Dynamo theory and GFD

Steve Childress

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1 Origins of the dynamo theory

Dynamo theory studies a conducting fluid moving in a magnetic field; the motion of the body through the field acts to generate new magnetic field, and the system is called a dynamo if the magnetic field so produced is self-sustaining.

1.1 Early ideas point the way

In the distant past, there was the idea that the earth was a permanent magnet. In the 1830s Gauss analyzed the structure of the Earth's magnetic field using potential theory, decomposing the field into harmonics. The strength of the dominant field was later found to change with time.

In 1919 Sir Joseph Larmor drew on the induction of currents in a moving conductor, to suggest that sunspots are maintained by magnetic dynamo action. P. M. Blackett proposed that magnetic fields should be produced by the rotation of fluid bodies.

The 'current consensus' is that the Earth's magnetic field is the result of a regenerating dynamo action in the fluid core. The mechanism of generation of the field is closely linked dynamically with the rotation of the Earth. Similar ideas are believed to apply to the solar magnetic field, to other planetary fields, and perhaps to the magnetic field permeating the cosmos.

1.2 Properties of the Earth and its Magnetic Field

The magnetic field observed at the Earth's surface changes polarity irregularly. The non-dipole components of the surface field also vary with time over many time scales greater than decades, and have a persistent drift to the west. The fluid core of the Earth is a spherical annulus, bounded by the solid inner core and the mantle. It is believed that the motion of the inner and outer core are sufficient to drive the geodynamo.

2 The homogeneous kinematic dynamo

The pre-Maxwell equations for a homogeneous moving conductor are

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu \, \mathbf{J} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \mathbf{J} &= \sigma \, (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{E} &= q/\epsilon \end{aligned} \tag{1}$$

Combined, these equations simplify to

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) - \eta \nabla^2 \mathbf{B} = 0 \tag{2}$$

where $\eta = (\mu\sigma)^{-1}$ is the magnetic diffusivity. This equation is called the magnetic induction equation.

2.1 Kinematic Dynamo Model of the Earth's Core

Neglecting the inner core, the conducting fluid is contained in a sphere of radius $r = r_c$. The exterior is regarded as free space. The equations are to be solved with a prescribed divergence-free (the core fluid is assumed incompressible) velocity field which is independent of time. On $r = r_c$, the magnetic field is continuous owing to the absence of magnetic monopoles and a concentrated surface current layer. The magnetic field on $r > r_c$ also matches with an external vacuum magnetic field

$$\mathbf{B}_e = \nabla \phi_e. \tag{3}$$

The tangential component of

$$\mathbf{E} = \eta \nabla \times \mathbf{B} - \mathbf{u} \times \mathbf{B} \tag{4}$$

is also continuous at $r = r_c$. We take the external field to decay like a potential dipole at $r = \infty$, i.e., $\mathbf{B} \approx O(r^{-3})$.

We pass to a dimensionless form using a characteristic length scale L, velocity scale U and time scale L/U. Using the vector identity

$$\nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{u} + \mathbf{u} \, \nabla \cdot \mathbf{B} - \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \, \nabla \cdot \mathbf{u}, \tag{5}$$

and since the velocity and magnetic fields are divergence free ($\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0$), the induction equation can be written in the form

$$\frac{D\mathbf{B}}{Dt} - \frac{1}{R} \nabla^2 \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} \tag{6}$$

in terms of the material derivative

$$\frac{D\mathbf{B}}{Dt} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B}. \tag{7}$$

Here R is the *magnetic Reynolds number*, $R = UL/\eta = UL\sigma\,\mu$. For $R << 1$, the magnetic field diffuses easily through the conductor. For $R >> 1$, the magnetic field is 'frozen into the moving conductor'. In this case, distortion and stretching of the field lines are caused by the term $\mathbf{B}\cdot\nabla\mathbf{u}$. Notice that eq. (6) corresponds to the vorticity equation upon substitution of $\nabla\times\mathbf{u}$ in place of $\mathbf{B}$.

2.2 Kinematic Dynamo as an Eigenvalue Problem

For a given velocity field $\mathbf{u}$, eq. (6), is linear for the magnetic field $\mathbf{B}$, and we may separate variables

$$\begin{aligned} \mathbf{B} &= e^{\lambda t}\,\mathbf{b}(\mathbf{x}), \quad r < 1 \\ \mathbf{B} &= e^{\lambda t}\,\nabla\phi, \qquad r > 1 \end{aligned} \tag{8}$$

Upon change of variable (8), the induction equation becomes

$$\boldsymbol{\mathcal{L}}\mathbf{b} = \frac{1}{R}\nabla^2\mathbf{b} - \mathbf{u}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{u} = \lambda\mathbf{b} \tag{9}$$

Here, length dimension L is $L = r_c$. The values of λ allowing acceptable solutions $\mathbf{b}$ are the eigenvalues, and depend on R. We say $\mathbf{u}(\mathbf{x})$ is a (steady) kinematic dynamo if for some $R > 0$ there exists an eigenvalue $\lambda(R)$ such that the real part of λ is greater than zero. Elsasser, Bullard, and others developed a theory of the spherical dynamo in this setting.

Note that although the kinematic equation is linear in $\mathbf{B}$, the dynamo problem has a nonlinear character, since the 'correct' velocity field $\mathbf{u}$ is not known *a priori*. Mathematically, we are trying to find the right function $\mathbf{u}(\mathbf{x})$ determining $\mathcal{L}$, so that the dynamo property is realized. More generally, we might consider a time periodic $\mathbf{x}$ and an analogous Floquet problem.

2.3 Expansion in eigenfunctions

Consider the space of complex-valued divergence-free vector fields $\mathbf{b}(\mathbf{x})$ in $r < 1$, matching continuously with a potential field in $r > 1$. The eigenfunctions $\mathbf{b}_n$ satisfy

$$\begin{aligned} \mathcal{L}\,\mathbf{b}_n &= \lambda_n\,\mathbf{b}_n, \quad r < 1 \\ \mathbf{b}_n &= \nabla\phi_n, \qquad r > 1 \end{aligned} \tag{10}$$

and are continuous on $r = 1$. The eigenvalues are known to be discrete and countable, and the eigenfunctions are complete in the above space. If the eigenfunctions were known to be mutually orthogonal, in the inner product

$$(\mathbf{b}_m, \mathbf{b}_n) = \int_{\mathcal{R}^3} \mathbf{b}_m^*\,\mathbf{b}_n\,dV \tag{11}$$

then, we would have

$$\int_{\mathcal{R}^3} |\mathbf{B}| = \sum_{n=1}^{\infty} e^{2\,\Re(\lambda_n)}||\mathbf{b}_n||^2 \tag{12}$$

Thus, if $\Re(\lambda_n) \neq 0$ for all n the magnetic energy would be bounded by its initial value. This is the case for a *normal* operator on a Hilbert space, one which commutes with its adjoint.

However, the operator $\mathcal{L}$ is not normal so the energy is not a simple sum of decaying non-negative terms. In fact, substantial transient growth of field energy is possible even if $\mathbf{u}$ is not a dynamo. This complicates numerical proofs of the dynamo property. The dynamo problem is one of the best examples of non-normality, and the effects become especially pronounced in the limit of large magnetic Reynolds number R. In the case of the Earth, R is of the order 10^3.

2.4 Free-decay modes for a rigid spherical core

In can be shown that in $\mathcal{R}^3$, the following decomposition of a divergence-free vector field $\mathbf{A}$ is possible

$$\mathbf{A} = \nabla \times (T\,\mathbf{r}) + \nabla \times \nabla \times (P\,\mathbf{r}) \tag{13}$$

where $T = T(\mathbf{r})$ and $P = P(\mathbf{r})$ are scalar functions called the toroidal and poloidal components of $\mathbf{A}$, respectively (figure 1). Consider this decomposition for $\mathbf{u} = 0$, so that we have

$$\frac{1}{R}\nabla^2 \mathbf{b}_n = \lambda_n\,\mathbf{b}_n \tag{14}$$

The eigenfunctions $\mathbf{b}_n$ are decomposed as follows:

$$\mathbf{b}_n = \nabla \times (T_n\,\mathbf{r}) + \nabla \times \nabla \times (P_n\,\mathbf{r}) \tag{15}$$

In the case of an axis-symmetric problem, one can express the eigenfunctions in terms of their components in spherical coordinates as follows:

$$\mathbf{b}_n = \frac{1}{r}L_2 P\,\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r} rP\,\mathbf{e}_\theta + \frac{\partial}{\partial\theta}T\,\mathbf{e}_\phi \tag{16}$$

where L_2 is the surface Laplacian. If $\mathbf{b}_n$ is expanded in spherical harmonics

$$\mathbf{b}_n = \sum_{l=1}^{\infty}\sum_{k=-l}^{l} b_l^m\, f(r) Y_l^m(\theta)\, e^{im\phi} \tag{17}$$

the surface operator has the property

$$L_2\,\mathbf{b}_n = -l(l+1)\,\mathbf{b}_n \tag{18}$$

The exterior of the dynamo is current free, and with the pre-Maxwell equation $\nabla \times \mathbf{B} = \mu\mathbf{J}$ one obtains $\nabla \times \mathbf{B} = 0$. In terms of eigenfunctions, the curl of the magnetic field can be written

$$\begin{aligned} \nabla \times \mathbf{b}_n &= \nabla \times (-\nabla^2(P_n\,\mathbf{r}) + \nabla\nabla\cdot(P_n\,\mathbf{r})) + \nabla \times \nabla \times (T_n\,\mathbf{r}) \\ &= \nabla \times (-\nabla^2(P_n\,\mathbf{r})) + \nabla \times \nabla \times (T_n\,\mathbf{r}) \end{aligned} \tag{19}$$

where we have used the identity $\nabla \times \nabla \times \mathbf{A} = \nabla\nabla\cdot\mathbf{A} - \nabla^2\mathbf{A}$. We reckon that the toroidal component of the magnetic field is the poloidal component of the curl of the field, and similarly $-\nabla^2 P_n$ becomes the toroidal component of the curl of the field:

$$\begin{aligned} (\nabla \times \mathbf{b}_n)_T &= -\nabla^2 P_n \\ (\nabla \times \mathbf{b}_n)_P &= T_n \end{aligned} \tag{20}$$

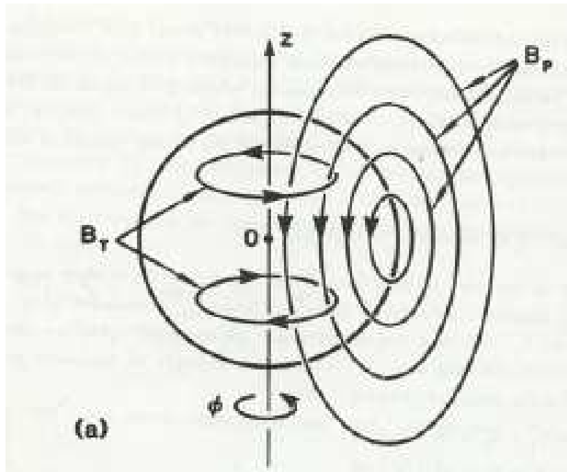

Figure 1: Toroidal and poloidal field lines for a sphere rotating about the z axis.

Using eq. (16) and considering the poloidal and toroidal components of the curl of the field, we obtain

$$\frac{1}{r} L_2 T_n = -\frac{1}{r} l(l+1) T_n = 0 \tag{21}$$

so the toroidal field is zero at the exterior ($T_n = 0$, $r > 1$). Looking at the ϕ-component of the field rotational, we have

$$\frac{\partial}{\partial \theta}(\nabla^2 P_n) = 0 \tag{22}$$

Using the matching condition on the core boundary, one obtains $\nabla^2 P_n = 0$, $r > 1$.

It then follows that we may consider separately problems for T_n and P_n

$$\begin{aligned} \nabla^2 T_n &= \lambda_n T_n \\ \nabla^2 P_n &= \lambda_n P_n \end{aligned} \quad r < 1$$
$$\begin{aligned} T_n &= 0 \\ \nabla^2 P_n &= 0 \end{aligned} \quad r \geq 1 \tag{23}$$

Here T_n, P_n, $\partial P_n / \partial r$ are continuous on $r = 1$, and $P_n \sim O(r^{-2})$ as $r \to \infty$.

The first few solutions are

$$T_1 = \frac{\sin \pi r}{r}, \quad \lambda_1 = -\pi^2, \tag{24}$$

where $r = |\mathbf{x}|$, (this mode is not believed to be relevant to the Earth's toroidal field except perhaps during reversals);

$$P_1 = \left(\frac{\pi \cos \pi r}{r} - \frac{\sin \pi r}{r^2} \right) \cos \theta, \quad r < 1, \quad P_1 = -\frac{\pi \cos \theta}{r^2}, \quad r > 1, \tag{25}$$

(this is the basic dipole component of the Earth's field with axis aligned with the rotation axis);

$$T_2 = \left(\frac{\mu \cos \mu r}{r} - \frac{\sin \mu r}{r^2} \right) \cos \theta, \quad \lambda_2 = -\mu^2 \approx -20.2, \tag{26}$$

where μ is the first zero of the spherical Bessel function j_1, given by $\tan\mu = \mu$ (this is a plausible dominant toroidal component of the Earth's field).

2.5 The Omega Effect

An example of transient growth of magnetic energy in a non-dynamo is the so called *Omega Effect*, in which differential rotation causes deformation of poloidal field lines, which induce a toroidal field (figure 2). Let $\mathbf{u} = U(r)\mathbf{e}_\phi$, where $r = \sqrt{x^2+y^2}$ in cylindrical polars, and represent the magnetic field, taken as symmetric with respect to the z-axis, in the alternative poloidal-toroidal decomposition

$$\mathbf{B} = B(r,z,t)\mathbf{e}_\phi + \nabla \times A(r,z,t)\mathbf{e}_\phi. \tag{27}$$

Now assume $A = rP$ determines the z-aligned dipole field and is independent of time. Taking the $\mathbf{e}_\phi$ components of the induction equation, B is then found to satisfy

$$\frac{\partial B}{\partial t} + r\frac{\partial A}{\partial z}\frac{\partial}{\partial r}\left(\frac{U(r)}{r}\right) - \frac{1}{R}\left(\nabla^2 - \frac{1}{r^2}\right)B = 0. \tag{28}$$

The second term here represents the omega effect - depending on $U(r)$ it can act as a source term for the toroidal component B, and shows how the poloidal field is coupled to the toroidal field and can cause the latter to grow. This however is not enough to create a dynamo, as we shall see below. Note a velocity $U(r)\mathbf{e}_\phi$ is easily realized in a rotating sphere of fluid as a *geostrophic flow*.

3 Establishing the possibility of a homogeneous dynamo

3.1 A negative result: Cowling's theorem

Cowling (1934) argued that no homogeneous dynamo can exist for axisymmetric $\mathbf{u}$ and $\mathbf{B}$. This is a result of the failure of the poloidal field to be maintained against ohmic dissipation. Taking

$$\mathbf{u} = U(r,z,t)\mathbf{e}_\phi + \nabla \times \psi(r,z,t)\mathbf{e}_\phi, \quad \mathbf{B} = B(r,z,t)\mathbf{e}_\phi + \nabla \times A(r,z,t)\mathbf{e}_\phi, \tag{29}$$

we find

$$\frac{\partial A}{\partial t} - \frac{\partial \psi}{\partial z}\frac{1}{r}\frac{\partial}{\partial r}(rA) + \frac{\partial A}{\partial z}\frac{1}{r}\frac{\partial}{\partial r}(r\psi) - \frac{1}{R}\left(\nabla^2 - \frac{1}{r^2}\right)A = 0, \tag{30}$$

$$\frac{\partial B}{\partial t} + r\frac{\partial A}{\partial z}\frac{\partial}{\partial r}\left(\frac{U}{r}\right) - \frac{1}{r}\frac{\partial}{\partial r}(rA)\frac{\partial U}{\partial z} - r\frac{\partial \psi}{\partial z}\frac{\partial}{\partial}\left(\frac{B}{r}\right) + \frac{1}{r}\frac{\partial}{\partial r}(r\psi)\frac{\partial B}{\partial z} - \frac{1}{R}\left(\nabla^2 - \frac{1}{r^2}\right)B = 0. \tag{31}$$

It is immediately clear from the equation for A that there is no coupling from the toroidal field to the poloidal field, and that the poloidal field will inevitably decay. Multiplying this equation for A by r^2A and integrating by parts, one finds

$$\frac{d}{dt}\int_{r<1}\frac{1}{2}(rA)^2\,\mathrm{d}V = -\frac{1}{R}\int_{R^2}|\nabla(rA)|^2\,\mathrm{d}V \le 0, \tag{32}$$

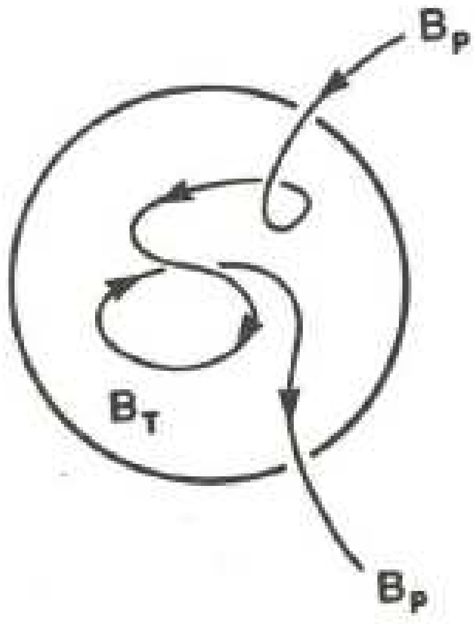

Figure 2: The Ω effect: differential rotation causes the twisting of poloidal field lines which generates toroidal field.

and this implies that the poloidal field must necessarily decay over time.

There are many such anti-dynamo theorems. The upshot is that although the velocity field may be simple, the magnetic field must be fully three-dimensional. It is thus significant that the observed planetry and stellar magnetic fields exhibit non-axisymmetric components.

3.2 A necessary condition for dynamo action

Note that the existence or non-existence of dynamo action will depend upon the value of $R = UL\sigma\mu$. The magnetic energy is

$$E_m = \int_V \frac{1}{2\mu}|\mathbf{B}|^2 \,\mathrm{d}V, \tag{33}$$

and its rate of change is

$$\begin{aligned} \mu\frac{dE_m}{dt} &= \int_V \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}\,\mathrm{d}V \\ &= -\int_V \mathbf{B}\cdot\nabla\times\mathbf{E}\,\mathrm{d}V \\ &= -\int_V \mathbf{E}\cdot\nabla\times\mathbf{B}\,\mathrm{d}V \\ &= \int_V \mathbf{u}\cdot(\mathbf{B}\times\nabla\times\mathbf{B}) - \frac{1}{\sigma\mu}|\nabla\times\mathbf{B}|^2\,\mathrm{d}V. \end{aligned} \tag{34}$$

For a dynamo to exist we need this rate of change of magnetic energy to be non-negative. Now if $U_m = \max_V |\mathbf{u}|$, the we can write

$$\begin{aligned} \int_V \mathbf{u} \cdot (\mathbf{B} \times \nabla \times \mathbf{B}) \, \mathrm{d}V &\le U_m \int_V |\mathbf{B}||\nabla \times \mathbf{B}| \, \mathrm{d}V \\ &\le U_m \left[\int_V |\mathbf{B}|^2 \, \mathrm{d}V \right]^{1/2} \left[\int_V |\nabla \times \mathbf{B}|^2 \, \mathrm{d}V \right]^{1/2}, \end{aligned} \tag{35}$$

using the Cauchy-Schwarz inequality. It is known that for a bounded homogeneous conductor V surrounded by vacuum there is a length L_V such that, for any $\mathbf{B}$ in the space of realizable magnetic fields,

$$\int_V |\nabla \times \mathbf{B}|^2 \, \mathrm{d}V \ge \frac{1}{L_V^2} \int_{R^3} |\mathbf{B}|^2 \, \mathrm{d}V. \tag{36}$$

Thus we have

$$\int_V \mathbf{u} \cdot (\mathbf{B} \times \nabla \times \mathbf{B}) \, \mathrm{d}V \le U_m L_V \int_V |\nabla \times \mathbf{B}|^2 \, \mathrm{d}V, \tag{37}$$

so

$$\mu \frac{dE_m}{dt} = \left(U_m L_V - \frac{1}{\sigma\mu} \right) \int_V |\nabla \times \mathbf{B}|^2 \, \mathrm{d}V. \tag{38}$$

This is negative if $U_m L_V \sigma\mu < 1$ in which case a dynamo cannot exist. Thus a necessary condition for dynamo action is that $U_m L_V \sigma\mu \ge 1$. In particular dynamo action occurs as a bifurcation from the state of no magnetic field, as the magnetic Reynolds number is increased.

3.3 The Bullard-Gellman computation

There was an early attempt to solve the eigenvalue problem utilizing direct expansion in the models of free decay of a spherical conductor. A steady velocity field was chosen as representative of rotating convection. The truncated system of modal equations produced a positive result, but in 1969 Gibbons and Roberts re-examined this choice of $\mathbf{u}$ and established that it was not in fact a kinematic dynamo. The original result was a numerical artifact.

3.4 Parker's model: The α effect

In 1955, Parker addressed the problem of producing poloidal field from toroidal field, to complement the omega effect. As we have seen, this is impossible if both the velocity field and the magnetic field are axisymmetric, but Parker envisaged averaging of the effects of small up-wellings within the core which could lift and twist poloidal field lines. The local magnetic Reynold's number was take to be large, so that the magnetic field was 'frozen' into the fluid as it was lifted and twisted (figure 3). Under averaging, the twisted loops produced a mean current, aligned with the local toroidal field line. In this way the poloidal field can be induced from the toroidal field, thus completing the dynamo cycle and maintaining it against dissipation. This mechanism is now know as the 'α effect'. Although Parker's supporting

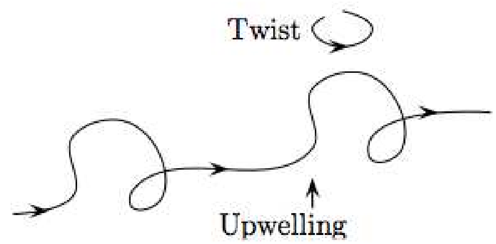

Figure 3: Toroidal field lines are lifted and twisted by small scale non-axisymmetric components of the velocity field. This generates a mean current in the toroidal field direction which induces a poloidal field, and is termed the α effect.

calculations left many gaps, these have since been filled in and the model provides a crucial element of the dynamo cycle, breaking the constraints of Cowling's theorem.

If $\mathbf{u} = U(s)\mathbf{e}_\phi$, the induction equation in this model is written as

$$\mathbf{B} = B(r,z,t)\mathbf{e}_\phi + \nabla \times A(r,z,t)\mathbf{e}_\phi, \tag{39}$$

$$\frac{\partial A}{\partial t} - \frac{1}{R}\left(\nabla^2 - \frac{1}{r^2}\right) A = \alpha B, \tag{40}$$

$$\frac{\partial B}{\partial t} + r\frac{\partial A}{\partial z}\frac{\partial}{\partial r}\left(\frac{U}{r}\right) - \frac{1}{R}\left(\nabla^2 - \frac{1}{r^2}\right) B = 0. \tag{41}$$

The added αB term here is crucial and completes the '$\alpha - \omega$ dynamo cycle'.

3.5 Early examples of dynamo action

The Herzenberg dynamo (1958), comprises two solid rotating spheres of radius a embedded inside a larger sphere of radius R, with $a \ll R$ and a also small compared to the distance between the two spheres. Each sphere produces an ω effect determined by the local field at that sphere. The resulting induced field increases the local field at the other sphere, and regenerative dynamo action can result. Herzenberg's theoretical analysis of this system relies on the smallness of the embedded spheres and the geometrical decay of the induced field components, leading to a spatial filtering. Such a dynamo was realized experimentally in solid iron by Lowes and Wilkinson (1963).

Backus (1958) gave a proof of dynamo action for a kinematic dynamo in a homogeneous spherical fluid conductor based upon temporal filtering of magnetic decay modes. This involves applying a velocity field $\mathbf{u}_1(\mathbf{x})$ for time $0 < t < T_1$, and then setting $\mathbf{u} = 0$ for time $T_1 < t < T_2$, during which time the magnetic field decays, filtering out modes with the faster decay. A new velocity field $\mathbf{u}_2(\mathbf{x})$ is then applied for $T_2 < t < T_3$ and then $\mathbf{u} = 0$ again. This sequence of applying velocity fields and then setting $\mathbf{u} = 0$ to filter out fast decaying modes is then continued.

If the 'basic mode' (involving the slower decaying poloidal and toroidal components) is $\mathbf{D}$ with $||\mathbf{D}|| = 1$, an initial magnetic field involving this basic mode with amplitude A, plus some arbitrary magnetic noise $\mathbf{N}(0)$, with $||\mathbf{N}(0)|| \leq \epsilon$ for some $\epsilon > 0$, is

$$\mathbf{B}(\mathbf{x}, 0) = A\mathbf{D}(\mathbf{x}) + \mathbf{N}(0). \tag{42}$$

If it can then be shown that at time $t = T_{2N}$ the magnetic field has the form

$$\mathbf{B}(\mathbf{x}, T_{2N}) = A_N\mathbf{D}(\mathbf{x}) + \mathbf{N}(T_{2N}), \tag{43}$$

where $|A_N| \geq |A|$ and $||\mathbf{N}(T_{2N})|| \leq \epsilon$, then dynamo action occurs. Thus the noise gets no bigger, and the basic field is at least maintained.

3.6 Smoothing Applied to the Kinematic Equation

The smoothing method consists in splitting the dynamo kinematic equation operator

$$\mathcal{L}\,\mathbf{B} = \frac{\partial}{\partial t}\mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) - \eta\nabla^2\mathbf{B} = 0, \tag{44}$$

and the velocity and magnetic field into some rough and smooth components:

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}_S + \mathbf{u}_R, \\
\mathbf{B} &= \mathbf{B}_R + \mathbf{B}_S, \\
\mathcal{L} &= \mathcal{L}_R + \mathcal{L}_S, \\
\mathcal{L}_R\,\mathbf{B} &= -\nabla \times (\mathbf{u}_R \times \mathbf{B}), \\
\mathcal{L}_S\,\mathbf{B} &= \frac{\partial}{\partial t}\mathbf{B} - \nabla \times (\mathbf{u}_S \times \mathbf{B}) - \eta\nabla^2\mathbf{B}.
\end{aligned} \tag{45}$$

Let P be a projection onto smooth fields ($P^2 = P$), such as

$$P\,\mathbf{B} = \mathbf{B}_S \tag{46}$$

and assume $P\,\mathcal{L}_S = \mathcal{L}_S\,P$. Subtracting $P\mathcal{L}$ from L, one gets

$$-(\mathcal{L}_R - P\mathcal{L}_R)\mathbf{B} = (\mathcal{L}_S - P\,\mathcal{L}_S)\mathbf{B} = \mathcal{L}_S\mathbf{B} - \mathcal{L}_S\mathbf{B}_S = \mathcal{L}_S\mathbf{B}_R \tag{47}$$

The smooth projection of $(\mathcal{L}_R - P\mathcal{L}_R)\mathbf{B}$ is

$$P(\mathcal{L}_R - P\mathcal{L}_R)\mathbf{B} = P\mathcal{L}_R - P^2\mathcal{L}_R\mathbf{B} = P\mathcal{L}_R - P\mathcal{L}_R\mathbf{B} = 0 \tag{48}$$

so $(\mathcal{L}_R - P\mathcal{L}_R)\mathbf{B}$ is a rough field. Assuming that $\mathcal{L}_S$ can be inverted on rough fields, we have

$$\mathbf{B}_R = -\mathcal{L}_S^{-1}(\mathcal{L}_R - P\mathcal{L}_R)\mathbf{B} \equiv M\,\mathbf{B} \tag{49}$$

So, one has

$$\mathbf{B} - \mathbf{B}_S = \mathbf{B}_R = M\,\mathbf{B} \tag{50}$$

or, if $I - M$ is invertible on smooth fields, we obtain $\mathbf{B}$ in terms of its smooth component,

$$\mathbf{B} = (I - M)^{-1}\mathbf{B}_S \tag{51}$$

Thus, one can write

$$\begin{aligned} P\,\mathcal{L}\,\mathbf{B} = P(\mathcal{L}_S + \mathcal{L}_R)\mathbf{B} &= \mathcal{L}_S\mathbf{B}_S + P\mathcal{L}_R(I-M)^{-1}\mathbf{B}_S \\ &= [\mathcal{L}_S + P\mathcal{L}_R(I-M)^{-1}]\mathbf{B}_S = 0 \end{aligned} \tag{52}$$

and it can be shown by substitution that if $\mathbf{B}_S$ satisfies

$$[\mathcal{L}_S + P\mathcal{L}_R(I-M)^{-1}]\mathbf{B}_S = 0 \tag{53}$$

then $\mathbf{B} = (I-M)^{-1}\,\mathbf{B}_S$ solves $\mathcal{L}\mathbf{B} = 0$. The smoothing method can be used to treat analytically the creation of the α-effect.

3.6.1 First-Order Smoothing

Assuming that $\mathcal{L}_R\mathbf{B}_S$ is a rough field such as $P\,\mathcal{L}_R\mathbf{B}_S = 0$, one can make the following approximations

$$0 = [\mathcal{L}_S + P\mathcal{L}_R(I-M)^{-1}]\mathbf{B}_S \approx [\mathcal{L}_S + P\mathcal{L}_R(I+M)]\mathbf{B}_S = [\mathcal{L}_S + P\mathcal{L}_R\,M]\mathbf{B}_S. \tag{54}$$

Here, we have

$$\mathcal{L}_S\,\mathbf{B}_S = \frac{\partial}{\partial t}\mathbf{B}_S - \nabla\times(\mathbf{u}_S\times\mathbf{B}_S) - \frac{1}{R}\nabla^2\mathbf{B}_S \tag{55}$$

and

$$P\,\mathcal{L}_R\,M\,\mathbf{B}_S = -P[\mathcal{L}_R\mathcal{L}_S^{-1}\mathcal{L}_R]\,\mathbf{B}_S \approx P\mathcal{L}_R\,\mathbf{B}_R \tag{56}$$

with $\mathbf{B}_R$ a linear function of $\mathbf{B}_S$. This is the basis of mean field electrodynamics developed by Krause, Radler and others in the 60's.

For example, set $\mathbf{u}_S = 0$ and let $\mathbf{u}_R$ be a real steady solenoidal field of the form

$$\mathbf{u}_R = \sum_{k\in K}\mathbf{a}_k e^{i\mathbf{k}.\mathbf{x}} \tag{57}$$

where K is all 3-vectors excluding the zero vector. If $\mathbf{B}_S$ is large scale relative to $\mathbf{u}_R$, then

$$\nabla\times(\mathbf{u}_R\times\mathbf{B}_S) = -\frac{1}{R}\nabla^2\mathbf{B}_R \tag{58}$$

implies

$$\mathbf{B}_R = R\,i\sum_{k\in K}k^{-2}\,\mathbf{k}\cdot\mathbf{B}_S\,\mathbf{a}_k e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{59}$$

and

$$P[\mathbf{u}_R\times\mathbf{B}_R] \approx R\,i\sum_{k\in K}k^{-2}(\mathbf{a}^*\times\mathbf{a}_k)\mathbf{B}_S\cdot\mathbf{k} \equiv \alpha\cdot\mathbf{B}_S. \tag{60}$$

It can be shown, by using the substitution $\mathbf{a}_k = \mathbf{k}\times\mathbf{b}_k$, that α is real and symmetric. For isotropic $\mathbf{u}_R$, of this form we may assume $\alpha_{ij} = \alpha\delta_{ij}$ where

$$\alpha = \frac{1}{3}R\,i\sum_{\mathbf{k}\in R}k^{-2}(\mathbf{a}_k^*\times\mathbf{a}_k)\cdot\mathbf{k}. \tag{61}$$

If all $\mathbf{k}$ have unit length we see that

$$\alpha = \frac{R}{3}\overline{\mathbf{u}_R \cdot (\nabla \times \mathbf{u}_R)}, \tag{62}$$

involving a spatial average of the helicity density $\mathbf{u}_R \cdot (\nabla \times \mathbf{u}_R)$ of the velocity field. A field satisfying all the above conditions is the Beltrami field (in which $\mathbf{u}$ is parallel to $\nabla \times \mathbf{u}$)

$$\mathbf{u}_R = (\sin z + \cos y, \sin x + \cos z, \sin y + \cos x), \tag{63}$$

with $\alpha = R/3$. The role of helicity is reminiscent of Parker's 'upwellings', which involved a rising, twisting flow.

3.6.2 Spatially periodic kinetic dynamos

This is a setting where complete analysis of the smoothing method may be carried out explicitly, with or without time dependence. Considering here only steady fields, the real velocity field has the form

$$\mathbf{u} = \mathbf{u}_R = \sum_{k \in K} \mathbf{a}_k e^{i\mathbf{k}\cdot\mathbf{x}} \tag{64}$$

The admissible magnetic fields have the complex form

$$\mathbf{B} = e^{i\mathbf{n}\cdot\mathbf{x}} \sum_{k \in K+(0,0,0)} \mathbf{B}_k e^{i\mathbf{k}\cdot\mathbf{x}}, |\mathbf{n}| < 1 \tag{65}$$

so $\mathbf{B}_0 = \mathbf{B}_S$ (the components for $k \neq 0$ are all rough). Kinematic dynamo action can be defined by growth of field if n is sufficiently small. This is the palindromic 'so many dynamos' theorem (G. O. Roberts): almost all such $\mathbf{u}$ are kinematic dynamos.

3.7 Braginskii's theory for nearly axisymmetric dynamos (1964)

Braginskii used a smoothing technique for nearly axisymmetric flows to derive the α effect. Here, the 'smooth' parts of the variables are the axisymmetric components, and the rough parts are the non-axisymmetric components. The smoothing operation is therefore an angle average over ϕ in cylindrical polar coordinates. Motivated by the fact that the magnetic Reynolds number R is large for the Earth and that the field is dominated by a dipole aligned with the rotation axis, Braginskii considers expanding the fields in powers of $\epsilon = R^{-1/2}$;

$$\mathbf{u} = U(r,z)\mathbf{e}_\phi + \epsilon \mathbf{u}' + \epsilon^2 \mathbf{u}_P, \tag{66}$$

$$\mathbf{B} = B(r,z,t)\mathbf{e}_\phi + \epsilon \mathbf{B}' + \epsilon^2 \mathbf{B}_P, \tag{67}$$

where $< \mathbf{u}' >=< \mathbf{B}' >= 0$ where $< \cdot >$ means an angle average. The relevant timescale is shorter, so writing $\tau = \epsilon^2 t$, the order ϵ terms in the induction equation give

$$\frac{\partial B}{\partial \tau} + r\mathbf{u}_P \cdot \nabla \left(\frac{B}{r}\right) = r\mathbf{B}_P \cdot \nabla \left(\frac{U}{r}\right) + (\nabla \times \mathcal{E})_\phi + \left(\nabla^2 - \frac{1}{r^2}\right) B = 0, \tag{68}$$

where $\mathcal{E} =< \mathbf{u}' \times \mathbf{B}' >$. If $\mathbf{B}_P = \nabla \times A(r,z,t)\mathbf{e}_\phi$, we also find

$$\frac{\partial A}{\partial \tau} + r\mathbf{u}_P \cdot \nabla \left(\frac{A}{r}\right) = \mathcal{E}_\phi + \left(\nabla^2 - \frac{1}{r^2}\right) A. \tag{69}$$

Braginskii found that the contributions from $\mathcal{E}$ can be mostly absorbed into 'effective' variables, denoted with a superscript e:

$$\frac{\partial B}{\partial \tau} + r\mathbf{u}_P^e \cdot \nabla \left(\frac{B}{r}\right) = r\mathbf{B}_P \cdot \nabla \left(\frac{U}{r}\right) + \left(\nabla^2 - \frac{1}{r^2}\right) B = 0, \tag{70}$$

$$\frac{\partial A^e}{\partial \tau} + r\mathbf{u}_P^e \cdot \nabla \left(\frac{A^e}{r}\right) = \alpha B + \left(\nabla^2 - \frac{1}{r^2}\right) A^e. \tag{71}$$

The term which cannot be absorbed gives rise to the α term, and in effective variables these equations are identical to the Parker model, the toroidal field exhibiting the α effect.

3.8 Soward's Lagrangian analysis of Braginskii's problem

The same situation of a nearly axisymmetric flow was analysed in a slightly different way by Soward (1972). The key idea of this method is to make use of an invariance of the induction equation for the diffusionless case $R = \infty$;

$$\mathbf{B}_t + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u}. \tag{72}$$

Consider two (prescribed) velocity fields $\mathbf{u}(\mathbf{x},t)$ and $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},t)$, and the Lagrangian variables $\mathbf{x}(\mathbf{a},t)$ and $\tilde{\mathbf{x}}(\mathbf{a},t)$ for these flows defined by

$$\left.\frac{d\mathbf{x}}{dt}\right|_{\mathbf{a}} = \mathbf{u}(\mathbf{x},t), \qquad \left.\frac{d\tilde{\mathbf{x}}}{dt}\right|_{\mathbf{a}} = \tilde{\mathbf{u}}(\mathbf{x},t). \tag{73}$$

Then

$$\tilde{u}_i = \left.\frac{\partial \tilde{x}_i}{\partial t}\right|_{\mathbf{x}} + \frac{\partial \tilde{x}_i}{\partial x_j} u_j. \tag{74}$$

The solutions to the induction equation for the two flows are

$$B_i(\mathbf{x},t) = B_j(\mathbf{a},0)\frac{\partial x_i}{\partial a_j}, \quad \tilde{B}_i(\tilde{\mathbf{x}},t) = \tilde{B}_j(\mathbf{a},0)\frac{\partial \tilde{x}_i}{\partial a_j}. \tag{75}$$

Using the chain rule

$$\frac{\partial \tilde{x}_i}{\partial a_j} = \frac{\partial \tilde{x}_i}{\partial x_k}\frac{\partial x_k}{\partial a_j} \tag{76}$$

we obtain

$$\tilde{B}_i = \frac{\partial \tilde{x}_i}{\partial x_k} B_k \tag{77}$$

Now the idea is to consider one of the velocity fields as including the 'rough' components while the other is the smooth velocity field without these rough components. For the Braginskii problem this means taking the tilde variables to include non-axisymmetric components of order ϵ, while the $\mathbf{x}$ variables do not include these. The $\mathbf{x}$ variables will therefore not be physical. We seek a near identity transformation $\mathbf{x}(\tilde{\mathbf{x}}, t) = \tilde{\mathbf{x}} + O(\epsilon)$ of R^3 which eliminates the $O(\epsilon)$ in the velocity, so that

$$\mathbf{u}(\mathbf{x}, t) = U(r, x)(1 + O(\epsilon^2))\mathbf{e}_\phi + \epsilon^2 \mathbf{u}_P^e(\mathbf{x}) + \epsilon^2 \mathbf{u}''(\mathbf{x}, t) + o(\epsilon^2). \tag{78}$$

The equations in $\mathbf{x}$ space can be angle averaged, and this results in Braginskii's result (70) and (71) with $A(\mathbf{x}, \tau)$ in place of $A^e(\tilde{\mathbf{x}}, \tau)$. The α effect is missing - it comes from the non-invariant diffusive term. The effective variable A^e occurs when we transform the unphysical $\mathbf{x}$ variables back into the real physical space $\tilde{\mathbf{x}}$.

The key point is that, by including the non-invariant diffusive term in this analysis, one obtains the α effect term in (71), along with terms which can be absorbed into effective variables.

4 Some examples of analytically accessible dynamos

There are some simple velocity fields for which dynamo action is known to occur. The Ponomarenko dynamo is a flow in a cylinder in which the fluid travels axially while rotating,

$$\mathbf{u} = U(r)\mathbf{e}_z + r\Omega(r)\mathbf{e}_\phi \tag{79}$$

The magnetic field is periodic in z,

$$\mathbf{B} = b(r)e^{\lambda t + im\phi + ikz}. \tag{80}$$

This can be analysed for large R. The dynamo mechanism here involves the Ω effect which maintains the toroidal field from the poloidal field, and diffusion which maintains the poloidal field from the toroidal field.

Another simple dynamo is the Roberts cell, which like the Beltrami flow is periodic in two dimensions

$$\mathbf{u} = (\cos y, \sin x, \cos x + \sin y). \tag{81}$$

By transforming the variables $x \to x - y$, $y \to \pi/2 + x + y$, the Roberts cell takes the form

$$\mathbf{u} = (\cos y \sin x, -\cos x \sin y, \sqrt{2} \sin x \sin y) \tag{82}$$

An illustration of this flow and the generation of magnetic field is shown in figure 4.

5 MHD dynamos

So far we have only talked about *kinetic* dynamo theory, when the velocity field is somehow prescribed and unaffected by the magnetic field. In magnetohydrodynamics the velocity and magnetic fields both interact with each other and some physical mechanism (such as convection) which drives the fluid velocity must be included. The momentum equation for the fluid in a rotating frame with angular velocity Ω includes the Coriolis force and the Lorentz force,

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p + 2\rho \mathbf{\Omega} \times \mathbf{u} + \frac{1}{\mu} \mathbf{B} \times (\nabla \times \mathbf{B}) - \rho \nu \nabla^2 \mathbf{u} = c\mathbf{g}. \tag{83}$$

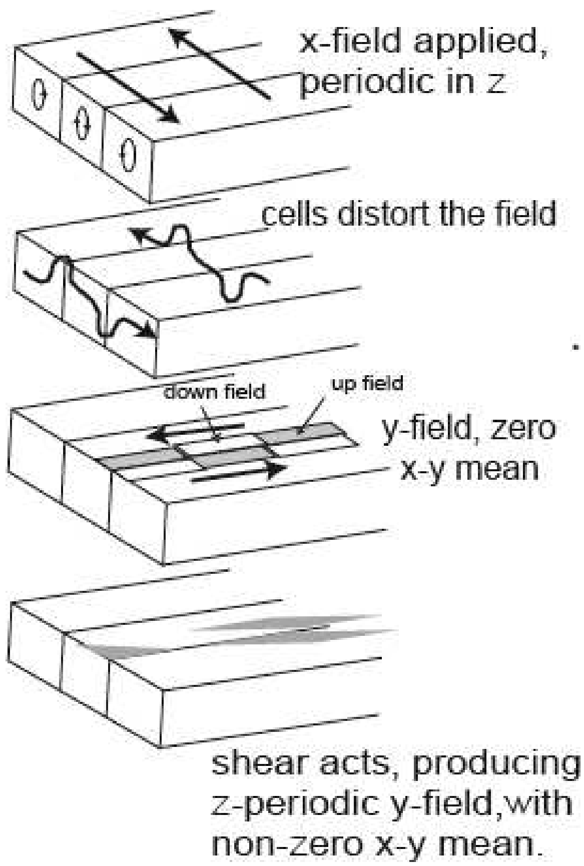

Figure 4: The Roberts cell with a z periodic field applied. The flow causes the production of z periodic y-field from x-field, and vice versa.

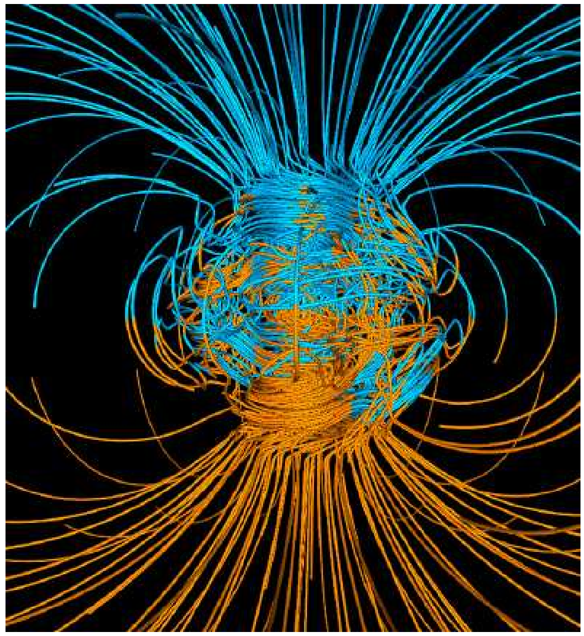

Figure 5: Magnetic field from a numerical MHD simulation of the Earth's core by Glatzmaier and Roberts.

The motion here is assumed to be driven by convection of some scalar c (temperature or composition, for instance),

$$\frac{Dc}{Dt} - D_c\nabla^2 c = Q. \tag{84}$$

If the convective forces are absent, and the velocity is axisymmetric, but the alpha-effect is included as in Braginskii's model, and the normal component of $\mathbf{u}$ vanishes at the core boundary, it is found that such an axisymmetric dynamo cannot be sustained in MHD. The microscale driving needed to produce the small scale fields associated with an alpha effect can not sustain the dynamo. The Earth's dynamo is now believed to be a result of larger scale driving as a result of thermal and compositional convection associated with the formation of the inner core.

Figure 5 shows a numerical simulation by Glatzmaier and Roberts, of a convective model of the Earth's core. This simulation exhibits the observed full reversals of the magnetic field on timescales of geophysical relevance. The role of the inner core is established as a reservoir of magnetic field during these reversals.

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