where f_1', g_1', h' are the hourly variations of f_1, g_1, h ; h', measured in radians/hour, is $(2\pi \times 1{\cdot}002738)/24 = 0{\cdot}2625$.

Thus $f'_1 = x' - 0{\cdot}2625\, Q_1 \qquad g'_1 = y' - 0{\cdot}2625\, \xi_1 \sin \delta_\star$

At the time of occultation, $t_1 + \Delta t$, if it is assumed that f'_1 and g'_1 are constant:

$$(f_1 + f'_1\, \Delta t)^2 + (g_1 + g'_1\, \Delta t)^2 - k^2 = 0$$

Since Δt is generally small, a first approximation is given by:

$$\Delta t_1 = \frac{k^2 - f_1^2 - g_1^2}{2\,(f_1 f'_1 + g_1 g'_1)}$$

and a second approximation by $\Delta t_1 + \Delta t_2$ where:

$$\Delta t_2 = -\,(\Delta t_1)^2\,\frac{f_1'^2 + g_1'^2}{2\,(f_1 f'_1 + g_1 g'_1)}$$

Defining for time t_1:

$$n_1^2 \equiv f_1'^2 + g_1'^2 \qquad k n_1 \cos \psi_1 \equiv f_1 f'_1 + g_1 g'_1$$

then at time $t_1 + \Delta t$:

$$f = f_1 + f'_1\, \Delta t \qquad g = g_1 + g'_1 \Delta t$$

$$kn \cos \psi = k n_1 \cos \psi_1 + n_1^2\, \Delta t$$

These quantities can be used to confirm that $f^2 + g^2 - k^2 = 0$, or if necessary to calculate a third correction to the time $t_1 + \Delta t_1 + \Delta t_2$ by:

$$\Delta t_3 = \frac{k^2 - f^2 - g^2}{2\, kn \cos \psi}$$

and to form the position angle of occultation by:

$$\sin P = -\, f/k \qquad \cos P = -\, g/k$$

The quantity n is a measure of the Moon's motion relative to the observer and ψ is the angle between the radius from the centre of the Moon to the star and the direction of this motion; ψ lies between 180° and 90° for disappearances and between 0° and 90° for reappearances. $kn \cos \psi$ is thus negative for disappearances and positive for reappearances; it is largest for central occultations but becomes small as ψ approaches 90°, or as the occultation becomes a graze. Any error in the position of the star or in the assumed limb of the Moon gives rise to an error in the predicted time of occultation that is inversely proportional to $kn \cos \psi$; the value of $kn \cos \psi$ has accordingly been used as the criterion for the rejection of grazing occultations.

In the determination of Δt, the second term Δt_2 is negligible in most cases but becomes appreciable as $kn_1 \cos \psi_1$ becomes small, that is as grazing conditions are approached. The preliminary time t_1 from the occultation machine is usually so close that the series for Δt converges rapidly; if a good preliminary time is not available, a second calculation, based on the time given by the first calculation, may be necessary.

After the formation of h_1, an intermediate step is inserted to check for altitude limit. This consists of forming:

$$s = \sin\,(\text{alt.}) - \sin 10° = \rho \sin \phi' \sin \delta_\star + Q_1 \cos \delta_\star - 0{\cdot}173$$

Those occultations for which s is negative are examined, and are rejected unless they are so near the limit that the subsequent correction to the time will make sufficient alteration, or are for stars brighter than magnitude 2·0. The doubtful cases are re-examined when the calculation is completed.

The position angle P, measured eastwards from the north point of the Moon's disk, is determined from the smaller of the two functions sin P and cos P, and compared, as a general check, with the value given by the occultation machine. The preliminary interval t_1 and the correction Δt, which are both in hours, are added to T_0 in hours, and the result converted to hours, minutes, and tenths of minutes.

Example 10.4. *The accurate prediction of an occultation*

During the examination of the star Z.C. 2833 on 1960 September 1 by means of the occultation machine, the disappearance of the star as seen from Strasbourg was found to take place $0^h.76$ before the time of conjunction in position angle 94°; the operator made no remarks as to Sun, altitude, or graze.

The Besselian elements are those for which the calculation has already been illustrated; both T_0 and H are used in hours. Also required are:

$$\sin\delta_\star = -0.3196 \qquad \cos\delta_\star = +0.9476$$

The station constants required for Strasbourg are:

$$\lambda = -0^h.518 \qquad \rho\sin\phi' = +0.7463 \qquad \rho\cos\phi' = +0.6628$$

$$h = H - \lambda + t_s$$
$$= 23^h.133 + 0^h.518 - 0^h.762 = 22^h.889$$

As a computational convenience h is taken as exact to $0^h.01$ and the preliminary time is adjusted to the appropriate three-decimal value. In this case the values to be used are:

$$h_1 = 22^h.89 \qquad t_1 = 0^h.759$$

$$\sin h_1 \; -0.2865 \qquad Q_1 \; +0.6350$$
$$\cos h_1 \; +0.9581$$

$s = \sin(\text{alt.}) - \sin 10° = +0.190$ This is satisfactory.

$$x_1 \; -0.4589 \qquad Y \; +0.9753 \qquad y_1 \; +0.9341$$
$$\xi_1 \; -0.1899 \qquad y't_1 \; -0.0412 \qquad \eta_1 \; +0.9101$$
$$f_1 = x_1 - \xi_1 \; -0.2690 \qquad g_1 = y_1 - \eta_1 \; +0.0240$$
$$x' \; +0.6046 \qquad y' \; +0.0543$$
$$-0.2625\,Q_1 \; -0.1667 \qquad -0.2625\,\xi_1\sin\delta_\star \; -0.0159$$
$$\text{sum} = f' \; +0.438 \qquad \text{sum} = g' \; +0.038$$

$$kn_1\cos\psi_1 = -0.117 \qquad \Delta t_1 = \frac{k^2 - f_1^2 - g_1^2}{2kn_1\cos\psi_1} = -0^h.006 \qquad \Delta t_2 = 0.000$$

For $\Delta t = \Delta t_1 + \Delta t_2 = -0^h.006$ $\qquad T_0 = 19^h.782$

$$f = -0.2716 \qquad t_1 = -0.759$$
$$g = +0.0238 \qquad \Delta t = -0.006$$
$$f^2 + g^2 - k^2 = +0.00008 \qquad T = 19.017$$
$$kn\cos\psi = -0.118 \qquad = 19^h\,01^m.0$$

The position angle is obtained from $\cos P = -0.0238/k$, giving $P = 95°$.

Preliminary time without the occultation machine

Of the various methods that can be devised for obtaining preliminary times the simplest is probably a graphical method. The outline of the projection of the Moon's disk on the fundamental plane is represented by a circle of radius 0.2725 units and the projections of the positions of the station are plotted for times at integral hours from the time of conjunction. If a point inside the circle is joined to one outside by a straight line, the point where this line intersects the circle gives a sufficiently good approximation to the projection of the station at the time of disappearance or reappearance. The preliminary time is obtained by assuming that the projection of the station moves uniformly along the straight line joining the two plotted values.

If the subscript o is used to denote the values of the functions at the time of conjunction (T_0) then:

$$\xi_0 = \rho \cos \phi' \sin (H - \lambda) \qquad Q_0 = \rho \cos \phi' \cos (H - \lambda)$$
$$f_0 = -\xi_0 \qquad g_0 = Y - \rho \sin \phi' \cos \delta_\star + Q_0 \sin \delta_\star$$
$$f_0' = x' - 0{\cdot}2625 Q_0 \qquad g_0' = y' - 0{\cdot}2625\, \xi_0 \sin \delta_\star$$
$$f_0'' = +0{\cdot}069\, \xi_0 \qquad g_0'' = -0{\cdot}069\, Q_0 \sin \delta_\star$$
$$f_0''' = +0{\cdot}018\, Q_0 \qquad g_0''' = +0{\cdot}018\, \xi_0 \sin \delta_\star$$

An inspection of the values of f_0, f_0' and g_0, g_0' will usually indicate which integral hours from T_0 are necessary to obtain points outside and inside the circle for the required phase. If the times are T_1 and $T_1 + 1^h$ and the distance between the points is d, and if the distance from the point at T_1 to the intersection of the line and circle is d_1, the preliminary time is $T_1 + d_1/d$ hour. This time may be used as a starting time for the more precise calculations already illustrated. The approximate position angle can be obtained from the diagram by measurement with a protractor. Sufficiently accurate values for plotting purposes are obtained from a calculation to three decimals.

Example 10.5. Direct evaluation of a preliminary time

A preliminary time for the disappearance of Z.C. 2833 on 1960 September 1 at Strasbourg is found as follows. Details of the initial calculation are not given.

At the time of conjunction, T_0:

$$H - \lambda = 23^h{\cdot}651 \qquad \sin (H - \lambda)\ -0{\cdot}091 \qquad \xi_0\ -0{\cdot}060$$
$$\cos (H - \lambda)\ +0{\cdot}996 \qquad Q_0\ +0{\cdot}660$$

f_0	+0·060	g_0	+0·057
f_0'	+0·431	g_0'	+0·049
f_0''	−0·004	g_0''	+0·015
f_0'''	+0·012	g_0'''	+0·000

An inspection of these figures indicates that the disappearance will occur between $T_0 - 1^h$ and T_0.

For time $T_0 - 1^h$, using Taylor's series:

$$f = -0{\cdot}375 \qquad g = +0{\cdot}015$$

The points (f_0, g_0) and (f, g) were plotted on squared paper, with a scale of 1 inch to 0·1 of the Earth's equatorial radius. The distance between the points was 4·37 inches, and the distance from the point for ($T_0 - 1^h$) to the intersection of the line joining them with the circle representing the Moon's disk was 1·02 inches. The preliminary time is therefore $T_0 - 1^h + \frac{1^h{\cdot}02}{4{\cdot}37} = 18^h{\cdot}78 + 0^h{\cdot}23 = 19^h{\cdot}01$, which may be compared with the time from the occultation machine of $T_0 - 0^h{\cdot}76 = 19^h{\cdot}02$ or the time of $19^h{\cdot}017$ derived from the precise calculation. The position angle measured by protractor is 94°.

Alternative methods of prediction

Another method, suitable for an electronic computer, consists of taking the time of conjunction as initial time, assuming that f_0' and g_0' are constants, and solving the quadratic equation:

$$(f_0 + f_0' \Delta t)^2 + (g_0 + g_0' \Delta t)^2 - k^2 = 0$$

by formal methods, similar to those used in eclipses (see section 9C). The two values of Δt, the earlier corresponding to disappearance and the later to reappearance, are then used to give initial times for further calculations. Considerable care is required, since the non-linearity of f' and g' may make the difference between a " graze " and a complete miss; in the latter case the time of closest approach, given by:

$$\Delta t = -(f_0 f_0' + g_0 g_0')/n_0^2$$

may be used as the new starting time for both phases.

The alternative method used for eclipses has much to commend it, if no other method of obtaining approximate times is available. Values of f, g for the five or six integral hours of E.T. bracketing both phases of the occultation are calculated; $f^2 + g^2 - k^2$ is formed for each hour and the times of disappearance and reappearance found by inverse interpolation. Only one set of values of x, y need be calculated for each conjunction of star and Moon, as for the occultation reduction elements that were given in *The Nautical Almanac* from 1937 to 1942; Besselian elements for the time of conjunction are not required. ξ, η have to be

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Example 10.6. *Alternative method of prediction, using Besselian elements*

For the purposes of illustration, the times of disappearance and reappearance of Z.C. 2833 on 1960 September 1 at Strasbourg are calculated by the alternative method, using the Besselian elements. An interval of 30 minutes is used, and four decimals are retained, though three would suffice.

T_0	$19^h{\cdot}7824$	$\rho \sin \phi'$	$+0{\cdot}7463$	$\sin \delta_\star$	$-0{\cdot}3196$
		$\rho \cos \phi'$	$+0{\cdot}6628$	$\cos \delta_\star$	$+0{\cdot}9476$
H	$23{\cdot}133$	x'	$+0{\cdot}6046$	y'	$+0{\cdot}0543$
λ	$-0{\cdot}518$				
$h = H - \lambda$	$23{\cdot}651$	$\sin h$	$-0{\cdot}0912$	$\cos h$	$+0{\cdot}9958$
		ξ_0	$-0{\cdot}0604$	Q_0	$+0{\cdot}6600$
Y	$+0{\cdot}9753$				
$-\rho \sin \phi' \cos \delta_\star$	$-0{\cdot}7072$	$\xi_0 \sin \delta_\star$	$+0{\cdot}0193$	$Q_0 \sin \delta_\star$	$-0{\cdot}2109$
		f_0	$+0{\cdot}0604$	g_0	$+0{\cdot}0572$

For $t = 0^h{\cdot}5$ $(1 - \cos t_s) = 0{\cdot}0086$ $\sin t_s = 0{\cdot}1309$
$1{\cdot}0$ $0{\cdot}0343$ $0{\cdot}2595$

t	f		g		$f^2 + g^2 - k^2$		
h							
$-1{\cdot}0$	$-0{\cdot}3751$		$+0{\cdot}0151$		$+0{\cdot}0667$		
		$+2191$		$+192$		-1154	
$-0{\cdot}5$	$-\ {\cdot}1560$		${\cdot}0343$		$-\ {\cdot}0487$		$+968$
		2164		229		$-\ 186$	
$0{\cdot}0$	$+\ {\cdot}0604$		${\cdot}0572$		$-\ {\cdot}0673$		948
		2155		264		$+\ 762$	
$+0{\cdot}5$	${\cdot}2759$		${\cdot}0836$		$+\ {\cdot}0089$		$+953$
		$+2158$		$+301$		$+1715$	
$+1{\cdot}0$	$+0{\cdot}4917$		$+0{\cdot}1137$		$+0{\cdot}1804$		

Inverse interpolation (section 16B) gives:

disappearance	$-1^h{\cdot}0$,	$p = 0{\cdot}471$,	$B_2 = -0{\cdot}0623$,	$T = 19^h{\cdot}018$
reappearance	$0{\cdot}0$,	$p = 0{\cdot}926$,	$B_2 = -0{\cdot}0171$,	$T = 20{\cdot}245$

The reappearance is not predicted because it occurs at the bright limb of the Moon and the star is faint.

calculated for each station. The interval of one hour, in which the Earth rotates through more than one-quarter of a radian, is very large and the method is practicable only to the low precision required for predictions.

If Besselian elements are available f, g can be calculated directly from the values at the time of conjunction by the rigorous formulae:

$$f_t = f_0 + x't + \xi_0 (1 - \cos t_s) - Q_0 \sin t_s$$
$$g_t = g_0 + y't - \sin \delta_\star \{ Q_0 (1 - \cos t_s) + \xi_0 \sin t_s \}$$

where t is the interval from conjunction, and t_s is its sidereal equivalent. These formulae may be used to calculate f, g and thus $f^2 + g^2 - k^2$ for integral hours before or after conjunction; inverse interpolation will then give the times of disappearance and reappearance. If approximate times are known these quantities may be tabulated at smaller intervals of time, thus making the inverse interpolation much easier.

It must be emphasised that methods suitable for a single prediction are not necessarily the best for the whole programme, which may well include over 8000 predictions a year; moreover, the many restrictive conditions (of sun, graze, altitude) limit the scope of general methods here described.

Longitude and latitude coefficients

The longitude and latitude coefficients, designated a and b, are the rates of change of time of occultation with the observing position; they enable observers in the neighbourhood of a station for which predictions are available to derive predictions for their own stations. They are determined, in minutes of time for each degree of longitude and latitude, from the formulae:

$$a = -\frac{K}{kn \cos \psi} (f Q + g \xi \sin \delta_\star)$$

$$b = -\frac{K}{kn \cos \psi} \{ C^2 \rho \sin \phi' (f \sin h - g \sin \delta_\star \cos h) - SC \rho \cos \phi' g \cos \delta_\star \}$$

where $K = 60^3 \sin 1'' = 1{\cdot}047$ is the numerical conversion factor necessary to express a, b in minutes of time per degree (instead of in hours per radian); $S = \rho \sin \phi' / \sin \phi$ and $C = \rho \cos \phi' / \cos \phi$ are tabulated in table 2.8. They may each be taken as unity in the above expressions as the maximum error that can be caused by this simplification is $0^m{\cdot}03$. The values of f, g, $kn \cos \psi$ used must be those at the time of occultation and not at a preliminary time.

These coefficients are based on linear variations and although they give good results for short distances, their inadequacy increases rapidly with distance particularly if $kn \cos \psi$ is small. For this reason they are not provided when $kn \cos \psi$ is numerically less than $0{\cdot}060$.

Use of a and b coefficients

An observer, whose position is $\Delta\lambda$ degrees west and $\Delta\phi$ degrees north of a standard station for which predictions and coefficients are given, can find times for his own position from the simple formula:

$$\text{approximate time} = \text{predicted time for station} + a\,\Delta\lambda + b\,\Delta\phi$$

Example 10.7. Calculation of a and b coefficients

From the precise calculation of the prediction of the disappearance of Z.C. 2833 at Strasbourg on 1960 September 1 the following quantities are obtained.

$\rho \sin \phi'$	$+0.746$	$\sin h$	-0.286	$\sin \delta_\star$	-0.320
$\rho \cos \phi'$	$+0.663$	$\cos h$	$+0.958$	$\cos \delta_\star$	$+0.948$
f	-0.272	ξ	-0.190	$kn \cos \psi$	-0.118
g	$+0.024$	Q	$+0.635$	$-K/kn \cos \psi$	$+8.9$

From table 2.8

S	$+0.995$	C	$+1.002$

whence

$C^2 \rho \sin \phi'$	$+0.749$	$SC \rho \cos \phi'$	$+0.661$	
$g \sin \delta_\star$	-0.008	$fQ + \xi g \sin \delta_\star$	-0.171	$a = -1^m.5$
$g \cos \delta_\star$	$+0.023$	$f \sin h - g \sin \delta_\star \cos h$	$+0.085$	

$C^2 \rho \sin \phi' (f \sin h - g \sin \delta_\star \cos h) - SC \rho \cos \phi' g \cos \delta_\star$ $+0.048$ $b = +0.4$

(The same value would be obtained if the factors C^2 and SC were omitted.)

For distances up to 200 miles the error in the times so obtained is unlikely to be greater than one minute, but it may rise to two minutes for distances of 300 miles.

A partial allowance for the neglected higher-order terms may be made by applying to the times calculated by using the above formula the correction $\pm 0^m.075 \cos^2 \phi \, (\Delta\phi)^2$; this, which is a mean value of the largest of the second-order terms, is positive for disappearances and negative for reappearances. With this correction times are rarely in error by more than one minute for distances up to about 250 miles, and for shorter distances the accuracy of the formula will be improved. The correction is tabulated below.

Additional correction for latitude

The correction ($= 0^m.075 \cos^2 \phi \, (\Delta\phi)^2$) to the approximate time of occultation is: to be *added* for a disappearance but is to be *subtracted* for a reappearance

$\Delta\phi$	Latitude of station 0°	10°	20°	30°	40°	50°	60°
°	m	m	m	m	m	m	m
1	0.1	0.1	0.1	0.1	0.0	0.0	0.0
2	0.3	0.3	0.3	0.2	0.2	0.1	0.1
3	0.7	0.7	0.6	0.5	0.4	0.3	0.2
4	1.2	1.2	1.1	0.9	0.7	0.5	0.3
5	1.9	1.8	1.7	1.4	1.1	0.8	0.5
6	3	3	2.4	2.0	1.6	1.1	0.7
7	4	4	3	3	2.2	1.5	0.9
8	5	5	4	4	3	2.0	1.2

When the observer lies between two standard stations, for both of which a and b coefficients are given, still greater accuracy can be obtained by working from the nearer station and using modified coefficients. The modification is made by assuming that the coefficients vary linearly with latitude and taking values of a, b appropriate to the mean of the latitudes of the nearer standard station and the observer. If a_1, b_1, ϕ_1, and a_2, b_2, ϕ_2 refer to the nearer and further standard stations respectively and ϕ is the observer's latitude, then the modified coefficients are:

$$a = a_1 + q\,(a_2 - a_1) \qquad b = b_1 + q\,(b_2 - b_1)$$
$$\text{where } q = \tfrac{1}{2}(\phi - \phi_1)/(\phi_2 - \phi_1)$$

The times so obtained are generally within one minute for distances up to 300 miles. This method should not be used if the difference of latitude between the two standard stations is less than one degree.

Example 10.8. *The use of a and b coefficients*

(i) An observer at Besançon who wished to observe the disappearance of Z.C. 2833 on September 1 would find the approximate time of the occurrence from the time and *a, b* coefficients published for Strasbourg.

$$\Delta\lambda \ +1^\circ{\cdot}78 \qquad \Delta\phi = -1^\circ{\cdot}33$$
$$a \ -1^{\mathrm{m}}{\cdot}5 \qquad b = +0^{\mathrm{m}}{\cdot}4$$

	h m
Predicted time at Strasbourg	19 01·0
$a\,\Delta\lambda$	−2·7
$b\,\Delta\phi$	−0·5
Correction from table	0·0
Approximate time at Besançon	18 57·8

(ii) The Boyden Station at Bloemfontein ($\lambda = -26^\circ{\cdot}40$, $\phi = -29^\circ{\cdot}04$) may derive predictions from those for Cape ($\lambda_2 = -18^\circ{\cdot}47$, $\phi_2 = -33^\circ{\cdot}93$) and Johannesburg ($\lambda_1 = -28^\circ{\cdot}07$, $\phi_1 = -26^\circ{\cdot}18$), the nearer station being the latter.

$$\Delta\lambda = \lambda - \lambda_1 = +1\overset{\circ}{\cdot}67 \qquad \Delta\phi = \phi - \phi_1 = -2\overset{\circ}{\cdot}86$$
$$\phi_2 - \phi_1 = -7{\cdot}75 \qquad q = \tfrac{1}{2}(-2{\cdot}86)/(-7{\cdot}75) = +0{\cdot}18$$

For the disappearance of Z.C. 1467 on 1960 April 7 the information given for Cape and Johannesburg is:

	U.T.	*a*	*b*
	h m	m	m
Cape	19 19·0	−2·2	−1·0
Johannesburg	19 43·2	−3·3	+0·9

$$a = -3^{\mathrm{m}}{\cdot}3 + 0{\cdot}18 \times 1^{\mathrm{m}}{\cdot}1 \qquad b = +0^{\mathrm{m}}{\cdot}9 + 0{\cdot}18 \times (-1^{\mathrm{m}}{\cdot}9)$$
$$= -3^{\mathrm{m}}{\cdot}1 \qquad = +0^{\mathrm{m}}{\cdot}6$$

	h m
Predicted time at Johannesburg	19 43·2
$a\,\Delta\lambda$	− 5·2
$b\,\Delta\phi$	− 1·7
Approximate time at Bloemfontein	19 36·3

Linking

The predicted times and the longitude and latitude coefficients may be checked by " linking " the results for standard stations comparatively near to each other. For linking, the results of the predictions for two stations are combined so that the second-order terms will tend to cancel out; this enables the check to be used for stations separated by up to 450 miles. If the subscripts 1, 2 are used to denote the two standard stations, then the criterion adopted is that:

$$(T_1 - T_2) - \tfrac{1}{2}\,(a_1\,\Delta\lambda + b_1\,\Delta\phi + a_2\,\Delta\lambda + b_2\,\Delta\phi) = 0 \text{ (approximately)}$$

where $\Delta\lambda = \lambda_1 - \lambda_2$ and $\Delta\phi = \phi_1 - \phi_2$. In practice no discrepancy greater than $0^m.3$ is acceptable without special investigation and explanation. The criterion used is a special case of the procedure described above, applied to the point midway between the two standard stations.

Stations *

A list of the stations for which predictions are made and the periodicals in which they are published is given below; some predictions are supplied in manuscript direct to the observatories, and for one station predictions are calculated for checking purposes only.

List of standard stations for occultation predictions

Station	Longitude	Latitude	Publication
	°	°	
Vancouver*	+123·100	+49·500	*Sky and Telescope*
	+121·000	+42·500	*Sky and Telescope*
	+120·000	+36·000	*Sky and Telescope*
Edmonton*	+113·075	+53·533	*Sky and Telescope*
	+109·000	+34·000	*Sky and Telescope*
Denver	+104·950	+39·677	*Sky and Telescope*
	+ 98·000	+31·000	*Sky and Telescope*
	+ 91·000	+40·000	*Sky and Telescope*
	+ 85·000	+33·000	*Sky and Telescope*
Toronto*	+ 79·400	+43·663	*Sky and Telescope*
Washington	+ 77·065	+38·920	*Sky and Telescope*
Jamaica	+ 76·775	+18·008	Manuscript
Montreal*	+ 73·575	+45·505	*Sky and Telescope*
	+ 72·500	+42·500	*Sky and Telescope*
Santiago	+ 70·550	−33·397	*Anuario* (Santiago)
Cordoba	+ 64·200	−31·422	Manuscript
La Plata	+ 57·925	−34·908	*Revista Astronomica*
Rio de Janeiro	+ 43·225	−22·895	Manuscript
Las Palmas	+ 15·425	+28·100	*Almanaque Nautico* †
Lisbon	+ 9·175	+38·708	Manuscript
Santiago de Compostela	+ 8·550	+42·875	*Almanaque Nautico*
Dublin	+ 6·350	+53·387	Manuscript
San Fernando	+ 6·200	+36·462	*Almanaque Nautico*
Madrid	+ 3·675	+40·408	*Almanaque Nautico*; *Anuario* (Madrid)
Edinburgh	+ 3·175	+55·925	*Handbook B.A.A.*
Almeria	+ 2·475	+36·838	*Almanaque Nautico*
Greenwich	0·000	+51·477	*Handbook B.A.A.*
Tortosa	− 0·500	+40·820	*Almanaque Nautico*
Toulouse	− 1·475	+43·612	*Connaissance des Temps*
Paris	− 2·325	+48·837	*Connaissance des Temps*
Brussels	− 4·350	+50·798	*Annuaire* (Brussels); *Hemel en Dampkring*
Utrecht	− 5·125	+52·087	*Hemel en Dampkring*
Berne	− 7·425	+46·953	*Der Sternenhimmel*

* Predictions for the brighter stars are also given in *The Observer's Handbook* published by the Royal Astronomical Society of Canada.

*Many changes have been made in this list since 1960.

†*Almanaque Nautico* was renamed *Efemérides Astronómicas* in 1961.

List of standard stations for occultation predictions—(*continued*)

Station	Longitude	Latitude	Publication
	°	°	
Turin	− 7·775	+45·038	*Annuario Astronomico* (Trieste)
Strasbourg	− 7·775	+48·583	*Connaissance des Temps*
Frankfurt am Main	− 8·650	+50·117	*Ast. Nach.*; *Kalender für Sternfreunde*
Munich	− 11·600	+48·147	*Ast. Nach.*; *Kalender für Sternfreunde*
Rome	− 12·450	+41·922	*Annuario Astronomico* (Trieste)
Copenhagen	− 12·575	+55·687	*Nordisk Astronomisk Tidsskrift*
Berlin	− 13·100	+52·407	*Ast. Nach.*; *Kalender für Sternfreunde*
Trieste	− 13·775	+45·643	*Annuario Astronomico* (Trieste)
Prague	− 14·400	+50·078	*Hvezdarska Rocenka*
Vienna	− 16·400	+48·212	*Himmelskalender für Österreich*
Poznan	− 16·875	+52·397	*Rocznik Astronomiczny*
Cape	− 18·475	−33·933	*Handbook B.A.A.*; *Handbook A.S.S.A.*
Budapest	− 18·975	+47·500	Manuscript
Cracow	− 19·950	+50·065	*Rocznik Astronomiczny*
Belgrade	− 20·525	+44·802	*Bulletin* (Belgrade)
Warsaw	− 21·025	+52·218	*Rocznik Astronomiczny*
Athens	− 23·725	+37·972	Manuscript
	− 24·000	−30·000	Checking station
Lvov	− 24·025	+49·833	Supplement to A.E. of U.S.S.R.
Riga	− 24·125	+56·952	Supplement to A.E. of U.S.S.R.
Helsinki	− 24·950	+60·155	Manuscript
Johannesburg	− 28·075	−26·182	*Handbook B.A.A.*; *Handbook A.S.S.A.*
Luanshya	− 28·400	−13·127	*Handbook A.S.S.A.*
Leningrad	− 30·325	+59·918	Supplement to A.E. of U.S.S.R.
Kiev	− 30·500	+50·453	Supplement to A.E. of U.S.S.R.
Odessa	− 30·750	+46·477	Supplement to A.E. of U.S.S.R.
Helwan	− 31·350	+29·858	Manuscript
Yavne	− 34·725	+31·815	Manuscript
Kharkov	− 36·225	+50·003	Supplement to A.E. of U.S.S.R.
Moscow	− 37·575	+55·755	Supplement to A.E. of U.S.S.R.
Abastuman	− 42·825	+41·755	Supplement to A.E. of U.S.S.R.
Kazan	− 48·825	+55·838	Supplement to A.E. of U.S.S.R.
Kitab	− 66·875	+39·133	Supplement to A.E. of U.S.S.R.
Dushanbe	− 68·775	+38·558	Supplement to A.E. of U.S.S.R.
Tashkent	− 69·300	+41·327	Supplement to A.E. of U.S.S.R.
Hyderabad	− 78·450	+17·432	Manuscript
Tomsk	− 84·950	+56·468	Supplement to A.E. of U.S.S.R.
Irkutsk	−104·350	+52·278	Supplement to A.E. of U.S.S.R.
Perth	−115·850	−31·953	Manuscript
Zô-Sè	−121·175	+31·097	Manuscript
Kyoto	−135·800	+34·995	Manuscript
Tokyo	−139·550	+35·673	Manuscript
Melbourne	−144·975	−37·832	*Handbook B.A.A.*; Supplement to *J.A.S. Victoria*
Sydney	−151·200	−33·862	*Handbook B.A.A.*; Supplement to *J.A.S. Victoria*
Dunedin	−170·500	−45·873	*Handbook B.A.A.*
Auckland	−174·775	−36·880	Manuscript
Wellington	−174·775	−41·285	*Handbook B.A.A.*

C. REDUCTION

An unreduced observation of an occultation consists of the time, the position of the observer, and the star name or Z.C. number. This information gives a relationship between the positions of the star, the limb of the Moon, and the observer, at the time of observation; but it is not in a form suitable for combination with other observations made from different places at different times, and has to be *reduced* to a more convenient form by the elimination of the place of observation. A convenient form in which to express this relation, for subsequent combination of observations, is to reduce each observation so that it yields the excess of the apparent distance of the star from the centre of the Moon over the adopted apparent semi-diameter of the Moon, together with the position angle of the star measured from the direction of the Moon's motion. The method of reduction to be described is adapted primarily to the determination of the difference between the observed and tabular values of the Moon's mean longitude and is in principle the method used for the reduction of all observations since 1943. The principal quantities are, however, obtained in a form suitable for other investigations; they may be expressed directly in terms of the errors in the adopted semi-diameter and other constants, and a solution made for the particular quantity or quantities required.

The reduction is illustrated by a fictitious example for 1960. No reductions have yet been performed using the improved lunar ephemeris, nor have corrections for the irregularities of the Moon's limb been applied systematically as it was decided to wait for the completion of the survey of the marginal zone of the Moon by Dr. C. B. Watts at the U.S. Naval Observatory. Later experience may show that it is desirable to make minor alterations to the methods used, although these changes are more likely to be in the method of analysis; in particular it may prove possible to introduce a system of weights in the observational equations.

It is clear that the process of reduction must be very similar to that of prediction. Besselian geometry is again used, but the coordinates are calculated more precisely; as a legacy from the period of " hand " calculation the unit of linear measurement, for which the equatorial radius of the Earth is used in predictions, is taken as the radius of the Moon. The value used for k, the ratio of the radius* of the Moon to that of the Earth, is 0·2724 953 and is the value adopted by R. T. A. Innes (*A.J.*, **35**, 155, 1924) for occultation reductions.

An advantage of the method used is that the quantities derived can be separated into those which depend on the position of the observer and those which depend on the position of the Moon; thus the quantities depending on the Moon's position can be derived for all the observations at the same time on the punched-card machines.

Preparation of initial information

The observations for one " year " are usually reduced at a time, the " year " consisting of an exact number of lunations, either twelve or thirteen, chosen to fit

*0·2725 026 used for 1968 onwards.

most closely to the calendar year. The initial information consists of the original lists of observations from the observers, and the punched cards containing the lunar ephemeris, the Besselian elements, and the predictions. Where possible the lists of observations are first compared with predictions and with lists of observations from neighbouring stations; this often enables recording errors, such as the wrong number for a star or a time recorded an hour out, to be corrected before the actual calculations are begun. The observations are then recorded on punched cards, permanent codes being used for station, telescope, observer, and remarks. The cards containing the Besselian elements of those conjunctions at which occultations have been observed are then selected.

Reduction of observer's position

The coordinates (ξ, η) of the position of the observer projected on the fundamental plane, in units of the radius of the Moon, are:

$$\xi = (\rho \cos \phi' \sin h)/k$$
$$\eta = (\rho \sin \phi' \cos \delta_\star - \rho \cos \phi' \sin \delta_\star \cos h)/k$$

where h is the hour angle of the star at the time of occultation and is given by:

$$h = \text{sidereal time at } 0^h \text{ U.T.} + t_s - \alpha_\star - \lambda$$

t_s being the sidereal equivalent of t, the observed U.T. of disappearance or reappearance.

The quantities λ, $(\rho/k) \sin \phi'$, $(\rho/k) \cos \phi'$ are constants for any particular station and are punched for permanent use on " Station Data " cards.

Reduction of Moon's position

The time t of observation is measured in U.T. but the tabular values of the Moon's ephemeris are given in terms of E.T. The latest approximation to ΔT must therefore be applied to t so that the lunar ephemeris can be entered with the best available value of E.T. of observation. A correction to the value of ΔT will be derived from the subsequent analysis of the observations.

The coordinates of the centre of the Moon on the fundamental plane are given by:

$$x = \frac{\cos \delta_☾ \sin (\alpha_☾ - \alpha_\star)}{k \sin \pi_☾}$$

$$y = \frac{\sin \delta_☾ \cos \delta_\star - \cos \delta_☾ \sin \delta_\star \cos (\alpha_☾ - \alpha_\star)}{k \sin \pi_☾}$$

or, since $\cos (\alpha_☾ - \alpha_\star)$ is approximately equal to $1 - \frac{1}{2} \sin^2 (\alpha_☾ - \alpha_\star)$:

$$y = \frac{\sin (\delta_☾ - \delta_\star)}{k \sin \pi_☾} + \tfrac{1}{2} x \sin \delta_\star \sin (\alpha_☾ - \alpha_\star)$$

As the range of the angles $(\alpha_☾ - \alpha_\star)$ and $(\delta_☾ - \delta_\star)$ is small, it is permissible to use, for computational convenience, the expressions:

$$x = 15 \cos \delta_☾ \{ (\alpha_☾ - \alpha_\star)^s - \text{correction} \}/s''$$
$$y = \{ (\delta_☾ - \delta_\star)'' - \text{correction} \}/s'' + 0{\cdot}3636\, x \sin \delta_\star (\alpha_☾ - \alpha_\star)^s \times 10^{-4}$$

where $s'' = k \sin \pi / \sin 1''$ is the Moon's geocentric semi-diameter and the corrections are those necessary to adjust for the difference between an angle and its sine. To the precision given, which is in general adequate, the corrections may be taken from the following critical tables; they are to be applied so as to diminish the numerical values of $(\alpha_{☾} - \alpha_{\star})^{s}$ and $(\delta_{☾} - \delta_{\star})''$.

$\alpha_{☾} - \alpha_{\star}$	Corr^n.	$\delta_{☾} - \delta_{\star}$	Corr^n.
s	s	′	″
0		0	
	0·00		0·0
178		2336	
	0·01		0·1
257		3370	
	0·02		0·2
304		3996	
	0·03		0·3
341		4470	
	0·04		0·4
370		4861	
	0·05		
396			
	0·06		
419			

Final part of the reduction

On the fundamental plane the distance of the observer from the centre of the tabular Moon is:

$$\{ (x - \xi)^2 + (y - \eta)^2 \}^{\frac{1}{2}}$$

and the semi-diameter of the Moon is unity, by virtue of the adopted unit of measurement. The distance between the observer and the Moon's limb is accordingly the small quantity:

$$\{ (x - \xi)^2 + (y - \eta)^2 \}^{\frac{1}{2}} - 1$$

At the distance of the Moon this corresponds to an angular separation of:

$$\Delta\sigma = s \{ (x - \xi)^2 + (y - \eta)^2 \}^{\frac{1}{2}} - s$$

where $\Delta\sigma$ is the excess of the apparent distance of the star from the centre of the Moon over the apparent semi-diameter of the Moon, reduced to the geocentric distance of the Moon.

The position angle of disappearance or reappearance (χ), with respect to the centre of the tabular Moon and measured eastwards from the north point, is given by:

$$\tan \chi = -(x - \xi)/-(y - \eta)$$

The position angle derived from this expression is used in the calculation of limb corrections. For the purpose of subsequent analysis less precision is required, and, at the moment of disappearance or reappearance, the value of $\{ (x - \xi)^2 + (y - \eta)^2 \}$ is sufficiently near unity to allow the position angle to be determined from:

$$\sin \chi = -(x - \xi) \qquad \cos \chi = -(y - \eta)$$

Hence the angle $\rho - \chi$ between the direction of motion of the Moon and the direction of the occulted star can be obtained from:

$$\sin (\rho - \chi) = +(x - \xi) \cos \rho - (y - \eta) \sin \rho$$
$$\cos (\rho - \chi) = -(x - \xi) \sin \rho - (y - \eta) \cos \rho$$

where ρ, $\sin \rho$, $\cos \rho$ are obtained from $\cot \rho = y'/x'$

The values of $\Delta\sigma$ are examined and the observations which give quite impossible results are investigated. In many of these cases it is found that the observer has apparently recorded the number of minutes wrongly, and that corrections of an exact number of minutes give reasonable values of $\Delta\sigma$. Such observations and the results of their reduction are punched on the cards with a code to show that they have been adjusted.

Limb corrections

The value of $\Delta\sigma$ found by the above method gives the distance of the star from the smooth limb of a spherical tabular Moon. Before this distance can be attributed to the difference, in position and size, between the tabular and observed Moon, a correction must be made to allow for the irregularities of the Moon's limb.

The amount of the irregularity will be taken from charts being prepared by Dr. C. B. Watts at the U.S. Naval Observatory. The arguments, which are in degrees, are the topocentric values of:

$$\Pi = \text{position angle from the central meridian of the Moon}$$
$$l = \text{libration in longitude}$$
$$b = \text{libration in latitude.}$$

The values of the geocentric librations in longitude and latitude and the position angle of the axis (l_0, b_0, C_0 respectively) are tabulated in the Ephemeris at intervals of one day with argument in U.T. These and two auxiliary quantities:

$$A = +\tan b_0 \cos C_0 + \tan \delta_{☾}$$
$$B = -\tan b_0 \sin C_0$$

which are precalculated at intervals of one day, are obtained by interpolation for the time of observation. The topocentric values are given by:

$$\Pi = \chi - C_0 + AX + BY$$
$$l = l_0 + \sec b_0 (-X \cos C_0 + Y \sin C_0)$$
$$b = b_0 + (X \sin C_0 + Y \cos C_0)$$

where X, Y are angular distances in degrees corresponding to ξ, η and are found from:

$$X = S\,\xi \qquad Y = S\,\eta$$

where S is strictly the topocentric semi-diameter in degrees, but is generally taken as $s''/3600$, where s'' is the geocentric value in seconds of arc.

With these arguments the charts give the amount of the irregularity on the Moon's limb in the direction of the star when the Moon is at its mean distance from the Earth. This quantity must accordingly be multiplied by the ratio of the actual to the mean semi-diameters, $s/s_0 = 0{\cdot}0010723\, s''$, and subtracted from the value of $\Delta\sigma$ obtained in the reduction.

Example 10.9. Reduction of an occultation, with limb correction

It is assumed that an observer at Strasbourg has reported that he has observed photoelectrically, under excellent conditions, the disappearance of Z.C. 2833 on 1960 September $1^d\ 19^h\ 01^m\ 07^s.81$ U.T.

From the Besselian elements, the station constants, and the ephemerides in the Ephemeris:

	h m s				
G.S.T. at 0^h U.T.	+22 40 37·61	x'	+0·6046	y'	+0·0543
t	+19 01 07·81	$\sin\rho$	+0·996	$\cos\rho$	+0·089
Correction to S.T.	+ 3 07·46				
$-\alpha_\star$	−19 22 50·24	$\sin\delta_\star$	−0·31960	$\cos\delta_\star$	+0·94755
$-\lambda$	+ 0 31 04·25	$(\rho\sin\phi')/k$	+2·7388	$(\rho\cos\phi')/k$	+2·4323
sum = h	+22 53 06·89	$\sin h$	−0·28772	$\cos h$	+0·95772
		ξ	−0·6998	η	+3·3397

It is assumed that the latest available approximation to ΔT is

$$\Delta T = +34^s.92 = +0^h.00970$$

whence E.T. of observation $= 19^h\ 01^m\ 07^s.81 + 34^s.92 = 19^h\ 01^m\ 42^s.73 = 19^h.02854$

	h m s		° ′ ″
$\alpha_☾$	19 20 53·72	$\delta_☾$	−17 42 10·3
$\alpha_\star$	19 22 50·24	$\delta_\star$	−18 38 18·9
$\alpha_☾ - \alpha_\star$	− 1 56·52	$\delta_☾ - \delta_\star$	+ 56 08·6
correction	0·00	correction	+ 0·1
$(\alpha_☾ - \alpha_\star)$ − corr^n.	− 116·52	$(\delta_☾ - \delta_\star)$ − corr^n.	+ 3368·5

$\pi_☾$	60′ 04″·77	$15\cos\delta_☾$	14·2897
s''	982·232	$1/s''$	0·0010 1809
x	−1·6952	y	+3·4271
ξ	−0·6998	η	+3·3397
$f = x - \xi$	−0·9954	$g = y - \eta$	+0·0874

$$\Delta\sigma = s\{(f^2+g^2)^{\frac{1}{2}} - 1\} = -0''.8$$

$\sin(\rho - \chi)$ −0·18 $\cos(\rho - \chi)$ +0·98

For the calculation of the arguments for the charts of limb irregularities:

$\cot\chi$	−0·0878	$X = S\xi$	−0°·191
χ	95°·02	$Y = S\eta$	+0 ·911

For 1960 September $1^d.792$, interpolating in the Ephemeris:

l_0	−0°·68	b_0	−5°·64	C_0	351°·24
$\tan b_0$	−0·099	$\sin C_0$	−0·152	A	−0·417
$\sec b_0$	+1·005	$\cos C_0$	+0·988	B	−0·015

$$\Pi = \chi - C_0 + AX + BY = 103°·85$$
$$l = l_0 + \sec b_0(-X\cos C_0 + Y\sin C_0) = -\ 0·63$$
$$b = b_0 + (X\sin C_0 + Y\cos C_0) = -\ 4·71$$

From the charts the irregularity on the Moon's limb at mean distance is $-0''.74$.

$$s/s_0 = 0·0010\ 723\ s'' = 1·05$$
$$\text{limb irregularity} = -1·05 \times 0''.74 = -0''.78$$
$$\text{corrected value of } \Delta\sigma = -0''.8 + 0''.8 = 0''.0$$

The values to be used in the subsequent analysis are thus

$$\Delta\sigma = 0''.0 \quad \sin(\rho - \chi) = -0·18 \quad \cos(\rho - \chi) = +0·98$$

The number of figures retained in the above example is the same as was used with the lunar ephemeris in the Ephemeris prior to 1960; with the improved lunar ephemeris, another decimal could be retained in x, y.

D. ANALYSIS AND DISCUSSION

As has been mentioned, the systematic reduction and discussion of lunar occultations has so far been done without the application of corrections for irregularities of the Moon's limb. In due course, limb corrections will be applied to all the observations made since 1943 and final analyses made. The observations have so far been compared with the positions of the Moon as tabulated in *The Nautical Almanac* and in *The American Ephemeris* before 1960, that is before the introduction of ephemeris time and of the improved lunar ephemeris.

The solutions are made by the method of least squares using the observational equation of condition:

$$\cos(\rho - \chi)\,\delta L + \sin(\rho - \chi)\,\delta B = \Delta\sigma$$

where δL and δB are the errors in the orbital longitude and latitude of the tabular Moon. If reductions were done using the improved lunar ephemeris, δL in this equation would be replaced by $\{0{\cdot}549\,\delta(\Delta T) + \delta L\}$. The method will however be reviewed before being applied to observations for 1960 onwards, and a somewhat different observational equation may be adopted.

Two sets of solutions are made, one restricted to disappearances at the dark limb only and the other including all observations of Z.C. stars. There have so far been insufficient observations of the other phases to make practicable separate solutions for each phase. Solutions are made for each lunation by the method of least squares and the arithmetic mean of the twelve or thirteen lunations is taken as the solution for the year, on the assumption that δL and δB may be regarded as constant for each lunation. The annual means for the years since 1943 are tabulated below; n is the number of observations and the errors are internal probable errors.

	Disappearances at dark limb					All observations				
Year	n	δL		δB		n	δL		δB	
		″	″	″	″		″	″	″	″
1943·5	586	+0·26	±0·028	−0·65	±0·043	714	+0·18	±0·027	−0·60	±0·043
1944·5	594	−0·11	·029	−0·44	·044	701	−0·15	·027	−0·34	·042
1945·5	623	−0·55	·033	−0·75	·050	729	−0·59	·030	−0·69	·044
1946·5	687	−0·76	·027	−0·73	·040	833	−0·80	·026	−0·55	·037
1947·5	765	−0·84	·028	−0·55	·040	931	−0·84	·024	−0·50	·036
1948·5	837	−1·15	·022	−0·77	·035	978	−1·14	·022	−0·68	·033
1949·5	1018	−1·40	·026	−0·44	·038	1179	−1·41	·022	−0·43	·033
1950·5	1229	−1·65	·021	−0·67	·030	1725	−1·68	·019	−0·56	·027
1951·5	996	−2·14	·023	−0·67	·035	1430	−2·16	·020	−0·64	·030
1952·5	1011	−2·34	·026	−0·53	·039	1325	−2·44	·022	−0·55	·034
1953·5	993	−2·58	·025	−0·61	·038	1359	−2·64	·022	−0·61	·034
1954·5	1169	−2·91	·018	−0·89	·029	1445	−2·97	·017	−0·78	·026
1955·5	1032	−3·36	·023	−0·70	·034	1236	−3·34	·020	−0·63	·031
1956·5	882	−3·63	·023	−0·84	·034	1025	−3·64	·021	−0·80	·032
1957·5	910	−4·29	·025	−0·81	·036	1024	−4·37	·024	−0·74	·034
1958·5	712	−4·60	·026	−0·53	·040	855	−4·71	·025	−0·34	·037
1959·5	689	−4·82	·031	−0·70	·047	831	−4·81	·028	−0·54	·043

The annual discussions are* published in the *Astronomical Journal*; the combined lists of individual observations, together with the results of the reductions and residuals from the adopted solutions, were published for the years 1943 to 1947 as an Appendix to *Greenwich Observations* for 1939.

Added note (1973). The results of an analysis of the observations from 1960 to 1966 are given by L. V. Morrison and F. McBain Sadler, *M.N.R.A.S.*, **144**, 129–141, 1969. A more extensive analysis covering all observations from 1943 to the present time is in progress. The technique for the automatic computation of limb corrections is described by L. V. Morrison and R. J. Martin, *The Moon*, **2**, 463–467, 1971.

E. OCCULTATIONS OF RADIO SOURCES

The observation of occultations of radio sources by the Moon offers a method of determining more precisely the positions of, and the intensity distributions across, the discrete radio sources. The need for predictions became apparent in 1955. At this time the second Cambridge survey of radio sources (*Memoirs R.A.S.*, **67**, 106–154, 1955) had just been completed, and this catalogue was used as the authority for source positions, predictions up to 1958 being made for the 26 brightest zodiacal sources in it. Commission 40 of the International Astronomical Union has prepared *Zodiacal Radio Sources—List* 2, which consists of 37 radio sources and gives the positions for 1950·0 based on all the observations available at the end of 1958; predictions are supplied for these 37 sources from 1959. The list is supplied to each observing station and corrections will be issued as the necessity for them becomes apparent from later observations.

A prediction normally gives the approximate times of disappearance and reappearance of the centre of the source together with approximate topocentric positions of the Moon for times which cover the duration of the occultation; when the source is known to be extended topocentric positions of the Moon are given when the centre is not occulted but there is a likelihood of part of the source being occulted.

The process of prediction is similar to that for visual occultations but as the positions of the radio sources are considerably less accurately determined than those of stars the following simplifications are made. In the selection of the sources and the preparation of the Besselian elements, the assumed positions are corrected for precession but not for nutation or aberration. The times of disappearance and reappearance are taken directly from the occultation machine without any subsequent calculation to obtain them more precisely. For up to six integral hours, to cover the duration of the occultations, approximate topocentric positions of the Moon are obtained from the geocentric positions by applying the correction:

$$\Delta\alpha = -4\,\pi_{☾}\,\rho\cos\phi'\sin h\sec\delta_{☾}$$
$$\Delta\delta = \pi_{☾}\,(\rho\cos\phi'\cos h\sin\delta_{☾} - \rho\sin\phi'\cos\delta_{☾})$$

where $\Delta\alpha$ is in seconds of time and $\Delta\delta$ and $\pi_{☾}$ in minutes of arc.

The times of disappearance and reappearance are given to $0^{h}{\cdot}01$, the topocentric right ascension to 1^{s}, and the declination to $0'{\cdot}1$; in each case the values may be in error by up to three units in the last figure. This is sufficiently accurate

*Up to year 1957. See *Royal Observatory Bulletins* No. 107 for the discussion of the occultations observed in 1958 and 1959.

for prediction at present. For successful observations, if this accuracy is inadequate for the reduction of the results, H.M. Nautical Almanac Office supplies on request values of the Moon's topocentric position to the accuracy required by that of the observations.

Predictions are supplied to the following radio astronomy observatories.

Place	Description	λ	φ	Alt.
		° ′	° ′ ″	m
Big Pine, Calif., U.S.A.	California Inst. of Technology	+118 17·6	+37 13 53·8	1216
Cambridge, England	Mullard Radio Astr. Obs.	− 0 02·4	+52 09 45	26
Clark Dry Lake, Calif., U.S.A.	Convair Scientific Res. Lab.	+116 15·7	+33 20·1	
Columbus, Ohio, U.S.A.	Ohio State University	+ 83 02·6	+40 01 00·2	245
Dwingeloo, Netherlands	Leiden University	− 6 23·8	+52 48 46·7	25
Helsinki, Finland	University of Helsinki	− 25 00·5	+60 13·4	2
Ithaca, New York, U.S.A.	Cornell University	+ 76 27·1	+42 29 18	341
Jodrell Bank, England	Nuffield Radio Astr. Lab.	+ 2 18·4	+53 14 11	70
Kislovodsk, U.S.S.R.	Mountain station of Pulkovo	− 42 40·2	+43 44 47	2130
Malvern, England	Royal Radar Establishment	+ 2 18·5	+52 07 30	58
Moscow, U.S.S.R.	Sternberg Institute	− 37 34·2	+55 45 19·8	166
Nançay, France	Observatory of Paris	− 2 11·8	+47 22 48	150
Ondřejov, Czechoslovakia	Astrophysical Obs.	− 14 47·0	+49 54 38·1	533
Pulkovo, U.S.S.R.	Astronomical Obs.	− 30 19·6	+59 46 18·5	75
Simeis, U.S.S.R.	Crimean Astrophysical Obs.	− 33 59·8	+44 24 11·6	346
Sydney, Australia	Fleurs Field Station	−150 46·4	−33 51·5	50
Washington, D.C., U.S.A.	Naval Research Laboratory	+ 77 01·6	+38 49 16·6	30

Added note. The predictions have been increased by a large factor, in respect of both the number of stations and the number of sources, since 1960.

F. PLANETARY OCCULTATIONS

The prediction of occultations of stars (or minor planets) by planets (or by * minor planets) largely centres on the search for conjunctions in right ascension within limits of difference of declination that make an occultation possible. These limits are very small, being approximately the sum of the horizontal parallax and the semi-diameter of the planet. Occultations by Mars and Jupiter are the most common, but the total numbers of such occultations, of stars brighter than $9^m{\cdot}0$, that have been predicted in recent years are given in the following table.

Predictions of occultations of stars by planets

Year	Mercury	Venus	Mars	Jupiter	Saturn
Magnitude limit	3·5	7·5	9·0	9·0	9·0
1955		1		2	
1956			3	2	
1957				1	1
1958			5		2
1959		1			
1960			3	3	1
1961			1	4	
Totals	0	2	12	12	4

*Predictions of occultations by natural satellites are also made by H.M. Nautical Almanac Office.

In addition one possible occultation by Pluto and two occultations of radio sources by Venus were predicted in 1958; an occultation by Mercury was predicted in 1953. The comparison of star catalogues with the ephemerides of the planets is a long and tedious task, yielding (as seen above) very few observable phenomena; it is done in successive stages, each corresponding to a closer approximation to the apparent position of the star. The final comparison must be made with the apparent positions of both star and planet, if possible supplemented by observationally determined corrections to the tabular positions of both planet and star.

The actual prediction follows the basic principles underlying those for eclipses, occultations, and transits; but more direct and less formal methods are used, both because the angles involved are much smaller and the prediction much less precise.

An occultation will take place as seen from some point on the Earth's surface, provided the difference $\delta - \delta_\star$ in apparent declination at the time of conjunction in right ascension satisfies the condition:

$$| \delta - \delta_\star | < (\pi + s) \operatorname{cosec} \rho$$

where π, s are respectively the equatorial horizontal parallax and semi-diameter of the planet, and ρ is the position angle of its direction of motion. The approximate area of visibility of the occultation can be determined by simple mental calculations, though a general picture of the circumstances at conjunction, as provided by the occultation machine, is of great assistance. In particular, the northern and southern latitude limits on the meridian (ϕ_N, ϕ_S), where they exist, may be found from:

$$\pi \sin (\phi_{N,S} - \delta_\star) = \delta - \delta_\star \pm s \operatorname{cosec} \rho$$

where the upper sign relates to ϕ_N. Note must be taken of the position of the Sun, as for lunar occultations, since it is generally useless to predict occultations of stars in daylight.

Local predictions for particular stations are done by a semi-graphical method. In principle (the details vary with the degree of precision required) a scale diagram of the planet (taking into account the oblateness of the disk where relevant) is drawn, and the relative geocentric positions of the star at one or more instants, including the time of conjunction, are plotted. At these times the parallactic shifts of the planet relative to the star, as seen from the stations for which predictions are required, are calculated in r, θ coordinates where:

$$r = \pi \cos (\text{altitude}) \qquad \theta = \text{parallactic angle}$$

The altitude and parallactic angle can either be calculated from the standard formulae or, preferably, taken from tabulated solutions of the spherical triangle in navigational tables, for example H.O. 214 (H.D. 486). The plotted geocentric positions are shifted accordingly, and the apparent paths of the star across the disk drawn in; the approximate direction ρ and speed n can be derived from the geocentric motion of the planet only, and suffice for most predictions.

The methods used are similar to those described by G. E. Taylor in *J.B.A.A.*, **65**, 84-90, 1955; in that paper is also described the determination of the actual limits of occultation. The predictions of planetary appulses and occultations are published in *The Handbook of the British Astronomical Association.*

11. EPHEMERIDES FOR PHYSICAL OBSERVATIONS OF THE SUN, MOON, AND PLANETS

A. INTRODUCTION

The ephemerides for physical observations represent the aspects of the apparent disks of the Sun, Moon, and planets; they are intended for making and reducing observations of the surface markings, and for reducing observations of positions for which the exact position of the central point of the disk must be determined from observations of the limbs. The tabular quantities enable an observer to determine, from the planetographic coordinates of the surface markings, those markings that are observable at any time, their positions on the apparent disk, the conditions of illumination, and the orientation of the disk on the celestial sphere; conversely, from measurements on the disk, the observer may determine the planetographic coordinates of surface markings and the values of the elements of the rotational motion.

These ephemerides are based on the fundamental ephemerides in the preceding part of the Ephemeris and on the additional data to which specific references are made. The tabular values are corrected for the effects of aberration, and should therefore be interpolated to the actual time of observation; but they are strictly geocentric. The value of the light-time for unit distance used in calculating them * is $498^s{\cdot}58$, corresponding to the adopted values of the solar parallax and the velocity of light. They are tabulated to an order of accuracy sufficient for the reduction of observations; any significant approximations made in calculating them are stated in the following explanations.

The elements of the rotational motion are: the period of rotation; the position of the axis of rotation in space, represented either by the coordinates of the point on the celestial sphere towards which the axis is directed, i.e., the pole of rotation, or by the inclination and node of the equator on an adopted reference plane; and the planetographic longitude of the central point of the apparent disk at an adopted epoch, which defines the central meridian on the disk.

To the order of accuracy of these ephemerides, the effects of the difference between ephemeris time and universal time are appreciable only in the longitudes of the central meridians of the Sun, Mars, and Jupiter, and in the selenographic colongitude of the Sun and the position angle of the lunar axis. The values of these quantities calculated from the fundamental ephemerides in ephemeris time

*$499^s{\cdot}012$ for 1968 onwards.

are reduced to universal time by corrections for ΔT at the tabular date; but no corrections are applied to the adopted central meridians at the epoch, although only in the case of the rotation period of Mars have any of the rotational elements been obtained from observations referred to ephemeris time.

When the rotation is direct, i.e., counter-clockwise as viewed from above the north pole of rotation, the motion of the surface markings as seen from the Earth is in the westward direction on the celestial sphere, from the east limb towards the west limb.

The apparent positions of points on the disk are in general most conveniently represented by apparent distance and position angle relative to the central point of the disk; position angles are ordinarily reckoned from the north point of the disk towards the east, but for some purposes they are reckoned from the vertex. The central point of the apparent disk is the subterrestrial point on the surface, and its position on the geocentric celestial sphere is diametrically opposite the apparent position of the Earth on the planetocentric celestial sphere.

The north point of the disk is on the apparent northern limb at its intersection with the celestial meridian that passes through the north celestial pole and the centre of the disk. The vertex of the disk is on the apparent upper limb at its intersection with the vertical circle that passes through the zenith and the centre of the disk.

B. EPHEMERIS FOR PHYSICAL OBSERVATIONS OF THE SUN

The ephemeris for physical observations of the Sun is calculated with the elements determined by Carrington (*Observations of the spots on the Sun*, pages 221 and 244, 1863):

Sidereal period of rotation	25·38 mean solar days
Inclination of the solar equator to the ecliptic	$I = 7° 15'$
Longitude of the ascending node of the solar equator on the ecliptic	$\Omega = 73° 40' + 50''{\cdot}25\, t$

where t is the time in years reckoned from 1850.

In the ephemeris: P denotes the position angle of the northern extremity of the axis of rotation, measured eastwards from the north point of the disk; B_0, the heliographic latitude; and L_0, the heliographic longitude of the central point of the disk. Heliographic longitudes on the surface of the Sun are measured from the solar meridian that passed through the ascending node of the solar equator on the ecliptic on 1854 January 1 at Greenwich mean noon, J.D. 239 8220·0; they are reckoned from 0° to 360°, in the direction of rotation, i.e., westwards on the apparent disk as viewed on the celestial sphere. Carrington's zero meridian passed the ascending node twelve hours earlier. Heliographic latitudes are reckoned from the solar equator, positive towards the north.

The synodic period of rotation is the interval of time during which L_0 decreases by 360°. The mean synodic period is $27^{d}{\cdot}2753$. The beginning of each synodic rotation is the instant at which L_0 passes through 0°; the rotations are numbered in continuation of Carrington's Greenwich photo-heliographic series, of which No. 1 commenced on 1853 November 9.

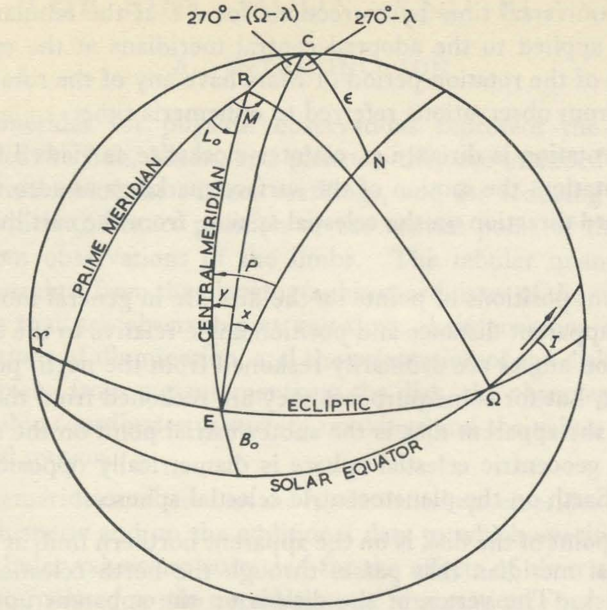

Figure 11.1. *Heliographic coordinates*

R North pole of rotation of Sun
C Pole of the ecliptic
N North celestial pole
E Subterrestrial point, i.e. projection of the centre of the disk

In computing the physical ephemeris, no allowance for the secular motion of the ecliptic is made in the values of the elements; and the latitude of the Sun is neglected.

The heliographic coordinates of the central or subterrestrial point of the disk are calculated from the spherical triangle (see figure 11.1) formed on the heliocentric celestial sphere by the apparent position of the Earth E, the north pole of rotation of the Sun R, and the ascending node of the solar equator on the ecliptic Ω. In this triangle, the arc of the ecliptic EΩ from the Earth to the node of the solar equator is $\Omega - (\lambda \pm 180°)$, where λ is the geocentric longitude of the Sun, and the opposite angle ERΩ is $M - L_0$ where M is the heliographic longitude of the node of the equator. Therefore:

$$\sin B_0 = \sin(\lambda - \Omega)\sin I$$
$$\cos B_0 \cos(L_0 - M) = -\cos(\lambda - \Omega)$$
$$\cos B_0 \sin(L_0 - M) = -\sin(\lambda - \Omega)\cos I$$

in which:

$$M = 360° - \frac{360°}{25{\cdot}38} \times (\text{J.D.} - 239\ 8220{\cdot}0)$$

In calculations the expression for M is used in the form:

$$M - 180° = 112°{\cdot}766 + (243\ 0000{\cdot}5 - \text{J.D.}) \times 14°{\cdot}1843\ 9716$$

The position angle of the axis of rotation is:

$$P = x + y$$

where x and y are the angles NEC and CER at the subterrestrial point E in the two triangles formed by this point and the pole of the ecliptic C with the north celestial pole N and with the north pole of rotation of the Sun R:

$$\tan x = -\cos\lambda \tan\epsilon$$
$$\tan y = -\cos(\lambda - \Omega)\tan I$$

in which ϵ is the obliquity of the ecliptic.

The north point of the solar disk from which P is measured is determined by the equator and equinox of date; therefore, in calculating x, ϵ is necessarily the true obliquity of date and λ is the apparent longitude of the Sun referred to the true equinox of date. Furthermore, by using the apparent longitude as the value of λ in all other formulae also, the aberration in longitude is completely included, and no further correction for aberration is required. No correction is applied to L_0 for rotation during the light-time, because presumably it is already included in Carrington's meridian; Carrington, in reducing his observations, *added* 20″ for aberration to the tabular longitude of the Sun taken from *The Nautical Almanac*, but he appears to have referred his measurements to the *apparent* central point of the disk.

For convenience of calculation, it has been the practice in the past to use procedures which only partially take account of the aberration in longitude. In the formulae in which $\lambda - \Omega$ occurs, λ may be referred to any equinox, provided Ω is referred to the same equinox; the fixed mean equinox of 1950·0 was therefore used, and the true longitude of the Sun referred to this equinox, as tabulated in the ephemeris of the Sun, was combined with the constant value 75°·063 for Ω to obtain $\lambda - \Omega$. The error introduced into P and B_0 by the neglect of aberration in $\lambda - \Omega$ is inappreciable, and the error in L_0, though at maximum it can amount to about 20″, was not considered important. This approximation is retained in the Ephemeris for 1960; but beginning with 1961, the aberration is completely included.

The times at which successive synodic rotations commence are determined by inverse interpolation of the ephemeris of L_0 with the aid of the interpolation table included in the Ephemeris (page 314).

The calculation of the physical ephemeris of the Sun does not involve any quantities that depend upon the rotation of the Earth. This ephemeris may therefore be calculated directly from the fundamental ephemerides with argument ephemeris time, and converted to universal time by interpolating every tabular entry for 0^h E.T. to a time ΔT *after* 0^h E.T., by applying the correction:

$$+\frac{\Delta T}{\text{tabular interval}}\,\delta_{\frac{1}{2}}$$

The tabular interval is 1^d, and the tabular entries are printed to 0°·01; to avoid a systematic error greater than 0°·001, this correction is applied, with an extrapolated value of ΔT, when $\delta_{\frac{1}{2}}$ in degrees exceeds $86{\cdot}4/(\Delta T)^s$.

Example 11.1. *The physical ephemeris of the Sun*
1960 March 7 at 0^h U.T.

Constants		Ω, equinox 1950·0	$75^\circ{\cdot}0625$
	$\sin I = 0{\cdot}12620$	Precession to 1960·0, *A.E.* page 20	0·1396
	$\tan I = 0{\cdot}12722$	Ω, equinox 1960·0	75·2021
	$\cos I = 0{\cdot}99200$		
	$\sec I = 1{\cdot}00806$	Daily motion of M	$14^\circ{\cdot}1843\ 9716$

1960 March $7^d\ 00^h$ E.T.

Longitude of Sun, 1960·0, page 20		$346^\circ{\cdot}4373$
Reduction to apparent longitude, page 20		−0·0034
Apparent longitude of Sun	$\lambda =$	346·4339
Precession from 1960·0 + nutation, page 20		+0·0023
Longitude affected by aberration, equinox 1960·0		346·4316
Ω, equinox 1960·0		75·2021
$\lambda - \Omega$		271·230

This value of $\lambda - \Omega$ is used in the following calculation; but as explained in the text, the value actually used in the calculation for *A.E.* 1960 was $271^\circ{\cdot}235 = 346^\circ{\cdot}437 - 75^\circ{\cdot}202$, in which aberration is neglected. The inclusion of aberration decreases L_0 by $0^\circ{\cdot}006$.

Apparent obliquity, page 20	$\epsilon =$	$23^\circ{\cdot}442$
$\cos\lambda$		+ 0·97210
$\tan\epsilon$		+ 0·43361
$-\tan\epsilon\cos\lambda = \tan x$		− 0·42151
$\cos(\lambda - \Omega)$		+ 0·02147
$-\tan I\cos(\lambda - \Omega) = \tan y$		− 0·00273
$x,\ -90^\circ < x < +90^\circ$		$-22^\circ{\cdot}856$
$y,\ -90^\circ < y < +90^\circ$		− 0·156
$x + y = P$		−23·01
$\sin(\lambda - \Omega)$		− 0·99977
$\sin I\sin(\lambda - \Omega) = \sin B_0$		− 0·12617
$B_0,\ -90^\circ < B_0 < +90^\circ$		$-7^\circ{\cdot}25$
$\cot(\lambda - \Omega)$		− 0·02147
$\sec I\cot(\lambda - \Omega) = \cot(L_0 - M)$		− 0·02164

The angle $L_0 - M$ is in the quadrant of $\lambda - \Omega \pm 180^\circ$; but for convenience in calculation, the corresponding angle in the same quadrant as $\lambda - \Omega$ is used to obtain L_0 by combining it with $M - 180^\circ$.

$L_0 - M + 180^\circ$	$271^\circ{\cdot}239$
$112^\circ{\cdot}766 - 7000 \times 14^\circ{\cdot}1843\ 9716$	−178·014
L_0, 0^h E.T.	93·225
Reduction from E.T. to U.T.	− 0·005
L_0, 0^h U.T.	93·22

C. PHASES OF THE MOON AND PLANETS, AND STELLAR MAGNITUDES OF THE PLANETS

Phases

The tabulations in the physical ephemerides of the Moon and planets include the data that determine the geometric aspect of the illuminated part of the apparent disk on the celestial sphere. The fraction (k) of the area of the apparent disk of the Moon or a planet that is illuminated by the Sun is called the *phase*; it depends upon the planetocentric elongation of the Earth from the Sun, called the *phase angle* (i).

Neglecting the oblateness of the body, the apparent disk is circular and the terminator is the orthogonal projection, onto a plane perpendicular to the line of sight, of the great circle that bounds the illuminated hemisphere of the body. The terminator is therefore in general an ellipse, reducing to a straight line at $i = 90°$ and becoming a circle at $i = 0°$ or $180°$. The line of cusps, joining the extremities of the terminator, is the major axis; it is a diameter of the apparent disk, of length $2s$ where s is the apparent semi-diameter. The minor axis lies on the diameter which passes through the midpoint of the bright limb and through the diametrically opposite point of greatest defect of illumination. Beginning with 1960, the physical ephemerides of the Moon, Mercury, and Venus contain the position angle (Θ) of the midpoint of the illuminated limb reckoned eastwards from the north point of the disk; in years preceding 1960, these ephemerides gave the position angle of the line of cusps. Mars, Jupiter, and Saturn show only very small departures from a fully illuminated disk, and hence in the physical ephemerides for these planets the position angle (Q) and angular amount (q) of the greatest defect of illumination are tabulated.

The radius of the Moon or of a planet that projects into the semi-minor axis of the terminator is perpendicular to the planetocentric direction of the Sun. (See figure 11.2). When $i > 90°$, which is only possible for the Moon, Mercury, or Venus, this radius is therefore at an angle $i - 90°$ to the line of sight and $180° - i$ to the plane of the apparent disk; it projects onto the dark area of the disk, and the length of the projection is $-s \cos i$. When $i < 90°$, this radius makes an angle of $90° - i$ with the line of sight, i with the plane of the disk, and projects onto the illuminated area into a length $+s \cos i$. In either case the length of the illuminated part of the diameter perpendicular to the line of cusps is $s\,(1 + \cos i)$; and since the area of an ellipse is the product of π and the two semi-axes, the total area of the illuminated part of the disk is $\frac{1}{2}\pi s^2\,(1 + \cos i)$. The phase is therefore:

$$k = \tfrac{1}{2}\,(1 + \cos i)$$

and is the value of both the ratio of the illuminated area of the disk to the total area, and the ratio of the illuminated length of the diameter perpendicular to the line of cusps to the complete diameter. The greatest defect of illumination is consequently:

$$q = 2s\,(1 - k)$$

The phase angle i is determined from the plane triangle formed in space by the Sun, the Earth, and the body, by means of the relations:

$$r \sin i = R \sin D$$
$$r \cos i = \Delta - R \cos D$$

in which R and r are the heliocentric distances of the Earth and the body, Δ the geocentric distance of the body, and D its geocentric elongation from the Sun. Note that i is less than or greater than 90° according as $r^2 + \Delta^2 - R^2$ is positive or negative.

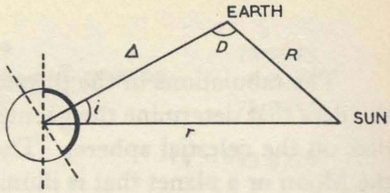

Figure 11.2. *The definition of phase angle*

The elongation (D) and the position angle (Θ) of the midpoint of the illuminated limb, are obtained from the relations in the triangle formed on the celestial sphere by the Sun, the north celestial pole, and the Moon or planet:

$$\sin D \sin \Theta = \cos \delta_\odot \sin (\alpha_\odot - \alpha)$$
$$\sin D \cos \Theta = \sin \delta_\odot \cos \delta - \cos \delta_\odot \sin \delta \cos (\alpha_\odot - \alpha)$$
$$\cos D = \sin \delta_\odot \sin \delta + \cos \delta_\odot \cos \delta \cos (\alpha_\odot - \alpha)$$

in which $\sin \Theta$ has the same sign as $\sin (\alpha_\odot - \alpha)$ and D is treated as positive; the direction of the elongation must be obtained from other considerations. In the case of the Moon, r and R are so nearly equal and parallel that it is sufficiently accurate to take $D = 180° - i$, and determine k directly from $\cos i = -\cos D$.

The position angle (Q) of the greatest defect of illumination is given by:

$$Q = \Theta + 180°$$

Before 1960, in the physical ephemeris for the Moon the direction of the line of cusps was represented by the position angle of the terminator, defined as the position angle of the northern cusp, which is always between $-90°$ and $+90°$; before full moon, it is 90° greater than Θ, while after full moon it is 90° less than Θ. For Mercury and Venus, however, the direction of the line of cusps was represented by defining as positive the direction along this line in which the illuminated area is to the right as seen from the Earth, and tabulating the position angle θ of the positive terminus; θ is given by:

$$\theta = \Theta + 90° = Q - 90°$$

where $-90° < D < +90°$, and $\cos \theta$ has the sign opposite to $\sin (\alpha_\odot - \alpha)$. See figure 11.3. The angle Θ is the position angle of the arc of the great circle from the planet to the Sun; θ is equal to the angle which this arc forms with a great circle passing through the planet and directed towards the west, measured from this westward-directed great circle, through north, east, and south, from 0° to 360°. At meridian transit, θ is the angle which the positive direction of the line of cusps forms with the northward direction of the meridian.

Stellar magnitudes

The stellar magnitudes of the planets are obtained from the formulae of G. Müller that are given in *Publicationen des Astrophysikalischen Observatoriums zu*

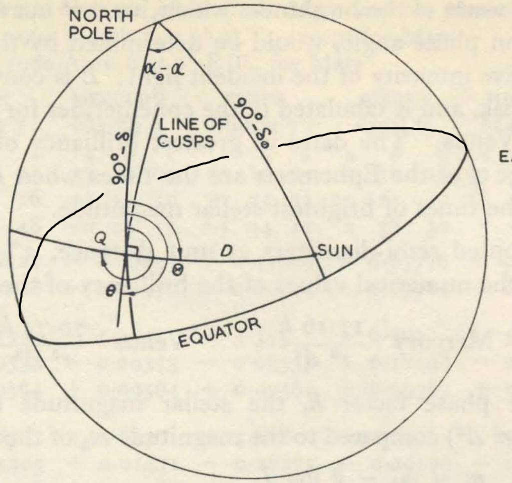

Figure 11.3. *The specification of the illuminated limb*

Potsdam, **8,** 366, 1893; see H. N. Russell, *Ap. J.*, **43**, 107-111, 1916, for an informative discussion.

The dependence of the brightness of a planet upon its geometric position relative to the Earth and the Sun does not completely determine the stellar magnitude. The brightness also depends upon the albedo of the surface as a whole in the different positions; but the reflection of light from the planets has not been satisfactorily represented by any theoretical formula. Consequently the actual relation between brightness and phase angle must be determined empirically, from observations extending over a long period of time.

The relation to the geometric circumstances, disregarding the dependence of albedo on phase angle or other conditions, is represented by the brightness relative to a certain standard of brightness. This is the brightness that the planet would have in opposition (if a superior planet), or in superior conjunction (if an inferior planet), at its mean heliocentric distance (a), with the Earth at mean heliocentric distance unity; in this configuration, the disk is fully illuminated, and the geocentric distance is $a \pm 1$, the upper sign applying to an inferior planet. At any geocentric distance Δ and phase angle i, the ratios of the semi-diameter and of the illuminated area to their values in the standard configuration are respectively $(a \pm 1)/\Delta$ and $k\,(a \pm 1)^2/\Delta^2$, where the phase $k = \frac{1}{2}(1 + \cos i)$; at heliocentric distance r, therefore, since the relative intensity of the incident light varies as $1/r^2$, the relative brightness is:

$$k\,\frac{a^2\,(a \pm 1)^2}{r^2\,\Delta^2} = ks^2\,\frac{a^2}{r^2}$$

where s is the apparent semi-diameter. Accordingly, since a is a constant, the quantity:

$$L = k\,\frac{s^2}{r^2}$$

is adopted as a measure of the brightness which, were it not for the dependence of apparent albedo on phase angle, would be determined by the visible illuminated area and the relative intensity of the incident light; L is conventionally called the brilliancy of the disk, and is tabulated in the ephemerides for the illuminated disks of Mercury and Venus. The dates of greatest brilliancy of Venus given in the *Diary* and on page 4 of the Ephemeris are the times when L is a maximum (see section 8C), not the times of brightest stellar magnitude.

With the adopted semi-diameters at unit distance, $3''\cdot34$ for Mercury and $8''\cdot41$ for Venus, the numerical values of the brilliancy of the disk (L) are:

$$\text{Mercury} \quad \frac{11\cdot16\,k}{r^2\,\Delta^2} \qquad \text{Venus} \quad \frac{70\cdot73\,k}{r^2\,\Delta^2}$$

Omitting the phase factor k, the stellar magnitude m at the brightness $J = a^2\,(a \pm 1)^2/(r^2\,\Delta^2)$ compared to the magnitude m_0 of the unit of brightness is:

$$m = m_0 - \tfrac{5}{2}\log J$$
$$= \{\, m_0 - 5\log(a^2 \pm a)\,\} + 5\log r\,\Delta$$

To obtain the actual stellar magnitude of a planet, a function $f(i)$ of the phase angle, or for Saturn of the position of the rings, is added to this expression, and both the function f and the first term of m are determined from observation.

Müller's formulae are based on observations which he made from 1877 September to 1891 February. The numerical expressions are as follows:

Mercury : $+1\cdot16 + 5\log r\,\Delta + 0\cdot02838\,(i - 50) + 0\cdot00010\,23\,(i - 50)^2$
Venus : $-4\cdot00 + 5\log r\,\Delta + 0\cdot01322\,i + 0\cdot00000\,04247\,i^3$
Mars : $-1\cdot30 + 5\log r\,\Delta + 0\cdot01486\,i$
Jupiter : $-8\cdot93 + 5\log r\,\Delta$
Saturn : $-8\cdot68 + 5\log r\,\Delta \pm 0\cdot044\,(U' + \omega - U) \mp 2\cdot60\sin B + 1\cdot25\sin^2 B$

in which i is measured in degrees.* For Saturn the stellar magnitude depends upon the aspect of the rings, and thus on the quantities U, U', B, ω which pertain to the rings and which are defined in section 12D. In the formula for the stellar magnitude of Saturn the signs used for the third and fourth terms are those which make these terms positive and negative respectively.

For Uranus and Neptune, and for the minor planets, Müller's formula is used in the form:

$$m = g + 5\log r\,\Delta$$

and the adopted values of g are:

Uranus	$-6\cdot85$	Ceres	$4\cdot0$	Juno	$5\cdot5$
Neptune	$-7\cdot05$	Pallas	$4\cdot5$	Vesta	$4\cdot0$

These values are used up to and including 1961. From 1962, for the minor planets, improved values of g have been used for the visual magnitudes given in *A.E.*, page 9† (*Phenomena*), and photographic magnitudes are given as footnotes to the ephemeris pages. The values of g have been supplied by Dr. T. Gehrels of Indiana University, and those corresponding to the photographic magnitudes are

*$(U' + \omega - U)$ is also measured in degrees.

†Page 7 in *A.E.* 1972–3, page 5 from 1974.

Example 11.2. *The bright limb and phase of the Moon, and the phases of the planets*

1960 March 7 at 0^h E.T., except for Mars
1960 April 6 at 0^h E.T., for Mars

0^h E.T. on 1960		MOON March 7	MERCURY March 7	VENUS March 7	MARS April 6	JUPITER March 7	SATURN March 7
		h m s	h m s	h m s	h m s	h m s	h m s
$\alpha_\odot$	(1)	23 10 04	23 10 04	23 10 04	0 59 42	23 10 04	23 10 04
α	(2)	6 13 16	23 31 19	21 25 53	22 21 12	18 03 06	19 10 38
$\alpha_\odot - \alpha$		16 56 48	−0 21 15	+1 44 11	2 38 30	5 06 58	3 59 26
$\sin(\alpha_\odot - \alpha)$		− 0·96222	− 0·09259	+ 0·43909	+ 0·63776	+ 0·97335	+ 0·86479
$\cos(\alpha_\odot - \alpha)$		− 0·27228	+ 0·99570	+ 0·89844	+ 0·77024	+ 0·22934	+ 0·50214
$\delta_\odot$	(1)	− 5°·355	− 5°·355	− 5°·355	+ 6°·372	− 5°·355	− 5°·355
$\sin\delta_\odot$		− 0·09333	− 0·09333	− 0·09333	+ 0·11098	− 0·09333	− 0·09333
$\cos\delta_\odot$		+ 0·99564	+ 0·99564	+ 0·99564	+ 0·99383	+ 0·99564	+ 0·99564
δ	(2)	+ 18°·183	+ 0°·843	− 15°·828	− 11°·642	− 23°·010	− 21°·956
$\sin\delta$		+ 0·31205	+ 0·01471	− 0·27275	− 0·20179	− 0·39089	− 0·37390
$\cos\delta$		+ 0·95006	+ 0·99989	+ 0·96209	+ 0·97942	+ 0·92044	+ 0·92747
$\cos\delta_\odot \cos(\alpha_\odot - \alpha)$		− 0·27109	+ 0·99136	+ 0·89452	+ 0·76549	+ 0·22834	+ 0·49995
$\sin D \sin\Theta$	(3)	− 0·95802	− 0·09219	+ 0·43718	+ 0·63382	+ 0·96911	+ 0·86102
$\sin D \cos\Theta$		− 0·00408	− 0·10790	+ 0·15419	+ 0·26316	+ 0·00336	+ 0·10037
$\cos D$		− 0·28668	+ 0·98988	+ 0·88606	+ 0·72734	+ 0·24666	+ 0·49858
$\sin D$	(4)	+ 0·95803	+ 0·14192	+ 0·46358	+ 0·68628	+ 0·96912	+ 0·86685
tan/cot Θ	(5)	+ 0·00426	+ 0·8543	+ 0·3527	+ 0·41520	+ 0·00347	+ 0·11657
Θ		269°·756	220°·51	70°·57	67°·452	89°·801	83°·351
$Q = \Theta + 180°$		89°·756	40°·51	250°·57	247°·452	269°·801	263°·351
R	(1)	—	0·99248	0·99248	1·00088	0·99248	0·99248
Δ	(2)	—	0·65895	1·44299	1·94736	5·44869	10·51784
r	(6)	—	0·35279	0·72750	1·39950	5·29205	10·05988
$\sin i$	(7)	—	+0·39925	+0·63243	+0·49081	+0·18175	+0·08553
$\cos i$		—	−0·91694	+0·77470	+0·87130	+0·98334	+0·99633
			156°·5	39°·23	29°·394	10°·472	4°·906
k	(8)	0·643	0·0416	0·8873	0·9356	0·9917	0·9982
Equatorial diameter	(9)				4″·806	36″·144	15″·845
Polar diameter	(10)					33·736	14·180
Defect of illumination	(11)				0·310	0·300	0·028

(1) *A.E.*, pages 21, 23. (2) *A.E.*, pages 84, 179, 187, 196, 203, 211.

(3) $\sin D \sin\Theta = \cos\delta_\odot \sin(\alpha_\odot - \alpha)$

$\sin D \cos\Theta = \sin\delta_\odot \cos\delta - \cos\delta_\odot \cos(\alpha_\odot - \alpha) \times \sin\delta$

$\cos D = \sin\delta_\odot \sin\delta + \cos\delta_\odot \cos(\alpha_\odot - \alpha) \times \cos\delta$

(4) $\sin^2 D = (\sin D \sin\Theta)^2 + (\sin D \cos\Theta)^2$; D is treated as positive.

(5) Θ is determined from the smaller of $\tan\Theta$ or $\cot\Theta$. (6) *A.E.*, pages 161, 168, 172, 174, 175

(7) $\sin i = (R \sin D)/r$; $\cos i = (\Delta - R\cos D)/r$; $\sin^2 i + \cos^2 i = 1$

(8) For the Moon, $k = \frac{1}{2}(1 - \cos D)$; for the planets $k = \frac{1}{2}(1 + \cos i)$

(9) For Mars, Jupiter, Saturn the constants are: 9″·36, 196″·94, 166″·66.

(10) For Jupiter, Saturn the constants are: 183″·82, 149″·14.

(11) The defect of illumination, $q = (1 - k) \times$ equatorial diameter.

No correction is required for the difference between E.T. and U.T. and the values obtained can be used for 0^h U.T. Five decimals are uniformly retained in the above calculations, although they are not always either required or justified.

also published in *Trans. I.A.U.*, **10**, 305, 1960. The adopted values of g are as follows.

	Visual	Photographic
Ceres	3·38	4·00
Pallas	4·51	5·06
Juno	5·58	6·33
Vesta	3·55	4·22

For Pluto a constant magnitude of 15 was adopted for 1960–1972; the adopted value of g for 1973 onwards is $-1{\cdot}01$.

Example 11.3. *The stellar magnitudes of the planets and the brilliancies of the disks of Mercury and Venus*

1960 March 7 at 0^h E.T., except for Mars
1960 April 6 at 0^h E.T., for Mars

The basic data are taken from example 11.2.

	MERCURY	VENUS	MARS	JUPITER	SATURN
r	0·35279	0·72750	1·39950	5·29205	10·05988
Δ	0·65895	1·44299	1·94736	5·44869	10·51784
$r\Delta$	0·23247	1·04978	2·72533	28·83474	105·80821
	°	°	°		
i	156·5	39·23	29·394	—	—
$5 \log r\Delta$	−3·17	+0·11	+2·177	+7·300	+10·123
constant + $f(i)$ (see note)	+5·34	−3·46	−0·86	−8·93	− 9·30
Stellar magnitude	+2·17	−3·35	+1·32	−1·63	+ 0·82

For the brilliancy of the disks:

Mercury : $k = 0{\cdot}0416$ $\qquad L = 11{\cdot}16\, k/r^2\Delta^2 = 8{\cdot}59$

Venus : $k = 0{\cdot}8873$ $\qquad L = 70{\cdot}73\, k/r^2\Delta^2 = 56{\cdot}95$

The quantity " constant + $f(i)$ " is usually taken directly from manuscript tables with argument i. For Saturn (Table 12.1 and *A.E.*, page 374): $(U' + \omega - U) = -5^\circ{\cdot}28$ and $B = +24^\circ{\cdot}22$, $\sin B = +0{\cdot}410$. Examples are not given for Uranus, Neptune, and the minor planets, since the calculations are identical in principle with that for Jupiter.

D. EPHEMERIS FOR PHYSICAL OBSERVATIONS OF THE MOON

The rotation of the Moon is a motion about its centre of mass that is characterised by small periodic variations from a mean rotational motion which conforms to the empirical laws formulated by J. Cassini in 1721 to describe the rotation as far as it had then become known from observation. The mean period of rotation is equal to the mean sidereal period of revolution around the Earth, and the mean plane of the lunar equator intersects the ecliptic at a constant inclination, in the line of nodes of the lunar orbit, with the descending node of the equator at the ascending node of the orbit. The oscillation of the actual rotational motion about this mean rotation is called the dynamical libration or physical libration.

On the average, therefore, the same hemisphere of the Moon is always turned towards the Earth; but, because of the periodic oscillation in the position of the lunar surface, due to the physical libration, and to the much larger apparent oscillations known as optical librations, which are due to variations in the geometric position of the Earth relative to the lunar surface during the course of the orbital motion of the Moon, about 59 per cent of the surface can be observed altogether.

The point on the surface of the Moon where it is intersected by the lunar radius that would be directed towards the centre of the Earth, were the Moon to be at the mean ascending node when the node coincided with either the mean perigee or mean apogee, defines the mean centre of the apparent disk. This point is the origin of the system of selenographic coordinates on the surface of the Moon. Selenographic longitudes are measured from the lunar meridian that passes through the mean centre of the apparent disk, positive in the direction towards *Mare Crisium*, i.e., towards the west on the geocentric celestial sphere.* Selenographic latitudes are measured from the lunar equator, positive towards the north limb; i.e., they are positive in the hemisphere containing *Mare Serenitatis*.

The position angle of the axis is the angle that the lunar meridian through the apparent central point of the disk towards the north lunar pole forms with the celestial meridian through the central point, measured eastwards from the north point of the disk.

The displacement, at any time, of the mean centre of the disk from the apparent centre, represents the amount of the libration, and is measured by the selenographic coordinates of the apparent centre of the disk at the time. These coordinates are the sums of the geocentric optical and physical librations in longitude and latitude. The tabular selenographic longitude and latitude of the Earth are the geocentric selenographic coordinates of the apparent central point of the disk; at this point on the surface of the Moon, the Earth is in the selenocentric zenith. When the libration in longitude, that is the selenographic longitude of the Earth, is positive, the mean central point of the disk is displaced eastwards on the celestial sphere, exposing to view a region on the west limb. When the libration in latitude, or selenographic latitude of the Earth, is positive, the mean central point of the disk is displaced towards the south, and a region on the north limb is exposed to view. Similarly for the tabular selenographic coordinates of the Sun, which determine the regions of the lunar surface that are illuminated.

The formulae for the optical librations are derived from the geometric definition of the mean centre of the disk. The physical librations are determined from the dynamical theory of the rotation of the Moon. In the calculation of the physical ephemeris, the formulae and constants for the physical librations and the value $1^\circ\ 32'\cdot 1$ for the inclination of the mean lunar equator to the ecliptic, that were determined by Hayn (*Abhandlungen der mathematisch-physischen Klasse der Königlichen Sächsischen Gesellschaft der Wissenchaften*, **30**, page 49, 1907) have been used. The ephemeris is calculated from the apparent coordinates of the Moon and the Sun, and therefore aberration is fully included, excepting the inappreciable difference between the light-times from the Sun to the Moon and from the Sun to the Earth.

The fraction illuminated and the position of the bright limb are calculated as explained in sub-section C. The age of the Moon is the number of days elapsed since the immediately preceding new moon.

*See note on page 523.

Geocentric librations

The optical librations are calculated from formulae introduced by Encke (*Berliner Astronomisches Jahrbuch für* 1843, pages 283-299).

On the selenocentric celestial sphere (see figure 11.4) the Earth is diametrically opposite the geocentric position of the Moon. From the definition of the mean centre of the disk, the condition that the centre of the apparent disk of the Moon be at the mean centre, and thus that the geocentric librations in longitude and latitude simultaneously vanish, is that $\lambda = ☾ = ☊$, where λ is the geocentric longitude of the Moon, ☾ is the mean longitude of the Moon, and ☊ is the longitude of the mean ascending lunar node. Under this condition, on the selenocentric celestial sphere the Earth is at the descending lunar node, and the prime meridian on the Moon is 180° from the ascending node.

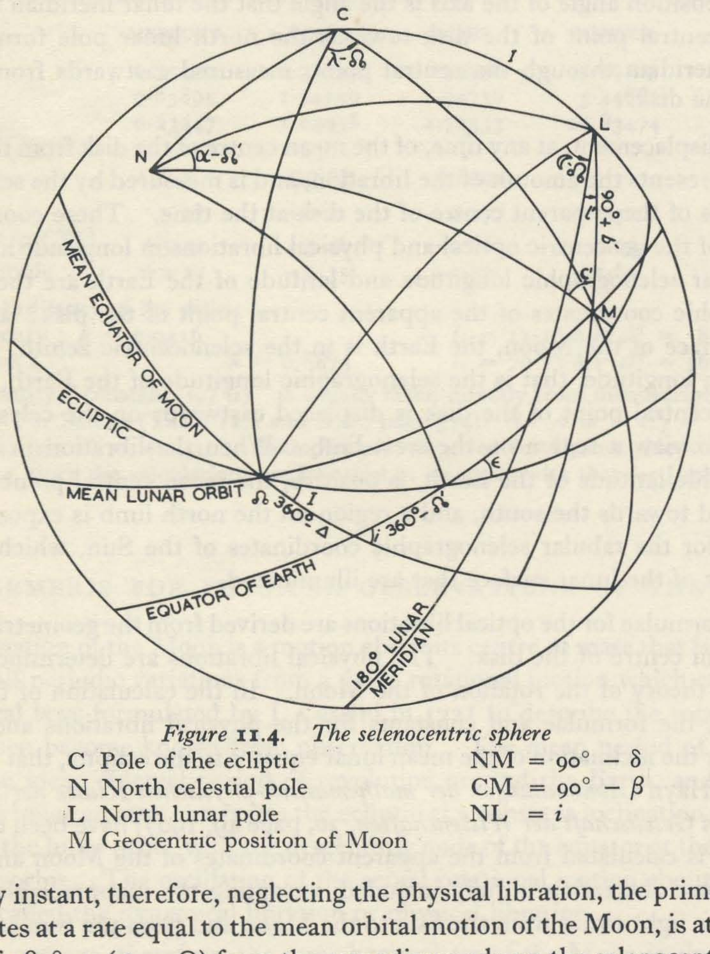

Figure 11.4. *The selenocentric sphere*

C Pole of the ecliptic	NM = 90° − δ
N North celestial pole	CM = 90° − β
L North lunar pole	NL = *i*
M Geocentric position of Moon	

At any instant, therefore, neglecting the physical libration, the prime meridian, which rotates at a rate equal to the mean orbital motion of the Moon, is at an angular distance of $180° + (☾ - ☊)$ from the ascending node on the selenocentric sphere; the Earth is at ecliptic longitude $180° + \lambda$ and latitude $-\beta$, where β is the geocentric latitude of the Moon, and the selenographic coordinates of the Earth represent the optical librations.

The relations which determine the optical librations are obtained from the two triangles MLC and MLN formed on the selenocentric celestial sphere (see figure 11.4) by the geocentric position of the Moon M and the north lunar pole L with the north pole of the ecliptic C and with the north celestial pole N. After some approximations, and the introduction of auxiliaries A, B, μ, defined by:

$$\sin \mu = \tan^2 \tfrac{1}{2}I \sin 2\,(\lambda - \Omega)$$
$$A = \sin I \cos (\lambda - \Omega)$$
$$\tan B = -\tan I \sin (\lambda - \Omega)$$

where I denotes the inclination of the mean lunar equator to the ecliptic, the formulae for the optical librations l' and b' in longitude and latitude, respectively, and for the position angle C' of the axis, may be expressed in the form:

$$l' = \lambda + \mu + Ab' - (☾ + \text{nutation})$$
$$b' = B - \beta$$
$$\sin C' = \sin i \cos (l' + \Delta + ☾ - \Omega) \sec \delta$$
$$= -\sin i \cos (\alpha - \Omega') \sec b'$$

in which i, Δ, Ω' are the elements of the mean lunar equator referred to the celestial equator that are explained in section 4C and tabulated in *A.E.*, page 51, and α, δ are the geocentric right ascension and declination of the Moon; in the lunar ephemeris λ is referred to the true equinox of date, but the mean longitude is referred to the mean equinox, and in the formula for l' the nutation must therefore be added to the tabular value. The auxiliaries are taken from table 11.1; with sufficient accuracy:

$$\mu = 0'{\cdot}617 \sin 2\,(\lambda - \Omega) = 0^\circ{\cdot}0103 \sin 2\,(\lambda - \Omega)$$

The optical librations may be obtained directly from the rigorous formulae:

$$\cos (☾ + l' - \Omega) \cos b' = +\cos (\lambda - \Omega - N) \cos \beta$$
$$\sin (☾ + l' - \Omega) \cos b' = +\sin (\lambda - \Omega - N) \cos \beta \cos I - \sin \beta \sin I$$
$$\sin b' = -\sin (\lambda - \Omega - N) \cos \beta \sin I - \sin \beta \cos I$$

in which N is the nutation in longitude.

Because of the physical libration, the actual inclination and descending node of the lunar equator on the ecliptic are $I + \rho$ and $\Omega + \sigma$, and the angular distance from the descending node of the lunar equator to the prime meridian is $(180^\circ + ☾ - \Omega) + (\tau - \sigma)$, where ρ, σ, τ are determined from the dynamical theory of the rotation. The expressions for ρ, σ, τ originally found by Hayn give for the physical librations in latitude, longitude, and position angle of the axis:

$$\delta b = 108'' \sin (\Gamma' - \Omega + l') + 37'' \sin (\Gamma' - \Omega - l') - 11'' \sin (☾ - \Omega - l')$$

$$\delta l = 12'' \sin (☾ - \Gamma') - 18'' \sin 2\,(\Gamma' - \Omega) - 59'' \sin g_\odot - \{ 108'' \cos (\Gamma' - \Omega + l') - 37'' \cos (\Gamma' - \Omega - l') + 11'' \cos (☾ - \Omega - l') \} \tan b'$$

$$\delta C = - \{ 108'' \cos (\Gamma' - \Omega + l') - 37'' \cos (\Gamma' - \Omega - l') + 11'' \cos (☾ - \Omega - l') \} \sec b'$$

from which the ephemeris is calculated. Putting:

$$M = 0^\circ{\cdot}040 \sin (\Gamma' - \Omega) - 0^\circ{\cdot}003 \sin (☾ - \Omega)$$
$$N = 0^\circ{\cdot}020 \cos (\Gamma' - \Omega) + 0^\circ{\cdot}003 \cos (☾ - \Omega)$$

λ − ☊	μ	A	B	λ − ☊	λ − ☊	μ	A	B	λ − ☊
°	°		°	°	°	°		°	°
0	0·000	+0·0268 −	0·000	180	45	+0·010 +	+0·0189 −	−1·085 +	225
1	·000	·0268	− ·027 +	181	46	·010	·0186	·104	226
2	+ ·001 +	·0268	·054	182	47	·010	·0183	·123	227
3	·001	·0268	·080	183	48	·010	·0179	·141	228
4	·001	·0267	·107	184	49	·010	·0176	·159	229
5	+0·002 +	+0·0267 −	−0·134 +	185	50	+0·010 +	+0·0172 −	−1·176 +	230
6	·002	·0266	·160	186	51	·010	·0169	·193	231
7	·002	·0266	·187	187	52	·010	·0165	·210	232
8	·003	·0265	·214	188	53	·010	·0161	·226	233
9	·003	·0265	·240	189	54	·010	·0157	·242	234
10	+0·004 +	+0·0264 −	−0·267 +	190	55	+0·010 +	+0·0154 −	−1·258 +	235
11	·004	·0263	·293	191	56	·010	·0150	·273	236
12	·004	·0262	·319	192	57	·009	·0146	·287	237
13	·005	·0261	·345	193	58	·009	·0142	·302	238
14	·005	·0260	·371	194	59	·009	·0138	·316	239
15	+0·005 +	+0·0259 −	−0·397 +	195	60	+0·009 +	+0·0134 −	−1·329 +	240
16	·005	·0257	·423	196	61	·009	·0130	·343	241
17	·006	·0256	·449	197	62	·009	·0126	·355	242
18	·006	·0255	·474	198	63	·008	·0122	·368	243
19	·006	·0253	·500	199	64	·008	·0117	·380	244
20	+0·007 +	+0·0252 −	−0·525 +	200	65	+0·008 +	+0·0113 −	−1·391 +	245
21	·007	·0250	·550	201	66	·008	·0109	·402	246
22	·007	·0248	·575	202	67	·007	·0105	·413	247
23	·007	·0247	·600	203	68	·007	·0100	·423	248
24	·008	·0245	·624	204	69	·007	·0096	·433	249
25	+0·008 +	+0·0243 −	−0·649 +	205	70	+0·007 +	+0·0092 −	−1·442 +	250
26	·008	·0241	·673	206	71	·006	·0087	·451	251
27	·008	·0239	·697	207	72	·006	·0083	·460	252
28	·009	·0237	·721	208	73	·006	·0078	·468	253
29	·009	·0234	·744	209	74	·005	·0074	·476	254
30	+0·009 +	+0·0232 −	−0·768 +	210	75	+0·005 +	+0·0069 −	−1·483 +	255
31	·009	·0230	·791	211	76	·005	·0065	·489	256
32	·009	·0227	·814	212	77	·005	·0060	·496	257
33	·009	·0225	·836	213	78	·004	·0056	·502	258
34	·010	·0222	·859	214	79	·004	·0051	·507	259
35	+0·010 +	+0·0219 −	−0·881 +	215	80	+0·004 +	+0·0047 −	−1·512 +	260
36	·010	·0217	·902	216	81	·003	·0042	·516	261
37	·010	·0214	·924	217	82	·003	·0037	·520	262
38	·010	·0211	·945	218	83	·002	·0033	·524	263
39	·010	·0208	·966	219	84	·002	·0028	·527	264
40	+0·010 +	+0·0205 −	−0·987 +	220	85	+0·002 +	+0·0023 −	−1·529 +	265
41	·010	·0202	1·007	221	86	·001	·0019	·531	266
42	·010	·0199	·027	222	87	·001	·0014	·533	267
43	·010	·0196	·047	223	88	+ ·001 +	·0009	·534	268
44	·010	·0193	·066	224	89	·000	+ ·0005 −	·535	269
45	+0·010 +	+0·0189 −	−1·085 +	225	90	0·000	0·0000	−1·535 +	270

The sign is to be taken from the same side as the argument.

λ − ☊	μ	A	B	λ − ☊	λ − ☊	μ	A	B	λ − ☊
°	°		°	°	°	°		°	°
90	0·000	0·0000	−1·535 +	270	135	−0·010 −	−0·0189 +	−1·085 +	315
91	·000	− ·0005 +	·535	271	136	·010	·0193	·066	316
92	− ·001 −	·0009	·534	272	137	·010	·0196	·047	317
93	·001	·0014	·533	273	138	·010	·0199	·027	318
94	·001	·0019	·531	274	139	·010	·0202	1·007	319
95	−0·002 −	−0·0023 +	−1·529 +	275	140	−0·010 −	−0·0205 +	−0·987 +	320
96	·002	·0028	·527	276	141	·010	·0208	·966	321
97	·002	·0033	·524	277	142	·010	·0211	·945	322
98	·003	·0037	·520	278	143	·010	·0214	·924	323
99	·003	·0042	·516	279	144	·010	·0217	·902	324
100	−0·004 −	−0·0047 +	−1·512 +	280	145	−0·010 −	−0·0219 +	−0·881 +	325
101	·004	·0051	·507	281	146	·010	·0222	·859	326
102	·004	·0056	·502	282	147	·009	·0225	·836	327
103	·005	·0060	·496	283	148	·009	·0227	·814	328
104	·005	·0065	·489	284	149	·009	·0230	·791	329
105	−0·005 −	−0·0069 +	−1·483 +	285	150	−0·009 −	−0·0232 +	−0·768 +	330
106	·005	·0074	·476	286	151	·009	·0234	·744	331
107	·006	·0078	·468	287	152	·009	·0237	·721	332
108	·006	·0083	·460	288	153	·008	·0239	·697	333
109	·006	·0087	·451	289	154	·008	·0241	·673	334
110	−0·007 −	−0·0092 +	−1·442 +	290	155	−0·008 −	−0·0243 +	−0·649 +	335
111	·007	·0096	·433	291	156	·008	·0245	·624	336
112	·007	·0100	·423	292	157	·007	·0247	·600	337
113	·007	·0105	·413	293	158	·007	·0248	·575	338
114	·008	·0109	·402	294	159	·007	·0250	·550	339
115	−0·008 −	−0·0113 +	−1·391 +	295	160	−0·007 −	−0·0252 +	−0·525 +	340
116	·008	·0117	·380	296	161	·006	·0253	·500	341
117	·008	·0122	·368	297	162	·006	·0255	·474	342
118	·009	·0126	·355	298	163	·006	·0256	·449	343
119	·009	·0130	·343	299	164	·005	·0257	·423	344
120	−0·009 −	−0·0134 +	−1·329 +	300	165	−0·005 −	−0·0259 +	−0·397 +	345
121	·009	·0138	·316	301	166	·005	·0260	·371	346
122	·009	·0142	·302	302	167	·005	·0261	·345	347
123	·009	·0146	·287	303	168	·004	·0262	·319	348
124	·010	·0150	·273	304	169	·004	·0263	·293	349
125	−0·010 −	−0·0154 +	−1·258 +	305	170	−0·004 −	−0·0264 +	−0·267 +	350
126	·010	·0157	·242	306	171	·003	·0265	·240	351
127	·010	·0161	·226	307	172	·003	·0265	·214	352
128	·010	·0165	·210	308	173	·002	·0266	·187	353
129	·010	·0169	·193	309	174	·002	·0266	·160	354
130	−0·010 −	−0·0172 +	−1·176 +	310	175	−0·002 −	−0·0267 +	−0·134 +	355
131	·010	·0176	·159	311	176	·001	·0267	·107	356
132	·010	·0179	·141	312	177	·001	·0268	·080	357
133	·010	·0183	·123	313	178	− ·001 −	·0268	·054	358
134	·010	·0186	·104	314	179	·000	·0268	− ·027 +	359
135	−0·010 −	−0·0189 +	−1·085 +	315	180	0·000	−0·0268 +	0·000	360

The sign is to be taken from the same side as the argument.

these formulae may be written:

$$\delta l = 0^\circ{\cdot}003 \sin (☾ - \Gamma') - 0^\circ{\cdot}005 \sin 2(\Gamma' - \Omega) - 0^\circ{\cdot}016 \sin g_\odot + \delta C \sin b'$$

$$\delta b = M \cos l' + N \sin l'$$

$$\delta C = (M \sin l' - N \cos l') \sec b'$$

in which $\cos l'$ and $\sec b'$ may be taken as unity, and $\delta C \sin b'$ as $(0{\cdot}018\ \delta C)\ b'$ with b' in degrees; Γ' denotes the longitude of the mean lunar perigee, and $g_\odot$ the mean anomaly of the Sun. The tabular values in the Ephemeris, under the headings of the Earth's selenographic longitude and latitude, are the sums of the optical and physical librations:

$$l = l' + \delta l \qquad b = b' + \delta b \qquad C = C' + \delta C$$

The values for the principal terms of the physical librations obtained from the dynamical theory depend upon six constants of integration, and upon the numerical values adopted for I and for the ratio f of $(C - B)/A$ to $(C - A)/B$ where A, B, C are the principal moments of inertia of the Moon. The terms containing the constants of integration represent a free libration, which has not been detected with certainty by observation and is therefore neglected. The other terms represent a forced libration. The exact value of f is uncertain; in deriving the expressions from which the physical ephemeris is calculated, Hayn adopted $f = 0{\cdot}75$. Hayn later derived improved and more complete expressions, with $f = 0{\cdot}73$ and $I = 1^\circ\ 32'\ 20''$, and also with a series of other values for f ranging from 0·5 to 0·8 (*Astr. Nach.*, **199**, 261, 1914 and **211**, 311, 1920); his results were confirmed and further improved by Koziel (*Acta Astronomica*, ser. a, **4**, 65, 1948). From more recent data, Jeffreys has determined $f = 0{\cdot}67$ (*M.N.R.A.S.*, **117**, 475, 1957), and Watts has found $I = 1^\circ\ 33'\ 50''$ (*A.J.*, **60**, 443, 1955).

Topocentric librations

The tabular librations and position angles of the axis are geocentric values; for precise reductions of observations, they should be reduced to the values at the location of the observer on the surface of the Earth. The differences may reach nearly 1°, and have important effects on the limb-contour. At a fixed point on the Earth the topocentric values undergo a daily variation which represents a diurnal libration due to the parallactic effect of the motion of the point as the Earth rotates. The variations in the geocentric horizontal parallax of the Moon also affect the limb-contour as much as do changes of several units in the last decimal of the tabular librations, an effect that could be regarded as a " parallactic libration " in addition to the other librations.

Topocentric librations and position angles may be obtained either by differential corrections of the tabular values, or by direct calculation. The topocentric *optical* librations in longitude and latitude, and the topocentric position angle of the axis affected by only optical libration, may be directly calculated by using the apparent topocentric coordinates of the Moon instead of the geocentric coordinates in identically the same formulae as already given for the geocentric values of l', b', and $\sin C'$; the tabular geocentric physical librations may be used without correction. For this purpose, the apparent topocentric right ascension and declination

Example 11.4. The optical and physical librations of the Moon, and the Earth's selenographic longitude and latitude

1960 March 7 at 0^h U.T.

The basic data are taken from the Ephemeris for 1960 March 7 at 0^h E.T.

Optical librations

Nutation in longitude (*A.E.*, p. 20)		− 0°·0002
Apparent longitude of Moon (*A.E.*, p. 54)	λ =	93·164
Mean node (*A.E.*, p. 51) plus nutation	☊ =	175·230
λ − ☊		277·934
B (Table 11.1)		+ 1·520
Latitude of Moon (*A.E.*, p. 54)	β =	− 5·222
$B - \beta = b'$, libration in latitude		+ 6·742
A (Table 11.1) = +0·0037		
μ (Table 11.1)		− 0·003
Ab'		+ 0·025
Mean longitude (*A.E.*, p. 51) plus nutation	☾ =	94·217
Δ (*A.E.*, p. 51)		355·506
$\lambda + \mu + Ab'$ − ☾ = l', libration in longitude		− 1·031
$l' + \Delta$ + ☾ − ☊		273·462
i (*A.E.*, p. 51)		24·972
Apparent declination of Moon (*A.E.*, p. 84)	δ =	+18·183
$\sin i$		+ 0·42217
$\cos (l' + \Delta$ + ☾ − ☊)		+ 0·06039
$\sec \delta$		+ 1·05256
Product = $\sin C'$		+ 0·02683
C', position angle of axis		+ 1°·538

Physical librations

☾ (*A.E.*, p. 51)	94°·2	Γ' − ☊	87°·8	sin (Γ' − ☊)	+1·00	cos (Γ' − ☊)	+0·04
Γ' (*A.E.*, p. 51)	263·0	☾ − ☊	279·0	sin (☾ − ☊)	−0·99	cos (☾ − ☊)	+0·16
☊ (*A.E.*, p. 51)	175·2			M	+0°·043	N	+0°·001
				sin l'	−0·02	cos l'	+1·00
☾ − Γ'	191·2	sin (☾ − Γ')	−0·19	δb	+0°·043	$\delta C \cos b'$	−0°·002
2 (Γ' − ☊)	175·6	sin 2 (Γ' − ☊)	+0·08				
$g_\odot$ (*A.E.*, p. 50)	62·5	$\sin g_\odot$	+0·89	$\tan b'$	+0·12	$\sec b'$	+1·01
$\delta C \sin b'$	−0°·000			δl	−0°·015	δC	−0°·002

$$\delta l = +0°{\cdot}003(-0{\cdot}19) - 0°{\cdot}005(+0{\cdot}08) - 0°{\cdot}016(+0{\cdot}89) + (-0°{\cdot}000) = -0°{\cdot}015$$

Selenographic coordinates of the Earth

Longitude	$l = l' + \delta l$	= −1°·031 − 0°·015	= −1°·046
Latitude	$b = b' + \delta b$	= +6·742 + 0·043	= +6·785
Position angle	$C = C' + \delta C$	= +1·538 − 0·002	= +1·536

The correction for the difference between E.T. and U.T. is insignificant.

of the Moon are obtained by applying corrections for geocentric parallax to the apparent geocentric values, and are transformed to ecliptic coordinates by the usual conversion formulae to obtain the apparent topocentric longitude and latitude. The table of μ, A, B has been included in the Ephemeris to facilitate this calculation. This table is being omitted as from the edition for 1962, but the method may still be used with table 11.1 included here; the physical libration continues to be tabulated separately. However, the application of differential corrections, as described below, should prove simpler.

Differential corrections for obtaining topocentric librations from the tabular geocentric values, in a form convenient for practical calculation, have been derived by, among others, Atkinson (*M.N.R.A.S.*, **111**, 448-454, 1951) in terms of the geocentric zenith distance (z) and parallactic angle (Q) of the Moon. From the geocentric right ascension and declination of the Moon, and the latitude (ϕ) and local sidereal time, the values of z and Q are calculated by the usual formulae:

$$\sin z \sin Q = \cos \phi \sin h$$
$$\sin z \cos Q = \cos \delta \sin \phi - \sin \delta \cos \phi \cos h$$
$$\cos z = \sin \delta \sin \phi + \cos \delta \cos \phi \cos h$$

where h is the local hour angle of the Moon. The topocentric parallax π' is obtained from the geocentric horizontal parallax π by:

$$\pi' = \pi (\sin z + 0{\cdot}0084 \sin 2z)$$

The corrections to the tabular geocentric librations (l, b) and position angle (C) inclusive of the physical librations are:

$$\Delta l = -\pi' \sin (Q - C) \sec b$$
$$\Delta b = +\pi' \cos (Q - C)$$
$$\Delta C = +\sin (b + \Delta b) \Delta l - \pi' \sin Q \tan \delta$$

The tabular values should be interpolated with second differences to the time of observation.

In the special case of a total solar eclipse, the selenographic coordinates of an observer who is on the axis of the shadow of the Moon are diametrically opposite the selenographic coordinates of the Sun at that instant. In the reduction of eclipse observations, therefore, the topocentric librations at any point where the eclipse is visible may be determined by differential corrections to the tabular selenographic coordinates of the Sun, and formulae for this purpose have been developed by Murray (*M.N.R.A.S.*, **114**, 676-679, 1954).

The selenographic position of the Sun

The selenographic longitude and latitude of the subsolar point on the lunar surface are obtained immediately by replacing the geocentric ecliptic coordinates of the Moon in the formulae for the selenographic coordinates of the Earth by the *heliocentric* ecliptic coordinates (λ_H, β_H) of the Moon.

Expressions for the heliocentric coordinates are readily found from the plane triangles formed in space by the Sun, Earth, and Moon, and the projections of the heliocentric and geocentric distances of the Moon on the plane of the ecliptic. With sufficient accuracy, they may be written:

Example 11.5. Differential corrections for obtaining topocentric librations

The calculation of the differential corrections for obtaining topocentric librations from the tabular geocentric values is illustrated for the time used as the disappearance of Z.C. 2833 on 1960 September 1 at Strasbourg in example 10.9, and some of the initial quantities are obtained directly from it.

U.T. $19^h\ 01^m\ 07^s{\cdot}81$ = Sept. $1^d{\cdot}792$

h	$+22^h\ 53^m\ 06^s{\cdot}89$	δ	$-17°\ 42'\ 10''{\cdot}3$	ϕ	$+48°\ 35'\ 02''{\cdot}0$
$\sin h$	$-0{\cdot}2877$	$\sin\delta$	$-0{\cdot}3041$	$\sin\phi$	$+0{\cdot}7499$
$\cos h$	$+0{\cdot}9577$	$\cos\delta$	$+0{\cdot}9527$	$\cos\phi$	$+0{\cdot}6615$
		$\tan\delta$	$-0{\cdot}3192$	$\cos\phi\cos h$	$+0{\cdot}6335$

The geocentric values are:

l_0	$-0°{\cdot}68$	b_0	$-5°{\cdot}64$	C_0	$351°{\cdot}24$

$\sin z \sin Q$	$-0{\cdot}1903$	$\sin z$	$+0{\cdot}9268$
$\sin z \cos Q$	$+0{\cdot}9071$	$\sin 2z$	$+0{\cdot}6960$
$\cos z$	$+0{\cdot}3755$		
$\tan Q$	$-0{\cdot}2098$		
Q	$348°{\cdot}15$	$\sin Q$	$-0{\cdot}205$
$Q - C_0$	$-3{\cdot}09$	$\sin(Q - C_0)$	$-0{\cdot}0539$
		$\cos(Q - C_0)$	$+0{\cdot}9985$
π	$1°{\cdot}0013$	π'	$0°{\cdot}934$
		$\sec b_0$	$+1{\cdot}005$
Δl	$+0°{\cdot}051$		
$l = l_0 + \Delta l$	$-0{\cdot}63$		
Δb	$+0{\cdot}93$		
$b = b_0 + \Delta b$	$-4{\cdot}71$	$\sin(b_0 + \Delta b)$	$-0{\cdot}082$
ΔC	$-0{\cdot}07$		
$C = C_0 + \Delta C$	$351{\cdot}17$		

These topocentric values l, b, C are in exact agreement with those derived in example 10.9.

$$\lambda_H = \lambda_\odot + 180° + \frac{8{\cdot}80}{60\,\pi R} \times 57°{\cdot}296 \cos\beta \sin(\lambda_\odot - \lambda) \quad *$$

$$\beta_H = \frac{8{\cdot}80}{60\,\pi R}\,\beta$$

in which π is the equatorial horizontal parallax of the Moon expressed in minutes of arc, $\lambda_\odot$ is the true longitude of the Sun referred to the true equinox of date, and R is the radius vector of the Sun. The selenographic longitude and latitude of the Sun are:

$$l_\odot = l'_\odot + \delta l_\odot \qquad b_\odot = b'_\odot + \delta b_\odot$$

where

$$l'_\odot = \lambda_H + \mu + Ab'_\odot - (☾ + \text{nutation}) \qquad b'_\odot = B - \beta_H$$

$$\delta l_\odot = 0°{\cdot}003 \sin(☾ - \Gamma') - 0°{\cdot}005 \sin 2(\Gamma' - \Omega) - 0°{\cdot}016 \sin g_\odot + (M \sin l'_\odot - N \cos l'_\odot) \tan b'_\odot$$

$$\delta b_\odot = M \cos l'_\odot + N \sin l'_\odot$$

in which A, B, μ are taken from table 11.1 as before, with argument $\lambda_H - \Omega$.

*8″·794 used for the solar parallax for 1968 onwards.

Subtracting $l_{\odot}$ from 90° or 450° gives the selenographic colongitude of the Sun ($c_{\odot}$) tabulated in the physical ephemeris instead of the longitude. The colongitude is convenient for determining the exact position of the terminator on the surface of the Moon. The subsolar point at $l_{\odot}$, $b_{\odot}$ is the pole of the great circle on the lunar surface that bounds the illuminated hemisphere. The morning terminator, where the Sun is rising on the Moon, is at selenographic longitude $l_{\odot} - 90° = 360° - c_{\odot}$; the colongitude of the Sun is therefore the *east* selenographic longitude of the morning terminator.† The evening terminator is at longitude $l_{\odot} + 90° = 180° - c_{\odot}$. When $c_{\odot} = 0°$, the Sun is rising at selenographic longitude 0°; therefore $c_{\odot}$ is approximately 0° at first quarter, when the morning terminator is approximately at longitude 0°. At full moon, last quarter, and new moon, respectively, $c_{\odot}$ is approximately 90°, 180°, and 270°, and the morning terminator is approximately at selenographic longitudes 270°, 180°, and 90°.

At a point on the lunar surface at selenographic longitude η and latitude θ, sunrise occurs approximately when $c_{\odot} = 360° - \eta$, noon when $c_{\odot} = 90° - \eta$, and sunset when $c_{\odot} = 180° - \eta$. The exact altitude H of the Sun above the lunar horizon at any time may be calculated from:

$$\sin H = \sin b_{\odot} \sin \theta + \cos b_{\odot} \cos \theta \sin (c_{\odot} + \eta)$$

Example 11.6. *The selenographic coordinates of the Sun*

The data are taken from the Ephemeris for 1960 March 7 at 0^h E.T. and from example 11.4; the subsequent correction from 0^h E.T. to 0^h U.T. is insignificant.

App. long. and r. v. of Sun (*A.E.*, p. 20, 21)	$\lambda_{\odot}$	346°·434			R	0·9925
App. long., lat., H.P. of Moon (*A.E.*, p. 54)	λ	93·164	β	−5°·222	π	54′·29
$F_0 = 8{\cdot}80/60\ \pi R$	$\lambda_{\odot} - \lambda$	253·270			F_0	0·00272
	$\sin(\lambda_{\odot} - \lambda)$	−0·958	$\cos\beta$	0·996		

For formulae, see text	λ_H	166°·285			β_H	−0°·014
Mean node, long. of Moon (*A.E.*, p. 51*)	☊	175·230	☾	94°·217		
λ_H − ☊ is argument for table 11.1	λ_H − ☊	351·055	μ	−0·003	B	+0·239
	A	+0·0265	$Ab'_{\odot}$	+0·007	$b'_{\odot}$	+0·253
$l'_{\odot} = \lambda_H + \mu + Ab'_{\odot}$ − ☾			$l'_{\odot}$	72·072		

From example 11.4.

sin (☾ − Γ')	−0·19	M	+0°·043	$\sin l'_{\odot}$	+0·95	$\tan b'_{\odot}$	0·00
sin 2(Γ' − ☊)	+0·08	N	+0·001	$\cos l'_{\odot}$	+0·31		
$\sin g_{\odot}$	+0·89	$(M \sin l'_{\odot} - N \cos l'_{\odot})$	+0°·041			$\delta b_{\odot}$	+0°·014

$$\delta l_{\odot} = 0°{\cdot}003(-0{\cdot}19) - 0°{\cdot}005(+0{\cdot}08) - 0°{\cdot}016(+0{\cdot}89) + 0°{\cdot}041(0{\cdot}00) = -0°{\cdot}015$$

Selenographic long., $l'_{\odot} + \delta l_{\odot}$, and lat., $b'_{\odot} + \delta b_{\odot}$	$l_{\odot}$	72°·057	$b_{\odot}$	+0°·267
Selenographic colongitude = 90° − $l_{\odot}$	$c_{\odot}$	17·943		

* Including nutation (*A.E.*, p. 20) −0°·0002

† See note on page 523.

E. THE ROTATIONS OF THE PLANETS

The numerical values adopted for the elements of the rotational motions of the planets in calculating the physical ephemerides are stated in sub-section G in the explanations for each individual planet. Referred to the plane of the equator of the Earth, the position of the equatorial plane of a planet is represented by its inclination $90° - \delta_0$ and the right ascension of its ascending node $90° + \alpha_0$, where α_0 and δ_0 are the right ascension and declination of the point on the celestial sphere towards which the axis of rotation of the planet is directed; this point is that pole of the equator of the planet from which the direction of rotation appears counter-clockwise. The inclination and node continually vary because of the precession and nutation of the axis of the Earth and because of the similar motion of the axis of the planet.

The variations of α_0 and δ_0 due to the precessional motions of the equator of the Earth and the equinox are:

$$\Delta\alpha_0 = (m + n \sin \alpha_0 \tan \delta_0)\,(t - t_0)$$

$$\Delta\delta_0 = (n \cos \alpha_0)\,(t - t_0)$$

where m and n are the general precessions in right ascension and declination.

On a fixed plane of reference, the rate of precession of the node of the equator of a planet under the action of the Sun is:

$$-\frac{3}{2}\frac{C - A}{C}\frac{n_0^2}{\omega\,(1 - e_0^2)^{\frac{3}{2}}}\cos\gamma$$

where $A = B < C$ are the principal moments of inertia of the planet, ω is its angular rate of rotation, n_0 is the mean orbital motion, e_0 is the orbital eccentricity, and γ is the inclination of the equator of the planet to the fixed reference plane. Under the action of a satellite of mass m, the rate of precession is:

$$-\frac{3}{2}\frac{C - A}{C}\frac{m}{M + m_0}\left(\frac{a_0}{a}\right)^3\frac{n_0^2}{\omega\,(1 - e^2)^{\frac{3}{2}}}\cos\gamma$$

in which: M is the mass of the Sun; m_0, a_0 are the mass and mean distance of the planet; and a, e are the mean distance and orbital eccentricity of the satellite.

Mercury and Venus

Surface details can be seen only with difficulty on Mercury and Venus (which is covered by cloud); the rotational motions of these planets have only been established by radar techniques in recent years. The rotation of Mercury is direct with period 58·66 days, and the rotation of Venus is retrograde with period 243 days. The physical ephemerides of these two planets in the Ephemeris are at present limited to the conditions of illumination of the disks, the phase, the phase angle, position of the bright limb, brilliancy, and stellar magnitude, for use in reducing observations of position.

Mars

Because of the detail that is observable on the solid surface of Mars, and the accuracy with which it enables the rotational motion to be determined, the physical ephemeris for Mars is more comprehensive than for any other planet. From the tabular quantities which it includes, the heliocentric and geocentric aspects of the disk and the conditions of illumination at every point may be completely determined (see sub-section G).

For the rate of precession, H. Struve (*Mém. Acad. Imp. Sci. St.-Pétersbourg*, ser. 8, **8,** 64-65, 1898), adopting $(C - A)/C = 0{\cdot}0050$, obtained $-7''{\cdot}07$ per Julian year on the 1880·0 orbit of Mars. Lowell (*A.J.*, **28,** 169-171, 1914), taking $(C - A)/C = 0{\cdot}004935$, likewise obtained $-7''{\cdot}08$ per terrestrial year, or $-13''{\cdot}31$ per Martian year, on the orbit of Mars.

Jupiter and Saturn

The visible surfaces of these planets are composed of clouds. On Jupiter, as on the Sun, the equatorial region rotates more rapidly than the polar regions; consequently, two different systems of planetographic coordinates have been established on its surface, one for the equatorial region and another for the other parts of the surface. The physical ephemeris for Jupiter is similar to that for Mars but it is not quite so comprehensive.

Sampson (*Mem. R. A. S.*, **63,** chap. IV, 1921), adopting $\frac{3}{2}(C - A)/C = 0{\cdot}111$, obtained for the mean value of the precession under the actions of the Sun and the four Galilean satellites:

$$\frac{d\Psi}{dt} = -0^\circ{\cdot}00000\ 1129\ \frac{3}{2}\frac{C - A}{C \sin I} = -0^\circ{\cdot}00000\ 231 \text{ per day}$$

where I is the inclination of the equator of Jupiter to the 1900·0 orbit of Jupiter, and Ψ denotes the longitude of the ascending node of the equator of Jupiter on this fixed reference plane, measured from the fixed mean equinox of 1900·0 along the fixed ecliptic of 1900·0 to the node of the orbit of Jupiter and then along the orbit to the equator of Jupiter.

A physical ephemeris for Saturn is given for the first time in the Ephemeris for 1960. The plane of the rings of Saturn is assumed to coincide with the equatorial plane of Saturn; the elements of the equator are therefore the same as the elements of the rings, but are not needed in calculating the physical ephemeris which is limited to the light-time, stellar magnitude, apparent equatorial and polar diameters, and the phase conditions.

H. Struve (*Pub. Obs. Cent. Nicolas*, ser. 2, **11,** 234-236, 1898), adopting $\frac{3}{2}(C - A)/C = 0{\cdot}10$, and 1/4700 for the mass of Titan, obtained $-0''{\cdot}46$ per Julian year for the average precessional motion, during the nineteenth century, of the equator of Saturn on the ecliptic, under the actions of the Sun and Titan.

Uranus, Neptune, and Pluto

The small disks of Uranus and Neptune do not show any markings from which the rotation can be directly determined, and Pluto has no observable disk. The rotations of these planets must be inferred from indirect evidence.

The plane of the equator of Uranus is presumably the same as the common orbital plane of the satellites, which is inclined at 98° to the ecliptic, since this orbital plane does not have the secular motion that otherwise would be caused by the large oblateness of Uranus shown by direct observation of the disk. The planet therefore rotates around an axis which is nearly parallel to the plane of the ecliptic; spectroscopic and photometric observations give a period of rotation of about 11 hours.

The elements of the equator of Neptune may be inferred from the secular variation of the orbital plane of Triton. According to the determination by Eichelberger and Newton (*A.P.A.E.*, **9**, part III, 325-326, 1926) the north pole of the equator of Neptune is at $\alpha_0 = 295°{\cdot}2$, $\delta_0 = +41°{\cdot}3$ (1900·0). The period of rotation is uncertain.

F. THE ILLUMINATED DISKS OF MERCURY AND VENUS

The quantities tabulated in these ephemerides are k, i, Θ, L, and the stellar magnitude; they are defined at the foot of each ephemeris and are calculated from the formulae developed in sub-section C.

As described in sub-section C, the terminator is a semi-ellipse whose major axis is a diameter of the planet in position angle $\Theta + 90°$ and whose semi-minor axis is of length (semi-diameter) $\times$ $(1 - 2k)$ in position angle Θ when k is less than 0·5, and of length (semi-diameter) $\times$ $(2k - 1)$ in position angle $\Theta + 180°$ when k is greater than 0·5.

When i is greater than 90°, k is less than 0·5, i.e. the planet is horned, and the correction to an observation of the cusp for defective illumination in declination is:

$$\text{semi-diameter} \times (1 \mp \sin \Theta)$$

the sign being taken to make the quantity inside the bracket less than unity.

When i is less than 90°, k is greater than 0·5, i.e. the planet is gibbous. If angles ϕ and ψ in the first quadrant are formed from:

$$\sin \phi = \sin i \sin \Theta \qquad \sin \psi = \sin i \cos \Theta$$

the correction for defective illumination in right ascension is:

$$\text{sidereal time of semi-diameter passing the meridian} \times (1 - \cos \phi)$$

and that in declination is:

$$\text{semi-diameter} \times (1 - \cos \psi)$$

When the corrections are very small they are sensibly equal to:

$$\text{sidereal time of semi-diameter passing the meridian} \times \tfrac{1}{2} \sin^2 i \sin^2 \Theta$$

and $$\text{semi-diameter} \times \tfrac{1}{2} \sin^2 i \cos^2 \Theta$$

In general dichotomy is not at greatest elongation, especially for Mercury, because of the orbital inclinations and eccentricities.

The formulae for the brilliancies of the disks and for the stellar magnitudes are given in sub-section C; but for the definition of the greatest brilliancy of Venus see section 8C.

G. EPHEMERIDES FOR PHYSICAL OBSERVATIONS OF MARS, JUPITER, AND SATURN

The physical ephemerides for Mars, Jupiter, and Saturn contain: the time required for light to travel from the planet to the Earth, calculated with the value * $498^s\cdot58$ for the time taken by light to travel unit distance, as deduced from the adopted solar parallax and the measured velocity of light; the apparent diameter, calculated from the same semi-diameter at unit distance as in the fundamental ephemerides; the stellar magnitude, the phase angle (i), the position angle (Q) and the angular amount (q) of the greatest defect of illumination; and for Mars the phase (k), calculated by the formulae developed in sub-section C. No other quantities are given for Saturn; but the ephemerides for Mars and Jupiter include quantities representing the geometric aspects of the rotating surface of the planet in relation to both the Sun and the Earth upon which depend the illumination over the surface by the Sun and the appearance in detail of the disk as seen from the Earth.

The variations in phase for these planets are small. The defect of phase is $\frac{1}{2}(1 - \cos i)$; approximately, it is a maximum when the planet is at quadrature and the phase angle is $\sin^{-1}(R/r)$. The extreme possible value of $\sin i$ is therefore about $1\cdot017/a\,(1 - e)$, and even for Mars the defect of phase cannot exceed about 0·16.

The observed aspect of the disk depends upon the positions of the Earth and the Sun relative to the different areas of the surface of the planet, or equivalently upon the apparent positions of the Earth and the Sun on the planetocentric celestial sphere at the different points of the surface. To represent these positions, coordinate systems are defined on the planetocentric sphere, by the plane of the equator of the planet and the plane of its orbit, in the same way as right ascension and declination, and celestial longitude and latitude, are defined on the geocentric celestial sphere by the equator of the Earth and the ecliptic. Because of the mathematically infinite radius of the celestial sphere, the same fundamental reference circles are defined on the geocentric sphere as on the planetocentric sphere by the orbital and equatorial planes of the Earth and the other planets.

On a planetocentric sphere (see figure 11.5) the apparent position of the Earth is diametrically opposite the geocentric position of the planet, and the Sun is opposite the heliocentric position of the planet. The planetocentric angular distance of the Earth from the equator of the planet, denoted by D_E, is numerically equal and opposite in sign to the geocentric angular distance of the planet from the plane of the equator of the planet. The ascending node of the orbit of the planet on its equator is the vernal equinox of the planet; the angular distance in the plane of the planetary equator from this point eastwards to the great circle through the Earth and the celestial pole of the planet, denoted by A_E, is equal to the geocentric longitude of the planet measured in the plane of its equator from its autumnal equinox, or descending node of its orbit on its equator. The coordinates A_E and

*$499^s\cdot012$ for 1968 onwards.

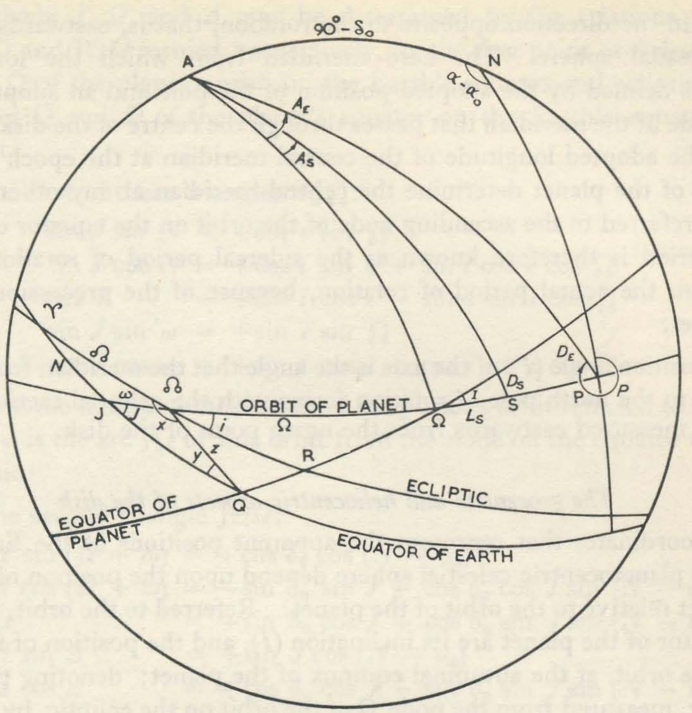

Figure 11.5. *Planetocentric sphere*

A North pole of rotation of the planet
N North celestial pole
P Geocentric position of the planet (α, δ)
S Heliocentric position of the planet $Q\Omega = \Delta$

D_E are known as the planetocentric right ascension and declination of the Earth; but the terms planetocentric right ascension and declination are also applied to the coordinates of objects on the planetocentric sphere that are referred to the equator and vernal equinox of the Earth, and care is necessary to avoid confusion.

Similarly, referred to the equator and vernal equinox of the planet, the planetocentric right ascension of the Sun, A_S, is equal to the heliocentric longitude of the planet measured in the plane of its equator from its autumnal equinox; and the planetocentric declination of the Sun, D_S, is numerically equal and opposite in sign to the heliocentric angular distance of the planet from the plane of its equator. The planetocentric longitude of the Sun, denoted by L_S and measured in the plane of the orbit of the planet from its vernal equinox, is equal to the heliocentric orbital longitude of the planet measured from its autumnal equinox; it is tabulated only for Mars.

These coordinates of the Earth and the Sun on the planetocentric sphere determine the geocentric and heliocentric aspects of the planetographic coordinate systems on the surface of the planet, to which the markings on the disk are referred. Planetographic longitudes on the surfaces of Mars and Jupiter are reckoned from

0° to 360° in the direction opposite to the rotation, that is, eastwards on the geocentric celestial sphere. The zero meridian from which the longitudes are measured is defined by the adopted position of the pole and an adopted value for the longitude of the meridian that passes through the centre of the disk at a selected epoch. The adopted longitude of the central meridian at the epoch and the rate of rotation of the planet determine the central meridian at any other time. The rotation is referred to the ascending node of the orbit on the equator of the planet, and the period is therefore known as the sidereal period of rotation; it differs slightly from the actual period of rotation, because of the precession of the axis of the planet.

The position angle (P) of the axis is the angle that the meridian from the centre of the disk to the north pole of rotation forms with the celestial meridian through the centre, measured eastwards from the north point of the disk.

The geocentric and heliocentric aspects of the disk

The coordinates that represent the apparent positions of the Earth and the Sun on the planetocentric celestial sphere depend upon the position of the equator of the planet relative to the orbit of the planet. Referred to the orbit, the elements of the equator of the planet are its inclination (I), and the position of its ascending node on the orbit, at the autumnal equinox of the planet; denoting the longitude of this node, measured from the node ☊ of the orbit on the ecliptic, by Ω, its longitude reckoned from the first point of Aries is Ω + ☊, where ☊ is the node of the orbit.

The elements I and Ω, and the arc Δ of the equator of the planet from its ascending node Q on the celestial equator to its ascending node Ω on the orbit, may be determined from the right ascension and declination of the pole A of the planet (α_0, δ_0) by the relations in the two triangles formed by the node Q of the equator of the planet on the celestial equator and the node ☊ of the orbit on the ecliptic with the first point of Aries ♈, and with the node Ω of the equator of the planet on the orbit. From the first of these triangles ☊Q♈, denoting the obliquity of the ecliptic by ϵ, the auxiliary angles x, y, z are calculated from:

$$\begin{aligned}
\sin z \sin x &= +\sin \epsilon \cos \alpha_0 \\
\sin z \cos x &= -\cos \epsilon \cos ☊ \cos \alpha_0 - \sin ☊ \sin \alpha_0 \\
\cos z &= +\cos \epsilon \sin ☊ \cos \alpha_0 - \cos ☊ \sin \alpha_0 \\
\sin z \sin y &= +\sin \epsilon \sin ☊ \\
\sin z \cos y &= +\cos \epsilon \sin ☊ \sin \alpha_0 + \cos ☊ \cos \alpha_0
\end{aligned}$$

From the second triangle ☊QΩ, in which the angles are I, $180° + (i - x)$, and $90° + (\delta_0 - y)$, where i is the inclination of the orbit to the ecliptic:

$$\begin{aligned}
\sin I \sin \Omega &= +\sin z \cos (y - \delta_0) \\
\sin I \cos \Omega &= +\sin (x - i) \sin (y - \delta_0) - \cos (x - i) \cos (y - \delta_0) \cos z \\
\cos I &= +\cos (x - i) \sin (y - \delta_0) + \sin (x - i) \cos (y - \delta_0) \cos z \\
\sin I \sin \Delta &= +\sin z \sin (x - i) \\
\sin I \cos \Delta &= -\cos (x - i) \cos (y - \delta_0) + \sin (x - i) \sin (y - \delta_0) \cos z
\end{aligned}$$

Alternatively, I, Ω, and Δ may be determined by the relations in the two triangles ♈J☊ and JQΩ formed, respectively, by the first point of Aries ♈ and the nodes J and ☊ of the planet's orbit on the Earth's equator and ecliptic, and by J and the nodes Q and Ω of the planet's equator on the Earth's equator and the planet's orbit.

From the first of these triangles ♈J☊:

$$\begin{aligned}
\sin J \sin N &= +\sin i \sin ☊ \\
\sin J \cos N &= +\cos i \sin \epsilon + \sin i \cos \epsilon \cos ☊ \\
\cos J &= +\cos i \cos \epsilon - \sin i \sin \epsilon \cos ☊ \\
\sin J \sin \omega &= +\sin \epsilon \sin ☊ \\
\sin J \cos \omega &= +\sin i \cos \epsilon + \cos i \sin \epsilon \cos ☊
\end{aligned}$$

where N, J are the node and inclination of the planet's orbit referred to the Earth's equator and ω is the arc J☊ of this orbit from the node on the equator to the node on the ecliptic.

From the second triangle JQΩ:

$$\begin{aligned}
\sin I \sin (\Omega + \omega) &= +\cos \delta_0 \cos (N - \alpha_0) \\
\sin I \cos (\Omega + \omega) &= -\sin \delta_0 \sin J + \cos \delta_0 \cos J \sin (N - \alpha_0) \\
\cos I &= +\sin \delta_0 \cos J + \cos \delta_0 \sin J \sin (N - \alpha_0) \\
\sin I \sin \Delta &= +\sin J \cos (N - \alpha_0) \\
\sin I \cos \Delta &= +\cos \delta_0 \cos J - \sin \delta_0 \sin J \sin (N - \alpha_0)
\end{aligned}$$

In calculating the physical ephemerides, α_0 and δ_0 must be referred to the true equinox and equator of date. The secular variation is the sum of the general precession of the first point of Aries and the precession of the vernal equinox of the planet; the coordinates of the pole referred to the mean equinox and equator at the beginning of the year are obtained from the secular variations, and further reduced to the equinox of date by means of the Besselian star reductions. Likewise, Δ is reckoned from the true equator of the Earth; the correction for the nutation at date is equal to the effect of nutation on the length QR of the segment of Δ between the celestial equator and the ecliptic, which by differentiation of the triangle ♈QR formed by the intersections of the ecliptic, the celestial equator, and the equator of the planet is found to be:

$$\Delta\psi \cos (90^\circ + \alpha_0) \sin \epsilon \sec \delta_0 + \Delta\epsilon \sin (90^\circ + \alpha_0) \sec \delta_0$$

in which $\Delta\psi$ is the nutation in longitude and $\Delta\epsilon$ the nutation in obliquity.

The reference circles are not affected by aberration; but corrections for light-time must be applied to all the quantities that depend upon the position of the planet.

On the planetocentric sphere, the Earth is at the same angular distance from the vernal equinox of the planet as the geocentric position of the planet is from the autumnal equinox, and at the same angular distance from the equator of the planet but on the opposite side. Therefore, in the triangle APN formed by the pole A of the planet at α_0, δ_0, the geocentric position P at α, δ, and the celestial pole N, the arc AP from the pole of the planet to its geocentric position is $90^\circ + D_E$ and the angle NAP at the pole of the planet is $90° - (\Delta + A_E)$; in this triangle APN:

$$\cos D_E \sin P = +\cos \delta_0 \sin (\alpha_0 - \alpha)$$
$$\cos D_E \cos P = +\sin \delta_0 \cos \delta - \cos \delta_0 \sin \delta \cos (\alpha_0 - \alpha)$$
$$\sin D_E = -\sin \delta_0 \sin \delta - \cos \delta_0 \cos \delta \cos (\alpha_0 - \alpha)$$
$$\cos D_E \sin (A_E + \Delta) = +\cos \delta_0 \sin \delta - \sin \delta_0 \cos \delta \cos (\alpha_0 - \alpha)$$
$$\cos D_E \cos (A_E + \Delta) = -\cos \delta \sin (\alpha_0 - \alpha)$$

from which the coordinates A_E and D_E of the Earth, and the position angle P of the axis of the planet may be determined.

The correction for aberration is implicitly included by using the apparent right ascension and declination of the planet in the calculations.

Similarly, on the planetocentric sphere the heliocentric position S of the planet is at an angular distance from its autumnal equinox equal to:

$$L_S = L - (☊ + \Omega)$$

where L is the orbital longitude of the planet, and is at the same angular distance D_S from its equator as the Sun but on the opposite side. In terms of the heliocentric ecliptic longitude of the planet (l), the arc ☊S $= L_S + \Omega$ may be deduced from $l -$ ☊ and the inclination i by:

$$\tan (L_S + \Omega) = \tan (l - ☊) \sec i$$

and, since i is small:

$$L_S + \Omega = (l - ☊) + \tan^2 \tfrac{1}{2}i \sin 2 (l - ☊)$$

From the right-angled triangle formed by the arcs D_S, A_S, and L_S:

$$\sin D_S = \sin L_S \sin I$$
$$\cos D_S \sin A_S = \sin L_S \cos I$$
$$\cos D_S \cos A_S = \cos L_S$$

The correction for light-time is made by applying to the orbital longitude, in the calculation of L_S, the product of the daily motion of the planet in heliocentric longitude by the time in days required by light to travel from the Sun to the planet and back to the Earth.

Mars

* The adopted rotation elements of Mars are:

North pole of Mars (Lowell, *M.N.R.A.S.*, **66**, 56, 1905).

At the beginning of the year t

$$\alpha_0 = 21^h\ 11^m\ 10^s{\cdot}42 + 1^s{\cdot}565\ (t - 1950{\cdot}0)$$
$$\delta_0 = +54°\ 39'\ 27'' + 12''{\cdot}60\ (t - 1950{\cdot}0)$$

Sidereal period of rotation (Ashbrook, *A.J.*, **58**, 145, 1953).

In ephemeris time, $24^h\ 37^m\ 22^s{\cdot}6689$

Central meridian, referred to the zero meridian of 1909.

Longitude of central meridian

1909 January 15·0 G.M.A.T. (J.D. 241 8322·0), 344°·41

Daily motion, 350°·891962

The position of the north pole was adopted in 1909. The zero meridian is defined by the first tabular longitude of central meridian calculated with this

**A.E.* 1968–1970	α_0	$316°{\cdot}55 + 0°{\cdot}006533\ (t - 1905{\cdot}0)$
	$\delta_0 +$	$52°{\cdot}85 + 0°{\cdot}003542\ (t - 1905{\cdot}0)$
A.E. 1971 onwards	α_0	$316°{\cdot}55 + 0°{\cdot}006750\ (t - 1905{\cdot}0)$
	$\delta_0 +$	$52°{\cdot}85 + 0°{\cdot}003479\ (t - 1905{\cdot}0)$

See de Vaucouleurs (*Icarus*, **3**, 236–247, 1964) position of the pole; this value of 344°·41, for Greenwich mean noon on 1909 January 15, was obtained by continuing the value 52°·01 given by Marth (*M.N.R.A.S.*, **56**, 403, 1896) for 1897 May 15·0 G.M.A.T., with the same period of rotation. Beginning with 1960, a period of rotation is adopted that differs from the value used in previous years; however, the same zero meridian is retained to avoid a further addition to the changes which, when neglected or overlooked in the reduction and discussion of observations, have sometimes led to erroneous results in the past. Consequently, from 1959 to 1960 there is a discontinuity in the tabular longitude of the central meridian, amounting to about $-1°$; to reduce the published ephemerides for 1909-1959 to the rate of rotation adopted in 1960, the following correction is to be applied to the tabular longitude of central meridian:

$$-0°{\cdot}000058\ (\text{J.D.} - 241\ 8322{\cdot}0)$$

in addition to any required reduction from ephemeris time to universal time.

For the orbit of Mars, Newcomb's elements as corrected by Ross are adopted; referred to the ecliptic and mean equinox of date:

$$☊ = 48°\ 47'\ 11''{\cdot}19 + 2775''{\cdot}57\ T - 0''{\cdot}005\ T^2$$

$$i = 1°\ 51'\ 01''{\cdot}20 - 2''{\cdot}430\ T + 0''{\cdot}0454\ T^2$$

in which T denotes Julian centuries from 1900 January 0·5 E.T. Their values at the beginning of each year are included in table 11.2 of the elements of the Martian equator.

Table 11.2 gives for the beginning of each year from 1950 to 1975 the values of α_0, δ_0, ☊, i, referred to the mean equinox, and of Δ, Ω, sin I, cos I. The reductions to be applied to α_0, δ_0 for the additional secular variation to date and for nutation, in terms of the Besselian day numbers A, B, are:

$$\Delta\alpha_0 = +0°{\cdot}000376\ A + 0°{\cdot}000291\ B - 0°{\cdot}001013\ \tau$$

$$\Delta\delta_0 = +0°{\cdot}000206\ A + 0°{\cdot}000186\ B - 0°{\cdot}000631\ \tau$$

where A, B are in seconds of arc and τ is the fraction of the year; when $\Delta\alpha_0$ is required in seconds of time:

$$\Delta\alpha_0 = +0^s{\cdot}0902\ A + 0^s{\cdot}0698\ B - 0^s{\cdot}243\ \tau$$

The values of the other quantities at date are obtained by interpolation. The tabular values of Δ include a correction for nutation at the beginning of the year, namely:

$$+0{\cdot}4619\ \Delta\psi + 1{\cdot}2829\ \Delta\epsilon$$

The variation of this correction during the year cannot exceed 0°·001 and therefore no further correction is necessary.

The correction for aberration which is applied to L_s to correct the planetocentric position of the Sun for light-time may computationally be included in Ω. A mean value of the correction of +0°·007, which is sufficiently accurate in practice, has been included in the tabular values of Ω. It is obtained by adopting 0·7 as the average distance of Mars from the Earth, giving a light time of $0^d{\cdot}00404$, and obtaining the total light-time by adding $0^d{\cdot}00879$ to allow for the mean distance of Mars from the Sun; multiplying by the mean daily motion of Mars gives $0{\cdot}01283 \times 0°{\cdot}52403 = 0°{\cdot}0067$.

The tabular longitude of central meridian of the apparent disk of Mars is for the geometric disk, not the illuminated disk; and the time of transit of the zero meridian is for the transit across the central point of the geometric disk. The longitude of the central meridian is the Martian hour angle of the Earth measured

Table 11.2. Elements for the physical ephemeris of Mars

Jan. 0·0	α_0	δ_0	☊	Δ	Ω	sin I	cos I
	h m s	°	°	°	°		
1950	21 11 10·4	54·658	49·172	46·022	39·031	0·40649	0·91365
51	12·0	·661	·180	·028	·035	·40649	·91365
52	13·6	·664	·187	·034	·039	·40650	·91365
53	15·1	·668	·195	·039	·043	·40650	·91365
54	16·7	·672	·203	·044	·047	·40650	·91365
1955	21 11 18·2	54·675	49·210	46·049	39·052	0·40650	0·91365
56	19·8	·678	·218	·053	·056	·40650	·91365
57	21·4	·682	·226	·057	·060	·40651	·91365
58	22·9	·686	·234	·061	·064	·40651	·91365
59	24·5	·689	·241	·066	·069	·40651	·91365
1960	21 11 26·1	54·692	49·249	46·070	39·073	0·40651	0·91364
61	27·6	·696	·257	·075	·077	·40651	·91364
62	29·2	·700	·264	·080	·081	·40652	·91364
63	30·8	·703	·272	·086	·085	·40652	·91364
64	32·3	·706	·280	·092	·090	·40652	·91364
1965	21 11 33·9	54·710	49·288	46·099	39·094	0·40652	0·91364
66	35·5	·714	·295	·106	·098	·40653	·91364
67	37·0	·717	·303	·112	·102	·40653	·91364
68‡	38·6	·720	·311	·119	·106	·40653	·91364
69	40·2	·724	·318	·125	·110	·40653	·91364
1970	21 11 41·7	54·728	49·326	46·131	39·115	0·40654	0·91363
71	43·3	·731	·334	·137	·119	·40654	·91363
72	44·8	·734	·342	·142	·123	·40654	·91363
73	46·4	·738	·349	·146	·127	·40654	·91363
74	48·0	·742	·357	·151	·131	·40654	·91363
1975	21 11 49·5	54·745	49·365	46·155	39·134	0·40654	0·91363

$i = 1°{\cdot}850$ throughout. ‡Values for 1968–75 are based on pole due to Lowell.

from the zero meridian of Mars; it is therefore $V - A_E$ where V is the Martian hour angle of the vernal equinox of Mars measured from the zero meridian. At the instant at which the tabular longitude of central meridian is taken to define the zero meridian, the value of V is the sum of the longitude of central meridian and the value of A_E at the instant, increased by the angular amount of rotation during the light-time; accordingly, in the adopted system of elements the value of V for
* 1909 January 15·0 G.M.A.T. is $145°{\cdot}845$, and at any date:

$$V + 180° = 325°{\cdot}845 + 350°{\cdot}891962\,(\text{J.D.} - 241\ 8322{\cdot}0) \quad *$$

The longitude of the central meridian obtained from this value of V is corrected for light-time by subtracting the amount of rotation during the light-time, namely
† ($498^s{\cdot}58\ \Delta$/period of rotation) × 360°, where Δ is the geocentric distance; thus the longitude of central meridian is given by:

$$(V + 180°) - (A_E + 180°) - 2°{\cdot}024858\ \Delta \quad †$$

The time of transit of the zero meridian is determined by inverse interpolation of the ephemeris of the longitude of the central meridian.

For the stellar magnitude of Mars see sub-section C.

*149°·475, 329°·475 for 1968–70; 149°·479, 329°·479 for 1971 onwards.

†$499^s{\cdot}012$, 2°·026612 for 1968 onwards.

Example 11.7. *The physical ephemeris of Mars*
1960 April 6 at 0^h U.T.

The data are taken either from the Ephemeris or from example 11.2, in which the calculation of the phase is illustrated; other data have been taken, *without* interpolation, from table 11.2.

τ (*A.E.*, p. 270)	$+0{\cdot}2619$
A (*A.E.*, p. 270)	$+4''{\cdot}294$
B (*A.E.*, p. 270)	$+8''{\cdot}794$
α_0, mean equinox 1960·0 from table 11.2	$317^\circ{\cdot}859$
$+0{\cdot}000376\,A + 0{\cdot}000291\,B - 0{\cdot}001013\,\tau$	$+0{\cdot}004$
α_0, true equinox of date	$317{\cdot}863$
α (*A.E.*, p. 196)	$335{\cdot}300$
$\alpha_0 - \alpha$	$-17{\cdot}437$
δ_0, mean equinox 1960·0 from table 11.2	$+54^\circ{\cdot}692$
$+0{\cdot}000206\,A + 0{\cdot}000186\,B - 0{\cdot}000631\,\tau$	$+0{\cdot}002$
δ_0, true equinox of date	$+54{\cdot}694$
δ (*A.E.*, p. 196)	$-11{\cdot}642$

$\sin(\alpha_0 - \alpha)$	$-0{\cdot}29966$	$\sin\delta_0$	$+0{\cdot}81608$	$\sin\delta$	$-0{\cdot}20179$
$\cos(\alpha_0 - \alpha)$	$+0{\cdot}95404$	$\cos\delta_0$	$+0{\cdot}57794$	$\cos\delta$	$+0{\cdot}97942$

Using the formulae for the planetocentric coordinates of the Earth:

$\cos\delta_0 \sin(\alpha_0 - \alpha)$	$-\ 0{\cdot}17319$	$\cos\delta_0 \cos(\alpha_0 - \alpha)$	$+\ 0{\cdot}55138$
$\cos D_E \sin P$	$-\ 0{\cdot}17319$	$\cos D_E \sin(A_E + \Delta)$	$-\ 0{\cdot}87918$
$\cos D_E \cos P$	$+\ 0{\cdot}91055$	$\cos D_E \cos(A_E + \Delta)$	$+\ 0{\cdot}29349$
$\sin D_E$	$-\ 0{\cdot}37536$	$\cot(A_E + \Delta)$	$-\ 0{\cdot}33382$
$\tan P$	$-\ 0{\cdot}19020$	$A_E + \Delta$	$288^\circ{\cdot}460$
P	$349^\circ{\cdot}231$	Δ (Table 11.2)	$46{\cdot}070$
		A_E	$242{\cdot}390$
D_E	$-22^\circ{\cdot}046$	$A_E + 180^\circ$	$62{\cdot}390$

For the planetocentric coordinates of the Sun:

L (*A.E.*, p. 172)	$303^\circ{\cdot}496$	$\sin D_S$	$-\ 0{\cdot}23417$
$☊ + \Omega$ (Table 11.2)	$88{\cdot}322$	$\cos D_S \sin A_S$	$-\ 0{\cdot}52631$
$L_S = L - (☊ + \Omega)$	$215{\cdot}174$	$\cos D_S \cos A_S$	$-\ 0{\cdot}81741$
$\sin L_S$	$-\ 0{\cdot}57606$	D_S	$-13^\circ{\cdot}543$
$\cos L_S$	$-\ 0{\cdot}81741$		
		$\tan A_S$	$+\ 0{\cdot}64388$
$\sin I$ (Table 11.2)	$+\ 0{\cdot}40651$	A_S	$212^\circ{\cdot}777$
$\cos I$ (Table 11.2)	$+\ 0{\cdot}91364$	$A_S - A_E$	$-29{\cdot}613$

The correction from 0^h E.T. to 0^h U.T. is insignificant.

Longitude of the central meridian:

J.D.	243 7030·5	$-(A_E + 180^\circ)$	$297^\circ{\cdot}610$	Δ	1·94736
J.D. of epoch	241 8322·0	$V_0 + 180^\circ$	$325{\cdot}845$		
Days elapsed	1 8708·5	Motion since epoch	$62{\cdot}271$		
		$-2^\circ{\cdot}024858\,\Delta$	$-3{\cdot}943$		

Longitude of central meridian at 0^h E.T. = sum	$321{\cdot}783$
Motion during $\Delta T = 36^s$	$+0{\cdot}146$
Longitude of central meridian at 0^h U.T.	$321{\cdot}929$

Following first difference of longitude of central meridian $350^\circ{\cdot}082$

U.T. of transit of zero meridian $= 24^h \times (360^\circ - 321^\circ{\cdot}929)/350^\circ{\cdot}082 = 2^h\,36^m{\cdot}6$

Jupiter

The adopted position of the pole is derived from the position for 1750 given by Damoiseau (*Tables Écliptiques des Satellites de Jupiter*, page 1, Paris, 1836). The longitude of the central meridian that defines the zero meridian, and the rate of rotation, are adopted from the ephemeris last published by Marth (*M.N.R.A.S.*, **56**, 523, 1896):

North pole of Jupiter

At the beginning of the year t,

$$\alpha_0 = 17^h\ 52^m\ 00^s{\cdot}84 + 0^s{\cdot}247\ (t - 1910{\cdot}0)$$
$$\delta_0 = +64^\circ\ 33'\ 34''{\cdot}6 - 0''{\cdot}60\ (t - 1910{\cdot}0)$$

	System I	System II
Sidereal period of rotation	$9^h\ 50^m\ 30^s{\cdot}003$	$9^h\ 55^m\ 40^s{\cdot}632$
Central meridian		
Longitude		
1897 July 14·0 G.M.A.T. (J.D. 241 4120·0)	47°·31	96°·58
Daily motion	877°·90	870°·27

System I applies to all points on or between the north component of the south equatorial belt and the south component of the north equatorial belt; System II applies north of the south component of the north equatorial belt, and south of the north component of the south equatorial belt.

The elements of the equator of Jupiter and the values of ☊ are given in table 11.3 for the beginning of each year. In the calculation of the table, the values used for the orbital elements of Jupiter are:

$$☊ = 99^\circ{\cdot}43798 + 1^\circ{\cdot}01053\ T + 0^\circ{\cdot}000352\ T^2$$
$$i = 1^\circ{\cdot}30876 - 0^\circ{\cdot}005696\ T$$

in which T is measured in Julian centuries from J.D. 241 5020·0; these elements are based on Hill's values for 1850 (*A.P.A.E.*, **7**, part 1, 1898) and on the variations determined by Leverrier and Gaillot (*Connaissance des Temps*).

The correction for aberration included in Ω is $+0^\circ{\cdot}0045$, obtained from a mean light-time of $0^d{\cdot}03002$ between the Sun and Jupiter, and $0^d{\cdot}02424$ between Jupiter and the Earth. The correction to the arc Δ for nutation is neglected. In calculating the ephemeris, the reductions for precession and nutation from the beginning of the year to date are also neglected for all the quantities in table 11.3.

The tabular longitudes of the central meridian are for the geometric disk; applying to them the corrections in the column headed *Correction for Phase* gives the longitudes of the central meridian of the illuminated disk. The numerical value of this correction is:

$$57^\circ{\cdot}3\ (1 - k) = 57^\circ{\cdot}3\ \sin^2 \tfrac{1}{2}i$$

and the sign is opposite the sign of $\sin (A_S - A_E)$. In addition, the longitude of the central meridian of the illuminated disk is tabulated at daily intervals in a separate ephemeris; the tables of the motion of the central meridian accompanying this ephemeris are based on the mean daily synodic rotations during the period when Jupiter is observable, which are 877°·95 for System I, and 870°·30 for

System II. An accuracy of 0°·1 for the longitude of the central meridian of the illuminated disk is usually sufficient, and may readily be obtained from the daily ephemeris; interpolation in the 4-day ephemeris is less convenient, but may be made in the infrequent cases when an accuracy of 0°·01 is needed.

The longitude of the central meridian of the geometric disk is calculated in the same way as explained for Mars. In the adopted system of elements:

$$V + 180° = 100°.974 + 877°.90 \text{ (J.D. } - 241\ 4120.0) \text{ for System I} \quad *$$
$$V + 180° = 149°.976 + 870°.27 \text{ (J.D. } - 241\ 4120.0) \text{ for System II} \quad *$$

The angular amount of rotation during the light-time at geocentric distance Δ is 5°·06601 Δ for System I, and 5°·02198 Δ for System II. *

For the stellar magnitude of Jupiter see sub-section C.

An ephemeris for physical observations of Jupiter during the intervals near conjunction that are omitted from the Ephemeris is given for 1960, 1961, and 1962 for the reduction of radio observations in *A.J.*, **65**, 104–106, 1960.

Table 11.3. Elements for the physical ephemeris of Jupiter

Jan. 0·0	α_0	δ_0	☊	Δ	Ω
	h m s	°	°	°	°
1960	17 52 13·2	64·551	100·044	318·052	216·262
61	13·4	·551	·054	·066	·266
62	13·7	·551	·065	·080	·272
63	13·9	·551	·075	·092	·274
64	14·2	·551	·085	·105	·278
1965	17 52 14·4	64·550	100·095	318·117	216·281
66	14·7	·550	·105	·131	·285
67	14·9	·550	·115	·144	·290
68	15·2	·550	·125	·156	·293
69	15·4	·550	·135	·170	·297
1970	17 52 15·7	64·550	100·146	318·184	216·302
71	15·9	·549	·156	·196	·306
72	16·2	·549	·166	·209	·309
73	16·4	·549	·176	·222	·313
74	16·6	·549	·186	·235	·317
1975	17 52 16·9	64·549	100·196	318·248	216·321
76	17·1	·549	·206	·261	·324
77	17·4	·548	·216	·274	·328
78	17·6	·548	·226	·287	·332
79	17·9	·548	·236	·300	·336
1980	17 52 18·1	64·548	100·247	318·314	216·340

$i = 1°.305$ $\qquad$ $\sin I = 0.05355$, $\cos I = 0.99856$

Saturn

The physical ephemeris for Saturn is very limited and contains no data that have not been fully described for other planets. For the stellar magnitude see sub-section C.

*101°·001, 150°·002, 5°·07040, 5°·02633 for 1968 onwards.

Example 11.8. ***The physical ephemeris of Jupiter***
1960 March 7 at 0^h U.T.

The data are taken from the Ephemeris, or from example 11.2, in which the calculation of the phase is illustrated, or from table 11.3. The calculation follows closely that for Mars in example 11.7.

α_0	268°·055				
α	270°·776				
$\alpha_0 - \alpha$	−2°·721	δ_0	+64°·551	δ	−23°·010
$\sin(\alpha_0 - \alpha)$	−0·04748	$\sin\delta_0$	+0·90297	$\sin\delta$	−0·39089
$\cos(\alpha_0 - \alpha)$	+0·99887	$\cos\delta_0$	+0·42970	$\cos\delta$	+0·92044

Using the formulae for the planetocentric coordinates of the Earth:

$\cos\delta_0 \sin(\alpha_0 - \alpha)$	−0·02040	$\cos\delta_0 \cos(\alpha_0 - \alpha)$	+0·42921
$\cos D_E \sin P$	−0·02040	$\cos D_E \sin(A_E + \Delta)$	−0·99816
$\cos D_E \cos P$	+0·99890	$\cos D_E \cos(A_E + \Delta)$	+0·04370
$\sin D_E$	−0·04210		
		$\cot(A_E + \Delta)$	−0·04378
$\tan P$	−0·02042		
		$A_E + \Delta$	272°·507
P	358°·830	Δ (Table 11.3)	318°·052
		A_E	314°·455
D_E	−2°·413	$A_E + 180°$	134°·455

For the planetocentric coordinates of the Sun:

L (*A.E.*, p. 174)	260°·242	$\sin D_S$	−0·04444
☊ + Ω (Table 11.3)	316°·306	$\cos D_S \sin A_S$	−0·82847
$L_S = L -$ (☊ + Ω)	303°·936	$\cos D_S \cos A_S$	+0·55827
$\sin L_S$	−0·82966	D_S	−2°·547
$\cos L_S$	+0·55827		
		$\cot A_S$	−0·67386
$\sin I$ (Table 11.3)	+0·05355	A_S	303°·974
$\cos I$ (Table 11.3)	+0·99856	$A_S + 180°$	123°·974

The correction from 0^h E.T. to 0^h U.T. is insignificant.

For the longitude of the central meridians in the two systems:

			System I	System II
J.D.	243 7000·5	$-(A_E + 180°)$	225°·545	225°·545
J.D. of epoch	241 4120·0	$V_0 + 180°$	100·974	149·976
Days elapsed	2 2880·5	Motion since epoch	230·950	252·735
Δ	5·44869	Rotation in light-time	−27·603	−27·363
Longitude of central meridian at 0^h E.T.		sums	169·866	240·893
Motion during $\Delta T = 36^s$			+ 0·366	+ 0·363
Longitude of central meridian at 0^h U.T.			170·232	241·256
Correction for phase 57°·3 $(1 - k)$, with opposite sign to $(A_S - A_E)$			+ 0·477	+ 0·477
Longitude of central meridian of illuminated disk			170·709	241·733

H. HISTORICAL NOTES

Sun

An ephemeris for physical observations of the Sun was first published in *The Nautical Almanac* for the year 1907. Previous to that time, observations were reduced with the aid of tables privately printed by Warren de la Rue (see the volumes of the Greenwich photoheliographic observations).

The ephemeris was first included in *The American Ephemeris* in the volume for 1913.

Moon

Ephemerides for physical observations of the Moon, calculated by Marth, appeared in *Monthly Notices of the Royal Astronomical Society* during the last quarter of the nineteenth century. The ephemeris was introduced into *The Nautical Almanac* in 1907. It was first included in *The American Ephemeris* in 1913; but formulae and tables for the calculation of the optical librations, and the times of the greatest librations, had been included with the ephemeris of the elements of the mean equator of the Moon, beginning in 1855.

Mercury and Venus

The physical ephemerides of Mercury and Venus were added to *The American Ephemeris* in 1882, and included in *The Nautical Almanac* in 1907. Previously, only a small table of the versed sine of the illuminated disk divided by the apparent diameter, for Venus and for Mars, had been given.

Mars and Jupiter

Ephemerides for physical observations of Mars, calculated by Marth, appeared in *Monthly Notices of the Royal Astronomical Society* beginning in 1869, and for Jupiter beginning in 1875. They were continued by Crommelin after Marth's death, and transferred to *The Nautical Almanac* in 1907; they were first included in *The American Ephemeris* in 1913.

Saturn

The physical ephemeris of Saturn first appeared in the Ephemeris in 1960. The stellar magnitude which is included in it had previously been given in the ephemeris for the rings.

12. SATELLITES

A. INTRODUCTION

The ephemerides of the satellites in the Ephemeris are intended only for search and identification, not for the exact comparison of theory with observation. They are therefore calculated only to an order of accuracy sufficient for the purpose of facilitating observations. The tabular values are corrected for light-time, and are directly comparable with observations at the tabular times; the value of the * light-time used in the calculations is $498^{\mathrm{s}}\cdot 58$ for unit distance, corresponding to the adopted values of the solar parallax and the velocity of light.

To the order of the precision given, corrections for the difference between E.T. and U.T. are significant only for the orbital longitudes of the satellites; the phenomena of Jupiter's satellites are tabulated only to the nearest minute of time. However, the elements on which the positions of the satellites are based are referred to epochs expressed in U.T. (or in G.M.A.T. = U.T. − $0^{\mathrm{d}}\cdot 5$ for the dates prior to 1925); if the epochs are to be regarded as expressed in E.T., the mean longitudes at the epoch should therefore be modified by the motion in ΔT at the epoch. No such modifications have been made, though corrections have been applied to the longitudes of the satellites of Saturn for the motion in ΔT at the current epoch. In most examples the reduction from 0^{h} E.T. to 0^{h} U.T. is not significant and has been ignored.

The apparent positions of the satellites are represented by their positions relative to the primary, expressed either by the apparent angular distance and position angle, or by the differential spherical (or rectangular) coordinates in right ascension and declination. These apparent coordinates on the celestial sphere relative to the primary are calculated from the geocentric position of the primary and the planetocentric position of the satellite that is obtained from the theory of its orbital motion around the primary.

The apparent orbit of a satellite on the geocentric celestial sphere is an ellipse which is the orthogonal projection, in the direction of the line of sight, of the actual orbit in space. In a circular orbit the orbital diameter that is perpendicular to the line of sight is projected into the major axis of the apparent ellipse; at its extremities, the satellite is at its greatest elongations from the primary, at an apparent angular distance a/Δ, where a is the apparent semi-major axis in seconds of arc at a distance of one astronomical unit and Δ is the geocentric distance of the

*$499^{\mathrm{s}}\cdot 012$ for 1968 onwards. See note on page 349.

primary. The orbital diameter that lies in the plane which passes through the line of sight, and which is perpendicular to the orbital plane, projects into the minor axis; at the farther extremity of this diameter, the satellite is in superior geocentric conjunction with the primary, and at the nearer extremity it is at inferior conjunction. The ratio of the semi-minor axis to the semi-major axis is the absolute value of sin B where B is the angle between the line of sight and the plane of the orbit.

On the planetocentric celestial sphere of the primary, the path of the satellite is the great circle in which the orbital plane intersects the sphere. The position of this great circle and the position of the satellite on the planetocentric sphere at any time are obtained from the orbital elements; to represent these positions, the same coordinate systems are adopted as are defined on the geocentric celestial sphere by the equator and the ecliptic. Because of the mathematically infinite radius of the celestial sphere, these reference circles are in identically the same positions on the planetocentric sphere as on the geocentric sphere. The Earth on the planetocentric sphere is diametrically opposite the geocentric position of the planet, and therefore at right ascension $\alpha \pm 180°$ and declination $-\delta$, where α and δ are the geocentric coordinates of the planet.

Referred to the celestial equator, the position of the great circle which the satellite describes on the planetocentric sphere is represented by its inclination (J) and the right ascension (N) of its ascending node, or by the right ascension ($N - 90°$) and declination ($90° - J$) of the pole of the orbit. Like the other circles of the sphere, this great circle is in the same position on the planetocentric and the geocentric celestial spheres; the major axis of the apparent elliptic orbit which the satellite describes on the geocentric sphere is parallel to the plane of this great circle.

Apparent distance and position angle

On the geocentric sphere, the satellite at any instant is on the great circle in which the sphere is intersected at the time by the plane through the Earth, the primary, and the satellite. Since this circle is likewise in the same position on both celestial spheres, the satellite is in the same position angle (p) on the geocentric sphere relative to the primary as it is on the planetocentric sphere relative to the point at α, δ, but at an apparent geocentric distance (s) from the primary that differs from its planetocentric angular distance (σ) from that point. In the plane triangle formed in space by the Earth, the primary, and the satellite:

$$\sin s = \frac{r}{\Delta_S} \sin \sigma$$

where r is the radius vector of the satellite and Δ_S is its geocentric distance.

The values of p and s are found from the position of the satellite in its orbit, and the geocentric position of the primary relative to the orbital plane.

The orbital position of the satellite is represented by the orbital longitude (u), measured along the orbit in the direction of motion from the ascending node on the equator. When the motion is direct, it appears counter-clockwise from the north pole of the orbit; when retrograde, counter-clockwise from the south pole of the orbit. On the geocentric sphere, the semi-minor axes of the apparent orbit are each directed towards one of the poles of the actual orbit; and the geocentric position angle (P) of the semi-minor axis that is directed towards the pole from which the motion appears counter-clockwise is the same as the position angle of the great circle arc on the planetocentric sphere from the geocentric position of the primary to this pole of the orbit.

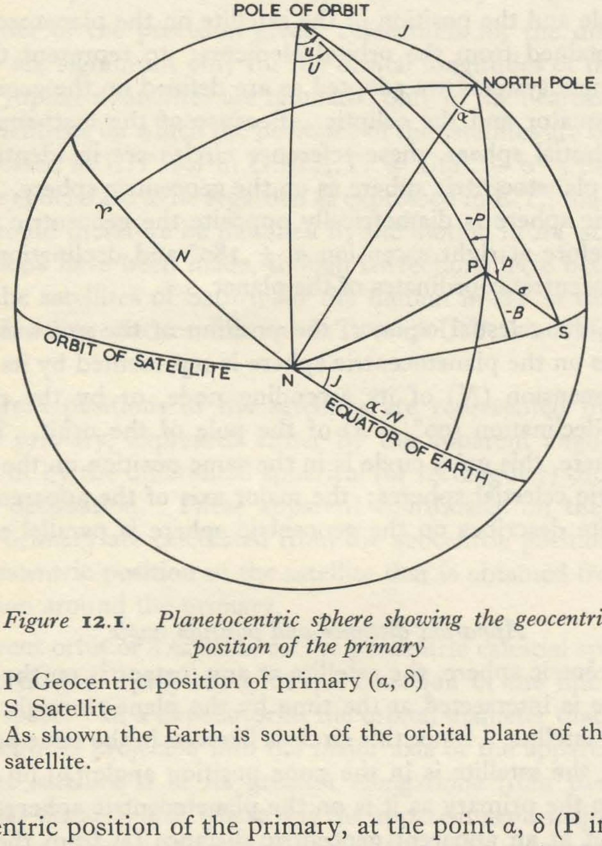

Figure 12.1. *Planetocentric sphere showing the geocentric position of the primary*

P Geocentric position of primary (α, δ)
S Satellite
As shown the Earth is south of the orbital plane of the satellite.

The geocentric position of the primary, at the point α, δ (P in figure 12.1) on the planetocentric sphere and diametrically opposite the planetocentric position of the Earth, is represented relative to the orbital plane of the satellite by: U, the geocentric longitude of the primary measured in the same way as u, along the orbit from the node on the equator in the direction of motion of the satellite; and $-B$, where $+B$ is the planetocentric latitude of the Earth referred to the orbital plane of the satellite, positive towards the pole of the orbit at position angle P from which the motion appears counter-clockwise.

From the triangle formed on the planetocentric sphere by this pole of the orbit, the north celestial pole, and the geocentric position P of the primary, we have for determining U, B, and P:

$$\begin{aligned}
\cos B \sin U &= +\cos J \cos \delta \sin (\alpha - N) + \sin J \sin \delta \\
\cos B \cos U &= +\cos \delta \cos (\alpha - N) \\
\sin B &= +\sin J \cos \delta \sin (\alpha - N) - \cos J \sin \delta \\
\cos B \sin P &= -\sin J \cos (\alpha - N) \\
\cos B \cos P &= +\sin J \sin \delta \sin (\alpha - N) + \cos J \cos \delta
\end{aligned}$$

Rigorous relations for p and σ in terms of the orbital position of the satellite may be derived from the triangle (see figure 12.1) formed on the planetocentric sphere by the satellite S, the geocentric position P of the primary, and the pole of the orbit:

$$\begin{aligned}
\sin \sigma \sin (p - P) &= \sin (u - U) \\
\sin \sigma \cos (p - P) &= \sin B \cos (u - U) \\
\cos \sigma &= \cos B \cos (u - U)
\end{aligned}$$

in which U, B, u are referred to the plane of the orbit. The geocentric distance s is derived from the planetocentric distance σ by the relations in the plane triangle formed in space by the satellite, the primary, and the Earth:

$$\begin{aligned}
\Delta_s \sin s &= r \sin \sigma \\
\Delta_s \cos s &= r \cos \sigma + \Delta
\end{aligned}$$

in which Δ is the geocentric distance of the primary. With sufficient accuracy the relation of s to σ may be written:

$$s = \frac{r}{\Delta \sin 1''} \sin \sigma = \frac{a}{\Delta} \sin \sigma$$

where a is the apparent radius of the orbit at unit distance, expressed in seconds of arc. Rigorous formulae for p and s in terms of the differential coordinates of the satellite, and also in terms of the planetocentric coordinates, are given later.

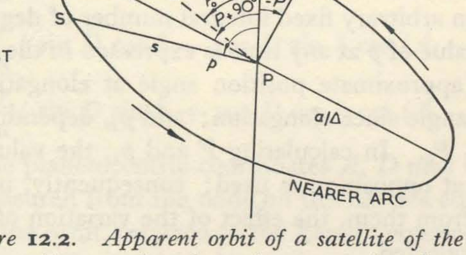

Figure 12.2. Apparent orbit of a satellite of the primary P as projected on the geocentric sphere
As shown the Earth is north of the orbital plane of the satellite.

At greatest elongation, $u - U = \pm 90°$, $\sigma = 90°$, and $s = a/\Delta$ in position angle $p = P \pm 90°$. At the extremities of the minor axis of the apparent orbit, $u - U$ is 0° or 180°; $\sigma = B$, and the position angle is P or $P + 180°$. Evidently, when the Earth is north of the orbital plane, the satellite is at inferior conjunction at the southern extremity of the minor axis, and is at superior conjunction at the northern extremity (see figure 12.2); B is then positive if the motion is direct, negative if retrograde. When the Earth is south of the orbital plane, superior conjunction is at the southern extremity of the minor axis, and B is negative in direct motion, positive in retrograde motion. Irrespective, therefore, of whether the motion is direct or retrograde, the position angle at superior conjunction when B is positive is $p = P$; when B is negative, it is $p = P + 180°$.

The apparent orbit becomes increasingly elliptical as the Earth approaches the orbital plane, and reduces to a straight line when the Earth is in this plane; as the Earth passes through the plane and B changes sign, each superior and inferior conjunction occurs at the opposite extremity of the minor axis from that at which it previously took place. Similarly, when the Earth is in the plane that is perpendicular to the orbital plane and contains the celestial pole, the minor axis is exactly in the north-south direction, the major axis exactly east-west; as the Earth passes through this plane, the extremity of the major axis which formerly was the more northerly becomes the more southerly.

Only when J is not too greatly different from 0° or 180° is the direction of the minor axis necessarily nearly enough north and south for the elongations to be strictly and unambiguously described as eastern and western. When J is in the neighbourhood of 90°, as in the case of the satellites of Uranus, the direction of the minor axis on the celestial sphere ranges from north-south to east-west, and introduces confusion in the terminology for the elongations, but in general they are more appropriately regarded as northern and southern than as eastern and western.

In the ephemerides for finding the apparent distance s and position angle p, the factor $\sin \sigma$ giving the ratio of s to the apparent distance at greatest elongation is denoted by F, and therefore the apparent distance of the satellite from the primary is given by:

$$s = F \frac{a}{\Delta}$$

With P_0 denoting an arbitrary fixed integral number of degrees near the value of P at opposition, the value of p at any time is expressed in the form $p_1 + p_2$, where p_1 is the sum of the approximate position angle at elongation and the amount of motion in position angle since elongation; and p_2, depending on the date, denotes the correction $P - P_0$. In calculating F and p_1, the value of the eccentricity of the apparent orbit at opposition is used; consequently, in the values of s and p which are derived from them, the effect of the variation of the eccentricity of the apparent orbit is neglected.

Differential coordinates

To a first approximation, the differences of right ascension and declination in the sense satellite *minus* planet are:

$$\Delta\alpha = s \sin p \sec (\delta + \Delta\delta)$$
$$\Delta\delta = s \cos p$$

in which $s \sin p$ and $s \cos p$ are the approximate rectangular coordinates of the satellite in the directions perpendicular to the circle of declination and along this circle, respectively:

$$x = s \sin p \qquad\qquad y = s \cos p$$
$$= (\alpha_s - \alpha) \cos \delta_s \qquad = \delta_s - \delta$$

Rigorous expressions, from which approximations to any desired order of accuracy can be obtained, may be derived directly in terms of the planetocentric coordinates of the satellite. Denoting the rectangular equatorial coordinates of the satellite in space, relative to the primary, by x_s, y_s, z_s, and the geocentric rectangular equatorial coordinates of the primary by X, Y, Z, we have for the geocentric rectangular coordinates of the satellite:

$$X_s = X + x_s \qquad Y_s = Y + y_s \qquad Z_s = Z + z_s$$

Transforming the rectangular coordinates to the geocentric spherical coordinates α, δ, Δ of the primary, α_s, δ_s, Δ_s of the satellite, and to the planetocentric spherical equatorial coordinates A, D, r of the satellite, gives equations from which rigorous expressions for the differential right ascension and declination are immediately obtained in the form:

$$\tan (\alpha_s - \alpha) = \frac{r \cos D \sin (A - \alpha)}{r \cos D \cos (A - \alpha) + \Delta \cos \delta}$$

$$\tan \delta_s = \frac{r \sin D + \Delta \sin \delta}{\{ r \cos D \cos (A - \alpha) + \Delta \cos \delta \} \sec (\alpha_s - \alpha)}$$

These equations may be written in an alternative form which affords some advantages. Thus:

$$\tan (\alpha_s - \alpha) = \frac{\xi}{(1 + \zeta) \cos \delta - \eta \sin \delta}$$

$$\tan (\delta_s - \delta) = \frac{\eta - \xi \tan \frac{1}{2} (\alpha_s - \alpha) \sin \delta}{(1 + \zeta) + \xi \tan \frac{1}{2} (\alpha_s - \alpha) \cos \delta}$$

where

$$\xi = \frac{a}{\Delta} \frac{r}{a} \cos D \sin (A - \alpha)$$

$$\eta = \frac{a}{\Delta} \frac{r}{a} \{ \sin D \cos \delta - \cos D \sin \delta \cos (A - \alpha) \}$$

$$\zeta = \frac{a}{\Delta} \frac{r}{a} \{ \sin D \sin \delta + \cos D \cos \delta \cos (A - \alpha) \}$$

In these expressions the planetocentric coordinates A, D may be derived from the orbital longitude u, measured from the node on the Earth's equator, and the node N and inclination J of the orbit, referred to the Earth's equator:

$$\cos D \cos (A - N) = \cos u$$
$$\cos D \sin (A - N) = \sin u \cos J$$
$$\sin D = \sin u \sin J$$

They may also be derived from similar expressions (see under the sixth and seventh satellites of Jupiter) in terms of the true longitude and latitude referred to the plane of the planet's orbit.

Unless A, D are specifically required for purposes of tabulation (and they are not so required for the Ephemeris) there is no need to form them. Writing:

$$l = \cos D \cos (A - N) \qquad m = \cos D \sin (A - N) \qquad n = \nu = \sin D$$

$$\lambda = \cos D \cos (A - \alpha) = l \cos (\alpha - N) + m \sin (\alpha - N)$$

$$\mu = \cos D \sin (A - \alpha) = - l \sin (\alpha - N) + m \cos (\alpha - N)$$

then:

$$\tan (\alpha_s - \alpha) = \frac{r\mu}{\Delta \cos \delta + r\lambda}$$

$$\tan \delta_s = \frac{\Delta \sin \delta + r\nu}{\Delta \cos \delta + r\lambda} \cos (\alpha_s - \alpha)$$

or, alternatively:

$$\xi = \frac{a}{\Delta} \frac{r}{a} \mu$$

$$\eta = \frac{a}{\Delta} \frac{r}{a} (- \lambda \sin \delta + \nu \cos \delta)$$

$$\zeta = \frac{a}{\Delta} \frac{r}{a} (\lambda \cos \delta + \nu \sin \delta)$$

Rigorous relations for determining p and s in terms of $\alpha_s - \alpha$ and δ_s may be obtained from the triangle formed on the geocentric celestial sphere by the primary, the satellite, and the celestial pole:

$$\sin s \sin p = \cos \delta_s \sin (\alpha_s - \alpha)$$

$$\sin s \cos p = \sin \delta_s \cos \delta - \cos \delta_s \sin \delta \cos (\alpha_s - \alpha)$$

$$\cos s = \sin \delta_s \sin \delta + \cos \delta_s \cos \delta \cos (\alpha_s - \alpha)$$

From these and from the previous equations may be derived convenient rigorous expressions for p and s in terms of the planetocentric coordinates:

$$\tan s \sin p = \frac{\xi}{1 + \zeta}$$

$$\tan s \cos p = \frac{\eta}{1 + \zeta}$$

where ξ, η, ζ have the same significance as before.

When s is small p is poorly determined, but is still given to the required precision.

Planetocentric coordinates

In constructing a theory of the motion of a satellite, the plane of the celestial equator is not usually the most advantageous fundamental reference plane. Frequently, therefore, transformations of the planetocentric coordinates obtained directly from the theory are necessary in order to calculate the apparent geocentric position by the preceding formulae.

In general, the orbital plane of a satellite is inclined at a nearly constant angle to a virtually fixed plane upon which the nodes steadily regress. This fixed plane is known as the Laplacian plane; it lies in the position where the components of the disturbing forces perpendicular to it balance one another, leaving no resultant orthogonal force. In most cases, the principal disturbing forces come from the oblateness of the primary and the action of the Sun; the Laplacian plane lies between the planes of the orbit of the primary and the equator of the primary, passing through their intersection at an angle depending on the ratio of the action of the oblateness to the action of the Sun. The greater this ratio, the nearer the Laplacian plane lies to the equatorial plane of the primary. The action of the oblateness is often the predominating influence, and the Laplacian plane of the satellite is nearly in the equatorial plane of the primary; the orbital elements of many of the satellites are therefore referred to the plane of the equator of the primary. In some cases the Laplacian plane is adopted as the reference plane (see figure 12.3); occasionally the plane of the ecliptic is used.

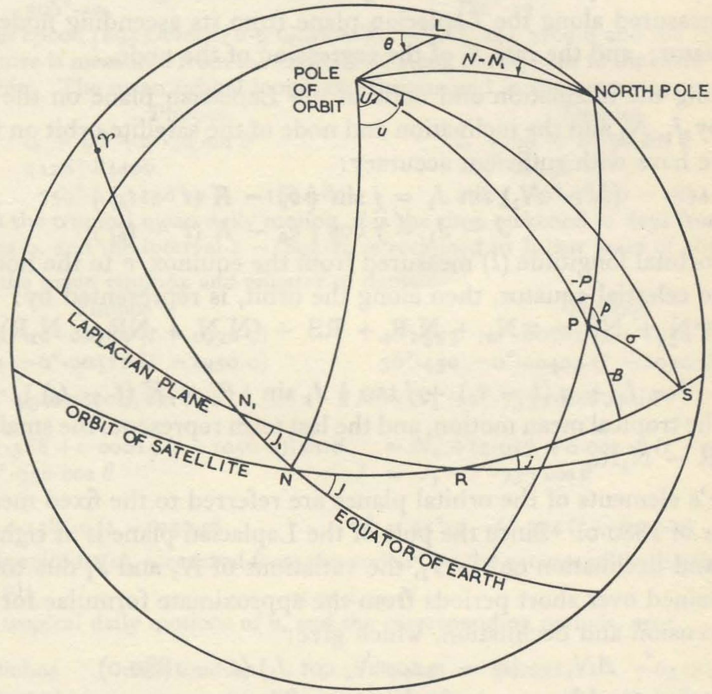

Figure 12.3. *Orbit of a satellite referred to the Laplacian plane*

L Pole of the Laplacian plane $\quad N_1R = \theta$

P Geocentric position of the primary (α, δ) $\quad K = -\dfrac{d\theta}{dt}$

S Satellite

The details of the transformations that are required in each case are given in the following explanations of the calculations for the ephemerides of the individual satellites.

Added note (1973)

Aberration is taken into account by using the apparent coordinates of the primary, calculating the geometric position of the satellite relative to this apparent position of the planet, then adding the light time $499^s{\cdot}012\Delta$ to the time to obtain the apparent position of the satellite relative to the apparent position of the planet.

B. THE SATELLITES OF MARS

The ephemerides of the satellites of Mars are computed from the orbital elements determined by H. Struve (*Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 1911, page 1073), although the eccentricities of the orbits are ignored.

These elements are referred to the fixed Laplacian planes (see figure 12.3) of the satellites. The orbital elements of a satellite referred to the Laplacian plane are: the inclination j; the longitude θ_0 of the ascending node R, at an adopted epoch t_0, measured along the Laplacian plane from its ascending node N_1 on the Earth's equator; and the rate K of the regression of the node.

Denoting the inclination and node of the Laplacian plane on the equator of the Earth by J_1, N_1 and the inclination and node of the satellite orbit on the equator by J, N, we have with sufficient accuracy:

$$(N - N_1) \sin J_1 = j \sin \{ \theta_0 - K(t - t_0) \}$$
$$J - J_1 = j \cos \{ \theta_0 - K(t - t_0) \}$$

The mean orbital longitude (l) measured from the equinox ♈ to the node N of the orbit on the celestial equator, then along the orbit, is represented by:

$$♈\mathrm{N} + \mathrm{NS} = ♈\mathrm{N_1} + \mathrm{N_1R} + \mathrm{RS} + (\mathrm{N_1N} + \mathrm{NR} - \mathrm{N_1R})$$

so that:

$$l = l_0 + n(t - t_0) + j \tan \tfrac{1}{2} J_1 \sin \{ \theta_0 - K(t - t_0) \}$$

where n is the tropical mean motion, and the last term represents the small difference $\mathrm{N_1N} + \mathrm{NR} - \mathrm{N_1R}$.

Struve's elements of the orbital planes are referred to the fixed mean equinox and equator of 1880·0. Since the pole of the Laplacian plane is at right ascension $N_1 - 90°$ and declination $90° - J_1$, the variations of N_1 and J_1 due to precession may be obtained over short periods from the approximate formulae for precession in right ascension and declination, which give:

$$\Delta N_1 = (m - n \cos N_1 \cot J_1)(t - 1880{\cdot}0)$$
$$\Delta J_1 = -n \sin N_1 (t - 1880{\cdot}0)$$

Over long periods the rigorous trigonometric reduction, as described in section 2B, is to be preferred to give directly the position of the pole referred to the equinox and equator of date.

The orbital elements of Phobos and Deimos

The node and inclination of the fixed Laplacian planes of Phobos and Deimos, referred to the equinox and equator of 1880·0, are:

		Phobos	Deimos
Right ascension of node	N_1	47° 03′·7	46° 01′·2
Inclination	J_1	37° 24′·0	36° 44′·0

The elements of the orbital plane referred to the Laplacian plane are:

		Phobos	Deimos
Inclination	j	0° 57′·5	1° 44′·0
Longitude of node	θ_0	359°·2	27°·3 at epoch 1894·80
Rate of regression of node	K	158°·0	6°·374 per Julian year of 365·25 days

Referred to the mean equinox and equator of 1880·0 the node and inclination of the orbital plane on the equator of the Earth at epoch t are therefore given by:

	Phobos	Deimos
$N - N_1$	$+1^\circ.578 \sin\{359^\circ.2 - 158^\circ.0\,(t - 1894{\cdot}80)\}$	$+2^\circ.898 \sin\{27^\circ.3 - 6^\circ.374\,(t - 1894{\cdot}80)\}$
$J - J_1$	$+0^\circ.958 \cos\{359^\circ.2 - 158^\circ.0\,(t - 1894{\cdot}80)\}$	$+1^\circ.733 \cos\{27^\circ.3 - 6^\circ.374\,(t - 1894{\cdot}80)\}$

The other orbital elements are:

	Phobos	Deimos
a	$12''.938$ at unit distance	$32''.373$ at unit distance
e	0·0170	0·0031
Π	$279^\circ.1 \ +158^\circ.0\,(t - 1894{\cdot}80)$	$231^\circ \ +6^\circ.374\,(t - 1894{\cdot}80)$
l_0	$296^\circ.40$	$186^\circ.17$

where l_0 is for epoch 1894 October 0·0 G.M.A.T. = J.D. 241 3102·0 and the longitude (Π) of the pericentre is measured from the equinox ♈, along the equator to the node N, and then along the orbit. The mean orbital longitude (l) measured in the same way, is given by:

	Phobos	Deimos
l	$l_0 + nd + 0^\circ.32 \sin\theta$	$l_0 + nd + 0^\circ.58 \sin\theta$
n	$1128^\circ.84406$	$285^\circ.16196$
θ	$359^\circ.2 - 158^\circ.0\,(t - 1894{\cdot}80)$	$27^\circ.3 - 6^\circ.374\,(t - 1894{\cdot}80)$

in which n is the tropical mean daily motion, d is the time reckoned in days from the epoch J.D. 241 3102·0, and the interval $t - 1894{\cdot}80$ is reckoned in Julian years of 365·25 days.

Referred to the mean equinox and equator of date:

	Phobos	Deimos
N_1	$47^\circ.609 + 0^\circ.00784\,(t - 1950{\cdot}0)$	$46^\circ.553 + 0^\circ.00762\,(t - 1950{\cdot}0)$
J_1	$37^\circ.114 - 0^\circ.00411\,(t - 1950{\cdot}0)$	$36^\circ.450 - 0^\circ.00404\,(t - 1950{\cdot}0)$
	$N = N_1 + 0^\circ.958 \operatorname{cosec} J_1 \sin\theta$	$N = N_1 + 1^\circ.733 \operatorname{cosec} J_1 \sin\theta$
	$= N_1 + \{1^\circ.588 + 0^\circ.00015\,(t - 1950{\cdot}0)\}\sin\theta$	$= N_1 + \{2^\circ.917 + 0^\circ.00028\,(t - 1950{\cdot}0)\}\sin\theta$
	$J = J_1 + 0^\circ.958 \cos\theta$	$J = J_1 + 1^\circ.733 \cos\theta$

where

Phobos	Deimos
$\theta = 277^\circ.6 - 158^\circ.0\,(t - 1950{\cdot}0)$	$\theta = 35^\circ.5 - 6^\circ.374\,(t - 1950{\cdot}0)$

The orbital longitude (u), measured from the node N on the equator of the Earth, is given by (taking $e = 0$):

$$u = l - N$$

The secular tropical daily motions of u, and the corresponding periods, are:

Phobos	daily motion	$1128^\circ.84404$	period	$7^h.653847$
Deimos	,,	$285^\circ.16194$	,,	$30^h.298573$

For current dates:

$$\text{Phobos } l = 263^\circ.62 + 1128^\circ.84406\,(\text{J.D.} - 243\ 6933{\cdot}5) + 0^\circ.32 \sin\theta$$
$$u = 215{\cdot}93 + 1128{\cdot}84404\,(\text{J.D.} - 243\ 6933{\cdot}5) - 1{\cdot}27 \sin\theta$$
$$\text{Deimos } l = 303{\cdot}42 + 285{\cdot}16196\,(\text{J.D.} - 243\ 6933{\cdot}5) + 0{\cdot}57 \sin\theta$$
$$u = 256{\cdot}79 + 285{\cdot}16194\,(\text{J.D.} - 243\ 6933{\cdot}5) - 2{\cdot}35 \sin\theta$$

The variations of the coefficients of the periodic terms are negligible over several decades from 1960.

Ephemerides of the greatest eastern elongations, and tables for determining the approximate apparent distance and position angle in the form described in sub-section A, are given for an interval of about 25 days on each side of the date of opposition of Mars. The eccentricities of the orbits are ignored throughout the calculations for the Ephemeris. The diagram of the apparent orbits is prepared from the values of U, B, P calculated, by the formulae given in sub-section A, for the date of opposition.

Example 12.1. *Orbital planes of Phobos and Deimos, and geocentric position of Mars referred to those planes*

The values of θ, J, N may differ in the end figure when calculated by the alternative formulae given.

			Opposition	
Date	1960 December 18·0		1960 December 30·0	
	Phobos	Deimos	Phobos	Deimos
$t - 1950{\cdot}0$	10·96		11·00	
θ	345°·9	325°·6	339°·6	325°·3
$\sin\theta$	−0·244	−0·565	−0·349	−0·569
$\cos\theta$	+0·970	+0·825	+0·937	+0·822
J_1	37°·069	36°·406	37°·069	36°·406
$j\cos\theta$	+0·929	+1·430	+0·898	+1·425
J	37·998	37·836	37·967	37·831
N_1	47°·695	46°·637	47°·695	46°·637
$\operatorname{cosec} J_1 \sin\theta$	−0·387	−1·650	−0·554	−1·662
N	47·308	44·987	47·141	44·975
Mars α	105°·065		99°·977	
(*A.E.*, p. 201) δ	+26·020		+26·799	

From the above values of J, N, α, δ values of U, B, P are deduced from the formulae of sub-section A; this standard transformation, which is illustrated in example 12.6, is not given here in detail.

U	61°·115	63°·117	57°·245	59°·060
B	+ 7·021	+ 7·544	+ 4·713	+ 5·299
P	−19·325	−17·978	−21·895	−20·688

For the systematic calculation of the elongations, U is calculated for 0^h of every fifth day during each of the 25-day intervals, and expressed in the form $U = U_0 + (U - U_0)$, where U_0 is the integral number of degrees nearest the average value of U for the interval. The orbital longitude (u) is calculated for 0^h of the first and the last date. The time of the apparent eastern elongation next following any time t_0 for which U and u have been calculated is:

$$t_0 + (U + 90^\circ - u)\,\frac{\text{period}}{360^\circ} + \text{light-time}$$

neglecting the variation of U during the interval from t_0 to elongation. The times of the elongations are therefore:

$$\left\{t_0 + (U_0 + 90^\circ - u)\,\frac{\text{period}}{360^\circ}\right\} + (U - U_0)\,\frac{\text{period}}{360^\circ} + \text{light-time}$$

where the first part, depending on t_0 and u, called the mean elongation, is obtained by successive additions of the period of u to the initial value, and the correction represented by the other part is obtained by interpolating the 5-day values to the times of the mean elongations. The light-time is $0^h{\cdot}13849\,\Delta$, where Δ is the geocentric distance of Mars. *

The tables for obtaining apparent distance and position angle are constructed

*$0^h{\cdot}13861$ for 1968 onwards.

Example 12.2. *Times of greatest elongation of Phobos and Deimos*

1960 December 18 at 0^h E.T. = J.D. 243 7286·5

The basic data are taken from example 12.1.

	Phobos	Deimos
	°	°
Constant term in l	263·62	303·42
Epoch 243 6933·5, $d = 353$ nd	321·953	222·172
$0°{\cdot}32$ or $0°{\cdot}57 \times \sin\theta$	−0·078	−0·322
sum = l	225·50	165·27
N	47·31	44·99
$l - N = u$	178·19	120·28
U	61·12	63·12
$u - U$	117·07	57·16
	h	h
Time for motion of $u - U$ through 1°	0·02126	0·08416
	°	°
Angle from eastern elongation = $u - U - 90°$	27·07	327·16
	h	h
Time from eastern elongation	0·576	27·534
Δ = 0·61236 (*A.E.*, p. 201) $0^h{\cdot}13849\Delta$	0·085	0·085
	d h	d h
Time of greatest eastern elongation	17 23·51	16 20·55

Example 12.3. *Table entries for distance and position angle of Phobos and Deimos*

1960 December 18

		Phobos	Deimos
		″	″
Δ = 0·61236	a/Δ	21·13	52·87
		°	°
Example 12.1	P	340·68	342·02
	P_0	338	339
	$P - P_0 = p_2$	+2·68	+3·02

At opposition, 1960 December 30

		Phobos	Deimos
		°	°
Example 12.1	B	+4·713	+5·299
	$\sin B$	+0·08216	+0·09236
	$\cos B$	+0·99662	+0·99572
Time from eastern elongation,	t	$4^h\ 00^m$	$4^h\ 00^m$
		°	°
	$90° + nt = u - U$	278·141	137·527
	$\sin\sigma\sin(p - P) = \sin(u - U)$	−0·98993	+0·67524
	$\cos(u - U)$	+0·14161	−0·73759
$\sin\sigma\cos(p - P)$	$= \cos(u - U)\sin B$	+0·01163	−0·06812
$\cos\sigma$	$= \cos(u - U)\cos B$	+0·14113	−0·73443
	$\cot(p - P)$	−0·01175	−0·10088
		°	°
	$p - P$	270·67	95·76
	P_0	338	339
	sum = p_1	248·67	74·76
	$\sin\sigma = F$	0·9900	0·6787

from the value of B at opposition, and daily values of a/Δ and P. The position angle of the satellite is expressed in the form:

$$p = (P + 90^\circ) + \{p - (P + 90^\circ)\}$$

The second term represents the amount of motion since elongation; at any interval of time t after elongation, the values of this term and of the factor F in the expression $F\ (a/\Delta)$ for the apparent distance s are calculated with the formulae in sub-section A by taking $u - U = 90^\circ + nt$, neglecting the variation of U during the interval t. In the systematic calculation, instead of P in the first term, a constant integral number of degrees P_0 near the value of P at opposition is used, giving p in the form:

$$p = \{(P_0 + 90^\circ) + p - (P + 90^\circ)\} + (P - P_0)$$

where the first part $p - (P - P_0) = p_1$ is tabulated with argument t, and the correction $P - P_0 = p_2$ is tabulated against the date.

C. THE SATELLITES OF JUPITER

The Galilean satellites

The ephemerides and phenomena of Satellites I-IV given in the Ephemeris are based on Sampson's *Tables of the four great satellites of Jupiter* (London, 1910). These tables include all theoretical terms having coefficients of 1″ or more, as well as some others which do not involve additional arguments, and give times of phenomena to $0^d.000001$ and positions of satellites to $0^\circ.00001$; corrections for light-time and for the phase of Jupiter are included. Since the data published in the Ephemeris are not intended for the comparison of observation with theory, such accuracy is not necessary, and they are computed by means of the simplified procedures developed by H. Andoyer (*Bulletin Astronomique*, **32**, 177, 1915). Only the main terms of Sampson's tables are used, and the resulting times of the phenomena are obtained to the nearest minute; the calculations are lengthy and complex, and it is impracticable to give examples here. Andoyer's method gives also the longitudes of the satellites in their orbits to $0^\circ.001$, and ephemerides for all four satellites may readily be obtained from the data, derived by this method, that are published annually in *Connaissance des Temps*.

The data tabulated in the Ephemeris consist only of: the approximate times of superior geocentric conjunction; the times of the geocentric phenomena; and the approximate configurations, in graphical form, of the satellites relative to the disk of Jupiter. These data are omitted for a period on each side of the date of conjunction of Jupiter with the Sun.

The U.T. of each superior geocentric conjunction is given for each satellite to the nearest minute. The phenomena for which times are given are eclipses, occultations, transits, and shadow transits; the U.T. of the beginning and end of each phenomenon (disappearance and reappearance for eclipses and occultations, ingress and egress for transits and shadow transits) are given to the nearest minute for all phenomena that are observable. Prior to 1960 the times for the eclipses were given to $0^m \cdot 1$. When Jupiter is in opposition the shadow may be hidden by the disk and no eclipses can be observed. In general, eclipses may be observed on the western side of Jupiter before opposition and on the eastern side after opposition. Before opposition the disappearance only of Satellite I into the shadow may be observed since it is occulted before it emerges from the shadow; after opposition only the reappearances from the shadow are visible. The same is true in general of Satellite II, although occasionally both phenomena can be seen. In the case of Satellites III and IV both phases of the eclipses are usually visible except near certain oppositions. Similarly the occultation disappearances and reappearances of a satellite cannot be observed if, at the time concerned, the satellite is eclipsed. For Satellites I and II there are therefore, in general, cycles of six phenomena consisting of both phases of both transit and shadow transit, of one phase of the eclipse, and the other phase of the occultation.

For Satellite IV none of the phenomena occur when the plane of the orbit of the satellite, essentially the same as that of Jupiter's equator, is inclined at more than about 2° to the line from Jupiter to the Earth (for occultations and transits) or to the Sun (for eclipses and shadow transits).

Owing to the finite disks of the satellites the phenomena do not take place instantaneously; the times given refer to the centre of the disk.

In certain favourable situations of the orbital planes of the satellites relative to the Earth, one satellite may be eclipsed by the shadow of another, or may occult another. No predictions of these phenomena are given in the Ephemeris, but predictions are given in *Handbook B.A.A.* at the relevant times; a description of the method of prediction is given by A. E. Levin (*Memoirs B.A.A.*, **30**, 149, 1934).

The configurations of the four satellites relative to the disk of Jupiter are shown in graphical form, on the pages facing the tabular ephemerides of the times of the eclipses and other phenomena. The central vertical band in each diagram represents the equatorial diameter of the disk of Jupiter; the relative positions of the satellites at any time with respect to the disk of Jupiter are given by the curves. Where a satellite is immersed in the shadow of Jupiter, or occulted by the disk, the curve is interrupted. In constructing these diagrams the coordinates of the satellites in the direction perpendicular to the equator of Jupiter are necessarily neglected, except in respect of their effect on the eclipses and occultations. The horizontal lines indicate 0^h U.T. on the day of the month stated; the relative positions of Jupiter and the four satellites at any instant of U.T. may be obtained by placing a ruler, or drawing a horizontal line, at a position on the vertical scale corresponding to that U.T.

For eclipses, the points d and r of disappearance into and reappearance from the shadow are shown pictorially at the foot of each right-hand page, for an eclipse near the middle of each month or the middle of the period covered; and at the foot of each left-hand page the rectangular coordinates (x, y) of these points are given, in units of the equatorial radius of Jupiter. The x-axis is parallel to the equator of Jupiter, positive towards the east, and the y-axis is positive towards the north pole of Jupiter. The subscript 1 refers to the beginning of an eclipse, and the subscript 2 to the end of an eclipse.

The fifth satellite

The data given in the Ephemeris consist of the universal times of every twentieth eastern and western elongation, from about 80 days before opposition to about 80 days after opposition. The elongations are computed from the following circular orbital elements given by A. J. J. van Woerkom (*A.P.A.E.*, **13**, part 1, pages 8, 14, and 16, 1950):

Epoch t_0 = 1903 September 1·5 U.T. = J.D. 241 6359·0

Mean elongation at unit distance	$a = 249''\cdot55$
Inclination to equator of Jupiter	$\gamma = 24'\cdot1$
Longitude of ascending node	$\theta = 82^\circ\cdot5 - 914^\circ\cdot62\ (t - t_0)$
Mean motion per mean solar day	$n = 722^\circ\cdot63175 = 2^r\cdot00731\ 0417$
Mean longitude at epoch	$u_0 = 194^\circ\cdot98$
Correction to mean longitude	$\delta u_0 = -0^\circ\cdot113 - 0^\circ\cdot0076\ (t - t_0) + 0^\circ\cdot00035\ (t - t_0)^2$

where $t - t_0$ is reckoned in Julian years. The longitudes are measured in the plane of the equator of Jupiter, from the ascending node of the mean orbital plane of Jupiter on the plane of the equator of Jupiter as in the physical ephemeris. From these orbital elements:

sidereal period	$0^d\cdot49817\ 905 = 11^h\cdot95629\ 7$
mean synodic period	$0^d\cdot49823\ 633 = 11^h\cdot95767\ 2 = 11^h\ 57^m\ 27^s\cdot62$

In determining these elements, van Woerkom adopted Souillart's elements of the equator of Jupiter, but for the calculation of the elongations in the Ephemeris the same elements are used as in the physical ephemeris; referred to the same epoch as the orbital elements, the right ascension and declination of the pole of Jupiter are:

$$\alpha_0 = 267^\circ\cdot99700 + 0^\circ\cdot00102\ 92\ (t - t_0)$$
$$\delta_0 = +64^\circ\cdot56067 - 0^\circ\cdot00016\ 67\ (t - t_0)$$

The right ascension (N) of the node of the orbit on the Earth's equator, the inclination (J) of the orbit to the Earth's equator, and the arc (ψ) along the orbit from the node on the Earth's equator to the node on Jupiter's equator are given by:

$$\begin{aligned}
\sin J \sin (N - \alpha_0) &= + \cos \delta_0 \cos \gamma + \sin \delta_0 \sin \gamma \cos (\theta - \Delta) \\
\sin J \cos (N - \alpha_0) &= - \sin \gamma \sin (\theta - \Delta) \\
\cos J &= + \sin \delta_0 \cos \gamma - \cos \delta_0 \sin \gamma \cos (\theta - \Delta) \\
\sin J \sin \psi &= + \cos \delta_0 \sin (\theta - \Delta) \\
\sin J \cos \psi &= + \sin \delta_0 \sin \gamma + \cos \delta_0 \cos \gamma \cos (\theta - \Delta)
\end{aligned}$$

in which $\theta - \Delta$, the arc along Jupiter's equator from its node on the Earth's equator to the node of the satellite's orbit (see section 11G), is given by:

$$\begin{aligned}
\theta - \Delta &= 219^\circ\cdot8 - 2^r\cdot54057\ 487\ (t - t_0) \\
&= 219^\circ\cdot8 - 2^\circ\cdot50405\ 737\ d
\end{aligned}$$

in which d is the number of days from the epoch (J.D. 241 6359·0).

The quantities N, J, ψ are calculated from the above formulae at a suitable interval and then interpolated; values of N, J at 0^h on any date are then used to form U (B and P are not required) from the standard formulae in sub-section A. The orbital longitude (u), measured from the node on the Earth's equator, at 0^h on that date is then derived from:

$$u = u_0 + nd - \theta + \psi + \delta u_0$$
$$= \psi + 112^\circ{\cdot}37 + 2^r{\cdot}01426\,6231\, d - 0^\circ{\cdot}0076\,(t - t_0) + 0^\circ{\cdot}00035\,(t - t_0)^2$$

Then the times of the elongations on that date are:

eastern elongation $0^h{\cdot}033212\,(U + 90^\circ - u) + 0^h{\cdot}13849\,\Delta$ *

western elongation $0^h{\cdot}033212\,(U + 270^\circ - u) + 0^h{\cdot}13849\,\Delta$

in which Δ is the geocentric distance of Jupiter. The terms in Δ are the corrections for light-time; for strict accuracy they should be interpolated to the times of geometric elongation which the first terms represent.

The calculation of the times of the elongations on a single date is shown in the following example; but this calculation need be made for only a few of the ephemeris dates, the others being obtained by means of multiples of the period. For strict accuracy, the calculation should be repeated with the values of the quantities at the calculated times instead of at 0^h; but for the tabular accuracy of $0^h{\cdot}1$, this is unnecessary, and no correction for reduction to U.T. is required.

Example 12.4. *The times of greatest elongation of Jupiter's fifth satellite*
1960 April 11 at 0^h E.T. = J.D. 243 7035·5

$d = 20676{\cdot}5$ $t - t_0 = 56{\cdot}6$ $(t - t_0)^2 = 3205$

γ	$24'{\cdot}1$	$\theta - \Delta$	$284^\circ{\cdot}658$	α_0	$268^\circ{\cdot}055$	δ_0	$+64^\circ{\cdot}551$
$\sin\gamma$	+0·00701 0	$\sin(\theta - \Delta)$	−0·96745	$\sin\gamma\cos(\theta - \Delta)$	+0·00177	$\sin\delta_0$	+ 0·90297
$\cos\gamma$	+0·99997 5	$\cos(\theta - \Delta)$	+0·25305	$\cos\gamma\cos(\theta - \Delta)$	+0·25304	$\cos\delta_0$	+ 0·42971

$\sin J \sin(N - \alpha_0)$	+0·43130	$\sin J$	+0·43135	$\sin J \sin\psi$	−0·41572
$\sin J \cos(N - \alpha_0)$	+0·00678	$\cos J$	+0·90219	$\sin J \cos\psi$	+0·11507
$\cot(N - \alpha_0)$	+0·01572			$\cot\psi$	−0·27680

$N - \alpha_0$	$89^\circ{\cdot}099$			ψ	$285^\circ{\cdot}472$
N	357·154	δ	$-22^\circ{\cdot}984$	constant in u	112·37
(*A.E.*, p. 204) α	273·783	Δ	4·8894	$2^r{\cdot}01426\,6231\,d$	351·261
$\alpha - N$	276·629	$0^h{\cdot}13849\,\Delta$	$0^h{\cdot}677$	secular terms in u	+0·691
				u	29·794
		Using the formulae of sub-section A		U	276·106
				$U - u$	246·312

$0^h{\cdot}03321\,2\,(U + 90^\circ - u) + 0^h{\cdot}13849\,\Delta = 11^h{\cdot}847$ = E.T. of eastern elongation

$+180 \times 0^h{\cdot}03321\,2 = 5^h{\cdot}978$ $= 17^h{\cdot}825$ = E.T. of western elongation

The number of figures retained in this, and similar examples, is not necessarily an indication of the precision to which the quantities are known; for example, γ is a constant and is treated as exact whereas the first decimal of $\theta - \Delta$ is uncertain.

*$0^h{\cdot}13861$ for 1968 onwards.

The sixth and seventh satellites

The differential right ascension and declination of Satellites VI and VII given in the Ephemeris are calculated from the tables constructed by J. Bobone, (*Astronomische Nachrichten*, **262**, 321, 1937; **263**, 401, 1937). Referred to the plane of the orbit of Jupiter, the orbital elements are:

Epoch 1935 January 1·0 U.T. = J.D. 242 7803·5

	Satellite VI	Satellite VII
Π	$307°\cdot359 + 1°\cdot3539\, t$	$162°\cdot69 + 1°\cdot460\, t$
☊	$143°\cdot994 - 1°\cdot2169\, t$	$203°\cdot64 - 1°\cdot305\, t$
i	$28°\cdot436$	$27°\cdot75$
g	$244°\cdot558 + 1°\cdot43307\,76\, d$	$266°\cdot48 + 1°\cdot38250\,8\, d$
a	0·07672 3	0·07845 5
e	0·15798	0·20719
n	$1°\cdot43674\,63$	$1°\cdot38646\,7$
Sidereal period	250·5662 days	259·6528 days

where t is reckoned in Julian years and d in days, from the epoch, and longitudes are measured from the vernal equinox, along the ecliptic, and then along the orbit of Jupiter.

The coordinates obtained immediately from the tables are: the elliptic orbital longitude, $V = \Pi + g + (v - g)$, in which v is the true anomaly, g is the mean anomaly, Π is the longitude of perijove, and $v - g$ is the equation of centre; the true longitude in the plane of the orbit of Jupiter, $l = V + R + \delta l$, in which R is the reduction to the orbit of Jupiter and δl is the sum of the periodic perturbations in longitude; the latitude referred to the orbital plane of Jupiter, $b = b_0 + \delta b$, in which b_0 is the elliptic latitude, given by $\sin b_0 = \sin i \sin (V - ☊)$, and δb is the sum of the periodic perturbations; the radius vector, $r = a/\{(a/r)_0 + \delta(a/r)\}$, in which $(a/r)_0 = (1 + e \cos v)/(1 - e^2)$ and $\delta(a/r)$ is the sum of the periodic perturbations.

In order to transform the coordinates l and b to the equator of the Earth, the position of the orbital plane of Jupiter referred to the plane of the celestial equator is represented by its inclination (J_0) and the right ascension (N_0) of its ascending node. (Bobone uses I, Ω for J_0, N_0.) These elements, and the arc (ω_0) of the orbit of Jupiter intercepted between the equator and the ecliptic, are also given in the tables; they may be determined from the triangle formed on the planetocentric sphere by the vernal equinox and the ascending nodes on the equator and the ecliptic:

$$\sin J_0 \sin N_0 = \sin i_0 \sin ☊_0$$
$$\sin J_0 \cos N_0 = \cos i_0 \sin \epsilon + \sin i_0 \cos \epsilon \cos ☊_0$$
$$\cos J_0 = \cos i_0 \cos \epsilon - \sin i_0 \sin \epsilon \cos ☊_0$$
$$\sin J_0 \sin \omega_0 = \sin \epsilon \sin ☊_0$$
$$\sin J_0 \cos \omega_0 = \cos \epsilon \sin i_0 + \sin \epsilon \cos i_0 \cos ☊_0$$

where ϵ is the obliquity of the ecliptic, and i_0, ☊$_0$ are the inclination and longitude of the node of the orbit of Jupiter on the ecliptic.

The true longitude (M) in the plane of Jupiter's orbit, measured from the

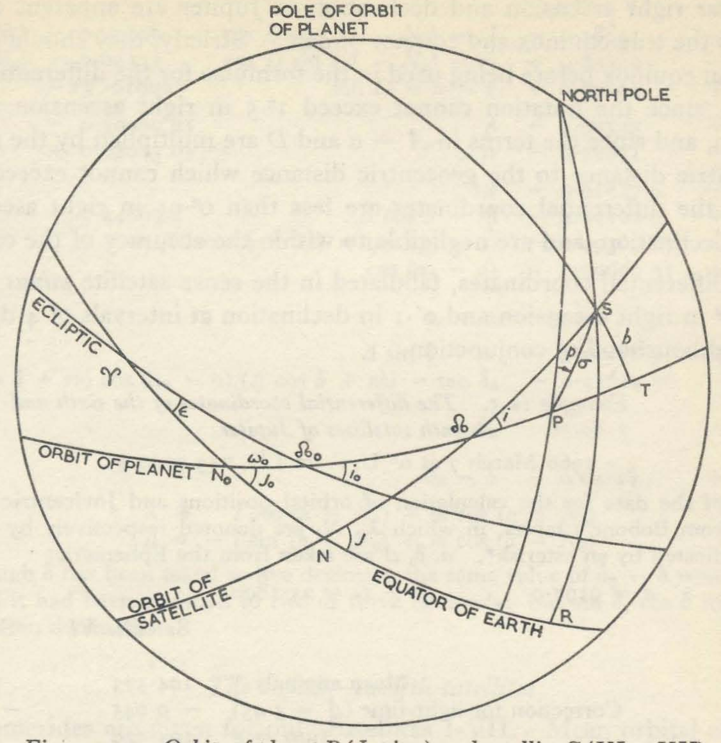

Figure 12.4. Orbits of planet P (Jupiter) and satellite S (VI or VII)

$$\mathrm{NS} = u \qquad \mathrm{N_0T} = M = l - ☊_0 + \omega_0$$
$$♈☊_0 + ☊_0\mathrm{T} = l \qquad \mathrm{NR} = A - N$$
$$♈\mathrm{R} = A \qquad \mathrm{N_0R} = A - N_0$$
$$\mathrm{SR} = D$$

---

ascending node on the Earth's equator is, as seen from figure 12.4:

$$M = l - ☊_0 + \omega_0$$

The Jovicentric right ascension (A) and declination (D) of the satellite are given by:

$$\cos D \cos (A - N_0) = \cos b \cos M$$
$$\cos D \sin (A - N_0) = \cos b \sin M \cos J_0 - \sin b \sin J_0$$
$$\sin D = \cos b \sin M \sin J_0 + \sin b \cos J_0$$

The geocentric differential coordinates are then found by the formulae of sub-section A.

The resulting values of A, D are the geometric coordinates referred to the mean equinox and equator of date. To correct for aberration, the light-time may either be added to the time for which the calculation is made or subtracted from the time of observation; in practice it is applied to the mean anomaly, by subtracting the mean motion during the light-time from the tabular value of g at the tabular time. With sufficient accuracy, the light-time is $0^d{\cdot}00577\ 060\ \Delta$, where Δ * is the geocentric distance of Jupiter, and the corrections to g are:

for Satellite VI $-0^\circ{\cdot}00826\ 97\ \Delta$ *

for Satellite VII $-0^\circ{\cdot}00797\ 79\ \Delta$ *

*$0^d{\cdot}00577\ 560$, $0^\circ{\cdot}00827\ 69$, $0^\circ{\cdot}00798\ 48$ for 1968 onwards.

The tabular right ascension and declination of Jupiter are apparent coordinates referred to the true equinox and equator of date. Strictly, they should be reduced to the mean equinox before being used in the formulae for the differential coordinates; but, since the nutation cannot exceed $1^s·5$ in right ascension nor $10''$ in declination, and since the terms in $A - \alpha$ and D are multiplied by the ratio of the planetocentric distance to the geocentric distance which cannot exceed 0·03, the effects on the differential coordinates are less than $0^s·05$ in right ascension and $0'·005$ in declination, and are negligible to within the accuracy of the calculations.

The differential coordinates, tabulated in the sense satellite *minus* planet, are given to 1^s in right ascension and $0'·1$ in declination at intervals of 4 days, except in the neighbourhood of conjunction.

Example 12.5. *The differential coordinates of the sixth and seventh satellites of Jupiter*

1960 March 7 at 0^h U.T. = J.D. 243 7000·5

Most of the data for the calculation of orbital positions and Jovicentric coordinates are taken from Bobone's tables, in which J_0, N_0 are denoted respectively by I, Ω; such data are indicated by an asterisk*. α, δ, Δ are taken from the Ephemeris.

$d = 9197·0$ $\qquad t - t_0 = 25·1800$

			Satellite VI	Satellite VII
		Mean anomaly *	° 104·573	° 21·41
		Correction for light-time (Δ = 5·45)	− 0·045	− 0·043
		sum = g	104·528	21·37
		Π *	341·450	199·45
		$v - g$ *	+16·468	+11·28
Elliptic orbital longitude		$\Pi + g + (v - g) = V$	102·446	232·10
		☊ *	113·352	170·78
		$V -$ ☊	349·094	61·32
Reduction to orbit of Jupiter		R *	+ 1·289	− 3·05
Perturbations		δl *	+ 0·163	− 7·54
Jovicentric longitude		$V + R + \delta l = l$	103·898	221·51
		$\omega_0 -$ ☊$_0$ *	− 2·992	− 2·99
		$l + (\omega_0 -$ ☊$_0) = M$	100·906	218·52
Elliptic latitude		b_0 *	− 5·169	+24·11
Perturbations		δb *	− 0·075	− 2·73
Jovicentric latitude		sum = b	− 5·244	+21·38
		$(a/r)_0$ *	0·94217	1·2272
Perturbations		$\delta(a/r)$ *	+ 0·01796	− 0·0045
		sum = a/r	0·96013	1·2227
Jovicentric radius vector		r	0·07990 9	0·06416 5
α	° 270·776	$\sin M$	+ 0·98194	− 0·62279
N_0 *	3·258	$\cos M$	− 0·18920	− 0·78239
$\alpha - N_0$	267·518			
J_0 *	23·252	$\cos b$	+ 0·99581	+ 0·93118
$\sin J_0$	+ 0·39477	$\sin b$	− 0·09140	+ 0·36455
$\cos J_0$	+ 0·91878	$\cos b \sin M$	+ 0·97783	− 0·57993

			Satellite VI	Satellite VII
$\sin(\alpha - N_0)$	$-0{\cdot}99906$	$\cos D \cos(A - N_0) = l$	$-0{\cdot}18841$	$-0{\cdot}72855$
$\cos(\alpha - N_0)$	$-0{\cdot}04331$	$\cos D \sin(A - N_0) = m$	$+0{\cdot}93449$	$-0{\cdot}67674$
δ	$-23^\circ{\cdot}01002$	$\sin D = n = \nu$	$+0{\cdot}30204$	$+0{\cdot}10600$
$\sin\delta$	$-0{\cdot}39089\ 21$	λ	$-0{\cdot}92545$	$+0{\cdot}70766$
$\cos\delta$	$+0{\cdot}92043\ 65$	μ	$-0{\cdot}22871$	$-0{\cdot}69856$
		$r\mu$	$-0{\cdot}01827\ 6$	$-0{\cdot}04482\ 3$
Δ	$5{\cdot}44869\ 2$	$\Delta\cos\delta + r\lambda$	$+4{\cdot}94122\ 3$	$+5{\cdot}06058\ 2$
		$r\mu/(\Delta\cos\delta + r\lambda) = \tan(\alpha_S - \alpha)$	$-0{\cdot}00369\ 87$	$-0{\cdot}00885\ 73$
		$\cos(\alpha_S - \alpha)$	$+0{\cdot}99999\ 31$	$+0{\cdot}99996\ 07$
		$\alpha_S - \alpha$	$-50^s{\cdot}86$	$-2^m\ 01^s{\cdot}79$
		$\Delta\sin\delta + r\nu$	$-2{\cdot}10571\ 5$	$-2{\cdot}12304\ 9$
		$(\Delta\sin\delta + r\nu)\cos(\alpha_S - \alpha)/(\Delta\cos\delta + r\lambda) = \tan\delta_S$	$-0{\cdot}42614\ 96$	$-0{\cdot}41951\ 02$
		δ_S	$-23^\circ{\cdot}0813$	$-22^\circ{\cdot}7585$
		$\delta_S - \delta$	$-0^\circ\ 04'{\cdot}28$	$+0^\circ\ 15'{\cdot}09$

$$\lambda = \quad l\cos(\alpha - N_0) + m\sin(\alpha - N_0)$$
$$\mu = -l\sin(\alpha - N_0) + m\cos(\alpha - N_0)$$

Although δ has been taken to five decimals, the same value of $\delta_S - \delta$ would have been obtained if it had been rounded to two or three decimals; but $\sin\delta$, $\cos\delta$ must be taken to six or seven decimals.

The eighth—twelfth satellites

Ephemerides are given for only Satellites I-VII. Mean orbital elements for the other five known satellites are listed below; but the motions are so greatly disturbed that in some cases it is hardly possible to represent them satisfactorily by elliptic elements. Ephemerides for some of these outer satellites, calculated in general by numerical integration, occasionally appear in astronomical journals or announcement circulars. The motions of Satellites VIII, IX, XI, and XII are retrograde.

Satellite VIII (H. R. J. Grosch, *A.J.*, **53**, 180-187, 1948)

Referred to the mean orbital plane of Jupiter, with longitudes measured from the ascending node of Jupiter on the ecliptic of 1950·0, the ranges of the elements during the interval 1930-1947 were:

a	0·1500 to 0·1650
e	0·291 to 0·660
i	155° to 146°
☊	Irregular increase from 259°·583 to 356°·964
ω	Irregular increase from 215°·392 to 305°·786

With $a = 0{\cdot}1570$, the sidereal period is 735 days, and the synodic period with respect to the Sun is 629 days.

Satellite IX (S. B. Nicholson, *Ap. J.*, **100**, 62, 1944)

The mean ecliptic elements and their ranges are:

a	$0{\cdot}1585 \pm 0{\cdot}008$
e	$0{\cdot}275 \pm 0{\cdot}15$
i	$157^\circ \pm 5^\circ$
☊	$\{ 61^\circ + 4^\circ{\cdot}44\,(t - 1940) \} \pm 7^\circ$
ω	$\{ 103^\circ + 3^\circ{\cdot}7\,(t - 1940) \} \pm 24^\circ$

The sidereal period is $758^d \pm 25^d$.

Satellite X (R. H. Wilson, Jr., *P.A.S.P.*, **51**, 241, 1939)
Satellite XI (P. Herget, *P.A.S.P.*, **50**, 347, 1938)
Satellite XII (S. Herrick, *P.A.S.P.*, **64**, 238, 1952)

The osculating ecliptic elements for the stated epochs of osculation are:

	Satellite X	Satellite XI	Satellite XII
Epoch (U.T.)	1938 March 6·5	1938 August 25·2575	1951 October 31·0
Equinox	1938·0	1950·0	1950·0
a	0·07705	0·15083 36	0·14177 3
e	0·14051	0·20678	0·16870 2
i	28° 24′·1	163°·377	146°·7338
☊	81° 29′·4	231°·753	227°·2804
ω	254° 01′·4	127°·948	313°·6193
n	1°·4240	0°·51989 0	0°·57051 5
P	253^d	692^d	631^d
M	206° 23′·9	66°·38825	
			T 1952 April 4·70254 U.T.

where P is the sidereal period and T is the time of pericentric passage.

D. THE RINGS AND SATELLITES OF SATURN

Authorities

The ephemeris of the rings of Saturn is computed from the elements of the plane of the rings determined by G. Struve (*Veröffentlichungen der Universitäts-sternwarte zu Berlin-Babelsberg*, **6**, no. 4, 49, 1930). The apparent outer dimensions of the outer ring are according to H. Struve (*Publications de l'Observatoire Central Nicolas*, ser. 2, **11**, 226, 1898); the factors for calculating the relative dimensions of the rings are from Bessel (*Abhandlungen*, Vol. I, pages 110, 150, 319, 1875), except those for the dusky ring which are based on the observations of various astronomers (in particular, O. Struve at Pulkovo, A. Hall at Washington, E. E. Barnard at Lick, and T. Lewis at Greenwich).

The ephemerides of the six inner satellites and of Iapetus are computed from the orbital elements determined by G. Struve (*Veröffentlichungen der Universitätssternwarte zu Berlin-Babelsberg*, **6**, no. 4, page 61, 1930; **6**, no. 5, 10-15, 1933). The ephemeris of Hyperion is computed from the elements given by J. Woltjer, Jr. (*Annalen van de Sterrewacht te Leiden*, **16**, part 3, 64, 1928); and that of Phoebe, from the theory by F. E. Ross (*Annals of Harvard College Observatory*, **53**, no. 6, 1905).

Eclipses, occultations, transits, and shadow-transits of the satellites occur during a limited period each time the Earth passes through the plane of the rings. Methods for calculating these phenomena are described by L. J. Comrie (*Mem. B.A.A.*, **30**, 97-106, 1934). See also S. W. Taylor, On the shadow of Saturn on its rings, *A.J.*, **55**, 229, 1951.

The rings of Saturn

The ephemeris of the rings contains the following quantities that determine the Saturnicentric positions of the Earth and Sun referred to the plane of the rings, upon which the appearance of the rings depends:

U = the geocentric longitude of Saturn, measured in the plane of the rings eastwards from its ascending node on the mean equator of the Earth; the Saturnicentric longitude of the Earth, measured in the same way, is $U + 180°$;

B = the Saturnicentric latitude of the Earth referred to the plane of the rings, positive towards the north; when B is positive, the visible surface of the rings is the northern surface;

P = the geocentric position angle of the northern semi-minor axis of the apparent ellipse of the rings, measured from the north towards the east;

U' = the heliocentric longitude of Saturn, measured in the plane of the rings eastwards from its ascending node on the ecliptic; the Saturnicentric longitude of the Sun, measured in the same way, is $U' + 180°$;

B' = the Saturnicentric latitude of the Sun referred to the plane of the rings, positive towards the north; when B' is positive, the northern surface of the rings is the illuminated surface;

P' = the heliocentric position angle of the northern semi-minor axis of the rings on the heliocentric celestial sphere, measured eastwards from the circle of latitude through Saturn.

In 1960 and preceding years, the ephemeris of the rings did not include the effect of aberration; but beginning with 1961, it is corrected for light-time and is immediately comparable with observation.

Referred to the ecliptic and mean equinox of 1889·25, G. Struve's values for the inclination of the plane of the rings and the longitude of the ascending node are:

$$i = 28° \, 04'\cdot 55$$
$$\Omega = 167° \, 58'\cdot 08$$

Adding the variations due to precession, calculated by the formulae in section 2B, gives the elements referred to the ecliptic and mean equinox of date; from them, the inclination (J) to the mean equator of date and the right ascension (N) of the ascending node measured from the mean equinox of date, together with the arc ω from the ascending node N on the mean equator to the ascending node Ω on the ecliptic, are obtained from the formulae (from triangle ♈N☊ of figure 12.5):

$$\begin{aligned}
\sin J \sin N &= \sin i \sin \Omega \\
\sin J \cos N &= \cos i \sin \epsilon + \sin i \cos \epsilon \cos \Omega \\
\cos J &= \cos i \cos \epsilon - \sin i \sin \epsilon \cos \Omega \\
\sin J \sin \omega &= \sin \epsilon \sin \Omega \\
\sin J \cos \omega &= \sin i \cos \epsilon + \cos i \sin \epsilon \cos \Omega
\end{aligned}$$

where ϵ is the mean obliquity of date. Table 12.1 gives the values of Ω, i, N, J, ω at intervals of 1000 days from 1954 to 1979.

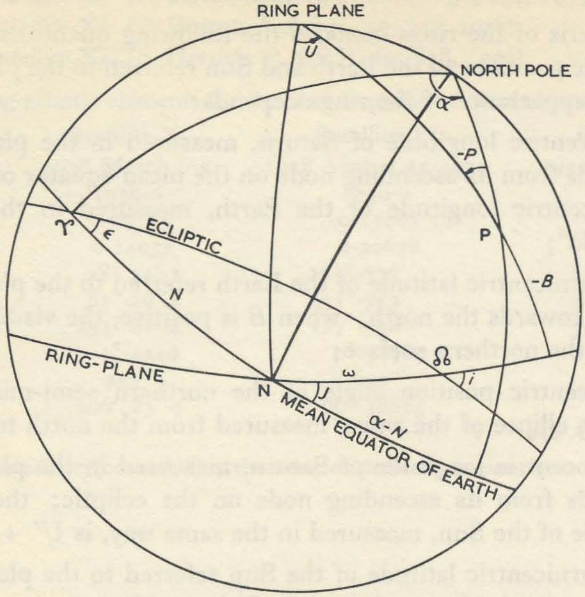

Figure **12.5**. *The ring-plane of Saturn*
P Geocentric position of Saturn (α, δ)

From the elements referred to the equator of the Earth, and from the geocentric equatorial coordinates of Saturn, the ephemeris of U, B, P is calculated by the formulae developed in sub-section A, since these quantities are defined with reference to the ring-plane in the same way as with reference to the orbital plane of a satellite (compare figure 12.5 with 12.1); but the effect of nutation must first be removed from the apparent right ascension and declination of Saturn, in order to refer the position of the planet to the same equinox as the elements of the reference plane.

Evidently, U', B', P' may be obtained by exactly analogous formulae from the ecliptic elements of the plane of the rings and the heliocentric longitude (l) and latitude (b) of Saturn, referred to the mean equinox of date:

$$\cos B' \sin P' = -\sin i \cos (l - \Omega)$$
$$\cos B' \cos P' = +\cos i \cos b + \sin i \sin b \sin (l - \Omega)$$
$$\sin B' = -\cos i \sin b + \sin i \cos b \sin (l - \Omega)$$
$$\cos B' \sin U' = +\sin i \sin b + \cos i \cos b \sin (l - \Omega)$$
$$\cos B' \cos U' = +\cos b \cos (l - \Omega)$$

In the following example of the calculation of the ephemeris of the rings of Saturn aberration is fully included; the corrections for light-time are determined in the same way as in the calculation of the planetocentric coordinates of the Earth and the Sun in the ephemerides for physical observations of Mars and Jupiter (see section 11G). In particular, the correction to be applied to the heliocentric

Table 12.1. Elements of the Rings of Saturn

J.D.	Date	☊	i	N	J	ω
		°	°	°	°	°
243 5000·5	1954 Sept. 15·0	168·8802	28·0673	128·6159	6·6692	41·3550
6000·5	1957 June 11·0	·9184	·0670	·7325	·6573	·2729
7000·5	1960 Mar. 7·0	·9566	·0666	·8494	·6454	·1903
8000·5	1962 Dec. 2·0	168·9947	·0662	128·9667	·6335	·1075
243 9000·5	1965 Aug. 28·0	169·0329	·0659	129·0844	·6217	41·0242
244 0000·5	1968 May 24·0	169·0711	28·0655	129·2024	6·6098	40·9406
1000·5	1971 Feb. 18·0	·1092	·0652	·3207	·5980	·8568
2000·5	1973 Nov. 14·0	·1474	·0648	·4395	·5863	·7725
3000·5	1976 Aug. 10·0	·1856	·0645	·5586	·5745	·6877
244 4000·5	1979 May 7·0	169·2237	28·0642	129·6781	6·5628	40·6027

Example 12.6. *The ephemeris of the rings of Saturn*
1960 March 7 at 0ʰ U.T.

The data are taken for 0ʰ E.T. from *A.E.*, pages 175 and 211, and from the preceding tables; the derivation of the correction for nutation is not illustrated.

As seen from the Earth

app. α	287°·6585	app. δ	−21°·9558	J	6°·6454
nutation	+0·0005	nutation	− 0·0023		
α	287·6590	δ	−21·9581	$\sin J$	+0·11572 42
N	128·8494			$\cos J$	+0·99328 14
$\alpha - N$	158·8096				
$\sin(\alpha - N)$	+0·36146 83	$\sin\delta$	−0·37392 84	$\sin\delta\sin(\alpha - N)$	−0·13516 33
$\cos(\alpha - N)$	−0·93238 44	$\cos\delta$	+0·92745 76	$\cos\delta\sin(\alpha - N)$	+0·33524 65
$\cos B\sin U$	+0·28972 15	$\sin B$	+0·41021 22	$\cos B\sin P$	+0·10789 94
$\cos B\cos U$	−0·86474 70	$\cos B$	+0·91199 01	$\cos B\cos P$	+0·90558 47
$\tan U$	−0·33503 61			$\tan P$	+0·11914 89
U	161°·4773	B	+24°·2182	P	+6°·7947

As seen from the Sun

tabular l	281°·4397	daily motion	+0°·03015	i	28°·0666
light-time	−0·0035 (mean value)	r	10·06		
l	281·4362	Δ	10·52	$\sin i$	+0·47049 75
☊	168·9566			$\cos i$	+0·88240 13
$l -$ ☊	112·4796	b	+0°·5122		
$\sin(l -$ ☊$)$	+0·92401 57	$\sin b$	+0·00893 95	$\sin b\sin(l -$ ☊$)$	+0·00826 02
$\cos(l -$ ☊$)$	−0·38235 45	$\cos b$	+0·99996 01	$\cos b\sin(l -$ ☊$)$	+0·92397 88
$\cos B'\sin U'$	+0·81952 61	$\sin B'$	+0·42684 15	$\cos B'\sin P'$	+0·17989 68
$\cos B'\cos U'$	−0·38233 92	$\cos B'$	+0·90432 64	$\cos B'\cos P'$	+0·88625 25
$\cot U'$	−0·46653 69			$\tan P'$	+0·20298 59
U'	115°·0108	B'	+25°·2673	P'	+11°·4743

It can be verified that the values printed in *A.E.* 1960, page 374, are reproduced using values of α = 287°·6635, δ = −21°·9553, l = 281°·4397 uncorrected for aberration. No correction is applied for the reduction from 0ʰ E.T. to 0ʰ U.T.

* longitude is $-0{\cdot}00577\ (\Delta + r) \times$ daily motion. The tabular values in *A.E.* 1960 are not so corrected for light-time, but beginning with 1961 these corrections are included as in this example.

The rings become invisible whenever:

(i) the ring-plane passes through the Sun, since neither side of the rings is then illuminated;

(ii) the ring-plane passes between the Sun and the Earth, since the unilluminated side of the rings is then facing the Earth;

(iii) the ring-plane passes through the Earth, since the rings are too thin to be visible edge on.

Twice during each revolution of Saturn around the Sun, near the times when the ring-plane passes through the Sun, the Earth crosses the ring-plane either once or, more often, three times. In 1936-1937, two of the three passages of the Earth through the ring-plane were coincident, at the time when the rings first disappeared, on 1936 June 30; the second disappearance was on December 29, and the rings remained continuously invisible until 1937 February 21 while the Earth was on the unilluminated side. In 1950, only one passage of the Earth through the ring-plane occurred, almost at the time of conjunction.

The five inner satellites of Saturn

The orbital elements of the first four satellites, Mimas, Enceladus, Tethys, and Dione, are referred to the ring-plane, which is assumed to coincide with the equator of Saturn. The longitude (Θ) of the node is measured from the equinox along the ecliptic to the ascending node of the ring-plane and then in this plane to the ascending node of the orbit; the inclination is denoted by γ. The longitude of perisaturnium, similarly measured from the equinox along the ecliptic to the node of the ring-plane, then in the ring-plane to the node of the orbit, and then along the orbit, is denoted by Π_1; the mean orbital longitude, measured in the same way, is denoted by l_1. (See figure 12.6).

The mean longitude is represented by:

$$l_1 = E_0 + n\,d + \delta l$$

in which E_0 is the mean longitude at the epoch, n is the tropical mean daily motion, d is the number of days from the epoch J.D. 241 1093·0, and δl denotes a libration in longitude that is characteristic of the motions of these satellites.

These librations are due to mutual perturbations depending on the near commensurabilities of the mean motions; the mean motions of Tethys and Mimas are nearly in the ratio 1 : 2, and those of Dione and Enceladus are likewise very nearly in the ratio 1 : 2.

Reckoning the time t from the epoch (1889·25) in tropical years, and expressing the date τ in Besselian years, G. Struve's values for the elements of these four satellites are:

*0·00578 for 1968 onwards.

Epoch 1889 April 0·0 G.M.A.T. = J.D. 241 1093·0

(Epoch for t is 1889·25)

	Mimas	Enceladus	Tethys	Dione
E_0	127° 05′·5	199° 25′·8	284° 28′·3	253° 52′·0
n	381°·99444 2	262°·73194 05	190°·69795 0	131°·53497 29
Θ	56°·1 − 365°·23 t	52° − 152°·7 t	110°·39 − 72°·25 t	201° − 31°·0 t
γ	1° 31′·0	1′·4	1° 05′·56	1′·4
Π_1	105°·0 + 365°·60 t	308°·38 + 123°·43 t	...	173°·4 + 30°·75 t
e	0·0201	0·00444	...	0·00221
a	255″·89	328″·29	406″·40	520″·51

Librations:

Mimas $\delta l = -44°{\cdot}390 \sin\{5°{\cdot}0864\,(\tau - 1866{\cdot}27)\} - 0°{\cdot}764 \sin 3\,\{5°{\cdot}0864\,(\tau - 1866{\cdot}27)\}$

Enceladus $\delta l = +14'{\cdot}39 \sin(63°{\cdot}75 + 32°{\cdot}51\,t) + 14'{\cdot}06 \sin(117°{\cdot}28 + 93°{\cdot}14\,t)$

Tethys $\delta l = +2°{\cdot}065 \sin\{5°{\cdot}0864\,(\tau - 1866{\cdot}27)\} + 0°{\cdot}036 \sin 3\,\{5°{\cdot}0864\,(\tau - 1866{\cdot}27)\}$

Dione $\delta l = -0'{\cdot}93 \sin(63°{\cdot}75 + 32°{\cdot}51\,t) - 0'{\cdot}91 \sin(117°{\cdot}28 + 93°{\cdot}14\,t)$

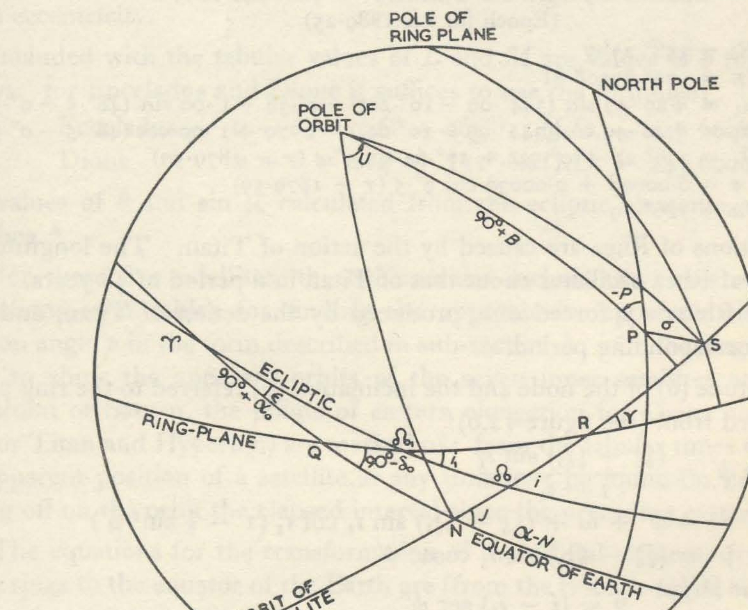

Figure 12.6. The inner satellites of Saturn

P Geocentric position of Saturn (α, δ)

S Planetocentric position of satellite

$$\Upsilon ☊_1 + ☊_1\mathrm{R} = \Theta \qquad ☊\mathrm{R} = \psi'$$

$$\mathrm{Q}☊_1 = \omega \qquad \mathrm{N}☊ = \omega'$$

$$\mathrm{QR} = \theta = \Theta - ☊_1 + \omega \qquad \mathrm{NR} = \psi = \psi' + \omega'$$

$$\mathrm{QR} + \mathrm{RS} = u \qquad \mathrm{NS} = u - \theta + \psi$$

$$\mathrm{RS} = u - \theta \qquad ☊_1\mathrm{R} = \theta - \omega$$

The longitude (θ) of the node of the orbit on the ring-plane, when measured from the ascending node of the ring-plane on the mean equator of the Earth instead of from the equinox, is given by:

$$\theta = \Theta - \Omega_1 + \omega$$

where Ω_1 is the longitude of the ascending node of the ring-plane on the ecliptic and ω is the arc of the ring-plane from its node on the equator to its node on the ecliptic.

Some of the elements of Rhea are not available in the same form as for the other inner satellites. Expressions for Θ and γ are not given, but instead there are expressions for the longitude (Ω) of the node of the orbital plane on the ecliptic and for the inclination (i) of the orbital plane to the ecliptic; in these expressions Ω_1 and i_1 denote the longitude of the node and the inclination of the ring-plane on the ecliptic. Further, the librations are not applied directly to the mean longitude ($l_1 = E_0 + nd$) but to Ω, i, e, Π_1. G. Struve's values for the elements of Rhea are thus:

Epoch 1889 April 0·0 G.M.A.T. = J.D. 241 1093·0
(Epoch for t is 1889·25)

$$E_0 = 358^\circ\ 23'{\cdot}7$$
$$n = 79^\circ{\cdot}69008\ 81$$
$$(\Omega - \Omega_1)\sin i_1 = +20'{\cdot}49 \sin(344^\circ{\cdot}09 - 10^\circ{\cdot}20t) - 0'{\cdot}38 + 1'{\cdot}00 \sin(48^\circ{\cdot}5 - 0^\circ{\cdot}50t)$$
$$i - i_1 = +20'{\cdot}49 \cos(344^\circ{\cdot}09 - 10^\circ{\cdot}20t) - 2'{\cdot}79 + 1'{\cdot}00 \cos(48^\circ{\cdot}5 - 0^\circ{\cdot}50t)$$
$$\Pi_1 = 276^\circ{\cdot}25 + 0^\circ{\cdot}53t + 17^\circ{\cdot}64 \sin 9^\circ{\cdot}5\,(\tau - 1879{\cdot}59)$$
$$e = 0{\cdot}00098 + 0{\cdot}00030 \cos 9^\circ{\cdot}5\,(\tau - 1879{\cdot}59)$$
$$a = 726''{\cdot}89$$

The librations of Rhea are caused by the action of Titan. The longitude of perisaturnium of Rhea oscillates about that of Titan in a period of 38 years. The eccentricity of Rhea is a forced one, produced by the action of Titan, and has variations of corresponding period.

The longitude (θ) of the node and the inclination (γ) referred to the ring plane may be obtained from (see figure 12.6):

$$\tan\psi' = \frac{(\Omega - \Omega_1)\sin i_1}{i - i_1}$$
$$\theta = \psi' + \omega + (\Omega - \Omega_1)\sin i_1 \cot i_1\,(1 - \tfrac{1}{2}\sin^2\psi')$$
$$\gamma = (\Omega - \Omega_1)\sin i_1 \operatorname{cosec}\psi'$$

or, if cosec ψ' is large:

$$\gamma = (i - i_1)\sec\psi'$$

The mean orbital longitude (L) of one of these satellites, measured along the ring-plane from the node of the ring-plane on the equator of the Earth to the ascending node of the orbit on the ring-plane and then along the orbit, is given by:

$$L = l_1 - \Omega_1 + \omega$$

The true orbital longitude (u) is measured in the same way, and is obtained by adding the equation of centre to L; thus:

$$u = L + 57^\circ{\cdot}2958\,(2e \sin M + \tfrac{5}{4}e^2 \sin 2M + \ldots)$$

where $M = l_1 - \Pi_1$ is the mean anomaly, measured from perisaturnium. Thus for these satellites, and for Titan, the definition of u differs from that used in sub-section A and for other satellites.

The radius vector is given by:

$$r/a = 1 + \tfrac{1}{2} e^2 - e \cos M - \tfrac{1}{2} e^2 \cos 2M - \dots$$

The periodic solar perturbations of the orbits of the five inner satellites are neglected in the computations for the Ephemeris. The mean longitude (L) and mean anomaly (M), calculated from the orbital elements by the preceding relations, are tabulated at intervals of 10 days; the tabular values are the actual values at the tabular time, not corrected for light-time. The true longitude and radius vector may be obtained from the formulae:

Mimas

$$u = L + 2^\circ{\cdot}303 \sin M + 0^\circ{\cdot}029 \sin 2M$$
$$r/a = 1{\cdot}0002 - 0{\cdot}0201 \cos M - 0{\cdot}0002 \cos 2M$$

Tethys

$$u = L$$
$$r/a = 1$$

Enceladus

$$u = L + 0^\circ{\cdot}509 \sin M$$
$$r/a = 1 - 0{\cdot}0044 \cos M$$

Dione

$$u = L + 0^\circ{\cdot}253 \sin M$$
$$r/a = 1 - 0{\cdot}0022 \cos M$$

Similar numerical expressions cannot be given for Rhea because of the variation of the eccentricity.

Included with the tabular values of L and M are values of θ for Mimas and Tethys; for Enceladus and Dione it suffices to use the formulae:

Enceladus $\quad u - \theta = 36^\circ + 263^\circ{\cdot}15$ (J.D. − 243 6000·5)

Dione $\quad u - \theta = 214^\circ + 131^\circ{\cdot}62$ (J.D. − 243 6000·5)

The values of θ and $\sin \gamma$, calculated from the ecliptic elements, are tabulated for Rhea.*

For these five satellites, the ephemerides include the times of the eastern elongations, and tables for finding the approximate apparent distance s and position angle p in the form described in sub-section A. On the diagram, which is given to show the apparent orbits of the seven inner satellites at the date of opposition of Saturn, the points of eastern elongation for Dione and Rhea (and also for Titan and Hyperion) are marked 0^d; from the tabular times of elongation, the apparent position of a satellite at any time may be found on the diagram by setting off on the orbit the elapsed interval since the preceding eastern elongation.

The equations for the transformation of the orbital elements from the plane of the rings to the equator of the Earth are (from the triangle QRN in figure 12.6):

$$\sin J \cos (N - \alpha_0) = -\sin \gamma \sin \theta$$
$$\sin J \sin (N - \alpha_0) = +\cos \gamma \cos \delta_0 + \sin \gamma \sin \delta_0 \cos \theta$$
$$\cos J = +\cos \gamma \sin \delta_0 - \sin \gamma \cos \delta_0 \cos \theta$$

where α_0, δ_0 are the right ascension and declination of the pole of the ring-plane. The arc ψ (NR in figure 12.6) of the orbit from the ascending node on the celestial equator to the ascending node on the plane of the rings is determined from:

$$\sin J \sin \psi = \cos \delta_0 \sin \theta$$
$$\sin J \cos \psi = \sin \gamma \sin \delta_0 + \cos \gamma \cos \delta_0 \cos \theta$$

and the orbital longitude reckoned from the ascending node of the satellite orbit on the celestial equator is $(u - \theta) + \psi$.

*γ, not $\sin \gamma$, is tabulated for 1971 onwards.

Example 12.7. The orbital positions of the five inner satellites of Saturn

1960 March 27 at 0^h E.T.

J.D.	243 7020·5	τ	1960·2345	i_1	28°·0666
Epoch	241 1093·0	$\tau - 1889{\cdot}25 = t$	70·9845	Ω_1	168·9574
d	2 5927·5	$\tau - 1866{\cdot}27$	93·9645	ω	41·1886
ΔT (assumed)	$+ 36^s$	$\tau - 1879{\cdot}59$	80·6445	$\omega - \Omega_1$	232·2312

	Mimas	Enceladus	Tethys	Dione	Rhea
	°	°	°	°	°
E_0	127·0917	199·4300	284·4717	253·8667	358·3950
nd	200·8950	62·3873	81·0986	93·0099	124·7592
δl	−39·1333	−0·3436	+1·8204	+0·0222	—
l_1	288·8534	261·4737	7·3907	346·8988	123·1542
Π_1	136·933	69·9968	—	196·173	326·587
$l_1 - \Pi_1 = M$	151·920	191·477	—	150·726	156·567
Θ	50·43	12·67	21·76	160·48	—
$\Theta + (\omega - \Omega_1) = \theta$	282·66	244·90	253·99	32·71	18·70
$l_1 + (\omega - \Omega_1) = L$	161·0846	133·7049	239·6219	219·1300	355·3854
γ	1·517	0·023	1·093	0·023	0·314

1960 March 27 at 0^h U.T.

	°	°	°	°	°
Motion in ΔT	0·1592	0·1095	0·0795	0·0548	0·0332
M	152·08	191·59	—	150·78*	156·60
L	161·2438	133·8144	239·7014	219·1848	355·4186

The librations are calculated from the formulae given. For Rhea:

$(\Omega - \Omega_1) \sin i_1$	−0°·11912	ω	41°·19
$i - i_1$	+0·29073	correction to θ	−0·21
$\tan \psi'$	−0·4097	ψ'	337·72
$\operatorname{cosec} \psi'$	−2·638	sum $= \theta$	18·70
γ	+0°·314	$\sin \gamma$	+0·00548

* The tabular values of M for Dione in *A.E.*, 1960, page 384, are in error owing to an error in the application of the correction from 0^h E.T. to 0^h U.T.

For the five inner satellites, the times of eastern elongation and the tables for apparent distance and position angle are calculated from the approximations:

$$U = U_R$$
$$B = B_R + \gamma \sin (U_R - \theta)$$
$$P = P_R - \gamma \cos (U_R - \theta) \sec B_R$$

where the subscript denotes the value for the ring-plane; these relations may be obtained by differentiating the spherical triangle (see figure 12.6) formed by the pole of the rings, the node of the orbit on the rings, and the geocentric position of Saturn. Further, a constant value B_0 is used for B, equal to the value at opposition of Saturn; and the orbital eccentricity is neglected, giving $u = L$.

Eastern elongation occurs when $u - U = +90°$. At any instant, the interval of time that has elapsed since the preceding apparent eastern elongation is the time equivalent of $(u - U) - 90° -$ (light-time), and the interval until the next one is the equivalent of $90° - (u - U) +$ (light-time). For the systematic calculation of the ephemerides, the value of $L - (U + 90°)$ is calculated at intervals of 10 days. With the rates of motion determined from these values, the elongations during each 10-day interval are obtained. The tables of F and p_1 are calculated from the angular equivalents of the tabular time intervals, using the formulae developed in sub-section A; these become in this particular case:

$$\begin{aligned} \sin\sigma \sin(p - P) &= \sin(L - U) \\ \sin\sigma \cos(p - P) &= \sin B_0 \cos(L - U) \\ \cos\sigma &= \cos B_0 \cos(L - U) \\ p_1 &= P_0 + (p - P) \\ p_2 &= P - P_0 \end{aligned}$$

where P, B_0 are defined above and $F = \sin\sigma$.

The apparent distance and position angle, and the rectangular coordinates referred to the plane of the rings, may also be obtained from the tabulated values of the actual orbital elements and the true orbital longitude (u) referred to the ring-plane. The formulae are derived by following the procedure outlined in the discussion of differential coordinates in sub-section A, but taking the ring-plane as the fundamental reference plane, and using the spherical triangle formed on the geocentric celestial sphere by Saturn, the satellite, and the pole of the ring-plane at position angle P. Neglecting higher-order terms in γ, the apparent rectangular coordinates referred to Saturnicentric axes, with the x-axis in the ring-plane and positive towards the east, the y-axis positive towards the north pole of Saturn, are:

$$\begin{aligned} x &= \frac{1}{1 + \zeta} \frac{a}{\Delta} \frac{r}{a} \sin(u - U) \\ &= s \sin(p - P) \\ y &= \frac{1}{1 + \zeta} \frac{a}{\Delta} \frac{r}{a} \left\{ \cos(u - U) \sin B + \sin\gamma \sin(u - \theta) \cos B \right\} \\ &= s \cos(p - P) \end{aligned}$$

where

$$\zeta = \frac{a}{\Delta} \frac{r}{a} \cos B \cos(u - U)$$

in which U and B refer to the ring-plane, and u is reckoned from the node of the ring-plane on the celestial equator to the node of the orbit on the ring-plane and then along the orbit.

Approximate values of $1/(1 + \zeta)$, which can never depart greatly from unity, are tabulated in the Ephemeris on the pages containing the orbital elements L, M, θ. The values are given in short critical tables with argument $u - U$; in *A.E.* 1960 and 1961 they are denoted by F, but (to avoid confusion with the use of F for the factor $\sin\sigma$ in the tabulations for the apparent distance) this notation will be changed in 1962.

Example 12.8. *The times of greatest eastern elongation of Mimas, Tethys, and Rhea*

1960 March 27 at 0^h E.T.

A.E., p. 374 $\quad U_R = 162°·843 \quad B_R = +23°·918 \quad P_R = +6°·840$

		Mimas	Tethys	Rhea
Example 12.7.	$u = L$	161°·0846	239°·6219	355°·3854
$U = U_R$	$u - U$	358·242	76·779	192·542
Time for motion of $u - U$ through 1°		0ʰ·06283	0ʰ·12589	0ʰ·30134
Angle to eastern elongation $90° - (u - U)$		91°·758	13°·221	257°·458
Equivalent time-interval		5ʰ·765	1ʰ·664	77ʰ·582
$0^h·13849\Delta$ (Δ = 10·20654)		1·414	1·414	1·414
E.T. of eastern elongation		27ᵈ 7ʰ·18	27ᵈ 3ʰ·08	30ᵈ 7ʰ·00

Example 12.9. *Table entries for apparent distance and position angle of Mimas, Tethys, and Rhea*

1960 March 27 at 0^h E.T.

	Mimas	Tethys	Rhea
Δ = 10·20654 (*A.E.*, p. 211); a/Δ	25″·07	39″·82	71″·22
$P_R - \gamma \cos (U_R - \theta) \sec B_R = P$	7°·66	6°·86	7°·12
Adopted constant P_0	7	7	7
$P - P_0 = p_2$	+0·66	−0·14	+0·12

1960 July 7 (date of opposition) at 0^h E.T.

A.E., p. 375 $\quad U_R = 160°·208 \quad B_R = +24°·590 \quad P_R = +6°·752$

	Mimas	Tethys	Rhea
$B_R + \gamma \sin (U_R - \theta) = B_0$	+24°·083	+23°·545	+24°·775
$\sin B_0$	+ 0·40806	+ 0·39947	+ 0·41906
$\cos B_0$	+ 0·91296	+ 0·91674	+ 0·90796
Time from eastern elongation t	10ʰ·0	1ᵈ 16ʰ	3ᵈ 06ʰ
(Note: $L = u$) $\quad 90° + nt = u - U$	249°·164	47°·830	348°·993
$\sin \sigma \sin (p - P) = \sin (u - U)$	− 0·93460	+ 0·74116	− 0·19093
$\cos (u - U)$	− 0·35570	+ 0·67133	+ 0·98160
$\sin \sigma \cos (p - P) = \cos (u - U) \sin B_0$	− 0·14515	+ 0·26818	+ 0·41135
$\cos \sigma = \cos (u - U) \cos B_0$	− 0·32474	+ 0·61544	+ 0·89125
$\cot/\tan (p - P)$	+ 0·15531	+ 0·36184	− 0·46416
$p - P$	261°·17	70°·11	335°·10
$P_0 + (p - P) = p_1$	268·2	77·1	342·1
$\sin \sigma = F$	0·946	0·788	0·453

Titan

Secular perturbations by the Sun and Iapetus are included in the mean elements of Titan, and periodic solar perturbations are added in calculating the ephemerides of L, M, e, sin γ, and θ that are given in the Ephemeris.* The effect of the periodic perturbations on the times of eclipses, occultations, and other phenomena of Titan may amount to 2^m.

The pole of the orbital plane describes a circle of radius 18′·35 on the celestial sphere, in a period of about 700 years, around the pole of the Laplacian plane of Titan. Referred to the ecliptic and equinox of 1890·0, the node of the Laplacian plane is at longitude 167° 51′·90 and the inclination is 27° 26′·33. The node of Titan on this fixed plane regresses at the rate 0°·506 per Julian year. Therefore, by relations of the same form as have been given in sub-section B for the satellites of Mars, the node and inclination of the orbit of Titan referred to the ecliptic and equinox of 1890·0 are:

$$\Omega = 167° \; 51'{\cdot}90 + 39'{\cdot}00 \sin (40°{\cdot}69 - 0°{\cdot}506\, t)$$
$$i = 27° \; 26'{\cdot}33 + 18'{\cdot}35 \cos (40°{\cdot}69 - 0°{\cdot}506\, t)$$

where t is reckoned in Julian years from 1890·0.

The variations of Ω and i due to precession and to the secular motion of the ecliptic are assumed to be the same as for the node Ω_1 and inclination i_1 of the ring-plane. With the value now adopted for Ω_1 in the ephemeris of the rings, its value for 1890·0 was 167° 58′·707; referred to the ecliptic and mean equinox of date, the elements of the orbital plane of Titan are therefore:

$$\Omega - \Omega_1 = -0°{\cdot}11345 + 0°{\cdot}65000 \sin (40°{\cdot}69 - 0°{\cdot}506\, t)$$
$$i = 27°{\cdot}43883 + 0°{\cdot}30583 \cos (40°{\cdot}69 - 0°{\cdot}506\, t) - 0°{\cdot}00013\, t$$

in which t is measured in Julian years from 1890·0.

The other orbital elements are:

$$a = 1684''{\cdot}35 \text{ at unit distance}$$
$$e = 0{\cdot}02910 + 0{\cdot}000186 (\cos 2g_0 - \cos 2g)$$
$$\Pi = 276° \; 07'{\cdot}7 + 31'{\cdot}41\, t + 22'{\cdot}0 (\sin 2g - \sin 2g_0)$$
$$g = \Pi - \Omega - \Psi, \text{ with } g = g_0 \text{ at } t = 0, \text{ i.e. at } 1890{\cdot}0$$
$$n = 22°{\cdot}57701\,508$$

where the longitude (Π) of the perisaturnium is measured from the mean equinox of date along the ecliptic to the node (Ω in figure 12.7) and then along the orbit, Ψ is the arc (ΩT in figure 12.7) of the orbit from the node on the ecliptic to the ascending node T on the orbit of Saturn, g is the arc of the orbit from the ascending node T on the orbit of Saturn to perisaturnium, and n is the tropical mean daily motion.

The mean longitude, measured from the mean equinox of date along the ecliptic to the node of the orbit and then in the orbit, is:

$$l = 260° \; 24'{\cdot}26 + 22°{\cdot}57701\,508 \, (\text{J.D.} - 241\,1368{\cdot}0)$$
$$+ 4'{\cdot}39 \sin (40°{\cdot}69 - 0°{\cdot}506\, t)$$

in which the second term represents the tropical motion in the orbit since 1890 January 0·0 G.M.A.T., and the periodic term is the reduction from the Laplacian plane to the ecliptic, as in the similar relation for the satellites of Mars in sub-section B.

The mean longitude (L), measured along the ring-plane from its node Q on

*γ, not sin γ, is tabulated for 1971 onwards.

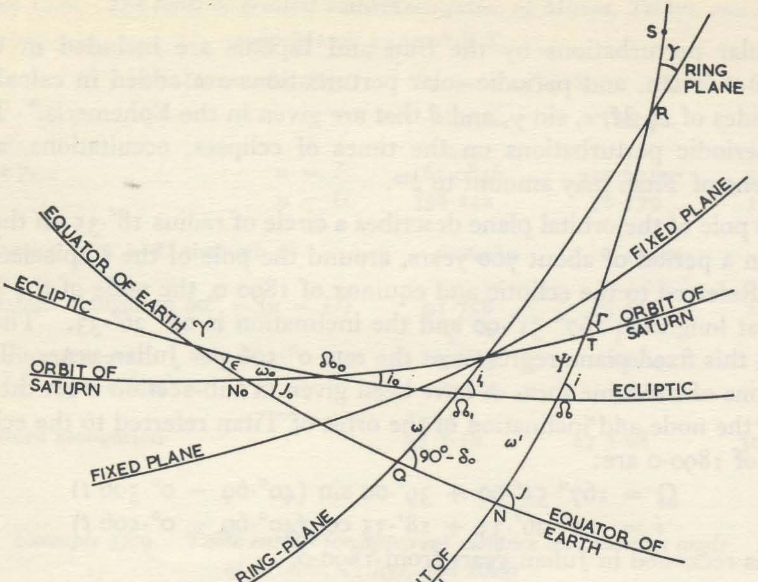

Figure 12.7. *Notation for Titan*

$\text{♈☊}_1 + \text{☊}_1\text{R} = \Theta_1$ $\qquad$ $\text{☊}_1\text{R} = \theta - \omega$

$\text{QR} = \theta = \Theta_1 - \text{☊}_1 + \omega$ $\qquad$ $\text{NR} = \psi = \psi' + \omega'$

$\text{☊R} = \psi'$ $\qquad$ $\text{QR} + \text{RS} = u$

$\text{♈☊}_0 + \text{☊}_0\text{T} = \Theta$ $\qquad$ $\text{NS} = \{l + (v - M)\} - \text{☊} + \omega'$

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the equator of the Earth to the node R of the orbit of Titan on the ring-plane and then in the orbit, is tabulated at intervals of 10 days and is given by:

$$L = l - (\text{☊} + \psi') + \theta$$

where ψ' is the arc ☊R of the orbit from the node on the ecliptic to the ascending node on the ring-plane, and θ is the arc QR of the ring-plane from its node Q on the equator of the Earth to the node R of the orbit of Titan on the ring-plane. The node (θ) and inclination (γ) of the orbital plane referred to the ring-plane and the arc (ψ') are calculated from the formulae (from the triangle ☊_1 R ☊ in figure 12.7):

$$\begin{aligned}
\sin\gamma\sin(\theta - \omega) &= \sin i \sin(\text{☊} - \text{☊}_1)\\
\sin\gamma\cos(\theta - \omega) &= -\cos i \sin i_1 + \sin i \cos i_1 \cos(\text{☊} - \text{☊}_1)\\
\cos\gamma &= \cos i \cos i_1 + \sin i \sin i_1 \cos(\text{☊} - \text{☊}_1)\\
\sin\gamma\sin\psi' &= \sin i_1 \sin(\text{☊} - \text{☊}_1)\\
\sin\gamma\cos\psi' &= \sin i \cos i_1 - \cos i \sin i_1 \cos(\text{☊} - \text{☊}_1)
\end{aligned}$$

where ω is the arc Q☊_1 of the ring-plane between the equator and ecliptic.

* The elements θ, $\sin\gamma$, e, the mean longitude (L), and the mean anomaly ($M = l - \Pi$) are tabulated at intervals of 10 days; the tabular L and M are the actual values at the tabular times, not corrected for light-time. The addition to the mean longitude of the equation of the centre, obtained from the tabular e and M by the usual elliptic formula, gives the true longitude.

*γ, not $\sin\gamma$, is tabulated for 1971 onwards.

Example 12.10. *The elements of the orbit of Titan referred to the plane of the rings*
1960 March 27 at 0^h E.T. = J.D. 243 7020·5

From epoch 1890·0, t = 70·23

From table 12.1	Ω_1	$168^\circ.9574$	i_1	$28^\circ.0666$	ω	$41^\circ.1886$
From the formulae	$\Omega - \Omega_1$	-0.0551	i	27.7343		

The routine solution of the triangle $\Omega_1\ R\ \Omega$ in figure 12.7 gives:

$$\theta - \omega = 184^\circ.4131 \quad \text{and thus} \quad \theta = 225^\circ.6017$$
$$\psi' = 184.4625 \qquad \sin\gamma = 0.00581\ 70$$

These are unperturbed values, whereas the values given in the Ephemeris include the effect of periodic solar perturbations.

The elements of the orbital plane of Titan referred to the orbit of Saturn, which are required in calculating the perturbations, are: the longitude (Θ) of the node, measured from the equinox along the ecliptic to the node Ω_0 of the orbit of Saturn and then along the orbit of Saturn; the inclination (Γ); and the arc (Ψ) of the orbit of Titan between the node Ω on the ecliptic and the node T on the orbit of Saturn. They are calculated from the formulae (from triangle $\Omega_0\ \Omega$ T in figure 12.7):

$$\begin{aligned}
\sin\Gamma\sin(\Theta - \Omega_0) &= \sin i\ \sin(\Omega - \Omega_0)\\
\sin\Gamma\cos(\Theta - \Omega_0) &= -\cos i\ \sin i_0 + \sin i\ \cos i_0\cos(\Omega - \Omega_0)\\
\cos\Gamma &= \cos i\ \cos i_0 + \sin i\ \sin i_0\cos(\Omega - \Omega_0)\\
\sin\Gamma\sin\Psi &= \sin i_0\sin(\Omega - \Omega_0)\\
\sin\Gamma\cos\Psi &= \sin i\ \cos i_0 - \cos i\ \sin i_0\cos(\Omega - \Omega_0)
\end{aligned}$$

where i_0 and Ω_0 are the ecliptic elements of Saturn.

In terms of the mean daily motion (n_0) of Saturn, and its mean longitude (l_0), perihelion (Π_0), and eccentricity (e_0), the periodic perturbations of Titan are represented by the expressions:

$$\begin{aligned}
\Delta e &= \frac{15}{8}\frac{n_0}{n}\, e\cos 2(l_0 - \Pi)\\
\Delta L &= \Delta E - 2\sin\tfrac{1}{2}i\sin\tfrac{1}{2}i_1\,\Delta\Omega\\
\Delta M &= \Delta E - \Delta\Pi\\
\sin\gamma\,\Delta\theta &= \sin i\cos\psi'\,\Delta\Omega - \sin\psi'\,\Delta i\\
\Delta\gamma &= \sin i\sin\psi'\,\Delta\Omega + \cos\psi'\,\Delta i
\end{aligned}$$

in which:

$$\Delta E = -3\frac{n_0}{n}\Big\{e_0\sin(l_0 - \Pi_0) + \tfrac{3}{4}e_0^2\sin 2(l_0 - \Pi_0) + \tfrac{15}{16}e^2\sin 2(l_0 - \Pi) + \tfrac{3}{16}\sin^2\Gamma\sin 2(l_0 - \Theta)\Big\}$$

$$\Delta\Omega = \frac{3}{8}\frac{n_0}{n}\frac{\sin\Gamma}{\sin i}\sin(2\,l_0 - 2\,\Theta + \Psi)$$

$$\Delta i = \frac{3}{8}\frac{n_0}{n}\sin\Gamma\cos(2\,l_0 - 2\,\Theta + \Psi)$$

$$\Delta\Pi = \frac{15}{8}\frac{n_0}{n}\sin 2(l_0 - \Pi)$$

In the calculation of the periodic perturbations of Titan, the values used for the elements of Saturn are based on Hill's values for 1850 (*A.P.A.E.*, **7**, part II, 1895) and the variations given in *Connaissance des Temps* which were determined by Leverrier and Gaillot. Referred to the ecliptic and mean equinox of date, with T measured in Julian centuries from 1900 January 0·0 G.M.A.T. = J.D. 241 5020·0:

$$i_0 = 2^\circ{\cdot}4923 - 0^\circ{\cdot}0039\,T$$
$$\Omega_0 = 112^\circ{\cdot}7836 + 0^\circ{\cdot}87320\,T$$
$$e_0 = 0{\cdot}05589 - 0{\cdot}000346\,T$$
$$\Pi_0 = 91^\circ{\cdot}0891 + 1^\circ{\cdot}9584\,T + 0^\circ{\cdot}0008\,T^2$$
$$l_0 = 266^\circ{\cdot}5653 + 1223^\circ{\cdot}5099\,T + 0^\circ{\cdot}0003\,T^2$$

With these expressions, using quantities already calculated, and with the time t reckoned in Julian years from 1890 January 0·0 G.M.A.T. = J.D. 241 1368·0:

$$\Omega - \Omega_0 = 55^\circ{\cdot}1687 + 0^\circ{\cdot}00521\,t + 0^\circ{\cdot}65000 \sin (40^\circ{\cdot}69 - 0^\circ{\cdot}506\,t)$$
$$l_0 - \Pi_0 = 53^\circ{\cdot}3378 + 12^\circ{\cdot}215515\,t$$
$$l_0 - \Pi = 228^\circ{\cdot}1028 + 11^\circ{\cdot}71160\,t - 0^\circ{\cdot}36667 (\sin 2g - \sin 2g_0)$$
$$= 227^\circ{\cdot}9398 + 11^\circ{\cdot}71160\,t - 0^\circ{\cdot}36667 \sin 2g$$
$$n_0/n = 0{\cdot}00148\,3716$$
$$i_1 = 28^\circ{\cdot}0757 - 0^\circ{\cdot}00013\,t$$

At $t = 0$, $g_0 = 103^\circ{\cdot}199$; and g may be determined by successive approximations from:

$$g = 108^\circ{\cdot}2633 + 0^\circ{\cdot}50956\,t - 0^\circ{\cdot}65000 \sin (40^\circ{\cdot}69 - 0^\circ{\cdot}506\,t) - \Psi + 0^\circ{\cdot}36667 (\sin 2g - \sin 2g_0)$$

Example 12.11. *The perturbed elements, referred to the plane of the rings, and the orbital position of Titan*

1960 March 27 at 0^h E.T. = J.D. 243 7020·5

From epoch 1900 January 0·0 G.M.A.T. = J.D. 241 5020·0 $\quad T = 0{\cdot}60234\,086$
From epoch 1890 January 0·0 G.M.A.T. = J.D. 241 1368·0 $\quad t = 70{\cdot}23271\,7$

The elements of Saturn are:

i_0 2°·4900 $\quad \Omega_0$ 113°·3096 $\quad e_0$ 0·05568 $\quad l_0$ 283°·535

From the formulae and example 12.10:

i 27°·7343 $\quad \Omega - \Omega_0$ 55°·5929 $\quad l_0 - \Pi_0$ 191°·267

The routine solution of triangle $\Omega_0\,\Omega$ T in figure 12.7 gives:

$\sin \Gamma$ +0·44464 $\quad \Theta - \Omega_0$ 59°·711 $\quad \Theta$ 173°·021 $\quad \Psi$ 4°·624

From this value of Ψ:

g	139°·169	$\sin 2g - \sin 2g_0$	−0·54482
g_0	103·199	$\cos 2g_0 - \cos 2g$	−1·04074

(unperturbed) Π 312·6953 $\quad l_0 - \Theta$ 110°·514 $\quad l_0 - \Pi$ 330·840
$\Psi + 2(l_0 - \Theta)$ 225·652

With these values, and those in example 12.10, using the formulae for the perturbations:

e	0·02890 6	$\Delta\Omega$	−0·00038 02	Δi	−0·00017 29
Δe	+0·00004 2	$\sin\gamma\,\Delta\theta$	+0·00016 29	$(\sin\gamma\,\Delta\theta)/\sin\gamma$	+0·02800 41
$e + \Delta e$	0·02894 8	$\Delta\theta$	+1°·6045	$\Delta\gamma$	+0·00018 61
		$\theta + \Delta\theta$	227·206	$\sin\gamma + \Delta(\sin\gamma)$	+0·00600 3

For the mean longitude and the mean anomaly:

ΔE	+0·00015 57	l	177°·2902	l	177°·2902
$\Delta\Pi$	−0·00236 75	Π	312·6953	$\theta - (\psi' + \Omega)$	232·2369
		$M = l - \Pi$	224·5949	sum = L	49·5271
ΔM	+0·00252 32	ΔM	+0·1446	ΔL	+0·0115
ΔL	+0·00019 99	Red. to 0^h U.T.	+0·0094	Red. to 0^h U.T.	+0·0094
		tabular M	224·749	tabular L	49·548

The orbital longitude $u = L + (v - M)$, measured along the ring-plane from its node on the celestial equator to the node of Titan on the ring-plane and then along the orbit, and the radius vector r/a may be obtained from the tabular values of L, M, e by the usual elliptic formulae. Because of the variation of e, numerical expressions for u and r/a can be given only for particular times. From these orbital coordinates and the ephemeris of U, B, P, the apparent distance and position angle may be calculated, and precise differential coordinates determined, in the same way as described for the five inner satellites or by the formulae developed in sub-section A. For Titan, the factor $1/(1 + \zeta)$, denoted by F in *A.E.* 1960-1961, is given by:

$$1 - \frac{0{\cdot}00817}{\Delta}\frac{r}{a}\cos(u - U)$$

for which a table with argument $u - U$ is given in the Ephemeris.

The practice of referring the tabular quantities L, M, θ to the plane of the rings follows the usage of *The Nautical Almanac* for the years prior to 1960, but differs from that of *Berliner Astronomisches Jahrbuch.* In this publication, L, M, θ and specially calculated values of U, B, P were referred to the plane of Titan's orbit. Since *Berliner Astronomisches Jahrbuch* is no longer published, some measure of continuity was maintained by including tables of U, B, P (referred to Titan's orbit) in *A.E.* 1960-1962 but these will not be given in *A.E.* 1963 onwards. It is important to note that the values of U, B, P for the rings should always be used with the tabulated values of L, M, θ for Titan and the five inner satellites. The procedure to be adopted for Hyperion and Iapetus is given below. *

The elements of the orbital plane of Titan referred to the equator of the Earth, which are required for the ephemerides of superior and inferior conjunction, eastern and western elongation, and the tables for apparent distance and position angle, are calculated from i, Ω, ϵ, with the formulae (from triangle ♈N☊ in figure 12.7):

$$\begin{aligned}
\sin J \sin N &= \sin i \sin \Omega \\
\sin J \cos N &= \cos i \sin \epsilon + \sin i \cos \epsilon \cos \Omega \\
\cos J &= \cos i \cos \epsilon - \sin i \sin \epsilon \cos \Omega \\
\sin J \sin \omega' &= \sin \epsilon \sin \Omega \\
\sin J \cos \omega' &= \sin i \cos \epsilon + \cos i \sin \epsilon \cos \Omega
\end{aligned}$$

These quantities, and the ephemeris of U, B, P, defined and measured relative to the orbital plane of Titan in the same way as the values relative to the plane of the rings, are calculated by the formulae developed in sub-section A, omitting the effects of the periodic perturbations. In these formulae, the value of the orbital longitude measured from the node N of the orbit on the equator of the Earth is denoted by u and is given by:

$$\{ l + (v - M) \} - \Omega + \omega'$$

The correction for light-time is applied to the mean anomaly. The apparent coordinates of Saturn are not corrected for nutation, as the error due to the neglect of this correction is insignificant for the purpose of the ephemerides.

The same quantities p_2, a/Δ and p_1, F are given for Titan as for other satellites;

*As from 1966 the elements (L, M etc.) of Hyperion and Iapetus are referred, as for Titan, to the ring-plane.

they are calculated in the same way (see sub-section A or example 12.9) and are not here illustrated. Differential coordinates are not tabulated, but they may be calculated from the tabulated elements (noting that these are always to be used with U, B, P for the rings) by the formulae given in sub-section A.

The apparent orbit at opposition of Saturn is included in the diagram, and the position may be found on the diagram in the same way as described for the five inner satellites.

In the Ephemeris there are tabulated the times (in U.T. to the nearest $0^h\cdot 1$) of the greatest eastern and western elongations, and also of inferior and superior conjunctions. These are calculated by precisely the same methods as for the inner satellites by evaluating the times at which $u - U$ attains the appropriate multiples of 90°.

Hyperion *

The orbital elements of Hyperion as given by Woltjer are referred partly to the ecliptic and partly to the equator of Saturn, that is, to the ring-plane. The longitude (Π) of perisaturnium and the mean longitude (l) are measured from the equinox along the ecliptic to the node and then along the orbit.

The motion of Hyperion is characterized by librations that are due to the action of Titan and depend upon the near 3:4 commensurability of the mean motions, and by long-period inequalities that are introduced by the large eccentricity of Titan. The libration argument is:

$$\sigma = 93^\circ\cdot 13 + 0^\circ\cdot 562039\,(\text{J.D.} - 241\ 5020\cdot 0)$$

with a period of about 1·8 years, and the argument of the long-period terms is:

$$\varpi = 148^\circ\cdot 72 - 19^\circ\cdot 184\,t$$

with a period of about 19 years, t being measured in tropical years from 1900·0.

Omitting solar perturbations and a few of the smaller perturbations by Titan, the elements of Hyperion are:

$$a = 2044''\cdot 4 - 7''\cdot 1 \cos\sigma \text{ at unit distance}$$

$$e = 0\cdot 10419 + 0\cdot 02414 \cos\varpi - 0\cdot 00401 \cos\sigma - 0\cdot 00183 \cos 2\varpi$$

$$\Pi = 70^\circ\cdot 05 - 18^\circ\cdot 6562\,t - 13^\circ\cdot 67 \sin\varpi + 0^\circ\cdot 93 \sin 2\varpi - 0^\circ\cdot 47 \sin\sigma$$

Tropical mean daily motion:

$$n = 16^\circ\cdot 91998\ 96$$

Mean longitude:

$$\begin{aligned} l = {} & 176^\circ\cdot 293 + n\,(\text{J.D.} - 241\ 5020\cdot 0) \\ & + 9^\circ\cdot 092 \sin\sigma + 0^\circ\cdot 211 \sin(\varpi + \sigma) \\ & + 0^\circ\cdot 192 \sin(\varpi - \sigma) - 0^\circ\cdot 077 \sin\varpi \end{aligned}$$

Orbital plane referred to the plane of the rings:

$$\begin{aligned} \gamma \sin(\theta - \omega) &= -0^\circ\cdot 061 + 0^\circ\cdot 574 \sin(95^\circ\cdot 9 - 2^\circ\cdot 392\,t) \\ &\quad + 0^\circ\cdot 315 \sin(42^\circ\cdot 78 - 0^\circ\cdot 500\,t) \\ \gamma \cos(\theta - \omega) &= -0^\circ\cdot 747 + 0^\circ\cdot 574 \cos(95^\circ\cdot 9 - 2^\circ\cdot 392\,t) \\ &\quad + 0^\circ\cdot 315 \cos(42^\circ\cdot 78 - 0^\circ\cdot 500\,t) \end{aligned}$$

where γ is the inclination to the ring-plane, θ is the longitude of the node on the ring-plane measured from the node of the ring-plane on the equator of the Earth, ω is the arc ($Q\Omega_1$ in figure 12·7) of the ring-plane from the equator to the ecliptic, and therefore $\theta - \omega$ is the longitude of the ascending node of the orbit on the ring-plane reckoned from the ascending node of the ring-plane on the ecliptic; t is measured in tropical years from 1900·0.

*See footnote on page 377.

The elements of the orbital plane referred to the equator of the Earth are determined by finding γ and θ from the preceding expressions, and calculating N, J, and ψ by the same formulae as for the five inner satellites, that is by the solution of the triangle QNR in figures 12.6 and 12.7.

The reduction of l to the mean orbital longitude (L) measured along the plane of the orbit from its node on the equator of the Earth is found from the longitude (Ω) of the node of the orbit on the ecliptic and the arc ψ' of the orbit from this node (Ω in figure 12.7) to its node R on the ring-plane; since γ is small (from the triangle Ω_1 ΩR of figure 12.7):

$$\begin{aligned}
\sin i \sin (\Omega - \Omega_1) &= \gamma \sin (\theta - \omega)\\
\sin i \cos (\Omega - \Omega_1) &= \gamma \cos (\theta - \omega) \cos i_1 + \sin i_1\\
\sin i \sin \psi' &= \sin i_1 \sin (\theta - \omega)\\
\sin i \cos \psi' &= \sin i_1 \cos (\theta - \omega) + \gamma \cos i_1
\end{aligned}$$

where i is the inclination of the orbit to the ecliptic, and Ω_1 and i_1 are the ecliptic elements of the ring-plane. Then:

$$L = l - \Omega + (\psi - \psi')$$

Example 12.12. *The orbital position of Hyperion*

1960 March 7 at 0^h U.T. = J.D. 243 7000·5

Days from epoch = 21980·5 Tropical years from 1900·0, t = 60·18061

From the elements:

ϖ	74°·215	a	2050″·7	l	201°·5708
σ	207·028	e	0·11589	Π	14·854
$\gamma \sin (\theta - \omega)$	−0·4187	$\cot (\theta - \omega)$	+0·13375	$\theta - \omega$	262·382
$\gamma \cos (\theta - \omega)$	−0·0560	γ	0°·4224		

From table 12.1:

α_0	38°·849	Ω_1	168°·957	ω	41°·190
δ_0	83·355	i_1	28·067	θ	303·572

From the solution of triangle QRN (figure 12.7):

J 6°·8880 N 125°·9137 ψ 306°·487

From the solution of triangle Ω_1RΩ (figure 12.7):

$\Omega - \Omega_1$	−0°·892	ψ'	263°·168
Ω	168·065	$\psi - \psi'$	43·319
		$l - \Omega$	33·506
$M = l - \Pi$	186°·717	sum = L	76·825
Corrections to 0^h U.T. ($\Delta T = 36^s$)	+ 0·007		+0·007
M	186·724	L	76·832

These values may be compared with those tabulated in *A.E.*, p. 385, by extrapolating the differences.

From elliptic motion: r/a 1·11525 $v - M$ −1°·360

r 0·01108 79 u 75·465 (for 0^h E.T.)

The values of L and of the mean anomaly $M = l - \Pi$ are tabulated in the Ephemeris at intervals of 10 days, together with the values of e and a; the tabular L and M are the actual values at the tabular times, not corrected for light-time. The equation of the centre $(v - M)$ and the radius vector (r/a) are calculated, using the value of e, by the usual elliptic formulae, and the orbital longitude (u) is obtained by adding the equation of the centre to the mean orbital longitude (L).

The other ephemerides for Hyperion are similar to those for Iapetus and are discussed below. The orbit of Hyperion, but not that of Iapetus, is included on the diagram with those of the five inner satellites and Titan.

Iapetus *

The orbital elements of Iapetus are referred to the ecliptic; the longitude (Π) of perisaturnium and the mean longitude (l) are measured along the ecliptic from the equinox to the node, then along the orbit.

Epoch 1885 September 1·0 G.M.A.T. = J.D. 240 9786·0

$$\begin{aligned}
\Omega &= 142^\circ\ 11'{\cdot}3 \quad - 1'{\cdot}375\ t \\
i &= 18^\circ\ 26'{\cdot}39 - 0'{\cdot}54\ t \\
\Pi &= 354^\circ\ 27'{\cdot}4 \quad + 8'{\cdot}1\ t \\
e &= 0{\cdot}02828 \\
a &= 4908''{\cdot}6 \\
E_0 &= 75^\circ\ 25'{\cdot}61 \\
n &= 4^\circ{\cdot}53799\ 536
\end{aligned}$$

where E_0 is the mean longitude at the epoch, n is the tropical mean daily motion, and t is measured in years from the epoch.

The orbital position is obtained directly from the elements by calculating the mean longitude (l) and the mean anomaly (M):

$$l = E_0 + nd$$
$$M = l - \Pi$$

and obtaining the equation of the centre $(v - M)$ and radius vector (r/a) by the usual elliptic formulae, as for the five inner satellites.

Example 12.13. *The orbital position of Iapetus*
1960 March 7 at 0^h U.T. = J.D. 243 7000·5

Days from epoch = 27214·5 Tropical years from epoch, t = 74·5108

From the elements:

a 4908″·6	Ω 140°·4808	Π 4°·516	l 94°·7015
e 0·02828	i 17·7692		$l - \Omega$ 314·221

From the solution of triangle ♈N☊ (figure 12.7):

J 14°·6830	N	50°·0103	ω'	92°·778
	$M = l - \Pi$	90·186	$L = l - \Omega + \omega'$	46·999
Corrections to 0^h U.T. ($\Delta T = 36^s$)		+0·002		+0·002
	M	90·188	L	47·001

These values of L, M may be compared with those tabulated in *A.E.*, p. 385, by extrapolating the differences.

From elliptic motion:	r/a	1·00089	$v - M$	+3°·239
	r	0·02381 87	u	50·238 (for 0^h E.T.)

*See footnote on page 377.

The orbit is referred to the equator of the Earth by calculating the inclination (J), the right ascension (N) of the node N, and the arc $\omega' = \psi - \psi'$ of the orbit from the ascending node N on the equator to the ascending node ☊ on the ecliptic, from the same formulae as for Titan (from the triangle ♈N☊ of figure 12.7). Then the mean orbital longitude (L) measured from the node of the orbit on the equator of the Earth is given by:

$$L = l - \Omega + \omega'$$

and the orbital longitude (u), measured in the same way, is given by:

$$u = L + (v - M)$$

The values of L and M in the ephemeris are the actual values at the tabular times, not corrected for light-time.

Hyperion and Iapetus *

In addition to the tabulation of the orbital positions of Hyperion and Iapetus at intervals of 10 days the Ephemeris contains the following tabulations for each of these satellites: the times of inferior and superior conjunctions, and of eastern and western elongations; tables for finding the approximate apparent distance and position angle, of the same form as for the other satellites, calculated without the inclusion of solar perturbations; an ephemeris of U, B, P at intervals of 8 days, defined and measured relative to the plane of the orbit of the satellite in the same way as the values relative to the ring-plane; and an ephemeris of differential right ascension and declination at intervals of 2 days.

The ephemerides of U, B, P are calculated from the formulae of sub-section A.

Example 12.14. *The coordinates of Saturn referred to the orbital planes of the satellites Hyperion and Iapetus*

1960 March 7 at 0^h E.T.

Coordinates of Saturn (*A.E.*, p. 211)

Apparent coordinates	α	$287^\circ.6585$	δ	$-21^\circ.9558$
Correction for nutation		$+0.0005$		-0.0023
Correction for aberration		$+0.0050$		$+0.0005$
Geometric coordinates		287.6640		-21.9576

U, B, P are obtained by the solution of the triangle formed by the pole of the orbit, the celestial pole, and the geocentric position of Saturn (see figure 12.6). N, J are taken from examples 12.12 and 12.13.

	Hyperion				Iapetus			
	N	$125^\circ.9137$	J	$6^\circ.8880$	N	$50^\circ.0103$	J	$14^\circ.6830$
	$\alpha - N$	161.7503	δ	-21.9576	$\alpha - N$	237.6537	δ	-21.9576
whence:	U	$164^\circ.5464$			U	$239^\circ.8040$		
	B	$+23.9572$			B	$+9.3871$		
	P	$+7.1596$			P	$+7.9007$		

No correction is applied to reduce to 0^h U.T.

*See footnote on page 377.

The rectangular coordinates of these satellites, referred also to the plane of the orbit, are given by:

$$x = \frac{1}{1 + \zeta} \frac{a}{\Delta} \frac{r}{a} \sin (u - U)$$

$$y = \frac{1}{1 + \zeta} \frac{a}{\Delta} \frac{r}{a} \cos (u - U) \sin B$$

where

$$\zeta = \frac{a}{\Delta} \frac{r}{a} \cos (u - U) \cos B$$

These formulae differ from those for the inner satellites, for which U, B, P, u are referred to the ring-plane.

The satellites are at greatest eastern elongation when $u - U = 90°$, and at western elongation when $u - U = 270°$; inferior conjunction is at 180°, superior conjunction at 0°. Since the value of U tabulated in the Ephemeris is not corrected * for light-time, u is similarly not corrected; a correction $+0^h.138494\ \Delta$ is applied to the resulting times of elongation. Two successive approximations are in general necessary to determine the times of these phenomena. In the following examples, the first approximation to the time of the phenomenon that occurs nearest March 7 is calculated; the second approximation is obtained by repeating the calculation with the values of $u - U$, its rate of change, and the light-time at this approximate time.

Successive approximations may be avoided by calculating $u - U$ at suitable equal intervals, and deducing the times at which it reaches multiples of 90° by inverse interpolation.

Example 12.15. *Times of conjunction and elongation of Hyperion and Iapetus*

1960 March 7 at 0^h E.T.

		Hyperion	Iapetus
		°	°
From examples 12.12 and 12.13	u	75·465	50·238
From example 12.14	U	164·546	239·804
	$u - U$	270·919	170·434
		h	h
Time for motion of $u - U$ through 1°		1·7792	5·3902
First approximation:			
		°	°
Angle from western elongation = $(u - U) - 270°$		0·919	—
Angle to inferior conjunction = $180° - (u - U)$		—	9·566
		h	h
Equivalent time-interval		−1·635	+51·563
Light-time (Δ = 10·5178)		+1·457	+ 1·457
		d h	d h
E.T. of western elongation		6 23·82	—
E.T. of inferior conjunction		—	9 05·02
		d h	d h
Second approximation gives:		March 6 23·79	9 05·25

*$0^h.138614$ for 1968 onwards.

In calculating the tables for finding the apparent distance and position angle the hourly motion of Hyperion in mean longitude is calculated from the formula:

$$m = 0^\circ{\cdot}7050 + 0^\circ{\cdot}0037 \cos \sigma$$

where σ is the libration argument; the formula is obtained by differentiating the expression for l. The tabular value of a/Δ is obtained with the value of a at opposition of Saturn. The tables of p_1 and F are calculated for the date of

Example 12.16. *Table entries for apparent distance and position angle of Hyperion and Iapetus*

		Hyperion	Iapetus
1960 July 7 (date of opposition) at 0^h E.T.			
From *A.E.*, p. 386	U	$163^\circ{\cdot}329$	$238^\circ{\cdot}641$
	$B = B_0$	$+24{\cdot}332$	$+9{\cdot}786$
	$\sin B_0$	$+0{\cdot}41203$	$+0{\cdot}16997$
	$\cos B_0$	$+0{\cdot}91118$	$+0{\cdot}98545$
Longitude of perisaturnium	$\Pi - \Omega + \omega'$	$244^\circ{\cdot}839$	$316^\circ{\cdot}868$
At eastern elongation			
	$u = U + 90^\circ$	$253^\circ{\cdot}329$	$328^\circ{\cdot}641$
True anomaly	v	$8{\cdot}490$	$11{\cdot}773$
From prepared tables	$v - M$	$+1{\cdot}780$	$+0{\cdot}648$
Mean anomaly	M_0	$6{\cdot}710$	$11{\cdot}125$
Motion in mean longitude	m	$0^\circ{\cdot}7054$/hour	$4^\circ{\cdot}537995$/day
1960 March 7 at 0^h E.T.			
$\Delta = 10{\cdot}5178$ (*A.E.*, p. 211)	a/Δ	$194''$	$467''$
From example 12.14	P	$+7^\circ{\cdot}160$	$+7^\circ{\cdot}901$
	P_0	$+7$	$+8$
	$P - P_0 = p_2$	$+0{\cdot}16$	$-0{\cdot}10$
Time from eastern elongation	t	$10^d\ 00^h$	18^d
	$M_0 + mt$	$176^\circ{\cdot}006$	$92^\circ{\cdot}809$
From prepared tables	r/a	$1{\cdot}114$	$1{\cdot}0018$
From prepared tables	$v - M$	$+0^\circ{\cdot}782$	$+3^\circ{\cdot}231$
Increase of $(v - M)$ since eastern elongation		$-0^\circ{\cdot}998$	$+2^\circ{\cdot}583$
	$90^\circ + mt$	$259{\cdot}296$	$171{\cdot}684$
	sum $= u - U$	$258{\cdot}30$	$174{\cdot}27$
	$\sin \sigma \sin (p - P) = \sin (u - U)$	$-0{\cdot}97922$	$+0{\cdot}09984$
	$\cos (u - U)$	$-0{\cdot}20279$	$-0{\cdot}99500$
	$\sin \sigma \cos (p - P) = \cos (u - U) \sin B_0$	$-0{\cdot}08356$	$-0{\cdot}16912$
	$\cos \sigma = \cos (u - U) \cos B_0$	$-0{\cdot}18478$	$-0{\cdot}98052$
	cot/tan $(p - P)$	$+0{\cdot}08533$	$-0{\cdot}59035$
	$p - P$	$265^\circ{\cdot}12$	$149^\circ{\cdot}44$
	$P_0 + (p - P) = p_1$	$272{\cdot}12$	$157{\cdot}44$
	$(r/a) \sin \sigma = F$	$1{\cdot}095$	$0{\cdot}197$

opposition, by the formulae developed in sub-section A, with the inclusion of a factor r/a in F for the effect of the orbital eccentricity. The approximate values obtained in this way are most accurate near the elongations, where observations are most often taken; near the conjunctions, the error may be rather large, but observations are in general impracticable for the close satellites.

In example 12.16 the values of U, B, P are uncorrected for light-time, i.e. they are as tabulated in *A.E.*, page 386; in forming $u - U$ no correction for light-time is to be included in u, which can thus be taken direct from examples 12.12 and 12.13.

No example is given of the calculation of the differential coordinates since this follows precisely the methods described in sub-section A and illustrated in example 12.5.

Phoebe

The only tabulation given for Phoebe in the Ephemeris is that of the differential right ascension and declination.

The orbital elements are referred to the plane of the ecliptic. The longitude (Π) of perisaturnium and the mean longitude are measured along the ecliptic from the equinox to the node and then along the orbit. The motion is retrograde; but angular coordinates are measured eastwards in the usual manner, opposite the direction of motion, and consequently the mean motion is negative. The ascending node is the node at which the satellite, moving in the retrograde direction, actually crosses the ecliptic from south to north; the argument of the latitude is measured eastwards from this node. Ross's elements are:

Epoch 1900 January 0·0 G.M.A.T. = J.D. 241 5020·0

$$\Omega = 224^\circ{\cdot}51 + 0^\circ{\cdot}4347\, t$$
$$i = 175^\circ{\cdot}08 - 0^\circ{\cdot}020\, t$$
$$\Pi = 291^\circ{\cdot}03 - 0^\circ{\cdot}2680\, t$$
$$E_0 = 343^\circ{\cdot}15$$
$$n = 0^\circ{\cdot}65398$$
$$e = 0{\cdot}1659$$
$$a = 17861'' \text{ at unit distance}$$

where E_0 is the mean longitude at the epoch, n is the tropical mean daily motion, and t is measured in Julian years from the epoch.

The position of Phoebe obtained directly from the theory developed by Ross is referred to the plane of the orbit of Saturn. The elliptic orbital longitude is $(\Pi_1 + M) + (v - M)$, where v is the true anomaly, M is the mean anomaly measured eastwards, that is opposite to the direction of motion, and Π_1 is the longitude of perisaturnium measured from the equinox along the ecliptic to the node of Saturn, then along the orbit of Saturn to the node of Phoebe, and then along the orbit of the satellite. From the ecliptic elements:

$$M = E_0 - nd - \Pi$$
$$= 52^\circ{\cdot}12 - 0^\circ{\cdot}653246\,(\text{J.D.} - 241\ 5020{\cdot}0)$$

where d is the number of days from the epoch. The inclination (i_1) of the orbit of

Phoebe to the orbit of Saturn, the longitude (Ω_1) of the node, measured from the equinox to the node of Saturn on the ecliptic, and then along the orbit of Saturn, and the value of Π_1 are obtained from the ecliptic elements (i, Ω) of Phoebe and the inclination (i_0) and node (Ω_0) of Saturn on the ecliptic by solving the triangle ($\Omega_0\Omega$T of figure 12.7) formed by the two orbits and the ecliptic. (Note that the symbols Γ and Θ are used instead of i_1 and Ω_1 for the corresponding quantities in the calculation of the perturbations of Titan, and are so marked in figure 12.7.) The Saturnicentric longitude (Λ) referred to the orbital plane of Saturn is measured from the equinox along the ecliptic to the node of Saturn, then on the orbit of Saturn to the perpendicular great circle through Phoebe. The tables constructed from the theory give the total amount to be added to the elliptic orbital longitude for the reduction to the orbit of Saturn and for the periodic solar perturbations in longitude, in order to obtain the longitude (Λ) on the orbit of Saturn.

The latitude referred to the orbital plane of Saturn is:

$$B = B_0 + \delta B$$

where the principal term is calculated from the disturbed longitude Λ by the formula:

$$\tan B_0 = \tan i_1 \sin(\Lambda - \Omega_1)$$

and the solar perturbations δB are taken from the tables. The Saturnicentric distance is given by the tables in the form of the inverse radius vector:

$$\frac{a}{r} = \frac{a}{r_0} + \delta\left(\frac{a}{r}\right)$$

in which a/r_0 is the elliptic value.

The coordinates Λ and B are transformed to the equator of the Earth in the same way as the planetocentric coordinates of the sixth and seventh satellites of Jupiter. Thus, the longitude (μ_S), measured in the plane of Saturn's orbit from the node on the Earth's equator, is given by:

$$\mu_S = \Lambda - \Omega_0 + \omega_0$$

where ω_0 is the arc $N_0\,\Omega_0$ in figure 12.7, and the Saturnicentric right ascension and declination (A, D) are given by:

$$\begin{aligned} \cos D \cos(A - N_0) &= \cos B \cos \mu_S \\ \cos D \sin(A - N_0) &= \cos B \sin \mu_S \cos J_0 - \sin B \sin J_0 \\ \sin D &= \cos B \sin \mu_S \sin J_0 + \sin B \cos J_0 \end{aligned}$$

where J_0 and N_0 are the inclination and node of the orbit of Saturn on the Earth's equator.

From A and D, the geocentric equatorial differential coordinates are calculated by the formulae developed in sub-section A. The correction for aberration is made by applying to the mean anomaly the correction $+0^\circ{\cdot}00377\,\Delta$, where Δ is the geocentric distance of Saturn. The apparent coordinates of Saturn are not corrected for nutation and errors as great as $1^s{\cdot}5/80 = 0^s{\cdot}02$ in right ascension and $10''/80 = 0''{\cdot}1$ in declination may sometimes be introduced by neglecting this correction; but such errors are admissible in a finding ephemeris that is given to only $0^s{\cdot}1$ and $1''$, and moreover they are well within the precision of the tables.

The computational precision of the tables of Phoebe is about 3″ or 4″; but, because of limitations of the theory and uncertainties of the elements, the actual precision is less.

Example 12.17. *The differential coordinates of Phoebe*
1960 March 7 at 0^h U.T. = J.D. 243 7000·5

Many of the data, indicated by an asterisk, are taken without detailed derivation from Ross's tables; the position of Saturn is taken from *A.E.*, page 211, and the elements of Saturn's orbit from page 177.

Epoch J.D. 241 5020·0 Days from epoch = 21980·5 t = 60·179

From the elements of Phoebe:

a	17861″	☊	250°·670	Π	274·902	M	93°·447
e	0·1659	i	173·876	E	343·15	+0·00377 Δ	+0·040
						M	93·487

From the elements of Saturn, and solution of triangles ♈N_0☊$_0$ and ☊$_0$☊T in figure 12.7:

☊$_0$	113°·318	i_0	2°·487	ω_0	107°·829
☊$_1$	229·292	i_1	−4·608	Π_1	274·991

From Ross's tables (indicated by *) and $\tan B_0 = \tan i_1 \sin (\Lambda -$ ☊$_1)$:

a/r* 0·96622 δB* −0°·042 $(v - M)$* + $\delta\lambda$* 17°·178

r 0·08962 0 B_0 −1°·851 $\Lambda -$ ☊$_1$ 156°·364 Λ 25°·656

J_0 22°·569 B −1°·893 $\Lambda -$ ☊$_0 + \omega_0 = \mu_S$ 20°·167

$\sin J_0$ + 0·38380 $\sin B$ −0·03303 $\sin \mu_S$ + 0·34476
$\cos J_0$ + 0·92342 $\cos B$ +0·99946 $\cos B \sin \mu_S$ +0·34457 $\cos \mu_S$ + 0·93869

α 287°·659 $\sin (\alpha - N_0)$ −0·97923 $l = \cos D \cos (\Lambda - N_0)$ + 0·93818
N_0 5·961 $\cos (\alpha - N_0)$ +0·20275 $m = \cos D \sin (\Lambda - N_0)$ + 0·33086
$\alpha - N_0$ 281·698 $n = \sin D = \nu$ + 0·10175

δ −21°·9558 $\sin \delta$ −0·37389 13 $l \cos (\alpha - N_0) + m \sin (\alpha - N_0) = \lambda$ − 0·13377
Δ 10·51784 $\cos \delta$ +0·92747 26 $- l \sin (\alpha - N_0) + m \cos (\alpha - N_0) = \mu$ + 0·98578

$r\,\mu$ +0·08834 6 $\Delta \sin \delta + r\,\nu$ − 3·92341 0
$\Delta \cos \delta + r\,\lambda$ +9·74302 0 $\tan \delta_S$ − 0·40267 28

$r\,\mu/(\Delta \cos \delta + r\,\lambda) = \tan (\alpha_S - \alpha)$ +0·00906 76 δ_S −21°·9333
$\cos (\alpha_S - \alpha)$ +0·99995 89 δ −21·9558

$\alpha_S - \alpha$ $+2^m\ 04^s{\cdot}68$ $\delta_S - \delta$ +1′ 21″·0

where $\tan \delta_S = (\Delta \sin \delta + r\,\nu) \cos (\alpha_S - \alpha) \,/\, (\Delta \cos \delta + r\,\lambda)$

The same value of $\delta_S - \delta$ would have been obtained if δ had been rounded to two decimals of a degree; but $\sin \delta$, $\cos \delta$ must be taken to six or seven decimals. No correction is necessary to reduce from 0^h E.T. of the calculation to 0^h U.T. of the tabulation.

E. THE SATELLITES OF URANUS

Ephemerides are given for the elongations, and for the apparent distance and position angle, of Satellites I-IV, Ariel, Umbriel, Titania, and Oberon. The orbits of these four satellites, as far as is yet shown by observations, are circular and lie in the same plane; the common orbital plane is presumably the plane of the equator of Uranus, since it does not have the secular motion that otherwise would be caused by the oblateness of Uranus. The ephemerides of Titania and Oberon are computed from the orbital elements determined by H. Struve (*Abhandlungen der Königlich Preussischen Akademie der Wissenschaften*, 1913). Struve's elements of the orbital plane are adopted for all four satellites, but for the other elements of Ariel and Umbriel the values determined by Newcomb (*Washington Observations* for 1873, appendix 1) are used.

No ephemeris for Miranda is given. The orbit of this satellite is approximately circular and in the same plane as the others. The sidereal period is about 1·4 days, and the mean distance about 80000 miles, or 0·0008 a.u. (Kuiper, G. P., *P.A.S.P.*, **61**, 129, 1949); the maximum apparent distance from the centre of Uranus is less than 10″.

The adopted inclination (J) of the orbital plane of Satellites I–IV to the equator of the Earth and the right ascension (N) of the ascending node are:

$$N = 166^\circ\cdot051 + 0^\circ\cdot0142\,(t - 1900\cdot0)$$
$$J = 75^\circ\cdot145 - 0^\circ\cdot0013\,(t - 1900\cdot0)$$

in which the unit of time is the Julian year. The other orbital elements are:

	Epoch 1910 January 0·0 G.M.A.T. = J.D. 241 8672·0		Epoch 1900 January −1·0 G.M.A.T. = J.D. 241 5019·0	
	Ariel	Umbriel	Titania	Oberon
a	264″·43	368″·38	604″·42	808″·29
u_0	244°·36	129°·90	80°·422	346°·867
n	142°·83544	86°·868793	41°·3514179	26°·7394710
Period	$2^d\cdot520383$	$4^d\cdot144181$	$8^d\cdot7058683$	$13^d\cdot4632432$
	$2^d\ 12^h\cdot489$	$4^d\ 03^h\cdot460$	$8^d\ 16^h\cdot941$	$13^d\ 11^h\cdot118$

Here u_0 is the value at the epoch of the orbital longitude (u) measured from the ascending node of the orbit on the equator of the Earth. Referred to the equator of the Earth the motions are direct; but referred to the ecliptic as the fundamental reference plane, they are retrograde, the inclination being about 98°.

The orbital plane is inclined at such a large angle to the equator that the semi-major axis of the apparent orbit usually lies nearly north and south, and consequently the greatest elongations are designated as northern and southern elongations instead of eastern and western. Only when the Earth is near the plane through the celestial pole perpendicular to the orbital plane, as during 1945-1946, is the minor axis of the apparent orbit directed approximately north and south; even then, the north pole of the orbital plane may lie to the south of the geocentric position of Uranus.

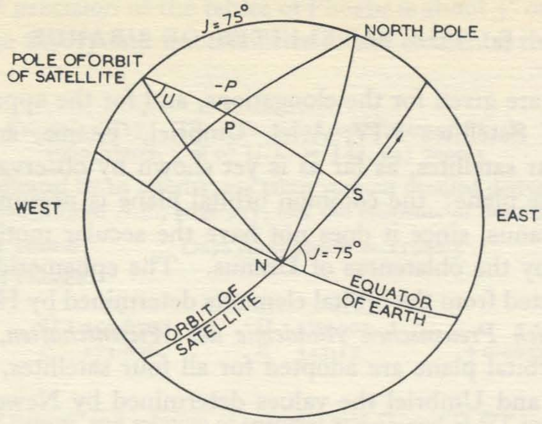

Figure 12.8. Orbit of a satellite of Uranus in relation to the equator of the Earth

The positions shown are those appertaining to opposition in 1949.
P Geocentric position of Uranus (α, δ)
S Satellite at superior conjunction

During the course of one revolution of Uranus, the Earth passes twice through the plane perpendicular to the orbital plane, when the apparent orbits of the satellites are almost circular; and likewise, it passes twice through the orbital plane, when the apparent orbits become straight lines. The Earth passed through the orbital plane from south to north in 1882, and the sequence of geometric relations during the interval from then until 1966 exemplifies the cycle during the 84-year period of Uranus.

1882–1924: Earth north of orbital plane; B positive
Superior conjunction at position angle P
1882–1902: Northern elongation at $P + 90°$, when $u - U = 90°$
1902: Earth passed through plane perpendicular to orbital plane, at $U = 270°$
1902–1924: Northern elongation at $P - 90°$, when $u - U = 270°$
1924: Earth passed through orbital plane, north to south; $B = 0°$
1924–1966: Earth south of orbital plane; B negative
Superior conjunction at position angle $P + 180°$
1924–1946: Northern elongation at $P - 90°$, when $u - U = 270°$
1946: Earth passed through plane perpendicular to orbital plane, at $U = 90°$
1946–1966: Northern elongation at $P + 90°$, when $u - U = 90°$

At the passage of the Earth through the plane perpendicular to the orbital plane, the position angle of the northern elongation changes by 180°, since the more northerly and more southerly extremities of the major axis are interchanged. The angle P is the position angle of the pole of the orbital plane that lies north of the celestial equator; but either the eastern or western extremity of the minor axis may be directed toward this pole, according to circumstances, and northern elongation may be to either the east or west of north. The elongations cannot be unambiguously designated as eastern and western.

The ephemerides are calculated with the formulae of sub-section A. The apparent right ascension and declination of Uranus are used in the calculations; the error from the inconsistency with the values of N and J referred to the mean equinox and equator is negligible for the purpose of the ephemerides.

The diagram of the apparent orbits is constructed from the values of U, B, P for the date of opposition of Uranus. These data, and the tables for obtaining apparent distance and position angle, are calculated by the methods used for the satellites of Mars.

Example 12.18. *Times of elongation, and table entries for the apparent distances and position angles of the satellites of Uranus*

1960 March 7 at 0^{h} E.T. = J.D. 243 7000·5

Uranus apparent α	140°·575	N 166°·906	U 10°·488
(*A.E.*, p. 219) apparent δ	+16·173	J 75·067	P −81·585
geocentric distance Δ	17·5206		Adopted P_0 −81

	Ariel	Umbriel	Titania	Oberon
Epoch t_0	J.D. 241 8672·0		J.D. 241 5019·0	
$(t - t_0)$		18328^{d}·5		21981^{d}·5
$P - P_0 = p_2$	−0°·58	−0°·58	−0°·58	−0°·58
a/Δ	15″·09	21″·03	34″·50	46″·13
$u_0 + n(t - t_0) = u$	283°·72	24°·63	46°·61	240°·55
$u - U$	273·23	14·14	36·12	230·06
Time for motion of $u - U$ through 1°	0^{h}·1680	0^{h}·2763	0^{h}·5804	0^{h}·8975
Angle from northern elong. = $(u - U) - 90°$	183·23	284°·14	306°·12	140°·06
Time from northern elongation	30·783	78^{h}·508	177^{h}·672	125·704
0^{h}·13849 Δ = light-time	2·426			
Time of northern elongation Mar. or Feb.	$5^{\mathrm{d}}\ 19^{\mathrm{h}}$·64	$3^{\mathrm{d}}\ 19^{\mathrm{h}}$·92	$28^{\mathrm{d}}\ 16^{\mathrm{h}}$·75	$1^{\mathrm{d}}\ 20^{\mathrm{h}}$·72

For 0^{h} E.T. on the date of opposition, 1960 February 8

B_0 −27°·757 sin B_0 −0·46572 cos B_0 +0·88493

	Ariel	Umbriel	Titania	Oberon
Time from northern elong. = t	$0^{\mathrm{d}}\ 12^{\mathrm{h}}$	$2^{\mathrm{d}}\ 00^{\mathrm{h}}$	$5^{\mathrm{d}}\ 10^{\mathrm{h}}$	$11^{\mathrm{d}}\ 00^{\mathrm{h}}$
$90° + nt = u - U$	161°·418	263°·738	313°·987	24°·134
$\sin\sigma \sin(p - P) = \sin(u - U)$	+ 0·31866	−0·99404	−0·71950	+0·40887
$\cos(u - U)$	− 0·94787	−0·10907	+0·69449	+0·91259
$\sin\sigma \cos(p - P) = \cos(u - U)\sin B_0$	+ 0·44144	+0·05080	−0·32344	−0·42501
$\cos\sigma = \cos(u - U)\cos B_0$	− 0·83880	−0·09652	+0·61458	+0·80758
$\cot(p - P)$	+ 1·3853	−0·05110	+0·44953	−1·0395
$p - P$	35°·82	272°·93	245°·79	136°·11
P_0	−81			
sum = p_1	314·82	191·93	164·79	55·11
$\sin\sigma = F$	0·5444	0·9953	0·7889	0·5898

At the opposition of Uranus, 1960 February 8, the latitude (B) of the Earth referred to the common orbital plane of the satellites is $B_0 = -27^\circ{\cdot}757$. The Earth is therefore on the opposite side of the orbital plane from the pole of the orbit from which the motion appears counter-clockwise, and hence the apparent motion is clockwise. The position angle $P = 278^\circ{\cdot}4$ of the extremity of the minor axis of the apparent orbit that is directed toward this pole is the position angle of the western extremity, a little to the north of west. Northern elongation is at position angle $P + 90^\circ$, to the east of north, when $u - U = 90^\circ$. Inferior conjunction is at position angle P, at the western extremity of the minor axis.

Example 12.18 has been chosen to illustrate the formation of position angles in different quadrants when, as in this case, B_0 is negative.

F. THE SATELLITES OF NEPTUNE

Ephemerides are given for the elongations and for the apparent distance and position angle of Triton, calculated from the orbital elements determined by W. S. Eichelberger and Arthur Newton (*A.P.A.E.*, **9**, part III, 1926).

No ephemeris for Nereid is included. The orbital equatorial elements of this satellite determined by van Biesbroeck (*A.J.*, **62**, 272-274, 1957) are:

a	0·03717 97	
e	0·74934	
i	$27^\circ{\cdot}807$	referred to
☊	$354^\circ{\cdot}563$	equator of
ω	$259^\circ{\cdot}119$	1950·0
Time of pericentric passage (T)		1951 February 2·468
Sidereal period (P)		359·881 days
Inclination to the ecliptic		$4^\circ{\cdot}971$

The orbit is unique among the satellites in the solar system, in respect of its extreme eccentricity and the great difference of its inclination from that of Triton.

The motion of Triton is retrograde. The orbit, as far as it has been determined from observations, is circular. The pole of the orbital plane is describing a small circle on the celestial sphere, with a radius of 20° and a period of about 585 years, showing that the orbital plane is inclined to the equatorial plane of Neptune at a constant angle of 20° while the nodes revolve on the equator of Neptune, due to the disturbing action of the oblateness of the planet.

The orbital elements of Triton referred to the plane of the equator of Neptune are:

Inclination $\gamma = 159^\circ{\cdot}945$

Longitude of the ascending node of the orbit, measured from the node of the equator of Neptune on the celestial equator of 1900·0

$$\theta = 127^\circ{\cdot}015 + 61^\circ{\cdot}494\, T$$

where T is reckoned in Julian centuries from 1900

Orbital longitude of Triton, measured from the ascending node of the orbit on the equator of Neptune, but in the retrograde direction

$$u_N = 186^\circ{\cdot}92 + 61^\circ{\cdot}25894\,94\,(\text{J.D.} - 241\,1368{\cdot}0)$$

in which the time is reckoned in days from 1890 January 0·0 G.M.A.T.

Greatest elongation at unit distance $489''.82$

Sidereal mean daily motion, obtained by subtracting from the motion of u_N the amount due to the variation of θ

$$n = 61°.25894\,94 - (61°.494/36525) \cos (180° - \gamma)$$
$$= 61°.25736\,79$$

Sidereal period $5^d\ 21^h\ 02^m\ 39^s$

The right ascension and declination of the north pole of rotation of Neptune, referred to the mean equinox and equator of date

$$\alpha_0 = 295°.153 + (46''.08 + 20''.05 \sin \alpha_0 \tan \delta_0)\, t$$
$$\delta_0 = +41°.348 + (20''.05 \cos \alpha_0)\, t$$

where t is reckoned in years from 1900·0.

The inclination (J) and the longitude (N) of the ascending node of the orbit of Triton on the equator of the Earth, and the arc ψ of the orbit *from* the node on the equator of the Earth *to* the node on the equator of Neptune are calculated by the methods used for the inner satellites of Saturn, using a value of θ corrected for precession:

$$\theta = 127°.015 + (0°.61494 - 0°.00557 \sin \alpha_0 \sec \delta_0)\, t$$

The orbital longitude (u), measured from the ascending node of the orbit on the celestial equator but in the retrograde direction, is given by:

$$u = u_N + \psi$$

Values of these quantities are given in table 12.2 for the years 1960 to 1980.

Table 12.2. Elements of Triton

	α_0	δ_0	N	J	θ	ψ	u_N
	°	°	°	°	°	°	°
1960*	295·655	41·492	199·915	112·027	164·314	12·618	3·850
1961	·663	·494	200·148	111·983	164·936	12·118	43·366
1962	·671	·497	200·380	·942	165·557	11·619	82·883
1963	·680	·499	200·614	·902	166·179	11·118	122·399
1964*	295·688	41·502	200·848	111·865	166·801	10·618	223·175
1965	·696	·504	201·082	·829	167·422	10·119	262·691
1966	·705	·506	201·317	·794	168·044	9·618	302·208
1967	·713	·509	201·552	·763	168·666	9·118	341·725
1968*	295·721	41·511	201·786	111·732	169·287	8·619	82·500
1969	·730	·514	202·022	·704	169·909	8·118	122·017
1970	·738	·516	202·257	·677	170·531	7·618	161·533
1971	·747	·518	202·493	·651	171·152	7·118	201·050
1972*	295·755	41·521	202·730	111·628	171·774	6·618	301·825
1973	·763	·523	202·965	·607	172·395	6·118	341·342
1974	·772	·526	203·203	·589	173·017	5·617	20·858
1975	·780	·528	203·438	·570	173·639	5·117	60·375
1976*	295·788	41·532	203·675	111·556	174·260	4·617	161·150
1977	·797	·535	203·912	·543	174·881	4·117	200·667
1978	·805	·537	204·149	·531	175·503	3·617	240·183
1979	·813	·540	204·386	·521	176·124	3·117	279·700
1980*	295·822	41·542	204·623	111·512	176·746	2·617	20·475

The values are for January 0·0 in common years and for January 1·0 in leap years (indicated by *).

Since the retrograde direction of the motion is represented by an orbital inclination greater than 90°, the ascending node of the orbit is the point at which the satellite crosses the equator from south to north. The pole of the orbit from which the motion appears counter-clockwise is the south pole, at position angle P; when the Earth is south of the orbital plane of Triton, B is positive. Eastern elongation is at position angle $P - 90°$, when $u - U = 270°$.

Twice during the course of one revolution of Neptune, about 165 years, the Earth passes through the orbital plane of Triton, when the apparent orbit becomes a straight line. The Earth crossed the orbital plane from north to south near the end of 1952; for an interval during 1950-1954, Triton transited the disk of Neptune and was occulted by the disk during each revolution. Before 1953, B was negative, and inferior conjunction was on the southern arc of the apparent orbit at position angle P; since the passage through the orbital plane, B has been positive, and inferior conjunction on the northern arc at $P + 180°$.

When the numerical value of B reaches a maximum as the Earth passes through the plane perpendicular to the orbital plane, as in 1905, the minor axis of the apparent orbit lies exactly north and south, the major axis lies east and west, and the position angle of the more northerly elongation changes by 180°. Before 1905, the eastern elongation was the more northerly; since then, the western elongation has been the more northerly.

The ephemeris, including the diagram of the apparent orbit, is calculated by the formulae in sub-section A, in the same way as described for the satellites of Uranus.

Example 12.19. *The time of elongation and the apparent distance and position angle of Triton*

1960 March 7 at 0^h E.T. = J.D. 243 7000·5

Neptune (*A.E.*, p. 227)	apparent α 217°·157	N 199°·957	U 341°·426
	apparent δ −12·772	J 112·019	P 244·299
	geocentric distance Δ 29·717		Adopted P_0 245
a 489″·82	a/Δ 16″·48		$p_2 = P - P_0$ −0°·70
u_N 86°·940	ψ 12°·528		u 99°·468
			$u - U$ 118·042
Time for motion of $u - U$ through 1°	0^h·39179		$(u - U) - 270°$ 208·042
Time from eastern elongation	81·509		
Light-time = 0^h·13849 Δ	4·116		

Time of eastern elongation = March 3^d 18^h·61

For 0^h E.T. on the date of opposition, April 28, $B_0 = +9°·776$

At $t = 4^d\ 00^h$ from eastern elongation, $u - U = 270° + nt = 155·029$

Whence $p - P = 110°·03$, $p_1 = 355°·03$, and $F = 0·4493$

G. HISTORICAL LIST OF AUTHORITIES

1. The Nautical Almanac, 1767–1900

No predictions for satellites, except for the Galilean satellites of Jupiter, were published until 1899, when diagrams of the apparent orbits at the time of opposition, and elongations (with a precision of $0^{h}\cdot 1$) for a limited period around opposition, were given for the satellites of Mars, Saturn, Uranus, and Neptune. The authorities for the latter are as for 1901, and are listed in sub-section G.3.

For Jupiter's satellites, diagrams of the configurations and predictions of eclipses have been published in every issue of *The Nautical Almanac*.

1767–1804: Eclipses to 1^{s}; based on Wargentin (1746).
1805–1823: Based on Lalande (1792) (quoting Delambre).
1824–1833: Based on Delambre (1817).
1834–1839: Eclipses to $0^{s}\cdot 1$, other phenomena to 1^{m}; based on Delambre (1817).
1840–1900: As 1834–1839; from 1877, eclipses to 1^{s}; from 1896, times of conjunction to $0^{m}\cdot 1$; based on Damoiseau (1836), and extensions by Adams and others.

2. The American Ephemeris, 1855–1900

In the volumes for 1855–1881, ephemerides were given only for the four great satellites of Jupiter, and the apparent elements of the rings of Saturn. The ephemerides of the satellites of Jupiter gave the superior geocentric conjunctions, the phenomena, the coordinates in the mean apparent ellipses, and diagrams of the phases of the eclipses.

In 1882, diagrams of the configurations of the four satellites of Jupiter were added, and the former ephemerides of the coordinates in the apparent orbits were omitted; a diagram of the apparent orbits was also added. Ephemerides of the elongations of the satellites of Mars, Saturn, Uranus, and Neptune, and diagrams of the apparent orbits, were introduced.

The elongations of the fifth satellite of Jupiter were added in 1898, but no statement of authority was given during 1898–1900. No authority was given for the satellites of Mars during 1882–1900. The authorities for the other satellites were:

Jupiter

1855–1881: Damoiseau (1836); extended to 1880 by Kendall (1877).
1882–1900: for elongations and eclipses, Todd's continuation (1876) of Damoiseau; for occultations, transits, etc., Woolhouse (1833), with Table II for each satellite adapted to Damoiseau.

Saturn

1855–1900: for rings, except the dusky ring, Bessel (1875a and 1875b).
1882–1900: for satellites, manuscript tables prepared by Newcomb.

Uranus and Neptune

1882–1900: Newcomb (1875).

3. The Nautical Almanac, and The American Ephemeris, 1901–1959

Mars

Diagrams of the apparent orbits, and times of elongations to a precision of $0^h.1$, have been given for a period of about a month on each side of opposition. From 1931 [in *A.E.*, from 1920], tables have been added for calculating the position angle and apparent distance of each satellite from the planet at any time during the same period.

1901–1902: Elements by Hall (1878). [In *A.E.*, those of Harshman.]
1903–1915: Elements by Harshman (1894).
1916–1959: Elements by Struve (1911, p. 1073).

Jupiter

Phenomena and configurations of Satellites I–IV are given throughout, with considerable variation from time to time in the precision, which is usually higher for eclipses than for other phenomena. Elongations (every twentieth on each side) of Satellite V have been given since 1906 [in *A.E.*, since 1901]; differential coordinates (satellite *minus* planet) of Satellites VI and VII have been given since 1931 [in *A.E.*, since 1912]. Sidereal periods of Satellites VIII to XI, and XII have been included since 1953 and 1957.

Satellites I–IV.
1901–1913: Damoiseau (1836) and later extensions. [In *A.E.*, also Woolhouse (1833).]
1914–1915: Sampson (1910). [In *A.E.*, Damoiseau and extensions and Woolhouse.]
1916–1959: Sampson, with Andoyer's (1915) modifications. [In *A.E.* until 1930, the configurations are attributed to Pottier's continuation (1896) of Damoiseau.]

Satellite V.
1901–1905: [In *A.E.* only, Robertson's elements (unpublished).]
1906–1915: Cohn (1897). [In *A.E.*, Robertson.]
1916–1959: Robertson (1924) [these are quoted in *A.E.* as—in 1930–1933 " from Connaissance des Temps for 1915 "; in 1934–1959 " from Connaissance des Temps every year beginning with 1919 "].

Satellites VI and VII.
1912–1930: [In *A.E.* only, Ross (1907a and 1907b).]
1931–1947: Ross.
1948–1959: Bobone (1937a and 1937b).

Saturn

Diagrams of the orbits of Satellites I–VII, and times of elongations and conjunctions of Satellites I–VIII to a precision of $0^h.1$, have been given for most of the year throughout the period. Differential coordinates of Satellite IX (Phoebe) were introduced in 1931 [in *A.E.*, in 1909] and those of Satellite VII (Hyperion) and Satellite VIII (Iapetus), [not in *A.E.*], also in 1931. Elements for determining the distance and position angle were added in 1931 [in *A.E.*, in 1912]. Since 1935 the Almanac [not *A.E.*] has contained, for about nine months in each opposition, quantities to assist in the calculation of the phenomena (eclipses, occultations, transits, and shadow-transits) of Satellites I–VI.

Satellites I–V—Mimas, Enceladus, Tethys, Dione, and Rhea.

1901–1915: Tables in manuscript "prepared by" Newcomb, except the elongations of Satellites I (Mimas) and III (Tethys), which are from H. Struve (1898). [In *A.E.*—1901–1903: Hall (1886); 1904: Hall, except the elongations of Satellites I and III (from Struve, 1898); 1905–1913: Struve (1898); 1914–1915: Struve (1888, 1898, and 1903).]

1916–1930: Struve (1888, 1898, and unpublished corrections).

1931–1935: H. Struve (1888, 1898, and 1903) and G. Struve (1924).

1936–1959: G. Struve (1924 and 1930).

Satellite VI—Titan.

1901–1915: Newcomb's "manuscript Tables" (see above). [In *A.E.*—1901–1904: Hall (1886); 1905–1913: Struve (1898); 1914–1915: Struve (1888, 1898 and 1903).]

1916–1930: Struve (as for Satellites I–V).

1931–1935: H. Struve (as for Satellites I–V).

1936–1937: G. Struve (1933). [In *A.E.*, H. Struve, as in 1931.]

1938–1959: G. Struve (1933).

Satellite VII—Hyperion.

1901–1915: Newcomb's "manuscript Tables". [In *A.E.*—1901–1902: Eichelberger (1892); 1903–1913: Struve (1898); 1914–1915: Struve (as for VI).]

1916–1935: Struve (as for Satellite VI).

1936–1959: Woltjer (1928).

Satellite VIII—Iapetus.

1901–1902: As for Satellite VII. [In *A.E.*, Hall (1885).]

1903–1935: As for Satellite VII.

1936–1937: H. Struve (1888, 1898, and 1903).

1938–1959: G. Struve (1933).

Satellite IX—Phoebe.

1909–1930: [In *A.E.* only, Ross (1905).]

1931–1959: Ross (1905).

Uranus

Diagrams of the orbits of Satellites I–IV, and times of elongations to a precision of $0^{h}.1$, have been given for about nine months in each opposition throughout this period. Elements for the calculation of distance and position angle were included in 1927 [in *A.E.*, in 1912]. The sidereal period of Satellite V (Miranda) was added in 1953.

1901–1915: Newcomb (1875).

1916–1959: Satellites I (Ariel) and II (Umbriel)—Newcomb; Satellites III (Titania) and IV (Oberon)—Struve (1913).

Neptune

Diagrams of the orbit of Triton, and times of elongations to a precision of $0^{h}.1$, have been given for about ten months in each opposition throughout this period. Elements for the calculation of distance and position angle were included in 1927 [in A.E., in 1912]. The sidereal period of Nereid was added in 1953.

1901: Newcomb (1875). [In *A.E.*, Hall (1898).]

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13. RISINGS, SETTINGS, AND TWILIGHT

A. INTRODUCTION

The astronomer is concerned with the phenomena of rising, setting, and twilight primarily in regard to the planning of observations. No great precision is required for this purpose, and the tables of these phenomena in the Ephemeris are accordingly restricted both in content and range of latitude. More extensive tables are available in *The Nautical Almanac* (*N.A.*) and *The Air Almanac* (*A.A.*), while the *Tables of Sunrise, Sunset, and Twilight* (*S.S.T.*) give permanent data from which all such phenomena may be calculated simply for all latitudes and all years.

The tabulated times of the phenomena refer to sea level with a clear horizon and normal meteorological conditions. The actual times of rising and setting may differ considerably, especially near extreme conditions when the altitude is changing slowly; the illumination at beginning or end of twilight varies greatly with meteorological conditions. Precise times have little real significance, except in special circumstances such as navigation at sea.

No data are given for the times of rising and setting of planets; they may be obtained fairly simply from navigation tables such as the *Sight Reduction Tables for Air Navigation* (H.O. 249 or A.P. 3270). Within their range of declination ($0° - 29°$) these tables also provide for stars. For fixed altitudes and depressions, more than $5°$ from the horizon, the more elaborate *Tables of Computed Altitude and Azimuth* (H.O. 214 or H.D. 486) may be used; these tables do not give altitudes within $5°$ of the horizon.

For most astronomical purposes by far the most suitable method of obtaining data of this nature is by the use of a star globe fitted within a simple framework of horizon and altitude circles in such a way that it can be set to any desired latitude and then rotated freely about the polar axis. Within its limitations of precision, depending on its scale and details of construction, it is possible for any selected latitude and hour angle to read off altitude (or depression) and azimuth of all objects marked on the globe. For illumination from the Sun the hour angles, and hence the local mean times, can be quickly read off for any date (corresponding to a marked position of the Sun on the globe) and for a series of depressions below the horizon. More subtle questions, such as "what is the earliest date in the year when a certain object will be at an altitude of at least $20°$

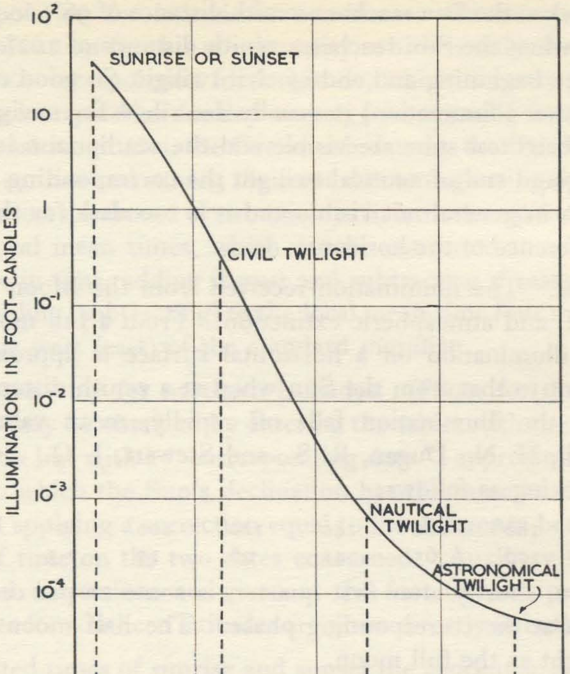

Figure 13.1. *Twilight illumination on a horizontal plane*

at the beginning of morning twilight?" can be answered immediately, and for a variety of twilights (depressions) corresponding to different observational requirements. Tables of the more important phenomena and dates may thus be prepared for each observatory.

When the Sun is below the horizon the sky is illuminated from the following sources.

(a) *Twilight.* This is caused by the scattering of sunlight from the upper layers of the Earth's atmosphere. It begins at sunset (ends at sunrise) and is conventionally taken to end (or begin) when the Sun reaches a zenith distance of 108°. The variation of the illumination on a horizontal surface, in clear conditions, is shown diagrammatically in figure 13.1 as a function of the zenith distance of the Sun; the rapid flattening of the curve from 105° onwards, after an almost linear relationship from 90° to 105°, is noteworthy. At a zenith distance of 108°, astronomical twilight, the indirect illumination from the Sun on a horizontal surface, is about 6×10^{-5} ft. candles, rather less than the contribution from star light and of the same order as that from the aurora, air glow, zodiacal light, and the gegenschein. The actual brightness of the sky depends on direction, as well as on meteorological conditions.

In navigational practice, and for some astronomical applications, two intermediate steps in the twilight period are recognised and tabulated: *civil twilight* ends (or begins) when the Sun reaches a zenith distance of 96°; and *nautical twilight* ends (or begins) when the Sun reaches a zenith distance of 102°. The degree of illumination at the beginning and end of civil twilight (in good conditions and in the absence of other illumination) is usually described for navigational purposes as such that the brightest stars are visible and the sea horizon is clearly defined; for the beginning and end of nautical twilight the corresponding statement is that the sea horizon is in general not visible and it is too dark for the observation of altitudes with reference to the horizon.

(b) *Moonlight.* The illumination received from the Moon varies according to phase, altitude, and atmospheric extinction. From a full moon in the zenith the intensity of illumination on a horizontal surface is approximately 0·02 ft. candles, equivalent to that from the Sun when at a zenith distance of about 98°. After full moon the illumination falls off rapidly, mean values from several observers (Russell, H. N., Dugan, R. S., and Stewart, J. Q., *Astronomy*, Vol. I, page 173, 1926) being as follows:

Elongation	180°	160°	140°	120°	100°	80°	60°	40°	20°
Illumination %	100	65	41	26	15	7·5	3·2	1·0	0·1

The waxing moon, shortly after first quarter, is some 20 per cent brighter than the waning moon at the corresponding phase. The half moon gives only one-ninth as much light as the full moon.

For astronomical observations the position of the Moon in the sky, rather than the general illumination, is the most important factor.

(c) *Star light.* The total illumination from the stars contributes about 2×10^{-4} ft. candles, rather more than the Sun at the beginning and end of astronomical twilight.

(d) *Aurora, air glow, zodiacal light, and gegenschein.* The illumination from these sources is very variable. That from the aurora may in rare cases be comparable with moonlight. The other sources are very faint and never give an illumination greatly exceeding that from star light. As with moonlight, the position of the source is the most important factor for astronomical observations.

B. SUNRISE, SUNSET, AND TWILIGHT

The data given in the Ephemeris enable the times of sunrise, sunset, and the beginning and end of astronomical twilight to be found for any position between latitudes 60° north and south. The times, tabulated for every fifth day, are the local mean times of the phenomena on the meridian of Greenwich and in the specified northern latitude; interpolation is necessary to obtain the local mean times for intermediate latitudes, for intermediate days, and for longitude. To an accuracy of about five minutes this interpolation can generally be done at sight; near limiting conditions, when interpolation becomes difficult, large changes of time correspond to only small changes in depression and accurate times have little real meaning.

Strictly, interpolation for latitude is non-linear; the interpolation table in *N.A.* includes a correction for this non-linearity, which is only of significance near extreme conditions. Interpolation for longitude, which is rarely justified, can be combined with the interpolation for date merely by increasing for west longitudes, or decreasing for east longitudes, the Greenwich date by the fraction (longitude in degrees/360); for sunrise and sunset, the error due to neglecting the variation with longitude amounts to a maximum of two minutes in latitudes up to 60°. The times so obtained are local mean times, which can be converted to universal time by applying longitude in time, adding if west and subtracting if east; standard times are obtained by adding (subtracting) to the local mean time four minutes for every degree of longitude west (east) of the standard meridian.

Times are given only for northern latitudes. Those for the corresponding southern latitudes may be obtained by entering the tables with the same numerical value of the latitude but with a " corresponding date ", approximately six months earlier or later, on which the Sun's declination has the same numerical value but opposite sign, and applying a correction equal to the difference between the values of the equation of time on the two dates concerned. Auxiliary tables of corresponding dates and corrections are given at the foot of each page of the main tabulation. Times for southern latitudes are given directly in both *N.A.* and *A.A.*

At the tabulated times of sunrise and sunset the geocentric zenith distance of the Sun is 90° 50′, 34′ being allowed for horizontal refraction and 16′ for semi-diameter; the Sun's upper limb is thus on the horizon. Corrections are necessary if some other value of the zenith distance is required, such as for the conventional meteorological value of 90° 34′ or to allow for the height of the observer and the elevation of the actual horizon. These may be obtained from *Tables of Sunrise, Sunset, and Twilight*, or from the data in *N.A.* and *A.A.* For small changes the formula and table given for moonrise and moonset in sub-section C may be used.

At the times given for astronomical twilight the geocentric zenith distance of the Sun is 108°, and the indirect illumination from the Sun is approximately equal to that of the night sky. Other twilights are in use for navigational purposes; times of civil (zenith distance 96°) and nautical twilight (zenith distance 102°) are given in both *N.A.* and *S.S.T.*, and duration of civil twilight (interval before sunrise or after sunset) is given in *A.A.* These tabulations enable times to be obtained corresponding to any desired zenith distance between 90° 50′ and 108°. In *A.A.* a table is given of the corrections to be applied to the time of sunrise or sunset to give the times at which the Sun has zenith distances between 90° and 102° (that is depressions down to 12°); one argument of this table is the (tabulated) duration of civil twilight. Sunrise or sunset at a height h feet above the level of the horizon occurs when the Sun's zenith distance is approximately:

$$90^\circ\ 50' + 1'{\cdot}17\, h^{\frac{1}{2}} \qquad (2'{\cdot}08, \text{ for } h \text{ in metres})$$

so that the same table is used to give corrections for height to the times of sunrise and sunset. In *A.A.* 1962 onwards, the corrections can be obtained from graphs.

Times of rising and setting and associated phenomena change rapidly from day to day in polar regions, or may not occur for long periods, the Sun being continuously above or below the horizon; accurate times are therefore difficult to tabulate. Diagrams are given in *A.A.* and *S.S.T.* that enable approximate times, sufficiently accurate for all practical purposes, to be obtained.

C. MOONRISE AND MOONSET

The tables in the Ephemeris give, for every day and for a range of latitudes from 60° north to 60° south, the local mean times of moonrise and moonset for the meridian of Greenwich. Interpolation for both latitude and longitude is necessary to obtain local mean times for other places. In practice times are rarely required more accurately than to within about five minutes and interpolation can be done mentally. Formal interpolation, using tables such as are given in *N.A.* and *A.A.*, yields times accurate to about two minutes. The times so obtained may be converted to universal time or standard time by applying longitude in time, adding if west and subtracting if east. Times for latitudes north of 60° are given in both *N.A.* and *A.A.* to 72° and in *A.A.*, in graphical form above 72°, right up to the north pole.

In calculating the times of moonrise and moonset, the zenith distance of the Moon is taken as:

$$90° \, 34' + \text{semi-diameter} - \text{horizontal parallax}$$

where 34′ is allowed for horizontal refraction; this zenith distance varies from 89° 49′ to 89° 55′. At these times the upper limb of the Moon is on the horizon; no allowance is made for phase.

The rate of change of hour angle with zenith distance, at a zenith distance of 90°, is:

$$(\cos^2 \phi - \sin^2 \delta)^{-\frac{1}{2}}$$

where ϕ is latitude and δ is declination; approximately, a change in zenith distance of Δz degrees causes a difference in the time of rising and setting of $A\Delta z$ minutes, where:

$$A = 4{\cdot}14 \, (\cos^2 \phi - \sin^2 \delta)^{-\frac{1}{2}}$$

The factor A is tabulated in the following table.

	Latitude, ϕ								
δ	0°	10°	20°	30°	40°	45°	50°	55°	60°
0°	4·1	4·2	4·4	4·8	5·4	5·9	6·4	7·2	8·3
10	4·2	4·3	4·5	4·9	5·5	6·0	6·7	7·6	8·8
20	4·4	4·5	4·7	5·2	6·0	6·7	7·6	9·0	11·3
23	4·5	4·6	4·8	5·4	6·3	7·0	8·1	10	13
26	4·6	4·7	5·0	5·5	6·6	7·5	8·8	11	17
29	4·7	4·8	5·1	5·8	7·0	8·0	9·8	14	34

The corrections are accurate to within a few minutes provided A does not exceed 20 and Δz a few degrees. At a height h feet above the horizon the zenith distance of the Moon when its upper limb is on the horizon is increased by $1' \cdot 17 h^{\frac{1}{2}}$ and the value of Δz to be used is:

$$\Delta z = 0^\circ{\cdot}019 \, h^{\frac{1}{2}} \qquad (2'{\cdot}08,\ 0^\circ{\cdot}035, \text{ for } h \text{ in metres})$$

The Moon revolves round the Earth and makes one complete revolution relative to the Sun, in a synodic month of mean length 29·53 days; in that time it therefore appears to lose one transit across any meridian and, in general, one rising and one setting. During each month there is therefore no moonrise on one local day (near last quarter) and no moonset on one day (near first quarter). In high latitudes the times of the phenomena change rapidly from day to day and may not occur for long periods, the Moon being continuously above or below the horizon; in these extreme conditions the times of moonrise and moonset sometimes decrease from day to day, instead of the usual increase in lower latitudes, and it is possible to have two moonrises or two moonsets during the same local day.

D. DERIVATION

The times of rising and setting, and of the beginning and end of twilight, may be calculated directly from the fundamental relation:

$$\cos h = -\tan \phi \tan \delta + \sec \phi \sec \delta \cos z$$

in which h and δ are the hour angle and declination at the time of the phenomenon, ϕ is the latitude, and z is the zenith distance. Negative, or eastern, hour angles correspond to the rising phenomena; positive, or western, hour angles to setting phenomena. The hour angle is:

$$h = \text{local sidereal time} - \text{right ascension}$$
$$= \text{G.H.A. Aries} - \lambda - \alpha$$

in which λ is the west longitude and α is the right ascension. The local mean time, or the universal time, of the phenomenon may be obtained (once α is known) from h by the methods described in section 3C. For a fast-moving object the declination should be interpolated to the approximate time of the phenomenon, before the hour angle is calculated.

In practice it is more convenient to use the tabulated values (for the Sun, Moon, and planets) of the time of transit instead of the right ascension; the time corresponding to any given hour angle is then obtained by interpolation of the time of transit to that hour angle. More precisely, in longitude λ (ephemeris longitude λ^*), the ephemeris time corresponding to a calculated hour angle h is the tabulated time of ephemeris transit interpolated towards the next value with an interpolating factor of $(h + \lambda^*)$ expressed as a fraction of 24^h or 360°. As shown in section 4F, the corresponding local mean time depends only slightly on ΔT; to the precision considered here, it is:

$$\text{E.T. of ephemeris transit} + h\,(1 + d) + \lambda d$$

where d is the mean value of the rate of increase of the tabulated times, expressed as a fraction of 24^h and strictly taken for the mid-point of the interval between the times of transit and the phenomenon.

Example 13.1. *Local times of sunrise, sunset, and the beginning and end of astronomical twilight*

1960 March 7, in latitude +52°

$\phi = +52°$ $\tan\phi = +1·2799$ $\sec\phi = 1·6243$

	Morning twilight	Sunrise	Sunset	Evening twilight
Approx. time	$4^h 30^m$	$6^h 30^m$	$18^h 00^m$	$19^h 30^m$
δ (*A.E.*, page 21)	−5° 17′	−5° 15′	−5° 04′	−5° 02′
$\tan\delta$	−0·0925	−0·0919	−0·0887	−0·0881
$\sec\delta$	+1·0043	+1·0042	+1·0039	+1·0039
z	108°	90° 50′	90° 50′	108°
$\cos z$	−0·3090	−0·0145	−0·0145	−0·3090
$-\tan\phi\tan\delta$	+0·1184	+0·1176	+0·1135	+0·1128
$\cos z \sec\phi \sec\delta$	−0·5041	−0·0237	−0·0237	−0·5039
$\cos h$	−0·3857	+0·0939	+0·0898	−0·3911
	h m	h m	h m	h m
h	7 30·7	5 38·5	5 39·4	7 32·1
12^h − Eqn. of Time	12 11·2	12 11·2	12 11·0	12 11·0
Time	4 40·5	6 32·7	17 50·4	19 43·1

The times given are for north latitude 52°. The Sun's declination at 0^h on March 7 is −5°·4; the " corresponding date " when the declination is of opposite sign but most nearly numerically equal is September 9. The times for south latitude 52° on March 7 are found from those for north latitude 52° for September 9 by applying a correction equal to the difference between the values of the equation of time on the two dates:

	m
Equation of time, September 9	+ 2·6
Equation of time, March 7	−11·2
	m
Difference = correction =	+13·8

This correction is tabulated in *A.E.*, pages 398 and 399.

Example 13.2. *Times of moonrise and moonset—direct method*

1960 March 7, in latitude +52°

	Moonrise	Moonset
Approx. time	$12^h 00^m$	$3^h 00^m$
δ (*A.E.*, page 84)	+18° 02′	+18° 10′
S.D. (*A.E.*, page 54)	14′·8	14′·8
H.P. (*A.E.*, page 54)	54′·4	54′·3
90° 34′ + S.D. − H.P. = z	89° 54′·4	89° 54′·5
$\tan\delta$	+0·3256	+0·3281
$\sec\delta$	+1·0517	+1·0525
$-\tan\phi\tan\delta$	−0·4167	−0·4199
$\cos z \sec\phi \sec\delta$	+0·0028	+0·0028
$\cos h$	−0·4139	−0·4171
	h m	h m
h	− 7 37·8	+ 7 38·6
Tabulated difference of transits	12·3973	12·3961
$1 + d$	1·0331	1·0330
	d h m	d h m
Time of upper transit	7 19 52·6	6 19 05·0
Moonrise, moonset	7 11 59·7	7 02 58·7

$1 + d$ is the mean value of the tabulated difference between upper and lower transits of the Moon, divided by 12^h.

On the Greenwich meridian the term λd is zero, so that the universal time is simply:

$$\text{E.T. of ephemeris transit} + h(1 + d)$$

The maximum value of h is 12^h so that, to a precision of one minute of time, d cannot be ignored except for the Sun. For the Moon $1 + d$ is about 1·03 or 1·04, and it is not quite sufficient to use the value of $1 + d$ at the time of transit for both rising and setting phenomena. In systematic calculation the approximate times of the phenomena are known beforehand to sufficient accuracy for the values of $1 + d$ and δ to be pre-calculated. For the Sun, the ephemeris transit is not tabulated in the Ephemeris until 1965; it suffices to add the hour angle to 12^h minus the tabulated equation of time, interpolated to the time of the phenomenon.

Some simplification can be achieved by using the ephemerides of Greenwich hour angle (G.H.A.) and declination (Dec.) in *N.A.* and *A.A.*; the argument of these ephemerides is universal time, which can thus be determined directly to correspond to the calculated value of h.

A more fundamental simplification can be achieved by calculating the zenith distances for a number of pre-determined times near rising and setting, and then using inverse interpolation to give the time corresponding to the zenith distance required. The fundamental relation is used in the form:

$$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos h$$

in which all quantities on the right-hand side are known for each time. When calculating times of moonrise and moonset systematically for a large number of latitudes on a high-speed electronic computing machine, this method has both practical and theoretical advantages; and it is the method actually used in practice.

The zenith distances are calculated for two or more integral hours near the

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Example 13.3. *Times of moonrise and moonset—alternative method*
1960 March 7, in latitude +52°

Moonrise				Moonset			
U.T.	Zenith distance	δ	δ^2	U.T.	Zenith distance	δ	δ^2
h	° ′	° ′	′	h	° ′	° ′	′
11	97 05·27			02	81 57·70		
		−7 13·11				+8 06·22	
12	89 52·16		−53·70	03	90 03·92		−54·49
		−8 06·81				+7 11·73	
13	81 45·35			04	97 15·65		
12^h	$\pi = 54'.392$		$z = 89° 54'.428$	03^h	$\pi = 54'.314$		$z = 89° 54'.485$

Inverse interpolation gives U.T. = $11^h\ 59^m.7$ — Inverse interpolation gives U.T. = $02^h\ 58^m.7$

The precision of 0′·01 in the calculated zenith distance is unnecessary in this example; a change of 1′ in zenith distance corresponds to a change of about $0^m.13$ ($A = 8$). But the additional precision involves no extra work, when calculated on an electronic computer, and it is required when the Moon rises and sets at a small inclination to the horizon; the extreme cases only arise for latitudes above 60°.

times of rising and setting, using the tabulated values of right ascension and declination. The Greenwich hour angle at H^h U.T. is:

$$h = \text{apparent sidereal time at } H^h \text{ U.T.} - \alpha \text{ at } (H^h + \Delta T) \text{ E.T.}$$

The motion in α in time ΔT is about $1^s{\cdot}0$ to $1^s{\cdot}6$ at the present time, and can either be neglected or applied later. The resulting zenith distances are interpolated to:

$$z = 90^\circ\ 34' - \text{horizontal parallax} + \text{semi-diameter}$$
$$= 90^\circ\ 34'{\cdot}001 - 0{\cdot}72755\ \pi$$

where π is the horizontal parallax, though a precision of $1'$ suffices to give times to 1^m in many cases. Linear interpolation is sufficient for the low latitudes, but for latitudes above 30° it is necessary to calculate three or more altitudes and to use second differences to obtain the required accuracy.

14. THE CALENDAR

A. INTRODUCTION

A calendar is a system of reckoning time over extended intervals by combining days into various periods adapted to purposes of civil life, to fixing religious observances, or to meeting scientific needs. Three of the periods used in calendars, namely days, months, and years, are based on astronomical periods that are of importance for the practical activities of daily life. Others, such as the week, are artificial.

The calendarial reckoning is according to conventional calendar years and adopted historical eras. In constructing and regulating civil calendars, and fixing ecclesiastical calendars, a number of auxiliary cycles and periods are used. The principal chronological eras and cycles are listed in *A.E.*, page 1, and are followed by the Gregorian calendar, giving day of week and Julian Date, for the current year.*

The complexity of calendars is due mainly to the incommensurability of the astronomical periods on which they are based. The supply of light by the Sun and Moon is governed by the solar day and the synodic month, while the return of the seasons depends on the tropical year. The length of the synodic month is 29·530589 days, and of the tropical year 365·242199 days, for the epoch 1900; the very small and somewhat uncertain secular variations in the lengths of these periods are unimportant for chronological purposes. The number of lunations in a tropical year is 12·368267 at the epoch 1900.

References

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Schram, R. *Kalendariographische und chronologische Tafeln*. Leipzig, 1908. This contains tables of the principal calendars of ancient and modern times.

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Fotheringham, J. K. *The calendar*. This scholarly article was first printed in *The Nautical Almanac* for 1931 and was reprinted (with revision in 1935) each year until 1938. Some paragraphs of this section of the Supplement have been taken directly from this article.

Annuaire du Bureau des Longitudes (Paris), 1959, pages 107–162. This contains a study of various calendars and many useful tables, some of which have been included in this section.

*All the calendarial information is given in *A.E.*, pages 10–11, for 1974 onwards.

B. HISTORICAL CALENDARS

The many calendars of historical times were lunar in origin, the year consisting usually of twelve months of about 30 days, with arbitrary or calculated intercalation of months or days to make the length of the year conform to the solar year. The Egyptian calendar was, up to the time of Julius Caesar's reform of the Roman calendar in 46 B.C., the only civil calendar in which the length of each month and year was fixed by rule, instead of being determined by the discretion of officials or by direct observation of some astronomical event.

1. The Egyptian calendar

The Egyptian year from an extremely remote date consisted of 12 months of 30 days each, followed by 5 additional days at the end of each year. This fixed calendar year of 365 days was not adjusted to the solar year by any intercalation; the Egyptian New Year consequently gradually retrograded through a complete circuit of the tropical year in a period of approximately 1460 years known as the Sothic cycle. The calendar year was divided into three seasons of four months each, called Flood time, Seed time, and Harvest time, corresponding to the annual cycle of the rise and fall of the Nile. The relation of the calendarial seasons to the natural seasons of the solar year was determined by the heliacal rising of Sirius (whose Egyptian name was Sothis), that is by the first appearance of the star in the morning sky after conjunction with the Sun; the mean interval between consecutive heliacal risings was $365^{d}\cdot2507$ according to Schoch.

This calendar originated from one of the variants of the earlier lunar calendar which regulated festivals in relation to the phases of the Moon, and which was eventually systematized to bring it into a fixed relation to the civil calendar.

The advantages of the fixed Egyptian calendar for astronomical calculations were recognized by the Hellenistic astronomers, and it became the standard astronomical system; it was still used by Copernicus in his lunar and planetary tables.

An attempt by Ptolemy Euergetes in 238 B.C. to introduce a sixth additional day once in four years failed, but a renewed attempt under Augustus (26–23 B.C.) was more successful. An additional day was inserted at the close of the Egyptian year 23–22 B.C. on August 29 of what we call the Julian calendar, and at the close of every fourth year afterwards, so that the reformed or Alexandrian year began on August 30 of the Julian calendar in the year preceding a Julian leap year and on August 29 in all other years. The effect of this reform was to keep each Egyptian month fixed to the place in the natural year which it happened to occupy under the old calendar in the years 26–22 B.C. But the old calendar was not easily suppressed, and we find the two used side by side until A.D. 238 at least. The old calendar was probably the more popular, and was preferred by astronomers and astrologers. Ptolemy always used it, except in his treatise on annual phenomena, for which the new calendar was obviously more convenient. Theon in the fourth century A.D., though mentioning the old calendar, habitually used the new.

The old Egyptian calendar was adopted by the Persians, perhaps about 500 B.C., in a form that cannot now be accurately restored, and survives in a slightly modified form in the Armenian calendar, the three first months of the old Egyptian year corresponding exactly with the three last months of the Armenian year. These are followed in the Armenian calendar by the five additional days, so that for the remainder of the year the Armenian months began five days later than those of the old Egyptian calendar. The Alexandrian calendar is still the calendar of Ethiopia and of the Coptic church, and is used for agricultural purposes in Egypt and other parts of northern Africa.

References

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Schoch, K. *Die Länge der Sothisperiode beträgt* 1456 *Jahre*. Berlin-Steglitz (by the author), 1928.

2. The Babylonian calendar

The Babylonian year consisted of 12 lunar months, each fixed by actual observation of the first appearance of the lunar crescent in the evening sky, with the intercalation of an additional month when necessary to keep the calendar year in a definite relation to the seasons. The year began in the spring with the month Nisannu.

Up to about 480 B.C., the intercalations show no regularity whatever; but attempts appear to have been made to formulate fixed rules, and at some time very close to 380 B.C. a regular cycle of 7 intercalations at fixed intervals during each 19 years came to be used. The 19-year cycle had been introduced by Meton at Athens about 50 years earlier, but whether the Babylonians obtained it from the Greeks or discovered it independently is not known. This cycle equates 19 years to 235 lunations; it still survives in the modern Jewish calendar, with the same value for the length of the mean synodic month as in the Babylonian calendar.

The conversion of dates in the Babylonian calendar to their exact equivalents in the Julian proleptic calendar is in general very difficult, and often uncertain or impossible. The ancient calendars that were regulated either arbitrarily or by observation of the lunar crescent cannot be completely restored with certainty and correlated with other calendars unless historical records are extant that give a sufficiently complete continuous record of the length of every month and attest to all the intercalary months.

References

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3. Greek calendars

Early Greek calendarial reckoning was rather chaotic. Each community had a separate calendar. All Greek calendars were lunar until the Roman period, and kept roughly in a fixed relation to the seasons by the intercalation of a thirteenth month when required; but the intercalations were determined by local public authorities, and were different in different calendars in addition to being irregular. There was also great variety in the season when the year began in different calendars.

From the sixth century B.C. onwards, a number of cycles were successively devised by the Greek astronomers as a basis for regulating the lunar calendar by fixed rules instead of by arbitrary intercalation. Among these, the Metonic and Callippic cycles came to be used by astronomers for dating observations, and appear to have been used over a period of several centuries extending into the Middle Ages to establish the dates of new moon for purposes of religious calendars. In the Metonic cycle, 19 years were equated to 235 months and to 6940 days; in the Callippic cycle, 76 years were equated to 940 lunations and to 27759 days, one day less than four Metonic cycles.

4. The Julian calendar

The Julian calendar was established in the Roman Empire by Julius Caesar in 46 B.C., by revising the ancient local calendar of the city of Rome, with the advice of the Alexandrian astronomer Sosigenes. Reaching its final form about A.D. 8, it was widely spread by the growth of the Empire; it remained in general use in the West during later centuries, until in 1582 it was further modified into the Gregorian calendar which has now come into almost worldwide use for civil purposes.

The Roman calendar was originally a lunar calendar, with arbitrary intercalation of months by the pontifical authorities. Under the pontificate of Julius Caesar, intercalation was neglected with such frequency that the calendar became about two months out of step with the solar year. To rectify the discrepancy, Caesar inserted intercalations into the year 46 B.C. that increased its length to 445 days, and instituted his reformed calendar beginning with 45 B.C.

In the Julian calendar, a mean length of 365·25 days for the year is adopted. The calendar year is adjusted to this mean value by inserting an intercalary day every fourth year; the intercalary year has 366 days, and each of the other three years has 365 days.

The year 45 B.C. was a Julian intercalary or leap year; but because of misunderstanding and confusion during the period following the adoption of the revised calendar, the intercalations were incorrectly made until the error was rectified in 8 B.C. by Augustus, who omitted further intercalations until A.D. 8. The adjustments actually made before the Augustan reform cannot be determined with certainty, and are ignored in the following sub-sections, but after A.D. 8 the Julian calendar was used without further change until the Gregorian reform in 1582.

The Christian era for the chronological reckoning of the years was first used by the Roman abbot Dionysius Exiguus, to designate the years in a table for determining the date of Easter that he prepared as a continuation of a previous table in which the years had been designated according to the era of Diocletian. In extending the table, he adopted 248 Diocletian era = A.D. 532. The year in which he prepared the table was six years before this, or A.D. 525; but how he determined the correspondence is unknown. His method for designating the years was adopted by others, and through increasing use during the next few centuries it became established in western Europe as a chronological era.

In this system, the Christian era begins with year A.D. 1; the immediately preceding year is designated 1 B.C. There is no year 0 in the chronological reckoning. For astronomical purposes, the year immediately preceding A.D. 1 is designated 0; the other years B.C. are denoted by negative numbers, each numerically one less than the designation in the historical reckoning. In the astronomical system the year preceding 0 is −1, and corresponds to 2 B.C. The year 0 was a leap year.

The first century of the Christian era ended with December 31 of A.D. 100, when the first one hundred years A.D. 1 to A.D. 100, inclusive, had been completed. Likewise, the nineteenth century ended with 1900 December 31; the twentieth century began with 1901 January 1, and the first half of the century ended with 1950 December 31. Considerable public controversy always attends these occasions. *

The Christian era was adopted at different times in different countries with a variety of dates for the beginning of the year. The most common initial dates were December 25, January 1, March 1, and March 25. These different reckonings of the year were known as *styles*. Traditionally in the ancient Roman calendar, March had been the first month of the year, as reflected in the numerical names which still survive for the months September to December and in the position of the intercalary day at the end of February; but in 153 B.C., with a change in the date of entry into office of the consuls and other magistrates to January 1, this became the first day of the official year and came to be widely adopted during later centuries in western Europe as the calendar New Year. In Italy, however, down to the eighteenth century the years of the Christian era began in the Venetian style on March 1, in the Pisan style on the preceding March 25, and in the Florentine style on the following March 25, while at Rome different styles were used for different purposes. In England the Nativity style beginning on December 25 was superseded in the fourteenth century by the Annunciation style beginning on March 25, but the Circumcision style beginning on January 1 was substituted in 1752 by the Act that introduced the Gregorian calendar. In Scotland the year

*The new century is popularly considered to begin one year earlier.

had begun officially on January 1 since 1600. The names old style and new style were, however, used to distinguish not the different dates for the beginning of the year, but the Julian and Gregorian calendars, each of which has been used with different initial dates.

The intercalary day was always inserted in every February which, if the years began with January 1, would fall in a year with a numerical designation divisible by 4. Consequently, when the actual beginning of the year was in March, the years divisible by 4 were not the leap years.

Preceding the Christian era, the rule that when the New Year is January 1 the years divisible by 4 are leap years is valid only if the *astronomical* designations of the years by negative numbers are used.

References

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C. THE GREGORIAN CALENDAR

The Gregorian calendar was instituted in 1582 by Pope Gregory XIII, primarily as a basis for regulating Easter and the ecclesiastical calendar. It is a solar calendar, distinguished principally by the system of intercalation adopted for keeping the calendar year in adjustment with the tropical year, and constructed by modifying the Julian calendar. The mean Julian calendar year of 365·25 days exceeds the length of the tropical year by about 11^{m} 14^{s}. The continual accumulation of this excess amounts to about 3 days every 400 years, and causes a gradual progressive change in the calendar dates of the seasons. This defect in the Julian calendar had produced a very noticeable effect on the date of Easter. Since Easter was the Christian continuation of the Jewish Passover, the date was fixed by rules that were intended to keep it near the vernal equinox, because the Passover was observed on 14 Nisan, and in the ancient Jewish calendar the beginning of this month was determined by observation of the lunar crescent nearest the vernal equinox. In practice, the date of Easter was determined from tables in which the lunar months were based on the Metonic cycle and March 21 was adopted as a fixed date for the equinox. Consequently, as the actual vernal equinox gradually occurred earlier in the calendar, the date of Easter became progressively later relative to the seasons; by the sixteenth century, the equinox had fallen back to about March 11, and Easter was tending nearer and nearer toward the summer.

The Gregorian reform of the Julian calendar consisted of:

(i) omitting 10 days from the calendar reckoning, the day next after 1582 October 4 being designated 1582 October 15, for the purpose of restoring the date of the actual vernal equinox to March 21;

(ii) adopting a different rule for leap year, by omitting the intercalary day in centurial years that are not divisible by 400, such as 1700, 1800, 1900, and 2100, in order to correct the error of the Julian calendar where an intercalary day is inserted every four years;

(iii) fixing rules for determining the date of Easter in the revised calendar.

The week was not modified in any way; special provision was made that the sequence of the days of the week was not broken.

The mean length of the Gregorian calendar year is 365·2425 days. At the completion of a 400-year calendar cycle, the cumulative discrepancy with the tropical year is only a few hours.

The authoritative treatise on the principles of this calendar and the associated ecclesiastical calendar is the book by Christoph Clavius, *Explicatio Romani Calendarii a Gregorio XIII P.M. restituti* (Rome, 1603), which is also included in Volume V of the collected works of Clavius published in 1612.

The Gregorian calendar was at once officially adopted for civil and religious purposes in Roman Catholic countries. During the following centuries, it came into almost universal use throughout the West, although with some diversity between civil and ecclesiastical practice; and it is widely used for some civil purposes in countries which have official native calendars.

The dates of the official adoption of the Gregorian calendar differed from country to country. In some regions, this calendar came into use gradually without official action. The introduction by legal action was in many cases not completely accepted among the people for a long period, and quite often did not affect ecclesiastical customs; for details, especially of the diverse church calendars, the references given at the end of this sub-section may be consulted, particularly Lange (**7**). In the Gregorian calendar, Easter has not in all cases been fixed strictly according to the Gregorian rules; in particular, it has occasionally been determined astronomically, e.g., by the German Protestants from 1700 to 1776, in Sweden from 1740 to 1844, and by the Eastern Orthodox Churches since 1923.

At a meeting of a Congress of the Orthodox Oriental Churches held in Constantinople in May, 1923, the Julian calendar was replaced by a modified Gregorian calendar in which century years are leap years only when division of the century number by 9 leaves a remainder of either 2 or 6, and Easter is determined by the astronomical Moon for the meridian of Jerusalem; see Milankovitch (**8**). The change was such that 1923 October 1, Julian calendar, became 1923 October 14 in the new calendar.

In the following list the dates of the official adoption of the Gregorian calendar are indicated in the form of double dates that give the corresponding Julian/Gregorian dates for the first day on which the Gregorian calendar was used. The authorities that were consulted are referred to by the numbers, in bold type, assigned to them in the list of references at the end of this sub-section. References **4** and **7** are considered to be the most reliable, while **6** and **12** should be reliable for the native countries of their authors; **10** is not documented.

List of dates of adoption of the Gregorian calendar

Alaska

1867 October 18, when Alaska was transferred to the United States under treaty of purchase from Russia, where the Julian calendar was still in use.

The Julian calendar dates had been in accordance with the reckoning to the west of the international date line. A further change was therefore made to conform to the reckoning east of the date line, and consequently the date was advanced by only 11 days instead of the 12 days by which the Gregorian calendar was then in advance of the Julian calendar.

Albania

1912 December, for civil purposes (Lange, **7**).

American Colonies

1752 September 3/14, at the same time as in Great Britain.

Austria. See German States.

Belgium

Different sources disagree:

4 1582 December 22/1583 January 1 in Flanders, Brabant, Hainaut, and other southern provinces; 1583 February 11/21 in Liege Bishopric.

10 1582 December 15/25 in Flanders, Hainaut, Luxembourg, and other southern provinces.

3 1583 in Flanders.

Bulgaria

Different sources disagree:

3 1915

7 1916 April 1, for civil purposes. Double dating had already been in use for some time, but was excluded by the law introducing the Gregorian calendar.

Chinese Republic

Different statements are given in different sources:

11 1912 January 1, by Sün Yat Sen.

6 1912; but during 1912–1928, both the Gregorian date and the Chinese calendar date were carried on official documents.

12 1929 January 1.

Czechoslovakia. See German States.

Denmark

1700 February 19/March 1 (Ginzel, **4**). Norway was then under Danish rule.

Egypt

1875, by ordinance of Ismail Pasha, for civil purposes (Lange, **7**).

Esthonia

1918 January (Lange, **7**).

Finland. See Sweden.

France. See also German States.

1582 December 10/20 in France and Lorraine, by edict of Henry III (Ginzel, **4**).

German States, listed according to the countries within which they now lie:

		Source
Austria		
Brixen †	1583 October 6/16	**10**
Carinthia	1583 December 15/25	**4**
Salzburg	1583 October 6/16	**10**
Styria	1583 December 15/25	**4, 10**
Tyrol	1583 October 6/16	**4**
Czechoslovakia		
Bohemia	1584 January 7/17	**4, 10**
Moravia	1584 January 7/17	**4**
France		
Alsace	after Peace of Munster (1648)	**4**
Strassburg (city of)	1682 February 6/16	**4, 10**
Strassburg (bishopric of)	1583 November 12/22	**10**
,, ,, ,,	1583 November 17/27	**4**

†Brixen is now known as Bressanone and is in Italy.

		Source
Germany		
Aachen	1583 November 4/14	4
Augsburg	1583 February 14/24	4
Augsburg (bishopric of)	1583 February 14/24	10
Baden (marquisate of)	1583 November 17/27	4
Bavaria	1583 October 6/16	10
Bavarian bishoprics	1583 October 6/16	4
Cologne (city of)	1583 November 4/14	4, 10
Eichstadt	1583 October 6/16	10
Freising	1583 October 6/16	10
Hildesheim (bishopric of)	1631 March 16/26	4
Julich	1583 November 3/13	4
Lausitz	1584 January 7/17	10
Mainz (archbishopric of)	1583 November 12/22	4, 10
Munster (city and county of)	1583 November 17/27	4
Neuburg Palatinate	1615 December 14/24	4, 10
Osnabruck (city of)	1624	4
Paderborn (bishopric of)	1585 June 17/27	4, 10
Prussia (duchy of)	1610 August 23/September 2	4, 10
Regensburg	1583 October 6/16	10
Silesia	1584 January 13/23	4, 10
Trier (archbishopric of)	1583 October 5/15	4, 10
Westphalia (duchy of)	1584 July 2/12	4, 10
Wurzburg (bishopric of)	1583 November 5/15	4, 10
Kaiser and Parliament	1584 January 7/17	4
Protestant Germany	1700 February 19/March 1	10
Under Frederick the Great, Gregorian reckoning was adopted in 1775 under the name of " improved calendar ".		4
Switzerland		
Appenzell (Protestant half)	see below	
Basel, Bern, and Biel	1701 January 1/12*	4
Fribourg	1584 January 12/22	4
Geneva	1701 January 1/12*	4
Graubunden	see below	
Lucerne	1584 January 12/22	4
Mulhausen	1701 January 1/12*	4
Neuchatel	1701 January 1/12	10
Prattigau (" Ten Districts ")	1812	4
Sargans	1701 January 1/12	4
Schaffhausen	1701 January 1/12*	4
Schwyz	1584 January 12/22	4
Solothurn	1584 January 12/22	4
Thurgau	1701 January 1/12	10
Uri	1584 January 12/22	4
Valais	see below	
Zug	1584 January 12/22	4
Zurich	1701 January 1/12*	4
Federal congress	1583 November 10	4
Appenzell (Protestant half): separated from the Roman Catholic half in 1597 and remained on Julian calendar.		4
Graubunden: Gregorian calendar adopted at first only by Roman Catholic districts in upper Rhine valley. The others retained Julian calendar until into 18th century.		4
Valais: 1622, except Sitten, Siders, Leuk, Raron, Visp, Brieg and Goms which changed in 1656.		4

* Improved Weigel calendar.

Great Britain and Dominions

1752 September 3/14, by Act of Parliament passed 1751 March 18; at the same time, the beginning of the year was changed from March 25 to January 1, commencing with the year 1752.

Greece

See Milankovitch (**8**); a slightly modified form of the Gregorian calendar was introduced 1924 March 10/23.

Hungary

1587 October 22/November 1. (Schram, **10**).

Italy

1582 October 5/15 (Ginzel, **4**).

Japan

1873 January 1 (van Wijk, **12**).

Jugoslavia

1919 (Fotheringham, **3**, but see also Milankovitch, **8**).

Latvia

The Gregorian calendar gradually came into use for civil purposes during the German occupation 1915–1918 (Lange, **7**).

Lithuania

1915, by the Catholic Church, which represented three quarters of the population (Lange, **7**).

Luxembourg

1582 December 15/25 (Schram, **10**).

Netherlands

In the Catholic States, 1582–1583; in the Protestant States, 1700–1701; but different sources disagree on the exact dates. For minute details, see van Wijk, **12**.

Norway. See Denmark.

Poland

1582 October 5/15 (Schram, **10**). In the Russian part of Poland, the Gregorian calendar was introduced by the German occupation troops 1915 March 21 (Lange, **7**).

Portugal

1582 October 5/15 (Ginzel, **4**).

Roumania

1919 April 1/14 (*L'Astronomie: Bull. Soc. Astr. de France*, **33**, 529, 1919).

Spain

1582 October 5/15 (Ginzel, **4**).

Sweden

1753 February 18/March 1 (Schram, **10**; Lange, **7**). Finland was then a part of Sweden.

Switzerland. See German States.

Turkey

1927 January 1 (*Astr. Jahresber.*, **29**, 48, 1927).

U.S.S.R.

1918 February 1/14 for civil purposes (Lange, **7**; *Observatory*, **41**, 146, 1918).

Equivalent dates in the Julian and the Gregorian calendars are frequently required. Both calendars were widely used for a long period after the Gregorian calendar was first introduced; for special purposes the Julian calendar is still of service, and occasionally the Gregorian proleptic calendar is used for dates before 1582. The 10 days difference between the two calendars at the time of the Gregorian reform increases by one at the bissextile, or intercalary, day in each centurial year after 1582 that is not divisible by 400; the difference is subtracted from a Gregorian date, added to a Julian date. Before 1582, the difference decreases. The year 0 is a leap year in the Gregorian proleptic calendar.

14.1—EQUIVALENT DATES IN THE JULIAN AND GREGORIAN CALENDARS

Year (astronomical)	Date Julian	Diff. (days)	Date Gregorian	Year (astronomical)	Date Julian	Diff. (days)	Date Gregorian
			Astronomical years −500 to +300				
−500	March 5		February 28	−100	March 2		February 28
−500	March 6	5	March 1	−100	March 3	2	March 1
−300	March 3		February 27	+100	February 29		February 27
−300	March 4		February 28	+100	March 1		February 28
−300	March 5	4	March 1	+100	March 2	1	March 1
−200	March 2		February 27	+200	February 28		February 27
−200	March 3		February 28	+200	February 29		February 28
−200	March 4	3	March 1	+200	March 1	0	March 1
−100	March 1		February 27	+300	February 28		February 28
			A.D. 300 March 1 to 1582 October 4/14				
300	February 29		March 1	1000	February 29		March 6
300	March 1	1	March 2	1000	March 1	6	March 7
500	February 28		March 1	1100	February 28		March 6
500	February 29		March 2	1100	February 29		March 7
500	March 1	2	March 3	1100	March 1	7	March 8
600	February 28		March 2	1300	February 28		March 7
600	February 29		March 3	1300	February 29		March 8
600	March 1	3	March 4	1300	March 1	8	March 9
700	February 28		March 3	1400	February 28		March 8
700	February 29		March 4	1400	February 29		March 9
700	March 1	4	March 5	1400	March 1	9	March 10
900	February 28		March 4	1500	February 28		March 9
900	February 29		March 5	1500	February 29		March 10
900	March 1	5	March 6	1500	March 1	10	March 11
1000	February 28		March 5	1582	October 4		October 14
			1582 October 5/15 onwards				
1582	October 5	10	October 15	1800	February 29		March 12
1582	October 6	10	October 16	1800	March 1	12	March 13
1700	February 18		February 28	1900	February 16		February 28
	February 19	10	March 1		February 17	12	March 1
	February 28		March 10		February 28		March 12
1700	February 29		March 11	1900	February 29		March 13
1700	March 1	11	March 12	1900	March 1	13	March 14
1800	February 17		February 28	2100	February 15		February 28
	February 18	11	March 1		February 16	13	March 1
1800	February 28		March 11	2100	February 28		March 13

The differences are constant between each pair of dates given in the table. The sign of the difference can be obtained by inspection.

Except in the centurial years that are given above, the leap years (astronomical year divisible by 4) are common to both calendars.

Equivalent dates in the Julian and Gregorian calendars, extending backwards to the year −500 (= 501 B.C.) are listed in table 14.1; it is clear that for the years before A.D. 200 the difference must be added to the Gregorian date, or subtracted from the Julian date. Care should be taken to assign to February the proper number of days in each calendar; the change points (especially after 1582) are, however, clearly indicated.

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D. THE WEEK

The week was not originally an integral part of any calendar; in its present form, it gradually became established in the Roman calendar during the one or two centuries preceding the Christian era. The Mosaic Law enjoining an abstinence from work on every seventh day had established the 7-day period as a Jewish measure of time, and this Jewish week later passed into the Christian Church. Meanwhile, shortly before the Christian era, an astrological practice had arisen of attaching the names of the seven " planets ", the term at that time including the Sun and Moon, in cyclic succession to successive days, in the order in which the planets were supposed to rule the days. The planetary designations for the days rapidly acquired a widespread popularity, and became the predominant usage throughout the Roman Empire. The coincidence in the number of days in this astrological cycle with the number of days in the entirely independent Jewish week led to the gradual establishment of the planetary week without official recognition, either civil or ecclesiastical.

Since first becoming established, the cyclic succession of the days of the week has not been altered, and no breaks in the sequence have occurred. In the Teutonic languages, the names of the Roman deities Mars, Mercury, Jupiter, and Venus have been replaced by their counterparts Tiu, Woden, Thor, and Freya.

The week, therefore, is a non-astronomical element of the calendar. The reckoning of time by weekly cycles in continuous succession is independent of the essentially astronomical reckoning by days, months, and years which is the principal basis of the calendar. The consequent complexity of the relation between the two reckonings causes difficulty in determining readily the day of the week that corresponds to any given calendar date. However, a table of dominical letters (see sub-section E) is essentially a calendar for the entire period covered by the table, and is easily used explicitly for this purpose with the aid of a simple auxiliary table (such as table 14.2). Nomograms have also been constructed; see, e.g., d'Ocagne, *Traité de Nomographie*, Paris, 1899; A. Saldaña, *Urania*, **38**, 20, 1953.

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Table 14.2. Perpetual calendar

This calendar gives the days of the week corresponding to the days of any month in the Julian or Gregorian calendars once the dominical letter for the year is known. [The dominical letter may be taken from table 14.5 (Julian) or table 14.9 (Gregorian); in leap years the first letter is to be used for January and February, the second for the remainder of the year.]

The column in which the dominical letter for the year is in the same line as the month for which the calendar is required gives the days of the week that correspond to the days of the month given at the left.

Month	Dominical Letter						
January, October	A	B	C	D	E	F	G
February, March, November	D	E	F	G	A	B	C
April, July	G	A	B	C	D	E	F
May	B	C	D	E	F	G	A
June	E	F	G	A	B	C	D
August	C	D	E	F	G	A	B
September, December	F	G	A	B	C	D	E

Day of Month					Day of Week						
1	8	15	22	29	Sun.	Sat.	Fri.	Thur.	Wed.	Tues.	Mon.
2	9	16	23	30	Mon.	Sun.	Sat.	Fri.	Thur.	Wed.	Tues.
3	10	17	24	31	Tues.	Mon.	Sun.	Sat.	Fri.	Thur.	Wed.
4	11	18	25		Wed.	Tues.	Mon.	Sun.	Sat.	Fri.	Thur.
5	12	19	26		Thur.	Wed.	Tues.	Mon.	Sun.	Sat.	Fri.
6	13	20	27		Fri.	Thur.	Wed.	Tues.	Mon.	Sun.	Sat.
7	14	21	28		Sat.	Fri.	Thur.	Wed.	Tues.	Mon.	Sun.

E. ECCLESIASTICAL CALENDARS

The date of Easter determines the dates of most of the movable festivals in Christian ecclesiastical calendars. During the early centuries of the Christian era many diverse practices were followed by the Churches of different countries in fixing the date of Easter, and practice did not become entirely uniform until during the 8th century; but the method used by the Alexandrian Church, which was widely recognized from the third century on and was favoured by the General Council of Nicaea in A.D. 325, eventually became generally accepted until the Gregorian reform in 1582.

In this method, a conventional 19-year cycle of dates for new moon in the Julian calendar is adopted, based on the Metonic cycle. March 21 is adopted for the date of the vernal equinox. The fourteenth day of each lunation of the cycle is the date adopted for the full moon, and Easter Day is the first Sunday after the full moon that occurs on or next after the tabular date of March 21 for the vernal equinox.

In the Gregorian calendar reform, in addition to the modifications of the Julian calendar to make it conform more closely to the tropical year, corrections were made to the tabular calendar of the 19-year cycle of dates for new moon to bring it into better accord with the actual lunations; March 21 was retained for the date of the vernal equinox. With the corrected tables, Easter is fixed by the same rule as previously used with the Julian calendar; but auxiliary rules are added which make March 22 the earliest possible date for Easter Day, and April 25 the latest.

The rule for determining Easter as commonly expressed in popular language is somewhat misleading because it is not a precise statement of the actual ecclesiastical rules. In order that the date should be incontrovertibly fixed, and determinable indefinitely in advance, tables based on the Metonic cycle were constructed to be used permanently for calculating the age of the Moon. Easter is determined by the " ecclesiastical moon " defined by these adopted tables, which is not strictly identical with the real Moon. In addition, the vernal equinox is fixed at March 21, not by the actual motion of the Sun. Moreover, the date of Easter is determined independently of any meridian of longitude, and is always the same in all time zones, unlike astronomical phenomena. The date of the full moon that occurs on or next after the vernal equinox is taken from the ecclesiastical tables, not from astronomical ephemerides; it is the fourteenth day of the tabular lunation, and Easter Day is the next following Sunday.

Inevitably, the date of Easter occasionally, though rarely, differs from the date that would be obtained from astronomical ephemerides by the same rule, as for example in 1954; but when this happens, it occurs only in part of the world, since two dates separated by the international date line are simultaneously in progress on the Earth. However, Easter has been determined astronomically in a few cases after the adoption of the Gregorian calendar, as already noted in the list of dates of the adoption.

The date of either the Julian or the Gregorian Easter in any past or future year may be quickly determined with a few simple tables that are readily constructed from the basic principles of the respective calendars.

The year that was adopted as the initial year of the continuous succession of 19-year cycles on which the ecclesiastical lunar calendars were based is 1 B.C. The successive years of each cycle are numbered consecutively from 1 to 19; the number of a year in the cycle has been known, since the late Middle Ages, as the *Golden Number* for the year. A table extending over any desired interval of time may be immediately written down, from which the golden number of any year may be taken out. The golden number for the year A.D. Y is one greater than the remainder after dividing Y by 19. The golden numbers in the twentieth century are directly given in table 14.3.

The Julian ecclesiastical lunar calendar is represented by a table (such as table 14.4) of the adopted 19-year cycle of dates for new moon, with the golden number as the argument.

As a means of readily identifying the dates of the Sundays in any year, the first seven letters of the alphabet are placed in cyclic succession against the days of each year, beginning with A on January 1; the letter that stands against the Sundays during the year is known as the *Dominical Letter* of that year. This device originated from the use of the eight letters A to H in the same way by the Roman calendar-makers to indicate market days. No letter is placed against an intercalary day; consequently, in leap years the dominical letter retrogrades by one place in the alphabet at the date of intercalation, and there are two letters for the year, one during January and February, the other during the remainder of the year. Otherwise, the dominical letter retrogrades by one place from one year to the next.

In the Julian calendar, January 1 of the year 0, which is a leap year, was Thursday; therefore, the Julian dominical letter for January and February is D, and for the remainder of the year C. From this starting point, a table extending over any desired period of years may be written down at once, and from it the dominical letter of any year may be taken out.

The dominical letter in the Julian calendar has a complete cycle of 28 years, and this is used as the basis of the short tabulation in table 14.5, although the dominical letters for the years of the twentieth century are given directly.

From the dominical letter (table 14.5) and the golden number (table 14.3), the date of the Julian Easter may be found from the lunar calendar (table 14.4) and a perpetual calendar (table 14.2); but for greater convenience, a further table (such as table 14.6) may be constructed, in several different forms, giving the date of Easter Day explicitly with the golden number and dominical letter as arguments.

Table 14.3. Golden number

Year A.D. (1900 + *y*)						*G*
1900	**1919**	**1938**	**1957**	**1976**	**1995**	1
04	**23**	**42**	**61**	**80**	**1999**	5
09	**28**	**47**	**66**	**85**		10
14	**33**	**52**	**71**	**90**		15
1918	**1937**	**1956**	**1975**	**1994**		19

The golden number (G) for the year A.D. Y is given by:

$$G = R + 1 \quad \text{if} \quad Y = 19N + R$$

and, if $Y = 100n + y$, by:

$$G \text{ for year } Y = G \text{ for year } (1900 + y) + R \text{ for year } 100n$$

less 19 if the sum exceeds 19.

	Year	*R*	Year	*R*	Year	*R*	Year	*R*
A.D.	**0**	0	**500**	6	**1000**	12	**1500**	18
	100	5	**600**	11	**1100**	17	**1600**	4
	200	10	**700**	16	**1200**	3	**1700**	9
	300	15	**800**	2	**1300**	8	**1800**	14
	400	1	**900**	7	**1400**	13	**1900**	0

Table 14.4. Julian ecclesiastical lunar calendar

DATE OF THE (ECCLESIASTICAL) NEW MOON

Golden Number	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1	23	*21*	23	*21*	21	*19*	19	*17*	16	*15*	14	*13*
2	12	*10*	12	*10*	10	*8*	8	*6*	5	*4*	3	*2*
3	*1*, 31		*1*, 31	*29*	29	*27*	27	*25*	24	*23*	22	*21*
4	20	*18*	20	*18*	18	*16*	16	*14*	13	*12*	11	*10*
5	9	*7*	9	*7*	7	*5*	5	*3*	*2*	2, *31*	30	*29*
6	28	*26(27)*	28	*26*	26	*24*	24	*22*	21	*20*	19	*18*
7	17	*15*	17	*15*	15	*13*	13	*11*	10	*9*	8	*7*
8	6	*4*	*6*	5	*4*	3	*2*	1, *30*	29	*28*	27	*26*
9	25	*23*	25	*23*	23	*21*	21	*19*	18	*17*	16	*15*
10	14	*12*	14	*12*	12	*10*	10	*8*	7	*6*	5	*4*
11	*3*	2	*3*	2	*1*, 31	*29*	29	*27*	26	*25*	24	*23*
12	22	*20*	22	*20*	20	*18*	18	*16*	15	*14*	13	*12*
13	11	*9*	11	*9*	9	*7*	7	*5*	4	*3*	2	*1, 31*
14	30	*28(29)*	30	*28*	28	*26*	26	*24*	23	*22*	21	*20*
15	19	*17*	19	*17*	17	*15*	15	*13*	12	*11*	10	*9*
16	8	*6*	8	*6*	6	*4*	4	*2*	*1*	1, *30*	29	*28*
17	27	*25(26)*	27	*25*	25	*23*	23	*21*	20	*19*	18	*17*
18	16	*14*	16	*14*	14	*12*	12	*10*	9	*8*	7	*6*
19	5	*3*	5	4	*3*	2	1, 30	*28*	27	*26*	25	*24*

In leap years use the date in parentheses.

Dates in *italic* type relate to the beginning of lunations of 30 days, the others to those of 29 days.

Table 14.5. Dominical letter—Julian calendar

For year A.D. *Y* enter table (upper argument) with remainder after dividing *Y* by 28.

Remainder	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Letter	DC	B	A	G	FE	D	C	B	AG	F	E	D	CB	A
Year of	04	05	06	07	08	09	10	11	12	13	14	15	16	17
twentieth	32	33	34	35	36	37	38	39	40	41	42	43	44	45
century	60	61	62	63	64	65	66	67	68	69	70	71	72	73
(1900 +)	88	89	90	91	92	93	94	95	96	97	98	99	100	

Remainder	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Letter	G	F	ED	C	B	A	GF	E	D	C	BA	G	F	E
Year of												01	02	03
twentieth	18	19	20	21	22	23	24	25	26	27	28	29	30	31
century	46	47	48	49	50	51	52	53	54	55	56	57	58	59
(1900 +)	74	75	76	77	78	79	80	81	82	83	84	85	86	87

For years in the twentieth century enter table (lower argument) with last two figures of the year.

In leap years two letters are given: the first is for January and February, the second is for the remainder of the year.

Table 14.6. Julian Paschal table

DATE OF EASTER DAY IN THE JULIAN CALENDAR

Golden Number	Dominical Letter A	B	C	D	E	F	G
1	April 9	April 10	April 11	April 12	April 6	April 7	April 8
2	March 26	March 27	March 28	March 29	March 30	March 31	April 1
3	April 16	April 17	April 18	April 19	April 20	April 14	April 15
4	April 9	April 3	April 4	April 5	April 6	April 7	April 8
5	March 26	March 27	March 28	March 29	March 23	March 24	March 25
6	April 16	April 17	April 11	April 12	April 13	April 14	April 15
7	April 2	April 3	April 4	April 5	April 6	March 31	April 1
8	April 23	April 24	April 25	April 19	April 20	April 21	April 22
9	April 9	April 10	April 11	April 12	April 13	April 14	April 8
10	April 2	April 3	March 28	March 29	March 30	March 31	April 1
11	April 16	April 17	April 18	April 19	April 20	April 21	April 22
12	April 9	April 10	April 11	April 5	April 6	April 7	April 8
13	March 26	March 27	March 28	March 29	March 30	March 31	March 25
14	April 16	April 17	April 18	April 19	April 13	April 14	April 15
15	April 2	April 3	April 4	April 5	April 6	April 7	April 8
16	March 26	March 27	March 28	March 22	March 23	March 24	March 25
17	April 16	April 10	April 11	April 12	April 13	April 14	April 15
18	April 2	April 3	April 4	April 5	March 30	March 31	April 1
19	April 23	April 24	April 18	April 19	April 20	April 21	April 22

The golden number is given in table 14.3.

The dominical letter is given in table 14.5; the second letter must be used in leap years.

The date of Easter Day in the Julian calendar may thus be determined from the dominical letter and the golden number by either of the alternative procedures:

(i) From the Julian ecclesiastical lunar table (table 14.4), with the golden number (table 14.3) as argument, find the date of the first day of the Paschal lunation.

With this date and the dominical letter (table 14.5), determine the date of the fourteenth day of the lunation, and the day of the week on which it falls; the day of the week may be found by use of a perpetual calendar (table 14.2).

The fourteenth day is the Paschal full moon, and the next following Sunday is Easter Day.

(ii) With the golden number (table 14.3) and the dominical letter (table 14.5) as arguments, take the date of Easter Day directly from the Julian Paschal table (table 14.6).

Example 14.1 *The date of Easter—Julian calendar*

The date of Easter in 1513 is involved in the history of the discovery of Florida. In historical literature, either through error or because of the diversity of calendar styles that were once used, confusion exists as to the year of the discovery; but since Florida was sighted by Ponce de Leon on Easter Sunday, March 27, the year may be identified as 1513 in the style in which January 1 is the beginning of the year.

For 1513, the Julian dominical letter (table 14.5) is B. The golden number (table 14.3) is 13 (= 18 + 14 − 19).

From the table of ecclesiastical new moons (table 14.4), with the golden number as argument, the first day of the Paschal lunation is found to be March 11. The 14th day, the date of the Paschal full moon, is therefore March 24.

From the dominical letter, the day of the week of the Paschal full moon is found, using table 14.2, to be Thursday. The next following Sunday is March 27.

This date of March 27, Julian calendar, for the date of Easter Day in 1513 checks with the date obtained directly from the Julian Paschal table (table 14.6) with arguments golden number 13 and dominical letter B.

In the Gregorian calendar, by using the Gregorian revision of the cycle of dates for new moon, the same method may be used to find the date of Easter as in the Julian calendar, and this method is retained in the English Prayer Book. However, in constructing the Gregorian lunar calendar, the modifications that were made in the Julian 19-year cycle of lunations included a cycle of forward and backward shifts of the dates of new moon in the centurial years, for the purpose of bringing the mean length of the lunations into closer agreement with the actual synodic month. Consequently, in order to use the golden number in calculating the Gregorian Easter, corresponding shifts of the cycle of golden numbers are necessary in some of the centurial years.

In the tables issued by authority of Pope Gregory XIII, the date of Easter is determined by means of the *Epact* instead of the golden number. The epact is the age of the Moon, diminished by one day, on January 1 in the Gregorian ecclesiastical lunar calendar. However, it is not always the same as the tabular

Table 14.7. Epact—Gregorian calendar

Year of century						Century 1582-1599	1600	1700	1800	1900
00	19	38	57	76	95	19	15	9	4	29
01	20	39	58	77	96	1	26	20	15	10
02	21	40	59	78	97	12	7	1	26	21
03	22	41	60	79	98	23	18	12	7	2
04	23	42	61	80	99	4	29	23	18	13
05	24	43	62	81		15	10	4	0	24
06	25	44	63	82		26	21	15	11	5
07	26	45	64	83		7	2	26	22	16
08	27	46	65	84		18	13	7	3	27
09	28	47	66	85		29	24	18	14	8
10	29	48	67	86		10	5	0	25	19
11	30	49	68	87		21	16	11	6	0
12	31	50	69	88		2	27	22	17	11
13	32	51	70	89		13	8	3	28	22
14	33	52	71	90		24	19	14	9	3
15	34	53	72	91		5	1	25	20	14
16	35	54	73	92		16	12	6	1	25*
17	36	55	74	93		27	23	17	12	6
18	37	56	75	94		8	4	28	23	17

* second epact 25

age on December 31 of the preceding year; a discontinuity of one day, or exceptionally of two days, is possible. The age is considered to be one day on the dates of new moon. The possible values of the epact are 0 to 29 inclusive; when it is 0, either no value at all is given, or else the value 30 or the symbol * is used. When epacts 24 and 25 both occur during the same 19-year cycle of golden numbers, the epact 25 is used as if it were 26; it is then called the "second epact 25", and is denoted by the symbol 25*.

From the involved rules that were formulated for constructing the lunar calendar, tables (such as table 14.7) may be prepared from which the epact for any year may be obtained directly.

The epact determines the dates of ecclesiastical new moon during the year. The Gregorian lunar calendar may therefore be represented by a table, such as table 14.8, of these dates with the epact as argument.

The Gregorian Easter is determined by the epact and the dominical letter; but the dominical letter in the Gregorian calendar differs from the Julian dominical letter for the same year, because the papal decree which put the calendar reform into effect included the explicit provision that the sequence of weekdays be continuous from the Julian to the Gregorian calendar. The Julian dominical letter for 1582 was G; October 4 in the Julian calendar was Thursday, and the immediately following day was Friday, October 15, in the Gregorian calendar. The dominical letter for the latter part of 1582 therefore became C in the Gregorian calendar.

In the Gregorian calendar the dominical letter has a cycle of 28 years within each century, but the cycle cannot readily be used to derive the dominical letter

Table 14.8. Gregorian ecclesiastical lunar calendar

DATE OF THE (ECCLESIASTICAL) NEW MOON

Epact	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
8	23	*21*	23	*21*	21	*19*	19	*17*	16	*15*	14	*13* [13]
19	12	*10*	12	*10*	10	*8*	8	*6*	5	*4*	3	*2* [2,*31*]
0	*1*, 31		*1*, 31	*29*	29	*27*	27	*25*	24	*23*	22	*21* [21]
11	20	*18*	20	*18*	18	*16*	16	*14*	13	*12*	11	*10* [10]
22	9	*7*	9	*7*	7	*5*	5	*3*	2	*1*, 31	*29*	*29* [29]
3	28	*26*(*27*)	28	*26*	26	*24*	24	*22*	21	*20*	19	*18* [18]
14	17	*15*	17	*15*	15	*13*	13	*11*	10	*9*	8	7 [7]
25	*6*	5	*6*	5	*4*	3	*2*	1, *30*	29	*28*	27	*26* [26]
25*	6	*4*	6	*4*	4	*2*	2, *31*	30	*28*	28	*26*	*26* [26]
6	25	*23*	25	*23*	23	*21*	21	*19*	18	*17*	16	*15* [15]
17	14	*12*	14	*12*	12	*10*	10	*8*	7	*6*	5	4 [4]
28	*3*	2	*3*	2	*1*, 31	*29*	29	*27*	26	*25*	24	*23* [23]
9	22	*20*	22	*20*	20	*18*	18	*16*	15	*14*	13	*12* [12]
20	11	*9*	11	*9*	9	*7*	7	*5*	4	*3*	2	*1*,*31* [*1*,*31*]
1	30	*28*(*29*)	30	*28*	28	*26*	26	*24*	23	*22*	21	*20* [20]
12	19	*17*	19	*17*	17	*15*	15	*13*	12	*11*	10	9 [9]
23	8	*6*	8	*6*	6	*4*	4	*2*	1, *30*	30	*28*	*28* [28]
4	27	*25*(*26*)	27	*25*	25	*23*	23	*21*	20	*19*	18	*17* [17]
15	16	*14*	16	*14*	14	*12*	12	*10*	9	*8*	7	6 [6]
26	*5*	4	*5*	4	*3*	2	*1*, 31	*29*	28	*27*	26	*25* [25]
7	24	*22*	24	*22*	22	*20*	20	*18*	17	*16*	15	*14* [14]
18	13	*11*	13	*11*	11	*9*	9	*7*	6	*5*	4	3 [3]
29	*2*	1	*2*	1, *30*	30	*28*	28	*26*	25	*24*	23	*22* [22]
10	21	*19*	21	*19*	19	*17*	17	*15*	14	*13*	12	*11* [11]
21	10	*8*	10	*8*	8	*6*	6	*4*	3	*2*	1, *30*	*30* [30]
2	29	*27*(*28*)	29	*27*	27	*25*	25	*23*	22	*21*	20	*19* [19]
13	18	*16*	18	*16*	16	*14*	14	*12*	11	*10*	9	8 [8]
24	7	*5*	7	*5*	5	*3*	3	*1*, 31	*29*	29	*27*	27 [27]
5	26	*24*(*25*)	26	*24*	24	*22*	22	*20*	19	*18*	17	*16* [16]
16	15	*13*	15	*13*	13	*11*	11	*9*	8	7	6	5 [5]
27	*4*	3	*4*	3	*2*	1, *30*	30	*28*	27	*26*	25	*24* [24]

In leap years use the date in parentheses (for February).

Dates in *italic* type relate to the beginning of lunations of 30 days, the others to those of 29 days.

Dates in brackets [for December] apply for the last years of the Metonic cycles, that is for years in which the golden number is 19.

for any year owing to the occurrence of leap years in some centurial years. There is an overall cycle of 400 years after which the dominical letters repeat: they are given directly in table 14.9.

The date of Easter Day in the Gregorian calendar may be tabulated, as in table 14.10, with the epact and dominical letter as arguments; with the epact, determined directly for the year (table 14.7), and the Gregorian dominical letter (table 14.9), the date of Easter Day may thus be obtained immediately. Alternatively,

Table 14.9. Dominical letter—Gregorian calendar

	Century				
Year of century	**1582-1599**	**1600**	**1700**	**1800**	**1900**
00		B A	C	E	G
01 29 57 85	F	G	B	D	F
02 30 58 86	E	F	A	C	E
03 31 59 87	D	E	G	B	D
04 32 60 88	C B	D C	F E	A G	C B
05 33 61 89	A	B	D	F	A
06 34 62 90	G	A	C	E	G
07 35 63 91	F	G	B	D	F
08 36 64 92	E D	F E	A G	C B	E D
09 37 65 93	C	D	F	A	C
10 38 66 94	B	C	E	G	B
11 39 67 95	A	B	D	F	A
12 40 68 96	G F	A G	C B	E D	G F
13 41 69 97	E	F	A	C	E
14 42 70 98	D	E	G	B	D
15 43 71 99	C	D	F	A	C
16 44 72		C B	E D	G F	B A
17 45 73		A	C	E	G
18 46 74		G	B	D	F
19 47 75		F	A	C	E
20 48 76		E D	G F	B A	D C
21 49 77		C	E	G	B
22 50 78		B	D	F	A
23 51 79		A	C	E	G
24 52 80		G F	B A	D C	F E
25 53 81		E	G	B	D
26 54 82	C	D	F	A	C
27 55 83	B	C	E	G	B
28 56 84	A G	B A	D C	F E	A G

In leap years the first letter relates to January and February, the second to the remainder of the year.

Example 14.2. The date of Easter—Gregorian calendar

In 1960, the dominical letter (table 14.9) for the months after February is B. The epact (table 14.7) is 2.

From the table of ecclesiastical new moons (table 14.8), with the epact as argument, the first day of the Paschal lunation is found to be March 29. The fourteenth day, the date of the Paschal full moon, is therefore April 11.

From the dominical letter, the day of the week of the Paschal full moon is found, using table 14.2, to be Monday. The next following Sunday is April 17.

This date of April 17 for Easter Day checks with the date obtained directly from the Gregorian Paschal table (table 14.10) with arguments dominical letter B and epact 2.

The date of Easter in 1954 is an instructive example. The epact is the second epact 25, or epact 25*; the dominical letter is C. The epact must be used as if it were 26; from the table of ecclesiastical new moons, the first day of the Paschal lunation is April 4. The Paschal full moon is therefore Saturday, April 17, and Easter Day is April 18, as may also be obtained from the Gregorian Paschal table with arguments C and 25*.

The Gregorian Easter was April 18 throughout the world; but when the astronomical full moon occurred (at $18^{d}\ 05^{h}\ 48^{m}$ U.T.), the date was already April 18 in the time zones from the international date line westward to the eastern standard time zone inclusive.

Table 14.10. Gregorian Paschal table

DATE OF EASTER DAY IN THE GREGORIAN CALENDAR

Epact	Dominical Letter A	B	C	D	E	F	G
0	April 16	April 17	April 18	April 19	April 20	April 14	April 15
1	16	17	18	19	13	14	15
2	16	17	18	12	13	14	15
3	16	17	11	12	13	14	15
4	16	10	11	12	13	14	15
5	April 9	April 10	April 11	April 12	April 13	April 14	April 15
6	9	10	11	12	13	14	8
7	9	10	11	12	13	7	8
8	9	10	11	12	6	7	8
9	9	10	11	5	6	7	8
10	April 9	April 10	April 4	April 5	April 6	April 7	April 8
11	9	3	4	5	6	7	8
12	2	3	4	5	6	7	8
13	2	3	4	5	6	April 7	1
14	2	3	4	5	April 6	March 31	1
15	April 2	April 3	April 4	April 5	March 30	March 31	April 1
16	2	3	April 4	March 29	30	31	1
17	2	April 3	March 28	29	30	31	1
18	April 2	March 27	28	29	30	31	1
19	March 26	27	28	29	30	31	April 1
20	March 26	March 27	March 28	March 29	March 30	March 31	March 25
21	26	27	28	29	30	24	25
22	26	27	28	29	23	24	25
23	March 26	March 27	March 28	March 22	March 23	March 24	March 25
24	April 23	April 24	April 25	April 19	April 20	April 21	April 22
25	April 23	April 24	April 25	April 19	April 20	April 21	April 22
25*	23	24	18	19	20	21	22
26	23	24	18	19	20	21	22
27	23	17	18	19	20	21	22
28	16	17	18	19	20	21	22
29	April 16	April 17	April 18	April 19	April 20	April 21	April 15

In leap years the second of the two dominical letters must be used.

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the date may be determined from the Gregorian lunar calendar (table 14.8): with the epact (table 14.7) as argument, find the first day after March 7 on which a new moon occurs; this is the first day of the Paschal lunation, and the fourteenth day is the date of the Paschal full moon. The day of the week may be found from a perpetual calendar (table 14.2), using the dominical letter for the year (table 14.9), and the next following Sunday is Easter Day.

A period of 5 700 000 years is required for the cyclical recurrence of Gregorian Easter dates. In the Julian calendar, the dates of Easter recur in cycles of 532 years. The days of the year recur on the same days of the week every

28 years in the Julian calendar; this period came to be known as the *Solar Cycle*, and could be used for the purpose of finding the day of the week for any particular Julian calendar date. The solar cycle, like the golden number, is the same in the Gregorian calendar as in the Julian calendar; but the dominical letters are different, and the days of the week recur cyclically on the same days of the year only every 400 years in the Gregorian calendar. The solar cycle is commonly given in almanacs, but is of little practical use. The initial year of the first cycle was 9 B.C.; from this starting point, a table for any desired interval of time may be constructed in the same way as for the golden number. Alternatively, it is evident that the solar cycle for any year of the Christian era may be obtained by adding 9 to the numerical designation of the year, and dividing the sum by 28; the remainder, if not zero, is the solar cycle, and if the remainder is zero the solar cycle is 28.

The dates of Easter Day, according to the Gregorian calendar, are given for the years 1961–2000 in table 14.11. The dates for more extended intervals, and also the dates according to the Julian calendar, are given by Ginzel (*Handbuch der . . . Chronologie*, Leipzig, 1906–1914), and in the *Annuaire* published by the Bureau des Longitudes, Paris.

Table 14.11. Date of Easter Day for the years 1961-2000

1961	April	2	1971	April	11	1981	April	19	1991	March	31
1962	April	22	1972	April	2	1982	April	11	1992	April	19
1963	April	14	1973	April	22	1983	April	3	1993	April	11
1964	March	29	1974	April	14	1984	April	22	1994	April	3
1965	April	18	1975	March	30	1985	April	7	1995	April	16
1966	April	10	1976	April	18	1986	March	30	1996	April	7
1967	March	26	1977	April	10	1987	April	19	1997	March	30
1968	April	14	1978	March	26	1988	April	3	1998	April	12
1969	April	6	1979	April	15	1989	March	26	1999	April	4
1970	March	29	1980	April	6	1990	April	15	2000	April	23

Among the days in the Christian ecclesiastical calendar that are given in *A.E.*, * page 1, Epiphany and Christmas Day are observed on fixed dates of the year. The First Sunday in Advent is the fourth Sunday before Christmas, therefore the Sunday nearest November 30. The others are on fixed days of the week, at fixed intervals before or after Easter Day:

	Days before Easter Day		Days after Easter Day
Septuagesima	63	Rogation Sunday	35
Quinquagesima	49	Ascension Day	39
Ash Wednesday	46	Whit Sunday	49
Palm Sunday	7	Trinity Sunday	56
Good Friday	2	Corpus Christi	60

Clavius, in his treatise on the Gregorian calendar, gives a table of the principal dates in the ecclesiastical calendar for the years from A.D. 1600 to A.D. 5000.

*Page 11 in *A.E.* 1974 onwards.

F. CHRONOLOGICAL ERAS

At the time of the Julian reform of the Roman calendar, it had long been the custom to reckon the calendar years from the legendary founding of Rome (A.U.C.). The most generally accepted date for this event was the one given by Varro, who assigned it to a time which in the Julian proleptic calendar was 753 B.C. With this epoch:

A.D. $(1900 + t)$, Julian calendar $= (2653 + t)$ A.U.C.

Among the other historically important chronological eras, the Seleucid era adopted in western Asia under the Seleucid monarchy was one of the most widely used. It was introduced into many different countries with a variety of different calendars, and consequently the correlation of dates in the Seleucid era is often difficult and uncertain. The most general practice among chronologists is to reckon this era by Julian years, from the epoch originally adopted in the region where the era was first introduced, which was 312 B.C. October 1. With this as the epoch, the year $(2212 + t)$, Seleucid era, begins on A.D. $(1900 + t)$ October 1, Julian calendar; but September 1 is also sometimes used for the beginning of the year.

The Byzantine era, used at Constantinople and elsewhere with a variety of dates for the beginning of the year, was reckoned from 5509 B.C., the supposed year of the Creation. With September 1 as the epoch, the year $(7409 + t)$ of the Byzantine era begins on A.D. $(1900 + t)$ September 1, Julian calendar.

At Alexandria, the era of Diocletian was established at the accession of the emperor Diocletian. The epoch is A.D. 284 August 29, Julian calendar, an Egyptian New Year; the era is reckoned by Alexandrian years. The Alexandrian calendar was a reformed Egyptian calendar, in which a sixth epagomenal, or intercalary, day was added at the end of every fourth year; this intercalary day is on August 29 of the Julian calendar, in the year immediately preceding a Julian leap year. In the Julian calendar, therefore, year $(1617 + t)$ of the Diocletian era begins in A.D. $(1900 + t)$, on August 30 in a year preceding a Julian leap year, and on August 29 in other years.

When the Gregorian calendar was adopted in Japan, the traditional reckoning of the years from the accession of the first human sovereign of Japan was retained; according to the Japanese Chronicles, this event was in 660 B.C. The Gregorian reckoning was begun with 1873 January 1, which was counted as the first day of year 2533 of the Japanese era; the immediately preceding 12th month of the year 2532 in the earlier calendar consisted of only two days. In this chronological system:

A.D. $(1900 + t)$, Gregorian calendar $= (2560 + t)$, Japanese era

The Gregorian calendarial year, with a different date for the beginning of the year, is used in the reformed Indian calendar introduced in 1957, for the reckoning of the native Saka era. The reckoning of the eras in the Jewish and the Moslem calendars is by years peculiar to the individual calendar. The details are given in the paragraphs on these calendars that follow in sub-section G.

In addition to these historical eras, others that are artificial have been introduced from time to time by chronologists. It is possible that the era of Nabonassar, which was extensively used by Ptolemy and other early astronomers for dating observations and constructing astronomical tables, is of this type; it is reckoned from the accession of Nabonassar to the throne of Babylon in 747 B.C., but there is no certain evidence that it was actually used as an historical era. For chronological purposes, the practice has been to reckon this era by Egyptian years from the epoch used by Ptolemy, which in the Julian proleptic calendar was 747 B.C. February 26, an Egyptian New Year. The Julian calendar date of the Egyptian New Year falls back by one day after every four years, completing a retrograde circuit every 1460 Julian years. The first Julian leap year following the epoch was 745 B.C., in which the intercalary day was subsequent to the beginning of the third year of Nabonassar; beginning with 4 Nabonassar, the Julian date of the commencement fell back to 744 B.C. February 25, where it remained for the four years 744–741 B.C. inclusive. From this starting point, the date may easily be traced until its return to February 26 in A.D. 713, at the beginning of 1461 Nabonassar, and then on to the present time. In the four years 1956–1959 inclusive, it fell on April 21 of the Julian calendar; in 1960, year 2709 Nabonassar begins with April 20, Julian calendar, or May 3, Gregorian calendar.

The chronological measure of time introduced in the sixteenth century by Josephus Justus Scaliger, under the name of the Julian period in honour of his father, strictly is a chronological cycle but practically is a continuous era. It is a period of 7980 Julian years, the least common multiple of the 28-year solar cycle, the 19-year lunar cycle, and an ancient non-astronomical cycle of 15 years known as the cycle of the indiction. The epoch is the year when all three cycles began together, which was 4713 B.C., and the reckoning of the period by days is the Julian day number described in section 3B.

The cycle of the indiction is of somewhat uncertain origin, but it became widely established during the fourth century A.D. as a conventional method of designating successive years. It appears to have arisen in connection with tax accounts, and several variants of the cycle were used in different regions. The Roman indiction still given in almanacs is the year of the cycle that is in progress on January 1; the epoch is supposed to be A.D. 312 December 25. The indiction for any year of the Christian era may therefore be found by adding 3 to the year and dividing the sum by 15; the remainder, if not zero, is the Roman indiction, and if the remainder is zero the indiction is 15.

G. OTHER MODERN CALENDARS

1. The Jewish calendar

The ancient Jewish calendar year contained twelve months, each beginning with the first visibility of the crescent Moon as determined by actual observation, and an intercalary month inserted at irregular intervals by repeating the twelfth month. The intercalations were determined by the public authorities, and in the early centuries of the Christian era by the Sanhedrin. The year began with either the spring month Nisan or the autumn month Tishri, according to the country.

This ancient empirical calendar was replaced, probably during the fourth century of the Christian era, by the fixed calendar which is still used. The fixed calendar is regulated by the same 19-year cycle as introduced by the Babylonians about 380 B.C. The calendar year depends upon the beginning of the month of Tishri, which is determined from the time of the mean new moon in the cycle by complicated rules designed to prevent certain solemn days from falling on inconvenient days of the week. As a consequence of these rules, a common year may contain 353, 354, or 355 days, and an embolismic or leap year 383, 384, or 385 days. In each 19-year cycle are 12 common years and 7 embolismic years. Each month has either 29 or 30 days. The complex rules governing the construction of this calendar are explained in detail by L. A. Resnikoff (Jewish calendar calculations. *Scripta Mathematica*, **9**, 191–195, 274–277, 1943).

The years are reckoned according to the era of the Creation, for which the adopted epoch is 3761 B.C. October 7.

The principal days of the Jewish ecclesiastical calendar are on fixed days of the months:

New Year (Rosh Hashanah)	Tishri 1
Day of Atonement (Yom Kippur)	Tishri 10
First Day of Tabernacles (Succoth)	Tishri 15
First Day of Passover (Pesach)	Nisan 15
Feast of Weeks (Shebuoth)	Sivan 6

Sivan 6 is always 50 days after Passover.

Dates in the Jewish calendar during the period before it had become a fixed calendar cannot be converted with certainty to dates in the Julian calendar unless contemporary historical records are extant that contain appropriate information.

References

Spier, A. *Comprehensive Hebrew calendar.* New York, 1952. Contains the corresponding Jewish and Gregorian calendars for the period A.M. 5660–5760 (A.D. 1900–2000).

Gandz, S. (translator). *Code of Maimonides, Book Three, Treatise Eight.* Sanctification of the New Moon. *Yale Judaica Series*, vol. XI, Yale University Press, 1956. With commentaries by Julian Obermann and Otto Neugebauer.

2. The Moslem calendar

The Islamic calendar year consists of 12 lunar months without intercalation; the Moslem New Year consequently makes a circuit of the seasons every 33 years.

This calendar is kept in adjustment with the Moon by a fixed cycle of 30 calendar years. The months have 30 days and 29 days alternately, except the twelfth month, which has 29 days in 19 of the years of the cycle, and 30 days in the other 11 years. There are two forms of the cycle, which give dates differing by one day in 348 of the 360 months of the cycle. The calendar equates 360 lunations to 10631 days; the discrepancy with the actual mean lunation amounts to only about one day after 2500 years.

For religious purposes, the fixed calendar is not used, but instead the beginning of the month is determined by observation of the lunar crescent, or, in localities where this is not practicable, by some method that gives the nearest practicable equivalent. In the religious calendar, the days each begin at sunset, on the evening preceding the civil calendar day.

The years are reckoned from the Hegira, the flight of Mohammed. The epoch of the era of the Hegira is A.D. 622 July 16 Julian calendar; with this epoch, the Gregorian dates of the New Year and of the first day of the principal religious month Ramadân that are given in the Ephemeris are obtained from the form of the 30-year cycle in which the sixteenth year is a leap year.

In the fixed calendar, the first day of Ramadân is the 237th day of the year. However, the day on which it is actually observed depends upon local practice, and may not coincide with the tabular date. Moreover, some Oriental chronologists begin the era of the Hegira with A.D. 622 July 15.

3. Indian calendars

In India, for official civil purposes, the Gregorian calendar has been used. The Islamic calendar is followed by the Moslems for religious purposes. In addition, about 30 different native calendars have been in use, principally for religious purposes. In 1952, a Calendar Reform Committee was appointed to study this calendar confusion, and to recommend a calendar for uniform use in the country.

In accordance with the recommendations of the Committee, a reformed Indian calendar was put in effect by the Government of India on 1957 March 22, an Indian New Year Day. In this calendar, the year begins with 1 Chaitra on March 22 of a Gregorian common year, March 21 of a Gregorian leap year. The years are reckoned according to the native historical Saka era, and

A.D. 1957 March 22 = 1 Chaitra, 1879 Saka.

References

The Indian Ephemeris and Nautical Almanac. Calcutta, Government of India Press.

Government of India. *Report of the Calendar Reform Committee.* Council of Scientific and Industrial Research, New Delhi, 1955.

14.12—CALENDAR FOR COMMON YEARS

Day of Month	January Day of Year	January Fraction of Year	February Day of Year	February Fraction of Year	March Day of Year	March Fraction of Year	April Day of Year	April Fraction of Year	May Day of Year	May Fraction of Year	June Day of Year	June Fraction of Year
0·0	− 1	−0·0027	30	0·0821	58	0·1588	89	0·2437	119	0·3258	150	0·4107
1·0	0	·0000	31	·0849	59	·1615	90	·2464	120	·3285	151	·4134
2·0	+ 1	+ ·0027	32	·0876	60	·1643	91	·2491	121	·3313	152	·4162
3·0	2	·0055	33	·0904	61	·1670	92	·2519	122	·3340	153	·4189
4·0	3	·0082	34	·0931	62	·1698	93	·2546	123	·3368	154	·4216
5·0	4	0·0110	35	0·0958	63	0·1725	94	0·2574	124	0·3395	155	0·4244
6·0	5	·0137	36	·0986	64	·1752	95	·2601	125	·3422	156	·4271
7·0	6	·0164	37	·1013	65	·1780	96	·2628	126	·3450	157	·4299
8·0	7	·0192	38	·1040	66	·1807	97	·2656	127	·3477	158	·4326
9·0	8	·0219	39	·1068	67	·1834	98	·2683	128	·3505	159	·4353
10·0	9	0·0246	40	0·1095	68	0·1862	99	0·2711	129	0·3532	160	0·4381
11·0	10	·0274	41	·1123	69	·1889	100	·2738	130	·3559	161	·4408
12·0	11	·0301	42	·1150	70	·1917	101	·2765	131	·3587	162	·4435
13·0	12	·0329	43	·1177	71	·1944	102	·2793	132	·3614	163	·4463
14·0	13	·0356	44	·1205	72	·1971	103	·2820	133	·3641	164	·4490
15·0	14	0·0383	45	0·1232	73	0·1999	104	0·2847	134	0·3669	165	0·4518
16·0	15	·0411	46	·1259	74	·2026	105	·2875	135	·3696	166	·4545
17·0	16	·0438	47	·1287	75	·2053	106	·2902	136	·3724	167	·4572
18·0	17	·0465	48	·1314	76	·2081	107	·2930	137	·3751	168	·4600
19·0	18	·0493	49	·1342	77	·2108	108	·2957	138	·3778	169	·4627
20·0	19	0·0520	50	0·1369	78	0·2136	109	0·2984	139	0·3806	170	0·4654
21·0	20	·0548	51	·1396	79	·2163	110	·3012	140	·3833	171	·4682
22·0	21	·0575	52	·1424	80	·2190	111	·3039	141	·3860	172	·4709
23·0	22	·0602	53	·1451	81	·2218	112	·3066	142	·3888	173	·4737
24·0	23	·0630	54	·1478	82	·2245	113	·3094	143	·3915	174	·4764
25·0	24	0·0657	55	0·1506	83	0·2272	114	0·3121	144	0·3943	175	0·4791
26·0	25	·0684	56	·1533	84	·2300	115	·3149	145	·3970	176	·4819
27·0	26	·0712	57	·1561	85	·2327	116	·3176	146	·3997	177	·4846
28·0	27	·0739	58	0·1588	86	·2355	117	·3203	147	·4025	178	·4873
29·0	28	·0767			87	·2382	118	·3231	148	·4052	179	·4901
30·0	29	0·0794			88	0·2409	119	0·3258	149	0·4079	180	0·4928
31·0	30	0·0821			89	0·2437			150	0·4107		

LEAP YEARS: For dates after February 28 in leap years the values for the following date must be used.

The Day of the Year and the Fraction of the Year (based on the tropical year of 365·2422 days) are measured from January 1·0.

14.13—BEGINNING OF THE BESSELIAN YEAR, 1900-1949

Year	Begins	Fraction to	Year	Begins	Fraction to	Year	Begins	Fraction to	Year	Begins	Fraction to	Year	Begins	Fraction to
19	Jan.	Jan. 1·0	19	Jan.	Jan. 1·0	19	Jan.	Jan. 1·0	19	Jan.	Jan. 1·0	19	Jan.	Jan. 1·0
00	0·313	+·0019	10	0·735	+·0007	20*	1·157	−·0004	30	1·079	−·0002	40*	1·501	−·0014
01	0·556	+·0012	11	0·978	+·0001	21	0·400	+·0016	31	1·322	−·0009	41	0·744	+·0007
02	0·798	+·0006	12*	1·220	−·0006	22	0·642	+·0010	32*	1·564	−·0015	42	0·986	·0000
03	1·040	−·0001	13	0·462	+·0015	23	0·884	+·0003	33	0·806	+·0005	43	1·228	−·0006
04*	1·282	−·0008	14	0·704	+·0008	24*	1·126	−·0003	34	1·048	−·0001	44*	1·470	−·0013
05	0·524	+·0013	15	0·946	+·0001	25	0·868	+·0004	35	1·290	−·0008	45	0·712	+·0008
06	0·767	+·0006	16*	1·189	−·0005	26	1·111	−·0003	36*	1·533	−·0015	46	0·955	+·0001
07	1·009	·0000	17	0·431	+·0016	27	1·353	−·0010	37	0·775	+·0006	47	1·197	−·0005
08*	1·251	−·0007	18	0·673	+·0009	28*	1·595	−·0016	38	1·017	·0000	48*	1·439	−·0012
09	0·493	+·0014	19	0·915	+·0002	29	0·837	+·0004	39	1·259	−·0007	49	0·681	+·0009

* Leap years—see footnote to table 14.12 for dates after February 28.

Jan. 0·0 denotes Greenwich noon on January 0 for years up to and including 1924, but the preceding midnight (0^h U.T.) from 1925 onwards.

Day of Month	July Day of Year	July Fraction of Year	August Day of Year	August Fraction of Year	September Day of Year	September Fraction of Year	October Day of Year	October Fraction of Year	November Day of Year	November Fraction of Year	December Day of Year	December Fraction of Year
0·0	180	0·4928	211	0·5777	242	0·6626	272	0·7447	303	0·8296	333	0·9117
1·0	181	·4956	212	·5804	243	·6653	273	·7474	304	·8323	334	·9145
2·0	182	·4983	213	·5832	244	·6680	274	·7502	305	·8351	335	·9172
3·0	183	·5010	214	·5859	245	·6708	275	·7529	306	·8378	336	·9199
4·0	184	·5038	215	·5887	246	·6735	276	·7557	307	·8405	337	·9227
5·0	185	0·5065	216	0·5914	247	0·6763	277	0·7584	308	0·8433	338	0·9254
6·0	186	·5093	217	·5941	248	·6790	278	·7611	309	·8460	339	·9282
7·0	187	·5120	218	·5969	249	·6817	279	·7639	310	·8488	340	·9309
8·0	188	·5147	219	·5996	250	·6845	280	·7666	311	·8515	341	·9336
9·0	189	·5175	220	·6023	251	·6872	281	·7694	312	·8542	342	·9364
10·0	190	0·5202	221	0·6051	252	0·6900	282	0·7721	313	0·8570	343	0·9391
11·0	191	·5229	222	·6078	253	·6927	283	·7748	314	·8597	344	·9418
12·0	192	·5257	223	·6106	254	·6954	284	·7776	315	·8624	345	·9446
13·0	193	·5284	224	·6133	255	·6982	285	·7803	316	·8652	346	·9473
14·0	194	·5312	225	·6160	256	·7009	286	·7830	317	·8679	347	·9501
15·0	195	0·5339	226	0·6188	257	0·7036	287	0·7858	318	0·8707	348	0·9528
16·0	196	·5366	227	·6215	258	·7064	288	·7885	319	·8734	349	·9555
17·0	197	·5394	228	·6242	259	·7091	289	·7913	320	·8761	350	·9583
18·0	198	·5421	229	·6270	260	·7119	290	·7940	321	·8789	351	·9610
19·0	199	·5448	230	·6297	261	·7146	291	·7967	322	·8816	352	·9637
20·0	200	0·5476	231	0·6325	262	0·7173	292	0·7995	323	0·8843	353	0·9665
21·0	201	·5503	232	·6352	263	·7201	293	·8022	324	·8871	354	·9692
22·0	202	·5531	233	·6379	264	·7228	294	·8049	325	·8898	355	·9720
23·0	203	·5558	234	·6407	265	·7255	295	·8077	326	·8926	356	·9747
24·0	204	·5585	235	·6434	266	·7283	296	·8104	327	·8953	357	·9774
25·0	205	0·5613	236	0·6461	267	0·7310	297	0·8132	328	0·8980	358	0·9802
26·0	206	·5640	237	·6489	268	·7338	298	·8159	329	·9008	359	·9829
27·0	207	·5667	238	·6516	269	·7365	299	·8186	330	·9035	360	·9856
28·0	208	·5695	239	·6544	270	·7392	300	·8214	331	·9062	361	·9884
29·0	209	·5722	240	·6571	271	·7420	301	·8241	332	·9090	362	·9911
30·0	210	0·5750	241	0·6598	272	0·7447	302	0·8268	333	0·9117	363	0·9939
31·0	211	0·5777	242	0·6626			303	0·8296			364	0·9966

LEAP YEARS: For dates after February 28 in leap years the values for the following date must be used.

The Day of the Year and the Fraction of the Year (based on the tropical year of 365·2422 days) are measured from January 1·0.

14.13—BEGINNING OF THE BESSELIAN YEAR, 1950-1999

Year 19	Begins Jan.	Fraction to Jan. 1·0	Year 19	Begins Jan.	Fraction to Jan. 1·0	Year 19	Begins Jan.	Fraction to Jan. 1·0	Year 19	Begins Jan.	Fraction to Jan. 1·0	Year 19	Begins Jan.	Fraction to Jan. 1·0
50	0·923	+·0002	**60***	1·345	−·0009	**70**	0·767	+·0006	**80***	1·189	−·0005	**90**	0·611	+·0011
51	1·166	−·0005	**61**	0·588	+·0011	**71**	1·010	·0000	**81**	0·432	+·0016	**91**	0·854	+·0004
52*	1·408	−·0011	**62**	0·830	+·0005	**72***	1·252	−·0007	**82**	0·674	+·0009	**92***	1·096	−·0003
53	0·650	+·0010	**63**	1·072	−·0002	**73**	0·494	+·0014	**83**	0·916	+·0002	**93**	0·338	+·0018
54	0·892	+·0003	**64***	1·314	−·0009	**74**	0·736	+·0007	**84***	1·158	−·0004	**94**	0·580	+·0011
55	1·134	−·0004	**65**	0·556	+·0012	**75**	0·978	+·0001	**85**	0·400	+·0016	**95**	0·822	+·0005
56*	1·377	−·0010	**66**	0·799	+·0006	**76***	1·221	−·0006	**86**	0·643	+·0010	**96***	1·065	−·0002
57	0·619	+·0010	**67**	1·041	−·0001	**77**	0·463	+·0015	**87**	0·885	+·0003	**97**	0·307	+·0019
58	0·861	+·0004	**68***	1·283	−·0008	**78**	0·705	+·0008	**88***	1·127	−·0003	**98**	0·549	+·0012
59	1·103	−·0003	**69**	0·525	+·0013	**79**	0·947	+·0001	**89**	0·369	+·0017	**99**	0·791	+·0006

The fraction of the Besselian year to any given following date is obtained by adding the Fraction to Jan. 1·0, from table 14.13, for the year concerned to the Fraction of the Year, from table 14.12, for the date concerned.

H. THE CALENDAR AND OTHER TABLES

The opportunity is taken of collecting together several tables associated with the calendar.

Table 14.12 gives, for a common or non-leap year, the day of the year and the fraction of the year for each day of each month; both are measured from January 1^d 00^h (January $1^d.0$) and the fraction of the year is based on a tropical year of 365·2422 days. For dates after February 28 in leap years the entries for the next *following* date must be used. The day of the year provides a convenient means of calculating the interval in days between two calendar dates when the corresponding Julian day numbers are not known. The fraction of the year can be used in conjunction with table 14.13 to give the fraction of the Besselian year for any year in the twentieth century. (*Warning*. In some publications, the days of the year are numbered starting from 1 on January 1.)

Table 14.13, on the same pages as table 14.12, provides for each year in the twentieth century the beginning of the Besselian (fictitious) solar year (see section 2B) and the fraction of the tropical year *to* January 1^d 00^h (January $1^d.0$) *from* which the fraction of the year in table 14.12 is measured. Thus the fraction of the tropical year, measured from the beginning of the Besselian year, may be obtained by the *addition* of the fractions from the two tables. Leap years are indicated by an asterisk; in these years, for dates after February 28, the following date must be used in table 14.12. The date-system used corresponds to that actually used in the ephemerides under various names: up to 1924 December 31 the system now unambiguously referred to as Greenwich mean astronomical time is used and 00^h represents noon; from 1925 January 1 onwards the system of universal time is used and 00^h represents the preceding midnight (see section 3D). For years before 1925 the fraction of the year in tables 14.12 and 14.13 is measured from (or to) noon on January 1; for years 1925 and onwards it is measured from (or to) the midnight on January 1 (i.e., 12^h earlier).

The three tables 14.14, 14.15, and 14.16 together provide means of ascertaining the Julian day number and the Greenwich sidereal day number for any calendar date from 2000 B.C. to A.D. 2000. The Julian day number is described and defined in sections 3B.1 and 14F, and the Greenwich sidereal day number in section 3B.2. Table 14.14 gives values for January 0 in each centurial year from 2000 B.C. to A.D. 2000, together with the differences between the values in the following century

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The tabulated value of the " Difference from 20th Century " is to be subtracted from the value obtained from table 14.15, or table 14.16, for the corresponding date in the twentieth century. (For example, A.D. 1234 May 5·5 U.T. is J.D. 217 1901·0, i.e., J.D. for 1934 May 5·5 U.T. less 25 5662; May 5·5 U.T., 1234 B.C. is J.D. 127 0829·0, i.e., J.D. for 1967 May 5·5 U.T. less 116 8787.) In the Julian calendar (and in 1600 of the Gregorian calendar) the " Difference " must be increased by 1 for dates before March 1 in the tabular centurial years.

The date of double transit of the first point of Aries varies by one or two days during the century.

Astronomical Year	Julian Day Number at Jan. 0·5 U.T.	Difference from 20th Century	Greenwich Sidereal Day Number	Beginning of G.S.D. (U.T.)	Difference from 20th Century	Date of Double Transit
Julian calendar						
−2000	99 0557	142 4462	99 3270	Jan. 0·771	142 8362	Oct. 9
1900	102 7082	138 7937	102 9895	·769	139 1737	8
1800	106 3607	135 1412	106 6520	·767	135 5112	8
1700	110 0132	131 4887	110 3145	·765	131 8487	7
1600	113 6657	127 8362	113 9770	·763	128 1862	6
−1500	117 3182	124 1837	117 6395	Jan. 0·761	124 5237	Oct. 6
1400	120 9707	120 5312	121 3020	·759	120 8612	5
1300	124 6232	116 8787	124 9645	·757	117 1987	4
1200	128 2757	113 2262	128 6270	·755	113 5362	3
1100	131 9282	109 5737	132 2895	·753	109 8737	3
−1000	135 5807	105 9212	135 9520	Jan. 0·751	106 2112	Oct. 2
900	139 2332	102 2687	139 6145	·749	102 5487	1
800	142 8857	98 6162	143 2770	·747	98 8862	Sept. 30
700	146 5382	94 9637	146 9395	·745	95 2237	29
600	150 1907	91 3112	150 6020	·742	91 5612	29
− 500	153 8432	87 6587	154 2645	Jan. 0·740	87 8987	Sept. 28
400	157 4957	84 0062	157 9270	·738	84 2362	27
300	161 1482	80 3537	161 5895	·736	80 5737	26
200	164 8007	76 7012	165 2520	·734	76 9112	26
− 100	168 4532	73 0487	168 9145	·732	73 2487	25
A.D. 0	172 1057	69 3962	172 5770	Jan. 0·730	69 5862	Sept. 24
100	175 7582	65 7437	176 2395	·728	65 9237	23
200	179 4107	62 0912	179 9020	·726	62 2612	23
300	183 0632	58 4387	183 5645	·724	58 5987	22
400	186 7157	54 7862	187 2270	·722	54 9362	21
500	190 3682	51 1337	190 8895	Jan. 0·719	51 2737	Sept. 20
600	194 0207	47 4812	194 5520	·717	47 6112	19
700	197 6732	43 8287	198 2145	·715	43 9487	19
800	201 3257	40 1762	201 8770	·713	40 2862	18
900	204 9782	36 5237	205 5395	·711	36 6237	17
1000	208 6307	32 8712	209 2020	Jan. 0·709	32 9612	Sept. 16
1100	212 2832	29 2187	212 8645	·707	29 2987	16
1200	215 9357	25 5662	216 5270	·705	25 6362	15
1300	219 5882	21 9137	220 1895	·703	21 9737	14
1400	223 2407	18 2612	223 8520	·700	18 3112	13
1500	226 8932	14 6087	227 5145	Jan. 0·698	14 6487	Sept. 13
1600	230 5457	10 9562	231 1770	·696	10 9862	12
1700	234 1982	7 3037	234 8395	·694	7 3237	11
1800	237 8507	3 6512	238 5020	·692	3 6612	10
1900	241 5032		242 1645	Jan. 0·690		Sept. 10
Gregorian calendar						
1500	226 8923	14 6097	227 5136	Jan. 0·723	14 6497	Sept. 22
1600	230 5447	10 9572	231 1760	·724	10 9872	22
1700	234 1972	7 3048	234 8385	·721	7 3248	21
1800	237 8496	3 6524	238 5009	·722	3 6624	21
1900	241 5020		242 1633	Jan. 0·723		Sept. 22

See footnote opposite.

14.15—JULIAN DAY NUMBER, 1900–1949

OF DAY COMMENCING AT GREENWICH NOON ON:

Year	Jan. 0	Feb. 0	Mar. 0	Apr. 0	May 0	June 0	July 0	Aug. 0	Sept. 0	Oct. 0	Nov. 0	Dec. 0
1900	241 5020	5051	5079	5110	5140	5171	5201	5232	5263	5293	5324	5354
1901	5385	5416	5444	5475	5505	5536	5566	5597	5628	5658	5689	5719
1902	5750	5781	5809	5840	5870	5901	5931	5962	5993	6023	6054	6084
1903	6115	6146	6174	6205	6235	6266	6296	6327	6358	6388	6419	6449
1904	6480	6511	6540	6571	6601	6632	6662	6693	6724	6754	6785	6815
1905	241 6846	6877	6905	6936	6966	6997	7027	7058	7089	7119	7150	7180
1906	7211	7242	7270	7301	7331	7362	7392	7423	7454	7484	7515	7545
1907	7576	7607	7635	7666	7696	7727	7757	7788	7819	7849	7880	7910
1908	7941	7972	8001	8032	8062	8093	8123	8154	8185	8215	8246	8276
1909	8307	8338	8366	8397	8427	8458	8488	8519	8550	8580	8611	8641
1910	241 8672	8703	8731	8762	8792	8823	8853	8884	8915	8945	8976	9006
1911	9037	9068	9096	9127	9157	9188	9218	9249	9280	9310	9341	9371
1912	9402	9433	9462	9493	9523	9554	9584	9615	9646	9676	9707	9737
1913	241 9768	9799	9827	9858	9888	9919	9949	9980	*0011	*0041	*0072	*0102
1914	242 0133	0164	0192	0223	0253	0284	0314	0345	0376	0406	0437	0467
1915	242 0498	0529	0557	0588	0618	0649	0679	0710	0741	0771	0802	0832
1916	0863	0894	0923	0954	0984	1015	1045	1076	1107	1137	1168	1198
1917	1229	1260	1288	1319	1349	1380	1410	1441	1472	1502	1533	1563
1918	1594	1625	1653	1684	1714	1745	1775	1806	1837	1867	1898	1928
1919	1959	1990	2018	2049	2079	2110	2140	2171	2202	2232	2263	2293
1920	242 2324	2355	2384	2415	2445	2476	2506	2537	2568	2598	2629	2659
1921	2690	2721	2749	2780	2810	2841	2871	2902	2933	2963	2994	3024
1922	3055	3086	3114	3145	3175	3206	3236	3267	3298	3328	3359	3389
1923	3420	3451	3479	3510	3540	3571	3601	3632	3663	3693	3724	3754
1924	3785	3816	3845	3876	3906	3937	3967	3998	4029	4059	4090	4120
1925	242 4151	4182	4210	4241	4271	4302	4332	4363	4394	4424	4455	4485
1926	4516	4547	4575	4606	4636	4667	4697	4728	4759	4789	4820	4850
1927	4881	4912	4940	4971	5001	5032	5062	5093	5124	5154	5185	5215
1928	5246	5277	5306	5337	5367	5398	5428	5459	5490	5520	5551	5581
1929	5612	5643	5671	5702	5732	5763	5793	5824	5855	5885	5916	5946
1930	242 5977	6008	6036	6067	6097	6128	6158	6189	6220	6250	6281	6311
1931	6342	6373	6401	6432	6462	6493	6523	6554	6585	6615	6646	6676
1932	6707	6738	6767	6798	6828	6859	6889	6920	6951	6981	7012	7042
1933	7073	7104	7132	7163	7193	7224	7254	7285	7316	7346	7377	7407
1934	7438	7469	7497	7528	7558	7589	7619	7650	7681	7711	7742	7772
1935	242 7803	7834	7862	7893	7923	7954	7984	8015	8046	8076	8107	8137
1936	8168	8199	8228	8259	8289	8320	8350	8381	8412	8442	8473	8503
1937	8534	8565	8593	8624	8654	8685	8715	8746	8777	8807	8838	8868
1938	8899	8930	8958	8989	9019	9050	9080	9111	9142	9172	9203	9233
1939	9264	9295	9323	9354	9384	9415	9445	9476	9507	9537	9568	9598
1940	242 9629	9660	9689	9720	9750	9781	9811	9842	9873	9903	9934	9964
1941	242 9995	*0026	*0054	*0085	*0115	*0146	*0176	*0207	*0238	*0268	*0299	*0329
1942	243 0360	0391	0419	0450	0480	0511	0541	0572	0603	0633	0664	0694
1943	0725	0756	0784	0815	0845	0876	0906	0937	0968	0998	1029	1059
1944	1090	1121	1150	1181	1211	1242	1272	1303	1334	1364	1395	1425
1945	243 1456	1487	1515	1546	1576	1607	1637	1668	1699	1729	1760	1790
1946	1821	1852	1880	1911	1941	1972	2002	2033	2064	2094	2125	2155
1947	2186	2217	2245	2276	2306	2337	2367	2398	2429	2459	2490	2520
1948	2551	2582	2611	2642	2672	2703	2733	2764	2795	2825	2856	2886
1949	243 2917	2948	2976	3007	3037	3068	3098	3129	3160	3190	3221	3251

14.15—JULIAN DAY NUMBER, 1950–1999

OF DAY COMMENCING AT GREENWICH NOON ON:

Year	Jan. 0	Feb. 0	Mar. 0	Apr. 0	May 0	June 0	July 0	Aug. 0	Sept. 0	Oct. 0	Nov. 0	Dec. 0
1950	243 3282	3313	3341	3372	3402	3433	3463	3494	3525	3555	3586	3616
1951	3647	3678	3706	3737	3767	3798	3828	3859	3890	3920	3951	3981
1952	4012	4043	4072	4103	4133	4164	4194	4225	4256	4286	4317	4347
1953	4378	4409	4437	4468	4498	4529	4559	4590	4621	4651	4682	4712
1954	4743	4774	4802	4833	4863	4894	4924	4955	4986	5016	5047	5077
1955	243 5108	5139	5167	5198	5228	5259	5289	5320	5351	5381	5412	5442
1956	5473	5504	5533	5564	5594	5625	5655	5686	5717	5747	5778	5808
1957	5839	5870	5898	5929	5959	5990	6020	6051	6082	6112	6143	6173
1958	6204	6235	6263	6294	6324	6355	6385	6416	6447	6477	6508	6538
1959	6569	6600	6628	6659	6689	6720	6750	6781	6812	6842	6873	6903
1960	243 6934	6965	6994	7025	7055	7086	7116	7147	7178	7208	7239	7269
1961	7300	7331	7359	7390	7420	7451	7481	7512	7543	7573	7604	7634
1962	7665	7696	7724	7755	7785	7816	7846	7877	7908	7938	7969	7999
1963	8030	8061	8089	8120	8150	8181	8211	8242	8273	8303	8334	8364
1964	8395	8426	8455	8486	8516	8547	8577	8608	8639	8669	8700	8730
1965	243 8761	8792	8820	8851	8881	8912	8942	8973	9004	9034	9065	9095
1966	9126	9157	9185	9216	9246	9277	9307	9338	9369	9399	9430	9460
1967	9491	9522	9550	9581	9611	9642	9672	9703	9734	9764	9795	9825
1968	243 9856	9887	9916	9947	9977	*0008	*0038	*0069	*0100	*0130	*0161	*0191
1969	244 0222	0253	0281	0312	0342	0373	0403	0434	0465	0495	0526	0556
1970	244 0587	0618	0646	0677	0707	0738	0768	0799	0830	0860	0891	0921
1971	0952	0983	1011	1042	1072	1103	1133	1164	1195	1225	1256	1286
1972	1317	1348	1377	1408	1438	1469	1499	1530	1561	1591	1622	1652
1973	1683	1714	1742	1773	1803	1834	1864	1895	1926	1956	1987	2017
1974	2048	2079	2107	2138	2168	2199	2229	2260	2291	2321	2352	2382
1975	244 2413	2444	2472	2503	2533	2564	2594	2625	2656	2686	2717	2747
1976	2778	2809	2838	2869	2899	2930	2960	2991	3022	3052	3083	3113
1977	3144	3175	3203	3234	3264	3295	3325	3356	3387	3417	3448	3478
1978	3509	3540	3568	3599	3629	3660	3690	3721	3752	3782	3813	3843
1979	3874	3905	3933	3964	3994	4025	4055	4086	4117	4147	4178	4208
1980	244 4239	4270	4299	4330	4360	4391	4421	4452	4483	4513	4544	4574
1981	4605	4636	4664	4695	4725	4756	4786	4817	4848	4878	4909	4939
1982	4970	5001	5029	5060	5090	5121	5151	5182	5213	5243	5274	5304
1983	5335	5366	5394	5425	5455	5486	5516	5547	5578	5608	5639	5669
1984	5700	5731	5760	5791	5821	5852	5882	5913	5944	5974	6005	6035
1985	244 6066	6097	6125	6156	6186	6217	6247	6278	6309	6339	6370	6400
1986	6431	6462	6490	6521	6551	6582	6612	6643	6674	6704	6735	6765
1987	6796	6827	6855	6886	6916	6947	6977	7008	7039	7069	7100	7130
1988	7161	7192	7221	7252	7282	7313	7343	7374	7405	7435	7466	7496
1989	7527	7558	7586	7617	7647	7678	7708	7739	7770	7800	7831	7861
1990	244 7892	7923	7951	7982	8012	8043	8073	8104	8135	8165	8196	8226
1991	8257	8288	8316	8347	8377	8408	8438	8469	8500	8530	8561	8591
1992	8622	8653	8682	8713	8743	8774	8804	8835	8866	8896	8927	8957
1993	8988	9019	9047	9078	9108	9139	9169	9200	9231	9261	9292	9322
1994	9353	9384	9412	9443	9473	9504	9534	9565	9596	9626	9657	9687
1995	244 9718	9749	9777	9808	9838	9869	9899	9930	9961	9991	*0022	*0052
1996	245 0083	0114	0143	0174	0204	0235	0265	0296	0327	0357	0388	0418
1997	0449	0480	0508	0539	0569	0600	0630	0661	0692	0722	0753	0783
1998	0814	0845	0873	0904	0934	0965	0995	1026	1057	1087	1118	1148
1999	245 1179	1210	1238	1269	1299	1330	1360	1391	1422	1452	1483	1513

14.16—GREENWICH SIDEREAL DAY NUMBER, 1900–1949

OF DAY COMMENCING AT UNIVERSAL TIME:

Year	Jan. 0·73	Feb. 0·64	Mar. 0·57	Apr. 0·48	May 0·40	June 0·32	July 0·23	Aug. 0·15	Sept.† 0·07	Oct. 0·98	Nov. 0·90	Dec. 0·81
1900	242 1633	1664	1692	1723	1753	1784	1814	1845	1876	1907	1938	1968
1901	1999	2030	2058	2089	2119	2150	2180	2211	2242	2273	2304	2334
1902	2365	2396	2424	2455	2485	2516	2546	2577	2608	2639	2670	2700
1903	2731	2762	2790	2821	2851	2882	2912	2943	2974	3005	3036	3066
1904	3097	3128	3157	3188	3218	3249	3279	3310	3341	3372	3403	3433
1905	242 3464	3495	3523	3554	3584	3615	3645	3676	3707	3738	3769	3799
1906	3830	3861	3889	3920	3950	3981	4011	4042	4073	4104	4135	4165
1907	4196	4227	4255	4286	4316	4347	4377	4408	4439	4470	4501	4531
1908	4562	4593	4622	4653	4683	4714	4744	4775	4806	4837	4868	4898
1909	4929	4960	4988	5019	5049	5080	5110	5141	5172	5203	5234	5264
1910	242 5295	5326	5354	5385	5415	5446	5476	5507	5538	5569	5600	5630
1911	5661	5692	5720	5751	5781	5812	5842	5873	5904	5935	5966	5996
1912	6027	6058	6087	6118	6148	6179	6209	6240	6271	6302	6333	6363
1913	6394	6425	6453	6484	6514	6545	6575	6606	6637	6668	6699	6729
1914	6760	6791	6819	6850	6880	6911	6941	6972	7003	7034	7065	7095
1915	242 7126	7157	7185	7216	7246	7277	7307	7338	7369	7400	7431	7461
1916	7492	7523	7552	7583	7613	7644	7674	7705	7736	7767	7798	7828
1917	7859	7890	7918	7949	7979	8010	8040	8071	8102	8133	8164	8194
1918	8225	8256	8284	8315	8345	8376	8406	8437	8468	8499	8530	8560
1919	8591	8622	8650	8681	8711	8742	8772	8803	8834	8865	8896	8926
1920	242 8957	8988	9017	9048	9078	9109	9139	9170	9201	9232	9263	9293
1921	9324	9355	9383	9414	9444	9475	9505	9536	9567	9598	9629	9659
1922	242 9690	9721	9749	9780	9810	9841	9871	9902	9933	9964	9995	*0025
1923	243 0056	0087	0115	0146	0176	0207	0237	0268	0299	0330	0361	0391
1924	0422	0453	0482	0513	0543	0574	0604	0635	0666	0697	0728	0758
1925	243 0789	0820	0848	0879	0909	0940	0970	1001	1032	1063	1094	1124
1926	1155	1186	1214	1245	1275	1306	1336	1367	1398	1429	1460	1490
1927	1521	1552	1580	1611	1641	1672	1702	1733	1764	1795	1826	1856
1928	1887	1918	1947	1978	2008	2039	2069	2100	2131	2162	2193	2223
1929	2254	2285	2313	2344	2374	2405	2435	2466	2497†	2528	2559	2589
1930	243 2620	2651	2679	2710	2740	2771	2801	2832	2863	2894	2925	2955
1931	2986	3017	3045	3076	3106	3137	3167	3198	3229	3260	3291	3321
1932	3352	3383	3412	3443	3473	3504	3534	3565	3596	3627	3658	3688
1933	3719	3750	3778	3809	3839	3870	3900	3931	3962	3993	4024	4054
1934	4085	4116	4144	4175	4205	4236	4266	4297	4328	4359	4390	4420
1935	243 4451	4482	4510	4541	4571	4602	4632	4663	4694	4725	4756	4786
1936	4817	4848	4877	4908	4938	4969	4999	5030	5061	5092	5123	5153
1937	5184	5215	5243	5274	5304	5335	5365	5396	5427	5458	5489	5519
1938	5550	5581	5609	5640	5670	5701	5731	5762	5793	5824	5855	5885
1939	5916	5947	5975	6006	6036	6067	6097	6128	6159	6190	6221	6251
1940	243 6282	6313	6342	6373	6403	6434	6464	6495	6526	6557	6588	6618
1941	6649	6680	6708	6739	6769	6800	6830	6861	6892	6923	6954	6984
1942	7015	7046	7074	7105	7135	7166	7196	7227	7258	7289	7320	7350
1943	7381	7412	7440	7471	7501	7532	7562	7593	7624	7655	7686	7716
1944	7747	7778	7807	7838	7868	7899	7929	7960	7991	8022	8053	8083
1945	243 8114	8145	8173	8204	8234	8265	8295	8326	8357	8388	8419	8449
1946	8480	8511	8539	8570	8600	8631	8661	8692	8723	8754	8785	8815
1947	8846	8877	8905	8936	8966	8997	9027	9058	9089	9120	9151	9181
1948	9212	9243	9272	9303	9333	9364	9394	9425	9456	9487	9518	9548
1949	243 9579	9610	9638	9669	9699	9730	9760	9791	9822	9853	9884	9914

† There are two transits of the first point of Aries on one mean solar day near September 22.

14.16—GREENWICH SIDEREAL DAY NUMBER, 1950–1999

OF DAY COMMENCING AT UNIVERSAL TIME:

Year	Jan. 0·73	Feb. 0·64	Mar. 0·57	Apr. 0·48	May 0·40	June 0·32	July 0·23	Aug. 0·15	Sept.† 0·07	Oct. 0·98	Nov. 0·90	Dec. 0·81
1950	243 9945	9976	*0004	*0035	*0065	*0096	*0126	*0157	*0188	*0219	*0250	*0280
1951	244 0311	0342	0370	0401	0431	0462	0492	0523	0554	0585	0616	0646
1952	0677	0708	0737	0768	0798	0829	0859	0890	0921	0952	0983	1013
1953	1044	1075	1103	1134	1164	1195	1225	1256	1287	1318	1349	1379
1954	1410	1441	1469	1500	1530	1561	1591	1622	1653	1684	1715	1745
1955	244 1776	1807	1835	1866	1896	1927	1957	1988	2019	2050	2081	2111
1956	2142	2173	2202	2233	2263	2294	2324	2355	2386	2417	2448	2478
1957	2509	2540	2568	2599	2629	2660	2690	2721	2752	2783	2814	2844
1958	2875	2906	2934	2965	2995	3026	3056	3087	3118	3149	3180	3210
1959	3241	3272	3300	3331	3361	3392	3422	3453	3484	3515	3546	3576
1960	244 3607	3638	3667	3698	3728	3759	3789	3820	3851	3882	3913	3943
1961	3974	4005	4033	4064	4094	4125	4155	4186	4217	4248	4279	4309
1962	4340	4371	4399	4430	4460	4491	4521	4552	4583	4614	4645	4675
1963	4706	4737	4765	4796	4826	4857	4887	4918	4949	4980	5011	5041
1964	5072	5103	5132	5163	5193	5224	5254	5285	5316	5347	5378	5408
1965	244 5439	5470	5498	5529	5559	5590	5620	5651	5682	5713	5744	5774
1966	5805	5836	5864	5895	5925	5956	5986	6017	6048	6079	6110	6140
1967	6171	6202	6230	6261	6291	6322	6352	6383	6414	6445	6476	6506
1968	6537	6568	6597	6628	6658	6689	6719	6750	6781	6812	6843	6873
1969	6904	6935	6963	6994	7024	7055	7085	7116	7147	7178	7209	7239
1970	244 7270	7301	7329	7360	7390	7421	7451	7482	7513	7544	7575	7605
1971	7636	7667	7695	7726	7756	7787	7817	7848	7879	7910	7941	7971
1972	8002	8033	8062	8093	8123	8154	8184	8215	8246	8277	8308	8338
1973	8369	8400	8428	8459	8489	8520	8550	8581	8612	8643	8674	8704
1974	8735	8766	8794	8825	8855	8886	8916	8947	8978	9009	9040	9070
1975	244 9101	9132	9160	9191	9221	9252	9282	9313	9344	9375	9406	9436
1976	9467	9498	9527	9558	9588	9619	9649	9680	9711	9742	9773	9803
1977	244 9834	9865	9893	9924	9954	9985	*0015	*0046	*0077	*0108	*0139	*0169
1978	245 0200	0231	0259	0290	0320	0351	0381	0412	0443	0474	0505	0535
1979	0566	0597	0625	0656	0686	0717	0747	0778	0809	0840	0871	0901
1980	245 0932	0963	0992	1023	1053	1084	1114	1145	1176	1207	1238	1268
1981	1299	1330	1358	1389	1419	1450	1480	1511	1542	1573	1604	1634
1982	1665	1696	1724	1755	1785	1816	1846	1877	1908	1939	1970	2000
1983	2031	2062	2090	2121	2151	2182	2212	2243	2274	2305	2336	2366
1984	2397	2428	2457	2488	2518	2549	2579	2610	2641	2672	2703	2733
1985	245 2764	2795	2823	2854	2884	2915	2945	2976	3007	3038	3069	3099
1986	3130	3161	3189	3220	3250	3281	3311	3342	3373	3404	3435	3465
1987	3496	3527	3555	3586	3616	3647	3677	3708	3739	3770	3801	3831
1988	3862	3893	3922	3953	3983	4014	4044	4075	4106	4137	4168	4198
1989	4229	4260	4288	4319	4349	4380	4410	4441	4472	4503	4534	4564
1990	245 4595	4626	4654	4685	4715	4746	4776	4807	4838	4869	4900	4930
1991	4961	4992	5020	5051	5081	5112	5142	5173	5204	5235	5266	5296
1992	5327	5358	5387	5418	5448	5479	5509	5540	5571	5602	5633	5663
1993	5694	5725	5753	5784	5814	5845	5875	5906	5937	5968	5999	6029
1994	6060	6091	6119	6150	6180	6211	6241	6272	6303	6334	6365	6395
1995	245 6426	6457	6485	6516	6546	6577	6607	6638	6669	6700	6731	6761
1996	6792	6823	6852	6883	6913	6944	6974	7005	7036	7067	7098	7128
1997	7159	7190	7218	7249	7279	7310	7340	7371	7402	7433	7464	7494
1998	7525	7556	7584	7615	7645	7676	7706	7737	7768	7799	7830	7860
1999	245 7891	7922	7950	7981	8011	8042	8072	8103	8134	8165	8196	8226

† There are two transits of the first point of Aries on one mean solar day near September 22.

and those in the twentieth century. Tables 14.15 and 14.16 give, respectively, the Julian day number and the Greenwich sidereal day number for the beginning of each month of the twentieth century. The Julian day number and the Greenwich sidereal day number for any calendar date may therefore be obtained by the addition of three numbers: the difference for the centurial year in table 14.14; the value for the corresponding month and year in the twentieth century in table 14.15 or 14.16; and the day of the month, allowing for double transit for the Greenwich sidereal day number. The date of double transit (on which there are two transits of the vernal equinox in one mean solar day) may differ by one or two days from the tabulated value during the course of the century.

The main purpose of the Julian day number and the Greenwich sidereal day number is to facilitate enumeration of days and of revolutions of the equinox. The Julian date and the Greenwich sidereal date are extensions of the same concepts; the corresponding decimal parts of these dates are best taken from the ephemerides * of universal and sidereal times in *A.E.*, pages 10–17, or calculated in a similar way. To an approximation adequate to give the corresponding day numbers:

$$\text{Greenwich sidereal date} = +0{\cdot}671 + 1{\cdot}00273\,79093 \times \text{Julian date}$$

$$\text{Julian date} = -0{\cdot}669 + 0{\cdot}99726\,95664 \times \text{Greenwich sidereal date}$$

The calendar explicitly refers to universal time and, in particular, the relationship with Greenwich sidereal date above is only true for Julian dates expressed in universal time. Dates and times in ephemeris time, expressed by the Julian ephemeris date, can only be obtained by applying ΔT to the date and time in universal time, expressed as the Julian date.

*Pages 12 to 19 in *A.E.* 1972 onwards.

15. THE DISTRIBUTION OF TIME*

A. RADIO TIME SIGNALS

The determination of time and the maintenance of a national standard of time is an essential duty of most national observatories. Astronomical observations made with a conventional transit instrument, an astrolabe, or a photographic zenith tube, are used to assess the performance of standard clocks and thus to establish a uniform time system (U.T.2.). Time so determined is then made available by means of radio time signals controlled, either directly or indirectly, by the observatory concerned and is used, not only for the general convenience of the community, but also in scientific and other applications in which the highest precision is demanded.

An adequate standard of accuracy for domestic and general requirements is usually most conveniently achieved by the radiation at selected times of a brief simple time signal on a national or regional broadcasting network. For example, the signal used in the United Kingdom, which is controlled by the Royal Greenwich Observatory and radiated in the programmes of the British Broadcasting Corporation, consists of " six pips " marking the last six seconds of each quarter-hour.

For purposes in which higher accuracy is required, most countries employ special transmissions from radio telephone or telegraph stations. Various standard forms of time signal are employed. In general they consist of a five-minute series of timing dots or dashes conforming to a standard sequence. The minutes are usually identified either by lengthening the dot which occurs at the exact minute, or by omitting some of the dots which precede the minute. For the convenience of users who require the maximum accuracy attainable without elaborate measuring equipment, some authorities transmit " rhythmic " signals which comprise a five-minute series of dots spaced at 61 to the minute, thus forming a time vernier: accurate comparisons between a rhythmic time signal and a clock or chronometer beating seconds or half-seconds may be made by observing the times of the coincidences. Full descriptions of the various forms of signal employed may be found in the national publications (e.g., for the U.K. in *The Admiralty List of Radio Signals*, volume V, published by the Hydrographic Department, Admiralty, London) or may be obtained from the observatories concerned. Details are also given in the *List of Special Service Stations*, published by the International Telecommunication Union.

*There have been many changes in the distribution of time since 1960. The principal changes are noted on pages vi, 95 and 453.

In recent years there has been an increase in the number of radio stations providing standard frequency transmissions with superposed seconds pulses which are adjusted within specified limits of U.T.2., and may thus be used as time signals. At the International Telecommunication and Radio Conference, Atlantic City, 1947, these transmissions were allocated the frequencies $2\frac{1}{2}$, 5, 10, 15, 20, and 25 Mc/s. The seconds pulses are usually of the form of five cycles of a 1,000 c/s tone locked to the standard radio frequency. This form of time signal is well suited for the accurate measurement of reception times. For technical reasons some standard frequency transmissions have also been established at low frequencies. Details of these services are given in the *Lists* that are referred to above. Coordination on a national basis between the conventional time signals and the signals superposed on standard frequency transmissions is clearly desirable and coordination between the services in different countries possesses many advantages. Such coordination has been arranged in the U.S.A. and the U.K., the services being based on astronomical observations made in the two countries.

Particularly on the higher frequencies the accuracy that can be obtained is severely restricted by unknown variations in the travel time of the radio signal, and the received signal may contain in varying quantity signals that have traversed different transmission paths. The seconds dots are thus liable to distortion, which may displace the reference point for measurement, and even a well-defined dot may have arrived by an unusual path containing an abnormal number of " hops ". Reception conditions may vary widely between different reception sites and, at any one site, there may be a considerable diurnal variation. The best result is generally achieved by making measures on transmissions at different times throughout the day and taking suitably weighted means. Anomalies are particularly troublesome when sunrise or sunset occurs on the transmission path, and these times should be avoided if possible. The most serious discordances may occur when the reception site is within the skip area of the transmitter as the predominating signal may be received by back-scatter from a distant point, either on the ground or in the ionosphere.

Time signals radiated by transmitters operating on very low frequencies are much less subject to variations in travel time, but the signal dots themselves rise slowly to their maximum amplitude and it is difficult to measure with sufficient accuracy the time of commencement of the dots. To minimise this difficulty the reference point for measurement is often taken as the point at which the dot rises to a specified fraction of its full amplitude. A memorandum on the techniques of the reception and measurement of radio time signals was prepared for the International Geophysical Year by a sub-committee under the chairmanship of B. Decaux. (" Instructions pour l'emploi de la radioélectricité dans l'opération mondiale des longitudes et des latitudes ". *Annals of the I.G.Y.*, **4**, part III, 155–194, 1957.)

The major observatories receive and record radio time signals from other countries in addition to their own and publish at regular intervals tabulated results showing the times of reception of the various signals in terms of the adopted U.T.2

at the observatory. Results are also communicated to the Bureau International de l'Heure at the Paris Observatory, and comprehensive tables are published in *Bulletin Horaire* giving the reception of time signals and the adopted U.T.2. of each participating observatory in terms of a time system (l'Heure Définitive) based on a mean of the results received. At the end of each year the results are analysed to determine corrections to adopted mean longitudes, errors of radio reception and measurement, and criteria for the assessment of the long- and short-term stability of the time systems of the various observatories.

Basis of radio time signals. The time system upon which the control of radio time signals is based and to which the times of reception are referred in the published corrections is required to be as uniform as possible. The performance of the standard clocks employed in a time service cannot be satisfactorily assessed or extrapolated on the basis of uncorrected astronomical observations. The irregularities in observed time may, for this purpose, be conveniently classified as arising from (i) the seasonal fluctuation in the rotation of the Earth, (ii) changes in the longitude of the observing station due to the polar motion, and (iii) irregular changes in the rotation of the Earth. The seasonal fluctuation is not strictly repetitive, but the effect on time observations may be largely compensated by assuming for the current year an amplitude and phase based on that observed during the preceding period. The instantaneous coordinates of the pole may be provisionally estimated on the basis of worldwide observations of latitude variation, and the corresponding longitude corrections may be calculated. Over short periods the motion may be regarded as periodic and extrapolated corrections used. The irregular variations are, by their very nature, unpredictable, and their effects cannot be satisfactorily estimated on the basis of current astronomical observations alone.

In all work involving intercomparisons between observatories it is convenient to apply to each observing station corrections derived from the same basis of extrapolation. For this reason the Bureau International de l'Heure was instructed by the International Astronomical Union (Dublin, 1955) to adopt and publish in advance each year corrections for the seasonal fluctuation, and all observatories were requested to employ these values in the derivation of *Universal Time* (U.T.). Corrections for the years 1956–1960 have been calculated from the formula:

$$+0^s{\cdot}022 \sin 2\pi t - 0^s{\cdot}017 \cos 2\pi t - 0^s{\cdot}007 \sin 4\pi t + 0^s{\cdot}006 \cos 4\pi t$$

where t is the fraction of the year and equal to zero on January 1. The Bureau International de l'Heure was also instructed to compute and distribute extrapolated longitude corrections based on provisional coordinates of the instantaneous pole. In order to ensure that the best possible data are available with the least possible delay, some seventeen observatories transmit latitude observations weekly to the Central Bureau of the International Latitude Service, and also to the Bureau International de l'Heure where provisional coordinates are computed (Rapid Latitude Service) and communicated to the participating observatories.

Thus at all the cooperating observatories the various systems of U.T. are derived in a strictly comparable manner. The uncorrected observations define U.T.0.; U.T.1 is obtained from U.T.0. by applying the provisional corrections of the Bureau International de l'Heure for the effect of polar motion; U.T.2 is obtained by applying in addition the predicted corrections for the seasonal fluctuation also adopted by the Bureau International de l'Heure. There is one minor source of divergence in that different astronomical observing instruments necessarily employ slightly differing adopted star places in the reduction of the observations. The International Astronomical Union (Zürich, 1948) recommended that all observatories should employ for time observations star places based on the FK3 system. This was the best catalogue then available, and a measure of uniformity was assured. Experience soon showed, however, that the periodic errors in right ascension were significant. Observations made with transit instruments of conventional design may thus give rise to spurious components in the observed seasonal fluctuation. This effect will be greatly reduced with the introduction of the FK4 system. But many observatories now employ instruments such as the P.Z.T. and Danjon astrolabe (see sub-section B), and, in order to achieve a standard of uniformity consonant with modern needs, the star places used are determined in relation to each other by means of observations extending over a period of a year or more. Thus the star places are mutually consistent, and almost entirely free from periodic errors in right ascension. The FK3 system has thus been abandoned, except that the zero of the system is tied to the average of the FK3 stars in the corresponding declination belt.

It will be clear from the manner in which U.T.2 is determined that it represents a provisional delineation of a time system free from periodic and quasi-periodic variation, but still subject to an irregular wandering arising from irregular changes in the rotation of the Earth. A " final " U.T.2 would employ definitive corrections for the seasonal fluctuation and for the effect of the polar motion, but would still show persistent deviations from a uniform time system such as ephemeris time. The general long-term trends of the divergences of U.T.2 from E.T. are indicated by comparisons between the lunar ephemeris (in which the argument is E.T.) and observations of the position of the Moon at specified times (U.T.2) as made with the transit circle or Moon camera, or derived from occultations. The short-term divergences may be studied by comparing the rate of U.T.2 with the frequency of an atomic or molecular standard. It was possible to show in this way a steady decrease in the rate of rotation of the Earth, lengthening the day by 0·5 milliseconds per year, throughout 1956 and 1957. This divergence is of considerable practical interest, since it is customary for the frequency of a standard frequency transmission to be maintained within specified limits of a constant value as indicated by an atomic or molecular standard, and to keep the seconds pulses in phase with this frequency except for occasional step adjustments to retain the seconds pulses within specified limits of U.T.2.

For operational work the current practice is to define the second of the atomic time scale as 9 192 631 770 cycles of the caesium resonance. This value has been chosen in order to bring the second of the atomic time scale into agreement with the recent determination of the second of ephemeris time which is the internationally adopted unit of time. Thus the nominal carrier frequencies of the transmission may be expressed as cycles per second of the atomic time scale. The frequency of the master oscillator, and thus the actual carrier frequencies of the transmissions, is offset by an amount S from its nominal value, so that the rate defined by the timing pulses is in general agreement with U.T.2. If the nominal frequency is F_0, the actual frequency in terms of the atomic time scale may be expressed as:

$$F = F_0 (1 + S)$$

The interval between successive time pulses is thus $(1 - S)$ seconds of the adopted time scale; in other words the time interval corresponds to $9\ 192\ 631\ 770\ (1 - S)$ cycles of the caesium resonance. For each year the adopted value of S is based upon comparisons between U.T.2 and atomic time during the preceding twelve months. The transmissions operate as closely as possible to F with the adopted value of S. Owing to unpredictable variations of U.T.2 the value of S based on the preceding year may not fit for the current year, thus giving rise to a divergence between the radio time signals and U.T.2. In order to retain satisfactory agreement between the radio time signals and U.T.2, a step adjustment of the phase of the time signals, usually by an integral number of milliseconds, is made as and when necessary and at pre-announced times.

It has been agreed to coordinate the time and frequency transmissions of the United Kingdom and the United States of America. The master oscillators controlling the transmissions are calibrated in terms of the caesium standards in the two countries, and are offset by an amount S as described above. The value of S adopted for 1960 and 1961 was -150×10^{-10}. The astronomical observations of the Royal Greenwich Observatory and of the U.S. Naval Observatory are used both in the determination of the value of S and in deciding when a 50-millisecond step correction is necessary; corrections are made simultaneously on all transmissions.

B. INSTRUMENTS AND EQUIPMENT

The transit circle is designed for the determination of fundamental, not relative, positions; that is to say, the observations refer the positions of the stars, Sun, Moon, and planets to the fundamental reference frame defined by the celestial equator and the equinox. The position of the celestial pole may be determined by observations of circumpolar stars at their transits above and below the pole, and thus the plane of the celestial equator may be established. By observations of the Sun it is possible to fix the position of the ecliptic among the stars and thence to determine the equinox. Transit circle observations thus form the basis for the preparation of fundamental star catalogues and the ephemerides of the Sun, Moon, and planets.

The same instrument may be used for time determination by comparing the clock times of transit of selected stars with the tabulated times of transit, due allowance being made for errors of collimation, level, and azimuth. It is more convenient, however, to use a smaller instrument which may be reversed in its bearings at each observation, thus eliminating such errors as collimation, and those arising from inequality of the pivots or lack of uniformity of the micrometer screw. Although the small instrument is inherently less stable, the drift in level and azimuth may be regarded as uniform throughout the short period required for the observation of the necessary 10 or 12 stars; the level is usually determined with a striding or hanging level mounted on the pivots, and the azimuth error is derived from observations of stars both north and south of the zenith. Because of the convenience and simplicity of this type of instrument, various modifications have been tried in order to achieve the highest possible accuracy. Considerable attention has been paid to the accuracy of the figure of the pivots, optical methods have been devised for the determination of the level, and methods have been developed for the photo-electric registration of the star transits. The external probable errors for one night for a time determination have been estimated as ±18 msecs for a small transit instrument with an impersonal micrometer, and ±12 msecs when photo-electric registration is employed. (*Trans. I.A.U.*, **10**, 488, 1960. Comm. 31, Moscow, 1958.).

The photographic zenith tube (P.Z.T.) is now used in some nine observatories for the determination of latitude variation and time. An important feature of this type of instrument is that the standard direction is not dependent on the accurate location of the instrument in its bearings, but is automatically defined by reflection of the stellar beam by the horizontal surface of a mercury pool at the base of the instrument. The observations are unaffected by any errors that are likely to occur in practice in the level and location of the reversible part of the instrument. Restriction of observations to stars that transit near the zenith minimises the errors due to atmospheric refraction, but limits the observing catalogue to some 100 stars whose positions may not be well known initially. Observations with the P.Z.T. provide a means of determining the relative positions of the stars, but these must be related by transit circle observations to the reference stars of the fundamental catalogue. Owing to precession, there will be a slow drift of stars from the visible belt, which must be compensated by the inclusion of different stars. The estimated external probable error of a typical P.Z.T. is ±4 msecs.

A similar accuracy is achieved with the Danjon astrolabe, which is a compact and easily transportable instrument for the determination of latitude variation and time. Stars are observed, not at meridian transit, but as they cross the almucantar at a zenith distance of 30°. The stellar beam is divided into two, one being viewed directly and the other by reflection from the horizontal surface of a mercury pool. In the field of view, two images of the star are seen; as the star approaches the almucantar, the two images converge, becoming coincident at the exact instant of transit. A Wollaston prism is placed in the convergent beam, and may be moved along the optical axis by means of a screw in such a way as to keep the two images coincident and to render the emergent beam parallel, thus providing an efficient impersonal micrometer. The optical path in the instrument is "folded" to secure compactness and convenience in use.

In most of the major time-keeping observatories pendulum clocks have now been entirely superseded by quartz clocks. The essential controlling element of a quartz clock is a bar, ring, or plate of quartz which is maintained in mechanical vibration in an electronic oscillator circuit. The frequency of oscillation is usually

100 kc/s or in the vicinity of 5 Mc/s; intercomparisons between standards are usually made at the oscillator frequency. Electronic dividers, or a combination of electronic and electro-mechanical devices, may be used to derive from the oscillator an accurately phase-locked electrical output of one pulse per second. In a nominal 100 kc/s standard, the frequency of oscillation will not be exactly 100 kc/s and, if the division ratio is exactly 100000 / 1, the clock provided by the output seconds pulses will exhibit a corresponding gaining or losing rate. It is moreover a general characteristic of quartz-crystal oscillators that there is a slight drift in the frequency, more rapid in the early stages but settling to an increase of the order of a few parts in 10^8 per year. The frequency error and drift (which correspond in time to rate and a uniform change of rate respectively) must be determined over a period. The adopted values are then used to compute an ephemeris of clock error, and subsequent astronomical observations serve to indicate departures from the ephemeris performance.

Various forms of atomic and molecular clocks are now coming into use. Instead of being controlled by the mechanical vibrations of a quartz crystal, the frequency of the electronic oscillator is compared or brought into average coincidence with the frequency characteristic of the absorption or emission when the atoms or molecules change between two selected energy levels. The first molecular standards were based on the absorption of a radio-frequency signal in passing through a wave-guide filled with ammonia gas at low pressure; maximum absorption occurs when the frequency corresponds to that of the $J = 3$, $K = 4$ line of the inversion spectrum of ammonia (23 870 Mc/s). Higher accuracies have since been achieved with an ammonia oscillator, of the " maser " type. (Microwave Amplification by Stimulated Emission of Radiation). A beam of ammonia molecules traverses a magnetic field which disperses the molecules in the low-energy absorbing state, and directs the high-energy molecules into a cavity resonator tuned to the appropriate frequency. The energy emitted by the molecules is reinforced and strong oscillations are maintained. An alternative form of standard, one of which has been running in England since June 1955, uses a hyperfine splitting line of caesium (9 192 M/cs). A beam of caesium atoms traverses a radio-frequency field: the low-energy state atoms absorb energy and undergo transition to the high-energy state. The change of energy level is accompanied by a reversal of the magnetic moment of the atom, and, by magnetic focusing, atoms in one state may be dispersed and atoms in the other state directed to a detector. A commercial standard operating on this principle has since been made in the U.S.A. Present developments include the use of a longer beam tube and the investigation of alternative resonances.

The stability of an oscillator employing a quartz crystal is of the order of 1 part in 10^{10} from day to day (0·01 msec/day) and of a few parts in 10^{10} from month to month. A caesium standard may be used to define frequency in terms of a spectral line with an accuracy of ± 2 parts in 10^{10} and the maser ammonia standard to ± 1 in 10^9, though the reproducibility of a carefully defined maser is considerably better.

C. LIST OF RADIO TIME SIGNALS

This short, illustrative list of radio time signals contains information on time signals that are widely used and are controlled by observatories communicating their results to the Bureau International de l'Heure. Since transmission times and frequencies are liable to change current schedules should be consulted to obtain up-to-date information.

The International Astronomical Union (Dublin, 1955) has recommended the cessation of ONOGO and rhythmic type signals; details of such signals have therefore been excluded from this list.

Country	Authority	U.T.	Call Sign	Frequency kc/s	Notes
		h m			
Argentine	Naval Observatory, Buenos Aires	01 00 13 00 21 00	LOL	8 110 17 180	
	Military Geographical Institute, Buenos Aires	10 05 22 05	LQC	17 550	
Australia	Mount Stromlo Observatory, Canberra	08 00 14 00 20 00	VHP VIX	44 6 428·5 8 478 12 907·5	
Brazil	National Observatory, Rio de Janeiro	00 30 13 30 20 30	PPE	8 721	
		01 30 14 30 21 30	PPR	6 421 8 634 17 194	
Canada	Dominion Observatory, Ottawa		CHU	3 330 7 335 14 670	Continuous transmissions.
China	Zi-Ka-Wei Observatory, Shanghai	11 00 13 00 15 00	BPV	9 368	
France	Observatory of Paris	08 00 09 00 09 30 13 00 20 00 21 00 22 30	FYP	91·15	
		09 00 21 00	FYA3	7 428	
		08 00 20 00	TQC9	10 775	
		09 30 13 00 22 30	TQG5	13 873	

Country	Authority	U.T.	Call Sign	Frequency kc/s	Notes
		h m			
Germany (Federal German Republic)	German Hydrographic Institute, Hamburg	00 00	DAM	4 265 6 475·5 8 638·5	
		12 00	DAM	8 638·5 16 980	
		11 00	DMR20 DMR27	3 970 6 075	
		08 10 11 10	DCF77	77·5	
Germany (German Democratic Republic)	Geodetic Institute, Potsdam		DIZ	4 525	Continuous transmissions.
Japan	Astronomical Observatory, Tokyo	12 30	JAS22	16 170	
Switzerland	Cantonal Observatory, Neuchâtel	08 15	HBB	96·05	
Union of Soviet Socialist Republics	Central Scientific Investigation Institute, Moscow	00 00 04 00 08 00 12 00 16 00 20 00	ROR	25	
		08 00 to 22 00	RWM	5 000 10 000 15 000 20 000	Signals are transmitted on one or more of these frequencies at intervals of 2^h.
	Astronomical Observatory, Tashkent	18 00	RPT	5 890 11 580	
United Kingdom	Royal Greenwich Observatory, Herstmonceux	10 00 18 00	GBR GBZ	16·0 19·6	GBZ used as a reserve transmitter for GBR.
		10 00 18 00	GIC27 GIC29 GIC33 GIC37 GPB30 GKU5	7 397·5 9 350 13 555 17 685 10 332·5 12 790	Signals are transmitted on two of these frequencies at the times quoted.
United States of America	United States Naval Observatory, Washington	00 00 02 00 06 00 08 00 12 00 14 00 18 00 20 00	NSS	121·95 5 870 9 425 13 575 17 050 23 650	162 kc/s replaces 121·95 on transmissions at $18^h\ 00^m$ and $20^h\ 00^m$ on Tuesday, Wednesday, and Thursday. Transmissions are on all frequencies at the times quoted.
			NBA	18	Continuous transmissions except between $13^h\ 00^m$ and $21^h\ 00^m$ U.T. on Wednesday.

D. LIST OF STANDARD FREQUENCY TRANSMISSIONS

Allocated Frequencies

This list contains information relating to standard frequency transmissions with a wide coverage and an extended period of continuous transmission. All transmissions are interrupted periodically at specified times for identification purposes.

Country	Authority	Call Sign	Frequency Mc/s	Transmission Times (U.T.)
Argentine	Naval Observatory, Buenos Aires	LOL	2·5	From
			5	$11^h\ 00^m - 12^h\ 00^m$
			10	$14^h\ 00^m - 15^h\ 00^m$
			15	$17^h\ 00^m - 18^h\ 00^m$
			20	$20^h\ 00^m - 21^h\ 00^m$
			25	$23^h\ 00^m - 24^h\ 00^m$
Czechoslovakia	Astronomical Institute of the Czechoslovak Academy of Sciences, Prague	OMA	2·5	Continuous
France	Observatory of Paris	FFH	2·5	9 hours per day between
			5	$07^h\ 30^m - 17^h\ 00^m$
			10	
Japan	Astronomical Observatory, Tokyo	JJY	2·5	Continuous
			5	
			10	
			15	
South Africa	Republic Observatory, Johannesburg	ZUO	5	Continuous
Switzerland	Cantonal Observatory, Neuchâtel	HBN	2·5	Continuous
			5	
Union of Soviet Socialist Republics	U.S.S.R. Committee of Measures and Standards	RWM	5	$17^h\ 00^m - 24^h\ 00^m$
			10	$01^h\ 00^m - 04^h\ 00^m$
			10	$13^h\ 00^m - 16^h\ 00^m$
			15	$05^h\ 00^m - 12^h\ 00^m$

Additional Frequencies

Country	Authority	Call Sign	Frequency kc/s	Transmission Times (U.T.)
Czechoslovakia	Astronomical Institute of the Czechoslovak Academy of Sciences, Prague	OMA	50	24 hours per day
Germany (Federal German Republic)	Physikalisch - Technische Bundesanstalt, Braunschweig	DCF77	77·5	19 hours per day between $07^h\ 00^m - 02^h\ 10^m$

E. LIST OF COORDINATED TIME AND FREQUENCY TRANSMISSIONS

This list contains information on the coordinated time and frequency transmissions of the United Kingdom and the United States of America. The co-operating authorities are: in the United Kingdom, the Royal Greenwich Observatory, the National Physical Laboratory, and the General Post Office; and in the United States, the U.S. Naval Observatory, the Naval Research Laboratory, and the National Bureau of Standards.

Country	Call Sign	Frequency kc/s	Transmission Times (U.T.)
United Kingdom	MSF	2 500 5 000 10 000	Continuous
		60	$14^h 30^m - 15^h 30^m$
	GBR	16	Continuous (traffic) except for daily maintenance between $13^h 00^m$ and $15^h 00^m$. Time signals at $10^h 00^m$ and $18^h 00^h$ only.
United States of America	WWV	2 500 5 000 10 000 15 000 20 000 25 000	Continuous
	WWVH	5 000 10 000 15 000	Continuous
	NBA	18	Continuous except between $13^h 00^m$ and $21^h 00^m$ on Wednesday.

Additional Note (1973)

Atomic time standards using the caesium resonance are now used to establish an internationally coordinated system of time scales that are uniform to a very high precision. Astronomical observations are used to determine the variations in the rate of rotation and the motion of the pole of rotation with respect to the principal axis of inertia, and hence to establish the UT1 time-scale. All major radio signals are referred to the scale of coordinated universal time (UTC) which since 1972 January 1 differs from the scale of international atomic time (IAT) by an exact number of seconds, and from UT1 by up to about $0^s.7$.

A short current list of radio time signals is given in *The Star Almanac for Land Surveyors*. Comprehensive lists of the standard times in use in most countries are published in *The Nautical Almanac* and *The Air Almanac*.

16. COMPUTATION AND INTERPOLATION

A. COMPUTING TECHNIQUES

In this sub-section attention is drawn to a few computing techniques of special relevance to astronomical calculations. For a more general treatment reference should be made to text-books on computing methods and numerical analysis. A short note on elementary computing principles is also included in *Interpolation and Allied Tables*; this covers the sources of mistakes, the nature of checks, and the nature and effect of the inevitable errors due to rounding-off and other causes; these topics, although of fundamental importance, are not discussed here.

An important principle of computation is that the maximum precision of a calculation is determined absolutely by that of the data entering into it. This precision can be reduced by poor computing, for example by a poor choice of formula or by the failure to retain an adequate number of figures in the intermediate stages, but it can never be increased. There is generally no simple relation between the errors (absolute or relative) of the final result and of the data, though it can readily be seen numerically by following through the calculation step by step, and often geometrically. It is misleading to quote a result to more figures than is justified by the data on which it is based, and it is erroneous to do so if an inadequate number of figures have been retained in the intermediate stages.

It is also wasteful of effort to retain more figures than necessary, if thereby a significant increase in work is involved: in some operations, however, especially in those done on a desk calculating machine without intermediate resetting, extra figures involve very little additional work; but in other operations an extra figure may require a more elaborate formula and more elaborate trigonometric tables, resulting in much more additional work. A suitable number of figures is that which offers the greatest convenience consistent with the building-up error, due to accumulation of rounding-offs, not exceeding the error of the data; this must be judged in relation to each calculation. It often happens that the desired precision does not justify the factor of 10 which an extra figure will give; such a ten-fold increase of precision may be avoided by: using the extra figure but deliberately allowing end-figure errors to rise above the normal (for example, by ignoring second differences in interpolation up to a limit of, say, 12 or 16 instead of 4); applying a simple factor (such as 2, 3 or 5) to all quantities concerned; modifying the formulae by, for example, normalising coordinates. All three of these devices are used in the calculations for the Ephemeris.

Mathematical formulae, if correct, are always adequate to give a desired precision in the result; but they may be inefficient, inconvenient, and misleading. The obvious example is the inefficiency of determining a small angle from its cosine: for a stated precision of angular measurement the number of decimals in the cosine is inversely proportional to the angle; and for a fixed number of decimals in the cosine (the usual and convenient case) the precision obtainable decreases as the angle decreases. Consider, for example, the formula for the third side of a spherical triangle, given two sides and the included angle:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \equiv z$$

in which a is to be determined from measured values of b, c, and A. When a is small it can be found to the same precision as the data only by the apparently incorrect procedure of retaining more decimals in each trigonometric function on the right-hand side (z) of the equation than the data appear to justify. This is legitimate because both $b - c$ and A must be small so that their cosines are known to extra figures. There are circumstances, however, in which the inconvenience of such a procedure is outweighed by other factors.

The apparent failure of a mathematical formula should not be confused with real geometrical limitations of precision. In the spherical triangle ABC:

$$\begin{aligned} \sin a \sin B &= \sin b \sin A &&\equiv x \\ \sin a \cos B &= \cos b \sin c - \sin b \cos c \cos A &&\equiv y \end{aligned}$$

both x and y are small (y by the cancellation of two nearly equal components) if a is small. It is not legitimate here to use extra figures in the trigonometric functions in x and y, and B can only be found with a precision proportional to cosec a; this precision is, however, clearly adequate to fix the point C (from BA). Similar arguments hold for the angle C. Neither B nor C can individually be determined as precisely as b, c, or A; but the sum $B + C$ can be so determined, as may be seen from geometrical considerations and from the equation:

$$\cot \tfrac{1}{2}(B + C) = \frac{\cos \frac{1}{2}(b + c)}{\cos \frac{1}{2}(b - c)} \tan \tfrac{1}{2} A$$

There are important problems in astronomy that require the sum (or difference) of two angles to be known more precisely than is possible with either (for example, the elements of a planetary orbit), and great care is required in handling them; the recommended method is to determine one of the angles and thenceforward to treat it as exact in finding the other, which must not be found independently.

Although there may be some uncertainty of precision in determining an angle, there should be no uncertainty of quadrant. This can always be achieved by adequate choice of formulae. For example, the following formulae for the solution of a spherical triangle, in which two sides and the included angle are given, are always adequate if the sides are less than 180°.

$$\begin{aligned} \sin a \sin B &= \sin b \sin A &&\equiv x \\ \sin a \cos B &= \cos b \sin c - \sin b \cos c \cos A &&\equiv y \\ \cos a &= \cos b \cos c + \sin b \sin c \cos A &&\equiv z \\ \sin a \cos C &= \sin b \cos c - \cos b \sin c \cos A && \\ \sin a \sin C &= \sin c \sin A && \end{aligned}$$

The sign of z determines the quadrant of a, and the signs of x, y that of B; similarly with C.

In astronomical problems, the sides and angles of triangles on the celestial sphere may be of any magnitude, and it is undesirable to solve them by methods that restrict the sides to arcs less than 180°. The above general formulae are all valid for triangles with sides of any length, and may be applied immediately to the general triangles of spherical astronomy without any restriction to values less than 180°; furthermore, to make the solution determinate, it is only necessary to find the algebraic sign of both the sine and the cosine of each arc or angle that may exceed 180°, in order to fix the quadrants in which they lie. Any of the cases of the general triangle is determinate when, in addition to the three given parts, the algebraic sign of the sine or the cosine of one of the required parts is also given, and in most practical problems it happens that the conditions of the problem supply this sign. In the general triangle the utmost care should be taken to specify unambiguously the direction of measurement of angles and arcs.

Collectively, the above formulae are sufficient for the solution of the general triangle without restriction on the magnitudes of the parts, but, in practice, the additional formulae:

$$\begin{aligned}
\cos A &= -\cos B \cos C + \sin B \sin C \cos a \\
\sin A \cos b &= \cos B \sin C + \sin B \cos C \cos a \\
\sin A \cos c &= \sin B \cos C + \cos B \sin C \cos a \\
\sin A \cot B &= \sin c \cot b - \cos c \cos A \\
\sin A \cot C &= \sin b \cot c - \cos b \cos A
\end{aligned}$$

are very useful. Collected formulae are given in table 16.4.

In practice, the trigonometric functions of b, c, A are taken out with a number of decimals depending on the precision of the data, the tables available, and the capacity of the calculating machine to be used. The following table gives corresponding precisions of angle and number of decimals:

$1'$	$\equiv 0{\cdot}00029$	1^s	$\equiv 0{\cdot}00007$	$0{\cdot}0001$	$\equiv 0'{\cdot}34$	$= 1^s{\cdot}4$
$1''$	$\equiv 0{\cdot}0000048$	$0^s{\cdot}1$	$\equiv 7 \times 10^{-6}$	1×10^{-6}	$\equiv 0''{\cdot}2$	$= 0^s{\cdot}014$
$0''{\cdot}1$	$\equiv 5 \times 10^{-7}$	$0^s{\cdot}01$	$\equiv 7 \times 10^{-7}$	1×10^{-7}	$\equiv 0''{\cdot}02$	$= 0^s{\cdot}0014$
$0''{\cdot}01$	$\equiv 5 \times 10^{-8}$	$0^s{\cdot}001$	$\equiv 7 \times 10^{-8}$	1×10^{-8}	$\equiv 0''{\cdot}002$	$= 0^s{\cdot}00014$

If no additional work is involved it is clearly advantageous to use more decimals than the precision warrants, as the effect of rounding-off errors is then much reduced. For instance, seven decimals give just sufficient coverage for data known to about $0''{\cdot}02$ and for results to be rounded off to $0''{\cdot}1$; but interpolation in eight-figure tables at interval $1''$ offers little, if any, more difficulty than interpolation in seven-figure tables at interval $10''$, and eight decimals can be used with little extra work.

In calculating expressions such as x, y, z on a desk calculating machine it is sometimes desirable in systematic calculations to record intermediate results, such as $\sin b \cos A$, even if each expression can be evaluated directly on the machine, using facilities for transferring a product to be the next multiplicand or multiplier;

the additional recordings are liable to transcription errors and, owing to rounding off, may give rise to apparent end-figure discrepancies when the work is checked. The check $x^2 + y^2 + z^2 - 1 = 0$ is essential after the formation of x, y, z.

In the solution of the above equations, a is found either from $\cos a = z$ or from $\sin a = (x^2 + y^2)^{\frac{1}{2}}$ according to which is the smaller; assuming a is less than 180°, the ambiguity of the second formula is resolved according to the sign of z. B is found from:

$$\tan B = \frac{x}{y} \text{ or } \cot B = \frac{y}{x}$$

according to which is less than unity. The quadrant is determined by the signs of x, y since, if a is less than 180°, $\sin a$ is positive.

Most trigonometric tables are arranged semi-quadrantally, with direct (D) and complementary (C) arguments. The following table shows the method of inverse use in finding B.

	0°–45°	45°–90°	90°–135°	135°–180°	180°–225°	225°–270°	270°–315°	315°–360°
x	+	+	+	+	−	−	−	−
y	+	+	−	−	−	−	+	+
tan or cot	tan	cot	cot	tan	tan	cot	cot	tan
D or C arg.	D	C	D	C	D	C	D	C

There is no difficulty in systematic computation.

The range of precision of the inverse determination of an angle from its sine (or cosine) and tangent (or cotangent) is indicated in the following table; the table is arranged to show the range of angles for which the alternative trigonometric function should be used, and gives the error in the angle corresponding to an error of 1×10^{-6} in the function.

Errors corresponding to 1×10^{-6} in function	0° / 180°		45° / 225°		90° / 270°		135° / 315°		180° / 360°
Use function		sin		cos		cos		sin	
Error in angle	0″·2		0″·29		0″·2		0″·29		0″·2
Use function		tan		cot		cot		tan	
Error in angle	0ˢ·014		0ˢ·007		0ˢ·014		0ˢ·007		0ˢ·014

The technique of inverse interpolation, referred to briefly in sub-section B.2, is a powerful tool in the solution of transcendental equations and of equations in which the algebraic solution is complicated. The fundamental principle of such methods lies in the tabulation of a *discriminant*, defined so that it attains a pre-determined value (usually zero) when the original equation is satisfied, and the calculation of the unknown argument corresponding to the pre-determined value of the discriminant by the process of inverse interpolation. Although these methods may involve more calculation than direct methods, they have two considerable advantages: they are usually independent of theoretical developments, of any approximations that may be necessary in such developments, and of extrapolations; the correctness of the calculations and of the required answer, and the precision of that answer, are directly under the control of the computer. Illustrations of the use of these methods are to be found in the preceding sections, particularly in the calculation of the local circumstances of eclipses (section 9D) and the derivation of the times of moonrise and moonset (section 13D). A detailed account of the technique of inverse interpolation is given in *Interpolation and Allied Tables.*

The availability of high-speed electronic computing machines has changed the relative importance to be attached to the various factors entering into a computation. It is no longer desirable to restrict the amount of calculation to a minimum: many repetitions of a simple iteration are often preferable to a more sophisticated direct calculation. In particular many of the transformations of coordinates arising in astronomy can be efficiently handled by the direct use of the accurate formulae, instead of by the approximate series expressions to the development of which so much ingenuity has been devoted; these often take the form of multiplication of a column matrix representing the direction cosines by a transformation matrix, as for the correction for precession and nutation. Close attention to the precision of computation is generally unnecessary as many extra figures can be kept without any extra work. But although such machines are used for the computation of the data in the Ephemeris, they are not generally available to the users; the foregoing remarks on the techniques of computation are therefore restricted to circumstances in which desk calculating machines are used.

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B. INTERPOLATION

Introduction

The purpose of this sub-section is to provide tables, and the necessary minimum explanation, for the interpolation of the data tabulated in the Ephemeris. Excluding a few quantities, such as the Moon's apparent longitude, not intended for interpolation, the maximum values of the differences concerned are approximately:

double second differences	10 000
third differences	1 200
double fourth differences	500

In some cases it is sufficiently accurate to interpolate linearly (i.e. by simple proportion) between the tabular values. In most other cases differences higher than the second can be ignored, and even in the extreme cases the maximum contributions from third and fourth differences are about 8 and 6 units respectively. Generally, first differences are printed whenever second differences are required for interpolation to full tabular precision; linear interpolation can often be used if less accuracy is required.

The following tables differ in some important details from those included in *A.E.* 1960; they give slightly greater precision and, since they are in a separate volume, may be more convenient to use. They also differ from the corresponding tables in *Interpolation and Allied Tables* (*I.A.T.*); but the latter is not restricted to the above ranges of differences and should certainly be used for general computation. *I.A.T.* contains an elementary introduction to interpolation and a discussion of the possible errors of an interpolate.

No attempt is made to provide for, or to illustrate, interpolation techniques required in the derivation of the data tabulated in the Ephemeris. These are all included in *I.A.T.* and in its companion booklet *Subtabulation.*

Use of the tables

The notation used is that of *I.A.T.*, with central differences; the interval is denoted by h and the interpolating factor by p.

	Func-	Differences			
Arg.	tion	First	Second	Third	Fourth
−2	f_{-2}		δ^2_{-2}		δ^4_{-2}
		$\delta_{-1\frac{1}{2}}$		$\delta^3_{-1\frac{1}{2}}$	...
−1	f_{-1}		δ^2_{-1}		δ^4_{-1}
		$\delta_{-\frac{1}{2}}$		$\delta^3_{-\frac{1}{2}}$	...
0	f_0		δ^2_0		δ^4_0
		$\delta_{\frac{1}{2}}$		$\delta^3_{\frac{1}{2}}$	...
+1	f_1		δ^2_1		...
		$\delta_{1\frac{1}{2}}$		...	...
+2	f_2		...		...

Note that any quantity is obtained by subtracting the upper from the lower of the two quantities to the left; and is the sum of the quantity immediately above it and that diagonally above it to the right, e.g.:

$$\delta^2_1 = \delta_{1\frac{1}{2}} - \delta_{\frac{1}{2}} = \delta^2_0 + \delta^3_{\frac{1}{2}}.$$

Also:

$$\delta^2_0 + \delta^2_1 = \delta_{1\frac{1}{2}} - \delta_{-\frac{1}{2}}, \text{ etc.}$$

Bessel's interpolation formula is used in its direct form:

$$f(t_0 + ph) \equiv f(t_p) \equiv f_p = f_0 + p\delta_{\frac{1}{2}} + B_2(\delta^2_0 + \delta^2_1) + B_3\,\delta^3_{\frac{1}{2}} + B_4(\delta^4_0 + \delta^4_1)$$

where B_2, B_3, B_4 are Bessel's interpolation coefficients and are simple functions of p. Tables are given to facilitate the calculation of the corrections for second, third, and fourth differences. The maximum numerical values of B_2, B_3, B_4, and the corresponding maximum adopted values of δ^2, δ^3, δ^4 that can contribute not more than 0·5 units of the end figure are:

Maximum values	$B_2 = 0{\cdot}0625$	$B_3 = 0{\cdot}008$	$B_4 = 0{\cdot}012$
Limits for differences	$\delta^2 < 4$	$\delta^3 < 60$	$\delta^4 < 20$

The maxima of B_2 and B_4 occur at $p \doteq 0{\cdot}5$ where B_3 is zero; the maxima of B_3 occur at $p = 0{\cdot}21$ and $0{\cdot}79$. Larger values of these differences may be ignored only if the corresponding larger errors can be accepted.

Linear interpolation, which is the first and most important step in any interpolation, is represented by the formula:

$$f_p = f_0 + p\delta_{\frac{1}{2}} = (1 - p)f_0 + pf_1$$

The first expression is recommended for normal use with a calculating machine as the settings of f_0 and $\delta_{\frac{1}{2}}$ can be checked by verifying, before calculating f_p, that their sum is f_1. The same technique can be used in inverse interpolation, i.e. for finding the value of p for which f_p takes a given value. If second differences up to 4 units are ignored the error of a rounded-off interpolate may reach 1·5 units, but it is usually much less than this.

When second differences are appreciable but third differences are negligible the formula is:

$$f_p = f_0 + p\delta_{\frac{1}{2}} + B_2(\delta^2_0 + \delta^2_1)$$

and the second-difference correction may usually be taken directly from table 16.1, which is entered with arguments p and $\delta^2_0 + \delta^2_1$. Since B_2 is symmetrical about

$p = 0{\cdot}50$ the argument p is read downwards on the left from 0·00 to 0·50 and upwards on the right from 0·50 to 1·00. The correction, which always has the opposite sign to $\delta_0^2 + \delta_1^2$, is given to 0·1 units; if the nearest values of the two arguments are used the correction may be in error by 0·6 but interpolation by inspection will considerably reduce the error from this source. The table may be used for values of $\delta_0^2 + \delta_1^2$ outside the range by adding two contributions (e.g. using $475 = 300 + 175$) or scaling down by a simple factor (e.g. $475 \doteqdot 2 \times 240 = 3 \times 160$). It is most unlikely that the error due to neglecting third and fourth differences will exceed 0·5 units if second differences are within the range of this table, so that the maximum error of an interpolate will be less than 2 units even if the nearest arguments are used.

Example 16.1. *Interpolation of the right ascension of the Sun*
1960 March 7 at $9^h\ 16^m\ 17^s{\cdot}6$ E.T.

Date 1960	Apparent Right Ascension	δ	δ^2	δ^3
	h m s			
Mar. 6	23 06 21·69			
		s +222·42		
7	23 10 04·11		−43	
		221·99		+5
8	23 13 46·10		−38	
		+221·61		
9	23 17 27·71			

The tabular values are for 0^h E.T. at an interval of one day so the interpolating factor is $9^h\ 16^m\ 17^s{\cdot}6$ expressed as a fraction of a day; i.e., using table 17.5 (*A.E.*, Table X), $p = 0{\cdot}386315$. The double second difference is −81, so table 16.1, entered with arguments $p = 0{\cdot}39$ and $\delta_0^2 + \delta_1^2 = 80$, gives the second-difference correction as +4·8, in units of $0^s{\cdot}01$. Hence the apparent right ascension of the Sun at $9^h\ 16^m\ 17^s{\cdot}6$ E.T. on 1960 March 7 is:

$$23^h\ 10^m\ 04^s{\cdot}11 + (+0{\cdot}386315) \times (+221{\cdot}99) + 0^s{\cdot}048 = 23^h\ 11^m\ 29^s{\cdot}92$$

For an inverse interpolation using table 16.1 a first approximation to p is found by linear interpolation; this value of p is used to find an approximate value of the second-difference correction, which can be applied mentally; a second approximation to p is then found and the procedure repeated until there is no change in p. Example 8.1 illustrates this procedure. Except near turning points where the first difference is small the precision of the value of p so found is largely determined by the size of the first difference.

If the range of table 16.1 is inadequate but third and fourth differences are negligible, the second-difference correction may be calculated by using the appropriate value of B_2 from table 16.2A which is entered with argument p. The table gives five-decimal values of B_2 at intervals of 0·001 in p, arranged so that, for example, the eleven values for $p = 0{\cdot}370$ to 0·380 are on the same horizontal line as the argument $p = 0{\cdot}37$. The maximum difference is 25, and interpolation by inspection will suffice to give values with errors less than, say, 0·00003, which is ample for the largest difference in the Ephemeris. If second differences do not exceed 2500 it suffices to take the nearest entry.

Double second difference $\delta_0^2 + \delta_1^2$

p	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	p
0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.00
.01	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3	0.4	0.99
.02	0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	0.7	0.7	.98
.03	0.1	0.1	0.2	0.3	0.4	0.4	0.5	0.6	0.7	0.7	0.8	0.9	0.9	1.0	1.1	.97
.04	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.2	1.3	1.4	.96
0.05	0.1	0.2	0.4	0.5	0.6	0.7	0.8	1.0	1.1	1.2	1.3	1.4	1.5	1.7	1.8	0.95
.06	0.1	0.3	0.4	0.6	0.7	0.8	1.0	1.1	1.3	1.4	1.6	1.7	1.8	2.0	2.1	.94
.07	0.2	0.3	0.5	0.7	0.8	1.0	1.1	1.3	1.5	1.6	1.8	2.0	2.1	2.3	2.4	.93
.08	0.2	0.4	0.6	0.7	0.9	1.1	1.3	1.5	1.7	1.8	2.0	2.2	2.4	2.6	2.8	.92
.09	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.3	2.5	2.7	2.9	3.1	.91
0.10	0.2	0.5	0.7	0.9	1.1	1.4	1.6	1.8	2.0	2.3	2.5	2.7	2.9	3.2	3.4	0.90
.11	0.2	0.5	0.7	1.0	1.2	1.5	1.7	2.0	2.2	2.4	2.7	2.9	3.2	3.4	3.7	.89
.12	0.3	0.5	0.8	1.1	1.3	1.6	1.8	2.1	2.4	2.6	2.9	3.2	3.4	3.7	4.0	.88
.13	0.3	0.6	0.8	1.1	1.4	1.7	2.0	2.3	2.5	2.8	3.1	3.4	3.7	4.0	4.2	.87
.14	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0	3.3	3.6	3.9	4.2	4.5	.86
0.15	0.3	0.6	1.0	1.3	1.6	1.9	2.2	2.6	2.9	3.2	3.5	3.8	4.1	4.5	4.8	0.85
.16	0.3	0.7	1.0	1.3	1.7	2.0	2.4	2.7	3.0	3.4	3.7	4.0	4.4	4.7	5.0	.84
.17	0.4	0.7	1.1	1.4	1.8	2.1	2.5	2.8	3.2	3.5	3.9	4.2	4.6	4.9	5.3	.83
.18	0.4	0.7	1.1	1.5	1.8	2.2	2.6	3.0	3.3	3.7	4.1	4.4	4.8	5.2	5.5	.82
.19	0.4	0.8	1.2	1.5	1.9	2.3	2.7	3.1	3.5	3.8	4.2	4.6	5.0	5.4	5.8	.81
0.20	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	4.4	4.8	5.2	5.6	6.0	0.80
.21	0.4	0.8	1.2	1.7	2.1	2.5	2.9	3.3	3.7	4.1	4.6	5.0	5.4	5.8	6.2	.79
.22	0.4	0.9	1.3	1.7	2.1	2.6	3.0	3.4	3.9	4.3	4.7	5.1	5.6	6.0	6.4	.78
.23	0.4	0.9	1.3	1.8	2.2	2.7	3.1	3.5	4.0	4.4	4.9	5.3	5.8	6.2	6.6	.77
.24	0.5	0.9	1.4	1.8	2.3	2.7	3.2	3.6	4.1	4.6	5.0	5.5	5.9	6.4	6.8	.76
0.25	0.5	0.9	1.4	1.9	2.3	2.8	3.3	3.8	4.2	4.7	5.2	5.6	6.1	6.6	7.0	0.75
.26	0.5	1.0	1.4	1.9	2.4	2.9	3.4	3.8	4.3	4.8	5.3	5.8	6.3	6.7	7.2	.74
.27	0.5	1.0	1.5	2.0	2.5	3.0	3.4	3.9	4.4	4.9	5.4	5.9	6.4	6.9	7.4	.73
.28	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.6	7.1	7.6	.72
.29	0.5	1.0	1.5	2.1	2.6	3.1	3.6	4.1	4.6	5.1	5.7	6.2	6.7	7.2	7.7	.71
0.30	0.5	1.1	1.6	2.1	2.6	3.2	3.7	4.2	4.7	5.3	5.8	6.3	6.8	7.4	7.9	0.70
.31	0.5	1.1	1.6	2.1	2.7	3.2	3.7	4.3	4.8	5.3	5.9	6.4	7.0	7.5	8.0	.69
.32	0.5	1.1	1.6	2.2	2.7	3.3	3.8	4.4	4.9	5.4	6.0	6.5	7.1	7.6	8.2	.68
.33	0.6	1.1	1.7	2.2	2.8	3.3	3.9	4.4	5.0	5.5	6.1	6.6	7.2	7.7	8.3	.67
.34	0.6	1.1	1.7	2.2	2.8	3.4	3.9	4.5	5.0	5.6	6.2	6.7	7.3	7.9	8.4	.66
0.35	0.6	1.1	1.7	2.3	2.8	3.4	4.0	4.6	5.1	5.7	6.3	6.8	7.4	8.0	8.5	0.65
.36	0.6	1.2	1.7	2.3	2.9	3.5	4.0	4.6	5.2	5.8	6.3	6.9	7.5	8.1	8.6	.64
.37	0.6	1.2	1.7	2.3	2.9	3.5	4.1	4.7	5.2	5.8	6.4	7.0	7.6	8.2	8.7	.63
.38	0.6	1.2	1.8	2.4	2.9	3.5	4.1	4.7	5.3	5.9	6.5	7.1	7.7	8.2	8.8	.62
.39	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4	5.9	6.5	7.1	7.7	8.3	8.9	.61
0.40	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4	6.0	6.6	7.2	7.8	8.4	9.0	0.60
.41	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4	6.0	6.7	7.3	7.9	8.5	9.1	.59
.42	0.6	1.2	1.8	2.4	3.0	3.7	4.3	4.9	5.5	6.1	6.7	7.3	7.9	8.5	9.1	.58
.43	0.6	1.2	1.8	2.5	3.1	3.7	4.3	4.9	5.5	6.1	6.7	7.4	8.0	8.6	9.2	.57
.44	0.6	1.2	1.8	2.5	3.1	3.7	4.3	4.9	5.5	6.2	6.8	7.4	8.0	8.6	9.2	.56
0.45	0.6	1.2	1.9	2.5	3.1	3.7	4.3	5.0	5.6	6.2	6.8	7.4	8.0	8.7	9.3	0.55
.46	0.6	1.2	1.9	2.5	3.1	3.7	4.3	5.0	5.6	6.2	6.8	7.5	8.1	8.7	9.3	.54
.47	0.6	1.2	1.9	2.5	3.1	3.7	4.4	5.0	5.6	6.2	6.9	7.5	8.1	8.7	9.3	.53
.48	0.6	1.2	1.9	2.5	3.1	3.7	4.4	5.0	5.6	6.2	6.9	7.5	8.1	8.7	9.4	.52
.49	0.6	1.2	1.9	2.5	3.1	3.7	4.4	5.0	5.6	6.2	6.9	7.5	8.1	8.7	9.4	.51
0.50	0.6	1.3	1.9	2.5	3.1	3.8	4.4	5.0	5.6	6.3	6.9	7.5	8.1	8.8	9.4	0.50

The correction has the opposite sign to $\delta_0^2 + \delta_1^2$.

Double second difference $\delta_0^2 + \delta_1^2$

p	160	170	180	190	200	210	220	230	240	260	280	300	p
0·00	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	1·00
·01	0·4	0·4	0·4	0·5	0·5	0·5	0·5	0·6	0·6	0·6	0·7	0·7	0·99
·02	0·8	0·8	0·9	0·9	1·0	1·0	1·1	1·1	1·2	1·3	1·4	1·5	·98
·03	1·2	1·2	1·3	1·4	1·5	1·5	1·6	1·7	1·7	1·9	2·0	2·2	·97
·04	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3	2·5	2·7	2·9	·96
0·05	1·9	2·0	2·1	2·3	2·4	2·5	2·6	2·7	2·9	3·1	3·3	3·6	0·95
·06	2·3	2·4	2·5	2·7	2·8	3·0	3·1	3·2	3·4	3·7	3·9	4·2	·94
·07	2·6	2·8	2·9	3·1	3·3	3·4	3·6	3·7	3·9	4·2	4·6	4·9	·93
·08	2·9	3·1	3·3	3·5	3·7	3·9	4·0	4·2	4·4	4·8	5·2	5·5	·92
·09	3·3	3·5	3·7	3·9	4·1	4·3	4·5	4·7	4·9	5·3	5·7	6·1	·91
0·10	3·6	3·8	4·1	4·3	4·5	4·7	5·0	5·2	5·4	5·9	6·3	6·8	0·90
·11	3·9	4·2	4·4	4·7	4·9	5·1	5·4	5·6	5·9	6·4	6·9	7·3	·89
·12	4·2	4·5	4·8	5·0	5·3	5·5	5·8	6·1	6·3	6·9	7·4	7·9	·88
·13	4·5	4·8	5·1	5·4	5·7	5·9	6·2	6·5	6·8	7·4	7·9	8·5	·87
·14	4·8	5·1	5·4	5·7	6·0	6·3	6·6	6·9	7·2	7·8	8·4	9·0	·86
0·15	5·1	5·4	5·7	6·1	6·4	6·7	7·0	7·3	7·7	8·3	8·9	9·6	0·85
·16	5·4	5·7	6·0	6·4	6·7	7·1	7·4	7·7	8·1	8·7	9·4	10·1	·84
·17	5·6	6·0	6·3	6·7	7·1	7·4	7·8	8·1	8·5	9·2	9·9	10·6	·83
·18	5·9	6·3	6·6	7·0	7·4	7·7	8·1	8·5	8·9	9·6	10·3	11·1	·82
·19	6·2	6·5	6·9	7·3	7·7	8·1	8·5	8·8	9·2	10·0	10·8	11·5	·81
0·20	6·4	6·8	7·2	7·6	8·0	8·4	8·8	9·2	9·6	10·4	11·2	12·0	0·80
·21	6·6	7·1	7·5	7·9	8·3	8·7	9·1	9·5	10·0	10·8	11·6	12·4	·79
·22	6·9	7·3	7·7	8·2	8·6	9·0	9·4	9·9	10·3	11·2	12·0	12·9	·78
·23	7·1	7·5	8·0	8·4	8·9	9·3	9·7	10·2	10·6	11·5	12·4	13·3	·77
·24	7·3	7·8	8·2	8·7	9·1	9·6	10·0	10·5	10·9	11·9	12·8	13·7	·76
0·25	7·5	8·0	8·4	8·9	9·4	9·8	10·3	10·8	11·3	12·2	13·1	14·1	0·75
·26	7·7	8·2	8·7	9·1	9·6	10·1	10·6	11·1	11·5	12·5	13·5	14·4	·74
·27	7·9	8·4	8·9	9·4	9·9	10·3	10·8	11·3	11·8	12·8	13·8	14·8	·73
·28	8·1	8·6	9·1	9·6	10·1	10·6	11·1	11·6	12·1	13·1	14·1	15·1	·72
·29	8·2	8·8	9·3	9·8	10·3	10·8	11·3	11·8	12·4	13·4	14·4	15·4	·71
0·30	8·4	8·9	9·5	10·0	10·5	11·0	11·6	12·1	12·6	13·7	14·7	15·8	0·70
·31	8·6	9·1	9·6	10·2	10·7	11·2	11·8	12·3	12·8	13·9	15·0	16·0	·69
·32	8·7	9·2	9·8	10·3	10·9	11·4	12·0	12·5	13·1	14·1	15·2	16·3	·68
·33	8·8	9·4	9·9	10·5	11·1	11·6	12·2	12·7	13·3	14·4	15·5	16·6	·67
·34	9·0	9·5	10·1	10·7	11·2	11·8	12·3	12·9	13·5	14·6	15·7	16·8	·66
0·35	9·1	9·7	10·2	10·8	11·4	11·9	12·5	13·1	13·7	14·8	15·9	17·1	0·65
·36	9·2	9·8	10·4	10·9	11·5	12·1	12·7	13·2	13·8	15·0	16·1	17·3	·64
·37	9·3	9·9	10·5	11·1	11·7	12·2	12·8	13·4	14·0	15·2	16·3	17·5	·63
·38	9·4	10·0	10·6	11·2	11·8	12·4	13·0	13·5	14·1	15·3	16·5	17·7	·62
·39	9·5	10·1	10·7	11·3	11·9	12·5	13·1	13·7	14·3	15·5	16·7	17·8	·61
0·40	9·6	10·2	10·8	11·4	12·0	12·6	13·2	13·8	14·4	15·6	16·8	18·0	0·60
·41	9·7	10·3	10·9	11·5	12·1	12·7	13·3	13·9	14·5	15·7	16·9	18·1	·59
·42	9·7	10·4	11·0	11·6	12·2	12·8	13·4	14·0	14·6	15·8	17·1	18·3	·58
·43	9·8	10·4	11·0	11·6	12·3	12·9	13·5	14·1	14·7	15·9	17·2	18·4	·57
·44	9·9	10·5	11·1	11·7	12·3	12·9	13·6	14·2	14·8	16·0	17·2	18·5	·56
0·45	9·9	10·5	11·1	11·8	12·4	13·0	13·6	14·2	14·9	16·1	17·3	18·6	0·55
·46	9·9	10·6	11·2	11·8	12·4	13·0	13·7	14·3	14·9	16·1	17·4	18·6	·54
·47	10·0	10·6	11·2	11·8	12·5	13·1	13·7	14·3	14·9	16·2	17·4	18·7	·53
·48	10·0	10·6	11·2	11·9	12·5	13·1	13·7	14·4	15·0	16·2	17·5	18·7	·52
·49	10·0	10·6	11·2	11·9	12·5	13·1	13·7	14·4	15·0	16·2	17·5	18·7	·51
0·50	10·0	10·6	11·3	11·9	12·5	13·1	13·8	14·4	15·0	16·3	17·5	18·8	0·50

If third and fourth differences are negligible: $f_p = f_0 + p\delta_{\frac{1}{2}} + B_2\,(\delta_0^2 + \delta_1^2)$.

In units of the fifth decimal—always negative

p	0·000		·001	·002	·003	·004	·005	·006	·007	·008	·009	·010
0·00	−0·00000	248	0025	0050	0075	0100	0124	0149	0174	0198	0223	0248
·01	0248	242	0272	0296	0321	0345	0369	0394	0418	0442	0466	0490
·02	0490	238	0514	0538	0562	0586	0609	0633	0657	0680	0704	0728
·03	0728	232	0751	0774	0798	0821	0844	0868	0891	0914	0937	0960
·04	0960	228	0983	1006	1029	1052	1074	1097	1120	1142	1165	1188
0·05	−0·01188	222	1210	1232	1255	1277	1299	1322	1344	1366	1388	1410
·06	1410	218	1432	1454	1476	1498	1519	1541	1563	1584	1606	1628
·07	1628	212	1649	1670	1692	1713	1734	1756	1777	1798	1819	1840
·08	1840	208	1861	1882	1903	1924	1944	1965	1986	2006	2027	2048
·09	2048	202	2068	2088	2109	2129	2149	2170	2190	2210	2230	2250
0·10	−0·02250	198	2270	2290	2310	2330	2349	2369	2389	2408	2428	2448
·11	2448	192	2467	2486	2506	2525	2544	2564	2583	2602	2621	2640
·12	2640	188	2659	2678	2697	2716	2734	2753	2772	2790	2809	2828
·13	2828	182	2846	2864	2883	2901	2919	2938	2956	2974	2992	3010
·14	3010	178	3028	3046	3064	3082	3099	3117	3135	3152	3170	3188
0·15	−0·03188	172	3205	3222	3240	3257	3274	3292	3309	3326	3343	3360
·16	3360	168	3377	3394	3411	3428	3444	3461	3478	3494	3511	3528
·17	3528	162	3544	3560	3577	3593	3609	3626	3642	3658	3674	3690
·18	3690	158	3706	3722	3738	3754	3769	3785	3801	3816	3832	3848
·19	3848	152	3863	3878	3894	3909	3924	3940	3955	3970	3985	4000
0·20	−0·04000	148	4015	4030	4045	4060	4074	4089	4104	4118	4133	4148
·21	4148	142	4162	4176	4191	4205	4219	4234	4248	4262	4276	4290
·22	4290	138	4304	4318	4332	4346	4359	4373	4387	4400	4414	4428
·23	4428	132	4441	4454	4468	4481	4494	4508	4521	4534	4547	4560
·24	4560	128	4573	4586	4599	4612	4624	4637	4650	4662	4675	4688
0·25	−0·04688	122	4700	4712	4725	4737	4749	4762	4774	4786	4798	4810
·26	4810	118	4822	4834	4846	4858	4869	4881	4893	4904	4916	4928
·27	4928	112	4939	4950	4962	4973	4984	4996	5007	5018	5029	5040
·28	5040	108	5051	5062	5073	5084	5094	5105	5116	5126	5137	5148
·29	5148	102	5158	5168	5179	5189	5199	5210	5220	5230	5240	5250
0·30	−0·05250	98	5260	5270	5280	5290	5299	5309	5319	5328	5338	5348
·31	5348	92	5357	5366	5376	5385	5394	5404	5413	5422	5431	5440
·32	5440	88	5449	5458	5467	5476	5484	5493	5502	5510	5519	5528
·33	5528	82	5536	5544	5553	5561	5569	5578	5586	5594	5602	5610
·34	5610	78	5618	5626	5634	5642	5649	5657	5665	5672	5680	5688
0·35	−0·05688	72	5695	5702	5710	5717	5724	5732	5739	5746	5753	5760
·36	5760	68	5767	5774	5781	5788	5794	5801	5808	5814	5821	5828
·37	5828	62	5834	5840	5847	5853	5859	5866	5872	5878	5884	5890
·38	5890	58	5896	5902	5908	5914	5919	5925	5931	5936	5942	5948
·39	5948	52	5953	5958	5964	5969	5974	5980	5985	5990	5995	6000
0·40	−0·06000	48	6005	6010	6015	6020	6024	6029	6034	6038	6043	6048
·41	6048	42	6052	6056	6061	6065	6069	6074	6078	6082	6086	6090
·42	6090	38	6094	6098	6102	6106	6109	6113	6117	6120	6124	6128
·43	6128	32	6131	6134	6138	6141	6144	6148	6151	6154	6157	6160
·44	6160	28	6163	6166	6169	6172	6174	6177	6180	6182	6185	6188
0·45	−0·06188	22	6190	6192	6195	6197	6199	6202	6204	6206	6208	6210
·46	6210	18	6212	6214	6216	6218	6219	6221	6223	6224	6226	6228
·47	6228	12	6229	6230	6232	6233	6234	6236	6237	6238	6239	6240
·48	6240	8	6241	6242	6243	6244	6244	6245	6246	6246	6247	6248
·49	6248	2	6248	6248	6249	6249	6249	6250	6250	6250	6250	6250
0·50	−0·06250											

$$f_p = f_0 + p\delta_{\frac{1}{2}} + B_2\,(\delta_0^2 + \delta_1^2) + B_3\,\delta_{\frac{1}{2}}^3 + B_4\,(\delta_0^4 + \delta_1^4)$$

p	Third difference $\delta^3_{\frac{1}{2}}$										$\delta^4_0 + \delta^4_1$					
	100	200	300	400	500	600	700	800	900	1000	100	200	300	400	500	600
0·00	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0
·01	0·1	0·2	0·2	0·3	0·4	0·5	0·6	0·6	0·7	0·8	0·0	0·1	0·1	0·2	0·2	0·2
·02	0·2	0·3	0·5	0·6	0·8	0·9	1·1	1·3	1·4	1·6	0·1	0·2	0·2	0·3	0·4	0·5
·03	0·2	0·5	0·7	0·9	1·1	1·4	1·6	1·8	2·1	2·3	0·1	0·2	0·4	0·5	0·6	0·7
·04	0·3	0·6	0·9	1·2	1·5	1·8	2·1	2·4	2·6	2·9	0·2	0·3	0·5	0·7	0·8	1·0
0·05	0·4	0·7	1·1	1·4	1·8	2·1	2·5	2·9	3·2	3·6	0·2	0·4	0·6	0·8	1·0	1·2
·06	0·4	0·8	1·2	1·7	2·1	2·5	2·9	3·3	3·7	4·1	0·2	0·5	0·7	1·0	1·2	1·4
·07	0·5	0·9	1·4	1·9	2·3	2·8	3·3	3·7	4·2	4·7	0·3	0·6	0·8	1·1	1·4	1·7
·08	0·5	1·0	1·5	2·1	2·6	3·1	3·6	4·1	4·6	5·2	0·3	0·6	1·0	1·3	1·6	1·9
·09	0·6	1·1	1·7	2·2	2·8	3·4	3·9	4·5	5·0	5·6	0·4	0·7	1·1	1·4	1·8	2·1
0·10	0·6	1·2	1·8	2·4	3·0	3·6	4·2	4·8	5·4	6·0	0·4	0·8	1·2	1·6	2·0	2·4
·11	0·6	1·3	1·9	2·5	3·2	3·8	4·5	5·1	5·7	6·4	0·4	0·9	1·3	1·7	2·1	2·6
·12	0·7	1·3	2·0	2·7	3·3	4·0	4·7	5·4	6·0	6·7	0·5	0·9	1·4	1·9	2·3	2·8
·13	0·7	1·4	2·1	2·8	3·5	4·2	4·9	5·6	6·3	7·0	0·5	1·0	1·5	2·0	2·5	3·0
·14	0·7	1·4	2·2	2·9	3·6	4·3	5·1	5·8	6·5	7·2	0·5	1·1	1·6	2·1	2·7	3·2
0·15	0·7	1·5	2·2	3·0	3·7	4·5	5·2	6·0	6·7	7·4	0·6	1·1	1·7	2·3	2·8	3·4
·16	0·8	1·5	2·3	3·0	3·8	4·6	5·3	6·1	6·9	7·6	0·6	1·2	1·8	2·4	3·0	3·6
·17	0·8	1·6	2·3	3·1	3·9	4·7	5·4	6·2	7·0	7·8	0·6	1·3	1·9	2·5	3·1	3·8
·18	0·8	1·6	2·4	3·1	3·9	4·7	5·5	6·3	7·1	7·9	0·7	1·3	2·0	2·6	3·3	4·0
·19	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·7	1·4	2·1	2·8	3·5	4·1
0·20	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·7	1·4	2·2	2·9	3·6	4·3
·21	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·7	1·5	2·2	3·0	3·7	4·5
·22	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·8	1·6	2·3	3·1	3·9	4·7
·23	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·8	1·6	2·4	3·2	4·0	4·8
·24	0·8	1·6	2·4	3·2	4·0	4·7	5·5	6·3	7·1	7·9	0·8	1·7	2·5	3·3	4·1	5·0
0·25	0·8	1·6	2·3	3·1	3·9	4·7	5·5	6·3	7·0	7·8	0·9	1·7	2·6	3·4	4·3	5·1
·26	0·8	1·5	2·3	3·1	3·8	4·6	5·4	6·2	6·9	7·7	0·9	1·8	2·6	3·5	4·4	5·3
·27	0·8	1·5	2·3	3·0	3·8	4·5	5·3	6·0	6·8	7·6	0·9	1·8	2·7	3·6	4·5	5·4
·28	0·7	1·5	2·2	3·0	3·7	4·4	5·2	5·9	6·7	7·4	0·9	1·8	2·8	3·7	4·6	5·5
·29	0·7	1·4	2·2	2·9	3·6	4·3	5·0	5·8	6·5	7·2	0·9	1·9	2·8	3·8	4·7	5·7
0·30	0·7	1·4	2·1	2·8	3·5	4·2	4·9	5·6	6·3	7·0	1·0	1·9	2·9	3·9	4·8	5·8
·31	0·7	1·4	2·0	2·7	3·4	4·1	4·7	5·4	6·1	6·8	1·0	2·0	3·0	3·9	4·9	5·9
·32	0·7	1·3	2·0	2·6	3·3	3·9	4·6	5·2	5·9	6·5	1·0	2·0	3·0	4·0	5·0	6·0
·33	0·6	1·3	1·9	2·5	3·1	3·8	4·4	5·0	5·6	6·3	1·0	2·0	3·1	4·1	5·1	6·1
·34	0·6	1·2	1·8	2·4	3·0	3·6	4·2	4·8	5·4	6·0	1·0	2·1	3·1	4·2	5·2	6·2
0·35	0·6	1·1	1·7	2·3	2·8	3·4	4·0	4·6	5·1	5·7	1·1	2·1	3·2	4·2	5·3	6·3
·36	0·5	1·1	1·6	2·2	2·7	3·2	3·8	4·3	4·8	5·4	1·1	2·1	3·2	4·3	5·4	6·4
·37	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·1	1·1	2·2	3·3	4·3	5·4	6·5
·38	0·5	0·9	1·4	1·9	2·4	2·8	3·3	3·8	4·2	4·7	1·1	2·2	3·3	4·4	5·5	6·6
·39	0·4	0·9	1·3	1·7	2·2	2·6	3·1	3·5	3·9	4·4	1·1	2·2	3·3	4·4	5·5	6·7
0·40	0·4	0·8	1·2	1·6	2·0	2·4	2·8	3·2	3·6	4·0	1·1	2·2	3·4	4·5	5·6	6·7
·41	0·4	0·7	1·1	1·5	1·8	2·2	2·5	2·9	3·3	3·6	1·1	2·3	3·4	4·5	5·6	6·8
·42	0·3	0·6	1·0	1·3	1·6	1·9	2·3	2·6	2·9	3·2	1·1	2·3	3·4	4·6	5·7	6·8
·43	0·3	0·6	0·9	1·1	1·4	1·7	2·0	2·3	2·6	2·9	1·1	2·3	3·4	4·6	5·7	6·9
·44	0·2	0·5	0·7	1·0	1·2	1·5	1·7	2·0	2·2	2·5	1·2	2·3	3·5	4·6	5·8	6·9
0·45	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·7	1·9	2·1	1·2	2·3	3·5	4·6	5·8	7·0
·46	0·2	0·3	0·5	0·7	0·8	1·0	1·2	1·3	1·5	1·7	1·2	2·3	3·5	4·7	5·8	7·0
·47	0·1	0·2	0·4	0·5	0·6	0·7	0·9	1·0	1·1	1·2	1·2	2·3	3·5	4·7	5·8	7·0
·48	0·1	0·2	0·2	0·3	0·4	0·5	0·6	0·7	0·7	0·8	1·2	2·3	3·5	4·7	5·8	7·0
0·49	0·0	0·1	0·1	0·2	0·2	0·2	0·3	0·3	0·4	0·4	1·2	2·3	3·5	4·7	5·9	7·0

The correction has the same sign as $\delta^3_{\frac{1}{2}}$.

The correction has the same sign as $\delta^4_0 + \delta^4_1$.

In units of the fifth decimal—always negative

p	0·000		·001	·002	·003	·004	·005	·006	·007	·008	·009	·010
0·50	−0·06250	2	6250	6250	6250	6250	6249	6249	6249	6248	6248	6248
·51	6248	8	6247	6246	6246	6245	6244	6244	6243	6242	6241	6240
·52	6240	12	6239	6238	6237	6236	6234	6233	6232	6230	6229	6228
·53	6228	18	6226	6224	6223	6221	6219	6218	6216	6214	6212	6210
·54	6210	22	6208	6206	6204	6202	6199	6197	6195	6192	6190	6188
0·55	−0·06188	28	6185	6182	6180	6177	6174	6172	6169	6166	6163	6160
·56	6160	32	6157	6154	6151	6148	6144	6141	6138	6134	6131	6128
·57	6128	38	6124	6120	6117	6113	6109	6106	6102	6098	6094	6090
·58	6090	42	6086	6082	6078	6074	6069	6065	6061	6056	6052	6048
·59	6048	48	6043	6038	6034	6029	6024	6020	6015	6010	6005	6000
0·60	−0·06000	52	5995	5990	5985	5980	5974	5969	5964	5958	5953	5948
·61	5948	58	5942	5936	5931	5925	5919	5914	5908	5902	5896	5890
·62	5890	62	5884	5878	5872	5866	5859	5853	5847	5840	5834	5828
·63	5828	68	5821	5814	5808	5801	5794	5788	5781	5774	5767	5760
·64	5760	72	5753	5746	5739	5732	5724	5717	5710	5702	5695	5688
0·65	−0·05688	78	5680	5672	5665	5657	5649	5642	5634	5626	5618	5610
·66	5610	82	5602	5594	5586	5578	5569	5561	5553	5544	5536	5528
·67	5528	88	5519	5510	5502	5493	5484	5476	5467	5458	5449	5440
·68	5440	92	5431	5422	5413	5404	5394	5385	5376	5366	5357	5348
·69	5348	98	5338	5328	5319	5309	5299	5290	5280	5270	5260	5250
0·70	−0·05250	102	5240	5230	5220	5210	5199	5189	5179	5168	5158	5148
·71	5148	108	5137	5126	5116	5105	5094	5084	5073	5062	5051	5040
·72	5040	112	5029	5018	5007	4996	4984	4973	4962	4950	4939	4928
·73	4928	118	4916	4904	4893	4881	4869	4858	4846	4834	4822	4810
·74	4810	122	4798	4786	4774	4762	4749	4737	4725	4712	4700	4688
0·75	−0·04688	128	4675	4662	4650	4637	4624	4612	4599	4586	4573	4560
·76	4560	132	4547	4534	4521	4508	4494	4481	4468	4454	4441	4428
·77	4428	138	4414	4400	4387	4373	4359	4346	4332	4318	4304	4290
·78	4290	142	4276	4262	4248	4234	4219	4205	4191	4176	4162	4148
·79	4148	148	4133	4118	4104	4089	4074	4060	4045	4030	4015	4000
0·80	−0·04000	152	3985	3970	3955	3940	3924	3909	3894	3878	3863	3848
·81	3848	158	3832	3816	3801	3785	3769	3754	3738	3722	3706	3690
·82	3690	162	3674	3658	3642	3626	3609	3593	3577	3560	3544	3528
·83	3528	168	3511	3494	3478	3461	3444	3428	3411	3394	3377	3360
·84	3360	172	3343	3326	3309	3292	3274	3257	3240	3222	3205	3188
0·85	−0·03188	178	3170	3152	3135	3117	3099	3082	3064	3046	3028	3010
·86	3010	182	2992	2974	2956	2938	2919	2901	2883	2864	2846	2828
·87	2828	188	2809	2790	2772	2753	2734	2716	2697	2678	2659	2640
·88	2640	192	2621	2602	2583	2564	2544	2525	2506	2486	2467	2448
·89	2448	198	2428	2408	2389	2369	2349	2330	2310	2290	2270	2250
0·90	−0·02250	202	2230	2210	2190	2170	2149	2129	2109	2088	2068	2048
·91	2048	208	2027	2006	1986	1965	1944	1924	1903	1882	1861	1840
·92	1840	212	1819	1798	1777	1756	1734	1713	1692	1670	1649	1628
·93	1628	218	1606	1584	1563	1541	1519	1498	1476	1454	1432	1410
·94	1410	222	1388	1366	1344	1322	1299	1277	1255	1232	1210	1188
0·95	−0·01188	228	1165	1142	1120	1097	1074	1052	1029	1006	0983	0960
·96	0960	232	0937	0914	0891	0868	0844	0821	0798	0774	0751	0728
·97	0728	238	0704	0680	0657	0633	0609	0586	0562	0538	0514	0490
·98	0490	242	0466	0442	0418	0394	0369	0345	0321	0296	0272	0248
0·99	−0·00248	248	0223	0198	0174	0149	0124	0100	0075	0050	0025	0000
1·00	0·00000											

$$f_p = f_0 + p\delta_{\frac{1}{2}} + B_2(\delta_0^2 + \delta_1^2) + B_3\,\delta_{\frac{1}{2}}^3 + B_4(\delta_0^4 + \delta_1^4)$$

p	Third difference $\delta^3_{\frac{1}{2}}$										$\delta^4_0 + \delta^4_1$					
	100	200	300	400	500	600	700	800	900	1000	100	200	300	400	500	600
0·50	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	1·2	2·3	3·5	4·7	5·9	7·0
·51	0·0	0·1	0·1	0·2	0·2	0·2	0·3	0·3	0·4	0·4	1·2	2·3	3·5	4·7	5·9	7·0
·52	0·1	0·2	0·2	0·3	0·4	0·5	0·6	0·7	0·7	0·8	1·2	2·3	3·5	4·7	5·8	7·0
·53	0·1	0·2	0·4	0·5	0·6	0·7	0·9	1·0	1·1	1·2	1·2	2·3	3·5	4·7	5·8	7·0
·54	0·2	0·3	0·5	0·7	0·8	1·0	1·2	1·3	1·5	1·7	1·2	2·3	3·5	4·7	5·8	7·0
0·55	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·7	1·9	2·1	1·2	2·3	3·5	4·6	5·8	7·0
·56	0·2	0·5	0·7	1·0	1·2	1·5	1·7	2·0	2·2	2·5	1·2	2·3	3·5	4·6	5·8	6·9
·57	0·3	0·6	0·9	1·1	1·4	1·7	2·0	2·3	2·6	2·9	1·1	2·3	3·4	4·6	5·7	6·9
·58	0·3	0·6	1·0	1·3	1·6	1·9	2·3	2·6	2·9	3·2	1·1	2·3	3·4	4·6	5·7	6·8
·59	0·4	0·7	1·1	1·5	1·8	2·2	2·5	2·9	3·3	3·6	1·1	2·3	3·4	4·5	5·6	6·8
0·60	0·4	0·8	1·2	1·6	2·0	2·4	2·8	3·2	3·6	4·0	1·1	2·2	3·4	4·5	5·6	6·7
·61	0·4	0·9	1·3	1·7	2·2	2·6	3·1	3·5	3·9	4·4	1·1	2·2	3·3	4·4	5·5	6·7
·62	0·5	0·9	1·4	1·9	2·4	2·8	3·3	3·8	4·2	4·7	1·1	2·2	3·3	4·4	5·5	6·6
·63	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·1	1·1	2·2	3·3	4·3	5·4	6·5
·64	0·5	1·1	1·6	2·2	2·7	3·2	3·8	4·3	4·8	5·4	1·1	2·1	3·2	4·3	5·4	6·4
0·65	0·6	1·1	1·7	2·3	2·8	3·4	4·0	4·6	5·1	5·7	1·1	2·1	3·2	4·2	5·3	6·3
·66	0·6	1·2	1·8	2·4	3·0	3·6	4·2	4·8	5·4	6·0	1·0	2·1	3·1	4·2	5·2	6·2
·67	0·6	1·3	1·9	2·5	3·1	3·8	4·4	5·0	5·6	6·3	1·0	2·0	3·1	4·1	5·1	6·1
·68	0·7	1·3	2·0	2·6	3·3	3·9	4·6	5·2	5·9	6·5	1·0	2·0	3·0	4·0	5·0	6·0
·69	0·7	1·4	2·0	2·7	3·4	4·1	4·7	5·4	6·1	6·8	1·0	2·0	3·0	3·9	4·9	5·9
0·70	0·7	1·4	2·1	2·8	3·5	4·2	4·9	5·6	6·3	7·0	1·0	1·9	2·9	3·9	4·8	5·8
·71	0·7	1·4	2·2	2·9	3·6	4·3	5·0	5·8	6·5	7·2	0·9	1·9	2·8	3·8	4·7	5·7
·72	0·7	1·5	2·2	3·0	3·7	4·4	5·2	5·9	6·7	7·4	0·9	1·8	2·8	3·7	4·6	5·5
·73	0·8	1·5	2·3	3·0	3·8	4·5	5·3	6·0	6·8	7·6	0·9	1·8	2·7	3·6	4·5	5·4
·74	0·8	1·5	2·3	3·1	3·8	4·6	5·4	6·2	6·9	7·7	0·9	1·8	2·6	3·5	4·4	5·3
0·75	0·8	1·6	2·3	3·1	3·9	4·7	5·5	6·3	7·0	7·8	0·9	1·7	2·6	3·4	4·3	5·1
·76	0·8	1·6	2·4	3·2	4·0	4·7	5·5	6·3	7·1	7·9	0·8	1·7	2·5	3·3	4·1	5·0
·77	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·8	1·6	2·4	3·2	4·0	4·8
·78	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·8	1·6	2·3	3·1	3·9	4·7
·79	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·7	1·5	2·2	3·0	3·7	4·5
0·80	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·7	1·4	2·2	2·9	3·6	4·3
·81	0·8	1·6	2·4	3·2	4·0	4·8	5·6	6·4	7·2	8·0	0·7	1·4	2·1	2·8	3·5	4·1
·82	0·8	1·6	2·4	3·1	3·9	4·7	5·5	6·3	7·1	7·9	0·7	1·3	2·0	2·6	3·3	4·0
·83	0·8	1·6	2·3	3·1	3·9	4·7	5·4	6·2	7·0	7·8	0·6	1·3	1·9	2·5	3·1	3·8
·84	0·8	1·5	2·3	3·0	3·8	4·6	5·3	6·1	6·9	7·6	0·6	1·2	1·8	2·4	3·0	3·6
0·85	0·7	1·5	2·2	3·0	3·7	4·5	5·2	6·0	6·7	7·4	0·6	1·1	1·7	2·3	2·8	3·4
·86	0·7	1·4	2·2	2·9	3·6	4·3	5·1	5·8	6·5	7·2	0·5	1·1	1·6	2·1	2·7	3·2
·87	0·7	1·4	2·1	2·8	3·5	4·2	4·9	5·6	6·3	7·0	0·5	1·0	1·5	2·0	2·5	3·0
·88	0·7	1·3	2·0	2·7	3·3	4·0	4·7	5·4	6·0	6·7	0·5	0·9	1·4	1·9	2·3	2·8
·89	0·6	1·3	1·9	2·5	3·2	3·8	4·5	5·1	5·7	6·4	0·4	0·9	1·3	1·7	2·1	2·6
0·90	0·6	1·2	1·8	2·4	3·0	3·6	4·2	4·8	5·4	6·0	0·4	0·8	1·2	1·6	2·0	2·4
·91	0·6	1·1	1·7	2·2	2·8	3·4	3·9	4·5	5·0	5·6	0·4	0·7	1·1	1·4	1·8	2·1
·92	0·5	1·0	1·5	2·1	2·6	3·1	3·6	4·1	4·6	5·2	0·3	0·6	1·0	1·3	1·6	1·9
·93	0·5	0·9	1·4	1·9	2·3	2·8	3·3	3·7	4·2	4·7	0·3	0·6	0·8	1·1	1·4	1·7
·94	0·4	0·8	1·2	1·7	2·1	2·5	2·9	3·3	3·7	4·1	0·2	0·5	0·7	1·0	1·2	1·4
0·95	0·4	0·7	1·1	1·4	1·8	2·1	2·5	2·9	3·2	3·6	0·2	0·4	0·6	0·8	1·0	1·2
·96	0·3	0·6	0·9	1·2	1·5	1·8	2·1	2·4	2·6	2·9	0·2	0·3	0·5	0·7	0·8	1·0
·97	0·2	0·5	0·7	0·9	1·1	1·4	1·6	1·8	2·1	2·3	0·1	0·2	0·4	0·5	0·6	0·7
·98	0·2	0·3	0·5	0·6	0·8	0·9	1·1	1·3	1·4	1·6	0·1	0·2	0·2	0·3	0·4	0·5
0·99	0·1	0·2	0·2	0·3	0·4	0·5	0·6	0·6	0·7	0·8	0·0	0·1	0·1	0·2	0·2	0·2

The correction has the opposite sign to $\delta^3_{\frac{1}{2}}$.

The correction has the same sign as $\delta^4_0 + \delta^4_1$.

Example 16.2. *Interpolation of the equatorial rectangular coordinates of the Sun*
1960 March 7 at $22^{\mathrm{h}}\ 26^{\mathrm{m}}\ 11^{\mathrm{s}}{\cdot}2$ E.T.

Date 1960	X	δ	δ^2	Y	δ	δ^2	Z	δ	δ^2
Mar.6	+0·960 3517		−2911	−0·228 8976		+ 680	−0·099 2672		+ 294
		+ 4 4560			+15 3669			+6 6639	
7	**·964 8077**		2920	**·213 5307**		632	**·092 6033**		277
		4 1640			15 4301			6 6916	
8	·968 9717		2927	·198 1006		586	·085 9117		256
		3 8713			15 4887			6 7172	
9	·972 8430		2932	·182 6119		539	·079 1945		235

The tabular values are for 0^{h} E.T. at intervals of one day so the interpolating factor is $22^{\mathrm{h}}\ 26^{\mathrm{m}}\ 11^{\mathrm{s}}{\cdot}2$ expressed as a fraction of a day, i.e., $p = 0{\cdot}934851$. Mental interpolation in table 16.2A (second opening) gives $B_2 = -0{\cdot}01522$. Third and fourth differences are smaller than the normal limits, so the values of the equatorial rectangular coordinates of the Sun at $22^{\mathrm{h}}\ 26^{\mathrm{m}}\ 11^{\mathrm{s}}{\cdot}2$ E.T. on March 7 are:

$$X = +0{\cdot}964\ 8077 + (+0{\cdot}934851) \times (+\ 4\ 1640) + (-0{\cdot}01522) \times (-5847) = +0{\cdot}968\ 7093$$

$$Y = -0{\cdot}213\ 5307 + (+0{\cdot}934851) \times (+15\ 4301) + (-0{\cdot}01522) \times (+1218) = -0{\cdot}199\ 1077$$

$$Z = -0{\cdot}092\ 6033 + (+0{\cdot}934851) \times (+\ 6\ 6916) + (-0{\cdot}01522) \times (+\ 533) = -0{\cdot}086\ 3485$$

Table 16.2 is primarily intended for use when third and fourth differences are also appreciable. Table 16.2A is entered with argument p to give B_2, and hence $B_2\,(\delta_0^2 + \delta_1^2)$ is calculated. Then table 16.2B, which is similar to table 16.1, is entered with arguments p and $\delta_{\frac{1}{2}}^3$ to give $B_3\delta_{\frac{1}{2}}^3$ directly, and then with arguments p and $\delta_0^4 + \delta_1^4$ to give $B_4\,(\delta_0^4 + \delta_1^4)$ directly. The first opening of table 16.2 gives all values for p less than 0·5 while the second gives all values for p greater than 0·5. Particular care should, however, be taken to ensure that the corrections are applied with the correct signs. The corrections are given to 0·1; if the nearest values of the arguments are taken the corrections may each be in error by about 0·7, but interpolation by inspection will considerably reduce the error from this source. When used to maximum accuracy the error of a rounded-off interpolate obtained by using these tables will not exceed 2 units. The tables cover directly the whole range of differences required.

Again inverse interpolation proceeds by successive approximation; the first approximation to the required value of p is obtained by linear interpolation, then estimates of B_2, $B_3\ \delta_{\frac{1}{2}}^3$, $B_4\,(\delta_0^4 + \delta_1^4)$ are obtained from the table and used to give a second approximation to p and so on. Except in extreme cases where the function is near a turning point, the first approximation to p is adequate for $B_3\ \delta_{\frac{1}{2}}^3$ and $B_4\,(\delta_0^4 + \delta_1^4)$.

Example 16.3. *Interpolation of the horizontal parallax of the Moon*
1960 March 7 at $9^h\ 53^m\ 24^s$ E.T.

Date 1960	Horizontal Parallax	δ	δ^2	δ^3	δ^4
	′ ″				
Mar. 6·0	54 13·294				
		″ + 0·842			
6·5	54 14·136		″ +2·596		
		3·438		− 82	
7·0	54 17·574		2·514		−40
		5·952		122	
7·5	54 23·526		2·392		−37
		8·344		159	
8·0	54 31·870		2·233		
		10·577			

The tabular values are for 0^h and 12^h E.T. so the interpolating factor is $9^h\ 53^m\ 24^s$ expressed as a fraction of 12^h, i.e., $p = 0{\cdot}82417$. Mental interpolation in table 16.2A gives $B_2 = -0{\cdot}03623$. Entry of table 16.2B with arguments $p = 0{\cdot}82$ and $\delta^3_{\frac{1}{2}} = 120$ gives 1·0 for the third-difference correction, which for $p > 0{\cdot}5$ has the opposite sign to $\delta^3_{\frac{1}{2}}$; and entry with $p = 0{\cdot}82$ and $\delta^4_0 + \delta^4_1 = 75$ gives 0·5 for the fourth-difference correction, which has the same sign as $\delta^4_0 + \delta^4_1$. Hence the horizontal parallax of the Moon at $9^h\ 53^m\ 24^s$ on 1960 March 7 is

$$54'\ 17''{\cdot}574 + (+0{\cdot}82417) \times (5''{\cdot}952) + (-0{\cdot}03023) \times (+4''{\cdot}906) + (+0''{\cdot}0010) + (-0''{\cdot}0005) = 54'\ 22''{\cdot}302$$

Example 16.4. *Inverse interpolation of the horizontal parallax of the Moon*

To find the time on 1960 March 7 when H.P. = 54′ 19″·000

The relevant data, using values from example 16.3, are in units of 0″·001:

$f_p - f_0 = +1426 \quad \delta_{\frac{1}{2}} = +5952 \quad \delta^2_0 + \delta^2_1 = +4906 \quad \delta^3_{\frac{1}{2}} = -122 \quad \delta^4_0 + \delta^4_1 = -77$

The first approximation to the required interpolating factor p is

$$(f_p - f_0) / \delta_{\frac{1}{2}} = +1426 / +5952 = 0{\cdot}24$$

This is used to obtain estimates of B_2 from table 16.2A, and $B_4\,(\delta^4_0 + \delta^4_1)$, from 16.2B; these are then used to give a second approximation to p as

$$\{ f_p - f_0 - B_2\,(\delta^2_0 + \delta^2_1) - B_3\,\delta_{\frac{1}{2}} - B_4\,(\delta^4_0 + \delta^4_1) \} / \delta_{\frac{1}{2}}$$

The value so obtained is used to correct the estimates of B_2 and, where necessary, of the third- and fourth-difference corrections as follows:

Approximation to p	B_2	$B_3\,\delta_{\frac{1}{2}}$	$B_4\,(\delta^4_0 + \delta^4_1)$	$\{ f_p - \dots \} / \delta_{\frac{1}{2}}$
0·24	−0·04560	−0·9	−0·6	0·277
0·277	−0·05007	,,	−0·7	0·2811
0·2811	−0·05052	,,	,,	0·2815
0·2815	−0·05056	,,	,,	,,

The required fraction p of the interval of 12^h is thus 0·2815, so that the time at which the horizontal parallax is 54′ 19″·000 is 1960 March 7 at $03^h\ 22^m{\cdot}7$.

This example illustrates the case in which $\delta_{\frac{1}{2}}$ is of the same order as $\delta^2_0 + \delta^2_1$, and convergence is slow. The number of approximations required may be reduced by intelligent anticipation; techniques suitable for desk calculating machines are described in *Interpolation and Allied Tables*.

Maximum differences in the fundamental geocentric ephemerides

Table 16.3 gives a rough guide to the maximum differences in the fundamental geocentric ephemerides tabulated in the Ephemeris for 0^h E.T. at intervals of one day, except for the Moon and Pluto. The maximum values of the differences are given in units of the end figure. First differences are tabulated for all these quantities except the longitude and latitude of the Sun and Moon, the nutation, the times of ephemeris transit of the planets, and the distances of the minor planets.

Auxiliary quantities, such as semi-diameters, and small horizontal parallaxes are not listed; they are usually linearly interpolable, although in some cases second differences may be significant.

Table 16.3. Maximum differences in the fundamental ephemerides

			Unit	δ^2	δ^3	δ^4	δ^6
SUN	(1^d)	Longitude (mean equinox)	$0''\cdot 1$	23	1		
		Rectangular coordinates (X, Y, Z)	10^{-7}	3000	60	12	
MOON	($0^d\cdot 5$)	Apparent longitude	$0''\cdot 01$	45000	9000	1500	100
		Apparent latitude	$0''\cdot 01$	33000	5300	1000	70
		Horizontal parallax	$0''\cdot 0001$	53000	8000	2000	200
NUTATION	(1^d)	in longitude	$0''\cdot 001$	73	42	20	
		in obliquity	$0''\cdot 001$	32	20	15	

	SUN	MOON (1^h)	MERCURY	VENUS	MARS	JUPITER	SATURN	URANUS	NEPTUNE	PLUTO (4^d)	MINOR PLANETS
Apparent right ascension (astrometric for Pluto and minor planets)											
Unit	$0^s\cdot 01$	$0^s\cdot 001$	$0^s\cdot 01$	$0^s\cdot 01$	$0^s\cdot 01$	$0^s\cdot 001$	$0^s\cdot 001$	$0^s\cdot 001$	$0^s\cdot 001$	$0^s\cdot 001$	$0^s\cdot 01$
δ^2	86	570	5100	1050	380	850	400	200	130	2000	220
δ^3	4	20	1200	110	25	30	12	7	5	180	14
δ^4			250	12						20	
δ^6			40								
Apparent declination (astrometric for Pluto and minor planets)											
Unit	$0''\cdot 1$	$0''\cdot 01$	$0''\cdot 1$	$0''\cdot 1$	$0''\cdot 1$	$0''\cdot 01$	$0''\cdot 01$	$0''\cdot 01$	$0''\cdot 01$	$0''\cdot 01$	$0''\cdot 1$
δ^2	285	1400	3600	950	200	500	270	140	70	1500	370
δ^3	6	20	800	90	15	20	10	4	3	130	14
δ^4			160	20						15	
δ^6			30								
Distance (in earth radii at interval $0^d\cdot 5$ for Moon)											
Unit	10^{-7}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-6}	10^{-6}	10^{-5}	10^{-5}	10^{-5}	10^{-6}
δ^2	65	78000	3200	300	200	320	300	35	30	500	300
δ^3	3	10000	350	12	7	10	8			40	6
δ^4		2500	10							6	
Ephemeris transit (equation of time for Sun)											
Unit	$0^s\cdot 01$	$0^h\cdot 0001$	1^s	1^s	1^s	1^s	1^s	1^s	1^s	1^m	1^s
δ^2	85	270	60	15	6						4
δ^3	3	95	15								
δ^4		30									

C. NUMERICAL DIFFERENTIATION

The following formulae for derivatives in terms of differences are used in other sections of the Supplement; reference should be made to *Interpolation and Allied Tables* for other formulae and notes.

Notation: f'_p denotes the value of df/dt at the point $t = t_0 + ph$.

Differentiation of Bessel's interpolation formula leads to:

$$hf'_p = \delta_{\frac{1}{2}} + \tfrac{1}{2}(p - \tfrac{1}{2})(\delta^2_0 + \delta^2_1) + \tfrac{1}{12}(1 - 6p + 6p^2)\delta^3_{\frac{1}{2}} + \dots$$

This formula is intended for use in the range $0 \leqslant p \leqslant 1$; the maximum value of the third-difference coefficient is $\frac{1}{12}$ and occurs at $p = 0$ and 1. At $p = \frac{1}{2}$ the formula reduces to:

$$hf'_{\frac{1}{2}} = \delta_{\frac{1}{2}} - \tfrac{1}{24}\delta^3_{\frac{1}{2}} + \tfrac{3}{640}\delta^5_{\frac{1}{2}} - \dots$$

Differentiation of Stirling's interpolation formula leads to:

$$hf'_p = \mu\delta_0 + p\delta^2_0 - \tfrac{1}{6}(1 - 3p^2)\mu\delta^3_0 + \dots$$

where, for example, $\mu\delta^3_0 = \frac{1}{2}(\delta^3_{-\frac{1}{2}} + \delta^3_{\frac{1}{2}})$. This formula is intended for use in the range $-\frac{1}{2} \leqslant p \leqslant \frac{1}{2}$; the maximum value of the coefficient of the mean third difference is $\frac{1}{6}$ and occurs at $p = 0$. At $p = 0$ the formula reduces to:

$$hf'_0 = \mu\delta_0 - \tfrac{1}{6}\mu\delta^3_0 + \tfrac{1}{30}\mu\delta^5_0 - \dots$$

The condition for a maximum or minimum is that $f'_p = 0$. For Stirling's formula this condition may be expressed as:

$$p = \{-\mu\delta_0 + \tfrac{1}{6}(1 - 3p^2)\mu\delta^3_0 - \dots\}/\delta^2_0$$

This equation must normally be solved by successive approximations. The maximum contribution from the third difference is $\frac{1}{6}\mu\delta^3_0/\delta^2_0$, and if this is negligible p may be evaluated directly from:

$$p = -\mu\delta_0/\delta^2_0 = -\tfrac{1}{2} - \delta_{-\frac{1}{2}}/\delta^2_0$$

Example 8.2 illustrates this procedure.

In astronomical usage, the terms " variation " and " motion " are synonymous with " derivative with respect to time ", when qualified by an adjective defining the unit of time, and are usually evaluated for the tabular points. The term " secular variation " usually implies a second derivative with respect to time.

The second derivative f''_p is obtained by differentiating the formulae for f'_p; for Stirling's formula this leads to:

$$h^2f''_p = \delta^2_0 + p\mu\delta^3_0 + \dots$$

and, when $p = 0$, to:

$$h^2f''_0 = \delta^2_0 - \tfrac{1}{12}\delta^4_0 + \tfrac{1}{90}\delta^6_0 - \dots$$

The angles of the triangle are denoted by A, B, C; the opposite sides, by a, b, c. Other formulae may be obtained by cyclic changes of A, B, C and a, b, c.

Plane triangle	*Spherical triangle*
$a \sin B = b \sin A$	$\sin a \sin B = \sin b \sin A$
$a \cos B = c - b \cos A$	$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$
$a^2 = b^2 + c^2 - 2\,bc \cos A$	$\cos a = \cos b \cos c + \sin b \sin c \cos A$
$a \cos C = b - c \cos A$	$\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A$
$a \sin C = c \sin A$	$\sin a \sin C = \sin c \sin A$
$\sin \tfrac{1}{2}(B + C) = \cos \tfrac{1}{2} A$	$\cos \tfrac{1}{2} a \sin \tfrac{1}{2}(B + C) = \cos \tfrac{1}{2} A \cos \tfrac{1}{2}(b - c)$
$\cos \tfrac{1}{2}(B + C) = \sin \tfrac{1}{2} A$	$\cos \tfrac{1}{2} a \cos \tfrac{1}{2}(B + C) = \sin \tfrac{1}{2} A \cos \tfrac{1}{2}(b + c)$
$a \sin \tfrac{1}{2}(B - C) = (b - c) \cos \tfrac{1}{2} A$	$\sin \tfrac{1}{2} a \sin \tfrac{1}{2}(B - C) = \cos \tfrac{1}{2} A \sin \tfrac{1}{2}(b - c)$
$a \cos \tfrac{1}{2}(B - C) = (b + c) \sin \tfrac{1}{2} A$	$\sin \tfrac{1}{2} a \cos \tfrac{1}{2}(B - C) = \sin \tfrac{1}{2} A \sin \tfrac{1}{2}(b + c)$
$s = \tfrac{1}{2}(a + b + c)$	$s = \tfrac{1}{2}(a + b + c)$
$r^2 = (s - a)(s - b)(s - c)/s$	$m^2 = \sin(s - a) \sin(s - b) \sin(s - c)/\sin s$
Area $= sr = \tfrac{1}{2}\,bc \sin A$	Area $= A + B + C - \pi$
$bc \sin^2 \tfrac{1}{2} A = (s - b)(s - c)$	$\sin b \sin c \sin^2 \tfrac{1}{2} A = \sin(s - b) \sin(s - c)$
$bc \cos^2 \tfrac{1}{2} A = s(s - a)$	$\sin b \sin c \cos^2 \tfrac{1}{2} A = \sin s \sin(s - a)$
$\tan \tfrac{1}{2} A = r/(s - a)$	$\tan \tfrac{1}{2} A = m/\sin(s - a)$

Spherical Triangle

Additional formulae	*Right-angled triangle* : $A = \pi/2$
$\sin A \sin b = \sin B \sin a$	$\sin a \sin B = \sin b$
$\sin A \cos b = \cos B \sin C + \sin B \cos C \cos a$	$\sin a \cos B = \cos b \sin c$
$\cos A = -\cos B \cos C + \sin B \sin C \cos a$	$\cos a = \cos b \cos c$
$\sin A \cos c = \sin B \cos C + \cos B \sin C \cos a$	$\sin a \cos C = \sin b \cos c$
$\sin A \sin c = \sin C \sin a$	$\sin a \sin C = \sin c$
$\cos a \cos B = \sin a \cot c - \sin B \cot C$	$\sin B \sin a = \sin b$
$\cos a \cos C = \sin a \cot b - \sin C \cot B$	$\sin B \cos a = \cos b \cos C$
	$\cos B = \cos b \sin C$
$\cos b \cos A = \sin b \cot c - \sin A \cot C$	$\sin B \cos c = \cos C$
$\cos c \cos A = \sin c \cot b - \sin A \cot B$	$\sin B \sin c = \sin b \sin C$
$S = \tfrac{1}{2}(A + B + C)$	$\tan a = \tan b \sec C$
$M^2 = -\cos(S - A) \cos(S - B) \cos(S - C)/\cos S$	$\tan a = \tan c \sec B$
$\sin B \sin C \sin^2 \tfrac{1}{2} a = -\cos S \cos(S - A)$	$\cos a = \cot B \cot C$
$\sin B \sin C \cos^2 \tfrac{1}{2} a = \cos(S - B) \cos(S - C)$	$\tan b = \sin c \tan B$
$\tan \tfrac{1}{2} a = \cos(S - A)/M$	$\tan c = \sin b \tan C$

Quadrantal triangle: $a = \pi/2$

$\sin A \sin b = \sin B$	$\sin b \sin A = \sin B$	$\tan A = -\tan B \sec c$
$\sin A \cos b = \cos B \sin C$	$\sin b \cos A = -\cos B \cos c$	$\tan A = -\tan C \sec b$
$\cos A = -\cos B \cos C$	$\cos b = \cos B \sin c$	$\cos A = -\cot b \cot c$
$\sin A \cos c = \sin B \cos C$	$\sin b \cos C = \cos c$	$\tan B = \sin C \tan b$
$\sin A \sin c = \sin C$	$\sin b \sin C = \sin B \sin c$	$\tan C = \sin B \tan c$

Spherical triangle in which b is small

$$a - c = -b \cos A + \tfrac{1}{2} b^2 \cot c \sin^2 A + \ldots$$

$$B \sin c = +b \sin A + \tfrac{1}{2} b^2 \cot c \sin 2A + \ldots$$

$$\pi - C - A = +b \cot c \sin A + \tfrac{1}{4} b^2 (1 + 2 \cot^2 c) \sin 2A + \ldots$$

17. CONVERSION TABLES

Introduction. The tables in this section are designed to facilitate the conversion of measures of time and of angle from one system of units to another. In using the tables it is usually only necessary to take out and add two quantities to obtain the required value, but occasionally three or more quantities are needed. Interpolation is not normally required; in a few cases small corrections are to be taken from critical tables.

Tables 17.1 *and* 17.2. These tables are for the conversion of intervals of mean sidereal time into equivalent intervals of mean solar time, and vice versa. In constructing the tables it has been assumed, following Newcomb, that:

24^h of mean sidereal time $= 23^h\ 56^m\ 04^s{\cdot}09054$ of mean solar time

24^h of mean solar time $= 24^h\ 03^m\ 56^s{\cdot}55536$ of mean sidereal time

An alternative set of tables is given in the Ephemeris (*A.E.* 1960, Tables VIII and IX); these tables give the corrections to be applied rather than the direct equivalents.

Table 17.3. This table is for the conversion of decimal fractions of a mean solar day into equivalent intervals of mean sidereal time, expressed in hours and minutes (to one decimal). It is primarily intended for use in computing parallax corrections (section 2F), and so the number of significant figures is small.

Tables 17.4 and 17.5. These tables are for the conversion of time intervals (or angles) expressed in decimals of a day (or revolution) into the equivalent measure in hours, minutes, and seconds, and vice versa. Table 17.5 is primarily intended for use in obtaining interpolating factors, and so the respondent is restricted to six decimals.

Tables 17.6 and 17.7. These tables are for the conversion of sexagesimal measure to decimal measure, and vice versa. Although the units indicated are degrees, minutes, and seconds the tables are equally applicable to hours, minutes, and seconds.

Tables 17.8 and 17.9. These tables are for the conversion of measures of time (hours, minutes, and seconds) to measures of arc (degrees, minutes, and seconds), and vice versa; they are the same as Tables XI and XII of the Ephemeris.

Table 17.10. This table is intended only for occasional use in the conversion, to low accuracy, of sexagesimal angular or time measure to decimals of a revolution (or day) or to radians, and vice versa. Conversion constants are given for use when greater accuracy is required.

Sidereal Time	Solar Time	Sidereal Time	Solar Time	Sidereal Time	Solar Time
h m	h m s	h m	h m s	h m	h m s
0 00	0 00 00·000	**8 00**	7 58 41·364	**16 00**	15 57 22·727
10	0 09 58·362	**10**	8 08 39·725	**10**	16 07 21·089
20	0 19 56·723	**20**	8 18 38·087	**20**	16 17 19·451
30	0 29 55·085	**30**	8 28 36·449	**30**	16 27 17·812
40	0 39 53·447	**40**	8 38 34·810	**40**	16 37 16·174
50	0 49 51·809	**50**	8 48 33·172	**50**	16 47 14·536
1 00	0 59 50·170	**9 00**	8 58 31·534	**17 00**	16 57 12·897
10	1 09 48·532	**10**	9 08 29·896	**10**	17 07 11·259
20	1 19 46·894	**20**	9 18 28·257	**20**	17 17 09·621
30	1 29 45·256	**30**	9 28 26·619	**30**	17 27 07·983
40	1 39 43·617	**40**	9 38 24·981	**40**	17 37 06·344
50	1 49 41·979	**50**	9 48 23·343	**50**	17 47 04·706
2 00	1 59 40·341	**10 00**	9 58 21·704	**18 00**	17 57 03·068
10	2 09 38·703	**10**	10 08 20·066	**10**	18 07 01·430
20	2 19 37·064	**20**	10 18 18·428	**20**	18 16 59·791
30	2 29 35·426	**30**	10 28 16·790	**30**	18 26 58·153
40	2 39 33·788	**40**	10 38 15·151	**40**	18 36 56·515
50	2 49 32·150	**50**	10 48 13·513	**50**	18 46 54·877
3 00	2 59 30·511	**11 00**	10 58 11·875	**19 00**	18 56 53·238
10	3 09 28·873	**10**	11 08 10·237	**10**	19 06 51·600
20	3 19 27·235	**20**	11 18 08·598	**20**	19 16 49·962
30	3 29 25·597	**30**	11 28 06·960	**30**	19 26 48·324
40	3 39 23·958	**40**	11 38 05·322	**40**	19 36 46·685
50	3 49 22·320	**50**	11 48 03·684	**50**	19 46 45·047
4 00	3 59 20·682	**12 00**	11 58 02·045	**20 00**	19 56 43·409
10	4 09 19·043	**10**	12 08 00·407	**10**	20 06 41·771
20	4 19 17·405	**20**	12 17 58·769	**20**	20 16 40·132
30	4 29 15·767	**30**	12 27 57·130	**30**	20 26 38·494
40	4 39 14·129	**40**	12 37 55·492	**40**	20 36 36·856
50	4 49 12·490	**50**	12 47 53·854	**50**	20 46 35·217
5 00	4 59 10·852	**13 00**	12 57 52·216	**21 00**	20 56 33·579
10	5 09 09·214	**10**	13 07 50·577	**10**	21 06 31·941
20	5 19 07·576	**20**	13 17 48·939	**20**	21 16 30·303
30	5 29 05·937	**30**	13 27 47·301	**30**	21 26 28·664
40	5 39 04·299	**40**	13 37 45·663	**40**	21 36 27·026
50	5 49 02·661	**50**	13 47 44·024	**50**	21 46 25·388
6 00	5 59 01·023	**14 00**	13 57 42·386	**22 00**	21 56 23·750
10	6 08 59·384	**10**	14 07 40·748	**10**	22 06 22·111
20	6 18 57·746	**20**	14 17 39·110	**20**	22 16 20·473
30	6 28 56·108	**30**	14 27 37·471	**30**	22 26 18·835
40	6 38 54·470	**40**	14 37 35·833	**40**	22 36 17·197
50	6 48 52·831	**50**	14 47 34·195	**50**	22 46 15·558
7 00	6 58 51·193	**15 00**	14 57 32·557	**23 00**	22 56 13·920
10	7 08 49·555	**10**	15 07 30·918	**10**	23 06 12·282
20	7 18 47·917	**20**	15 17 29·280	**20**	23 16 10·644
30	7 28 46·278	**30**	15 27 27·642	**30**	23 26 09·005
40	7 38 44·640	**40**	15 37 26·004	**40**	23 36 07·367
50	7 48 43·002	**50**	15 47 24·365	**50**	23 46 05·729
8 00	7 58 41·364	**16 00**	15 57 22·727	**24 00**	23 56 04·091

Sidereal Time	0m Solar Time	1m Solar Time
s	m s	m s
0	0 00·000	0 59·836
1	0 00·997	1 00·833
2	0 01·995	1 01·831
3	0 02·992	1 02·828
4	0 03·989	1 03·825
5	0 04·986	1 04·823
6	0 05·984	1 05·820
7	0 06·981	1 06·817
8	0 07·978	1 07·814
9	0 08·975	1 08·812
10	0 09·973	1 09·809
11	0 10·970	1 10·806
12	0 11·967	1 11·803
13	0 12·965	1 12·801
14	0 13·962	1 13·798
15	0 14·959	1 14·795
16	0 15·956	1 15·792
17	0 16·954	1 16·790
18	0 17·951	1 17·787
19	0 18·948	1 18·784
20	0 19·945	1 19·782
21	0 20·943	1 20·779
22	0 21·940	1 21·776
23	0 22·937	1 22·773
24	0 23·934	1 23·771
25	0 24·932	1 24·768
26	0 25·929	1 25·765
27	0 26·926	1 26·762
28	0 27·924	1 27·760
29	0 28·921	1 28·757
30	0 29·918	1 29·754
31	0 30·915	1 30·752
32	0 31·913	1 31·749
33	0 32·910	1 32·746
34	0 33·907	1 33·743
35	0 34·904	1 34·741
36	0 35·902	1 35·738
37	0 36·899	1 36·735
38	0 37·896	1 37·732
39	0 38·894	1 38·730
40	0 39·891	1 39·727
41	0 40·888	1 40·724
42	0 41·885	1 41·721
43	0 42·883	1 42·719
44	0 43·880	1 43·716
45	0 44·877	1 44·713
46	0 45·874	1 45·711
47	0 46·872	1 46·708
48	0 47·869	1 47·705
49	0 48·866	1 48·702
50	0 49·863	1 49·700
51	0 50·861	1 50·697
52	0 51·858	1 51·694
53	0 52·855	1 52·691
54	0 53·853	1 53·689
55	0 54·850	1 54·686
56	0 55·847	1 55·683
57	0 56·844	1 56·681
58	0 57·842	1 57·678
59	0 58·839	1 58·675

FRACTIONS OF A SECOND

The mean solar equivalent of a fraction of a sidereal second is equal to that fraction *diminished* by the amount in the accompanying critical table.

In critical cases ascend.

Fraction of a second	Amount to be subtracted
s	s
0·000	
	0·000
0·183	
	·001
0·549	
	·002
0·915	
	0·003
1·000	

Sidereal Time	2^m	3^m	4^m	5^m	6^m	7^m	8^m	9^m
	Solar Time							
s	m s	m s	m s	m s	m s	m s	m s	m s
0	1 59·672	2 59·509	3 59·345	4 59·181	5 59·017	6 58·853	7 58·689	8 58·526
1	2 00·670	3 00·506	4 00·342	5 00·178	6 00·014	6 59·850	7 59·687	8 59·523
2	2 01·667	3 01·503	4 01·339	5 01·175	6 01·012	7 00·848	8 00·684	9 00·520
3	2 02·664	3 02·500	4 02·337	5 02·173	6 02·009	7 01·845	8 01·681	9 01·517
4	2 03·661	3 03·498	4 03·334	5 03·170	6 03·006	7 02·842	8 02·678	9 02·515
5	2 04·659	3 04·495	4 04·331	5 04·167	6 04·003	7 03·840	8 03·676	9 03·512
6	2 05·656	3 05·492	4 05·328	5 05·164	6 05·001	7 04·837	8 04·673	9 04·509
7	2 06·653	3 06·489	4 06·326	5 06·162	6 05·998	7 05·834	8 05·670	9 05·506
8	2 07·651	3 07·487	4 07·323	5 07·159	6 06·995	7 06·831	8 06·668	9 06·504
9	2 08·648	3 08·484	4 08·320	5 08·156	6 07·992	7 07·829	8 07·665	9 07·501
10	2 09·645	3 09·481	4 09·317	5 09·154	6 08·990	7 08·826	8 08·662	9 08·498
11	2 10·642	3 10·478	4 10·315	5 10·151	6 09·987	7 09·823	8 09·659	9 09·496
12	2 11·640	3 11·476	4 11·312	5 11·148	6 10·984	7 10·820	8 10·657	9 10·493
13	2 12·637	3 12·473	4 12·309	5 12·145	6 11·982	7 11·818	8 11·654	9 11·490
14	2 13·634	3 13·470	4 13·306	5 13·143	6 12·979	7 12·815	8 12·651	9 12·487
15	2 14·631	3 14·468	4 14·304	5 14·140	6 13·976	7 13·812	8 13·648	9 13·485
16	2 15·629	3 15·465	4 15·301	5 15·137	6 14·973	7 14·810	8 14·646	9 14·482
17	2 16·626	3 16·462	4 16·298	5 16·134	6 15·971	7 15·807	8 15·643	9 15·479
18	2 17·623	3 17·459	4 17·296	5 17·132	6 16·968	7 16·804	8 16·640	9 16·476
19	2 18·620	3 18·457	4 18·293	5 18·129	6 17·965	7 17·801	8 17·638	9 17·474
20	2 19·618	3 19·454	4 19·290	5 19·126	6 18·962	7 18·799	8 18·635	9 18·471
21	2 20·615	3 20·451	4 20·287	5 20·124	6 19·960	7 19·796	8 19·632	9 19·468
22	2 21·612	3 21·448	4 21·285	5 21·121	6 20·957	7 20·793	8 20·629	9 20·465
23	2 22·610	3 22·446	4 22·282	5 22·118	6 21·954	7 21·790	8 21·627	9 21·463
24	2 23·607	3 23·443	4 23·279	5 23·115	6 22·952	7 22·788	8 22·624	9 22·460
25	2 24·604	3 24·440	4 24·276	5 24·113	6 23·949	7 23·785	8 23·621	9 23·457
26	2 25·601	3 25·438	4 25·274	5 25·110	6 24·946	7 24·782	8 24·618	9 24·455
27	2 26·599	3 26·435	4 26·271	5 26·107	6 25·943	7 25·779	8 25·616	9 25·452
28	2 27·596	3 27·432	4 27·268	5 27·104	6 26·941	7 26·777	8 26·613	9 26·449
29	2 28·593	3 28·429	4 28·266	5 28·102	6 27·938	7 27·774	8 27·610	9 27·446
30	2 29·590	3 29·427	4 29·263	5 29·099	6 28·935	7 28·771	8 28·607	9 28·444
31	2 30·588	3 30·424	4 30·260	5 30·096	6 29·932	7 29·769	8 29·605	9 29·441
32	2 31·585	3 31·421	4 31·257	5 31·093	6 30·930	7 30·766	8 30·602	9 30·438
33	2 32·582	3 32·418	4 32·255	5 32·091	6 31·927	7 31·763	8 31·599	9 31·435
34	2 33·580	3 33·416	4 33·252	5 33·088	6 32·924	7 32·760	8 32·597	9 32·433
35	2 34·577	3 34·413	4 34·249	5 34·085	6 33·921	7 33·758	8 33·594	9 33·430
36	2 35·574	3 35·410	4 35·246	5 35·083	6 34·919	7 34·755	8 34·591	9 34·427
37	2 36·571	3 36·407	4 36·244	5 36·080	6 35·916	7 35·752	8 35·588	9 35·425
38	2 37·569	3 37·405	4 37·241	5 37·077	6 36·913	7 36·749	8 36·586	9 36·422
39	2 38·566	3 38·402	4 38·238	5 38·074	6 37·911	7 37·747	8 37·583	9 37·419
40	2 39·563	3 39·399	4 39·235	5 39·072	6 38·908	7 38·744	8 38·580	9 38·416
41	2 40·560	3 40·397	4 40·233	5 40·069	6 39·905	7 39·741	8 39·577	9 39·414
42	2 41·558	3 41·394	4 41·230	5 41·066	6 40·902	7 40·739	8 40·575	9 40·411
43	2 42·555	3 42·391	4 42·227	5 42·063	6 41·900	7 41·736	8 41·572	9 41·408
44	2 43·552	3 43·388	4 43·225	5 43·061	6 42·897	7 42·733	8 42·569	9 42·405
45	2 44·549	3 44·386	4 44·222	5 44·058	6 43·894	7 43·730	8 43·567	9 43·403
46	2 45·547	3 45·383	4 45·219	5 45·055	6 44·891	7 44·728	8 44·564	9 44·400
47	2 46·544	3 46·380	4 46·216	5 46·053	6 45·889	7 45·725	8 45·561	9 45·397
48	2 47·541	3 47·377	4 47·214	5 47·050	6 46·886	7 46·722	8 46·558	9 46·395
49	2 48·539	3 48·375	4 48·211	5 48·047	6 47·883	7 47·719	8 47·556	9 47·392
50	2 49·536	3 49·372	4 49·208	5 49·044	6 48·881	7 48·717	8 48·553	9 48·389
51	2 50·533	3 50·369	4 50·205	5 50·042	6 49·878	7 49·714	8 49·550	9 49·386
52	2 51·530	3 51·367	4 51·203	5 51·039	6 50·875	7 50·711	8 50·547	9 50·384
53	2 52·528	3 52·364	4 52·200	5 52·036	6 51·872	7 51·709	8 51·545	9 51·381
54	2 53·525	3 53·361	4 53·197	5 53·033	6 52·870	7 52·706	8 52·542	9 52·378
55	2 54·522	3 54·358	4 54·195	5 54·031	6 53·867	7 53·703	8 53·539	9 53·375
56	2 55·519	3 55·356	4 55·192	5 55·028	6 54·864	7 54·700	8 54·536	9 54·373
57	2 56·517	3 56·353	4 56·189	5 56·025	6 55·861	7 55·698	8 55·534	9 55·370
58	2 57·514	3 57·350	4 57·186	5 57·023	6 56·859	7 56·695	8 56·531	9 56·367
59	2 58·511	3 58·347	4 58·184	5 58·020	6 57·856	7 57·692	8 57·528	9 57·364

Solar Time	Sidereal Time	Solar Time	Sidereal Time	Solar Time	Sidereal Time
h m	h m s	h m	h m s	h m	h m s
0 00	0 00 00·000	**8 00**	8 01 18·852	**16 00**	16 02 37·704
10	0 10 01·643	**10**	8 11 20·495	**10**	16 12 39·346
20	0 20 03·285	**20**	8 21 22·137	**20**	16 22 40·989
30	0 30 04·928	**30**	8 31 23·780	**30**	16 32 42·632
40	0 40 06·571	**40**	8 41 25·423	**40**	16 42 44·275
50	0 50 08·214	**50**	8 51 27·066	**50**	16 52 45·917
1 00	1 00 09·856	**9 00**	9 01 28·708	**17 00**	17 02 47·560
10	1 10 11·499	**10**	9 11 30·351	**10**	17 12 49·203
20	1 20 13·142	**20**	9 21 31·994	**20**	17 22 50·846
30	1 30 14·785	**30**	9 31 33·636	**30**	17 32 52·488
40	1 40 16·427	**40**	9 41 35·279	**40**	17 42 54·131
50	1 50 18·070	**50**	9 51 36·922	**50**	17 52 55·774
2 00	2 00 19·713	**10 00**	10 01 38·565	**18 00**	18 02 57·417
10	2 10 21·356	**10**	10 11 40·207	**10**	18 12 59·059
20	2 20 22·998	**20**	10 21 41·850	**20**	18 23 00·702
30	2 30 24·641	**30**	10 31 43·493	**30**	18 33 02·345
40	2 40 26·284	**40**	10 41 45·136	**40**	18 43 03·988
50	2 50 27·927	**50**	10 51 46·778	**50**	18 53 05·630
3 00	3 00 29·569	**11 00**	11 01 48·421	**19 00**	19 03 07·273
10	3 10 31·212	**10**	11 11 50·064	**10**	19 13 08·916
20	3 20 32·855	**20**	11 21 51·707	**20**	19 23 10·558
30	3 30 34·498	**30**	11 31 53·349	**30**	19 33 12·201
40	3 40 36·140	**40**	11 41 54·992	**40**	19 43 13·844
50	3 50 37·783	**50**	11 51 56·635	**50**	19 53 15·487
4 00	4 00 39·426	**12 00**	12 01 58·278	**20 00**	20 03 17·129
10	4 10 41·069	**10**	12 11 59·920	**10**	20 13 18·772
20	4 20 42·711	**20**	12 22 01·563	**20**	20 23 20·415
30	4 30 44·354	**30**	12 32 03·206	**30**	20 33 22·058
40	4 40 45·997	**40**	12 42 04·849	**40**	20 43 23·700
50	4 50 47·640	**50**	12 52 06·491	**50**	20 53 25·343
5 00	5 00 49·282	**13 00**	13 02 08·134	**21 00**	21 03 26·986
10	5 10 50·925	**10**	13 12 09·777	**10**	21 13 28·629
20	5 20 52·568	**20**	13 22 11·420	**20**	21 23 30·271
30	5 30 54·211	**30**	13 32 13·062	**30**	21 33 31·914
40	5 40 55·853	**40**	13 42 14·705	**40**	21 43 33·557
50	5 50 57·496	**50**	13 52 16·348	**50**	21 53 35·200
6 00	6 00 59·139	**14 00**	14 02 17·991	**22 00**	22 03 36·842
10	6 11 00·782	**10**	14 12 19·633	**10**	22 13 38·485
20	6 21 02·424	**20**	14 22 21·276	**20**	22 23 40·128
30	6 31 04·067	**30**	14 32 22·919	**30**	22 33 41·771
40	6 41 05·710	**40**	14 42 24·562	**40**	22 43 43·413
50	6 51 07·353	**50**	14 52 26·204	**50**	22 53 45·056
00	7 01 08·995	**15 00**	15 02 27·847	**23 00**	23 03 46·699
10	7 11 10·638	**10**	15 12 29·490	**10**	23 13 48·342
20	7 21 12·281	**20**	15 22 31·133	**20**	23 23 49·984
30	7 31 13·924	**30**	15 32 32·775	**30**	23 33 51·627
40	7 41 15·566	**40**	15 42 34·418	**40**	23 43 53·270
50	7 51 17·209	**50**	15 52 36·061	**50**	23 53 54·913
8 00	8 01 18·852	**16 00**	16 02 37·704	**24 00**	24 03 56·555

Solar Time	0m Sidereal Time	1m Sidereal Time
s	m s	m s
0	0 00·000	1 00·164
1	0 01·003	1 01·167
2	0 02·005	1 02·170
3	0 03·008	1 03·172
4	0 04·011	1 04·175
5	0 05·014	1 05·178
6	0 06·016	1 06·181
7	0 07·019	1 07·183
8	0 08·022	1 08·186
9	0 09·025	1 09·189
10	0 10·027	1 10·192
11	0 11·030	1 11·194
12	0 12·033	1 12·197
13	0 13·036	1 13·200
14	0 14·038	1 14·203
15	0 15·041	1 15·205
16	0 16·044	1 16·208
17	0 17·047	1 17·211
18	0 18·049	1 18·214
19	0 19·052	1 19·216
20	0 20·055	1 20·219
21	0 21·057	1 21·222
22	0 22·060	1 22·225
23	0 23·063	1 23·227
24	0 24·066	1 24·230
25	0 25·068	1 25·233
26	0 26·071	1 26·235
27	0 27·074	1 27·238
28	0 28·077	1 28·241
29	0 29·079	1 29·244
30	0 30·082	1 30·246
31	0 31·085	1 31·249
32	0 32·088	1 32·252
33	0 33·090	1 33·255
34	0 34·093	1 34·257
35	0 35·096	1 35·260
36	0 36·099	1 36·263
37	0 37·101	1 37·266
38	0 38·104	1 38·268
39	0 39·107	1 39·271
40	0 40·110	1 40·274
41	0 41·112	1 41·277
42	0 42·115	1 42·279
43	0 43·118	1 43·282
44	0 44·120	1 44·285
45	0 45·123	1 45·287
46	0 46·126	1 46·290
47	0 47·129	1 47·293
48	0 48·131	1 48·296
49	0 49·134	1 49·298
50	0 50·137	1 50·301
51	0 51·140	1 51·304
52	0 52·142	1 52·307
53	0 53·145	1 53·309
54	0 54·148	1 54·312
55	0 55·151	1 55·315
56	0 56·153	1 56·318
57	0 57·156	1 57·320
58	0 58·159	1 58·323
59	0 59·162	1 59·326

FRACTIONS OF A SECOND

The sidereal equivalent of a fraction of a mean solar second is equal to that fraction *increased* by the amount in the accompanying critical table.

In critical cases ascend.

Fraction of a second	Amount to be added
s	s
0·000	
	0·000
0·182	
	·001
0·547	
	·002
0·913	
	0·003
1·000	

Solar Time	2^m	3^m	4^m	5^m	6^m	7^m	8^m	9^m
	Sidereal Time							
s	m s	m s	m s	m s	m s	m s	m s	m s
0	2 00·329	3 00·493	4 00·657	5 00·821	6 00·986	7 01·150	8 01·314	9 01·478
1	2 01·331	3 01·496	4 01·660	5 01·824	6 01·988	7 02·153	8 02·317	9 02·481
2	2 02·334	3 02·498	4 02·663	5 02·827	6 02·991	7 03·155	8 03·320	9 03·484
3	2 03·337	3 03·501	4 03·665	5 03·830	6 03·994	7 04·158	8 04·322	9 04·487
4	2 04·340	3 04·504	4 04·668	5 04·832	6 04·997	7 05·161	8 05·325	9 05·489
5	2 05·342	3 05·507	4 05·671	5 05·835	6 05·999	7 06·164	8 06·328	9 06·492
6	2 06·345	3 06·509	4 06·674	5 06·838	6 07·002	7 07·166	8 07·331	9 07·495
7	2 07·348	3 07·512	4 07·676	5 07·841	6 08·005	7 08·169	8 08·333	9 08·498
8	2 08·350	3 08·515	4 08·679	5 08·843	6 09·008	7 09·172	8 09·336	9 09·500
9	2 09·353	3 09·517	4 09·682	5 09·846	6 10·010	7 10·175	8 10·339	9 10·503
10	2 10·356	3 10·520	4 10·684	5 10·849	6 11·013	7 11·177	8 11·342	9 11·506
11	2 11·359	3 11·523	4 11·687	5 11·851	6 12·016	7 12·180	8 12·344	9 12·509
12	2 12·361	3 12·526	4 12·690	5 12·854	6 13·019	7 13·183	8 13·347	9 13·511
13	2 13·364	3 13·528	4 13·693	5 13·857	6 14·021	7 14·186	8 14·350	9 14·514
14	2 14·367	3 14·531	4 14·695	5 14·860	6 15·024	7 15·188	8 15·353	9 15·517
15	2 15·370	3 15·534	4 15·698	5 15·862	6 16·027	7 16·191	8 16·355	9 16·520
16	2 16·372	3 16·537	4 16·701	5 16·865	6 17·029	7 17·194	8 17·358	9 17·522
17	2 17·375	3 17·539	4 17·704	5 17·868	6 18·032	7 18·196	8 18·361	9 18·525
18	2 18·378	3 18·542	4 18·706	5 18·871	6 19·035	7 19·199	8 19·363	9 19·528
19	2 19·381	3 19·545	4 19·709	5 19·873	6 20·038	7 20·202	8 20·366	9 20·530
20	2 20·383	3 20·548	4 20·712	5 20·876	6 21·040	7 21·205	8 21·369	9 21·533
21	2 21·386	3 21·550	4 21·715	5 21·879	6 22·043	7 22·207	8 22·372	9 22·536
22	2 22·389	3 22·553	4 22·717	5 22·882	6 23·046	7 23·210	8 23·374	9 23·539
23	2 23·392	3 23·556	4 23·720	5 23·884	6 24·049	7 24·213	8 24·377	9 24·541
24	2 24·394	3 24·559	4 24·723	5 24·887	6 25·051	7 25·216	8 25·380	9 25·544
25	2 25·397	3 25·561	4 25·726	5 25·890	6 26·054	7 26·218	8 26·383	9 26·547
26	2 26·400	3 26·564	4 26·728	5 26·893	6 27·057	7 27·221	8 27·385	9 27·550
27	2 27·402	3 27·567	4 27·731	5 27·895	6 28·060	7 28·224	8 28·388	9 28·552
28	2 28·405	3 28·569	4 28·734	5 28·898	6 29·062	7 29·227	8 29·391	9 29·555
29	2 29·408	3 29·572	4 29·736	5 29·901	6 30·065	7 30·229	8 30·394	9 30·558
30	2 30·411	3 30·575	4 30·739	5 30·904	6 31·068	7 31·232	8 31·396	9 31·561
31	2 31·413	3 31·578	4 31·742	5 31·906	6 32·071	7 32·235	8 32·399	9 32·563
32	2 32·416	3 32·580	4 32·745	5 32·909	6 33·073	7 33·238	8 33·402	9 33·566
33	2 33·419	3 33·583	4 33·747	5 33·912	6 34·076	7 34·240	8 34·405	9 34·569
34	2 34·422	3 34·586	4 34·750	5 34·914	6 35·079	7 35·243	8 35·407	9 35·572
35	2 35·424	3 35·589	4 35·753	5 35·917	6 36·081	7 36·246	8 36·410	9 36·574
36	2 36·427	3 36·591	4 36·756	5 36·920	6 37·084	7 37·248	8 37·413	9 37·577
37	2 37·430	3 37·594	4 37·758	5 37·923	6 38·087	7 38·251	8 38·415	9 38·580
38	2 38·433	3 38·597	4 38·761	5 38·925	6 39·090	7 39·254	8 39·418	9 39·583
39	2 39·435	3 39·600	4 39·764	5 39·928	6 40·092	7 40·257	8 40·421	9 40·585
40	2 40·438	3 40·602	4 40·767	5 40·931	6 41·095	7 41·259	8 41·424	9 41·588
41	2 41·441	3 41·605	4 41·769	5 41·934	6 42·098	7 42·262	8 42·426	9 42·591
42	2 42·444	3 42·608	4 42·772	5 42·936	6 43·101	7 43·265	8 43·429	9 43·593
43	2 43·446	3 43·611	4 43·775	5 43·939	6 44·103	7 44·268	8 44·432	9 44·596
44	2 44·449	3 44·613	4 44·778	5 44·942	6 45·106	7 45·270	8 45·435	9 45·599
45	2 45·452	3 45·616	4 45·780	5 45·945	6 46·109	7 46·273	8 46·437	9 46·602
46	2 46·454	3 46·619	4 46·783	5 46·947	6 47·112	7 47·276	8 47·440	9 47·604
47	2 47·457	3 47·622	4 47·786	5 47·950	6 48·114	7 48·279	8 48·443	9 48·607
48	2 48·460	3 48·624	4 48·789	5 48·953	6 49·117	7 49·281	8 49·446	9 49·610
49	2 49·463	3 49·627	4 49·791	5 49·956	6 50·120	7 50·284	8 50·448	9 50·613
50	2 50·465	3 50·630	4 50·794	5 50·958	6 51·123	7 51·287	8 51·451	9 51·615
51	2 51·468	3 51·632	4 51·797	5 51·961	6 52·125	7 52·290	8 52·454	9 52·618
52	2 52·471	3 52·635	4 52·799	5 52·964	6 53·128	7 53·292	8 53·457	9 53·621
53	2 53·474	3 53·638	4 53·802	5 53·966	6 54·131	7 54·295	8 54·459	9 54·624
54	2 54·476	3 54·641	4 54·805	5 54·969	6 55·133	7 55·298	8 55·462	9 55·626
55	2 55·479	3 55·643	4 55·808	5 55·972	6 56·136	7 56·301	8 56·465	9 56·629
56	2 56·482	3 56·646	4 56·810	5 56·975	6 57·139	7 57·303	8 57·468	9 57·632
57	2 57·485	3 57·649	4 57·813	5 57·977	6 58·142	7 58·306	8 58·470	9 58·635
58	2 58·487	3 58·652	4 58·816	5 58·980	6 59·144	7 59·309	8 59·473	9 59·637
59	2 59·490	3 59·654	4 59·819	5 59·983	7 00·147	8 00·311	9 00·476	10 00·640

17.3—DECIMALS OF A SOLAR DAY TO SIDEREAL TIME

Solar Time	Sidereal Time	Solar Time	Sidereal Time
d	h m	d	h m
0·00	0 00·0	0·50	12 02·0
·01	0 14·4	·51	12 16·4
·02	0 28·9	·52	12 30·9
·03	0 43·3	·53	12 45·3
·04	0 57·8	·54	12 59·7
0·05	1 12·2	0·55	13 14·2
·06	1 26·6	·56	13 28·6
·07	1 41·1	·57	13 43·0
·08	1 55·5	·58	13 57·5
·09	2 10·0	·59	14 11·9
0·10	2 24·4	0·60	14 26·4
·11	2 38·8	·61	14 40·8
·12	2 53·3	·62	14 55·2
·13	3 07·7	·63	15 09·7
·14	3 22·2	·64	15 24·1
0·15	3 36·6	0·65	15 38·6
·16	3 51·0	·66	15 53·0
·17	4 05·5	·67	16 07·4
·18	4 19·9	·68	16 21·9
·19	4 34·3	·69	16 36·3
0·20	4 48·8	0·70	16 50·8
·21	5 03·2	·71	17 05·2
·22	5 17·7	·72	17 19·6
·23	5 32·1	·73	17 34·1
·24	5 46·5	·74	17 48·5
0·25	6 01·0	0·75	18 03·0
·26	6 15·4	·76	18 17·4
·27	6 29·9	·77	18 31·8
·28	6 44·3	·78	18 46·3
·29	6 58·7	·79	19 00·7
0·30	7 13·2	0·80	19 15·2
·31	7 27·6	·81	19 29·6
·32	7 42·1	·82	19 44·0
·33	7 56·5	·83	19 58·5
·34	8 10·9	·84	20 12·9
0·35	8 25·4	0·85	20 27·4
·36	8 39·8	·86	20 41·8
·37	8 54·3	·87	20 56·2
·38	9 08·7	·88	21 10·7
·39	9 23·1	·89	21 25·1
0·40	9 37·6	0·90	21 39·5
·41	9 52·0	·91	21 54·0
·42	10 06·5	·92	22 08·4
·43	10 20·9	·93	22 22·9
·44	10 35·3	·94	22 37·3
0·45	10 49·8	0·95	22 51·7
·46	11 04·2	·96	23 06·2
·47	11 18·7	·97	23 20·6
·48	11 33·1	·98	23 35·1
·49	11 47·5	0·99	23 49·5
0·50	12 02·0	1·00	24 03·9

Solar Time	Sidereal Time	Solar Time	Sidereal Time	Solar Time	Sidereal Time
d	m	d	m	d	m
0·00000	0·0	0·00328	4·8	0·00661	9·6
·00003	0·1	·00335	4·9	·00668	9·7
·00010	0·2	·00342	5·0	·00675	9·8
·00017	0·3	·00349	5·1	·00682	9·9
·00024	0·4	·00356	5·2	·00689	10·0
·00031	0·5	·00363	5·3	·00696	10·1
·00038	0·6	·00370	5·4	·00702	10·2
·00045	0·7	·00377	5·5	·00709	10·3
·00051	0·8	·00384	5·6	·00716	10·4
·00058	0·9	·00391	5·7	·00723	10·5
·00065	1·0	·00398	5·8	·00730	10·6
·00072	1·1	·00405	5·9	·00737	10·7
·00079	1·2	·00412	6·0	·00744	10·8
·00086	1·3	·00418	6·1	·00751	10·9
·00093	1·4	·00425	6·2	·00758	11·0
·00100	1·5	·00432	6·3	·00765	11·1
·00107	1·6	·00439	6·4	·00772	11·2
·00114	1·7	·00446	6·5	·00779	11·3
·00121	1·8	·00453	6·6	·00786	11·4
·00128	1·9	·00460	6·7	·00792	11·5
·00135	2·0	·00467	6·8	·00799	11·6
·00141	2·1	·00474	6·9	·00806	11·7
·00148	2·2	·00481	7·0	·00813	11·8
·00155	2·3	·00488	7·1	·00820	11·9
·00162	2·4	·00495	7·2	·00827	12·0
·00169	2·5	·00502	7·3	·00834	12·1
·00176	2·6	·00509	7·4	·00841	12·2
·00183	2·7	·00515	7·5	·00848	12·3
·00190	2·8	·00522	7·6	·00855	12·4
·00197	2·9	·00529	7·7	·00862	12·5
·00204	3·0	·00536	7·8	·00869	12·6
·00211	3·1	·00543	7·9	·00876	12·7
·00218	3·2	·00550	8·0	·00882	12·8
·00225	3·3	·00557	8·1	·00889	12·9
·00232	3·4	·00564	8·2	·00896	13·0
·00238	3·5	·00571	8·3	·00903	13·1
·00245	3·6	·00578	8·4	·00910	13·2
·00252	3·7	·00585	8·5	·00917	13·3
·00259	3·8	·00592	8·6	·00924	13·4
·00266	3·9	·00599	8·7	·00931	13·5
·00273	4·0	·00605	8·8	·00938	13·6
·00280	4·1	·00612	8·9	·00945	13·7
·00287	4·2	·00619	9·0	·00952	13·8
·00294	4·3	·00626	9·1	·00959	13·9
·00301	4·4	·00633	9·2	·00966	14·0
·00308	4·5	·00640	9·3	·00973	14·1
·00315	4·6	·00647	9·4	·00979	14·2
·00322	4·7	·00654	9·5	·00986	14·3
·00328	4·8	·00661	9·6	·00993	14·4
0·00335		0·00668		0·01000	

In critical cases ascend.

d	Equivalent time interval of			*d*	Equivalent time interval of		
	d	*d*/100	*d*/10000		*d*	*d*/100	*d*/10000
d	h m s	m s	s	d	h m s	m s	s
0·00	0 00 00	0 00·00	0·00	**0·50**	12 00 00	7 12·00	4·32
·01	0 14 24	0 08·64	0·09	**·51**	12 14 24	7 20·64	4·41
·02	0 28 48	0 17·28	0·17	**·52**	12 28 48	7 29·28	4·49
·03	0 43 12	0 25·92	0·26	**·53**	12 43 12	7 37·92	4·58
·04	0 57 36	0 34·56	0·35	**·54**	12 57 36	7 46·56	4·67
0·05	1 12 00	0 43·20	0·43	**0·55**	13 12 00	7 55·20	4·75
·06	1 26 24	0 51·84	0·52	**·56**	13 26 24	8 03·84	4·84
·07	1 40 48	1 00·48	0·60	**·57**	13 40 48	8 12·48	4·92
·08	1 55 12	1 09·12	0·69	**·58**	13 55 12	8 21·12	5·01
·09	2 09 36	1 17·76	0·78	**·59**	14 09 36	8 29·76	5·10
0·10	2 24 00	1 26·40	0·86	**0·60**	14 24 00	8 38·40	5·18
·11	2 38 24	1 35·04	0·95	**·61**	14 38 24	8 47·04	5·27
·12	2 52 48	1 43·68	1·04	**·62**	14 52 48	8 55·68	5·36
·13	3 07 12	1 52·32	1·12	**·63**	15 07 12	9 04·32	5·44
·14	3 21 36	2 00·96	1·21	**·64**	15 21 36	9 12·96	5·53
0·15	3 36 00	2 09·60	1·30	**0·65**	15 36 00	9 21·60	5·62
·16	3 50 24	2 18·24	1·38	**·66**	15 50 24	9 30·24	5·70
·17	4 04 48	2 26·88	1·47	**·67**	16 04 48	9 38·88	5·79
·18	4 19 12	2 35·52	1·56	**·68**	16 19 12	9 47·52	5·88
·19	4 33 36	2 44·16	1·64	**·69**	16 33 36	9 56·16	5·96
0·20	4 48 00	2 52·80	1·73	**0·70**	16 48 00	10 04·80	6·05
·21	5 02 24	3 01·44	1·81	**·71**	17 02 24	10 13·44	6·13
·22	5 16 48	3 10·08	1·90	**·72**	17 16 48	10 22·08	6·22
·23	5 31 12	3 18·72	1·99	**·73**	17 31 12	10 30·72	6·31
·24	5 45 36	3 27·36	2·07	**·74**	17 45 36	10 39·36	6·39
0·25	6 00 00	3 36·00	2·16	**0·75**	18 00 00	10 48·00	6·48
·26	6 14 24	3 44·64	2·25	**·76**	18 14 24	10 56·64	6·57
·27	6 28 48	3 53·28	2·33	**·77**	18 28 48	11 05·28	6·65
·28	6 43 12	4 01·92	2·42	**·78**	18 43 12	11 13·92	6·74
·29	6 57 36	4 10·56	2·51	**·79**	18 57 36	11 22·56	6·83
0·30	7 12 00	4 19·20	2·59	**0·80**	19 12 00	11 31·20	6·91
·31	7 26 24	4 27·84	2·68	**·81**	19 26 24	11 39·84	7·00
·32	7 40 48	4 36·48	2·76	**·82**	19 40 48	11 48·48	7·08
·33	7 55 12	4 45·12	2·85	**·83**	19 55 12	11 57·12	7·17
·34	8 09 36	4 53·76	2·94	**·84**	20 09 36	12 05·76	7·26
0·35	8 24 00	5 02·40	3·02	**0·85**	20 24 00	12 14·40	7·34
·36	8 38 24	5 11·04	3·11	**·86**	20 38 24	12 23·04	7·43
·37	8 52 48	5 19·68	3·20	**·87**	20 52 48	12 31·68	7·52
·38	9 07 12	5 28·32	3·28	**·88**	21 07 12	12 40·32	7·60
·39	9 21 36	5 36·96	3·37	**·89**	21 21 36	12 48·96	7·69
0·40	9 36 00	5 45·60	3·46	**0·90**	21 36 00	12 57·60	7·78
·41	9 50 24	5 54·24	3·54	**·91**	21 50 24	13 06·24	7·86
·42	10 04 48	6 02·88	3·63	**·92**	22 04 48	13 14·88	7·95
·43	10 19 12	6 11·52	3·72	**·93**	22 19 12	13 23·52	8·04
·44	10 33 36	6 20·16	3·80	**·94**	22 33 36	13 32·16	8·12
0·45	10 48 00	6 28·80	3·89	**0·95**	22 48 00	13 40·80	8·21
·46	11 02 24	6 37·44	3·97	**·96**	23 02 24	13 49·44	8·29
·47	11 16 48	6 46·08	4·06	**·97**	23 16 48	13 58·08	8·38
·48	11 31 12	6 54·72	4·15	**·98**	23 31 12	14 06·72	8·47
·49	11 45 36	7 03·36	4·23	**0·99**	23 45 36	14 15·36	8·55
0·50	12 00 00	7 12·00	4·32	**1·00**	24 00 00	14 24·00	8·64

m	0^h d	1^h d	2^h d	3^h d	4^h d	5^h d	s	Seconds d
0	0·000 000	0·041 667	0·083 333	0·125 000	0·166 667	0·208 333	0	0·000 000
1	·000 694	·042 361	·084 028	·125 694	·167 361	·209 028	1	·000 012
2	·001 389	·043 056	·084 722	·126 389	·168 056	·209 722	2	·000 023
3	·002 083	·043 750	·085 417	·127 083	·168 750	·210 417	3	·000 035
4	·002 778	·044 444	·086 111	·127 778	·169 444	·211 111	4	·000 046
5	0·003 472	0·045 139	0·086 806	0·128 472	0·170 139	0·211 806	5	0·000 058
6	·004 167	·045 833	·087 500	·129 167	·170 833	·212 500	6	·000 069
7	·004 861	·046 528	·088 194	·129 861	·171 528	·213 194	7	·000 081
8	·005 556	·047 222	·088 889	·130 556	·172 222	·213 889	8	·000 093
9	·006 250	·047 917	·089 583	·131 250	·172 917	·214 583	9	·000 104
10	0·006 944	0·048 611	0·090 278	0·131 944	0·173 611	0·215 278	10	0·000 116
11	·007 639	·049 306	·090 972	·132 639	·174 306	·215 972	11	·000 127
12	·008 333	·050 000	·091 667	·133 333	·175 000	·216 667	12	·000 139
13	·009 028	·050 694	·092 361	·134 028	·175 694	·217 361	13	·000 150
14	·009 722	·051 389	·093 056	·134 722	·176 389	·218 056	14	·000 162
15	0·010 417	0·052 083	0·093 750	0·135 417	0·177 083	0·218 750	15	0·000 174
16	·011 111	·052 778	·094 444	·136 111	·177 778	·219 444	16	·000 185
17	·011 806	·053 472	·095 139	·136 806	·178 472	·220 139	17	·000 197
18	·012 500	·054 167	·095 833	·137 500	·179 167	·220 833	18	·000 208
19	·013 194	·054 861	·096 528	·138 194	·179 861	·221 528	19	·000 220
20	0·013 889	0·055 556	0·097 222	0·138 889	0·180 556	0·222 222	20	0·000 231
21	·014 583	·056 250	·097 917	·139 583	·181 250	·222 917	21	·000 243
22	·015 278	·056 944	·098 611	·140 278	·181 944	·223 611	22	·000 255
23	·015 972	·057 639	·099 306	·140 972	·182 639	·224 306	23	·000 266
24	·016 667	·058 333	·100 000	·141 667	·183 333	·225 000	24	·000 278
25	0·017 361	0·059 028	0·100 694	0·142 361	0·184 028	0·225 694	25	0·000 289
26	·018 056	·059 722	·101 389	·143 056	·184 722	·226 389	26	·000 301
27	·018 750	·060 417	·102 083	·143 750	·185 417	·227 083	27	·000 312
28	·019 444	·061 111	·102 778	·144 444	·186 111	·227 778	28	·000 324
29	·020 139	·061 806	·103 472	·145 139	·186 806	·228 472	29	·000 336
30	0·020 833	0·062 500	0·104 167	0·145 833	0·187 500	0·229 167	30	0·000 347
31	·021 528	·063 194	·104 861	·146 528	·188 194	·229 861	31	·000 359
32	·022 222	·063 889	·105 556	·147 222	·188 889	·230 556	32	·000 370
33	·022 917	·064 583	·106 250	·147 917	·189 583	·231 250	33	·000 382
34	·023 611	·065 278	·106 944	·148 611	·190 278	·231 944	34	·000 394
35	0·024 306	0·065 972	0·107 639	0·149 306	0·190 972	0·232 639	35	0·000 405
36	·025 000	·066 667	·108 333	·150 000	·191 667	·233 333	36	·000 417
37	·025 694	·067 361	·109 028	·150 694	·192 361	·234 028	37	·000 428
38	·026 389	·068 056	·109 722	·151 389	·193 056	·234 722	38	·000 440
39	·027 083	·068 750	·110 417	·152 083	·193 750	·235 417	39	·000 451
40	0·027 778	0·069 444	0·111 111	0·152 778	0·194 444	0·236 111	40	0·000 463
41	·028 472	·070 139	·111 806	·153 472	·195 139	·236 806	41	·000 475
42	·029 167	·070 833	·112 500	·154 167	·195 833	·237 500	42	·000 486
43	·029 861	·071 528	·113 194	·154 861	·196 528	·238 194	43	·000 498
44	·030 556	·072 222	·113 889	·155 556	·197 222	·238 889	44	·000 509
45	0·031 250	0·072 917	0·114 583	0·156 250	0·197 917	0·239 583	45	0·000 521
46	·031 944	·073 611	·115 278	·156 944	·198 611	·240 278	46	·000 532
47	·032 639	·074 306	·115 972	·157 639	·199 306	·240 972	47	·000 544
48	·033 333	·075 000	·116 667	·158 333	·200 000	·241 667	48	·000 556
49	·034 028	·075 694	·117 361	·159 028	·200 694	·242 361	49	·000 567
50	0·034 722	0·076 389	0·118 056	0·159 722	0·201 389	0·243 056	50	0·000 579
51	·035 417	·077 083	·118 750	·160 417	·202 083	·243 750	51	·000 590
52	·036 111	·077 778	·119 444	·161 111	·202 778	·244 444	52	·000 602
53	·036 806	·078 472	·120 139	·161 806	·203 472	·245 139	53	·000 613
54	·037 500	·079 167	·120 833	·162 500	·204 167	·245 833	54	·000 625
55	0·038 194	0·079 861	0·121 528	0·163 194	0·204 861	0·246 528	55	0·000 637
56	·038 889	·080 556	·122 222	·163 889	·205 556	·247 222	56	·000 648
57	·039 583	·081 250	·122 917	·164 583	·206 250	·247 917	57	·000 660
58	·040 278	·081 944	·123 611	·165 278	·206 944	·248 611	58	·000 671
59	0·040 972	0·082 639	0·124 306	0·165 972	0·207 639	0·249 306	59	0·000 683

m	6^h	7^h	8^h	9^h	10^h	11^h	s	Seconds
m	d	d	d	d	d	d	s	d
0	0·250 000	0·291 667	0·333 333	0·375 000	0·416 667	0·458 333	0	0·000 000
1	·250 694	·292 361	·334 028	·375 694	·417 361	·459 028	1	·000 012
2	·251 389	·293 056	·334 722	·376 389	·418 056	·459 722	2	·000 023
3	·252 083	·293 750	·335 417	·377 083	·418 750	·460 417	3	·000 035
4	·252 778	·294 444	·336 111	·377 778	·419 444	·461 111	4	·000 046
5	0·253 472	0·295 139	0·336 806	0·378 472	0·420 139	0·461 806	5	0·000 058
6	·254 167	·295 833	·337 500	·379 167	·420 833	·462 500	6	·000 069
7	·254 861	·296 528	·338 194	·379 861	·421 528	·463 194	7	·000 081
8	·255 556	·297 222	·338 889	·380 556	·422 222	·463 889	8	·000 093
9	·256 250	·297 917	·339 583	·381 250	·422 917	·464 583	9	·000 104
10	0·256 944	0·298 611	0·340 278	0·381 944	0·423 611	0·465 278	10	0·000 116
11	·257 639	·299 306	·340 972	·382 639	·424 306	·465 972	11	·000 127
12	·258 333	·300 000	·341 667	·383 333	·425 000	·466 667	12	·000 139
13	·259 028	·300 694	·342 361	·384 028	·425 694	·467 361	13	·000 150
14	·259 722	·301 389	·343 056	·384 722	·426 389	·468 056	14	·000 162
15	0·260 417	0·302 083	0·343 750	0·385 417	0·427 083	0·468 750	15	0·000 174
16	·261 111	·302 778	·344 444	·386 111	·427 778	·469 444	16	·000 185
17	·261 806	·303 472	·345 139	·386 806	·428 472	·470 139	17	·000 197
18	·262 500	·304 167	·345 833	·387 500	·429 167	·470 833	18	·000 208
19	·263 194	·304 861	·346 528	·388 194	·429 861	·471 528	19	·000 220
20	0·263 889	0·305 556	0·347 222	0·388 889	0·430 556	0·472 222	20	0·000 231
21	·264 583	·306 250	·347 917	·389 583	·431 250	·472 917	21	·000 243
22	·265 278	·306 944	·348 611	·390 278	·431 944	·473 611	22	·000 255
23	·265 972	·307 639	·349 306	·390 972	·432 639	·474 306	23	·000 266
24	·266 667	·308 333	·350 000	·391 667	·433 333	·475 000	24	·000 278
25	0·267 361	0·309 028	0·350 694	0·392 361	0·434 028	0·475 694	25	0·000 289
26	·268 056	·309 722	·351 389	·393 056	·434 722	·476 389	26	·000 301
27	·268 750	·310 417	·352 083	·393 750	·435 417	·477 083	27	·000 312
28	·269 444	·311 111	·352 778	·394 444	·436 111	·477 778	28	·000 324
29	·270 139	·311 806	·353 472	·395 139	·436 806	·478 472	29	·000 336
30	0·270 833	0·312 500	0·354 167	0·395 833	0·437 500	0·479 167	30	0·000 347
31	·271 528	·313 194	·354 861	·396 528	·438 194	·479 861	31	·000 359
32	·272 222	·313 889	·355 556	·397 222	·438 889	·480 556	32	·000 370
33	·272 917	·314 583	·356 250	·397 917	·439 583	·481 250	33	·000 382
34	·273 611	·315 278	·356 944	·398 611	·440 278	·481 944	34	·000 394
35	0·274 306	0·315 972	0·357 639	0·399 306	0·440 972	0·482 639	35	0·000 405
36	·275 000	·316 667	·358 333	·400 000	·441 667	·483 333	36	·000 417
37	·275 694	·317 361	·359 028	·400 694	·442 361	·484 028	37	·000 428
38	·276 389	·318 056	·359 722	·401 389	·443 056	·484 722	38	·000 440
39	·277 083	·318 750	·360 417	·402 083	·443 750	·485 417	39	·000 451
40	0·277 778	0·319 444	0·361 111	0·402 778	0·444 444	0·486 111	40	0·000 463
41	·278 472	·320 139	·361 806	·403 472	·445 139	·486 806	41	·000 475
42	·279 167	·320 833	·362 500	·404 167	·445 833	·487 500	42	·000 486
43	·279 861	·321 528	·363 194	·404 861	·446 528	·488 194	43	·000 498
44	·280 556	·322 222	·363 889	·405 556	·447 222	·488 889	44	·000 509
45	0·281 250	0·322 917	0·364 583	0·406 250	0·447 917	0·489 583	45	0·000 521
46	·281 944	·323 611	·365 278	·406 944	·448 611	·490 278	46	·000 532
47	·282 639	·324 306	·365 972	·407 639	·449 306	·490 972	47	·000 544
48	·283 333	·325 000	·366 667	·408 333	·450 000	·491 667	48	·000 556
49	·284 028	·325 694	·367 361	·409 028	·450 694	·492 361	49	·000 567
50	0·284 722	0·326 389	0·368 056	0·409 722	0·451 389	0·493 056	50	0·000 579
51	·285 417	·327 083	·368 750	·410 417	·452 083	·493 750	51	·000 590
52	·286 111	·327 778	·369 444	·411 111	·452 778	·494 444	52	·000 602
53	·286 806	·328 472	·370 139	·411 806	·453 472	·495 139	53	·000 613
54	·287 500	·329 167	·370 833	·412 500	·454 167	·495 833	54	·000 625
55	0·288 194	0·329 861	0·371 528	0·413 194	0·454 861	0·496 528	55	0·000 637
56	·288 889	·330 556	·372 222	·413 889	·455 556	·497 222	56	·000 648
57	·289 583	·331 250	·372 917	·414 583	·456 250	·497 917	57	·000 660
58	·290 278	·331 944	·373 611	·415 278	·456 944	·498 611	58	·000 671
59	0·290 972	0·332 639	0·374 306	0·415 972	0·457 639	0·499 306	59	0·000 683

	0′	1′	2′	3′	4′	5′
″	°	°	°	°	°	°
0	0·000 000	0·016 667	0·033 333	0·050 000	0·066 667	0·083 333
1	00 278	16 944	33 611	50 278	66 944	83 611
2	00 556	17 222	33 889	50 556	67 222	83 889
3	00 833	17 500	34 167	50 833	67 500	84 167
4	01 111	17 778	34 444	51 111	67 778	84 444
5	0·001 389	0·018 056	0·034 722	0·051 389	0·068 056	0·084 722
6	01 667	18 333	35 000	51 667	68 333	85 000
7	01 944	18 611	35 278	51 944	68 611	85 278
8	02 222	18 889	35 556	52 222	68 889	85 556
9	02 500	19 167	35 833	52 500	69 167	85 833
10	0·002 778	0·019 444	0·036 111	0·052 778	0·069 444	0·086 111
11	03 056	19 722	36 389	53 056	69 722	86 389
12	03 333	20 000	36 667	53 333	70 000	86 667
13	03 611	20 278	36 944	53 611	70 278	86 944
14	03 889	20 556	37 222	53 889	70 556	87 222
15	0·004 167	0·020 833	0·037 500	0·054 167	0·070 833	0·087 500
16	04 444	21 111	37 778	54 444	71 111	87 778
17	04 722	21 389	38 056	54 722	71 389	88 056
18	05 000	21 667	38 333	55 000	71 667	88 333
19	05 278	21 944	38 611	55 278	71 944	88 611
20	0·005 556	0·022 222	0·038 889	0·055 556	0·072 222	0·088 889
21	05 833	22 500	39 167	55 833	72 500	89 167
22	06 111	22 778	39 444	56 111	72 778	89 444
23	06 389	23 056	39 722	56 389	73 056	89 722
24	06 667	23 333	40 000	56 667	73 333	90 000
25	0·006 944	0·023 611	0·040 278	0·056 944	0·073 611	0·090 278
26	07 222	23 889	40 556	57 222	73 889	90 556
27	07 500	24 167	40 833	57 500	74 167	90 833
28	07 778	24 444	41 111	57 778	74 444	91 111
29	08 056	24 722	41 389	58 056	74 722	91 389
30	0·008 333	0·025 000	0·04 1667	0·058 333	0·075 000	0·091 667
31	08 611	25 278	41 944	58 611	75 278	91 944
32	08 889	25 556	42 222	58 889	75 556	92 222
33	09 167	25 833	42 500	59 167	75 833	92 500
34	09 444	26 111	42 778	59 444	76 111	92 778
35	0·009 722	0·026 389	0·043 056	0·059 722	0·076 389	0·093 056
36	10 000	26 667	43 333	60 000	76 667	93 333
37	10 278	26 944	43 611	60 278	76 944	93 611
38	10 556	27 222	43 889	60 556	77 222	93 889
39	10 833	27 500	44 167	60 833	77 500	94 167
40	0·011 111	0·027 778	0·044 444	0·061 111	0·077 778	0·094 444
41	11 389	28 056	44 722	61 389	78 056	94 722
42	11 667	28 333	45 000	61 667	78 333	95 000
43	11 944	28 611	45 278	61 944	78 611	95 278
44	12 222	28 889	45 556	62 222	78 889	95 556
45	0·012 500	0·029 167	0·045 833	0·062 500	0·079 167	0·095 833
46	12 778	29 444	46 111	62 778	79 444	96 111
47	13 056	29 722	46 389	63 056	79 722	96 389
48	13 333	30 000	46 667	63 333	80 000	96 667
49	13 611	30 278	46 944	63 611	80 278	96 944
50	0·013 889	0·030 556	0·047 222	0·063 889	0·080 556	0·097 222
51	14 167	30 833	47 500	64 167	80 833	97 500
52	14 444	31 111	47 778	64 444	81 111	97 778
53	14 722	31 389	48 056	64 722	81 389	98 056
54	15 000	31 667	48 333	65 000	81 667	98 333
55	0·015 278	0·031 944	0·048 611	0·065 278	0·081 944	0·098 611
56	15 556	32 222	48 889	65 556	82 222	98 889
57	15 833	32 500	49 167	65 833	82 500	99 167
58	16 111	32 778	49 444	66 111	82 778	99 444
59	0·016 389	0·033 056	0·049 722	0·066 389	0·083 056	0·099 722

′	°	′	°
0	0·0	**30**	0·5
6	0·1	**36**	0·6
12	0·2	**42**	0·7
18	0·3	**48**	0·8
24	0·4	**54**	0·9

In units of the sixth decimal of a degree

″		″	
0·00	000	**0·50**	139
·01	003	**·51**	142
·02	006	**·52**	144
·03	008	**·53**	147
·04	011	**·54**	150
0·05	014	**0·55**	153
·06	017	**·56**	156
·07	019	**·57**	158
·08	022	**·58**	161
·09	025	**·59**	164
0·10	028	**0·60**	167
·11	031	**·61**	169
·12	033	**·62**	172
·13	036	**·63**	175
·14	039	**·64**	178
0·15	042	**0·65**	181
·16	044	**·66**	183
·17	047	**·67**	186
·18	050	**·68**	189
·19	053	**·69**	192
0·20	056	**0·70**	194
·21	058	**·71**	197
·22	061	**·72**	200
·23	064	**·73**	203
·24	067	**·74**	206
0·25	069	**0·75**	208
·26	072	**·76**	211
·27	075	**·77**	214
·28	078	**·78**	217
·29	081	**·79**	219
0·30	083	**0·80**	222
·31	086	**·81**	225
·32	089	**·82**	228
·33	092	**·83**	231
·34	094	**·84**	233
0·35	097	**0·85**	236
·36	100	**·86**	239
·37	103	**·87**	242
·38	106	**·88**	244
·39	108	**·89**	247
0·40	111	**0·90**	250
·41	114	**·91**	253
·42	117	**·92**	256
·43	119	**·93**	258
·44	122	**·94**	261
0·45	125	**0·95**	264
·46	128	**·96**	267
·47	131	**·97**	269
·48	133	**·98**	272
0·49	136	**0·99**	275

°	′	″	°	′	″
0·00	0	00	0·50	30	00
·01	0	36	·51	30	36
·02	1	12	·52	31	12
·03	1	48	·53	31	48
·04	2	24	·54	32	24
0·05	3	00	0·55	33	00
·06	3	36	·56	33	36
·07	4	12	·57	34	12
·08	4	48	·58	34	48
·09	5	24	·59	35	24
0·10	6	00	0·60	36	00
·11	6	36	·61	36	36
·12	7	12	·62	37	12
·13	7	48	·63	37	48
·14	8	24	·64	38	24
0·15	9	00	0·65	39	00
·16	9	36	·66	39	36
·17	10	12	·67	40	12
·18	10	48	·68	40	48
·19	11	24	·69	41	24
0·20	12	00	0·70	42	00
·21	12	36	·71	42	36
·22	13	12	·72	43	12
·23	13	48	·73	43	48
·24	14	24	·74	44	24
0·25	15	00	0·75	45	00
·26	15	36	·76	45	36
·27	16	12	·77	46	12
·28	16	48	·78	46	48
·29	17	24	·79	47	24
0·30	18	00	0·80	48	00
·31	18	36	·81	48	36
·32	19	12	·82	49	12
·33	19	48	·83	49	48
·34	20	24	·84	50	24
0·35	21	00	0·85	51	00
·36	21	36	·86	51	36
·37	22	12	·87	52	12
·38	22	48	·88	52	48
·39	23	24	·89	53	24
0·40	24	00	0·90	54	00
·41	24	36	·91	54	36
·42	25	12	·92	55	12
·43	25	48	·93	55	48
·44	26	24	·94	56	24
0·45	27	00	0·95	57	00
·46	27	36	·96	57	36
·47	28	12	·97	58	12
·48	28	48	·98	58	48
·49	29	24	0·99	59	24
0·50	30	00	1·00	60	00

°	″	°	″
0·0000	0·00	0·0050	18·00
01	0·36	51	18·36
02	0·72	52	18·72
03	1·08	53	19·08
04	1·44	54	19·44
0·0005	1·80	0·0055	19·80
06	2·16	56	20·16
07	2·52	57	20·52
08	2·88	58	20·88
09	3·24	59	21·24
0·0010	3·60	0·0060	21·60
11	3·96	61	21·96
12	4·32	62	22·32
13	4·68	63	22·68
14	5·04	64	23·04
0·0015	5·40	0·0065	23·40
16	5·76	66	23·76
17	6·12	67	24·12
18	6·48	68	24·48
19	6·84	69	24·84
0·0020	7·20	0·0070	25·20
21	7·56	71	25·56
22	7·92	72	25·92
23	8·28	73	26·28
24	8·64	74	26·64
0·0025	9·00	0·0075	27·00
26	9·36	76	27·36
27	9·72	77	27·72
28	10·08	78	28·08
29	10·44	79	28·44
0·0030	10·80	0·0080	28·80
31	11·16	81	29·16
32	11·52	82	29·52
33	11·88	83	29·88
34	12·24	84	30·24
0·0035	12·60	0·0085	30·60
36	12·96	86	30·96
37	13·32	87	31·32
38	13·68	88	31·68
39	14·04	89	32·04
0·0040	14·40	0·0090	32·40
41	14·76	91	32·76
42	15·12	92	33·12
43	15·48	93	33·48
44	15·84	94	33·84
0·0045	16·20	0·0095	34·20
46	16·56	96	34·56
47	16·92	97	34·92
48	17·28	98	35·28
49	17·64	0·0099	35·64
0·0050	18·00	0·0100	36·00

°	″
0·000000	
	0·00
01	
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	0·24
68	
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73	
	0·27
76	
	0·28
79	
	0·29
81	
	0·30
84	
	0·31
87	
	0·32
90	
	0·33
93	
	0·34
95	
	0·35
0·000098	
	0·36
0·000101	

In critical cases ascend.

m	0^h ° ′	1^h ° ′	2^h ° ′	3^h ° ′	4^h ° ′	5^h ° ′
0	0 00	15 00	30 00	45 00	60 00	75 00
1	0 15	15 15	30 15	45 15	60 15	75 15
2	0 30	15 30	30 30	45 30	60 30	75 30
3	0 45	15 45	30 45	45 45	60 45	75 45
4	1 00	16 00	31 00	46 00	61 00	76 00
5	1 15	16 15	31 15	46 15	61 15	76 15
6	1 30	16 30	31 30	46 30	61 30	76 30
7	1 45	16 45	31 45	46 45	61 45	76 45
8	2 00	17 00	32 00	47 00	62 00	77 00
9	2 15	17 15	32 15	47 15	62 15	77 15
10	2 30	17 30	32 30	47 30	62 30	77 30
11	2 45	17 45	32 45	47 45	62 45	77 45
12	3 00	18 00	33 00	48 00	63 00	78 00
13	3 15	18 15	33 15	48 15	63 15	78 15
14	3 30	18 30	33 30	48 30	63 30	78 30
15	3 45	18 45	33 45	48 45	63 45	78 45
16	4 00	19 00	34 00	49 00	64 00	79 00
17	4 15	19 15	34 15	49 15	64 15	79 15
18	4 30	19 30	34 30	49 30	64 30	79 30
19	4 45	19 45	34 45	49 45	64 45	79 45
20	5 00	20 00	35 00	50 00	65 00	80 00
21	5 15	20 15	35 15	50 15	65 15	80 15
22	5 30	20 30	35 30	50 30	65 30	80 30
23	5 45	20 45	35 45	50 45	65 45	80 45
24	6 00	21 00	36 00	51 00	66 00	81 00
25	6 15	21 15	36 15	51 15	66 15	81 15
26	6 30	21 30	36 30	51 30	66 30	81 30
27	6 45	21 45	36 45	51 45	66 45	81 45
28	7 00	22 00	37 00	52 00	67 00	82 00
29	7 15	22 15	37 15	52 15	67 15	82 15
30	7 30	22 30	37 30	52 30	67 30	82 30
31	7 45	22 45	37 45	52 45	67 45	82 45
32	8 00	23 00	38 00	53 00	68 00	83 00
33	8 15	23 15	38 15	53 15	68 15	83 15
34	8 30	23 30	38 30	53 30	68 30	83 30
35	8 45	23 45	38 45	53 45	68 45	83 45
36	9 00	24 00	39 00	54 00	69 00	84 00
37	9 15	24 15	39 15	54 15	69 15	84 15
38	9 30	24 30	39 30	54 30	69 30	84 30
39	9 45	24 45	39 45	54 45	69 45	84 45
40	10 00	25 00	40 00	55 00	70 00	85 00
41	10 15	25 15	40 15	55 15	70 15	85 15
42	10 30	25 30	40 30	55 30	70 30	85 30
43	10 45	25 45	40 45	55 45	70 45	85 45
44	11 00	26 00	41 00	56 00	71 00	86 00
45	11 15	26 15	41 15	56 15	71 15	86 15
46	11 30	26 30	41 30	56 30	71 30	86 30
47	11 45	26 45	41 45	56 45	71 45	86 45
48	12 00	27 00	42 00	57 00	72 00	87 00
49	12 15	27 15	42 15	57 15	72 15	87 15
50	12 30	27 30	42 30	57 30	72 30	87 30
51	12 45	27 45	42 45	57 45	72 45	87 45
52	13 00	00	43 00	58 00	73 00	88 00
53	13 15	28 15	43 15	58 15	73 15	88 15
54	13 30	28 30	43 30	58 30	73 30	88 30
55	13 45	28 45	43 45	58 45	73 45	88 45
56	14 00	29 00	44 00	59 00	74 00	89 00
57	14 15	29 15	44 15	59 15	74 15	89 15
58	14 30	29 30	44 30	59 30	74 30	89 30
59	14 45	29 45	44 45	59 45	74 45	89 45

Seconds

s	′ ″	s	″	s	″
0	0 00	**0·00**	0·00	**0·50**	7·50
1	0 15	**·01**	0·15	**·51**	7·65
2	0 30	**·02**	0·30	**·52**	7·80
3	0 45	**·03**	0·45	**·53**	7·95
4	1 00	**·04**	0·60	**·54**	8·10
5	1 15	**0·05**	0·75	**0·55**	8·25
6	1 30	**·06**	0·90	**·56**	8·40
7	1 45	**·07**	1·05	**·57**	8·55
8	2 00	**·08**	1·20	**·58**	8·70
9	2 15	**·09**	1·35	**·59**	8·85
10	2 30	**0·10**	1·50	**0·60**	9·00
11	2 45	**·11**	1·65	**·61**	9·15
12	3 00	**·12**	1·80	**·62**	9·30
13	3 15	**·13**	1·95	**·63**	9·45
14	3 30	**·14**	2·10	**·64**	9·60
15	3 45	**0·15**	2·25	**0·65**	9·75
16	4 00	**·16**	2·40	**·66**	9·90
17	4 15	**·17**	2·55	**·67**	10·05
18	4 30	**·18**	2·70	**·68**	10·20
19	4 45	**·19**	2·85	**·69**	10·35
20	5 00	**0·20**	3·00	**0·70**	10·50
21	5 15	**·21**	3·15	**·71**	10·65
22	5 30	**·22**	3·30	**·72**	10·80
23	5 45	**·23**	3·45	**·73**	10·95
24	6 00	**·24**	3·60	**·74**	11·10
25	6 15	**0·25**	3·75	**0·75**	11·25
26	6 30	**·26**	3·90	**·76**	11·40
27	6 45	**·27**	4·05	**·77**	11·55
28	7 00	**·28**	4·20	**·78**	11·70
29	7 15	**·29**	4·35	**·79**	11·85
30	7 30	**0·30**	4·50	**0·80**	12·00
31	7 45	**·31**	4·65	**·81**	12·15
32	8 00	**·32**	4·80	**·82**	12·30
33	8 15	**·33**	4·95	**·83**	12·45
34	8 30	**·34**	5·10	**·84**	12·60
35	8 45	**0·35**	5·25	**0·85**	12·75
36	9 00	**·36**	5·40	**·86**	12·90
37	9 15	**·37**	5·55	**·87**	13·05
38	9 30	**·38**	5·70	**·88**	13·20
39	9 45	**·39**	5·85	**·89**	13·35
40	10 00	**0·40**	6·00	**0·90**	13·50
41	10 15	**·41**	6·15	**·91**	13·65
42	10 30	**·42**	6·30	**·92**	13·80
43	10 45	**·43**	6·45	**·93**	13·95
44	11 00	**·44**	6·60	**·94**	14·10
45	11 15	**0·45**	6·75	**0·95**	14·25
46	11 30	**·46**	6·90	**·96**	14·40
47	11 45	**·47**	7·05	**·97**	14·55
48	12 00	**·48**	7·20	**·98**	14·70
49	12 15	**·49**	7·35	**0·99**	14·85
50	12 30	**0·50**	7·50	**1·00**	15·00
51	12 45				
52	13 00				
53	13 15				
54	13 30				
55	13 45				
56	14 00				
57	14 15				
58	14 30				
59	14 45				

$6^h = 90°$

$12^h = 180°$

$18^h = 270°$

Degrees						Minutes		Seconds					
°	h m	°	h m	°	h m	′	m s	″	s	″	s	″	s
0	0 00	**60**	4 00	**120**	8 00	**0**	0 00	**0**	0·000	**0·00**	0·000	**0·50**	0·033
1	0 04	**61**	4 04	**121**	8 04	**1**	0 04	**1**	0·067	**·01**	·001	**·51**	·034
2	0 08	**62**	4 08	**122**	8 08	**2**	0 08	**2**	0·133	**·02**	·001	**·52**	·035
3	0 12	**63**	4 12	**123**	8 12	**3**	0 12	**3**	0·200	**·03**	·002	**·53**	·035
4	0 16	**64**	4 16	**124**	8 16	**4**	0 16	**4**	0·267	**·04**	·003	**·54**	·036
5	0 20	**65**	4 20	**125**	8 20	**5**	0 20	**5**	0·333	**0·05**	0·003	**0·55**	0·037
6	0 24	**66**	4 24	**126**	8 24	**6**	0 24	**6**	0·400	**·06**	·004	**·56**	·037
7	0 28	**67**	4 28	**127**	8 28	**7**	0 28	**7**	0·467	**·07**	·005	**·57**	·038
8	0 32	**68**	4 32	**128**	8 32	**8**	0 32	**8**	0·533	**·08**	·005	**·58**	·039
9	0 36	**69**	4 36	**129**	8 36	**9**	0 36	**9**	0·600	**·09**	·006	**·59**	·039
10	0 40	**70**	4 40	**130**	8 40	**10**	0 40	**10**	0·667	**0·10**	0·007	**0·60**	0·040
11	0 44	**71**	4 44	**131**	8 44	**11**	0 44	**11**	0·733	**·11**	·007	**·61**	·041
12	0 48	**72**	4 48	**132**	8 48	**12**	0 48	**12**	0·800	**·12**	·008	**·62**	·041
13	0 52	**73**	4 52	**133**	8 52	**13**	0 52	**13**	0·867	**·13**	·009	**·63**	·042
14	0 56	**74**	4 56	**134**	8 56	**14**	0 56	**14**	0·933	**·14**	·009	**·64**	·043
15	1 00	**75**	5 00	**135**	9 00	**15**	1 00	**15**	1·000	**0·15**	0·010	**0·65**	0·043
16	1 04	**76**	5 04	**136**	9 04	**16**	1 04	**16**	1·067	**·16**	·011	**·66**	·044
17	1 08	**77**	5 08	**137**	9 08	**17**	1 08	**17**	1·133	**·17**	·011	**·67**	·045
18	1 12	**78**	5 12	**138**	9 12	**18**	1 12	**18**	1·200	**·18**	·012	**·68**	·045
19	1 16	**79**	5 16	**139**	9 16	**19**	1 16	**19**	1·267	**·19**	·013	**·69**	·046
20	1 20	**80**	5 20	**140**	9 20	**20**	1 20	**20**	1·333	**0·20**	0·013	**0·70**	0·047
21	1 24	**81**	5 24	**141**	9 24	**21**	1 24	**21**	1·400	**·21**	·014	**·71**	·047
22	1 28	**82**	5 28	**142**	9 28	**22**	1 28	**22**	1·467	**·22**	·015	**·72**	·048
23	1 32	**83**	5 32	**143**	9 32	**23**	1 32	**23**	1·533	**·23**	·015	**·73**	·049
24	1 36	**84**	5 36	**144**	9 36	**24**	1 36	**24**	1·600	**·24**	·016	**·74**	·049
25	1 40	**85**	5 40	**145**	9 40	**25**	1 40	**25**	1·667	**0·25**	0·017	**0·75**	0·050
26	1 44	**86**	5 44	**146**	9 44	**26**	1 44	**26**	1·733	**·26**	·017	**·76**	·051
27	1 48	**87**	5 48	**147**	9 48	**27**	1 48	**27**	1·800	**·27**	·018	**·77**	·051
28	1 52	**88**	5 52	**148**	9 52	**28**	1 52	**28**	1·867	**·28**	·019	**·78**	·052
29	1 56	**89**	5 56	**149**	9 56	**29**	1 56	**29**	1·933	**·29**	·019	**·79**	·053
30	2 00	**90**	6 00	**150**	10 00	**30**	2 00	**30**	2·000	**0·30**	0·020	**0·80**	0·053
31	2 04	**91**	6 04	**151**	10 04	**31**	2 04	**31**	2·067	**·31**	·021	**·81**	·054
32	2 08	**92**	6 08	**152**	10 08	**32**	2 08	**32**	2·133	**·32**	·021	**·82**	·055
33	2 12	**93**	6 12	**153**	10 12	**33**	2 12	**33**	2·200	**·33**	·022	**·83**	·055
34	2 16	**94**	6 16	**154**	10 16	**34**	2 16	**34**	2·267	**·34**	·023	**·84**	·056
35	2 20	**95**	6 20	**155**	10 20	**35**	2 20	**35**	2·333	**0·35**	0·023	**0·85**	0·057
36	2 24	**96**	6 24	**156**	10 24	**36**	2 24	**36**	2·400	**·36**	·024	**·86**	·057
37	2 28	**97**	6 28	**157**	10 28	**37**	2 28	**37**	2·467	**·37**	·025	**·87**	·058
38	2 32	**98**	6 32	**158**	10 32	**38**	2 32	**38**	2·533	**·38**	·025	**·88**	·059
39	2 36	**99**	6 36	**159**	10 36	**39**	2 36	**39**	2·600	**·39**	·026	**·89**	·059
40	2 40	**100**	6 40	**160**	10 40	**40**	2 40	**40**	2·667	**0·40**	0·027	**0·90**	0·060
41	2 44	**101**	6 44	**161**	10 44	**41**	2 44	**41**	2·733	**·41**	·027	**·91**	·061
42	2 48	**102**	6 48	**162**	10 48	**42**	2 48	**42**	2·800	**·42**	·028	**·92**	·061
43	2 52	**103**	6 52	**163**	10 52	**43**	2 52	**43**	2·867	**·43**	·029	**·93**	·062
44	2 56	**104**	6 56	**164**	10 56	**44**	2 56	**44**	2·933	**·44**	·029	**·94**	·063
45	3 00	**105**	7 00	**165**	11 00	**45**	3 00	**45**	3·000	**0·45**	0·030	**0·95**	0·063
46	3 04	**106**	7 04	**166**	11 04	**46**	3 04	**46**	3·067	**·46**	·031	**·96**	·064
47	3 08	**107**	7 08	**167**	11 08	**47**	3 08	**47**	3·133	**·47**	·031	**·97**	·065
48	3 12	**108**	7 12	**168**	11 12	**48**	3 12	**48**	3·200	**·48**	·032	**·98**	·065
49	3 16	**109**	7 16	**169**	11 16	**49**	3 16	**49**	3·267	**·49**	·033	**0·99**	·066
50	3 20	**110**	7 20	**170**	11 20	**50**	3 20	**50**	3·333	**0·50**	0·033	**1·00**	0·067
51	3 24	**111**	7 24	**171**	11 24	**51**	3 24	**51**	3·400				
52	3 28	**112**	7 28	**172**	11 28	**52**	3 28	**52**	3·467				
53	3 32	**113**	7 32	**173**	11 32	**53**	3 32	**53**	3·533				
54	3 36	**114**	7 36	**174**	11 36	**54**	3 36	**54**	3·600				
55	3 40	**115**	7 40	**175**	11 40	**55**	3 40	**55**	3·667				
56	3 44	**116**	7 44	**176**	11 44	**56**	3 44	**56**	3·733				
57	3 48	**117**	7 48	**177**	11 48	**57**	3 48	**57**	3·800				
58	3 52	**118**	7 52	**178**	11 52	**58**	3 52	**58**	3·867				
59	3 56	**119**	7 56	**179**	11 56	**59**	3 56	**59**	3·933				

$90^\circ = 6^h$

$180^\circ = 12^h$

$270^\circ = 18^h$

Arc	Time	Revolutions	Radians	Arc	Time	Revolutions	Radians
°	h m	r		°	h m	r	
0	0 00	0·00000	0·00000	50	3 20	0·13889	0·87266
1	04	0278	·01745	51	3 24	·14167	0·89012
2	08	0556	·03491	52	3 28	·14444	0·90757
3	12	0833	·05236	53	3 32	·14722	0·92502
4	16	1111	·06981	54	3 36	·15000	0·94248
5	0 20	0·01389	0·08727	55	3 40	0·15278	0·95993
6	24	1667	·10472	56	3 44	·15556	0·97738
7	28	1944	·12217	57	3 48	·15833	0·99484
8	32	2222	·13963	58	3 52	·16111	1·01229
9	36	2500	·15708	59	3 56	·16389	1·02974
10	0 40	0·02778	0·17453	60	4 00	0·16667	1·04720
11	44	3056	·19199	61	4 04	·16944	1·06465
12	48	3333	·20944	62	4 08	·17222	1·08210
13	52	3611	·22689	63	4 12	·17500	1·09956
14	0 56	3889	·24435	64	4 16	·17778	1·11701
15	1 00	0·04167	0·26180	65	4 20	0·18056	1·13446
16	1 04	4444	·27925	66	4 24	·18333	1·15192
17	1 08	4722	·29671	67	4 28	·18611	1·16937
18	1 12	5000	·31416	68	4 32	·18889	1·18682
19	1 16	5278	·33161	69	4 36	·19167	1·20428
20	1 20	0·05556	0·34907	70	4 40	0·19444	1·22173
21	1 24	5833	·36652	71	4 44	·19722	1·23918
22	1 28	6111	·38397	72	4 48	·20000	1·25664
23	1 32	6389	·40143	73	4 52	·20278	1·27409
24	1 36	6667	·41888	74	4 56	·20556	1·29154
25	1 40	0·06944	0·43633	75	5 00	0·20833	1·30900
26	1 44	7222	·45379	76	5 04	·21111	1·32645
27	1 48	7500	·47124	77	5 08	·21389	1·34390
28	1 52	7778	·48869	78	5 12	·21667	1·36136
29	1 56	8056	·50615	79	5 16	·21944	1·37881
30	2 00	0·08333	0·52360	80	5 20	0·22222	1·39626
31	2 04	8611	·54105	81	5 24	·22500	1·41372
32	2 08	8889	·55851	82	5 28	·22778	1·43117
33	2 12	9167	·57596	83	5 32	·23056	1·44862
34	2 16	9444	·59341	84	5 36	·23333	1·46608
35	2 20	0·09722	0·61087	85	5 40	0·23611	1·48353
36	2 24	·10000	·62832	86	5 44	·23889	1·50098
37	2 28	·10278	·64577	87	5 48	·24167	1·51844
38	2 32	·10556	·66323	88	5 52	·24444	1·53589
39	2 36	·10833	·68068	89	5 56	·24722	1·55334
40	2 40	0·11111	0·69813	90	6 00	0·25000	1·57080
41	2 44	·11389	·71558	91	6 04	·25278	1·58825
42	2 48	·11667	·73304	92	6 08	·25556	1·60570
43	2 52	·11944	·75049	93	6 12	·25833	1·62316
44	2 56	·12222	·76794	94	6 16	·26111	1·64061
45	3 00	0·12500	0·78540	95	6 20	0·26389	1·65806
46	3 04	·12778	·80285	96	6 24	·26667	1·67552
47	3 08	·13056	·82030	97	6 28	·26944	1·69297
48	3 12	·13333	·83776	98	6 32	·27222	1·71042
49	3 16	·13611	·85521	99	6 36	·27500	1·72788
50	3 20	0·13889	0·87266	100	6 40	0·27778	1·74533

The equivalents in revolutions end in recurring decimals.

Arc	Revolutions	Radians
′	r	
10	0·000463	0·00291
20	0926	0582
30	1389	0873
40	1852	1164
50	2315	1454
60	0·002778	0·01745
70	3241	2036
80	3704	2327
90	4167	2618
100	0·004630	0·02909
″	r	
10	0·000008	0·00005
20	15	10
30	23	15
40	31	19
50	39	24
60	0·000046	0·00029
70	54	34
80	62	39
90	69	44
100	0·000077	0·00048

Time	Days	Radians
m s	d	
0 10	0·000116	0·00073
0 20	231	145
0 30	347	218
0 40	463	291
0 50	579	364
1 00	0·000694	0·00436
1 10	0810	509
1 20	0926	582
1 30	1042	654
1 40	1157	727
1 50	1273	800
2 00	0·001389	0·00873
2 10	1505	0945
2 20	1620	1018
2 30	1736	1091
2 40	1852	1164
2 50	1968	1236
3 00	0·002083	0·01309
3 10	2199	1382
3 20	2315	1454
3 30	2431	1527
3 40	2546	1600
3 50	0·002662	0·01673

revolutions

1° = 0·00277 77778
$1'$ = 0·00004 62963
$1''$ = 0·00000 07716
1^h = 0·04166 66667
1^m = 0·00069 44444
1^s = 0·00001 15741

radians

1° = 0·01745 32925
$1'$ = 0·00029 08882
$1''$ = 0·00000 48481
1^h = 0·26179 93878
1^m = 0·00436 33231
1^s = 0·00007 27221

Revs	Arc	Arc	Time	Time	Radians	Arc	Time
r	°	′	h m	m		° ′ ″	h m s
0·1	36	2160	2 24	144	0·1	5 43 46	0 22 55·1
·2	72	4320	4 48	288	·2	11 27 33	0 45 50·2
·3	108	6480	7 12	432	·3	17 11 19	1 08 45·3
·4	144	8640	9 36	576	·4	22 55 06	1 31 40·4
0·5	180	10800	12 00	720	0·5	28 38 52	1 54 35·5
·6	216	12960	14 24	864	·6	34 22 39	2 17 30·6
·7	252	15120	16 48	1008	·7	40 06 25	2 40 25·7
·8	288	17280	19 12	1152	·8	45 50 12	3 03 20·8
0·9	324	19440	21 36	1296	0·9	51 33 58	3 26 15·9
r	° ′	′	h m s	m		° ′ ″	m s
0·01	3 36	216	0 14 24	14·4	0·01	0 34 23	2 17·5
·02	7 12	432	0 28 48	28·8	·02	1 08 45	4 35·0
·03	10 48	648	0 43 12	43·2	·03	1 43 08	6 52·5
·04	14 24	864	0 57 36	57·6	·04	2 17 31	9 10·0
0·05	18 00	1080	1 12 00	72·0	0·05	2 51 53	11 27·5
·06	21 36	1296	1 26 24	86·4	·06	3 26 16	13 45·1
·07	25 12	1512	1 40 48	100·8	·07	4 00 39	16 02·6
·08	28 48	1728	1 55 12	115·2	·08	4 35 01	18 20·1
0·09	32 24	1944	2 09 36	129·6	0·09	5 09 24	20 37·6
r	° ′ ″	″	m s	s		′ ″	m s
0·001	0 21 36	1296	1 26·4	86·4	0·001	3 26	0 13·8
·002	0 43 12	2592	2 52·8	172·8	·002	6 53	0 27·5
·003	1 04 48	3888	4 19·2	259·2	·003	10 19	0 41·3
·004	1 26 24	5184	5 45·6	345·6	·004	13 45	0 55·0
0·005	1 48 00	6480	7 12·0	432·0	0·005	17 11	1 08·8
·006	2 09 36	7776	8 38·4	518·4	·006	20 38	1 22·5
·007	2 31 12	9072	10 04·8	604·8	·007	24 04	1 36·3
·008	2 52 48	10368	11 31·2	691·2	·008	27 30	1 50·0
0·009	3 14 24	11664	12 57·6	777·6	0·009	30 56	2 03·8
r	′ ″	″	m s	s		′ ″	s
0·0001	2 09·6	129·6	0 08·64	8·64	0·0001	0 21	1·4
·0002	4 19·2	259·2	0 17·28	17·28	·0002	0 41	2·8
·0003	6 28·8	388·8	0 25·92	25·92	·0003	1 02	4·1
·0004	8 38·4	518·4	0 34·56	34·56	·0004	1 23	5·5
0·0005	10 48·0	648·0	0 43·20	43·20	0·0005	1 43	6·9
·0006	12 57·6	777·6	0 51·84	51·84	·0006	2 04	8·3
·0007	15 07·2	907·2	1 00·48	60·48	·0007	2 24	9·6
·0008	17 16·8	1036·8	1 09·12	69·12	·0008	2 45	11·0
0·0009	19 26·4	1166·4	1 17·76	77·76	0·0009	3 06	12·4

Above values are exact. *Above values are rounded-off.*

Conversion constants

$$1^r = 1^d = 360^\circ = 21600' = 1296000'' = 24^h = 1440^m = 86400^s$$

$$1 \text{ radian} = 57^\circ{\cdot}29577\,951 = 57^\circ\,17'\,44''{\cdot}81 = 3437'{\cdot}74677 = 206264''{\cdot}806$$

$$= 3^h{\cdot}81971\,863 = 3^h\,49^m\,10^s{\cdot}99 = 229^m{\cdot}18312 = 13750^s{\cdot}987$$

°		h		r		radians	radians		r		°		h
15	=	1	=	0·0416...	=	0·26179 93878	1	=	0·15915 49431	≑	57·3	≑	3·8
30		2		·083...		0·52359 87756	2		·31830 98862		114·6		7·6
45		3		·125		0·78539 81634	3		·47746 48293		171·9		11·5
60		4		·166...		1·04719 75512	4		·63661 97724		229·2		15·3
90		6		·250		1·57079 63268	5		·79577 47155		286·5		19·1
180		12		·500		3·14159 26536	6		0·95492 96586		343·8		22·9
270		18		0·750		4·71238 89804	7		1·11408 46016		401·1		26·7
360		24		1·000		6·28318 53072	8		1·27323 95447		458·4		30·6

18. REFERENCE DATA

The following list of constants and physical data has been compiled from various sources, and is not intended to be definitive. Reference should be made to the appropriate sections (indicated in parentheses after the headings) of this Supplement, or to the Explanation in the relevant volume, for the basic data used in each ephemeris. See section 6, especially table 6.1 on page 169, for the system of astronomical constants used prior to 1968, and see the reprint of the *Supplement to the A.E. 1968*, especially the reference list on pages 498–9, for details of the IAU system of astronomical constants used from 1968 onwards.

Units

For the computation of motions in the solar system it is customary to use the Gaussian system of astronomical units of mass, time and distance (length). The astronomical unit of mass is the mass of the Sun, the astronomical unit of time is the ephemeris day, and the astronomical unit of length (a.u.) is such that the Gaussian gravitational constant k has the exact value 0·017 202 098 950 (see also page 493). These astronomical units are related to the SI units of mass (kilogram, kg), time (second, s) and length (metre, m) as follows:

1 astronomical unit of mass $= (1{\cdot}990 \pm 0{\cdot}002) \times 10^{30}$ kg
1 astronomical unit of time $= 86\,400 \times (1{\cdot}0 \pm 2 \times 10^{-9})$ s
1 astronomical unit of length $= (1{\cdot}495\,979 \pm 1 \times 10^{-6}) \times 10^{11}$ m

where the given standard errors are based on the quoted errors of the observational determinations (up to 1971) of relevant constants. In the rest of the section the term second (s), unless otherwise qualified, can be considered to refer either to the ephemeris second or to the SI second.

Time

1 day = 24 hours = 1440 minutes = 86400 seconds
1 Julian year = 365·25 days = 8766 hours = 525 960 minutes = 31 557 600 seconds
1 mean tropical year at 1900·0 = 31 556 925·975 ephemeris seconds

Length of the year at 1960 (4B)

	d	d	h	m	s
Tropical (equinox to equinox)	365·24220	365	05	48	46
Sidereal (fixed star to fixed star)	365·25636	365	06	09	10
Anomalistic (perihelion to perihelion)	365·25964	365	06	13	53
Eclipse (Moon's node to Moon's node)	346·62005	346	14	52	52
Gaussian (Kepler's law for $a = 1$)	365·25690	365	06	09	56
Julian	365·25	365	06	00	00

Length of the month (4C)

	d	d	h	m	s
Synodic (new moon to new moon)	29·53059	29	12	44	03
Tropical (equinox to equinox)	27·32158	27	07	43	05
Sidereal (fixed star to fixed star)	27·32166	27	07	43	12
Anomalistic (perigee to perigee)	27·55455	27	13	18	33
Draconic (node to node)	27·21222	27	05	05	36

Length of the day (3B)

1^d of mean solar time $= 1^d{\cdot}00273\,79093$ of mean sidereal time
$= 24^h\,03^m\,56^s{\cdot}55536$ of mean sidereal time
$= 86636{\cdot}55536$ mean sidereal seconds

1^d of mean sidereal time $= 0^d{\cdot}99726\,95664$ of mean solar time
$= 23^h\,56^m\,04^s{\cdot}09054$ of mean solar time
$= 86164{\cdot}09054$ mean solar seconds

See figure 3.2 (b) on page 91 for the variations in the length of the mean solar day during the past three centuries.

Standard epochs

1900 January 0, Greenwich mean noon = 1900 January 0·0 G.M.A.T.
= J.D. 241 5020·0 = 1900 January 0·5 U.T.

1925 January 0, Greenwich mean noon = 1924 December 31·0 G.M.A.T.
= J.D. 242 4151·0 = 1925 January 0·5 U.T.

Beginning of Besselian year (see also table 14.13)

B.Y.	Julian date	B.Y.	Julian date	B.Y.	Julian date
1850·0	239 6758·203	1935·0	242 7803·790	1960·0	243 6934·845
1875·0	240 5889·258	1940·0	242 9630·001	1965·0	243 8761·056
1900·0	241 5020·313	1945·0	243 1456·212	1970·0	244 0587·267
1925·0	242 4151·368	1950·0	243 3282·423	1975·0	244 2413·478
1930·0	242 5977·579	1955·0	243 5108·634	1980·0	244 4239·689

J.D. = 243 3282·423 + 365·2422 (B.Y. − 1950·0)
1950·0 = 1950 January $0^d·923$

Greenwich sidereal date (3B and 14H)

G.S.D. ≑ +0·671 + 1·00273 79093 × J.D.
J.D. ≑ −0·669 + 0·99726 95664 × G.S.D.

Sun, Earth, and Moon

Sun (11B)

Radius	$6·96 \times 10^8$ m	
Semi-diameter at mean distance	15′ 59″·63 = 959″·63	circular to 0″·01
Mass	$1·99 \times 10^{30}$ kg	
Mean density	1·41 g/cm³	
Surface gravity	$2·74 \times 10^2$ m/s² = 27·9 *g*	

Inclination of solar equator to ecliptic	7° 15′
Longitude of ascending node (*T* in centuries from 1900)	74° 22′ + 84′ *T*
Period of synodic rotation (ϕ = latitude)	$26^d·90 + 5^d·2 \sin^2 \phi$
Period of sidereal rotation adopted for heliographic longitudes	25·38 days

Motion relative to near stars — apex: $\alpha = 271°$ $\delta = +30°$; speed: $1·94 \times 10^4$ m/s = 0·0112 a.u./d

Figure and gravity field of the Earth (2F)*

	Hayford spheroid	IAU system
Equatorial radius (*a*)	6 378 388 m	6 378 160 m
Flattening (*f*)	1/297 = 0·003 367 0	1/298·25 = 0·003 352 9
Polar radius (*b*)	6 356 912 m	6 356 774·5 m
Square of eccentricity (e^2)	0·006 722 67	0·006 694 6

$b = a(1 - f)$ $b^2 = a^2(1 - e^2)$ $e^2 = 2f - f^2$

For a point on the spheroid of the IAU system at geodetic latitude ϕ:

1° of latitude $110·575 + 1·110 \sin^2\phi$ km
1° of longitude $(111·320 + 0·373 \sin^2\phi) \cos\phi$ km
Height of sphere of radius *a* above the spheroid $21·4 \sin^2\phi$ km
Geodetic latitude (ϕ) − geocentric latitude (ϕ') $692''·74 \sin 2\phi - 1''·16 \sin 4\phi$
Geocentric gravitational constant $398\,603 \times 10^9$ m³/s² J_2 0·001 0827
Mass of the Earth $5·98 \times 10^{24}$ kg Mean density 5·52 g/cm³
Normal gravity (*g*) $9·780 + 0·052 \sin^2\phi$ m/s²

*See note on page 523.

Orbit of the Earth (4B)

	Values in use 1900–1967	IAU system 1968 onwards
Solar parallax	8″·80	8″·79405
Constant of aberration	20″·47	20″·4958
Light-time for 1 a.u.	498ˢ·38; 498ˢ·58	499ˢ·012
1 astronomical unit of length	not specified	$1{\cdot}496 \times 10^{11}$ m
Mass ratio – Sun/Earth	333 432	332 958
Mass ratio – Sun/(Earth + Moon)	329 390	328 912
Mass ratio – Earth/Moon	81·53; 81·45	81·30

	At 1900:	At 2000:
Mean eccentricity	0·01675	0·01671
Mean obliquity of the ecliptic	23°·45229	23°·43928
Annual rate of rotation of the ecliptic	0″·4711	0″·4704

* Mean distance of Earth from Sun	1·000 000 23 a.u. = 23455 earth radii
Mean orbital speed	29 800 m/s = 0·0172 a.u./d
Mean centripetal acceleration	0·00594 m/s² = 0·0006 *g*

Rotation of the Earth (2B, 2C, 3B)

Period with respect to fixed stars = $24^h\ 00^m\ 00^s{\cdot}0084$ of mean sidereal time
= $23^h\ 56^m\ 04^s{\cdot}0989$ of mean solar time

Rate of rotation $7{\cdot}292\ 115 \times 10^{-5}$ rad/s = 15″·041 07 s^{-1}
= 1·002 738 rev/d = 6·300 387 rad/d

Annual rates of precession: (adopted values; *T* in centuries from 1900)		
	general precession (p)	50″·2564 + 0″·0222 T
	luni-solar precession (ψ')	50″·3708 + 0″·0050 T
	planetary precession (λ')	0″·1247 − 0″·0188 T
	in right ascension (m)	3ˢ·07234 + 0ˢ·00186 T
	in declination (n)	20″·0468 − 0″·0085 T

Period of precession is about 25 730 years

Constant of nutation 9″·210 (recent determination: 9″·207)
Maximum value of nutation in longitude is about 19″
Maximum value of nutation in obliquity is about 10″
Period of principal term in nutation is about 6798 days

At equator: speed 465 m/s centripetal acceleration 0·0339 m/s² = 0·0035 *g*

Moon

Mean radius	$1{\cdot}738 \times 10^6$ m	Mass	$7{\cdot}35 \times 10^{22}$ kg
Semi-diameter at mean distance	15′ 32″·6	Mean density	3·34 g/cm³
		Surface gravity	1·62 m/s² = 0·17 *g*

Orbit of Moon about the Earth (4C)

Sidereal mean motion of Moon (1900) $2{\cdot}661\ 699\ 489 \times 10^{-6}$ rad/s
Mean distance of Moon from Earth $3{\cdot}844 \times 10^8$ m = 60·27 earth radii
= 0·002 570 a.u.

Equatorial horizontal parallax at mean distance 57′ 02″·608 = 3422″·608
Mean distance of centre of Earth from Earth–Moon barycentre $4{\cdot}671 \times 10^6$ m

Mean eccentricity	0·05490	Mean inclination to ecliptic	5° 09′
Mean inclination to lunar equator	6° 41′	Limits of geocentric declination	± 29°

Saros ≑ 223 lunations ≑ 19 passages of Sun through node = 6585⅓ days

Period of revolution of node	6798 days
Period of revolution of perigee	3232 days
Mass ratio – Earth/Moon	81·301 (See above for adopted values)
Mean orbital speed	1023 m/s = 0·000 591 a.u./d
Mean centripetal acceleration	0·002 72 m/s² = 0·0003 *g*

*See note on page 523.

Mean elements of planetary orbits

For epoch 1960 January 1·5 E.T.

	Inclination i	Mean Longitude of Node Ω	Mean Longitude of Perihelion ϖ	Mean Longitude at Epoch L	Eccentricity e
	°	°	°	°	
Mercury	7·00399	47·85714	76·83309	222·62165	0·205627
Venus	3·39423	76·31972	131·00831	174·29431	0·006793
Earth	0·0	0·0	102·25253	100·15815	0·016726
Mars	1·84991	49·24903	335·32269	258·76729	0·093368
Jupiter	1·30536	100·04444	13·67823	259·83112	0·048435
Saturn	2·48991	113·30747	92·26447	280·67135	0·055682
Uranus	0·77306	73·79630	170·01083	141·30496	0·047209
Neptune	1·77375	131·33980	44·27395	216·94090	0·008575
Pluto*	17·1699	109·88562	224·16024	181·64632	0·250236

	Mean Distance a	Mean Distance 10^6 km	Sidereal Period (tropical years)	Synodic Period	Mean Daily Motion n	Orbital Velocity (km/s)
				d	°	
Mercury	0·387099	57·9	0·24085	115·88	4·092339	47·8
Venus	0·723332	108·1	0·61521	583·92	1·602130	35·0
Earth	1·000000	149·5	1·00004		0·985609	29·8
Mars	1·523691	227·8	1·88089	779·94	0·524033	24·2
Jupiter	5·202803	778	11·86223	398·88	0·083091	13·1
Saturn	9·538843	1426	29·45772	378·09	0·033460	9·7
Uranus	19·181951	2868	84·01331	369·66	0·011732	6·8
Neptune	30·057779	4494	164·79345	367·48	0·005981	5·4
Pluto*	39·43871	5896	247·686	366·72	0·003979	4·7

* The elements for Pluto are osculating values for epoch 1960 September 23·0 E.T. = J.D. 243 7200·5.

See page 112 for the masses of the planets adopted in the Ephemeris.

Dimensions and rotations of the planets and Moon

	S.D. at Unit Distance	Radius* on scale Earth = 1	Reciprocal of Flattening	Mass on scale Earth = 1	Density g/cm³	Surface Gravity Earth = 1	Rotation Period	Inclination of Equator to Orbit
	″							
Mercury	3·34	0·39	∞	0·056	5·13	0·36	$58^{d}\ 16^{h}$	0° 00′
Venus	8·41	0·97	∞	0·817	4·97	0·87	$242^{d}\ 23^{h}$	177° 50′
Earth	8·79	1·00	298	1·000	5·52	1·00	$23^{h}\ 56^{m}$	23° 27′
Moon	2·40	0·27	—	0·0123	3·34	0·17	$27^{d}\ 07^{h}$	1° 32′
Mars	4·68	0·53	192	0·108	3·94	0·38	$24^{h}\ 37^{m}$	25° 12′
Jupiter	98·47	11·19	16·1	318·0	1·33	2·64	$9^{h}\ 50^{m}$	3° 04′
Saturn	83·33	9·47	10·4	95·2	0·69	1·13	$10^{h}\ 14^{m}$	26° 44′
Uranus	34·28	3·69	16	14·6	1·56	1·07	$10^{h}\ 49^{m}$	97° 53′
Neptune	36·56	3·50	50	17·3	2·27	1·41	$15^{h}\ 40^{m}$	28° 48′
Pluto	—	0·5 ?	?	0·1 ?	4 ?	0·4	$6^{d}\ 10^{h}$	90° ?

* The radii of the planets are based on recent values for the angular semi-diameters; the equatorial radius of the Earth is 6378 km = 3963 miles. The tabulated semi-diameters are the values adopted in the Ephemeris.

Satellites

		Mean distance 10³ km	Angular at mean oppⁿ. dist.	Sidereal period (days)	Synodic period (days)	Inclination of orbit to planet's equator	Eccentricity of mean orbit	Magnitude at mean opposition distance
Earth								
	Moon	384·4		27·32166	29·531	Var.	0·05490	−12·5
Mars			° ′ ″			° ′		
I	Phobos	9·4	25	0·31891	0·319	0 57	0·0210	11
II	Deimos	23·5	1 02	1·26244	1·265	1 18	0·0028	12
Jupiter								
V		181	59	0·49818	0·498	0 24	0·003	13
I	Io	422	2 18	1·76914	1·770	0	0	5·5*
II	Europa	671	3 40	3·55118	3·554	0	0	6·1*
III	Ganymede	1071	5 51	7·15455	7·166	0	0	5·1*
IV	Callisto	1884	10 18	16·68902	16·754	0	0	6·2*
VI		11480	1 02 45	250·57	266	27 38	0·15798	14·7
VII		11740	1 04 10	259·65	276	24 46	0·20719	18
X		11860	1 04 48	263·55	281	29 01	0·13029	19
XII		21200	1 55 58	631·1		147	0·16870	18
XI		22600	2 03 24	692·5		164	0·20678	19
VIII		23500	2 08 35	738·9		145	0·378	17·0
IX		23700	2 09 39	758		153	0·275	18·6
Saturn								
X	Janus	159	25	0·7490	0·749	0	0	14·0
I	Mimas	186	30	0·94242	0·943	1 31	0·0201	12·1
II	Enceladus	238	38	1·37022	1·370	0 01	0·00444	11·7
III	Tethys	295	48	1·88780	1·888	1 06	0	10·6
IV	Dione	378	1 01	2·73692	2·738	0 01	0·00221	10·7
V	Rhea	527	1 25	4·51750	4·519	0 21	0·00098	10·0
VI	Titan	1222	3 17	15·94545	15·969	0 20	0·0289	8·3
VII	Hyperion	1481	3 59	21·27666	21·319	0 26	0·104	15
VIII	Iapetus	3562	9 35	79·33082	79·920	14 43	0·02828	10·8
IX	Phoebe	12960	34 56	550·45	523·7	150	0·16326	14
Uranus								
V	Miranda	124	9	1·414	1·4	0	< 0·01	17
I	Ariel	192	14	2·52038	2·521	0	0·0028	14
II	Umbriel	267	20	4·14418	4·145	0	0·0035	14
III	Titania	438	33	8·70588	8·708	0	0·0024	14
IV	Oberon	587	44	13·46326	13·469	0	0·0007	14
Neptune								
I	Triton	354	17	5·87683	5·877	159 57	0	14
II	Nereid	5570	4 24	359·4	362	27 27	0·76	19

* Variable

Saturn's rings

	10³ km	At unit distance	At mean oppⁿ. dist.	Ratios	
		″	″		
Outer diameter of outer ring	272·1	375·4	43·96	1·0000	2·252
Inner diameter of outer ring	239·5	330·4	38·69	0·8801	1·982
Outer diameter of inner ring	234·0	322·8	37·80	0·8599	1·936
Inner diameter of inner ring	180·9	249·6	29·24	0·6650	1·498
Inner diameter of dusky ring	149·2	205·9	24·12	0·5486	1·236
Equatorial diameter of Saturn	120·8	166·7	19·52	0·4440	1·000

Gravitational constants

Gaussian gravitational constant

$$k = 0{\cdot}01720\ 20989\ 50000$$
$$k^2 = 0{\cdot}00029\ 59122\ 08286$$
$$1/k = 58{\cdot}13244\ 08670$$
$$2\pi/k = 365{\cdot}25689\ 83263$$
$$k^\circ = 360k/2\pi = 0{\cdot}98560\ 76686$$

The value of k is treated as exact, and defines the astronomical unit (a.u.) as the radius of a circular orbit in which a body of negligible mass, and free of perturbations, would revolve round the Sun in a Gaussian year of $2\pi/k$ ephemeris days.

When the unit of mass is the mass of the Sun the gravitational attraction between two particles of mass M and m separated by a distance r is k^2Mm/r^2; further the mean motion (n in radians per day, n° in degrees per day), period (P in days), and speed (v in a.u. per day) of a body of mass m revolving in an elliptic orbit about a centre of attraction of mass M at mean distance a (in a.u.) are given by:

$$n^2a^3 = k^2(M + m) \qquad n^\circ = k^\circ (M + m)^{\frac{1}{2}} a^{-\frac{3}{2}}$$
$$P = (2\pi/k)(M + m)^{-\frac{1}{2}} a^{\frac{3}{2}} \qquad v = k\sqrt{(M + m)}\left(\frac{2}{r} - \frac{1}{a}\right)$$

Values of the constants in these expressions, when m is negligible, are given below for motion about each of the major planets and the Moon.

Values of the gravitational constant G, corresponding to k^2, may be obtained for other units from:

$$G = k^2 L^3/MT^2$$

where M is the mass of the Sun, T is one ephemeris day, and L is one astronomical unit expressed in terms of the new units; the precision of G obtained in this way is limited by the precision to which L, M, T are known. By direct measurement:

$$G = 6{\cdot}670 \times 10^{-11}$$

when the units are the metre, the kilogram, and the second.

For motion about the Sun $GM = 13{\cdot}246 \times 10^{19}$ when the units of length and time are the metre and the second; values for the planets are given below.

Units	Mass (M) Sun = 1	$k^2 M$	$k\sqrt{M}$	$k^\circ\sqrt{M}$	$2\pi/k\sqrt{M}$	GM m³/s²
		a.u., days				
		$\times 10^{-10}$	$\times 10^{-5}$	$\times 10^{-5}$	$\times 10^{5}$	$\times 10^{19}$
Mercury	0·00000 017	0·49	0·71	40	8·9	0·00000 22
Venus	·00000 245	7·25	2·69	154	2·33	·00003 25
Earth + Moon	·00000 304	8·98	3·00	172	2·09	·00004 02
Earth	·00000 2999	8·87	2·98	171	2·11	·00003 97
Moon	·00000 0037	0·11	0·33	19	19	·00000 049
Mars	·00000 032	0·96	0·97	56	6·5	·00000 43
Jupiter	·00095 4786	2825·33	53·1538	3045·49	0·11821	·01264 3
Saturn	·00028 5584	845·08	29·0702	1665·60	0·21614	·00378 2
Uranus	·00004 3727	129·39	11·375	651·74	0·55236	·00057 9
Neptune	0·00005 1776	153·21	12·378	709·20	0·50761	0·00068 6

For motion about the Earth it is appropriate to use the mass of the Earth, the radius of the Earth, and the minute as the units and to denote GM_E by k_E^2; it may be determined accurately from the observed acceleration due to gravity at the Earth's surface. If k_E is treated as exact it defines a unit of distance, which may be called the gravitational radius of the Earth, as the radius of an equatorial circular orbit in which a particle of negligible mass, and free of perturbations, would revolve round the Earth in a period of $2\pi/k_E$ ephemeris minutes.

$$k_E = 0{\cdot}07436\ 574 \qquad k_E^2 = 0{\cdot}00553\ 02633$$
$$1/k_E = 13{\cdot}44705 \qquad 2\pi/k_E = 84{\cdot}49032$$
$$k_E^\circ = 360\,k_E/2\pi = 4{\cdot}26084\ 3 \qquad \text{Unit of distance} = 6378270\ \text{m}$$

Units of length, speed, and mass

	Metres (m)	Miles (mi.)	Astronomical units (a.u.)	Light years l.y.	Parsecs pc
1 m	= 1	6.214×10^{-4}	6.684×10^{-12}	1.057×10^{-16}	3.241×10^{-17}
1 mi.	= 1.609×10^{3}	1	1.076×10^{-8}	1.701×10^{-13}	5.215×10^{-14}
1 a.u.	= 1.496×10^{11}	9.296×10^{7}	1	1.581×10^{-5}	4.848×10^{-6}
1 l.y.	= 9.461×10^{15}	5.879×10^{12}	6.324×10^{4}	1	0·3066
1 pc	= 3.086×10^{16}	1.917×10^{13}	2.063×10^{5}	3·262	1

1 (statute) mile = 5280 feet 1 foot = 0·30480 metres

	Metres per second (m/s)	Miles per hour (mi./hr.)	Astronomical units per day (a.u./day)	Parsecs per century (pc/cent.)	Light years per year (velocity of light)
1 m/s	= 1	2·237	5.775×10^{-7}	1.023×10^{-7}	3.336×10^{-9}
1 mi./hr.	= 0·4470	1	2.582×10^{-7}	4.572×10^{-8}	1.491×10^{-9}
1 a.u./day	= 1.731×10^{6}	3.873×10^{6}	1	0·1771	0·005776
1 pc/cent.	= 9.778×10^{6}	2.187×10^{7}	5·647	1	0·03262
velocity of light	= 2.998×10^{8}	6.706×10^{8}	173·14	30·66	1

1 kilogram (kg) = 2·205 pounds (lb) 1 lb = 0·4536 kg 1 (long) ton = 2240 lb = 1016 kg

Greek alphabet

α	A	alpha	η	H	eta	ν	N	nu	τ	T	tau
β	B	beta	θ	Θ	theta	ξ	Ξ	xi	υ	Υ	upsilon
γ	Γ	gamma	ι	I	iota	o	O	omicron	ϕ	Φ	phi
δ	Δ	delta	κ	K	kappa	π	Π	pi*	χ	X	chi
ϵ	E	epsilon	λ	Λ	lambda	ρ	P	rho	ψ	Ψ	psi
ζ	Z	zeta	μ	M	mu	σ	Σ	sigma†	ω	Ω	omega

* ϖ (' curly pi ') is an alternative form of π.

† ς is an alternative form of σ.

Constellation names and abbreviations

The following list of constellation names and abbreviations is in accordance with the resolutions of the International Astronomical Union (*Trans. I.A.U.*, **1**, 158; **4**, 221; **9**, 66 and 77). The boundaries of the constellations are listed by E. Delporte, on behalf of the I.A.U., in *Délimitation scientifique des constellations* (*tables et cartes*), Cambridge University Press, 1930; the areas of the constellations are given in *Handbook B.A.A.*, 1961.

ominative		Genitive	Nominative		Genitive
dromeda	And	Andromedae	Lacerta	Lac	Lacertae
tlia	Ant	Antliae	Leo	Leo	Leonis
us	Aps	Apodis	Leo Minor	LMi	Leonis Minoris
uarius	Aqr	Aquarii	Lepus	Lep	Leporis
uila	Aql	Aquilae	Libra	Lib	Librae
a	Ara	Arae	Lupus	Lup	Lupi
go	Arg	Argus	Lynx	Lyn	Lyncis
ies	Ari	Arietis	Lyra	Lyr	Lyrae
riga	Aur	Aurigae	Mensa	Men	Mensae
otes	Boo	Bootis	Microscopium	Mic	Microscopii
elum	Cae	Caeli	Monoceros	Mon	Monocerotis
melopardalis	Cam	Camelopardalis	Musca	Mus	Muscae
ncer	Cnc	Cancri	Norma	Nor	Normae
nes Venatici	CVn	Canum Venaticorum	Octans	Oct	Octantis
nis Major	CMa	Canis Majoris	Ophiuchus	Oph	Ophiuchi
nis Minor	CMi	Canis Minoris	Orion	Ori	Orionis
pricornus	Cap	Capricorni	Pavo	Pav	Pavonis
rina	Car	Carinae	Pegasus	Peg	Pegasi
ssiopeia	Cas	Cassiopeiae	Perseus	Per	Persei
ntaurus	Cen	Centauri	Phoenix	Phe	Phoenicis
pheus	Cep	Cephei	Pictor	Pic	Pictoris
tus	Cet	Ceti	Pisces	Psc	Piscium
amaeleon	Cha	Chamaeleontis	†Piscis Austrinus	PsA	Piscis Austrini
rcinus	Cir	Circini	Puppis	Pup	Puppis
lumba	Col	Columbae	Pyxis	Pyx	Pyxidis
ma Berenices	Com	Comae Berenices	Reticulum	Ret	Reticuli
rona Austrina	CrA	Coronae Austrinae	Sagitta	Sge	Sagittae
rona Borealis	CrB	Coronae Borealis	Sagittarius	Sgr	Sagittarii
rvus	Crv	Corvi	Scorpius	Sco	Scorpii
ater	Crt	Crateris	Sculptor	Scl	Sculptoris
ux	Cru	Crucis	Scutum	Sct	Scuti
gnus	Cyg	Cygni	‡Serpens	Ser	Serpentis
lphinus	Del	Delphini	Sextans	Sex	Sextantis
rado	Dor	Doradus	Taurus	Tau	Tauri
aco	Dra	Draconis	Telescopium	Tel	Telescopii
quuleus	Equ	Equulei	Triangulum	Tri	Trianguli
idanus	Eri	Eridani	Triangulum Australe	TrA	Trianguli Australis
rnax	For	Fornacis	Tucana	Tuc	Tucanae
emini	Gem	Geminorum	Ursa Major	UMa	Ursae Majoris
us	Gru	Gruis	Ursa Minor	UMi	Ursae Minoris
ercules	Her	Herculis	Vela	Vel	Velorum
orologium	Hor	Horologii	Virgo	Vir	Virginis
ydra	Hya	Hydrae	Volans	Vol	Volantis
ydrus	Hyi	Hydri	Vulpecula	Vul	Vulpeculae
dus	Ind	Indi			

* In modern usage Argo is divided into Carina, Puppis, and Vela.
† Australis is sometimes used, in both nominative and genitive.
‡ Serpens may be divided into Serpens Caput and Serpens Cauda.

Short table of $a^{-\frac{3}{2}}$ and $a^{\frac{3}{2}}$

a	$a^{-\frac{3}{2}}$	$a^{\frac{3}{2}}$	a	$a^{-\frac{3}{2}}$	$a^{\frac{3}{2}}$	a	$a^{-\frac{3}{2}}$	$a^{\frac{3}{2}}$
0·1	31·623	0·03162	**1**	1·00000	1·000	**10**	0·03162	31·6
·2	11·180	·08944	**2**	0·35355	2·828	**20**	·01118	89·4
·3	6·086	·16432	**3**	·19245	5·196	**30**	·00609	164·3
·4	3·953	·25298	**4**	·12500	8·000	**40**	·00395	253·0
0·5	2·828	0·35355	**5**	0·08944	11·180	**50**	0·00283	353·6
·6	2·152	·46476	**6**	·06804	14·697	**60**	·00215	464·8
·7	1·707	·58566	**7**	·05399	18·520	**70**	·00171	585·7
·8	1·398	·71554	**8**	·04419	22·627	**80**	·00140	715·5
0·9	1·171	0·85381	**9**	·03704	27·000	**90**	·00117	853·8
1·0	1·000	1·00000	**10**	0·03162	31·623	**100**	0·00100	1000·0

Mathematical constants and other data

π	3·14159 265	$1/\pi$	0·31830 989	$\sqrt{\pi}$	1·77245 385	$1/\sqrt{\pi}$	0·56418 958
$\pi/2$	1·57079 633	$2/\pi$	0·63661 977	$\sqrt{(\pi/2)}$	1·25331 414	$\sqrt{(2/\pi)}$	0·79788 456
2π	6·28318 531	$1/(2\pi)$	0·15915 494	$\sqrt{(2\pi)}$	2·50662 827	$1/\sqrt{(2\pi)}$	0·39894 228
π^2	9·86960 440	$1/\pi^2$	0·10132 118	$\pi/4$	0·78539 816	$4/\pi$	1·27323 954
e	2·71828 183	1/e	0·36787 944	$\sqrt{e}$	1·64872 127	$1/\sqrt{e}$	0·60653 066
e^{π}	23·14069 263	$e^{-\pi}$	0·04321 392	$e^{\pi/2}$	4·81047 738	$e^{-\pi/2}$	0·20787 958
$\sqrt{2}$	1·41421 356	$1/\sqrt{2}$	0·70710 678	$\sqrt{3}$	1·73205 081	$1/\sqrt{3}$	0·57735 027
$\sqrt{10}$	3·16227 766	$1/\sqrt{10}$	0·31622 777	$10^{\frac{3}{2}}$	31·62277 660	$10^{-\frac{3}{2}}$	0·03162 278

$\log_{10} x = 0{\cdot}43429\,448 \log_e x \qquad \log_e x = 2{\cdot}30258\,509 \log_{10} x$

1 radian =	57°·29577 951	1° =	0·01745 32925 199 radians
1 ,, =	3437′·74677 078	1′ =	0·00029 08882 087 ,,
1 ,, =	206 264″·80624 710	1″ =	0·00000 48481 368 ,,

For other angular conversion constants see table 17.10.

Solid angles

1 steradian =	3283 square degrees	1 square degree =	$0{\cdot}305 \times 10^{-3}$ steradians
1 ,, =	$1{\cdot}18 \times 10^{7}$ square minutes	1 square minute =	$0{\cdot}846 \times 10^{-7}$,,
1 ,, =	$4{\cdot}25 \times 10^{11}$ square seconds	1 square second =	$0{\cdot}235 \times 10^{-11}$,,

A sphere subtends 4π steradians = 41253 square degrees
= $1{\cdot}485 \times 10^{8}$ square minutes = $5{\cdot}35 \times 10^{11}$ square seconds

Accuracy of use of tables of sin x

Change in sin x	Corresponding change in x: near 0°	near 60°	near 80°	near 90°
0·005	0°·29 = 17′	0°·57 = 34′	1°·66 = 99′	5°·73 = 344′
0·00000 5	0°·0003 = 1″	0°·0006 = 2″	0°·0016 = 6″	0°·18 = 11′
0·00000 0005	$0^\circ{\cdot}0^6 3$ = 0″·001	$0^\circ{\cdot}0^6 6$ = 0″·002	$0^\circ{\cdot}0^5 16$ = 0″·006	0°·006 = 21″

Convergence of differences in tables of sin x and cos x

$$\delta^{2n} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (a + ph) = (-1)^n k^n \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (a + ph) \text{ where } k = 4 \sin^2 (h/2)$$

h =	90°	80°	70°	60°	50°	40°	30°	20°	10°	5°	1°
k	2	1·65	1·32	1	0·714	0·468	0·268	0·121	0·0304	0·00761	$0{\cdot}0^3 305$
k^2	4	2·73	1·73	1	·510	·219	·0718	·0145	$\cdot 0^3 923$	$\cdot 0^4 579$	$\cdot 0^7 928$
k^3	8	4·51	2·28	1	·365	·102	·0192	·00175	$\cdot 0^4 281$	$\cdot 0^6 441$	$\cdot 0^{10} 28$
k^4	16	7·46	3·00	1	·261	·0479	·00515	$\cdot 0^3 212$	$\cdot 0^6 852$	$\cdot 0^8 335$	$\cdot 0^{14} 86$
k^5	32	12·33	3·95	1	·186	·0224	·00138	$\cdot 0^4 255$	$\cdot 0^7 259$	$\cdot 0^{10} 26$	$\cdot 0^{17} 26$

For $h = 180^\circ$, $k = 4$; for $h = 120^\circ$, $k = 3$

SUPPLEMENT TO THE A.E. 1968*

THE INTRODUCTION OF THE IAU SYSTEM OF ASTRONOMICAL CONSTANTS

A. General considerations

1. *Purpose*

At its session on 3 September 1964 in Hamburg, the General Assembly of the International Astronomical Union adopted the following resolution. (*Trans. IAU* **12B**, 95, 1966.)

The International Astronomical Union endorses the final list of constants prepared by the Working Group on the System of Astronomical Constants and recommends that it be used in the national and international ephemerides at the earliest practicable date.

Commission 4 (Ephemerides) had earlier recommended, in the following resolution (*Trans. IAU* **12B**, 105, 1966), that the effects of the new system be introduced into the national and international ephemerides in 1968.

Commission 4 recommends that the new system of astronomical constants should be introduced into the national and international ephemerides as for the year 1968, as far as this is practicable. It recognises the difficulty of introducing into the planetary and lunar ephemerides the full effects of the changes of mass of the Earth and Moon, and recommends that the compilers of the relevant ephemerides should decide the precise procedures to be followed. Further, where it is impracticable to correct the current ephemerides, it recommends that differential corrections or formulae from which such corrections can be calculated should be given.

The main purpose of this Supplement is to describe précisely how these resolutions are being implemented. A brief analysis of how the IAU System of Astronomical Constants can best be introduced into the ephemerides is followed by a detailed derivation of the corrections that are necessary to convert the printed ephemerides, based on the old system, into those based on the newly-adopted system. A list is also given of the changes required in the *Explanatory Supplement*.

2. *The IAU System of Astronomical Constants*

The IAU System of Astronomical Constants was formally proposed by a Working Group (W. Fricke, *Chairman*; D. Brouwer; J. Kovalevsky; A. A. Mikhailov; G. A. Wilkins, *Secretary*) set up by the Executive Committee of the Union in response to a resolution adopted at IAU Symposium No. 21 (The System of Astronomical Constants) held in Paris in May 1963. The full report of the Working Group is published as an appendix to the report of the Joint Discussion on 'The IAU System of Astronomical Constants' in *Trans. IAU* **12B**,

Continued on page 502

* The abbreviation "A.E." is used for practical convenience to denote the two publications, identical other than for title, *The Astronomical Ephemeris* and *The American Ephemeris and Nautical Almanac*.

REFERENCE LIST OF RECOMMENDED CONSTANTS*

Defining constants: Constantes de définition

1. Number of ephemeris seconds in 1 tropical year (1900) Nombre de secondes de temps des éphémérides pour l'année tropique (1900)	s = 31 556 925·974 7
2. Gaussian gravitational constant, defining the A.U. Constante de la gravitation universelle, définissant l'unité astronomique (U.A.)	k = 0·017 202 098 95

Primary constants: Constantes primaires

3. Measure of 1 A.U. in metres Longueur de l'U.A. en mètres	A = 149 600 × 10^6
4. Velocity of light in metres per second Vitesse de la lumière, en mètres par seconde	c = 299 792·5 × 10^3
5. Equatorial radius for Earth in metres Rayon équatorial terrestre, en mètres	a_e = 6 378 160
6. Dynamical form-factor for Earth Facteur d'ellipticité géopotentielle	J_2 = 0·001 082 7
7. Geocentric gravitational constant (units: $m^3 s^{-2}$) Constante géocentrique de la gravitation	GE = 398 603 × 10^9
8. Ratio of the masses of the Moon and Earth Rapport de la masse de la Lune à celle de la Terre	μ = 1/81·30
9. Sidereal mean motion of Moon in radians per second (1900) Moyen mouvement sidéral de la Lune en radians par seconde (1900)	$n^*_{☾}$ = 2·661 699 489 × 10^{-6}
10. General precession in longitude per tropical century (1900) Précession générale en longitudes par siècle tropique (1900)	p = 5025″·64
11. Obliquity of the ecliptic (1900) Obliquité de l'écliptique (1900)	ϵ = 23° 27′ 08″·26
12. Constant of nutation (1900) Constante de la nutation (1900)	N = 9″·210

Auxiliary constants and factors

k/86400, for use when the unit of time is 1 second	k' = 1·990 983 675 × 10^{-7}
Number of seconds of arc in 1 radian	206 264·806
Factor for constant of aberration (note 15)	F_1 = 1·000 142
Factor for mean distance of Moon (note 20)	F_2 = 0·999 093 142
Factor for parallactic inequality (note 23)	F_3 = 49 853″·2

* The above list was confirmed by the Working Group, adopted as definitive at the Joint Discussion, and endorsed by the Twelfth General Assembly (Resolution no. 4, page 95); it now represents the 'IAU System of Astronomical Constants'.

* La liste ci-dessus a été confirmée par le Groupe de Travail, adoptée définitivement à la Discussion Commune, et appuyée par la Douzième Assemblée Générale (Résolution no. 4, page 94); elle représente désormais le 'Système UAI des Constantes Astronomiques'.

REFERENCE LIST OF RECOMMENDED CONSTANTS*

Derived constants: Constantes secondaires

13. Solar parallax
Parallaxe solaire — $\arcsin (a_e/A) = \pi_\odot = 8''.794\,05\ (8''.794)$

14. Light-time for unit distance
Temps de lumière relatif à l'U.A. — $A/c = \tau_A = 499^s.012 = 1^s/0{\cdot}002\,003\,96$

15. Constant of aberration
Constante de l'aberration — $F_1 k' \tau_A = \kappa = 20''.4958\ (20''.496)$

16. Flattening factor for Earth
Aplatissement terrestre — $f = 0{\cdot}003\,352\,9 = 1/298{\cdot}25$

17. Heliocentric gravitational constant (units: $m^3\ s^{-2}$)
Constante héliocentrique de la gravitation — $A^3 k'^2 = GS = 132\,718 \times 10^{15}$

18. Ratio of masses of Sun and Earth
Rapport de la masse du Soleil à celle de la Terre — $(GS)/(GE) = S/E = 332\,958$

19. Ratio of masses of Sun and Earth + Moon
Rapport de la masse du Soleil à celle du système Terre-Lune — $S/E(1 + \mu) = 328\,912$

20. Perturbed mean distance of Moon, in metres
Demi-grand axe perturbé de l'orbite de la Lune, en mètres — $F_2(GE(1 + \mu)/n_{\text{☾}}^{*2})^{1/3} = a_{\text{☾}} = 384\,400 \times 10^3$

21. Constant of sine parallax for Moon
Sinus de la parallaxe de la Lune — $a_e/a_{\text{☾}} = \sin \pi_{\text{☾}} = 3422''.451$

22. Constant of lunar inequality
Constante de l'inégalité lunaire — $\frac{\mu}{1 + \mu}\frac{a_{\text{☾}}}{A} = L = 6''.439\,87\ (6''.440)$

23. Constant of parallactic inequality
Constante de l'inégalité parallactique — $F_3 \frac{1 - \mu}{1 + \mu}\frac{a_{\text{☾}}}{A} = P_{\text{☾}} = 124''.986$

System of planetary masses

	Reciprocal mass		Reciprocal mass
24. Mercury	6 000 000	Jupiter	1 047·355
Venus	408 000	Saturn	3 501·6
Earth + Moon	329 390	Uranus	22 869
Mars	3 093 500	Neptune	19 314
		Pluto	360 000

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* La liste ci-dessus a été confirmée par le Groupe de Travail, adoptée définitivement à la Discussion Commune, et appuyée par la Douzième Assemblée Générale (Résolution no. 4, page 94); elle représente désormais le 'Système UAI des Constantes Astronomiques'.

NOTES ON THE CONSTANTS

1. The value given for the number of ephemeris seconds in the tropical year at 1900 is taken from the definition of the ephemeris second that was adopted by the Comité International des Poids et Mesures (*Procès Verbaux des Séances* deuxième série, **25**, 77, 1957). It is, in fact, derived from the coefficient of T, measured in Julian centuries of 36 525 days, in Newcomb's expression for the geometric mean longitude of the Sun referred to the mean equinox of date. In the list '1900' refers to the fundamental epoch of ephemeris time, namely 1900 January 0 at 12^h E.T., or to 1900·0, as appropriate; the values for constants 20-23 also refer to the fundamental epoch. Throughout the list and this Report the term 'second' must be understood to mean the 'ephemeris second'.

2. The value of the Gaussian gravitational constant (k) is that adopted by the IAU in 1938, and serves to define the astronomical unit of length (A.U.) since the corresponding (astronomical) units of mass and time are already defined. (The unit of mass is that of the Sun and the unit of time is the ephemeris day of 86 400 ephemeris seconds. The units of k are: $(\text{A.U.})^{3/2}$ (ephemeris day)$^{-1}$ (Sun's mass)$^{-1/2}$.) To simplify the later equations an auxiliary constant k', defined as $k/86\,400$, is introduced and a rounded value is given in the list.

3. The value for the measure of the A.U. in metres is a rounded value of recent radar determinations.

4. The value for the velocity of light is that recommended by the International Union of Pure and Applied Physics in September 1963.

5. The term 'equatorial radius for Earth' refers to the equatorial radius of an ellipsoid of revolution that approximates to the geoid. (See also note 16.)

6. The term 'dynamical form-factor for Earth' refers to the coefficient of the second harmonic in the expression for the Earth's gravitational potential as adopted by IAU Commission 7 in 1961. (See also note 16.)

7. The geocentric gravitational constant (GE) is appropriate for use for geocentric orbits when the units of length and time are the metre and the second; E denotes the mass of the Earth including its atmosphere. Kepler's third law for a body of mass M moving in an unperturbed elliptic orbit around the Earth may be written

$$GE\,(1 + M/E) = n^2a^3$$

where n is the sidereal mean motion in radians per second and a is the mean distance in metres. The value of GE is based on gravity measurements and observations of satellites.

8. Again the mass of Earth includes the mass of the atmosphere. The reciprocal of 81·30 is 0·0123 001.

9. The value for the sidereal mean motion of the Moon is consistent with the value of the tropical mean motion used in the improved lunar ephemeris, less the general precession in longitude.

10–12. The values of the principal constants defining the relative positions and motions of the equator and ecliptic are those in current use. Secular terms and derived quantities are already tabulated elsewhere.

13. The rounded value 8″·794 for the solar parallax should be used except where extra figures are required to ensure numerical consistency.

14. The value of the light-time for unit distance is numerically equal to the number of light-seconds in 1 A.U. Its reciprocal is equal to the velocity of light in A.U. per second.

15. Apart from the factor F_1 the constant of aberration is equal to the ratio of the speed of a hypothetical planet of negligible mass moving in a circular orbit of unit radius to the velocity of light; it is conventionally expressed in seconds of arc by multiplying by the number of seconds of arc in one radian. The factor F_1 is the ratio of the mean speed of the Earth to the

speed of the hypothetical planet and is given by

$$F_1 = \frac{n_\odot}{k'} \frac{a_\odot}{(1 - e^2)^{1/2}}$$

where $n_\odot$ is the sidereal mean motion of the Sun in radians per second, $a_\odot$ is the perturbed mean distance of the Sun in A.U., and e is the mean eccentricity of the Earth's orbit. Newcomb's values for $n_\odot$, $a_\odot$ and e are of ample accuracy for this purpose. The factor F_1 and the constant of aberration take the following values

	F_1	κ''
1800	1·000 142 7	20·495 83
1900	1·000 142 0	20·495 82
2000	1·000 141 3	20·495 81

The rounded value 20″·496 should be used except where the extra figures are required to ensure numerical consistency.

16. The condition that the reference ellipsoid of revolution for the Earth shall be an equipotential surface implies that three parameters are sufficient to define its geometrical form and external gravitational field, provided that the angular velocity (ω) of the Earth and the relative mass of the atmosphere (μ_a) are assumed to be known. The variability of the rate of rotation of the Earth can be ignored, and the mass of the atmosphere is only just significant; the required values are:

$$\omega = 0{\cdot}000\ 072\ 921 \text{ radians per second}; \quad \mu_a = 0{\cdot}000\ 001$$

The expressions for the flattening (f) and the apparent gravity at the equator (g_e) in terms of the primary constants are, to second order:

$$f = \tfrac{3}{2} J_2 + \tfrac{1}{2} m + \tfrac{9}{8} J_2^2 + \tfrac{15}{28} J_2 m - \tfrac{39}{56} m^2$$

$$g_e = (GE/a_e^2)\left(1 - \mu_a + \tfrac{3}{2} J_2 - m + \tfrac{27}{8} J_2^2 - \tfrac{6}{7} J_2 m + \tfrac{47}{56} m^2\right)$$

where $m = a_e \omega^2 / g_e$ is obtained by successive approximations. The new value of f is given here only for astronomical use (parallax corrections, etc).

17. The heliocentric gravitational constant corresponds to GE, but is appropriate for heliocentric orbits when the units are the metre and the second.

18–19. The derived values of the masses of the Earth and of the Earth + Moon differ from those currently in use, but will not supersede them completely until the system of planetary masses is revised as a whole. (See note 24.)

20. The perturbed mean distance of the Moon is the semi-major axis of Hill's variational orbit, and differs from that calculated from Kepler's law by the factor F_2, which depends on the well-determined ratio of the mean motions of the Sun and Moon. (E. W. Brown, *Mem. R. astr. Soc.* **53**, 89, 1897).

21. The constant of sine parallax for the Moon is conventionally expressed in seconds of arc by multiplying by the number of seconds of arc in one radian. The corresponding value of $\pi_{\text{☾}}$ itself is 3422″·608.

22. The constant of the lunar inequality is defined by the expression given and is conventionally expressed in seconds of arc.

23. The constant of the parallactic inequality is defined by the expression given; the coefficient F_3 is consistent with the corresponding quantities in Brown's Tables.

24. The system of planetary masses is that adopted in the current ephemerides and the values given for the reciprocals of the masses include the contributions from atmospheres and satellites. The value for Neptune is that adopted in the numerical integration of the motions of the outer planets; the value used in Newcomb's theories of the inner planets is 19 700. In planetary theory the adopted ratio of the mass of the Earth to the mass of the Moon is 81·45 (compared with 81·53 in the lunar theory), and the ratio of the mass of the Sun to the mass of the Earth alone is 333 432. This system of masses should be revised within the next few years when improved values for the inner planets are available from determinations based on space-probes.

593, 1966. The proceedings of IAU Symposium No. 21 have been printed in *Bulletin Astronomique*, Tome XXV, Fascicules I, II and III, July-September 1965, and are now available as a bound reprint under the title *IAU Symposium No. 21, The System of Astronomical Constants.*

The adopted list of constants, together with the explanatory notes prepared by the Working Group, is given on pages 498-501. They have been reproduced photographically, by kind permission of the International Astronomical Union, from *Trans. IAU* **12B**.

3. *Summary of the main changes*

The main changes from the system of constants in use immediately prior to the introduction of the new system are as follows:

(*a*) *Scale or size.* The measure of the astronomical unit in metres (not previously used) is adopted as a primary constant (constant 3); and a revised value (constant 5) is adopted for the equatorial radius of the Earth in metres. From these there is derived a changed value of the solar parallax.

(*b*) *Aberration.* The velocity of light, in metres per second, is adopted as a primary constant (constant 4). From this, in conjunction with other changes, there are derived changed values of the light-time for unit distance and of the constant of aberration.

(*c*) *Flattening.* Instead of the flattening factor previously used, a dynamical form-factor (constant 6) is adopted as the primary constant to specify the figure of the Earth.

(*d*) *The geocentric gravitational constant.* The value of the geocentric gravitational constant in metres^3 seconds^{-2} is adopted (constant 7) as a new primary constant. From this there are derived a changed value of the mass-ratio S/E and, in conjunction with the change of the mass-ratio M/E, a changed value of the mean distance, and thus of the corresponding parallax, of the Moon.

(*e*) *The mass-ratio Moon/Earth.* A revised value (constant 8) for the mass-ratio M/E is adopted as a primary constant.

(*f*) *Unchanged primary constants.* The remaining primary constants are unchanged:

1: the number of ephemeris seconds in a tropical year at 1900·0
2: the Gaussian gravitational constant
9: the sidereal mean motion of the Moon
10: the general precession in longitude per tropical century at 1900·0
11: the obliquity of the ecliptic at 1900·0
12: the constant of nutation at 1900·0

4. *The epoch of Ephemeris Time*

The epoch 1900 January 0^d 12^h E.T. is assigned by definition (*Trans. IAU* **10**, 72, 1960) to the measure of Ephemeris Time at the instant at which the Sun's *geometric* mean longitude, referred to the mean equinox, was 279° 41′ 48″·04. A change in the adopted value of the constant of aberration thus involves a change in the measure of E.T. (see *Trans. IAU* **10**, 72, 1960 and *Explanatory Supplement*, page 69); any such change would involve consequential changes in the mean longitudes of the Sun, Moon and planets, if the fit with observation is not to be altered.

Pending international discussion of, and agreement on, the implications of such a change, with the possibility of a re-definition of the epoch of E.T., it has been decided not to introduce the effect of the change either into the measure of E.T. or into the ephemerides.

5. *Effect of the changes*

In so far as they affect the ephemerides in the A.E., the changes may be grouped as follows:

(*a*) Scale or size:	constants 3 and 5; 13
(*b*) Aberration:	constants 3 and 4; 14 and 15
(*c*) Flattening:	constant 6; 16
(*d*) Mass-ratio S/E:	constants 3 and 7; 18
Mass-ratio M/E:	constant 8; 19, 20, 21, 22 and 23

As discussed above in paragraph 4, no changes are being introduced into the measure of Ephemeris Time as a consequence of the change in the adopted value of the constant of aberration.

The system of planetary masses, as given in constant 24, is inconsistent with the new value of the ratio of the combined mass of Earth and Moon to that of the Sun (constant 19). But, as suggested by the Working Party in its report, no changes are being made in the ephemerides of the planets due to the changed value of $S/(E + M)$. With this understanding the general effect of the changes are described in the following paragraphs. A detailed derivation of the corrections required is given in section B on page 508.

(*a*) *Scale or size.* Although these changes constitute some of the most important features of the new system, their direct effect on the tabulated ephemerides is small. Heliocentric and geocentric distances, expressed in astronomical units, are unchanged; but, of course, the tabulated values of the horizontal parallax of the Sun and planets must be changed proportionally to the change in the adopted value for the solar parallax (constant 13, 8″·794 instead of 8″·80). Other, indirect and small, effects arise in connection with other changes, particularly the changed mass-ratios S/E and M/E.

(*b*) *Aberration.* The change in the adopted value of the constant of aberration (constant 15, 20″·496 instead of 20″·47) affects all apparent ephemerides (other than the Moon for which it is negligible), the aberrational day numbers, the physical ephemerides (especially the light-times in the calculation of which the light-time for unit distance—constant 14—is now $499^s·012$ instead of $498^s·58$), and the ephemerides of the satellites. In all cases the corrections are small, and in many cases are not significant to the precision of the ephemerides.

(*c*) *Flattening.* The change in the adopted value of the flattening (constant

16, 1/298·25 instead of 1/297) has a small direct effect on all quantities related to topographic positions on the Earth, such as predictions and reductions of eclipses and occultations, data for observatories, tables of S and C etc.

The adopted value of the dynamical form-factor for the Earth (constant 6, J_2 = 0·001 082 7) differs from that used in Brown's theory of the motion of the Moon; the resulting correction affects the longitude and latitude of the Moon by terms which may reach ±0″·2.

(*d*) *Mass-ratios S/E and M/E.* The heliocentric motion of the centre of gravity of the Earth and Moon is clearly affected by a change in the adopted value of the mass-ratio $S/(E + M)$ (constant 19, 328 912 instead of 329 390); but, since the mean motion is unchanged, the only change is a very small (actually insignificant) increase in the mean distance measured in astronomical units. As mentioned earlier, the complicated changes in the heliocentric motion of the planets due to the changed value of $S/(E + M)$, which may give rise to periodic corrections of the order of 0″·04 in the longitude of Venus, are being ignored.

The change in the mass-ratio M/E (constant 8, 1/81·30 instead of 1/81·45), coupled with the changed values of the solar and lunar parallaxes, gives rise to a change in the constant of lunar inequality (constant 22, 6″·440 instead of 6″·425), which affects significantly (about 0″·015 in longitude) the lunar perturbations in the Earth's orbit; and this change is, of course, enhanced in the geocentric ephemerides of the inner planets.

The consequential changes in the orbit of the Moon arise through the change in the constant of parallactic inequality (constant 23, 124″·986 instead of 125″·154); but the tabulated values of the horizontal parallax will also be directly affected by the change in the adopted value of the constant of sine parallax of the Moon (constant 21, 3422″·451 instead of 3422″·54).

Any changes in the lunar ephemeris will give rise to consequential changes in the prediction and reduction of eclipses and occultations, in the value of the correction ΔT to be applied to Universal Time to give Ephemeris Time, and (if significant) in the physical ephemeris of the Moon.

It must, however, be emphasised that it is only the numerical values of ΔT, as determined in practice from the motion of the Moon, that are changed; the introduction of the new system of constants leaves unchanged the definitions of both U.T. and E.T. (See paragraph 4 above in respect of the effect on the measure of Ephemeris Time of the change in the adopted value of the constant of aberration.)

6. *Correction of the ephemerides*

New ephemerides completely consistent with the new system of constants, and with a revised system of planetary masses, will eventually be required. However, difficulties will arise in correcting the current ephemerides of the Sun and planets for changes in the adopted masses of the planets, and it will almost certainly be preferable to wait until new theories of their motion are available. With modern electronic computers ephemerides may be calculated for long periods with little additional effort, so that such new ephemerides can easily be made available for the precise comparison of theory and current observation. It is a matter of detail whether such new ephemerides should be printed in full year-by-year as in the present A.E., or given for each body for many years at a time in a separate volume, or made available as computer programs from which positions, in any form desired, may be calculated for any time. Certainly, ephemerides based on the present authorities and corrected to the new system of constants will soon be superseded. It is extremely unlikely that they would ever be used for their main purpose of comparison of observation with theory; moreover, the introduction of a discontinuity in the basis of tabulation would be a disadvantage for any current investigations.

With the exception of the lunar ephemeris, and the day numbers, the corrections are so small that they may be ignored for all purposes other than the precise comparison of observation with theory; moreover, they are of the same order as the rounding-off error of the tabulated ephemerides so that corrected ephemerides, if to be of the same standard of precision, must be recalculated and not corrected differentially.

There is therefore a strong case for retaining the current ephemerides unchanged in the A.E., until such time as they can be replaced by new ephemerides based on new theories as well as new constants. This case is strengthened for the year 1968, and immediately following years, by the fact that the fundamental ephemerides are already in print and have been distributed, under international agreements, to the offices of the national ephemerides.

These considerations do not necessarily apply to the lunar ephemeris, which is tabulated to a precision considerably greater than that of optical observation, to the day numbers, and to certain other data which are required for other ephemerides or for the reduction of observations. But it is clearly adequate to publish differential corrections to be applied to the tabulated lunar ephemeris, until such time as a corrected ephemeris can be introduced, possibly for the year 1972. The publication of differential corrections has the advantage that, while individual tabulated values may be corrected if desired, the effect of the corrections on a group of observations or on other deduced quantities may be evaluated directly.

Where data, such as the day numbers, are to be used as a basis for other calculations in which the new system of constants may be used, either corrected values or full differential corrections must be given; the choice is largely a matter of practicability and convenience.

The introduction of the IAU System of Astronomical Constants into the tabulations in the A.E. for 1968 and future years is based on these considerations, bearing in mind the terms of the resolution by IAU Commission 4. In every case, one of the following procedures has been adopted:

(i) the tabulations have been corrected to agree with the new system of constants;

(ii) differential corrections to be applied to the uncorrected tabulations, based on the old system, are included in an annual appendix to the A.E., on page ix onwards;

(iii) simple formulae for the calculation of the corrected values or of differential corrections are given in the annual appendix.

None of the corrections are sufficiently large to affect significantly the navigational ephemerides derived from the fundamental ephemerides in the A.E.

7. *Correction of the Explanation*

It is not practicable to include in the Explanation full details of all the changes required by the introduction of the new system of constants, or of the derivation of the corrections. The main changes occur in the section *Fundamental Units and Astronomical Constants*, but consequential alterations are required in many other sections, particularly in those involving aberration, corrections for light-time, and topocentric positions. The information in the Explanation is, in fact, confined to material concerning the ephemerides as printed in the main tabulations; as far as practicable the bases of the differential corrections, and of the ephemerides resulting from their application, are summarised in the appendix.

8. *Correction of the Explanatory Supplement*

The number of amendments required to the *Explanatory Supplement* is so large that the only satisfactory solution is a revised edition; but for many reasons it is desirable to wait until the new ephemerides have been fully introduced into the A.E. Reprints of the present Supplement will be available to serve as a correcting supplement to the *Explanatory Supplement*; and for this reason, a comprehensive list of changes has been included in section C.

B. Theory of the corrections

In this section, there are given the formulae for the corrections to be applied to the current tabulations to convert them to the new system of constants, together with their derivation. They are arranged under a few general headings.

1. *Scale or size*

The change in the adopted value of the solar parallax (constant 13) has a direct effect on the tabulated values of the equatorial horizontal parallaxes of the Sun and planets; the correction is

$$-0''\cdot006\ \Delta^{-1} \quad \text{or} \quad -0\cdot0007 \times \text{H.P.} \qquad (1)$$

where Δ is the geocentric distance and H.P. is the tabulated value.

The semi-diameters of the Sun, Moon and planets, being based on observed values reduced to unit or mean distance, are not affected by the changes.

The associated change in the constant of sine parallax for the Moon (constant 21) requires a proportional change in the tabulated values of the horizontal parallax of the Moon; the correction is

$$-0''{\cdot}089 \times \frac{\text{H.P. in }''}{3422{\cdot}54} = -0''{\cdot}00156 \times \text{H.P. in} \qquad (2)$$

The semi-diameter of the Moon at mean distance is an observed quantity and its adopted value, 15′ 32″·58, is unchanged; but it now corresponds to a slightly reduced value, 57′ 02″·608, of the equatorial horizontal parallax at mean distance so that the coefficient k in the formula

$$\sin \text{S.D.} = k \sin \text{H.P.} \qquad (3)$$

becomes

$$0{\cdot}272488 \text{ instead of } 0{\cdot}272481 \qquad (4)$$

The correction to the tabulated values of the semi-diameter is thus (making legitimate approximations)

$$+0{\cdot}000007 \times \text{H.P.} - k \times 0''{\cdot}00156 \times \text{H.P. in }' = 0$$

as must be the case. The change (4) in the value of k will, however, give rise to small changes in the prediction and reduction of occultations and eclipses.

2. *Aberration*

All apparent ephemerides are affected by the change in the constant of aberration (κ), constant 15, or by the corresponding change in the light-time for unit distance (τ_A), constant 14.

The apparent geocentric coordinates of the Sun, Moon and planets contain the aberrational term

$$-0{\cdot}0057683\, \Delta \times \text{daily motion}$$

where the coefficient is the fraction of the day corresponding to a light-time for unit distance of $498^s{\cdot}38$, and Δ is the geocentric distance. The correction to be applied for a change to $499^s{\cdot}012$ is thus

$$-0{\cdot}0000073\, \Delta \times \text{daily motion} \qquad (5)$$

Astrometric ephemerides are given for Pluto and the minor planets, and these include corrections only for that part of the aberration due to the actual motion of the planet in the time that it takes light from the planet to reach the Earth. If the inclination to the ecliptic be ignored, and if it be assumed that the direction of motion of the planet be within 20° or 30° of the perpendicular to the line joining the Earth and the planet, then the correction to be applied can be easily shown to be

$$-0''{\cdot}026\, a^{\frac{1}{2}}\, r^{-1} \text{ to the geocentric longitude} \qquad (6)$$

where a, r are the semi-major axis and radius vector respectively. If, further, the eccentricity is ignored and r is taken to be equal to a (as is just within the permissible limits of approximation), the correction may be regarded as constant for each planet, namely

$$-0''{\cdot}026\, a^{-\frac{1}{2}} \text{ to the geocentric longitude} \qquad (7)$$

The aberrational day numbers C and D, which are essentially directly proportional to the constant of aberration, have been recalculated; the values given in the A.E. for 1968 and future years are thus in accord with the new system of constants and no corrections are necessary.

In the physical ephemerides of the Sun, Moon and planets, and in the ephemerides of the satellites, the phenomenon of aberration does not enter directly, but allowance is made for the appropriate motions during the time taken by light to reach the Earth. Corrections must therefore be made for the change in the light-time for unit distance from the currently adopted value (for these ephemerides) of $498^s{\cdot}58$ to the new value of $499^s{\cdot}012$. The correction to be subtracted from any tabulated quantity is thus

$$\text{its motion in } 0^s{\cdot}43\,\Delta \tag{8}$$

where Δ is the geocentric distance, with the possible revision of the zero point.

3. *Flattening*

The direct effect of the change in the adopted value of the flattening of the Earth (constant 16) is to change the tabulated values of S and C in Table VII, and all quantities (such as $\rho \sin \phi'$ and $\rho \cos \phi'$ in the list of Observatories) derived from them. The necessary corrections to pass from the present value of $1/297$ to $1/298{\cdot}25$ are:

$$\begin{aligned} &\text{in } S && + 71 \times 10^{-7} (3 + \cos 2\phi) \\ &\text{in } C && - 71 \times 10^{-7} (1 - \cos 2\phi) \\ &\text{in } \rho && + 71 \times 10^{-7} (1 - \cos 2\phi) \\ &\text{in } \phi' - \phi && + 141 \times 10^{-7} \sin 2\phi \end{aligned} \tag{9}$$

where ϕ is the geographic latitude.

The most important changes are the consequential ones in the topographical positions, or local predictions, associated with eclipses and occultations, though most of these are not published in the A.E. All quantities depending on the parallax of the Moon, such as the reduction of meridian observations, will also be affected.

4. *The mass-ratio $S/(E + M)$*

The heliocentric motion of the centre of gravity of the Earth and Moon is affected by a change in the adopted value of the mass-ratio $S/(E + M)$. Since both the mean motion (constant 1) and the Gaussian gravitational constant (constant 2) are unchanged, the mean distance a must be changed to correspond. With $S/(E + M) = 328\,912$ (constant 19) as compared with the present value of 329 390, it is easily seen that the correction to the semi-major axis, a, is

$$+1{\cdot}47\, a \times 10^{-9} \tag{10}$$

which is insignificant, and which is neglected.

As suggested by the Working Group in its report the effect of the change (and of changes in the adopted values of the masses of other planets) on the heliocentric motions of the planets is being ignored; it could introduce periodic corrections of amplitude about $0''{\cdot}035$ into the longitude of Venus.

No corrections are applicable, on this account, to the heliocentric ephemerides of the planets, including the Earth.

5. *The mass-ratio E/M*

The change in the mass-ratio $\mu^{-1} = E/M$ (constant 8) will affect the lunar perturbations of the geocentric motion of the Sun, due to the transfer of origin from the centre of gravity of Earth and Moon to the centre of the Earth. The lunar perturbations are proportional to the constant of lunar inequality, which is very nearly the coefficient of the principal term (sin D) in the Sun's longitude; the adopted value is $6''\cdot440$ (constant 22) as compared with the value $6''\cdot425$ used in Newcomb's tables. A direct calculation, using Newcomb's values $\mu^{-1} = 81\cdot45$, $\pi_{☾} = 3422''\cdot68$, $\pi_{\odot} = 8''\cdot79$ confirms that the correction to be applied to an ephemeris based on Newcomb's tables is $+0\cdot00230$ times the actual lunar perturbations.

With an error of less than $0''\cdot001$, the correction in longitude (λ) is

$$\Delta\lambda = +0''\cdot015 \sin D \tag{11}$$

where D is the mean elongation of the Moon from the Sun. The correction in latitude cannot exceed $0''\cdot0015$ and can be neglected. The correction to the radius vector (R), neglecting terms less than $0\cdot2 \times 10^{-8}$, is

$$\Delta R = +7\cdot1 \times 10^{-8} \cos D - 0\cdot7 \times 10^{-8} \cos (D - l) \tag{12}$$

where l, in the notation of Brown's tables, is the mean anomaly of the Moon.

To sufficient precision, the corrections may thus be represented by:

$$\Delta\lambda = +0''\cdot015 \sin D \quad \Delta\beta = 0 \quad \Delta R = +7 \times 10^{-8} \cos D \tag{13}$$

Alternatively, the effect of the change is to increase, by a factor of $1\cdot0023$, the distance of the Earth from the unchanged position of the centre of gravity of the Earth and Moon. To a sufficient approximation, the Moon may be regarded as moving in a circular orbit in the ecliptic, so that the correction consists of a shift of the Earth's position in the ecliptic of mean magnitude $7\cdot2 \times 10^{-8}$ astronomical units rotating with the mean motion of the Moon. In terms of the equatorial rectangular coordinates of the Sun the corrections are:

$$\Delta X = +7 \times 10^{-8} \cos ☾ \quad \Delta Y = +7 \times 10^{-8} \sin ☾ \quad \Delta Z = +3 \times 10^{-8} \sin ☾ \tag{14}$$

where ☾ is the mean longitude of the Moon. The error due to the various approximations is unlikely to exceed 1×10^{-8}.

Simple geometry shows that the consequential corrections to be applied to the geocentric ecliptic coordinates of the planets (which are not tabulated in the A.E.) are, with the same order of approximation,

to the geocentric longitude, λ_P: $\Delta\lambda = +0''\cdot015\, \Delta^{-1} \sin (☾ - L_P)$ (15)
to the geocentric distance, Δ: $+7 \times 10^{-8} \cos (☾ - L_P)$ (16)
to the geocentric latitude, β_P: less than $0''\cdot0015\, \Delta^{-1}$

where L_P is the mean longitude of the planet.

Approximately, the corresponding corrections in geocentric right ascension, α, and declination, δ, are

to the right ascension: $\Delta\lambda$ (17)
to the declination : $0\cdot4\, \Delta\lambda \cos \alpha$ (18)

To the precision required the difference between the two mean longitudes, $(☾ - L_P)$, may be replaced by $(D - E)$, where D is the mean elongation of the Moon from the Sun and E (positive to the east) is the elongation of the planet from the Sun.

6. *The ephemeris of the Moon*

The direct effect on the ephemeris of the Moon of the changes in the values of the constants concerning the Moon (constants 8, 9, 20, 21, 22 and 23) consists of:

(*a*) *Constant of sine parallax.* The corrections to the tabulated values of the horizontal parallax, due to the change of the constant of sine parallax (constant 21), already dealt with in paragraph 1 above.

Formula (2) of paragraph 1 is designed to facilitate a mental calculation of the main correcting term to the tabulated horizontal parallax of the Moon; it can be expanded as, in units of 0″·0001,

$$- [890 + 7 \cos 2D + 49 \cos l + 9 \cos (l - 2D) + 1 \cos (l + 2D) + 3 \cos 2l] \qquad (19)$$

where l is, in the notation of Brown's tables, the mean anomaly of the Moon.

(*b*) *Constant of parallactic inequality.* The correction to the longitude, latitude and horizontal parallax due to the changed value of the constant of parallactic inequality (constant 23).

The value of the constant of parallactic inequality used in Brown's *Tables of the Motion of the Moon* (and in the *Improved Lunar Ephemeris*) is 125″·154 as compared with 124″·986 now to be adopted. In terms of the notation used by Brown, the new value of α_1 is 0·002 509 35 as compared with Brown's 0·002 512 73. The correction to be applied thus consists of −0·00134 times all terms in Brown's theory in which α_1 occurs in the principal characteristic. The terms are listed in *Memoirs of the Royal Astronomical Society*, Vol. 57, pages 130-145 and again (with revised values of the constants) in Lists iα-iθ in the Tables; but, in List iβ, α_1 is omitted from the principal characteristic, and it is convenient to refer to the earlier expression for the terms in the latitude dependent on α_1.

The following list includes all correcting terms with coefficients greater than 0″·0010 in longitude and latitude, and 0″·00010 in parallax; the notation used is that of Brown's tables, with $F = L - \Omega$ being the argument of latitude and l' the mean anomaly of the Sun.

Longitude (20) unit 0″·001	Latitude (21) unit 0″·001	Parallax (22) unit 0″·0001
$+168 \sin D$	$+7 \sin (F + D)$	$+13 \cos D$
$-\ 25 \sin (l - D)$	$-6 \sin (F - D)$	$+\ 1 \cos (l + D)$
$+\ 11 \sin (l + D)$	$-1 \sin (F + l' + D)$	$-\ 2 \cos (l' + D)$
$-\ 4 \sin (l - 3D)$	$+1 \sin (F - l' - D)$	
$-\ 24 \sin (l' + D)$		
$-\ 2 \sin (2l - D)$		
$-\ 2 \sin (2l - 3D)$		
$-\ 2 \sin (l + l' + D)$		
$+\ 1 \sin (l - l' - D)$		

(*c*) *Flattening of the Earth.* The value of the ellipticity, f, of the Earth's figure as used in Brown's tables, and thus in the *Improved Lunar Ephemeris*, is 1/294.

The perturbations in the motion of the Moon are, however, proportional to the dynamical form factor J_2, where

$$\tfrac{3}{2}J_2 = f - \tfrac{1}{2}\sigma \qquad (23)$$

and σ is approximately the ratio of the centrifugal force to gravity on the Earth's equator.

According to the new system of constants the coefficients should be calculated with $\tfrac{3}{2}J_2 = 0{\cdot}001\ 624\ 05$ (constant 6). It would appear that the values given in the Tables are based on $\tfrac{3}{2}J_2 = 0{\cdot}001\ 667\ 36$ corresponding to a value of $\sigma = 0{\cdot}003\ 468$, slightly different from the new value (constants 6 and 16) of $0{\cdot}003\ 457\ 8$. The correcting factor is thus not directly proportional to the difference between the old, $1/294$, and the new, $1/298{\cdot}25$, values of f.

Moreover, Brown includes the main terms of these perturbations as additions to the elements so that it is not easy to derive explicit corrections, corresponding to a change in the adopted value of f or $\tfrac{3}{2}J_2$, to the true longitude and latitude. It is convenient to base such corrections on the expressions given by G. W. Hill in *Astronomical Papers of the American Ephemeris*, Vol. III, Part II, 1884. Hill states that he used $\tfrac{3}{2}J_2 = 0{\cdot}001\ 759\ 5$, so that the factor by which Hill's coefficients must be multiplied to correct the Improved Lunar Ephemeris to the new system of constants is

$$(0{\cdot}001\ 624\ 05 - 0{\cdot}001\ 667\ 36)/0{\cdot}001\ 759\ 5 = -0{\cdot}0246 \qquad (24)$$

The following list includes all correcting terms with coefficients greater than $0''{\cdot}0010$; the correction to the horizontal parallax is negligible. The notation is that of Brown's tables; but Hill used ξ for the mean longitude $\Omega + F$ for which Brown used L.

Longitude (25) unit $0''{\cdot}001$	Latitude (26) unit $0''{\cdot}001$
$-189 \sin \Omega$	$+215 \sin(\Omega + F)$
$-\ 13 \sin(\Omega - l)$	$-\ 12 \sin(\Omega + F - l)$
$-\ 13 \sin(\Omega + l)$	$+\ 11 \sin(\Omega + F + l)$
$-\ 10 \sin(\Omega + 2F)$	$-\ 9 \sin(\Omega - F)$
$-\ 2 \sin(\Omega - 2D)$	$-\ 8 \sin(\Omega + F - 2D)$
$-\ 2 \sin(\Omega + 2D)$	$+\ 2 \sin(\Omega + F - l + 2D)$
$-\ 2 \sin(\Omega - l + 2D)$	$-\ 2 \sin(\Omega + F + l - 2D)$
$-\ 1 \sin(\Omega + l - 2D)$	$-\ 2 \sin(2\Omega + F)$
$-\ 1 \sin(\Omega - l + 2F)$	$-\ 1 \sin(\Omega - F - l)$
$-\ 1 \sin(\Omega + l + 2F)$	$+\ 1 \sin(\Omega + F + 2D)$
$-\ 1 \sin(\Omega + 2F - 2D)$	

(*d*) *Correction to term No.* 182. In the course of a comprehensive recomputation of the solar terms in Brown's theory of the motion of the Moon, Dr. W. J. Eckert (*Trans. IAU*, **12B**, in press, 1966) discovered an error in term No. 182. The term* should be:

$$-1''{\cdot}370 \sin(2l - 2F) \quad \text{instead of} \quad -1''{\cdot}298 \sin(2l - 2F) \qquad (27)$$

*A later (April 1966) determination suggests that the coefficient should be $-1''{\cdot}372$.

The existence and magnitude of the error have been verified by B. L. Klock and D. K. Scott (*Astronomical Journal*, Vol. 70, p. 335, 1965) by the analysis of meridian observations of the Moon made at the U.S. Naval Observatory. Adopting the theoretical value of the coefficient, a correction of

$$-0''\cdot072 \sin (2l - 2F) \qquad (28)$$

has to be applied to the longitude of the Moon. It seems desirable to include this correction at the same time as the others.

Total corrections to the ephemeris of the Moon

Collecting together the four separate contributions, ignoring terms whose coefficients are less than 0″·0025 in longitude and latitude, or 0″·00025 in parallax, and converting to the notation of the A.E., the corrections to be applied to the ephemerides as tabulated in the A.E. are:

To the true longitude, λ unit 0″·001 (29)	To the latitude, β unit 0″·001 (30)	To the horizontal parallax, π unit 0″·0001 (31)
$\Delta\lambda =$	$\Delta\beta =$	$\Delta\pi =$
$-189 \sin ☊$	$+215 \sin ☾$	-890
$+168 \sin D$	$-12 \sin \Gamma'$	$-49 \cos (☾ - \Gamma')$
$+72 \sin (2\Gamma' - 2☊)$	$+11 \sin (2☾ - \Gamma')$	$+13 \cos D$
$-25 \sin (☾ - \Gamma' - D)$	$+9 \sin (☾ - 2☊)$	$-9 \cos (☾ - \Gamma' - 2D)$
$-24 \sin (g + D)$	$-8 \sin (☾ - 2D)$	$-7 \cos 2D$
$+13 \sin (☾ - \Gamma' - ☊)$	$+7 \sin (☾ - ☊ + D)$	$-3 \cos (2☾ - 2\Gamma')$
$-13 \sin (☾ - \Gamma' + ☊)$	$-6 \sin (☾ - ☊ - D)$	
$+11 \sin (☾ - \Gamma' + D)$		
$-10 \sin (2☾ - ☊)$		
$-4 \sin (☾ - \Gamma' - 3D)$		

where ☾ is the mean longitude of the Moon, ☊, Γ' are the mean longitudes of the node and perigee, D is the mean elongation from the Sun and g is the mean anomaly of the Sun. The notation differs from that of Brown's tables, as used earlier, in that ☾ is used instead of L, Γ' instead of $\varpi = L - l$, and g instead of l'; thus

$$l \equiv ☾ - \Gamma'; \quad l' \equiv g; \quad F \equiv ☾ - ☊; \quad ☊ \equiv ☊ \text{ and } D \equiv D.$$

These corrections, together with the consequential corrections to the right ascension, and declination, are tabulated in the annual appendix in the A.E. for 0^h E.T. on each day. The tabulated values of the semi-diameter and the ephemeris transit are unchanged to the precision given.

It can be mentioned here that:

(i) the precepts in Brown's *Tables*, Section I, p. 140, for the correction resulting from a change in the constant of parallactic inequality are erroneous;

(ii) the precepts for the correction resulting from a change in the ellipticity are only approximate, since the tables P 34 and P 36 contain other terms not depending on the ellipticity;

(iii) the coefficients of the periodic terms in the perturbations due to the flattening, as given by Brown in *Memoirs R.A.S.*, Vol. 59, pp. 79-81, 96, 1908, are based, contrary to his statement, on the value (α) of $f^{-1} = 292{\cdot}9$; the coefficients of the secular terms are, however, calculated with the stated value (β) of $f^{-1} = 296{\cdot}3$. (This was pointed out by Brown in *M.N.R.A.S.* Vol. 70, 3, 1909.)

C. Changes in the Explanatory Supplement

1. *Introduction*

The main purpose of this section is to serve as a convenient source of reference to the changes in, and corrections to, the *Explanatory Supplement* (to *The Astronomical Ephemeris and The American Ephemeris and Nautical Almanac*) consequent upon the introduction of the IAU System of Astronomical Constants.

The changes and corrections are given without any indication of the year in which they have been, or will be, incorporated into the printed ephemerides in the A.E.; this information can readily be obtained from the A.E. itself.

One of the main purposes of the *Explanatory Supplement* is to illustrate, by numerical examples, the derivation of all quantities tabulated in the A.E.; the principles of such illustrations will not be affected by changes in the numerical values of the constants. Further, the uncorrected formulae and derivations are still required to reproduce those ephemerides in the A.E. which have not yet been corrected but for which differential corrections are provided. In fact, it would not be sufficient at the present time to produce a revised edition of the *Explanatory Supplement* based solely on the new constants.

The formulae for the differential corrections are given in section B, and are not repeated here. They can be used, if required, to check the application of the corrections to the formulae and derivations in the *Explanatory Supplement*. No real difficulty should be encountered in deriving any printed quantity in the A.E., both before and after incorporation of the changes.

The opportunity is taken of listing all the known errors in, and corrections to, the *Explanatory Supplement*.

2. *Form of the changes and corrections*

The general considerations in section A, and the theory of the corrections in section B, indicate that the changes fall into two quite distinct categories:

(*a*) Those, such as the changes in the constants of solar parallax and aberration, which affect a very large number of ephemerides and derivations but which are trivial to correct. Generally the only correction required is the substitution of one number for another.

(*b*) Those, such as the consequential changes arising in the ephemerides of the Sun, Moon and planets, which are in themselves rather involved but which do not affect, in principle, any derived ephemerides or quantities. The bases of these ephemerides are generally given in the *Explanatory Supplement* only through references to the appropriate authorities; and, in such cases, the only corrections required are the expansions of these references to include the substance of section B of this Supplement.

It is not practicable to list *every* occurrence of the constants in category (*a*), particularly those in the numerical examples; but the list in para. 3 is, in fact, fairly comprehensive. A general indication of where such corrections must be made is given in the list of changes and corrections in para. 3. The principal corrections are:

	For	Substitute
Solar parallax: constant	8″·80	8″·794
Aberration: constant	20″·47	20″·496
light-time for unit distance	498^s·58	499^s·012
,, ,, ,, ,,	498^s·38, or 0^d·0057683	499^s·012, or 0^d·0057756
Figure of the Earth: equatorial radius	6 378 388 m	6 378 160 m
flattening	1/297	1/298·25
Mass-ratio, Earth to Moon	81·45 or 81·53	81·30
Mass-ratio, Sun to Earth and Moon	329 390	328 912
Lunar parallax: horizontal parallax	3422″·70	3422″·608
constant of sine	3422″·54	3422″·451
ratio of semi-diameter, Moon to Earth	0·272481	0·272488
Velocity of light, m s^{-1}	299 860 × 10^3	299 792·5 × 10^3

Adequate references to the changes in category (*b*) are given under the appropriate chapter headings in the following list of changes and corrections.

The changes should be interpreted in relation to their contexts; for example, a statement "that a certain constant was used" cannot be changed, and the listed change in the value of the constant should be interpreted as indicating the value that would have been used if the IAU system had then been in use.

3. *List of changes and corrections*

The following list is arranged in order of chapter, section and page number. As mentioned in para. 2 above, not necessarily all corrections are listed, but an attempt has been made to include all major occurrences.

Chapter 1. Introduction

Sections A, B. No change.

Section C. It is appropriate to add here the relevant historical material from section A of this Supplement.

Sections D, E, F and G. No change.

Chapter 2. Coordinates and Reference Systems

Sections A, B, C. No change.

Section D (Aberration). The only changes arise through the change in the adopted value (20″·496 instead of 20″·47) of the constant of aberration.

p. 48. $k = 20''{\cdot}496$

p. 49. The constant $k = 20''{\cdot}496$ is equivalent to a light-time for unit distance of $499^s{\cdot}012 = 0^d{\cdot}0057756$; it corresponds to a value of $c = 299\,792{\cdot}5 \times 10^3$ metres per second for the velocity of light.

p. 51. The constant should be 0·0057756.

Section E. No change.

Section F (Parallax). The following numerical values, calculated with $f = 0{\cdot}003\,352\,89$, should be substituted on pages 57 and 58.

$$a = 6378{\cdot}160 \text{ km} \quad f = 1/298{\cdot}25 \quad b = 6356{\cdot}775 \text{ km}$$

$$S = 0{\cdot}99497\,418 - 0{\cdot}00167\,082 \cos 2\phi + 0{\cdot}00000\,210 \cos 4\phi$$

$$C = 1{\cdot}00167\,997 - 0{\cdot}00168\,208 \cos 2\phi + 0{\cdot}00000\,212 \cos 4\phi$$

$$\rho = 0{\cdot}99832\,707 + 0{\cdot}00167\,644 \cos 2\phi - 0{\cdot}00000\,352 \cos 4\phi + 0{\cdot}00000\,001 \cos 6\phi$$

$$\phi - \phi' = 692''{\cdot}74 \sin 2\phi - 1''{\cdot}16 \sin 4\phi$$

$$\tan \phi' = (0{\cdot}9933054 + 0{\cdot}0011\,h \times 10^{-6}) \tan \phi$$

The values in Table 2.8 on page 59 should be replaced by those in A.E. Table VII; however, it suffices to good approximation to reduce $\phi - \phi'$ and the defect or excess over unity of S, C and ρ, by the factor $1 - 0{\cdot}00419$ (approx. $1 - \frac{1}{240}$).

No changes are required in the paragraph concerning the parallax of the Moon.

For the parallaxes of the Sun and planets on pages 63 and 64, the numerical values should be changed as follows:

p. 63, for $8''{\cdot}80$ substitute $8''{\cdot}794$

p. 63, for $0^s{\cdot}587$ substitute $0^s{\cdot}586$

p. 64, for 426·64 substitute 426·35

p. 65, for $20''{\cdot}47$ substitute $20''{\cdot}496$

Examples 2.4 and 2.5, and the illustrative table on p. 64, require corrections for the changed value of the flattening, as well as the solar parallax.

Chapter 3. Systems of Time Measurement

The only change required in this chapter is to the numerical values of ΔT in Table 3.1 and Figure 3.3. They should be changed, as foreseen at the foot of p. 87, to accord with the corrections to the Improved Lunar Ephemeris.

Attention is drawn to the fourth paragraph on page 69; see section A, para. 4 above for a discussion on the effect of a change in the adopted value of the constant of aberration.

Chapter 4. Fundamental Ephemerides

Section A (Introduction). The only changes are:

p. 96. The adopted values of the fundamental constants have *now* been changed.

p. 97, for $498^s{\cdot}38$ substitute $499^s{\cdot}012$.

Section B (The Sun). The basis of the ephemeris is Newcomb's *Tables of the Sun*, modified by the changes in the lunar perturbations (see section B, para. 5, of this Supplement) primarily due to the change (81·30 instead of 81·45) in the mass-ratio Earth to Moon. Otherwise the only changes are:

p. 100, for $20''{\cdot}47$ substitute $20''{\cdot}496$; also in Example 4.2.

p. 101, for $8''{\cdot}80$ substitute $8''{\cdot}794$; also in Example 4.4.

p. 103. Example 4.6, for 0·0057 683 and 0·0028 841 substitute 0·0057 756 and 0·0028 878.

Section C (The Moon). The basis of the ephemeris is that of the Improved Lunar Ephemeris, as described on pp. 106 and 107, modified by the changes described in section B, para. 6, of this Supplement. The only numerical changes required are:

p. 107. Constant of sine parallax: for $3422''{\cdot}5400$ substitute $3422''{\cdot}451$

p. 107. The equatorial horizontal parallax at mean distance (60·2682 equatorial radii of the Earth) is $57'\,02''{\cdot}608$.

p. 107. Mass-ratio Earth to Moon: for 81·53 substitute 81·30

p. 109. Formula for semi-diameter:
for 57′ 02″·70 substitute 57′ 02″·608
for 0·272481 substitute 0·272488
and $s'' = 0{\cdot}0799 + 0{\cdot}272453\,\pi''$; also in Example 4.12.

Section D (The planets). The authorities for the heliocentric ephemerides of the planets, including the adopted values for the masses of the planets on p. 112, are unchanged. The only changes occur in the geocentric apparent ephemerides and are due to the changes in the lunar perturbations of the Earth, in the constant of aberration, and in the solar parallax. The few numerical changes are:

p. 125, for 0·0057683 substitute 0·0057756; also in Example 4.22.
p. 125, for 8″·80 substitute 8″·794; also in Example 4.23.
p. 127, for 0·0057683 substitute 0·0057756; also in Example 4·25.

Section E. No change.

Section F (Ephemerides at transit). The only changes are those arising in the parallax corrections, due to the changed figure of the Earth. Also in Example 4.26.

Section G. No change.

Chapter 5. Mean and Apparent Places of Stars

Section A, B. No change.

Section C (Reduction from mean to apparent place). The only change arises in the correction for aberration.

p. 155, for 20″·47 substitute 20″·496

Section D (Day numbers). Changes arise solely through the change in the value of the constant of aberration.

p. 158, for 20″·47 substitute 20″·496
p. 158, for 1189″·80 substitute 1191″·30; also in Example 5.2.

Section E. No change.

Chapter 6. The System of Astronomical Constants

Clearly the whole chapter requires to be rewritten to incorporate the new system and the remarks of the Working Group, as given in section A of this Supplement. All references to authorities, most of the remarks concerning the derivation of and the inconsistencies between the values of the constants, and some other comments are now superseded; and it is not practicable to list all such changes. However, a few important changes of principle, and the principal numerical changes, are given below:

Table 6.1 on page 169

a :	for 6 378 388	substitute 6 378 160
f :	for 1/297	substitute 1/298·25
b :	for 6 356 911·946	substitute 6 356 774·7
g :	for 978·049	substitute 978·031
	for 0·00528 84	substitute 0·00530 24
π :	for 8″·80	substitute 8″·794
k :	for 20″·47	substitute 20″·496
	Equatorial horizontal parallax of the Moon:	
	for 60·2665	substitute 60·2682
	for 57′ 02″·70	substitute 57′ 02″·608
c :	for 299 860	substitute 299 792·5
	for 186 324	substitute 186 282·6

Light-time: for 498^s·58 and 498^s·38 substitute 499^s·012
Mass-ratio Earth to Moon: for 81·45 and 81·53 substitute 81·30
Mass-ratio Sun to Earth plus Moon: for 329 390 substitute 328 912

p. 171. It is rather misleading, even in the special case considered, to use the mechanical ellipticity H in the formula

$$H = f - \tfrac{1}{2}\sigma$$

and it is better to write

$$\tfrac{3}{2}J_2 = f - \tfrac{1}{2}\sigma \quad \text{or} \quad f = \tfrac{3}{2}J_2 + \tfrac{1}{2}\sigma$$

where J_2 is the dynamical form factor for the Earth.

p. 171. The two values of the light-time for unit distance should both be replaced by $499^s \cdot 012$, and the resulting discussion on the discrepancies ignored. It should be noted that the values of k, c and π give 1·27006 16 for their product, the difference from 1·27010 64 being accounted for by the change in the adopted value of the equatorial radius of the Earth. The inconsistencies, including that between the values of the solar parallax and the mass-ratio Sun to Earth plus Moon, have been entirely removed in the new system.

p. 173. The final remarks (before the section on references), should be interpreted in the light of the new system of constants, in which the theoretical relationship between the constants are rigorously satisfied; thus the conventional (*a*) and the adjusted (*c*) values are generally the same. However, there are still a few purely conventional values (such as the constants of precession and nutation) for which no exact theoretical relationships exist.

Chapter 7. Historical List of Authorities

Clearly a few additions are required, particularly in the section on Constants on p. 193.

Chapter 8. Configurations of the Sun, Moon, and Planets

The only changes required are those arising from the changed value of the light-time for unit distance.

p. 207. Example 8.3: for 0·002884 substitute 0·002888.

Chapter 9. Eclipses and Transits

Apart from the changes in the ephemerides of the Sun, Moon and planets (which will, of course, be incorporated through their tabulated values), corrections are necessary on account of the changes in the figure of the Earth, the solar and lunar parallaxes, and the constant of aberration. These are trivial in principle but require fairly extensive alterations. Starting in 1968 the following substitutions should be used in describing the eclipse computations.

Section A (Introduction)

p. 213. k: for 0·2724 807 substitute 0·2724 880
for 0·2722 74 substitute 0·2722 81

Section B (Solar eclipses—fundamental equations)

p. 215. Table 9.1: for 57′ 02″·70 substitute 57′ 02″·608
for 8″·80 substitute 8″·794

p. 216. Example 9.2: for 0·2722 74 substitute 0·2722 81
for 0·2722 39 substitute 0·2722 46

p. 217. Numerical coefficients in $\sin f_1$ and $\sin f_2$:
replace 8″·80 for $\pi_\odot$ by 8″·794 (p. 218) and the coefficients by:
for $k = 0{\cdot}2722\ 81$: in f_1 0·0046 6400 9; in f_2 0·0046 4079 2
for $k = 0{\cdot}2724\ 880$: in f_1 0·0046 6401 8; in f_2 0·0046 4078 3

p. 219. Example 9.3. The value of k used should be 0·2724 880 for 1963 onwards, with the corresponding coefficients for $\sin f_1$ and $\sin f_2$.

p. 220. Ellipticity, e^2: for 0·0067 2267 substitute 0·0066 9454. The reference to Hayford's spheroid should be deleted. This affects many examples, particularly Example 9.4.

Section C (Solar eclipses—predicted data)

p. 223, for a flattening of 1/298·25, $(1 - e^2)^{-\frac{1}{2}} = 1{\cdot}0033\ 64$; also in Examples 9.6, 9.7, 9.13, 9.14, 9.15 and 9.16.

Section D (Solar eclipses—local circumstances)

p. 241. Ellipticity, e^2: for 0·0067 2267 substitute 0·0066 9454. This affects almost all examples (see Examples 9.19 and 9.22) in principle.

Section E (Lunar eclipses)

p. 257. π_1: for 0·9983 33 substitute 0·9983 40

p. 260. Example 9.23:
Moon's semi-diameter: for 0·2722 74 substitute 0·2724 88
: for 0·9983 33 substitute 0·9983 40
Solar parallax : for 8″·80 substitute 8″·794

Section F (Transits of Mercury)

p. 265. The coefficient used in calculating the correction for aberration: for 0·138439 substitute 0·138614; also in Example 9.27.

p. 271. Horizontal parallax of the Sun: for 8″·80 substitute 8″·794

Chapter 10. Occultations

As for eclipses, corrections are required for changes in the adopted values for the figure of the Earth and for the lunar parallax.

Sections A, B. No change.

Section C. (Reduction)

p. 297. k: for 0·2724 953 substitute 0·2725 026, and amend the reference to Innes, if it is desired to retain the same value for the observed semi-diameter at mean distance. Also in Example 10.9.

Sections D, E and F. No change.

Chapter 11. Ephemerides for Physical Observations of the Sun, Moon, and Planets

The only corrections which are not introduced through the tabulated ephemerides are those due to the changed value of the light-time for unit distance.

Section A (Introduction). For the light-time for unit distance of 498^s·58 substitute 499^s·012. The consequences of this change are most marked in the longitudes of the central meridians of the planets, particularly those of Mars and Jupiter.

Sections B, C. No change.

Section D (Ephemeris for physical observations of the Moon)

p. 325. Formulae for λ_H, β_H: for 8·80 substitute 8·794; also in Example 11.6.

Sections E, F. No change.

Section G (Ephemerides for physical observations of Mars, Jupiter and Saturn)

p. 330. Light-time for unit distance: for 498^s·58 substitute 499^s·012.

p. 336. **Constant in $V + 180°$: for 325°·845 substitute 325°·847**
Rotation during light-time: for 498^s·58 substitute 499^s·012.
Longitude of central meridian: for 2°·024858 substitute 2°·026612; also in Example 11.7.

p. 339. **Constants in $V + 180°$: for 100°·974 substitute 101°·001; for 149°·976 substitute 150·002**
Longitudes of central meridians:
for 5°·06601 substitute 5°·07040; for 5°·02198 substitute 5°·02633; also in Example 11.8.

Section H. No change.

Chapter 12. Satellites

The only corrections required are those due to the changed value of the light-time for unit distance, which affects the light-time from the planet, and thus the times at which the satellites are observed.

Section A (Introduction). The value of the light-time for unit distance should be changed as follows:

p. 342, for $498^s \cdot 58$ substitute $499^s \cdot 012$

Section B (The satellites of Mars)

p. 352. for $0^h \cdot 13849$ substitute $0^h \cdot 13861$; also in Example 12.2.

Section C (The satellites of Jupiter)

p. 354. It should be noted that corrections for light-time are already included in Sampson's tables for the four Galilean satellites.

p. 357, for $0^h \cdot 13849$ substitute $0^h \cdot 13861$; also in Example 12.4.

p. 359, for $0^d \cdot 00577\ 060$ substitute $0^d \cdot 00577\ 560$; for $0^\circ \cdot 00826\ 97$ substitute $0^\circ \cdot 00827\ 69$; and for $0^\circ \cdot 00797\ 79$ substitute $0^\circ \cdot 00798\ 48$; also in Example 12.5.

Section D (The rings and satellites of Saturn)

p. 366, for 0·00577 substitute 0·00578; also in Example 12.6.

p. 372. Example 12.8: for $0^h \cdot 13849$ substitute $0^h \cdot 13861$.

p. 382, for $0^h \cdot 138494$ substitute $0^h \cdot 138614$; also in Example 12.15.

p. 385. $0^\circ \cdot 00377$ is unchanged; also in Example 12.17.

Section E (The satellites of Uranus)

p. 389. Example 12.18: for $0^h \cdot 13849$ substitute $0^h \cdot 13861$.

Section F (The satellites of Neptune)

p. 392. Example 12.19: for $0^h \cdot 13849$ substitute $0^h \cdot 13861$.

Section G. No change.

Chapters 13, 14, 15, 16 and 17

No changes are required.

Chapter 18. Reference Data

In general, changes are required to bring the "adopted" values into accord with the new system of constants.

Sun, Earth and Moon

p. 489. *Figure and Mass of the Earth*

a	: for 6378·388	substitute	6378·160
f	: for 1/297	substitute	1/298·25
b	: for (6356·912)	substitute	(6356·775)
e	: for (1/12·1963)	substitute	(1/12·2219)
1° of latitude	: for 111·137	substitute	111·133
1° of longitude	: for 111·418	substitute	111·413
$\phi - \phi'$	: for $695'' \cdot 66$ substitute $692'' \cdot 74$; for $-1'' \cdot 17$ substitute $-1'' \cdot 16$		
g	: for 980·64	substitute	980·62

p. 490. *Orbit of the Earth*

Solar parallax: for $8'' \cdot 80$ substitute $8'' \cdot 794$

Constant of aberration: for $20'' \cdot 47$ substitute $20'' \cdot 496$

Light-time for 1 a.u.: for $498^s \cdot 38$; $498^s \cdot 58$ substitute $499^s \cdot 012$

Mean distance of Earth from Sun:

for 149 500 000 (92 900 000) substitute 149 600 000 (92 957 000)

Mass-ratio—Sun/Earth: for 333 432 substitute 332 958

Mass-ratio—Sun/(Earth + Moon): for 329 390 substitute 328 912

p. 490. *Orbit of the Moon about the Earth*

Equatorial horizontal parallax: for 57′ 02″·54 substitute 57′ 02″·608 (the printed value is in error; it should have been 57′ 02″·70).

The mean distance given agrees (fortuitously) with the new value.

Mass-ratio—Earth/Moon: for 81·31 or 81·53 substitute 81·30

Other data

There are many consequential changes in the other tables of data: no list is given, as these are mainly trivial end-figure changes in values that are only given approximately. The value of the reciprocal of the flattening of the Earth in the table of "Dimensions and rotations of the planets" on p. 491 should, however, be changed to 298·25. Also, the value of k_E and the unit of distance on p. 493 need to be changed.

4. *List of Errata in and corrections to the Explanatory Supplement*

In the following list corrections are given in respect of all known errors in the first edition of the *Explanatory Supplement*; those preceded by an asterisk (*) were corrected in the 1962 reprint. The opportunity has been taken to add a few changes since introduced into the A.E., but no attempt has been made to bring up-to-date the various data concerning occultation predictions, radio time signals, etc.

Page No.	
4	Line −6: insert after 1950·0 "were given for the years 1928 and 1929"
15	*Royal Observatory Annals*: delete reference to Number 2.
*18	Volume XVI: for 1959 read 1958.
22	Line 8: for 0ᵈ·814 read 0ᵈ·813.
37	Formula for ω_0: for cosec read cosec i.
38	Formula for c should read $c = 180° - \Pi_m + \frac{1}{2}a$. ,, ,, c' ,, ,, $c' = 180° - \Pi_m - \frac{1}{2}a$.
41	Line −13: for circular read elliptical.
*95	Line 22: for B.2 read B.1.
*98	Mean anomaly: the second expression for g requires a sign of equality.
115	Jupiter, L: for 259 48 52·05 read 259 49 52·05.
127	Line 2: after catalogues, insert ", corrected for proper motion and annual parallax."
171	See comment in para 3.
175	Chapter 7: A significant omission is that of the authorities for the mean places of the stars.
197	Add: Nautical Almanac Office, H.M. 1958. *Planetary Co-ordinates for the years* 1960-1980 *referred to the equinox of* 1950·0 4to, London.
200	Add 1929: Co-ordinates of the Sun for 1950·0 for 1928 and 1929.
200	Add 1950: Ephemeris of Pluto.
202	Add 1950: Ephemeris of Pluto.
219	Line 21: to k 0·2722 74 add a footnote "k = 0·2724 807 adopted from 1963 onwards"; see also para. 3.
230	Line −4: to $(2L_1 - 0·5459)$ add a footnote "0·5464 adopted from 1963 onwards".
239	Lines 20 and −6: to 0·5459 add a footnote "0·5464 adopted from 1963 onwards".
251	Lines −8 and −6: to 0·5459 add a footnote "0·5464 adopted from 1963 onwards".
256	Line −13: after l add "(elements)".
260	Line −17: to 0·2722 74 add a footnote "0·2724 807 adopted from 1963 onwards".

*264	The expressions given for $\Delta \log r$ are those for $\Delta \log_e r$; values calculated from them must be multiplied by 0·4343 before being applied to log *r* (that is, $\log_{10} r$). In Example 9·24, the value of $\Delta \log r$ should in consequence be changed from +0·0000 026 to +0·0000 011.
279	Line 23: delete the reference to *Royal Observatory Annals*.
295	*Almanaque Nautico* has since 1961 been renamed *Efemérides Astronómicas*.
295	Many changes have been made in the list of standard stations; it is not practicable to give full details here.
*296	For Stalinabad read Dushanbe.
303, 304	The prediction of occultations of radio sources has been increased by a large factor, in respect both of the number of stations and of the number of radio sources. It is not practicable to give full details here.
334	A new position of the North pole of Mars has been adopted as from 1968. Details are given in the Explanation.
366	*The five inner satellites of Saturn.* The epoch 1889·25 from which *t* is measured in tropical years differs from that (J.D. 241 1093·0) from which *d* is measured in days; *t* is thus *not* equal to *d*/365·2422, and the last paragraph on the page should be corrected accordingly.
*367	Line −6: for S Geocentric read S Planetocentric.
377	End of second paragraph; add a footnote "As from 1966 the elements (*L*, *M* etc.) of Hyperion and Iapetus are referred, as for Titan, to the ring-plane."
378	To Hyperion add a footnote "See page 377."
380	To Iapetus add a footnote "See page 377."
381	To Hyperion and Iapetus add a footnote "See page 377."
*396	Kuiper, G. P. 1956: for *Vistas in Astronomy*, **1** read *Vistas in Astronomy*, **2**.
442	Line 12: for Earth read equinox.
443–453	There have been many changes in respect of the distribution of time, but it is not practicable to give full details here.
*452	South Africa: for Union Observatory read Republic Observatory.
*459	Line 14: for *Achstellige* read *Achtstellige*.
470	Table 16.3: Sun (*X*, *Y*, *Z*), δ^4, for 2 read 12. Nutation in obliquity, for 82 32 20 read 32 20 15. Ephemeris transit, under unit 0^h·0001 for 160 read 270 ,, 20 ,, 95 ,, 10 ,, 30
490	*Rotation of the Earth*: for $23^h\ 56^m\ 04^s$·09895 read $23^h\ 56^m\ 04^s$·09890. *Orbit of Moon about the Earth*: for 57′ 02″·54 = 3422″·54 read 57′ 02″·70 = 3422″·70. (See also para. 3).
*491	Last line: after 3963 miles insert "The tabulated semi-diameters are the values adopted in the Ephemeris".
494	1 a.u. in Light years: for 0·158 read 1·580. 1 a.u./day: for 1732 read 1730.

SECTION 1F. The series *Astronomical Papers . . . of the American Ephemeris . . .* now includes:

Volume XVI. II. G. M. Clemence. "Theory of Mars—Completion". 1961.

III. Paul Herget. "Rectangular coordinates of Ceres, Pallas, Juno, Vesta 1960–1980". 1962.

Volume XVII. C. B. Watts. "The marginal zone of the Moon". 1963.

Volume XVIII. Milton P. Jarnagin, Jr. "Expansions in elliptic motion". 1965.

Volume XIX. I. Otto G. Franz and Betty F. Mintz. "Tables of X and Y, elliptic rectangular coordinates". 1964.

Volume XX. I. Edward S. Jackson, "Determination of the equinox and equator from meridian observation of the minor planets". 1968.

II. Raynor L. Duncombe. "Heliocentric coordinates of Ceres, Pallas, Juno, Vesta 1928–2000". 1969.

III. Douglas A. O'Handley. "Determination of the mass of Jupiter from the motion of 65 Cybele". 1969.

Volume XXI. I. Paul M. Janiczek. "The orbit of Polyhymnia and the mass of Jupiter". 1971.

The series of *Circulars* of the U.S. Naval Observatory includes the following issues:

No.

90. Provisional ephemeris of Mars 1950–2000.
91. Rectangular coordinates of the Moon 1952–1971.
92. Ephemeris of the radio longitude of the central meridian of Jupiter, System III (1957.0). 1961–1963.
93. GC and DM numbers of FK3 stars.
94. Ephemeris of . . . central meridian of Jupiter, System III . . . 1964–1967.
95. Provisional ephemeris of Mars 1800–1950.
96. Geocentric distance and velocity of Venus 1961–1970.
98. Physical ephemeris of Mars 1877–1967.
104. Durchmusterung and Henry Draper Numbers of Albany General Catalog Stars.
106. Rectangular coordinates of Mercury 1800–2000.
107. Sunlight, moonlight, and twilight for Antarctica 1966–1968.
117. Ephemeris of . . . central meridian of Jupiter, System III . . . 1968–1971.
120. Sunlight, moonlight, and twilight for Antarctica 1969–1971.
132. Sunlight, moonlight, and twilight for Antarctica 1972–1974.
134. Normalized observations of Venus 1901–1949.
137. Ephemeris of . . . central meridian of Jupiter, System III . . . 1972–1975.

Details of total and annular solar eclipses are given in the following *Circulars*:

No.	No.	No.
101. 1971–1975.	116. 1968 Sept. 22.	129. 1972 Jan. 16.
102. 1965 May 30.	122. 1969 Mar. 18.	131. 1972 July 10.
109. 1966 May 20.	123. 1969 Sept. 11.	135. 1973 June 30.
110. 1966 Nov. 12.	125. 1970 Mar. 7.	
113. 1976–1980.	126. 1970 Aug. 31–Sept. 1.	

The following points of clarification may be noted:

Page 44. The published values of the nutation are based on the arguments printed here, but see note for page 107, below.

Pages 90–91. The principal differences between the values of ΔT tabulated on page 90 and those plotted in the graph on page 91 are due to the addition to the values prior to 1956 of the correction

$$0^s.15 - 2^s.55\ (T - 0.63) + 18^s.21\ (T - 0.63)^2$$

where T is measured in Julian centuries from 1900.0, corresponding to the correction to the mean longitude of the Moon (which changes by 1″ in $1^s.821$). Prior to 1861, the basic data were taken from the thesis by C. F. Martin (Yale University, 1969). Further small corrections have also been applied in an endeavour to put all values on a common basis. The quadratic term in the correction is not yet well established.

Values of ΔT (A), providing a first approximation to ET − UT, and of UT1 − UTC, are given at an interval of 3 months from 1956 January 1 in the current tables on page vii of *A.E. 1974* onwards.

Page 107. Expressions for ☾, Γ, D after removal of the implicit partial correction for aberration (*Astron. Jour.*, **57**, 46–7, 1952) are given in the Explanation of *A.E. 1973* onwards.

Pages 152–3. The expressions for the second-order terms and for J and J' assume that f, g, h are measured in radians. Each expression should be multiplied by sin 1″ if the coefficients (f, g, h, J, J') are expressed in seconds of arc; a further factor of $\frac{1}{15}$ is required to obtain J in seconds of time.

Page 223. The value of $(1 - e^2)^{-\frac{1}{2}}$ used for 1968 onwards is based on a flattening of 1/298.25.

Page 241. The value of e^2 used for 1968 onwards is based on the rounded value 298.25 for the reciprocal of the flattening of the reference spheroid. The difference from the precise value (see below) is of no significance in the eclipse predictions.

Page 317, line 10 and *page* 326, line 7. Selenographic longitude increases in the direction of rotation of the Moon, i.e. positively to the east on the selenocentric celestial sphere, but this implies that the selenographic longitudes of the Sun and of the terminators are decreasing functions of time. It is therefore convenient to tabulate co-longitudes that are measured in the opposite sense, i.e. positively to the east on the *geocentric* celestial sphere. The origin of co-longitude here differs from that of longitude in such a way that the co-longitudes of points on the hemisphere visible from the Earth normally lie between 0° and 180°.

Page 489. The parameters defining the figure of the current spheroid of reference have been derived from the adopted values of the primary constants using as many guarding figures as are necessary to ensure formal self-consistency of the system; rounded values, such as those on this page, may not satisfy the relations precisely. The precise value of the flattening corresponding to the adopted value of J_2 is

$$f = 1/298.247\ 167\ 427 = 0.003\ 352\ 923\ 713$$

and correspondingly

$b = 6\ 356\ 774.516$ m, $e^2 = 0.006\ 694\ 605\ 329$, $e = 0.081\ 820\ 567\ 882$.

1° of latitude	$111.133\ 35 - 0.559\ 84 \cos 2\phi + 0.001\ 17 \cos 4\phi$ km
1° of longitude	$111.413\ 28 \cos \phi - 0.093\ 51 \cos 3\phi + 0.000\ 12 \cos 5\phi$ km
Geodetic latitude (ϕ) − Geocentric latitude (ϕ')	
	$692''.750 \sin 2\phi - 1''.163\ 3 \sin 4\phi + 0''.002\ 6 \sin 6\phi$
Normal gravity (g)	$9.806\ 21 - 0.025\ 93 \cos 2\phi + 0.000\ 03 \cos 4\phi$ m/s^2

Page 490. The value given here for the mean distance (in a.u.) of the Earth from the Sun is that discussed on page 98 rather than the value (1.000 000 03) derived from Kepler's law, discussed on page 96, where n is expressed in radians per day.

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$$-0''\cdot 15 - 2''\cdot 35\,(T - 0\cdot 03) - 16''\cdot 21\,(T - 0\cdot 03)^2$$

where T is measured in Julian centuries from 1900·0, corresponding to the correction to the mean longitude of the Moon (which changes by 1″ in 0^s·821). Prior to 1861, the basic data were taken from the *Tables* by C. F. Brouwer (Yale University, 1952). Further small corrections have also been applied in an endeavour to put all values on a common basis. The quadratic term in the correction is not yet well established.

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$$f = 1/298\cdot 257\ 167\ 427 = 0\cdot 003\ 352\ 813\ 713$$

and correspondingly

$a = 6\ 378\ 140$ m, $\quad e^2 = 0\cdot 006\ 694\ 385\ 999$, $\quad e = 0\cdot 081\ 819\ 218\ 048$.

1° of latitude	$111\cdot 132\ 95 - 0\cdot 559\ 84\cos 2\phi + 0\cdot 001\ 17\cos 4\phi$ km
1° of longitude	$111\cdot 412\ 88\cos\phi - 0\cdot 093\ 50\cos 3\phi + 0\cdot 000\ 12\cos 5\phi$ km
Geodetic latitude (ϕ) − Geocentric latitude (ϕ')	$692''\cdot 730\sin 2\phi - 1''\cdot 163\sin 4\phi + 0''\cdot 026\sin 6\phi$
Normal gravity (g)	$9\cdot 806\ 21 - 0\cdot 025\ 93\cos 2\phi + 0\cdot 000\ 03\cos 4\phi$ m/s²

Page 490. The value given here for the mean distance (in a.u.) of the Earth from the Sun is that discussed on page 98 rather than the value (1·000 000 03) derived from Kepler's law, discussed on page 96, where n is expressed in radians per day.

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