LIE GROUPS: HISTORY, FRONTIERS AND APPLICATIONS

EDITORS: NOLAN WALLACH & ROBERT HERMANN

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LIE GROUPS: HISTORY, FRONTIERS AND APPLICATIONS

VOLUME II

RICCI AND LEVI-CIVITA'S TENSOR ANALYSIS PAPER

Translation, Comments and
Additional Material by
ROBERT HERMANN

MATH SCI PRESS
53 Jordan Road
B ookline, assachusetts 02146 (USA)

Contains a translation of
"Methodes de Calcul Différentiel
Absolu et Leurs Application,"
by M. M. G. Ricci and T. Levi-Civita,
Mathematishe Annalen,
Vol. 54 (1900)

ISBN 0-915692-11-2
Library of Congress Catalog Card
Number 75-26206

MATH SCI PRESS
53 Jordan Road
Brookline, Mass. 02146 (USA)

PRINTED IN THE UNITED STATES OF AMERICA

TRANSLATOR'S PREFACE

This is a translation of "Methodes de calcul differential absolu et leurs applications," by G. Ricci and T. Levi-Civita, Mathematische Annalen, vol. 54 (1900). This memoire is one of the most influential and important in the history of *both* differential geometry and mathematical physics. For example, it seems to have been the basic document from which Einstein learned the tensor analysis that he used in the creation of General Relativity.

My immediate aim is, of course, to make this key paper - which is still very readable and informative - available to the scientific community. To enhance its usefulness, I have added Remarks which link the material with contemporary differential geometry and physics. I have also modernized the notations and terminology, e.g. using the summation convention, and substituting the term "Tensor Analysis" for "Absolute Differential Calculus." I have also added a few topics to the main text, e.g. the notion of "mixed tensor," which seemed useful.

This is the first of a series of translations and edited impressions of classic work in differential geometry, Lie group and differential equation theory of the late 19th and early 20th centuries. I have always found this work a great inspiration, and much of it very "modern" in spirit. What is particularly important is that this group of geometers (e.g. Lie, Cartan, Ricci, Levi-Civita) thought of mathematics in close relation to physics, and their work can serve as a model of the sort of synthesis and interaction between mathematics and the disciplines which use mathematics that I am trying to develop in this series.

I have changed the system used for references to conform to contemporary fashion, and to eliminate footnotes, which are cumbersom to print and to read. The references are referred to by author and date, e.g. Ricci [1886] refers to Ricci's article dated 1886, listed in the Bibliography at the end. I refer to my own books by abbreviations, given in the Bibliography. For example, DGCV refers to my book "Differential Geometry and the Calculus of Variations." (This book, together with "Geometry, Physics and Systems" and some of the other volumes in this series can serve as to furnish general background detail, and notation for my Remarks.) "Volume II" refers to the Interdisciplinary Mathematics series.

I again thank Mrs. Alta Zapf for her superb typing!

PREFACE

Poincaré has written that a good notation has the same philosophical importance in Mathematics as a good classification system has in the Natural Sciences. One may extend this remark, with even greater force, to cover methods, since they determine the possibility of grouping diverse facts which have no obvious interconnection according to certain natural relations.

One may also say that a theorem is only half-proved when it is proved by a tortuous route or by using artifices with no essential links with the material. Almost always the same theorem can be developed in a more complete and general manner, if one approaches it by a more direct route and with appropriate methods.

For example, let us cite the case of the proof given by Jacobi and extended by Beltrami of the invariance of the expression $\Delta(U)$ (which is now called the Laplace-Beltrami operator). It is certainly elegant, and testifies to the profundity of thought of its discoverers, but it is surprising that to prove the theorem - which involves the algebraic theory of elimination - they use the variation of an integral. The application of this methodological principle to this example has led to the development of methods that we call the Absolute Differential Calculus (see Ricci [1886, 1889]), and discovery of a chain of differential invariants generalizing the Laplace-Beltrami operator.

The algorithm that we call Absolute Differential Calculus, the subject of this work, may be found in a remark by Christoffel [1899] - but the methods are founded on the notion of "n-dimensional manifold" that we owe to the genus of Gauss and Riemann.

The metric properties of such manifolds are defined intrinsically by n independent variables and by an equivalance class of quadratic differential forms in these variables, with two such forms equivalent if one can be transformed into the other by a change of variables. As a consequence a manifold, denoted by M_n, remains invariant under all transformation of coordinates. The Absolute Differential Calculus, which acts on covariant and contravariant forms of M_n to derive others of the same nature, is also - in formulas and results - independent of changes of coordinates. Since it is essentially attached to M_n, (not to a choice of coordinates), it is a natural tool for all research which deals with such manifolds, or where one meets positive quadratic differential forms and their derivatives.

The brief exposition that we give here of these methods and their applications is intended to convince the reader of their advantages - that we think are great and self-evident - and to reduce as much as possible the effort that is required - as for any new technique - from those who want to apply it. We think that, after having surmounted the initial difficulties, one will readily find that the generality and independence of choice of coordinates leads not only to elegance, but also to agility and insight into proofs and conclusions.

Remark: *The first two paragraphs are a classic statement of the practical importance of the development of elegant and general tools. Here, the authors clearly set the tone for much of twentieth-century mathematics. Physicists and engineers who like to complain about mathematicians becoming too fancy and abstract for their taste should keep this in mind! Presumably, a point that Ricci and Levi-Civita would take for granted (since it is clear in all their work) is that this thrust towards elegant and general methods is to be tempered by a broad perspective and insight into the really important topics, not the least of which are those done in contact with science!*

From now on, I translate Absolute Differential Calculus by Tensor Analysis, although there may be subtle differences in what they meant and what we now mean by the terms.

Nowadays, the notion of "manifold" is of course different from that which Ricci and Levi-Civita ascribe to Gauss and Riemann. We now think of a manifold more concretely as a set of points with certain definite properties, or satisfying certain definite "axioms." However, their emphasis on the use of local coordinates, provided the concepts are developed in a way which is "manifestly invariant" of the choice of any one coordinate system, means that most of the concepts developed by the "local" Tensor Analysts carry over to manifolds as we now know them. Of course this more classical viewpoint also played a critically important role in Einstein's mind in his development of General Relativity. To this day most physicists and engineers find the older ideas more congenial than those of modern differentiable manifold theory.

It is interesting to note that they refer to positive quadratic differential forms (we now say: positive Riemannian metrics) in the fifth paragraph. In fact, everything works also in the more general case that the form is non-degenerat , and this enabled Einstein to apply their methods to gravitation. This is an even greater vindication for the methodological principle they state in the first two paragraphs than the example - now forgotten - that they cite.

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Chapter I

THE ALGORITHM OF TENSOR ANALYSIS

1. CHANGE OF VARIABLES AND SYSTEMS OF FUNCTIONS

Denote by T a general coordinate transformation

$$x^i = x^i(y^1, \ldots, y^n), \qquad (1.1)$$
$$1 \le i, j, k \le n,$$

which is invertible and regular in the domains we encounter. Further, let Ω be a system whose elements consist of functions

$$(f_1, \ldots, f_p),$$

where $f_1, \ldots, f_p$ are functions of the variables x, functions that we call *elements* of the system Ω. Denote also by S a substitution which acts on the system Ω by substituting for the elements $f_1, \ldots, f_p$ a set $g_1, \ldots, g_p$ of functions of the variables y.

Let us think of S as a function of T, i.e. suppose that, for each transformation 1.1 which acts on the independent variables one is given a well-defined substitution S. Further, assume that S, considered as a function of T, satisfies the following conditions:

1) If T = identity, then S = identity

2) If T, T_1, T_2 are three transformations of type 1.1, with S, S_1, S_2 the corresponding substitutions, and if one has

$$T = T_2 T_1,$$

then also:

$$S = S_2 S_1.$$

There are different ways to determine S as function of T. One may, for example, take as elements of the transformed system the function obtained by substituting y for the variables x, according to the given formulas (1.1). We will say in this case that the system transforms by invariance, or that it is invariant.

But often the nature of the given system may make us prefer another transformation law. For example, if $f_1, \ldots, f_n$ are the derivatives of a function f with respect to $x_1, \ldots, x_n$, one finds it natural to take as transformed system the derivatives $f_1', \ldots, f_n'$ of the function f' which is the transform of f by the transformation T, instead of the functions which one obtains from the $f_1, \ldots, f_n$ by applying the transformation T.

In this case, the substitution S will be defined by formulas:

$$f_i' = f_j \frac{\partial x^j}{\partial y^i}, \tag{1.2}$$

where the functions $f_1, \ldots, f_n$ ought to be expressed in terms of the variables y.

Remark: The ***summation convention*** *is used in (1.2), and will be used from now on. It was not known when the paper was written, but was invented by Einstein, and enormously extended the calculational usefulness and simplicity of the formalism.*

If the given system consists of a function f and its derivatives up to a given order, one may require that it transform in the same way. In this case the formulas which represent the transformation law of the system are, analytically, fairly complicated. The function transforms by invariance, its first derivatives by formula (1.2), and its second derivatives by the following formulas:

$$\frac{\partial^2 f}{\partial y^i \partial y^j} = \frac{\partial^2 f}{\partial x^k \partial x^\ell} \frac{\partial x^k}{\partial y^i} \frac{\partial x^\ell}{\partial y^j} + \frac{\partial f}{\partial x^k} \frac{\partial^2 x^k}{\partial y^i \partial y^j} \tag{1.3}$$

One also obtains important examples by considering the coefficients of an expression which is linear and homogeneous in the first derivatives of a function, such as

$$A^i \frac{\partial f}{\partial x^i}, \tag{1.4}$$

or the coefficients of an expression which is quadratic in the differentials of the independent variables, such as:

$$a_{ij}dx^i dx^j \tag{1.5}$$

When one makes the transformation (1.1) on the independent variables, at the same time one passes from the expressions (1.4) and (1.5) to their transforms:

$$B^i \frac{\partial f}{\partial y^i}$$
$$b_{ij}dy^i dy^j .$$

The new coefficients B^i and b_{ij} are given by the following formulas:

$$B^i = A^j \frac{\partial y^i}{\partial x^j} \tag{1.6}$$

$$b_{ij} = a_{k\ell} \frac{\partial x^k}{\partial y_i} \frac{\partial x^\ell}{\partial y_j} \tag{1.7}$$

It is then natural to perform on the system of coefficients of the expressions (1.4) or (1.5) the substitutions (1.6) and (1.7) each time that one transforms the independent variables by the formulas (1.1).

We conclude that the nature of the systems we study often suggests to us transformation laws which are different from the simplest transformation by invariance.

Remarks: *This section develops the idea of general "covariance," which is the main algebraic feature of tensor analysis, and which is also very important physically, especially in General Relativity.*

Recall that, in the Special Theory of Relativity, Einstein first expressed the "laws of physics" in a form which is "covariant," i.e. has a well-defined transformation law, according to changes of "reference frames" corresponding to "observers" in uniform rectilinear motion with respect to each other. This corresponds (when the variables x are identified with space-time variables) to allowing only changes of variables (1.1) of a certain special type. (Precisely, belonging to a subgroup of the group of diffeomorphisms of R^4, usually either the group of Lorentz or Galilean transformations on space-time). To accomplish this required some interesting analysis of the physical (and mathematical) "nature" of space and time, and physical laws, but no really new physics beyond that described by Newton's laws of mechanics and Maxwell's equations of electromagnetism.

However, when Einstein turned his attention to "covariance" under general coordinate transformations on space-time - which was natural to him both because of his own private physical and philosophical ideas and his under-

standing of this paper by Ricci and Levi-Civita - he was led to something fundamentally new, a completely "covariant" theory of gravitation to replace Newton's (and which reproduced Newton's to an approximation). Ever since then, this idea of expressing the "laws of physics" in a "covariant form" has been one of the most powerful mathematical tools in the physicist's arsenal, particularly in the process of discovering new physical laws. In many areas, the way to do this is still unknown - for example, there is no quantum gravitational theory which is completely covariant.

The profound ideas of Section 1 may be readily described in terms of modern mathematical ideas. I refer to Volume I in this series, "General Algebraic Ideas," for the terminology to be used now.

Let R^n denote the "space" of n-real variables, i.e. a point of R^n is denoted by an n-tuple

$$x = (x^1, \ldots, x^n)$$

A transformation

$$T: R^n \to R^n$$

is a mapping satisfying the following conditions:

1) *The functions (1.1) describing the transformation are differentiable an arbitrary number of times. One says that T is infinitely differentiable or C^∞.*

2) *T is one-one and onto. The inverse map T^{-1} (which is then well-defined as a set-theoretic mapping) is <u>also</u> C^∞. (Note that there are such mappings which are one-one and onto, for which T^{-1} is <u>not</u> C^∞; e.g., the map $x \to x^3$ of $R \to R$).*

Such a mapping is called a <u>diffeomorphism</u> in modern differential geometry, but I will use "transformation" in order to stay closer to their terminology.

Two such transformations can be composed and their inverse taken, i.e. the set of all such transformations forms a <u>group</u>, which we denote by

$$G(R^n).$$

$G(R^n)$ is of course <u>defined</u> as a <u>transformation group</u> acting on R^n: The transform of a point $x \in R^n$ by a $T \in G(R^n)$ is the point

$$T(x).$$

However, $G(R^n)$ may act on other spaces. What Ricci and Levi-Civita mean by a "transformation" law involves such actions on other spaces Ω, (usually infinite dimensional ones).

For example, consider Ω as forming what they call (in the first paragraph) a <u>system</u> of functions. In this case, Ω may be taken as the space of all C^∞ mappings

$$f\colon R^n \to R^m, \tag{1.8}$$

where m is another integer. Such a mapping may be defined as a set

$$(f_1, \ldots, f_m)$$

of m, real-valued, C^∞ functions on R^n, i.e. the mapping (1.8 is defined explicitly as follows:

$$f(x) = (f_1(x), \ldots, f_m(x)) \tag{1.9}$$
$$\text{for } x = (x^1, \ldots, x^n) \in R^n.$$

What they mean by a transformation law is a transformation group action of $G(R^n)$ on Ω, i.e. a mapping

$$G(R^n) \times \Omega \to \Omega$$

satisfying the following conditions:

1) $T_1(T_2(f)) = (T_1T_2)(f)$
   for $T_1, T_2 \in G(R^n)$, $f \in \Omega$

(The image of $(T, f) \in G(R^n) \times \Omega$ in Ω is denoted by $T(f)$).

2) *If T = identity map, then $T(f) = f$ for all $f \in \Omega$.*

Although the authors do not include it explicitly, it would be appropriate to add a "locality" condition. Here is one way of phrasing it:

3) *Suppose that $f, f' \in \Omega$ are two elements, and $x \in R^n$ is a point at which all derivatives of the corresponding components of f and f' agree, then all derivatives of $T(f)$, $T(f')$ agree at $T(x)$.*

To appreciate the full geometric meaning, it would be necessary to go into the theory of "jets" of mappings. See VB and GPS.

Various generalizations are needed and useful in different situations in mathematics and physics. One generalization is to replace $G(R^n)$ by a subgroup G; for example, Special Relativity involves:

$n = 4$

G = *group of Lorentz transformations on* R^4.

Another generalization is to replace R^n by a general differentiable manifold N, and $G(R^n)$ with the group of diffeomorphism of N, and to take Ω to be the space of cross-sections of a fiber space, with N as base. (See DGCV, VB, GPS for more detail). Another possibility is to relax the global nature of the transformations, i.e. to allow $G(R^n)$ to include coordinate transformations defined on open subsets of R^n, but only to require that elements of $G(R^n)$ be compared when their domains and ranges matchup. (Such a mathematical object is called a pseudo group.) Finally, and most important (at least for the purposes of physics) is the following generalization:

G *is an abstract group.*

A group homomorphism (not necessarily one-one)

$G \to G(R^n)$

is given, a transformation group action of G on Ω then determines the "transformation law."

This set-up allows the possibility of a non-unique transformation law for a system of functions $f_1, \ldots, f_m$ *(and hence for physical objects). The most prominent example is the case of spinors in Special Relativity: Given a Lorentz transformation on* R^4, *there are two possible transformation laws for "spinor fields." Algebraically, this means that the postulated group homomorphism*

$$G \to G(R^4)$$

has C_2, *the cyclic group with two elements, as kernel.*

We can now describe more explicitly the action of $G(R^n)$ *on Ω that Ricci and Levi-Civita call - in paragraph 3 - transformation by invariance. Let*

$$T: R^n \to R^n$$

be an element of $G(R^n)$, *and let*

$$f: R^n \to R^m$$

be an element of Ω, with:

$$f = (f_1, \ldots, f_m).$$

Let

$$T(f): x \to f(T^{-1}x) \equiv T(f)(x) \tag{1.10}$$

$T(f)$ *defined by (1.10) is the transform by invariance. (In modern differential geometry, this is also denoted by* $T^{-1*}(f)$

As the next example, given in Section 1, suppose:

$$f_i = \frac{\partial f}{\partial x^i}, \ 1 \le i, \ j \le n.$$

Set:

$$f' = T(f).$$

Then,

$$f'_i = T(f_j) \frac{\partial x^i}{\partial y^i}. \tag{1.11}$$

It often requires a certain care with logic to correlate this "active" way of looking at things - regarding a "transformation" as acting on <u>points</u>, with the "passive" way <u>implicitly</u> used by the authors, where the "points" remain the same, and only the way of labelling them changes. For example, consider formulas (1.11). If

$$y = (y^1, \ldots, y^n), \ x = (x^1, \ldots, x^n)$$

denote <u>points</u> of R^n, then formula (1.11) assigns to each y an x. This must be interpreted as:

$$T^{-1}x = y.$$

Thus, if $x \to f(x)$ is an element of Ω, its transform "by invariance" is

$$y \to f'(y) \equiv f(T^{-1}x).$$

Thus,

$$f'(y) = f(x(y)).$$

Hence, using the chain rule for differentiation:

$$\frac{\partial f'}{\partial y^i} = \frac{\partial f}{\partial x^j}(x(y))\,\frac{\partial x^j}{\partial y^i}, \tag{1.12}$$

which is identical to formula (1.11).

To summarize, the "active" point of view - points change, while "coordinates" remain the same - is the more customary in modern differential geometry (hence we use it here), while the "passive" one - points remain the same but coordinates change - is used more in physics and the older differential-geometric literature. In quantum mechanics, they "active" approach is called the ***Schrödinger picture****, while the "passive" one is called the* ***Heisenberg picture****.*

Example. Change of reference frame in mechanics

Consider

$$R^4$$

as our space. A point is denoted by:

$$x = (x^1, x^2, x^3, x^4).$$

Let:

$$\vec{x} = (x^1, x^2, x^3),$$

the *position vector,*

$$t = x^4,$$

the *time coordinate.* *Let* $\vec{v} = (v^1, v^2, v^3)$ *be a fixed vector in* R^3*. Using it, define a transformation*

$$T: R^4 \to R^4$$

as follows:

$$T(\vec{x},\ t) = (\vec{x} + \vec{v}t,\ t). \tag{1.13}$$

This is a Galilean transformation representing a "change of frame" corresponding to an "observer" moving with constant velocity $\vec{v}$.

From the point of view of physics, it is natural to associate two types of "systems". First,

$$f(\vec{x},\ t) = (f_1,\ f_2,\ f_3),$$

with

$$f_1 = x^1,\ f_2 = x^1,\ f_3 = x^3$$

f defined in this way represents the position vector. Thus,

$$f(\vec{x},\ t) = \vec{x}.$$

Now,

$$T^{-1}(\vec{x},\ t) = (\vec{x} - \vec{v}t,\ t).$$

Hence,

$$\begin{aligned} f'(\vec{x},\ t) &= f(T^{-1}(\vec{x},\ t)) \\ &= f(\vec{x} - \vec{v}t,\ t) \\ &= \vec{x} - \vec{v}t. \end{aligned}$$

This represents the position vector of the system as seen by the observer moving with constant velocity $\vec{v}$.

Similarly, let

$$f(\vec{x},\ t) = t.$$

This system represents time. It satisfies:

$$f' = f = t,$$

i.e. *time is invariant* *for Galilean* *transformations.* *Here is the important point:*

Both "position" and "time" systems transform "by invariance" under Galilean transformations, i.e. by the law (1.10). However, the time system is invariant, i.e. transforms into itself under Galilean transformation.

Exercise. *Work out the analogous concepts in case T is a Lorentz transformation. Is there a system which "transforms by invariance" and transforms into itself, i.e. is "invariant" in the usual sense?*

Now, I want to explain the summation convention. In contrast to the convention used in my previous work (e.g. in DGCV), we now use the form used in Tensor Analysis, namely:

> *An index occurring in two places in a formula, one upper the other lower, is assumed to be summed over its natural (i.e. given) range of values. It is assumed also that indices do not occur in a repeated form except in pairs, one upper, the other lower.*

For example, formula (1.2) reads, without the summation convention:

$$f_i' = \sum_{j=1}^{n} f_j \frac{\partial x^j}{\partial y^i}$$

If (a_{ij}) *is a 2-indexed symbol, then*

$$\sum_{i=1}^{n} a_{ii}$$

is not

$$a_{ii}.$$

Using the summation convention, it is:

$$\delta^{ij}a_{ij},$$

where

$$(\delta^{ij})$$

is the Kronecker-Delta symbol, i.e.

$$\delta^{ij} = \begin{Bmatrix} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{Bmatrix}.$$

As we go along, we shall see further rules for using the summation convention.

We have emphasized the group-theoretic meaning of the ideas of Tensor Analysis. In fact, E. Cartan has emphasized that this way of looking at it brings it within the influence of Klein's Erlangen Program, with infinite dimensional Lie groups as the basic group-theoretic objects. Another direction for abstraction is towards the modern theory of Categories and Functors. See MacLane [1].

2. COVARIANT AND CONTRAVARIANT TENSORS. EXAMPLES

Among all the transformation laws that one may conceive, there are two which play a predominant role in mathematics. We call them the covariant and contravariant transformation laws.

Fix the integer n and the indices as follows:

$$1 \le i, j, k, \ldots, i_1, i_2, \ldots, j_1, j_2, \ldots, \ldots \le n.$$

Let

$$x = (x^1, \ldots, x^n)$$

be independent variables, representing

$$R^n.$$

An m-th order system is a system (of the type described in Section 1), which is labelled by m-tuples

$$(i_1, \ldots, i_m)$$

of indices ranging over the values from 1 to n, i.e. to each

$$(i_1, \ldots, i_m) \in \underbrace{Z_n \times Z_n \times \ldots \times Z_n}_{m},$$

there is a unique function

$$f(i_1, \ldots, i_n)$$

in the system Ω.

Remark. Z_n *denotes the first n integers.* $\underbrace{Z_n \times \ldots \times Z_n}_{m}$ *denotes the Cartesian product of m copies of* Z_n. *Thus, an "m-th order system," as defined by Ricci and Levi-Civita, is a mapping*

$$\underbrace{Z_n \times \ldots \times Z_n}_{m} \to \text{(space of real-valued } C^\infty \text{ functions on } R^n)$$

Various types of m-th order systems are defined by putting some of the indices "below", some "above". For example, for m = 2, we might label them as

$$f_{i_1 i_2}$$

This would be "covariant". Alternately,

$$f^{i_1 i_2}.$$

This would be "contravariant."

$$f^{i_1}_{i_2}$$

would be "mixed".

<u>Definition</u>. The <u>covariant m-th order system</u> consists of the sets of functions indexed (in order to make this transformation law natural from the point of view of the summation convention) as follows:

$$X_{i_1 \ldots i_m}, \tag{2.1}$$

$$1 \le i_1, \ldots, i_m \le n.$$

The transformation law of the covariant system, for a transformation

$$x \to T(x) = y$$

of $R^n \to R^n$ (given again by formulas (1.1)), is the following one:

$$Y_{i_1 \ldots i_m} = X_{j_1 \ldots j_m} \frac{\partial x^{j_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{j_m}}{\partial y^{i_m}}. \tag{2.2}$$

Remark. $Y_{i_1 \ldots i_m}$ *denotes the components of the system transformed by T. (It was denoted by X' in Section 1).*

The elements of this system are called *m-th order covariant tensor fields on* R^n.

Definition. The *m-th order contravariant system* consists of the sets of functions indexed as follows:

$$X^{i_1 \ldots i_m},$$

with the following transformation law:

$$Y^{i_1 \ldots i_m} = X^{j_1 \ldots j_m} \frac{\partial y^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{i_m}}{\partial x^{j_m}}. \tag{2.3}$$

The elements of this system are called *contravariant tensor fields.*

In (2.3), it is assumed that the y's are functions of the x's, which requires the inversion of relations (1.1), then expressing everything back in terms of y.

Denoting by X a function of the variables x, by Y the same function expressed in terms of y, the formula

$$Y = X$$

may be regarded as a particular case of both (2.2) and (2.3).

Thus, a system of order 0 besides being an "invariant", may be considered as a limiting case of a covariant or contravariant system.

From now on, when we introduce a symbol such as

$$X_{i_1 \dots i_m}$$

(or $X^{i_1 \dots i_m}$) we understand that it belongs to a covariant (or contravariant) system of order m, that we call system $X_{i_1 \dots i_m}$ (or $X^{i_1 \dots i_m}$).

The first order derivatives of a function and the coefficients of a quadratic differential form φ provide examples of covariant systems of, respectively, first and second order. The inverse of the coefficients of φ provides an example of a contravariant system of second order. Similarly, the formulas

$$dy^i = dx^j \frac{\partial y^i}{\partial x^j}$$

tell us that the differentials of the independent variables are examples of a contravariant systems.

The systems which are formed of the derivatives of order $m > 1$ of a function

$$f(x)$$

are neither covariant nor contravariant. The transformation law of these systems are more complex - which is the source of the difficulty one meets in Differential Calculus in transforming the partial derivatives of order greater than the first.

We will see that one may avoid these difficulties by replacing ordinary differentiation with another operation.

Remark: This operation is now called covariant differentiation.

It is useful to note that covariant or contravariant systems of the theory of algebraic forms are particular cases of those we just defined, since in this theory one considers transformations of type (1.1) which are linear and homogeneous.

Remark: Here is how some of these basic ideas may be described in more contemporary algebraic language. Let

$$F(R^n)$$

denote the set of real-valued, C^∞ functions

$$x \to f(x)$$

on R^n. (x, as always, denotes a point of R^n, with coordinates $(x^1, \ldots, x^n)$.) One can add and multiply two such functions, i.e. $F(R^n)$ forms a commutative ring. (See Vol. I for the general algebraic ideas used here).

A *derivation* is a mapping

$$X: F(R^n) \to F(R^n)$$

such that:

$$X(f_1 f_2) = X(f_1) f_2 + f_1 X(f_2). \tag{2.4}$$

Such derivations can be added and multiplied by elements of $F(R^n)$, i.e. they form an $F(R^n)$ module which is denoted by

$$V(R^n).$$

The elements of $V(R^n)$ are also called *vector fields*. It turns out that they are also naturally identified with *1-contravariant tensor fields*, as defined above, as we shall see in a moment. Now, set:

$$\begin{aligned} V^2(R^n) &= V(R^n) \otimes V(R^n) \\ V^3(R^n) &= V^2(R^n) \otimes V(R^n). \end{aligned} \tag{2.5}$$

and so forth. It turns out that the elements of $V^m(R^n)$ are the *m-contravariant tensor fields*. (In (2.5), the symbol $\otimes$ denotes *tensor product of the two $F(R^n)$-modules*. Now, in IM, Vol. II, we have defined tensor products of two vector spaces, with "fields" (e.g. the real or complex numbers) as scalars. The ideas generalize, and one can define the tensor product of modules as well.)

To define the "co" objects, set:

$$\begin{aligned} F^1(R^n) &= \text{dual module of } V(R^n) \\ &\equiv \text{set of maps} \\ &\qquad \theta: V(R^n) \to F(R^n) \end{aligned}$$

such that:

$$\theta(X_1 + X_2) = \theta(X_1) + \theta(X_2)$$

$$\theta(fX_1) = f\theta(X_1)$$

for $X_1, X_2 \in V(R^n)$, $f \in F(R^n)$.

$F^1(R^n)$ *is also an* $F(R^n)$*-module, hence its tensor products may be defined. The m-covariant tensors are then to be identified with elements of*

$$\underbrace{F^1(R^n) \otimes \ldots \otimes F^1(R^n)}_{m \text{ times}}.$$

We must now show that these algebraic definitions reproduce those of Ricci and Levi-Civita. To this end, define the coordinate differential operators

$$\frac{\partial}{\partial x^i} \colon F(R^n) \to F(R^n),$$

$$1 \le i,\ j \le n.$$

They satisfy the "derivation" rule (2.4), hence define elements of $V(R^n)$.

__Theorem 1__. The $\frac{\partial}{\partial x^i}$ *form a basis for the* $F(R^n)$*-module,* $V(M)$, *i.e. each* $X \in V(R^n)$ *can be written in a unique way as*

$$X = X^i \frac{\partial}{\partial x^i}, \tag{2.6}$$

with $(X^1, \ldots, X^n) \in F(R^n)$.

The proofs of this and other results in these Remarks may be found in the standard differential-geometric references *listed in the Bibliography. See especially DGCV and Bishop and Goldberg [1].*

Theorem 2. *The assignment*

$$X \to (X^i)$$

defined by (2.6) sets up a one-one, onto correspondence between the elements of $V(R^n)$ and the 1-contravariant tensor fields.

This correspondence will only be useful and natural if it leads to the characteristic "contravariant tensor" transformation law. We sketch how this goes.

Given a transformation

$$T: R^n \to R^n,$$

the transform of $f \in F(R^n)$ by T is given by the formula:

$$T(f)(x) = f(T^{-1}(x))$$

for $x \in R^n$.

(This is the "transformation by invariance," which is the first sort of transformation described by Ricci and Levi-Civita). Given $X \in V(R^n)$, transform it by T as follows:

$$T(X)(f) = T(X(T^{-1}(f))). \tag{2.7}$$

Notice that $X \to T(X)$ defines a transformation group action of $G(R^n)$ on $V(R^n)$.

Theorem 3. The correspondence

$$X \to (X^i)$$

between $V(R^n)$ and 1-contravariant tensors intertwines the action of $G(R^n)$ in both spaces. In other words, if

$$T(X) = Y^i \frac{\partial}{\partial x_i},$$

then (Y^i) is the transformation of (X^i) according to the contravariant transformation law.

According to the principles of tensor algebra (see Vol. II), an element of

$$\underbrace{V(R^n) \otimes \ldots \otimes V(R^n)}_{M \text{ times}} \tag{2.8}$$

is of the form:

$$X = X^{i_1 \ldots i_m} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_m}} .$$

It is now readily verified that the correspondence

$$X \leftrightarrow (X^{i_1 \ldots i_n})$$

sets up a 1-1 station between elements of the module (2.8) and the contravariant m-fold tensors, in the sense of Ricci and Levi-Civita.

Now, for the covariant ones. Define a mapping

$$d\colon F(R^n) \to F^1(R^n)$$

by the following formula:

$$df(X) = X(f) \tag{2.9}$$

for $f \in F(R^n)$, $X \in V(R^n)$.

df is called the <u>*differential*</u> *or* <u>*exterior derivative*</u> *of f.*

In particular, one can apply d to the coordinate functions x^i. *The resulting elements of* $F^1(R^n)$

$$dx^1, \ldots, dx^n,$$

form a <u>*basis*</u> *for the module* $F^1(R^n)$. *Let* $G(R^n)$ *act as a transformation group on* $F^1(R^n)$ *in the following way:*

$$T(\omega)(X) = \omega(T^{-1}X) \tag{2.10}$$

for $T \in G(R^n)$, $\omega \in F^1(R^n)$, $X \in V(R^n)$.

<u>*Theorem 4*</u>. *Set up a correspondence*

$$\omega \leftrightarrow (X_i)$$

between elements of $F^1(R^n)$ *(called* <u>*differential forms of degree*</u>*) and "systems" of functions, in the sense defined by Ricci and Levi-Civita. This correspondence then intertwines the action of* $G(R^n)$ *on* $F^1(R^n)$ *(defined via formula (2.10) and the natural action on 1-covariant tensors.*

Similarly, a element of

$$\underbrace{F^1(R^n) \otimes \ldots \otimes F^1(R^n)}_{M\ \text{times}} \tag{2.11}$$

may be written in the form:

$$X_{i_1 \dots i_m}\, dx^{i_1} \otimes \dots \otimes dx^{i_m} \qquad (2.12)$$

This sets up a natural correspondence between the module (2.11) and the space of m-fold covariant tensors in the Ricci and Levi-Civita sense.

We can also introduce the <u>*moving frame*</u> *viewpoint (due to E. Cartan) to serve as a bridge between classical tensor analysis and modern differential geometry.*

A <u>*moving frame*</u> *for* R^n *is a set*

$$(y^1, \dots, y^n)$$

of functions which form a new <u>*coordinate system*</u> *for* R^n, *i.e. which satisfy the following condition:*

The map

$$x \to (y^1(x), \dots, y^n(x))$$
of $R^n \to R^n$ *is a diffeomorphism.* (2.12)

Each such moving frame determines bases of $V(R^n)$ *and* $F^1(R^n)$, *namely the following elements:*

$$\frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j} \qquad (2.13)$$

$$dy^j = \frac{\partial y^j}{\partial x^i} dx^i. \qquad (2.14)$$

Each moving frame then sets up a correspondence of $V(R^n) \otimes \dots \otimes V(R^n)$ *and* $F^1(R^n) \otimes \dots \otimes F^1(R^n)$ *between m-fold* *contra and co-variant indexed quantities:*

$$Y^{i_1 \ldots i_m} \frac{\partial}{\partial y^{i_1}} \ldots \frac{\partial}{\partial y^{i_m}} \in V(R^n) \otimes \ldots \otimes V(R^n)$$

$$Y_{i_1 \ldots i_m} \, dy^{i_1} \otimes \ldots \otimes dy^{i_m} \in F^1(R^n) \otimes \ldots \otimes F^1(R^n).$$

Again, one sees that this labelling is the correct one to ***derive*** *the covariant transformation law for* $(Y_{i_1 \ldots i_m})$ *and the contravariant transformation law for* $(Y^{i_1 \ldots i_m})$ *from the moving frame transformation law (2.13) -(2.14).*

3. ADDITION, MULTIPLICATION AND CONTRACTION OF TENSOR FIELDS. RIEMANNIAN-METRICS. RECIPROCAL SYSTEMS MIXED TENSORS

Addition. If

$$X_{i_1 \ldots i_m}, \; Z_{i_1 \ldots i_m}$$

are two m-covariant tensor fields

$$Y = X_{i_1 \ldots i_m} + Z_{i_1 \ldots i_m}$$

is also a m-covariant tensor field. We say it is the sum of the two given fields. Similiarly, one defines the sum of two m-contravariant tensor fields, which will be one of the same type.

Tensor Multiplication. If

$$X_{i_1 \ldots i_m}, \; Z_{j_1 \ldots j_p}$$

are two covariant tensor fields, of order m and p, respectively,

$$Y_{i_1 \ldots i_m j_1 \ldots j_p} = X_{i_1 \ldots i_m} Z_{j_1 \ldots j_p}$$

is a covariant tensor field of order m + p, that one calls the (tensor) product of the two tensor fields. Substituting the word "contravariant" for "covariant" defines the product of two covariant tensor fields.

Contraction. If

$$X_{i_1 \ldots i_m j_1 \ldots j_p}$$

is a covariant tensor field of order (m + p), and $Z^{j_1 \ldots j_p}$ is a m-contravariant tensor field, then:

$$Y_{i_1 \ldots i_m} = Z^{j_1 \ldots j_p} X_{i_1 \ldots i_m j_1 \ldots j_p}$$

is covariant of order m. Similiarly, given tensor fields

$$X^{i_1 \ldots i_m j_1 \ldots j_p}, \; Z_{j_1 \ldots j_p},$$

form the following m-contravariant tensor field:

$$Y^{i_1 \ldots i_m} = X^{i_1 \ldots i_m j_1 \ldots j_p} Z_{j_1 \ldots j_p}$$

We say that $Y_{i_1 \ldots i_m}$ and $Y^{i_1 \ldots i_m}$ are the contractions of the two given tensor fields.

In particular, for $m = 0$, one obtains by contraction a system of order zero, i.e. an invariant, which results from the contraction of two tensor fields of opposite type and the same order.

The reader will perceive that these concepts, which are frequently used in calculation, are derived from a unified principle, called the saturation of indices.

The fundamental Riemannian metric

The methods of Tensor Analysis essentially require that we be given a positive quadratic differential form in the n-variables $x^1, \ldots, x^n$, i.e. an expression of the form:

$$\varphi = g_{ij} dx^i dx^j .$$

Remark. Later on, the techniques of tensor analysis were completely freed of the need to hypothesize such a metric. See Schouten [1] and Vranceanu [1].

The coefficients (g_{ij}) of this form occur everywhere in our formulas, and give them a remarkable symmetry and simplicity.

Reciprocal systems.

Let

$$(g^{ij})$$

denote the inverse matrix to (g_{ij}), i.e.

$$g^{ij}g_{jk} = \delta_k{}^i, \tag{3.1}$$

where $\delta_k{}^i$, the Kronecker delta symbol, is zero if $i \neq k$, equal to +1 if $i = k$.

In general, given an m-covariant tensor field $X_{i_1 \ldots i_m}$, construct an m-contravariant tensor field denoted by $X^{i_1 \ldots i_m}$, by the following formula:

$$X^{i_1 \ldots i_m} = g^{i_1 j_1} \ldots g^{i_m j_m} X_{j_1 \ldots j_m}. \tag{3.2}$$

In the same way, starting from a contravariant tensor field $Z^{i_1 \ldots i_m}$, define a covariant one, $Z_{i_1 \ldots i_m}$, by the following formulas:

$$Z_{i_1 \ldots i_m} = g_{i_1 j_1} g_{i_2 j_2} \ldots g_{i_m j_m} Z^{j_1 \ldots j_m}. \tag{3.3}$$

The succession of the operations (3.2) and (3.3) is the identity, hence we say that pairs of tensor fields

$$X^{i_1 \ldots i_m},\ X_{i_1 \ldots i_m} \text{ and } Z_{i_1 \ldots i_m},\ Z^{i_1 \ldots i_m}$$

are reciprocal with respect to the Riemannian metric.

From (3.1)-(3.3), one derives the following identity:

$$X^{i_1 \ldots i_m} Z_{i_1 \ldots i_m} = X_{i_1 \ldots i_m} Z^{i_1 \ldots i_m}. \tag{3.4}$$

It can be interpreted in words as follows:

"Each function resulting from contracting a covariant and contravariant tensor field of the same order is equal to the contraction of their reciprocals."

After fixing the Riemannian metric, it suffices to give a covariant or contravariant tensor field and their reciprocals are determined. This fact is reflected in the convention, which we have already used in the examples, that the same LETTER represents a covariant tensor or its reciprocal, according to whether the indices are placed below or above the letter.

We shall now define an n-covariant tensor

$$\varepsilon_{i_1 \ldots i_n} \tag{3.5}$$

by the following rules:

$\varepsilon_{i_1 \ldots i_n}$ changes sign when two adjacent indices are permuted.

$$\varepsilon_{12 \ldots n} = \pm \sqrt{g},$$

where

g = absolute value of $\det(g_{ij})$, the determinant of the matrix formed by the coefficients of the metric tensor, and +1 or -1 according to whether the determinant of the coordinate system in which the components of the tensor are being described has a positive or negative Jacobian with respect to the original coordinate system of R^n.

The elements of the reciprocal tensor $\varepsilon^{i_1 \dots i_n}$ have simi-liar properties with respect to permutation of the indices, and have values

$$0, \ \pm \frac{1}{\sqrt{g}} \ .$$

Thus, if

$$X_{ij}$$

is a 2-covariant tensor field,

$$\varepsilon^{i_1 \dots i_m} \ X_{1i_1} \dots X_{ni_n}$$

is

$$\det(X_{ij})/\pm\sqrt{g},$$

where $\det(X_{ij})$ is the determinant of the matrix (X_{ij}). In particular, if $z^1, \dots, z^n$ are functions of n-variables $x^1, \dots, x^n$, and

$$\Delta(z^1, \dots, z^n)$$

is the Jacobian determinant of the functions, then we have the following identity:

$$\pm\sqrt{g} \ \Delta(z^1, \dots, z^n) = \varepsilon^{i_1 \dots i_n} \ \frac{\partial z^1}{\partial x^{i_1}} \dots \frac{\partial z^n}{\partial x^{i_n}}.$$

This identity makes intuitive the invariant property of Δ, and at the same time makes it accessible to the methods of tensor analysis. We denote the tensors

$$\varepsilon_{i_1 \cdots i_n} \text{ and } \varepsilon^{i_1 \cdots i_n}$$

by the names of <u>covariant</u> and <u>contravariant tensors E</u>.

<u>Mixed tensors</u>

Let

$$(X^i) \text{ and } (X_i')$$

be a 1-contravariant and covariant tensor. If

$$(y^i)$$

is another coordinate system, recall that the components

$$(Y^i),\ (Y_i')$$

in this new coordinate system transforms as follows:

$$Y^i = X^j \frac{\partial y^i}{\partial x^j}$$

$$Y_i' = X_j' \frac{\partial x^j}{\partial y^i}.$$

Set:

$$X_j^{\ i} = X^i X_j'$$

$$Y_j^{\ i} = Y^i Y_j'$$

=, using the transformation laws given above.

$$X^k \frac{\partial y^i}{\partial x^k} X_\ell{}' \frac{\partial x}{\partial y_j}$$

$$= (X^k X_\ell{}') \frac{\partial y'}{\partial x^k} \frac{\partial x}{\partial y^j} .$$

We see that the products

$$(X_j{}^i)$$

form a system (in the sense of Section 1) which transforms in a linear homogeneous way under a change of coordinates. Such a system we call a <u>1-contravariant, 1-covariant</u> mixed tensor, and this way of forming such a tensor from a contravariant and covariant one we will call the <u>tensor product</u>.

In general, a mixed <u>m-contravariant, p-covariant</u> tensor assigns, to each coordinate system

$$(x^i),$$

a system of function labelled

$$(X_{j_1 \ldots j_p}^{i_1 \ldots i_m}),$$

which transform to a new coordinate system (y^i) according to the following rule:

$$Y_{j_1 \ldots j_p}^{i_1 \ldots i_m} = X_{k_1 \ldots i_p}^{\ell_1 \ldots \ell_m} \frac{\partial y^{i_1}}{\partial x^{\ell_1}} \cdots \frac{\partial y^{i_m}}{\partial x^{\ell_m}} \frac{\partial x^{k_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{k_p}}{\partial y^{j_p}} .$$

The products of two mixed tensors, e.g. an (m, p) and (m', p') one, form another mixed tensor, of type

$$(m + m', p + p').$$

(This material on mixed tensors is not in the original paper).

<u>Remarks</u>: This section is purely algebraic, and may be readily described in terms of modern "Tensor Algebra". (See IM, Volume II). The basic idea is that contravariant and covariant tensors of a given order are <u>dual modules</u>. I will briefly sketch some of the material required:

Let F be a <u>ring</u>, i.e. a set with an abelian addition and multiplication operation

$$(f_1, f_2) \to f_1 + f_2 \text{ and } f_1 f_2,$$

satisfying the usual rules of algebra. We also suppose that F has a multiplicative unit element, denoted by

$$1.$$

A set M is an <u>F-module</u> if it has the following pair of binary algebraic operation:

<u>Addition</u>: $M \times M \to M$, denoted by

$$X \times Y \to X + Y$$

for $X, Y \in M$.

<u>Scalar multiplication</u>: $F \times M \to M$, denoted by

$$(f, X) \to fX.$$

The following laws hold:

$$f_1(f_2X) = (f_1f_2)X$$

$$f(X + Y) = fX + fY$$

$$1X = X.$$

The dual module, denoted by

$$M^d,$$

consists of the set of all F-linear maps

$$\theta\colon M \to F,$$

i.e. θ *satisfies*

$$\theta(fX) = f\theta(X).$$

A set $X_1, \ldots, X_r$ *of elements of M forms a basis of M if each* $X \in M$ *can be written as:*

$$X = f_1X_1 + \ldots + f_rX_r,$$

and the coefficients $(f_1, \ldots, f_n) \in F$ *are uniquely determined by X.*

M is said to be a free module if it has at least one basis.

If M, M' are modules, one can form a third module

$$M \otimes M',$$

called the tensor product of M and M'. The elements of

$M \otimes M'$ are the linear combination

$$\Sigma f_{ij} X_i \otimes X_j^{1}$$

of elements $X_i \in M$, $X_j^{1} \in M^{1}$, subject to the following rules:

$$(X_1 + X_2) \otimes X^{1} = X_1 \otimes X^{1} + X_2 \otimes X^{1}$$

$$X \otimes (X_1^{1} + X_2^{1}) = X \otimes X_1^{1} + X_1 \otimes X_2^{1}.$$

__*Theorem 1*__. *If M, M^{1} are free modules, so are:*

$$M^{d} \text{ and } M \otimes M^{1}.$$

__*Theorem 2*__. *If M is a free module, and*

$$\underbrace{M^{m} = M \otimes \ldots \otimes M}_{m\text{-times}}$$

$$\underbrace{M_{m} = M^{d} \otimes \ldots \otimes M^{d}}_{m\text{-times}},$$

then M_m is the dual module to M^{m}. The duality between them is as follows:

$$(\theta_1 \otimes \ldots \otimes \theta_m)(X_1 \otimes \ldots \otimes X_m)$$

$$= \theta_1(X_1) \ \ldots \ \theta_m(X_m).$$

__*Theorem 3*__. *If M is a free module,*

$$M^m \otimes M^p \text{ is } M^{m+p}$$

$$M_m \otimes M_p \text{ is } M_{m+p}.$$

The identification involved are as follows:

$$(X_1 \otimes \ldots \otimes X_m) \otimes (Y_1 \otimes \ldots \otimes Y_p)$$
$$= X_1 \otimes \ldots \otimes X_m \otimes Y_1 \otimes \ldots \otimes Y_p$$

$$\theta_1 \otimes \ldots \otimes \theta_m) \otimes (\eta_1 \otimes \ldots \otimes \eta_p)$$
$$= \theta_1 \otimes \ldots \otimes \theta_m \otimes \eta_1 \otimes \ldots \otimes \eta_p.$$

Notational remark. The use of indices here, e.g. $X_1, \ldots$ *is not tensorial.* $X_1, X_2, \ldots$ *denote particular elements of M. The indices then are simple "counting" indices, and the reader must keep their role distinct from the far more essential role that indices play in classical tensor analysis! One reason Tensor Analysis is no longer in fashion in mathematics is that it depends critically on this elaborate "technology" of indices, whereas the "coordinate free" methods of modern differential geometry are much more in tune with tendencies in algebra, and the rest of mathematics.*

We can apply these algebraic results to understand the algebraic operations Ricci and Levi-Civita define in this Section. Set:

T^m = *set of m-contravariant tensor fields* $X^{i_1 \dots i_m}$.

T_m = *set of m-covariant tensor fields* $X_{i_1 \dots i_m}$.

Two elements of T_m *or* T^m *may be added and multiplied by a function of x, i.e. they form* *F(Rⁿ)-modules*. *We have already seen, in the Remarks to Section 2, that:*

$$T_m = \underbrace{F^1(R^n) \otimes \dots \otimes F^1(R^n)}_{m\text{-times}}$$

$$T^m = \underbrace{V(R^n) \otimes \dots \otimes V(R^n)}_{m\text{-times}}$$

We see then that:

$$T^m \otimes T^p = T^{m+p}$$

$$T_m \otimes T_p = T_{m+p}.$$

This defines the *tensor multiplication* *presented in this Section. The contraction operation is based on the fact that:*

$$(T^m)^d = T_m$$

$$(T_m)^d = T^m.$$

Contraction *is defined by bilinear maps*

$$T^{m+p} \times T_p \to T^m$$

$$T_{m+p} \times T^p \to T_m,$$

or by <u>*linear*</u> *maps*

$$T^{m+p} \otimes T_p \to T^m$$

$$T_{m+p} \otimes T^p \to T_m.$$

Explicitly, if:

$$X_1, \ldots, X_{m+p} \in V(R^n),$$

$$\theta_1, \ldots, \theta_p \in F^1(R^n),$$

then the contraction map is defined as follows:

$$(X_1 \otimes \ldots \otimes X_{m+p}) \times (\theta_1 \otimes \ldots \otimes \theta_p)$$

$$\to (X_1 \otimes \ldots \otimes X_p)(\theta_1(X_{m+1}) \ \ldots \ \theta_p(X_{m+p}).$$

A <u>*Riemannian metric*</u> *is a symmetric, $F(R^n)$-bilinear, non-degenerate map*

$$\varphi\colon V(R^n) \times V(R^n) \to F(R^n).$$

It sets up an isomorphism between $V(R^n)$ and its dual space, i.e. $F^1(R^n)$, and this isomorphism extends to an isomorphism between

$$T_m \text{ and } T^m.$$

This is what Ricci and Levi-Civita call the <u>*reciprocal*</u> *operation.*

Consider the tensor E defined by (3.5). In DGCV and Vol. IV I have defined and used the <u>*volume element differential form associated to the Riemannian metric* φ</u>. *(There are no problems with orientation of the manifold, since we always work with* R^n*, which comes with its natural Cartesian coordinate system* $(x^1,\ldots, x^n)$*). It is defined to be that n-form which has inner product +1 with itself (with respect to the natural metric on differential forms defined by* φ*) and* <u>*which is positively oriented with respect to the Cartesian coordinate system* $(x^1,\ldots, x^n)$ *for* R^n</u>. *Thus, if*

$$\varphi = g_{ij}dx^i dx^j$$

is the Riemannian metric,

$$\omega = \sqrt{g}\ dx^1 \wedge\ldots\wedge dx^n$$

is the explicit formula for the volume element differential form, where:

$$g = \det(g_{ij}).$$

In any other coordinate system (y^i),

$$\varphi = h_{ij}dy^i dy^j,$$

we have:

$$\omega = \pm \sqrt{h}\ dy^1 \wedge\ldots\wedge dy^n,$$

where the sign +1 is chosen if $(y^1,\ldots, y^n)$ *is* <u>*positively oriented*</u> *with respect to the original coordinate system*

$(x^1, \ldots, x^n)$ (i.e. f the Jacobian determinant $\det(\frac{\partial y^i}{\partial x^i})$, is positive) and -1 is chosen otherwise. Thus, we see that the *tensor* *E defined in the text is essentially* *defined* *to be the tensor such that, in* *every* *coordinate system $(y^1, \ldots, y^n)$ for R^n, the volume element differential form ω for the given Riemannian metric is given by the following formula:*

$$\omega = \frac{1}{n!} \varepsilon_{i_1 \ldots i_n} dy^{i_1} \wedge \ldots \wedge dy^{i_n}.$$

The *mixed tensors* *which are m-contravariant, p-covariant can now readily be defined as the elements of*

$$T_p \otimes T^m.$$

4. APPLICATIONS TO VECTOR ANALYSIS

Remark: The material in this section does not seem closely related to the rest of the paper, nor of great current interest, hence I have omitted it.

5. COVARIANT DERIVATIVES AND RIEMANNIAN METRICS. GENERALIZATION OF THE RULES OF ORDINARY DIFFERENTIAL CALCULUS COVARIANT DERIVATIVE OF COVARIANT TENSORS

Let

$$\varphi = g_{ij} dx^i dx^j$$

be the fundamental metric tensor. Set:

$$\left\{ {j \atop ik} \right\} = g^{j\ell} \frac{1}{2}\left(\frac{\partial g_{i\ell}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^\ell} \right). \tag{5.1}$$

$\left\{ {j \atop ik} \right\}$ are called the <u>Christoffel symbols</u>.

<u>Remark</u>: *Ricci and Levi-Civita seem to take for granted that the reader knew what the Christoffel symbols were. In these notations, I have more closely followed Eisenhart [1], except that I do not define all the possible Christoffel symbols.*

Christoffel has been the first to remark [1869] that, if

$$X_{i_1 \ldots i_m}$$

is an m-covariant tensor, the following system of order m + 1,

$$\begin{aligned} X_{i_1 \ldots i_m,\, i_{m+1}} = {} & \frac{\partial X_{i_1 \ldots i_m}}{\partial x_{i_{m+1}}} \\ & - \left\{ {j \atop i_1 i_{m+1}} \right\} X_{j i_2 \ldots i_m} \\ & - \left\{ {j \atop i_2 i_{m+1}} \right\} X_{i_1 j i_3 \ldots i_m} \\ & - \ldots \end{aligned} \tag{5.2}$$

is also a covariant (m+1)-tensor. We call the operation which assigns to a given tensor $(X_{i_1 \dots i_m})$ the tensor $(X_{i_1 \dots i_m,\, i_{m+1}})$ the <u>covariant derivative with respect to the metric φ</u>. We say that $(X_{i_1 \dots i_m,\, i_{m+1}})$ is the <u>first derived tensor</u> of $(X_{i_1 \dots i_m})$.

As limiting case, for $m = 0$ we see that the first derived tensor of a scalar tensor X consists of the derivatives of this function, which is independent of the metric. Set:

$$X_i = \frac{\partial X}{\partial x_i}. \tag{5.3}$$

Similiarly, one finds the following formula for the covariant derivative of a 1-tensor;

$$X_{i,j} = \frac{\partial X_i}{\partial x_j} - \left\{ {k \atop ij} \right\} X_k \tag{5.4}$$

and for a 2-tensor:

$$X_{ij,k} = \frac{\partial X_i}{\partial x_k} - \left\{ {\ell \atop ik} \right\} X_{\ell j} - \left\{ {\ell \atop jk} \right\} X_{i\ell}. \tag{5.5}$$

For

$$X_{ij} \equiv g_{ij},$$

we have the following identities:

$$g_{ij,k} = 0,$$

which tell us that:

<u>The covariant derivative of the coefficients of the metric φ is identically zero</u>.

Applying the formula (5.2) to the covariant tensor E defined in Section 3, one verifies that:

The covariant derivative of the tensor E is identically zero.

If a symbol with m indices represents a covariant tensor, it will be understood that the same letter, with another covariant index, after a comma, represents its covariant derivative (with respect to the given Riemannian metric).

Of course, applying the covariant derivative p-times, one may associate to an m-tensor one of order (m+p), called the *p-th derived system*.

For example, starting with a scalar X, one may define the covariant derivatives

$$X_i,\ X_{i,j},\ X_{i,j,k},\ldots$$

of the function X.

From the well-known properties of the Christoffel symbols and formula (5.4) one deduces that:

A 1-covariant tensor results from the derivatives of a scalar if and only if its covariant derivative is a symmetric tensor.

Using (5.2), we see that the derivatives of a covariant tensor with respect to an arbitrary coordinate system are linear functions of the tensor itself and its covariant derivatives. One may therefore replace, in many calculations, the ordinary derivatives by the covariant derivatives.

Remark: *This principle turned out to be extremely important for physics. After deriving a system of differential equations which expresses a physical law in a Cartesian coordinate system, one may often simply substitute covariant derivatives for the ordinary ones to write the same law in arbitrary coordinate systems* or *in an arbitrary Riemannian metric. For example, with $n = 3$, $\varphi = \delta_{ij}dx^i dx^j$, i.e. Euclidean geometry, Poisson's equation (a typical partial differential equation of mathematical physics) is:*

$$\frac{\partial^2 f}{\partial^2 x^1} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^3} = \rho.$$

In terms of covariant derivative this equation reads:

$$g^{ij} f_{i,j} = \rho,$$

or

$$f_{,i}{}^{i} = \rho.$$

This is important practically where one wants to compute the Laplacian operator

$$f \to f_{,i}{}^{i} \text{ in general}$$

coordinates, e.g. polar or cylindrical coordinates. Maxwell's electromagnetic equations are another important example. Here, $n = 4$. They read:

$$F_{ij} = A_{i,j} - A_{j,i}$$

$$F^{j}_{i,j} = J_i .$$

Here (F_{ij}) is the electromagnetic field tensor, (A_i) is the electromagnetic potential, (J_i) is the current. Now, when the metric tensor (g_{ij}) defines the Lorentz metric on R^4, the Maxwell equations in their classical form are obtained. When one substitutes for this a general Riemannian metric (e.g. one which, physically, defines the gravitational field) one obtains a set of equations which are called (when one adds to them the Einstein equations for the metric) the Maxwell-Einstein equations. They are very important in Cosmology and Astrophsics.

In Mathematics, one has the advantage (in working with covariant instead of ordinary derivatives) of dealing with systems of differential equations in a form which has a uniform and simple transformation law under change of co-ordinates.

We will see later that it is precisely the tensorial form of the transformation law for covariant derivatives that is responsible for the invariant nature of the formulas and equations that are developed using Tensor Analysis:

Covariant derivative of contravariant tensors and contravariant derivatives.

Suppose given a contravariant tensor

$$X^{i_1 \ldots i_m}.$$

Its covariant derivative will be defined as a mixed tensor:

$$X^{i_1 \ldots i_m}_{,j},$$

m-times contravariant, 1-time covariant. As definition,

$$X^{i_1 \ldots i_m}_{,j} = \frac{\partial X^{i_1 \ldots i_m}}{\partial x^j} + X^{i i_2 \ldots i_m} \begin{Bmatrix} i_1 \\ ij \end{Bmatrix}$$

$$+ \ldots + X^{i_1 \ldots i_{m-1} i} \begin{Bmatrix} i_m \\ ij \end{Bmatrix}.$$

The contravariant derivative is then defined by raising the covariant index to a contravariant one, via the metric tensor:

$$X^{i_1 \ldots i_m,\, i_{m+1}} = g^{i_{m+1} j} X^{i_1 \ldots i_m}_{,j}.$$

One may treat this operation in a way which is analogous to the previous treatment of covariant derivative.

Remark: Since Ricci and Levi-Civita do not deal with mixed tensors, they do not define the covariant derivative of a contravariant tensor. I have added it because it is now *such a standard idea.*

One may say, in general, that there exists a law of "reciprocity" or "duality" which enables one to derive from each theorem or formula of Tensor Analysis a reciprocal theorem or formula by interchanging the words <u>covariant</u> and <u>contravariant</u>, and using the metric tensor to raise or lower indices.

Calculational rules

The well-known rules for ordinary differentiation of sums and products of functions generalize to covariant differentiation of sums and products of tensors. Suppose that:

$$Y_{i_1 \ldots i_m} = X_{i_1 \ldots i_m} + Z_{i_1 \ldots i_m}$$

$$Y_{i_1 \ldots i_m j_1 \ldots j_p} = X_{i_1 \ldots i_m} Z_{j_1 \ldots j_p}.$$

Then,

$$Y_{i_1 \ldots i_m, i} = X_{i_1 \ldots i_m, i} + Z_{i_1 \ldots i_m, i}$$

$$Z_{i_1 \ldots i_m j_1 \ldots j_p, i} = X_{i_1 \ldots i_m, i} Z_{j_1 \ldots j_p} + X_{i_1 \ldots i_m} Z_{j_1 \ldots j_p, i}.$$

Analogous rules hold for contravariant or mixed tensors, or for sums and products of an arbitrary number of tensors.

Consider a contracted tensor:

$$Y_{i_1 \ldots i_m} = X_{i_1 \ldots i_m j_1 \ldots j_p} Z^{j_1 \ldots j_p}.$$

Then,

$$Y_{i_1 \ldots i_m, i} = X_{i_1 \ldots i_m j_1 \ldots j_p, i} Z^{j_1 \ldots j_p}$$
$$+ X_{i_1 \ldots i_m j_1 \ldots j_p} Z^{j_1 \ldots j_p}_{,i}.$$

For example, for a scalar such as

$$Y = Z^i X_i,$$

one has:

$$Y_j = Z^i_{,j} X_i + Z^i X_{i,j}.$$

For a function f of $x^1, \ldots, x^n$, consider the scalar

$$\Delta_1 f = f^i f_i.$$

One has:

$$(\Delta_1 f)_j = f_j{}^i f_i + f^i f_{i,j}$$
$$= 2f^i f_{i,j}.$$

<u>Remark</u>: *This material has undergone extensive generalization, abstraction and algebrazation since 1900, both by the classical Tensor Analysis school (see Schouten [1]) and the modern geometers who base differential calculus on manifold theory.*

From the Tensor Analysis viewpoint, in order to free the notion of "covariant derivative" from its dependence on the choice of metric, one defines an affine connection to be a generalization of a tensor, namely as a system of functions assigned to each coordinate system, labelled as

$$\left\{ {i \atop jk} \right\},$$

which have the same transformation law under change of variables as do the Christoffel symbols derived from a metric tensor.

In modern differential geometry, an affine connection, typically denoted by ∇, $(X, Y) \to \nabla_X Y$, *is an R-bilinear mapping*

$$V(R^n) \times V(R^n) \to V(R^n),$$

such that:

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y$$

$$\nabla_{fX} Y = f\nabla_X Y,$$

$$\text{for } f \in F(R^n);\ X,\ Y \in V(R^n).$$

To see the relation between the two definitions, suppose first that we are given such a ∇. *For an arbitrary coordinate system* (x^i) *of* R^n, *let* $\left\{ {i \atop jk} \right\}$ *be the system of functions such that:*

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{\partial}{\partial x^i} .$$

One readily verifies that the system of functions

$$\begin{Bmatrix} i \\ jk \end{Bmatrix}$$

defined in this way transforms under change of coordinates precisely in the way required by tensor analysis.

The $\begin{Bmatrix} i \\ jk \end{Bmatrix}$ *are called the components of the connection.*

Given a Riemannian metric

$$\varphi: V(R^n) \times V(R^n) \to F(R^n),$$

there is a unique affine connection - called the Levi-Civita connection - such that:

$$X(\varphi(Y, Z)) = \varphi(\nabla_X Y, Z) + \varphi(Y, \nabla_X Z)$$

for $X, Y, Z \in V(R^n)$.

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for $X, Y \in V(R^n)$,

where $[X, Y]$ *is the Jacobi bracket of the two vector fields. (See DGCV). It turns out that the components* $\begin{Bmatrix} i \\ jk \end{Bmatrix}$ *of this connection are the Christoffel symbols, i.e. are defined in terms of the metric tensor by formula (5.1).*

6. RIEMANN CURVATURE TENSOR. SECOND COVARIANT DERIVATIVES

Let

$$\varphi = g_{ij}dx^i dx^j$$

be the given Riemannian metric. Let

$$\left\{ {i \atop jk} \right\}$$

be the Christoffel symbols, given by formula (5.1) in terms of the metric tensor

$$(g_{ij}).$$

Set:

$$\begin{aligned} R^{\ell}_{ijk} = \frac{\partial}{\partial x^j} \left\{ {\ell \atop ik} \right\} - \frac{\partial}{\partial x^k} \left\{ {\ell \atop ij} \right\} \\ + \left\{ {m \atop ik} \right\} \left\{ {\ell \atop mj} \right\} - \left\{ {m \atop ij} \right\} \left\{ {\ell \atop mk} \right\}. \end{aligned} \tag{6.1}$$

$$R_{\ell ijk} = g_{\ell m}R^{m}_{ijk}. \tag{6.2}$$

The (R^{ℓ}_{ijk}) form a mixed tensor, called the <u>Riemann curvature tensor</u>, which is of great importance in the theory of Riemannian metrics. It may be found, up to a factor, in <u>Commentatio Mathematica</u> by Riemann (See his Collected Works, p. 270). Its fundamental properties were worked out by Christoffel.

The number of independent components of the covariant

Riemann curvature tensor (i.e. (6.2)) is

$$N = \frac{n^2(n^2-1)}{12} .$$

In particular, for n = 2,

$$N = 1.$$

The only component is

$$R_{1212},$$

or the ratio

$$\frac{R_{1212}}{g},$$

that we call K, which is the well-known Gaussian curvature function for surfaces.

Let $X_{i_1 \ldots i_m}$ be an arbitrary covariant tensor. Consider the second covariant derivatives $X_{i_1 \ldots i_m, i, j}$. One can prove the following identities:

$$X_{i_1 \ldots i_m, i, j} - X_{i_1 \ldots i_m, j, i} = R^k_{i_1 ij} X_{k i_2 \ldots i_m} + R^k_{i_2 ij} X_{i_1 k i_3 \ldots i_m} + \ldots \tag{6.3}$$

They show that $X_{i_1 \ldots i_m, i, j}$ is not, in general, equal to $X_{i_1 \ldots i_m, j, i}$.

Remark. *Identity (6.3) is now called the Ricci identity. I shall sketch in the Remarks how it can be proved.*

If the Riemannian metric φ is flat, i.e. if the coordinate system (x^i) can be chosen so that

$$\varphi = \delta_{ij} dx^i dx^j ,$$

then the Riemann curvature tensor is identically zero, hence the left hand side of (6.3) is zero, i.e. covariant derivatives commute, in the same way as ordinary derivatives. Here is one consequence:

<u>If the Riemann curvature tensor vanishes, then an (m+1)-tensor</u>

$$X_{i_1 \ldots i_{m+1}}$$

<u>is the covariant derivative of an m-tensor if and only if</u>

$$X_{i_1 \ldots i_{m+1}, j} = X_{i_1 \ldots j, m+1} .$$

<u>Remark</u>: *To emphasize the geometric meaning of these ideas, one should note a fundamental property of tensors:*

> *If the components of a given tensor <u>all</u> vanish in one coordinate system, then they vanish in all coordinate systems.*

Of course, this is a trivial consequence of the postulated linear, homogeneous way that tensors transform. For example, note that affine connections $\left\{ {i \atop jk} \right\}$ *do not behave in this way - their components may vanish in one coordinate* *system, but not in another. However, the curvature tensor, which is formed via formula (6.1) from the Christoffel symbols, is a tensor.*

Here is the way that the Ricci identity (6.3) is proved in the modern algebraic way. Recall (see the Remarks of Section 5) that an affine connection is a map

$$(X, Y) \to \nabla_X Y$$

from pairs of vector fields to vector fields, which is R-bilinear, and such that:

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y \tag{6.4}$$

$$\nabla_{fX} Y = f\nabla_X Y \tag{6.5}$$

for $f \in F(R^n)$; $X, Y \in V(R^n)$.

Now, (6.4) is the "obstacle" to ∇ being of "tensorial" nature. We can get rid of the "non-tensorial" term $X(f)Y$ by iterating covariant derivatives. Here is the technique for doing this:

For $X, Y, Z \in V(R^n)$, set:

$$R(X, Y)(Z) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]} Z. \tag{6.6}$$

Now, verify that the map

$$(X, Y, Z) \to R(X, Y)(Z)$$

of $\quad V(R^n) \times V(R^n) \times V(R^n) \to V(R^n)$

is <u>*$F(R^n)$-multilinear*</u>. *(Notice again that* ∇ *itself is not; the first term on the right hand side of (6.4) is the obstacle. The magic is that this term is eliminated by the specific form taken by the right hand side of (6.6)). As explained in DGCV (see also Hicks [1], Bishop-Goldberg [1] for this approach) this "module linearity" is the algebraic equivalent of the "tensorial" transformation property introduced by Ricci and Levi-Civita.*

Given R defined by (6.6), define the "tensor" $R^i_{jk\ell}$ *by the following rule:*

$$R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right)\left(\frac{\partial}{\partial x^j}\right) = R^i_{jk\ell}\,\frac{\partial}{\partial x^i}. \qquad (6.7)$$

What we must now verify is that, with ∇ *satisfying (6.4), (6.5), and being the "Levi-Civita" affine connection associated with the metric, as explained in Section 5, and with* $R(\ ,\ )(\ )$ *defined by (6.6),* $R^i_{jk\ell}$ *by (6.7), the* $R^i_{jk\ell}$ *are given by the Riemann-Christoffel formula (6.1).*

Completing this calculation (which is left to the reader) will verify the Ricci identity (6.3) <u>*in case X is a 1-contravariant tensor field*</u>. *In fact, notice that what we have done is use a typical trick of modern mathematics, namely take a property of something that was discovered by the old-fashioned, calculational, way, and, turning it upside down, make it into a* <u>*definition*</u>. *In the case of the Ricci identity, what we have done is to "axiomatise",* *in this way only a special case. We must show that the proof in the general case follows from the special case and the properties of the covariant derivative operation.*

I have already explained, in these Remarks, how general tensors are defined in terms of the algebra of tensor products of modules. Now I want to discuss an alternate useful way of defining them.

Recall that a vector field $X \in V(R^n)$ *is a derivation map:* $F(R^n) \to F(R^n)$. *It can be written, in local coordinates* (x^i), *as*

$$X = X^i \frac{\partial}{\partial x^i}.$$

$F^1(R^n)$ *is defined as the dual module to* $V(R^n)$, *i.e. each* $\theta \in F^1(R^n)$ *is an* $F(R^n)$*-linear map*

$$V(R^n) \to F(R^n).$$

In coordinates (x^i), *the* <u>*components*</u> *of* θ *are the functions* θ_i *defined by:*

$$\theta(\frac{\partial}{\partial x^i}) = \theta_i.$$

This assignment $\theta \to (\theta_i)$ *identifies* $F^1(R^n)$ *with the 1-covariant tensors.*

Consider now an m-multilinear map

$$\underbrace{\omega\colon V(R^n) \times \ldots \times V(R^n) \to F(R^n)}_{m\text{-times}}. \tag{6.8}$$

Assign to such a map an m-covariant tensor, with coefficients

$$\omega_{i_1 \dots i_m}$$

in coordinates (x^i), *by the following rule:*

$$\omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_m}}\right) = \omega_{i_1 \dots i_m}.$$

one readily verifies that this assignment

$$\omega \to (\omega_{i_1 \dots i_m})$$

identifies (with the automatically correct rules for the transformation under change of variables!) the space of such multilinear maps and the space of m-covariant tensors.

Given an ω *indicated by (6.8), and an affine connection* $(X, Y) \to \nabla_X Y$, *one may now define the covariant derivative of* ω *by* X *as another m-multilinear map, via the following formula:*

$$\begin{aligned}\nabla_X(\omega)(X_1, \dots, X_m) &= X(\omega(X_1, \dots, X_m)) - \omega(\nabla_X X_1, X_2, \dots, X_m) \\ &\quad - \dots - \omega(X_1, X_2, \dots, \nabla_X X_m).\end{aligned}$$

It turns out (again, a small "miracle", considering that ∇ *is not* $F(R^m)$*-bilinear) that the map*

$$(X, X_1, \dots, X_m) \to \nabla_X(\omega)(X_1, \dots, X_m)$$

of

$$\underbrace{V(R^n) \times \ldots \times V(R^n) \to F(R^n)}_{(m+1)\text{-times}}$$

<u>*is*</u> *$F(R^n)$-multilinear, i.e. defines an $(m+1)$-covariant tensor field. This tensor field, when written in terms of a coordinate system, is the covariant derivative as defined by Ricci and Levi-Civita, i.e.*

$$\omega_{i_1 \ldots i_m, i} = \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_m}}\right). \tag{6.10}$$

The full Ricci identity, (6.3), now follows from (6.1), (6.7), and (6.6). One can also readily extend the Ricci identity to mixed tensors, using this method. Identify an m-covariant, p-contravariant tensor with an $F(R^n)$-multilinear map

$$\tau: \underbrace{V(R^n) \times \ldots \times V(R^n)}_{m\text{-times}} \times \underbrace{F^1(R^n) \times \ldots \times F^1(R^n)}_{p\text{-times}} \to F(R^n),$$

and define the covariant derivative

$$\nabla_X(\tau)$$

by a formula analogous to (6.6).

7. THE INVARIANT NATURE OF THE EQUATIONS OF TENSOR ANALYSIS

The equations which define the transformation law of tensors tell us that the property of all the components of a tensor equalling zero is independent of the choice of variables $(x^1, \ldots, x^n)$. It is this property that is meant when one says that a system of equations

$$X_{i_1 \ldots i_n} = 0$$

has an <u>invariant</u> or <u>absolute</u> nature.

When one meets a new problem, to put the equations in invariant form it suffices to express its main features in terms of general coordinates, and then to substitute covariant differentiation for ordinary differentiation, in a way that is almost always evident by the nature of the problem. As we shall see in various applications, this path should be followed when dealing with general theories and when attempting a systematic development of such theories.

Here is a more practical general problem. One has derived a system of equations (E) associated to a problem, expressed in terms of variables y. One wants to transform the equations using general coordinates without repeating the steps which led to the derivation of the equations (E). In order to do this it suffices to determine a tensor X such that, in the y-variables, the components of X coincide - perhaps only up to a common factor - with the terms of the equations (E). It is then evident that, to have the equivalent of equations (E) in coordinates (x), it suffices to equate to zero the components of tensor X in the coordinates x.

Certainly, this method will not succeed in all cases, but it often works in a quick and easy way. As we shall see, this is particularly so for the equations of mathematical physics. In fact, we are astonished to see the difficult and devious routes that were formerly used to arrive at the same goal.

Remark: *This has indeed turned out to be a prophetic statement! The method they describe here has served not only to write known equations in elegant form but to derive new equations. The most prominent example of this is the Einstein gravitational equations. Today, this approach is most alive in the field of Continuum Mechanics.*

It would be interesting to formalize this argument more precisely, using the better philosophical and metamathematical tools that we have now. The ideas of the theory of "categories" and "functors" should play some role here. For example, "physical theories" should be some sort of "category". "Equations" should be another category. The assignment

$$\text{"physical theories"} \to \text{"equations"}$$

should be a "functor". The equations described in terms of *tensor analysis should be a "subcategory". I have found it even more useful (e.g. in the other books in this series) to express these, in Cartan's way, in terms of vector fields and differential forms. Of course, in principle this is a special case of Ricci and Levi-Civita's method, but in practice the two are more distinct.*

Chapter II

INTRINSIC GEOMETRY AS A CALCULATIONAL TOOL

1. GENERALITIES ON ORTHOGONAL CONGRUENCES

As basic references for this material, see Ricci [1895] and [1896].

In this chapter we make use of geometric language, with a Riemannian metric tensor φ defining the basic "geometry."

Let (λ^i) be a 1-contravariant tensor, which is non-zero at each point.

Consider the differential equations:

$$\frac{dx^1}{\lambda^1} = \frac{dx^2}{\lambda^2} = \ldots = \frac{dx^n}{\lambda^n}. \tag{1.1}$$

These equations define in R^n a <u>congruence of curves</u>.

<u>Remark</u>: From now on, I will use the term <u>vector field</u> or <u>contravariant vector</u> as an alternate to 1-contravariant tensor. I will also use the terms <u>1-differential form</u> or <u>covariant vector field</u> as synonyms for "1-covariant tensor fields."

Notice that the transformation properties of contravariant vectors are precisely those which guarantee that equations (1.1) be invariant in nature, i.e. independent of coordinates.

<u>Remark</u>: *Here is what this means more explicitly. Suppose* (x^i) *is one coordinate system for* R^n. *Consider a curve in* R^n, *parameterized <u>in these coordinates</u> by functions*

$$t \to x^i(t).$$

To say that this curve satisfies (1.1) is to say that:

$$\frac{dx^1}{dt}/\lambda^1(x(t)) = \frac{dx^2}{dt}/\lambda^2(x(t)) = \ldots \tag{1.2}$$

Now, choose different coordinates (y^i). *The <u>same curve</u> will have coordinates*

$$t \to y^i(t).$$

The vector field will have components

$$(\alpha^i)$$

in these coordinates, which of course are related to λ^i *via the contravariant transformation law:*

$$\alpha^i = \lambda^j \frac{\partial y^i}{\partial x^j}$$

Then, equations (1.2) will be satisfied if and only if:

$$\frac{dy^1}{dt}/\alpha^1(y(t)) = \frac{dy^2}{dt}/\alpha^2(y(t)) = \ldots$$

Of course, a more formal and elegant way of deducing this *"invariance", without any calculation, is to notice that both*

$$dx^i \text{ and } \lambda^i$$

change contravariantly, hence that their quotient

$$\frac{dx^i}{\lambda^i} \text{ (not summed)}$$

is "invariant." This is the method indicated by Ricci and Levi-Civita in the text.

Since the equations (1.1) do not change when the vector field λ is multiplied by a common factor, we suppose this factor to be chosen so that:

$$g_{ij}\lambda^i\lambda^j = 1 = \lambda^i\lambda_i. \tag{1.3}$$

We will say that the vector field (λ^i) is the <u>contravariant vector coordinate</u> of the congruence of curves represented by the equations (1.1), and that its reciprocal differential form (λ_i) is its <u>covariant vector coordinate</u>.

Denote by ds the element of arc-length of a curve of the congruence, i.e. the positive value of $\sqrt{\varphi}$. From (1.1) and (1.3) we see that ds is the absolute value of the ratios which appear in (1.1). We see that:

$$\frac{dx^i}{ds} = \pm\lambda^i(x(s)), \qquad i = 1, \ldots, n. \tag{1.4}$$

If one takes the positive sign in (1.4) as we shall do from now on, this determines at each point of R^n a direction pointing tangent to the congruence, that we call the <u>positive direction</u>.

If the metric φ is Euclidean, and if (x^i) are orthogonal Cartesian coordinates, the λ^i (which coincide with λ_i) are just the cosines of the angles that the curves of the conguence make with the coordinate axes.

By definition (due to Beltrami), the angle α between time tangent directions dx^i and δx^i leaving from the same point P of R^n is given (in terms of the metric φ) by the following formula:

$$\cos \alpha = \frac{g_{ij} dx^i \delta x^j}{(g_{ij} dx_i{}^i dx^j)^{1/2} (g_{ij} \delta x^i \delta x^j)^{1/2}}. \tag{1.5}$$

If one is given by two congruences, defined by contravariant vector fields

$$(\lambda^i) \text{ and } (\mu^i),$$

and if α denotes the angle between the curves of the congruence leaving from the same point P of R^n, (1.3)-(1.5) imply that:

$$\cos \alpha = \lambda^i \mu_i = \lambda_i \mu^i = g_{ij} \lambda^i \mu^j. \tag{1.6}$$

The condition of <u>orthogonality</u> between the two congruences is then:

$$\lambda^i \mu_i = 0. \tag{1.7}$$

Now, we must consider n separate congruences, defined by n unit-length contravariant vector fields

$$\lambda^i[1], \ldots, \lambda^i[n].$$

Suppose them pairwise orthogonal. Then, the following conditions are satisfied:

$$\lambda^i[j]\lambda_i[k] = \delta_k^{\,j}, \tag{1.8}$$

where $\delta_k^{\,j}$ is the Kronecker symbol, zero if $j \neq k$, equal to 1 if $j = k$.

<u>Remark</u>: Here is where tensor analysis notations begin to be awkward and inconvenient, leading to "un debauch d'indices," that Cartan complained about in his book "Géométrie des éspaces de Riemann." Here, the j, k indices are <u>not</u> tensorial in nature, but simple "counting" indices. Ricci and Levi-Civita handle this by denoting

$$\lambda_i[k] \text{ as } \lambda_{k/i}.$$

I find this awkward, hence have substituted the one indicated. In the final Remarks to this section I will indicate how these notations <u>are</u> enormously simplified in modern differential geometry.

We call each set of n congruences with the properties just described on <u>orthonormal moving frame</u>. We denote by [1], [2],..., [n] the congruences which make up the moving frame; by 1, 2,..., n the curves of the congruences passing through a given point of R^n; and by $s^1, \ldots, s^n$ the arc-length parameter along these curves.

<u>Remark</u>: The term used in the text is <u>ennuple orthogonale</u>. I have substituted the term ("rêpère mobile") used by Cartan in his exposition of Riemannian geometry.

Expansion of a Tensor Field in Terms of an Orthonormal Moving Frame

Let $X_{i_1 \ldots i_m}$ be a covariant tensor field, and let

$$\lambda^i[j]$$

be an orthonormal moving frame. One can then write:

$$X_{i_1 \ldots i_m} = C_{j_1 \ldots j_m} \lambda_{i_1}[j_1] \ldots \lambda_{i_m}[j_m]. \tag{1.9}$$

<u>Remark</u>: It is meant that the summation convention applies to $j_1, \ldots, j_m$.

The coefficient functions C are determined by the following formulas:

$$C_{j_1 \ldots j_m} = X_{i_1 \ldots i_m} \lambda^{i_1}[j_1] \ldots \lambda^{i_m}[j_m], \qquad (1.10)$$

which tell us that they are invariants. One can extend them easily to describe contravariant or mixed tensors.

In particular, if one considers the metric tensor g_{ij}, one has:

$$g_{ij} = \sum_{k=1}^{n} \lambda_i[k] \lambda_j[k] \qquad (1.11)$$

$$g^{ij} = \sum_{k=1}^{n} \lambda^i[k] \lambda^j[k]. \qquad (1.12)$$

<u>Remark</u>: *Notice that the summation convention breaks down at this point, and we are forced to use summation signs!*

The determinants

$$\det(\lambda_i[j])$$

$$\det(\lambda^i[j])$$

are then equal, respectively, to $\sqrt{g}$ and $\sqrt{g^{-1}}$.

Returning to equations (1.9) and (1.10), we see that each system of tensorial equations

$$X_{i_1 \ldots i_m} = 0$$

may be replaced by equations:

$$C_{i_1 \ldots i_m} = 0,$$

i.e. each tensorial system of equations may be transformed in such a way that its left hand side is composed of invariants. We shall often make advantageous use of this transformation.

Remark: It is confusing to understand exactly what is meant here, so I will try to explain further. Recall that by an "invariant" they mean quantities which remain the same when the coordinate system is changed. The m-covariant tensor fields form an $F(R^n)$-module T_m. Each tensor field $X_{i_1 \ldots i_m}$ determines an element of T_m. Now, an orthogonal moving frame, once fixed, determines a basis for T_m (via tensor-product of modules, as we shall explain in the final Remarks to this section), which is independent of the choice of coordinates, and the $C_{i_1 \ldots i_m}$ are the coefficients of the expansion of the tensor field in terms of this moving frame basis for T_m. So, the coefficients do not involve choice of the coordinates (x^i), i.e. they are "invariants." This trick is that involved in the "Schrödinger-Heisenberg picture," which is familiar to physicists from quantum mechanics.

Let us remark also that, since

$$\frac{dx^i}{ds^j} = \lambda^i[j], \tag{1.13}$$

if f is a function of $(x^1, \ldots, x^n)$, we have:

$$\lambda^i[j]f_{,i} = \frac{d}{ds^j}(f(x(s))) \qquad (1.14)$$

$$j = 1, \ldots, n.$$

Remark: Here is what is meant. Suppose $s \to x^i(s)$ are the coordinates of a curve, parameterized by the arc-length of the Riemannian metric $\varphi \equiv ds^2$, which belongs to one of the congruences - say the j-th - which make up the moving frame. Then, (1.13) are the ordinary differential equations which the coordinates of the curve must satisfy. Equation (1.14) evaluates the direction derivative

$$\frac{d}{ds} f(x(s))$$

of any function f along this curve.

First Order Properties of the Metric

The metric properties of the lines 1, 2,..., n - which are related to what one ordinarily calls the curvature of space curves - are described by the derivatives of the $\lambda_i[j]$. These derivatives are not all independent; they should satisfy the $n^2(n+1)/2$ equations obtained by differentiating the relations (1.11).

Set:

$$\gamma^j_{k\ell} = \lambda^h[k]\lambda^i[\ell]\lambda_{h,i}[j], \tag{1.15}$$

and apply the covariant derivative. We find first the $n^2(n+1)/2$ equations mentioned above, in the following form:

$$\lambda^i[k]\lambda_{i,j}[h] + \lambda^i[h]\lambda_{i,j}[k] = 0, \tag{1.16}$$

and one sees readily that they may be replaced by:

$$\gamma^j_{k\ell} + \gamma^k_{j\ell} = 0, \tag{1.17}$$

which includes as special case

$$\gamma^i_{i\ell} = 0 \tag{1.18}$$

(no summation).

The number of the independent invariants among the $\gamma^j_{k\ell}$ is then equal to

$$\frac{n^2(n-1)}{2}.$$

Since this number is equal to

$$n^3 - \frac{n^2(n+1)}{2},$$

with n^2 the number of derivatives of $\lambda_i[j]$ and $n^2(n+1)/2$ the number of relations among these derivatives, one may express the

$$\lambda_{i,j}[k]$$

in terms of the $\lambda_i[j]$ and the invariants γ. Using the equations (1.15), we obtain these expressions in the following form:

$$\gamma_{k,\ell}[h] = \sum_{i,j} \gamma^{h}_{ij} \lambda_k[i] \lambda_\ell[j]. \tag{1.19}$$

In order to study the metric properties of the lines 1, 2,..., n, it then suffices to consider the invariants γ^{i}_{jk}. In fact, the relation linking the metric properties to the invariants are very direct and simple. We will not at the moment examine in detail the geometric or kinematic significance of each of the γ; we have said enough for the applications which follow. Let us add that, because of their kinematical meaning, the invariants γ will be called the rotational coefficients of the moving frame

[1], [2],..., [n].

Remarks: *Despite their final claim of "directness and simplicity," in fact we would now say that their formalism is awkward for the analytic description of this material, and that Cartan's (which will now be explained) is much simpler. However, the underlying geometric ideas are the same - choose objects (orthogonal congruences in Ricci and Levi-Civita's framework, bases of differential forms in Cartan's) which are "naturally" adapted to the Riemannian metric.*

Begin with a modern definition of "congruence." Consider $V(R^n)$, the vector fields on R^n (or 1-contravariant tensors). Recall that they are <u>defined</u> as the space of <u>derivations</u> or <u>first order linear homogeneous differential operators</u>: $F(R^n) \to F(R^n)$. $V(R^n)$ is an $F(R^n)$-module. $F^1(R^n)$, the space of 1-differential forms (or 1-covariant tensor fields) is the dual module to $V(R^n)$.

An element $f \in F(R^n)$ is said to be <u>invertible</u> if:

$$f^{-1} \in F(R^n). \tag{1.20}$$

(Of course, the condition for (1.20) is that $f(x) \neq 0$ for all $x \in R^n$). Define an equivalence relation on $V(R^n)$ as follows:

> *$X \sim Y$ iff. there exists an invertible $f \in F(R^n)$ such that $X = fY$.*

<u>Definition</u>. A <u>congruence</u> is an equivalence class of vector fields. A vector field X is said to <u>belong</u> to the congruence if it belongs to the equivalence class. (Thus, algebraically a "congruence" is an element if the "projective space" associated with the $F(R^n)$-module $V(R^n)$).

To describe what Ricci and Levi-Civita mean by the curves of the congruence, let us recall the basic notion *of* _*integral curve*_ *of a vector field. (See DGCV). In order to avoid confusion with the letter "x" standing for coordinate, I will denote a* _*point*_ *of* R^n *(in the geometric sense) by P. (The authors use this device also). t or s usually denote curve parameters, varying over some interval of real numbers.*

*Definition.* *Given* $X \in V(R^n)$, *a curve*

$$t \to P(t)$$

in R^n *is an* _*integral curve*_ *of X if the following conditions are satisfied:*

$$\frac{d}{dt} f(P(t)) = X(f)(P(t)) \qquad (1.21)$$

for all t, all $f \in F(R^n)$.

Thus, if (x^i) *are coordinates for* R^n, *if*

$$X = \lambda^i \frac{\partial}{\partial x^i},$$

i.e. (λ^i) *is the 1-contravariant tensor, and if* $t \to x^i(t) (\equiv x^i(P(t)))$ *are the coordinates of the curve, then conditions (2.1) are equivalent to:*

$$\frac{dx^i}{dt} = \lambda^i(x(t)). \qquad (1.22)$$

Notice that these differential equations - when freed of the dependence on the parameter t, are precisely equations

(1.21), which define what Ricci and Levi-Civita mean by "curve of the congruence." Let us formalize this in the following way.

Definition: Suppose given a congruence on R^n, as formalized in the previous Definition. A curve $t \to P(t)$ is then said to be a *curve belonging to the congruence* if there is a vector field X belonging to the congruence such that the curve is an integral curve of X.

Another way of doing this is to say that a *congruence is a 1-dimensional foliation*.

So far, the Riemannian metric φ has not been used. Recall that φ is a positive-definite, symmetric, $F(R^n)$-bilinear map

$$V(R^n) \times V(R^n) \to F(R^n).$$

A congruence is said to be *regular* if there is a vector field X belonging to the congruence such that:

$$\varphi(X, X) > 0$$

at all point of R^n.

(Congruences well assumed to be regular, unless mentioned otherwise.) One may then normalize X, multiplying if necessary by a scalar factor, so that

$$\varphi(X, X) = 1. \tag{1.22}$$

(Condition (1.22) is equivalent to (1.3)). X is then determined up to ±1. Fixing the "orientation" or "direction" of integral curves then uniquely determines X.

The integral curves of a vector field X satisfying (1.22) are automatically in arc-length parameterization with respect to the metric φ. The authors' convention is that "s" is to be used as parameter for such curves. (This is the meaning of (1.4)).

Two congruence are <u>orthogonal</u> if

$$\varphi(X, Y) = 0$$

for X in the first congruence, Y in the second. Geometrically, this means that curves of the two congruences always meet at right angles.

A <u>moving frame</u> is a set

$$(X_1, \ldots, X_n)$$

of vector fields which form an $F(R^n)$-basis of $V(R^n)$. We can now denote a moving frame by

$$(X_i).$$

Here "i" is a <u>counting index</u>. We shall see that there is a good reason for putting it below, i.e. treating it as a "covariant" index. Now, this counting index has a different geometric nature from the indices (which range over *the same values, i.e. 1 to n) which parameterize the components of tensors with respect to coordinate systems on R^n. This fact is what forces Ricci and Levi-Civita to various notational contortions.*

For example, suppose (x^i) is a coordinate system for R^n. They denote the 1-contravariant tensor components of X_i in this coordinate system by

$$\lambda_i^{(j)}.$$

Thus, we have:

$$X_i = \lambda_i^{(j)} \frac{\partial}{\partial x^j} \tag{1.23}$$

Apparently, Ricci and Levi-Civita were nervous about introducing "mixed" tensors, so they did not use the more natural notation

$$\lambda_i^{\ j}.$$

So far, this is not too bad. However, let

$$\omega^i$$

be the <u>reciprocal</u> 1-covariant tensors (i.e. 1-differential forms) with respect to the metric φ. They form a basis for $F^1(R^n)$. Ricci and Levi-Civita would denote them as follows:

$$\omega_{i/} = \lambda_{i/j} dx^j \tag{1.24}$$

Here the slash / signifies somehow that i and j have a different geometric significance, which is true enough. However, the slash / is _sometimes_ *(admittedly, rather rarely) used as notation for covariant derivative. At any rate, it is too close for comfort to the covariant derivative notation. That is why I changed the notation in my translation to*

$$\lambda_j[i].$$

To be consistent, I also changed

$$\lambda_i^{(j)} \text{ to } \lambda_i[j].$$

This will require that the reader be careful when reading the translation - but this is unfortunately in the nature of things, since it is linked to the notational limitations of classical tensor analysis.

However, Cartan (notably in his book "Géométrie des éspaces de Riemann", but also in his other differential geometry work) brilliantly resolved these notational problems by saying: Forget about the coordinates, and use _moving frames_ *to describe the geometry, and tensors as well. This restores the "purity" of the index notation for tensors.*

For example, suppose (X_i) *is a basis for* $V(R^n)$*, and*

(ω^i) *is the reciprocal basis for 1-forms.*

Warning. ω^i *is not the dual basis of differential forms, i.e. the 1-forms* θ^i *such that*

$$\theta^i(X_j) = \delta_j{}^i.$$

In fact, we have:

$$\omega^i(X_j) = \theta(X_i, X_j).$$

We see that $\omega^i = \theta^i$ *iff.* $\theta(X_i, X_j) = \delta_{ij}$, *i.e.* (X_i) *is an orthonormal moving frame.*

An m-covariant tensor T *can then be written as follows:*

$$T = c_{i_1 \ldots i_m} \omega^{i_1} \otimes \ldots \otimes \omega^{i_m},$$

whereas a contravariant one can be written as:

$$T = c^{i_1 \ldots i_m} X_{i_1} \otimes \ldots \otimes X_{i_m}.$$

Now, Cartan reasons, as long as one is expanding tensor analysis by allowing such "non-holonomic representations" (this is the terminology that is sometimes used in the classical literature) why not choose them in the most convenient way. In the case of Riemannian geometry, this convenient choice is where (X_i) *forms a basis of* $V(R^n)$ *which is orthogonal with respect to the form* φ, *i.e. which satisfies:*

$$\varphi(X_i, X_j) = \delta_{ij}. \tag{1.25}$$

This is called an *<u>orthonormal moving frame</u>*. *Its geometric significance in terms of congruences should be clear.*

The reciprocal basis ω^i *of 1-forms then satisfies the following relations:*

$$\varphi = \sum_{k=1}^{n} \omega_k \cdot \omega_k \tag{1.26}$$

(• means the *<u>symmetric product</u>* *of differential forms.)*

We shall now show that equation (1.26) is equivalent to equation (1.11). This provides a key link between the two formalisms. Suppose (x^i) *is a coordinate system for* R^n. *Then,*

$$\omega_k = \lambda_i[k]dx^i. \tag{1.27}$$

Now, the metric tensor (g_{ij}) *is determined by:*

$$\varphi = g_{ij}dx^i \cdot dx^j. \tag{1.28}$$

Substitute (1.27) into (1.26), and compare with (1.28):

$$g_{ij}dx^i \cdot dx^j = \sum_{k=l}^{n} \lambda_i[k]\lambda_j[k]dx^i \cdot dx^j.$$

Comparing coefficients on both sides gives relation (1.11).

In Cartan's theory the rotational coefficients γ, *defined by (1.15), take a much more direct and important meaning. Let* (ω^i) *be a basis for 1-forms satisfying (1.26). It can be proved (see DGCV) that there are a set*

$$(\omega_j^{\ i})$$

of 1-forms satisfying (and uniquely determined by) the following conditions:

$$d\omega^i = \omega_j^{\ i} \wedge \omega^j \qquad (1.29)$$

$$\omega_j^{\ i} + \omega_j^{\ i} = 0. \qquad (1.30)$$

Here, d and $\wedge$ are the exterior derivative and exterior product operation, which are basic in Cartan's theory and explained in detail in all modern differential geometry books, e.g. DGCV. The $\omega_j^{\ i}$ are called the __*connection forms,*__ *since the affine connection ∇ associated to the Riemannian metric (hence also the covariant derivative) is determined by the following relation:*

$$\omega_j^{\ i}(X) = \omega^i(\nabla_X X_j) \qquad (1.31)$$

for all $X \in V(R^n)$.

The rotational coefficients are now determined as follows:

$$\omega_j^{\ i} = \gamma_{jk}^{\ \ i}\omega^k. \qquad (1.32)$$

They are "invariants", in the sense that they are independent of __*coordinates,*__ *although they are of course dependent on the choice of moving frame, - and in fact transform in a non-tensorial way on change of moving frame. However, the* __*curvature forms*__

$$\Omega_j^{\ i} = d\omega_j^{\ i} - \omega_j^{\ k} \wedge \omega_k^{\ i} \qquad (1.33)$$

have the property that their coefficients $R_{jk\ell}^{\ \ \ j}$,

$$\Omega_j^{\ i} = R_{jk\ell}^{\ \ \ i}\omega^k \wedge \omega^\ell,$$

do change in a tensorial, i.e. linear-homogeneous, way on change of orthogonal moving frame.

2. INTRINSIC DERIVATIVES AND THEIR RELATIONS

We must establish the relations between

$$\frac{\partial}{\partial s^k}\frac{\partial f}{\partial s^j} \text{ and } \frac{\partial}{\partial s^j}\frac{\partial f}{\partial s^k},$$

because one may not commute the operations represented by the symbols

$$\frac{\partial}{\partial s^j} \text{ and } \frac{\partial}{\partial s^k}.$$

In fact, if one differentiates the identity (1.14), one has first:

$$\frac{\partial}{\partial x^i}\frac{\partial f}{\partial s^j} = \lambda^k[j]f_{k,i} + f^k\lambda_{k,i}[j],$$

and then:

$$\frac{\partial}{\partial s^\ell}\frac{\partial f}{\partial s^j} = \lambda^i[\ell]\,\frac{\partial}{\partial x^i}\frac{\partial f}{\partial s^j}$$
$$= \lambda^i[\ell]\lambda^k[j]f_{k,i} + \lambda^i[\ell]f^k\lambda_{k,i}[j].$$

We derive from this the following relations, which determine how $\frac{\partial}{\partial s^k}$ and $\frac{\partial}{\partial s^j}$ commute:

$$\frac{\partial}{\partial s^j}\frac{\partial f}{\partial s^k} - \frac{\partial}{\partial s^k}\frac{\partial f}{\partial s^j} = \sum_i (\gamma_{jk}{}^i - \gamma_{kj}{}^i)\frac{\partial f}{\partial s_i}. \qquad (2.1)$$

__Remarks:__ I have not given the full details of the argument given by the authors leading to (2.1), since it is much more readily derived by Cartan's methods.

First, let us interpret what

$$\frac{\partial f}{\partial s^j}$$

__means__. Recall that (X_j) is an orthonormal moving frame. The curve

$$s^j \to P(s^j)$$

is the integral curve of X_j. Since X_j has length +1 in the metric, s^j is automatically the arc-length parameter. Thus

$$\frac{\partial f}{\partial s^j} \equiv X_j(f). \qquad (2.2)$$

Hence, the left hand side of (2.1) is:

$$X_j(X_k(f)) - X_k(X_j(f)). \qquad (2.3)$$

Now, it is well-known (see DGCV again) that (2.3) equals

$$[X_j, X_k](f),$$

where $[X_j, X_k]$ *is the Jacobi bracket of the two vector fields. Further, using the relations between Jacobi bracket and exterior derivative,*

$$d\omega^i(X_j, X_k) = X_j(\omega^i(X_k)) - X_k(\omega^i(X_j)) - \omega^i([X_j, X_k]) = -\omega^i([X_j, X_k]).$$

Hence,

$$[X_j, X_k] = -d\omega^i(X_j, X_k)X_i. \tag{2.4}$$

Recall now, from the Remarks of Section 1, that:

$$d\omega^i = \omega_j{}^i \wedge \omega^j, \tag{2.5}$$

$$\omega_j{}^i = \gamma_{jk}{}^i\omega_k. \tag{2.6}$$

Relation (2.1) results from putting together these relations.

3. NORMAL AND GEODESIC CONGRUENCES. ISOTHERMAL FAMILIES OF SURFACES. CANONICAL SYSTEM OF A CONGRUENCE

A congruence of curves in a Riemannian metric space is said to be <u>normal</u> if it is composed of the orthogonal trajectories to a one-parameter family of surfaces

$$f(x^1, \ldots, x^n) = \text{constant}.$$

Suppose given an orthogonal moving frame, composed of congruences [1],..., [n], we propose to determine the necessary and sufficient conditions that the congruence [n] is orthogonal.

Clearly, for this to happen it is necessary and sufficient that each tangent direction

$$\delta x^i$$

normal to a curve of [n] be tangent to the curve f(x) = constant, i.e. that:

$$\frac{\partial f}{\partial x^i} \delta x^i = 0.$$

As before, let $\lambda^i[j]$ be the components of the vector field which makes up the j-th congruence. Since the congruences [1],..., [n] are <u>orthogonal</u>, the vectors

$$\lambda^i[1], \ldots, \lambda^i[n-1]$$

fill up the orthogonal space to $\lambda^i[n]$. Hence, we must have:

$$\lambda^i[j] f_i = 0$$

$$\text{for } j = 1, \ldots, n-1.$$

Let X_j be the vector fields which make up the moving frame, i.e.

$$X_j = \lambda^i[j] \frac{\partial}{\partial x_i}.$$

Then, the condition that [n] be normal is that:

$$X_j(f) = 0$$

for $j = 1, \ldots, n-1$.

The condition that such an f exists is that the first order differential operators represented by the differential operators $X_1, \ldots, X_{n-1}$ be completely integrable, i.e. that:

The Jacobi brackets

$[X_j, X_k]$ are linear combinations of $X_1, \ldots, X_{n-1}$, for $1 \leq j, k \leq n-1$. (3.1)

Rewrite (2.1) as

$$[X_j, X_k] = (\gamma_{jk}{}^i - \gamma_{kj}{}^i)X_i. \tag{3.2}$$

Compare (3.1) and (3.2). It implies the following $\frac{(n-1)(n-2)}{2}$ conditions:

$$\gamma_{jk}{}^n = \gamma_{kj}{}^n \tag{3.3}$$

for $1 \leq j, k \leq n-1$.

Let us sum up as follows:

<u>The necessary and sufficient conditions that the congruence [n] be normal is that (3.3) be satisfied.</u>

The following condition also holds:

<u>Each congruence of the orthogonal moving frame is normal if and only if:</u>
$\gamma_{jk}{}^{i} = 0$ <u>for each triple</u> (i, j, k) <u>of indices which are all distinct</u>.

If conditions (3.3) are satisfied, then the $\lambda_j[n]$ are proportional to the derivatives f_j of a function f, i.e. there is a function μ such that:

$$f_j = \mu\lambda_j[n],$$

which also satisfies:

$$f_{j,k} = f_{k,j}.$$

Using formula (1.19), we have:

$$f_{j,k} = \mu_k\lambda_j[n] + \mu\gamma_{i\ell}{}^{n} f_j[i] f_k[\ell]. \tag{3.4}$$

Set:

$$\psi = \log \mu.$$

The function ψ satisfies the following equations:

$$\psi_j\lambda_k[n] + \gamma_{i\ell}{}^{n}\lambda_j[i]\lambda_k[\ell]$$
$$= \psi_k\lambda_j[n] + \gamma_{i\ell}{}^{n}\lambda_k[i]\lambda_j[\ell]$$

After multiplying these equations by $\lambda^j[n]$ and summing on j, one obtains the following equation:

$$\psi_k = \nu\lambda_k[n] + \sum_{i=1}^{n-1} \gamma_{in}{}^{n}\lambda_k[n], \tag{3.5}$$

with ν remaining indeterminate.

Isothermal Families of Surfaces

A one-parameter family of surfaces $f(x^1,\ldots, x^n) =$ (constant) in R^n is said to be <u>isothermal</u>, and f is said to be an <u>isothermal parameter</u>, if it satisfies the following equation:

$$g^{ij}f_{i,j} = 0. \tag{3.6}$$

(We shall see later on that this equation is a generalization of the equation for harmonic functions in Euclidean space).

Now, any family of surfaces

$$f(x) = \text{constant}$$

determines the congruence of orthogonal trajectories, i.e. the curves which are perpendicular to the surfaces. Thus one can always find an orthonormal moving frame, whose rotational coefficients satisfy the following relations:

$$\gamma_{jk}{}^{n} = \gamma_{kj}{}^{n} \tag{3.7}$$

for $1 \le j,\ k \le n-1$.

<u>*Remark:*</u> *Here is what is meant: Choose the basis*

$(X_1, \ldots, X_n)$ of $V(R^n)$ to be orthonormal with respect to the metric φ, and so that:

$$X_n = \frac{grad\ f}{\varphi(grad\ f,\ grad\ f)^{1/2}},$$

where "grad f" is the gradient vector field associated with f, as defined by the metric φ. (See DGCV for definition of "gradient").

We now propose to establish the necessary and sufficient conditions that a family of surfaces be isothermal, and to determine its isothermal parameter.

Substitute in (3.6) the expressions (3.4) for the covariant derivatives of f_j; we obtain the following equivalent equation:

$$\frac{\partial\psi}{\partial s^n} = - \sum_{i=1}^{n} \gamma_{ii}^{\ \ n}$$

Remark: Recall that $\frac{\partial\psi}{\partial s^n} = X_n(\psi)$.

This determines the function ν of formula (3.5) as follows:

$$\nu = - \sum_{i=1}^{n-1} \gamma_{ii}^{\ \ n}. \qquad (3.8)$$

The necessary and sufficient condition that a family of surfaces whose orthogonal trajectory congruence is [n] be isothermal is that, after substituting the value of ν given by (3.8) into the right hand side of (3.5), equation (3.5) be solvable for ψ. After determining ψ, f is determined by:

$$f_j = Ce^{\psi}\lambda_j[n],$$

i.e.

$$f = C\int e^{\psi}\lambda_j[n]dx^j + c,$$

with C and c arbitrary constants.

One sees easily that the integrability conditions that (3.5) be solvable for ψ are the following equations:

$$X_j(\nu) + X_n(\gamma_{nn}{}^j) + \nu\gamma_{nn}{}^j + \sum_{i=1}^{n-1} \gamma_{nn}{}^i(\gamma_{jn}{}^i - \gamma_{nj}{}^i) = 0 \tag{3.9}$$

$$X_k(\gamma_{nn}{}^j) + \sum_{i=1}^{n-1} \gamma_{nn}{}^i\gamma_{jk}{}^i = X_j(\gamma_{nn}{}^k) + \sum_{i=1}^{n-1} \gamma_{nn}{}^i\gamma_{kj}{}^i, \quad 1 \le j,\ k \le n-1. \tag{3.10}$$

If the congruences [1],..., [n] are all normal, i.e. if the curves of the congruences are intersections of n orthogonal surfaces in R^n, these equations reduce to the following simpler form:

$$X_j(\nu) + X_n(\gamma_{nn}{}^j) + \nu\gamma_{nn}{}^j = 0 \tag{3.11}$$

$$X_k(\gamma_{nn}{}^j) = X_j(\gamma_{nn}{}^k) \tag{3.12}$$

for $1 \leq j,\ k \leq n-1$.

Geodesic congruences. See Ricci [1896a] and [1898, Part I, Chapter IV].

A curve in R^n is a geodesic of the metric given by the form φ if the first variation of the integral

$$\int ds = \int (g_{ij}dx^i dx^j)^{1/2}$$

vanishes. We will say that a congruence [n] is a geodesic congruence if all the curves belonging to the congruence are geodesics. If [n] belongs to an orthonormal moving frame [1],..., [n], the conditions that it be geodesic are:

$$\gamma_{nn}{}^i = 0. \tag{3.13}$$

Notice that this equation indicates geometrically that γ's have an invariant character. In particular, if the metric is Euclidean, equations (3.13) give the intrinsic characterization of rectilinear congruences.

Geodesic Curvature of a Congruence

If the congruence [n] is not geodesic, and if one considers R^n as a submanifold of R^{n+m}, with the metric form on R^n just that induced from the Euclidean metric on R^{n+m}, we may represent in the following way the geodesic curvature of a curve of [n] passing through the point P of R^n:

The length of the geodesic curvature vector is given by the formula

$$\gamma^2 = \sum_{i=1}^{n-1} (\gamma_{nn}{}^{i})^2. \tag{3.14}$$

The direction of the geodesic tangent vector is

$$\mu_j = \sum_{i=1}^{n-1} \gamma_{nn}{}^{i} \lambda_j[i]. \tag{3.15}$$

This vector has the following properties:

1) It is identically zero if and only if the congruence [n] is a geodesic congruence.
2) Its projection on the tangent plane spanned by the tangent vectors to the curves [i] and [n] is equal to the curvature of the projection of the tangent vector to [n] on the same plane.
3) It is perpendicular to [n].

Because of these properties, we will call this vector the <u>geodesic curvature</u> and the curves which belong to congruence generated by the covariant vector field μ_j the <u>curves of geodesic curvature</u> of the congruence n.

Canonical Systems with Respect to a given congruence

Given a congruence [n], one may, and in many ways, choose (n-1) other congruences which form with [n] an orthonormal moving frame. Among all the possible choices of these (n-1) other congruences, we are going to define a certain subset, whose elements we will designate as <u>canonical</u> with respect to the congruence [n].

Let $\lambda_i[n]$ denote the normalized 1-covariant tensor field which defines the congruence [n]. Set

$$2Z_{ij} = \lambda_{i,j}[n] + \lambda_{j,i}[n],$$

and consider the following systems of algebraic equations:

$$\lambda_i[n]\sigma^i = 0. \tag{3.16}$$

$$\lambda_j[n]\mu + (Z_{jk} + \omega g_{jk})\sigma^k = 0. \tag{3.17}$$

In the equations (3.16)-(3.17), the

$$\mu,\ \omega,\ \sigma^1, \ldots,\ \sigma^n$$

unknowns. Now, together, they form a system of (n+1) linear, homogeneous equations in (n+1)-unknowns

$$\mu,\ \sigma^1, \ldots,\ \sigma^n,$$

with ω a parameter. Let $\Delta(\omega)$ denote the determinant of the system, a polynomial of degree (n-1) in ω. Consider the equation

$$\Delta(\omega) = 0, \tag{3.18}$$

which has real roots. Denote these roots by $\omega_1, \ldots, \omega_{n-1}$, and suppose first that all the roots are simple. If one substitutes into (3.16) and (3.17) the value

$$\omega = \omega_i,$$

and solve for the corresponding $\sigma^1, \ldots, \sigma^n$, they form the components of a 1-contravariant tensor field, which we denote by:

$$\sigma^j[i].$$

They are determined - up to a sign - by the condition that they be of unit length. As i varies, they determine (n-1) congruences [1],..., [n-1], which are orthonormal, and together with the given congruence [n], form an orthonormal moving frame for R^n. In this case, the canonical system is completely determined.

If the roots of the equation (3.18) are all equal, every orthogonal system of (n-1) congruences forms with n a moving frame which satisfies equations (3.16) and (3.17), and hence every orthonormal moving frame, such that the given congruence is the n-th element, may be regarded as a canonical system.

In general, let

$$\omega_1, \ldots, \omega_m$$

be the distinct roots of the equation (3.18), and $p_1, \ldots, p_m$ denote their multiplicities, and set in equations (3.16)-(3.17) the values:

$$\omega = \omega_h, \; h = 1, 2, \ldots, m.$$

One may for each h, determine p_h orthonormal congruences whose contravariant components are solutions of (3.16)-(3.17), with $\omega = \omega_h$. Of course, such congruences are only determined up to an orthogonal transformation of order p_h, i.e. the general solution of these equations depends on

$$p_h(p_h-1)/2$$

arbitrary functions. One may now consider one as canonical orthonormal moving frames consisting of p_1 orthonormal congruences satisfying (3.16)-(3.17) for $\omega = \omega_1$, p_2 satisfying (3.16)-(3.17) for $\omega = \omega_2$, etc. The family of such canonical moving frames thus depends on

$$\sum_{h=1}^{m} p_h(p_h-1)/2$$

arbitrary functions.

When [1],..., [n] are orthonormal congruences such that [1],..., [n-1] are elements of a canonical system with respect to [n], the following characteristic equations are satisfied:

$$\gamma_{jk}{}^{n} + \gamma_{kj}{}^{n} = 0. \tag{3.19}$$

If the congruence [n] is normal, the $\gamma_{jk}{}^{n}$ are all zero for $j \neq k$ in the canonical system for [n]. In this case, the congruences of this system have a simple geometric interpretation: They determine the principal curvatures of the surfaces which are orthogonal to the curves of [n]. (For the theory of curvatures of hypersurfaces see Lipschitz [1870].)

It is possible to give a simple geometric interpretation for the canonical system when n = 3. (See Levi-Civita [1897). It would be feasible to generalize this interpretation to manifolds of arbitrary dimension, but we pass on to other topics.

Remarks:

I shall now describe how these ideas may be described in the modern way. Let

$$\varphi\colon\ V(R^{n}) \times V(R^{n}) \to F(R^{n})$$

be a positive definite, symmetric $F(R^{n})$*-bilinear form which defines a Riemannian metric on* R^{n}. *We shall extend* φ *to define* $F(R^{n})$*-linear and bilinear maps*

$$\varphi\colon\ V(R^{n}) \to F^{1}(R^{n})$$

$$\varphi\colon F^1(R^n) \times F^1(R^n) \to F(R^n),$$

by means of the following formulas:

$$\varphi(X)(Y) = \varphi(X, Y) \tag{3.20}$$

for $X,\ Y \in V(R^n)$.

$$\varphi(\omega_1, \omega_2) = \varphi(\varphi^{-1}(\omega_1), \varphi^{-1}(\omega_2)) \tag{3.21}$$

In terms of coordinates (x^i)*, here is the description of these maps:*

$$\varphi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}$$

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = g_{ij}dx^j$$

$$\varphi(dx^i, dx^j) = g^{ij}.$$

If $f \in F(R^n)$*, the vector field*

$$\text{grad } f = \varphi^{-1}(df) \tag{3.22}$$

is called the *__gradient__* *of* f*. In coordinates,*

$$\text{grad } f = \varphi^{-1}\left(\frac{\partial f}{\partial x^i}\, dx^i\right)$$

$$= \frac{\partial f}{\partial x^i}\, g^{ij}\, \frac{\partial}{\partial x^j},$$

i.e. grad f *is a 1-contravariant tensor field whose components are:*

$$f^j = g^j\, \frac{\partial f}{\partial x^i} \equiv g^{ji} f_i,$$

i.e. grad f is the reciprocal tensor to df.

Wee see that a given congruence is <u>normal</u> if and only if there is a function f such that

$$\text{grad } f \text{ is a vector field of the congruence.} \tag{3.23}$$

To see what the condition for this is, Let $X \in V(R^n)$ be a vector field of the congruence normalized so that

$$\varphi(X,\ X) = 1.$$

Set:

$$\omega = \varphi(X) \in F^1(R^n).$$

Then,

$$\varphi(\omega,\ \omega) = 1.$$

ω is the covariant tensor field which is reciprocal to X. Condition (3.23) is equivalent to the following one:

$$df = h\omega,$$

for some function h. In turn, this equation means that the Pfaffian equation

$$\omega = 0$$

is completely integrable. The condition for this is:

$$\omega \wedge d\omega = 0 \tag{3.24}$$

This can be made explicit as follows:

$$d\omega = \frac{1}{2}\,(X_{i,j} - X_{j,i})dx^i \wedge dx^j,$$

where $(X_{i,j})$ *is the covariant derivative of the tensor field* X_i. *Alternately, one can follow the route suggested by the authors: Set*

$$H = \{Y \in V(R^n)\colon \varphi(Y, X) = 0\},$$

i.e. H *is the* *orthogonal complement* *to the congruence in* $V(R^n)$. *"Normality" means complete integrability of* H, *i.e.*

$$[H, H] \subset H.$$

Isothermal Surfaces and the Laplace-Beltrami Operator

Given $Y \in V(R^n)$, *set:*

$$\operatorname{div} Y = g^{ij}(\nabla_{X_i} Y, X_j), \tag{3.25}$$

where (X_i) *is an arbitrary* $F(R^n)$*-bases of* $V(R^n)$, *and:*

$$g_{ij} = g(X_i, X_j),$$

(g^{ij}) = *inverse matrix to* (g_{ij}). $\operatorname{div} Y$ *is called the divergence of* Y. *In terms of classical tensors, if*

$$Y = (Y^i),$$

then

$$div\ Y = Y^i{}_{,i} . \tag{3.26}$$

Given $f \in F(R^n)$, *set*

$$\Delta(f) = div\ (grad\ f)$$

$$= div\ (g^{ij} f_j)$$

$$= (g^{ij} f_j)_i \ = g^{ij} f_{j,i}.$$

The second order linear differential operator $f \to \Delta(f)$ *is called the* *Laplace-Beltrami operator* *of* f. *A function* f *is said to be* *harmonic* *(with respect to the given Riemannian metric) if*

$$\Delta(f) = 0.$$

The calculation given in the text determines the condition that a vector field $X \in V(R^n)$ *such that*

$$\varphi(X, X) = 1$$

must satisfy in order that:

$$X = h\ grad\ f, \tag{3.27}$$

for some pair (h, f) *of functions in* $F(R^n)$ *such that* $\Delta(f) = 0$.

It is important to keep in mind the physical meaning of the term "isothermal." In case

$$n = 3,$$

$\varphi = \delta_{ij} dx^i dx^j$, *a solution f of the harmonic equation*

$$\Delta f = 0$$

may be thought of as a solution of the Heat or Diffusion equation:

$$\frac{\partial f}{\partial t} = c \Delta f,$$

which is independent of t. For example, "temperature" satisfies such an equation, and harmonic functions may be thought of as time-independent temperature functions. *Points lying on the surface*

$$f(x) = \text{constant},$$

where f is a harmonic function, may then be thought of as having "equal temperature," whence the term "isothermal" for such surfaces.

There are several papers (e.g. #166, 167, 168, 172) in Vol. 2 of Part III of Cartan's Collected Works which deal with the theory of isothermal surfaces. They are recommended both as a summary of the classical results and for their interesting new detail, particularly the links with the theory of Lie groups.

Geodesic Congruences

Let X be a vector field, belonging to a given con-

gruence, normalized so that

$$\varphi(X, X) = 1.$$

The condition that the congruence is <u>*geodesic*</u>*, in the sense that all curves of the congruence are also geodesics of the metric* φ*, is then that:*

$$\nabla_X X = 0.$$

<u>*Geodesic Curvature of a Congruence*</u>

Let X *be a vector field, belonging to a given congruence, normalized so that*

$$\varphi(X, X) = 1.$$

Let:

$$H = \{Y \in V(R^n): \varphi(X, Y) = 0\}. \tag{3.28}$$

In words, H *consists of the vector fields which are perpendicular to* X*.* $V(R^n)$ *is thus a direct sum of* H *and the one-dimensional submodule spanned by* X*.*

Now, set:

$$Y = \text{projection of } \nabla_X X \text{ in } H. \tag{3.29}$$

The congruence determined by Y *is then the* <u>*geodesic curvature congruence of the congruence X*</u>*.*

Canonical System of a Congruence

Let X continue as a unit-length vector field generating a congruence, and let H be defined by (3.28). Let α *be the map*

$$\alpha : H \to H$$

with the following properties:

$$\varphi(\alpha(Y), Z) = \frac{1}{2}\varphi(X, \nabla_Y Z + \nabla_Z Y) \qquad (3.30)$$

$$\varphi(\alpha(Y), Z) = \varphi(Y, \alpha(Z)) \qquad (3.31)$$

for $Y, Z \in H$.

It is readily seen that α *is an* $F(R^n)$*-linear map. Further, it is (by (3.31)) symmetric with respect to the positive definite form (3.31). (This accounts for the fact that its eigenvalues, which are the numbers* $\omega_1, \ldots, \omega_p$ *of the text, are real numbers).*

Definition. An orthonormal moving frame of vector fields $(X_1, \ldots, X_n)$ *is said to be a canonical system for the congruence X if the following conditions are satisfied:*

$$X_n = X.$$

X_i *is an eigenvalue for* α,

for $1 \le i \le n-1$.

As Ricci and Levi-Civita remark, the canonical system is essentially uniquely determined if all the eigenvalues of α *are distinct. In the case some are degenerate, the canonical system is determined up to* **an** *$F(R^n)$-linear map*

$$\beta : H \to H$$

which is an isomorphism of the metric form φ*, and which commutes with* α*.*

If H is completely integrable (i.e. X is a normal congruence), then α *is essentially the second fundamental form (with respect to the metric* φ*; see DGCV) of the maximal integral submanifolds of H.*

This geometric concept has reappeared recently in General Relativity and Cosmology. Here, the metric φ *is non-positive, and the* α *may not be a diagonalizable map. However, it is often applied to the case where* φ *is a hyperbolic metric, i.e. its normal form has 1 plus and (n = 1)-minus signs, and X is a time-like congruence, i.e.*

$$\varphi(X, X) = 1.$$

In this important case, φ *restricted to H is negative definite, and the authors' ideas will carry over. Physically, the curves of the congruence X are thought of as defining a "fluid flow"; the rotational coefficients* $\gamma_{jk}^{\ i}$ *defined* *by the canonical systems are then important physical invariants of this fluid flow.*

4. PROPERTIES OF THE ROTATIONAL COEFFICIENTS AND RELATIONS TO DARBOUX' THEORY OF MOVING FRAMES

We have seen in Section 2 that there are $n^2(n-1)/2$ rotational coefficients $\gamma_{jk}{}^i$ associated to an orthonormal moving frame (X_i) of vector fields.

<u>Remark</u>: *Keep in mind that they can be defined in the modern notations as follows:*

$$\nabla_{X_j} X_k = \gamma_{jk}{}^i X_i. \tag{4.1}$$

This identity enormously simplifies the calculations.

These functions $\gamma_{jk}{}^i$ satisfy certain first order differential equations. Set:

$$\begin{aligned}\gamma_{ik\ell}{}^h = X_\ell(\gamma_{ik}{}^h) - X_k(\gamma_{i\ell}{}^h) \\ + \gamma_{ij}{}^h(\gamma_{k\ell}{}^j - \gamma_{\ell k}{}^j) \\ + \sum_j(\gamma_{h\ell}{}^j\gamma_{ik}{}^j - \gamma_{hk}{}^j\gamma_{i\ell}{}^j).\end{aligned} \tag{4.2}$$

Suppose that $\lambda^i[j]$ are the components of X_j. Then, one has:

$$\gamma_{ik\ell}{}^{h} = R_{rstu}\lambda^{r}[h]\lambda^{s}[i]\lambda^{t}[k]\lambda^{u}[\ell], \tag{4.3}$$

where $R_{(\ )}$ is the (completely covariant) Riemann curvature tensor. These equations give us the necessary and sufficient conditions that the functions $\gamma_{jk}{}^{i}(x)$ given <u>in advance</u> on R^n may be regarded as the rotational coefficients of an orthogonal moving frame with respect to the given Riemannian metric.

<u>Remark</u>: In other words, that there exist vector field (X_i) which are orthonormal and which satisfy (4.1). In the Remarks at the end of this section I will describe the modernized version of Cartan's way of dealing with this question.

For n = 2, equations (4.1) reduce to a single equation:

$$X_2(\gamma_{21}{}^{1}) + X_1(\gamma_{12}{}^{2}) = (\gamma_{21}{}^{1})^2 + (\gamma_{12}{}^{2})^2 + K, \tag{4.4}$$

where K is the Gaussian curvature. This formula is well-known in the theory of surfaces, since $\gamma_{21}{}^{1}$ and $\gamma_{12}{}^{2}$ are the <u>geodesic curvatures</u> of the curves belonging to the congruences [1] and [2].

For n = 3, these equations are the generalization of those which link the components p, q, r of the rotations in Darboux' theory of moving frames. (See Darboux [1894; T. I, Chapter V). If the metric φ is the Euclidean metric for R^3, the tangents to the curves 1, 2, 3 determine at each point P of space a set of three orthonormal vectors in R^3. As the point P "moves" the three orthonormal vectors "move", and this geometric idea is the reason for calling them "moving frames." The rotational coefficients $\gamma_{jk}^{\ i}$ give us the infinitesimal rotations p_i, q_i, r_i ($i = 1, 2, 3$), which define the infinitesimal displacements along the curves 1, 2, 3. See Levi-Civita [1899] for further developments.

One may see in this example how the methods described in this paper subsume and possess all the advantages of those procedures already known.

Remarks: In order to see the relation between the ideas in this section and modern ideas it is most convenient to describe, in the language of manifold theory, the notion of "frame bundle."

Let N be a manifold of dimension n. (This now replaces R^n). $F(N)$ denotes the ring of C^∞, real-valued functions on N. $V(N)$ denotes its derivations, i.e. the vector fields. For $p \in N$, N_p denotes the tangent vector space to N at p.

Each $X \in V(N)$ *determines an* $X(p) \in N_p$, *called its* *value* *at* p.

Suppose given a Riemannian metric on N, *i.e. an* $F(N)$*-bilinear, positive-definite, symmetric map*

$$\varphi\colon V(N) \times V(N) \to F(N)$$

It possesses a *value* *at each point* $p \in N$, *which is an* R*-bilinear, symmetric, positive definite map*

$$\varphi\colon N_p \times N_p \to R.$$

In fact,

$$\varphi(X, Y)(p) = \varphi(X(p), Y(p))$$

for $X, Y \in V(N)$.

Definition. *A set* $(v_1, \ldots, v_n)$ *of vectors in* N_p *is* *an orthonormal frame* *at the point* $p \in N$ *if:*

$$\varphi(v_i, v_j) = \delta_{ij}. \tag{4.5}$$

Definition. *The* *(orthonormal) frame bundle* *to* N, *denoted by* $FR(N)$, *consists of the set of all ordered* $(n+1)$*-tuples*

$$(p, v_1, \ldots, v_n),$$

where:

$$p \in N,$$

$v_1, \ldots, v_n \in N_p$

$(v_1, \ldots, v_n)$ *form orthonormal frame at* p, *i.e. satisfy (4.5).*

Let

$$\pi: FR(N) \to N$$

be the map defined as follows:

$$\pi(p, v_1, \ldots, v_n) = p.$$

π, *called the* <u>*projection map*</u>, *defines* $FR(N)$ *as a (local product)* <u>*fiber space*</u> *over* N. *It is, in fact, a* <u>*principal fiber bundle over* N*, with structure group* $O(n, R)$</u>, *in the sense described in Steenrod's book* [*1951*]. *(Indeed, this example was the prototype for the "principal fiber bundle" notion!)*

Now, we have already defined an (orthonormal) moving frame as a set (X_i) *of vector fields such that*

$$\varphi(X_i, X_j) = \delta_{ij}.$$

Such an object defines a cross-section map

$$\sigma: N \to FR(N),$$

since we can map

$$p \to (p, X_1(p), \ldots, X_n(p)) \equiv \sigma(p)$$

Thus, the notion of "orthonormal moving frame" and "cross-

sections of FR(N)" are essentially identical.

One proves now that there are 1-forms (ω^i, ω_i^j) *on FR(N) with the following properties:*

1) (ω^i, ω_i^j) *forms an* *absolute parallelism* *for FR(N), i.e. an F(FR(N))-basis for* $F^1(FR(N))$.

2) $\omega_i^j + \omega_j^i = 0$

3) $d\omega^i = \omega_j^i \wedge \omega^j$

4) $\sigma^*(\omega^i)(X_j) = \delta_j^i$, *where* $\sigma(p) = (p, X_1(p), \ldots, X_n(p))$ *for all* $p \in N$, *i.e. is the cross-section map assigned to the moving frame*

$$(X_1, \ldots, X_n).$$

Condition 4 then means that $\sigma^*(\omega^i)$ *are the differential forms which are associated via the metric to the vector fields* X_i.

Let $R_{jk\ell}^i$ *be the functions* *on FR(N)* *such that:*

$$d\omega_j^i - \omega_k^i \wedge \omega_j^k = R_{jk\ell}^i \omega^k \wedge \omega^\ell. \qquad (4.6)$$

Here is a statement of the problem considered in this section. Suppose that γ_{jk}^i *are* *functions given on N*.

$$\sigma: N \to FR(N)$$

such that:

$$\sigma^*(\omega_j^{\ i}) = \gamma_{jk}^{\ \ i}\sigma^*(\omega^k) \qquad (4.7)$$

This is a problem which is best set up in terms of E. Cartan's theory of Exterior Differential Systems. The Integrability Conditions are derived by applying the exterior derivative operation to both sides of (4.7), and using relations (4.5) and (4.6).

5. CANONICAL FORMS FOR TENSORS ASSOCIATED TO THE RIEMANNIAN METRIC

In problems of Geometry, Physics, Mechanics, etc., one is almost always led to system of equations which have an invariant nature, (See Section 7 of Chapter I), and in which one encounters a Riemannian metric, and several associated 1 or 2-tensor field and their covariant derivatives. For simplicity, we restrict attention to the case of one associated tensor field.

Suppose first that this tensor field is a 1-covariant tensor field

$$\tau_i.$$

We associate to it a congruence [n], whose curves are defined by the equations:

$$\frac{dx^1}{\tau^1} = \dots = \frac{dx^n}{\tau^n}.$$

The normalized covariant component of this congruence is:

$$\lambda_i[n] = \tau_i/\rho,$$

with

$$\rho^2 = \tau_i.$$

We will say then that the following formula:

$$\tau_i = \rho\lambda_i[n] \qquad (5.1)$$

gives the <u>canonical form</u> for the tensor τ.

The next step is to give canonical forms for the covariant derivatives of τ. To do this, proceed as follows. Start by defining (n-1) congruences which form, with [n], an orthonormal moving frame. Further, one may suppose - as explained in Section 3 - that they form a canonical system with respect to the congruence [n].

One obtains in this way a system of equations which is closely linked to the essential features of the problem. The geometric interpretation of the problem, which is nearly always simple and natural, usually characterizes the equations in an efficient and useful manner. Often, writing the system in this way will provide suggestions for its solution and for a choice of independent variables.

If the equations can be solved, one may at the end introduce ordinary notations, which will give the canonical solution of the problem.

We are the first to admit that these methods (i.e. a "canonical" choice of moving frame) will not eliminate the essential difficulties of the problems to which they are applied. On the contrary, after transforming the equations in this way all the difficulties remain. These methods only teach us how to avoid the accidental obstacles. Often, starting from a relatively complicated set of equations one ends up with a simple and tractable canonical system, leading to interesting and unexpected successes, where the ordinary methods would have almost certainly failed.

Here is a method for dealing with a symmetric 2-tensor (α_{ij}). Consider the eigenvalue equations:

$$(\alpha_{ij} - \rho g_{ij})\lambda^j = 0. \qquad (5.2)$$

Solving them for a non-zero set of λ's requires that ρ satisfy a polynomial equation of degree n (the characteristic polynomial of the matrix $\alpha_j{}^i \equiv g^{ik}\alpha_{kj}$) with well-known properties. All the roots of this equation are real, and their substitution into (5.2), then solution for the λ's, determines an orthonormal moving frame

[1],..., [n] such that:

$$\alpha_{ij} = \sum_k \rho_k \lambda_i[k] \lambda_j[k].$$

This is the canonical form for the tensor field α_{ij}.

Starting with these expressions, one transforms the equations of problems involving such tensors, and one arrives often at canonical solutions, analogously to what has been done for 1-tensor fields.

We can deduce certain general rules from these examples. We have seen that the components of an m-covariant tensor field may be expressed as homogeneous functions of degree m of the components of an arbitrarily given reference moving frame. In the case m = 1, we have obtailed a canonical form of a tensor field τ_i by choosing a moving frame [1],..., [n] such that, in the general formula for τ,

$$\tau_i = \sum_j c_j \lambda_i[j],$$

we have:

$$c_1 = c_2 = \dots = c_{n-1} = 0.$$

The rule is then that the canonical form is determined by the choice of moving frame so that a maximal number of coefficients of τ vanish.

Similiarly, we have determined the canonical form for a symmetric 2-tensor τ_{ij} by choosing the moving frame so that, in the general formula

$$\tau_{ij} = \sum_{k\ell} c_{k\ell}\lambda_i[h]\lambda_j[\ell],$$

one has:

$$c_{k\ell} = 0 \text{ for } k \neq \ell.$$

Again, notice that the canonical form is determined by choosing the moving frame so that a maximal number of components vanish.

In general, to treat a covariant tensor of order m, choose the moving frame so that the components of the tensor field with respect to that moving frame take the simplest or most convenient form. After that, to obtain the intrinsic equations of the problem one has only to follow very simple and uniform procedures.

Remarks: This Section sketches a method for dealing with geometric and physical problems that was later extensively developed by Cartan. He called it "the method of the moving frame." In modern fiber bundle language, it can roughly be described as follows: (It is notoriously difficult to formalize precisely. See my paper [1965]).

Describe the situation first in terms of a principal fiber bundle with a certain structure group G. (In the case here, $G = O(n, R)$, so that the bundle is the frame bundle of the Riemannian metric.) Then, try to reduce the structure group to a subgroup in a "natural" geometric way, choosing the subgroup to be as small as possible.

Of course, "reducing the structure group" is a concept that also appears in topology. (See Steenrod [1951]). However, in this case there are not usually topological obstructions (in fact, the base space of the fiber bundle is usually R^n), but one wants to find reductions which satisfy certain sets of differential equations. Despite the confidence displayed here by Ricci and Levi-Civita about the ease in application of the method, it has turned out to have certain mysterious features which have inhibited full development. However, I do believe that understanding better how the method works will lead to new progress in differential geometry - the promise anticipated by Ricci and Levi-Civita is still there!

There is also a more intelligible purely algebraic problem involved here, involving what is now known as invariant theory. (See Dieudonné and Carrel [1971]). Let G be a group of linear transformations on a vector space V.

(In the situation discussed by the authors, $V = R^n$, $G = O(n, R)$, the group of $n \times n$ real orthogonal matrices.) *Let*

$$T_p^{\ m}(V)$$

denote the space of mixed tensors on V which are m-times contravariant, p-times covariant. The action of G on V determines a linear action of G on $T_p^{\ m}(V)$. What is involved is a canonical form or fundamental domain for the action of G on $T_p^{\ m}(V)$, a subset of $T_p^{\ m}(V)$ which slices across the orbits of G, meeting the "general" orbit in precisely one point. In the case at hand, where G is compact, it is not difficult to show the existence of such a subset, and even to prove useful facts about its properties. (See DGCV, Chapter 25). In the case G is non-compact (which would be of interest for relativity, e.g. G = O(1, 3)), the relevant theory does not exist, although with existing tools a fair amount could be done.

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There is a suggestion here of a general method for constructing such "fundamental domains" that has, to the best of my knowledge, not been worked on systematically.

Namely, on each orbit of G acting on $T_p^m(V)$, choose the element such that *__a maximal number of components with respect to a fixed basis of V vanish__*.

Chapter III

ANALYTIC APPLICATIONS

1. CLASSIFICATION OF QUADRATIC DIFFERENTIAL FORMS

Let φ be a positive definite quadratic differential form in the n variables $x^1,\ldots, x^n$. If μ is sufficiently large it is possible to choose $n + \mu$ functions

$$y^1,\ldots, y^{n+\mu}$$

of the x's such that:

$$\varphi = (dy^1)^2 +\ldots+ (dy^n)^2 +\ldots+ (dy^{n+\mu})^2.$$

Let m denote the smallest possible value of μ. One has:

$$0 \le m \le \frac{n(n-1)}{2}.$$

m is called the class of the Riemannian metric φ. m can serve to classify metrics.

For example, if $n = 2$, the class is zero or one.

The metrics of class zero (of any number of variables) are characterized by the condition that their Riemannian curvature tensor vanishes identically. Here is a result (see Ricci [1884]) which characterizes the metrics of class one.

<u>φ is of class one if and only if there exists a symmetric tensor</u> b_{ij} <u>such that:</u>

1) $R_{ijk\ell} = b_{ik}b_{j\ell} - b_{i\ell}b_{jk}$

2) $b_{ij,k} + b_{ik,j} = 0.$

When these conditions are satisfied, the functions $y^1, \ldots, y^{n+1}$ may be determined as solutions of a differential system whose integrability conditions are satisfied.

For forms of higher class there is an analogous theorem.

But, we do not pursue this generalization; another important application of Tensor Analyses awaits our attention.

<u>Remarks</u>: *The problem briefly alluded to here is now called the <u>isometric embedding problem</u>. Of course, the authors mean to work only <u>locally</u>. The global isometric embedding of a Riemannian manifold into a higher dimensional Euclidean space was only proved within the last twenty years, by John Nash [1]. In fact, Nash's work is one of the most brilliant and important results in 20th century differential geometry.*

It is interesting that Ricci and Levi-Civita took for *granted the existence of a local isometric embedding. To the best of my knowledge, this was in fact only proved in the 1920's by Janet and Cartan.*

2. ABSOLUTE INVARIANTS. GEOMETRIC REMARKS. DIFFERENTIAL OPERATORS

See Ricci [1884, 1898], Levi-Civita [1894].

The classical work of Jacobi, Lamé and Beltrami, which introduced into analysis the invariants known as <u>differential parameters</u>, is based on the first variation of certain integrals. Despite the elegance and ingenuity of this approach, it leads to methods which are indirect and far away from those which the nature of the problem would seem to suggest. In fact the study of such differential parameters is part of the following general problem, which after all only involves algebraic elimination:

> <u>Given a Riemannian metric φ and a number of associated tensors S</u> (covariant or contravariant), <u>determine all the absolute invariants which may be formed from the coefficients of φ and S, and their derivatives up to an order μ fixed in advance</u>.

If one did not have to take into account the derivatives, this would be a well-known algebraic problem. The involvement of derivatives seems at first to severely complicate the situation. Happily, it really does not. Tensor analysis leads us back to the algebraic problem, because we can substitute covariant for ordinary derivatives. More precisely, we have the following Theorem:

> To obtain the absolute differential invariants of order μ, it suffices to determine the algebraic invariants of the following form

a) The fundamental form

b) The forms associated to S and their covariant derivatives (with respect to φ) up to order μ.

c) For $\mu > 1$, the Riemann curvature tensor and its covariant derivatives up to order $\mu - 2$.

The first problem is that of determining the differential invariants of φ alone. From the preceding Theorem, we deduce the following Corollaries:

> The metrics φ of class zero have no non-zero differential invariants. The metrics of non-zero class have no differential invariants of the first order. The invariants of order

> greater than one are the invariants of φ, the Riemann curvature tensor, and its covariant derivatives.

These results are of the simplest form for the case

$n = 2$ or 3.

For $n = 2$, the Riemann curvature tensor reduces to the Gaussian curvature K, which is the only second order invariant of 2-dimensional Riemannian metrics.

We might remark at this point that, when we regard φ as the metric of a surface, the value of K is the product of the principal radii of curvature. This is the reason for calling K the total curvature of the metric φ. We know that $K = 0$ is the necessary and sufficient condition that the metric φ be of class zero. In geometric language, this is the well-known fact that developable surfaces are the only ones which are applicable on a plane.

For $K = 0$, our metric form has no non-zero differential invariants. In general:

> The invariants of a 2-dimensional metric of order $\mu > 2$ are obtained as the algebraic invariants of the form φ and the covariant derivatives of K up to order $\mu - 2$.

This result is implicitly contained in a memoir by Casorati [1860].

Turn now to the study of the case

$$n = 3.$$

In place of the Riemann curvature tensor $R_{ijk\ell}$, substitute the contravariant tensor α^{ij} defined by the following formula:

$$\alpha^{ij} = \frac{R_{i+1,\ i+2,\ j+1,\ j+2}}{g}.$$

where we use the convention that two indices are identified if they are equal modulo 3. Thus,

$$\alpha^{12} = \frac{R_{2334}}{g} = \frac{R_{2331}}{\sqrt{g}}$$

(See the remarks at the end of this section for the coordinate-free way of defining α.)

Now, $\alpha = 0$ is the necessary and sufficient condition that φ be of class zero. When α is non-zero, consideration of the quadratic forms g_{ij} and α_{ij} gives us all the differential invariants of these two forms, we may take the roots of the equation

$$\det(\alpha_{ij} - \rho g_{ij}) = 0,$$

and call them the <u>fundamental differential invariants</u> of the metric φ. We are led to this choice by the general process of reduction of the 2-tensor α_{ij} to its canonical form. (Section 5, Chapter II). It leads very naturally to an orthonormal moving frame which is very important for the study of the geometric properties which generalize those associated to the total curvature in the 2-dimensional case.

We will come back to these geometric applications (Chapter IV, Section 8). For the moment, we limit ourselves to the remark that the field of eigenvectors of α are called the principal congruences, and their values at points are called principal directions.

It is hardly necessary to add that, to obtain the differential invariants of the 3-dimensional metric φ up to order $\mu > 2$, it suffices to take into consideration, in addition to the forms α_{ij}, g_{ij}, those which one obtains by covariant differentiation of α up to order $\mu - 2$.

Having dealt with the invariants of a metric φ alone, let us now examine several simple examples of the general case where one has also associated tensors.

First, suppose that we are given two functions U and V associated to the metric φ in n-variables. Suppose that:

$$\varphi = g_{ij}dx^i dx^j.$$

The differential invariants of first order of the system (φ, U, V) are described by the following differential operators:

$$\Delta_1(U) = g^{ij}U_i U_j$$

$$\Delta_1(V) = g^{ij}V_i V_j$$

$$\nabla(U, V) = g^{ij}U_i V_j.$$

Remark: Recall that

$$U_i = \frac{\partial U}{\partial x^i}.$$

When the system is formed by the metric φ and a single function U, the only first order invariant is $\Delta_1(U)$. To treat the second order, consider the invariants of the following three differential forms:

$$g_{ij}dx^i dx^j, \; U_i dx^i, \; U_{i,j}dx^i dx^j.$$

In particular, the invariants of the pair

$$g_{ij}dx^i dx^j, \; U_{i,j}dx^i dx^j$$

are the roots of the equation

$$\det(U_{i,j} - \rho g_{ij}) = 0,$$

and will be of degree 1, 2,..., n in the second covariant derivatives in U. The invariant of the first degree,

$$\Delta_2(U) = g^{ij}U_{i,j},$$

is the well-known second degree operator defined by Beltrami.

Remark: Δ_2 is now called the ***Laplace-Beltrami operator*** *associated with the metric φ.*

It completely characterizes the metric φ.

Suppose now that we associate with the metric φ a 1-covariant tensor

$$(\tau_i)$$

The pair (φ, τ) defines first order invariants, which are the algebraic invariants of the differential form

$$\varphi, \ \tau_i dx^i, \ \tau_{i,j} dx^i \otimes dx^j.$$

Among these invariants, the most important for applications is:

$$\theta(\tau) = g^{ij}\tau_{i,j}.$$

From the point of view of applications, it is important to note that the following alternate form for θ:

$$\theta(\tau) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g}\ \tau^i).$$

This expression is often most useful in calculations, while the first form is best suited to theoretical deductions. In the particular case where n = 2, one may replace the bilinear form $\tau_{i,j} dx^i \otimes dx^j$ by the symmetric bilinear form

$$(\tau_{i,j} + \tau_{j,i}) dx^i dx^j,$$

provided one adds the invariant formed from the tensor $\tau_{i,j}$ and the contravariant tensor E defined in Chapter I, Section 3. Here is its explicit formula:

$$\epsilon^{ij}\tau_{i,j} = \frac{1}{\sqrt{g}} (\tau_{1,2} - \tau_{2,1})$$

$$= \text{also}$$

$$\frac{1}{\sqrt{g}} \left\{\frac{\partial \tau_1}{\partial x^2} - \frac{\partial \tau_2}{\partial x^1}\right\}$$

Similiarly, for n = 3, one associates to (φ, τ) the symmetric tensor

$$\tau_{i,j} + \tau_{j,i},$$

and the 1-covariant tensor

$$\mu^i = \frac{1}{2}\ \epsilon^{ijk} X_{j,k}.$$

This can also be written as follows:

$$\mu^i = \frac{1}{2\sqrt{g}} \left\{ \frac{\partial \tau_{i+2}}{\partial x^{i+1}} - \frac{\partial \tau_{i+1}}{\partial x^{i+2}} \right\}$$

where again one adopts the convention that indices equal modulo three are the same. This formula is particularly effective for calculations in applications.

__Remarks:__ *This Section certainly covers ground! In fact, it proposes ideas which have not been adequately formulated or developed even to this day. Accordingly, I shall now try to explain what is involved, using the ideas and notations of manifold and vector bundle theory.*

First, there is a purely "algebraic" part. The authors aim to reduce the study of "geometric" invariants to "algebraic" ones. I will describe some of these algebraic problems from the viewpoint of modern group and vector space theory.

Let V be a real finite dimensional vector space. Let V^d be its dual space. For each pair of integers (r, s), let:

$$V_r^s = \underbrace{V \otimes \ldots \otimes V}_{s \text{ times}} \otimes \underbrace{V^d \otimes \ldots \otimes V^d}_{r \text{ times}}$$

Thus, an element of V_r^s is an r-fold covariant, s-fold *contravariant tensor, in terms of V.*

Let W be the vector space direct sum of a certain number of copies of such tensor spaces. Say, that:

$$W = V_{r_1}^{s_1} \oplus \ldots \oplus V_{r_m}^{s_m}.$$

Typically, one is given an element w of W in a geometric problem. What is wanted is a notion of a set of algebraic "invariants" associated with w. This may be thought of as a "mapping" (in the sense of algebraic geometry; see Volume VIII) from W to another vector space W'.

Consider the following example:

$$m = 2;\ r_1 = 1 = r_2 = s_1 = s_2.$$

Giving w amounts then to giving a pair

$$\alpha_1,\ \alpha_2$$

of linear maps: V → V. (What is involved here is the identification of the space of linear maps: V → V with the set of tensors a_i^j*, i.e. matrices, i.e. 1-fold covariant, 1-contravariant.)*

What are the "invariants" of such pairs of linear maps? The answer comes from the Weirstrass-Kronecker "elementary divisor" theory. Let λ be a new variable, and form the "pencil" of linear maps:

$$\alpha(\lambda) = \alpha_1 + \lambda\alpha_2.$$

See Volumes III, VIII, IX and Gantmacher [1964] for the full story of what is involved here.

Alternately, one can define

$$a_0(\alpha_1, \alpha_2), \ldots, a_n(\alpha_1, \alpha_2)$$

as the real numbers such that:

$$\det(\alpha(\lambda)) = a_0 + a_1\lambda + \ldots + a_n\lambda^n.$$

The $a_0, \ldots, a_n$ *are* ***polynomial*** *functions in the terms of linear parameters for* α_1, α_2. *Hence, we can map*

$$W \to R^{n+1}$$

by assigning:

$$(\alpha_1, \alpha_2) \to (a_0(\alpha_1, \alpha_2), \ldots, a_n(\alpha_1, \alpha_2)).$$

This map defines, in a obvious sense, the "algebraic invariants" of the elements of W. Unfortunately, such a simple and natural construction of "algebraic invariants" is not known for most more complicated collections of tensors!

This "algebraic invariant" problem also has a group-theoretic side. Here is one typical situation.

Let G be a group of linear transformations on V. G extends in a tensorial way to act on V_r^s, *hence on W, which* *is a direct sum of a number of copies of such tensor spaces. The problem can be stated in several interrelated ways:*

> *Find the orbits of G acting on W. Find rational functions on W which are invariant under G. More generally, find vector spaces W' on which G acts irreducibly, and find maps (in the sense of algebraic geometry again)*
>
> $W \to W'$
>
> *which intertwine the action of G.*

Study of these general questions form a modern version of what was classically known as invariant theory. See the book by Dieudonné and Carrell [1971] for a recent treatment.

I shall now try to give an idea, in modern language, of what 19th century geometers meant by a differential invariant or covariant. Let M be a manifold. F(M) denotes the ring of C^∞, real-valued functions on M. Let V(M) be the C^∞ vector fields on M, i.e. the derivations of F(M). V(M) is a module over the ring F(M). Let $F^1(M)$ be the dual module, i.e. the 1-differential forms on M. Set:

$$T_r^{\,s}(M) = \underbrace{F^1(M) \otimes \dots \otimes F^1(M)}_{r\text{-times}} \otimes \underbrace{V(M) \otimes \dots \otimes V(M)}_{s\text{-times}}$$

($\otimes$ denotes tensor product of F(M)-modules). The elements of $T_r^s(M)$ are the r-fold covariant, s-fold contravariant tensor fields on M.

Let G(M) be the group of diffeomorphisms of M. G(M) acts, in the natural "tensorial" way, as a transformation group on each $T_r^s(M)$.

Now, let Γ be a submodule of the direct sum of a certain number of tensor field spaces. Say, that

$$\Gamma \subset T_{r_1}^{s_1}(M) \oplus \ldots \oplus T_{r_m}^{s_m}(M).$$

Suppose that the action of G(M) leaves the submodule Γ invariant.

A k-th order differential invariant is a mapping (not necessarily linear, and not necessarily even defined everywhere)

$$\Gamma \to T_r^s(M)$$

which:

a) Intertwines the action of G(M)

b) Is defined, in each local coordinate system for M, by formulas which involve differential operators of order k in the components of the elements of Γ.

The main point of this Section may be stated as *follows: Suppose*

$$\Gamma \subset ST_2(M) \oplus T_{r_2}^{s_2}(M) \oplus \ldots \oplus T_{r_m}^{s_m},$$

i.e. the first component of $\gamma \in \Gamma$ *is a symmetric, two-fold covariant tensor field* φ. *(ST denotes "symmetric tensors".) Generally, this* φ *will be* *non-degenerate*, *i.e. will define a Riemannian metric (possibly non-positive, of course) for M. Thus, the* *covariant derivative* *with respect to* φ *may be used - as indicated by the examples given in the text - together with the algebraic invariants constructed on tensors at one point, to define differential invariants.*

Chapter IV

GEOMETRIC APPLICATIONS

1. STUDY OF TWO DIMENSIONAL MANIFOLDS (GEOMETRY ON A SURFACE): GENERALITIES - CURVATURE - CONGRUENCES - BUNDLES OF CONGRUENCES - INVARIANCE OF A BUNDLE - BELTRAMI'S THEOREM

The theory of surfaces and curves on surfaces - founded by Gauss - is now developed to form in itself a vast and rich scientific domain. But, even in the best expositions of this subject, unified methods are lacking. It is not developed naturally from simple, well-determined principles. Tensor Analysis, by contrast, gives the theory a form which seems as simple as possible.

It also leads to a rational separation of those properties of two dimensional surfaces which are intrinsic and those which depend on the embedding into our three dimensional Euclidean space R^3. The intrinsic properties are derived from the ds^2 of the surface (first fundamental form) induced from the Euclidean metric on R^3, while the embedding properties are defined by another quadratic differential form, called, by Bianchi, the second fundamental form.

Let us begin with the first form. Choose indices and summation convention as follows:

$$1 \leq i, j \leq 2.$$

Let M be a manifold, with coordinates

$$(x^1, x^2)$$

and metric

$$ds^2 = g_{ij}dx^i dx^j = \varphi.$$

Let us regard this form as the fundamental one. If its Gaussian curvature K vanishes, we have seen that the manifold will be linear. If K does not vanish, the assignment

$$\varphi \to K$$

gives rise to all the invariants of the form φ, i.e. to all the (differential) expressions linked to intrinsic properties of the Riemannian manifold M.

Let

$$\lambda_i[1]$$

$$\lambda_i[2]$$

be the covariant components of two <u>orthonormal congruences</u>, defined on our manifold. They will be referred to as congruence [1] and

congruence [2].

Let us use, for this case n = 2, the general orthonormal frame formalism of Chapter 2. Set:

$$\varphi_i = \gamma_{1j}{}^{2}\lambda_i[j].$$

Then, we have:

$$\lambda_{i,j}[1] = -\lambda_i[2]\varphi_j$$
$$\lambda_{i,j}[2] = \lambda_i[1]\varphi_j. \tag{1.1}$$

The rotational coefficients of the moving frame [1], [2] have, in this case, two independent components. We shall take them as:

$$\gamma_{21}{}^{1},\ \gamma_{12}{}^{2}.$$

They represent the geodesic curvature of the integral curves of the vector fields [1] and [2].

Set:

$$\bar{\varphi}_i = \epsilon_{ij}\varphi^j. \tag{1.2}$$

The formulas of Chapter 2 specialize to:

$$K = g^{ij}\bar{\varphi}_{i,j}. \tag{1.3}$$

We have:

$$\lambda_i[1] = \epsilon_{ij}\lambda^j[2]. \tag{1.4}$$

Substitute (1.4) into (1.1). This gives a system of differential equations for the $\lambda_i[2]$. Equation (1.3) is the integrability condition for these equations, when one adds the algebraic condition:

$$\lambda_i[2]\lambda^i[2] = 1. \tag{1.5}$$

The general solution of this system has the form:

$$\lambda_i = \sin(\alpha)\lambda_i[1] + \cos\alpha\lambda_i[2], \tag{1.6}$$

where α is a constant.

For a particular value of α, the λ_i are components of a congruence whose integral curves meet the integral curves of $\lambda_i[2]$ in the angle α.

Such a system of congruences is called a bundle of congruences.

Remark. They use the term "faisceau".

φ_i is called the covariant vector field of the bundle.

Equation (1.3) then is the condition that a vector field given in advance be associated in this way with a bundle.

If φ_i and ψ_i are vector fields associated with two bundles, the differences $\varphi_i - \psi_i$ have a remarkable geometric significance. They are the derivatives of the angle between two vector fields of the different bundles.

Following the rules of the preceding section, suppose that one has constructed all the differential invariants obtainable by the association of the vector field φ to the congruence [2]. One obtains in this way all the intrinsic properties of a congruence in the 2-dimensional manifold M.

We obtain a single algebraic invariant, represented by equation (1.5).

Because of equations (1.1), the differential invariants of the first order are the algebraic invariants common to the metric tensor and to the two vector fields $\lambda_i[2]$ and φ_i. There are two such invariants:

$$J_1 = \lambda^i[2]\varphi_i = \gamma_{12}{}^2$$

$$J_2 = \varphi^i\varphi_i = (\gamma_{21}{}^1)^2 + (\gamma_{12}{}^2)^2.$$

The second order invariants are the Gaussian curvature K and the quadratic form:

$$\psi_{ij} = \frac{1}{2}(\varphi_{i,j} + \varphi_{j,i}).$$

The invariants of order greater than the second are the covariant derivatives of K and of the tensor ψ_{ij}.

As we have said, the differential invariants of the metric tensor represent intrinsic geometric properties of the manifold M. Similiarly, the invariants of φ_i, ψ_{ij}, and their covariant derivatives represent the invariants of the bundle. Thus, the invariant J_2 represents the sum of the squares of the geodesic curvature of the two orthogonal curves belonging to one of the pair of congruences of the bundle. Similiarly, the function

$$g^{ij}\varphi_{i,j}$$

in an invariant of the bundle. Its vanishing is the necessary and sufficient condition that each congruence of the buneld is isothermal. This proves in a very natural way the following theorem of Beltrami:

> If a congruence is isothermal, so is each congruence which belongs to the same bundle, i.e. which makes a constant angle with the given congruence.

Remarks: I will now redo this material in Cartan's form. Let

$$(\omega^j)$$

be an orthonormal moving frame of 1-differential forms. (If (x^i) is a coordinate system, the $\lambda_i[j]$ can be defined by the relation:

$$\omega^j = \lambda_i[j]dx^i).$$

Then, we have:

$$d\omega^j = \omega_k{}^j \wedge \omega^k$$

$$\omega_k{}^j = \gamma_{ki}{}^j dx^i.$$

Thus,

$$\varphi_i dx^i = \gamma_{1,j}{}^2 \lambda_i[2] dx^i = \gamma_{1j}{}^2 \omega^j.$$

Hence, we have:

$$\varphi_i dx^i = \omega_1{}^2.$$

We see that, in Cartan's language, what Ricci and Levi-Civita are saying is that the assignment

$$(\omega^1, \omega^2) \to \omega_1{}^2$$

defines the geometry on the 2-dimensional Riemannian manifold. The 1-form $\omega_1{}^2$ *determines the Riemannian connection, and the second fundamental form (i.e. the "geodesic curvature") of the integral curves of the moving frame.*

Here is the relation to the Gaussian curvature K:

$$d\omega_1{}^2 = K\omega_1 \wedge \omega_2.$$

Definition. *Let* (ω^1, ω^2), (θ^1, θ^2) *be two orthonormal moving frames for the metric. They are said to belong to the same bundle of moving frames if there is a constant real number such that:*

$$\theta^1 = \cos \alpha\omega^1 + \sin \alpha\omega^2$$

$$\theta^2 = -\sin \alpha\omega^1 + \cos \alpha\omega^2.$$

Another way of putting this is to say that these formulas define an action of SO(2, R) on the moving frames. The __*bundles*__ *are the orbits of this action.*

Then, we have

$$d\theta^1 = \cos \alpha\omega^1{}_2 \wedge \omega^2 - \sin \alpha\omega^1{}_{-2} \wedge \omega^1$$

$$= \omega_2{}^1 \wedge \theta^2.$$

Similiarly,

$$d\theta^2 = -\ \omega_2{}^1 \wedge \theta^1.$$

Hence:

The assignment

$$(\omega^1,\ \omega^2) \to (\omega_2{}^1)$$

is __*invariant*__ *under the action of SO(2, R), hence defines a* __*differential invariant*__ *of the bundles of moving frames.*

2. SURFACES OF 3-SPACE-FUNDAMENTAL EQUATIONS - PARTICULAR REMARKABLE FORMS - GENERALIZATION OF THE FORMULAS OF GAUSS AND CODAZZI

As indicated in Section 1 of the preceding Chapter, in order to determine all surfaces of R^3 which admit a given metric ds^2, it suffices to determine all tensors

$$b_{ij},\ i \leq 1,\ j \leq 2,$$

which satisfy the following algebro - differential system of equations:

$$b_{ij,k} = b_{ik,j} \tag{2.1}$$

$$\frac{b}{g} = K, \tag{2.2}$$

where:

$$b = b_{11}b_{22} - b_{12}^{\ 2}$$

$$g = g_{11}g_{22} - g_{12}^{\ 2}.$$

Choose another set of indices as follows:

$$1 \leq a,\ b \leq 3.$$

Let (y^a) be functions which define the 2-dimensional manifold as a surface in R^3. Then, we have:

$$g_{ij} = \sum_{a=1}^{3} y_i^{\ a} y_j^{\ a} \tag{2.3}$$

$$y_{i,j}^{\ \ a} = z^a b_{ij}, \tag{2.4}$$

where the z^a are defined by the following relations:

$$\sum_{a=1}^{3} z^a y_i^{\ a} = 0 \tag{2.5}$$

$$\sum_{a=1}^{2} (z^a)^2 = 1. \tag{2.6}$$

($y_i{}^a$, $y_{i,j}{}^a$ denote the covariant derivatives of the functions y^a with respect to the connection associated with the metric g_{ij}).

Considered as a system of partial differential equations for the y^a, the system formed by (2.3) and (2.4) is completely integrable. (Conditions (2.1) and (2.2) are the integrability conditions.) The general solution of this system depends on six arbitrary constants; they fix the position of the coordinate axes of R^3 with respect to the surface.

Remark: Here is what is meant. Consider (y^a) as defining a map

$$\underline{y}: M \to R^3,$$

where M is the 2-dimensional Riemannian manifold. Let G be the group of rigid motions acting on R^3. It is a 6-dimensional Lie group (whence the "six arbitrary constants"). Each $g \in G$ acts on $\underline{y}$:

$$\underline{y} \to g\underline{y}.$$

Clearly, $g\underline{y}$ is again a solution of (2.3)-(2.4).

We see that to each particular solution of equations

(2.3)-(2.4) there corresponds a surface on R^3, uniquely determined up to a rigid motion, which admits the given metric ds^2 as the metric induced from the flat metric on R^3. Equations (2.3)-(2.4) may be called the intrinsic equations of the surface. Equations (2.1)-(2.4) (which we will call the fundamental equations of the theory of surfaces) are much better suited to the study of geometric properties of the surface than the literal equations defining the surface, which involve objects which are not naturally defined in terms of the surface itself.

As outlined in Chapter 2, Section 1, equations (2.1)-(2.4) may be written in terms of an orthonormal moving frame in the following form:

$$b_{ij} = \beta_{k\ell}\lambda_i[k]\lambda_j[\ell], \tag{2.7}$$

where

$$\beta_{k\ell} = \beta_{\ell k},$$

and $\lambda_i[1]$, $\lambda_i[2]$ denote the covariant components of an orthonormal moving frame. One sees that β_{11}, β_{22}, β_{12} measure (up to a sign) the normal curvature and the geodesic torsion of the integral curves of the orthonormal congruence [1], [2].

In order to write these equations in a form which is intrinsic to the orthonormal moving frame, let us adopt the following indicial convention:

$$i + 2 = i.$$

Then, (2.1) and (2.2) take the following form:

$$\frac{\partial\beta_{ii}}{\partial s_{i+1}} - \frac{\partial\beta_{i(i+1)}}{\partial s_{i+2}} = \sum_{h=1}^{2} \beta_{ih}\gamma^{i}{}_{(i+2)h} + \beta_{(i+1)h}\gamma^{h}{}_{(h+1)(\gamma+1)} \tag{2.8}$$

$$\beta_{11}\beta_{22} - \beta_{12}{}^{2} = K. \tag{2.9}$$

As we have described in Chapter 2, there is an orthonormal moving frame [1], [2] for which the expression (2.7) takes its <u>canonical form</u>. This frame is that tangent to the <u>lines of curvature</u> of the surface. We have then

$$\beta_{12} = 0,$$

which gives the known theorem that the lines of curvature have zero geodesic torsion. β_{11}, β_{12}, after a change of sign, are the <u>principal curvatures</u>. Equations (2.8) and (2.9) reduce, in this case, to the well known <u>Gauss-Codazzi equations</u>.

One may also choose the orthonormal frame so that:

$$\beta_{22} = 0.$$

(It may be necessary to use <u>complex</u> frames. Real frames are sufficient if $K \leq 0$). The curves of the congruence [2] are then asymptotic curves of the surface. Equation (2.9) then defines β_{12}, and (2.8) leads to relations which have been already mentioned by M. Raefy [1892].

The reader who wants further detail on how these techniques give the most important theorems of surface theory should see Ricci's "Lezioni sulla teoria delle superficie," to which we have already frequently referred. Instead of going in this direction, in the next section we deal with a problem in the theory of isometry of surfaces, where Tensor Analysis has completely solved the problem.

Remarks: Cartan's methods are ideally suited to this material. Let

$$(\theta^a)$$

be a moving frame of 1-forms on R^3 *which is orthonormal with respect to the flat Riemannian metric on* R^3. *(In other words, if* (y^1, y^2, y^3) *are Cartesian coordinates on* R^3, *then*

$$\theta^1 \cdot \theta^1 + \theta^2 \cdot \theta^2 + \theta^3 \cdot \theta^3 = dy^1 \cdot dy^1 + dy^2 \cdot dy^2 + dy^3 \cdot dy^3,$$

where $\cdot$ *denotes the symmetric product of differential forms.)*

Let M be a 2-dimensional submanifold of R^3. *Choose the moving frame* (θ^a) *so that:*

$$\theta^3 = 0 \text{ on } M. \qquad (2.10)$$

Set:

$$\omega^1 = \theta^1 \text{ restricted to } M$$

$$\omega^2 = \theta^2 \text{ restricted to } M$$

$$ds^2 = \omega^1 \cdot \omega^1 + \omega^2 \cdot \omega^2. \qquad (2.11)$$

This quadratic differential form on M is then the <u>*Riemannian metric induced from the flat metric on* R^3</u>.

Suppose that

$$\theta_a{}^b$$

are 1-forms on R^3 *such that:*

$$d\theta^a = \theta_b{}^a \wedge \omega^b \qquad (2.12)$$

$$\theta_b{}^a + \theta_a{}^b = 0. \qquad (2.13)$$

Set:

$$\omega_i{}^j = \theta_i{}^j \text{ restricted to } M. \qquad (2.14)$$

Then,

$$d\omega^i = \omega_j{}^i \wedge \omega^j. \qquad (2.15)$$

This relation tells us that the

$$(\omega_j^{\ i})$$

are the connection forms of the metric ds^2*, given by formula (2.11) with respect to the orthonormal moving frame* (ω^i).

Equation (2.12) implies one additional relation when restricted to M. *Namely, set* $a = 3$ *in (2.12), and use (2.10):*

$$\bar{\theta}_i^{\ 3} \wedge \omega^i = 0, \tag{2.16}$$

where:

$$\bar{\theta}_i^{\ 3} = \theta_i^{\ 3} \text{ restricted to } M. \tag{2.17}$$

Now, let (x^i) *be an arbitrary coordinate system on* M. *Set:*

$$\bar{\theta}_i^{\ 3} = b_{ij} dx^j. \tag{2.18}$$

(The b_{ij} *are the functions used in the text). Also,*

$$\bar{\theta}_i^{\ 3} = \beta_{ij} \omega^j. \tag{2.19}$$

Set:

$$\beta = \bar{\theta}_i^{\ 3} \cdot \omega^i. \tag{2.20}$$

This quadratic differential form on M *is called its* <u>*second fundamental form*</u>. *(The "first" fundamental form is the metric* ds^2.*) It is* <u>*extrinsic*</u>*, i.e. it depends on the embedding of* M *as a submanifold of* R^3.

Now, condition (2.18) is equivalent to the following one:

$$\beta_{ij} = \beta_{ji}. \tag{2.21}$$

In turn, (2.21) implies:

$$\beta = \beta_{ij}\omega^i \cdot \omega^j, \tag{2.22}$$

i.e. (β_{ij}) *is the matrix of the quadratic differential form with respect to the moving frame* (ω^i).

We can now develop the curvature relations. R^3 *has zero curvature, hence:*

$$d\theta_a{}^b = \theta_a{}^c \wedge \theta_c{}^b. \tag{2.23}$$

Restrict this relation to M, and set $a = i$, $b = j$:

$$\begin{aligned} d\omega_i{}^j &= \omega_i{}^k \wedge \omega_k{}^j \\ &\quad - \bar{\theta}_i{}^3 \wedge \bar{\theta}_j{}^3. \end{aligned} \tag{2.24}$$

Now,

$$d\omega_i{}^j - \omega_j{}^k \wedge \omega_k{}^j = K\omega^i \wedge \omega^j, \tag{2.25}$$

where K denotes the Gaussian curvature of M. Combining (2.24) and (2.25) gives the following relation:

$$K\omega^i \wedge \omega^j = - \bar{\theta}_i{}^3 \wedge \bar{\theta}_j{}^3. \tag{2.26}$$

Use (2.19)

$$\begin{aligned} \bar{\theta}_i{}^3 \wedge \bar{\theta}_j{}^3 &= \beta_{ij}\omega^k \wedge \beta_{j\ell}\omega^\ell \\ &= \tfrac{1}{2}(\beta_{ik}\beta_{j\ell} - \beta_{i\ell}\beta_{jk})\omega^k \wedge \omega^\ell. \end{aligned}$$

Thus, we have:

$$K(\delta_i^{\ k}\delta_j^{\ \ell} - \delta_i^{\ \ell}\delta_j^{\ k}) = \beta_{ik}\beta_{j\ell} - \beta_{i\ell}\beta_{jk}. \tag{2.27}$$

Relation (2.27) has been obtained from (2.23) by setting

$$a = i,\ b = j.$$

We also obtain relations by setting

$$a = i,\ b = 3.$$

Then,

$$d\bar{\theta}_i^{\ 3} = \omega_i^{\ j} \wedge \bar{\theta}_j^{\ 3} \tag{2.28}$$

Formula (2.27) is called Gauss' formula. (2.28) are called the Mainardi-Codazzi formulas. They are the Fundamental Formulas of surface theory. It is readily verified that (2.28) is equivalent to the relation (2.4), i.e.

$$b_{ij,k} = b_{ik,j}. \tag{2.29}$$

Suppose now conversely that M is considered as a Riemannian manifold, with its metric ds^2 given intrinsically. Suppose that β is a quadratic differential form on M, which satisfies (2.29), i.e. that the covariant derivative of β with respect to the metric ds^2 is a completely symmetric 3-tensor. We shall show that there is an embedding of M as a submanifold of R^3 such that the metric ds^2 is induced *on M by the flat metric on* R^3. *(Such an embedding is called an* *isometric embedding.)*

To do this, let (ω^i) *be an arbitrary orthonormal moving frame for the metric* ds^2 *on M, and let* (θ^a) *be an orthonormal moving frame for the flat metric on* R^3.

Let $\omega_i^{\ j}$ *be the 1-form such that* $d\omega^i = \omega_j^{\ i} \wedge \omega^j$, $\omega_j^{\ i} + \omega_i^{\ j} = 0$. *Let* $\theta_a^{\ b}$ *be the 1-forms such that*

$$d\theta^a = \theta_b^{\ a} \wedge \theta^b, \ \theta_b^{\ a} + \theta_a^{\ b} = 0.$$

Suppose:

$$\beta = \beta_{ij}\omega^i \cdot \omega^j.$$

Set:

$$\bar{\theta}_i^{\ 3} = \beta_{ij}\omega^j.$$

(Thus, we are just reversing the preceding definitions, starting off now with (ω^i, θ^a) *as arbitrary moving frames.*

Consider the manifold

$$M \times R^3,$$

and consider the forms $(\omega^i, \theta^a, \omega_i^{\ j}, \theta_a^{\ b})$ *on* $M \times R^3$, *pulled back with Cartesian projection maps.*

Consider the following differential forms on $M \times R^3$.

$$\eta^i = \theta^i - \omega^i,$$

$$\eta^3 = \theta^3,$$

$$\eta_i{}^j = \omega_i{}^j - \theta_i{}^j,$$

$$\eta_i{}^3 = \theta_i{}^3 - \bar{\theta}_i{}^3.$$

Let η denote the $F(M \times R^3)$-submodule of $F^1(M \times R^3)$ spanned by these 1-forms. Consider it as defining an exterior differential system. An integral submanifold is a submanifold of $M \times R^3$ such that all the forms in η are zero when restricted to that submanifold.

It is now readily seen that (as a consequence of the Gauss and Mainardi-Codazzi equations) this system is completely integrable, i.e. defines a foliation of $M \times R^3$. (See DGCV) Further, the leaves of this foliation are 2-dimensional submanifolds, which locally are the graphs of submanifold maps

$$M \to R^3.$$

These maps are the isometric embeddings of the Riemannian manifold M, which are determined by giving the second fundamental form β.

3. SURFACES WITH GIVEN PROPERTIES - QUADRICS

Suppose given a quadratic differential form φ on a 2-dimensional manifold M. One general problem is to recognize whether embeddings of M into R^3 exist, such that the flat ds^2 of R^3 restrict to φ, and which satisfy certain conditions which are fixed in advance. In order to do this, equations must be added to the isometric embedding equations. The integrability conditions of the resulting system must be found. If they are compatible, the resulting system of equations is the one which determines the unknown function.

This is the classical method. Typical problems are to decide whether surfaces with a given metric exist which are ruled, of constant mean curvature, etc. We restrict ourselves here to pointing out that the known theorems on the deformation of such types of surfaces may be proved very naturally using our methods.

In particular, we have used these methods to determine when a given metric admits an embedding as a second degree surface, i.e. as a quadric. (See Ricci 1895, "Lezioni", Second Part, Chapter VI). This problem, which had been solved only for the sphere, is now finished for an arbitrary quadric. One may, using simple finite operations, decide if a given form φ may be the metric of a second degree surface. Up to a rigid motion, there is at most one surface with this property.

Remark: Here is what is meant by the problem of defor-

mation in the classical literature.

Let M be a 2-dimensional submanifold of R^3, and let φ be the Riemannian metric induced from the flat metric ds^2 on R^3. A deformation of this surface is another submanifold mapping

$$\pi: M \to R^3$$

such that:

$$\pi^*(ds^2) = \varphi, \text{ but}$$

such that π does not result from a rigid motion of R^3. For example, it is a famous theorem that a compact, 2-dimensional Riemannian manifold of positive Gaussian curvature admits, up to a rigid motion, exactly one isometric embedding into R^3. In the classical language, it is indeformable.

4. GENERALIZATIONS OF THE THEORY OF SURFACES TO n-DIMENSIONAL SPACES

The general ideas of the preceding sections generalize easily to the case of n dimensional submanifolds of R^{n+1}, which are called hypersurfaces. The formulas are a specialization of those given in the preceding chapter, which express that the metric is of the first class.

Remark: *Ricci and Levi-Civita now proceed to write down these generalizations. Since these formulas are quite complicated in their formalism, will present them in Cartan's way.*

Choose the following range of indices and summation conventions:

$$1 \le i,\ j \le n$$

$$1 \le a,\ b \le n+1.$$

Let M be an n-dimensional submanifold of R^{n+1}. *Let* (θ^a) *be an orthonormal moving frame of 1-forms in* R^{n+1}, *such that:*

$$\theta^{n+1} = 0 \text{ on } M.$$

Set:

$$\omega^i = \theta^i \text{ restricted to } M.$$

The (ω^i) *are an orthonormal moving frame of the induced metric on M.*

Let $(\theta_a^{\ b})$ *be the forms on* R^{n+1} *such that:*

$$d\theta^a = \theta_b^{\ a} \wedge \theta^b$$

$$\theta_a^{\ b} + \theta_b^{\ a} = 0.$$

Set:

$$\omega_i^{\ j} = \theta_i^{\ j} \text{ restricted to } M.$$

$$\bar{\theta}_i^{\ n+1} = \theta_i^{\ n+1} \text{ restricted to } M.$$

$$\bar{\theta}_{n+1}^{\ \ i} = \theta_{n+1}^{\ \ i} \text{ restricted to } M = -\bar{\theta}_i^{\ n+1}$$

$$\beta = \bar{\theta}_i^{\ n+1} \cdot \omega^i = \text{second fundamental form of } M$$

$$\equiv \beta_{ij}\omega^i \cdot \omega^j.$$

The conditions expressing the flatness of the metric on R^{n+1} are:

$$d\theta_a^{\ b} - \theta_a^{\ c} \wedge \theta_c^{\ b} = 0.$$

Restrict these conditions to M, obtaining the following relations:

$$d\omega_i^{\ j} - \omega_i^{\ k} \wedge \omega_k^{\ j} = \bar{\theta}_i^{\ n+1} \wedge \bar{\theta}_{n+1}^{\ \ j}.$$

Set:

$$\Omega_i^{\ j} = d\omega_i^{\ j} - \omega_i^{\ k} \wedge \omega_k^{\ j}$$

$$= R_{ik\ell}^{\ \ \ j}\omega^k \wedge \omega^\ell$$

$\equiv$ *<u>Riemann curvature tensor of the induced metric on M</u>.*

Thus,

$$R_{ik\ell}^{\ \ \ j}\omega^k \wedge \omega^\ell = -\beta_{ik}\omega^k \wedge \beta_{j\ell}\omega^\ell,$$

or

$$R_{ik\ell}{}^{j} = \frac{1}{2}\,(\beta_{i\ell}\beta_{jk} - \beta_{ik}\beta_{j\ell}). \qquad (4.1)$$

This is the generalization of <u>*Gauss' formula*</u>. *It can be put into the form given in the text by Ricci and Levi-Civita by transforming frames from* (ω^i) *to* (dx^i), *where* (x^i) *is a coordinate system for M.*

To define the <u>*Mainardi-Codazzi equations*</u>, *start with the following formula on* R^{n+1}, *part of the conditions expressing the fact that it has zero curvature.*

$$d\theta_i^{\,n+1} = \theta_j^{\,n+1} \wedge \theta_i^{\,j}.$$

Restrict it to M:

$$d\bar{\theta}_i^{\,n+1} = \bar{\theta}_j^{\,n+1} \wedge \omega_i^{\,j}. \qquad (4.2)$$

These are a set of differential equations which must be satisfied by the second fundamental form β. *They can be written in tensor analysis language as:*

$$\beta_{ij,k} = \beta_{ik,j},$$

i.e. the 3-covariant tensor

$$(\beta_{ij,k})$$

is <u>*completely symmetric*</u>.

There is another approach to this question which goes back to Darboux' methods in his "Theorie des surfaces."

Replace R^{n+1} *by* G*, the Lie group of* <u>*rigid motions of*</u> R^{n+1}. R^{n+1} *is a coset space of* G*. This method has the great advantage of generalizing to other homogenous spaces, thus enabling one to develop a theory of submanifolds of other geometries, e.g. projective and conformal. Cartan himself extensively developed this approach. I shall now briefly indicate how it goes for the particular case:*

G *= group of rigid motions of* R^{n+1}.

G *is of dimension*

$$\frac{(n+1)(n)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

The vector space of 1-differential forms on G *which are invariant under left-translation has a basis labelled as follows:*

$$(\theta^a, \theta_b^{\ a}).$$

These 1-forms satisfy the following structure relations:

$$d\theta^a = \theta_b^{\ a} \wedge \theta^b$$

$$d\theta_b^{\ a} = \theta_c^{\ a} \wedge \theta_b^{\ c}.$$

The Pfaffian system

$$\theta^a = 0$$

is then completely integrable. Its maximal integral manifold which passes through the identity element of G

is a subgroup H, which is isomorphic to

$$SO(n+1, R).$$

(H is the subgroup of rigid motions of R^{n+1} *which leaves the origin fixed - hence is linear - and which have determinant +1).* R^{n+1} *is the coset space G/H. We shall write this as*

$$G/H = R^{n+1}.$$

Now, let M be a manifold of dimension n. Let α *be a quadratic differential form on M which defines a Riemann metric for M. Let* β *be another quadratic differential form on M.*

Let (ω^i) *be a moving frame for M which is orthonormal relative to the metric* φ*. Let* (ω_i^j) *be the corresponding connection forms. Let:*

$$\beta = \beta_{ij}\omega^i\cdot\omega^j.$$

$$\bar{\theta}_i^{\,n+1} = \beta_{ij}\omega^j.$$

Now, on M × G, consider the following exterior differential system:

$$\omega^i - \theta^i = 0$$

$$\theta^{n+1} = 0$$

$$\theta_i^{\,n+1} - \bar{\theta}_i^{\,n+1} = 0$$

$$\theta_j{}^i - \omega_j{}^i = 0.$$

It is readily verified that this system is <u>*completely integrable*</u> *(i.e. defines a* <u>*foliation*</u> *of $M \times G$) if and only if φ and β are related as follows:*

> *φ and β satisfy both the generalized* <u>*Gauss*</u> *and* <u>*Mainardi-Codazzi Equations*</u>.

If these conditions are satisfied, then the following conclusions can be derived:

> *The leaves of the foliation are (at least locally) graphs of mappings $M \to G$. Follow such a mapping with the projection $G \to G/H = R^{n+1}$, and obtain an isometric embedding of M as a submanifold of R^{n+1}, with β as the second fundamental form.*

There is also an important group-invariance property. The forms θ^a, $\theta_b{}^a$ are, by definition, invariant under left translation by elements of G. Let G act on $M \times G$ as follows:

> *The translation of $(p,g) \in M \times G$ by $g_0 \in G$ is $(p, g_0 g)$.*

Then, we see that the exterior differential system is *invariant under this action of G. In particular, if Gauss-Mainard-Codazzi is satisfied:*

> *The transformation of a leaf of the foliation by a rigid notion is again a leaf.*

Projected down to R^n, this implies that:

> *Two isometric embedding of M with β as second fundamental form differ by a rigid motion.*

5. GROUPS OF MOTIONS OF A RIEMANNIAN MANIFOLD

Let φ be the quadratic differential form which defines a Riemannian metric on a manifold M of dimension n. Let (x^i), $1 \leq i, j \leq n$, be a coordinate system for M. Let

$$X = X^i \frac{\partial}{\partial x^i}$$

be a vector field on M. It generates a one parameter group of motions of M. We say this group is <u>rigid</u> or <u>deformation-free</u> if each element of the group preserves distances. The condition for this is that the Lie derivative of φ by X is zero, or, alternately, that the following relations be satisfied:

$$X_{,j}{}^{i} + X_{,i}{}^{j} = 0. \tag{5.1}$$

These relations are due to Killing [1892], and a vector field X satisfying them is called a Killing vector field.

Remark. *Recall that* $X^i_{,j}$ *denotes the covariant derivative of the contravariant vector field with respect to the metric tensor* φ. *In modern notations, relation (5.1) can be written as:*

$$\varphi(\nabla_Y X, Z) + \varphi(\nabla_Z X, Y) = 0 \qquad (5.2)$$

for $Y, Z \in V(M)$.

Let

$$X = \rho Y, \qquad (5.3)$$

with Y a normalized vector field (i.e. $\varphi(Y, Y) = 1$), ρ a function. Y defines a congruence.

Setting relation (5.3) into (5.1), we have the following theorem, which is a natural generalization of the situation for surfaces.

Let C be a given congruence on the manifold M. In order that the curves of C be orbits (after change in parameterization) of a Killing vector field it is necessary and sufficient that the following conditions be satisfied:

a) Each system of (n-1) congruences which forms, together with C, an orthonormal system of congruences is canonical with respect to C.

b) Let C' be a congruence which is orthogonal to C. Then, either C' is geodesic, or its geodesic curvature at each point is perpendicular to the curve of C passing through this point.

c) C is a normal congruence, and the one-parameter family of hypersurfaces which is perpendicular to C is isothermal.

Consider the case where M is 3-dimensional and X is a Killing vector field. Let

$$X_i$$

be the covariant components of X, and let $X_{i,j}$ be their covariant derivative. The Killing equations (5.1) then assert that $X_{i,j}$ is a skew-symmetric tensor. Hence, it may be written in the form

$$X_{i,j} = \epsilon_{ijk}Z^k. \tag{5.4}$$

(ϵ_{ijk}) is the completely skew-symmetric tensor defined in Chapter I, Section 3.

Introduce an orthonormal moving frame [1], [2], [3] into M. Let X_i', Z_i' be the components of X and Z with respect to this moving frame, defined by the following equation:

$$X_i' = X^j \lambda_j[i]$$
$$Z_i' = Z^j \lambda_j[i]. \tag{5.5}$$

Remark: Recall that $(\lambda_j[i])$ are the components (with respect to the coordinates (x^i) of the vector field $[i]$.

Conditions (5.4) then take the following form:

$$\frac{\partial X_i'}{\partial s_i} = \sum_{j=1}^{3} \gamma_{ji}{}^{i} X_j' \tag{5.6}$$

$$\frac{\partial X_i'}{\partial s_{i+1}} = \sum_{j=1}^{3} \gamma^{i}_{j(i+1)} X_j' + Z'_{i+2} \tag{5.7}$$

$$\frac{\partial X_i'}{\partial s_{i+2}} = \sum_{j=1}^{3} \gamma^{i}_{j(i+2)} X_j' - Z'_{i+1} \tag{5.8}$$

We also have the following integrability conditions:

$$\frac{\partial Z_i'}{\partial s_j} = \sum_{k=1}^{3} (\gamma_{kj}{}^{i} Z_i' + \gamma^{i}_{(i+2)j} X'_{j+1} - \gamma^{j}_{j(i+1)} X_{j+2}) \tag{5.9}$$

Remark: In (5.6)-(5.9), the summation convention is __not__ in force. The authors are using the special convention (for $n = 3$) that indices differing mod 3 are identical.

Remarks: I will discuss some of the general problems *suggested by this material.*

Let M be a (positive definite) Riemannian manifold and let G be a group of isometries of M, i.e. each $g \in G$ is a diffeomorphism of M which preserves the length of curves. In order to be able to use certain technical tools of differential geometry and Lie group theory, we make the following assumption:

$$\text{G is a closed subgroup of the group of all isometries of the metric.} \tag{5.10}$$

(It is known (see Helgason [1]) that the group of isometries can be made into a Lie group, which acts in a smooth way on M. "Closed" refers to the Lie group topology on the group).

Hence, by a theorem of Cartan, G itself is a Lie group and acts smoothly on M, i.e. the map

$$G \times M \to M$$

defined by the transformation group action is C^∞.

The following property is very useful in deducing general properties. It is only true if G acts as a group of isometries of a positive metric:

The transformation group map $G \times M \to M$ is proper, i.e. the inverse image of a compact subset of M is compact. (5.11)

As for any transformation group, the action of G on M defines an equivalence relation:

Points p, $p' \in M$ are equivalent *if there is a $g \in G$ such that*

$$p' = gp.$$

(See "Interdisciplinary Mathematics," vol. I, for general algebraic concepts related to transformation group actions.) The equivalence classes are called orbits of G. *The space of equivalence classes,*

$$G\backslash M,$$

is called the orbit space. *In this section, Ricci and Levi-Civita are concerned with properties of the orbits and the orbit space in case G is a one-parameter group, and the orbits are one-dimensional submanifolds. I shall attempt to cover the general case.*

The first question is: What type of topological-manifold structure is it natural to put on $G\backslash M$? In DGCV, I have described a metric space *structure on $G\backslash M$ - this of course defines a topology.*

Let us examine the manifold structure question. Let G_o be the connected component of the identity of G. G_o is an invariant subgroup *of G. The orbits of G_o define a* *foliation* of M. (A *foliation* of a manifold M is an equivalence relation on M, whose equivalence classes are connected submanifolds of M). The quotient group $G\backslash G_o$ acts on the orbit space $G_o\backslash M$. It is readily seen that this action preserves the *metric space structure* on $G_o\backslash M$ defined - as explained in DGCV - by the given Riemannian metric on M. Thus we can seperate the problem of studying orbit spaces into two parts - first orbit spaces of connected groups, then discrete ones.

Suppose then that G is *connected*. In the classical literature, one usually assumes everything is "non-singular." In this case, this means that the orbits are of *constant dimension*. *Let us also suppose that this condition is satisfied*. (See DGCV for what can be said more generally.)

$\mathcal{G}$, the Lie algebra of G, is a Lie algebra of vector fields on M. The foliation defined by $\mathcal{G}$ is non-singular.

Definition. The foliation is said to be *regular* if the space of leaves (in this case, $G\backslash M$), can be made into a manifold in such a way that the projection map $\pi: M \to G\backslash M$ which assigns to each point $p \in M$ the orbit on which it lies is a *submersion* map, i.e.

$$\pi_*(M_p) = (G\backslash M)_{\pi(p)}$$

for all $p \in M$.

H. Sussman has given a very useful and convenient necessary and sufficient set of conditions that a non-singular foliation be regular. He requires that:

a) *The set of points of* $M \times M$ *of the form* (p, gp), *with* $p \in M$, $g \in G$, *is a* ***closed*** *subset of* $M \times M$.

b) *Let* $\underset{\sim}{S}$ *be the set of vector fields* X *on* M *such that the one parameter group generated by* X *maps each leaf of the foliation into another leaf. (*$\underset{\sim}{S}$ *is called the* ***symmetry Lie algebra*** *of the foliation). Sussman's condition is then that:* $\underset{\sim}{S}(p) = M_p$ *for all* $p \in M$, *i.e. that* $\underset{\sim}{S}$ *act* ***transitively*** *on* M.

Sussman's conditions apply to general non-singular foliations. In this case - where the leaves are orbits of a ***closed, connected group of isometries***, *we can show that a) is always satisfied. For, suppose*

$$(p_1, q_1), (p_2, q_2), \ldots$$

is a sequence of points of $M \times M$, *each of which lies in the same orbit, and which approaches a limit* (p, q) *as* $j \to \infty$, *i.e.*

$$\lim_{j\to\infty} p_j = p$$

$$\lim_{j\to\infty} q_j = 1.$$

By hypotheses that each (p_j, q_j) *lies on the same orbit, there is a* $q_j \in G$ *such that*

$$q_j = g_j p_j.$$

From the theorem that the transformation group action $G \times M \to M$ *is proper, we see that a subsequence of the* $\{g_j\}$ *must converge, say to a* $g \in G$. *Then, we have:* $q = gp$, *i.e.* (p, q) *lies on an orbit, and we have proved that condition a) is satisfied.*

Unfortunately, I do not see any useful general conditions which imply that condition b) is satisfied. Perhaps this is a suitable topic for further research.

Let us now suppose that the foliation is regular, in the sense that $G\backslash M$ *is a manifold. What are the differential geometric properties of* $G\backslash M$? *Especially, can one give a set of properties which characterize orbit spaces? (Notice that Ricci and Levi-Civita do this here, in case* G *is a one-parameter group.) Now, one general property of Riemannian manifolds that has been isolated in recent times is the notion of a Riemannian submersion mapping.*

(Their properties have been discussed by, among others, B. Reinhart (who was the first), the author and B. O'Neill.) It is easy to see that the natural projection map

$$M \to G \backslash M$$

has (in the regular case) this Riemannian submersion property. Ricci and Levi-Civita's remarks suggest (to me at least) that an interesting geometric problem might be:

> *Find the further properties that a Riemannian submersion must satisfy to imply that it be is the orbit space of an isometry group action.*

According to the statements in this Section, Ricci has done this for foliations with 1-dimensional leaves.

In order to develop some insight into this problem, I will develop, in Section 8 at the end of this chapter, the general geometric properties of Riemannian submersions.

6. COMPLETE STUDY OF THE GROUPS OF MOTION OF THREE DIMENSIONAL RIEMANNIAN MANIFOLDS - RESOLUTION OF THE PROBLEM OF RECOGNIZING IF A THREE DIMENSIONAL RIEMANNIAN MANIFOLD ADMITS A GROUP OF MOTIONS, AND OF DETERMINING THE GROUP

Intransitive groups. In a 3-dimensional Riemannian manifold M a one-parameter family of surfaces may be represented (as explained in Chapter 2, Section 3) by a covariant vector field which represents the congruence of its orthogonal trajectories. Let us be given such an object, and ask whether M admits rigid motions which transform each of the surfaces into itself. First, we remark that, from previous work, one may regard the problem as solved when the motion group has one-dimensional orbits; this of course happens for one parameter groups.

This said, consider, in equations (5.7)-(5.9) of the preceding section, the congruence [3] as the orthogonal trajectory of the family of surfaces. One will have:

$$X_3{}' = 0. \tag{6.1}$$

<u>*Remark*</u>. *Recall the notations, using Cartan's moving frame notations. Let* (ω^i) *be an orthonormal moving frame for M.*

$$\omega^3 = 0$$

determines the family of surfaces. X is a Killing vector field resulting from the group of motions. $X_i{}'$ *denote the* <u>*covariant*</u> *components with respect to the* (ω^i) *of the vector field X.*

Equations (5.7)-(5.8) give the following relations:

$$\sum_{j=1}^{3} \gamma_{j3}{}^{3} X_j{}' = 0 \tag{6.2}$$

$$Z_1{}' = \sum_{j=1}^{3} \gamma_{j2}{}^{3} X_j{}' \tag{6.3}$$

$$Z_2{}' = \sum_{j=1}^{3} \gamma_{j1}{}^{3} X_j{}'. \tag{6.4}$$

If (6.2) is not satisfied identically, it determines the orbit of the group as one dimensional submanifolds. If, to the contrary, (6.2) is satisfied identically, one has:

$$\gamma_{13}{}^{3} = 0 = \gamma_{23}{}^{3}. \tag{6.5}$$

These equations tell us that the congruence [3] is geodesic. Hence, we have the following result:

> In order that a 3-dimensional Riemannian manifold M admit a rigid motion group of more than one parameter, which leaves invariant each surface of a one-parameter family of surfaces, it is necessary that these surfaces be geodesically parallel.

Remark: The theory of Riemannian submersions, described in Section 8, covers this result. In this language, Ricci and Levi-Civita assume given a submersion map

$$\pi: M' \to M,$$

with:

$$\dim M' = 1,$$

i.e. the fibers of π *are 2-dimensional submanifolds of M. They assume given a connected Lie group G, acting on M as a group of isometries, which maps into itself each of the fibers of* π*. They (implicitly) assume that the action of G is "non-singular," in the sense that its orbits are of constant dimension.*

The first case is that where the orbits are all 1-dimensional. Here is the result which covers that situation.

Theorem. Suppose G has all one-dimensional orbits. Then G is itself one-dimensional.

Proof. Let p be a point of M. If G were not one-dimensional, there would be a one-parameter subgroup $t \to g(t)$ *which acts as the identity on the orbit*

$$Gp$$

of G through p. Now, by assumption, G maps into itself a 2-dimensional submanifold N (the fiber of π*) passing* *through p. Each g(t) acts as* ***isometries****, hence leaves invariant the tangent vectors to N which are perpendicular to the orbit Gp. Hence, g(t) leaves each point of N invariant. Again, g(t) leaves invariant the tangent vectors to M which are perpendicular to N, hence also leaves fixed the geodesics they determine. Hence, each g(t) acts as the identity on M, which is the contradiction.*

Consider the case now where the orbits of G are 2-dimensional. They then fill up the fibers of π*, i.e. the fibers of* π *are orbits of an isometry group. As proved in Section 8, the submersion map* π *is then Riemannian, and the fibers of* π *are geodesically parallel.*

Let us remark that, because of Equations (6.1)-(6.4), the unknown functions reduce to three, namely:

$$X_1', X_2', Z_3.$$

These must also satisfy Equations (5.7)-(5.9), which express the partial derivatives of these three unknown functions in terms of the functions themselves and known quantities. One concludes that:

> A group of isometries which leaves invariant each fiber of π is at most 3-dimensional.

Remark: Here is a more precise result.

Theorem. Let G be a connected group of isometries of a connected, 3-dimensional Riemannian manifold. Suppose also that G leaves invariant a 2-dimensional connected submanifold N of M. Then, G is at most 3-dimensional. If G is 2 or 3-dimensional, then N is (in the Riemannian metric induced from that on M) of constant curvature.

Proof. Again, we will show that every one-parameter subgroup

$$t \to g(t)$$

of G must act in a non-trivial way on N. Suppose otherwise, i.e. each $g(t)$ leaves each point of N fixed. $g(t)$ then leaves fixed each perpendicular tangent vector to N, hence leaves invariant each perpendicular geodesic. Hence, $g(t)$ leaves fixed each point of an open neighborhood of N. Since M is connected, $g(t)$ acts as the identity on M, contradiction.

Hence, G is determined by its action on N. Now, one can readily prove that a Lie group of isometries of a 2-dimensional Riemannian manifold is at most 3-dimensional, and if it act transitively, (which will happen if *dim G = 2 or 3) N must be of constant curvature.*

To finish our work, it would be necessary to discuss the full system of equations determining the group. A convenient choice of moving frame [1], [2] would make this discussion relatively easy. We cannot go into it here. Among the results to which one is led we restrict ourselves to citing the following:

> If a manifold M admits a group G of rigid motions of the kind considered just before, i.e. which leaves invariant a 1-parameter family of surfaces, then at a point of M the principal directions are given by the normal and principal tangent vectors to the surface passing through this point. The principal invariants of M, are invariant under the group and have a constant value on each surface of the family.

<u>Transitive groups</u>

Here are the results obtained in case G acts transitively on the 3-dimensional Riemannian manifold M.

In order that G act transitively, it is necessary that the principal invariants (*of the Riemannian metric*) be constants. Denote these invariants by

c_1, c_2, c_3

<u>Remark</u>: *c_1, c_2, c_3 are the eigenvalues of the Ricci tensor of the metric on M.*

There are three cases to consider:

1) $c_1 = c_2 = c_3$

2) $c_2 = c_3$; $c_1 \neq c_2$

3) $c_1 \neq c_2$; $c_3 \neq c_1$; $c_1 \neq c_2$.

In the first case, the manifold M is of constant curvature, and G has at most six parameters.

In the second case, the invariant c_1 corresponds to a principal congruence [1] which is unique and determined; c_2, c_3 correspond to congruences [2], [3] such that [1], [2], [3] determine an orthonormal moving frame for M. G will be transitive and 4-dimensional provided that the following additional conditions be satisfied:

a) The congruence [1] is geodesic; For each orthogonal congruence [2], its geodesic curvature is perpendicular to the lines of [1], [2].

b) The rotational coefficients $\gamma_{32}{}^1$, $\gamma_{23}{}^1$ have constant values, and their sum is zero.

In the last case, the orthonormal frame [1], [2], [3] is completely determined. In order that G be transitive, it is necessary and sufficient that the rotational coefficients of the moving frame all be constants. If this condition is satisfied, then G has exactly 3 dimensions.

Remarks: *The modern version of this material would be to classify groups of isometries of all 3-dimensional Riemannian manifolds. I will now describe some results which can be obtained by modern techniques, and which are in the spirit of the material sketched by Ricci and Levi-Civita in this section.*

Let M be a Riemannian manifold, and let G be a connected Lie group of isometries of M. Suppose, in addition, that G is a closed subgroup of the group of all isometries of M.

A point $p \in M$ is said to be a maximal point of the action of G if:

$$\dim (Gp) \geq \dim (Gp')$$

for all $p' \in M$.

The set of all maximal orbits forms an open subset of M. (See DGCV, Chapter 25, for the basic material.)

Given a point $p \in M$, let

$$G^p = \{g \in G: gp = p\}.$$

G^p *is called the* *isotropy subgroup* *of* G *at* p.

A point $p \in M$ *is called a* *principal point* *(relative to the action of* G*) if* p *is a maximal point, and if* G^p *has a minimal number of connected components, compared with the other maximal orbits.*

Let N *be an orbit of* G *at a principal point* p. *Here is a basic result. (See DGCV):*

$$G^p \text{ leaves fixed each geodesic of } M \text{ which is perpendicular to } N. \tag{6.6}$$

From this, we deduce:

Theorem 1. *Suppose a* $g \in G$ *acts as the identity on* N. *Then,* g *acts as the identity on* M. *In particular,* G *is isomorphic to a transitive transformation group on* N.

Corollary to Theorem 1. *If the principal orbits are one-dimensional, then* G *is a one parameter group. If the principal orbits are two dimensional,* G *is isomorphic to a group which can act transitively on a 2-dimensional Riemannian manifold of positive curvature.*

$\mathcal{G}$, *the Lie algebra of* G, *is identified with a Lie* *algebra of vector fields on M, i.e. with a Lie subalgebra of V(M). Denote the Riemannian metric by* φ. *Let*

$$d_\varphi p$$

denote the 3-dimensional differential form on M which is the <u>*volume element*</u> *relative to* φ. *(*$d_\varphi p$ *is characterized by the property that its inner product under* φ *is +1).*

Given $X \in \mathcal{G}$, *it defines a* <u>*Killing vector field*</u> *on M. For* $Y, Z \in V(M)$, *set:*

$$\omega_X(Y, Z) = \varphi(\nabla_Y X, Z). \tag{6.7}$$

The Killing equations imply that ω_X *is a 2-differential form on M. This defines an R-linear map*

$$X \to \omega_X$$

of $\mathcal{G} \to F^2(M)$. *Let us investigate its properties, relative to the Lie algebra structure on* $\mathcal{G}$.

Suppose X_1, X_2 *are elements of* $\mathcal{G}$. *Lie derivation by* X_1 *is an infinitesimal automorphism of the affine connection. This means that:*

$$[X_1, \nabla_Y Z] = \nabla_{[X_1, Y]} Z + \nabla_Y([X_1, Z]) \tag{6.8}$$

for $Y, Z \in V(M)$.

Hence,

$$\nabla_Y([X_1, X_2]) = [X_1, \nabla_Y X_2] - \nabla_{[X_1, Y]} X_2. \tag{6.9}$$

Thus, using (6.9) and (6.7):

$$\omega_{[X_1, X_2]}(Y, Z) = \varphi(\nabla_Y[X_1, X_2], Z)$$

$$= \varphi([X_1, \nabla_Y X_2], Z) - \varphi(\nabla_{[X_1, Y]}X_2, Z)$$

=, *using the fact that the Lie derivative of* φ *by* X_1 *is zero.*

$$- \varphi(\nabla_Y X_2, [X_1, Z]) + X_1(\varphi(\nabla_Y X_2, Z))$$

$$- \varphi(\nabla_{[X_1, Y]}X_2, Z).$$

We see then that we have:

$$X_1(\omega_{X_2}) = \omega_{[X_1, X_2]}. \tag{6.10}$$

This means that the map

$$\underset{\sim}{G} \to F^2(M)$$

intertwines the action of $\underset{\sim}{G}$.

Let us now specialize to the case considered by Ricci and Levi-Civita, namely:

$$\dim M = 3.$$

Given $X \in \underset{\sim}{G}$, *let*

$$\delta(X)$$

be the vector field on M such that:

$$\delta(X) \lrcorner\, d_\varphi p = \omega_X. \tag{6.11}$$

(Here we use the fact - which is basic for the success of classical vector analysis - that the space of tangent vectors is isomorphic to the space of 2-covectors.) It is this map

$$\delta \colon \underset{\sim}{G} \to V(X)$$

which the authors use in the text. It intertwines the action of $\underset{\sim}{G}$, *i.e.*

$$[X_1, \delta(X_2)] = \delta([X_1, X_2]) \qquad (6.12)$$

for $X_1, X_2 \in G$.

Now, let us suppose, as in the text, that an orbit of G is a 2-dimensional submanifold N of M. Let S be the second fundamental form of S. The action of G leaves S invariant. Now, S and φ *are both quadratic differential forms on N which are invariant under G. In particular, the trace of S is invariant under G, i.e.*

<u>*N has constant mean curvature*</u>

(For the significance of this fact, see Section 8.)

Let us now consider the case where:

<u>*G acts transitively on M.*</u>

Given a point $p \in M$, *the Riemann curvature tensor* $\underline{R}$

is a quadratic form on the vector space

$$M_p \wedge M_p .$$

Since <u>*M is 3-dimensional,*</u> *$M_p \wedge M_p$ is isomorphic to M_p. Thus,* <u>*R*</u> *is identified with a quadratic form on M_p. (This is, in fact, essentially identical with the* <u>*Ricci curvature tensor*</u> *of the metric). Let c_1, c_2, c_3 denote the eigenvalues of quadratic from an M_p determined by* <u>*R*</u>*.*

Now, <u>*R*</u> *is invariant under the action of G. Hence,*

$$c_1,\ c_2,\ c_3 \text{ are } \underline{\text{constant}} \text{ on } M. \tag{6.13}$$

To derive the consequences of (6.13), let

$$(\omega^i),\ 1 \le i,\ j \le 3, \tag{6.14}$$

be an orthonormal moving frame of 1-forms on M. Let

$$(\omega_i^{\,j})$$

be the corresponding connection forms, and let

$$\Omega_i^{\,j}$$

be the curvature forms.

Suppose that the moving frame (ω^i) is chosen to be the eigenvectors of <u>*R*</u>*, with eigenvalues c_1, c_2, c_3. Then, the following conditions are satisfied:*

$$\Omega_1^{\,2} = c_3 \omega^1 \wedge \omega^2 \tag{6.15}$$

$$\Omega_1{}^3 = c_2\omega^1 \wedge \omega^3 \tag{6.16}$$

$$\Omega_2{}^3 = c_1\omega^2 \wedge \omega^3 \tag{6.17}$$

As we have seen, the *transitivity* *of G, as a group of isometries, implies that the c_1, c_2, c_3 are* *constants*. *Combine relations (6.15)-(6.17) with the following Riemannian structure relations:*

$$d\omega_i{}^j - \omega_i{}^k \wedge \omega_k{}^j = \Omega_i{}^j. \tag{6.18}$$

We see that they imply that there are *constants* *$c_{jk}{}^i$ such that:*

$$d\omega^i = c_{jk}{}^i\omega^j \wedge \omega^k. \tag{6.19}$$

We recognize that (6.19) means that M is diffeomorphic to a 3-dimensional Lie group H such that the forms (ω^i) are identified with the *left invariant* *1-forms on H. Since the metric φ is*

$$\delta_{ij}\omega^i\cdot\omega^j,$$

we see that the metric φ is invariant under *left translation* *by H.*

We can now sum up as follows:

Theorem. *Every homogeneous 3-dimensional Riemannian manifold is isometric to a left-translation-invariant metric* *on a 3-dimensional Lie group H.*

One can prove further that the group of isometries of M is a subgroup of H × H, acting via left and right translation on itself.

7. RELATIONS BETWEEN THE PRECEDING RESULTS AND THE RESEARCH OF LIE AND BIANCHI

The research that we have just described is closely linked with that of Lie on the problem of Riemann-Helmholtz and that of Bianchi [1897] on 3-dimensional spaces which admit a connected group of rigid motions. He has determined, in convenient coordinate systems, all the isomorphism classes of Lie groups that appear as motions and the invariant metrics.

In Section 6 we have considered the following question:

Given the metric tensor φ of a 3-dimensional Riemannian manifold, determine <u>if</u> it admits rigid motions, at all, and also determine the full group of such motions.

Our results also give a new contribution to what Lie calls the <u>Problem of Riemann-Helmholtz</u>. (See Lie and Engel's "Transformationsgruppen," vol. III). To see this connection, recall that in Section 6 we have considered two Killing vector fields X and Z whose components satisfy the following relations:

$$X_{i,j} = \epsilon_{ijk} Z_k.$$

When the manifold M is R^3, with the Euclidean metric, Z can be considered as an infinitesimal generator of a one-parameter translation group, while X is the infinitesimal generator of a group of rotations. Following the ideas of Section 4 of the preceding Chapter, we may form these equations in an arbitrary Riemannian manifold M. We see that:

> M has constant curvature if and only if there is a Killing vector field for which the components of translation and rotation take on arbitrary values.

This gives a precise kinematic meaning to the words of Riemann (Gesamelte Werke, p. 264) on the subject:

> The common property of constant curvature manifolds can be expressed by saying that figures may be moved arbitrarily without deformation.

If we consider a 3-dimensional Riemannian manifold which possesses a 4-dimensional transitive group G of rigid motions, the translation part may be chosen arbitrarily, but the rotation is only about one axis. When the group is 3-dimensional, there is no rotational component.

Finally, we remark that these results answer - at least for the case of 3-dimensional manifolds - the problem which has been posed as a competition by the Jablonowski faculty for the year 1901.

We also point out that these results may be described without using group theory,' although we do prefer to use its language in order to present to the reader the full spirit of the work and to point out the connection with what we already known.

8. RIEMANNIAN AND ISOPARAMETRIC-ISOTHERMAL SUBMERSIONS

Let M, M' be Riemannian manifolds. (For simplicity, we suppose that the Riemannian metric is <u>*positive*</u>*, although it is possible, using the ideas developed in Vol. V of IM to extend the development to the non-positive case.) Let*

$$\pi: M \to M'$$

be a submersion mapping. Suppose φ *and* φ' *denote the*

Riemannian metrics on M and M'. Take φ and φ' in their <u>*contravariant form*</u>*, i.e. φ is a bilinear, symmetric form on 1-differential forms. π is said to be a* <u>*Riemannian submersion*</u> *if the following condition is satisfied:*

$$\pi_*(\varphi) = \varphi'. \tag{8.1}$$

Recall from IM, vol. V, that (8.1) means the following condition:

$$\varphi(\pi^*(\theta_1), \pi^*(\theta_2)) = \pi^*(\varphi'(\theta_1, \theta_2)) \tag{8.2}$$

for $\theta_1, \theta_2 \in F^1(M')$.

This condition is also said to define π as a <u>*metric homomorphism*</u>*. Suppose from now on that it is satisfied.*

Choose indices and the summation convention as follows:

$1 \le i, j \le n = \dim M'$.

$n + 1 \le v, u \le m = \dim M$

$1 \le a, b \le m = \dim M$.

Let (θ^i) be a moving frame of 1-forms for M' which is orthonormal with respect to the metric φ'. This means that:

$$\varphi'(\theta^i, \theta^j) = \delta^{ij}. \tag{8.3}$$

Set:

$$\omega^i = \pi^*(\theta^i). \tag{8.4}$$

Condition (8.2) implies that:

$$\varphi(\omega^i, \omega^j) = \pi^*(\varphi'(\theta^i, \theta^j)) = \delta^{ij},$$

i.e. the (ω^i) *are* *<u>orthonormal with respect to the metric</u>* *<u>φ on M</u>*.

One can now find additional 1-forms (ω^u) *on* M*, such that the* (ω^i, ω^u) *form an orthonormal moving frame for the metric* φ*. A moving frame of this type is said to be* *<u>adapted to the submersion</u>*. *(M has a* *<u>G-structure</u>*, *in the sense of Cartan, where* $G = O(n) \times O(m-n)$*. The* (ω^i, ω^u) *are moving frames adapted to this G-structure.)*

Let $(\theta_i^{\ j})$ *be the connection forms of the moving frame* (θ^i)*, and let* $(\omega_a^{\ b})$ *be the connection formes of the moving frame* $(\omega^a) \equiv (\omega^i, \omega^u)$*. By definition, we have the following relations:*

$$d\theta^i = \theta_j^{\ i} \wedge \theta^j$$

$$\theta_j^{\ i} + \theta_i^{\ j} = 0$$

$$d\omega^a = \omega_b^{\ a} \wedge \omega_b$$

$$\omega_b^{\ a} + \omega_a^{\ b} = 0.$$

Set:

$$\omega_b^{\ a} = \gamma_{bc}^{\ \ a}\omega^c. \tag{8.5}$$

Using (8.4), we have:

$$(\gamma_{jk}{}^{i}\omega^{i} + \gamma_{ju}{}^{i}\omega^{u}) \wedge \omega^{j}$$
$$+ (\gamma_{vk}{}^{i}\omega^{i} + \gamma_{vu}{}^{i}\omega^{u}) \wedge \omega^{v}$$
$$= \pi^{*}(\theta_{j}^{i}) \wedge \omega^{j}.$$

We see from this relation that:

$$\pi^{*}(\theta_{j}^{i}) = \gamma_{jk}{}^{i}\omega^{k} \qquad (8.6)$$

$$\gamma_{vu}{}^{i} = \gamma_{uv}{}^{i} \qquad (8.7)$$

$$\gamma_{ju}{}^{i} = -\gamma_{uj}{}^{i}. \qquad (8.8)$$

Conditions (8.6)-(8.8) are the first order conditions implied by the Riemannian submersion relations.

We can transform relations (8.6)-(8.8) into a basis-independent form. To do this, let

$$(X_{a})$$

be the basis of vector fields on M which are dual to the basis (ω^{a}) *of 1-forms. This means that*

$$\omega^{a}(X_{b}) = \delta_{b}^{a}.$$

Hence,

$$\pi_{*}(X_{u}) = 0 \qquad (8.9)$$

$$\pi_{*}(X_{i}) \equiv Y_{i}$$

are the basis of vector fields on M dual to the (θ^i). *The vector fields* X_u *are* vertical *with respect to the submersion map* π, *while the* X_a *are* horizontal. *Let*

$$V: V(M) \to V(M)$$

$$H: V(M) \to V(M)$$

be the projection maps of vector fields on the vertical and horizontal ones.

Let ∇ *be the affine connection associated with the metric* φ, *and let* ∇' *be the affine connection associated with the metric* φ'. *Then*

$$\nabla_{X_a} X_b = \gamma_{ab}{}^{c} X_c. \tag{8.10}$$

From (8.6), we see that:

$$\omega^k(\nabla_{X_i} X_j) = \theta^k(\nabla_{Y_i}{}' Y_j). \tag{8.11}$$

We can rewrite this as follows:

$$\pi_*(H(\nabla_{X_i} X_j)) = \nabla'_{\pi_*(X_i)} \pi_*(X_j) \tag{8.12}$$

We can interpret this formula in the following basis independent way:

Theorem 1. *Let* Y_1, Y_2 *be vector fields on* M', *and let* X_1, X_2 *be their* horizontal lifts *to* M. *Then,*

$$\begin{array}{l} H(\nabla_{X_1} X_2) \text{ is the horizontal} \\ \text{lift of } \nabla'_{X_1} Y_2. \end{array} \tag{8.13}$$

As a corollary, we have the following basic results on the geometric properties of geodesics of the metric on M:

Theorem 8.2. *Let σ be a curve in M which is horizontal in the sense that it is perpendicular to each fiber of π, and which is a geodesic of the metric on M. Then, $\pi(\sigma)$ is a geodesic of the metric on M'.*

Proof. *Since σ is a geodesic of M, there is (at least locally) a vector field X on M such that:*

σ *is an orbit curve of X.*

The condition that σ is a geodesic is that:

$$\nabla_X X = 0 \text{ on } \sigma.$$

Since σ is horizontal, we can suppose without loss in generality that X is horizontal. Consider

$$H(\nabla_X X).$$

Projected into M' via π, it is a vector field Y such that $\pi(\sigma)$ is one of its orbit curves. Hence, $\nabla_Y{}'Y = 0$ on $\pi(\sigma)$. This shows that $\pi(\sigma)$ is an geodesic of M'.

Theorem 8.3. *Let σ be a curve in M which is horizontal, in the sense that it is perpendicular to each fiber of π.*

Suppose that $\pi(\sigma)$ is a geodesic of M'. Then, σ is a geodesic of the metric on M.

Proof. A *Hamilton Jacobi function* on M' is a function f' such that:

$$\varphi'(df', df') = 1. \tag{8.14}$$

Let f' be such a function, and let

$$Y = \text{grad } f',$$

i.e.

$$Y(h) = \varphi'(df', dh) \tag{8.15}$$

for each $h \in F(M')$. Then, the orbits of Y are geodesics of the metric φ', and, locally, each such geodesic arises in this way from Hamilton-Jacobi functions. (See DGCV).

In particular, choose f' so that $\pi(\sigma)$ is an orbit of Y. Set:

$$f = \pi^*(f'). \tag{8.16}$$

Then,

$$\varphi(df, df) = \varphi(\pi^*(df'), \pi^*(df'))$$

=, using the Riemannian submersion property for π. (See IM, vol. V),

$$\pi^*(\varphi(df'), df')) = 1.$$

We see that f is a Hamilton-Jacobi function with respect to the metric φ.

Set:

$$X = \text{grad } f.$$

Then, if Z is a vertical vector field on M,

$$\varphi(Z, X) = Z(f) = Z(\pi^*(f'))$$

$$= \pi^*(\pi_*(Z)(f'))$$

$$= 0,$$

since Z satisfies:

$$\pi_*(Z) = 0.$$

In particular,

<u>*X is horizontal*</u>.

Now, for $p \in M$, $h \in F(M')$,

$$\pi_*(X(p))(h) = X(p)(\pi^*(h))$$

$$= \varphi(df, \pi^*(dh))(p)$$

$$= \varphi(d\pi^*(f'), \pi^*(dh))(p)$$

$$= \pi^*(\varphi(df', dh))(p)$$

$$= \pi^*(Y(h))(p).$$

Hence,

$$\pi_*(X(p)) = Y(\pi(p)).$$

Now, a tangent vector to M' admits a <u>*unique*</u> *horizontal lifting to a tangent vector to M. The tangent vector to* σ *is one such horizontal lifting for the tangent vector to* $\pi(\sigma)$. *X is a horizontal lifting for Y. Hence, by* <u>*uniqueness*</u>, σ *is an orbit curve of X, hence is a geodesic of the metric* φ, *by the Hamilton-Jacobi Theorem.*

<u>*Theorem 8.4*</u>. *Let* σ *be a geodesic of the metric* φ *which is perpendicular to one fiber of* π. *Then, it is perpendicular to each fiber that it meets.*

<u>*Proof*</u>. *Suppose that* σ *is parameterized by* $0 \leq t \leq 1$, *and the* $\sigma(0)$ *is the point at which* σ *is perpendicular to the fiber of* π. $\sigma'(0)$, *the tangent vector to* σ *at* $t = 0$, *is then* <u>*horizontal*</u>. *By Theorem 8.4, there exists a* <u>*horizontal*</u> *geodesic of* φ *which passes through* $\sigma(0)$ *and which has* $\sigma'(0)$ *as tangent vector. Since geodesics are uniquely determined by their tangent vectors at one point, this horizontal geodesic must* <u>*equal*</u> σ, *hence* σ *is* <u>*horizontal*</u>, *which means that* σ *is perpendicular to each fiber that it touches.*

<u>*Theorem 8.5*</u>. *Let* $\pi\colon M \to M'$ *be a Riemannian submersion map*

such that the following condition is satisfied:

$$\text{All the fibers of } \pi \text{ are totally geodesic submanifolds of } M. \tag{8.17}$$

Let $X \in V(M)$ be a vector field satisfying the following conditions:

a) *X is horizontal*

b) *X is projectable under π to a vector field Y on M', i.e.*

$$\pi_*(X) = Y. \tag{8.18}$$

Let $t \to \alpha_t$ be the one parameter group of diffeomorphism of M generated by X. Then, each α_t maps a fiber of π into another fiber, and is an <u>isometry</u> of the metric on the fibers induced from the metric φ on M.

<u>Proof</u>. If Z is a vector field on M, we have:

$$\frac{\partial}{\partial t}\, \alpha_{t*}(Z) = [X\alpha_{t*}(Z)]. \tag{8.19}$$

Let Z be a vertical vector field. Then,

$$\frac{\partial}{\partial t}\, \varphi(\alpha_{t*}(Z),\ \alpha_{t*}(Z)) = 2\varphi([X,\ \alpha_{t*}(Z)],\ \alpha_{t*}(Z)) \tag{8.20}$$

Also,

$\alpha_{t*}(Z)$ *is vertical.*

Now, the right hand side of (8.20) is:

$$2\varphi(\nabla_X \alpha_{t*}(Z) - \nabla_{\alpha_{t*}(Z)} X, \ \alpha_{t*}(Z))$$
$$= X(\varphi(\alpha_{t*}(Z), \ \alpha_{t*}(Z)) \tag{8.21}$$
$$+ \ 2\varphi(\nabla_{\alpha_{t*}(Z)} \alpha_{t*}(Z), \ X).$$

Now, if all the fibers of π *are totally geodesic, then the covariant derivative of one vertical vector field with respect to another vertical vector field is again vertical. In particular, we see that the second term on the right hand side of (8.21) vanishes. Combining (8.20) and (8.21), we have:*

$$\frac{\partial}{\partial t} \alpha_{-t}{}^*(\varphi(\alpha_{t*}Z, \ \alpha_{t*}Z)) = 0,$$

which implies that α_t *is indeed an isometry of the fibers of* π.

Theorems 8.1 - 8.5 are now standard properties of Riemannian submersions. They involve the *<u>first order</u>* *properties of the metrics. In order to better understand the material presented by Ricci and Levi-Civita, let us investigate the* *<u>second order properties</u>*. *The most convenient way to do this is to study the behavior of the*

Laplace-Beltrami operators of the metrics on M and M'.

Let (Y_i) be an orthonormal moving frame of vector fields on M'. Let (X_a) be, similiarly, an orthonormal moving frame on M. Recall that we have divided in the indices (a) into two groups

$$1 \le i,\ j \le n = \dim M'$$

$$n + 1 \le u,\ v \le m = \dim M.$$

Suppose that:

$$\pi_*(X_i) = Y_i,$$

i.e. the $X_1, \ldots, X_n$ are the horizontal lifting of the vector fields $Y_1, \ldots, Y_n$.

Let $f' \in F(M')$. The value of the Laplace-Beltrami operator on it is given by the following formula:

$$\begin{aligned}\Delta'(f') &= \delta^{ij}(\nabla'_{Y_i} df')(Y_j)\\ &= \delta^{ij}(Y_i Y_j(f) = df'(\nabla'_{Y_i} Y_j)).\end{aligned}$$

Now,

$$\begin{aligned}\varphi^*(\Delta'(f')) = \delta^{ij}&(X_i X_j(\pi^*(f))\\ &- d(\pi^*(f')(\nabla_{X_i} X_j)).\end{aligned} \tag{8.22}$$

(To derive the second term on the right hand side of (8.22), use the fact that

$$\pi_*(\nabla_{X_i} X_j) = \nabla'_{Y_i} Y_j)$$

Similiarly, let $\Delta: F(M) \to F(M)$ *be the Laplace-Beltrami operator associated to the metric* φ. *Then,*

$$\Delta(\pi^*(f')) = \delta^{ab}(X_a X_b(\pi^*(f')) - d\pi^*(f')(\nabla_{X_a} X_b)$$

Since the X_u *are vertical, we have*

$$X_u(\pi^*(f')) = 0.$$

Hence, we have:

$$\pi^*(\Delta'(f')) - \Delta(\pi^*(f')) = - \delta^{uv} d\pi^*(f')(\nabla_{X_u} X_v) \qquad (8.23)$$

To interpret this formula geometrically, for a point $p \in M$. $d\pi^*(f')$ *is a 1-covector which annihilates the tangent vectors to the fiber of* π *through* p. *Hence:*

> *The right hand side of (8.23), evaluated at* p, *is the* <u>*trace of the second fundamental form of the fiber of* π *at* p</u> *in the direction of the covector* $d\pi^*(f^*)(p)$. (8.24)

<u>*Remark:*</u> *Here is the general definition of second fundamental form. (See DGCV). Let* M *be a manifold, with a torsion-free affine connection* ∇. *Let*

$$N \subset M$$

be a submanifold of M. For $p \in N$, let θ be a one-covector to M at p such that:

$$\theta(N_p) = 0.$$

The _*second fundamental form*_

$$S_\theta \colon N_p \times N_p \to R$$

is a _*symmetric bilinear form*_*, such that:*

$$S_\theta(Z_1(p),\ Z_2(p)) = \theta(\nabla_{Z_1} Z_2) \qquad (8.25)$$

for every pair (Z_1, Z_2) of vector fields of M which are _*tangent to N*_*.*

With these notations, we can write (8.23) and (8.24) as:

$$\pi^*(\Delta'(f')) - \Delta(\pi^*(f')) = \mathrm{trace}(S_{d\pi^*(f')}), \qquad (8.26)$$

where Δ, Δ' are the Laplace-Beltrami associated with the metrics φ, φ' and $S_{(\ )}(\ ,\ )$ is the second fundamental form of the fibers of the map π. In particular, this formula proves the following result:

Theorem 8.6.

$$\pi^*\Delta' = \Delta\pi^* \tag{8.27}$$

if and only if the fibers of π *are minimal submanifolds of the Riemannian metric* φ, *i.e. the traces of their fundamental forms all vanish.*

Here is a weaker property than minimality.

Definition. *The Riemannian submersion* $\pi: M \to M'$ *is said to be isoparametric if, for each function* f' *on* M', *the function*

$$\mathrm{trace}(S_{d\pi^*(f')}(\ ,\ )) \tag{8.28}$$

is constant on the fibers of π.

Theorem 8.7. *Suppose*

$$\dim M' = 1, \tag{8.29}$$

i.e. the fibers of π *are hypersurfaces of* M. *Then,* π *is isoparametric if and only if the mean curvature of each fiber of* π *is constant.*

Proof. *The mean curvature at a point* $p \in M$ *is the absolute value of*

$$trace(S_\theta(\ ,\ )),$$

where S denotes the second fundamental form of the fiber through p, and θ *is a covector at p perpendicular to the fiber which is of* ***length one***.

Now, if $f' \in F(M')$*, the Riemannian submersion property of* π *implies that the length of*

$$d\pi^*(f')$$

is ***constant*** *over the fibers of* π*. That this implies the statement of Theorem 8.7 should be obvious.*

Remarks: *E. Cartan, in papers No. 166, 167, 168 and 172 in Part 3, vol. 2 of his Collected Works, has discussed this "isoparametric" notion in case condition (8.29) is satisfied and the metric* ***M is of constant curvature***. *Thus, our material is a natural generalization to the case of foliations with lower dimensional leaves. Here is a result which relates our definition to Ricci and Levi-Civita's definition of "geodesic" and "isothermal" congruences:*

Theorem 8.8. *Let* $\pi: M \to M'$ *be a Riemannian submersion mapping, such that:*

$$\dim M' = 1.$$

Let Z be a horizontal vector field of unit length. (In terms used by Ricci and Levi-Civita in Chapter 2, Z defines a normal, geodesic congruence). Then, the congruence Z is isothermal in Ricci and Levi-Civita's sense if and only if π is an isoparametric map.

Proof. Z is isothermal if and only if there is an $f' \in F(M')$ such that:

$$\Delta(\pi^*(f')) = 0.$$

Given (8.26), this implies that:

$$\text{trace}(S_{d\pi^*(f')}) = \pi^*(\Delta'(f')),$$

which shows that trace $S_{d\pi^*(f')}$ is constant on the fibers of π.

Conversely, suppose that π is isoparametric. We can suppose the coordinate x for M' is chosen so that:

$$\Delta' = \frac{d^2}{dx^2} .$$

We must show that we can find a function $f'(x)$ of this variable so that:

$$\Delta\pi^*(f') = 0. \tag{8.30}$$

Using (8.29), (8.30) will be satisfied if and only if:

$$\pi^*(\frac{d^2 f'}{dx^2}) = \pi^*(\frac{df'}{dx}) \ \text{trace}(S_{d\pi^*(x)}). \tag{8.31}$$

Since $\mathrm{trace}(S_{d\pi^*(x)})$ *is* π^* *applied to a function of* x*, to find* f' *is now only a matter of solving an* <u>*ordinary*</u> *differential equation.*

Finally, here is the way we can specialize this material to cover the ideas sketched by Ricci and Levi-Civita in Section 5.

<u>*Theorem 8.9*</u>. *Let M be a Riemannian manifold, and let G be a connected Lie group of isometries of M. Let*

$$\pi: M \to M'$$

be a submersion mapping, such that the fibers of π *are the orbits of G. Then,* π *is a Riemannian, isoparametric submersion.*

<u>*Proof*</u>. *To show that the submersion map* π *is a Riemannian, it suffices to prove that horizontal lifting of tangent vectors to points of* M' *all have the same length. But, this follows from the transitivity of G on the fibers of* π*, and the fact that G preserves length of tangent vectors.*

Let us prove the isoparametric property. Let $f' \in F(M')$*, and let* $p' \in M'$*. Then,*

$$g^*(\pi^*(f')) = \pi^*(f')$$

for all $g \in G$.

Now, g preserves the second fundamental form $S_{(\ )}(\ ,\ )$, since it acts as an isometry. Hence,

$$\mathrm{trace}(S_{d\pi^*(f')})$$

is invariant under the action of G, hence is constant on the fibers of π.

q.e.d.

Chapter V

APPLICATIONS TO MECHANICS

1. INTEGRAL FUNCTIONS OF THE EQUATIONS OF MECHANICS - LINEAR INTEGRALS

Consider a mechanical system, with (holonomic) constraints which are time-independent. Let

$$2T = g_{ij}\dot{x}^i\dot{x}^j \tag{1.1}$$

be the kinetic energy of the system. (As usual, dots above observables denote time-derivatives. i,j are indices which range from 1 to n, the number of degrees of freedom of the system.)

The Lagrange equations, determining the motion of the system under the action of given forces, are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} = X_i, \tag{1.2}$$

which the X_i are related to the forces by well-known relations.

We see easily that, when the coordinates (x^i) are chosen, the X_i transform as covariant 1-tensors. Introduce the reciprocal covariant tensor, using the following metric tensor:

$$ds^2 \equiv 2Tdt^2 = g_{ij}dx^i dx^j \tag{1.3}$$

After solving the Lagrange equations (1.2) with respect to the second time derivatives of the coordinates, we have:

$$\ddot{x}^i = X^i - \left\{ {i \atop jk} \right\} \dot{x}^j \dot{x}^k. \tag{1.4}$$

(The $\left\{ {i \atop jk} \right\}$ are the Christoffel symbols. See formula (5.1) of Chapter I). This is the form of Lagrange's equations which is best suited to our goal.

Suppose that f is a function of the variables x^i and $\dot{x}^i$. In order that f be an integral of motion, it is necessary and sufficient that

$$\frac{df}{dt} = \frac{\partial f}{\partial x^i} \dot{x}^i + \frac{\partial f}{\partial \dot{x}^i} \ddot{x}^i$$

be identically zero when one replaces the $\ddot{x}^i$ by their values, as given by (1.4). This condition is then:

$$\frac{\partial f}{\partial \dot{x}^i} X^i + \frac{\partial f}{\partial x^i} X^i - \frac{\partial f}{\partial \dot{x}^i} \left\{ {i \atop jk} \right\} \dot{x}^j \dot{x}^k = 0. \tag{1.5}$$

It is known (see Levi-Civita [1896]) that to each integral which is algebraic with respect to the $\dot{x}$ there corresponds an integral (which is homogeneous with respect to the $\dot{x}$) for the system obtained by setting the forces X_i equal to zero.

For this reason, the study of integrals of force-free systems is particularly important. Geometrically, it corresponds to the study of homogeneous integrals of equations of geodesics, for the trajectories of motion with zero forces are geodesics of the Riemannian metric

$$ds^2 = g_{ij}dx^i dx^j. \tag{1.6}$$

Let us apply formula (1.5) to determine the conditions that a homogeneous form of degree m

$$f = c_{i_1 \ldots i_m}\dot{x}^{i_1}\ldots\dot{x}^{i_m} \tag{1.7}$$

be an integral of the geodesic equations. Notice that the coefficients of f form an m-th degree symmetric tensor field. Assuming $X_i = 0$, we see that equations (1.5) can be written in terms of covariant derivatives with respect to the metric (1.6), as follows:

$$c_{i_1 \ldots i_m, i_{m+1}}\dot{x}^{i_1}\ldots\dot{x}^{i_m}\dot{x}^{i_{m+1}} = 0. \tag{1.8}$$

The covariant derivative tensor of the tensor $\{c_{i_1 \ldots i_m}\}$ thus appears very naturally. We immediately see the simplications that the covariant derivative operation can make in this type of research!

If it is not assumed that the forces X_i vanish, the conditions that an f of form (1.7) be an integral is (1.8) plus the following equation:

$$c_{i_1 \ldots i_m} X^{i_1} \dot{x}^{i_2} \ldots \dot{x}^{i_m} = 0 \qquad (1.9)$$

From (1.9), one derives (using the symmetry in the indices) the condition:

$$c_{i_1 \ldots i_m} X^{i_1} = 0. \qquad (1.10)$$

To illustrate these general facts, let us discuss the conditions for the existence of integrals

$$f = c_i \dot{x}^i \qquad (1.11)$$

which are linear with respect to the velocity variables $\dot{x}^i$. The identity (1.8) becomes:

$$c_{i,j} \dot{x}^i \dot{x}^j = 0, \text{ or}$$

$$c_{i,j} + c_{j,i} = 0. \qquad (1.12)$$

In the case there is a non-zero force field X_i, one adds to this the condition:

$$c_i X^i = 0. \qquad (1.13)$$

We have already met equations (1.12) in Chapter IV. They express the fact that the one-parameter group generated by the infinitesimal transformation (or vector field)

$$c^i \frac{\partial}{\partial x^i} \qquad (1.14)$$

is a group of isometries of the metric ds^2 given by (1.6). This link between linear integrals of geodesics and groups of isometries is well-known, and hence merits only this brief comment

Remark: The standard terminology is now to say that a vector field of form (1.14), satisfying (1.12), is a Killing vector field. Although Ricci and Levi-Civita did not use this terminology, I will do so. (1.12) expresses the fact that the Lie derivative of the metric ds^2 with respect to the vector field (1.14) is zero.

A linear integral of form (1.11) determines a canonical congruence

$$\lambda_i = \rho c_i \tag{1.15}$$

Conditions (1.13) then have the following simple geometric interpretation:

The canonical congruence of a linear integral is perpendicular to the lines of force.

When the force is derived from a potential function U, i.e.

$$X_i = \frac{\partial U}{\partial x^i},$$

this condition means that:

The canonical congruence of a linear integral is perpendicular to the lines of force.

When the force is derived from a potential function U, i.e.

$$X_i = \frac{\partial U}{\partial x^i},$$

this condition means that:

The canonical congruence is equipotential.

Because of their importance, we defer the study of integrals which are quadratic in the velocities to the next section.

We want to say a few words about invariant hypersurfaces.

Remark: Their term is integrals particulariseés or equations invariants.

We mean by this an equation

$$f(x, \dot{x}) = 0 \tag{1.16}$$

which is such that a solution of (1.14) which satisfies it at one value of t satisfies it for all values of t. (In other words, the submanifold of R^{2n} defined by (1.16) is invariant under the one-parameter group whose orbits are the solutions of (1.14)). The condition for this is that

$$\frac{df}{dt}$$

be a consequence of the equations (1.4) and the equation $f = 0$ itself. We are thus led to the following identity:

$$\frac{\partial f}{\partial x^i} \dot{x}^i + \frac{\partial f}{\partial \dot{x}^i} \ddot{x}^i = Mf, \tag{1.17}$$

where the x^i are replaced by their values in terms of x, $\dot{x}$, as given by equations (1.4), and M is same function.

We shall only consider here the case where:

f is linear in the $\dot{x}^i$.

It is then permissable to suppose that the invariant hypersurface is determined by the following equation:

$$\varphi_i[n]\dot{x}^i = 0, \tag{1.18}$$

where $(\lambda_i[j])$ determine an orthonormal moving frame of the metric ds^2.

The left hand side of (1.17) we have already calculated in our work on integrals of motion. We have then:

$$\lambda_{i,j}[n]\dot{x}_i\dot{x}_j + X^i\lambda_i[n] = M\lambda_i[n]\dot{x}^i. \tag{1.19}$$

We conclude that the multiplier M must be linear in the $\dot{x}^i$, say of the following form

$$M = \nu_i \dot{x}^i. \tag{1.20}$$

(1.19) and (1.20) combine to give the following conditions:

$$X^i \lambda_i[n] = 0 \tag{1.21}$$

$$\lambda_{i,j}[n] + \lambda_{j,i}[n] = \nu_i \lambda_j[n] + \nu_j \lambda_i[n] \tag{1.22}$$

Equation (1.21) tells us that the lines of force are perpendicular to the curves of the n-th congruence. As before, when the forces are conservative, this implies that the congruence [n] is equipotential. In order to discuss equation (1.22), set:

$$\omega_i = \nu_j \lambda^j[i]. \tag{1.23}$$

Condition (1.22) may be written in the following equivalent form:

$$\gamma_{nij} + \gamma_{nji} = \epsilon_{jn}\omega_i + \epsilon_{in}\omega_j. \tag{1.24}$$

For n = 2, there are three equations of form (1.24). The third reduces to

$$\gamma_{211} = 0, \tag{1.25}$$

which means that the congruence [2] is a <u>geodesic congruence</u>. But, we have seen that the lines of this congruence are lines of force, i.e. are integral curves of the contravariant vector field (X^i). Here is the conclusion:

> The equations of motion of a mechanical system with two degrees of freedom admit an invariant hypersurface only if the lines of force are geodesics of the metric ds^2. The notions which lie on this hypersurface have the property that their velocity and forces have the same direction.

It would be interesting to work out the conditions that a system with an arbitrary number of degrees of freedom admit such an invariant.

2. QUADRATIC INTEGRALS OF FORCE-FREE SYSTEMS - INTRINSIC CONDITIONS FOR THEIR EXISTENCE - PARTICULAR CONDITIONS WHICH LEAD TO THE DYNAMIC SYSTEMS OF STÄCKEL

Consider a mechanical system with no force, whose notions are then geodesics of the Riemannian metric

$$ds^2 = g_{ij}dx^i dx^j. \qquad (2.1)$$

Suppose it has a quadratic integral of the following form.

$$H = c_{ij}\dot{x}^i\dot{x}^j. \qquad (2.2)$$

As we have seen in Section 1, the conditions for this are that:

$$c_{ij,k}\dot{x}^i\dot{x}^j\dot{x}^k = 0,$$

i.e. that:

$$c_{ij,k} + c_{jk,i} + c_{ki,j} = 0. \qquad (2.3)$$

In order to study these relations, it is natural to introduce the canonical form of the tensor c_{ij} (Chapter II, Section 5):

$$c_{ij} = \sum_{k=1}^{n} \rho_k \lambda_i[k]\lambda_j[k], \qquad (2.4)$$

where the ρ_k are the roots of the eigenvalue equation

$$\det(c_{ij} - \rho g_{ij}) = 0, \qquad (2.5)$$

and $(\lambda_i[k])$ determine a orthonormal moving frame. Combining (2.3) and (2.4) gives the following relations:

$$0 = (\rho_k - \rho_i)\gamma_{kij} + (\rho_i - \rho_j)\gamma_{ijk}$$
$$+ (\rho_j - \rho_k)\gamma_{jki},$$

$k \neq i \neq j$, no <u>summation</u>:

$$\frac{\partial \rho_k}{\partial s^i} = 2(\rho_k - \rho_i)\gamma_{ikk}, \qquad (2.6)$$

also no summation on the indices.

<u>Remark</u>: Recall that

$$\frac{\partial k}{\partial s^i} = Y_i(\rho_k),$$

where

$$Y_i = \lambda^j[i] \frac{\partial}{\partial x^j}$$

is the i-th vector field of the moving frame.

Equations (2.5) and (2.8) give the intrinsic geometric form to the problem of finding all types of kinetic energy forms whose geodesics admit at least one quadratic integral. To obtain these various types, start off with the metric ds^2, considering (2.5) as equations for the orthonormal moving frame, with the ρ_i as auxillary unknowns. The first assumption is to consider all the ρ_i as equal. Equations (2.5) are then satisfied identically, and equations (2.6) imply that the ρ_i are constant. There exists then, for any metric, at least one quadratic integral, determined by the metric form itself. This is just the kinetic energy itself, which is conserved, because there are no external forces.

Putting aside this obvious case, it would appear plausible to study the system of equations (2.5) and (2.6) by classifying the type of ρ which may appear. For example, first consider the case where all the ρ_i are distinct, then the case where (n-1) are distinct, and so forth. Such a study has not been carried through. It would be of the greatest interest if it were done, but at the moment it seems very arduous.

We do have particular solutions of the system, which correspond to the kinetic energy forms discovered by Stäckel [1897]. These examples generalize the classical examples found by Hamilton and Liouville. (See Di Pirro [1896], Stäckel [1897], Painlevé [1897]). One may find Stäckel's examples from equations (2.5) and (2.6) by making the special assumption that the orthonormal moving frame ($\lambda_i[k]$) is normal. (To be exact, one assumes the normality of all the vector fields in the moving frame when all the ρ_i are distinct. When several coincide, the relevant hypothesis is slightly less restrictive. See Levi-Civita [1897a]).

One might be tempted to conjecture that Stäckel's examples exhaust the solutions of examples (2.5) and (2.6), i.e. that all moving frames which are solutions of (2.5)-(2.6) are normal. This is true for n = 2 (trivially, since all vector fields are normal in dimension 2), but as soon as one passes to a larger number of variables one sees easily that there are new types of solutions. See Levi-Civita [1897b]. The true difficulty lies in the problem of finding all solutions. The first step in this direction should be the solution of the system. (2.5)-

(2.6) in the next case beyond the case where all the ρ_i are equal, namely the case where only two of the ρ_i are distinct.

We point out to the reader this possibly interesting research direction, which we have reduced to a comparatively simple form.

3. SURFACES WHOSE GEODESICS POSSESS A QUADRATIC INTEGRAL (LIOUVILLE SURFACES). CLASSIFICATION OF THESE SURFACES BY THE NUMBER OF THEIR DISTINCT GEODESIC INTEGRAL FUNCTIONS

The basic reference of this section is Ricci [1894], and [1898], Part I, Chapter VI, VII.

For n = 2, equations (2.5) are satisfied automatically, and the only relevant conditions are equations (2.6), which become:

$$\begin{aligned} &\frac{\partial \rho_1}{\partial s^1} = \frac{\partial \rho_2}{\partial s^2} = 0 \\ &\frac{\partial \rho_1}{\partial s^2} = 2(\rho_1 - \rho_2)\gamma_{211} \\ &\frac{\partial \rho_2}{\partial s^1} = 2(\rho_2 - \rho_1)\gamma_{122}. \end{aligned} \tag{3.1}$$

Assume that:

$$\rho_1 \neq \rho_2.$$

Then, equations (3.1) imply the following integrability conditions:

$$\frac{\partial \gamma_{211}}{\partial s^1} = \frac{\partial \gamma_{122}}{\partial s^2} = -3\gamma_{211}\gamma_{122}. \tag{3.2}$$

Each orthonormal moving frame [1], [2] which satisfies (3.2) provides a function H (different from the kinetic energy) which is quadratic in the velocity variables and which is conserved under the geodesic flow of the metric. Denote by θ the angle between the curves of the congruence [2] and the curves of an arbitrary geodesic congruence. One sees readily that the conservation condition on H are equivalent to the following geometric property:

$$\rho_1 \sin^2\theta + \rho_2 \cos^2\theta = \text{constant} \tag{3.3}$$

along each geodesic

Conditions (3.2) imply that the moving frame [1], [2] belongs to an <u>isothermal bundle</u>. (See Chapter IV, Section 1.)

From (3.1) it follows without difficulty that, one may choose coordinates (u, v) of the surface such that:

$$ds^2 = (\rho_2 - \rho_1)(du^2 + dv^2). \tag{3.4}$$

Vector field [1] is proportional to $\frac{\partial}{\partial u}$ (3.5)

Vector field [2] is proportional to $\frac{\partial}{\partial v}$ (3.6)

Combining these conditions with the first set of equations of (3.1), we see that:

$$\begin{aligned} &\rho_1 \text{ is a function } \rho_1(u) \text{ of } u \text{ only} \\ &\rho_2 \text{ is a function } \rho_2(v) \text{ of } v \text{ only.} \end{aligned} \tag{3.7}$$

Here is the classical

<u>Definition</u>. A two dimensional Riemannian manifold is said to be a <u>Liouville manifold</u> (or <u>Liouvillé surface</u>) if it admits a coordinate system (u, v) satisfying (3.4) and (3.7).

We have then proved the following result:

<u>In order that the geodesics of a two dimensional Riemannian manifold admit a conserved quadratic function it is necessary and sufficient that the manifold be Liouville</u>.

Now, the following problem is suggested: How to recognize if a metric, given in advance, is of Liouville form, and in how many different ways may it be written in Liouville form? This problem is naturally equivalent to that of determining the number of quadratic conserved functions of the geodesic equations.

Here are the results found by Ricci [1894]. Let us say that an orthonormal frame [1], [2] which satisfies

(3.2) is an isothermal Liouville system.

1) Only manifolds of constant curvature possess isothermal Liouville systems which depend on four parameters.

2) Manifolds of non-constant curvature admit at most two parameter families of Liouville systems. There is one class of manifolds which admits precisely this number. They are the surfaces in R^3 which are isometric to surfaces of revolution and also have parallel lines of curvature.

3) There exist manifolds which have 1-parameter families of Liouville systems, and others which have a unique liouville system.

Koenigs, in a Prize Memoir of the Academie des Sciences de Paris [1894], has dealt with a problem which is closely related to the one just described, but not exactly identical. He proposes to find all metrics which admit at least two Liouville systems. With this viewpoint, he can establish some of the results described above. Koenigs announced them at the same time as Ricci.

4. PROJECTIVE TRANSFORMATIONS OF THE EQUATIONS OF DYNAMICS

Suppose given two dynamical systems, (A) and (A_1), whose forces are velocity independent. We say that (A) and (A_1) are <u>projectively equivalent</u> if the solution curves of A and A_1 in configuration space are the same <u>up to a change in parameterization</u>.

<u>Remark</u>: The terminology used by Ricci and Levi-Civita is that the systems <u>correspond</u>.

Painlevé [1894] has proposed the following problem:

> <u>Given a dynamical system (A), determine the conditions that it admit projectively equivalent systems, and find all of them</u>.

One can prove that, if the external forces are zero for (A), they are also for (A_1). Then, the geometric side of the problem of projective equivalence takes the following form:

> <u>Determine all the Riemannian metrics which are projectively equivalent to a given one</u>.

This problem has been studied by R. Liouville [1895]. He has proved general and remarkable results, without giving a definitive answer. (Perhaps it was not possible without the help of Tensor Analysis). We now give an idea of the methods of attack on this problem, following work of Levi-Civita [1896].

Let

$$g_{ij}dx^i dx^j$$

$$h_{ij}dx^i dx^j$$

be two metrics which are projectively equivalent, i.e. which have the same geodesic curves up to a change in parameterization. Consider the first metric as given (and determining covariant differentiation). Here are the equations which the second metric must satisfy:

$$2\mu h_{ij,k} + 2\mu_k h_{ij} + \mu_j h_{ik} + \mu_i h_{jk} = 0, \tag{4.1}$$

where μ is same function of x. Denote by g and h the determinant of the matrices (g_{ij}), (h_{ij}). Set:

$$A_{ij} = \mu^2 h_{ij}. \tag{4.2}$$

From the preceding equations one readily derives the following relation:

$$\mu = C(\tfrac{g}{h})^{\frac{1}{n+1}}, \tag{4.3}$$

with C a constant; and

$$A_{ij,k} + A_{jk,i} + A_{ki,j} = 0. \tag{4.4}$$

This shows (see Section 2) that

$$A_{ij}\dot{x}_i\dot{x}_j \tag{4.5}$$

is conserved under the geodesic flow of the metric $g_{ij}dx^i dx^j$.

To investigate these relations further, consider the canonical form for the tensor h_{ij}:

$$h_{ij} = \sum_k \rho_k \lambda_i[k]\lambda_j[k].$$

Equations (4.1) imply the following relations:

$$(\rho_k - \rho_i)\gamma_{kij} = 0, \quad k \neq i \neq j. \tag{4.6}$$

$$2(\rho_i - \rho_j)\gamma_{iji} = \frac{\partial \rho_i}{\partial s^j} \tag{4.7}$$

(i ≠ j; no summation)

$$\frac{\partial(\mu\rho_i)}{\partial s^j} = 0 \quad (i \neq j) \tag{4.8}$$

$$\frac{\partial(\mu\rho_i)}{\partial s^i} + \rho_i \frac{\partial \mu}{\partial s^i} = 0, \tag{4.9}$$

(no summation.)

The form of the system of equations (4.6)-(4.9) indicate to us that the number and nature of the independent conditions involved depends on the number of distinct roots and multiplicities of the eigenvalue equation

$$\det(h_{ij} - \rho g_{ij}) = 0.$$

Suppose first that all the ρ are distinct. The orthonormal reference frame is in this case completely determined, and, because of equations (4.6), the rotational coefficients γ_{ijk} with i,j,k distinct must vanish.

It follows (Chapter II, Section 3) that all the congruences of the moving frame are normal, and one is naturally led to take the corresponding orthogonal hypersurfaces as coordinates. With the choice of such a coordinate system, the metric takes the following form:

$$ds^2 = H_i (dx^i)^2. \tag{4.10}$$

With this form for the metric, equations (4.6) are automatically satisfied, and conditions (4.7)-(4.9) take the following form:

$$2(\rho_i - \rho_j) \frac{\partial \log H_i}{\partial x^j} + \frac{\partial \rho_i}{\partial x^j} = 0 \tag{4.11}$$

$$(i \neq j;\ \text{no summation})$$

$$\frac{\partial(\mu\rho_i)}{\partial x^j} = 0,\ i \neq j. \tag{4.12}$$

$$\frac{\partial(\mu\rho_i)}{\partial x^i} + \rho_i \frac{\partial \mu}{\partial x^i} = 0;\ \text{no summation} \tag{4.13}$$

The solution of this system is easy. We are led to the following result:

Theorem. For each index i between 1 and n, let ψ_i *be an arbitrary function of* x_i *and let c, C be arbitrary constants, Set:*

$$ds^2 = \sum_{i=1}^{n} \Big(\prod_{\substack{j=1 \\ j \neq i}}^{n} (\psi_j - \psi_i)\Big)(dx^j)^2 \tag{4.14}$$

$$ds_1{}^2 = \frac{c}{(\psi_1+c)(\psi_2+c)\ldots(\psi_n+c)} \sum_{i=1}^{n} \frac{1}{(\psi_i+c)} \Big(\prod_{\substack{j=1 \\ j \neq i}}^{n} (\psi_j - \psi_i)\Big)(dx^i)^2. \tag{4.15}$$

Then, the metric $ds_1{}^2$ *is projectively equivalent to* ds^2, *and every metric which is projectively equivalent to* ds^2 *is of this form.*

Let us pass to the other extreme case, where all the ρ are equal. We see that h_{ij} is then a constant multiple of g_{ij}, which is the trivial sort of projective equivalence.

The intermediate cases, where some of the ρ are equal, some unequal, combined with equations (4.6)-(4.9), lead to well-defined systems of equations which determine the projectively equivalent metrics. As we have seen for the case covered by the above Theorem, the geometric interpretation suggests the choice of variables in which the system is easiest to solve.

Returning for a moment to the case where all the ρ_i are unequal, let us remark that the corresponding conserved function for the geodesic flow of the metric (4.14) takes the following form:

$$f = \sum_{i=1}^{n} (\psi_1 + c)\ldots(\psi_{i-1} + c)(\psi_{i+1} + c)$$
$$\ldots(\psi_n + c)\left(\prod_{\substack{j=1 \\ j\neq i}}^{n} (\psi_i - \psi_2)\right)(\dot{x}^i)^2 \qquad (4.16)$$

As the right hand side of (4.16) is conser ed <u>for each value of c</u>, the coefficients of the expansion in powers of c are also conserved, and are quadratic in the velocity variables. They give n distinct conserved functions. (In general, there will be as many distinct conserned quadratic functions obtained in this way as there are distinct eigenvalues among the $\rho_1, \ldots, \rho_n$.

It would be important to characterize invariantly the metrics which can be reduced to the form (4.14). (They are called <u>generalized Liouville metrics</u>.) We have already done this in Section 3 for the case n = 2; the case n = 3 is the next situation to study.

More generally, it should be kept in mind that the problem of projective equivalence for non-zero forces is still unsolved. Painlevé [1896] has made very interesting contributions, and has solved the problem for n = 2.

See also the work of Viterbi [1900]. Will Tensor Analysis enable us to push this problem to a conclusion? At the moment, we can only hope so.

<u>Remark</u>: *In case the forces admit a potential (which of course includes the case n = 2) the solutions of the Lagrange equations are, after reparameterization, geodesics of a metric. (This is the "Principle of Maupertuis." See DGCV, Chapter 16). Hence, the essence of this last problem lies in the case where the forces do not admit a potential.*

In addition, the reader will notice that a vast area of research is still open. For example, Stäckel has extended [1898] the projective equivalence problem by requiring that two dynamical systems (A) and (A_1) have $k(< 2n-1)$ parameter families of orbits which differ only by a change in parameterization. In an article which will appear soon, Malipiero considers the geodesic case from this point of view, and presents some remarks which are not without interest.

<u>Remarks</u>: *There has been of course a considerable amount of work on these problems since 1900, successfully using (as Ricci and Levi-Civita had suggested) the methods of*

Tensor Analysis. I am not very familier with this work - the treatises by Schouten [1954] and Vranceanu [1957] would be a good place to start. Of course, highlights of the post - 1900 work were the introduction of the projective curvature tensor *by Weyl, and the development of the theories of* projective connections *by Weyl and Cartan.*

I would imagine that there is interesting work to be done in non-classical directions; for example, the study of global *properties of projective equivalence and the relations with quantum mechanics. For example, equations of a force-free rigid rotation (or more generally a dynamical system on a Lie group invariant under left translation) admit quadratic conserved quantities. I suspect that they are related to "projective" symmetries. What is the role of these projective symmetries in quantum mechanics?*

Chapter VI

PHYSICAL APPLICATIONS

1. REDUCIBILITY TO TWO VARIABLES OF THE HARMONIC EQUATION (BINARY POTENTIALS)

Consider Laplace's equation in Cartesian coordinates:

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{1.1}$$

If we suppose that the function u is independent of z, it must satisfy:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.2}$$

This equation defines an extended class of potentials, which are constant along the lines in R^3 which are parallel to the z-axis. (C. Neumann calls them logarithmic potentials).

Similiarly, consider Laplace's equation in spherical coordinates (r, θ, φ). Assume that u is a solution which does not depend on φ. It must satisfy the following equation:

$$\frac{1}{r^2 \sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2 \sin^2\theta \frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right)\right\} = 0 \tag{1.3}$$

There is no trace of φ remaining in the coefficients. The solutions of (1.3) form a very important class of harmonic functions, the symmetric potentials, which are well-known after the work of Beltrami [1881]. They are constant on the circles r = constant, θ = constant.

Similiarly, we may look for functions which are independent of r. The corresponding potentials are the solutions of

$$\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin\theta}\,\frac{\partial^2 u}{\partial\varphi^2} = 0. \tag{1.4}$$

Their equipotential curves are the lines in R^3 through the origin.

It is not possible to treat θ on the same footing. For, if u is a solution of (1.1) which is independent of θ, we have two equations

$$\frac{\partial}{\partial r}\left(r^2\,\frac{\partial u}{\partial r}\right) = 0$$

$$\frac{\partial^2 u}{\partial\varphi^2} = 0,$$

which have the solutions:

$$u = \left(c_1 + \frac{c_2}{r}\right) + \left(c_3 + \frac{c_4}{r}\right),$$

where the c's are arbitrary constants. This solution does not have the same degree of generality as the previous cases.

<u>Remark</u>: Physically, the trouble with this family of solutions is that there are not enough to provide solutions of boundary value problems.

These remarks led Volterra [1883] to pose the following problem:

> Consider the harmonic equation $\Delta u = 0$, transformed in terms of an arbitrary coordinate system (x^1, x^2, x^3) for R^3. In general, when one sets $\frac{\partial u}{\partial x^3} = 0$, one cannot eliminate x_3 from Δ (i.e., the two equations $\Delta(u) = 0$, $\frac{\partial u}{\partial x^3} = 0$ do not form a completely integrable system). There are some cases - we have already encountered some simple examples - where x^3 can be eliminated. Here is Volterra's problem: <u>Determine all such cases</u>.

To each such coordinate system, there is a class of potential functions depending on two variables, called <u>binary potentials</u>. In applications they may be used in the same way as logorithmic or symmetric potentials. Volterra has studied [1883] then from a general point of view.

A problem which remains is to find if there are types in addition to the ones which are known, and to determine them.

Riemann has solved this problem (Collected Works, p. 370) for the heat equation

$$\frac{\partial u}{\partial t} + k\Delta u = 0,$$

but his method is extremely cumbersome. It is necessary to clear away the complicating material; Tensor Analysis can do this. (See Levi-Civita [1899]). We shall now cover the highlights of this research.

Notice that one type of bilinear potentials is characterized by its associated <u>equipotential congruence,</u> i.e. the congruence whose curves satisfy:

$$x^1 = \text{constant},\ x^2 = \text{constant},$$

formed by the curves along which all the elements of the class are constant. In fact, when this congruence is given, it suffices to choose coordinates arbitrarily (x^1, x^2, x^3) such that the curves of the congruence are:

$$x^1 = \text{constant},\ x^2 = \text{constant}.$$

The equation which defines the corresponding binary potentials is obtained by writing the harmonic equation

$$\Delta u = 0$$

in these coordinates, and then setting $\frac{\partial u}{\partial x^3} = 0$. (The hypothesis that the congruence is equipotential is precisely equivalent to the fact that one can set $\frac{\partial u}{\partial x^3} = 0$ in $\Delta u = 0$ without being bothered by x^3).

The problem then comes down to finding all the equipotential congruences of space. These congruences (assumed to be real) fall into the four following types:

1) Isotropic straight-line congruences, called Ribaucour congruences. (See Bianchi, "Lezioni di geometra differenziale," Chap. X, Levi-Civita [1899]).
2) Congruence of circles with the same axis.
3) Congruences of helices.
4) Congruences of spirals.

We derive from this a corresponding classification of binary potentials. They are isotropic, symmetric, helicoidical, and spiral.

Remarks: *This material is still of great current interest. Here is one general way of formulating the problem, using the theory of linear differential operators. (See GPS, Chapter I).*

Let M be a manifold, $F(M)$ the C^∞ real-valued functions on M. Let

$$\Delta : F(M) \to F(M)$$

be a differential operator. (The case where Δ is the Laplace operator is typical and important, but is by no means the only interesting example!)

Let M' be another manifold,

$$\Delta': F(M') \to F(M)$$

a differential operator on M'.

Definitions. *A map*

$$\varphi: M \to M'$$

is said to intertwine Δ and Δ' if:

$$\Delta(\varphi^*(f')) = \varphi^*(\Delta'(f)) \quad \text{for all } f' \in F(M'). \tag{1.5}$$

φ is said to be a homomorphism from Δ to Δ' if the following condition is satisfied:

$$\text{If } \Delta'(f') = 0, \text{ then } \Delta(\varphi^*(f)) = 0. \tag{1.6}$$

Of course, condition (1.5) implies (1.6). The concepts discussed by Ricci and Levi-Civita are really "localizations" of this one.

Here is another formulation in terms of foliations.

Let M be a manifold,

$$\Delta: F(M) \to F(M)$$

a differential operator. Let $V \subset V(M)$ be a vector field system on M, i.e. an $F(M)$-submodule of $V(M)$. Suppose it is <u>*integrable,*</u> *i.e. that it defines a* <u>*foliation*</u> *on M, i.e.*

$$[V,\ V] \subset V.$$

An <u>*integral*</u> *of V is a real-valued function f defined on an open subset of M such that*

$$X(f) = 0 \text{ for all } X \in V.$$

Let $I(O,\ V)$ denote the set of integral functions defined in each open subset O of M.

<u>*Definition*</u>. *The foliation V is said to* <u>*reduce*</u> *the differential operator Δ if:*

$$\Delta(I(O,\ V)) \subset I(O,\ V). \tag{1.7}$$

Ricci and Levi-Civita consider the case where

$M = R^3$,

Δ = *Laplace operator.*

V = *1-dimensional foliation.*

As in so much of the rest of this paper, I would say that this material should be reworked from a modern point of view (perhaps with a global outlook), with special attention *to new problems of this sort which arise in the theory of Lie group representation theory, differential equations, and General Relativity.*

Here is another example, of an important geometric situation which is covered by (1.5). Suppose

$$\beta\colon V(M) \times V(M) \to F(M) \tag{1.8}$$

defines a Riemannian metric on M. Δ can be chosen as the first Beltrami operator, considered earlier:

$$\Delta(f) = \beta(df, df). \tag{1.9}$$

(Of course, β must be "dualized" first, to define an inner product on differential forms). Suppose also that M′ is another manifold, β′ a Riemannian metric on M′, Δ′ its first Beltrami operator. Condition (1.5) then means that Δ is a homomorphism *between the Riemannian metrics β and β′, as defined in Volume V. (In the terminology used earlier by Reinhart and O'Neill, β is a* bundle-like metric *with respect to φ, and φ is a* Riemannian submersion *between M and M′). See Chapter IV for more detail.*

2. VECTOR FIELDS

For general ideas on vector fields, from our point of view, one might consult with profit (in addition to the well-known treatise by Tait) the posthumous memoir by Ferraris [1897], and the recent memoir by Donati [1898].

By a vector field we mean a correspondence which assigns to each point of a domain of space (i.e. R^3) a vector V whose origin is at P.

Let y^1, y^2, y^3 be the Cartesian coordinates of P; Y^1, Y^2, Y^3 the coordinates of V with respect to the coordinate axes. The law of coorespondence between points and vectors means that the components Y^1, Y^2, Y^3 of V are functions of y^1, y^2, y^3. We suppose, of course, that these functions are continuous and possess as many derivatives as is necessary.

Given such a vector field, the scalar quantity

$$\frac{\partial Y^1}{\partial y^1} + \frac{\partial Y^2}{\partial y^2} + \frac{\partial Y^3}{\partial y^3}$$

is naturally associated. One calls it the divergence of V at the point P:

$$\text{div } V = \frac{\partial Y^1}{\partial y^1} + \frac{\partial Y^2}{\partial y^2} + \frac{\partial Y^3}{\partial y^2} \tag{2.1}$$

It appears quite often in physical theories. For example, if the vector V represents the displacement of P in an elastic deformation, div(V) is a measure of the stretching of the particles in a neighborhood of the point P. More generally, when V is a flux of any physical nature, div(V) measures its condensation.

If the components (Y_i) are derivatives of a function U(in which case we say that the flow represented by V is of potential type), then

$$\operatorname{div} V = \Delta(U).$$

There is another vector closely linked to the field, the curl 2ω. Its components are given by the following formulas:

$$\frac{\partial Y^3}{\partial y^2} - \frac{\partial Y^2}{\partial y^3}$$

$$\frac{\partial Y^1}{\partial y^3} - \frac{\partial Y^3}{\partial y^1}$$

$$\frac{\partial Y^2}{\partial y^1} - \frac{\partial Y^1}{\partial y^2}\ .$$

To see its physical interpretation, think, for example, of hydrodynamics. If V is the velocity of the fluid, then the "rotation" of the fluid is defined by the vector ω. It is identically zero for the velocity fields of potential type.

Now, suppose that the physical space, R^3, is described by arbitrary coordinates (x^1, x^2, x^3). One naturally encounters the problem representing the vector field and the above operations in these coordinates.

To do this, associate with the vector field V a 1-covariant tensor (X_i), $1 \le i, j \le 3$, whose components reduce in Cartesian coordinates (y^i) to the components (Y^i) of V. The principles of Tensor Analysis immediately enable us to write down the above operations. (The same method enables one to immediately write down the standard integral formulas, such as those of Green and Stokes, in arbitrary coordinate systems. See Ricci [1897]).

Let (g_{ij}) be the metric tensor of the Euclidean metric on R^3, i.e.

$$ds^2 = \delta_{ij} dy^i dy^j$$
$$= g_{ij} dx^i dx^j .$$

Then,

$$\text{div } V = g^{ij} X_{i,j} \tag{2.1}$$

$$(\text{curl } V)^i = \epsilon^{ijk} X_{j,k} . \tag{2.2}$$

(For the definition of the ϵ tensor, see Chapter I, Section 5). To prove these formulas, notice that the right hand sides are invariants under change of coordinates, which reduce, in Cartesian coordinates, y^1, y^2, y^3, to the values they should to make (2.1)-(2.2) identities. For vector fields of potential type, i.e.

$$X_i = U_i = \frac{\partial U}{\partial x^i},$$

it follows that

$$\text{div } V = g^{ij} U_{i,j},$$

which is the general form of the Laplace operator ΔU, as expected.

We have already remarked (Chapter I, Section 5) that one may write (2.1) in the following form:

$$\text{div } V = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{a}\ X^i) \tag{2.3}$$

$$(\text{curl } V)^i = \frac{1}{\sqrt{g}} \left(\frac{\partial X_{i+}}{\partial x^{i+1}} - \frac{\partial X_{i+1}}{\partial x^{i+2}}\right.$$

(Again, in (2.4) regard two indices which differ modulo three as the same). These formulas are useful for calculations.

In elasticity and, especially, in electrodynamics, one encounters the vector field

$$\Omega = - \text{curl } (\text{curl } V). \tag{2.5}$$

Here are the formulas for this operation in terms of Tensor Analysis: Start with Cartesian coordinates (y^i):

$$\Omega_i = \frac{\partial}{\partial y^{i+2}} \left(\frac{\partial Y_i}{\partial y^{i+2}} - \frac{\partial Y_{i+2}}{\partial y^{i}}\right) - \frac{\partial}{\partial y^{i+1}} \left(\frac{\partial Y_{i+1}}{\partial y^{i}} - \frac{\partial Y_i}{\partial y^{i+1}}\right)$$

This may be written as:

$$\Omega_i = \delta^{jk} \frac{\partial^2 Y_i}{\partial y^j \partial y^k} - \frac{\partial}{\partial y^i} \frac{\partial Y^j}{\partial y^j}$$

$$- \delta^{jk} \frac{\partial^2 Y_i}{\partial y^j \partial y^k} - \frac{\partial}{\partial y^i} (\text{div } V). \qquad (2.6)$$

We can now write down the formula in a arbitrary coordinate system.

$$\Omega_i = g^{jk} X_{i,jk} - \frac{\partial}{\partial x^i} (\text{div } V). \qquad (2.7)$$

To prove formula (2.6), note that it is an invariant formula, which takes the form (2.6) in the Cartesian coordinate system.

3. DIVERSE EXAMPLES - EQUATIONS IN GENERAL COORDINATES OF ELECTRODYNAMICS, THEORY OF HEAT, AND ELASTICITY

Electrodynamics. The electromagnetic field is defined by two vector fields

$$E, B,$$

called the electric and magnetic field vectors. This depends on time. For fixed time, they are vector fields on R^3, as considered in Section 2. We denote by

$$\frac{\partial E}{\partial t}, \frac{\partial B}{\partial t},$$

the vector fields obtained by taking partial derivatives with respect to the time, t.

<u>Remark</u>: *I follow the modern notations, now used in physics books. (See Jackson [1961] or Volume IV, Chapter IV) Ricci and Levi-Civita use F_e, F_m, (which actually is not a bad notation!) To be precise, in terms of arbitrary coordinates (x^i) for R^3, $1 \le i, j \le 3$,*

$$E = E^i \frac{\partial}{\partial x^i},$$

where E^i are functions of the form $E^i(x, t)$, then

$$\frac{\partial E}{\partial t} = \left(\frac{\partial E^i}{\partial t}\right) \frac{\partial}{\partial x^i}.$$

With these notations, the equations for a homogeneous, isotropic dialectric are, in Hertz' form:

$$A\mu \frac{\partial B}{\partial t} = \text{curl } E \tag{3.1}$$

$$A\epsilon \frac{\partial E}{\partial t} = \text{curl } B \tag{3.2}$$

A, μ, ϵ are constants.

We may translate these equations into an explicit form, in terms of an arbitrary coordinate system (x^i) for R^3, by using the formulas of the preceding section. Let

(E_i), (B_i) denote the covariant 1-tensors corresponding to the vector fields E and B. Equations (3.1) and (3.2) become:

$$A\mu \frac{\partial B_i}{\partial t} = \frac{1}{\sqrt{g}} \left\{ \frac{\partial E_{i+2}}{\partial x^{i+1}} - \frac{\partial E_{i+1}}{\partial x^{i+2}} \right\} \tag{3.3}$$

$$A\epsilon \frac{\partial E_i}{\partial t} = \frac{1}{\sqrt{g}} \left\{ \frac{\partial B_{i+1}}{\partial x^{i+2}} - \frac{\partial B_{i+2}}{\partial x^{i+1}} \right\}. \tag{3.4}$$

It may be useful (for the study of waves, for example) to separate the differential equations satisfied by E and B. This may be done by eliminating first E_1 then B_1 from equations (3.1)-(3.2). One finds in this way:

$$A^2\mu\epsilon \frac{\partial^2 B}{\partial t^2} = A\epsilon \frac{\partial}{\partial t} (\text{curl } E)$$

$$= A\epsilon \text{ curl } \left(\frac{\partial E}{\partial t}\right)$$

$$= - A\epsilon \text{ curl } (\text{curl } B).$$

Similiarly,

$$A^2\mu\epsilon \frac{\partial^2 E}{\partial t^2} = - \text{curl } (\text{curl } E).$$

Formula (2.7) leads to the following relations:

$$A^2\mu\epsilon \frac{\partial^2 B_i}{\partial t^2} = g^{jk} B_{i,jk} - \frac{\partial}{\partial x^i} (\text{div } B) \tag{3.5}$$

$$A^2\mu\epsilon \frac{\partial^2 E_i}{\partial t^2} = g^{jk} E_{i,jk} - \frac{\partial}{\partial x^i} (\text{div } E). \tag{3.6}$$

For empty space, one has, in particular,

$$\mu = \epsilon = 1 \tag{3.7}$$

$$\operatorname{div}(B) = \operatorname{div}(E) = 0. \tag{3.8}$$

One may now translate the limit conditions into generalized coordinates, then, introducing polarisation and current, consider the case of dielectrics and conductors.

It would be interesting to present some applications of these general formulas, but this would lead us too far afield. We have simply given enough to guide the reader.

Remark: After the development of Special and General Relativity it became clear that the equations of electrodynamics were even more elegantly and usefully describable in the context of four dimensional tensor analysis. See Jackson [1962], Landau-Lifschitz [1959]. Then, Cartan showed that the ultimate framework was in terms of differential forms. See Volume IV.

Heat. The movement of heat in a conducting body is determined (when one neglects phenomena of absorption and mechanical work) by the following equation:

$$C\rho \frac{\partial T}{\partial t} = \operatorname{div}(F). \tag{3.9}$$

C and ρ represent specific heat and density; T is temperature, (a scalar), while F, a vector field, is the heat flux. All are time-dependent objects defined over space.

The vector field F is defined in isotropic bodies by the fact that its component in any direction is proportional to the derivative of temperature in that direction, i.e.

$$F = c \text{ grad } T \tag{3.10}$$

c may depend on the coordinates. If (F_i) denotes the covariant coordinates of F in an arbitrary coordinate system (x^i), these equations can be written as follows:

$$C\rho \frac{\partial T}{\partial t} = \frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{\partial}{\partial x^i} \left(c \sqrt{g} \frac{\partial T}{\partial x^i}\right). \tag{3.11}$$

When C is a constant, we have the well-known result:

$$C\rho \frac{\partial T}{\partial t} = c\Delta T. \tag{3.12}$$

Remark: Equation (3.12) is now called the heat or diffusion equation.

Notice that (3.10) is unnatural from the point of view of tensor analysis, since relation (3.9) seems to indicate that F is a contravariant vector field. Physically, the origin of this identification is our assumption that the conductor was isotropic. For an arbitrary conductor, the following general relation should be assumed:

$$F^i = c^{ij} \frac{\partial T}{\partial x^j}. \tag{3.13}$$

The coefficients c^{ij} (which are symmetric, and are called conductivity coefficients) are functions of the variables. They define a 2-covariant tensor.

Remark: In modern continuum mechanics, relations (3.13) are typical constitutive relations.

We can now write down the general version of (3.11) that is correct from the viewpoint of tensor-analysis:

$$C\rho \frac{\partial T}{\partial t} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(c^{ij} \frac{\partial T}{\partial x^j}\right). \tag{3.14}$$

If the heat conductor is homogeneous, the coefficient c^{ij} are constant in Cartesian coordinates. Of course, if they are regarded as forming the component of a tensor, they are not constant in arbitrary coordinate systems. In particular, one cannot take c^{ij} outside the differentiation sign in (3.14). Another form of (3.14) is:

$$c\rho \frac{\partial T}{\partial t} = c^{ij} T_{i,j}. \tag{3.15}$$

To see that both (3.14) and (3.15) are legitimate verions of the heat equation in general coordinates, note that both have an invariant nature, and reduce to the standard equation in Cartesian coordinates.

<u>Remark</u>: *This is a typical application of Tensor Analysis in physics! Finding the "correct" or "invariant" form (from the viewpoint of Tensor Analysis) of system of Equation is often a great help in investigating new physical phenomena. For example, quantum field theory carries over this viewpoint to describe elementary particles in terms of quantum fields. One must regard such equations as mathematically inspired guesses, whose consequences are compared with experiment to check the correctness of the mathematical guesses. Unfortunately, in quantum field theory it is, so far, too hard mathematically to derive enough consequences to check with experiment!*

<u>Elasticity</u>. Let (y^i) be Cartesian coordinates. Let u_i be the coordinates (in the y-coordinates) of points in the elastic body.

$$2\alpha_{ij} = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} \tag{3.16}$$

(α_{ij}) measures the <u>strain</u> of the body.

The potential of the elastic forces acting on the body is a function 2Π of the α_{ij}, which is typically quadratic and homogeneous. Set:

$$2\Pi = c^{ijk\ell}\alpha_{ij}\alpha_{k\ell}. \tag{3.17}$$

The coefficients $c^{ijk\ell}$ are called elasticity coefficients, and may be functions of the y.

Denote by F the vector field representing the force which acts on a unit amount of mass. ρ is the density of matter in the body. Set:

$$\Pi^{ij} = \frac{\partial\Pi}{\partial\alpha_{ij}} = c^{ijk\ell}\alpha_{k\ell}. \qquad (3.18)$$

The equations of elastic equilibrium are then:

$$\frac{\partial\Pi^{ij}}{\partial y^j} = \rho F^i. \qquad (3.19)$$

It is now easy to write these equations in a form which is valid in an arbitrary coordinate system

(x^1, x^2, x^3) for R^3. Regard:

(u_i) as a 1-covariant tensor

$(c^{ijk\ell})$ as a 4-contravariant tensor.

F as a 1-contravariant tensor.

Set:

$$2\alpha_{ij} = u_{i,j} + u_{j,i} \qquad (3.20)$$

$$\Pi^{ij} = c^{ijk\ell}\alpha_{k\ell} \qquad (3.21)$$

$$\Pi_{,}{}^{ij} = \rho F^i. \qquad (3.22)$$

Again, the equations (3.20)-(3.22) are independent of choice of coordinates - because they are written in terms of Tensor Analysis - and reduce to the usual relations (3.16)-(3.19) when the coordinates for R^3 are orthogonal Cartesian.

This is not the place to go further, but we might mention that the theory of elasticity is one of the areas where Tensor Analysis may be called on to serve as a mathematical language and framework.

Remark: This last prophecy - as for so many others in the book - indeed became true - see any current treatise on elasticity and continuum mechanics. (Unfortunately, they are all in the engineering or applied mathematics literature - a contemporary version of Ricci and Levi-Civita's work is very badly needed in this field!)

It is also historically appropriate that the article ends with equation (3.22), showing that the "divergence" operation on symmetric 2-tensors is the appropriate invariant form of one of the classical partial differential equations of mathematical physics. This fact was a key idea - even a dramatic clue, in the detective story sense - in Einstein's discovery of General Relativity. See his book, "The Meaning of Relativity" [1].

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