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On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields.*

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§ 1. The formalism of the ordinary quantum theory of wave fields.

Recently Yukawa[1] has made a comprehensive consideration about the basis of the quantum theory of wave fields. In his article he has pointed out the fact that the existing formalism of the quantum field theory is not yet perfectly relativistic.

Let $v(xyz)$ be the quantity specifying the field, and $\lambda(xyz)$ denote its canonical conjugate. Then the quantum theory requires the commutation relations of the form:

$$\begin{cases} [v(xyzt),\ v(x'y'z't')]=[\lambda(xyzt),\ \lambda(x'y'z't')]=0 \\ [v(xyzt),\ \lambda(x'y'z't')]=i\hbar\delta(x-x')\delta(y-y')\delta(z-z'), \end{cases} \quad (1)^{**}$$

but these have quite non-relativistic forms.

The equations (1) give namely the commutation relations between the quantities at different points (xyz) and $(x'y'z')$ at the same instant of time t. The concept "same instant of time at different points" has, however, a definite meaning only one specifies some definite Lorentz frame of reference. Thus this is not a relativistically invariant concept.

Further, the Schrödinger equation for the ψ-vector representing the state of the system has the form;

$$\left(\bar{H}+\frac{\hbar}{i}\frac{\partial}{\partial t}\right)\psi=0, \quad (2)$$

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* Translated from the paper, Bull. I. P. C. R. (Riken-iho), **22** (1943), 545, appeared originally in Japanese.

** $[A, B]=AB-BA$. We assume that the field obeys the Bose statistics. Our considerations apply also to the case of Fermi statistics.

where $\bar{H}$ is the operator representing the total energy of the field which is given by the space integral of a function of v and λ. As we adopt here the Schrödinger picture, v and λ are operators independent of time. The vector representing the state is in this picture a function of the time, and its dependence on t is determined by (2).

Also the differential equation (2) is no less non-relativistic. In this equation the time variable t plays a role quite distinguished from the space coordinates x, y and z. This situation is closely connected with the fact that the notion of probability amplitude does not fit with the relativity theory.

As is well known, the vector ψ has, as the probability amplitude, the following physical meaning: Suppose the representation which makes the field quantity $v(xyz)$ diagonal. Let $\psi[v'(xyz)]$ denote the representative of ψ in this representation.* Then the representative $\psi[v'(xyz)]$ is called probability amplitude, and its absolute square

$$W[v'(xyz)] = |\psi[v'(xyz)]|^2 \qquad (3)$$

gives the relative probability of $v(xyz)$ having the specified functional form $v'(xyz)$ at the instant of time t. In other words: Suppose a plane** which is parallel to the xyz-plane and intersepts the time axis at t. Then the probability that the field has the specified functional form $v'(xyz)$ on this plane is given by (3).

As one sees, a plane parallel to the xyz-plane plays here a significant role. But such a plane is only defined by referring to a certain frame of reference. Thus the probability amplitude is not a relativistically invariant concept in the space-time world.

§ 2. Four-dimensinal form of the commutation relations.

As stated above, the laws of the quantum theory of wave fields are

* We use the square blackets to indicate a functional. Thus $\psi[v'(xyz)]$ means that ψ is a functional of the variable function $v'(xyz)$. When we use ordinary blackets ( ), as $\psi(v'(xyz))$, we consider ψ as an ordinary function of the function $v'(xyz)$. For example: the energy density is written as $H(v(xyz), \lambda(xyz))$ and this is also a function of x, y and z, whereas the total energy $H = \int H(v(xyz), \lambda(xyz))dv$ is a functional of $v(xyz)$ and $\lambda(xyz)$ and is written as $\bar{H}[v(xyz), \lambda(xyz)]$.

** We call a three-dimensional minifold in the four-dimensional space-time world simply "surface".

usually expressed as mathematical relations between quantities having their meanings only in some specified Lorentz frame of reference. But since it is proved that the whole contents of the theory are of cource relativistically invariant, it must be certainly possible to build up the theory on the basis of concepts having relativistic space-time meanings. Thus, in his consideration, Yukawa has required with Dirac[2] to generalize the notion of probability amplitude so that it fits with the relativity theory. We shall now show below that the generalization of the theory on these lines is in fact possible to the relativistically necessary and sufficient extent. Our results are, however, not so general as expected by Dirac and by Yukawa, but are already sufficiently general in so far as it is required by the relativity theory.

Let us suppose for simplicity that there are only two fields interacting with each other. The case of more number of fields can also be treated in the same way. Let v_1 and v_2 denote the quantities specifying the fields. The canonically conjugate quantities be λ_1 and λ_2 respectively. Then between these quantities the commutation relations

$$\begin{cases} [v_r(xyzt),\ v_s(x'y'z't')]=0 \\ [\lambda_r(xyzt),\ \lambda_s(x'y'z't')=0 \qquad r, s=1, 2 \\ [v_r(xyzt),\ \lambda_s(x'y'z't')]=i\hbar\delta(x-x')\delta(y-y')\delta(z-z')\delta_{rs} \end{cases} \tag{4}$$

must hold. The ψ-vector satisfies the Schrödinger equation

$$\left(\overline{H}_1+\overline{H}_2+\overline{H}_{12}+\frac{\hbar}{i}\frac{\partial}{\partial t}\right)\psi=0. \tag{5}$$

In this equation $\overline{H}_1$ and $\overline{H}_2$ mean respectively the energy of the first and the second field. $\overline{H}_1$ is given by the space integral of a function of v_1 and λ_1, $\overline{H}_2$ by the space integral of a function of v_2 and λ_2. Further, $\overline{H}_{12}$ is the interaction energy of the fields and is given by the space integral of a function of both v_1, λ_1 and v_2, λ_2. We assume (i) that the integrand of $\overline{H}_{12}$, i. e. the interaction-energy density, is a scalar quantity, and (ii) that the energy densities at two different points (but at the same instant of time) commute with eath other. In general, these two facts follow from the single assumption: the interaction term in the Lagrangean does not contain the time derivatives of v_1 and v_2.

If this energy density is denoted by H_{12}, then we have

$$\overline{H}_{12}=\int H_{12}dx\,dy\,dz. \tag{6}$$

As we adopt here the Schrödinger picture, the quantities v and λ in H_1, H_2 and H_{12} are all operators independent of time.

Thus far we have merely summarized the well known facts. Now, as the first stage of making the theory relativistic, we suppose the unitary operator

$$U=\exp\left\{\frac{i}{\hbar}(\overline{H}_1+\overline{H}_2)t\right\} \tag{7}$$

and introduce the following unitary transformations ot v and λ, and the corresponding transformation of ψ:

$$\begin{cases} V_r=Uv_rU^{-1}, \quad \Lambda_r=U\lambda_rU^{-1} \\ \Psi=U\psi. \end{cases} \qquad r=1,\ 2 \tag{8}$$

As stated above, v and λ in (5) are quantities independent of time. But V and Λ obtained from them by means of (8) contain t through U. Thus they depend on t by

$$\begin{cases} i\hbar\dot{V}_r=V_r\overline{H}_r-\overline{H}_rV_r \\ i\hbar\dot{\Lambda}_r=\Lambda_r\overline{H}_r-\overline{H}_r\Lambda_r. \end{cases} \qquad r=1,\ 2 \tag{9}$$

These equations must necessarily have covariant forms against Lorentz transformations, because they are just the field equations for the fields when they are left alone without interacting with eath other.

Now, the solutions of these " vacuum equations ", the equations which the fields must satisfy when they are left alone, together with the commutaion relations (4), give rise to the relations of the following forms:

$$\begin{cases} [V_r(xyzt),\ V_s(x'y'z't')]=A_{rs}(x-x',\ y-y',\ z-z'\ t-t') \\ [\Lambda_r(xyzt),\ \Lambda_s(x'y'z't')]=B_{rs}(x-x',\ y-y',\ z-z',\ t-t') \\ [V_r(xyzt),\ \Lambda_s(x'y'z't')]=C_{rs}(x-x',\ y-y',\ z-z',\ t-t') \end{cases} \tag{10}$$

where A_{rs}, B_{rs} and C_{rs} are functions which are combinations of the so-called four-dimensional δ-functions and their derivatives.[3] One denotes usually these four dimensional δ-functions by $D_r(xyzt)$, $r=1,\ 2$. They are defined by

$$D_r(xyzt) = \frac{1}{16\pi^3}\iiint\left\{\frac{e^{i(k_x x+k_y y+k_z z+ck_r t)}}{ik_r} - \frac{e^{i(k_x x+k_y y+k_z z-ck_r t)}}{ik_r}\right\} dk_x dk_y dk_z \quad (11)$$

with

$$k_r = \sqrt{k_x^2+k_y^2+k_z+k_r^2}\,, \quad (12)$$

$\varkappa_r$ being the constant characteristic to the field r. It can be easily proved that these functions are relativistically invariant.*

Since (10) gives, in contrast with (4), the commutation relations between the fields at two different world points $(xyzt)$ and $(x'y'z't')$, it contains no more the notion of same instant of time. Therefore, (10) is sufficiently relativistic presupposing no special frame of reference. We call (10) fourdimensional form of the commutation relations.

One property of $D(xyzt)$ will be mentioned here: When the world point $(xyzt)$ lies outside the light cone whose vertex is at the origin, then $D(xyzt)$ vanishes identically:

$$D(xyzt)=0 \quad \text{for} \quad x^2+y^2+z^2-c^2t^2>0. \quad (13)$$

It follows directly from (13) that, if the world point $(x'y'z't')$ lies outside the light cone whose vertex is at the world point $(xyzt)$, the right-hand sides of (10) always vanish. In words: Suppose two world points P and P'. When these points lie outside each other's light cones, the field quantities at P and field quantities at P' commute with eath other.

§ 3. Generalization of the Schrödinger equation.

Next we observe the vector Ψ obtained from ψ by means of the unitary transformation U. We see from (5), (7) and (8) that this Ψ, considered as

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* Suppose that a surface in the $k_x\,k_y\,k_z$ k-space is defined by means of the equation $k^2=k_x^2+k_y^2+k_z^2+\varkappa^2$. Then this surface has the invariant meaning in this space, since $k_x^2+k_y^2+k_z^2-k^2$ is invariant against Lorentz transformations. The area of the surface element of this surface is given by $dS=\sqrt{\left(\frac{\partial k}{\partial k_x}\right)^2+\left(\frac{\partial k}{\partial k_y}\right)^2+\left(\frac{\partial k}{\partial k_z}\right)^2-1}\,dk_x\,dk_y\,dk_z = \varkappa\frac{dk_x\,dk_y\,dk_z}{k}$. Now, since dS has the invariant meaning, we can thus conclude that $\frac{dk_x\,dk_y\,dk_z}{k}$ is an invariant, and this results that the function defined by (11) is invariant.

a function of t, satisfies

$$\left\{\int H_{12}(V_1(xyzt),\ \Lambda_1(xyzt),\ V_2(xyzt),\ \Lambda_2(xyzt))\,dx\,dy\,dz + \frac{\hbar}{i}\frac{\partial}{\partial t}\right\}\Psi=0. \qquad (14)$$

One sees that t plays also here a role distinguished from x, y and z: also here a plane parallel to the xyz-plane has a special significance. So we must in some way remove this unsatisfactory feature of the theory.

This improvement can be attained in the way similar to that in which Dirac[4] has built up the so-called many-time formalism of the quantum mechanics. We will now recall this theory.

The Schrödinger equation for the system containing N charged particles interacting with the electromagnetic field is given by

$$\left\{\bar{H}_{el}+\sum_{n=1}^{N} H_n(q_n,\ p_n,\ \mathfrak{a}(q_n))+\frac{\hbar}{i}\frac{\partial}{\partial t}\right\}\psi=0. \qquad (15)$$

Here $\bar{H}_{el}$ means the energy of the electromagnetic field, H_n the energy of the n-th particle. H_n contains, besides the kinetic energy of the n-th particle, the interaction energy between this particle and the field through $\mathfrak{a}(r_n)$, q_n being the coordinates of the particle and $\mathfrak{a}$ the potential of the field. p_n in (15) means as usual the momentum of the n-th particle.

We consider now the unitary operator

$$u=\exp\left\{\frac{i}{\hbar}\bar{H}_{el}t\right\} \qquad (16)$$

and introduce the unitary transformation of $\mathfrak{a}$:

$$\mathfrak{A}=u\mathfrak{a}u^{-1} \qquad (17)$$

and the corresponding transformation of ψ:

$$\Phi=u\psi. \qquad (18)$$

Then we see that Φ satisfies the equation

$$\left\{\sum_n H_n(q_n,\ p_n,\ \mathfrak{A}(q_n,\ t))+\frac{\hbar}{i}\frac{\partial}{\partial t}\right\}\Phi=0. \qquad (19)$$

In contrast with $\mathfrak{a}$, which was independent of times (Schrödinger picture), $\mathfrak{A}$

contains t through u. To emphsize this, we have written t explicitely as argument of $\mathfrak{A}$. We can prove that $\mathfrak{A}$ satisfles the maxwell equations in vacuum (accurately speaking, we need special considerattons for the equation div $\mathfrak{E}=0$).

The equation (19) is the starting point of the many-time theory. In this theory one introduces then the function $\Phi(q_1t_1, q_2t_2, \ldots, q_N, t_N)$ containing so many time variables $t_1, t_2, \ldots\ldots t_N$ as the number of the particles in place of the function $\Phi(q_1, q_2, \ldots, q_N, t)$ containing only one time variable,* and suppose that this $\Phi(q_1t_1, q_2t_2, \ldots, q_Nt_N)$ satisfies simultaneously the following N equations;

$$\left\{H_n(q_n, p_n, \mathfrak{A}(q_n, t_n)) + \frac{\hbar}{i}\frac{\partial}{\partial t_n}\right\}\Phi(q_1t_1, q_2t_2, \ldots, q_Nt_N)=0$$
$$n=1, 2. \ldots, N. \qquad (20)$$

This $\Phi(t_1, t_2, \ldots, t_N)$, which is a fundamental quantity in the many-time theory, is related to the ordinary probability amplitude $\Phi(t)$ by

$$\Phi(t)=\Phi(t, t, \ldots, t). \qquad (21)$$

Now, the simultaneous equations (20) can be solved when and only when the N^2 conditions

$$(H_nH_n'-H_n'H_n)\Phi(q_1t_1, q_2t_2, \ldots, q_Nt_N)=0 \qquad (22)$$

are satisfied for all pairs of n and n'. If the world point (q_nt_n) lies outside the light cone whose vertex is at the point $(q_n't_n')$, we can prove $H_nH_n'-H_n'H_n=0$. As the result, the function satisfying (20) can exist in the region where

$$(q_n-q_n')^2-c^2(t_n-t_n')^2\gtrless 0 \qquad (23)$$

is satisfied simultaneously for all values of n and n'.

According to Bloch[5] we can give $\Phi(q_1t_1\ q_2t_2, \ldots, q_Nt_N)$ a physical meaning when its arguments lie in the region given by (23). Namely

$$W(q_1t_1, q_2t_2, \ldots. q_Nt_N)=|\Phi(q_1t_1, q_2t_2, \ldots, q_Nt_N)|^2 \qquad (24)$$

gives the relative probability that one finds the value q_1 in the measurement of the position of the first particle at the instant of time t_1, the value q_2 in

* Here we suppose the representation which makes the coordinates $q_1, q_2, \ldots, q_N$ diagonal. Thus the vector Φ is represented by a function of these coordinates.

the measurement of the position of the second particle at the instant of time t_2, ... and the value q_N in the measurement of the position of the N-th particale at the instant of time t_N.

This is the outline of the many-time formalism of the quantum mechanics.

We will now return to our main subject. If we compare our equation (14) with the equation (19) of the many-time theory, we notice a marked similarity between these two equations. In (19) stands the suffix n, which designates the particle, while in (14) stand the variables x, y and z, which designate the position in space. Further, Φ is a function of the N independent variables q_1, q_2, ..., q_N, q_n giving the position of the n-th particle, while Ψ is a functional of the infinitely many "independent variables" $v_1(xyz)$ and $v_2(xyz)$, $v_1(xyz)$ and $v_2(xyz)$ giving the fields at the position (xyz). Corresponding to the sum $\sum_n H_n$ in (19) the integral $\int H_{12} dx dy dz$ stands in (14). In this way, to the suffix n in (19) which takes the values 1, 2, 3, ..., N correspond the variables x, y and z whice take continuously all values from $-\infty$ to $+\infty$

Such a similarity suggests us to introduce infinitely many time variables t_{xyz}, which we may call local time * each for one position (xyz) in the space as we have introduced N time variables, particle times, t_1, t_2, ... t_N, each for one particle. The only difference consist in that we use in our case infinitely many time variables whereas we have used N time variables in the ordinary many-time theory

Corresponding to the transition from the use of the function with one time variable to the use of the function of N time variables, we must now consider the transition from the use of $\Psi(t)$ to the use of a functional $\Psi[t_{xyz}]$ of infinitely many time variables t_{xyz}.

We regard now t_{xyz} as a function of (xyz) and consider its variation ε_{xyz} which differs from zero only in a small domain V_0 in the neighbourhood of the point $(x_0 y_0 z_0)$. We will define the partical differential coefficient of the functional $\Psi[t_{xyz}]$ with respect to the variable $t_{x_0 y_0 z_0}$ in the following manner:

$$\frac{\delta' \Psi}{\delta t_{x_0 y_0 z_0}} = \lim_{\substack{\varepsilon \to 0 \\ V_0 \to 0}} \frac{\Psi[t_{xyz} + \varepsilon_{xyz}] - \Psi[t_{xyz}]}{\iiint \varepsilon_{xyz}\, dx\, dy\, dz} \qquad (25)$$

* The notion of local time of this kind has been occasionary introduced by Stueckerberg.(6)

We then generalize (14), and regard

$$\left\{H_{12}(x,\ y,\ z,\ t)+\frac{\hbar}{i}\frac{\delta}{\delta t_{xyz}}\right\}\Psi=0 \qquad (26)$$

the infinitely many simultaneous equations corresponding to the N equations (20), as the fundamental equations of our theory. In (26) we have written, for simplicity. $H_{12}(x,\ y,\ z,\ t)$ in place of $H_{12}(V_1(xyz,\ t),\ V_2(xyz,\ t),\ \ldots\ldots)$. In general, when we have a function $F(V,\ A)$ of V and A, we will write simply $F(x,\ y,\ z,\ t)$ for $F(V(xyz,\ t_{xyz}),\ A(xyz,\ t_{xyz}))$, or still simpler $F(P)$ P denoting the world point with the coordinates $(xyz,\ t_{xyz})$. Thus $F(P')$ means $F(x',\ y',\ z',\ t')$ or, more precisely, $F(V(x'y'z',\ t_{x'y'z'}),\ (x'y'z',\ {}_{x'y'z'}))$.

We will now adopt the equation (26) as the basis of our theory. For $V_1(P)$, $V_2(P)$, $A_1(P)$ and $A_2(P)$ in H_{12} the commutation relations (10) hold, where $D(xyzt)$ has the property (13). As the consequence, we have

$$H_{12}(P)H_{12}(P')-H_{12}(P')H_{12}(P)=0 \qquad (27)$$

when the point P lies a finite distance apart from P' and outside the light cone whose vertex is at P. Further, from our assumption (ii) the relation (27) holds also when P and P' are two adjacent points approaching in a space-like direction. Thus our system of equations (26) is integrable when the surface defined by the equations $t=t_{xyz}$, considering t_{xyz} as a function of x, y and z, is space-like.

In this way, a functional of the variable surface in the space-time world is determined by the functional partial differential equations (26). Corresponding to the relation (21) in case of many-time theory, $\Psi[t_{xyz}]$ reduces to the ordinary $\Psi(t)$ when the surface reduces to a plane parallel to the xyz-plane.

The dependent variable surface $t=t_{xyz}$ can be of any (space-like) form in the space-time world, and we need not presuppose any Lorenz frame of reference to define such a surface. Therefore, this $\Psi[t_{xyz}]$ is a relativistically invariant concept. The restriction that the surface must be space-like makes no harm since the property that a surface is space-like or time-like does not depend on a special choise of the reference system. It is not necessary, from the stand-point of the relativity theory, to admit also time-like surfaces for the variable surface, what was required by Dirac and by Yukawa. Thus we consider that $\Psi[t_{xyz}]$ introduced above is already the sufficient generalization of the ordinary ψ-vector, and assume that the quantum-

theoretical state* of the fields is represented by this functional vector.

Let C denote the surface defined by the equation $t=t_{xyz}$. Then Ψ is a functional of the surface C. We write this as $\Psi[C]$. On C we take a point P, whose coordinates are (xyz, t_{xyz}), and suppose a surface C' which overlap C except in a small domain about P. We denote the volume of the small world lying between C and C' with $d\omega_P$. Then we may write (25) also in the form:

$$\frac{\delta\Psi[C]}{\delta C_P}=\lim_{C'\to C}\frac{\Psi[C']-\Psi[C]}{d\omega_P}. \tag{28}$$

Then (26) can be written in the form:

$$\left\{H_{12}(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Psi[C]=0. \tag{29}$$

This equation (29) has now a perfect space-time form. In the first place, H_{12} is a scalar according to our assumption (i); in the second place, the commutation relations between $V(P)$ and $A(P)$ contained in H_{12} has the four-dimensional forms as (10), and finally the differentiation $\frac{\delta}{\delta C_P}$ is defined by (28) quite independently of any frame of reference.

A direct conclusion obrained from (29) is that $\Psi[C']$ is obtained from $\Psi[C]$ by the following infinitesimal transformation:

$$\Psi[C']=\left\{1-\frac{i}{\hbar}H_{12}(P)d\omega_P\right\}\Psi[C]. \tag{30}$$

When there exist in the space-time world two surfaces C_1 and C_2 a finite distance apart, we need only to repeat the infinitesimal transformations in order to obtain $\Psi[C_2]$ from $\Psi[C_1]$. Thus

$$\Psi[C_2]=\prod_{C_1}^{C_2}\left\{1-\frac{i}{\hbar}H_{12}(P)d\omega_P\right\}\Psi[C_1] \tag{31}$$

The meaning of this equation is as follows: We devide the world region lying between C_1 and C_2 in small elemdnts $d\omega_P$ (it is necessary that each world element is surrounded by two space-like surfaces). We consider for

* The word state is here used in the relativistic space-time meaning. Cf. Dirac's book (second eddition) § 6.

each world element the infinitesimal transformation $1-\frac{i}{\hbar}H_{12}(P)d\omega_P$. Then we take the product of these transformations, the order of the factor being taken from C_1 to C_2. This product transforms then $\Psi[C_1]$ into $\Psi[C_2]$.

The surfaces C_1 and C_2 must be here both space-like, but otherwise they may have any form and any configuration. Thus C_2 does not necessarily lie afterward against C_1; C_1 and C_2 may even cross with each other.

The relation of the form (31) has been already introduced by Heisenberg.[7] It can be regarded as the integral form of our generalized Schrödinger equation (29)

§ 4. Generalized probability amplitude.

We must now find the physical meaning of the functional $\Psi[C]$. As regards this we can make a similar consideration as Bloch has done for the case of ordinary many-time theory. Besides the fact that in our case there appear infinitely many time variables, one point differs from Bloch's case that in (16) the unitary operator u is commutable with the coordinates q_1, q_2, ... q_N, our U is not commutable with the field quantities $v_1(xyz)$ and $v_2(xyz)$. Noting this difference and treating the continuum infinity as the limit of an ennumerable infinity by some artifice, for instance, by the procedure of Heisenberg and Pauli,[8] Bloch's consideration can be applied also here almost without any alteration. We shall give here only the results.

Let us suppose that the fields are in the state represented by a vector $\Psi[C]$. We suppose that we make measurments of a function $f(v_1, v_2, \lambda_1, \lambda_2)$ at every point on a surface C_1 in the space-time world. Let P_1 denote the variable point on C_1, then, if $f(P_1)$ at any two "values" of P_1 commute with each other, the measurement of f at each of these two points do not interfere with each other. Our first conclusion says that in this case the expectation value of $f(P_1)$ is given by

$$\overline{f(P_1)}=((\Psi[C_1], f(P_1)\Psi[C_1])) \qquad (32)$$

where $f(P_1)$ means $f(V_1(P_1), ...)$ according to our convention on page 35, and the symbol $((A, B))$ with double blackets is the scalar product of two vectors A and B. It is impossible in case of continuously many degree of freedom to represent this scalar product by an integral of the product of two functions. For this purpose we must replace the continuum infinity by an at least ennumerable infinity.

More generally, we suppose a functional $F[f(P_1)]$ of the independent variable function $f(P_1)$, regarding $f(P_1)$ as a function of P_1. Then the expectation value of this F is given by

$$\overline{F[f(P_1)]}=((\Psi[C_1],\ F[f(P_1)]\Psi[C_1])). \qquad (33)$$

A physically interesting F is the projective operator $M[v_1'(P_1), v_2'(P_1), V_1(P_1), V_2(P_1)]$ belonging to the "eigen-value" $v_1'(P_1)$, $v_2'(P_1)$ of $V_1(P_1)$, $V_2(P_1)$. Then its expectation value

$$\overline{M[v_1'(P_1), v_2'(P_1); V_1(P_1), V_2(P_1)]}$$
$$=((\Psi[C_1],\ M[v_1'(P_1), v_2'(P_1)\ ;\ V_1(P_1), V_2(P_1)]\Psi[C_1])) \quad (34)$$

gives the probability that the field 1 and the field 2 have respectively the functional form $v_1'(P_1)$ and $v_2'(P_1)$ on the surface C_1. As C_1 is assumed to be space-like, the measurement of the functional M is possible (the measurements of $V_1(P_1)$ and $V_2(P_1)$ at all points on C_1 mean just the measurement of M).

Thus far we have made no mention of the representation of $\Psi[C]$. We use now the special representation in which $V_1(P_1)$ at all points on C_1 are simultaneously diagonal. It is always possible to make all $V_1(P_1)$ and $V_2(P_1)$ diagonal when the surface C_1 is space-like. In this representation $\Psi[C_1]$ is represented by a functional $\Psi[v_1'(P_1), v_2'(P_1)\ ;\ C_1]$ of the eigenvalues $v_1'(P_1)$ and $v_2'(P_1)$ of $V_1(P_1)$ and $V_2(P_1)$. The projection operator M has in this representation such diagonal form that (34) is simplified as follows

$$W[v_1'(P_1), v_2'(P_1)]=\overline{M[v_1'(P_1), v_2'(P_1); V_1(P_1), V_2(P_1)]}$$
$$=|\Psi[v_1'(P_1), v_2'(P_1);\ C_1]|^2. \qquad (35)$$

In this sence we can call $\Psi[v_1'(P_1), v_2'(P_1);\ C_1]$ "generalized probability amplitude".

§ 5. Generalized transformation functional.

We have stated adove that between $\Psi[C_1]$ and $\Psi[C_2]$ the relation (31) holds, where C_1 and C_2 are two spece-like surfaces in the space-time world. We see thus that the transformation operator

$$T[C_2;\ C_1]=\prod_{C_1}^{C_2}\left(1-\frac{i}{\hbar}H_{12}d\omega\right) \qquad (36)$$

plays an important role. It is evident that also this operator has a space-time meaning.

Similarly as the special representative of the ψ-vector, the probability amplitude, has a distinct physical meaning, there is a special representation in which the representative of the transformation operator $T[C_2;\ C_1]$ has a distinct physical meaning.

We introduce namely the mixed representative of $T[C_2;\ C_1]$ whose rows refer to the representation in which $V_1(P_1)$ and $V_2(P_1)$ at all points on C_1 become diagonal and whose column refer to the representation in which $V_1(P_2)$ and $V_2(P_2)$ at all potnts on C_2 become diagonal. We denote this representation by

$$[v_1''(P_2),\ v_2''(P_2)\,|\,I[C_2;\ C_1]\,|\,v_1'(P_1),\ v_2'(P_1)], \quad (37)^*$$

or simpler:

$$[v_1''(P_2),\ v_2''(P_2)\,|\,v_1'(P_1),\ v_2'(P_1)]. \qquad (38)^*$$

If we note here the relation (35), we see that we can give the matrix elements of this representation the following meaning: One measures the field quantities V_1 and V_2 at all points on C_2 when the fields are prepared in such a way that they have certainly the values $v_1'(P_1)$ and $v_2'(P_1)$ at all points on C_1. Then

$$W[v_1''(P_2),\ v_2''(P_2)\ ;\ v_1'(P_1),\ v_2'(P_1)]$$
$$=|\,[v_1''(P_2),\ v_2''(P_2)|\,v_1'(P_1),\ v_2'(P_1)]\,|^2 \qquad (39)$$

gives the probability that one obiains the result $v_1''(P_2)$ and $v_2''(P_2)$ in this measurement. In this proposition we have assumed that C_2 lies afterward against C_1.

From this physical interpretation we may regard the matrix element (37), or (38), considered as a functional of $v_1''(P_2)$, $v_2''(P_2)$ and $v_1'(P_1)$, $v_2'(P_1)$, as the generalization of the ordinary transformation function $(q_{t_2}''\,|\,q_{t_1}')$.

* As the matrix elements are functionals of $v(P)$, we use here the square blackets.

As a special case it may happen that C_2 lies apart from C_1 only in a portion S_2 and a portion S_1 of C_2 and C_1 respectively, the other parts of C_1 and C_2 overlapping with each other (see Fig. 1).

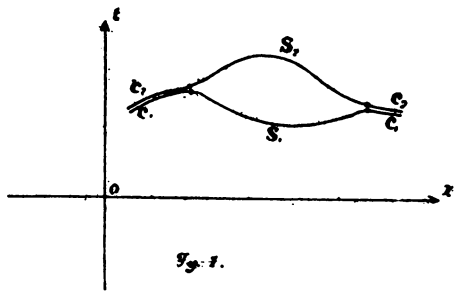

Fig. 1.

In this case the matrix elements of $T[C_2; C_1]$ depend only on the values of the fields on the portions S_1 and S_2 of the surfaces C_1 and C_2. In this case we need for calculating $T[C_2; C_1]$ to take the product in (36) only in the closed domain surrounded by S_1 and S_2, thus

$$T[S_2; S_1] = \overset{S_2}{\underset{S_1}{\Pi}}\left(1 - \frac{i}{\hbar} H_{12} d\omega\right). \qquad (40)$$

The matrix elements of the mixed representation of this T is a functional of $v_1'(p_1)$, $v_2'(p_1)$ and $v_1''(p_2)$, $v_2''(p_2)$ where p_1 denotes the moving point on the portion S_1, and p_2 the moving point on the portion S_2. This matrix is independent on the field quantities on the other portions of the surfaces C_1 and C_2.

The matrix element of $T[S_2; S_1]$ regarded as a functional of $v_1'(p_1)$, $v_2'(p_1)$ and $v_1''(p_2)$, $v_2''(p_2)$ has the properties of g. t. f. (generalized transformation functional) of Dirac. But in defining our g.t.f. we had to restrict the surfaces S_1 and S_2 to be space-like, while Dirac has required his g. t. f. to be defined also referring to the time-like surfaces. As mentioned above, however, such a generalization as required by Dirac is superflous so far as the relativity theory concerns.

It is to be noted that for the physical interpretation of $[v_1''(P_2), v_2''(P_2)|$ $v_1'(P_1), v_2'(P_1)]$ it is not necessary to assume C_2 to lie afterward against C_1. Also when the inverse is the case, we can as well give the physical meaning for W of (39): One measures the field quantities V_1 and V at all points on C_2 when the fields are prepared in such a way that they would have certainly the values $v_1'(P_1)$ and $v_2'(P_1)$ at all points on C_1 if the fields were left alone until C_1 without being measured before on C_2. Then W gives the probability that one finds the results $v_1''(P_2)$ and $v_2''(P_2)$ in this measurement on C_2.

§ 6. Concluding remark.

We have thus shown that the quantum tneory of wave fields can be really brought into a form which reveals directly the invariance of the theory against Lorentz transformatrons. The reason why the ordinary formalism of the quantum field theory is so unsatisfactory lies in the fact that one has built up this theory in the way which is too much analogous to the ordinary non-relativtstic mechanics. In this ordinary formalism of the quantum theory of fields the theory is devided into two distinct sections: the section giving the kinematical relations between various quantities at the same instant of time, and the section determining the causal relations between quantities at different instants of time. Thus the commutation relations (1) belong to the first section and the Schrodinger equation (2) to the second.

As stated defore, this way of separating the theory into two sections is very unrelativistic, since here the concept "same instant of time" plays a distinct role.

Also in our formalism the theory is devided into two sections. But now the separation is introduced in another place: In our formalism the theory consists of two sections, one of which gives the laws of behavior of the fields when they are left alone, and the other of which gives the laws determining the deviation from this behavior due to interactions. This way of separating the theory can be carried out relativistically.

Although in this way the theory can be brought into more satisfactory form, no new contents are added thereby. So, the well known divergence difficulties of the theory are inherited also by our theory. Indeed, our fundamental equations (29) admit only catastrophal solutions as can be seen directly in the fact that the unavoidable infinity due to non-vanishing zero-point amplitudes of the fields inheres in the operator $H_{12}(P)$. Thus, a more profound modification of the theory is required in order to remove this fundamental difficulty.

It is expected that such a modification of the theory would possiblly be introduced by some revision of the concept of interaction, because we meet no such difficulty when we deal with the non-interacting fields. This revision would then result that in the separability of the theory into two sections, one for free fields and one for interactions, some uncertainty would be introduced. This seems to be implied by the very fact that, when we formulate the quantum field theory in a relativistically satisfactory manner,

this way of separation has revealed itself as the fundamental element of the theory.

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