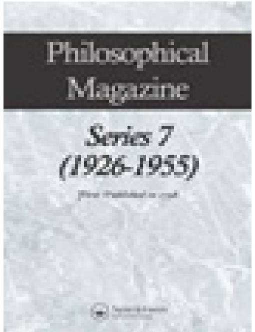

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T.L. Eckersley M.A.

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XX. *The Compton Scattering and the Structure of Radiation.* *By* T. L. Eckersley, *M.A.* *

COMPTON †, in discussing the scattering of X-rays by free electrons, has evolved a theory which frankly discards the laws of electromagnetism in favour of purely quantum considerations. In justification, he points out the inability of the classical theory to account for the change in wave-length in the scattered radiation, which experiments have shown undoubtedly occurs when very hard γ- or X-rays are used as the exciting source.

In making this statement he must have been unaware of an investigation by Sir J. J. Thomson ‡, who showed that ions or electrons are set in motion in the direction of propagation of electric waves, so that the scattering is effected by electrons which are moving along with the wave. To a fixed observer the scattered radiation will therefore appear of longer wave-length than that of the exciting source, in agreement with experimental results.

The formula evolved by Sir J. J. Thomson is:

$$c^2-(c-v)^2=\frac{e^2c^2H^2}{m^2p^2} \quad \text{or} \quad \frac{e^2H^2\lambda^2}{4\pi^2m^2},$$

where c is the velocity of light, v the average drift produced in the direction of propagation, λ the wave-length, e and m the charge and mass of the electron respectively, and H the magnetic force in the wave-front.

A numerical examination of this formula shows that for wave-length of 0·1 Å. the change of wave-length is infinitesimal unless enormous values of H are involved.

In the following investigation the classical methods are employed, and Sir J. J. Thomson's analysis is extended to take account of the relativity variation of mass.

In accordance with quantum ideas, however, the steady state is calculated neglecting the reaction due to re-radiation or scattering, this process being assumed to take place in a sudden transition from one quantum state to another. The steady state is thus unaffected by re-radiation.

In the classical theory the change of wave-length is a function of the intensity of the incident radiation. If in accordance with the experimental evidence the scattered

* Communicated by the Author.
† Phys. Review, xxi. p. 483 (1923).
‡ J. J. Thomson, Phil. Mag. ser. 6, vol. iv. p. 253 (August 1902).

frequency is only a function of the incident frequency, there must be a restriction on the intensity of the incident radiation. The assumption that this is in quanta is insufficient to determine this relation, for this only fixes the total energy $h\nu$ in the quantum, and not the intensity of E, which determines the change of frequency.

In the present theory the relation is provided by quantizing the orbit of the scattering electrons, which is found to be periodic. Such a process is perhaps rather outside the scope of the usual quantum theory, which to my mind merely exhibits the four-dimensional (dynamic) structure of the atom; but the theory must stand by the test of results, and these appear to be in accordance with the experimental evidence.

To accept the process outlined here seems to me to be more logical than to discard, with Compton, the whole structure of the electromagnetic laws, which are undoubtedly valid in the lower frequency ranges.

The method which uses the classical theory to calculate the steady states, and only involves the quantum theory to restrict the possible states (or calculate the arbitrary constants), has the merit of retaining the classical theory as far as possible, and is logically self-consistent if we consider the classical theory as exactly applicable to all periodic or quasi-periodic states of motion, and the quantum theory as a guide to define the possible steady states and the probability of transfer of one state to another.

Before entering into mathematical details of the problem, it may be worth while to attempt to give a mental picture of the physical processes which result in the production of a drift of electrons in the direction of propagation, according to the classical theory. Thus an isolated electron in the field of a plane electromagnetic wave will, under the influence of the electric force, in the first place, execute a vibration to and fro along the lines of electric force, *i. e.* at right angles to the direction of propagation of the waves. But in doing so it cuts across the lines of magnetic force H in the wave-front, and is given a resultant forward impulse. It will be found that the mean force is zero when averaged over a large number of periods, so that there is no resultant acceleration, and the average velocity given in the first impulse remains unaltered.

This average velocity v is a function of the frequency and intensity of the incident radiation, and $\frac{v}{c}$ measures the change of frequency of the scattered radiation.

To state the problem explicitly, let us suppose a plane wave:

$$E = E_0 \sin p(t - x/c), \quad t > x/c, \quad E = 0 \text{ for } t < x/c,$$
$$H = H_0 \sin p(t - x/c), \quad ,, \quad H = 0 \quad ,,$$

moving from left to right along the x axis to meet a stationary electron at $x=0$, $y=0$, and $z=0$.

Let the electric force be in the z direction and the magnetic force in the y direction.

The general equations for the movement of the electron are:

$$-m_0 \left\{ \frac{d^2 x_\mu}{ds^2} + \{\alpha\beta, \mu\} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} \right\} = F_\nu^\mu J^\nu, \quad . \quad * (1.1)$$

where J^ν is the current vector

$$\rho \frac{dx}{dt}, \quad \rho \frac{dy}{dt}, \quad \rho \frac{dz}{dt}, \quad \rho c.$$

F_μ is the tensor

$$\left. \begin{matrix} 0 & -\gamma & \beta & X \\ \gamma & 0 & -\alpha & Y \\ -\beta & \alpha & 0 & Z \\ +X & -Y & -Z & 0 \end{matrix} \right| , \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

assuming Galilean coordinates.

Also

$$\left. \begin{aligned} \gamma = \alpha &= 0, \\ X = Y &= 0, \\ \{\alpha\beta, \mu\} &= 0. \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

We get the equation:

$$\left. \begin{aligned} m_0 \frac{d^2 z}{ds^2} &= \rho Z - \rho\beta \frac{dx}{dt}, \\ m_0 \frac{d^2 x}{ds^2} &= \rho\beta \frac{dz}{dt}, \\ m_0 \frac{d^2 ct}{ds^2} &= Z\rho v_2. \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (1.4)$$

* Eddington, 'Mathematical Theory of Relativity,' pp. 118, 181.

For our purpose, since β is a function of t, $\beta=\beta_0 e^{ip\left(t-\frac{x}{2}\right)}$, it is best to modify the equation as follows:

$$m_0 \frac{d^2x}{ds^2} = m_0 \frac{d}{ds}\left(\frac{dt}{ds}\cdot\frac{dx}{dt}\right). \quad . \quad . \quad . \quad . \quad (1.5)$$

Now $m_0 \frac{dt}{ds} = m$, the value of the mass of the moving electron referred to fixed axes.

$$\frac{d^2x}{ds^2} = \frac{dM_x}{ds}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.6)$$

where M_x is the X component of the momentum of the electron, and

$$\frac{dM_x}{ds} = \frac{dt}{ds}\cdot\frac{dM_x}{dt}; \quad . \quad . \quad . \quad . \quad . \quad (1.7)$$

so that

$$\left.\begin{aligned} \frac{dM_x}{dt} &= \rho\frac{ds}{dt}\cdot Z - \rho\frac{ds}{dt}\cdot\frac{dx}{dt}\cdot\beta, \\ \frac{dM_z}{dt} &= \rho\frac{ds}{dt}\cdot\frac{dz}{dt}\cdot\beta, \\ \frac{dmc}{dt} &= Z\rho\frac{ds}{dt}\cdot\frac{dz}{dt}. \end{aligned}\right\} \quad . \quad . \quad . \quad (1.8)$$

Now $\rho\frac{ds}{dt}$ = invariant charge e, so that

$$\left.\begin{aligned} \frac{dM_x}{dt} &= eZ - e\beta v_x, \\ \frac{dM_x}{dt} &= e\beta v_z, \\ \frac{dme}{dt} &= Zev_z, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.9)$$

or, expressing everything in terms of the four vectors M_x, M_y, M_z, mc,

$$\left.\begin{aligned} \frac{dM_z}{dt} &= eZ - \frac{e\beta}{m}\cdot M_x, \\ \frac{dM_x}{dt} &= \frac{e\beta}{m} M_z, \\ m\frac{dme}{dt} &= Zev_z. \end{aligned}\right\} \quad . \quad . \quad . \quad (1.10)$$

If we use units for which E and H are of the same dimensions (as a matter of convenience), the equations are obviously :

$$\left.\begin{aligned} \frac{dM_z}{dt} &= eZ - \frac{e\beta}{mc} M_x, \\ \frac{dM_x}{dt} &= \frac{e\beta}{mc} M_x, \\ m\frac{dmc}{dt} &= ZeM_z, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1.11)$$

the last being the activity equation.

§ 2. We have

$$M_z = \frac{mc}{e\beta} \cdot \frac{dM_x}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

and $$\frac{dM_z}{dt} = \frac{d}{dt}\frac{mc}{e\beta}\frac{dM_x}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

Substituting in (1·9) for $\frac{dM_z}{dt}$, we get

$$\frac{d}{dt}\frac{mc}{e\beta}\frac{dM_x}{dt} = eZ - \frac{c\beta}{me} M_x. \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

Multiplying both sides by $\frac{mc}{e\beta}\frac{dM_x}{dt}$, and denoting this quantity by U, we get

$$U\frac{dU}{dt} = \frac{Zmc}{\beta}\frac{dM_x}{dt} - M_x\frac{dM_x}{dt}, \quad . \quad . \quad (2.4)$$

$$\frac{1}{2}\frac{dU^2}{dt} = \frac{Zme}{\beta}\frac{dM_x}{dt} - \frac{1}{2}\frac{dM_x}{dt}, \quad . \quad . \quad (2.5)$$

or $$\frac{1}{2}\frac{d}{dt}(U^2 + M_x^2) = mc\frac{dM_x}{dt}, \text{ since } Z = \beta. \quad . \quad . \quad (2.6)$$

If we could consider m to be constant, we could get an approximate integral by integrating through with respect to t, but we can proceed more accurately as follows :—

The quantity

$$M_x,\ M_y,\ M_z,\ -mc \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.7)$$

is a four-vector, and the invariant

$$M_x^2 + M_y^2 + M_z^2 - m^2c^2 = -m^2 . c^2 \quad . \quad . \quad . \quad (2.8)$$

in the special case of relativity. In this case, $M_y = 0$,

$$M_x^2 + M_z^2 - m^2c^2 = -m^2 . c^2, \quad . \quad . \quad . \quad . \quad (2.9)$$

or differentiating,

$$\frac{d}{dt}(M_x^2+M_z^2)=\frac{d}{dt}m^2c^2 \quad . \quad . \quad . \quad . \quad (2.10)$$

Now the energy equation derived from 1 and 2 gives

$$\frac{d}{dt}\left(\frac{M_x^2+M_z^2}{2}\right)=eZM_z. \quad . \quad . \quad . \quad . \quad (2.11)$$

Now

$$m\frac{dM_x}{dt}=\frac{dmM_x}{dt}-M_x\frac{dm}{dt} \quad . \quad (2.12)$$

and

$$M_x\frac{dm}{dt}=\frac{M_x}{m}\frac{d}{dt}\frac{m^2}{2}=\frac{M_x}{m}\frac{d}{dt}(M_x^2+M_z^2), \quad (2.13)$$

$$=\frac{M_x}{m}\frac{eZ}{c^2}M_z. \quad . \quad . \quad (2.14)$$

Substituting for M_z from (2.1),

$$M_x\frac{dm}{dt}=\frac{M_x}{m}\cdot\frac{eZ}{c^2}\frac{mc}{c\beta}\frac{dM_x}{dt}=\frac{M_x}{c}\frac{dM_x}{dt}; \quad (2.15)$$

so

$$cm\frac{dM_x}{dt}=\frac{d}{dt}\left(cmM_x-\frac{M_x^2}{2}\right). \quad . \quad . \quad . \quad . \quad (2.16)$$

We have therefore expressed the right-hand side of (a) as a perfect differential with respect to t, which we may write now as

$$\tfrac{1}{2}U^2=cmM_x-M_x^2+c, \quad . \quad . \quad . \quad . \quad (2.17)$$

where c is a constant.

A result which is useful is contained in equation (2.15), *i. e.*:

$$M_x\frac{dm}{dt}=\frac{M_x}{c}\frac{dM_x}{dt} \quad . \quad . \quad . \quad . \quad . \quad (2.18)$$

or

$$m=m_0+\frac{M_x}{c}. \quad . \quad . \quad . \quad . \quad . \quad (2.20)$$

It is to be remarked that U in our equation (2.4),

$$U=\frac{mc}{e\beta}\frac{dM_x}{dt}=M_z, \quad . \quad . \quad . \quad . \quad (2.21)$$

so that 2·4 may be re-written:

$$\tfrac{1}{2}M_z^2+M_x^2=cmM_x+c. \quad . \quad . \quad . \quad (2.22)$$

Now M_x and M_z are zero together (initially), so that the arbitrary constant c must be zero.

We have

$$\frac{1}{2}\left(\frac{mc}{e\beta}\cdot\frac{dM_x}{dt}\right)^2 = cmM_x - M_x^2 \quad . \quad . \quad . \quad . \quad (2.23)$$

or

$$\left(\frac{dM_x}{dt}\right)^2 = \frac{2e^2\beta^2}{c^2}\left\{\frac{cM_x}{m} - \frac{M_x^2}{m^2}\right\} \quad . \quad . \quad (2.24)$$

Now

$$\frac{M_x}{m} = v_x = \frac{M_x}{m_0 + \frac{M_x}{c}};$$

so that

$$M_x = \frac{m_0 v_x}{(1 - v_{x/c})} \text{ and } m = \frac{m_0}{1 - v_{x/c}}. \quad . \quad . \quad (2.25)$$

We therefore get

$$\frac{d}{dt}\frac{m_0 v_x}{1 - v_{x/c}} = \frac{\sqrt{2}e\beta}{c}\{cv_x - v_x^2\}^{1/2}, \quad (2.26)$$

$$\frac{1}{\{cv_x - v_x^2\}^{1/2}}\frac{d}{dt}\frac{m_0 v_x}{(1 - v_{x/c})} = \frac{\sqrt{2}e\beta}{c}. \quad . \quad . \quad . \quad . \quad . \quad (2.27)$$

Now

$$\beta = \beta_0 e^{ip(t - x/c)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.28)$$

and

$$\frac{d\beta}{dt} = \beta_0 e^{ip(t - x/c)} ip(1 - v_{x/c})$$

since x is a function of t; . (2.29)

so that

$$\frac{1}{(v_{x/c})^{\frac{1}{2}}(1 - v_{x/c})^{\frac{1}{2}}}\frac{d}{dt}\frac{v_{x/c}}{(1 - v_{x/c})} = \frac{\sqrt{2}e\beta}{cip(1 - v_{x/c})}\frac{d\beta}{dt}, \quad (2.30)$$

or

$$\frac{(1 - v_{x/c})^{\frac{1}{2}}}{(v_{x/c})^{\frac{1}{2}}}\frac{d}{dt}\frac{v_{x/c}}{(1 - v_{x/c})} = \frac{\sqrt{2}e\beta}{m_0 cip}\cdot\frac{d\beta}{dt}; \quad . \quad . \quad (2.31)$$

and integrating,

$$\left(\frac{v_{x/c}}{1 - v_{x/c}}\right)^{\frac{1}{2}} = \frac{e\beta}{\sqrt{2}m_0 cip}\beta. \quad . \quad . \quad (2.32)$$

Taking real parts,

$$\left.\begin{aligned}\left(\frac{v_{x/c}}{1 - v_{x/c}}\right)^{\frac{1}{2}} &= \frac{e\beta}{\sqrt{2}m_0 cp}\cdot\sin\left(p(t - x/c)\right),\\ \left(\frac{v_{x/c}}{1 - v_{x/c}}\right) &= \frac{e^2\beta^2}{2m_0^2c^2p^2}\sin^2 p(t - x/c).\end{aligned}\right\} \quad . \quad . \quad (2.33)$$

§(3) In this process we have made no mention of the type of wave-train involved. In any case we must suppose the electromagnetic disturbance to be zero until $t=0$ (at $x=0$).

The subsequent history of the wave-train, *i. e.* whether finite or extending to infinity, can only be conjectured.

If it is to be considered as a quantum of radiation, no doubt it is finite. The actual circumstances of the motion of the electron depend to a certain extent upon our choice of the type of wave-train, in particular upon whether the wave-front is a surface of discontinuity or not. At the outset let us take the simplest case, and suppose that the disturbance at $x=0$ is zero for negative values of $t-x/c$ and $\mathrm{R}.\beta_0 e^{ip(t-x/c)}$ for positive values of $t-x/c$.

Sommerfeld has shown that in this case the integral

$$\beta = \frac{\mathrm{R}}{2\pi}\beta_0\int_{-\infty}^{+\infty}\frac{e^{ip(t-x/c)}}{(p-p_1)}dp \quad . \quad . \quad . \quad . \quad (3.1)$$

represents the complete disturbance if the integral is taken along the real axis, except in the neighbourhood of p_2p_1, this point being encircled by a semicircle in the 4th quadrant. With this form of wave-train, Z and β are continuous at the wave-front, although $\frac{\partial\beta}{\partial x}$ and $\frac{\partial \mathrm{Z}}{\partial x}$ are not.

The continuity is associated with the fact that the Fourier integral is convergent at $t=x/c$.

The equation (2.32) still holds, even if we use this form, for

$$\beta = \frac{\mathrm{R}}{2\pi}\beta_0\int_{-\infty}^{+\infty}\frac{e^{ip(t-x/c)}}{p-p_1}dp \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

and

$$\frac{d\beta}{dt} = \frac{\mathrm{R}}{2\pi}\beta_0(1 \quad v_{x/c})\int\frac{ip\,dp\,e^{ip(t-x/c)}}{p-p_1}, \quad t-\frac{x}{c}\neq 0, (3.3)$$

and

$$= (1-v_{x/c})\frac{\mathrm{R}}{2\pi}\beta ip, \quad t-\frac{x}{c}\neq 0,$$

so long as

$$t-\frac{x}{c}\neq 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

As before,

$$\frac{v_{x/c}}{1-v_{x/c}} = \frac{e^2\beta^2}{2m_0{}^2c^2p^2}\sin^2(p)(t-x/c). \quad . \quad . \quad (3.5)$$

The more general case where Z and β are discontinuous at the wave-front can be formed by the integral:

$$\beta = \frac{\mathrm{R}}{2\pi}\beta_0\int\frac{e^{i\phi}\,e^{ip(t-x/c)}\,p\,dp}{p-p_1}, \quad . \quad . \quad . \quad . \quad (3.6)$$

which is not convergent at $t-x/c=0$.

This quantity is zero if $t-x/c<0$, and is equal to

$$-\mathrm{R}\beta_0 i\, e^{i\phi} e^{ip(t-x/c)},$$

$$i.\,e.,\quad \beta_0 \sin p(t-x/c+\phi),$$

when $t>x/c$.

Equation (3.5) is now modified by the addition of a constant:

$$\left(\frac{v_{x/c}}{1-v_{x/c}}\right)^{\frac{1}{2}}=\frac{e\beta_0}{\sqrt{2mcp}}\cdot\sin p(t-x/c+\phi)-\mathrm{C}, \quad (3.8)$$

and the constant C must equal $\sin\phi$, so that $v_x=0$ when $t=0$, as required.

Thus

$$\left(\frac{v_{x/c}}{1-v_{x/c}}\right)=\frac{e^2\beta_0{}^2}{2m_0{}^2c^2p^2}\left\{\sin\left(p(t-x/c)+\phi\right)-\sin\phi\right\}^2 \quad (3.9)$$

when the wave-front is discontinuous.

Reverting to our former expression, we can get the complete integral of the motion; *i. e.* the relation between x and t is as follows:—

If we put

$$p(t-x/c) = y, \quad \ldots \ldots \quad (3.10)$$

then

$$\frac{dy}{dt}=p(1-v_{x/c})$$

and

$$\frac{v_{x/c}}{1-v_{x/c}}=\frac{e^2\beta_0{}^2}{2m_0{}^2c^2p^2}\sin^2 y=\zeta^2\sin^2 y,\ \text{say}, \quad (3.11)$$

or

$$pv_{x/c}=\zeta^2\sin^2 y\frac{dy}{dt}, \quad \ldots \ldots \quad (3.12)$$

$$\frac{p}{c}\frac{dx}{dt}=\frac{\zeta^2}{2}\frac{d}{dt}\left(y-\frac{\sin 2y}{2}\right); \quad \ldots \ldots \quad (3.13)$$

and integrating,

$$\frac{px}{c}=\frac{\zeta^2}{2}\left(y-\frac{\sin 2y}{2}\right), \quad \ldots \ldots \quad (3.14)$$

$$\frac{x}{c}=\frac{\zeta^2}{2c}\left(t-\frac{x}{c}\right)-\frac{\zeta^2}{4p}\sin 2p\left(t-\frac{x}{c}\right). \quad (3.15)$$

No constant of integration is needed, since the condition $x=0$ when $t=0$ is satisfied identically by this relation. We thus have the required relation between x and t.

The corresponding values of v_z and Z may be obtained from (3), *i. e.*:

$$M_z = \frac{me}{e\beta}\frac{dM_x}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.16)$$

$$= \frac{me}{e\beta}\frac{\sqrt{2}e\beta}{c}\{cv_x - v_x^2\}^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.17)$$

or

$$v_z = \sqrt{2}c\left(\frac{v_x}{c}\right)^{\frac{1}{2}}\left\{1-\frac{v_x}{c}\right\}^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.18)$$

$$v_z = \sqrt{2}(1-v_{x/c})\frac{(v_{x/c})^{\frac{1}{2}}}{(1-v_{x/c})^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.19)$$

Thus

$$v_z = c(1-v_x/c)\frac{e\beta_0}{m_0cp}.\sin(p)(t-x/c)$$

$$= \frac{e\beta_0}{m_0p^2}\sin p(t-x/c)\frac{d}{dt}(t-x/c) \quad . \quad . \quad . \quad (3.20)$$

and

$$z = -\frac{e\beta_0}{m_0p^2}\{\cos p(t-x/c)-1\}$$

$$= -c\frac{\zeta}{p}\sqrt{2}\{\cos p(t-x/c)-1\}, \quad . \quad . \quad . \quad (3.21)$$

since $z=0$ when $t=0$.

§ (4) This forms the conclusion of the calculation of the steady state on the classical theory.

It will be observed that on this basis there is a definite drift of the electron with an average velocity $\frac{1}{4}\frac{e^2\beta_0^2}{m_0^2p^2c^2}$ ($v_{x/c}$ small) in the direction of propagation.

Combined with the periodic v_z motion this gives a scattered wave of a frequency which differs from that of the existing source by the amount $\delta\nu$, where $\frac{\delta\nu}{\nu}=\frac{v_x}{c}(1-\cos\theta)$, the scattering angle θ being the angle made with the positive normal to the wave-front.

In respect of the factor $1-\cos\theta$, the result agrees with that determined experimentally by Compton. The value, however, is a function of the intensity β of the incoming wave. There should therefore be no fixed value of the change of frequency unless there is a restriction on the value of β, occasioned, for instance, by the quantum structure of the incident radiation.

It might be thought sufficient at first sight to postulate the energy of the incident radiation as $h\nu$, where ν is the frequency; but this determines only the total energy, and not the value of β, which determines the energy per unit volume. The procedure which naturally suggests itself is to treat the momentum M_z—which is periodic—according to the rules of the quantum theory, *i. e.* to suppose that the periodic motion of the electron is one of the specially determined steady quantum states.

We should then put

$$\oint M_z\,dz = nh,$$

and the process is more natural in that $\oint M_z\,dz$ expressed as a function of z turns out to be exactly an inverse periodic function, associated with which is the modulus of periodicity obtained by the integration round the two branch points of M_z.

Thus

$$\oint M_z\,dz = \oint \frac{mc}{e\beta}\cdot\frac{dM_z}{dt}\,dz \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

$$= \oint \frac{mc}{e\beta}\,\frac{\sqrt{2}e\beta}{c}(cv_x)^{\frac{1}{2}}(1-v_{x/c})^{\frac{1}{2}}. \quad . \quad (4.2)$$

Now

$$m = \frac{m_0}{1-v_{x/c}} \quad \text{by} \quad (2.25),$$

$$\oint M_z\,dz = \oint \frac{m_0\sqrt{2}c(v_{x/c})^{\frac{1}{2}}}{(1-v_{x/c})^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

$$= \oint \frac{e\beta}{pc}.\sin(p(t-x/c))\,dz. \quad . \quad . \quad (4.4)$$

Now, by (3.21),

$$\cos p(t-x/c) = \frac{zp}{c\zeta\sqrt{2}}+1 \qquad = z',\ \text{say}, \quad . \quad . \quad . \quad (4.5)$$

$$\sin p(t-x/c) = \sqrt{1-z'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

$$dz = c\zeta\sqrt{2}\,dz',$$

$$\oint M_z\,dz = \oint \frac{m_0 c\zeta\sqrt{2}}{p}\sqrt{1-z'^2}\,dz' c\zeta\sqrt{2} \quad (4.7)$$

$$= \frac{2m_0c^2\zeta^2}{p}\oint\sqrt{1-z'^2}\,dz$$

$$= \frac{2m_0c^2\zeta^2\pi}{p} = n_1h, \quad . \quad . \quad . \quad . \quad . \quad (4.8)$$

$$\zeta^2 = \frac{n_1h\nu}{m_0c^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.9)$$

for $n=1$, *i. e.* the 1-quantum state

$$\zeta^2 = \frac{h\nu}{m_0 c^2}. \quad \ldots \ldots \quad (4.10)$$

A guarantee of the general reasonableness of our choice in the method of quantizing the orbit of the electron is contained in the following relation :—

The total kinetic energy of the electron can be expressed, in accordance with the relativity theory, in the form :

$$(m-m_0)c^2 = \text{KE}. \quad \ldots \ldots \quad (4.11)$$

Now

$$m = m_0 \frac{1}{1-v_{x/c}}. \quad \ldots \ldots \quad (4.12)$$

$$\therefore \quad \text{KE} = m_0 c^2 \left(\frac{v_{x/c}}{1-v_{x/c}}\right) \quad \ldots \ldots \quad (4.13)$$

$$= m_0 c^2 \zeta^2 \sin^2 y$$

$$\text{KE}_{\text{max.}} = h\nu.$$

Now the energy equation gives

$$c^2 m \frac{dm}{dt} = m v_z \text{Z}e, \quad \ldots \ldots \quad (4.14)$$

$$c^2 \frac{dm}{dt} = v_z \text{Z}e, \quad \ldots \ldots \quad (4.15)$$

$$\frac{d\text{KE}}{dt} = \text{Z}e \frac{dz}{dt}$$

$$\text{or} \quad \frac{d\text{T}+\text{W}}{dt} = 0 \text{ and } \text{T}+\text{W} = h\nu, \quad . \quad (4.16)$$

where

$$\text{T} = \text{KE} + \text{W} = -\int \text{Z}e \frac{dz}{dt}\, dt, \quad \ldots \quad (4.17)$$

which is identically satisfied on using the expressions for Z and $\frac{dz}{dt}$ we have found in eq. (3·21)

The orbital energy is $h\nu$. This is in accordance with the ordinary notion of the quantum theory, since the initial state of the electron was one of zero energy level, and $h\nu$ is here given as the difference in energy level of the zero and 1-quantum state.

§ (5) This solution just obtained I would term "The Selected Semi-Classical Solution," to distinguish it from Compton's purely quantum solution. It corresponds exactly to the solutions of the orbital motions of electrons under central forces in the atom.

I call it "selected," as quantum considerations are used in fixing the amplitude of the motions ; it is semi-classical because in the equations of motion no account is taken of re-radiation or scattering, which, according to quantum principles, must be considered to be a separate process involving a change from one quantum state to another: this change, involving re-radiation, may be considered to take place when the train of waves has passed over the electron. We are therefore led to consider a second quantum state, and in accordance with the idea presented here, it should, if possible, be one which satisfies the field equations, the passage from the first to the second state being, of course, accompanied by a scattered radiation. The procedure which is perhaps least open to criticism from the quantum point of view, and for which there is strong experimental evidence, is to assume that, at least formally, this final state is the one calculated by Compton. The conservation of energy and momentum are used to calculate the path and velocity of the electron and the direction, intensity, and frequency of the scattered quantum, which is supposed to be a directed quantity.

This process is consistent on the average with the field equations.

The equations which fix this state are :

$$\left.\begin{aligned} \frac{h\nu}{c} &= \frac{h\nu_x'}{c} + \mathrm{M}_x, \\ 0 &= h\nu_\rho' - \mathrm{M}_\rho, \\ \nu_x'^2 + \nu_\rho'^2 &= \nu'^2. \end{aligned}\right\} \text{ Momentum Equations, } \quad (5.1)$$

$$h\nu - h\nu' = (m - m_0)c^2, \quad \text{Energy Equation,} \quad (5.2)$$

$$\mathrm{M}_x^2 + \mathrm{M}_y^2 = (m^2 - m_0^2)c^2. \quad (5.3)$$

These equations give rise to the following frequency relation:

$$\frac{\delta\nu}{\nu} = \frac{\chi^2(1-\cos\theta')}{(1-\chi^2\cos\theta')}, \text{ where } \chi^2 = \frac{\frac{h\nu}{m_0c^2}}{1+\frac{h\nu}{m_0c^2}}, \quad (5{\cdot}4)$$

which is approximately

$$\frac{\delta\nu}{\nu} = \frac{h\nu}{m_0c^2}(1-\cos\theta'), \quad (5.5)$$

Correspondence.

This method exhibits the change of frequency on scattering as a purely quantum process, involving the idea of frequency as a determinant of the energy according to the relation $W=h\nu$; but it is the purpose of this paper to exhibit this change of frequency as a Doppler effect, in order to bridge over the gap between the Compton theory on one side and the classical theory on the other side, which undoubtedly applies to long-wave phenomena, and to use the idea of selected statical states for this purpose.

In this manner the solution (1) may be considered as a detailed expression of an intermediate state in the Compton analysis, introduced so as to bring this more in line with the usual methods of quantum analysis and to help to bridge over the gap just referred to.

In order to bring in the classical idea of Doppler effect, the following method may be used:—As we have seen, the process of re-radiation must occur in the jump from the first to the second quantum state. Since there is no re-radiation during the passage of the train of waves over the electron, the velocity of drift occasioned does not determine the Doppler change of frequency of the scattered radiation; but this latter must be determined by velocity after the change. By a suitable choice of the conditions at the end of the train we can continue the solution without discontinuity, so as to give an x velocity from which the Doppler effect corresponding to the Compton effect may be deduced.

Thus the x velocity determined from solution (1) is:

$$\frac{v_x}{c}=\frac{\frac{h\nu}{m_0c^2}\sin^2 p\left(t-\frac{x}{c}\right)}{\left\{1+\frac{h\nu}{m_0c^2}\sin^2 p\left(t+\frac{x}{c}\right)\right\}}. \quad . \quad . \quad . \quad (5.6)$$

If the train is chosen to end when

$$\sin p\left(t-\frac{x}{c}\right)=1, \quad . \quad . \quad . \quad . \quad (5.7)$$

the value of $\frac{v_x}{c}$ will be

$$\frac{v_x}{c}=\frac{\frac{h\nu}{m_0c^2}}{1+\frac{h\nu}{m_0c^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.8)$$

and the corresponding Doppler effect will be

$$\frac{\delta\nu}{\nu} = \frac{v_x}{c}(1-\cos\theta) \doteq \frac{h\nu}{m_0c^2}(1-\cos\theta), \quad . \quad . \quad (5.9)$$

which corresponds with the Compton change of frequency.

Or we may arrive at the Doppler effect idea as follows:—

In developing the relation (5.5) from the equations (5.1), (5.2), (5.3) according to Compton's method, we get the relation

$$\mathrm{M}_x = \frac{h\nu}{c}(1-\cos\theta) \text{ approx.}, \quad . \quad . \quad (5.10)$$

or

$$\frac{v_x}{c} = \frac{h\nu}{mc^2}(1-\cos\theta),$$

and the corresponding Doppler effect would be

$$\frac{v_x}{c} = \frac{h\nu}{mc^2}(1-\cos\theta), \quad . \quad . \quad . \quad . \quad (5.11)$$

which is the relation already obtained.

Thus the change of frequency on scattering may either be considered as an effect due to the energy removed from the incident wave by the recoil electron, *i.e.* as a purely quantum energy change, or as a Doppler effect associated with the velocity v_x of recoil.

These two aspects enable one to pass over from the quantum aspect as developed by Compton to the classical aspect, which follows as a result of the application of the ordinary field laws, by the process of building up a statistical state from the individual quantum process, the relative probability of emission in various directions being determined by the field equations. The details of such a process must be left to a future paper.

The results obtained by the method of this paper are formally just the same as those obtained by Compton ; but the method outlined here seems to bring relation between the field equation and the quantum processes into a clearer light, and to pave the way to the understanding of the relation between the individual quantum processes on the one hand and the electromagnetic state as calculated from the field laws which apply on the longer waves on the other. The exhibiting of the quantum process as a selected classical solution, and the identification of the quantum and Doppler aspect of the change of frequency, ensures that a proper choice of the statistical aggregate of quantum processes must conform to the ordinary field equations.

It should be noted that the Einstein conception of a quantum of radiation having a definite direction of travel in space is used throughout, and it seems to me that this is a logical necessity if quantum and relativity notions are to be reconciled.

For suppose the quantum were a spherical type wave, such as, for instance, is radiated from a source at O vibrating in a vertical direction, if the frequency were ν, the corresponding energy would be $h\nu$.

Now refer this system to a set of axes moving relatively to source with the velocity v: the various parts of the spherically-radiated quantum will have different frequencies, and the relation $W = h\nu$ will no longer have any meaning.

I have been guided throughout in fixing the quantum rules by the consideration that any quantity which can be defined in the classical theory—for instance, $\delta\nu/\nu = v/c$—should not clash with the quantum definition of $\delta\nu$, *i. e.* the same quantity. In this manner we have been led to a *definite* value of the change of frequency in scattering which has been observed.

It must be admitted, of course, that the work has been carried out with half an eye on the expected result; but the real value lies in the fact that the quantum conditions must be introduced, and these are naturally and inevitably introduced through the only really periodic quantity v_z. In using this we find the orbited energy to be in accordance with the known quantum principles.

We can picture the process outlined here as the arrival of a quantum $h\nu$ of radiation at the stationary electron, an absorption of this energy by the electron, and a consequent transference of this to the 1-quantum state with energy $h\nu$ (fixing the amplitude of oscillation M_z) ; finally, a transition from the 1-quantum state to the 2-quantum state, with an emission of scattered radiation with frequency and energy determined by the difference of energy level in the 1 and 2 states, *i. e.*,

$$h\nu' = h\nu - \frac{mv_x^2}{2} = W_1 - W_2,$$

each steady state being rigidly defined by the classical theory (of course neglecting the effect of re-radiation). It is then found that the change in frequency on scattering may either be regarded as a change of energy content of the radiation, or as a Doppler effect associated with the recoil velocity v_x, at least to the first approximation.

§ 6. The further consequences of this theory have a bearing on the structure of radiation.

This is rather to be expected, as this example illustrates perhaps the simplest case of interaction between the quantum and the electron.

The quantity ζ^2 has two interpretations. From the quantum point of view it is the ratio of the quantum of energy $h\nu$ to the potential energy of the electron m_0c^2. From the classical point of view it involves β^2, the energy density reckoned in the ordinary manner, thus:

$$\zeta^2 = \frac{h\nu}{m_0c^2} = \frac{e^2\beta^2}{m_0{}^2p^2c^2} \text{ (energy/vol.} = \beta^2 \text{ in our units)},$$

so that the energy density in the incident radiation is determined by the relation:

$$\beta^2 = \frac{m_0p^2h\nu}{e^2} = \frac{4\pi^2m_0h\nu^3}{e^2}. \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

Now we know the total energy in the incident radiation, i.e. $h\nu$; so that if β^2 is supposed to be uniformly distributed in space, the ratio $\frac{h\nu}{\beta^2} = \frac{e^2}{4\pi^2\nu^2m_0}$ should be the volume in which the quantum is contained, and in calculating this we should get an insight into the structure of radiation.

Now the quantity $\frac{e^2}{m_0c^2}$ has the dimensions of a length. In the usual electromagnetic theory it expresses the linear dimensions of the electron, according to the relation:

$$m_0 = \frac{e^2}{6\pi ac^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.2)$$

$$\therefore \quad \text{Vol.} = \frac{e^2}{4\pi^2\nu^2\frac{e^2}{6\pi ac^2}} = \frac{3ac^2}{2\pi\lambda^2}, \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

or

$$\text{Vol.} = \frac{3a\lambda^2}{2\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

where "a" is the radius of the electron.

Also we may put it in the form:

$$\text{Vol.} = \frac{4\pi a^3}{3}\frac{1}{2}\left(\frac{3}{2}\frac{\lambda}{2\pi a}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.5)$$

so that the volume tends to that of an electron when $\lambda = \sqrt{2}\frac{2\pi a}{3}$; *i. e.*, the quantum apparently shrinks to electronic dimensions when the wave-length is of the order of the electronic radius.

A quantum no doubt consists of a finite train of oscillations. We may perhaps be allowed to consider the electron as remaining in the 1-quantum state during the time that this quantum train passes over the electron. Now by combining classical principles with Einstein's probability coefficients—*i. e.* $A_{1\to 2}$, $B_{2\to 1}$—E. A. Milne* has arrived at a formula which expresses the mean duration of a given quantum state. It is

$$\tau = \frac{c}{4\pi^2 a \nu^2}, \quad \ldots \ldots \quad (6.6)$$

an expression which, as has been pointed out, does not involve h and is therefore independent of the atomic structure, and should be applicable to the type of electronic vibration envisaged in this case.

In the present analysis exactly the same result can be obtained by assuming that on the average the re-radiation is what it would have been on the classical theory; thus if τ is the average duration of the quantum state 1, $\frac{h\nu}{\tau}$ will be equal to the rate of radiation calculated on the classical lines. Thus

$$\frac{h\nu}{\tau} = \frac{1}{2}\frac{e^2}{6\pi}\frac{\Gamma^2}{c^3}, \quad \ldots \ldots \quad (6.7)$$

where Γ is the acceleration.

$$\Gamma = \frac{dv_z}{dt} = \frac{d}{dt}\frac{e\beta}{mp}\sin pt \text{ approx.}$$

$$= \frac{e\beta}{m}\cos pt. \quad \ldots \ldots \quad (6.8)$$

The mean rate of radiation of energy is therefore

$$\frac{1}{2}\frac{e^2 p^2}{6\pi c^3}\left(\frac{e\beta}{mp}\right)^2,$$

and is independent of p.

* Monthly Notices R. A. S. p. 743, No. 8, June 1925.

$$\frac{h\nu}{\tau}=\frac{1}{2}\frac{e^2p^2}{6\pi c}\left(\frac{\beta e}{m_0pc}\right)^2=\frac{e^2p^2h\nu}{6\pi m_0c^3}, \quad . \quad . \quad . \quad (6.9)$$

so that

$$\frac{1}{\tau}=\frac{e^2p^2}{6\pi m_0c^3};$$

and since $m_0=\frac{e^2}{6\pi ac^2}$,

$$\frac{1}{\tau}=\frac{4\pi^2\nu^2a}{c} \text{ as before.} \quad . \quad . \quad . \quad (6.10)$$

But if we are to retain the same mental picture of the process as before, $c\tau$ must represent the length of a train of oscillations.

One curious consequence of this formula is the following:—

The number of oscillations in the train is $\nu\tau$, or $\frac{c}{4\pi^2a\nu}=\frac{\lambda}{4\pi^2a}$. The number of oscillations in a train tends to unity as $\lambda\rightarrow4\pi^2a$. This is in agreement with the fact that the quantum appears to shrink up to the size of an electron for this frequency. The quantum appears to lose its periodic nature at these high frequencies. What the meaning of $h\nu$ in the formula $W=h\nu$ is, very obscure! This result is perhaps only apparent and due to some approximation introduced into the last stage of the calculation, and is only true if $c\tau$ represents the actual length of the train of oscillations.

On this basis, the length of the train of oscillations is $c\tau$. We may now express the volume V in the form :

$$V=\frac{3ac^2}{2\pi\nu^2}=\frac{6a^2c^2}{4\pi^2\nu^2a}=6a^2c\tau. \quad . \quad . \quad . \quad (6.11)$$

From this point of view, the quantum appears to be contained within a cylinder or tube of radius $\sqrt{\frac{6a}{\pi}}$, or practically the electronic radius and length of $c\tau$. It is perhaps as well to note that this volume is not a fixed quantity; it seems to adjust itself to the radius of the particle acted upon by the quantum. For instance, if the electrified particle had been a proton, the equivalent value of "a" in formula would have been proportionally smaller, and the quantum would have to adjust itself to this smaller radius. These results inevitably suggest the tube of force attached to the electron or proton, the quantum being a disturbance propagated along this tube, as a state of distortion say.

The tube ends in its appropriate electron or proton, and its approximate diameter is correctly given if some latitude is allowed in fixing the structure of the electron or proton.

The motion of the tube must be fixed by the electromagnetic laws. It is well known, for instance, that the second set of circuital equations can be deduced from purely kinematic considerations of the movement of discrete tubes of force, if the magnetic force is interpreted as the velocity of the tube at right angles to itself.

The whole would form a very compelling picture, were it not that the interference effects are still wholly unexplained, and that the nature of the jump from one steady state to another is also physically difficult to picture.

If we take the view that it is the function of theory to provide an accurate mathematical description of the physical facts, the relativity quantum theory outlined here is singularly adequate—not that it is complete, since the framing of the quantum rules has certainly not been achieved yet.

A theory must pass a very strict test nowadays; it must not only be accurate, it must also be a convenient and powerful instrument of thought. It is a commonplace reflexion, for instance, that certain problems are much more easily solved in terms of certain coordinates than others, and that they may take on an aspect of artificial complexity if inconvenient coordinates are chosen. I have been struck throughout the foregoing analysis with the ease with which the solution unfolded itself; and although it would be foolish to take this as a guarantee of correctness, it certainly seems to indicate that it is a simple mode of expression of the realities involved in one of the simplest of quantum processes.

It will be seen that this scheme really differs very little from Compton's theory. The quantum of energy as portrayed by this theory is very similar to the material particle envisaged by Compton in his theory (for frequencies comparable with $\frac{m_0c^2}{h}$), the contraction in volume being a striking example of this similarity. Compton's theory short-circuits the analytical details of this by using the principles of conservation of energy and momentum which are, of course, implicit in the present classical quantum theory. The present method may be perhaps looked upon as strengthening the assumption upon which Compton's theory is based—*i. e.*, that the quantum acts as a particle which exchanges its energy (in accordance with the ordinary laws of momentum and energy exchange) in a collision.

It is perhaps significant, in view of recent speculations concerning the transformation of matter into radiation (for example, at intensely high temperatures in the interior of stars), that at these frequencies, *i.e.* of the order of $\frac{m_0 c^2}{h}$, the distinction between the material element, *i.e.* electrons and protons, and the quantum of radiation seems to disappear, for the energy and volume are of the same order; and if we accept the approximate formula for the number of oscillations in a quantum, a relation which, however, seems to me of very doubtful validity, the quantum loses its periodic structure at these frequencies, and there is but little left to distinguish it from a proton or electron, say.

It is hoped that this paper may pave the way to a fuller understanding of the relation between the quantum and classical theories of radiation.

Perhaps too much stress should not be laid on the physical picture of the volume of a quantum as portrayed here; but if we are to retain our present ideas of time and space as extended by Einstein, we must regard this quantity at least as dimensionally a volume.

XXI. *Note on the Molecular Association of Benzoic Acid in Benzene.* By J. A. WILCKEN, *B.Sc., Ph.D., Lecturer in Electrical Engineering at Armstrong College, Newcastle-upon-Tyne* *.

THE question as to the connexion between dielectric and other physical and chemical properties of matter, which has an important bearing on problems of cable insulation, has led to the investigation into the molecular nature of solutions, and as a type, benzoic acid in benzene has been examined.

Auwers † and his co-workers, and Beckmann ‡ have dealt with a large number of solutions of this class, but whereas the former attributes the abnormal behaviour (freezing-point depression) to mutual influence of solute and solvent, the latter inclines to the view that complex molecules are formed. Turner § leaves the question open.

This part of the work had been temporarily discontinued, but Prof. Partington's and Mr. Rule's valuable contribution to the subject in the May issue of this Journal suggested that the preliminary results here given might be of some interest.

* Communicated by the Author.

† *Zeitschr. Physikal. Chemie*, xii. pp. 689–722 (1893); *ibid.* xxx. pp. 300–340 (1899).

‡ *Ibid.* xxii. pp. 609–618 (1897).

§ 'Molecular Association' (Longmans).