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A Class of Self-Sustaining Dissipative Spherical Dynamos

GEORGE BACKUS

Project Matterhorn, Princeton University, Princeton, New Jersey

The present paper treats rigorously the dynamo equations describing the effects of the internal motion of a bounded volume of incompressible fluid with nonzero ohmic resistivity on the magnetic field produced by electric currents in that fluid. The procedure involves representing an arbitrary solenoidal vector field in terms of two scalars, analogous to the representation of an arbitrary irrotational field as the gradient of a single scalar. The dynamo equations are reduced to scalar heat equations for the two field scalars, the coupling between them taking the form of a heat source term. Precise results about the magnetic field can be obtained from these heat equations with the help of several variational inequalities analogous to Rayleigh's variational estimate for the fundamental frequency of a vibrating system.

The main result is the explicit construction of a large class of continuously differentiable fluid velocities capable of indefinitely maintaining or amplifying the dipole moment of the external magnetic field. These motions all involve periods of stasis in the fluid, and cannot, therefore, be expected to occur in the earth's core. It is believed that it will be possible eventually to obtain more exact bounds than those presented here for the magnetic field components with high wave number, thus eliminating the need for such periods of stasis. The fluid motions shown capable of dynamo maintenance are of this sort: a toroidal shear symmetric about the $\hat{z}$ axis proceeds long enough to produce from $\boldsymbol{P}_{1z1}$, the lowest poloidal free-decay mode symmetric about that axis, a very large energy in $\boldsymbol{T}_{1z1}$, the lowest toroidal free-decay mode with such symmetry. During a period of stasis, everything else almost dies out, leaving a field which is largely $\boldsymbol{T}_{1z1}$. Then almost any velocity which has a radial component and is not axisymmetric about the $\hat{z}$ axis will regenerate $\boldsymbol{P}_{1z1}$ and the external dipole moment.

A critique of some previous attempts to produce dissipative self-regenerative spherical dynamos is included.

The techniques which lead to the existence of self-sustaining dynamos produce other results about the dynamo equations, most of which are to the author's knowledge either new or not previously precisely formulated. These results are listed below.

(i) Fluid motions in a sphere can be regarded as bounded linear operators on the Hilbert space of magnetic fields with finite total energy.

(ii) The free-decay modes in the rigid sphere are complete in that space.

(iii) The magnetic effect of a given fluid motion on a given initial field depends continuously on the resistivity ρ of the fluid even at $\rho = 0$.

(iv) The magnetic effect of any motion can be approximated with arbi-

trary accuracy by replacing it by a series of rapid jerks interpersed with periods of rest.

(v) The effect of a rigid rotation of the fluid is to rotate the magnetic field while that field decays as if the fluid were motionless. This effect can be approximated with arbitrary accuracy by rotating all but a sufficiently thin shell at the surface, even if every point on the surface remains fixed and a large shear develops in the shell.

(vi) If the fluid velocity has no radial component, the poloidal magnetic field decays as rapidly as if the fluid were at rest. If, further, no poloidal field is initially present, the toroidal field decays as rapidly as if the fluid were at rest.

(vii) Dynamo maintenance is impossible if the local strain-rate of the fluid is always and everywhere less than the decay rate of $\boldsymbol{P}_{1,1}$ when the velocity of the fluid is zero.

1. STATEMENT OF THE PROBLEM

(A) The Origin of the Geomagnetic Field

The present paper is addressed to one part of the question of the origin of the earth's magnetic field. Gauss (*1*) in 1838 used what was then known about the earth's surface magnetic field to conclude that the electric currents (or other sources) which produced it were inside the earth. Recent geomagnetic surveys indicate that no more than 2% of the earth's surface field can be ascribed to external electric currents (*2*). As Elsasser (*3*) has pointed out, if the currents inside a sphere of radius R and uniform electrical conductivity σ are not driven by any source of electromotive force, the external dipole moment of the magnetic field produced by those currents will decay exponentially with a mean life of no more than $\pi^{-2}\mu_0\sigma R^2$ seconds, where μ_0 is the magnetic permeability of free space. For a sphere the size of the sun with a conductivity as large as copper's at room temperature, this mean life is 10^{11} years; thus it is not out of the question that the present solar dipole field (*4*) was produced at the birth of the sun and has had no time to decay. If, as Elsasser (*5*) has suggested, turbulent convection destroys stellar magnetic fields, then the sun would need some sort of regenerative mechanism. The earth certainly needs such a mechanism, since for it $\pi^{-2}\mu_0\sigma R^2$ is of the order of 15000 years (*3*), while the paleomagnetic evidence (*6*) indicates that the earth's surface field has never been orders of magnitude stronger than it is now. Therefore, a source of electromotive force must be sought, capable of driving the internal currents which maintain the earth's external dipole field.

There is considerable evidence (*7*) that the fluid in the earth's core is moving relative to the mantle, so that as Larmor (*8*) proposed, one source of electromotive force might be the Lorentz electric field $\boldsymbol{u} \times \boldsymbol{B}$ seen by the fluid in the core as its velocity $\boldsymbol{u}$ carries it across the lines of force of the magnetic field $\boldsymbol{B}$. For a finite volume of fluid whose electrical conductivity is finite, it is still unknown whether there can exist such a "self-regenerative dynamo", a source-free fluid motion capable of maintaining a magnetic field indefinitely against ohmic losses. It is the purpose of the present paper to answer this question rigorously in the affirmative. There is a large class of solenoidal velocity fields inside a sphere which leave all points on the surface of that sphere fixed, are periodic in time (except for short intervals), are bounded and continuously differentiable in space and time, and are capable of maintaining or amplifying indefinitely the external dipole moment of the magnetic field produced by electric currents in the sphere.

(B) The Equations to be Solved

Although the magnetic dynamo equation giving the effects of a fluid velocity $\boldsymbol{u}$ on a magnetic field $\boldsymbol{B}$ is well known, the complete system of equations for the electromagnetic field in the presence of fluid motion is widely scattered in the literature, and some question has been raised about whether the magnetic dynamo equation alone is all that need be considered (*3*). For completeness, a derivation is given below of the formal procedure for obtaining the whole electromagnetic field once the magnetic dynamo equation has been solved.

Let V be a bounded volume of fluid with surface S outside of which is vacuum. Denote all of three dimensional space by $\mathcal{E}$. Suppose the fluid has finite isotropic ohmic electrical conductivity σ and is incompressible. Suppose it moves with velocity $\boldsymbol{u}(\boldsymbol{y}, t)$ and that the outward normal component $\boldsymbol{n}\cdot\boldsymbol{u}$ vanishes on S so that the fluid always remains inside the fixed volume V. Then Ohm's law is $\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$ inside V, $\boldsymbol{j} = \boldsymbol{0}$ outside V. Neglecting displacement current, Maxwell's equations are $\nabla \times \boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$, $\epsilon_0\nabla\cdot\boldsymbol{E} = \rho$, $\nabla\cdot\boldsymbol{B} = 0$. At first sight these equations, which imply $\nabla\cdot\boldsymbol{j} = 0$, appear to contradict the continuity equation $\partial\rho/\partial t + \nabla\cdot\boldsymbol{j} = 0$. However, the term $\partial\rho/\partial t$ is of order u/c times the term $\nabla\cdot\boldsymbol{j}$, where c is the velocity of light. If u/c is small enough to justify the neglect of the displacement current it is small enough to justify the neglect of $\partial\rho/\partial t$ in the continuity equation. As Bullard and Gellman (*9*) point out, the fact that ρ can be neglected in the continuity equation does not mean that its effect on $\boldsymbol{E}$ can be neglected. The usual argument that inside a metallic conductor ρ dies very rapidly [like $e^{-(\sigma t/\epsilon_0)}$] to zero fails here because of the extra term $\boldsymbol{u} \times \boldsymbol{B}$ in Ohm's law. Volume and surface charges will accumulate during the motion and will influence $\boldsymbol{E}$. But Elsasser (*10*) has shown that the extra current produced by the motion of these charges is small of order u/c compared with the current computed from Ohm's law and the Maxwell equations deprived of the displacement current.

The boundary conditions on the electromagnetic field are that $\boldsymbol{B}$ be continuous across S (no surface current), that $\boldsymbol{n} \times \boldsymbol{E}$ be continuous across S, that $\boldsymbol{n}\cdot\boldsymbol{j} = 0$ on S (with an error of order u/c), and that $r^3\boldsymbol{B}$ and $r^2\boldsymbol{E}$ be bounded at infinity. If the conductivity σ is infinite, then surface currents must be allowed, and only $\boldsymbol{n}\cdot\boldsymbol{B}$ need be continuous across S, while $\boldsymbol{n}\cdot\boldsymbol{j}$ need not vanish there since currents flowing into S from V can flow away as surface currents. If σ is finite, so that

$\boldsymbol{B}$ is continuous across S, then $\boldsymbol{n}\cdot\nabla \times \boldsymbol{B}$ is continuous across S, and, since it vanishes just outside S, it does so just inside S as well; therefore the condition $\boldsymbol{n}\cdot\boldsymbol{j} = 0$ on S is a consequence of the boundary conditions on $\boldsymbol{B}$ and can be omitted from the statement of the problem.

Eliminating $\boldsymbol{E}$ and $\boldsymbol{j}$ from Maxwell's equations and Ohm's law, one obtains the so-called dynamo equations (*9*):

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \frac{1}{\mu_0\sigma}\nabla \times \Delta \times \boldsymbol{B} \qquad \text{in } V; \tag{1a}$$

$$\mathit{0} = \nabla \times \boldsymbol{B} \qquad \text{in } \mathcal{E} - V; \tag{1b}$$

$$\nabla\cdot\boldsymbol{B} = \mathit{0} \qquad \text{in } \mathcal{E}; \tag{1c}$$

$$r^3\boldsymbol{B} \text{ is bounded in } \mathcal{E}; \tag{1d}$$

$$\boldsymbol{B} \text{ is continuous across } S \text{ if } \sigma < \infty; \tag{1e}$$

$$\boldsymbol{n}\cdot\boldsymbol{B} \text{ is continuous across } S \text{ if } \sigma = \infty. \tag{1f}$$

It is now necessary to show that if a solution $\boldsymbol{B}$ of Eqs. (1) has been obtained for some prescribed velocity field $\boldsymbol{u}$ then that solution generates a unique solution of the whole system of Maxwell's equations.

Since $\boldsymbol{B}$ is known everywhere, a unique vector potential $\boldsymbol{A}$ can be found such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, $\nabla\cdot\boldsymbol{A} = 0$, $r^2\boldsymbol{A}$ is bounded, and $\boldsymbol{A}$ is continuous. The equation

$$\boldsymbol{E} = \sigma^{-1}\boldsymbol{j} - \boldsymbol{u} \times \boldsymbol{B} \tag{2}$$

uniquely specifies $\boldsymbol{E}$ in V and in consequence of the first of the dynamo equations (1) there is a scalar ϕ such that in V

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi. \tag{3}$$

In $\mathcal{E} - V$ let ϕ be defined by the demands that $r^2\phi$ be bounded, $\nabla^2\phi = 0$, and that ϕ be continuous across S. If ϕ is any solution of equation (3) in V, $\phi + C$ is another, where C is any constant. Then from equations (2) and (3) in V, $\phi = f(r) + C$ on S, where f is a known function and C is an unspecified constant. If ϕ_0 is the unique harmonic function in $\mathcal{E} - V$ which takes the value $f(r)$ on S and vanishes at infinity, and if ψ is the unique harmonic function in $\mathcal{E} - V$ which takes the value 1 on S and vanishes at infinity, then in $\mathcal{E} - V$, $\phi = \phi_0 + C\psi$. If the constant C were known, Eq. (3) would give $\boldsymbol{E}$ in $\mathcal{E} - V$. This constant can be determined from the total electric charge Q on the body V, since

$$Q = -\epsilon_0 \int_S (\boldsymbol{n}\cdot\nabla\phi)\, dS = -\epsilon_0 \int_S (\boldsymbol{n}\cdot\nabla\phi_0)\, dS - \epsilon_0 C \int_S (\boldsymbol{n}\cdot\nabla\psi)\, dS,$$

and it is easy to show that $\int_S (\boldsymbol{n}\cdot\nabla\psi)\, dS$ cannot vanish.

The value of $\boldsymbol{E}$ being now determined in all of $\mathcal{E}$, the volume charge density ρ can be found from $\rho = \epsilon_0 \nabla \cdot E$ in V and the surface charge density on S is $\epsilon_0 \boldsymbol{n} \cdot (\boldsymbol{E}^+ - \boldsymbol{E}^-)$, the superscripts $+$ and $-$ referring to values, respectively, just outside and inside S.

These considerations indicate that any solution of the dynamo equations (1) generates a consistent solution of the whole set of Maxwell's equations with displacement current neglected, Ohm's law, and the continuity equation with $\partial\rho/\partial t$ neglected, assuming that the total electrostatic charge Q on the fluid volume V is known. Of course Q is a constant of the motion.

The general dynamo problem can now be formulated thus: solve equations (1) in conjunction with the equations of motion and continuity for the fluid. The restricted dynamo problem, the subject of the present paper, ignores the source of the fluid's motion and asks simply for the solution $\boldsymbol{B}(\boldsymbol{y}, t)$ of Eqs. (1) when $\boldsymbol{B}(\boldsymbol{y}, 0)$ and $\boldsymbol{u}(\boldsymbol{y}, t)$ are given. In particular, are there "physically reasonable" fluid motions for which $\boldsymbol{B}(\boldsymbol{y}, t)$ does not decay with time? By a "physically reasonable" motion is meant a velocity field $\boldsymbol{u}$ continuously differentiable at all times at all places in V, for which a bounded positive scalar ρ exists such that $\partial\rho/\partial t + \nabla \cdot \rho \boldsymbol{u} = 0$. This paper will make the more restrictive demand $\nabla \cdot \boldsymbol{u} = 0$, or that the fluid is incompressible. The volume V will be assumed to be a sphere of radius 1.

In the preceding section, the symbol ρ has been used first for the charge density, and then, in the paragraph above, for the matter density. Both these meanings will be dropped, and henceforth ρ will mean $(\mu_0\sigma)^{-1}$, which differs from the fluid's resistivity by the factor μ_0^{-1}, but for brevity will be called the resistivity throughout this paper.

2. PREVIOUS WORK ON THE DYNAMO PROBLEM

The present author has given a short survey of previous attempts to construct self-sustaining dynamos (*11*). This discussion will not be repeated here. However, further remarks are warranted about three dynamo attempts.

First, Cowling (*12*) writes that he is convinced of the existence of self-sustaining dynamos by the numerical computations carried out by Bullard and Gellman (*9*) in an attempt to solve the eigenvalue problem $\nabla \times \nabla \times \boldsymbol{B} = W\nabla \times (\boldsymbol{u} \times \boldsymbol{B})$ for the eigenvalue W and the eigenfunction $\boldsymbol{B}$, given $\boldsymbol{u}$. Such a solution would represent a steady dynamo. The Bullard and Gellman scheme was to expand $\boldsymbol{B}$ as a sum of fields $f_l^m(r)\boldsymbol{B}_l^m(\theta, \phi)$ where the angular part $\boldsymbol{B}_l^m$ is a vector spherical harmonic of the form $\nabla \times rY_l^m$ or $\nabla \times \nabla \times rY_l^m$, Y_l^m being a scalar spherical harmonic of order l. The partial differential equation $\nabla \times \nabla \times \boldsymbol{B} = W\nabla \times (\boldsymbol{u} \times \boldsymbol{B})$ gives rise to an infinite system of coupled ordinary differential equations for the $f_l^m(r)$. Bullard and Gellman approximated this system by the sequence of finite systems obtained by setting all f_l^m equal to zero above a certain value of l. They took $l = 1, 2, 3$, and obtained successive approximations W_l to the true eigenvalue W. Typical values for the W_l they obtained for various velocity fields $\boldsymbol{u}$ are $W_1 = 47.5$, $W_2 = 63.9$, W_3 not computed; $W_1 = 22.06$, W_2 not computed, $W_3 = 67.4$. These sequences are supposed to converge to the true values of W; perhaps what Cowling finds convincing about them is that at least they are real. But as Chandrasekhar (*13*) has pointed out the steady increase of these approximate values of W as l increases and the approximation improves may indicate that in the exact solution an infinite value of W is required, or in other words that the particular velocities $\boldsymbol{u}$ chosen for the calculation cannot maintain a steady dynamo.

Parker's paper (*14*) is an attempt to exploit explicitly the suggestion made and rejected by Elsasser (*10*) and further examined by Bullard (*7*) that the main poloidal magnetic field may be generated from a much larger toroidal field by means of a poloidal fluid motion (poloidal and toroidal are here used in the sense of Elsasser (*3*)). The toroidal field itself would be generated by an axisymmetric toroidal shearing motion in the fluid (*10*). Parker gives a detailed calculation of the effect of a cyclonic vortex motion in an infinite perfectly conducting fluid on a magnetic field which was originally uniform. He tries to show that the resistivity of the fluid can in fact be neglected, but his method, a perturbation calculation in the small parameter ρ, the resistivity, is not adequate to the problem unless a convergence proof can be supplied, probably a troublesome task and one he does not attempt.

A minor difficulty in Parker's work is his failure to fit his velocity fields explicitly into a sphere. This would cause no misgivings were it not well known that singular velocity fields, velocities with point sources for example, can maintain dynamos. The difficulty is minor because Parker's cyclonic vortices can easily be fitted into a sphere.

The principal difficulty the present author sees in Parker's approach is that the real question at issue is the long-term behavior of the magnetic field. Since a successful dynamo cannot be axisymmetric (*15*, *11*), the poloidal flow will generate other fields besides the desired axisymmetric poloidal field, and the toroidal shear flow will transform these in a fashion which may eventually destroy the whole process, and whose understanding constitutes the real difficulty in an attempt to use Elsasser's and Bullard's suggestion. Parker ignores all these stray fields.

Bullard and Gellman (*9*) point out a less specific objection to Parker's attempt to construct a dynamo: in the absence of corroborative experimental evidence, no such qualitative argument can carry conviction on the dynamo question. What is needed is either a proof or a numerical calculation with every appearance of convergence.

Batchelor (*16*) has argued that there is in fact experimental evidence on the dynamo question. He points out that the equations for the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ in a fluid of kinematic viscosity ν moving with velocity $\boldsymbol{u}$ are $\partial\boldsymbol{\omega}/\partial t = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) - \nu\nabla \times \nabla \times \boldsymbol{\omega}$ and $\nabla\cdot\boldsymbol{\omega} = 0$. If $\boldsymbol{\omega}$ is identified with the magnetic field $\boldsymbol{B}$ and $\boldsymbol{u}$ with the vector potential $\boldsymbol{A}$ these equations are identical with the dynamo equations (1). The experimental observation that there are fluid motions in which $\boldsymbol{\omega}$ does not decay is then to be taken to show that velocity fields $\boldsymbol{u}$ exist for which the dynamo equations (1) have a nondecaying solution.

Batchelor advanced this argument only for fluids of infinite extent, and used it to conclude that turbulent generation of magnetic fields was possible. Bullard and Gellman (*9*) tried to extend it to finite fluids. For a fluid of finite extent, however, the analogy between $\boldsymbol{\omega}$ and $\boldsymbol{B}$ and between $\boldsymbol{u}$ and $\boldsymbol{A}$ fails because of differences in the boundary conditions at the surface of the fluid. These differences are presumably irrelevant, as Batchelor has assumed, for times of the order of a few mean lives of a turbulent magnetic disturbance whose spatial extent is much less than that of the whole fluid [although Cowling (Ref. *12*, p. 96) disputes even this], but that the boundary conditions can be ignored for times longer than the slowest magnetic free decay time for the whole fluid is not so clear.

3. A HEURISTIC DESCRIPTION OF A DYNAMO

In the published attempts to show that velocity fields $\boldsymbol{u}$ exist for which the dynamo equations (1) have nondecaying solutions $\boldsymbol{B}$ their authors have usually demanded that $\boldsymbol{u}$ be a velocity which might at least qualitatively resemble the actual motion in the earth's core (*10*, *17*, *9*, *14*). Since none of these attempts was successful, and since the motion of the core is very imperfectly known, it would appear expedient to relax this restriction for the time being. In the present paper any solenoidal velocity $\boldsymbol{u}$ will be admitted which is bounded and continuously differentiable everywhere for all time, and which vanishes on S, the surface of the fluid.

With such a wide class of velocities available it turns out to be possible to carry out in detail Elsasser's (*10*) and Bullard's (*7*) suggestion: using an axisymmetric toroidal shear flow to produce a large axisymmetric toroidal from a small axisymmetric poloidal magnetic field, and then using a poloidal flow to transfer some of the energy of the toroidal field back into the poloidal field. Specifically, suppose that initially the magnetic field has the form $\boldsymbol{P}_1 + \boldsymbol{R}$ where $\boldsymbol{P}_1$ is an axisymmetric poloidal free decay mode with longest mean life in a rigid sphere (*3*). The field $\boldsymbol{P}_1$ is taken to have unit energy, and the energy in the remaining field $\boldsymbol{R}$ is much less than 1. A rapid axisymmetric shear flow with $\boldsymbol{P}_1$'s axis of symmetry will produce from $\boldsymbol{P}_1$ a very large axisymmetric toroidal field $\boldsymbol{T}_1$, along with some unwanted fields produced from $\boldsymbol{R}$. It will be shown that by stopping the fluid motion after the shear has been completed, these unwanted fields can be made to decay to a much smaller energy than that of $\boldsymbol{T}_1$, either because they have shorter free decay times or because they were not produced in such large amounts as $\boldsymbol{T}_1$. What remains is a still large and almost pure axisymmetric toroidal field $\boldsymbol{T}_1$. A nonaxisymmetric flow applied for a short time will transfer some of the energy of this $\boldsymbol{T}_1$ back into $\boldsymbol{P}_1$. If the fluid motion is stopped again for a time $\boldsymbol{P}_1$ will decay more slowly than any other fields present, and eventually the field will be $\alpha(\boldsymbol{P}_1 + \boldsymbol{R}')$ where the energy of $\boldsymbol{R}'$ is no greater than that of the original stray field $\boldsymbol{R}$ and the constant α can be made arbitrarily large by using a sufficiently rapid and protracted shear flow at the stage of the motion where $\boldsymbol{T}_1$ is produced from $\boldsymbol{P}_1$. If all these assertions can be proved, then it is clear that repetition of the motion described above will indefinitely maintain or amplify the external dipole field, since $\boldsymbol{P}_1$ is a pure dipole field in the vacuum outside the fluid.

When the argument is presented in detail it will be clear that the axisymmetric toroidal field $\boldsymbol{T}_1$ need not be regarded as contamination; the argument will work even if the second rigid decay time is so short that a large $\boldsymbol{T}_1$ is always present throughout the whole motion, as long as this second decay time is long enough to remove all the stray fields except $\boldsymbol{T}_1$ and $\boldsymbol{P}_1$.

The particular motion used in this dynamo is clearly indefensible as a reasonable imitation of the actual motion in the earth's core. However, the methods used to prove that this motion does maintain $\boldsymbol{B}$ are of sufficient generality that the author believes they can be applied to any fluid motion, and he expects to return to this problem in a future publication. The simple dynamo presented here will be useful primarily because of the clarity with which it represents at least one physical mechanism for maintaining an external magnetic field by means of fluid motions.

4. THE REPRESENTATION OF SOLENOIDAL FIELDS

Previous authors (*3*, *9*) have represented solenoidal fields as infinite series of products of radial function and vector spherical harmonics $\nabla \times \boldsymbol{r}Y_l^m$ and $\nabla \times \nabla \times \boldsymbol{r}Y_l^m$, where Y_l^m is a scalar spherical harmonic. Elsasser (*3*) has already observed that every vector field of the form $-\nabla \times \nabla \times \boldsymbol{r}p - \nabla \times \boldsymbol{r}q$, where p and q are any scalars, is solenoidal. In this section, the converse will be proved. It will be shown that if $\nabla \cdot \boldsymbol{B} = 0$ in $\mathcal{E}$, then for every choice of origin there exist unique scalars p and q such that $\boldsymbol{B} = -\nabla \times (\nabla \times \boldsymbol{r}p) - \nabla \times \boldsymbol{r}q$ while p and q average to zero on every spherical surface concentric with the origin.

Choose a fixed origin in $\mathcal{E}$ and let $\boldsymbol{r}$ denote the position vector in $\mathcal{E}$ while $\hat{\boldsymbol{r}}$ is the unit vector in the direction of $\boldsymbol{r}$. Let $\boldsymbol{\Lambda}$ denote the operator $\boldsymbol{r} \times \nabla$. Then $-i\boldsymbol{\Lambda}$ is the usual quantum mechanical angular momentum operator. The following properties of $\mathbf{\Lambda}$ follow easily from its representation in Cartesian coordinates:

$$\boldsymbol{\nabla}^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\Lambda^2; \tag{4a}$$

$$\mathbf{\Lambda}\Lambda^2 = \Lambda^2\mathbf{\Lambda}; \qquad \mathbf{\Lambda}\nabla^2 = \nabla^2\mathbf{\Lambda}; \qquad \nabla^2\Lambda^2 = \Lambda^2\nabla^2; \tag{4b}$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{\Lambda} = \mathbf{\Lambda}\cdot\boldsymbol{\nabla} = \boldsymbol{r}\cdot\mathbf{\Lambda} = \mathbf{\Lambda}\cdot(\boldsymbol{\nabla}\times\mathbf{\Lambda}) = (\boldsymbol{\nabla}\times\mathbf{\Lambda})\cdot\mathbf{\Lambda} = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{\Lambda})\\ = (\mathbf{\Lambda}\times\boldsymbol{\nabla})\cdot\mathbf{\Lambda} = 0;\end{aligned} \tag{4c}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{\Lambda} = -\mathbf{\Lambda}\nabla^2. \tag{4d}$$

If r, θ, ϕ are the radius, polar angle, and azimuthal angle in a system of spherical polar coordinates whose origin is that already chosen, and if $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ denote unit vectors in the local directions of increase of r, θ, and ϕ, then

$$\mathbf{\Lambda} = -\hat{\boldsymbol{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}; \tag{5a}$$

$$\boldsymbol{\nabla}\times\mathbf{\Lambda} = \hat{\boldsymbol{R}}\frac{1}{r}\Lambda^2 - \hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta}\frac{1}{r}\frac{\partial}{\partial r}r - \hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\frac{1}{r}\frac{\partial}{\partial r}r; \tag{5b}$$

$$\Lambda^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin^2\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}; \tag{5c}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin^2\theta\Lambda^2 = \Lambda^2\sin\theta\frac{\partial}{\partial\theta}; \tag{5d}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2A_r + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta A_\theta + \frac{\partial A_\phi}{\partial\phi}\right); \tag{5e}$$

$$\mathbf{\Lambda}\cdot\boldsymbol{A} = \boldsymbol{r}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\phi - \frac{1}{\sin\theta}\frac{\partial A_\theta}{\partial\phi}; \tag{5f}$$

$$\mathbf{\Lambda}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} = -\frac{1}{r}\Lambda^2A_r + \frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\theta + \frac{1}{\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right); \tag{5g}$$

here $\boldsymbol{A} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$ is an arbitrary vector field. Let Y_l^m be a normalized spherical harmonic,

$$Y_l^m(\theta,\phi) = (-1)^m\left(\frac{2l+1}{4\pi}\right)^{1/2}\left(\frac{(l-m)!}{(l+m)!}\right)^{1/2}P_l^m(\cos\theta)e^{im\phi}, \tag{6a}$$

where P_l^m is an associated Legendre function,

$$\begin{aligned}P_l^m(x) &= \frac{(1-x^2)^{m/2}}{2^l l!}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \quad \text{if} \quad l \geq 0, |m| \leq l;\\ &= 0 \quad \text{if} \quad l < 0 \quad \text{or} \quad m < -l.\end{aligned} \tag{6b}$$

Then, as is well known from the theory of Laplace's equation,

$$\Lambda^2 Y_l^m = -l(l+1) Y_l^m. \tag{7}$$

If S_r denotes the spherical surface of radius r concentric with the origin, and if $\boldsymbol{A}$ is any vector field defined on S_r, f any scalar field defined on S_r, then

$$\int_{S_r} (\boldsymbol{A} \cdot \boldsymbol{\Lambda} f)\, dS = -\int_{S_r} f \boldsymbol{\Lambda} \cdot \boldsymbol{A}\, dS. \tag{8}$$

The representation of the arbitrary solenoidal field $\boldsymbol{B}$ which is to be obtained in the present section can now be written

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{\Lambda} p + \boldsymbol{\Lambda} q. \tag{9}$$

If such a representation is possible for a given field $\boldsymbol{B}$, Eqs. (4c) and (5b) show that the scalars p and q satisfy, respectively,

$$\Lambda^2 p = \boldsymbol{r} \cdot \boldsymbol{B}; \tag{10a}$$

$$\Lambda^2 q = \boldsymbol{\Lambda} \cdot \boldsymbol{B} = \boldsymbol{r} \cdot \boldsymbol{\nabla} \times \boldsymbol{B}. \tag{10b}$$

To find these scalars, it will be necessary to invert the operator Λ^2, that is, to find f when g is known and f satisfies

$$\Lambda^2 f = g. \tag{11}$$

Since Λ^2 is independent of r there is no loss of generality in assuming f and g to be defined on S_1, the surface of the unit sphere concentric with the origin. An arbitrary point on this surface will be denoted by $\boldsymbol{\omega}$. It is a vector of length 1 and is determined by θ and ϕ. Elements of area on S_1 will be denoted by $d^2\omega$.

Equation (8) shows that if Eq. (11) is to have a solution f then g must satisfy

$$\int_{S_r} g\, d^2\omega = 0.$$

If $\mathcal{H}_1$ is the Hilbert space of square integrable functions on S_1, with inner product

$$(g_1, g_2) = \int_{S_r} g_1^* g_2\, d^2\omega,$$

the asterisk denoting complex conjugation, then the functions Y_l^m of Eq. (6a) form a complete orthonormal set in $\mathcal{H}_1$. The set $\mathcal{G}_1$ of functions in $\mathcal{H}_1$ orthogonal to Y_0^0 is a closed linear subspace of $\mathcal{H}_1$, and, as has just been remarked, a necessary condition for the solubility of Eq. (11) is that g lie in this subspace $\mathcal{G}_1$. Then, always assuming that g is square integrable on S_1, it can be written in the form

$$g(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} g_l^n Y_l^m(\theta, \phi), \tag{12}$$

the series being convergent in the mean square. By Eq. (7), there is a unique f in $\mathcal{G}_1$ satisfying Eq. (11), and this f is

$$f(\theta, \phi) = -\sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{g_l^m}{l(l+1)} Y_l^m(\theta, \phi), \tag{13}$$

a series which is also convergent in the mean square. To be quite precise, an operator Λ^{-2} can be defined on the linear space $\mathcal{G}_1$: if g is given by Eq. (12), then $\Lambda^{-2}g$ is defined to be the f of Eq. (13). This operator Λ^{-2} is linear, and if g is sufficiently smooth, $\Lambda^2\Lambda^{-2}g = g$ and $\Lambda^{-2}\Lambda^2 g = g$.

Although the above argument shows how Eq. (11) can be solved in principle, it is a somewhat clumsy way of investigating the smoothness of f. Fortunately, the generalized Green's function for Eq. (11) can easily be determined [see, for example, Courant and Hilbert (*18*), pp. 327–328], and gives the following explicit formula for f in terms of g:

$$f(\boldsymbol{\omega}) = \frac{1}{4\pi} \int_{S_1} g(\boldsymbol{\omega}') \ln(1 - \boldsymbol{\omega} \cdot \boldsymbol{\omega}')\, d^2\omega'. \tag{14}$$

If ω is fixed, then

$$\int_{S_1} \ln^2(1 - \boldsymbol{\omega} \cdot \boldsymbol{\omega}')\, d^2\omega' = 4\pi\kappa^2$$

where $\kappa = [(\ln 2)^2 - 2 \ln 2 + 2]^{1/2} = 1.04603 \cdots$. An application of Schwarz's inequality to Eq. (14) then gives

$$| f(\boldsymbol{\omega}) | \leq \kappa \left[\frac{1}{4\pi} \int_{S_1} | g(\boldsymbol{\omega}') |^2\, d^2\omega' \right]^{1/2}. \tag{15}$$

In particular, if g is bounded on S_1, then

$$| f(\boldsymbol{\omega}) | \leq \kappa \sup \{ | g(\boldsymbol{\omega}') | : \boldsymbol{\omega}' \text{ on } S_1 \}. \tag{16}$$

Suppose now that $g(r, \theta, \phi)$ is defined in all of space $\mathcal{E}$ and that on each spherical surface S_r concentric with the origin g averages to zero. Then, for each fixed r, g regarded as a function of θ and ϕ is a member of $\mathcal{G}_1$, and consequently a function $f(r, \theta, \phi)$ can be found satisfying Eq. (11) and given explicitly in terms of g by Eq. (14). This function $f(r, \theta, \phi)$ is defined for every r and hence in all of space; it averages to zero on every S_r. Equation (14) immediately implies that if g is continuous (continuously differentiable) inside any S_r then except possibly at the single point $r = 0$ the same is true of f. The exceptional point $r = 0$ must be examined separately and in some detail, since the smoothness of the solutions of Eq. (11) is critical in later arguments.

First, if g is continuous at $r = 0$, since it averages to zero on each S_r, it must actually vanish at $r = 0$. Then inequality (16) implies that $\lim_{r\to 0} f(r, \theta, \phi) = 0$, while Eq. (14) gives $f(0, \theta, \phi) = 0$. Thus f is continuous at $r = 0$.

Second, if when expressed in rectangular coordinates x, y, z, g is differentiable at $r = 0$, then $g = \alpha x + \beta y + \gamma z + h(r, \boldsymbol{\omega})$ where $r^{-1}h(r, \boldsymbol{\omega})$ approaches zero uniformly in $\boldsymbol{\omega}$ as r approaches zero. Then $f = \Lambda^{-2}g = -\frac{1}{2}(\alpha x + \beta y + \gamma z) + \Lambda^{-2}h$, and $r^{-1}\Lambda^{-2}h = \Lambda^{-2}r^{-1}h$ approaches zero uniformly in $\boldsymbol{\omega}$ as r approaches zero, again in consequence of inequality (16). Thus f is differentiable at $r = 0$.

Finally, suppose g is continuously differentiable at $r = 0$. Differentiation of Eq. (14) gives

$$\sin\theta\,\frac{\partial f}{\partial\theta} = \frac{1}{4\pi}\int_{S_1} d^2\omega'\left(\sin\theta'\,\frac{\partial g'}{\partial\theta'} - 2\cos\theta' g'\right)\ln\left(\frac{1 - \boldsymbol{\omega}\cdot\boldsymbol{\omega}'}{\sin\theta\sin\theta'}\right),$$

$$\frac{\partial f}{\partial\phi} = \frac{1}{4\pi}\int_{S_1} d^2\omega'\,\frac{\partial g'}{\partial\phi'}\ln\left(1 - \boldsymbol{\omega}\cdot\boldsymbol{\omega}'\right),$$

$$\frac{\partial f}{\partial r} = \frac{1}{4\pi}\int_{S_1} d^2\omega'\,\frac{\partial g'}{\partial r}\ln\left(1 - \boldsymbol{\omega}\cdot\boldsymbol{\omega}'\right),$$

where g' means $g(r, \theta', \phi')$. From these facts there must be a constant M such that for all sufficiently small r

$$\left|\frac{\partial f}{\partial r}\right| + \left|\frac{\partial f}{\partial\phi}\right| + \left|\sin\theta\,\frac{\partial f}{\partial\theta}\right| \leq M\left(\sup|g| + \sup|\nabla g|\right).$$

Since this inequality is true at all points in all coordinate systems with the same origin, $\boldsymbol{\omega}$ can be fixed on S_1 and a coordinate system chosen in which this fixed $\boldsymbol{\omega}$ has polar angle $\theta = \pi/2$. Therefore, at the given $\boldsymbol{\omega}$, which is an arbitrary point on S_1,

$$|\nabla f| \leq M(\sup|g| + \sup|\nabla g|). \tag{17}$$

If g is continuously differentiable at $r = 0$, $g = \alpha x + \beta y + \gamma z + h(r, \boldsymbol{\omega})$, where $r^{-1}h(r, \boldsymbol{\omega})$ and $\nabla h(r, \boldsymbol{\omega})$ approach zero uniformly in $\boldsymbol{\omega}$ as r approaches zero. Then $f = \Lambda^{-2}g = -\frac{1}{2}(\alpha x + \beta y + \gamma z) + \Lambda^{-2}h$ and by inequality (16) $r^{-1}\Lambda^{-2}h$ approaches zero uniformly in $\boldsymbol{\omega}$ as r approaches zero, while by inequality (17), $\nabla\Lambda^{-2}h$ does likewise. Therefore f is continuously differentiable at $r = 0$ if g is so.

The method just developed for solving Eq. (11) can now be applied to Eqs. (10a) and (10b). Given an arbitrary solenoidal field $\boldsymbol{B}$, unique scalars p and q can always be found to satisfy Eqs. (10a) and (10b) and average to zero on every S_r. Furthermore, inside any S_r, p will be at least as smooth as $\boldsymbol{r}\cdot\boldsymbol{B}$ and q at least as smooth as $\boldsymbol{r}\cdot\nabla\times\boldsymbol{B}$. There remains the question whether $\boldsymbol{B}$ is given in terms of p and q by Eq. (9). The following theorem settles this question:

Theorem 1: If a vector field $\boldsymbol{A}$ is defined on every S_r in some range $r_0 < r < r_1$ and in that range $A_r = 0$ while $A_\theta(r, \theta, \phi)$ and $A_\phi(r, \theta, \phi)$ are bounded for each fixed r and are continuously differentiable except possibly at $\theta = 0$ and $\theta = \pi$, and if further $\boldsymbol{\Lambda}\cdot\boldsymbol{A} = \nabla\cdot\boldsymbol{A} = 0$, then $\boldsymbol{A} = \boldsymbol{o}$.

The proof of this theorem is straightforward. Let $\xi = -\ln(\csc\theta + \cot\theta)$ so that the mapping $(\theta, \phi) \to (\xi, \phi)$ is the Mercator projection of the surface of the sphere S_1 onto a plane. Then $\sin\theta(\partial/\partial\theta) = \partial/\partial\xi$, so since $A_r = 0$, $\nabla \cdot \boldsymbol{A} = 0$ is equivalent to $\partial(\sin\theta A_\theta)/\partial\xi + \partial(\sin\theta A_\phi)/\partial\phi = 0$, while $\boldsymbol{\Lambda} \cdot \boldsymbol{A} = 0$ is equivalent to $\partial(\sin\theta A_\phi)/\partial\xi - \partial(\sin\theta A_\theta)/\partial\phi = 0$. In the plane of the complex variable $z = \xi + i\phi$ these are the Cauchy-Riemann equations for the function $f(z) = \sin\theta(A_\phi + iA_\theta)$, which must therefore be an entire function of z. Since f is bounded, by the Liouville theorem it is constant, and since as $\xi \to +\infty$, $f \to 0$, that constant must be zero. Hence $A_\phi = A_\theta = 0$.

Applying theorem 1 to the vector field $\boldsymbol{A} = \boldsymbol{B} - \nabla \times \boldsymbol{\Lambda} p - \boldsymbol{\Lambda} q$, if the scalars p and q are defined by Eqs. (10a) and (10b), Eq. (9) follows immediately. Following Elsasser (*3*) we call a field $\boldsymbol{T}$ toroidal if it has the form $\boldsymbol{T} = \boldsymbol{\Lambda} q$ and a field $\boldsymbol{P}$ poloidal if it has the form $\boldsymbol{P} = \nabla \times \boldsymbol{\Lambda} p$. The theory of Λ^{-2} shows that if q and p are required to average to zero on every S_r they are uniquely determined by their fields $\boldsymbol{T}$ and $\boldsymbol{P}$. Theorem 1 and Eq. (10b) show that a field $\boldsymbol{T}$ is toroidal if and only if $\nabla \cdot \boldsymbol{T} = \hat{\boldsymbol{r}} \cdot \boldsymbol{T} = 0$, while theorem 1 and Eq. (10a) show that a field $\boldsymbol{P}$ is poloidal if and only if $\nabla \cdot \boldsymbol{P} = \boldsymbol{\Lambda} \cdot \boldsymbol{P} = 0$. The representation (9) can be summarized by saying that every solenoidal field is uniquely expressible as the sum of a poloidal and a toroidal field.

5. THE SPACE $\mathcal{B}$ OF REALIZABLE MAGNETIC FIELDS

(A) Fluid Motions as Linear Operators

Suppose that the Lagrangian description of a certain fluid motion is given: that is, the position $\boldsymbol{y}(\boldsymbol{x}, t)$ at time t of the fluid element which was at position $\boldsymbol{x}$ at time zero is given for all $\boldsymbol{x}$ in V and all t in some finite interval $0 \leq t \leq t_0$. If the resistivity ρ of the fluid is zero and the initial magnetic field $\boldsymbol{B}(\boldsymbol{x}, 0)$ in the fluid is given, the final field $\boldsymbol{B}(\boldsymbol{y}, t_0)$ produced by the fluid motion is completely determined by the function $\boldsymbol{y}(\boldsymbol{x}, t_0)$ and is independent of $\boldsymbol{y}(\boldsymbol{x}, t)$ for $0 < t < t_0$ (*19*). If ρ differs from zero, $\boldsymbol{B}(\boldsymbol{y}, t)$ depends on the whole fluid motion. A useful way of visualizing this situation is as follows: let $\mathfrak{D}$ denote the space of all continuously differentiable volume-preserving transformations $\boldsymbol{y}(\boldsymbol{x})$ of the region V onto itself [the fluid point $\boldsymbol{x}$ is moved to the point $\boldsymbol{y}(\boldsymbol{x})$]. Then the fluid motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq t_0$, is a continuous path in $\mathfrak{D}$ whose endpoints are the transformations $\boldsymbol{y}(\boldsymbol{x}, 0) = \boldsymbol{x}$ and $\boldsymbol{y}(\boldsymbol{x}, t_0)$. In a fluid of zero resistivity ρ the effect of such a motion on magnetic fields depends only on the endpoints of the path in $\mathfrak{D}$; if ρ is positive, that effect depends on the whole path.

If $\boldsymbol{B}(\boldsymbol{x}, 0)$ is an initial magnetic field, the final magnetic field $\boldsymbol{B}(\boldsymbol{y}, t_0)$ produced from it by the fluid motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq t_0$, is obtained by solving the dynamo equations (1) for $\boldsymbol{B}(\boldsymbol{y}, t)$ using

$$\boldsymbol{u}(\boldsymbol{y}, t) = \frac{\partial \boldsymbol{y}(\boldsymbol{x}, t)}{\partial t} \tag{18}$$

as the velocity in those equations. The spatial differential operators in the dynamo equations refer to $\boldsymbol{y}$, the instantaneous position of a fluid element, rather than to $\boldsymbol{x}$, its initial position. Since the final field $\boldsymbol{B}(\boldsymbol{y}, t_0)$ depends only on the initial field $\boldsymbol{B}(\boldsymbol{x}, 0)$, the resistivity ρ of the fluid, and the motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq t_0$, that fluid motion can be regarded as an operator $\mathfrak{M}_\rho$ which transforms the initial field into the final one. This operator is defined by the equation

$$\mathfrak{M}_\rho \boldsymbol{B}(0) = \boldsymbol{B}(t_0). \tag{19}$$

Since the dynamo equations are linear in $\boldsymbol{B}$ when the motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq t_0$, is given a priori, the operator $\mathfrak{M}_\rho$ corresponding to that motion is linear. Regarding the motion as a path in $\mathfrak{D}$, $\mathfrak{M}_0$ depends only on the endpoints while $\mathfrak{M}_\rho$ depends on the whole path.

Closer attention must now be given to the space on which $\mathfrak{M}_\rho$ operates. This space will be denoted by $\mathfrak{B}$ and will consist of all magnetic fields which are allowable initial fields for the dynamo equations (1). A field $\boldsymbol{B}(\boldsymbol{x})$ will be in the space $\mathfrak{B}$ if it satisfies all the following conditions:

$$r^3\boldsymbol{B} \quad \text{is bounded in } \mathcal{E}; \tag{20a}$$

$\boldsymbol{B}$ is continuous in $\mathcal{E}$ and continuously differentiable in

$$\mathcal{E} - V \text{ and } V \text{ separately}; \tag{20b}$$

$$\nabla \times \boldsymbol{B} = 0 \quad \text{in } \mathcal{E} - V; \tag{20c}$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{in } \mathcal{E}. \tag{20d}$$

If $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are any two such fields in $\mathfrak{B}$, an inner product

$$(\boldsymbol{B}_1, \boldsymbol{B}_2) = \int_{\mathcal{E}} \boldsymbol{B}_1^* \cdot \boldsymbol{B}_2 \tag{21}$$

can be defined since the integral is finite. The asterisk denotes complex conjugation, it being expedient to admit complex-valued $\boldsymbol{B}$'s. In terms of this inner product, the usual norm may be defined:

$$\| \boldsymbol{B} \| = (\boldsymbol{B}, \boldsymbol{B})^{1/2}. \tag{22}$$

For obvious reasons, $\| \boldsymbol{B} \|^2$ will be called the "energy" of the field $\boldsymbol{B}$, even though it differs from the usual energy by a factor $2\mu_0$. This norm has the expected properties of a length:

$$\| \boldsymbol{B} \| \geq 0 \quad \text{and} \quad \| \boldsymbol{B} \| = 0 \quad \text{if and only if} \quad \boldsymbol{B} = \boldsymbol{0}; \tag{23a}$$

$$\| \alpha\boldsymbol{B} \| = | \alpha | \, \| \boldsymbol{B} \| \quad \text{for any complex scalar } \alpha; \tag{23b}$$

$$\| \boldsymbol{B}_1 + \boldsymbol{B}_2 \| \leq \| \boldsymbol{B}_1 \| + \| \boldsymbol{B}_2 \| \quad \text{(the triangle inequality)}. \tag{23c}$$

Finally, the inner product and norm are related by the usual Schwarz inequality (Ref. *18*, p. 2):

$$| (\boldsymbol{B}_1, \boldsymbol{B}_2) | \leq \| \boldsymbol{B}_1 \| \| \boldsymbol{B}_2 \|. \tag{23d}$$

By means of the inner product (21) the space $\mathcal{B}$ can be completed to a Hilbert space, a fact which will be used only to invoke much of the elementary terminology of Hilbert space theory (*20*). In particular, throughout the present paper two fields $\boldsymbol{B}_1(\boldsymbol{x})$ and $\boldsymbol{B}_2(\boldsymbol{x})$ will be called orthogonal when $(\boldsymbol{B}_1, \boldsymbol{B}_2) = 0$, rather than when $\boldsymbol{B}_1(\boldsymbol{x}) \cdot \boldsymbol{B}_2(\boldsymbol{x}) = 0$ at every point $\boldsymbol{x}$ of space.

(B) The Three Spaces Used in the Present Paper

To avoid confusion it is necessary to list the three different spaces of functions which will be used in what follows, and to make clear the relations among them. First there is the space $\mathcal{G}_1$ defined in Section 4, consisting of all square-integrable scalar functions $g(\omega)$ defined and averaging to zero on the surface of the unit sphere S_1. Second there is the space $\mathcal{G}$ of all scalar functions $g(\boldsymbol{r})$ defined and square integrable in the interior V of S_1 and averaging to zero on every S_r for which $0 < r \leq 1$. Finally there is the space $\mathcal{B}$ of vector functions defined in Section 5a. In each of these spaces an inner product is defined:

$$(g_1, g_2)_1 = \int_{S_1} g_1^* g_2 \, d^2\omega \quad \text{if} \quad g_1 \text{ and } g_2 \text{ are in } \mathcal{G}_1;$$

$$(g_1, g_2) = \int_V g_1^* g_2 \, d^3r \quad \text{if} \quad g_1 \text{ and } g_2 \text{ are in } \mathcal{G};$$

$$(\boldsymbol{B}_1, \boldsymbol{B}_2) = \int_{\mathcal{E}} \boldsymbol{B}_1^* \cdot \boldsymbol{B}_2 \, d^3r \quad \text{if} \quad \boldsymbol{B}_1 \text{ and } \boldsymbol{B}_2 \text{ are in } \mathcal{B}.$$

The norms $\| g \| (g, g)^{1/2}$ can be defined in $\mathcal{G}_1$ and $\mathcal{G}$. In terms of these inner products and norms, $\mathcal{G}_1$ and $\mathcal{G}$ are Hilbert spaces while $\mathcal{B}$ can be completed to a Hilbert space. Equations (23) apply to all three spaces.

Two elementary concepts from quantum mechanics or Hilbert space theory will be essential in what follows, namely the bound of a linear operator and decompositions of a space into orthogonal subspaces by means of the orthogonal projections onto those spaces. These ideas apply to any Hilbert space, and in particular to $\mathcal{G}_1$, $\mathcal{G}$, and $\mathcal{B}$. Since only the definitions are required, these are stated in a short space below for readers unacquainted with them.

(C) Bounds of Linear Operators

Let $\mathcal{H}$ be any linear space with complex scalars on which a norm $\| h \|$ is defined having the three properties (23a), (23b), (23c). Let $\mathcal{M}$ be any linear operator on $\mathcal{H}$. The "bound" of this operator is conventionally defined as the smallest positive number m such that for every vector h in $\mathcal{H}$.

$$\| \mathfrak{M}h \| \leq m \| h \|.$$

The number m is usually denoted by $\| \mathfrak{M} \|$; clearly it is the least upper bound of the values attained by $\| \mathfrak{M}h \|$ for any vector h such that $\| h \| = 1$. If $\| \mathfrak{M} \|$ is finite, $\mathfrak{M}$ is called a "bounded" linear operator.

As an immediate and well-known consequence of this definition, Eqs. (23a), (23b), (23c) are true if the vectors in those equations are replaced by operators. Furthermore, for every h in $\mathcal{H}$

$$\| \mathfrak{M}h \| \leq \| \mathfrak{M} \| \, \| h \|. \tag{24}$$

Finally, if $\mathfrak{M}$ and $\mathfrak{N}$ are both linear operators on $\mathcal{H}$ and $\mathfrak{M}\mathfrak{N}$ is their operator product, the operator obtained by applying first $\mathfrak{N}$ and then $\mathfrak{M}$, it is another well-known and easily derived consequence of the definition of the bound of an operator that

$$\| \mathfrak{M}\mathfrak{N} \| \leq \| \mathfrak{M} \| \, \| \mathfrak{N} \|. \tag{25}$$

There is a useful relation between the operators on the spaces $\mathcal{G}_1$ and $\mathcal{G}$ defined in Section 5b. If $\mathfrak{M}$ is a bounded linear operator on $\mathcal{G}_1$ it may be regarded as an operator on $\mathcal{G}$ in the following sense: if $g(r, \theta, \phi)$ is any function in $\mathcal{G}$, then for almost every fixed r it is in $\mathcal{G}_1$ as a function of θ and ϕ. Then for every such fixed r the function $f(r, \theta, \phi) = \mathfrak{M}g(r, \theta, \phi)$ is well defined and in $\mathcal{G}_1$ as a function of θ and ϕ. Then

$$\begin{aligned} \| f \|^2 &= \int_V | f |^2 = \int_0^1 r^2 \, dr \int_{S_1} d^2\omega \, | \mathfrak{M}g(r, \omega) |^2 = \int_0^1 r^2 \, dr \, \| \mathfrak{M}g \|^2 \\ &\leq \int_0^1 r^2 \, dr \, \| \mathfrak{M} \|_1^2 \, \| g(r, \omega) \|_1^2 = \| \mathfrak{M} \|_1^2 \int_0^1 r^2 \, dr \int_{S_1} d^2\omega \, | g |^2 \\ &= \| \mathfrak{M} \|_1^2 \int_V | g |^2. \end{aligned}$$

Thus $\| f \| = \| \mathfrak{M}g \| \leq \| \mathfrak{M} \|_1 \| g \|$. Therefore f is in $\mathcal{G}$ and $\mathfrak{M}$ is a bounded linear operator on $\mathcal{G}$ whose bound $\| \mathfrak{M} \|$ is no greater than its bound $\| \mathfrak{M} \|_1$ on $\mathcal{G}_1$. As a matter of fact, it is not difficult to construct examples to show that

$$\| \mathfrak{M} \| = \| \mathfrak{M} \|_1 . \tag{26}$$

(D) Orthogonal Subspaces and Orthogonal Projection Operators

If $\mathcal{H}$ is any complex linear space on which is defined a positive-definite complex-valued inner product (h_1 , h_2) which is linear in h_2 and satisfies $(h_1 , h_2) =$

$(h_2, h_1)^*$, then two vectors h_1 and h_2 in $\mathcal{H}$ are called orthogonal if their inner product vanishes. Two linear subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}$ are called orthogonal if any vector from the first is orthogonal to every vector in the second. If $\mathcal{H}_1$, $\mathcal{H}_2$, $\cdots$ is a sequence of mutually orthogonal subspaces of $\mathcal{H}$ such that every vector h in $\mathcal{H}$ can be written in the form $h = h_1 + h_2 + \cdots$ where h_n is in $\mathcal{H}_n$, the series being convergent in the norm $\| h \| = (h, h)^{1/2}$, then $\mathcal{H}$ is called the "direct sum" of $\mathcal{H}_1$, $\mathcal{H}_2$, $\cdots$, and is written $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$. The orthogonality of the spaces $\mathcal{H}_n$ implies that the vectors h_n are unique. The mapping $\mathcal{Q}_n$ of $\mathcal{H}$ onto $\mathcal{H}_n$ which sends the vector h into the vector h_n is called the orthogonal projection operator of $\mathcal{H}$ onto $\mathcal{H}_n$. Since clearly $\| h_n \| \leq \| h \|$, $\mathcal{Q}_n$ is a bounded linear operator and $\| \mathcal{Q}_n \| \leq 1$. Since $\mathcal{Q}_n h_n = h_n$,

$$\| \mathcal{Q}_n \| = 1. \tag{27}$$

The fact that every h has the form $h = h_1 + h_2 + \cdots$ can be expressed by the equation

$$I = \mathcal{Q}_1 + \mathcal{Q}_2 + \cdots \tag{28}$$

where I is the identity operator on $\mathcal{H}$.

6. IMMEDIATE CONSEQUENCES OF THE DYNAMO EQUATIONS

Some straightforward applications of the techniques already developed will now yield considerable information about the solutions of the dynamo equations (1). Some of this information will be used later in the construction of a particular dynamo, and all of it illuminates the general behavior of dynamos.

(A) The Boundedness of $\mathfrak{M}_\rho$

The reason for the discussion of boundedness in Section (5c) was that the operators $\mathfrak{M}_\rho$ corresponding to fluid motions $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq t_0$, whose velocities $\boldsymbol{u}(\boldsymbol{y}, t)$ are continuously differentiable functions of $\boldsymbol{y}$ which vanish on the fluid surface S are in fact bounded linear operators on $\mathcal{B}$. The present subsection is devoted to proving this fact.

If $\boldsymbol{B}(\boldsymbol{x}, 0)$ is an initial magnetic field and $\boldsymbol{B}(\boldsymbol{y}, t)$ is the field produced from it by the fluid motion at time t, then there is a scalar $\phi(\boldsymbol{y}, t)$ defined in $\mathcal{E} - V$ such that $\boldsymbol{B}(\boldsymbol{y}, t) = \nabla\phi(\boldsymbol{y}, t)$ there. This scalar can always be extended into V so as to be continuously differentiable in all of $\mathcal{E}$. Of course, inside V there will be no relation between $\boldsymbol{B}(\boldsymbol{y}, t)$ and $\phi(\boldsymbol{y}, t)$, and $\nabla^2\phi$ cannot vanish everywhere in V, since it vanishes in $\mathcal{E} - V$. The argument below is for real $\boldsymbol{B}$; the modification required to extend it to complex $\boldsymbol{B}$ is clear. If $\boldsymbol{n}$ is the outward normal on the fluid surface S,

$$\frac{d}{dt}\frac{1}{2}\int_{\mathcal{E}-V} | \boldsymbol{B} |^2 = \int_{\mathcal{E}-V} \boldsymbol{B}\cdot\frac{\partial \boldsymbol{B}}{\partial t} = \int_{\mathcal{E}-V} \nabla\phi\cdot\frac{\partial \boldsymbol{B}}{\partial t} = \int_{\mathcal{E}-V} \nabla\cdot\left(\phi\frac{\partial \boldsymbol{B}}{\partial t}\right) = -\int_S \phi\left(\boldsymbol{n}\cdot\frac{\partial \boldsymbol{B}}{\partial t}\right).$$

Even if $\rho = 0$, $\boldsymbol{n}\cdot\partial\boldsymbol{B}/\partial t$ is continuous across S, so the last of the integrals above is

$$-\int_V \nabla\cdot\left(\phi\,\frac{\partial \boldsymbol{B}}{\partial t}\right) = -\int_V \nabla\phi\cdot\frac{\partial \boldsymbol{B}}{\partial t} = -\int_V \nabla\phi\cdot\nabla\times(\boldsymbol{u}\times\boldsymbol{B} - \rho\nabla\times\boldsymbol{B}).$$

Applying the vector identity $\boldsymbol{A}\cdot\nabla\times\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\times\boldsymbol{A} - \nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})$, this last integral is $\int_S \boldsymbol{n}\cdot[\nabla\phi\times(\boldsymbol{u}\times\boldsymbol{B} - \rho\nabla\times\boldsymbol{B})^-]$ where the superscript $-$ means that the term in parenthesis, not being continuous across S, is to be evaluated just inside S. Since $\nabla\phi$ is continuous across S and $\boldsymbol{u} = \boldsymbol{0}$ on S, the integral is $-\rho\int_S [\boldsymbol{B}^+\times(\nabla\times\boldsymbol{B})^-]\cdot\boldsymbol{n}$, where $\boldsymbol{B}^+$ is the value of $\boldsymbol{B}$ just outside S. If $\rho \neq 0$, $\boldsymbol{B}^+ = \boldsymbol{B}^-$, and if $\rho = 0$ the whole term vanishes, so in either case

$$\frac{d}{dt}\frac{1}{2}\int_{\mathcal{E}-V} |\boldsymbol{B}|^2 = -\rho\int_S [\boldsymbol{B}^-\times(\nabla\times\boldsymbol{B})^-]\cdot\boldsymbol{n}.$$

In rectangular coordinates, since $\nabla\cdot\boldsymbol{u} = 0$,

$$\frac{d}{dt}\frac{1}{2}\int_V |\boldsymbol{B}|^2 = \int_V \boldsymbol{B}\cdot\frac{D\boldsymbol{B}}{Dt}$$

where $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\nabla$ is the substantial derivative. Therefore

$$\frac{d}{dt}\frac{1}{2}\int_V |\boldsymbol{B}|^2 = \int_V \boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - \rho\int_V \boldsymbol{B}\cdot\nabla\times\nabla\times\boldsymbol{B}.$$

The last integral on the right is $\int_V |\nabla\times\boldsymbol{B}|^2 - \int_S [\boldsymbol{B}^-\times(\nabla\times\boldsymbol{B})^-]\cdot\boldsymbol{n}$. Therefore

$$\frac{d}{dt}\frac{1}{2}\int_{\mathcal{E}} |\boldsymbol{B}|^2 = \int_V \boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - \rho\int_V |\nabla\times\boldsymbol{B}|^2. \quad (29)$$

This equation is valid even if $\rho = 0$. It has been derived by Bullard and Gellman (Ref. *9*, Eq. (11)), and a different proof is given above because later a generalization of Eq. (29) will be needed which is somewhat less easy to derive by the method of Bullard and Gellman. The above proof appears longer than that of Bullard and Gellman because they use Poynting's theorem without including the justification of it when the displacement current is dropped.

Using the Einstein summation convention, $\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{u} = B_iB_j(\partial u_i/\partial y_j) = \frac{1}{2}B_iB_j(\partial u_i/\partial y_j + \partial u_j/\partial y_i)$. Let $m(t)$ be the algebraically largest value that any characteristic root of the symmetric matrix $\frac{1}{2}(\partial u_i/\partial y_j + \partial u_j/\partial y_i)$ ever takes anywhere in V at time t. Then $\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{u} \leq m(t)\,|\boldsymbol{B}|^2$. Therefore Eq. (29) implies

$$\frac{d}{dt}\frac{1}{2}\int_{\mathcal{E}} |\boldsymbol{B}|^2 \leq m(t)\int_{\mathcal{E}} |\boldsymbol{B}|^2 - \rho\int_V |\nabla\times\boldsymbol{B}|^2. \quad (30)$$

In Section 7d it will be shown that if V is a sphere of radius 1 and $\boldsymbol{B}$ is in $\mathcal{B}$, $\int_V |\nabla \times \boldsymbol{B}|^2 \geq \pi^2 \int_{\mathcal{E}} |\boldsymbol{B}|^2$. It follows that for a sphere of radius R,

$$\int_V |\nabla \times \boldsymbol{B}|^2 \geq \frac{\pi^2}{R^2} \int_{\mathcal{E}} |\boldsymbol{B}|^2. \tag{31}$$

Assuming that V is such a sphere, inequalities (30) and (31) imply

$$\frac{d}{dt} \| \boldsymbol{B} \|^2 \leq 2m(t) \| \boldsymbol{B} \|^2 - \frac{\pi^2 \rho}{R^2} \| \boldsymbol{B} \|^2.$$

Recall that $\rho = (\mu_0\sigma)^{-1}$ and that $\rho\nu_1 = (\pi^2/R^2\mu_0\sigma)$ is the inverse of the mean life of the longest lived free decay mode for a rigid sphere of radius R and conductivity σ. Then

$$\| \boldsymbol{B}(t) \|^2 \leq \| \boldsymbol{B}(0) \|^2 \exp 2 \int_0^t [m(\tau) - \rho\nu_1] \, d\tau. \tag{32}$$

This inequality has been proved (subject to the verification of inequality (31) in Section 7) only for spheres. That it is true for bounded fluids of arbitrary shape is a consequence of a variational method for computing the slowest exponential decay rate $\rho\nu_1$ for a rigid conductor V of any shape. This method will not be developed here, since it is a simple extension of work already published (Ref. *11*, Section VI). The result is merely to replace π^2/R^2 by ν_1 in inequality (31), thus proving inequality (32) for a V of any shape.

Since $\boldsymbol{B}(t_0)$ is by definition $\mathfrak{M}_\rho \boldsymbol{B}(0)$, Eq. (32) can be restated in the language of bounded operators as

$$\| \mathfrak{M}_\rho \| \leq \exp \int_0^t [m(\tau) - \rho\nu_1] \, d\tau. \tag{33}$$

Wthout inequality (31), inequality (30) implies inequality (33) directly if ν_1 is omitted from the latter. The presence of ν_1 in inequality (33) is interesting in that it gives a necessary condition for a dynamo to be self-sustaining. The rate $\frac{1}{2}(\partial u_i/\partial y_j + \partial u_j/\partial y_i)$ of local stretching of the fluid (and the magnetic lines of force) in a self-sustaining dynamo cannot be always and everywhere less than the slowest rigid decay rate $\rho\nu_1$. That some such result would be true was suggested by Bullard and Gellman (Ref. *9*, p. 217) on the basis of a dimensional argument.

(B) The Effects of a Superposed Rigid Rotation

It might appear that caution was necessary in applying the foregoing necessary condition for field maintenance to the earth's core, since the boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on S is met only in a frame of reference rotating rigidly with the earth's mantle. In fact, no such caution is necessary. Let $\mathcal{R}(t)$ be a proper 3×3 orthogonal matrix whose entries $\mathcal{R}_{ij}(t)$ depend only on time. Then $\mathcal{R}(t)$ describes a rigid rotation with the angular velocity $\boldsymbol{\omega}(t)$ whose instantaneous rectangular components are $\omega_i = -\frac{1}{2}\epsilon_{ijk}(d/dt)\mathcal{R}_{jk}(t)$, ϵ_{ijk} being the alternating tensor in three dimensions. For an observer whose reference frame at time t is obtained from some fixed reference frame via the rotation $\mathcal{R}(t)$, the rectangular coordinates y_i' of the position vector whose coordinates are y_i in the fixed frame can be computed as $\boldsymbol{y}' = \mathcal{R}^{-1}(t)\boldsymbol{y}$. A fluid velocity $\boldsymbol{u}(\boldsymbol{y}, t)$ in the fixed frame is, in the rotating frame,

$$\begin{aligned} \boldsymbol{u}'(\boldsymbol{y}', t) &= \mathcal{R}^{-1}[\boldsymbol{u}(\boldsymbol{y}, t) - \boldsymbol{\omega} \times \boldsymbol{y}] \\ &= \mathcal{R}^{-1}[\boldsymbol{u}(\mathcal{R}\boldsymbol{y}', t) - \boldsymbol{\omega} \times \mathcal{R}\boldsymbol{y}'] \\ &= \mathcal{R}^{-1}\boldsymbol{u}(\mathcal{R}\boldsymbol{y}', t) - (\mathcal{R}^{-1}\boldsymbol{\omega}) \times \boldsymbol{y}'. \end{aligned}$$

The magnetic field $\boldsymbol{B}(\boldsymbol{y}, t)$ in the fixed frame of reference becomes in the rotating frame $\boldsymbol{B}'(\boldsymbol{y}', t) = \mathcal{R}^{-1} \boldsymbol{B}(\mathcal{R}\boldsymbol{y}', t)$, if all the terms in the Lorentz transformation of the electromagnetic field which are of the order u/c or smaller are neglected. It is now a matter simply of substitution to verify that if $\boldsymbol{B}(\boldsymbol{y}, t)$ and $\boldsymbol{u}(\boldsymbol{y}, t)$ satisfy the dynamo equations (1) when spatial derivatives refer to $\boldsymbol{y}$, then $\boldsymbol{B}'(\boldsymbol{y}', t)$ and $\boldsymbol{u}'(\boldsymbol{y}', t)$ satisfy those equations when spatial derivatives refer to $\boldsymbol{y}'$. Therefore the theory of the magnetic dynamo equations (1) is invariant to arbitrary time-dependent rigid rotations of the frame of reference, and if on any fluid motion $\boldsymbol{y}(\boldsymbol{x}, t)$ an arbitrary time-dependent rigid rotation is superposed, its effect is simply to make the magnetic field due to the original velocity rotate in the same way.

The corresponding result for the electric field is false, and the effect on $\boldsymbol{E}$ of a superposed rigid rotation has been worked out elsewhere (*21*).[1]

(c) The Equivalence of High Velocity and Low Resistivity

It will occasionally be useful in what follows to shorten the time scale for a fluid motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq t_0$, by some large factor κ, that is, to replace that motion by the motion $\boldsymbol{y}(\boldsymbol{x}, \kappa t)$, $0 \leq t \leq \kappa^{-1} t_0$. The introduction of an extra parameter to describe such scaling can be avoided by observing that the effect of the motion $\boldsymbol{y}(\boldsymbol{x}, \kappa t)$, $0 \leq t \leq \kappa^{-1}t_0$ on an initial magnetic field $\boldsymbol{B}(0)$ in a fluid of resistivity ρ is identical with the effect of the original motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq t_0$, on $\boldsymbol{B}(0)$ in a fluid of resistivity $\kappa^{-1}\rho$. The operator $\mathfrak{M}_\rho$ for the accelerated motion is identical with the operator $\mathfrak{M}_{\kappa^{-1}\rho}$ for the original motion. This fact can be seen immediately from the first of the dynamo equations (1) and the definition $\boldsymbol{u}(\boldsymbol{y}, t) = \partial\boldsymbol{y}(\boldsymbol{x}, t)/\partial t$; it amounts to writing the dynamo equations in dimensionless form.

[1] A correction is necessary in that paper. The constant C of its Eq. (16) cannot be determined, as there asserted, simply from the demand that the electric potential vanish at infinity. It must be determined from the total charge on the body, like the constant C in Section 1b of the present paper.

(D) The Generality of Jerky Motions

The motion proposed in Section 3, bounded and differentiable though it is, looks quite unphysical, since it consists of very rapid motions followed by periods of rest. In this subsection it will be shown that such a motion is the first step in an approximation scheme by which the magnetic effect of any motion whatever can be computed.

Suppose a fluid motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \le t \le t_0$, is given. Let $\boldsymbol{y}_\epsilon(\boldsymbol{x}, t)$, $0 \le t \le t_0$, be any other motion with the property that $| y_i' - y_i |$,

$$\left| \frac{\partial y_i'}{\partial x_j} - \frac{\partial y_i}{\partial x_j} \right|, \quad \left| \frac{\partial^2 y_i'}{\partial x_j \partial x_k} - \frac{\partial^2 y_i}{\partial x_j \partial x_k} \right|, \quad \text{and} \quad \left| \frac{\partial^3 y_i'}{\partial x_j \partial x_k \partial x_l} - \frac{\partial^3 y_i}{\partial x_j \partial x_k \partial x_l} \right|$$

are all less than ϵ for all $\boldsymbol{x}$ in V and all t in $0 \le t \le t_0$. If $\boldsymbol{y}(\boldsymbol{x}, t)$ is regarded as a path in the space $\mathfrak{D}$ of fluid displacements, $\boldsymbol{y}_\epsilon(\boldsymbol{x}, t)$ is another nearby path, and the points on the two paths at a given time t are always close, even though the velocities of those points may be widely different. The fluid velocities $\boldsymbol{u}(\boldsymbol{y}, t) = \partial \boldsymbol{y}(\boldsymbol{x}, t)/\partial t$ and $\boldsymbol{u}_\epsilon(\boldsymbol{y}, t) = \partial \boldsymbol{y}_\epsilon(\boldsymbol{x}, t)/\partial t$ can be quite different. Then as $\epsilon \to 0$, the operator $\mathfrak{M}_\rho^\epsilon$ giving the effect of the motion $\boldsymbol{y}_\epsilon'(\boldsymbol{x}, t)$, $0 \le t \le t_0$, on magnetic fields approaches the operator $\mathfrak{M}_\rho$ which gives the effect of $\boldsymbol{y}$ on those fields.

No attempt will be made to prove this result formally, since its value in the present paper is only the heuristic one of indicating that the dynamo of Section 3 is not as special as it seems. The essential idea of the proof is suggested by Lundquist's (*19*) integral for the resistance-free fluid.

If the initial positions $\boldsymbol{x} = \boldsymbol{y}(\boldsymbol{x}, 0)$ of the fluid points are used as a system of curvilinear coordinates at time t, the rectangular Cartesian coordinates y^i at that time are given by the Lagrangian description of the fluid motion: $y^i = y^i(\boldsymbol{x}, t)$. In Cartesian coordinates, Lundquist's integral for an incompressible resistance-free fluid is

$$B^i(\boldsymbol{y}, t) = \frac{\partial y^i}{\partial x^j} B^j(\boldsymbol{x}, 0).$$

This equation says that in the system of curvilinear coordinates $\boldsymbol{x}$ the contravariant components of $\boldsymbol{B}$ are constants of the motion. This suggests that the dynamo equation be written in the curvilinear coordinates $\boldsymbol{x}$ even when ρ is positive.

Denote by $B^i(\boldsymbol{y}, t)$ the Cartesian components of $\boldsymbol{B}$ at time t and by $b^i(\boldsymbol{x}, t)$ the contravariant components of $\boldsymbol{B}$ at time t in the curvilinear coordinates $\boldsymbol{x}$. Then

$$B^i(\boldsymbol{y}, t) = \frac{\partial y^i}{\partial x^j} b^j(\boldsymbol{x}, t).$$

It is a matter of ordinary tensor analysis (*22*) to show that in terms of the curvilinear coordinates $\boldsymbol{x}$ the first of the dynamo equations (1) becomes

$$\frac{\partial b^i(\boldsymbol{x}, t)}{\partial t} = \rho g^{jk} b^i{}_{;j;k}\,, \tag{34}$$

where g^{jk} is the contravariant metric tensor for the coordinates $\boldsymbol{x}$ while $b^i{}_{;j}$ denotes a covariant derivative of b^i. The condition $\nabla \cdot \boldsymbol{B} = 0$ becomes, of course, $b^i{}_{;i} = 0$.

The right side of Eq. (34) involves g^{ij}, the Christoffel symbols $\left\{ {i \atop jk} \right\}$ of the second kind, and their derivatives with respect to $\boldsymbol{x}$. Thus it involves the first, second and third derivatives of $\boldsymbol{y}(\boldsymbol{x}, t)$ with respect to $\boldsymbol{x}$. It does not involve any derivatives of $\boldsymbol{y}(\boldsymbol{x}, t)$ with respect to t. If $\boldsymbol{y}'(\boldsymbol{x}, t)$ is any other motion which, with all possible $\boldsymbol{x}$ derivatives up to and including those of order three, is always close to the motion $\boldsymbol{y}$, then the operators on the right side of Eq. (34) will be practically the same for the two motions. Since $| \boldsymbol{y}' - \boldsymbol{y} |$ is small, the boundary conditions on b^i will be almost the same for the two motions, and in fact will be identical if the velocities of both motions vanish on the surface of the fluid. Therefore, the magnetic fields $b^i(\boldsymbol{x}, t)$ and $b'^i(\boldsymbol{x}, t)$ produced by the two motions from the same initial field $b^i(\boldsymbol{x}, 0)$ will be practically identical. This is the (rather feeble) generalization of Lundquist's integral to fluids of finite resistivity ρ.

Now given the motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \le t \le t_0$, define the motion $\boldsymbol{y}'(\boldsymbol{x}, t)$ as follows: divide the interval $0 \le t \le t_0$ by n points $t_1 < t_2 < \cdots < t_n$. Let κ be some fixed number very much larger than 1. Then $\boldsymbol{y}'(\boldsymbol{x}, t) = \boldsymbol{y}(\boldsymbol{x}, \kappa t)$ if $0 \le t \le \kappa^{-1} t_1$; $\boldsymbol{y}'(\boldsymbol{x}, t) = \boldsymbol{y}(\boldsymbol{x}, t_1)$ if $\kappa^{-1} t_1 \le t \le t_1$; $\boldsymbol{y}'(\boldsymbol{x}, t) = \boldsymbol{y}[\boldsymbol{x}, t_1 + \kappa(t - t_1)]$ if $t_1 \le t \le t_1 + \kappa^{-1}(t_2 - t_1)$; $\boldsymbol{y}'(\boldsymbol{x}, t) = \boldsymbol{y}(\boldsymbol{x}, t_2)$ if $t_1 + \kappa^{-1}(t_2 - t_1) \le t \le t_2$; etc. Then $\boldsymbol{y}'$ approximates $\boldsymbol{y}$ by a series of short, rapid jerks interspersed with long periods of rest. From the form of Eq. (34) it now follows in the manner remarked above that if the number n of points of subdivision of the interval $0 \le t \le t_0$ approaches infinity in such a way that the maximum distance $| t_{i+1} - t_i |$ approaches zero, then the magnetic field produced by $\boldsymbol{y}'$ from an initial field $b^i(\boldsymbol{x}, 0)$ becomes very close to that produced by $\boldsymbol{y}$ from the same initial field.

To make the above proof complete, it would be necessary to show that the solution b^i of Eq. (34) depends continuously on the boundary conditions and on the coefficients in that equation. No such completeness will be attempted here.

Physically speaking, what has been proved is that smooth motions of the fluid involve no new magnetic effects beyond a distortion of the magnetic lines of force by the fluid as if it were a perfect conductor and the decay of the field as if the fluid were rigid.

(E) THE CONTINUITY OF $\mathfrak{M}_\rho$ AT $\rho = 0$

A rather touchy point was skirted in the preceding subsection. It was shown that the effect of the motion $\boldsymbol{y}'(\boldsymbol{x}, t)$ on a particular initial field $b^i(\boldsymbol{x}, 0)$ approached that of the motion $\boldsymbol{y}$ as the motion $\boldsymbol{y}'$ approached $\boldsymbol{y}$ in the sense of that subsection. It was not shown, and the author is not sure it is true, that the rate of approach is independent of $b^i(\boldsymbol{x}, 0)$. This point will not be discussed further.

A similar difficulty, which must be examined in some detail, arises in connection with the effects of a given fluid motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq t_0$, in a succession of fluids whose resistivities ρ are approaching zero. As ρ approaches zero, does the magnetic effect of the motion $\boldsymbol{y}$ approach its effect when $\rho = 0$, and, if so, is the rate of approach independent of the initial magnetic field?

Let $\boldsymbol{B}_\rho(\boldsymbol{y}, t)$ be the magnetic field produced by the given fluid motion from the initial field $\boldsymbol{B}(\boldsymbol{x}, 0)$ when the fluid has resistivity ρ. As usual, suppose that the fluid velocity vanishes on the surface S. Then $(d/dt)\frac{1}{2} \int_{\mathcal{E}} | \boldsymbol{B}_0 |^2$ and $(d/dt)\frac{1}{2} \int_{\mathcal{E}} | \boldsymbol{B}_\rho |^2$ are given by Eq. (29) while $(d/dt) \int_{\mathcal{E}} \boldsymbol{B}_0 \cdot \boldsymbol{B}_\rho$ can be computed in the same way as was that equation. Combining these three time derivatives in the obvious fashion gives the following equation for the energy of the difference field $\mathfrak{B}_\rho(\boldsymbol{y}, t) = \boldsymbol{B}_\rho(\boldsymbol{y}, t) - \boldsymbol{B}_0(\boldsymbol{y}, t)$:

$$\frac{d}{dt}\frac{1}{2}\int_{\mathcal{E}} | \mathfrak{B} |^2$$

$$= \int_V \mathfrak{B}\cdot(\mathfrak{B}\cdot\nabla)\boldsymbol{u} - \rho \int_V (\nabla \times \mathfrak{B})\cdot(\nabla \times \boldsymbol{B}_\rho) + \rho \int_S [(\boldsymbol{B}_0^- - \boldsymbol{B}_0^+) \times (\nabla \times \boldsymbol{B}_\rho)^-]\cdot \boldsymbol{n}$$

where the superscripts have their usual meaning. If besides the velocity $\boldsymbol{u}$ all its first derivatives $\partial u_i/\partial y_j$ vanish at the surface of the fluid, and if initially $\boldsymbol{B}_0^- = \boldsymbol{B}_0^+$ (as must be the case if $\boldsymbol{B}(\boldsymbol{x}, 0)$ is in $\mathfrak{B}$) then $\boldsymbol{B}_0^- = \boldsymbol{B}_0^+$ at all times. The identity $(\nabla \times \mathfrak{B})\cdot(\nabla \times \boldsymbol{B}_\rho) = \frac{1}{2} | \nabla \times \mathfrak{B} |^2 + \frac{1}{2} | \nabla \times \boldsymbol{B}_\rho |^2 - \frac{1}{2} | \nabla \times \boldsymbol{B}_0 |^2$ allows the above equation to take the form

$$\frac{d}{dt}\frac{1}{2}\int_{\mathcal{E}} | \mathfrak{B} |^2$$
$$= \int_V \mathfrak{B}\cdot(\mathfrak{B}\cdot\nabla)\boldsymbol{u} + \frac{1}{2}\rho\int_V | \nabla \times \boldsymbol{B}_0 |^2 - \frac{1}{2}\rho\int_V (| \nabla \times \mathfrak{B} |^2 + | \nabla \times \boldsymbol{B}_\rho |^2). \tag{35}$$

If $m(t)$ is defined, as in subsection 6a, to be the maximum strain rate in the fluid at time t, then Eq. (35) implies

$$\frac{d}{dt}\int_{\mathcal{E}} | \mathfrak{B} |^2 \leq 2m(t)\int_{\mathcal{E}} | \mathfrak{B} |^2 + \rho\int_V | \nabla \times \boldsymbol{B}_0 |^2,$$

an inequality which can be integrated immediately, using the initial condition $\int_{\mathcal{E}} |\beta(0)|^2 = 0$, to yield

$$\int_{\mathcal{E}} |\beta_\rho(\boldsymbol{y}, t)|^2 \leq \rho e^{2M(t)} \int_0^t e^{-2M(\tau)} \left(\int_V |\nabla \times \boldsymbol{B}_0|^2 \right) d\tau, \tag{36}$$

where $M(t) = \int_0^t m(\tau)\, d\tau$. If $\mathfrak{M}_\rho$ is the operator on $\mathcal{B}$ corresponding to the motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq t_0$, in a fluid of resistivity ρ, then inequality (36) shows that for any fixed $\boldsymbol{B}(0)$

$$\lim_{\rho \to 0} \| \mathfrak{M}_\rho \boldsymbol{B}(0) - \mathfrak{M}_0 \boldsymbol{B}(0) \| = 0. \tag{37}$$

This is not enough to warrant the stronger conclusion that

$$\lim_{\rho \to 0} \| \mathfrak{M}_\rho - \mathfrak{M}_0 \| = 0,$$

and the author doubts that this stronger conclusion is true, although he has been unable either to prove it or to produce a counter-example.

(In the language of Hilbert space it has been shown that $\mathfrak{M}_\rho$ is a continuous function of ρ at $\rho = 0$ in the weak operator topology but nothing has been proved about its continuity in the topology of the operator norms; the author conjectures that it is not continuous in the latter topology.)

Incidentally, the foregoing argument can easily be generalized to show that

$$\lim_{\rho \to \rho_0} \| \mathfrak{M}_\rho \boldsymbol{B}(0) - \mathfrak{M}_{\rho_0} \boldsymbol{B}(0) \| = 0$$

for any $\rho_0 \geq 0$, and if $\rho_0 > 0$ this conclusion remains true for velocities $\boldsymbol{u}$ whose derivatives $\partial u_i / \partial y_j$ do not vanish at the surface of the fluid, as long as the velocities themselves vanish there.

7. THE FREE DECAY OF CURRENTS IN A RIGID SPHERE

The modes of free decay of the electric currents in a rigid sphere of positive resistivity ρ when the displacement current is neglected have been obtained by Elsasser (*3, 10*), who used the vector spherical harmonics first applied to the problem of the electromagnetic behavior of a conducting sphere by Debye (*23*) and Mie (*24*). In problems where they apply, these vector spherical harmonics are usually introduced as vector fields which can be shown to satisfy the vector Helmholtz equation (*25, 3, 26*). In order to make their origin somewhat clearer, to establish a notation, and to illustrate on a simple problem techniques later used in more complicated problems, the first two subsections below are devoted to a discussion *de novo* of freely decaying currents in a rigid sphere, even though this problem may now be said to have been exhaustively treated in the literature. The present section contains no new results, and is included simply to collect the many widely scattered results about the problem of free decay which will be useful in what follows.

(A) THE NORMAL MODES OF FREE DECAY

As usual, the volume V of fluid will be taken to have radius 1 and in the whole of the present section its resistivity ρ will also be taken to be 1. Since the sphere is rigid, the dynamo equations (1) with $\boldsymbol{u} = \boldsymbol{0}$ completely describe the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$. Let $p(\boldsymbol{x}, t)$ and $q(\boldsymbol{x}, t)$ be the scalars of Eq. (9) for this magnetic field: $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{\Lambda} p + \boldsymbol{\Lambda} q$. Then, in consequence of Eq. (4d), $\boldsymbol{\nabla} \times \boldsymbol{B} = -\boldsymbol{\Lambda}\nabla^2 p + \boldsymbol{\nabla} \times \boldsymbol{\Lambda} q$ and $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{\Lambda}\nabla^2 p - \boldsymbol{\Lambda}\nabla^2 q$. The dynamo equation (1a) becomes

$$\boldsymbol{\Lambda}\left(\frac{\partial q}{\partial t} - \nabla^2 q\right) - \boldsymbol{\nabla} \times \boldsymbol{\Lambda}\left(\frac{\partial p}{\partial t} - \nabla^2 p\right) = \boldsymbol{0} \quad \text{in } V. \tag{38}$$

Since p and q average to zero on every S_r for which $0 < r < 1$, the same is true of the two scalar functions $\partial q/\partial t - \nabla^2 q$ and $\partial p/\partial t - \nabla^2 p$. Equation (38) may be regarded as giving a representation of the solenoidal vector $\boldsymbol{0}$ in the form (9). The uniqueness of the scalars in equation (9) then establishes that, in V, $\partial q/\partial t = \nabla^2 q$ and $\partial p/\partial t = \nabla^2 p$. A similar argument applied to Eq. (1b) establishes that, in $\mathcal{E} - V$, $q = 0$ and $\nabla^2 p = 0$. The boundary conditions which $\boldsymbol{B}$ must satisfy at S_1 and at infinity finally reduce the dynamo equations (1) for a rigid sphere to the two following sets of scalar equations: For the poloidal scalar p,

$$\frac{\partial p}{\partial t} = \nabla^2 p \quad \text{in } V; \tag{39a}$$

$$\nabla^2 p = 0 \quad \text{in } \mathcal{E} - V; \tag{39b}$$

$$p \text{ and } \boldsymbol{\nabla} p \text{ are continuous in } \mathcal{E}; \tag{39c}$$

$$r^2 p \text{ is bounded in } \mathcal{E}; \tag{39d}$$

$$p \text{ averages to zero on every } S_r. \tag{39e}$$

For the toroidal scalar q,

$$\frac{\partial q}{\partial t} = \nabla^2 q \quad \text{in } V; \tag{40a}$$

$$q = 0 \text{ in } \mathcal{E} - V; \tag{40b}$$

$$q \text{ is continuous in } \mathcal{E}; \tag{40c}$$

$$q \text{ averages to zero on every } S_r. \tag{40d}$$

The two problems (39) and (40) are both heat flows problems in a sphere of radius 1, although the boundary condition in problem (39) is not usual. Equations (39) have a system of solutions $p_{lmn}(r, \theta, \phi)e^{-\lambda_{ln}t}$ and Eqs. (40) have a system $q_{lmn}(r, \theta, \phi)e^{-\mu_{ln}t}$, where

$$\begin{aligned} p_{lmn} &= \left(\frac{2}{l(l+1)}\right)^{1/2} \frac{j_l(\alpha_{l-1,n}r)}{\alpha_{l-1,n}j_l(\alpha_{l-1,n})} Y_l^m(\theta, \phi) \quad \text{if} \quad 0 \le r \le 1, \\ &= \left(\frac{2}{l(l+1)}\right)^{1/2} \frac{1}{\alpha_{l-1,n}r^{l+1}} Y_l^m(\theta, \phi) \quad \text{if} \quad 1 \le r < \infty; \end{aligned} \tag{41}$$

and

$$\begin{aligned} q_{lmn} &= \left(\frac{2}{l(l+1)}\right)^{1/2} \frac{j_l(\alpha_{ln}r)}{j_{l+1}(\alpha_{ln})} Y_l^m(\theta, \phi) \quad \text{if} \quad 0 \le r \le 1, \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad 1 \le r < \infty. \end{aligned} \tag{42}$$

Here $j_l(r)$ is the lth spherical Bessel function $(\pi/2r)^{1/2}J_{l+1/2}(r)$, α_{ln} is its nth positive zero, and Y_l^m is the normalized spherical harmonic (6a). The decay constants are

$$\lambda_{ln} = \alpha_{l-1,n}^2 \tag{43a}$$

and

$$\mu_{ln} = \alpha_{ln}^2. \tag{43b}$$

The indices take the following values: $l = 1, 2, 3, \cdots$; $m = -l, \cdots, l$; $n = 1, 2, 3, \cdots$.

The two sets of functions p_{lmn} and q_{lmn} are well-known to be each complete in the space $\mathcal{G}$ of square-integrable scalar functions defined inside the unit sphere. Consequently they can be used to solve initial value problems for the two heat equations, (39) and (40). Because of the representation of an arbitrary solenoidal $\boldsymbol{B}$ in terms of scalars p and q this amounts to solving the initial value problem for the dynamo equation (1) when $\boldsymbol{u} = \boldsymbol{0}$.

The vector fields

$$\boldsymbol{P}_{lmn}(r, \theta, \phi) = \nabla \times \boldsymbol{\Lambda} p_{lmn}, \tag{44a}$$

$$\boldsymbol{T}_{lmn}(r, \theta, \phi) = \boldsymbol{\Lambda} q_{lmn} \tag{44b}$$

are, except for normalization factors, Elsasser's (*3*) poloidal and toroidal fundamental decay modes, or normal modes. The field $\boldsymbol{P}_{lmn}$ satisfies $\nabla \cdot \boldsymbol{P}_{lmn} = 0$ and these conditions:

$$\begin{aligned} &\nabla \times \nabla \times \boldsymbol{P}_{lmn} = \lambda_{ln}\boldsymbol{P}_{lmn} \quad \text{in } V; \\ &\nabla \times \boldsymbol{P}_{lmn} = \boldsymbol{0} \quad \text{in } \mathcal{E} - V; \\ &\boldsymbol{P}_{lmn} \text{ and } \nabla(\boldsymbol{r} \cdot \boldsymbol{P}_{lmn}) \text{ are continuous in } \mathcal{E}; \\ &r^{l+2}\boldsymbol{P}_{lmn} \text{ is bounded in } \mathcal{E}. \end{aligned} \tag{45}$$

The field $\boldsymbol{T}_{lmn}$ satisfies $\nabla \cdot \boldsymbol{T}_{lmn} = 0$ and these conditions:

$$\nabla \times \nabla \times \boldsymbol{T}_{lmn} = \mu_{ln} \boldsymbol{T}_{lmn} \quad \text{in } V;$$
$$\boldsymbol{T}_{lmn} = \boldsymbol{o} \quad \text{in } \mathcal{E} - V;$$
$$\boldsymbol{T}_{lmn} \quad \text{is continuous in } \mathcal{E}. \tag{46}$$

(B) The Poloidal and Toroidal Normal Modes as a Complete Orthonormal Set in $\mathcal{B}$

Suppose that $\boldsymbol{Q}$ is any continuous vector field which satisfies the equations

$$\nabla \times \nabla \times \boldsymbol{Q} = \nu \boldsymbol{Q} \quad \text{in } V;$$
$$\nabla \times \boldsymbol{Q} = \boldsymbol{0} \quad \text{in } \mathcal{E} - V;$$
$$\nabla \cdot \boldsymbol{Q} = 0 \quad \text{in } \mathcal{E}, \tag{47}$$

where ν is some real number. Let $\boldsymbol{B}$ be any vector field in $\mathcal{B}$, the space of admissible magnetic fields defined in Section 5a. Then by introducing a scalar potential ϕ for $\boldsymbol{B}$ in the region $\mathcal{E} - V$ and extending ϕ into V as in the proof of Eq. (29), it is a matter of successive integrations by parts to show that

$$\nu \int_{\mathcal{E}} \boldsymbol{Q}^* \cdot \boldsymbol{B} = \int_V (\nabla \times \boldsymbol{Q}^*) \cdot (\nabla \times \boldsymbol{B}). \tag{48}$$

The vector fields $\boldsymbol{P}_{lmn}$ and $\boldsymbol{T}_{lmn}$ are themselves in $\mathcal{B}$ and Eqs. (8) and (4c) imply that any poloidal and any toroidal field are orthogonal in the sense of $\mathcal{B}$'s inner product (21), while Eq. (48) implies that

$$(\boldsymbol{T}_{lmn}, \boldsymbol{T}_{l'm'n'}) = \delta_{ll'} \delta_{mm'} \delta_{nn'} = (\boldsymbol{P}_{lmn}, \boldsymbol{P}_{l'm'n'})$$

on account of the normalization factors chosen in Eqs. (41) and (42). Therefore the vector fields $\boldsymbol{P}_{lmn}$ and $\boldsymbol{T}_{lmn}$ are an orthonormal set in $\mathcal{B}$. From Eq. (48) it follows that if $\boldsymbol{B}$ is any member of $\mathcal{B}$ which is orthogonal to all the fields $\boldsymbol{P}_{lmn}$ and $\boldsymbol{T}_{lmn}$, then the scalars p and q in $\boldsymbol{B}$'s representation (9) are, as members of $\mathcal{G}$ (Section 5b) orthogonal, respectively, to all the p_{lmn} and the q_{lmn}. Since both these sets of scalars are complete orthogonal sets in $\mathcal{G}$, p, and q vanish, so $\boldsymbol{B}$ vanishes. Therefore the vector fields $\boldsymbol{P}_{lmn}$ and $\boldsymbol{T}_{lmn}$ form a complete orthonormal set in $\mathcal{B}$.

(C) Projections onto the Spaces of Free Decay

Let the exponential decay rates λ_{ln} and μ_{ln} of the normal modes $\boldsymbol{P}_{lmn}$ and $\boldsymbol{T}_{lmn}$ be relabelled ν_k, in order of increasing size: $\nu_1 < \nu_2 < \nu_3 < \cdots$. Then the decay rate ν_1 is $\lambda_{01} = \pi^2$, and only the three poloidal modes $\boldsymbol{P}_{1m1}$, $m = -1, 0, 1$, decay at this rate. The decay rate ν_2 is $\lambda_{21} = \mu_{11} = \alpha_{11}^2 = 20.19 \cdots$; to this decay rate belong the three toroidal modes $\boldsymbol{T}_{1m1}$, $m = -1, 0, 1$, and the five poloidal modes $\boldsymbol{P}_{2m1}$, $m = -2, -1, 0, 1, 2$. Table I gives the first seven decay

TABLE I

THE FIRST SEVEN RATES OF DECAY IN A RIGID SPHERE

k	1	2	3	4	5	6	7
ν_k	α_{01}^2	α_{11}^2	α_{21}^2	α_{02}^2	α_{31}^2	α_{12}^2	α_{41}^2
Approximate value of ν_k	9.87	20.19	33.22	39.48	48.83	59.68	66.95
Poloidal modes	$\boldsymbol{P}_{1m1}$	$\boldsymbol{P}_{2m1}$	$\boldsymbol{P}_{3m1}$	$\boldsymbol{P}_{1m2}$	$\boldsymbol{P}_{4m1}$	$\boldsymbol{P}_{2m2}$	$\boldsymbol{P}_{5m1}$
Toroidal modes	—	$\boldsymbol{T}_{1m1}$	$\boldsymbol{T}_{2m1}$	—	$\boldsymbol{T}_{3m1}$	$\boldsymbol{T}_{1m2}$	$\boldsymbol{T}_{4m1}$
Total	3	8	12	3	16	8	20

rates ν_k, together with the normal modes which decay at those rates, and the total number of such modes belonging to each ν_k. In every case, $m = -l, \cdots, l$.

Denote by $\mathcal{B}_k$ the subspace of $\mathcal{B}$ consisting of all linear combinations of normal modes with decay rate ν_k. The last row of Table I gives the dimension of $\mathcal{B}_k$ for $k = 1, \cdots, 7$. If $k \neq k'$, $\mathcal{B}_k$ and $\mathcal{B}_{k'}$ are orthogonal subspaces of $\mathcal{B}$. The normal modes being complete in $\mathcal{B}$, every $\boldsymbol{B}$ in $\mathcal{B}$ can be written in the form $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2 + \cdots$ with $\boldsymbol{B}_k$ in $\mathcal{B}_k$. Therefore, in the sense of Section (5d), $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots$ and the projection operators $\mathcal{Q}_k$ onto the subspaces $\mathcal{B}_k$ are well defined and satisfy Eq. (28). Denote by $\mathcal{B}_k^P$ that part of $\mathcal{B}_k$ consisting of linear combinations of poloidal free decay modes, and by $\mathcal{B}_k^T$ that part of $\mathcal{B}_k$ consisting of linear combinations of toroidal free decay modes. Then clearly, in the sense of Section (5d), $\mathcal{B}_k = \mathcal{B}_k^P \oplus \mathcal{B}_k^T$, so $\mathcal{B} = \mathcal{B}_1^P \oplus \mathcal{B}_1^T \oplus \mathcal{B}_2^P \oplus \mathcal{B}_2^T \oplus \cdots$. Therefore the projection operators $\mathcal{P}_k$ and $\mathfrak{I}_k$ onto the spaces $\mathcal{B}_k^P$ and $\mathcal{B}_k^T$ are well defined, and $\mathcal{Q}_k = \mathcal{P}_k + \mathfrak{I}_k$. Note that for some k, as at $k = 1$, $\mathcal{B}_k^T = 0$ so that $\mathfrak{I}_k = 0$. The projection operator $\mathcal{P}$ will be defined as $\mathcal{P}_1 + \mathcal{P}_2 + \cdots$, while $\mathfrak{I} = \mathfrak{I}_1 + \mathfrak{I}_2 + \cdots$. The poloidal part of $\boldsymbol{B}$ is $\mathcal{P}\boldsymbol{B}$; its toroidal part is $\mathfrak{I}\boldsymbol{B}$.

The meanings of all these projection operators are quite simple. Suppose an arbitrary field $\boldsymbol{B}$ in $\mathcal{B}$ is expanded in terms of the free decay modes (44):

$$\boldsymbol{B}(r, \theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \sum_{n=1}^{\infty} [a_{ln}{}^m \boldsymbol{P}_{lmn}(r, \theta, \phi) + b_{ln}{}^m \boldsymbol{T}_{lmn}(r, \theta, \phi)]. \tag{49}$$

For a particular decay rate ν_k let l and n be chosen so that $\nu_k = \alpha_{l-1,n}^2 = \lambda_{ln} = \mu_{l-1,n}$. Then ν_k is the decay rate of all the poloidal normal modes $\boldsymbol{P}_{lmn}$, $m = -l, \cdots, l$ and of all the toroidal normal modes $\boldsymbol{T}_{l-1,mn}$, $m = -(l-1), \cdots, (l-1)$. The projection operators $\mathcal{Q}_k$, $\mathcal{P}_k$, $\mathfrak{I}_k$ act on $\boldsymbol{B}$ as follows:

$$\mathcal{Q}_k\boldsymbol{B} = \sum_{m=-l}^{l} a_{ln}{}^m \boldsymbol{P}_{lnm} + \sum_{m=-(l-1)}^{l-1} b_{l-1,n}{}^m \boldsymbol{T}_{l-1,mn}; \tag{50a}$$

$$\mathcal{P}_k\boldsymbol{B} = \sum_{m=-l}^{l} a_{ln}{}^m \boldsymbol{P}_{lmn}; \tag{50b}$$

$$\mathfrak{I}_k\boldsymbol{B} = \sum_{m=-(l-1)}^{l-1} b_{l-1,n}{}^m \boldsymbol{T}_{l-1,mn}. \tag{50c}$$

If the initial field $\boldsymbol{B}(\boldsymbol{x}, 0)$ in a rigid conducting sphere of radius 1 and resistivity 1 has the form (49), then by time t the field $\boldsymbol{B}(\boldsymbol{x}, t)$ will have become

$$\boldsymbol{B}(r, \theta, \phi, t) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \sum_{n=1}^{\infty} [a_{ln}{}^m \boldsymbol{P}_{lmn} e^{-\lambda_{ln} t} + b_{ln}{}^m \boldsymbol{T}_{lmn} e^{-\mu_{ln} t}]. \tag{51}$$

This equation may be written succinctly as

$$\boldsymbol{B}(t) = \sum_{k=1}^{\infty} e^{-\nu_k t} \mathcal{Q}_k \boldsymbol{B}(0).$$

If the operator $\mathfrak{D}_t$ on the space $\mathcal{B}$ is defined as that which carries the initial field $\boldsymbol{B}(0)$ into the field $\boldsymbol{B}(t)$ that it has become after a time t of free decay in the rigid sphere, then clearly

$$\mathfrak{D}_t = \sum_{k=1}^{\infty} e^{-\nu_k t} \mathcal{Q}_k = \sum_{k=1}^{\infty} e^{-\nu_k t} (\mathcal{P}_k + \mathfrak{J}_k). \tag{52}$$

Equation (52) gives the complete solution of the initial value problem for the dynamo equation (1) when the velocity of the fluid is zero.

Every projection operator is equal to its own square, so $\mathcal{Q}_k^2 = \mathcal{Q}_k$, $\mathcal{P}_k^2 = \mathcal{P}_k$, $\mathfrak{J}_k^2 = \mathfrak{J}_k$, results which are clear also from Eqs. (50). And since $\mathcal{B}_k^P$ and $\mathcal{B}_{k'}^T$ are mutually orthogonal subspaces, $\mathcal{Q}_k \mathcal{Q}_{k'} = \mathcal{P}_k \mathcal{P}_{k'} = \mathfrak{J}_k \mathfrak{J}_{k'} = 0$ if $k \neq k'$, while $\mathcal{P}_k \mathfrak{J}_{k'} = \mathfrak{J}_{k'} \mathcal{P}_k = 0$ for any k and k'. Therefore

$$\| \mathfrak{D}_t \boldsymbol{B} \|^2 = \sum_{k=1}^{\infty} \| \mathcal{Q}_k \boldsymbol{B} \|^2 e^{-2\nu_k t},$$

and

$$\| (I - \mathcal{Q}_1 - \cdots \mathcal{Q}_s) \mathfrak{D}_t \boldsymbol{B} \|^2 = \sum_{k=s+1}^{\infty} e^{-2\nu_k t} \| \mathcal{Q}_k \boldsymbol{B} \|^2$$

$$\leq e^{-2\nu_{s+1} t} \sum_{k=s+1}^{\infty} \| \mathcal{Q}_k \boldsymbol{B} \|^2 \leq e^{-2\nu_{s+1} t} \| \boldsymbol{B} \|^2.$$

From this fact, for any s,

$$\| (I - \mathcal{Q}_1 - \cdots - \mathcal{Q}_s) \mathfrak{D}_t \| \leq e^{-\nu_{s+1} t}. \tag{53}$$

Inequality (53) is simply another way of stating the fact that if a field is decaying freely in a rigid sphere, the energy of that field contained in modes with decay rate faster than ν_s has a decay rate at least as fast as ν_{s+1}, a result which can also be seen immediately by comparing Eqs. (49) and (51). Similar arguments establish the inequalities

$$\|(\mathcal{P} - \mathcal{P}_1 - \cdots - \mathcal{P}_s) \mathfrak{D}_t \| \leq e^{-\nu_{s+1} t}; \tag{54a}$$

$$\|(\mathfrak{J} - \mathfrak{J}_1 - \cdots - \mathfrak{J}_s) \mathfrak{D}_t \| \leq e^{-\nu_{s+1} t}. \tag{54b}$$

Finally, from Eq. (52) and the fact that all the projection operators $\mathcal{P}_k$, $\mathfrak{J}_k$, $\mathcal{Q}_k$ commute with one another,

$$\mathcal{P}_k\mathfrak{D}_t = \mathfrak{D}_t\mathcal{P}_k = e^{-\nu_k t}\mathcal{P}_k ; \tag{55a}$$

$$\mathfrak{J}_k\mathfrak{D}_t = \mathfrak{D}_t\mathfrak{J}_k = e^{-\nu_k t}\mathfrak{J}_k ; \tag{55b}$$

$$\mathcal{Q}_k\mathfrak{D}_t = \mathfrak{D}_t\mathcal{Q}_k = e^{-\nu_k t}\mathcal{Q}_k . \tag{55c}$$

(D) Variational Inequalities

Several ineqularities will be needed later which are analogous to Rayleigh's (Ref. *27*, p. 110) inequality for the fundamental frequency of a vibrating body. These inequalities are as follows: let $\mathcal{W}_q$ be the space of continuous, piecewise continuously differentiable functions q defined in the unit sphere V, vanishing on its surface S_1, and averaging to zero on every spherical surface S_r for which $0 \leq r \leq 1$. Then if q is in $\mathcal{W}_q$,

$$\int_V |\nabla q|^2 \geq \alpha_{11}^2 \int_V |q|^2; \tag{56}$$

$$\int_V |\mathbf{\Lambda} q|^2 \geq 2 \int_V |q|^2. \tag{57}$$

If the Cartesian components $\Lambda_i q$ of $\mathbf{\Lambda} q$ are in $\mathcal{W}_q$, as will be true, for example, when q is continuously differentiable, twice piecewise continuously differentiable, and constant on S_1, then

$$\int_V \Lambda^2 q \nabla^2 q \geq \alpha_{11}^2 \int_V |\mathbf{\Lambda} q|^2. \tag{58}$$

Further, let $\mathcal{W}_p$ be the space of continuous, piecewise continuously differentiable functions p defined in all of $\mathcal{E}$, averaging to zero on every S_r, and for which $r^2 p$ is bounded. Then if p is in $\mathcal{W}_p$,

$$\int_{\mathcal{E}} |\nabla p|^2 \geq \alpha_{01}^2 \int_V |p|^2. \tag{59}$$

Finally, if $\boldsymbol{B}$ is in $\mathcal{B}$,

$$\int_V |\nabla \times \boldsymbol{B}|^2 \geq \alpha_{01}^2 \int_{\mathcal{E}} |\boldsymbol{B}|^2. \tag{60}$$

Inequality (60) was assumed in the proof of inequality (33).

To prove inequality (56), let $\nabla\mathcal{W}_q$ denote the space of all vector functions ∇q for which q is in $\mathcal{W}_q$. Introduce on $\mathcal{W}_q$ the usual inner product, $(q_1, q_2) = \int_V q_1^* q_2$. Introduce on $\nabla\mathcal{W}_q$ the inner product $(\nabla q_1, \nabla q_2) = \int_V \nabla q_1^* \cdot \nabla q_2$. It is well known that the functions $[l(l+1)]^{1/2}q_{lmn}$ with $l \geq 1$ constitute a complete orthonormal basis for $\mathcal{W}_q$. If ∇q is any vector field in $\nabla\mathcal{W}_q$ which is orthogonal to all the normalized vector fields $\boldsymbol{h}_{lmn} = \alpha_{ln}^{-1}\nabla[l(l+1)]^{1/2}q_{lmn}$ then q is orthogonal to all the $[l(l+1)]^{1/2}q_{lmn}$ and hence vanishes. So, therefore, does ∇q. Thus the $\boldsymbol{h}_{lmn}$ with $l \geq 1$ constitute a complete orthonormal basis for $\nabla\mathcal{W}_q$. The equation $\mathcal{O}(\nabla q) = q$ unambiguously defines a linear transformation $\mathcal{O}$ from $\nabla\mathcal{W}_q$ to $\mathcal{W}_q$. Its effect on the basis vectors is $\mathcal{O}\boldsymbol{h}_{lmn} = \alpha_{ln}^{-1}[l(l+1)]^{1/2}q_{lmn}$, so $\mathcal{O}$ is a bounded linear operator and $\| \mathcal{O} \| = \alpha_{11}^{-1}$. This means that for any q in $\mathcal{W}_q$, $\int_V | q |^2 \leq \alpha_{11}^{-2}\int_V | \nabla q |^2$, which is inequality (56).

To prove inequality (57), let $\mathbf{\Lambda}\mathcal{W}_q$ be the space of vector functions $\mathbf{\Lambda}q$ where q is in $\mathcal{W}_q$, and define on $\mathbf{\Lambda}\mathcal{W}_q$ the inner product $(\mathbf{\Lambda}q_1, \mathbf{\Lambda}q_2) = \int_V \mathbf{\Lambda}q_1^* \cdot \mathbf{\Lambda}q_2$. Then the vectors $\boldsymbol{h}_{lmn} = [l(l+1)]^{-1/2}\mathbf{\Lambda}[l(l+1)]^{1/2}q_{lmn}$, by an argument like that of the preceding paragraph, constitute a complete orthonormal basis in $\mathbf{\Lambda}\mathcal{W}_q$. A linear transformation $\mathcal{O}$ from $\mathbf{\Lambda}\mathcal{W}_q$ to $\mathcal{W}_q$ can be unambiguously defined by the equation $\mathcal{O}(\mathbf{\Lambda}q) = q$; its effect on the basis vectors is

$$\mathcal{O}\boldsymbol{h}_{lmn} = [l(l+1)]^{-1/2}([l(l+1)]^{1/2}q_{lmn}),$$

so, since $l \geq 1$, $\mathcal{O}$ is a bounded linear operator and $\| \mathcal{O} \| = 2^{-1/2}$. This is inequality (57).

If in inequality (56) the function q is replaced by $\Lambda_i q$ where Λ_i is any of the three Cartesian components of $\mathbf{\Lambda}$, and the index i is summed from 1 to 3, inequality (58) is the result.

To prove inequality (59), let $\mathcal{W}_p^0$ be the subspace of $\mathcal{W}_p$ consisting of all functions in $\mathcal{W}_p$ which are harmonic in $\mathcal{E} - V$. Let $\nabla\mathcal{W}_p^0$ be the space of all vector fields ∇p where p is in $\mathcal{W}_p^0$. On $\nabla\mathcal{W}_p^0$ define the inner product $(\nabla p_1, \nabla p_2) = \int_{\mathcal{E}} \nabla p_1^* \cdot \nabla p_2$. The functions $\alpha_{l-1,n}[l(l+1)]^{1/2}p_{lmn}$ with $l \geq 1$ are well known to be a complete orthonormal set in $\mathcal{W}_p^0$ with the inner product $(p_1, p_2) = \int_V p_1^* p_2$. By an argument like that used to prove inequality (56), it follows that the functions $\boldsymbol{h}_{lmn} = [l(l+1)]^{1/2}\nabla p_{lmn}$ are a complete orthonormal basis in $\nabla\mathcal{W}_p^0$. A linear transformation $\mathcal{O}$ from $\nabla\mathcal{W}_p^0$ to $\mathcal{W}_p^0$ is well-defined by the equation $\mathcal{O}(\nabla p) = p$, and its effect on basis vectors is

$$\mathcal{O}(\boldsymbol{h}_{lmn}) = \alpha_{l-1,n}^{-1}(\alpha_{l-1,n}[l(l+1)]^{1/2}p_{lmn}).$$

Since $l \geq 1$, $\mathcal{O}$ is bounded and $\| \mathcal{O} \| = \alpha_{01}^{-1}$. This is inequality (59) when p is in $\mathcal{W}_p^0$. If p is in $\mathcal{W}_p$ but not $\mathcal{W}_p^0$, define p_0 as p in V and in $\mathcal{E} - V$ as that harmonic function which is equal to p on S_1, the boundary of V, and for which $r^2 p_0$ is bounded. Then

$$\int_V | p_0 |^2 = \int_V | p |^2$$

and

$$\int_V |\nabla p_0|^2 = \int_V |\nabla p|^2$$

while

$$\int_{\mathcal{E}-V} (|\nabla p|^2 - |\nabla p_0|^2) = \int_{\mathcal{E}-V} |\nabla p - \nabla p_0|^2 + 2\int_{\mathcal{E}-V} \nabla p_0 \cdot \nabla(p - p_0)$$

$$= \int_{\mathcal{E}-V} |\nabla p - \nabla p_0|^2 > 0.$$

Thus

$$\int_{\mathcal{E}} |\nabla p|^2 > \int_{\mathcal{E}} |\nabla p_0|^2.$$

But p_0 is in $\mathcal{W}_p^0$ and hence obeys inequality (59). Therefore so does p.

Finally, to prove inequality (60), define the space $\nabla \times \mathcal{B}$ to consist of all vector fields $\nabla \times \boldsymbol{B}$ for which $\boldsymbol{B}$ is in $\mathcal{B}$. It was shown in Section 7b that the normal modes $\boldsymbol{P}_{lmn}$ and $\boldsymbol{T}_{lmn}$ with $l \geq 1$ are a complete orthonormal basis for $\mathcal{B}$; therefore, by the argument used to prove inequality (56), the normalized vector fields $\alpha_{l-1,n}^{-1}\nabla \times \boldsymbol{P}_{lmn}$ and $\alpha_{ln}^{-1}\nabla \times \boldsymbol{T}_{lmn}$ are a complete orthonormal basis for $\nabla \times \mathcal{B}$. A linear transformation $\mathcal{O}$ from $\nabla \times \mathcal{B}$ to $\mathcal{B}$ is well-defined by the equation $\mathcal{O}(\nabla \times \boldsymbol{B}) = \boldsymbol{B}$, and its effect on the basis vectors is as follows:

$$\mathcal{O}(\alpha_{l-1,n}^{-1}\nabla \times \boldsymbol{P}_{lmn}) = \alpha_{l-1,n}^{-1}\boldsymbol{P}_{lmn}, \qquad \mathcal{O}(\alpha_{ln}^{-1}\nabla \times \boldsymbol{T}_{lmn}) = \alpha_{ln}^{-1}\boldsymbol{T}_{lmn}.$$

Hence $\mathcal{O}$ is a bounded linear operator and $\|\mathcal{O}\| = \alpha_{01}^{-1}$. This is inequality (60).

8. THE EFFECTS OF FLUID MOTION ON THE POLOIDAL FIELD

If the solution $\boldsymbol{B}$ of the dynamo equations (1) is represented in the form $\boldsymbol{B} = \nabla \times \boldsymbol{\Lambda} p + \boldsymbol{\Lambda} q$, those equations lead to equations for the two scalars p and q. In the present section a discussion will be given of the equation for p or, strictly speaking, for $\Lambda^2 p$ since that turns out to be a more convenient poloidal scalar. In particular, it will be shown that if $\boldsymbol{u}$ has no radial component then the poloidal field dies out as rapidly as if $\boldsymbol{u}$ were zero.

(A) The General Poloidal Equation

The fluid in the sphere V is assumed to have an arbitrary solenoidal velocity $\boldsymbol{u}$. Let w be defined as

$$w = \boldsymbol{r}\cdot\boldsymbol{B} = \Lambda^2 p. \tag{61}$$

Then $\nabla \times \nabla \times \boldsymbol{B} = -\nabla \times \boldsymbol{\Lambda}\nabla^2 p - \boldsymbol{\Lambda}\nabla^2 q$ so

$$\boldsymbol{r}\cdot\nabla \times \nabla \times \boldsymbol{B} = -\Lambda^2\nabla^2 p = -\nabla^2\Lambda^2 p = -\nabla^2 w.$$

From the dynamo equation (1a), in V

$$\frac{\partial}{\partial t}\boldsymbol{r}\cdot\boldsymbol{B} + \rho\boldsymbol{r}\cdot\nabla \times \nabla \times \boldsymbol{B} = \boldsymbol{r}\cdot\nabla \times (\boldsymbol{u} \times \boldsymbol{B}).$$

If $\boldsymbol{u}$ and $\boldsymbol{B}$ are arbitrary solenoidal fields,

$$\boldsymbol{r}\cdot\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)(\boldsymbol{r}\cdot\boldsymbol{u}) - (\boldsymbol{u}\cdot\nabla)(\boldsymbol{r}\cdot\boldsymbol{B}).$$

Therefore

$$\frac{\partial w}{\partial t} + \boldsymbol{u}\cdot\nabla w - \rho\nabla^2 w = (\boldsymbol{B}\cdot\nabla)(ru_r) \quad \text{in } V. \tag{62a}$$

From dynamo equation (1b), $\nabla^2 p = 0$ in $\mathcal{E} - V$, so $\Lambda^2\nabla^2 p = \nabla^2\Lambda^2 p = 0$, or

$$\nabla^2 w = 0 \quad \text{in} \quad \mathcal{E} - V. \tag{62b}$$

The boundary conditions on $\boldsymbol{B}$ imply further that

$$r^2 w \text{ is bounded in } \mathcal{E}; \tag{62c}$$

$$w \text{ and } \nabla w \text{ are continuous in } \mathcal{E}. \tag{62d}$$

Equations (62) are the poloidal part of the dynamo equations (1); if Eqs. (62) have been solved for w, the poloidal part $\boldsymbol{P}$ of $\boldsymbol{B}$ can be obtained immediately as $\boldsymbol{P} = \nabla \times \boldsymbol{\Lambda}(\Lambda^{-2}w)$. Equations (62) are also the equations of a certain heat transfer problem: w is regarded as a temperature, and the region $\mathcal{E} - V$ has finite heat conductivity but no heat capacity, so that any temperature distribution w on the surface S immediately establishes in $\mathcal{E} - V$ the steady state temperature distribution appropriate to the given temperature on S. The region $\mathcal{E} - V$ is held at temperature zero at large distance. Thus Eqs. (62b), (62c), (62d) simply describe a particular way of losing heat from the spherical surface S of the fluid V. That fluid itself has thermometric conductivity ρ, is stirred with velocity $\boldsymbol{u}$, and contains a volume source of heat of strength $(\boldsymbol{B}\cdot\nabla)(ru_r)$ per unit volume. It is only through this "heat source" that the toroidal scalar q appears in Eqs. (62), so no purely toroidal velocity is able to generate poloidal from toroidal fields.

From the temperature analogy it is clear that if $u_r = 0$ the scalar w dies out at least as fast as if the fluid were not stirred at all. This is a generalization of the observation of Bullard and Gellman (Ref. *9*, p. 228) that toroidal velocities cannot support steady dynamos. Curiously enough, the dynamo presented in Section 11 of the present paper depends for its success primarily on a precise statement of how the poloidal magnetic field decays when $\boldsymbol{u}$ is purely toroidal. This precise statement is developed in subsections 8b and 8c.

(B) A Formal Bound on the Poloidal Field Generated by a Toroidal Flow

In the rest of Section 8 it will be assumed that $u_r = 0$. If Eq. (62a) is multiplied by w and the result integrated over V, then after an integration by parts

$$\frac{d}{d\tau}\frac{1}{2}\int_V |w|^2 = -\rho\int_V |\boldsymbol{\nabla} w|^2 + \rho\int_S w\boldsymbol{n}\cdot\boldsymbol{\nabla} w.$$

Since $\nabla^2 w = 0$ in $\mathcal{E} - V$,

$$0 = -\rho\int_{\mathcal{E}-V} |\boldsymbol{\nabla} w|^2 - \rho\int_S w\boldsymbol{n}\cdot\boldsymbol{\nabla} w.$$

The continuity of w and $\boldsymbol{\nabla} w$ across S allows these two equations to be added to give the result

$$\frac{d}{d\tau}\frac{1}{2}\int_V |w|^2 = -\rho\int_{\mathcal{E}} |\boldsymbol{\nabla} w|^2. \tag{63}$$

This equation has been proved only for real w; an obvious modification of the proof extends it to complex w. From inequality (59),

$$\left(\frac{d}{d\tau} + 2\rho\nu_1\right)\int_V |w|^2 \le 0,$$

and integration of this inequality from 0 to τ gives

$$\| w(\tau) \| = \left(\int_V |w|^2\right)^{1/2} \le \| w(0) \| e^{-\rho\nu_1\tau}. \tag{64}$$

Inequality (64) in its full strength will not be needed. Since

$$\| w(0) \|^2 = \int_V |\boldsymbol{r}\cdot\boldsymbol{B}|^2 \le \int_V |\mathcal{P}\boldsymbol{B}|^2 \le \int_{\mathcal{E}} |\mathcal{P}\boldsymbol{B}|^2 = \|\mathcal{P}\boldsymbol{B}(0)\|^2,$$

therefore

$$\| w(\tau) \| \le \| \mathcal{P}\boldsymbol{B}(0) \| e^{-\rho\nu_1\tau}. \tag{65}$$

Neither of the inequalities (64) and (65) directly conveys information about the energy in the poloidal part of the field $\boldsymbol{B}(\tau)$. To obtain such information, let $\mathcal{V}_\tau$ be the operator on $\mathcal{B}$ which gives the effect on magnetic fields $\boldsymbol{B}(0)$ of the persistence of the toroidal velocity $\boldsymbol{u}$ for a time $\tau{:}\ \boldsymbol{B}(\tau) = \mathcal{V}_\tau\boldsymbol{B}(0)$. What is needed is a bound on $\| \mathcal{P}\mathcal{V}_\tau\boldsymbol{B}(0) \|^2$, the energy in the poloidal part $P(\tau) = \mathcal{P}\boldsymbol{B}(\tau)$ of the magnetic field at the end of the motion. Observe that

$$\| \boldsymbol{P}(\tau) \|^2 = \sum_{lmn} | [\boldsymbol{P}_{lmn}, \boldsymbol{P}(\tau)] |^2$$

and that, from Eqs. (48) and (8),

$$[\boldsymbol{P}_{lmn}, \boldsymbol{P}(\tau)] = -\lambda_{ln} \int_V p_{lmn}^* w(\tau).$$

In the notation of Section 5b, Schwarz's inequality implies

$$| [\boldsymbol{P}_{lmn}, \boldsymbol{P}(\tau)] | \leq \lambda_{ln} \| p_{lmn} \| \| w(\tau) \|.$$

The norm $\| p_{lmn} \|$ is, from the definition (41), $[l(l+1)\lambda_{ln}]^{-1/2}$, so

$$| [\boldsymbol{P}_{lmn}, \boldsymbol{P}(\tau)] |^2 \leq \frac{\lambda_{ln}}{l(l+1)} \| w(\tau) \|^2. \tag{66}$$

This inequality is not directly useful in bounding $\| \boldsymbol{P}(\tau) \|^2$, since the resulting infinite sum diverges; $\| \boldsymbol{P}(\tau) \|^2$ could, of course, be bounded by the general argument of Section 6a, but the bound so obtained grows exponentially with τ and is not strong enough for subsequent arguments in which τ becomes very large. Whether $\| \boldsymbol{P}(\tau) \|^2$ can grow exponentially as the result of an appropriately chosen toroidal velocity field is not known to the author. One way out of this difficulty is to observe that inequalities (65) imply that such exponential growth, if it occurs at all, must result from a gradual accumulation of energy in normal modes with ever larger decay rates. Therefore, if, after the toroidal motion has been completed at time τ, the fluid is held motionless for a further time t_1, all this exponentially accumulated energy will disappear. That is, if $\mathfrak{D}_{t_1}$ is the free decay operator defined in Section 7c, it should be possible to bound $\| \mathcal{P}\mathfrak{D}_{t_1}\mathcal{V}_\tau \boldsymbol{B}(0) \|^2$. Whether such a device can be avoided is not at present known to the author, and on this question hinges the possibility of obtaining a simple sufficient condition on arbitrary velocity fields to test whether they can maintain dynamos. The author proposes to pursue this subject further in a subsequent paper. For the moment, the device will be accepted.

At the end of the toroidal motion, the total energy in the poloidal components of $\boldsymbol{B}(\tau)$ with free decay rate $\nu_k = \lambda_{ln}$ is

$$\| \mathcal{P}_k \boldsymbol{B}(\tau) \|^2 = \sum_{m=-l}^{l} | [\boldsymbol{P}_{lmn}, \boldsymbol{P}(\tau)] |^2 \leq \frac{(2l+1)}{l(l+1)} \lambda_{ln} \| \mathcal{P}\boldsymbol{B}(0) \|^2 \exp(-2\rho\nu_1\tau).$$

The inequality follows from inequalities (65) and (66). If the fluid is now held motionless for a time t_1, the resulting field $\mathfrak{D}_{t_1}\boldsymbol{B}(\tau)$ has altogether in the poloidal components with free decay rate ν_k the energy

$$\| \mathcal{P}_k \mathfrak{D}_{t_1}\mathcal{V}_\tau \boldsymbol{B}(0) \|^2 \leq \frac{(2l+1)}{l(l+1)} \lambda_{ln} \exp(-2\rho\lambda_{ln}t_1 - 2\rho\nu_1\tau) \| \mathcal{P}\boldsymbol{B}(0) \|^2. \tag{67}$$

The quantity which will be needed later is the total energy in poloidal modes with decay rates larger than ν_2. This is

$$\| (\mathcal{P} - \mathcal{P}_1 - \mathcal{P}_2)\mathfrak{D}_{t_1}\boldsymbol{B}(\tau) \|^2 = \sum_{k=3}^{\infty} \| \mathcal{P}_k\mathfrak{D}_{t_1}\boldsymbol{B}(\tau) \|^2,$$

so, by inequality (67), it has the bound

$$\| (\mathcal{P} - \mathcal{P}_1 - \mathcal{P}_2)\mathfrak{D}_{t_1}\mathcal{V}_\tau\boldsymbol{B}(0) \|^2 \leq \| \mathcal{P}\boldsymbol{B}(0) \|^2 \sum_{\lambda_{ln} \geq \nu_3} \frac{(2l+1)}{l(l+1)} \lambda_{ln} \exp(-2\rho\lambda_{ln}t_1 - 2\rho\nu_3\tau)$$

The sum is over all l and n for which $\lambda_{ln} \geq \nu_3$, or $\alpha_{l-1,n} \geq \nu_3^{1/2}$. In terms of operators, this inequality implies

$$\| (\mathcal{P} - \mathcal{P}_1 - \mathcal{P}_2)\mathfrak{D}_{t_1}\mathcal{V}_\tau \|^2$$

$$\leq \sum_{\alpha_{ln} \geq \nu_3^{1/2}} \frac{(2l+3)}{(l+1)(l+2)} \alpha_{ln}^2 \exp(-2\rho\alpha_{ln}^2 t_1 - 2\rho\alpha_{01}^2\tau) \tag{68}$$

where the variable of summation has been changed from l to $l + 1$.

(c) A Numerical Bound on the Poloidal Field Generated by a Toroidal Flow

The bound (68) is formal until the series on the right has been shown to converge and an upper bound has been produced for its sum. Such a demonstration of course demands information about the distribution of the roots α_{ln} of the spherical Bessel functions. Denote the sum on the right of inequality (68) by Y. Define the function $F(\nu)$ for $\nu \geq \nu_3$ as

$$F(\nu) = \sum_{\nu_3^{1/2} \leq \alpha_{ln} < \nu^{1/2}} \frac{(2l+3)}{(l+1)(l+2)} \alpha_{ln}^2, \tag{69}$$

the sum being over all values of l and n for which $\nu_3^{1/2} \leq \alpha_{ln} < \nu^{1/2}$. The function $F(\nu)$ is constant between two successive values of ν_k and at each ν_k it jumps by a finite amount. [Incidentally, for each ν_k there is only one term in Y, that is, only one pair l, n such that $\alpha_{ln}^2 = \nu_k$; see Watson (Ref. *28*, Section 15.28).] The derivative of F is a linear combination of Dirac delta functions $\delta(\nu - \nu_k)$, and if these are taken to be asymmetrical, that is $\int_0^\infty \delta(\nu)d\nu = 1$, then

$$Y = \int_{\nu_3}^{\infty} e^{-2\rho t_1\nu} F'(\nu)\, d\nu.$$

Integrating this result by parts,

$$Y = 2\rho t_1 \int_{\nu_3}^{\infty} e^{-2\rho t_1\nu} F(\nu)\, d\nu.$$

If $G(\nu)$ is any function such that $F(\nu) \leq G(\nu)$ for all $\nu \geq \nu_3$, then

$$Y \leq 2\rho t_1 \int_{\nu_3}^{\infty} e^{-2\rho t_1 \nu} G(\nu)\, d\nu.$$

After another integration by parts,

$$Y \leq G(\nu_3) e^{-2\rho\nu_3 t_1} + \int_{\nu_3}^{\infty} e^{-2\rho t_1 \nu} G'(\nu)\, d\nu. \tag{70}$$

To allay suspicion about the use of delta functions in this argument, the terms $F'(\nu)\, d\nu$ and $G'(\nu)\, d\nu$ which arise in the integrals can be replaced by $dF(\nu)$ and $dG(\nu)$, those integrals being regarded as Stieltjes integrals (Ref. *29*, p. 64 ff.). Therefore the sum Y on the right of inequality (68) can be bounded if a bound $G(\nu)$ can be found for the function $F(\nu)$ of Eq. (69).

To obtain such a bound, perform the summation (69) first over n for a fixed l. What is needed is then

$$\sum_{\nu_3^{1/2} \leq \alpha_{ln} < \nu^{1/2}} \alpha_{ln}^2$$

for a fixed l. The following lemma bounds this sum:

Lemma: Let $y(x)$ be a positive, convex function of x defined in the interval $a - h/2 \leq x \leq b + h/2$. Suppose all the n points $x_1, \cdots, x_n$ lie between a and b and $x_{i+1} - x_i \geq h > 0$ for $i = 1, \cdots, n - 1$. Then

$$\sum_{i=1}^{n} y(x_i) \leq \frac{1}{h} \int_{a-h/2}^{b+h/2} y(x)\, dx.$$

The proof of this lemma is elementary. Because y is convex ($y'' \geq 0$ if y'' exists),

$$hy(x_i) \leq \int_{x_i - h/2}^{x_i + h/2} y(\xi)\, d\xi.$$

Since y is positive and $x_{i+1} - x_i$ and $x_i - x_{i-1}$ are both larger than h, it follows that if $i \neq 1, n$ then

$$hy(x_i) \leq \int_{(x_{i-1}+x_i)/2}^{(x_i+x_{i+1})/2} y(\xi)\, d\xi.$$

Adding these inequalities for $i = 1, \cdots, n$,

$$h \sum_{i=1}^{n} y(x_i) \leq \int_{x_1 - h/2}^{x_n + h/2} y(\xi)\, d\xi.$$

Since y is positive and $a \leq x_1 < x_n \leq b$, the conclusion of the lemma follows immediately.

To apply this lemma to

$$\sum_{\nu_3^{1/2} \leq \alpha_{ln} < \nu^{1/2}} \alpha_{ln}^2,$$

let $y(x) = x^2$ and $h = \pi$. Because of the fact that, for any l and n, $\alpha_{l,n+1} - \alpha_{ln} > \pi$ [Ref. *28*, Section 15.83. In that section let $u_1 = C \sin(x - \alpha_{ln})$ and $u_2 = xj_l(x)$, with C chosen so that $u_1'(\alpha_{ln}) = u_2'(\alpha_{ln})$], the above lemma implies

$$\sum_{\nu_3^{1/2} \leq \alpha_{ln} < \nu^{1/2}} \alpha_{ln}^2 \leq \frac{1}{\pi} \int_{\nu_3^{1/2} - \pi/2}^{\nu^{1/2} + \pi/2} x^2 \, dx \leq \frac{1}{3\pi} \left(\nu^{1/2} + \frac{\pi}{2}\right)^3. \tag{71}$$

Another well-known fact (Ref. *28*, Section 15.3) is that $\alpha_{l1} > l + \frac{1}{2}$. Since $\alpha_{ln} > \alpha_{l1}$, in the sum (69) only those l's can occur for which $l + \frac{1}{2} > \nu^{1/2}$. From this fact and inequality (71), the sum (69) is bounded as follows:

$$F(\nu) \leq \frac{1}{3\pi} \left(\nu^{1/2} + \frac{\pi}{2}\right)^3 \sum_{l=1}^{\nu^{1/2} - 1/2} \left[\frac{2}{l+1} - \frac{1}{(l+1)(l+2)}\right].$$

The usual method of bounding sums by integrals shows that

$$\sum_{l=1}^{\nu^{1/2} - 1/2} \frac{1}{l+1} \leq \int_{3/2}^{\nu^{1/2} - 1} \frac{dl}{l} \leq \ln(\nu^{1/2} - 1).$$

Consequently,

$$F(\nu) \leq \frac{2}{3\pi} \left(\nu^{1/2} + \frac{\pi}{2}\right)^3 \ln(\nu^{1/2} - 1).$$

Since

$$\nu^{-2} \left(\nu^{1/2} + \frac{\pi}{2}\right)^3 \ln(\nu^{1/2} - 1) \leq \nu_3^{-2} \left(\nu_3^{1/2} + \frac{\pi}{2}\right)^3 \ln(\nu_3^{1/2} - 1) = 0.559$$

if $\nu \geq \nu_3$, in that range of ν

$$F(\nu) \leq G(\nu) = 0.119\nu^2. \tag{72}$$

Combining inequalities (68), (70), and (72)

$$\| (\mathcal{P} - \mathcal{P}_1 - \mathcal{P}_2) \mathfrak{D}_{t_1} \mathcal{V}_\tau \| \leq 2e^{-\rho\nu_1\tau - \rho\nu_3 t_1} \left(1 + \frac{2}{(2\rho\nu_3 t_1)} + \frac{2}{(2\rho\nu_3 t_1)^2}\right). \tag{73}$$

For completeness, note also the following special cases of inequality (67):

$$\| \mathcal{P}_1 \mathfrak{D}_{t_1} \mathcal{V}_\tau \| \leq \left(\frac{3\nu_1}{2}\right)^{1/2} e^{-\rho\nu_1\tau - \rho\nu_1 t_1}; \tag{74}$$

$$\| \mathcal{P}_2 \mathfrak{D}_{t_1} \mathcal{V}_\tau \| \leq \left(\frac{5\nu_2}{6}\right)^{1/2} e^{-\rho\nu_1\tau - \rho\nu_2 t_1}. \tag{75}$$

The whole purpose of subsections 8b and 8c has been to obtain inequalities (73), (74), (75). They are valid in the following circumstances: the velocity $\boldsymbol{u}$ which produces the magnetic operator $\mathcal{V}_\tau$ is purely toroidal ($u_r = 0$) and proceeds for a time τ. Thereafter the fluid is held motionless for a time t_1. In the present approach, this rigid decay cannot be avoided, since the bound (73) approaches infinity as t_1 approaches zero. As already remarked, the author does not at present know whether this reflects the physical situation or is a defect in the argument. Of course the energy given by inequality (73) cannot in reality be infinite, as is shown in subsection 6a, but as t_1 approaches zero that energy may conceivably become exponentially large in τ.

9. THE EFFECTS OF AXISYMMETRIC TOROIDAL VELOCITIES ON THE TOROIDAL FIELD

(A) The Toroidal Field Equation in General and in Special Cases

Representing the solution of the dynamo equations (1) in the form $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{\Lambda} p + \boldsymbol{\Lambda} q$, the equation for the toroidal scalar q analogous to Eq. (62a) for the poloidal scalar $w = \Lambda^2 p$ can be obtained by dotting $\boldsymbol{\Lambda}$ into both sides of Eq. (1a), that is, by equating the radial components of the curls of the two sides of that equation. The result is

$$\Lambda^2 \left(\frac{\partial q}{\partial t} - \rho \nabla^2 q \right) = \boldsymbol{\Lambda} \cdot \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}). \tag{76}$$

Let $\boldsymbol{A} = \boldsymbol{u} \times \boldsymbol{B}$ in Eq. (5g). Define ω_ϕ, ω_θ, and D as

$$\omega_\phi = \frac{u_\phi}{r \sin \theta}; \qquad \omega_\theta = \frac{u_\theta}{r \sin \theta}; \qquad D = \frac{1}{r} \frac{\partial}{\partial r} r. \tag{77}$$

Several judicious applications of Eq. (5d) then permit the conclusion

$$\begin{aligned} \boldsymbol{\Lambda} \cdot \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \Lambda^2 \Big[& -\boldsymbol{u} \cdot \boldsymbol{\nabla} q - q D u_r + \sin \theta \frac{\partial \omega_\phi}{\partial r} \frac{\partial p}{\partial \theta} \\ & + \omega_\theta \frac{\partial}{\partial \phi} D p - D \sin \theta p \frac{\partial \omega_\phi}{\partial \theta} \Big] + D \Big[(\boldsymbol{\Lambda} u_r) \cdot (\boldsymbol{\Lambda} q - \boldsymbol{\nabla} \times \boldsymbol{\Lambda} p) \\ & + q \Lambda^2 u_r - \frac{\partial}{\partial \phi} (w \omega_\theta) + \sin \theta \frac{\partial p}{\partial \theta} \Lambda^2 \omega_\phi \\ & + p \Lambda^2 \left(\sin \theta \frac{\partial \omega_\phi}{\partial \theta} \right) + \frac{2}{\sin \theta} \frac{\partial}{\partial \phi} \left(\frac{\partial \omega_\phi}{\partial \theta} \frac{\partial p}{\partial \phi} - \frac{\partial \omega_\phi}{\partial \phi} \frac{\partial p}{\partial \theta} \right) \Big]. \end{aligned}$$

Since when q averages to zero on every S_r so does $\partial q / \partial t + u \cdot \nabla q - \rho \nabla^2 q$, Eq. (76) can be written

$$\frac{\partial q}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = h \quad \text{in } V, \tag{78a}$$

where

$$h = \Lambda^{-2}\Lambda^2\left[-qDu_r + \sin\theta\,\frac{\partial\omega_\phi}{\partial r}\frac{\partial p}{\partial\theta} + \omega_\theta\frac{\partial}{\partial\phi}Dp - \sin\theta\, D\left(p\,\frac{\partial\omega_\phi}{\partial\theta}\right)\right]$$
$$+ \Lambda^{-2}D\left[(\mathbf{\Lambda}u_r)\cdot(\mathbf{\Lambda}q - \mathbf{\nabla}\times\mathbf{\Lambda}p) + q\Lambda^2u_r - \frac{\partial}{\partial\phi}(w\omega_\theta)\right. \tag{78b}$$
$$\left. + \sin\theta\,\frac{\partial p}{\partial\theta}\Lambda^2\omega_\phi + p\Lambda^2\left(\sin\theta\,\frac{\partial\omega_\phi}{\partial\theta}\right) + \frac{2}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial\omega_\phi}{\partial\theta}\frac{\partial p}{\partial\phi} - \frac{\partial\omega_\phi}{\partial\phi}\frac{\partial p}{\partial\theta}\right)\right].$$

As in Section 7, the boundary condition on q is that it vanish in $\mathcal{E} - V$ and be continuous everywhere. Equations (78) are the toroidal analogue of the poloidal equations (62) and, like the latter, are applicable for any solenoidal velocity $\boldsymbol{u}$. It should be noted that the operator $\Lambda^{-2}\Lambda^2$ is the identity only when it operates on functions which average to zero on every S_r. For an arbitrary function f, $\Lambda^{-2}\Lambda^2 f = f - f_r$ where f_r is the average of f on S_r.

Clearly Eqs. (78) are the equations for the temporal behavior of the temperature q in a fluid V with thermometric conductivity ρ, stirred at velocity $\boldsymbol{u}$, whose boundary S_1 is maintained at temperature zero, and which contains a source of heat h per unit volume. In case $u_r = 0$ and initially $p = 0$, then from Eqs. (62) $p = 0$ at all times, so $h = 0$, and equation (78a) is $\partial q/\partial t + \boldsymbol{u}\cdot\mathbf{\nabla}q - \rho\nabla^2 q = 0$. Therefore in this special case the toroidal scalar q dies out at least as rapidly as if the velocity $\boldsymbol{u}$ were zero.

Formula (78b) becomes much simplified in the one case in which it will be employed in the present paper, that of an axisymmetric toroidal velocity. In this case, $u_r = u_\theta = 0$ and ω_ϕ depends only on r and θ. Since ω_ϕ is the only angular velocity remaining in h, it will hereafter be denoted simply by $\omega(r, \theta)$. For an axisymmetric toroidal velocity, $\boldsymbol{u} = r\sin\theta\,\omega(r, \theta)\hat{\boldsymbol{\phi}}$, formula (78b) for h becomes

$$h = \Lambda^{-2}\Lambda^2\left[\sin\theta\,\frac{\partial\omega}{\partial r}\frac{\partial p}{\partial\theta} - \sin\theta\,\frac{1}{r}\frac{\partial}{\partial r}\left(rp\,\frac{\partial\omega}{\partial\theta}\right)\right]$$
$$+ \Lambda^{-2}\frac{1}{r}\frac{\partial}{\partial r}r\left[\sin\theta\,\frac{\partial p}{\partial\theta}\Lambda^2\omega + p\,\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\Lambda^2\omega) + \frac{2}{\sin\theta}\frac{\partial\omega}{\partial\phi}\frac{\partial^2 p}{\partial\phi^2}\right]. \tag{79}$$

For the purposes of the present paper, it suffices to consider an even more special case, that in which $\omega(r, \theta) = f(r)\cos\theta$ and

$$\boldsymbol{u} = r\sin\theta\cos\theta f(r)\hat{\boldsymbol{\phi}}. \tag{80}$$

With this further specialization, Eq. (79) becomes

$$h = \frac{df}{dr}\left(\Lambda^{-2}\Lambda^2\cos\theta\sin\theta\,\frac{\partial}{\partial\theta}\Lambda^{-2}w\right) + \left(\frac{1}{r}\frac{d}{dr}rf\right)\Xi w + f\,\Xi\,\frac{\partial w}{\partial r}, \tag{81a}$$

where

$$\Xi = \Lambda^{-2}\left[\Lambda^2 \sin^2\theta - 2\left(\frac{\partial^2}{\partial\phi^2} + 3\cos^2\theta - 1 + \cos\theta\sin\theta\frac{\partial}{\partial\theta}\right)\right]\Lambda^{-2}. \quad (81b)$$

In order to make use of this formula it is necessary to find the average over each S_r of the operand of $\Lambda^{-2}\Lambda^2$. To this end, define

$$g_l^m = \left[\frac{(l+m)(l-m)}{(2l+1)(2l-1)}\right]^{1/2} \quad (82)$$

for $l = 1, 2, \cdots, m = -l, \cdots, l$. Define

$$R_l^m = g_l^m g_{l+1}^m, \quad (83a)$$

$$H_l^m = (g_l^m)^2. \quad (83b)$$

Then (30)

$$\begin{aligned}\sin\theta e^{-i\phi} Y_l^m &= \left[\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}\right]^{1/2} Y_{l+1}^{m-1} \\ &\quad - \left[\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}\right]^{1/2} Y_{l-1}^{m-1},\end{aligned} \quad (84a)$$

$$\cos\theta Y_l^m = g_l^m Y_{l-1}^m + g_{l+1}^m Y_{l+1}^m, \quad (84b)$$

$$\begin{aligned}\sin\theta e^{i\phi} Y_l^m &= \left[\frac{(l-m)(l-m-1)}{(2l+1)(2l-1)}\right]^{1/2} Y_{l-1}^{m+1} \\ &\quad - \left[\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}\right]^{1/2} Y_{l+1}^{m+1},\end{aligned} \quad (84c)$$

where Y_l^m are the normalized spherical harmonics (6a). Therefore

$$\cos^2\theta Y_l^m = R_{l-1}^m Y_{l-2}^m + (H_l^m + H_{l+1}^m) Y_l^m + R_{l+1}^m Y_{l+2}^m. \quad (85)$$

As is well known in the quantum theory of angular momentum, if $\boldsymbol{L} = -i\boldsymbol{\Lambda}$ and $L^+ = L_x + iL_y$, $L^- = L_x - iL_y$ in Cartesian coordinates, then

$$\begin{aligned}L^+ Y_l^m &= [(l-m)(l+m+1)]^{1/2} Y_l^{m+1}, \\ L^- Y_l^m &= [(l+m)(l-m+1)]^{1/2} Y_l^{m-1}.\end{aligned} \quad (86)$$

Since

$$\frac{\partial}{\partial\theta} = \frac{1}{2}[e^{-i\phi}L^+ - e^{i\phi}L^-],$$

Eqs. (84a), (84c), and (86) imply

$$\sin\theta\frac{\partial}{\partial\theta} Y_l^m = l g_{l+1}^m Y_l^m - (l+1) g_l^m Y_{l-1}^m. \quad (87)$$

Invoking Eq. (84b),

$$\cos\theta\sin\theta\frac{\partial}{\partial\theta}Y_l^m = -(l+1)R_{l-1}^mY_{l-2}^m + [lH_{l+1}^m - (l+1)H_l^m]Y_l^m + lR_{l+1}^mY_{l+2}^m. \tag{88}$$

Write

$$p(r,\theta,\phi,t) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} p_l^m(r,t)Y_l^m(\theta,\phi). \tag{89}$$

Then the operand of $\Lambda^{-2}\Lambda^2$ in Eq. (81) has for each fixed r an expansion in spherical harmonics Y_l^m of which the term with $l = 0$ is

$$-R_1^0Y_0^0\left[f(r)\frac{\partial p_2^0(r,t)}{\partial r} + \left(\frac{1}{r}\frac{d}{dr}rf\right)p_2^0(r,t) + 3\frac{df}{dr}p_2^0(r,t)\right].$$

If this quantity is subtracted from the operand of $\Lambda^{-2}\Lambda^2$ in Eq. (81), the truth of that equation is unaffected, while the new operand of $\Lambda^{-2}\Lambda^2$ averages to zero on every S_r. The operator $\Lambda^{-2}\Lambda^2$ has no effect on this new operand, so

$$h = \frac{df}{dr}\cos\theta\sin\theta\frac{\partial p}{\partial\theta} + \sin^2\theta\left(f\frac{\partial p}{\partial r} + p\frac{df}{dr} + \frac{f}{r}p\right) + R_1^0Y_0^0\left[f\frac{\partial p_2^0}{\partial r} + p_2^0\left(4\frac{df}{dr} + \frac{f}{r}\right)\right] - 2\left(\frac{df}{dr} + \frac{f}{r}\right)\Lambda^{-2}\Gamma p - 2f\Lambda^{-2}\Gamma\frac{\partial p}{\partial r}, \tag{90a}$$

where

$$\Gamma = \frac{\partial^2}{\partial\phi^2} + (3\cos^2\theta - 1) + \cos\theta\sin\theta\frac{\partial}{\partial\theta}.$$

To summarize, if the fluid velocity has the toroidal axisymmetric form (80), then the equation for the toroidal scalar q is the heat equation (78a), which, in this simple case, is

$$\frac{\partial q}{\partial t} + \omega\frac{\partial q}{\partial\phi} - \rho\nabla^2 q = h. \tag{90b}$$

The heat source h is given by Eq. (90a) in terms of the poloidal scalar p. The boundary condition on q is that it vanish on the surface S of the fluid.

(B) The Effect of a Toroidal Flow on $\boldsymbol{P}_{101}$

The fact that from an initial poloidal field axisymmetric toroidal shearing motions can produce large toroidal fields was pointed out by Elsasser (*10*).

To construct a dynamo it will be necessary to see how much of the $\boldsymbol{T}_{101}$ mode can be produced from the initial field $\boldsymbol{B}(0) = \boldsymbol{P}_{101}$ by the persistence of the fluid velocity (80) for a time τ. If the magnetic operator corresponding to this motion is denoted by $\mathfrak{V}_\tau$, what is wanted is $\mathfrak{J}_2\mathfrak{V}_\tau \boldsymbol{P}_{101}$.

This problem could be solved by the axisymmetric techniques introduced by Lüst and Schlüter (*31*) and Chandrasekhar (*32*), but Eqs. (90), being ready to hand, will be used instead. The poloidal field scalar at time t will be simply $p = p_{101}e^{-\nu_1 t}$, and the toroidal scalar q will vanish initially and will always be axisymmetric. The solution of Eq. (90) in this situation is straightforward, and leads to the result that if $b_{ln}{}^m(t)$ are the expansion coefficients in the series (49) then

$$b_{ln}{}^m(t) = \frac{4}{5}\,\delta_{m0}C_{ln}\left[\frac{e^{-\rho\nu_1 t} - e^{-\rho\mu_{ln}t}}{\rho(\mu_{ln} - \nu_1)}\right],$$

where

$$C_{ln} = \frac{\delta_{l1}}{j_2(\alpha_{1n})}\int_0^1 dr\, r^2 j_1(\alpha_{1n}r)\left[2\alpha_{01}fj_1'(\alpha_{01}r) + j_1(\alpha_{01}r)\left(f' + \frac{2}{r}f\right)\right]$$
$$+ \frac{\delta_{l3}}{j_4(\alpha_{3n})}\left(\frac{2}{7}\right)^{1/2}\int_0^1 dr\, r^2 j_3(\alpha_{3n}r)\left[\left(2f' - \frac{f}{r}\right)j_1(\alpha_{01}r) - \alpha_{01}fj_1'(\alpha_{01}r)\right].$$

In particular, if

$$f = r - r^3, \tag{91}$$

then

$$(\boldsymbol{T}_{101}, \mathfrak{J}_2\mathfrak{V}_\tau\boldsymbol{P}_{101}) = \frac{4}{5}\,C_{11}\left[\frac{e^{-\rho\nu_1\tau} - e^{-\rho\nu_2\tau}}{\rho(\nu_2 - \nu_1)\tau}\right]\tau, \tag{92a}$$

where

$$j_2(\alpha_{11})C_{11} = -\int_0^1 dr\,(r^2 + r^4)j_1(\alpha_{11}r)j_1(\alpha_{01}r) + 2\alpha_{01}\int_0^1 (r^3 - r^5)j_1(\alpha_{11}r)j_0(\alpha_{01}r)\,dr.$$

Since the integrands here are products of trigonometric function and powers of r, the integrals can be evaluated exactly. The result is

$$\tfrac{4}{5}C_{11} = 0.0976. \tag{92b}$$

Equations (92) give the amount of $\boldsymbol{T}_{101}$ generated from $\boldsymbol{P}_{101}$ by the motion $\boldsymbol{u} = r^2(1 - r^2)\sin\theta\cos\theta\hat{\boldsymbol{\phi}}$. A bound on the total toroidal field $\boldsymbol{T}$ produced from $\boldsymbol{P}_{101}$ by this motion can also be obtained. Multiply Eq. (90c) by $-\Lambda^2 q$ and integrate over V. Since $\partial q/\partial\phi = 0$, the result is

$$\frac{1}{2}\frac{d}{dt}\int_V |\boldsymbol{\Lambda} q|^2 + \rho\int_V \Lambda^2 q\nabla^2 q = \int_V \boldsymbol{\Lambda} q\cdot\boldsymbol{\Lambda} h.$$

Then Schwarz's inequality and inequality (58) imply

$$\frac{d}{dt} \| \boldsymbol{T} \| + \rho\nu_2 \| \boldsymbol{T} \| \leq \| \boldsymbol{\Lambda} h \|,$$

where $\boldsymbol{T} = \boldsymbol{\Lambda} q = \mathfrak{I}\boldsymbol{B}(t)$. For the particular velocity under discussion, if $p_{101}(r, \theta)$ is written as $\tilde{p}(r) \cos \theta$,

$$e^{\rho\nu_1 t} h = \frac{2}{5} P_1 (\cos \theta) \left[\tilde{p} \left(\frac{df}{dr} + \frac{2f}{r} \right) + 2f \frac{d\tilde{p}}{dr} \right] + \frac{2}{15} P_3 (\cos \theta) \left[\left(2 \frac{df}{dr} - \frac{f}{r} \right) \tilde{p} - f \frac{d\tilde{p}}{dr} \right]$$

where P_l is lth Legendre polynomial. It is a matter of straightforward computation to show that, when $f = r - r^3$,

$$\| \boldsymbol{\Lambda} h(t) \| = 0.31308 e^{-\rho\nu_1 t}$$

so that

$$\| \mathfrak{I}\mathfrak{V}_\tau \boldsymbol{P}_{101} \| = \| \boldsymbol{T}(\tau) \| \leq 0.31308\tau \left[\frac{e^{-\rho\nu_1\tau} - e^{-\rho\nu_2\tau}}{\rho(\nu_2 - \nu_1)\tau} \right]. \tag{92c}$$

(c) A Bound on the Toroidal Field Scalar Generated by a Toroidal Flow

Elsasser (*10*) asserts the the effects of an axisymmetric toroidal velocity persisting for any time τ are obvious: the poloidal field decays and the toroidal components grow at most linearly with τ. However, in justifying this assertion, Elsasser essentially assumes the result

$$\lim_{\rho \to 0} \| \mathfrak{M}_\rho - \mathfrak{M}_0 \| = 0$$

discussed in Section (6e) of the present paper. Elsasser does not prove this result, and since the present author has been unable to do so (and in fact doubts that it is true), a new approach must be devised.

If Eq. (78a) is multiplied by q and the result integrated over V, it follows that

$$\frac{1}{2} \frac{d}{dt} \| q \|^2 + \rho \int_V | \boldsymbol{\nabla} q |^2 = (q, h).$$

From Schwarz's inequality and the variational inequality (56),

$$\left(\frac{d}{dt} + \rho\nu_2 \right) \| q \| \leq \| h \|.$$

This last inequality can be integrated from zero to t to yield

$$\| q(t) \| \leq \| q(0) \| e^{-\rho\nu_2 t} + e^{-\rho\nu_2 t} \int_0^t e^{\rho\nu_2\tau} \| h(\tau) \| d\tau. \tag{93}$$

Inequality (93) is true for any solenoidal velocity whatever. When the velocity has the special form (80), h is given by Eq. (81) and is independent of q. Therefore h can be bounded by means of inequality (64), so that Eq. (93) becomes a bound on q.

As a first step in bounding the h of Eq. (81), a bound will be obtained for the linear operator Ξ. From Eqs. (85) and (88) it is not difficult to show that if $l \geq 1$

$$\Xi Y_l^m = a_{l-2}^m Y_{l-2}^m + b_l^m Y_l^m + c_{l+2}^m Y_{l+2}^m,$$

where

$$a_l^m = \frac{R_{l+1}^m}{(l+1)(l+2)}, \qquad b_l^m = \frac{2[3m^2 - l(l+1)]}{l(l+1)(2l-1)(2l+3)}, \qquad c_l^m = \frac{R_{l-1}^m}{l(l-1)}.$$

Now if

$$f(\theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f_l^m Y_l^m,$$

then

$$\| \Xi f \|_1^2 = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} | a_l^m f_{l+2}^m + b_l^m f_l^m + c_l^m f_{l-2}^m |^2.$$

Since Schwarz's inequality is no less valid in three-dimensional spaces than in infinite dimensional ones,

$$\| \Xi f \|_1^2 \leq \sum_{l=1}^{\infty} \sum_{m=-l}^{l} (| f_{l+2}^m |^2 + | f_l^m |^2 + | f_{l-2}^m |^2) (| a_l^m |^2 + | b_l^m |^2 + | c_l^m |^2)$$

$$\leq 3 \| f \|_1^2 \sup_{lm} (| a_l^m |^2 + | b_l^m |^2 + | c_l^m |^2).$$

From the formulas for a_l^m, b_l^m, and c_l^m

$$\sup_{lm} [| a_l^m |^2 + | b_l^m |^2 + | c_l^m |^2] = 17/105 ,$$

so $\| \Xi \|_1 \leq 0.6967$. Then by Eq. (26)

$$\| \Xi \| \leq 0.6967. \tag{94}$$

When $f = r - r^3$, $\max | df/dr | = 2$, $\max | f | = \tfrac{2}{3}^{3/2}$, $\max | r^{-1} d(rf)/dr | = 2$, so from Eq. (81) and inequality (94),

$$\| h \| \leq 2.1005 \| w \| + 0.2682 \left\| \frac{\partial w}{\partial r} \right\|. \tag{95}$$

Combining inequalities (64), (93), and (95),

$$\| q(t) \| \leq \| q(0) \| e^{-\rho\nu_2 t} + 2.1005 \| w(0) \| \left[\frac{e^{-\rho\nu_1 t} - e^{-\rho\nu_2 t}}{\rho(\nu_2 - \nu_1)} \right] + 0.2682 e^{-\rho\nu_2 t} \int_0^t e^{\rho\nu_2\tau} \left\| \frac{\partial w}{\partial r} \right\| d\tau.$$

Clearly $\| \partial w/\partial r \| \leq \| \nabla w \|$ and from Schwarz's inequality

$$\int_0^t e^{\rho\nu_2\tau} \| \nabla w \| \, d\tau \leq \left[\int_0^t e^{2\rho\nu_2\tau} \, d\tau \right]^{1/2} \left[\int_0^t \| \nabla w \|^2 \, d\tau \right]^{1/2} = \left[\frac{e^{2\rho\nu_2 t} - 1}{2\rho\nu_2} \right]^{1/2} \left[\frac{1}{2\rho} \left(\| w(0) \|^2 - \| w(t) \|^2 \right) \right]^{1/2},$$

the last equality being obtained via Eq. (63). Thus

$$\| q(\tau) \| \leq \| q(0) \| e^{-\rho\nu_2\tau} + 2.1005\tau \| w(0) \| \left[\frac{e^{-\rho\nu_1\tau} - e^{-\rho\nu_2\tau}}{\rho(\nu_2 - \nu_1)\tau} \right] + 0.2682\tau \| w(0) \| \frac{1}{(2\rho\tau)^{1/2}} \left(\frac{1 - e^{-2\rho\nu_2\tau}}{2\rho\nu_2\tau} \right)^{1/2}. \tag{96}$$

Since $\| q(0) \| \leq 2^{-1/2} \| \boldsymbol{T}(0) \|$, $\| w(0) \| \leq \| \boldsymbol{P}(0) \|$, and $\| \boldsymbol{B}(0) \|^2 = \| \boldsymbol{P}(0) \|^2 + \| \boldsymbol{T}(0) \|^2$, if the right-hand side of inequality (96) is regarded as the inner product of the two-dimensional vector $\| \boldsymbol{T}(0) \| \hat{\boldsymbol{x}} + \| \boldsymbol{P}(0) \| \hat{\boldsymbol{y}}$ with another two-dimensional vector, Schwarz's inequality gives finally

$$\frac{\| q(\tau) \|^2}{\| \boldsymbol{B}(0) \|^2} \leq \frac{1}{2} e^{-2\rho\nu_2\tau} + \tau^2 \left[2.1005 \left(\frac{e^{-\rho\nu_1\tau} - e^{-\rho\nu_2\tau}}{\rho(\nu_2 - \nu_1)\tau} \right) + \frac{0.2682}{(2\rho\tau)^{1/2}} \left(\frac{1 - e^{-2\rho\nu_2\tau}}{2\rho\nu_2\tau} \right)^{1/2} \right]^2.$$

This inequality is true of the toroidal magnetic field scalar if the velocity $\boldsymbol{u} = r^2(1 - r^2)(\sin\theta\cos\theta)\hat{\boldsymbol{\phi}}$ has been extant for a time τ. A similar inequality, with different numerical coefficients, could be obtained without difficulty for the slightly more general velocity $\boldsymbol{u} = rf(r)\sin\theta\cos\theta\hat{\boldsymbol{\phi}}$, but any change in the angular behavior of $\boldsymbol{u}$ will complicate the analysis considerably.

(D) A Formal Bound on the Toroidal Field Generated by a Toroidal Velocity

Just as in Section 8b the bound on $\| w(\tau) \|$ was converted to a bound on $\| \mathcal{P}\boldsymbol{B}(\tau) \| = \| \mathcal{P}\mathcal{V}_\tau\boldsymbol{B}(0) \|$, so here the bound (97) on $\| q(\tau) \|$ must be converted to a bound on $\| \mathcal{T}\boldsymbol{B}(\tau) \|$. In order to obtain such a bound, it will be necessary, as it was in Section 8b, to hold the fluid stationary for a time t_1 after the motion has been in progress for a time τ; this will allow the possibly large amounts of energy which have accumulated in normal modes with high decay rates to die out.

Clearly, if $\mathcal{V}_\tau$ is the magnetic operator of the motion produced by the persistence of velocity $\boldsymbol{u}$ for time τ,

$$\begin{aligned} \| \mathfrak{D}_{t_1}\mathfrak{I}\mathcal{V}_\tau \boldsymbol{B}(0) \|^2 &= \| \mathfrak{D}_{t_1}\boldsymbol{T}(\tau) \|^2 = \sum_{lmn} | [\boldsymbol{T}_{lmn}, \boldsymbol{T}(\tau)] |^2 e^{-2\rho\mu_{ln}t_1} \\ &= \sum_{lmn} l^2(l+1)^2((q_{lmn}, q(\tau))]^2 e^{-2\rho\mu_{ln}t_1} \\ &\leq \| q(\tau) \|^2 \sum_{lmn} l^2(l+1)^2 \| q_{lmn} \|^2 e^{-2\rho\mu_{ln}t_1} \\ &= \| q(\tau) \|^2 \sum_{ln} l(l+1)(2l+1) e^{-2\rho\mu_{ln}t_1}. \end{aligned} \tag{98}$$

In particular,

$$\| (\mathfrak{I} - \mathfrak{I}_2)\mathfrak{D}_{t_1}\mathcal{V}_\tau \boldsymbol{B}(0) \|^2 \leq \| q(\tau) \|^2 \sum_{\mu_{ln} \geq \nu_3} l(l+1)(2l+1) e^{-2\rho\mu_{ln}t_1},$$

the summation being over all l and n such that $\mu_{ln} = \alpha_{ln}^2 \geq \nu_3$. Therefore, if $H^2(\tau)$ is defined as the quantity on the right-hand side of inequality (97),

$$\| (\mathfrak{I} - \mathfrak{I}_2)\mathfrak{D}_{t_1}\mathcal{V}_\tau \|^2 \leq H^2(\tau) \sum_{\mu_{ln} \geq \nu_3} l(l+1)(2l+1) e^{-2\rho\mu_{ln}t_1}. \tag{99}$$

This is the formal bound on the toroidal energy analogous to the bound (68) on the poloidal energy.

(E) A Numerical Bound on the Toroidal Field Generated by a Toroidal Field

The bound (99) is useless without a bound for the sum on the right in that inequality. Since the procedure for obtaining such a bound is formally the same as that adopted in Section 8c, many of the details will be omitted here. If

$$F(\nu) = \sum_{\nu_3^{1/2} \leq \alpha_{ln} < \nu^{1/2}} l(l+1)(2l+1),$$

then

$$\begin{aligned} F(\nu) &\leq \sum_{l=1}^{\nu^{1/2}-1/2} l(l+1)(2l+1) \left[\frac{2\nu^{1/2} - (2l+1)}{2\pi} + 1 \right] \\ &\leq \left(\frac{\nu^{1/2}}{\pi} + 1 \right) \int_{1/2}^{\nu^{1/2}} x(x+1)(2x+1)\,dx - \frac{1}{2\pi} \int_0^{\nu^{1/2}-1/2} x(x+1)(2x+1)^2\,dx \\ &\leq 0.173\nu^{5/2} = G(\nu). \end{aligned}$$

Therefore

$$\sum_{\mu_{ln} \geq \nu_3} l(l+1)(2l+1)e^{-2\rho\mu_{ln}t_1} = \int_{\nu_3}^{\infty} F'(\nu)e^{-2\rho t_1 \nu}\,d\nu$$

$$\leq G(\nu_3)e^{-2\rho\nu_3 t_1} + \int_{\nu_3}^{\infty} G'(\nu)e^{-2\rho t_1\nu}\,d\nu$$

$$\leq 0.173\nu_3^{5/2}e^{-2\rho\nu_3 t_1}\left[1 + \frac{5}{4\rho\nu_3 t_1} + \frac{5}{(2\rho\nu_3 t_1)^2} + \frac{5}{(2\rho\nu_3 t_1)^3}\right].$$

Finally, if $\mathcal{V}_\tau$ is the magnetic operator produced by the persitence for time τ of the particular velocity $\boldsymbol{u} = r^2(1 - r^2)\sin\theta\cos\theta\hat{\boldsymbol{\phi}}$,

$$\|{}_{\bullet}^{2}\mathfrak{D}_{t_1}(\mathfrak{J} - \mathfrak{J}_2)\mathcal{V}_\tau\|^2$$

$$\leq H^2(\tau)(0.173)\nu_3^{5/2}e^{-2\rho\nu_3 t_1}\left[1 + \frac{5}{4\rho\nu_3 t_1} + \frac{5}{(2\rho\nu_3 t_1)^2} + \frac{5}{(2\rho\nu_3 t_1)^3}\right], \qquad (100a)$$

where

$$H^2(\tau) = \frac{1}{2}e^{-2\rho\nu_2\tau} + \tau^2\left[2.1005\left(\frac{e^{-\rho\nu_1\tau} - e^{-\rho\nu_2\tau}}{\rho(\nu_2 - \nu_1)\tau}\right) + \frac{0.2682}{(2\rho\tau)^{1/2}}\left(\frac{1 - e^{-2\rho\nu_2\tau}}{2\rho\nu_2\tau}\right)^{1/2}\right]^2. \qquad (100b)$$

Equation (100a) is true for any motion $\mathcal{V}_\tau$ whatever if $H(\tau)$ is interpreted as $\| q(\tau) \| / \| \boldsymbol{B}(0) \|$. The function $H(\tau)$ can be readily computed only for axisymmetric toroidal velocities, so for velocities outside this class inequality (100a) will not be useful.

For completeness, note the following special case of inequality (98):

$$\| \mathfrak{J}_2\mathcal{V}_\tau \|^2 \leq 6H^2(\tau). \qquad (100c)$$

(F) Special Cases for Which a Free Rigid Decay is Unnecessary

Inequalities like (100) can be proved for the magnetic operator giving the effect of the persistence for a time τ of any axisymmetric toroidal fluid velocity, $\boldsymbol{u} = r\sin\theta\omega(r, \theta)\hat{\boldsymbol{\phi}}$, as long as $\omega(r, \theta)$ is sufficiently smooth. As already remarked, the author does not know whether, when the free decay $\mathfrak{D}_{t_1}$ is omitted, a bound can be obtained which, like the bound (100), grows only linearly with τ or whether in consequence of this omission nothing better than the exponential bound (33) can be obtained. However, there are two special cases in which a bound linear in τ can be obtained even if the free decay is omitted.

For an arbitrary axisymmetric toroidal velocity field $\boldsymbol{u}$ Eq. (78a) for the toroidal field scalar q can be multiplied by $-\Lambda^2 q$ and integrated over V to give

$$\frac{1}{2}\frac{d}{dt}\int_V |\boldsymbol{\Lambda} q|^2 + \int_V \frac{\partial q}{\partial\theta}\frac{\partial q}{\partial\phi}\frac{\partial\omega}{\partial\theta} + \rho\int_V \Lambda^2 q\nabla^2 q = \int_V \boldsymbol{\Lambda} h\cdot\boldsymbol{\Lambda} q. \qquad (101)$$

If the initial field was axisymmetric, $\partial q/\partial \phi = 0$ at all times. In this special case and also in the special case where $\partial \omega/\partial \theta = 0$ the second integral in Eq. (101) vanishes. Then Schwarz's inequality and inequality (58) imply that

$$\frac{d}{dt} \| \boldsymbol{T} \| + \rho \nu_2 \| \boldsymbol{T} \| \leq \| \boldsymbol{\Lambda} h \|,$$

and

$$\| \boldsymbol{T}(\tau) \| \leq \| \boldsymbol{T}(0) \| e^{-\rho \nu_2 \tau} + e^{-\rho \nu_2 \tau} \int_0^\tau e^{\rho \nu_2 t} \| \boldsymbol{\Lambda} h(t) \| \, dt. \tag{102}$$

Equation (81) for h together with the bound (64) on $w = \Lambda^2 p$ gives a bound on $\| \boldsymbol{\Lambda} h \|$ from which it follows that there are constants H and K such that

$$\| \mathfrak{I}\mathfrak{V}_\tau \boldsymbol{B}(0) \| \leq \| \boldsymbol{B}(0) \| (e^{-\rho \nu_2 \tau} + H\tau^{1/2} + K\tau). \tag{103}$$

If $\partial \omega/\partial \theta = 0$, inequality (103) is true for any $\boldsymbol{B}(0)$, and therefore $\| \mathfrak{I}\mathfrak{V}_\tau \| \leq e^{-\rho \nu_2 \tau} + H\tau^{1/2} + K\tau$. Furthermore, when $\partial \omega/\partial \theta = 0$, the equation $\partial p/\partial t + \omega \partial p/\partial \phi - \rho \nabla^2 p = 0$ for the poloidal field scalar p can be operated on by $\boldsymbol{\nabla} \times \boldsymbol{\Lambda}$ to give, in V,

$$\frac{\partial \boldsymbol{P}}{\partial t} + \omega \boldsymbol{\nabla} \times \boldsymbol{\Lambda} \frac{\partial p}{\partial \phi} + \rho \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{P} = \boldsymbol{0}.$$

Integrate the dot product of the left hand side of this equation and $\boldsymbol{P}$ over V. Integrate once by parts, using the condition $\boldsymbol{\nabla} \times \boldsymbol{P} = \boldsymbol{0}$ in $\mathcal{E} - V$ to evaluate the boundary term. There results

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{E}} | \boldsymbol{P} |^2 + \rho \int_V | \boldsymbol{\nabla} \times \boldsymbol{P} |^2 = 0.$$

The variational inequality (60) then implies that $\| \boldsymbol{P}(\tau) \| \leq \| \boldsymbol{P}(0) \| e^{-\rho \nu_1 \tau}$ or $\| \mathcal{P}\mathfrak{V}_\tau \| \leq e^{-\rho \nu_1 \tau}$.

In the other special case, when $\partial \omega/\partial \theta \neq 0$ but $\boldsymbol{B}(0)$ is axisymmetric, the poloidal field behaves as if the fluid were motionless, so

$$\| \mathcal{P}\mathfrak{V}_\tau \boldsymbol{B}(0) \| \leq \| \boldsymbol{B}(0) \| e^{-\rho \nu_1 \tau}. \tag{104}$$

For an arbitrary $\omega(r, \theta)$, inequalities (103) and (104) have been proved only if $\boldsymbol{B}(0)$ is axisymmetric; in inequality (103) $H = 0$ if $\boldsymbol{B}(0)$ is axisymmetric.

10. THE TRANSFER OF ENERGY FROM $\boldsymbol{T}_{101}$ to $\boldsymbol{P}_{101}$

(A) The Possibility of Such Transfer

Let the magnetic field $\boldsymbol{B}(t)$ produced from the initial field $\boldsymbol{B}(0)$ by the solenoidal velocity $\boldsymbol{u}$ be expanded as

$$\boldsymbol{B}(t) = \sum_{lmn} [a_{ln}{}^m(t) \boldsymbol{P}_{lmn} + b_{ln}{}^m(t) \boldsymbol{T}_{lmn}]. \tag{105}$$

If $\boldsymbol{B}(0) = \boldsymbol{T}_{101}$, then for what velocities $\boldsymbol{u}$ will there be times at which $a_{11}{}^{m}$ does not vanish? Elsasser (*10*) has given some examples of such velocities. Since $a_{11}{}^{m}(0) = 0$ when $\boldsymbol{B}(0) = \boldsymbol{T}_{101}$, any velocity will be of the desired type if it makes at least one of the time derivatives of $a_{11}{}^{m}(t)$ initially differ from zero. One would expect that this class of velocities is large; how large it is will be shown in the present subsection.

In what follows it will be convenient to define

$$\boldsymbol{P}_{1x1} = -2^{-1/2}(\boldsymbol{P}_{111} + \boldsymbol{P}_{1-11}) = \boldsymbol{\nabla} \times \boldsymbol{\Lambda} p_{1x1}, \tag{106a}$$

$$\boldsymbol{P}_{1y1} = i2^{-1/2}(\boldsymbol{P}_{111} - \boldsymbol{P}_{1-11}) = \boldsymbol{\nabla} \times \boldsymbol{\Lambda} p_{1y1}, \tag{106b}$$

$$\boldsymbol{P}_{1z1} = \boldsymbol{P}_{101} = \boldsymbol{\nabla} \times \boldsymbol{\Lambda} p_{1z1}. \tag{106c}$$

The function $\tilde{p}(r) = \sec\theta p_{101}(r, \theta)$ depends only on r, and if ξ is any of x, y, and z, $p_{1\xi1} = \xi r^{-1}\tilde{p}(r)$, so the fields $\boldsymbol{P}_{1x1}$ and $\boldsymbol{P}_{1y1}$ are obtained by rotating the field $\boldsymbol{P}_{1z1}$ until its external dipole moment points along the $\hat{\boldsymbol{x}}$ or $\hat{\boldsymbol{y}}$ axis instead of the $\hat{\boldsymbol{z}}$ axis.

Define $a_{11}{}^{x}$, $a_{11}{}^{y}$, $a_{11}{}^{z}$ by the equation

$$\sum_{m=-1}^{1} a_{11}{}^{m}\boldsymbol{P}_{1m1} = a_{11}{}^{x}\boldsymbol{P}_{1x1} + a_{11}{}^{y}\boldsymbol{P}_{1y1} + a_{11}{}^{z}\boldsymbol{P}_{1z1}.$$

It is not difficult to show from Eqs. (1) (Ref. *9*) that if ξ is any of x, y, z, then

$$\left(\frac{d}{dt} + \rho\nu_1\right) a_{1\xi1} = -\int_V \boldsymbol{B}(t)\cdot[\boldsymbol{u} \times \boldsymbol{\nabla} \times \boldsymbol{P}_{1\xi1}]. \tag{107}$$

Since $\boldsymbol{u}$ is solenoidal, it has a representation in the form

$$\boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{\Lambda} U + \boldsymbol{\Lambda} V, \tag{108}$$

where if $U = V = \partial U/\partial r = 0$ on S, $\boldsymbol{u} = \boldsymbol{0}$ on S, and if $U = V = \partial U/\partial r = \partial V/\partial r = \partial^2 U/\partial r^2$ on S, $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{\nabla u} = \boldsymbol{0}$ on S, while if U and V are analytic in x, y, z then so are u_x, u_y, and u_z. From the definitions (41) and (42) of p_{101} and q_{101}, Eqs. (107) and (108) imply that when $t = 0$

$$\frac{d}{dt} a_{11}{}^{x} = \frac{-3\alpha_{01}}{4\pi j_1(\alpha_{01})j_2(\alpha_{11})} \int_V \left(\frac{\Lambda^2 U}{r}\right) j_1(\alpha_{11}r)j_1(\alpha_{01}r) \sin\theta \sin\phi, \tag{109a}$$

$$\frac{d}{dt} a_{11}{}^{y} = \frac{3\alpha_{01}}{4\pi j_1(\alpha_{01})j_2(\alpha_{11})} \int_V \left(\frac{\Lambda^2 U}{r}\right) j_1(\alpha_{11}r)j_1(\alpha_{01}r) \sin\theta \cos\phi, \tag{109b}$$

$$\frac{d}{dt} a_{11}{}^{z} = 0, \tag{109c}$$

and

$$\frac{d^2}{dt^2} a_{11}^z = \frac{-3\alpha_{11}^2}{2\pi\alpha_{01} j_1(\alpha_{01}) j_2(\alpha_{11})} \int_V \left(\frac{\Lambda^2 U}{r}\right)\left(\frac{\partial^2 V}{\partial\phi^2} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial\phi} \sin\theta \frac{\partial U}{\partial\theta}\right) \cdot j_1(\alpha_{11} r) j_1(\alpha_{01} r). \tag{109d}$$

Any choice of the scalars U and V for which the appropriate integral (109) fails to vanish gives a velocity (108) capable of transferring energy from $\boldsymbol{T}_{101}$ into one of the modes $\boldsymbol{P}_{1x1}$, $\boldsymbol{P}_{1y1}$, $\boldsymbol{P}_{1z1}$.

As an example, let $V = 0$ and $U = f(r) \sin\theta \cos\phi$, where $f(r) = r$ when $0 \leq r \leq 1 - \epsilon$ while in the thin shell $1 - \epsilon \leq r \leq 1$ the function $f(r)$ is brought smoothly down so that $f(1) = f'(1) = 0$. Then $\boldsymbol{u}$ is continuously differentiable and vanishes on the surface of the fluid. Inside the sphere $S_{1-\epsilon}$, $\boldsymbol{u} = -2\hat{\boldsymbol{x}}$, and the fluid thus translated in the negative $\hat{\boldsymbol{x}}$ direction inside $S_{1-\epsilon}$ is returned in the opposite direction in the outer part of the shell $1 - \epsilon \leq r \leq 1$. From formulas (109), $[da_{11}^x/dt]_{t=0} = [da_{11}^z/dt]_{t=0} = 0$ while

$$\left[\frac{da_{11}^y}{dt}\right]_{t=0} = \frac{-2\alpha_{01}}{j_1(\alpha_{01}) j_2(\alpha_{11})} \int_0^1 r f(r) j_1(\alpha_{11} r) j_1(\alpha_{01} r)\, dr$$

$$\approx \frac{-2\alpha_{01}}{j_1(\alpha_{01}) j_2(\alpha_{11})} \int_0^1 r^2 j_1(\alpha_{11} r) j_1(\alpha_{01} r)\, dr = \frac{-2\alpha_{01}\alpha_{11}}{\alpha_{11}^2 - \alpha_{01}^2}.$$

Therefore a purely poloidal flow which carries most of the interior of the fluid in the positive $\hat{\boldsymbol{x}}$ direction will initially transfer energy from the mode $\boldsymbol{T}_{101}$ [which might be called $\boldsymbol{T}_{1z1}$ by analogy with Eqs. (106)] into the mode $\boldsymbol{P}_{1y1}$ but not into $\boldsymbol{P}_{1x1}$ or $\boldsymbol{P}_{1z1}$.

As another example, let $U = V = f(r) \sin\theta \cos\phi$ where $f(r)$ is the function described in the preceding paragraph. Then $[d^2 a_{11}^z/dt^2]_{t=0} \neq 0$ if $\boldsymbol{B}(0) = \boldsymbol{T}_{101}$. Inside the sphere $S_{1-\epsilon}$, $\boldsymbol{u} = -2\hat{\boldsymbol{x}} + z\hat{\boldsymbol{y}} - y\hat{\boldsymbol{z}}$. Here the translation along the $\hat{\boldsymbol{x}}$ axis produces $\boldsymbol{P}_{1y1}$ from $\boldsymbol{T}_{1z1}$ while the rigid rotation about the $\hat{\boldsymbol{x}}$ axis transforms $\boldsymbol{P}_{1y1}$ into $\boldsymbol{P}_{1z1}$. As is to be expected in such a second order process, $[da_{11}^z/dt]_{t=0} = 0$.

The foregoing remarks prove that there is a large class of fluid motions capable of transforming the initial field $\boldsymbol{B}(0) = \boldsymbol{T}_{101}$ into a field with energy in the $\boldsymbol{P}_{101}$ mode. However, Eqs. (109) are not useful except for times so short that the total energy produced in the $\boldsymbol{P}_{101}$ mode (and all others) is much less than the amount initially present in $\boldsymbol{T}_{101}$. For a useful estimate of the velocity at which a self-sustaining dynamo must be operated, it is desirable to be able to treat larger energy transfers. Furthermore, as Eqs. (109) make clear, the initial production of $\boldsymbol{P}_{101}$ from $\boldsymbol{T}_{101}$ is accompanied by a much larger production of $\boldsymbol{P}_{1x1}$ and $\boldsymbol{P}_{1y1}$. This raises the question of whether dynamos can be constructed in which the external dipole moment does not shift through large angles during one decay time of $\boldsymbol{P}_{101}$.

In Section 10b below, the amount of energy transferred from $\boldsymbol{T}_{1z1}$ to $\boldsymbol{P}_{1y1}$ by a particular large fluid displacement will be estimated, and in Section 10c it will be shown that an arbitrarily large amount of this energy can be transferred to $\boldsymbol{P}_{1z1}$ so that each cycle of the dynamo to be constructed in Section 11 regenerates $\boldsymbol{P}_{1z1}$ without production of large amounts of $\boldsymbol{P}_{1y1}$ and $\boldsymbol{P}_{rx1}$.

(B) Generation of $\boldsymbol{P}_{1y1}$ from $\boldsymbol{T}_{1z1}$ by a Particular Fluid Displacement

To simplify the notation, a new coordinate system will be chosen, in which the new $\hat{\boldsymbol{x}}$ axis is the old $\hat{\boldsymbol{z}}$ axis and the new $\hat{\boldsymbol{z}}$ axis is the old axis $-\hat{\boldsymbol{x}}$. In the rest of Section 10b, x, y, z, r, θ, and ϕ will refer to the new coordinate system instead of the old one. In the new system let $\varpi = (x^2 + y^2)^{1/2}$. Stated in terms of the new coordinate axes, the problem is as follows: find the matrix element $(\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1})$ where $\mathfrak{U}$ is the magnetic operator corresponding to some as yet unspecified finite fluid motion and $\boldsymbol{T}_{1x1}$ is the field $\boldsymbol{B}(0)$ at the outset of this motion. In Cartesian components

$$\boldsymbol{B}(0) = \frac{B(r)}{r}(-z\hat{\boldsymbol{y}} + y\hat{\boldsymbol{z}}), \tag{110a}$$

where

$$B(r) = -\left(\frac{3}{4\pi}\right)^{1/2}\frac{j_1(\alpha_{11}r)}{j_2(\alpha_{11})}. \tag{110b}$$

Suppose the fluid motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq 1$, which produces finally the displacement of the fluid point $\boldsymbol{x}$ to $\boldsymbol{y}(\boldsymbol{x}, 1)$ has a velocity $\boldsymbol{u}$ and velocity gradient $\nabla\boldsymbol{u}$ which vanish on the fluid surface S. If the same final displacement is effected by the more rapid motion $\boldsymbol{y}(\boldsymbol{x}, \kappa t)$, $0 \leq t \leq \kappa^{-1}$, whose magnetic operator is $\mathfrak{U}_\kappa$, then Sections 6c and 6e make clear that

$$\lim_{\kappa\to\infty}(\boldsymbol{P}_{1y1}, \mathfrak{U}_\kappa\boldsymbol{T}_{1x1}) = (\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1})$$

where $\mathfrak{U}$ is the magnetic operator for the motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq 1$ in a fluid of resistivity zero.

Suppose that $\boldsymbol{u} = \nabla \times \boldsymbol{\Lambda}U + \boldsymbol{\Lambda}V$ vanishes on S but $\nabla\boldsymbol{u}$ does not; suppose also that $(\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1})$ is particularly easy to evaluate, $\mathfrak{U}$ being the magnetic operator of the motion resulting from the persistence of the velocity $\boldsymbol{u}$ for unit time in a fluid of resistivity zero. The remarks of the preceding paragraph are not directly applicable to this motion. Define the function $h_\epsilon(r)$ for small ϵ as follows: $h_\epsilon(r) = 1$ if $0 \leq r \leq 1 - \epsilon$, and $h_\epsilon(r)$ drops smoothly to zero in $1 - \epsilon \leq r \leq 1$. Then if $\boldsymbol{u}_\epsilon = \nabla \times \boldsymbol{\Lambda}(h_\epsilon U) + \boldsymbol{\Lambda}(h_\epsilon V)$ both $\boldsymbol{u}_\epsilon$ and $\nabla\boldsymbol{u}_\epsilon$ vanish on S, so the preceding paragraph does apply to the magnetic operator $\mathfrak{U}^\epsilon$ of the motion $\boldsymbol{y}_\epsilon(\boldsymbol{x}, t)$, $0 \leq t \leq 1$, produced by the persistence of $\boldsymbol{u}_\epsilon$ for unit time in a fluid of resistivity zero. From the expression for $(\boldsymbol{P}_{1y1}, \mathfrak{U}^{\epsilon}\boldsymbol{T}_{1x1})$ as a volume integral it is clear that

$$\lim_{\epsilon \to 0}(\boldsymbol{P}_{1y1}, \mathfrak{U}^{\epsilon}\boldsymbol{T}_{1x1}) = (\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1}).$$

Therefore, by choosing ϵ small enough, $(\boldsymbol{P}_{1y1}, \mathfrak{U}^{\epsilon}\boldsymbol{T}_{1x1})$ can be made very close to the easily computed $(\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1})$ and if the motion $\boldsymbol{y}_{\epsilon}(\boldsymbol{x}, t)$, $0 \leq t \leq 1$, is executed rapidly enough, its magnetic operator $\mathfrak{U}_{\rho}^{\epsilon}$ in a fluid of nonzero resistivity ρ produces a matrix element $(\boldsymbol{P}_{1y1}, \mathfrak{U}_{\rho}^{\epsilon}\boldsymbol{T}_{1x1})$ which is very close to $(\boldsymbol{P}_{1y1}, \mathfrak{U}^{\epsilon}\boldsymbol{T}_{1x1})$. The remarks of this and the preceding paragraph make the following clear: let $\mathfrak{U}$ be the magnetic operator for a fluid motion $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq 1$, in a fluid of resistivity zero whose velocity $\boldsymbol{u}$ vanishes on the surface S. Then fluid motions $\boldsymbol{y}_{\epsilon}(\boldsymbol{x}, t)$, $0 \leq t \leq 1$, not very different from $\boldsymbol{y}(\boldsymbol{x}, t)$, $0 \leq t \leq 1$, can be found which, if sufficiently speeded up, lead to magnetic operators $\mathfrak{U}_{\rho}^{\epsilon}$ in a fluid of fixed nonzero resistivity ρ whose matrix elements $(\boldsymbol{P}_{1y1}, \mathfrak{U}_{\rho}^{\epsilon}\boldsymbol{T}_{1x1})$ are arbitrarily close to $(\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1})$. Therefore the rest of the present Section 10b, is devoted to the evaluation of $(\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1})$ for a particularly simple motion in a fluid whose resistivity vanishes.

The motion to be considered is that produced by the persistence for unit time of the steady axisymmetric velocity $\boldsymbol{u} = \varpi^{-1}\nabla s \times \hat{\boldsymbol{\phi}}$ whose axisymmetric stream function s is

$$s = \tfrac{1}{4}\varpi^4(r^2 - 1)^2. \tag{111a}$$

Then

$$\boldsymbol{u} = \hat{\boldsymbol{r}}r^2(1 - r^2)^2 \sin^2\theta \cos\theta - \hat{\boldsymbol{\theta}}r^2(1 - r^2)(1 - 2r^2)\sin^3\theta. \tag{111b}$$

The symmetric tensor whose Cartesian components are $\frac{1}{2}(\partial u_i/\partial y_j + \partial u_j/\partial y_i)$ becomes the covariant dyadic $\frac{1}{2}(u_{i;j} + u_{j;i})$ in spherical polar coordinates, where now i and j take the values r, θ, and ϕ. In spherical coordinates it is not difficult to show that the largest characteristic root of the tensor $\frac{1}{2}(u_{i;j} + u_{j;i})$ obtained from the velocity (111b) is 1 and occurs at $r = 1$, $\theta = \pi/2$. If $\mathfrak{U}$ is the magnetic operator on $\mathfrak{B}$ produced by allowing the velocity (111b) to persist for unit time, then inequality (33) implies

$$\| \mathfrak{U} \| \leq e = 2.71828 \cdots. \tag{111c}$$

This inequality is true a fortiori for the operators $\mathfrak{U}_{\rho}^{\epsilon}$ of the preceding paragraph.

To compute $(\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1})$ it will be necessary to have an expression for the final position r, θ, ϕ of a fluid element which at the onset of the velocity (111b) was at the initial position r', θ', ϕ'. To find such an expression, introduce the new coordinates σ, χ, ϕ defined in terms of ϖ, z, ϕ by these equations:

$$2\varpi z = \sigma \sin\chi, \tag{112a}$$

$$2\varpi^2 - 1 = \sigma \cos\chi. \tag{112b}$$

Then $\sigma = 4\varpi^2(r^2 - 1) + 1$, so $0 \leq \sigma \leq 1$, and $\sigma = 0$ only at the point $\varpi = 2^{-1/2}$, $z = 0$. Since $\partial s/\partial \chi = 0$, the level lines of σ are the flow lines of the fluid velocity. The level lines of σ and χ are shown in Fig. 1. The motion (111) simply decreases the coordinate χ of every fluid particle, without affecting σ or ϕ. To see the details of this decrease in χ, define still another system of coordinates,

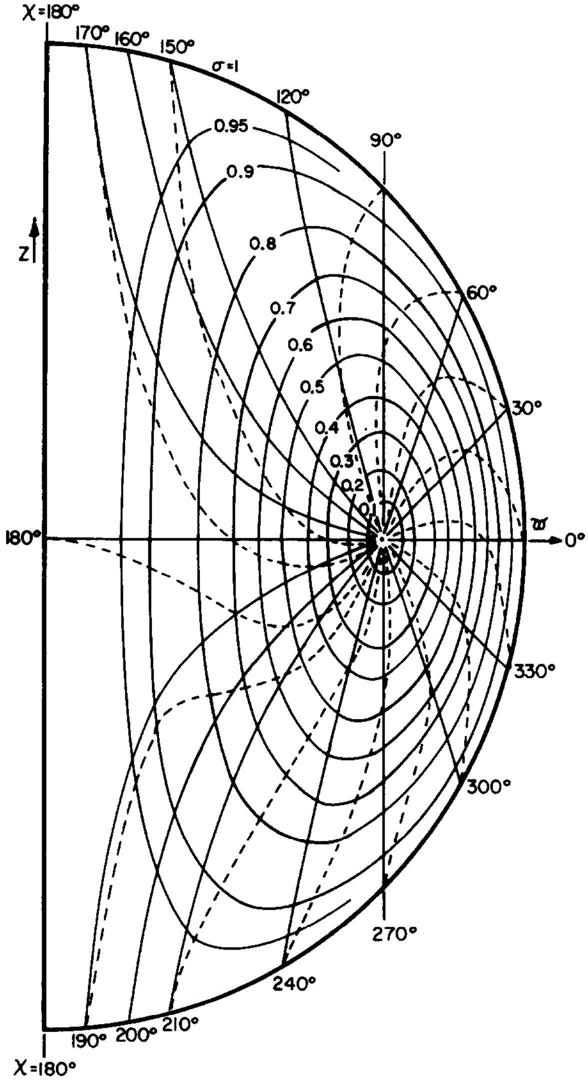

FIG. 1. Level lines of σ and χ in a meridian plane. The dotted lines are the initial positions of the fluid elements whose final positions are the solid level lines of χ.

σ, ε, ϕ: σ and ϕ are as before, while ε is given in terms of χ and σ by the equations

$$\sin \tfrac{1}{2}\chi = \mathrm{sn}_k\, \varepsilon, \qquad \cos \tfrac{1}{2}\chi = \mathrm{cn}_k\, \varepsilon, \tag{113a}$$

$$k^2 = \frac{2\sigma}{1+\sigma}. \tag{113b}$$

Here sn_k and cn_k are the Jacobian elliptic functions (*33*) defined by the equations $\mathrm{sn}_k^2\varepsilon + \mathrm{cn}_k^2\varepsilon = 1$ and

$$\varepsilon = \int_0^{\mathrm{sn}_k \varepsilon} \frac{dy}{(1-y^2)^{1/2}(1-k^2y^2)^{1/2}}.$$

The position at time t of a particle in the fluid moving with velocity (111) is

$$\sigma(t) = \sigma(0), \qquad \phi(t) = \phi(0), \qquad \varepsilon(t) = \varepsilon(0) - \frac{(1-\sigma^2)t}{2^{5/2}}. \tag{114}$$

Therefore the initial position r', θ', ϕ' and the final position r, θ, ϕ of a fluid particle are related by Eqs. (112), (113), and

$$\varepsilon' = \varepsilon + \frac{1-\sigma^2}{2^{5/2}}, \qquad \sigma' = \sigma, \qquad \phi' = \phi. \tag{115}$$

Now that the fluid displacement has been explicitly obtained, Lundquist's (*19*) integral of the dynamo equation (1a) can be used to compute the field $\mathfrak{U}\boldsymbol{T}_{1y1}$. If B^σ, B^χ, B^ϕ are the contravariant components of $\boldsymbol{B}$ in the curvilinear coordinate system σ, χ, ϕ, then

$$B^\sigma(\sigma,\chi,\phi) = B^{\sigma'}(\sigma',\chi',\phi'), \qquad B^\chi(\sigma,\chi,\phi) = B^{\chi'}(\sigma',\chi',\phi'),$$
$$B^\phi(\sigma,\chi,\phi) = B^{\phi'}(\sigma',\chi',\phi'). \tag{116}$$

From Eqs. (110) for the initial field $\boldsymbol{B}'(0) = \boldsymbol{T}_{1x1}$ it is not difficult to find the contravariant components of that field in the coordinates σ', χ', ϕ':

$$-\left(\frac{4\pi}{3}\right)^{1/2} j_2(\alpha_{11})B^{\sigma'}(\sigma',\chi',\phi') = \frac{j_1(\alpha_{11}r')}{r'} \sin\phi' \sin\chi' \left(\frac{1-\sigma'^2}{1+\sigma'\cos\chi'}\right),$$
$$-\left(\frac{4\pi}{3}\right)^{1/2} j_2(\alpha_{11})B^{\chi'}(\sigma',\chi',\phi') = \frac{j_1(\alpha_{11}r')}{r'} \sin\phi' \left(\frac{\cos\chi' + 2\sigma' + \sigma'^2\cos\chi'}{\sigma'(1+\sigma'\cos\chi')}\right), \tag{117}$$
$$-\left(\frac{4\pi}{3}\right)^{1/2} j_2(\alpha_{11})B^{\phi'}(\phi',\chi',\phi') = \frac{-j_1(\alpha_{11}r')}{r'} \cos\phi' \sin\chi' \left(\frac{\sigma'}{1+\sigma'\cos\chi'}\right),$$

where

$$r' = \left[\frac{1+\sigma'^2+2\sigma'\cos\chi'}{2(1+\sigma'\cos\chi')}\right]^{1/2}. \tag{118}$$

Now if $\boldsymbol{B} = \mathfrak{U}\boldsymbol{B}(0)$,

$$(\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{B}(0)) = (\boldsymbol{P}_{1y1}, \boldsymbol{B}) = -\alpha_{01}^2 \int_V p_{1y1} r B_r.$$

The covariant and contravariant components B_r and B^r are the same in spherical coordinates. Therefore, it follows from Eqs. (115), (116), and (117) that

$$\begin{aligned} -4\left(\frac{4\pi}{3}\right)^{1/2} & j_2(\alpha_{11})\varpi^2 B^r(r, \theta, \phi) \\ &= \frac{(1 - r^2) j_1(\alpha_{11} r')}{r r'} \sin \phi \left\{\frac{(1 + \sigma^2) \sin (\chi' - \chi) + 2\sigma(\sin \chi' - \sin \chi)}{1 + \sigma \cos \chi'}\right\}. \end{aligned} \tag{119}$$

Finally, since the Jacobian determinant $| \partial(x, y, z)/\partial(\sigma, \chi, \phi) | = 8\varpi/\sigma$,

$$\begin{aligned} (\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1}) &= \frac{3\pi^2}{64 j_2(\alpha_{11})} \int_0^1 \sigma \, d\sigma \int_0^{2\pi} \frac{j_1(\alpha_{01} r)}{r} \frac{j_1(\alpha_{11} r')}{r'} \\ &\quad \cdot (1 - r^2) \left\{\frac{(1 + \sigma^2) \sin (\chi' - \chi) + 2\sigma(\sin \chi' - \sin \chi)}{(1 + \sigma \cos \chi)(1 + \sigma \cos \chi')}\right\} d\chi \end{aligned} \tag{120a}$$

where r' and r are given by formula (118) with and without primes, and the primed variables are obtained from the unprimed ones via the coordinate transformation (113) and the fluid displacement (115).

With the help of the addition formulas for the Jacobian elliptic functions (*33*) the integrand in Eq. (26) can be expressed in terms of σ and χ using only square roots of rational functions of σ and trigonometric functions of χ. The author did not attempt to obtain this expression since the chances were that the integral would have to be evaluated numerically in any case, and that integral is in a very convenient form in Eq. (120) for numerical evaluation with the help of trigonometric tables and tables of $\mathrm{sn}_k\varepsilon$ (see Ref. *34*). Mrs. Joan Peskin carried out such a numerical evaluation, and obtained

$$(\boldsymbol{P}_{1y1}, \mathfrak{U}\boldsymbol{T}_{1x1}) = 0.277 \tag{120b}$$

corresponding to an energy transfer of 7.67%. Larger displacements within limits will give larger energy transfers. Mrs. Peskin and the author found one which transferred 20% of the energy of $\boldsymbol{T}_{1x1}$ into $\boldsymbol{P}_{1y1}$. Such large displacements are objectionable in constructing a dynamo because the magnetic operators they produce have norms exponentially large in the amplitude of the displacement when the finite resistivity of the fluid is taken into account, and may produce large stray fields. Incidentally from Eq. (119) it is clear that

$$(\boldsymbol{P}_{1x1}, \mathfrak{U}\boldsymbol{T}_{1x1}) = (\boldsymbol{P}_{1z1}, \mathfrak{U}\boldsymbol{T}_{1z1}) = 0.$$

Before leaving the operator $\mathfrak{U}$ it will be convenient to point out that better bounds on its matrix elements than inequality (111c) can be obtained. In particular,

$$\| \mathcal{P}_1 \mathcal{U} \| \leq 0.447. \tag{121}$$

To prove inequality (121), observe that in consequence of Eqs. (107), if $\xi = x, y$, or z,

$$\frac{d}{dt} \alpha_{11}{}^{\xi} + \rho\nu_1 \alpha_{11}{}^{\xi} = -\lambda_{11} \int_V \boldsymbol{B}(t) \cdot [\boldsymbol{u} \times \boldsymbol{\Lambda} p_{1\xi 1}]$$

so

$$\left| \left(\frac{d}{dt} + \rho\nu_1 \right) \alpha_{11}{}^{\xi} \right| \leq \lambda_{11} \| \boldsymbol{u} \times \boldsymbol{\Lambda} p_{1\xi 1} \| \, \| \boldsymbol{B}(t) \|. \tag{122}$$

For the particular velocity (111b), as has been remarked, the $m(t)$ occurring in Eq. (33) is 1, so $\| \boldsymbol{B}(t) \| \leq \| \boldsymbol{B}(0) \| e^t$, and therefore from inequality (122)

$$| \alpha_{11}{}^{\xi} | \leq \| \boldsymbol{B}(0) \| \left(\frac{e^t - e^{-\rho\nu_1 t}}{1 + \rho\nu_1} \right) \nu_1 \| \boldsymbol{u} \times \boldsymbol{\Lambda} p_{1\xi 1} \|. \tag{123}$$

The velocity (111b) and the functions $p_{1\xi 1}$ are simple enough that the norms $\| \boldsymbol{u} \times \boldsymbol{\Lambda} p_{1\xi 1} \|$ can be computed exactly:

$$\| \boldsymbol{u} \times \boldsymbol{\Lambda} p_{1z1} \|^2 = \frac{3}{4\pi} \int_V r^4 (1 - r^2)^2 \sin^6 \theta j_1{}^2(\alpha_{01} r) \cdot [(1 - r^2)^2 \cos^2 \theta + (1 - 2r^2)^2 \sin^2 \theta]; \tag{124}$$

if $\xi = x$ or y

$$2 \| \boldsymbol{u} \times \boldsymbol{\Lambda} p_{1\xi 1} \|^2 = \frac{3}{4\pi} \int_V r^4 (1 - r^2)^2 \sin^4 \theta \cos^2 \theta j_1{}^2(\alpha_{01} r) \cdot [(1 - r^2)^2 (1 + \cos^2 \theta) + (1 - 2r^2)^2 \sin^2 \theta]. \tag{125}$$

From the values of these integrals

$$\sum_{\xi} | a_{11}{}^{\xi} |^2 \leq \| \boldsymbol{B}(0) \|^2 \left[\frac{e^t - e^{-\rho\nu_1 t}}{1 + \rho\nu_1} \right]^2 (0.0687). \tag{126}$$

If the motion is performed very rapidly, ρ is very small (see Section 6c) so the term in brackets in inequality (126) is essentially $e - 1$. Since the sum on the left of that inequality is $\| \mathcal{P}_1 \mathcal{U} \boldsymbol{B}(0) \|^2$, inequality (121) follows immediately.

(c) Rotating the External Dipole Moment without Moving the Fluid Surface

The "new coordinate axes" introduced at the beginning of subsection 10b will be used also in the present subsection. In subsection 10b a motion was exhibited which from an initial field $\boldsymbol{B}(0) = \boldsymbol{T}_{1x1}$ produced energy in the $\boldsymbol{P}_{1y1}$ mode and none in the $\boldsymbol{P}_{1x1}$ or $\boldsymbol{P}_{1z1}$ modes. The initial field $\boldsymbol{T}_{1x1}$ was itself generated in subsection 9b from a field $\boldsymbol{P}_{1x1}$ (or $\boldsymbol{P}_{1z1}$ in terms of the "old coordinate axes"), and the question now arises whether the field $\boldsymbol{P}_{1y1}$ just produced from $\boldsymbol{T}_{1x1}$ can be converted to the original poloidal field $\boldsymbol{P}_{1x1}$.

Once a dipole moment in the $\hat{\boldsymbol{y}}$ direction has been produced, it is clear intuitively or from subsection 6b that if the whole fluid is rotated rigidly through 90° about the $\hat{\boldsymbol{z}}$ axis, the dipole moment will then point in the $\hat{\boldsymbol{x}}$ direction, and all of the energy in the $\boldsymbol{P}_{1y1}$ mode will have been transferred into $\boldsymbol{P}_{1x1}$. However, since the fluids dealt with in this paper cannot move at their surfaces, they cannot perform such a rigid rotation. Can the effects of a rigid rotation be duplicated by allowing the interior of the sphere $S_{1-\epsilon}$ to rotate rigidly while the angular velocity in the thin shell $1 - \epsilon \leq r \leq 1$ drops smoothly from its value $-\omega_1$ at $r = 1 - \epsilon$ to zero at $r = 1$? Since to duplicate a rigid rotation, ϵ would presumably have to be small, leading to a large shear in the outer shell, the answer is not obviously yes. It is yes, nonetheless.

By subsection 6b, the whole process can be viewed from a reference frame rotating with the same angular velocity $-\omega_1$ as the interior of the sphere $S_{1-\epsilon}$. Therefore, the question is as follows: let $\mathcal{R}_\epsilon$ be the magnetic operator on $\mathcal{B}$ produced by the persistence for some fixed time t of the velocity $\boldsymbol{u} = r \sin\theta\omega(r)\hat{\boldsymbol{\phi}}$ where $\omega(r) = 0$ if $0 \leq r \leq 1 - \epsilon$ and $\omega(r)$ rises smoothly from zero at $r = 1 - \epsilon$ to ω_1 at $r = 1$. Let $\mathcal{R}_0$ denote the free decay operator $\mathfrak{D}_t$. Then can $\mathcal{R}_\epsilon$ be made close to $\mathcal{R}_0$ by choosing ϵ small? As has happened so often already in this paper, it will be necessary to follow the rigid rotation by a short period t_2 of free decay in order to achieve the desired conclusion, which is

$$\lim_{\epsilon \to 0} \| \mathfrak{D}_{t_2}(\mathcal{R}_\epsilon - \mathcal{R}_0) \| = 0 \quad \text{if} \quad t_2 > 0. \tag{127}$$

For concreteness it will be assumed that in $1 - \epsilon \leq r \leq 1$,

$$\omega_\epsilon(r) = \omega_1 \left(\frac{r - 1 + \epsilon}{\epsilon} \right)^2, \tag{128}$$

[and, of course, $\omega(r) = 0$ if $0 \leq r \leq 1 - \epsilon$] although the results would be the same except for numerical coefficients if $\omega_\epsilon(r)$ were any piecewise continuously differentiable function whose derivative did not become very much larger than $\epsilon^{-1}\omega_1$, the minimum required to get from zero to ω_1 in the short interval $1 - \epsilon \leq r \leq 1$. If the angular velocity (128) persists for a time t, it will be shown that

$$\| \mathfrak{D}_{t_2}\mathcal{P}(\mathcal{R}_\epsilon - \mathcal{R}_0) \| \leq 66\epsilon^{1/2} e^{-\rho\nu_1 t_2} \left(\frac{\omega_2 t}{\pi} \right) \left[1 + \frac{19}{6\kappa_1} + \frac{19}{2\kappa_1^2} + \frac{19}{\kappa_1^3} + \frac{19}{\kappa_1^4} \right]^{1/2} \tag{129a}$$

where $\kappa_1 = 2\rho\nu_1 t_2$, and

$$\| \mathfrak{D}_{t_2}\mathfrak{T}(\mathfrak{R}_\epsilon - \mathfrak{R}_0) \| \leq 1314\epsilon^{1/2}e^{-\rho\nu_2 t_2}\left(\frac{\omega_1 t}{\pi}\right)\left(\frac{\omega_1 t + \epsilon}{\pi}\right) \cdot\left[1 + \frac{14}{3\kappa_2} + \frac{56}{3\kappa_2^2} + \frac{56}{\kappa_2^3} + \frac{112}{\kappa_2^4} + \frac{112}{\kappa_2^5}\right]^{1/2} \quad (129b)$$

where $\kappa_2 = 2\rho\nu_2 t_2$.

To prove relation (129a), let $\boldsymbol{B}(t)$ be the field produced from $\boldsymbol{B}(0)$ by the actual motion (128), while $\boldsymbol{B}^0(t) = \mathfrak{R}_0\boldsymbol{B}(0)$ is obtained from $\boldsymbol{B}(0)$ by free decay for the same time t. After motion (128) and a free decay for time t_2, the poloidal energy remaining in the difference field $\mathfrak{D}_{t_2}[\boldsymbol{B}(t) - \boldsymbol{B}^0(t)]$ is, in an obvious notation,

$$\| \mathfrak{D}_{t_2}\mathcal{P}(\mathfrak{R}_\epsilon - \mathfrak{R}_0)\boldsymbol{B}(0) \|^2 = \sum_{lmn} | [\boldsymbol{P}_{lmn}, \boldsymbol{P}(t) - \boldsymbol{P}^0(t)] |^2 e^{-2\rho\lambda_{ln}t_2}.$$

If $w = \boldsymbol{r}\cdot\boldsymbol{B} = \Lambda^2 p$, then $| (\boldsymbol{P}_{lmn}, \boldsymbol{P} - \boldsymbol{P}^0) | = \lambda_{ln} | (p_{lmn}, w - w^0) |$. Defining

$$y_{lmn}(t) = (p_{lmn}, w - w^0) \quad \text{and} \quad k_{ln}^2(t) = \sum_{m=-l}^{l} | y_{lmn} |^2,$$

then

$$\| \mathfrak{D}_{t_2}\mathcal{P}(\mathfrak{R}_\epsilon - \mathfrak{R}_0)\boldsymbol{B}(0) \|^2 \leq \sum_{ln} \lambda_{ln}^2 k_{ln}^2(t) e^{-2\rho\lambda_{ln}t_2}.$$

To estimate $k_{ln}(t)$ multiply the equation for the poloidal scalar of the difference field,

$$\frac{\partial(w - w^0)}{\partial t} - \rho\nabla^2(w - w^0) = -\omega\frac{\partial w}{\partial\phi},$$

by p_{lmn}^* and integrate over V, obtaining

$$\left(\frac{d}{dt} + 2\rho\lambda_{ln}\right) y_{lmn}(t) = -\int_V \omega\frac{\partial w}{\partial\phi} p_{lmn}^* = -im\int_V \omega w p_{lmn}^*.$$

Multiply this last equation by y_{lmn}^*, add the complex conjugate equation, and sum over m. The result is

$$\begin{aligned}\left(\frac{d}{dt} + 2\rho\lambda_{ln}\right) k_{ln}^2(t) &= -i\int_V \omega w \sum_{m=-l}^{l} m(y_{lmn}^* p_{lmn}^* - y_{lmn}p_{lmn}) \\ &\leq \| \omega \| \, \| w \| \sup\left| \sum_{m=-l}^{l} m(y_{lmn}^* p_{lmn}^* - y_{lmn}p_{lmn}) \right| \\ &\leq 2k_{ln} \| \omega \| \, \| w \| \sup\left[\sum_{m=-l}^{l} m^2 | p_{lmn} |^2 \right]^{1/2}\end{aligned}$$

so

$$\left(\frac{d}{dt} + \rho\lambda_{ln}\right) k_{ln}(t) \leq l \parallel \omega \parallel \parallel w \parallel \sup \left[\sum_{m=-l}^{l} \mid p_{lmn} \mid^2 \right]^{1/2}.$$

Since $k_{ln}(0) = 0$, inequality (65) implies that

$$k_{ln}(t) \leq t \parallel \omega \parallel \parallel \mathcal{P}\boldsymbol{B}(0) \parallel l \sup \left[\sum_{m=-l}^{l} \mid p_{lmn} \mid^2 \right]^{1/2}$$

and

$$\parallel \mathfrak{D}_{t_2}\mathcal{P}(\mathcal{R}_\epsilon - \mathcal{R}_0)\boldsymbol{B}(0) \parallel^2$$
$$\leq t^2 \parallel \omega \parallel^2 \parallel \mathcal{P}\boldsymbol{B}(0) \parallel^2 \sum_{ln} \lambda_{ln}^2 l^2 \left[\sup \sum_{m=-l}^{l} \mid p_{lmn} \mid^2 \right] e^{-2\rho\lambda_{ln}t_2}. \quad (130)$$

From the definition (128), $\parallel \omega \parallel^2 \leq 4\pi\omega_1^2\epsilon/5$, so there remains only to evaluate the sums in the expression above.

In consequence of the addition theorem for spherical harmonics,

$$\sum_{m=-l}^{l} \mid Y_l^m \mid^2 = \frac{2l+1}{4\pi},$$

so

$$\sum_{m=-l}^{l} \mid p_{lmn} \mid^2 = \frac{2}{l(l+1)} \left(\frac{2l+1}{4\pi}\right) \frac{j_l^2(\alpha_{l-1,n}r)}{\alpha_{l-1,n}^2 j_l^2(\alpha_{l-1,n})}.$$

Since (Ref. *28*, p. 50) for any x and $l \geq 0$

$$j_l(x) = \frac{(-i)^l}{2} \int_{-1}^{1} e^{itx} P_l(t)\, dt$$

an application of Schwarz's inequality gives $\mid j_l(x) \mid^2 \leq (2l+1)^{-1}$. It is shown in Appendix I that, if $l \geq 1$,

$$\alpha_{l-1,n}^2 j_l^2(\alpha_{l-1,n}) \geq \frac{1}{1.48 l^{1/3}}. \quad (131)$$

Therefore

$$\sum_{m=-l}^{l} \mid p_{lmn} \mid^2 \leq \left(\frac{1.48}{2\pi}\right) \frac{1}{l^{2/3}(l+1)}$$

and from inequality (130),

$$\parallel \mathfrak{D}_{t_2}\mathcal{P}(\mathcal{R}_\epsilon - \mathcal{R}_0) \parallel^2 \leq \left(\frac{\omega_1 t}{\pi}\right)^2 \left(\frac{2\epsilon}{5}\right) (1.48) \sum_{l=1}^{\infty} \frac{l^{4/3}}{(l+1)} \sum_{n=1}^{\infty} \alpha_{l-1,n}^4 \exp -2\rho\alpha_{l-1,n}^2 t_2 .$$

The sum in this inequality can be bounded by means of an argument essentially the same as that used to bound the sum in inequality (68). Therefore the details of this argument will be omitted; its result is inequality (129a).

To prove inequality (129b) is somewhat more troublesome. The toroidal energy in the difference field $\mathfrak{D}_{t_2}[\boldsymbol{B}(t) - \boldsymbol{B}^0(t)]$ resulting after the motion (128) has persisted for a time t and then the fluid has been motionless for a time t_2 is

$$\| \mathfrak{D}_{t_2}\mathfrak{J}(\mathfrak{R}_\epsilon - \mathfrak{R}_0)\boldsymbol{B}(0) \|^2 = \sum_{lmn} | [\boldsymbol{T}_{lmn}, \boldsymbol{T}(t) - \boldsymbol{T}^0(t)] |^2 e^{-2\rho\mu_{ln}t_2}$$

$$= \sum_{lmn} l^2(l+1)^2 | [q_{lmn}, q(t) - q^0(t)] |^2 e^{-2\rho\mu_{ln}t_2}$$

in an obvious notation. Now let $y_{lmn}(t) = [q_{lmn}, q(t) - q^0(t)]$ and

$$k_{ln}^2(t) = \sum_{m=-l}^{l} | y_{lmn}(t) |^2.$$

The equation for the toroidal scalar of the difference field is, in the present situation,

$$\frac{\partial}{\partial t}(q - q^0) - \rho\nabla^2(q - q^0) = \frac{d\omega}{dr}\left[\sin\theta\frac{\partial p}{\partial\theta} + \frac{2}{3^{1/2}} p_1^0(r, t) Y_0^0\right] - \omega\frac{\partial q}{\partial\phi}$$

where

$$p = \sum_{lm} p_l^m(r, t) Y_l^m(\theta, \phi).$$

Multiplying the differential equation by q_{lmn}^* and integrating over V gives

$$\left(\frac{d}{dt} + \rho\mu_{ln}\right) y_{lmn} = \int_V \left[\frac{d\omega}{dr} j_0(\pi r)\sin\theta\right]\left[\frac{q_{lmn}^*}{j_0(\pi r)}\right]\left(\frac{\partial p}{\partial\theta}\right) - im\int_V [\omega j_0(\pi r)]\left[\frac{q_{lmn}^*}{j_0(\pi r)}\right] q.$$

If this last equation is multiplied by y_{lmn}^* and the result summed over m and added to its complex conjugate, one finds

$$\left(\frac{d}{dt} + 2\rho\mu_{ln}\right) k_{ln}^2 = \int_V \left(\frac{\partial p}{\partial\theta}\right)\left[\sin\theta\, j_0(\pi r)\frac{d\omega}{dr}\right]\left[\sum_{m=-l}^{l}\frac{q_{lmn}^* y_{lmn}^* + q_{lmn} y_{lmn}}{j_0(\pi r)}\right]$$

$$- i\int_V q\omega j_0(\pi r)\left[\sum_{m=-l}^{l} m\frac{q_{lmn}^* y_{lmn}^* - q_{lmn} y_{lmn}}{j_0(\pi r)}\right].$$

Therefore

$$\left(\frac{d}{dt} + 2\rho\mu_{ln}\right) k_{ln}^2 \le \left\|\frac{\partial p}{\partial\theta}\right\| \left\|\sin\theta\frac{d\omega}{dr} j_0(\pi r)\right\| \sup\left|\sum_{m=-l}^{l}\frac{q_{lmn}^* y_{lmn}^* + q_{lmn} y_{lmn}}{j_0(\pi r)}\right|$$

$$+ \| q \| \, \| \omega j_0(\pi r) \| \sup\left|\sum_{m=-l}^{l} m\frac{q_{lmn}^* y_{lmn}^* - q_{lmn} y_{lmn}}{j_0(\pi r)}\right|.$$

Schwarz's inequality for $(2l + 1)$-dimensional spaces then implies

$$\left(\frac{d}{dt} + \rho\mu_{ln}\right) k_{ln} \leq \left\{\left\|\frac{\partial p}{\partial \theta}\right\| \left\|\sin\theta \frac{d\omega}{dr} j_0(\pi r)\right\| + l \, \| q \| \, \| \omega j_0(\pi r) \|\right\} \sup \left[\sum_{m=-l}^{l} \left|\frac{q_{lmn}}{j_0(\pi r)}\right|^2\right]^{1/2}. \tag{132}$$

Bounds must now be found for all the terms on the right-hand side of inequality (132). First $\| \partial p/\partial\theta \| \leq \| \mathbf{\Lambda} p \| \leq 2^{-1/2} \| w \|$ so

$$\left\|\frac{\partial p}{\partial \theta}\right\| \leq \frac{1}{2^{1/2}} \| \boldsymbol{B}(0) \| \, e^{-\rho\nu_1 t}$$

in consequence of inequality (65). Second, q satisfies the equation

$$\frac{\partial q}{\partial t} + \omega \frac{\partial q}{\partial \phi} - \rho\nabla^2 q = \frac{d\omega}{dr}\left[\sin\theta \frac{\partial p}{\partial \theta} + \frac{2}{3^{1/2}} p_1^{\,0}(r, t) Y_0^{\,0}\right],$$

from which follows by a now familiar argument that

$$\left(\frac{d}{dt} + \rho\nu_2\right) \| q \| \leq \sup\left|\frac{d\omega}{dr}\right| \left\|\sin\theta \frac{\partial p}{\partial\theta}\right\|,$$

and, since $\| q(0) \| \leq 2^{-1/2} \| \mathbf{\Lambda} q(0) \| \leq 2^{-1/2} \| \boldsymbol{B}(0) \|$,

$$\| q(t) \| \leq \frac{1}{2^{1/2}} \| \boldsymbol{B}(0) \| \left[e^{-\rho\nu_2 t} + t \sup\left|\frac{d\omega}{dr}\right|\right].$$

Third, from the form (128) of ω,

$$\| \omega j_0(\pi r) \| \leq \left(\frac{4\pi}{105}\right)^{1/2} \omega_1 \epsilon^{3/2},$$

$$\left\|\sin\theta \frac{d\omega}{dr} j_0(\pi r)\right\| \leq \frac{4}{3}\left(\frac{\pi}{5}\right)^{1/2} \omega_1 \epsilon^{1/2},$$

and

$$\sup\left|\frac{d\omega}{dr}\right| = \frac{2\omega_1}{\epsilon}.$$

Fourth and last,

$$\sum_{m=-l}^{l}\left|\frac{q_{lmn}}{j_0(\pi r)}\right|^2 = \left(\frac{2l + 1}{4\pi}\right)\frac{2}{l(l + 1)}\left|\frac{j_l(\alpha_{ln} r)}{j_0(\pi r)}\right|^2 \frac{1}{j_{l+1}^{\,2}(\alpha_{ln})}.$$

It is shown in Appendix I that if $l \geq 0$ and $0 \leq r \leq 1$,

$$\left|\frac{j_l(\alpha_{ln} r)}{j_0(\pi r)}\right| \leq 1. \tag{133}$$

This and inequality (131) imply that

$$\left[\sum_{m=-l}^{l}\left|\frac{q_{lmn}}{j_0(\pi r)}\right|^2\right]^{1/2} \le \left(\frac{1.48}{\pi}\right)^{1/2}\frac{\left(l+\frac{1}{2}\right)^{1/2}\alpha_{ln}}{l^{1/2}(l+1)^{1/3}}.$$

The bounds obtained above for the terms on the right side of inequality (132) lead, since $k_{ln}(0) = 0$, to the following inequality:

$$k_{ln}(t) \le \epsilon^{1/2}\, \| \boldsymbol{B}(0) \| \left(\frac{1.48}{\pi}\right)^{1/2}\frac{\alpha_{ln}\left(l+\frac{1}{2}\right)^{1/2}}{l^{1/2}(l+1)^{1/3}}\left\{\frac{4\pi}{3}\left(\frac{\pi}{10}\right)^{1/2}\left(\frac{\omega_1 t}{\pi}\right) + l\left(\frac{2\pi}{105}\right)^{1/2}\pi^2\left(\frac{\omega_1 t}{\pi}\right)\left(\frac{\omega_1 t+\epsilon}{\pi}\right)\right\}.$$

Since

$$\| \mathfrak{D}_{t_2}\mathfrak{I}(\mathfrak{R}_\epsilon - \mathfrak{R}_0)\boldsymbol{B}(0) \|^2 = \sum_{ln} l^2(l+1)^2 k_{ln}^2(t)e^{-2\rho\mu_{ln}t_2},$$

and since for any a and b, $(a+b)^2 \le 2a^2 + 2b^2$, it follows that

$$\| \mathfrak{D}_{t_2}\mathfrak{I}(\mathfrak{R}_\epsilon - \mathfrak{R}_0) \|^2 \le \left(\frac{16\pi^2}{45}\right)(1.48\epsilon)\left(\frac{\omega_1 t}{\pi}\right)^2 A + \left(\frac{4\pi^2}{105}\right)(1.48\epsilon)\left(\frac{\omega_1 t}{\pi}\right)^2\left(\frac{\omega_1 t+\epsilon}{\pi}\right)^2 B,$$

where

$$A = \sum_{ln} \alpha_{ln}^2 l\,(l+\tfrac{1}{2})\,(l+1)^{4/3}\exp\,(-2\rho\alpha_{ln}^2 t_2)$$

and

$$B = \sum_{ln} \alpha_{ln}^2 l^3\,(l+\tfrac{1}{2})\,(l+1)^{4/3}\exp\,(-2\rho\alpha_{ln}^2 t_2)\,.$$

The sums A and B can be bounded via the methods used to bound the sum on the right of inequality (68). Inequality (129b) is the result of such a calculation in which no great effort was made to obtain a close bound; the reader could produce smaller bounds without great difficulty.

In rotating a dipole moment into a desired direction, it will never be necessary to use an angle of rotation $\omega_1 t$ larger than π radians, so $\omega_1 t/\pi$ may be replaced by 1 in inequalities (129). Equation (127) then follows immediately. Therefore, even if the fluid's surface must be held stationary, all the magnetic effects of a rigid rotation can be obtained by rigidly rotating the interior of a sphere $S_{1-\epsilon}$, allowing a large shear to develop in a thin outer shell, and afterwards leaving the fluid motionless for a short time.

11. A CLASS OF SELF-SUSTAINING DYNAMOS

In this section a set of conditions on a fluid motion in a sphere of unit radius and unit resistivity will be stated which are sufficient to insure that that motion can maintain or amplify the external magnetic dipole moment due to electric currents in the fluid. The results of Sections 4 through 10 will then be used to show that motions exist which satisfy these sufficient conditions, and such a motion will be constructed.

(A) Some Conditions Sufficient for Self-Regeneration in a Dynamo

Suppose that all the modes of free decay except those in $\mathcal{B}_1^P$ are regarded as "contamination." To be precise, if a field $\boldsymbol{B}$ has the form

$$\boldsymbol{B} = \kappa(\boldsymbol{P}_1 + \boldsymbol{R}), \tag{134a}$$

where $\boldsymbol{P}_1$ is in $\mathcal{B}_1^P$ and

$$\| \boldsymbol{P}_1 \| = 1, \qquad \| \boldsymbol{R} \| \leq r, \tag{134b}$$

then the field $\boldsymbol{B}$ will be said to have a "level of contamination" no greater than r.

Consider a fluid motion whose magnetic operator $\mathcal{K}$ amplifies $\boldsymbol{P}_{1z1}$ without raising the level of contamination. That is, if r is small enough, there exist numbers $\kappa > 1$ and $r' < r$ such that if $\boldsymbol{R}$ is any field for which $\| \boldsymbol{R} \| \leq r$ then

$$\mathcal{K}(\boldsymbol{P}_{1z1} + \boldsymbol{R}) = \kappa(\boldsymbol{P}_1 + \boldsymbol{R}')$$

where $\boldsymbol{P}_1$ is in $\mathcal{B}_1^P$ and $\| \boldsymbol{P}_1 \| = 1$, while $\| \boldsymbol{R}' \| \leq r'$. Any such motion permits the maintenance of an external dipole moment in the $\hat{\boldsymbol{z}}$ direction forever. This fact is obvious if rigid rotations of the fluid are permitted, since $\boldsymbol{P}_1$ can then be rotated into the position $\boldsymbol{P}_{1z1}$ by the magnetic operator $\mathfrak{D}_{t_2}\mathcal{R}$ corresponding to such a rigid rotation requiring a time t_2. And as pointed out in Section 10c, even when the points of the fluid boundary must remain fixed, fluid motions can be found with magnetic operators $\mathcal{R}_\epsilon$ such that

$$\lim_{\epsilon \to 0} \| \mathcal{R}_\epsilon - \mathfrak{D}_{t_2}\mathcal{R} \| = 0.$$

Therefore ϵ may be chosen so small that $\mathcal{R}_\epsilon \mathcal{K}(\boldsymbol{P}_{1z1} + \boldsymbol{R}) = \kappa'(\boldsymbol{P}_{1z1} + \boldsymbol{R}'')$ where $\kappa' > 1$ and $\| \boldsymbol{R}'' \| \leq r$. The motion whose magnetic operator is $\mathcal{R}_\epsilon \mathcal{K}$ can be repeated indefinitely (the axis of rotation of the operator $\mathcal{R}_\epsilon$ may change with each repetition, but the angle of rotation will never exceed π) and after every repetition the external dipole moment will have increased in magnitude while preserving its direction and the level of contamination of the magnetic field will have decreased.

But are there any motions whose magnetic operators $\mathcal{K}$ increase the external dipole moment while decreasing the contamination level? Suppose that a continuum of motions $\boldsymbol{y}_1(\boldsymbol{x}, t)$, $0 \leq t \leq \tau$, is given, one for each τ (these motions might, but need not, be obtained from the persistence of some steady velocity for various times τ). Let $\mathcal{V}_\tau$ be the magnetic operator of the motion $\boldsymbol{y}_1(\boldsymbol{x}, \rho^{-1}t)$, $0 \leq t \leq \rho\tau$, where ρ is a small number. Suppose $\boldsymbol{y}_2(\boldsymbol{x}, t)$, $0 \leq t \leq$ t, is another motion, with magnetic operator $\mathcal{U}$. Suppose that the operators $\mathcal{U}$ and $\mathcal{V}_\tau$ satisfy these conditions:

$$\mathcal{P}\mathcal{V}_\tau \boldsymbol{P}_{101} = e^{-\rho\nu_1\tau}\boldsymbol{P}_{101} ; \tag{135a}$$

$$\mathfrak{I}_2\mathcal{V}_\tau \boldsymbol{P}_{101} = \alpha\tau \boldsymbol{T}_{101} , \alpha \quad \text{a nonzero constant;} \tag{135b}$$

$$\| \mathcal{P}_1\mathcal{V}_\tau \| \leq pe^{-\rho\nu_1\tau}, p \quad \text{a constant;} \tag{135c}$$

$$\| \mathfrak{I}\mathcal{V}_\tau \boldsymbol{P}_{101} \| \leq \beta\tau, \beta \quad \text{a nonzero constant;} \tag{135d}$$

$$\| \mathfrak{D}_{t_1}(I - \mathcal{P}_1)\mathcal{V}_\tau \|^2 \leq [q_1^2(t_1 , \tau) + \tau^2 q_2^2(t_1 , \tau)]e^{-2\nu_2 t_1}, \tag{135e}$$

where $q_1(t_1 , \tau)$ and $q_2(t_1 , \tau)$ are functions of t_1 and τ which remain bounded as t_1 and τ become large, but may be unbounded for small t_1 and τ;

$$\| \mathcal{P}_1\mathcal{U}\boldsymbol{T}_{101} \| = \gamma, \gamma \quad \text{a nonzero constant;} \tag{135f}$$

$$\| \mathcal{U} \| \leq \mu, \mu \quad \text{a constant;} \tag{135g}$$

$$\| \mathcal{P}_1\mathcal{U} \| \leq \lambda, \lambda \quad \text{a constant.} \tag{135h}$$

Then r, t_1 , t_2 , and τ can be chosen so that the operator

$$\mathcal{K} = \mathfrak{D}_{t_2}\mathcal{U}\mathfrak{D}_{t_1}\mathcal{V}_\tau \tag{136}$$

decreases contamination levels and increases external dipole moments for all fields $\boldsymbol{P}_{101} + \boldsymbol{R}$ with contamination levels below r. To see this, write the field $\mathcal{K}(\boldsymbol{P}_{101} + \boldsymbol{R})$ in the form

$$\begin{aligned} &\mathfrak{D}_{t_2}\mathcal{U}\mathfrak{D}_{t_1}\mathcal{V}_\tau(\boldsymbol{P}_{101} + \boldsymbol{R}) \\ &\quad = \mathfrak{D}_{t_2}\mathcal{P}_1\mathcal{U}\mathfrak{D}_{t_1}\mathfrak{I}_2\mathcal{V}_\tau\boldsymbol{P}_{101} + \mathfrak{D}_{t_2}\mathcal{P}_1\mathcal{U}\mathfrak{D}_{t_1}[\mathcal{P}_1\mathcal{V}_\tau(\boldsymbol{P}_{101} + \boldsymbol{R}) + (\mathfrak{I} - \mathfrak{I}_2)\mathcal{V}_\tau\boldsymbol{P}_{101} \\ &\qquad + (I - \mathcal{P}_1)\mathcal{V}_\tau\boldsymbol{R}] + \mathfrak{D}_{t_2}(I - \mathcal{P}_1)\mathcal{U}\mathfrak{D}_{t_1}[\mathcal{P}_1\mathcal{V}_\tau(\boldsymbol{P}_{101} + \boldsymbol{R}) \\ &\qquad + \mathfrak{I}_2\mathcal{V}_\tau\boldsymbol{P}_{101} + (\mathfrak{I} - \mathfrak{I}_2)\mathcal{V}_\tau\boldsymbol{P}_{101} + (I - \mathcal{P}_1)\mathcal{V}_\tau\boldsymbol{R}]. \end{aligned} \tag{137}$$

Equation (137) is simply an identity, except that the terms $(\mathcal{P} - \mathcal{P}_1)\mathcal{V}_\tau\boldsymbol{P}_{101}$ which ought to appear there have been set equal to zero on account of Eq. (135a). The first term on the right in Eq. (137) is $\alpha\gamma\tau e^{-\nu_1 t_2 - \nu_2 t_1}\boldsymbol{P}_1'$ where $\boldsymbol{P}_1' = \gamma^{-1}\mathcal{P}_1\mathcal{U}\boldsymbol{T}_{101}$ is a field in $\mathcal{B}_1^P$ for which $\| \boldsymbol{P}_1' \| = 1$. The first of the two terms involving brackets on the right in Eq. (137) is a stray field in $\mathcal{B}_1^P$ and the second is a stray field orthogonal to $\mathcal{B}_1^P$. The operators before any term in Eq. (137) display its origin and subsequent history, and permit the application of inequalities (135) in order to estimate the size of the stray fields. In fact, it is relatively simple to show from inequalities (135) that

$$\mathfrak{D}_{t_2}\mathfrak{U}\mathfrak{D}_{t_1}\mathfrak{V}_\tau(\boldsymbol{P}_{101} + \boldsymbol{R}) = e^{-\nu_1 t_2 - \nu_2 t_1}\alpha\gamma\tau[\boldsymbol{P}_1' + \boldsymbol{P}_1'' + \boldsymbol{X}], \tag{138a}$$

where $\boldsymbol{P}_1'$ and $\boldsymbol{P}_1''$ are in $\mathcal{B}_1^P$, $\boldsymbol{X}$ is orthogonal to $\mathcal{B}_1^P$, $\| \boldsymbol{P}_1' \| = 1$,

$$\| \boldsymbol{P}_1'' \|^2 \leq \left(\frac{\lambda}{\alpha\gamma}\right)^2 \left\{\frac{e^{2(\nu_2-\nu_1)t_1 - 2\rho\nu_1\tau}(1 + rp)}{\tau^2} + \left[r\left(\frac{q_1^2}{\tau^2} + q_2^2\right)^{1/2} + \beta e^{(\nu_2-\nu_3)t_1}\right]^2\right\}, \tag{138b}$$

and

$$\| \boldsymbol{X} \|^2 \leq e^{2(\nu_1-\nu_2)t_2}\left(\frac{\mu}{\alpha\gamma}\right)^2 \left\{\frac{e^{2(\nu_2-\nu_1)t_1-2\rho\nu_1\tau}(1 + rp)}{\tau^2} + \left[r\left(\frac{q_1^2}{\tau^2} + q_2^2\right)^{1/2} + \beta e^{(\nu_2-\nu_3)t_1}\right]^2 + \left[r\left(\frac{q_1^2}{\tau^2} + q_2^2\right)^{1/2} + \alpha\right]^2\right\}. \tag{138c}$$

Let f be any number between zero and one. If it can be arranged that

$$\| \boldsymbol{P}_1'' \| \leq f, \tag{139a}$$

$$\| \boldsymbol{X} \| \leq r'(1 - f),\ r' < r, \tag{139b}$$

$$(1 - f)\alpha\gamma\tau e^{-\nu_1 t_2 - \nu_2 t_1} > 1, \tag{139c}$$

then the field $\boldsymbol{P}_1 = (1 - f)^{-1}(\boldsymbol{P}_1' + \boldsymbol{P}_1'')$ is in $\mathcal{B}_1^P$ and

$$1 \leq \| \boldsymbol{P}_1 \| \leq (1 + f)/(1 - f),$$

while $\| (1 - f)^{-1}\boldsymbol{X} \| \leq r'$ and

$$\mathfrak{K}(\boldsymbol{P}_{101} + \boldsymbol{R}) = e^{-\nu_1 t_2 - \nu_2 t_1}\alpha\gamma\tau(1 - f)[\boldsymbol{P}_1 + (1 - f)^{-1}\boldsymbol{X}];$$

thus indeed $\mathfrak{K}$ decreases the contamination level and increases the dipole moment. Therefore, it remains only to show that r, t_1, t_2, and τ can be chosen so that relations (139) are satisfied. To see that this is possible, let g and h be any numbers between zero and one. Choose t_1 so large that

$$\frac{\nu\beta}{\alpha\gamma}\, e^{(\nu_2-\nu_3)t_1} < fgh^{1/2}. \tag{140a}$$

Then require τ to be larger than some lower limit τ_0 and choose r so small that if $\tau \geq \tau_0$

$$r\left(\frac{q_1^2}{\tau^2} + q_2^2\right)^{1/2} < f(1 - g)h^{1/2}. \tag{140b}$$

With this choice of t_1 and r now choose τ so large that $\tau \geq \tau_0$ and

$$\frac{e^{2(\nu_2-\nu_1)t_1-2\rho\nu_1\tau}(1 + rp)}{\tau^2} < 1 - h. \tag{140c}$$

Inequalities (138b) and (140) now imply (139a), and r, t_1 are fixed while τ must be larger than some lower limit τ_1. Let r' be any number less than the r just obtained. Then fix t_2 at a value so large that inequality (139b) is satisfied. Finally, choose τ so large that $\tau \geq \tau_1$ and that inequality (139c) is satisfied. This completes the proof that if $\mathcal{U}$ and $\mathcal{V}_\tau$ satisfy the relations (135), then the fluid motion (136) purifies and amplifies fields $\boldsymbol{P}_{101} + \boldsymbol{R}$ whose initial levels of contamination are sufficiently low; consequently, the motion (136) constitutes a self-sustaining dissipative dynamo.

(B) The Existence of Fluid Motions Which Satisfy the Conditions (135) Sufficient for Dynamo Maintenance

Motions for which the coefficients in inequalities (135) have been computed in Sections 4 through 10 are as follows: $\mathcal{V}_\tau$ is the magnetic operator of the motion resulting from the persistence for time $\rho\tau$ of the velocity

$$\boldsymbol{u} = \rho^{-1} r^2(1 - r^2) \sin\theta \cos\theta \hat{\boldsymbol{\phi}}$$

in the given fluid of unit resistivity, or the velocity $\boldsymbol{u} = r^2(1 - r^2) \sin\theta \cos\theta\hat{\boldsymbol{\phi}}$ for time τ in a fluid of resistivity ρ. The magnetic operator $\mathcal{U}$ is one of the operators $\mathcal{U}_\rho^\epsilon$ discussed in subsection 10b, whose norms are less than e and whose matrix elements are very close to those of the operator $\mathcal{U}$ of that subsection obtained by allowing the velocity (111b) to persist for unit time in a perfectly conducting fluid. With these magnetic operators, the constants and functions appearing in relations (135) are as follows:

$$\alpha = (0.0976)\left(\frac{e^{-\rho\nu_1\tau} - e^{-\rho\nu_2\tau}}{\rho(\nu_2 - \nu_1)\tau}\right) \quad \text{[see Eq. (92a, b)]}; \tag{141a}$$

$$p = 3.85 \quad \text{[see Eq. (74)]}; \tag{141b}$$

$$\beta = (0.31308)\left(\frac{e^{-\rho\nu_1\tau} - e^{-\rho\nu_2\tau}}{\rho(\nu_2 - \nu_1)\tau}\right) \quad \text{[see Eq. (92c)]}; \tag{141c}$$

$$q_1^2(t_1, \tau) = e^{-2\rho\nu_1\tau}\left[16.8 + \frac{A(t_1)}{2} + 4e^{2(\nu_2-\nu_3)t_1}\left(1 + \frac{2}{2\nu_3 t_1} + \frac{2}{(2\nu_3 t_1)^2}\right)\right], \tag{141d}$$

and

$$q_2(t_1, \tau) = A^{1/2}(t_1)\left[2.1005\left(\frac{e^{-\rho\nu_1\tau} - e^{-\rho\nu_2\tau}}{\rho(\nu_2 - \nu_1)\tau}\right) + \frac{0.2682}{(\rho\tau)^{1/2}}\left(\frac{1 - e^{-2\rho\nu_2\tau}}{2\rho\nu_2\tau}\right)^{1/2}\right], \tag{141e}$$

where

$$A(t_1) = 6 + 1100e^{2(\nu_2-\nu_3)t_1}\left[1 + \frac{5}{2(2\nu_3 t_1)} + \frac{5}{(2\nu_3 t_1)^2} + \frac{5}{(2\nu_3 t_1)^3}\right]$$

[see Eqs. (73), (75), and (100)];

$$\gamma = 0.277 \quad \text{[see Eq. (120b)]}; \tag{141f}$$

$$\mu = e = 2.71828 \cdots \quad \text{[see Eq. (111c)}; \tag{141g}$$

$$\lambda \leq 0.477 \quad \text{[see Eq. (121)]}. \tag{141h}$$

The following choice of r, t_1, t_2, τ, ρ will be found to conform to the demands (140) with $f = 5/9$, $g = 3/5$, and h close to 1: $\rho\tau = 1.5 \times 10^{-2}$, $t_1 = 0.2105$, $r = 1.047 \times 10^{-3}$, $t_2 = 0.985$, $\tau = 1.2 \times 10^{8}$. Then the factor by which the velocity (80) is speeded up is $\rho^{-1} = 8 \times 10^{9}$. Since the maximum value of the velocity (80) is $1/8$, the maximum velocity achieved in the dynamo is 10^{9} in dimensionless units. This velocity is maintained for a time $\rho\tau$ during the whole cycle of length $\rho\tau + t_1 + t_2$ so the time average of the maximum velocity is 1.25×10^{7}. The root-mean-square of velocity (80) is about half its maximum, giving a time- and space-averaged velocity of 6×10^{6} dimensionless units. By way of comparison, from inequality (33) the velocity below which dynamo maintenance has been proved impossible is $\nu_1 \approx 10$ dimensionless units. If the mean life T_0 of $\boldsymbol{P}_{101}$ in a rigid earth's core of radius $R = 3000$ kilometers is taken to be 15000 years, the unit of velocity is $R/\nu_1 T_0 = 6.67 \times 10^{-5}$ cm/sec, so the largest velocity which has been proved incapable of maintaining a dynamo in the earth's core is about 6×10^{-4} cm/sec while the smallest mean velocity which has been proved capable of dynamo maintenance is 4×10^{2} cm/sec.

12. CONCLUSIONS

(A) Improving the Lower Bound on Dynamo-Maintaining Velocities

For the motion (136) for which numerical results have been obtained there is a gap of almost six orders of magnitude in which it has not been shown whether a dynamo can be maintained. Most of this large gap is produced by the loss of information which occurred every time an equality was replaced by an inequality in the argument. And the most serious such loss of information occurred through the decision to treat only $\mathcal{B}_1$ as worth observing, everything orthogonal to it being called "contamination". The minimum velocity which can be shown capable of maintaining a dynamo is materially lowered by scrutinizing spaces with higher decay rates.

In particular, if the stigma of "contamination" is removed from $\mathcal{B}_2$, then to obtain a dynamo from the motions considered in Section 11 one must start with a field of the form $\boldsymbol{P}_{1z1} + A\boldsymbol{P}_2 + B\boldsymbol{T}_2 + \boldsymbol{R}$ where A and B are constants agreed on before hand, $\boldsymbol{P}_2$ and $\boldsymbol{T}_2$ are fields of unit energy in $\mathcal{B}_2^{P}$ and $\mathcal{B}_2^{T}$, and only the field $\boldsymbol{R}$ is regarded as contamination and required to have a small norm. The effect of the magnetic operator

$$\mathfrak{D}_{t_2}\mathfrak{U}\mathfrak{D}_{t_1}\mathfrak{V}_\tau$$

must then be computed in two parts: first, all the parts of

$$\boldsymbol{G} = (\mathcal{P}_1 + \mathcal{P}_2 + \mathfrak{I}_2)\mathfrak{D}_{t_2}\mathfrak{U}\mathfrak{D}_{t_1}\mathfrak{V}_\tau(\boldsymbol{P}_{1z1} + A\boldsymbol{P}_2 + B\boldsymbol{T}_2)$$

which grow linearly with τ must be computed exactly (except such terms as can be shown by symmetry arguments not to interfere with the regeneration of $\boldsymbol{P}_{1z1}$) and bounds for the remainder of this field must be obtained. Second, bounds must be obtained on the fields

$$\boldsymbol{H} = (\mathcal{P}_1 + \mathcal{Q}_2)\mathfrak{D}_{t_2}\mathfrak{U}\mathfrak{D}_{t_1}\mathfrak{V}_\tau\boldsymbol{R}$$

and

$$\boldsymbol{K} = (I - \mathcal{P}_1 - \mathcal{Q}_2)\mathfrak{D}_{t_2}\mathfrak{U}\mathfrak{D}_{t_1}\mathfrak{V}_\tau(\boldsymbol{P}_{1z1} + A\boldsymbol{P}_2 + B\boldsymbol{T}_2 + \boldsymbol{R}).$$

The bound on $\boldsymbol{H}$ will determine the first decay time t_1 and the level r of contamination which can be allowed, since $\boldsymbol{H}$ must be so small that when added to $\boldsymbol{G}$ it cannot cancel $\mathcal{P}_1\boldsymbol{G}$. Then the demand that $\| \boldsymbol{K} \|$ be so small that the final field has a contamination level no greater than r will determine t_2. Finally, τ is determined by the demand that at the end of the motion the energy in $\mathcal{B}_1^P$ is no less than it was at the beginning. (There is a lower bound on τ arising from the bounds on $\boldsymbol{G}$ and $\boldsymbol{H}$ but this is much less than the τ required to give amplification, and can be ignored.) This program looks onerous, since $\mathcal{B}_1 \oplus \mathcal{B}_2$ is an 11-dimensional space: however, for the motions considered in subsection 11b only three of the possible 121 matrix elements of $\mathfrak{U}$ and only nine of $\mathfrak{V}_\tau$ need be computed exactly because of the symmetries of those motions. A very preliminary estimate indicates that, by scrutinizing the relevant part of $\mathcal{B}_1 \oplus \mathcal{B}_2$ instead of just $\mathcal{B}_1$, the minimal velocity proved capable of dynamo maintenance in the motion (136) can be lowered by about two orders of magnitude, to about 6×10^4 dimensionless units; in an earth's core with longest rigid decay time of 15,000 years, this is 4 cm/sec. It is possible that elevating higher $\mathcal{B}_k$'s from the incompletely observed contamination into the company of the observed fields will lower this minimum by another one or two orders of magnitude, but the author believes that the techniques of this paper, carried out with however large a space of observed fields, will leave a gap of at least two orders of magnitude between the minimum velocity proved capable of maintaining a dynamo and the maximum velocity which inequality (33) proves incapable of dynamo maintenance. This question will have to be examined further at a later date.

(B) The Amplification Factor as a Function of Velocity

If $\mathfrak{U}$ and $\mathfrak{V}_\tau$ are the magnetic operators of any motions satisfying relations (135), the relation between the mean velocity $\langle u \rangle$ of the whole motion and the average growth rate $\langle\kappa\rangle$ of the dipole moment per unit time can be found from inequality (139c). The time for one full cycle of the motion (136) is $\rho\tau + t_1 + t_2$ while the factor by which the field $\boldsymbol{P}_{101}$ has been amplified during that time lies between $(1 - f)\alpha\gamma\tau e^{-(\nu_1 t_2 + \nu_2 t_1)}$ and $(1 + f)\alpha\gamma\tau e^{-(\nu_1 t_2 + \nu_2 t_1)}$, where f is the number between 0 and 1 chosen for inequality (139a). The average velocity is

$$\langle u\rangle = V\tau(\rho\tau + t_1 + t_2)^{-1}$$

where V is a constant (about 0.05 in the numerical example of subsection 11b). If $\rho\tau$, t_1, and t_2 are fixed, as they must be in the approach of Section 11a, then this amplification factor is proportional to $\langle u\rangle$, and if the amplitude of $\boldsymbol{P}_{101}$ is written in the form $e^{\kappa t}$ the average value of κ is $\langle\kappa\rangle = \ln C\langle u\rangle$, where C is a constant depending on V, $\rho\tau$, t_1, and t_2. In his disc-and-loop dynamo, Bullard (35) found $\langle\kappa\rangle = C'(\langle u\rangle - a)$ where C' and a were constants. The large comparative loss of efficiency which occurs at high $\langle u\rangle$ in the dynamos presented here is a result of the drastic decays required to enable the stray fields to be kept under control by the crude estimates of this paper.

(c) The Generality of the Class of Motions Treated

In this paper the attempt to produce dynamos by means of a velocity believed to be like that in the earth's core has been explicitly eschewed. Nevertheless, it is interesting to ask what motions can, by the methods presented here, be proved capable of dynamo maintenance.

Any motion whose magnetic operator has the form (136), its components satisfying relations (135), has been shown to maintain a dynamo if τ is sufficiently large (the toroidal velocity is sufficiently high). For reasons already pointed out, the free decays $\mathfrak{D}_{t_1}$ and $\mathfrak{D}_{t_2}$, during which the fluid is motionless, are essential in the present approach. No such stasis can be expected of the earth's core, so this limitation must be removed before the present approach becomes rigorously applicable to motions which might be expected in that core. There are two lines along which the difficulty might be attacked: it might be possible that approximating a motion by a series of jerks interspersed with periods of free decay as suggested in Section 6d would lead to a criterion for testing the ability of arbitrary motions to maintain dynamos. It might also be possible to obtain bounds on stray fields generated by arbitrary motions which discard so little information that no period of free decay is needed to assure that those stray fields do not grow.

The magnetic operators $\mathfrak{V}_\tau$ which can be proved by the methods of this paper to satisfy relations (135) must come from toroidal shears symmetric about the $\hat{\boldsymbol{z}}$ axis whose angular velocities $\omega(r, \theta)$ involve only a finite sum of Legendre polynomials in $\cos\theta$. If $\omega(r, \theta)$ is symmetric about the equatorial plane $\theta = \pi/2$, $\mathfrak{J}_2\mathfrak{V}_\tau\boldsymbol{P}_{101} = 0$ and the least rapidly decaying toroidal field produced from $\boldsymbol{P}_{101}$ by $\mathfrak{V}_\tau$ is $\mathfrak{J}_3\mathfrak{V}_\tau\boldsymbol{P}_{101} = \alpha'_\tau\boldsymbol{T}_{201}$. This equation together with $\mathfrak{J}_2\mathfrak{V}_\tau\boldsymbol{P}_{101} = \boldsymbol{0}$ replaces equation (135b) for such angular velocities, and then $\mathcal{P}_2\boldsymbol{B}$ can never be treated as a stray field. Otherwise the analysis goes through as in subsection 11b.

The magnetic operator $\mathcal{U}$ obtained from the velocity (111) has to recommend it only that its effects can be calculated explicitly in a fluid of zero resistivity. The motion is axisymmetric, but about an axis perpendicular to $\hat{z}$, the presumed axis of symmetry of any motions with large scale organization in the earth's core. Furthermore, $\mathcal{U}$ produces a $\mathcal{P}_1\mathcal{U}\boldsymbol{T}_{1z1} = \boldsymbol{P}_{1y1}$ which has to be rotated back to the $\hat{z}$ direction, a physically unlikely motion. But the demands (135f, g, h) on $\mathcal{U}$ are very weak. Inequalities (135g) and (135h) are an automatic consequence of inequality (33) for any motion whatever as long as its final displacement is fixed. The only real demand on $\mathcal{U}$ is Eq. (135f). If this demand is strengthened and it is required that $\mathcal{P}_1\mathcal{U}\boldsymbol{T}_{1z1}$ have most of its energy in $\boldsymbol{P}_{1z1}$ and very little in $\boldsymbol{P}_{1x1}$ or $\boldsymbol{P}_{1y1}$, then the magnetic operator (136) just as it stands regenerates $\boldsymbol{P}_{1z1}$, and no rotation is required. Parker's (*14*) vortices, which the reader will be able without difficulty to fit into a sphere using the formation of Eq. (108), and whose magnetic effects can be calculated from Eq. (109) when the displacements involved are very small, are an example of such a motion. In this example, a small region of the fluid is made to move poloidally and simultaneously to rotate about $\hat{r}$, the rest of the fluid remaining stationary. From Eqs. (109) the main effect of this motion is to produce $\boldsymbol{P}_{1x1}$ and $\boldsymbol{P}_{1y1}$, but a small amount of $\boldsymbol{P}_{1z1}$ is also produced. A large number of such small disjoint regions is distributed through the sphere. If the fluid were a perfect conductor, the magnetic operator of the whole motion would be the sum of the operators of the individual vortices, and this is approximately true if the motion is executed fast enough when $\rho > 0$. But then if the vortices are more or less axisymmetrically distributed about $\hat{z}$, their individual $\boldsymbol{P}_{1x1}$ and $\boldsymbol{P}_{1y1}$ productions will almost cancel, and only the $\boldsymbol{P}_{1z1}$ will remain. With this sort of scheme for regnerating $\boldsymbol{P}_{1z1}$ from $\boldsymbol{T}_{1z1}$ there is no necessity for wild fluctuations in the direction of the external dipole moment and the axis of symmetry of the internal toroidal field, and the latter field need not be small at any time during the motion.

It should now be clear that the methods of the present paper are sufficiently powerful to treat axisymmetric toroidal shears protracted indefinitely, and arbitrary motions of fixed finite total displacement. The most serious limitation of these methods is their dependence on occasional stasis in the fluid in order to eliminate insufficiently scrutinized "contamination" fields.

APPENDIX 1. SOME INEQUALITIES FOR BESSEL FUNCTIONS

Despite the extensive asymptotic theory of Bessel functions, very little work seems to have been done on strict inequalities associated with that theory. Therefore, it is necessary to provide proofs of inequalities (131) and (133). These proofs involve Sturm's theorem in a slightly stronger form than that proved by

Watson (Ref. *28*, p. 518), but his proof can easily be modified to give this stronger result, so it will not be proved here. The result needed is

Sturm's Theorem: Suppose that for all x larger than some fixed a, $\omega_2(x) \geq \omega_1(x)$, and $d^2y_i/dx^2 + \omega_i y_i = 0$, $i = 1, 2$. Suppose also that $0 < y_2(a) \leq y_1(a)$ and $y_2'(a) \leq y_1'(a)$. Then in any interval $a < x < c$ in which $y_2(x)$ is positive, $y_1(x) \geq y_2(x)$.

Inequality (131) will be proved first. To conform to Watson's notation, j_ν will denote the first positive zero of J_ν, and the lth spherical Bessel function $(\pi/2x)^{1/2}J_{l+1/2}(x)$ will always be written $j_l(x)$ to avoid confusion. For the moment, consider inequality (131) only when $n = 1$. Then that inequality can be rewritten as

$$\left(\nu + \frac{1}{2}\right)^{1/3} j_\nu J_{\nu+1}^2(j_\nu) \geq \frac{2}{1.48\pi}.$$

Since (Ref. *28*, p. 487) $j_\nu^2 \leq \frac{4}{3}(\nu + 1)(\nu + 5)$, this inequality is a consequence of

$$\nu^{-2/3} j_\nu^2 J_{\nu+1}^2(j_\nu) \geq 1.215, \tag{142}$$

a result which will be proved immediately. Observe that

$$j_\nu^2 J_{\nu+1}^2(j_\nu) = \nu^2 J_\nu'(\nu)^2 + 2\int_\nu^{j_\nu} xJ_\nu^2(x)\,dx$$

(Ref. *28*, p. 135). If $\nu \leq \nu \sec\beta \leq j_\nu$ and if ξ is defined as $\xi = \nu(\tan\beta - \beta)$ then Watson (Ref. *28*, p. 521) proves from Sturm's theorem that

$$\frac{J_\nu(\nu\sec\beta)}{\cos^{1/2}\beta} \geq \nu^{-1/3}A(\nu)[\xi^{1/3}J_{-1/3}(\xi) + B(\nu)\xi^{1/3}J_{1/3}(\xi)] = \nu^{-1/3}F_\nu(\xi), \tag{142a}$$

where

$$A(\nu) = \left(\frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{2\pi 3^{1/6}}\right)\left(\frac{2^{2/3}3^{1/6}\pi\nu^{1/3}J_\nu(\nu)}{\Gamma(\frac{1}{3})}\right) \tag{142b}$$

and

$$B(\nu) = \left(\frac{\Gamma(\frac{1}{3})}{6^{1/3}\Gamma(\frac{2}{3})}\right)\left(\frac{\nu^{1/3}J_\nu'(\nu)}{J_\nu(\nu)}\right). \tag{142c}$$

In consequence, if ξ_0 is the first positive zero of $F_\nu(\xi)$ and β_0 is defined by $\xi_0 = \nu(\tan\beta_0 - \beta_0)$ then, as Watson (Ref. *28*, p. 521) observes, $j_\nu^2 > \frac{1}{3}\nu\beta_0^3$, so

$$\int_\nu^{j_\nu} xJ_\nu^2(x)\,dx \geq \frac{1}{2}\int_{\nu^2}^{1/3\nu\beta_0^3} J_\nu^2(x)\,d(x^2) = \nu\int_0^{\xi_0}\frac{J_\nu^2(\nu\sec\beta)}{\sin\beta\cos\beta}\,d\xi.$$

Since $\cos\beta \geq (\nu/3\xi)^{1/3}$,

$$\int_{\nu}^{j_\nu} xJ_\nu^2(x)\,dx \geq \frac{\nu^{2/3}}{3^{1/3}}\int_0^{\xi_0} \xi^{-1/3}F_\nu^2(\xi)\,d\xi.$$

Thus

$$\nu^{-2/3}j_\nu^2 J_{\nu+1}^2(j_\nu) \geq [\nu^{2/3}J_\nu'(\nu)]^2 + \frac{2}{3^{1/3}}\int_0^{\xi_0} \xi^{-1/3}F_\nu^2(\xi)\,d\xi. \tag{143}$$

Since both $A(\nu)$ and $B(\nu)$ are monotonically increasing functions of ν (Ref. *28*, p. 260) while $\xi^{1/3}J_{1/3}(\xi)$ and $F_\nu(\xi)$ are positive between zero and ξ_0, it follows that $F_\nu^2(\xi)$ is a monotonically increasing function of ν; then so is its first zero, $\xi_0(\nu)$. Since $\nu^{2/3}J_\nu'(\nu)$ also increases monotonically with ν (Ref. *28*, p. 260) the right-hand side of inequality (143) does likewise. Inequality (141) was proved by evaluating its left hand side from a table of zeroes of spherical Bessel functions for $\nu = 1/2, 3/2, \cdots, 39/2$, and then computing the right-hand side of inequality (143) for $\nu = 41/2$, so below $41/2$ inequality (141) has been proved only for half-integral values of ν.

The left side of inequality (141) turned out to be a monotonically decreasing function of ν from $\nu = 1/2$ to $\nu = 39/2$. The author has not tried to prove that this situation continues for all ν, but if it does then inequality (141) can be strengthened: the left-hand side is greater than its limit as $\nu \to \infty$, namely $1.24716 \cdots$. This limit can be computed from various limits given by Watson (Ref. *28*, p. 260) or from the asymptotic expansion for the left side of inequality (141) given by Olver (*36*), who shows, incidentally, that when $\nu = \infty$ the integral on the right in inequality (143) can be evaluated in terms of Airy functions.

There still remains the comparatively simple task of proving inequality (131) when $n > 1$. Define

$$y_l(x) = xj_l(x) \tag{144a}$$

and

$$\omega_l(x) = 1 - \frac{l(l+1)}{x^2} \tag{144b}$$

so that Bessel's equation becomes

$$\frac{d^2y_l}{dx^2} + \omega_l(x)y_l = 0. \tag{144c}$$

Multiply Eq. (144c) by dy_l/dx and integrate from a to b, obtaining

$$\left[\left(\frac{dy}{dx}\right)^2 + \omega(x)y^2\right]_a^b = \int_a^b y^2\frac{d\omega}{dx}\,dx. \tag{145}$$

If a and b are both zeroes of $y_l(x)$, since $d\omega_l/dx > 0$, $y_l'(a)^2 < y_l'(b)^2$ if $a < b$. Since inequality (131) can be written as $y_{l-1}'(\alpha_{l-1,n})^2 \geq (1.48l^{1/3})^{-1}$, its truth for $n = 1$ implies its truth for all higher n.

Inequality (131) having been proved, inequality (133) must now be dealt with. Two lemmas will be useful.

Lemma 1: If $l \geq 1$, $\alpha_{l1}^2 \geq l(l + 1) + \pi^2$.

From the tables of roots of Bessel functions, $\alpha_{l1}^2 \geq (l + 1)(l + 2) + \pi^2$ if $l = 1, 2, 3, 4$. Since $l(l + 1) + \pi^2$ and α_{l1}^2 are monotonically increasing functions of l (Ref. *28*, p. 508) the lemma is true if $1 \leq l \leq 5$. The inequality $\alpha_{l1}^2 \geq (l + \frac{1}{2})(l + \frac{5}{2})$ (Ref. *28*, p. 486) proves the lemma for $l \geq 5$.

Lemma 2: If $d^2y/dx^2 + \omega(x)y = 0$, $d\omega/dx \geq 0$, and $y > 0$ when $a < x < b$, while $y(x)$ and $\omega(x)$ are continuous when $a \leq x \leq b$, and $y(a) = y(b) = 0$, then the unique point c between a and b at which $y'(c) = 0$ is larger than $\frac{1}{2}(a + b)$.

To prove lemma 2, let $y_2(x) = y(x)$ and $\omega_2(x) = \omega(x)$ when $c \leq x \leq b$, while $y_1(x) = y(2c - x)$ and $\omega_1(x) = \omega(2c - x)$ when $c \leq x \leq 2c - a$. Then at $x = c$, $y_1 = y_2$ and $y_1' = y_2'$, while in the interval $0 \leq x \leq \min [c, 2c - a]$, $\omega_1 < \omega_2$. Hence, by Sturm's theorem the zero of y_1, $2c - a$, is larger than b, the zero of y_2. This proves lemma 2.

Again defining $y_l(x)$ as $xj_l(x)$, inequality (133) can be written

$$| y_l(x) | \leq \frac{\alpha_{ln}}{\pi} \sin \frac{\pi x}{\alpha_{ln}} \equiv S_{ln}(x) \tag{146a}$$

when

$$0 \leq x \leq \alpha_{ln} . \tag{146b}$$

The case $n = 1$ is again the hardest and must be settled first. As was shown in Section 10c, $| j_l(x) | \leq (2l + 1)^{-1/2}$, so $| y_l(x) | \leq x(2l + 1)^{-1/2}$, so $| y_l(x) | \leq S_{l1}(x)$ if $0 \leq x \leq \alpha_{l1}/2$. To dispose of the other half of the interval (146b) let κ_l be the point x at which the $\omega_l(x)$ of Eq. (144b) becomes equal to $(\pi/\alpha_{l1})^2$. Lemma 1 insures that $\kappa_l < \alpha_{l1}$, so $\omega_l(x) < (\pi/\alpha_{l1})^2$ if $0 \leq x < \kappa_l$, and $\omega_l(x) > (\pi/\alpha_{l1})^2$ if $\kappa_l < x \leq \alpha_{l1}$. The well-known asymptotic expression for $j_l(x)$ when x is large (Ref. *28*, p. 199) shows that

$$\lim_{n\to\infty} y_l'(\alpha_{ln})^2 = 1,$$

and it has already been shown in this appendix that $y_l'(\alpha_{ln})^2$ increases monotonically with n. Hence $y_l'(\alpha_{l1})^2 < 1 = S_{l1}'(\alpha_{l1})^2$. Since $y_l(\alpha_{l1}) = S_{l1}(\alpha_{l1}) = 0$ and

$$\frac{d^2}{dx^2} S_{ln} + \left(\frac{\pi}{\alpha_{ln}}\right)^2 S_{ln} = 0, \tag{147}$$

Sturm's theorem implies that $| y_l(x) | \leq S_{l1}(x)$ if $\kappa_l \leq x \leq \alpha_{l1}$.

Now let x' be the point between zero and α_{l1} at which $y_l'(x) = 0$. If $\kappa_l \leq x'$, then from the preceding paragraph $|y_l(x)| \leq S_{l1}(x)$ when $x' \leq x \leq \alpha_{l1}$. And if $x' < \kappa_l$, then at least $|y_l(x')| < S_{l1}(x')$. To see this, suppose the contrary: $|y_l(x')| \geq S_{l1}(x')$. Lemma 2 implies that $\alpha_{l1}/2 < x'$, so $y_l'(x') = 0 > S_{l1}'(x')$. Sturm's theorem applied to Eqs. (144c) and (147) then implies that $|y(\kappa_l)| > S_{l1}(\kappa_l)$, contradicting the result of the preceding paragraph. But now suppose there is any point x at all for which $x' < x < \kappa_l$ and $|y_l(x)| \geq S_{l1}(x)$. Then there is a least such point, a. At a, $d|y_l(x)|/dx \geq dS_{l1}/dx$ and $|y_l(a)| = S_{l1}(a)$, so another application of Sturm's theorem leads to the false result $|y_l(\kappa_l)| > S_{l1}(\kappa_l)$. *In fine*, regardless of the relative positions of x' and κ_l, $|y_l(x)| \leq S_{l1}(x)$ if $x' \leq x \leq \alpha_{l1}$.

Since $S_{l1}(x)$ is symmetric about the line $x = \alpha_{l1}/2$ and since $|y_l(x)| \leq |y_l(x')|$ if $0 \leq x \leq \alpha_{l1}$, it follows that $|y_l(x)| \leq S_{l1}(x)$ if $\alpha_{l1} - x' \leq x \leq \alpha_{l1}$. But $\alpha_{l1} - x' < \alpha_{l1}/2$, so $|y_l(x)| \leq S_{l1}(x)$ in the whole range $0 \leq x \leq \alpha_{l1}$.

For higher n, the argument is quite simple, and proceeds by induction on n. Suppose inequality (146a) in the range (146b) has been proved for n. Since $S_{ln}(x) \leq S_{l,n+1}(x)$ if $0 \leq x \leq \alpha_{ln}$, inequality (146a) is true for $n + 1$ in this interval. And since $\kappa_l < \alpha_{l1}$, $\omega_l > (\pi/\alpha_{l1})^2 > (\pi/\alpha_{ln})^2$ when $\alpha_{ln} \leq x \leq \alpha_{l,n+1}$, while $y_l'(\alpha_{l,n+1})^2 < 1 = S_{l,n+1}'(\alpha_{l,n+1})^2$ and $y_l(\alpha_{l,n+1}) = S_{l,n+1}(\alpha_{l,n+1}) = 0$. Hence by Sturm's theorem $|y_l(x)| \leq S_{l,n+1}(x)$ when $\alpha_{ln} \leq x \leq \alpha_{l,n+1}$. This completes the proof of inequality (146a) in the range (146b) for all $l \geq 1$ and all $n \geq 1$. When $l = 0$, that inequality is obvious. Therefore inequality (133) is proved.

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