Conservation Laws and Gravitational Waves in General Relativity (1915–1918)

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1. Introduction

This chapter deals with two closely related debates in general relativity in 1916–1918, one on gravitational waves, the other on the correct formulation of conservation laws. Both issues involve the definition of a quantity representing the stress-energy of the gravitational field. Such definitions were typically proposed in the context of deriving the gravitational field equations from a variational principle. A proper understanding of the debates on gravitational waves and conservation laws therefore requires some discussion of the rather complicated history of attempts to derive gravitational field equations from a variational principle.[1]

We will trace Einstein's work on gravitational waves and his work on conservation laws during the years 1916–1918 in this more complex network. We will look at objections to Einstein's approach from Levi-Civita, Schrödinger, and Bauer; at alternative approaches suggested by Lorentz and Levi-Civita; and at Einstein's response to all of them. In particular, we will examine the 1917 correspondence between Einstein and Levi-Civita. We will see how Levi-Civita's criticism of Einstein's formulation of conservation laws strengthened Einstein in his conviction that physical considerations force one to adopt a noncovariant formulation of conservation laws for matter plus gravitational field.

2. The Importance of the Conservation Laws in Einstein's 1914 Gravitational Theory

In Einstein and Grossmann 1914 and Einstein 1914, Einstein used a variational method to derive the field equations of limited covariance of his so-called *Entwurf* theory (Einstein and Grossmann 1913). He used conservation of energy-momentum of matter plus gravitational field—the stress-energy of the latter being represented by a pseudotensor rather than a tensor—to define the Lagrangian H for the gravitational field and to restrict the covariance of his theory. Einstein believed he had found a very general argument to fix the Lagrangian for the gravitational field. This Lagrangian leads to the field equations of the *Entwurf* theory.

By substituting the gravitational tensor into the law of conservation of energy-momentum of matter (with stress-energy tensor $\mathcal{T}_\mu{}^\nu$), Einstein was able to derive certain constraints on H that he thought uniquely fixed its form. Imposing conservation of energy-momentum of matter and unaware of the contracted Bianchi identities, he obtained a set of equations to be satisfied by the gravitational field:

$$\frac{\partial}{\partial x^\nu} S_\sigma{}^\nu - \mathcal{B}_\sigma = 0, \qquad (\sigma, \nu, \ldots = 0, 1, 2, 3)$$

where[2]

$$\mathcal{B}_\mu \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x^\sigma \partial x^\alpha}\left(g^{\nu\alpha} \frac{\partial H\sqrt{-g}}{\partial g_\sigma^{\mu\nu}}\right) \tag{1}$$

and

$$S_\sigma{}^\nu \stackrel{\text{def}}{=} g^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g^{\sigma\tau}} + g_\mu^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g_\mu^{\sigma\tau}} + \tfrac{1}{2}\delta_\sigma^\nu H\sqrt{-g} - \tfrac{1}{2} g_\sigma^{\mu\tau} \frac{\partial H\sqrt{-g}}{\partial g_\nu^{\mu\tau}}. \tag{2}$$

Then Einstein showed that both $\mathcal{B}_\mu$ and $S_\sigma{}^\nu$ must vanish:

$$\mathcal{B}_\mu = 0, \qquad S_\sigma{}^\nu = 0, \tag{3}$$

and used these conditions to define the form of H. He finally obtained the *Entwurf* field equations in the form[3]

$$\frac{\partial}{\partial x^\alpha}(\sqrt{-g}\, g^{\alpha\beta}\Gamma^\nu_{\sigma\beta}) = -\chi(\mathcal{T}_\sigma{}^\nu + t_\sigma{}^\nu), \tag{4}$$

where the stress-energy tensor[4] $t_\sigma{}^\nu$ for the gravitational field is defined as

$$t_\sigma{}^\nu \stackrel{\text{def}}{=} \frac{\sqrt{-g}}{\chi}\left(g^{\nu\tau}\Gamma^\rho_{\mu\sigma}\,\Gamma^\mu_{\rho\tau} - \tfrac{1}{2}\,\delta^\nu_\sigma\, g^{\tau\alpha}\,\Gamma^\rho_{\mu\tau}\,\Gamma^\mu_{\rho\alpha}\right), \tag{5}$$

$\Gamma^\rho_{\mu\sigma}$ being the Christoffel symbols. Differentiating equation (4) with respect to x^ν, Einstein obtained the conservation law for matter plus gravitational field in the form

$$\frac{\partial}{\partial x^\nu}(\mathcal{T}_\sigma{}^\nu + t_\sigma{}^\nu) = 0. \tag{6}$$

It must be stressed, however, that, already in 1914, Einstein noticed that

$t_\sigma{}^\nu$ does not transform as a tensor under arbitrary justified transformations, but only under linear transformations; nevertheless, we will call $t_\sigma{}^\nu$ the [stress-]energy tensor[5] of the gravitational field. Something analogous holds for the components $\Gamma^\nu_{\sigma\beta}$ of the gravitational field strength. (Einstein 1914, p. 1077)

In the spring of 1915, in private correspondence with Einstein, Levi-Civita sharply attacked Einstein's proofs of the covariance of certain fundamental quantities of his *Entwurf* theory (Cattani and De Maria 1989b); however, he did not explicitly criticize the pseudotensor character of $t_\sigma{}^\nu$.

3. Lorentz's Variational Approach (1915)

In 1915, Lorentz published a paper (Lorentz 1915) in which he criticized both the *Entwurf* theory and the variational formulation Einstein had given to it in 1914. In the second part of his paper, Lorentz proposed a more general variational derivation of gravitational field equations. Lorentz did not specify the form of the Lagrangian; he just assumed it to be a function of the metric tensor and its first-order derivatives. Requiring that the action integral be stationary not only for arbitrary infinitesimal variations of the coordinates, as Einstein had required, but also for arbitrary infinitesimal variations of the components of the metric tensor, Lorentz obtained the gravitational field equations in the form

$$\frac{\partial \mathcal{R}^*}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\sigma}\left(\frac{\partial \mathcal{R}^*}{\partial g^{\mu\nu}_\sigma}\right) = -\chi \frac{\partial \mathcal{M}}{\partial g_{\mu\nu}}, \tag{7}$$

where $\mathcal{R}^*$ and $\mathcal{M}$ are the Lagrangians for the gravitational field and matter, respectively. Furthermore, Lorentz showed that equations (7) turn into the *Entwurf* field equations when the function H chosen by Einstein is substituted for $\mathcal{R}^*$. As is well known, Einstein himself later realized that his choice of a Lagrangian was, in fact, quite arbitrary (Cattani and De Maria 1989b). Unlike Levi-Civita, Lorentz at this point was unaware of the mathematical mistakes Einstein made in his early variational approach, and praised him for "his ingenious mode of reasoning" (Lorentz 1915, p. 1089).

4. Hilbert's Variational Approach (1915)

On November 20, 1915, Hilbert presented a paper, entitled "The Foundations of Physics" (Hilbert 1915), in which he discussed a variational principle for general relativity. Hilbert cited both Einstein (1914, 1915a, 1915b, 1915c) and Mie (1912), the former for his gravitational field equations, the latter for his work on nonlinear electrodynamics and his electromagnetic theory of matter. Like Mie, Hilbert restricted his investigation to the situation of an electromagnetic field in the presence of a gravitational field.

Hilbert was critical of Einstein's 1914 variational approach as the following quotation from his paper illustrates:

> Einstein gave the fundamental original idea of general invariance a simple expression; however, for Einstein the Hamilton principle only plays a subordinate role and his function H is not at all generally invariant. Moreover, the electrical potentials are not included [in his theory]. (Hilbert 1915, I, p. 396, footnote)

Hilbert proceeded as follows. He assumed that the quantities characterizing the fields are the ten gravitational potentials $g_{\mu\nu}$ and the four electromagnetic potentials q_μ. He defined a unique invariant world function according to the following axioms:

> Axiom 1 (of Mie about the world function). The law of physical events is determined through a world function [Lagrangian] $\mathcal{H} = \sqrt{-g}H$ that contains the following arguments:
>
> $$g_{\mu\nu}, \ \frac{\partial g_{\mu\nu}}{\partial x^\alpha}, \ \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}; \qquad q_\sigma, \ \frac{\partial q_\sigma}{\partial x^\alpha}$$
>
> and specifically the variation of the action integral must vanish for [changes in] every one of the 14 potentials $g_{\mu\nu}, q_\sigma$.
>
> Axiom 2 (of general invariance). The world function $\mathcal{H}$ is invariant with respect to arbitrary transformations of the world parameters [coordinates] x^α. (Hilbert 1915, I, p. 396)

He then defined two Lagrangian functions, one for the gravitational field and one for matter. For the gravitational field he used the Riemann curvature scalar $\mathcal{R}$. For the matter part he introduced a function $\mathcal{M}$. As long as the gravitational field equations contain no derivatives of $g_{\mu\nu}$ higher than of second order, the total Lagrangian $\mathcal{H}$ must be the direct sum of these two functions:

$$\mathcal{H} = \mathcal{R} + \mathcal{M}. \tag{8}$$

By evaluating the "Lagrangian derivatives" (Hilbert 1915, I, p. 397) of $\mathcal{H}$ with respect to the various variables, Hilbert obtained the evolution equations for both gravitational and electromagnetic potentials. His next step was to show that Axiom 2 allows one to give an explicit proof of the covariance of these evolution equations. Splitting the Lagrangian into two parts, the scalar curvature invariant for the gravitational field and a Lagrangian for the electromagnetic field, Hilbert arrived at the correct gravitational field equations:

$$G_{\mu\nu} = -\chi \frac{1}{\sqrt{-g}} \mathcal{T}_{\mu\nu}, \tag{9}$$

where

$$G_{\mu\nu} \overset{\text{def}}{=} R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}. \tag{10}$$

Finally, Hilbert obtained the evolution equations for electrodynamics in a curved space-time by generalizing Mie's derivation for flat Minkowski space-time.

In conclusion, we want to stress the limits of Hilbert's method:

(1) Hilbert derived the field equations in the context of Mie's electromagnetic theory of matter. As a consequence, his variational method could not readily be generalized to other matter. To accomplish that, one would have to specify how the matter Lagrangian depends on the gravitational potentials $g_{\mu\nu}$.

(2) Although Hilbert obtained generally covariant field equations, he made use of Lagrangian derivatives that were not generally covariant.

(3) Hilbert was unaware of the contracted Bianchi identities, so that he arrived at the explicit form of the gravitational tensor in a rather clumsy way.

5. Lorentz's Variational Approach (1916)

In 1916, Lorentz published a long paper in four parts on general relativity (Lorentz 1916, I–IV). In part III, he derived the correct gravitational field equations and an expression for the "stress energy complex" for the gravitational field. In part IV, he discussed the conservation law for the gravitational field.

As opposed to the unspecified Lagrangian of his 1915 article, Lorentz now chose the Riemann curvature scalar $\mathcal{R}$ as the Lagrangian for the gravitational field. He had come to realize that the Lagrangian has to be a generally covariant scalar (Lorentz 1916, I, p. 248, p. 251; see also Janssen 1992).

Lorentz split the variation of the action $\mathcal{R}$ into two parts. The first part, which is no longer a scalar quantity, leads to gravitational field equations; the second part vanishes identically on account of the boundary conditions. Moreover, he showed that the form of his gravitational tensor coincided with Einstein's "only *for one special* choice of coordinates" (Lorentz 1916,

p. 281, italics in the original). Lorentz obtained the correct gravitational field equations (Lorentz 1916, III, p. 285). We want to stress, however, that Lorentz made some mathematically unwarranted assumptions in deriving his results. He assumed that the infinitesimal variations of the components of the metric tensor have tensor character. Moreover, he had to make a special choice of coordinates.

Lorentz also discussed the conservation of energy-momentum of matter plus gravitational field, and arrived at the equations (6) obtained by Einstein in 1914 (Lorentz 1916, III, p. 292). Lorentz too was aware of the fact that the complex $t_\sigma{}^\nu$ is not a tensor (Lorentz 1916, III, p. 294). Whereas this was perfectly acceptable to Einstein, Lorentz wrote that

> [e]vidently it would be more satisfactory if we could ascribe a stress-energy-*tensor* to the gravitation field. Now this can really be done. (Lorentz 1916, III, p. 295, italics in the original)

A "natural" candidate for this tensor, according to Lorentz, was the gravitational tensor $G_{\mu\nu}$ of Einstein's generally covariant field equations. Therefore he suggested one interpret these equations as conservation laws. In Lorentz's opinion this interpretation of the field equations

> and the conception to which they have led, may look somewhat startling. According to it we should have to imagine that behind the directly observable world with its stresses, energy etc. the gravitation field is hidden with stresses, energy etc. that are everywhere equal and opposite to the former; evidently this is in agreement with the interchange of momentum and energy which accompanies the action of gravitation. On the way of a lightbeam, e.g., there would be everywhere in the gravitation field an energy current equal and opposite to the one existing in the beam. If we remember that this hidden energy-current can be fully described mathematically by the quantities g_{ab} and that only the interchange just mentioned makes it perceptible to us, this mode of viewing the phenomena does not seem unacceptable. At all events we are forcibly led to it if we want to preserve the advantage of a stress-energy-*tensor* also for the gravitation field. (Lorentz 1916, III, p. 296, italics in the original)

In part IV of his paper, Lorentz compared his definition of the stress-energy components of the gravitational field with the definition given by Einstein. While his expression contained first and second order derivatives of the metric, "Einstein on the contrary has given values for the stress-energy components which contain the first derivatives only and which therefore are in many respects much more fit for application" (Lorentz 1916, IV, p. 297). Thus Lorentz defined a stress-energy complex with components $t_\sigma{}^{\nu'}$ that are homogeneous and quadratic functions of the first-order derivatives of the metric and do not contain any higher-order derivatives. The divergence of

Lorentz's complex coincides with the divergence of Einstein's $t_\sigma{}^\nu$. Lorentz showed that when $\sqrt{-g} = 1$ and $g_{\alpha\beta} = \delta_{\alpha\beta}$ his complex is the same as Einstein's. He added that "it seems very probable that the agreement will exist in general" (Lorentz 1916, IV, p. 299).

In conclusion, we want to stress that Lorentz showed, for the first time, that the quantity representing gravitational stress-energy was not uniquely defined.

6. Einstein's Variational Approach (1916)

In 1916, Einstein returned to a variational approach to derive his gravitational field equations. He remarked that both Lorentz and Hilbert had succeeded in giving general relativity a clear form by deriving the field equations from a single variational principle. His aim now was to present the basic relations of the theory as clearly as possible and in a more general way. In fact, he considered his new approach more general and "in contrast especially with Hilbert's treatment" (Einstein 1916b, p. 1111), since he rejected some of Hilbert's restrictive hypotheses on the nature of matter.

His starting point was the universal function $\mathcal{H} \overset{\text{def}}{=} H\sqrt{-g}$, assumed to be a function of the metric tensor and its first-order derivatives and a linear function of its second-order derivatives. Furthermore, he generalized the variational principle to any physical phenomenon by assuming $\mathcal{H}$ to be dependent on matter variables q_ρ (not necessarily of electromagnetic origin) and their first-order derivatives. Thus, he replaced his 1914 Lagrangian by

$$\mathcal{H} = \mathcal{H}\left(g^{\mu\nu}, \frac{\partial g^{\mu\nu}}{\partial x^\sigma}, \frac{\partial^2 g^{\mu\nu}}{\partial x^\rho \partial x^\sigma}; \; q_\rho, \frac{\partial q_\rho}{\partial x^\alpha}\right). \tag{11}$$

Integrating a Lagrangian of this form with the usual boundary conditions, one arrives at the variational principle

$$\delta \int \mathcal{H}^* \, \mathrm{d}\tau = 0, \tag{12}$$

where $\mathcal{H}^*$ no longer depends on the second-order derivatives of the metric. Einstein had to start from a function of the form of (11) because, according to his principle of general relativity, the Lagrangian $\mathcal{H}$ must be invariant under arbitrary coordinate transformations. However, the reduction of $\mathcal{H}$ to $\mathcal{H}^*$ (i.e., the reduction to a quadratic function of the metric's first-order derivatives) enabled Einstein to make use of the mathematical machinery developed in his 1914 paper. Meanwhile, the problems he had struggled with in 1914 had been overcome: the theory was now generally covariant and his choice of a Lagrangian was no longer arbitrary (Norton 1984; Cattani and De Maria 1989b).

Einstein's next step was to split, like Hilbert, the Lagrangian into a gravitational and a matter part (see equation (8) above). Einstein concluded that in order to satisfy his principle of general relativity, the gravitational part of the Lagrangian "(up to a constant factor) must be the scalar of the Riemann curvature tensor; since there is no other invariant with the required properties" (Einstein 1916b, p. 1113). Closely following his 1914 variational approach, Einstein showed, using an infinitesimal coordinate transformation $x^{\mu\prime} = x^{\mu} + \Delta x^{\mu}$, that the condition $\mathcal{B}_{\mu} = 0$ (see equation (3) above) still holds. In fact, Einstein proved that this condition could be obtained by showing that $\Delta \int \mathcal{R}\, d\tau = \Delta \int \mathcal{R}^*\, d\tau$ where

$$\mathcal{R}^* = \sqrt{-g}\, g^{\mu\nu}\left(\Gamma^{\beta}_{\mu\alpha}\,\Gamma^{\alpha}_{\nu\beta} + \Gamma^{\alpha}_{\mu\nu}\,\Gamma^{\beta}_{\alpha\beta}\right).$$

Therefore, the relation $\mathcal{B}_{\mu} = 0$ now holds in every coordinate system, due to the invariance of $\mathcal{R}$ and to the principle of general relativity. $\mathcal{B}_{\mu}$ played a fundamental role in Einstein's new derivation of the conservation laws. In fact, according to Einstein, the gravitational equations could be explicitly written as equations (7). These equations allowed him to obtain, in a very straightforward way, the conservation laws. By multiplying equations (7) by $g^{\mu\nu}$ he obtained

$$\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial \mathcal{R}^*}{\partial g^{\sigma\mu}_{\alpha}}\, g^{\nu\mu}\right) = \chi\,(\mathcal{T}_{\sigma}{}^{\nu} + t_{\sigma}{}^{\nu}), \tag{13}$$

where

$$\mathcal{T}_{\sigma}{}^{\nu} = g^{\mu\nu}\frac{\partial \mathcal{M}}{\partial g^{\mu\sigma}} \tag{14}$$

and

$$t_{\sigma}{}^{\nu} \overset{\text{def}}{=} \frac{1}{\chi}\left(\frac{\partial \mathcal{R}^*}{\partial g^{\mu\sigma}_{\alpha}}\, g^{\mu\nu}_{\alpha} + \frac{\partial \mathcal{R}^*}{\partial g^{\mu\sigma}}\, g^{\mu\nu}\right).$$

When conditions (2)–(3) are imposed, it follows that

$$t_{\sigma}{}^{\nu} = \tfrac{1}{2}\left(\mathcal{R}^*\delta^{\nu}_{\sigma} - \frac{\partial \mathcal{R}^*}{\partial g^{\mu\nu}_{\alpha}}\, g^{\sigma}_{\mu\alpha}\right). \tag{15}$$

When equation (13) is differentiated with respect to x^{ν}, the left-hand side turns into $\mathcal{B}_{\mu}$. Since $\mathcal{B}_{\mu}$ vanishes, the relation obtained in this way is just equation (6), expressing conservation of total energy-momentum.

As in his previous theory, Einstein identified $\mathcal{T}_\sigma{}^\nu$ as representing the stress-energy density for matter and $t_\sigma{}^\nu$ as representing the stress-energy density of the gravitational field (Einstein 1916b, p. 1116). He concluded that although $t_\sigma{}^\nu$ was not a tensor, the equations expressing the conservation of total energy-momentum are generally covariant, since they were obtained directly from the principle of general relativity (Einstein 1916b, p. 1116). As we shall see, this claim led Levi-Civita, in 1917, to dispute not only the tensor character of $t_\sigma{}^\nu$ but also the equations Einstein used as his conservation laws for matter plus gravitational field (Cattani and De Maria 1989a).

7. Einstein's First Paper on Gravitational Waves (1916)

In another paper from 1916, Einstein tried to compute the components of $t_\sigma{}^\nu$ for the special case of a weak field, and in doing so discovered the existence of gravitational waves. The metric for the weak field is written, as usual, in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \tag{16}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $\gamma_{\mu\nu}$ (and its first-order derivatives) are infinitesimal quantities. In the weak-field approximation the field equations reduce to

$$\sum_{\alpha=1}^{4} \frac{\partial^2 \gamma'_{\mu\nu}}{\partial x^{\alpha 2}} = 2\chi \mathcal{T}_{\mu\nu}, \tag{17}$$

where

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \tfrac{1}{2}\gamma\delta_{\mu\nu}, \qquad \gamma \stackrel{\text{def}}{=} \gamma^\mu_\mu. \tag{18}$$

The quantities $\gamma'_{\mu\nu}$ are defined only up to a gauge transformation. Einstein therefore imposed the gauge condition

$$\sum_{\nu=1}^{4} \frac{\partial \gamma'_{\mu\nu}}{\partial x^\nu} = 0.$$

In this way, he found solutions of the weak-field equations, vanishing at infinity, that are the analogs of retarded potentials in electrodynamics. Therefore, according to Einstein, "gravitational fields propagate as waves with the speed of light" (Einstein 1916a, p. 692). Multiplying equation (17) by $\partial\gamma'_{\mu\nu}/\partial x^\sigma$, Einstein obtained the conservation law for the total energy-momentum in the usual form (6), where

$$t_{\mu\nu} = \frac{1}{4\chi}\left[\sum_{\alpha\beta} \frac{\partial \gamma'_{\alpha\beta}}{\partial x^\mu}\frac{\partial \gamma'_{\alpha\beta}}{\partial x^\nu} - \tfrac{1}{2}\delta_{\mu\nu}\sum_{\alpha\beta\tau}\left(\frac{\partial \gamma'_{\alpha\beta}}{\partial x^\tau}\right)^2\right]. \tag{19}$$

In deriving the conservation law, however, Einstein made a trivial mathematical error (he used $\gamma'^{\alpha\beta}$ instead of $\gamma^{\alpha\beta}$ in the conservation law for matter). As we shall see, two years elapsed before Einstein discovered this "regrettable error in computation" (Einstein 1918b, p. 154). The error caused some "strange results" (Einstein 1916a, p. 696). Einstein obtained three different types of gravitational waves compatible with equation (17): not just longitudinal and transversal ones but also a "new type" of wave (Einstein 1916a, p. 693). Using equation (19) to compute the energy carried by these waves, he found the paradoxical result that no energy transport was associated with either the longitudinal or the transversal waves. He tried to explain this absurdity by treating these waves as fictitious:

> The strange result that there should exist gravitational waves without energy transport... can easily be explained. They are not "real" waves, but "apparent" ones, because we have chosen as the coordinate system the one vibrating as the waves. (Einstein 1916a, p. 696)

Einstein found that only the third kind of waves transport energy. He concluded, however, that the mean value of the energy radiated by this new type of waves was very small, because of a damping factor $1/c^4$ and because of the small value of the gravitational constant χ $(= 1.87 \cdot 10^{-27})$ that entered into its expression. Still, the possibility of gravitational radiation was bothersome. As Einstein stated in his paper:

> Nevertheless, due to the motion of the electrons in the atom, the atoms should radiate not only electromagnetic energy, but also gravitational energy, though in a little quantity. Since, this does not happen in nature, it seems that the quantum theory should modify not only the electrodynamics of Maxwell, but also the new theory of gravitation. (Einstein 1916a, p. 696)

8. Levi-Civita's 1917 Article

Einstein's choice of a noncovariant stress-energy complex (Einstein 1916b) and his strange results on gravitational waves (Einstein 1916a) motivated Levi-Civita to try and find a satisfactory definition of a gravitational stress-energy tensor in Einstein's theory (Levi-Civita 1917). In Levi-Civita's opinion, it was Einstein's use of pseudotensor quantities that led to his physically unacceptable results on gravitational waves. He wrote:

> The idea of a gravitational [stress-energy] tensor belongs to the majestic construction of Einstein. But the definition proposed by the author is unsatisfactory. First of all, from the mathematical point of view, it lacks the invariant character it should have in the spirit of general relativity.

> More serious is the fact, noticed also by Einstein, that it leads to a clearly unacceptable physical result regarding gravitational waves. He thought that the way out of this last problem was through the quantum theory.... Indeed, the explanation is closer at hand: everything depends on the correct form of the gravitational [stress-energy] tensor. (Levi-Civita 1917, p. 381)

In Levi-Civita's opinion, general relativity called for a generally covariant gravitational stress-energy tensor. Since no differential invariants of the first order exist, one cannot have a stress-energy tensor containing only first-order derivatives of the metric; and, since the definition of $t_\sigma{}^\nu$ in (Einstein 1916b) only contains first-order derivatives, Levi-Civita concluded that "Einstein's choice of the gravitational tensor is not justified" (Levi-Civita 1917, p. 391). Levi-Civita, in fact, showed that Einstein's stress-energy complex was covariant under linear transformations only. He proposed a new candidate for the gravitational stress-energy tensor, and, consequently, a new candidate for the conservation law.

Starting from the Ricci tensor $R_{\mu\nu}$, Levi-Civita, like Hilbert in 1915, defined $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R$ and wrote the gravitational field equations in the form of (9). Using, for the first time, the contracted Bianchi identities, Levi-Civita showed that the covariant divergence of $G_\mu{}^\nu$ vanishes: $\nabla_\nu G_\mu{}^\nu = 0$. Consequently, $\nabla_\nu \mathcal{T}_\mu{}^\nu = 0$. This conservation law for matter will hold, Levi-Civita pointed out, since "$\mathcal{T}_\mu{}^\nu$ includes the complete contribution of all phenomena (but gravitation) which take place at the point in time under consideration" (Levi-Civita 1917, p. 389).

Levi-Civita now made a move similar to the one we saw Lorentz make earlier: he proposed to interpret equation (9) both as field equations and as conservation laws. Defining the stress-energy tensor for the gravitational field as

$$A_{\mu\nu} \stackrel{\text{def}}{=} \frac{1}{\chi}\mathcal{G}_{\mu\nu} = -\mathcal{T}_{\mu\nu} \quad \Rightarrow \quad A_{\mu\nu} + \mathcal{T}_{\mu\nu} = 0, \tag{20}$$

he identified

> $A_{\mu\nu}$ as the components of a [stress-]energy tensor of the space–time domain, i.e., depending only on the coefficients of ds^2. Such a tensor can be called both gravitational and inertial, since gravity and inertia simultaneously depend on ds^2. (Levi-Civita 1917, p. 389)

According to Levi-Civita, $A_{\mu\nu}$ completely characterizes the contribution of gravity to the local mechanical behavior. With this interpretation, it follows from equation (20) that no net flux of energy can exist. This equilibrium is guaranteed by the "real" existence of both quantities which, being tensors, are independent of the choice of coordinates. Hence,

> [n]ot only the total force applied to every single element vanishes, but also (taking into account the inertia of the $A_{\mu\nu}$) the total stress, the flux, and the energy density. (Levi-Civita 1917, p. 389)

So, for Levi-Civita, the gravitational stress-energy is characterized by the only element independent of the coordinates, the Riemann tensor.

In Levi-Civita's approach, the problems that Einstein ran into are avoided. Einstein had to admit the possibility that gravitational waves transporting energy are generated in the absence of sources. Einstein's weak-field equations have solutions for $\mathcal{T}_{\mu\nu} = 0$ representing such spontaneous gravitational waves. Moreover, the energy flux, computed on the basis of equation (17), could be zero in one coordinate system and nonzero in another. Einstein invoked the help of quantum theory to solve these problems. Levi-Civita claimed that it was enough to define the gravitational stress-energy tensor the way he suggested and to reinterpret the field equations accordingly. This precludes all counterintuitive situations of the sort Einstein encountered, for, according to (20), the gravitational stress-energy tensor $A_{\mu\nu}$ vanishes whenever the stress-energy tensor $\mathcal{T}_{\mu\nu}$ for matter vanishes.

9. Einstein's Response to Levi-Civita

In the summer of 1917, the Great War still raging on, Einstein went on a vacation trip to his home country, neutral Switzerland. While there, the mathematician Adolf Hurwitz gave him a copy of Levi-Civita's paper (Levi-Civita 1917), which had just been published in *Rendiconti dell'Accademia dei Lincei*. From Lucerne, on August 2, 1917, Einstein wrote a long letter to Levi-Civita,[6] still in Padua (which was very close to the war front), in order to rebut the latter's criticism of his theory, especially his use of a pseudotensor to represent gravitational stress-energy. Einstein gave some physical considerations to show that the stress-energy of the gravitational field cannot be represented by a generally covariant tensor.

Einstein began his letter expressing his admiration for Levi-Civita's "beautiful new work":

> I admire the elegance of your method of calculation. It must be nice to ride through these fields upon the horse of true mathematics, while people like me have to make their way laboriously on foot.... I still don't understand your objections to my view of the gravitational field. I would like to tell you again what causes me to persist in my view. (Einstein to Levi-Civita, August 2, 1917, p. 1)

He then proceeded to discuss the example of a counterweight pendulum clock to show that Levi-Civita's choice of a tensor to represent the stress-energy of the gravitational field is problematic from a physical point of view:

> I start with a Galilean space, i.e., one with constant $g_{\mu\nu}$. Merely by changing the reference system [i.e., by introducing an accelerated reference system], I obtain a gravitational field. If in K' a pendulum clock driven by a weight is set up in a state in which it is not working, gravitational energy is transformed into heat, while relative to the original system K, certainly no gravitational field and thereby no energy of this field is present.[7] Since, in K, all components of the energy "tensor" in question vanish identically, all components would also have to vanish in K', if the energy of gravitation could actually be expressed by a tensor. (Einstein to Levi-Civita, August 2, 1917, p. 1)

If gravitational stress-energy could be expressed by a tensor, no gravitational process could occur in K', in which case, contrary to experience, gravitational energy could not be transformed into heat. In short, the pendulum clock example shows that it should be possible for the components of gravitational stress-energy to be zero in one reference frame and nonzero in another. Therefore, gravitational stress-energy cannot be represented by a generally covariant tensor. Notice how Einstein's reasoning here is deeply rooted in his conception of the equivalence principle.

To the physical argument of the pendulum clock, Einstein adds an argument against the tensor character of gravitational stress-energy of a more mathematical nature:

> In general, it seems to me that the energy components of the gravitational field should only depend upon the first-order derivatives of $g_{\mu\nu}$, because this is also valid for the *forces* exerted by the fields.[8] *Tensors* of the first order (depending only on $\partial g_{\mu\nu}/\partial x^\sigma = g_\sigma^{\mu\nu}$), however, do not exist. (Einstein to Levi-Civita, August 2, 1917, pp. 1–2)

In his letter, Einstein went on to criticize Levi-Civita's interpretation of the gravitational field equations (20) as conservation laws. Einstein gave some examples showing that such conservation laws would have strange and undesired consequences. He wrote to Levi-Civita,

> You think that the field equations ... should be conceived of as energy equations, so that $\left[\mathcal{G}_\mu^\sigma\right]$ would be the [stress-]energy components of the gravitational field. However, with this conception it is quite incomprehensible how something like the energy law could hold in spaces where gravity can be disregarded. Why, for example, should it not be possible on your view for a body to cool off without giving off heat to the outside? (Einstein to Levi-Civita, August 2, 1917, p. 2)

On Levi-Civita's proposed definition of the conservation laws, the only way for matter to lose energy, it seems, is to transfer it locally to the gravitational field. It does not seem to allow for the possibility of energy transfer from one place to another.

At the same time, Levi-Civita's proposal did seem to allow for processes one would like to rule out. Einstein wrote:

> The equation
> $$\mathcal{G}_4^4 + \mathcal{T}_4^4 = 0 \tag{21}$$
> allows $\mathcal{T}_4^4$ to decrease everywhere, in which case this change is compensated for by a decrease of the, physically not perceived, absolute value of the quantity $\mathcal{G}_4^4$.... I maintain, therefore, that what you [Levi-Civita] call the energy law has nothing to do with what is otherwise so designated in physics. (Einstein to Levi-Civita, August 2, 1917, p. 2)

On these grounds, Einstein rejected Levi-Civita's interpretation of the field equations as conservation laws, and held on to his earlier formulation of the conservation laws (6). He argued that this formulation was perfectly sensible from a physical point of view, even though it involved a pseudotensor representing gravitational stress-energy:

> [My] conclusions are correct, whether or not one admits that the t_ν^σ are "really" the components of the gravitational [stress-]energy. That is to say, the relation
> $$\frac{\mathrm{d}}{\mathrm{d}x^4}\left\{\int \left(\mathcal{T}_4{}^4 + t_4{}^4\right) \mathrm{d}V\right\} = 0$$
> holds true with the vanishing of $\mathcal{T}_\sigma{}^\nu$ and $t_\sigma{}^\nu$ at [spatial] infinity, where the integral is extended over the whole three-dimensional space. For my conclusions, it is only necessary that $\mathcal{T}_4{}^4$ be the energy density of matter, which neither one of us doubts. (Einstein to Levi-Civita, August 2, 1917, p. 2)

Finally, Einstein pointed out that, in his definition, the gravitational stress-energy exhibits the desired behavior at spatial infinity:

> ... (in the static case) the field at infinity must be completely determined by the energy of matter and of the gravitational field (taken together). This is the case with my interpretation.... (Einstein to Levi-Civita, August 2, 1917, p. 2)

10. Levi-Civita's Response to Einstein

At the end of August 1917, Einstein received Levi-Civita's answer,[9] full of flattery as well as criticism:

> I am very grateful that you kindly appreciate the mathematics of my last articles but the credit of having discovered these new fields of research goes to you. (Levi-Civita to Einstein, August 1917, draft, p. 1)

In his letter, Levi-Civita criticized Einstein's definition of the gravitational field energy, wondering why a function of first-order derivatives of the metric tensor should be taken as stress-energy (pseudo)tensor, and asking for a more convincing motivation of this choice.

On the other hand, Levi-Civita granted Einstein that his interpretation of the field equations as conservation laws was not very *fecund*:

> I recognize the importance of your objection that, in doing so, the energy principle would lose all its heuristic value, because no physical process (or almost none) could be excluded a priori. In fact, [in order to get any physical process] one only has to associate with it a suitable change of the ds^2. (Levi-Civita to Einstein, August 1917, draft, p. 1)

Levi-Civita seems to be referring to Einstein's example of a stress-energy tensor for matter whose energy component decreases everywhere. Einstein's conservation laws (4) rule out such a stress-energy tensor. It looks as if Levi-Civita's conservation laws, i.e., the gravitational field equations, do not. It looks as if it would be possible for almost any matter stress-energy tensor to find a metric field such that the field equations are satisfied. The conservation laws thus seem to lose their "heuristic value" of restricting the range of acceptable matter stress-energy tensors. Of course, through the contracted Bianchi identities, the field equations do, in fact, restrict the range of acceptable matter stress-energy tensors.

In his letter, Levi-Civita stressed having no prejudice against a definition of gravitational stress-energy dependent on the choice of coordinates, or, as he put it,

> dependent on the expression of ds^2, in analogy with what happens for the notion of force of the field.... In the case of the equations of motion, written in the form
>
> $$\frac{d^2x^\nu}{ds^2} = -\left\{ \begin{matrix} \nu \\ \sigma\ \mu \end{matrix} \right\} \frac{dx^\sigma}{ds}\frac{dx^\nu}{ds},$$
>
> one can explicitly connect the right-hand side (which does *not* define either a covariant or a contravariant system) with the ordinary notion of force. According to you, the same should happen for your $t_\sigma{}^\nu$ (which do not constitute a tensor). I am not in principle opposed to your point of view. On the contrary, I am inclined to presume that it is right as are all intuitions of geniuses. But I would like to see each conceptual step [canceled: logical element] to be clearly explained and described, as is done (or, at least, as is known can be done) in the case of the equation above, where we know how to recover the ordinary notion of force. (Levi-Civita to Einstein, August 1917, draft, pp. 1–2)

At the same time, Levi-Civita insisted that, at least from a logical point of view, there was nothing wrong with his own choice of a generally covariant tensor to represent gravitational stress-energy:

> [canceled: Let me add some opinions for a *logical* defense]. While I maintain an attitude of prudent reserve and wait, I still want to defend the *logical flawlessness* of my tensor $\mathcal{G}_{\mu\nu}$. (Levi-Civita to Einstein, August 1917, draft, p. 2)

Next, Levi-Civita attacked the counterweight pendulum-clock example:

> I want to stress that, contrary to what you claim, there is no contradiction between the accounts of the pendulum-clock in the two systems K and K', the first one fixed (in the Newtonian sense), the second one moving with constant acceleration. You say that:
>
> (a) in K, the energy tensor is zero because the $g_{\mu\nu}$ *are constant*;
> (b) in K', this is not the case; instead, there is a physical phenomenon with an observable transformation of energy into heat;
> (c) due to the invariant character of a null tensor, the simultaneous validity of (a) and (b) implies that there is something wrong with the premises.
>
> I contest (a), since we can assume that $g_{\mu\nu}$ are constant outside of the ponderable bodies, but [not] in the space taken up by your pendulum-clock. (Levi-Civita to Einstein, August 1917, draft, p. 2)

In other words, Levi-Civita denied that Einstein's pendulum clock example is incompatible with the tensor character of $A_{\mu\nu}$, observing that since the pendulum is not massless, strictly Euclidean coordinates cannot be assumed in K. Therefore, the energy tensor for gravitational field is different from zero both in K and in K'.

Finally, Levi-Civita responded to Einstein's comment on the behavior of the gravitational field at infinity:

> With regard to the last consideration of your letter (point 4), if I am not wrong, it [the behavior of the gravitational field at infinity] is not a consequence of the special form of your $t_\sigma{}^\nu$, but is equally valid for my $A_{\mu\nu}$. It seems to me that the behavior at infinity can be obtained from [our equation (20)] by using the circumstance that the divergence of the tensor $A_{\mu\nu}$ is identically zero; therefore, the divergence of $\mathcal{T}_{\mu\nu}$ also vanishes, and it reduces asymptotically to $\frac{\partial}{\partial x^\nu}\mathcal{T}_{\sigma\nu} = 0$, because the $g_{\mu\nu}$ tend to the values $\epsilon_{\mu\nu}$ [i.e., the constant Minkowski values of the metric tensor]. (Levi-Civita to Einstein, August 1917, draft, p. 2)

So, Levi-Civita invoked the contracted Bianchi identities to show that his conservation laws, like Einstein's, exhibit the desired behavior at spatial infinity.

In an addendum, Levi-Civita finally remarked:

> An indication in favor [of our equation (20)] is the negative value of the energy density of the gravitational field A_{00} (assuming $\mathcal{T}_{00} > 0$). This is in agreement with the old attempts to localize the potential energy of a Newtonian body, and explains the minus sign as due to the exceptional role of gravity compared to all other physical phenomena. (Levi-Civita to Einstein, August 1917, draft, p. 2)

11. Einstein's Second Paper on Gravitational Waves (1918)

After Levi-Civita's August 1917 letter, the polemic between the two scientists stopped until Einstein in 1918 published a new paper on gravitational waves (Einstein 1918b). In the introduction, he recognized that his earlier approach to gravitational waves (in Einstein 1916a)

> was not transparent enough, and it was marred by a regrettable error in computation. Therefore, I have to turn back to the same argument. (Einstein 1918b, p. 154)

Because of this error, he had obtained the wrong expression for his stress-energy complex. Correcting the error, Einstein could easily derive the correct expression for the stress-energy complex. As a consequence, he obtained only two kinds of waves, thereby resolving all the physical paradoxes of his previous results. Einstein could now assert with confidence that

> [a] mechanical system which always maintains its spherical symmetry cannot radiate, contrary to the result of my previous paper, which was obtained on the basis of an erroneous calculation. (Einstein 1918b, p. 164)

In the last section of (Einstein 1918b), entitled "Answer to an objection advanced by Mr. Levi-Civita,"[10] Einstein publicly gave his final reply to Levi-Civita's old objections. Einstein gave improved versions of some of the arguments already given in his August 1917 letter to Levi-Civita. He stressed that at least the time component of equation (6) must be looked upon as the energy equation, even if the $t^{\nu}{}_{\sigma}$ cannot be considered components of a tensor.

In this section of his paper, Einstein gave ample credit to Levi-Civita for his contributions to general relativity:

> In a recent series of highly interesting studies, Levi-Civita has contributed significantly to the clarification of some problems in general relativity. In one of these papers [Levi-Civita 1917], he defends a point

> of view regarding the conservation laws different from mine, and disputes my conclusions about the radiation of energy through gravitational waves. Although we have already settled the issue to the satisfaction of both of us in private correspondence, I think it is fitting, because of the importance of the problem, to add some further considerations concerning conservation laws.... There are different opinions on the question whether or not $t^{\nu}{}_{\sigma}$ should be considered as the components of the [stress-]energy of the gravitational field. I consider this disagreement to be irrelevant and merely a matter of words. But I have to stress that [our equation (6)], about which there are no doubts, implies a simplification of views that is important for the significance of the conservation laws. This has to be underscored for the fourth equation ($\sigma = 4$), which I want to define as the energy equation. (Einstein 1918b, p. 166)

Without entering into the mathematical details of $t_{\sigma}{}^{\nu}$, Einstein defended his energy equation with the following argument:

> Let us consider a spatially bounded material system, whose matter density and electromagnetic field vanish outside some region. Let $\mathcal{S}$ be the boundary surface, at rest, which encloses the entire material system. Then, by integration of the fourth equation over the domain inside $\mathcal{S}$, we get
>
> $$-\frac{\mathrm{d}}{\mathrm{d}x^4}\int_{\mathcal{V}}\left(\mathcal{T}_4{}^4 + t_4{}^4\right)\mathrm{d}V = \int_{\mathcal{S}}\left(t_4{}^1\cos(nx_1) + t_4{}^2\cos(nx_2) + t_4{}^3\cos(nx_3)\right)\mathrm{d}\sigma.$$
>
> One is not entitled to define $t_4{}^4$ as the energy density of the gravitational field and $(t_4{}^1,\ t_4{}^2,\ t_4{}^3)$ as the components of the flux of gravitational energy. But one can certainly maintain, in cases where the integral of $t_4{}^4$ is small compared to the integral of the matter energy density $\mathcal{T}_4{}^4$, that the right-hand side represents the material energy loss of the system. It was only this result that was used in this paper and in my first article on gravitational waves. (Einstein 1918b, pp. 166–167)

Einstein then considered Levi-Civita's main objection against his choice of conservation laws:

> Levi-Civita (and prior to him, although less sharply, H.A. Lorentz) proposed a different formulation... of the conservation laws. He (as well as other specialists) is against emphasizing [equations (6)] and against the above interpretation because $t_{\sigma}{}^{\nu}$ is not a tensor. (Einstein 1918b, p. 166)

Although Einstein obviously had to admit that $t_{\sigma}{}^{\nu}$ is not a tensor, he concluded:

> I have to agree with this last criticism, but I do not see why only those quantities with the transformation properties of the components of a tensor should have a physical meaning. (Einstein 1918b, p. 167)

Finally, Einstein stressed that, even though there is no "logical objection" (Einstein 1918b, p. 167) against Levi-Civita's proposal, it has to be dismissed on physical grounds.

> I find, on the basis of [equation (20)], that the components of the total energy vanish everywhere. [Equation (20)], (contrary to [equation (6)]), does not exclude the possibility that a material system disappears completely, leaving no trace of its existence. In fact, the total energy in [equation (20)] (but not in [equation (6)]) is zero from the beginning; the conservation of this value of the energy does not guarantee the persistence of the system in any form. (Einstein 1918b, p. 167)

In fact, this result is due to the algebraic form of Levi-Civita's "conservation law" (according to which the total stress-energy is equal to zero everywhere). In Levi-Civita's opinion, the local vanishing of the matter stress-energy does not allow any energy flux. From a mathematical point of view, Levi-Civita's approach, with a generally covariant gravitational stress-energy tensor, was certainly more general than Einstein's, and apparently more in line with the spirit of general relativity. Einstein's choice, on the other hand, was more convincing on the basis of physical arguments, as Levi-Civita himself admitted. At the time, Einstein stood alone in his defense of a noncovariant definition of gravitational energy. Modern general relativists, however, follow Einstein's rather than Levi-Civita's approach to conservation laws.

12. Schrödinger's Example against Einstein's Stress-Energy Complex and Einstein's Reply

Lorentz and Levi-Civita were not the only two scientists to criticize Einstein's definition of gravitational stress-energy. In November 1917, Erwin Schrödinger showed, in a straightforward calculation, that, given a symmetrical distribution of matter, Einstein's gravitational stress-energy complex $t_\sigma{}^\nu$ can be zero in a suitable coordinate system. Schrödinger evaluated the stress-energy complex, starting from the Schwarzschild metric for the case of an incompressible fluid sphere of matter, and noticed that

> to determine $t_\sigma{}^\nu$, we must always specify the coordinate system, since their values do not have tensor character and do not vanish in every system, but only in some of them. The result we get in this particular case, i.e. the possibility of reducing $t_\sigma{}^\nu$ to be identically zero, is so surprising that I think it will need a deeper analysis.... Our calculation shows that there are some real gravitational fields whose [stress-]energy components vanish; in these fields not only the momentum and the energy flow but also the energy density and the analogs of the Maxwell

> stresses can vanish, in some finite region, as a consequence of a suitable choice of the coordinate system. (Schrödinger 1918, p. 4)

Thus, Schrödinger concluded,

> This result seems to have, in this case, some consequences for our ideas about the physical nature of the gravitational field. Since we have to renounce the interpretation of $t_\sigma{}^\nu \ldots$ as the [stress-]energy components of the gravitational field, the conservation law is lost, and it will be our duty to somehow replace this essential part in the foundation [of the theory]. (Schrödinger 1918, pp. 6–7)

About two and a half months later (on February 5, 1918), Einstein replied to Schrödinger in the same journal (Einstein 1918a). Oddly enough, Einstein started by raising further doubts about his choice of the quantities $t_\sigma{}^\nu$ to represent gravitational stress-energy:

> Schrödinger's calculations have shown that in a suitably chosen coordinate system all [stress-]energy components $t_\alpha{}^\sigma$ of the gravitational field [generated by a] sphere vanish outside of this sphere. Understandably, he was puzzled by this result, and so was I at first; in particular, he wondered whether $t_\alpha{}^\sigma$ should really be interpreted as [stress-]energy components.... To these doubts I can add two more:
>
> (1) the [stress-]energy components of matter $\mathcal{T}_\sigma{}^\nu$ represent a tensor, while this is not true for the "[stress-]energy components" of the gravitational field $t^\sigma{}_\nu$;
>
> (2) the quantities $\mathcal{T}_{\sigma\tau} = \sum_\nu \mathcal{T}_\sigma{}^\nu g_{\nu\tau}$ are symmetric in the indices σ and τ, while this not true for $t_{\sigma\tau} = \sum_\nu t_\sigma{}^\nu g_{\nu\tau}$.
>
> For the same reason as mentioned in point (1), Lorentz and Levi-Civita also raised doubts about interpreting $t_\alpha{}^\sigma$ as the [stress-]energy components of the gravitational field. Even though I can share their doubts, I am still convinced that it is helpful to give a more convenient expression for the energy components of the gravitational field. (Einstein 1918a, p. 115)

Einstein then offered the following explanation for Schrödinger's apparently strange result. He pointed out that a gravitational field generated by only one body, as in Schrödinger's example, is different from physical gravitational fields that always involve more than one body: "in gravitational fields mediating exchange effects between different bodies the quantities $t^\sigma{}_\nu$ cannot vanish identically" (Einstein 1918a, p. 115). As an example, Einstein considered two material bodies, M_1 and M_2, connected by a rigid rod. Using his conservation law, he found that since the stresses for matter are nonzero, the gravitational energy flux is nonzero as well. Therefore,

> [t]hese considerations hold *mutatis mutandis* in all those cases where the field transmits exchange effects between different bodies. But this is not the case for the field considered by Schrödinger. (Einstein 1918a, p. 116)

He concluded peremptorily:

> Hence, the formal doubts (1) and (2) cannot lead to a rejection of my proposal for the expression of the energy-momentum. It does not seem justified to put any further formal demands [on the properties of a quantity representing gravitational stress-energy]. (Einstein 1918a, p. 116)

13. Bauer's Example against Einstein's Stress-Energy Complex and Einstein's Final Reply

About one month after Einstein's reply to Schrödinger, Hans Bauer attacked Einstein's choice of $t^{\sigma}{}_{\nu}$ (Bauer 1918). He discussed an example complementary to Schrödinger's. Schrödinger had shown that Einstein's gravitational stress-energy sometimes vanishes despite the presence of a gravitational field. Bauer now showed that it does not always vanish in the absence of a gravitational field. He stressed that

> the partial nonvanishing of the [stress-]energy components has nothing to do with the presence of a gravitational field, but it is due only to the choice of a coordinate system.... This behavior is not surprising, since $t^{\sigma}{}_{\nu}$ is not a tensor. (Bauer 1918, p. 165)

So, Bauer thought he had thrown another stone at the physical plausibility of Einstein's proposal:

> we have to conclude that the "[stress-]energy components" $t^{\sigma}{}_{\nu}$ are not related to the presence of a gravitational field as they depend only on the choice of coordinates. They can vanish in presence of a field, as shown by Schrödinger, and do not always vanish in absence of a field, as shown below. Hence, their physical significance seems to be very dubious. (Bauer 1918, p. 165)

Einstein replied to Bauer's criticism without delay. In May 1918, he published a new reply to Schrödinger and Bauer (Einstein 1918c). He once again justified his choice with physical arguments. In his opinion,

> the theory of general relativity has been accepted by most theoretical physicists and mathematicians, even though almost all colleagues stand against my formulation of the energy-momentum law. Since I am convinced that I am right, I will in the following present my point of view on these matters in more detail. (Einstein 1918c, p. 448)

Einstein reminded his readers how special relativity combines the ordinary conservation laws of energy and momentum into one differential equation (i.e., the vanishing of the four-divergence of the stress-energy tensor) which is equivalent to the integral form of these conservation laws verified in experience. The generalization of this conservation law to general relativity, he explained, was particularly delicate. Einstein showed how, with his choice, "the classical concepts of energy and momentum are established as concisely as we are accustomed to expect in classical mechanics" (Einstein 1918c, p. 449). Then he demonstrated that the energy and momentum of a closed system are uniquely determined only when the motion of the system (considered as a whole) is expressed "with respect to a given coordinate system" (Einstein 1918c, pp. 449–450). In particular, he showed that the stress-energy of such closed systems can only be expected to transform as a tensor under certain coordinate transformations, viz. those coordinate transformations that reduce to the identity transformation at infinity. The transformations used in Schrödinger and Bauer's examples do not meet this requirement, so they do not count as counterexamples.

After this article by Einstein, the debate on the correct formulation of conservation laws in general relativity apparently came to the end.

14. Conclusions

In this chapter, we have described the polemic between Einstein and Levi-Civita on the correct formulation of conservation laws in general relativity during the years 1917–1918. Prompted by a mistake Einstein made in his first paper on gravitational waves, Levi-Civita criticized the use of non-covariant quantities in a generally covariant theory. This, in turn, stimulated Einstein to give a new and correct description of gravitational waves. Meanwhile, Lorentz had shown that there is no unique definition of the stress-energy of the gravitational field in general relativity. Following up on this insight, Lorentz proposed to interpret the field equations as conservation laws. Levi-Civita independently made the same proposal in a mathematically more satisfactory way, using the contracted Bianchi identities. Einstein held on to his old formulation of the conservation laws involving the pseudotensor $t_\sigma{}^\nu$ to represent the gravitational stress-energy. Schrödinger and Bauer showed that, in certain cases, Einstein's choice of $t_\sigma{}^\nu$ led to paradoxical results.

This episode makes for an interesting case study in the history of general relativity for at least two reasons: (1) it clarifies the connections between variational methods and conservation laws in general relativity and their cross-fertilization; (2) it shows the extent of Einstein's scientific isolation in his efforts to complete the edifice of general relativity during 1916–1918. Some of the most celebrated mathematical physicists, such as Lorentz and Levi-Civita, attacked his choice of a pseudotensor to represent gravitational stress-energy on the basis of formal mathematical arguments very much in the spirit of general relativity. Moreover, two young theoretical physicists, Schrödinger and Bauer, came up with some apparently damning counterexamples against Einstein's choice. Yet Einstein, masterfully exploiting the equivalence principle as a heuristic tool, stubbornly defended his choice and justified it with strong physical arguments. By today's standards, he was right.

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NOTES

[1] See also Cattani's chapter "Levi-Civita's Influence on Palatini's Contribution to General Relativity" in this volume.

[2] With his 1914 choice of H, $\mathcal{B}_\mu$ explicitly is

$$\mathcal{B}_\mu = \frac{\partial^2}{\partial x^\nu \partial x^\alpha}\Big((-g)^{1/2} g^{\alpha\beta} g_{\sigma\mu} \frac{\partial g^{\mu\nu}}{\partial x^\beta}\Big).$$

[3] For a more extensive discussion of these calculations, see Norton (1984).

[4] Einstein defined the pseudotensor t_σ^ν as (Einstein 1914, p. 1077)

$$t_\sigma{}^\nu \stackrel{\text{def}}{=} \frac{1}{\chi}\Big(-g^{\nu\tau}\frac{\partial H(-g)^{1/2}}{\partial g^{\sigma\tau}} - g_\alpha^{\nu\tau}\frac{\partial H(-g)^{1/2}}{\partial g_\alpha^{\sigma\tau}}\Big),$$

in order to show explicitly its dependence on H.

[5] In this period physicists meant stress-energy tensor when they said energy-tensor.

[6] Einstein to Levi-Civita, August 2, 1917, Einstein Archive, Boston (EA 16-253). English translation by J. Goldstein and E.G. Straus with some modifications.

[7] Let us examine Einstein's pendulum clock example a little more closely. In K, the reference frame in which there is no gravitational field, the clock is not working since the counterweight that should drive it is not subjected to a gravitational field. Let us take a concrete example. Suppose our clock is in a spacecraft far from any masses with its engines turned off (frame K). In this case, the clock is in a situation of "absence of weight," and consequently cannot work. When the engines are turned on, the spacecraft accelerates (frame K'). Consequently, all objects inside the spacecraft experience an apparent gravitational field. Our clock will want to start working under the influence of this field. If, in K', we want to prevent this, the clock's gravitational energy will be transformed into heat.

[8] Here Einstein presumably alludes to the fact that in general relativity gravitational forces are expressed in terms of the Christoffel symbols, which contain first-order derivatives of the metric only.

[9] Levi-Civita to Einstein, August 1917. Only a draft of this letter survives (Levi-Civita Papers, Accademia dei Lincei, Rome). It seems reasonable, though, to assume that the actual letter was not all that different from the draft.

[10] "Antwort auf einen von Hrn. Levi-Civita herrührenden Einwand," Einstein 1918b, pp. 166–167.

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