CHAPTER 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

SECTION 1.1

DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELING

The main purpose of Section 1.1 is simply to introduce the basic notation and terminology of differential equations, and to show the student what is meant by a solution of a differential equation. Also, the use of differential equations in the mathematical modeling of real-world phenomena is outlined.

Problems 1-12 are routine verifications by direct substitution of the suggested solutions into the given differential equations. We include here just some typical examples of such verifications.

3. If $y_1 = \cos 2x$ and $y_2 = \sin 2x$, then $y_1' = -2\sin 2x$ and $y_2' = 2\cos 2x$ so

$$y_1'' = -4\cos 2x = -4y_1 \quad \text{and} \quad y_2'' = -4\sin 2x = -4y_2.$$

Thus $y_1'' + 4y_1 = 0$ and $y_2'' + 4y_2 = 0$.

4. If $y_1 = e^{3x}$ and $y_2 = e^{-3x}$, then $y_1 = 3e^{3x}$ and $y_2 = -3e^{-3x}$ so

$$y_1'' = 9e^{3x} = 9y_1 \quad \text{and} \quad y_2'' = 9e^{-3x} = 9y_2.$$

5. If $y = e^x - e^{-x}$, then $y' = e^x + e^{-x}$ so $y' - y = \left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) = 2e^{-x}$. Thus $y' = y + 2e^{-x}$.

6. If $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$, then $y_1' = -2e^{-2x}$, $y_1'' = 4e^{-2x}$, $y_2' = e^{-2x} - 2xe^{-2x}$, and $y_2'' = -4e^{-2x} + 4xe^{-2x}$. Hence

$$y_1'' + 4y_1' + 4y_1 = \left(4e^{-2x}\right) + 4\left(-2e^{-2x}\right) + 4\left(e^{-2x}\right) = 0$$

and

$$y_2'' + 4y_2' + 4y_2 = \left(-4e^{-2x} + 4xe^{-2x}\right) + 4\left(e^{-2x} - 2xe^{-2x}\right) + 4\left(xe^{-2x}\right) = 0.$$

8. If $y_1 = \cos x - \cos 2x$ and $y_2 = \sin x - \cos 2x$, then $y_1' = -\sin x + 2\sin 2x$, $y_1'' = -\cos x + 4\cos 2x$, and $y_2' = \cos x + 2\sin 2x$, $y_2'' = -\sin x + 4\cos 2x$. Hence

$$y_1'' + y_1 = (-\cos x + 4\cos 2x) + (\cos x - \cos 2x) = 3\cos 2x$$

and

$$y_2'' + y_2 = (-\sin x + 4\cos 2x) + (\sin x - \cos 2x) = 3\cos 2x.$$

11. If $y = y_1 = x^{-2}$ then $y' = -2x^{-3}$ and $y'' = 6x^{-4}$, so

$$x^2 y'' + 5x y' + 4y = x^2(6x^{-4}) + 5x(-2x^{-3}) + 4(x^{-2}) = 0.$$

If $y = y_2 = x^{-2}\ln x$ then $y' = x^{-3} - 2x^{-3}\ln x$ and $y'' = -5x^{-4} + 6x^{-4}\ln x$, so

$$\begin{aligned} x^2 y'' + 5x y' + 4y &= x^2(-5x^{-4} + 6x^{-4}\ln x) + 5x(x^{-3} - 2x^{-3}\ln x) + 4(x^{-2}\ln x) \\ &= (-5x^{-2} + 5x^{-2}) + (6x^{-2} - 10x^{-2} + 4x^{-2})\ln x = 0. \end{aligned}$$

13. Substitution of $y = e^{rx}$ into $3y' = 2y$ gives the equation $3re^{rx} = 2e^{rx}$ that simplifies to $3r = 2$. Thus $r = 2/3$.

14. Substitution of $y = e^{rx}$ into $4y'' = y$ gives the equation $4r^2 e^{rx} = e^{rx}$ that simplifies to $4r^2 = 1$. Thus $r = \pm 1/2$.

15. Substitution of $y = e^{rx}$ into $y'' + y' - 2y = 0$ gives the equation $r^2 e^{rx} + re^{rx} - 2e^{rx} = 0$ that simplifies to $r^2 + r - 2 = (r+2)(r-1) = 0$. Thus $r = -2$ or $r = 1$.

16. Substitution of $y = e^{rx}$ into $3y'' + 3y' - 4y = 0$ gives the equation $3r^2 e^{rx} + 3re^{rx} - 4e^{rx} = 0$ that simplifies to $3r^2 + 3r - 4 = 0$. The quadratic formula then gives the solutions $r = (-3 \pm \sqrt{57})/6$.

The verifications of the suggested solutions in Problems 17-36 are similar to those in Problems 1-12. We illustrate the determination of the value of C only in some typical cases.

17. $C = 2$

18. $C = 3$

19. If $y(x) = Ce^x - 1$ then $y(0) = 5$ gives $C - 1 = 5$, so $C = 6$.

20. If $y(x) = Ce^{-x} + x - 1$ then $y(0) = 10$ gives $C - 1 = 10$, so $C = 11$.

21. $C = 7$

22. If $y(x) = \ln(x+C)$ then $y(0)=0$ gives $\ln C = 0$, so $C = 1$.

23. If $y(x) = \frac{1}{4}x^5 + Cx^{-2}$ then $y(2)=1$ gives the equation $\frac{1}{4}\cdot 32 + C\cdot\frac{1}{8} = 1$ with solution $C = -56$.

24. $C = 17$

25. If $y(x) = \tan(x^2+C)$ then $y(0)=1$ gives the equation $\tan C = 1$. Hence one value of C is $C = \pi/4$ (as is this value plus any integral multiple of π).

26. Substitution of $x=\pi$ and $y=0$ into $y = (x+C)\cos x$ yields the equation $0 = (\pi+C)(-1)$, so $C = -\pi$.

27. $y' = x+y$

28. The slope of the line through (x,y) and $(x/2,0)$ is $y' = (y-0)/(x-x/2) = 2y/x$, so the differential equation is $x y' = 2y$.

29. If $m=y'$ is the slope of the tangent line and m' is the slope of the normal line at (x,y), then the relation $mm'=-1$ yields $m' = 1/y' = (y-1)/(x-0)$. Solution for y' then gives the differential equation $(1-y)y' = x$.

30. Here $m=y'$ and $m'=D_x(x^2+k) = 2x$, so the orthogonality relation $mm'=-1$ gives the differential equation $2xy' = -1$.

31. The slope of the line through (x,y) and $(-y,x)$ is $y' = (x-y)/(-y-x)$, so the differential equation is $(x+y)y' = y-x$.

In Problems 32-36 we get the desired differential equation when we replace the "time rate of change" of the dependent variable with its derivative, the word "is" with the = sign, the phrase "proportional to" with k, and finally translate the remainder of the given sentence into symbols.

32. $dP/dt = k\sqrt{P}$

33. $dv/dt = kv^2$

34. $dv/dt = k(250-v)$

35. $dN/dt = k(P-N)$

36. $dN/dt = kN(P-N)$

37. $y(x) = 1$ or $y(x) = x$

38. $y(x) = e^x$

39. $y(x) = x^2$

40. $y(x) = 1$ or $y(x) = -1$

41. $y(x) = e^x/2$

42. $y(x) = \cos x$ or $y(x) = \sin x$

43. **(a)** $y(10) = 10$ yields $10 = 1/(C-10)$, so $C = 101/10$.

(b) There is no such value of C, but the constant function $y(x) \equiv 0$ satisfies the conditions $y' = y^2$ and $y(0) = 0$.

(c) It is obvious visually that one and only one solution curve passes through each point (a,b) of the *xy*-plane, so it follows that there exists a unique solution to the initial value problem $y' = y^2$, $y(a) = b$.

44. **(b)** Obviously the functions $u(x) = -x^4$ and $v(x) = +x^4$ both satisfy the differential equation $x y' = 4y$. But their derivatives $u'(x) = -4x^3$ and $v'(x) = +4x^3$ match at $x = 0$, where both are zero. Hence the given piecewise-defined function $y(x)$ is differentiable, and therefore satisfies the differential equation because $u(x)$ and $v(x)$ do so (for $x \le 0$ and $x \ge 0$, respectively).

(c) If $a \ge 0$ (for instance), chose C_0 so that $C_0 a^4 = b$. Then the function

$$y(x) = \begin{cases} C x^4 & \text{if } x \le 0, \\ C_0 x^4 & \text{if } x \ge 0 \end{cases}$$

satisfies the given differential equation for every value of C.

SECTION 1.2

INTEGRALS AS GENERAL AND PARTICULAR SOLUTIONS

This section introduces **general solutions** and **particular solutions** in the very simplest situation — a differential equation of the form $y' = f(x)$ — where only direct integration and evaluation of the constant of integration are involved. Students should review carefully the elementary concepts of velocity and acceleration, as well as the fps and mks unit systems.

1. Integration of $y' = 2x+1$ yields $y(x) = \int (2x+1)\,dx = x^2 + x + C$. Then substitution of $x = 0,\ y = 3$ gives $3 = 0 + 0 + C = C$, so $y(x) = x^2 + x + 3$.

2. Integration of $y' = (x-2)^2$ yields $y(x) = \int (x-2)^2\,dx = \frac{1}{3}(x-2)^3 + C$. Then substitution of $x = 2,\ y = 1$ gives $1 = 0 + C = C$, so $y(x) = \frac{1}{3}(x-2)^3$.

3. Integration of $y' = \sqrt{x}$ yields $y(x) = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$. Then substitution of $x = 4,\ y = 0$ gives $0 = \frac{16}{3} + C$, so $y(x) = \frac{2}{3}(x^{3/2} - 8)$.

4. Integration of $y' = x^{-2}$ yields $y(x) = \int x^{-2}\,dx = -1/x + C$. Then substitution of $x = 1,\ y = 5$ gives $5 = -1 + C$, so $y(x) = -1/x + 6$.

5. Integration of $y' = (x+2)^{-1/2}$ yields $y(x) = \int (x+2)^{-1/2}\,dx = 2\sqrt{x+2} + C$. Then substitution of $x = 2,\ y = -1$ gives $-1 = 2 \cdot 2 + C$, so $y(x) = 2\sqrt{x+2} - 5$.

6. Integration of $y' = x(x^2+9)^{1/2}$ yields $y(x) = \int x(x^2+9)^{1/2}\,dx = \frac{1}{3}(x^2+9)^{3/2} + C$. Then substitution of $x = -4,\ y = 0$ gives $0 = \frac{1}{3}(5)^3 + C$, so $y(x) = \frac{1}{3}\left[(x^2+9)^{3/2} - 125\right]$.

7. Integration of $y' = 10/(x^2+1)$ yields $y(x) = \int 10/(x^2+1)\,dx = 10\tan^{-1}x + C$. Then substitution of $x = 0,\ y = 0$ gives $0 = 10 \cdot 0 + C$, so $y(x) = 10\tan^{-1}x$.

8. Integration of $y' = \cos 2x$ yields $y(x) = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$. Then substitution of $x = 0,\ y = 1$ gives $1 = 0 + C$, so $y(x) = \frac{1}{2}\sin 2x + 1$.

9. Integration of $y' = 1/\sqrt{1-x^2}$ yields $y(x) = \int 1/\sqrt{1-x^2}\,dx = \sin^{-1}x + C$. Then substitution of $x = 0,\ y = 0$ gives $0 = 0 + C$, so $y(x) = \sin^{-1}x$.

10. Integration of $y' = xe^{-x}$ yields

$$y(x) = \int xe^{-x}\,dx = \int ue^{u}\,du = (u-1)e^{u} = -(x+1)e^{-x} + C$$

(when we substitute $u = -x$ and apply Formula #46 inside the back cover to the textbook). Then substitution of $x = 0$, $y = 1$ gives $1 = -1 + C$, so $y(x) = -(x+1)e^{-x} + 2$.

11. If $a(t) = 50$ then $v(t) = \int 50\,dt = 50t + v_0 = 50t + 10$. Hence

$$x(t) = \int (50t+10)\,dt = 25t^2 + 10t + x_0 = 25t^2 + 10t + 10.$$

12. If $a(t) = -20$ then $v(t) = \int (-20)\,dt = -20t + v_0 = -20t - 15$. Hence

$$x(t) = \int (-20t-15)\,dt = -10t^2 - 15t + x_0 = -10t^2 - 15t + 5.$$

13. If $a(t) = 3t$ then $v(t) = \int 3t\,dt = \frac{3}{2}t^2 + v_0 = \frac{3}{2}t^2 + 5$. Hence

$$x(t) = \int (\tfrac{3}{2}t^2 + 5)\,dt = \tfrac{1}{2}t^3 + 5t + x_0 = \tfrac{1}{2}t^3 + 5t.$$

14. If $a(t) = 2t+1$ then $v(t) = \int (2t+1)\,dt = t^2 + t + v_0 = t^2 + t - 7$. Hence

$$x(t) = \int (t^2 + t - 7)\,dt = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + x_0 = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + 4.$$

15. If $a(t) = 2t+1$ then $v(t) = \int (2t+1)\,dt = t^2 + t + v_0 = t^2 + t - 7$. Hence

$$x(t) = \int (t^2 + t - 7)\,dt = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + x_0 = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + 4.$$

16. If $a(t) = 1/\sqrt{t+4}$ then $v(t) = \int 1/\sqrt{t+4}\,dt = 2\sqrt{t+4} + C = 2\sqrt{t+4} - 5$ (taking $C = -5$ so that $v(0) = -1$). Hence

$$x(t) = \int (2\sqrt{t+4} - 5)\,dt = \tfrac{4}{3}(t+4)^{3/2} - 5t + C = \tfrac{4}{3}(t+4)^{3/2} - 5t - \tfrac{29}{3}$$

(taking $C = -29/3$ so that $x(0) = 1$).

17. If $a(t) = (t+1)^{-3}$ then $v(t) = \int (t+1)^{-3}\,dt = -\frac{1}{2}(t+1)^{-2} + C = -\frac{1}{2}(t+1)^{-2} + \frac{1}{2}$ (taking $C = \frac{1}{2}$ so that $v(0) = 0$). Hence

$$x(t) = \int \left[-\tfrac{1}{2}(t+1)^{-2} + \tfrac{1}{2}\right]dt = \tfrac{1}{2}(t+1)^{-1} + \tfrac{1}{2}t + C = \tfrac{1}{2}\left[(t+1)^{-1} + t - 1\right]$$

(taking $C = -\frac{1}{2}$ so that $x(0) = 0$).

18. If $a(t) = 50\sin 5t$ then $v(t) = \int 50\sin 5t\,dt = -10\cos 5t + C = -10\cos 5t$ (taking $C = 0$ so that $v(0) = -10$). Hence

$$x(t) = \int(-10\cos 5t)\,dt = -2\sin 5t + C = -2\sin 5t + 10$$

(taking $C = -10$ so that $x(0) = 8$).

19. $v = -9.8t + 49$, so the ball reaches its maximum height $(v = 0)$ after $t = 5$ seconds. Its maximum height then is $y(5) = -4.9(5)^2 + 49(5) = 122.5$ meters.

20. $v = -32t$ and $y = -16t^2 + 400$, so the ball hits the ground $(y = 0)$ when $t = 5$ sec, and then $v = -32(5) = -160$ ft/sec.

21. $a = -10\text{ m/s}^2$ and $v_0 = 100\text{ km/h} \approx 27.78\text{ m/s}$, so $v = -10t + 27.78$, and hence $x(t) = -5t^2 + 27.78t$. The car stops when $v = 0$, $t \approx 2.78$, and thus the distance traveled before stopping is $x(2.78) \approx 38.59$ meters.

22. $v = -9.8t + 100$ and $y = -4.9t^2 + 100t + 20$.

(a) $v = 0$ when $t = 100/9.8$ so the projectile's maximum height is $y(100/9.8) = -4.9(100/9.8)^2 + 100(100/9.8) + 20 \approx 530$ meters.

(b) It passes the top of the building when $y(t) = -4.9t^2 + 100t + 20 = 20$, and hence after $t = 100/4.9 \approx 20.41$ seconds.

(c) The roots of the quadratic equation $y(t) = -4.9t^2 + 100t + 20 = 0$ are $t = -0.20,\ 20.61$. Hence the projectile is in the air 20.61 seconds.

23. $a = -9.8$ m/s2 so $v = -9.8\,t - 10$ and

$$y = -4.9\,t^2 - 10\,t + y_0.$$

The ball hits the ground when $y = 0$ and

$$v = -9.8\,t - 10 = -60,$$

so $t \approx 5.10$ s. Hence

$$y_0 = 4.9(5.10)^2 + 10(5.10) \approx 178.57\text{ m}.$$

24. $v = -32t - 40$ and $y = -16t^2 - 40t + 555$. The ball hits the ground $(y = 0)$ when $t \approx 4.77$ sec, with velocity $v = v(4.77) \approx -192.64$ ft/sec, an impact speed of about 131 mph.

25. Integration of $dv/dt = 0.12\,t^3 + 0.6\,t$, $v(0) = 0$ gives $v(t) = 0.3\,t^2 + 0.04\,t^3$. Hence $v(10) = 70$. Then integration of $dx/dt = 0.3\,t^2 + 0.04\,t^3$, $x(0) = 0$ gives $x(t) = 0.1\,t^3 + 0.04\,t^4$, so $x(10) = 200$. Thus after 10 seconds the car has gone 200 ft and is traveling at 70 ft/sec.

26. Taking $x_0 = 0$ and $v_0 = 60$ mph $= 88$ ft/sec, we get

$$v = -at + 88,$$

and $v = 0$ yields $t = 88/a$. Substituting this value of t and $x = 176$ in

$$x = -at^2/2 + 88t,$$

we solve for $a = 22$ ft/sec^2. Hence the car skids for $t = 88/22 = 4$ sec.

27. If $a = -20$ m/sec^2 and $x_0 = 0$ then the car's velocity and position at time t are given by

$$v = -20t + v_0, \quad x = -10\,t^2 + v_0 t.$$

It stops when $v = 0$ (so $v_0 = 20t$), and hence when

$$x = 75 = -10\,t^2 + (20t)t = 10\,t^2.$$

Thus $t = \sqrt{7.5}$ sec so

$$v_0 = 20\sqrt{7.5} \approx 54.77 \text{ m/sec} \approx 197 \text{ km/hr}.$$

28. Starting with $x_0 = 0$ and $v_0 = 50$ km/h $= 5\times10^4$ m/h, we find by the method of Problem 24 that the car's deceleration is $a = (25/3)\times10^7$ m/h^2. Then, starting with $x_0 = 0$ and $v_0 = 100$ km/h $= 10^5$ m/h, we substitute $t = v_0/a$ into

$$x = -at^2 + v_0 t$$

and find that $x = 60$ m when $v = 0$. Thus doubling the initial velocity quadruples the distance the car skids.

29. If $v_0 = 0$ and $y_0 = 20$ then

$$v = -at \text{ and } y = -\tfrac{1}{2}at^2 + 20.$$

Substitution of $t = 2$, $y = 0$ yields $a = 10$ ft/sec^2. If $v_0 = 0$ and $y_0 = 200$ then

$$v = -10t \text{ and } y = -5t^2 + 200.$$

Hence $y = 0$ when $t = \sqrt{40} = 2\sqrt{10}$ sec and $v = -20\sqrt{10} \approx -63.25$ ft/sec.

30. **On Earth:** $v = -32t + v_0$, so $t = v_0/32$ at maximum height (when $v = 0$). Substituting this value of t and $y = 144$ in

$$y = -16t^2 + v_0 t,$$

we solve for $v_0 = 96$ ft/sec as the initial speed with which the person can throw a ball straight upward.

On Planet Gzyx: From Problem 27, the surface gravitational acceleration on planet Gzyx is $a = 10$ ft/sec^2, so

$$v = -10t + 96 \quad \text{and} \quad y = -5t^2 + 96t.$$

Therefore $v = 0$ yields $t = 9.6$ sec, and thence $y_{max} = y(9.6) = 460.8$ ft is the height a ball will reach if its initial velocity is 96 ft/sec.

31. If $v_0 = 0$ and $y_0 = h$ then the stone's velocity and height are given by

$$v = -gt, \quad y = -0.5\,gt^2 + h.$$

Hence $y = 0$ when $t = \sqrt{2h/g}$ so

$$v = -g\sqrt{2h/g} = -\sqrt{2gh}\,.$$

32. The method of solution is precisely the same as that in Problem 30. We find first that, on Earth, the woman must jump straight upward with initial velocity $v_0 = 12$ ft/sec to reach a maximum height of 2.25 ft. Then we find that, on the Moon, this initial velocity yields a maximum height of about 13.58 ft.

33. We use units of miles and hours. If $x_0 = v_0 = 0$ then the car's velocity and position after t hours are given by

$$v = at, \quad x = \tfrac{1}{2}t^2.$$

Since $v = 60$ when $t = 5/6$, the velocity equation yields $a = 72$ mi/hr^2. Hence the distance traveled by 12:50 pm is

$$x = (0.5)(72)(5/6)^2 = 25 \text{ miles.}$$

34. Again we have

$$v = at, \quad x = \tfrac{1}{2}t^2.$$

But now $v = 60$ when $x = 35$. Substitution of $a = 60/t$ (from the velocity equation) into the position equation yields

$$35 = (0.5)(60/t)(t^2) = 30t,$$

whence $t = 7/6$ hr, that is, 1:10 p.m.

35. Integration of $y' = (9/v_S)(1 - 4x^2)$ yields

$$y = (3/v_S)(3x - 4x^3) + C,$$

and the initial condition $y(-1/2) = 0$ gives $C = 3/v_S$. Hence the swimmer's trajectory is

$$y(x) = (3/v_S)(3x - 4x^3 + 1).$$

Substitution of $y(1/2) = 1$ now gives $v_S = 6$ mph.

36. Integration of $y' = 3(1 - 16x^4)$ yields

$$y = 3x - (48/5)x^5 + C,$$

and the initial condition $y(-1/2) = 0$ gives $C = 6/5$. Hence the swimmer's trajectory is

$$y(x) = (1/5)(15x - 48x^5 + 6),$$

so his downstream drift is $y(1/2) = 2.4$ miles.

SECTION 1.3

SLOPE FIELDS AND SOLUTION CURVES

As pointed out in the textbook, the instructor may choose to delay covering Section 1.3 until later in Chapter 1. However, before proceeding to Chapter 2, it is important that students come to grips at some point with the question of the existence of a unique solution of a differential equation — and realize that it makes no sense to look for the solution without knowing in advance that it exists. The instructor may prefer to combine existence and uniqueness by simplifying the statement of the existence-uniqueness theorem as follows:

> Suppose that the function $f(x, y)$ and the partial derivative $\partial f / \partial y$ are both continuous in some neighborhood of the point (a, b). Then the initial value problem
>
> $$\frac{dy}{dx} = f(x,y), \qquad y(a) = b$$

has a unique solution in some neighborhood of the point a.

Slope fields and geometrical solution curves are introduced in this section as a concrete aid in visualizing solutions and existence-uniqueness questions. Solution curves corresponding to the slope fields in Problems 1–10 are shown in the answers section of the textbook and will not be duplicated here.

11. Each isocline $x - 1 = C$ is a vertical straight line.

12. Each isocline $x + y = C$ is a straight line with slope $m = -1$.

13. Each isocline $y^2 = C \geq 0$, that is, $y = \sqrt{C}$ or $y = -\sqrt{C}$, is a horizontal straight line.

14. Each isocline $\sqrt[3]{y} = C$, that is, $y = C^3$, is a horizontal straight line.

15. Each isocline $y/x = C$, or $y = Cx$, is a straight line through the origin.

16. Each isocline $x^2 - y^2 = C$ is a hyperbola that opens along the x-axis if $C > 0$, along the y-axis if $C < 0$.

17. Each isocline $xy = C$ is a rectangular hyperbola that opens along the line $y = x$ if $C > 0$, along $y = -x$ if $C < 0$.

18. Each isocline $x - y^2 = C$, or $y^2 = x - C$, is a translated parabola that opens along the x–axis.

19. Each isocline $y - x^2 = C$, or $x^2 = y - C$, is a translated parabola that opens along the y–axis.

20. Each isocline is an exponential graph of the form $y = Ce^x$.

21. Because both $f(x,y) = 2x^2y^2$ and $\partial f / \partial y = 4x^2y$ are continuous everywhere, the existence-uniqueness theorem of Section 1.3 in the textbook guarantees the existence of a unique solution in some neighborhood of $x = 1$.

22. Both $f(x,y) = x \ln y$ and $\partial f / \partial y = x/y$ are continuous in a neighborhood of $(1, 1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 1$.

23. Both $f(x,y) = y^{1/3}$ and $\partial f / \partial y = (1/3)y^{-2/3}$ are continuous near $(0, 1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x = 0$.

24. $f(x,y) = y^{1/3}$ is continuous in a neighborhood of $(0, 0)$, but $\partial f/\partial y = (1/3)y^{-2/3}$ is not, so the theorem guarantees existence but not uniqueness in some neighborhood of $x = 0$.

25. $f(x,y) = (x - y)^{1/2}$ is not continuous at $(2, 2)$ because it is not even defined if $y > x$. Hence the theorem guarantees neither existence nor uniqueness in any neighborhood of the point $x = 2$.

26. $f(x,y) = (x - y)^{1/2}$ and $\partial f/\partial y = -(1/2)(x - y)^{-1/2}$ are continuous in a neighborhood of $(2, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 2$.

27. Both $f(x,y) = (x - 1/y$ and $\partial f/\partial y = -(x - 1)/y^2$ are continuous near $(0, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 0$.

28. Neither $f(x,y) = (x - 1)/y$ nor $\partial f/\partial y = -(x - 1)/y^2$ is continuous near $(1, 0)$, so the existence-uniqueness theorem guarantees nothing.

29. Both $f(x,y) = \ln(1 + y^2)$ and $\partial f/\partial y = 2y/(1 + y^2)$ are continuous near $(0, 0)$, so the theorem guarantees the existence of a unique solution near $x = 0$.

30. Both $f(x,y) = x^2 - y^2$ and $\partial f/\partial y = -2y$ are continuous near $(0, 1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x = 0$.

31. If $f(x,y) = -(1 - y^2)^{1/2}$ then $\partial f/\partial y = y(1 - y^2)^{-1/2}$ is not continuous when $y = 1$, so the theorem does not guarantee uniqueness.

32. The two solutions are $y_1(x) = 0$ (constant) and $y_2(x) = x^3$.

35. The isoclines of $y' = y/x$ are the straight lines $y = Cx$ through the origin, and $y' = C$ at points of $y = Cx$, so it appears that these same straight lines are the solution curves of $xy' = y$. Then we observe that there is

(i) a unique one of these lines through any point not on the y-axis;
(ii) no such line through any point on the y-axis other than the origin; and
(iii) infinitely many such lines through the origin.

36. $f(x,y) = 4xy^{1/2}$ and $\partial f/\partial y = 2xy^{-1/2}$ are continuous if $y > 0$, so for all a and all $b > 0$ there exists a unique solution near $x = a$ such that $y(a) = b$. If $b = 0$ then the theorem guarantees neither existence nor uniqueness. For any a, both $y_1(x) = 0$ and $y_2(x) = (x^2 - a^2)^2$ are solutions with $y(a) = 0$. Thus we have existence but not uniqueness near points on the x-axis.

SECTION 1.4

SEPARABLE EQUATIONS AND APPLICATIONS

Of course it should be emphasized to students that the possibility of separating the variables is the first one you look for. The general concept of natural growth and decay is important for all differential equations students, but the particular applications in this section are optional. Torricelli's law in the form of Equation (24) in the text leads to some nice concrete examples and problems.

1. $\int \frac{dy}{y} = -\int 2x\,dx; \quad \ln y = -x^2 + c; \quad y(x) = e^{-x^2+c} = Ce^{-x^2}$

2. $\int \frac{dy}{y^2} = -\int 2x\,dx; \quad -\frac{1}{y} = -x^2 - C; \quad y(x) = \frac{1}{x^2+C}$

3. $\int \frac{dy}{y} = \int \sin x\,dx; \quad \ln y = -\cos x + c; \quad y(x) = e^{-\cos x + c} = Ce^{-\cos x}$

4. $\int \frac{dy}{y} = \int \frac{4\,dx}{1+x}; \quad \ln y = 4\ln(1+x) + \ln C; \quad y(x) = C(1+x)^4$

5. $\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{2\sqrt{x}}; \quad \sin^{-1} y = \sqrt{x} + C; \quad y(x) = \sin\left(\sqrt{x} + C\right)$

6. $\int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x}\,dx; \quad 2\sqrt{y} = 2x^{3/2} + 2C; \quad y(x) = \left(x^{3/2} + C\right)^2$

7. $\int \frac{dy}{y^{1/3}} = \int 4x^{1/3}\,dx; \quad \frac{3}{2}y^{2/3} = 3x^{4/3} + \frac{3}{2}C; \quad y(x) = \left(2x^{4/3} + C\right)^{3/2}$

8. $\int \cos y\,dy = \int 2x\,dx; \quad \sin y = x^2 + C; \quad y(x) = \sin^{-1}\left(x^2 + C\right)$

9. $\int \frac{dy}{y} = \int \frac{2\,dx}{1-x^2} = \int \left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx$ (partial fractions)

$\ln y = \ln(1+x) - \ln(1-x) + \ln C; \quad y(x) = C\frac{1+x}{1-x}$

10. $\displaystyle\int \frac{dy}{(1+y)^2} = \int \frac{dx}{(1+x)^2}; \quad -\frac{1}{1+y} = -\frac{1}{1+x} - C = -\frac{1+C(1+x)}{1+x}$

$$1+y = \frac{1+x}{1+C(1+x)}; \quad y(x) = \frac{1+x}{1+C(1+x)} - 1 = \frac{x-C(1+x)}{1+C(1+x)}$$

11. $\displaystyle\int \frac{dy}{y^3} = \int x\,dx; \quad -\frac{1}{2y^2} = \frac{x^2}{2} - \frac{C}{2}; \quad y(x) = \left(C-x^2\right)^{-1/2}$

12. $\displaystyle\int \frac{y\,dy}{y^2+1} = \int x\,dx; \quad \tfrac{1}{2}\ln\left(y^2+1\right) = \tfrac{1}{2}x^2 + \tfrac{1}{2}\ln C; \quad y^2+1 = Ce^{x^2}$

13. $\displaystyle\int \frac{y^3\,dy}{y^4+1} = \int \cos x\,dx; \quad \tfrac{1}{4}\ln\left(y^4+1\right) = \sin x + C$

14. $\displaystyle\int \left(1+\sqrt{y}\right)dy = \int \left(1+\sqrt{x}\right)dx; \quad y + \tfrac{2}{3}y^{3/2} = x + \tfrac{2}{3}x^{3/2} + C$

15. $\displaystyle\int \left(\frac{2}{y^2} - \frac{1}{y^4}\right)dy = \int \left(\frac{1}{x} - \frac{1}{x^2}\right)dx; \quad -\frac{2}{y} + \frac{1}{3y^3} = \ln|x| + \frac{1}{x} + C$

16. $\displaystyle\int \frac{\sin y\,dy}{\cos y} = \int \frac{x\,dx}{1+x^2}; \quad -\ln(\cos x) = \tfrac{1}{2}\ln\left(1+x^2\right) + \ln C$

$$\sec y = C\sqrt{1+x^2}; \quad y(x) = \sec^{-1}\left(C\sqrt{1+x^2}\right)$$

17. $y' = 1+x+y+xy = (1+x)(1+y)$

$$\int \frac{dy}{1+y} = \int (1+x)\,dx; \quad \ln|1+y| = x + \tfrac{1}{2}x^2 + C$$

18. $x^2y' = 1-x^2+y^2-x^2y^2 = (1-x^2)(1+y^2)$

$$\int \frac{dy}{1+y^2} = \int \left(\frac{1}{x^2} - 1\right)dx; \quad \tan^{-1} y = -\frac{1}{x} - x + C; \quad y(x) = \tan\left(C - \frac{1}{x} - x\right)$$

19. $\displaystyle\int \frac{dy}{y} = \int e^x\,dx; \quad \ln y = e^x + \ln C; \quad y(x) = C\exp(e^x)$

$y(0) = 2e$ implies $C = 2$ so $y(x) = 2\exp(e^x)$

20. $\int \frac{dy}{1+y^2} = \int 3x^2\,dx; \quad \tan^{-1} y = x^3 + C; \quad y(x) = \tan\left(x^3 + C\right)$

$y(0) = 1$ implies $C = \tan^{-1} 1 = \pi/4$ so $y(x) = \tan\left(x^3 + \pi/4\right)$

21. $\int 2y\,dy = \int \frac{x\,dx}{\sqrt{x^2-16}}; \quad y^2 = \sqrt{x^2-16} + C$

$y(5) = 2$ implies $C = 1$ so $y^2 = 1 + \sqrt{x^2-16}$

22. $\int \frac{dy}{y} = \int \left(4x^3 - 1\right)dx; \quad \ln y = x^4 - x + \ln C; \quad y(x) = C\exp(x^4 - x)$

$y(1) = -3$ implies $C = -3$ so $y(x) = -3\exp(x^4 - x)$

23. $\int \frac{dy}{2y-1} = \int dx; \quad \frac{1}{2}\ln(2y-1) = x + \frac{1}{2}\ln C; \quad 2y - 1 = Ce^{2x}$

$y(1) = 1$ implies $C = e^{-2}$ so $y(x) = \frac{1}{2}\left(1 + e^{2x-2}\right)$

24. $\int \frac{dy}{y} = \int \frac{\cos x\,dx}{\sin x}; \quad \ln y = \ln(\sin x) + \ln C; \quad y(x) = C\sin x$

$y(\frac{\pi}{2}) = \frac{\pi}{2}$ implies $C = \frac{\pi}{2}$ so $y(x) = \frac{\pi}{2}\sin x$

25. $\int \frac{dy}{y} = \int \left(\frac{1}{x} + 2x\right); \quad \ln y = \ln x + x^2 + \ln C; \quad y(x) = Cx\exp(x^2)$

$y(1) = 1$ implies $C = e^{-1}$ so $y(x) = x\exp(x^2 - 1)$

26. $\int \frac{dy}{y^2} = \int \left(2x + 3x^2\right); \quad -\frac{1}{y} = x^2 + x^3 + C; \quad y(x) = \frac{-1}{x^2 + x^3 + C}$

$y(1) = -1$ implies $C = -1$ so $y(x) = \frac{1}{1 - x^2 - x^3}$

27. $\int e^y\,dy = \int 6e^{2x}\,dx; \quad e^y = 3e^{2x} + C; \quad y(x) = \ln\left(3e^{2x} + C\right)$

$y(0) = 0$ implies $C = -2$ so $y(x) = \ln\left(3e^{2x} - 2\right)$

28. $\int \sec^2 y\,dy = \int \frac{dx}{2\sqrt{x}}; \quad \tan y = \sqrt{x} + C; \quad y(x) = \tan^{-1}\left(\sqrt{x} + C\right)$

$y(4) = \frac{\pi}{4}$ implies $C = -1$ so $y(x) = \tan^{-1}\left(\sqrt{x} - 1\right)$

29. The population growth rate is $k = \ln(30000/25000)/10 \approx 0.01823$, so the population of the city t years after 1960 is given by $P(t) = 25000e^{0.01823t}$. The expected year 2000 population is then $P(40) = 25000e^{0.01823\times40} \approx 51840$.

30. The population growth rate is $k = \ln(6)/10 \approx 0.17918$, so the population after t hours is given by $P(t) = P_0 e^{0.17918t}$. To find how long it takes for the population to double, we therefore need only solve the equation $2P = P_0 e^{0.17918t}$ for $t = (\ln 2)/0.17918 \approx 3.87$ hours.

31. As in the textbook discussion of radioactive decay, the number of ${}^{14}C$ atoms after t years is given by $N(t) = N_0 e^{-0.0001216t}$. Hence we need only solve the equation $\frac{1}{6}N_0 = N_0 e^{-0.0001216t}$ for $t = (\ln 6)/0.0001216 \approx 14735$ years to find the age of the skull.

32. As in Problem 31, the number of ${}^{14}C$ atoms after t years is given by $N(t) = 5.0\times10^{10} e^{-0.0001216t}$. Hence we need only solve the equation $4.6\times10^{10} = 5.0\times10^{10} e^{-0.0001216t}$ for the age $t = \left(\ln(5.0/4.6)\right)/0.0001216 \approx 686$ years of the relic. Thus it appears not to be a genuine relic of the time of Christ 2000 years ago.

33. The amount in the account after t years is given by $A(t) = 5000e^{0.08t}$. Hence the amount in the account after 18 years is given by $A(20) = 5000e^{0.08\times20} \approx 21{,}103.48$ dollars.

34. When the book has been overdue for t years, the fine owed is given in dollars by $A(t) = 0.30e^{0.05t}$. Hence the amount owed after 100 years is given by $A(100) = 0.30e^{0.05\times100} \approx 44.52$ dollars.

35. To find the decay rate of this drug in the dog's blood stream, we solve the equation $\frac{1}{2} = e^{-5k}$ (half-life 5 hours) for $k = (\ln 2)/5 \approx 0.13863$. Thus the amount in the dog's bloodstream after t hours is given by $A(t) = A_0 e^{-0.13863t}$. We therefore solve the equation $A(1) = A_0 e^{-0.13863} = 50\times45 = 2250$ for $A_0 \approx 2585$ mg, the amount to anesthetize the dog properly.

36. To find the decay rate of radioactive cobalt, we solve the equation $\frac{1}{2} = e^{-5.27k}$ (half-life 5.27 years) for $k = (\ln 2)/5.27 \approx 0.13153$. Thus the amount of radioactive cobalt left after t years is given by $A(t) = A_0 e^{-0.13153t}$. We therefore solve the equation

$A(t) = A_0 e^{-0.13153t} = 0.01 A_0$ for $t = (\ln 100)/0.13153 \approx 35.01$ and find that it will be about 35 years until the region is again inhabitable.

37. Taking $t = 0$ when the body was formed and $t = T$ now, the amount $Q(t)$ of ^{238}U in the body at time t (in years) is given by $Q(t) = Q_0 e^{-kt}$, where $k = (\ln 2)/(4.51\times10^9)$. The given information tells us that

$$\frac{Q(T)}{Q_0 - Q(T)} = 0.9.$$

After substituting $Q(T) = Q_0 e^{-kT}$, we solve readily for $e^{kT} = 19/9$, so $T = (1/k)\ln(19/9) \approx 4.86\times10^9$. Thus the body was formed approximately 4.86 billion years ago.

38. Taking $t = 0$ when the rock contained only potassium and $t = T$ now, the amount $Q(t)$ of potassium in the rock at time t (in years) is given by $Q(t) = Q_0 e^{-kt}$, where $k = (\ln 2)/(1.28\times10^9)$. The given information tells us that the amount $A(t)$ of argon at time t is

$$A(t) = \tfrac{1}{9}[Q_0 - Q(t)]$$

and also that $A(T) = Q(T)$. Thus

$$Q_0 - Q(T) = 9\,Q(T).$$

After substituting $Q(T) = Q_0 e^{-kT}$ we readily solve for

$$T = (\ln 10/\ln 2)(1.28\times10^9) \approx 4.25\times10^9.$$

Thus the age of the rock is about 1.25 billion years.

39. Because $A = 0$ the differential equation reduces to $T' = kT$, so $T(t) = 25e^{-kt}$. The fact that $T(20) = 15$ yields $k = (1/20)\ln(5/3)$, and finally we solve

$$5 = 25e^{-kt} \quad \text{for} \quad t = (\ln 5)/k \approx 63 \text{ min.}$$

40. The amount of sugar remaining undissolved after t minutes is given by $A(t) = A_0 e^{-kt}$; we find the value of k by solving the equation $A(1) = A_0 e^{-k} = 0.75 A_0$ for $k = -\ln 0.75 \approx 0.28768$. To find how long it takes for half the sugar to dissolve, we solve the equation $A(t) = A_0 e^{-kt} = \frac{1}{2} A_0$ for $t = (\ln 2)/0.28768 \approx 2.41$ minutes.

41. **(a)** The light intensity at a depth of x meters is given by $I(x) = I_0 e^{-1.4x}$. We solve the equation $I(x) = I_0 e^{-1.4x} = \frac{1}{2} I_0$ for $x = (\ln 2)/1.4 \approx 0.495$ meters.

(b) At depth 10 meters the intensity is $I(10) = I_0 e^{-1.4\times 10} \approx (8.32\times 10^{-7})\, I_0$.

(c) We solve the equation $I(x) = I_0 e^{-1.4x} = 0.01 I_0$ for $x = (\ln 100)/1.4 \approx 3.29$ meters.

42. (a) The pressure at an altitude of x miles is given by $p(x) = 29.92\, e^{-0.2x}$. Hence the pressure at altitude 10000 ft is $p(10000/5280) \approx 20.49$ inches, and the pressure at altitude 30000 ft is $p(30000/5280) \approx 9.60$ inches.

(b) To find the altitude where $p = 15$ in., we solve the equation $29.92\, e^{-0.2x} = 15$ for $x = (\ln 29.92/15)/0.2 \approx 3.452$ miles $\approx 18{,}200$ ft.

43. (a) $A' = rA + Q$

(b) The solution of the differential equation with $A(0) = 0$ is given by

$$rA + Q = Qe^{rt}.$$

When we substitute $A = 40$ (thousand), $r = 0.11$, and $t = 18$, we find that $Q = 0.70482$, that is, \$704.82 per year.

44. Let $N_8(t)$ and $N_5(t)$ be the numbers of ^{238}U and ^{235}U atoms, respectively, at time t (in billions of years after the creation of the universe). Then $N_8(t) = N_0 e^{-kt}$ and $N_5(t) = N_0 e^{-ct}$, where N_0 is the initial number of atoms of each isotope. Also, $k = (\ln 2)/4.51$ and $c = (\ln 2)/0.71$ from the given half-lives. We divide the equations for N_8 and N_5 and find that when t has the value corresponding to "now",

$$e^{(c-k)t} = \frac{N_8}{N_5} = 137.7.$$

Finally we solve this last equation for $t = (\ln 137.7)/(c-k) \approx 5.99$. Thus we get an estimate of about 6 billion years for the age of the universe.

45. The cake's temperature will be 100° after 66 min 40 sec; this problem is just like Example 6 in the text.

46. (b) By separating the variables we solve the differential equation for

$$c - r\,P(t) = (c - r\,P_0)\, e^{rt}.$$

With $P(t) = 0$ this yields

$$c = r P_0 e^{rt} / (e^{rt} - 1).$$

With $P_0 = 10{,}800$, $t = 60$, and $r = 0.010$ we get \$239.37 for the monthly payment at 12% annual interest. With $r = 0.015$ we get \$272.99 for the monthly payment at 18% annual interest.

47. If $N(t)$ denotes the number of people (in thousands) who have heard the rumor after t days, then the initial value problem is

$$N' = k(100 - N), \quad N(0) = 0$$

and we are given that $N(7) = 10$. When we separate variables ($dN/(100-N) = k\,dt$) and integrate, we get $\ln(100-N) = -kt + C$, and the initial condition $N(0) = 0$ gives $C = \ln 100$. Then $100 - N = 100e^{-kt}$, so $N(t) = 100\left(1 - e^{-kt}\right)$. We substitute $t = 7$, $N = 10$ and solve for the value $k = \ln(100/90)/7 \approx 0.01505$. Finally, 50 thousand people have heard the rumor after $t = (\ln 2)/k \approx 46.05$ days.

48. With $A(y)$ constant, Equation (19) in the text takes the form

$$\frac{dy}{dt} = k\sqrt{y}$$

We readily solve this equation for $2\sqrt{y} = kt + C$. The condition $y(0) = 9$ yields $C = 6$, and then $y(1) = 4$ yields $k = 2$. Thus the depth at time t (in hours) is $y(t) = (3 - t)^2$, and hence it takes 3 hours for the tank to empty.

49. With $A = \pi(3)^2$ and $a = \pi(1/12)^2$, and taking $g = 32$ ft/sec^2, Equation (20) reduces to $162\,y' = -\sqrt{y}$. The solution such that $y = 9$ when $t = 0$ is given by $324\sqrt{y} = -t + 972$. Hence $y = 0$ when $t = 972$ sec $= 16$ min 12 sec.

50. The radius of the cross-section of the cone at height y is proportional to y, so $A(y)$ is proportional to y^2. Therefore Equation (20) takes the form

$$y^2 y' = -k\sqrt{y},$$

and a general solution is given by

$$2y^{5/2} = -5kt + C.$$

The initial condition $y(0) = 16$ yields $C = 2048$, and then $y(1) = 9$ implies that $5k = 1562$. Hence $y = 0$ when

$$t = C/5k = 2048/1562 = 1.31 \text{ hr}.$$

51. The solution of $y' = -k\sqrt{y}$ is given by

$$2\sqrt{y} = -kt + C.$$

The initial condition $y(0) = h$ (the height of the cylinder) yields $C = 2\sqrt{h}$. Then substitution of $t = T,\ y = 0$ gives $k = (2\sqrt{h})/T$. It follows that

$$y = h(1 - t/T)^2.$$

If r denotes the radius of the cylinder, then

$$V(y) = \pi r^2 y = \pi r^2 h(1 - t/T)^2 = V_0(1 - t/T)^2.$$

52. Since $x = y^{3/4}$, the cross-sectional area is $A(y) = \pi x^2 = \pi y^{3/2}$. Hence the general equation $A(y)y' = -a\sqrt{2gy}$ reduces to the differential equation $yy' = -k$ with general solution

$$(1/2)y^2 = -kt + C.$$

The initial condition $y(0) = 12$ gives $C = 72$, and then $y(1) = 6$ yields $k = 54$. Upon separating variables and integrating, we find that the the depth at time t is

$$y(t) = \sqrt{144 - 108t}\ y(t).$$

Hence the tank is empty after $t = 144/108$ hr, that is, at 1:20 p.m.

53. **(a)** Since $x^2 = by$, the cross-sectional area is $A(y) = \pi x^2 = \pi by$. Hence the equation $A(y)y' = -a\sqrt{2gy}$ reduces to the differential equation

$$y^{1/2}y' = -k = -(a/\pi b)\sqrt{2g}$$

with the general solution

$$(2/3)y^{3/2} = -kt + C.$$

The initial condition $y(0) = 4$ gives $C = 16/3$, and then $y(1) = 1$ yields $k = 14/3$. It follows that the depth at time t is

$$y(t) = (8 - 7t)^{2/3}.$$

(b) The tank is empty after $t = 8/7$ hr, that is, at 1:08:34 p.m.

(c) We see above that $k = (a/\pi b)\sqrt{2g} = 14/3$. Substitution of $a = \pi r^2$, $b = 1$, $g = (32)(3600)^2$ ft/hr^2 yields $r = (1/60)\sqrt{7/12}$ ft ≈ 0.15 in for the radius of the bottom-hole.

54. With $g = 32$ ft/sec^2 and $a = \pi(1/12)^2$, Equation (24) simplifies to

$$A(y)\frac{dy}{dt} = -\frac{\pi}{18}\sqrt{y}.$$

If z denotes the distance from the center of the cylinder down to the fluid surface, then $y = 3 - z$ and $A(y) = 10(9 - z^2)^{1/2}$. Hence the equation above becomes

$$10(9-z^2)^{1/2}\frac{dz}{dt} = \frac{\pi}{18}(3-z)^{1/2},$$
$$180(3+z)^{1/2}dz = \pi\, dt,$$

and integration yields

$$120(3+z)^{1/2} = \pi t + C.$$

Now $z = 0$ when $t = 0$, so $C = 120(3)^{3/2}$. The tank is empty when $z = 3$ (that is, when $y = 0$) and thus after

$$t = (120/\pi)(6^{3/2} - 3^{3/2}) \approx 362.90 \text{ sec.}$$

It therefore takes about 6 min 3 sec for the fluid to drain completely.

55. $A(y) = \pi(8y - y^2)$ as in Example 7 in the text, but now $a = \pi/144$ in Equation (24), so the initial value problem is

$$18(8y - y^2)y' = -\sqrt{y}, \qquad y(0) = 8.$$

We seek the value of t when $y = 0$. The answer is $t \approx 869$ sec $=$ 14 min 29 sec.

56. The cross-sectional area function for the tank is $A = \pi(1 - y^2)$ and the area of the bottom-hole is $a = 10^{-4}\pi$, so Eq. (24) in the text gives the initial value problem

$$\pi(1-y^2)\frac{dy}{dt} = -10^{-4}\pi\sqrt{2\times 9.8y}, \quad y(0) = 1.$$

Simplification gives

$$\left(y^{-1/2} - y^{3/2}\right)\frac{dy}{dt} = -1.4\times 10^{-4}\sqrt{10}$$

so integration yields

$$2y^{1/2} - \frac{2}{5}y^{5/2} = -1.4\times10^{-4}\sqrt{10}\,t + C.$$

The initial condition $y(0) = 1$ implies that $C = 2 - 2/5 = 8/5$, so $y = 0$ after $t = (8/5)/(1.4\times10^{-4}\sqrt{10}) \approx 3614$ seconds. Thus the tank is empty at about 14 seconds after 2 pm.

57. **(a)** As in Example 8, the initial value problem is

$$\pi(8y - y^2)\frac{dy}{dt} = -\pi k\sqrt{y}, \qquad y(0) = 4$$

where $k = 0.6r^2\sqrt{2g} = 4.8r^2$. Integrating and applying the initial condition just in the Example 8 solution in the text, we find that

$$\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2} = -kt + \frac{448}{15}.$$

When we substitute $y = 2$ (ft) and $t = 1800$ (sec, that is, 30 min), we find that $k \approx 0.009469$. Finally, $y = 0$ when

$$t = \frac{448}{15k} \approx 3154 \text{ sec} = 53 \text{ min } 34 \text{ sec}.$$

Thus the tank is empty at 1:53:34 pm.

(b) The radius of the bottom-hole is

$$r = \sqrt{k/4.8} \approx 0.04442 \text{ ft} \approx 0.53 \text{ in, thus about a half inch.}$$

58. The given rate of fall of the water level is $dy/dt = -4$ in/hr $= -(1/10800)$ ft/sec. With $A = \pi x^2$ and $a = \pi r^2$, Equation (24) is

$$(\pi x^2)(1/10800) = -(\pi r^2)\sqrt{2gy} = -8\pi r^2\sqrt{y}.$$

Hence the curve is of the form $y = kx^4$, and in order that it pass through $(1, 4)$ we must have $k = 4$. Comparing $\sqrt{y} = 2x^2$ with the equation above, we see that

$$(8r^2)(10800) = 1/2,$$

so the radius of the bottom hole is $r = 1/(240\sqrt{3})$ ft $\approx 1/35$ in.

59. Let $t = 0$ at the time of death. Then the solution of the initial value problem

$$T' = k(70 - T), \qquad T(0) = 98.6$$

is

$$T(t) = 70 + 28.6e^{-kt}.$$

If $t = a$ at 12 noon, then we know that

$$T(t) = 70 + 28.6e^{-ka} = 80,$$

$$T(a+1) = 70 + 28.6e^{-k(a+1)} = 75.$$

Hence

$$28.6e^{-ka} = 10 \quad \text{and} \quad 28.6e^{-ka}e^{-k} = 5.$$

It follows that $e^{-k} = 1/2$, so $k = \ln 2$. Finally the first of the previous two equations yields

$$a = (\ln 2.86)/(\ln 2) \approx 1.516 \text{ hr} \approx 1 \text{ hr } 31 \text{ min},$$

so the death occurred at 10:29 a.m.

60. Let $t = 0$ when it began to snow, and $t = t_0$ at 7:00 a.m. Let x denote distance along the road, with $x = 0$ where the snowplow begins at 7:00 a.m. If $y = ct$ is the snow depth at time t, w is the width of the road, and $v = dx/dt$ is the plow's velocity, then "plowing at a constant rate" means that the product wyv is constant. Hence our differential equation is of the form

$$k\frac{dx}{dt} = \frac{1}{t}.$$

The solution with $x = 0$ when $t = t_0$ is

$$t = t_0 e^{kx}.$$

We are given that $x = 2$ when $t = t_0 + 1$ and $x = 4$ when $t = t_0 + 3$, so it follows that

$$t_0 + 1 = t_0 e^{2k} \quad \text{and} \quad t_0 + 3 = t_0 e^{4k}.$$

Elimination of t_0 yields the equation

$$e^{4k} - 3e^{2k} + 2 = (e^{2k} - 1)(e^{2k} - 2) = 0,$$

so it follows (since $k > 0$) that $e^{2k} = 2$. Hence $t_0 + 1 = 2t_0$, so $t_0 = 1$. Thus it began to snow at 6 a.m.

61. We still have $t = t_0 e^{kx}$, but now the given information yields the conditions

$$t_0 + 1 = t_0 e^{4k} \quad \text{and} \quad t_0 + 2 = t_0 e^{7k}$$

at 8 a.m. and 9 a.m., respectively. Elimination of t_0 gives the equation

$$2e^{4k} - e^{7k} - 1 = 0,$$

which we solve numerically for $k = 0.08276$. Using this value, we finally solve one of the preceding pair of equations for $t_0 = 2.5483$ hr $\approx$ 2 hr 33 min. Thus it began to snow at 4:27 a.m.

SECTION 1.5

LINEAR FIRST-ORDER EQUATIONS

1. $\rho = \exp\left(\int 1\,dx\right) = e^x;\quad D_x\left(y \cdot e^x\right) = 2e^x;\quad y \cdot e^x = 2e^x + C;\quad y(x) = 2 + Ce^{-x}$

   $y(0) = 0$ implies $C = -2$ so $y(x) = 2 - 2e^{-x}$

2. $\rho = \exp\left(\int (-2)\,dx\right) = e^{-2x};\quad D_x\left(y \cdot e^{-2x}\right) = 3;\quad y \cdot e^{-2x} = 3x + C;\quad y(x) = (3x + C)e^{2x}$

   $y(0) = 0$ implies $C = 0$ so $y(x) = 3x e^{2x}$

3. $\rho = \exp\left(\int 3\,dx\right) = e^{3x};\quad D_x\left(y \cdot e^{3x}\right) = 2x;\quad y \cdot e^{3x} = x^2 + C;\quad y(x) = (x^2 + C)e^{-3x}$

4. $\rho = \exp\left(\int (-2x)\,dx\right) = e^{-x^2};\quad D_x\left(y \cdot e^{-x^2}\right) = 1;\quad y \cdot e^{-x^2} = x + C;\quad y(x) = (x + C)e^{x^2}$

5. $\rho = \exp\left(\int (2/x)\,dx\right) = e^{2\ln x} = x^2;\quad D_x\left(y \cdot x^2\right) = 3x^2;\quad y \cdot x^2 = x^3 + C$

   $y(x) = x + C/x^2;\quad y(1) = 5$ implies $C = 4$ so $y(x) = x + 4/x^2$

6. $\rho = \exp\left(\int (5/x)\,dx\right) = e^{5\ln x} = x^5;\quad D_x\left(y \cdot x^5\right) = 7x^6;\quad y \cdot x^5 = x^7 + C$

   $y(x) = x^2 + C/x^5;\quad y(2) = 5$ implies $C = 32$ so $y(x) = x^2 + 32/x^5$

7. $\rho = \exp\left(\int (1/2x)\,dx\right) = e^{(\ln x)/2} = \sqrt{x};\quad D_x\left(y \cdot \sqrt{x}\right) = 5;\quad y \cdot \sqrt{x} = 5x + C$

$y(x) = 5\sqrt{x} + C/\sqrt{x}$

8. $\rho = \exp\left(\int (1/3x)\,dx\right) = e^{(\ln x)/3} = \sqrt[3]{x};\quad D_x\left(y\cdot\sqrt[3]{x}\right) = 4\sqrt[3]{x};\quad y\cdot\sqrt[3]{x} = 3x^{4/3} + C$

$y(x) = 3x + Cx^{-1/3}$

9. $\rho = \exp\left(\int (-1/x)\,dx\right) = e^{-\ln x} = 1/x;\quad D_x\left(y\cdot 1/x\right) = 1/x;\quad y\cdot 1/x = \ln x + C$

$y(x) = x\ln x + Cx;\quad y(1) = 7$ implies $C = 7$ so $y(x) = x\ln x + 7x$

10. $\rho = \exp\left(\int (-3/2x)\,dx\right) = e^{(-3\ln x)/2} = x^{-3/2};\quad D_x\left(y\cdot x^{-3/2}\right) = 9x^{1/2}/2;\quad y\cdot x^{-3/2} = 3x^{3/2} + C$

$y(x) = 3x^3 + Cx^{3/2}$

11. $\rho = \exp\left(\int (1/x - 3)\,dx\right) = e^{\ln x - 3x} = xe^{-3x};\quad D_x\left(y\cdot xe^{-3x}\right) = 0;\quad y\cdot xe^{-3x} = C$

$y(x) = Cx^{-1}e^{3x};\quad y(1) = 0$ implies $C = 0$ so $y(x) \equiv 0$ (constant)

12. $\rho = \exp\left(\int (3/x)\,dx\right) = e^{3\ln x} = x^3;\quad D_x\left(y\cdot x^3\right) = 2x^7;\quad y\cdot x^3 = \frac{1}{4}x^8 + C$

$y(x) = \frac{1}{4}x^5 + Cx^{-3};\quad y(2) = 1$ implies $C = 56$ so $y(x) = \frac{1}{4}x^5 + 56x^{-3}$

13. $\rho = \exp\left(\int 1\,dx\right) = e^x;\quad D_x\left(y\cdot e^x\right) = e^{2x};\quad y\cdot e^x = \frac{1}{2}e^{2x} + C$

$y(x) = \frac{1}{2}e^x + Ce^{-x};\quad y(0) = 1$ implies $C = \frac{1}{2}$ so $y(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

14. $\rho = \exp\left(\int (-3/x)\,dx\right) = e^{-3\ln x} = x^{-3};\quad D_x\left(y\cdot x^{-3}\right) = x^{-1};\quad y\cdot x^{-3} = \ln x + C$

$y(x) = x^3\ln x + Cx^3;\quad y(1) = 10$ implies $C = 10$ so $y(x) = x^3\ln x + 10x^3$

15. $\rho = \exp\left(\int 2x\,dx\right) = e^{x^2};\quad D_x\left(y\cdot e^{x^2}\right) = xe^{x^2};\quad y\cdot e^{x^2} = \frac{1}{2}e^{x^2} + C$

$y(x) = \frac{1}{2} + Ce^{-x^2};\quad y(0) = -2$ implies $C = -\frac{5}{2}$ so $y(x) = \frac{1}{2} - \frac{5}{2}e^{-x^2}$

16. $\rho = \exp\left(\int \cos x\,dx\right) = e^{\sin x};\quad D_x\left(y\cdot e^{\sin x}\right) = e^{\sin x}\cos x;\quad y\cdot e^{\sin x} = e^{\sin x} + C$

$y(x) = 1 + Ce^{-\sin x};\quad y(\pi) = 2$ implies $C = 1$ so $y(x) = 1 + e^{-\sin x}$

17. $\rho = \exp\left(\int 1/(1+x)\,dx\right) = e^{\ln(1+x)} = 1 + x;\quad D_x\left(y\cdot(1+x)\right) = \cos x;\quad y\cdot(1+x) = \sin x + C$

$$y(x) = \frac{C+\sin x}{1+x}; \quad y(0)=1 \text{ implies } C=1 \text{ so } y(x) = \frac{1+\sin x}{1+x}$$

18. $\rho = \exp\left(\int(-2/x)\,dx\right) = e^{-2\ln x} = x^{-2}; \quad D_x\left(y\cdot x^{-2}\right) = \cos x; \quad y\cdot x^{-2} = \sin x + C$

$y(x) = x^2(\sin x + C)$

19. $\rho = \exp\left(\int \cot x\,dx\right) = e^{\ln(\sin x)} = \sin x; \quad D_x\left(y\cdot\sin x\right) = \sin x\cos x$

$y\cdot\sin x = \frac{1}{2}\sin^2 x + C; \quad y(x) = \frac{1}{2}\sin x + C\csc x$

20. $\rho = \exp\left(\int(-1-x)\,dx\right) = e^{-x-x^2/2}; \quad D_x\left(y\cdot e^{-x-x^2/2}\right) = (1+x)e^{-x-x^2/2}$

$y\cdot e^{-x-x^2/2} = -e^{-x-x^2/2} + C; \quad y(x) = -1 + Ce^{-x-x^2/2}$

$y(0)=0$ implies $C=1$ so $y(x) = -1 + e^{-x-x^2/2}$

21. $\rho = \exp\left(\int(-3/x)\,dx\right) = e^{-3\ln x} = x^{-3}; \quad D_x\left(y\cdot x^{-3}\right) = \cos x; \quad y\cdot x^{-3} = \sin x + C$

$y(x) = x^3\sin x + Cx^3; \quad y(2\pi)=0$ implies $C=0$ so $y(x) = x^3\sin x$

22. $\rho = \exp\left(\int(-2x)\,dx\right) = e^{-x^2}; \quad D_x\left(y\cdot e^{-x^2}\right) = 3x^2; \quad y\cdot e^{-x^2} = x^3 + C$

$y(x) = \left(x^3+C\right)e^{-x^2}; \quad y(0)=5$ implies $C=5$ so $y(x) = \left(x^3+5\right)e^{-x^2}$

23. $\rho = \exp\left(\int(2-3/x)\,dx\right) = e^{2x-3\ln x} = x^{-3}e^{2x}; \quad D_x\left(y\cdot x^{-3}e^{2x}\right) = 4e^{2x}$

$y\cdot x^{-3}e^{2x} = 2e^{2x} + C; \quad y(x) = 2x^3 + Cx^3e^{-2x}$

24. $\rho = \exp\left(\int 3x/(x^2+4)\,dx\right) = e^{3\ln(x^2+4)/2} = (x^2+4)^{3/2}; \quad D_x\left(y\cdot(x^2+4)^{3/2}\right) = x(x^2+4)^{1/2}$

$y\cdot(x^2+4)^{3/2} = \frac{1}{3}(x^2+4)^{3/2} + C; \quad y(x) = \frac{1}{3} + C(x^2+4)^{-3/2}$

$y(0)=1$ implies $C = \frac{16}{3}$ so $y(x) = \frac{1}{3} + \frac{16}{3}(x^2+4)^{-3/2}$

25. First we calculate

$$\int\frac{3x^3\,dx}{x^2+1} = \int\left[3x - \frac{3x}{x^2+1}\right]dx = \frac{3}{2}\left[x^2 - \ln(x^2+1)\right].$$

It follows that $\rho = (x^2+1)^{-3/2}\exp(3x^2/2)$ and thence that

$$D_x\left(y\cdot(x^2+1)^{-3/2}\exp(3x^2/2)\right) = 6x(x^2+4)^{-5/2},$$
$$y\cdot(x^2+1)^{-3/2}\exp(3x^2/2) = -2(x^2+4)^{-3/2}+C,$$
$$y(x) = -2\exp(3x^2/2)+C(x^2+1)^{3/2}\exp(-3x^2/2).$$

Finally, $y(0)=1$ implies that $C=3$ so the desired particular solution is

$$y(x) = -2\exp(3x^2/2)+3(x^2+1)^{3/2}\exp(-3x^2/2).$$

26. With $x' = dx/dy$, the differential equation is $y^3x'+4y^2x=1$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int(4/y)\,dy\right) = e^{4\ln y} = y^4; \quad D_y\left(x\cdot y^4\right) = y$$

$$x\cdot y^4 = \frac{1}{2}y^2+C; \quad x(y) = \frac{1}{2y^2}+\frac{C}{y^4}$$

27. With $x' = dx/dy$, the differential equation is $x'-x = ye^y$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int(-1)\,dy\right) = e^{-y}; \quad D_y\left(x\cdot e^{-y}\right) = y$$

$$x\cdot e^{-y} = \tfrac{1}{2}y^2+C; \quad x(y) = \left(\tfrac{1}{2}y^2+C\right)e^y$$

28. With $x' = dx/dy$, the differential equation is $(1+y^2)x'-2yx=1$. Then with y as the independent variable we calculate

$$\rho(y) = \exp\left(\int(-2y/(1+y^2)\,dy\right) = e^{-\ln(y^2+1)} = (1+y^2)^{-1}$$

$$D_y\left(x\cdot(1+y^2)^{-1}\right) = (1+y^2)^{-2}$$

An integral table (or trigonometric substitution) now yields

$$\frac{x}{1+y^2} = \int\frac{dy}{\left(1+y^2\right)^2} = \frac{1}{2}\left(\frac{y}{1+y^2}+\tan^{-1}y+C\right)$$

$$x(y) = \tfrac{1}{2}\left[y+\left(1+y^2\right)\left(\tan^{-1}y+C\right)\right]$$

29. $\rho = \exp\left(\int(-2x)\,dx\right) = e^{-x^2}; \quad D_x\left(y \cdot e^{-x^2}\right) = e^{-x^2}; \quad y \cdot e^{-x^2} = C + \int_0^x e^{-t^2}\,dt$

$y(x) = e^{-x^2}\left(C + \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right)$

30. After division of the given equation by $2x$, multiplication by the integrating factor $\rho = x^{-1/2}$ yields

$$x^{-1/2}y' - \tfrac{1}{2}x^{-3/2}y = x^{-1/2}\cos x,$$
$$D_x\left(x^{-1/2}y\right) = x^{-1/2}\cos x,$$
$$x^{-1/2}y = C + \int_1^x t^{-1/2}\cos t\,dt.$$

The initial condition $y(1) = 0$ implies that $C = 0$, so the desired particular solution is

$$y(x) = x^{1/2}\int_1^x t^{-1/2}\cos t\,dt.$$

31. (a) $y_c' = Ce^{-\int P\,dx}(-P) = -P\,y_c$, so $y_c' + P\,y_c = 0.$

(b) $y_p' = (-P)e^{-\int P\,dx}\cdot\left[\int\left(Qe^{\int P\,dx}\right)dx\right] + e^{-\int P\,dx}\cdot Qe^{\int P\,dx} = -Py_p + Q$

32. (a) If $y = A\cos x + B\sin x$ then

$$y' + y = (A+B)\cos x + (B-A)\sin x = 2\sin x$$

provided that $A = -1$ and $B = 1$. These coefficient values give the particular solution $y_p(x) = \sin x - \cos x$.

(b) The general solution of the equation $y' + y = 0$ is $y(x) = Ce^{-x}$ so addition to the particular solution found in part (a) gives $y(x) = Ce^{-x} + \sin x - \cos x$.

(c) The initial condition $y(0) = 1$ implies that $C = 2$, so the desired particular solution is $y(x) = 2e^{-x} + \sin x - \cos x$.

33. The amount $x(t)$ of salt (in kg) after t seconds satisfies the differential equation $x' = -x/200$, so $x(t) = 100e^{-t/200}$. Hence we need only solve the equation $10 = 100e^{-t/200}$ for $t = 461$ sec $= 7$ min 41 sec (approximately).

34. Let $x(t)$ denote the amount of pollutants in the lake after t days, measured in millions of cubic feet. Then $x(t)$ satisfies the linear differential equation $dx/dt = 1/4 - x/16$ with solution $x(t) = 4 + 16e^{-t/16}$ satisfying $x(0) = 20$. The value of t such that $x = 8$ is

$t = 16\ln 4 \approx 22.2$ days. For a complete solution see Example 4 in Section 7.6 of Edwards and Penney, *Calculus with Analytic Geometry* (5th edition, Prentice-Hall, 1998).

35. The only difference from the Example 4 solution in the textbook is that $V = 1640\text{ km}^3$ and $r = 410\text{ km}^3/\text{yr}$ for Lake Ontario, so the time required is

$$t = \frac{V}{r}\ln 4 = 4\ln 4 \approx 5.5452 \text{ years.}$$

36. **(a)** The volume of brine in the tank after t min is $V(t) = 60 - t$ gal, so the initial value problem is

$$\frac{dx}{dt} = 2 - \frac{3x}{60-t}, \qquad x(0) = 0.$$

The solution is

$$x(t) = (60-t) - \frac{(60-t)^3}{3600}.$$

(b) The maximum amount ever in the tank is $40/\sqrt{3} \approx 23.09$ lb. This occurs after $t = 60 - 20\sqrt{3} \approx 25/36$ min.

37. The volume of brine in the tank after t min is $V(t) = 100 + 2t$ gal, so the initial value problem is

$$\frac{dx}{dt} = 5 - \frac{3x}{100+2t}, \qquad x(0) = 50.$$

The integrating factor $\rho(t) = (100+2t)^{3/2}$ leads to the solution

$$x(t) = (100+2t) - \frac{50000}{(100+2t)^{3/2}}.$$

such that $x(0) = 50$. The tank is full after $t = 150$ min, at which time $x(150) = 393.75$ lb.

38. **(a)** $dx/dt = -x/20$ and $x(0) = 50$ so $x(t) = 50e^{-t/20}$.

(b) The solution of the linear differential equation

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200} = \frac{5}{2}e^{-t/20} - \frac{1}{40}y$$

with $y(0) = 50$ is

$$y(t) = 150e^{-t/40} - 100e^{-t/20}.$$

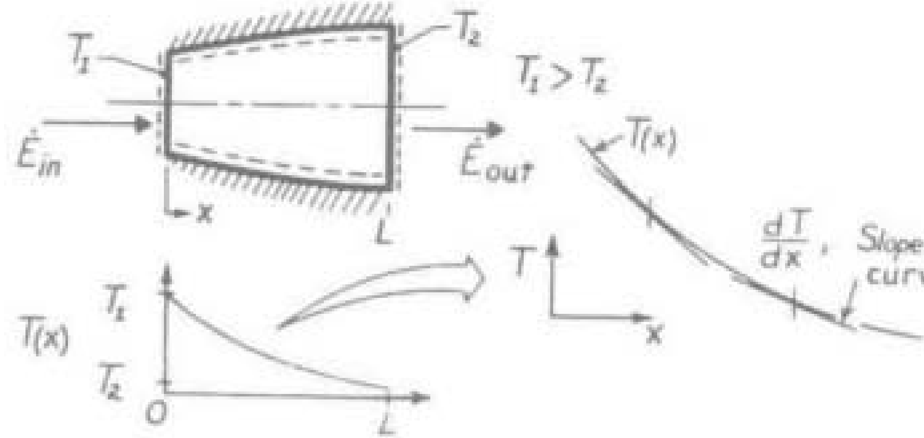
(c) The maximum value of y occurs when

PROBLEM 2.1

KNOWN: Steady-state, one-dimensional heat conduction through an axisymmetric shape.

FIND: Sketch temperature distribution and explain shape of curve.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, one-dimensional conduction, (2) Constant properties, (3) No internal heat generation.

ANALYSIS: Performing an energy balance on the object according to Eq. 1.11a, $\dot{E}_{in} - \dot{E}_{out} = 0$, it follows that

$$\dot{E}_{in} = \dot{E}_{out} = q_x$$

and that $q_x \neq q_x(x)$. That is, the heat rate within the object is everywhere constant. From Fourier's law,

$$q_x = -kA_x \frac{dT}{dx},$$

and since q_x and k are both constants, it follows that

$$A_x \frac{dT}{dx} = \text{Constant}.$$

That is, the product of the cross-sectional area normal to the heat rate and temperature gradient remains a constant and independent of distance x. It follows that since A_x increases with x, then dT/dx must decrease with x. Hence, the temperature distribution appears as shown above. Note the gradient decreases with increasing x.

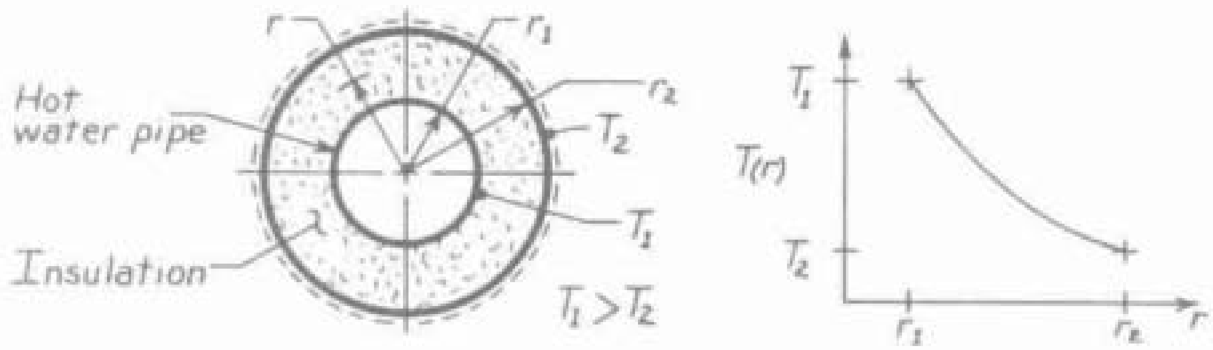
COMMENTS: (1) Be sure to recognize that dT/dx is the slope of the temperature distribution. (2) What would the distribution be when $T_2 > T_1$? (3) Show on the above plot how the heat flux, q''_x, varies with distance.

PROBLEM 2.2

KNOWN: Hot water pipe covered with thick layer of insulation.

FIND: Sketch temperature distribution and give brief explanation to justify shape.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional (radial) conduction, (3) No internal heat generation, (4) Insulation has uniform properties independent of temperature and position.

ANALYSIS: Fourier's law, Eq. 2.1, for this one-dimensional (cylindrical) radial system has the form

$$q_r = -kA_r \frac{dT}{dr} = -k\,(2\pi r \ell)\,\frac{dT}{dr}$$

where $A_r = 2\pi r\ell$ and ℓ is the axial length of the pipe-insulation system. Recognize that for steady-state conditions with no internal heat generation, an energy balance on the system requires $\dot{E}_{in} = \dot{E}_{out}$ since $\dot{E}_g = \dot{E}_{st} = 0$ and hence

$$q_r = \text{Constant.}$$

That is, q_r is independent of radius (r). Since the thermal conductivity is also constant, it follows that

$$r\left[\frac{dT}{dr}\right] = \text{Constant.}$$

This relation requires that the product of the radial temperature gradient, dT/dr, and the radius, r, remains constant throughout the insulation. For our situation, the temperature distribution must appear as shown in the above, right sketch.

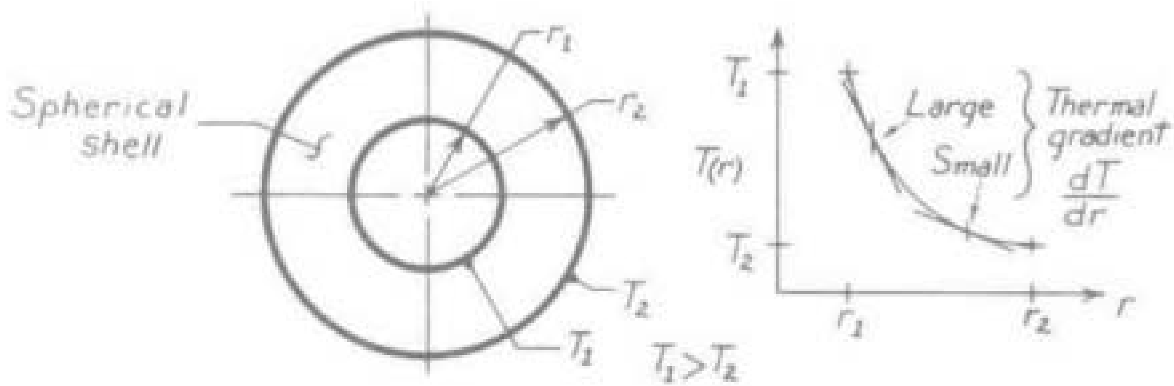
COMMENTS: (1) Note that while q_r is a constant and independent of r, q_r'' is not a constant. How does $q_r''(r)$ vary with r? (2) Recognize that the radial temperature gradient, dT/dr, decreases with increasing radius.

PROBLEM 2.3

KNOWN: A spherical shell with prescribed geometry and surface temperatures.

FIND: Sketch temperature distribution and explain shape of the curve.

SCHEMATIC:



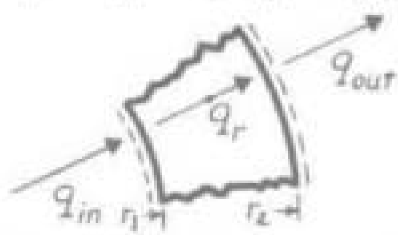
ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in radial (spherical coordinates) direction, (3) No internal generation, (4) Constant properties.

ANALYSIS: Fourier's law, Eq. 2.1, for this one-dimensional, radial (spherical coordinate) system has the form

$$q_r = -k\,A_r\,\frac{dT}{dr} = -k\,(4\pi r^2)\,\frac{dT}{dr}$$

where A_r is the surface area of a sphere given as $A_r = 4\pi r^2$. For steady-state conditions, an energy balance on the system requires that since $\dot{E}_g = \dot{E}_{st} = 0$, $\dot{E}_{in} = \dot{E}_{out}$ and thus

$$q_{in} = q_{out} = q_r \neq q_r(r)\,.$$



That is, q_r is a constant, independent of the radial coordinate. Since the thermal conductivity is constant, it follows that

$$r^2\left[\frac{dT}{dr}\right] = \text{Constant.}$$

This relation requires that the product of the radial temperature gradient, dT/dr, and the radius squared, r^2, remains constant throughout the shell. Hence, the temperature distribution appears as shown in the above, right sketch.

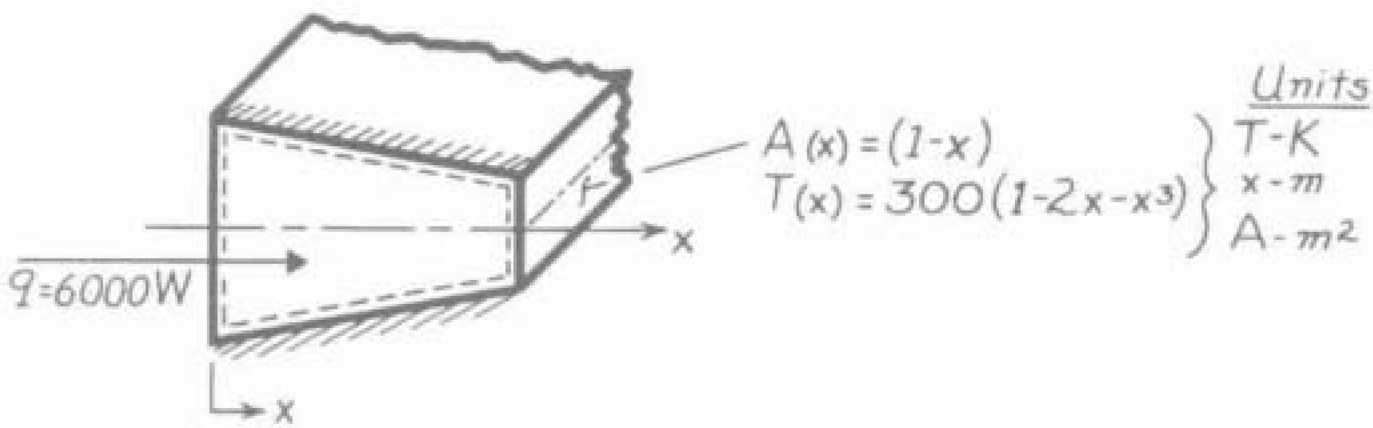
COMMENTS: Note that for the above conditions, $q_r \neq q_r(r)$; that is, q_r is everywhere constant. But how does q_r'' vary as a function of radius?

PROBLEM 2.4

KNOWN: Axisymmetric shape with prescribed cross-sectional area, temperature distribution and heat rate.

FIND: Expression for the thermal conductivity, k.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in x-direction, (3) No internal heat generation.

ANALYSIS: Application of the energy balance relation, Eq. 1.11a, to the system, it follows that since $\dot{E}_{in} = \dot{E}_{out}$,

$$q_x = \text{Constant} \neq f(x) .$$

Using Fourier's law, Eq. 2.1, with appropriate expressions for A_x and T, yields

$$q_x = -k\, A_x\, \frac{dT}{dx}$$

$$6000\text{W} = -k \cdot (1-x)\text{m}^2 \cdot \frac{d}{dx}\left[300(1-2x-x^3)\right] \frac{\text{K}}{\text{m}} .$$

Solving for k and recognizing its units are W/m · K,

$$k = \frac{-6000}{(1-x)[300(-2-3x^2)]} = \frac{20}{(1-x)(2+3x^2)} . \qquad \triangleleft$$

COMMENTS: (1) Note that at x=0, k=10W/m·K and that indeed the units are correctly obtained.

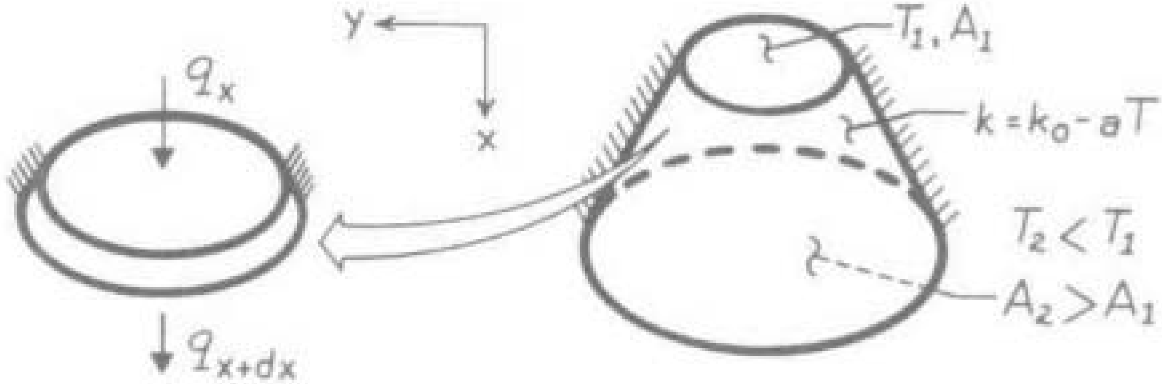
(2) Recognize that the 1-D assumption is an approximation which is more appropriate as the area change with distance x is less.

PROBLEM 2.5

KNOWN: End-face temperatures and temperature dependence of k for a truncated cone.

FIND: Variation with axial distance along the cone of q_x, q''_x, k, and dT/dx.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in x (negligible temperature gradients along y), (2) Steady-state conditions, (3) Adiabatic sides, (4) No internal heat generation.

ANALYSIS: For the prescribed conditions, it follows from conservation of energy, Eq. 1.11a, that for a differential control volume, $\dot{E}_{in} = \dot{E}_{out}$ or $q_x = q_{x+dx}$. Hence

q_x is independent of x.

Since A(x) *increases* with *increasing* x, it follows that $q''_x = q_x/A(x)$ *decreases* with *increasing* x. Since T *decreases* with *increasing* x, k *increases* with *increasing* x. Hence, from Fourier's law, Eq. 2.2,

$$q''_x = -k\,\frac{dT}{dx},$$

it follows that | dT/dx | *decreases* with increasing x.

PROBLEM 2.6

KNOWN: Temperature dependence of the thermal conductivity, k(T), for heat transfer through a plane wall.

FIND: Effect of k(T) on temperature distribution, T(x).

ASSUMPTIONS: (1) One-dimensional conduction, (2) Steady-state conditions, (3) No internal heat generation.

ANALYSIS: From Fourier's law and the form of k(T),

$$q_x'' = -k\,\frac{dT}{dx} = -(k_o + aT)\,\frac{dT}{dx}\,. \qquad (1)$$

The shape of the temperature distribution may be inferred from knowledge of $d^2T/dx^2 = d(dT/dx)/dx$. Since q_x'' is independent of x for the prescribed conditions,

$$-\frac{d}{dx}\left[(k_o + aT)\,\frac{dT}{dx}\right] = \frac{dq_x''}{dx} = 0$$

$$-(k_o + aT)\,\frac{d^2T}{dx^2} - a\left(\frac{dT}{dx}\right)^2 = 0\,.$$

Hence,

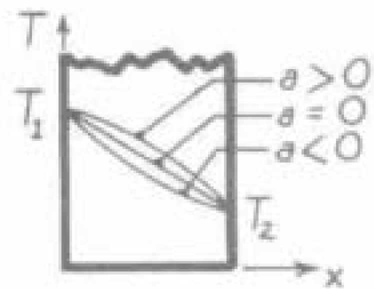
$$\frac{d^2T}{dx^2} = \frac{a}{k_o + aT}\left(\frac{dT}{dx}\right)^2 \qquad \text{where} \begin{cases} k_o + aT = k > 0 \\ \left(\frac{dT}{dx}\right)^2 > 0 \end{cases}$$

from which it follows that for

$a > 0$: $d^2T/dx^2 < 0$

$a = 0$: $d^2T/dx^2 = 0$

$a < 0$: $d^2T/dx^2 > 0$.



where the curvature for the temperature distribution T(x) is negative, zero, and positive, respectively.

COMMENTS: The shape of the distribution could also be inferred from Eq. (1). Since T decreases with increasing x,

$a > 0$: k decreases with increasing x => | dT/dx | increases with increasing x

$a = 0$: $k = k_o$ => dT/dx is constant

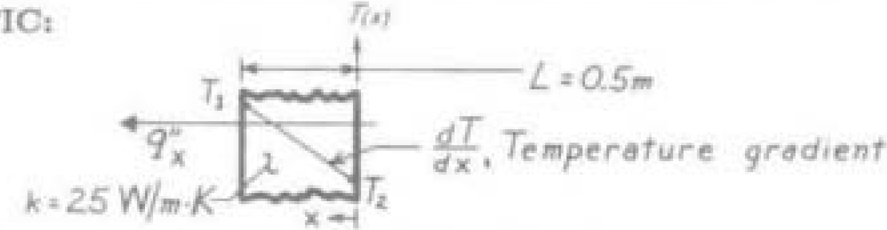
$a < 0$: k increases with increasing x => | dT/dx | decreases with increasing x.

PROBLEM 2.7

KNOWN: Thermal conductivity and thickness of a one-dimensional system with no internal heat generation and steady-state conditions.

FIND: Unknown surface temperatures, temperature gradient or heat flux.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional heat flow, (2) No internal heat generation, (3) Steady-state conditions, (4) Constant properties.

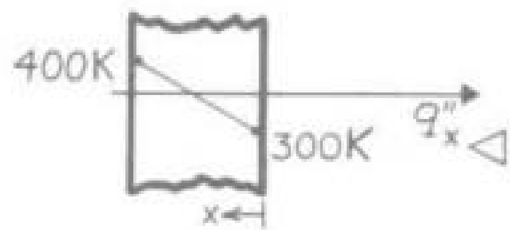
ANALYSIS: The rate equation and temperature gradient for this system are

$$q''_x = -k\frac{dT}{dx} \qquad \text{and} \qquad \frac{dT}{dx} = \frac{T_1 - T_2}{L}. \tag{1,2}$$

Using Eqs. (1) and (2), the unknown quantities can be determined.

(a) $\frac{dT}{dx} = \frac{(400-300)K}{0.5m} = 200\ K/m$

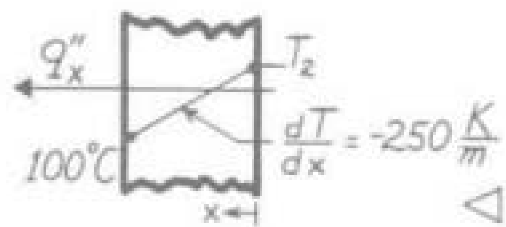
$q''_x = -25\ \frac{W}{m\cdot K} \times 200\frac{K}{m} = -5000\ W/m^2.$



(b) $q''_x = -25\ \frac{W}{m\cdot K} \times \left[-250\frac{K}{m}\right] = 6250\ W/m^2$

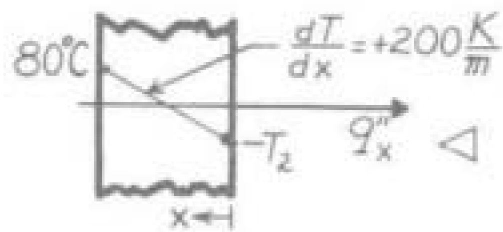
$T_2 = T_1 - L\left[\frac{dT}{dx}\right] = 1000^\circ C - 0.5m\left[-250\frac{K}{m}\right]$

$T_2 = 225^\circ C.$



(c) $q''_x = -25\ \frac{W}{m\cdot K} \times 200\frac{K}{m} = -5000\ W/m^2$

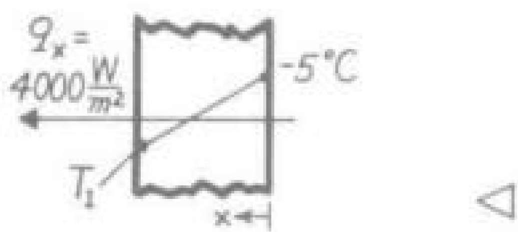
$T_2 = 80^\circ C - 0.5m\left[200\frac{K}{m}\right] = -20^\circ C.$



(d) $\frac{dT}{dx} = -\frac{q''_x}{k} = -\frac{4000\ W/m^2}{25\ W/m\cdot K} = -160\frac{K}{m}$

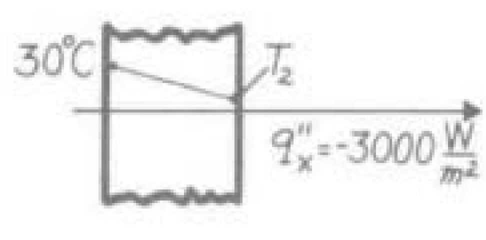
$T_1 = L\left[\frac{dT}{dx}\right] + T_2 = 0.5m\left[-160\frac{K}{m}\right] + (-5^\circ C)$

$T_1 = -85^\circ C.$



(e) $\frac{dT}{dx} = -\frac{q''_x}{k} = -\frac{(-3000\ W/m^2)}{25\ W/m\cdot K} = 120\frac{K}{m}$

$T_2 = 30^\circ C - 0.5m\left[120\frac{K}{m}\right] = -30^\circ C.$

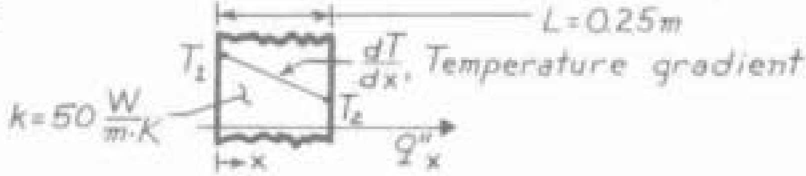


PROBLEM 2.8

KNOWN: One-dimensional system with prescribed thermal conductivity and thickness.

FIND: Unknowns for various temperature conditions and sketch distribution.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) No internal heat generation, (4) Constant properties.

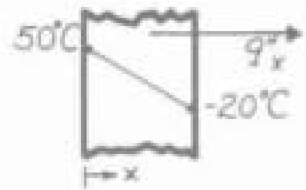
ANALYSIS: The rate equation and temperature gradient for this system are

$$q''_x = -k \frac{dT}{dx} \qquad \text{and} \qquad \frac{dT}{dx} = \frac{T_2 - T_1}{L}. \qquad (1,2)$$

Using Eqs. (1) and (2), the unknown quantities for each case can be determined.

(a) $$\frac{dT}{dx} = \frac{(-20 - 50)\ K}{0.25m} = -280\ K/m$$

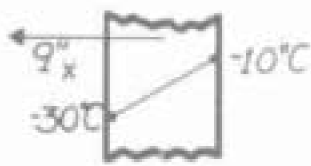
$$q''_x = -50 \frac{W}{m \cdot K} \times \left| -280 \frac{K}{m} \right| = 14.0\ kW/m^2.$$



◁

(b) $$\frac{dT}{dx} = \frac{(-10 - (-30))K}{0.25m} = 80\ K/m$$

$$q''_x = -50 \frac{W}{m \cdot K} \times \left| 80 \frac{K}{m} \right| = -4.0\ kW/m^2.$$

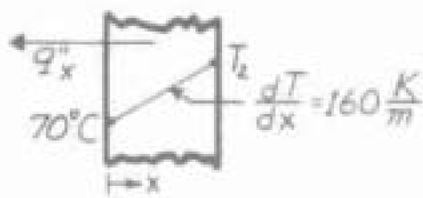


◁

(c) $$q''_x = -50 \frac{W}{m \cdot K} \times \left| 160 \frac{K}{m} \right| = -8.0\ kW/m^2$$

$$T_2 = L \cdot \frac{dT}{dx} + T_1 = 0.25m \times \left| 160 \frac{K}{m} \right| + 70\ ^\circ C.$$

$$T_2 = 110\ ^\circ C.$$

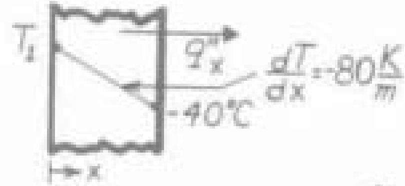


◁

(d) $$q''_x = -50 \frac{W}{m \cdot K} \times \left| -80 \frac{K}{m} \right| = 4.0\ kW/m^2$$

$$T_1 = T_2 - L \cdot \frac{dT}{dx} = 40\ ^\circ C - 0.25m \left| -80 \frac{K}{m} \right|.$$

$$T_1 = 60\ ^\circ C.$$



◁

(e) $$q''_x = -50 \frac{W}{m \cdot K} \times \left| 200 \frac{K}{m} \right| = -10.0\ kW/m^2$$

$$T_1 = T_2 - L \cdot \frac{dT}{dx} = 30\ ^\circ C - 0.25m \left| 200 \frac{K}{m} \right| = -20\ ^\circ C.$$

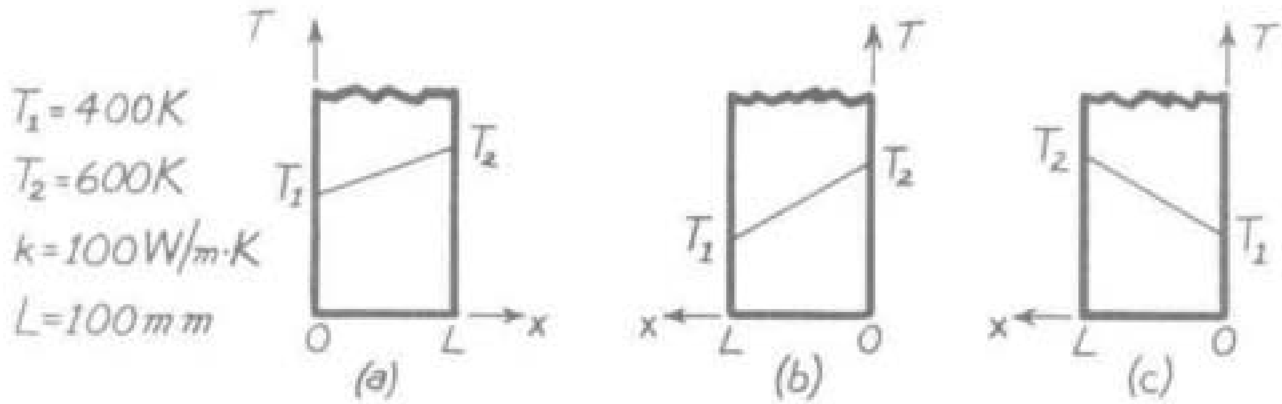
◁

PROBLEM 2.9

KNOWN: Plane wall with prescribed thermal conductivity, thickness, and surface temperatures.

FIND: Heat flux, q''_x, and temperature gradient, dT/dx, for the three different coordinate systems shown.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional heat flow, (2) Steady-state conditions, (3) No internal generation, (4) Constant properties.

ANALYSIS: The rate equation for conduction heat transfer is

$$q''_x = -k \frac{dT}{dx}, \tag{1}$$

where the temperature gradient is constant throughout the wall and of the form

$$\frac{dT}{dx} = \frac{T(L) - T(0)}{L}. \tag{2}$$

Substituting numerical values, find the temperature gradients,

(a) $\frac{dT}{dx} = \frac{T_2 - T_1}{L} = \frac{(600-400)K}{0.100m} = 2000\ K/m$ ◁

(b) $\frac{dT}{dx} = \frac{T_1 - T_2}{L} = \frac{(400-600)K}{0.100m} = -2000\ K/m$ ◁

(c) $\frac{dT}{dx} = \frac{T_2 - T_1}{L} = \frac{(600-400)K}{0.100m} = 2000\ K/m$. ◁

The heat rates, using Eq. (1) with $k = 100\ W/m\cdot K$, are

(a) $q''_x = -100 \frac{W}{m\cdot K} \times 2000\ K/m = -200\ kW/m^2$ ◁

(b) $q''_x = -100 \frac{W}{m\cdot K} (-2000\ K/m) = +200\ kW/m^2$ ◁

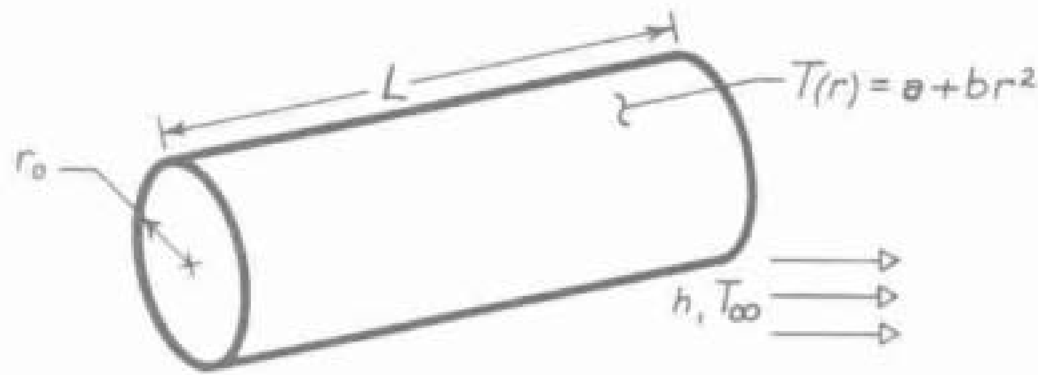
(c) $q''_x = -100 \frac{W}{m\cdot K} \times 2000\ K/m = -200\ kW/m^2$. ◁

PROBLEM 2.10

KNOWN: Temperature distribution in solid cylinder and convection coefficient at cylinder surface.

FIND: Expressions for heat rate at cylinder surface and fluid temperature.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, radial conduction, (2) Steady-state conditions, (3) Constant properties.

ANALYSIS: The heat rate from Fourier's law for the radial (cylindrical) system has the form

$$q_r = -kA_r \frac{dT}{dr}.$$

Using the expression for the temperature distribution, $T(r) = a + br^2$, evaluate the temperature gradient, dT/dr, to find the heat rate,

$$q_r = -k(2\pi rL)\, 2br = -4\pi kbLr^2.$$

At the outer surface ($r = r_o$), the conduction heat rate is

$$q_{r=r_o} = -4\pi kbLr_o^2. \qquad \triangleleft$$

From a surface energy balance at $r = r_o$,

$$q_{r=r_o} = q_{conv} = h(2\pi r_o L)\left[T(r_o) - T_\infty\right],$$

and solving for T_∞ using the heat rate at $r = r_o$, find

$$T_\infty = T(r_o) + \frac{2kbr_o}{h}$$

$$T_\infty = a + br_o^2 + \frac{2kbr_o}{h}$$

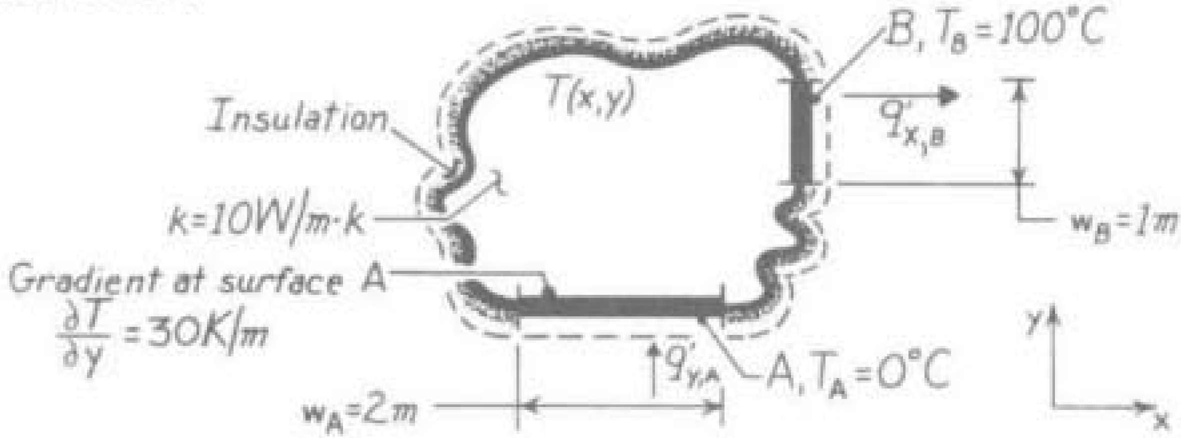
$$T_\infty = a + br_o\left[r_o + \frac{2k}{h}\right]. \qquad \triangleleft$$

PROBLEM 2.11

KNOWN: Two-dimensional body with specified thermal conductivity and two isothermal surfaces of prescribed temperatures; one surface, A, has a prescribed temperature gradient.

FIND: Temperature gradients, $\partial T/\partial x$ and $\partial T/\partial y$, at the surface B.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Steady-state conditions, (3) No heat generation, (4) Constant properties.

ANALYSIS: At the surface A, the temperature gradient in the x-direction must be zero. That is, $(\partial T/\partial x)_A = 0$. This follows from the requirement that the heat flux vector must be normal to an isothermal surface. The heat rate at the surface A is given by Fourier's law written as

$$q'_{y,A} = -k \cdot w_A \left.\frac{\partial T}{\partial y}\right)_A = -10\frac{W}{m\cdot K} \times 2m \times 30\frac{K}{m} = -600 W/m \, .$$

On the surface B, it follows that

$$(\partial T/\partial y)_B = 0 \qquad \triangleleft$$

in order to satisfy the requirement that the heat flux vector be normal to the isothermal surface B. Using the conservation of energy requirement, Eq. 1.11a, on the body, find

$$q'_{y,A} - q'_{x,B} = 0 \qquad \text{or} \qquad q'_{x,B} = q'_{y,A} \, .$$

Note that,

$$q'_{x,B} = -k \cdot w_B \left.\frac{\partial T}{\partial x}\right)_B$$

and hence

$$(\partial T/\partial x)_B = \frac{-q'_{y,A}}{k \cdot w_B} = \frac{-(-600 \; W/m)}{10/W/m\cdot K \times 1m} = 60 \; K/m \, . \qquad \triangleleft$$

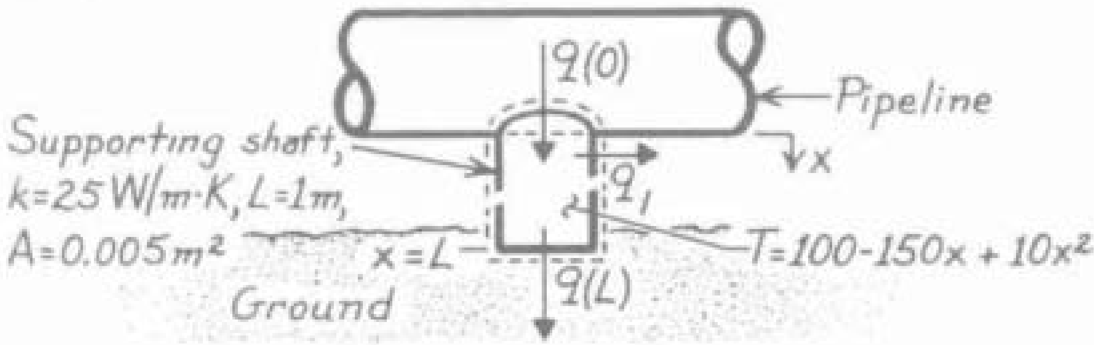
COMMENT: Note that in using the conservation requirement, $q'_{in} = +q'_{y,A}$ and $q'_{out} = +q'_{x,B}$.

PROBLEM 2.12

KNOWN: Length and thermal conductivity of a shaft. Temperature distribution along shaft.

FIND: Temperatures and heat rates at ends of shaft.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in x, (3) Constant properties.

ANALYSIS: Temperatures at the top and botton of the shaft are, respectively,

$$T(0) = 100\,^\circ C \qquad T(L) = -40\,^\circ C. \qquad \triangleleft$$

Applying Fourier's law, Eq. 2.1,

$$q_x = -kA\frac{dT}{dx} = -25\ W/m{\cdot}K(0.005\ m^2)(-150 + 20x)\,^\circ C/m$$

$$q_x = 0.125(150 - 20x)W.$$

Hence,

$$q_x(0) = 18.75\ W \qquad q_x(L) = 16.25\ W. \qquad \triangleleft$$

The difference in heat rates, $q_x(0) > q_x(L)$, is due to heat losses q_ℓ from the side of the shaft.

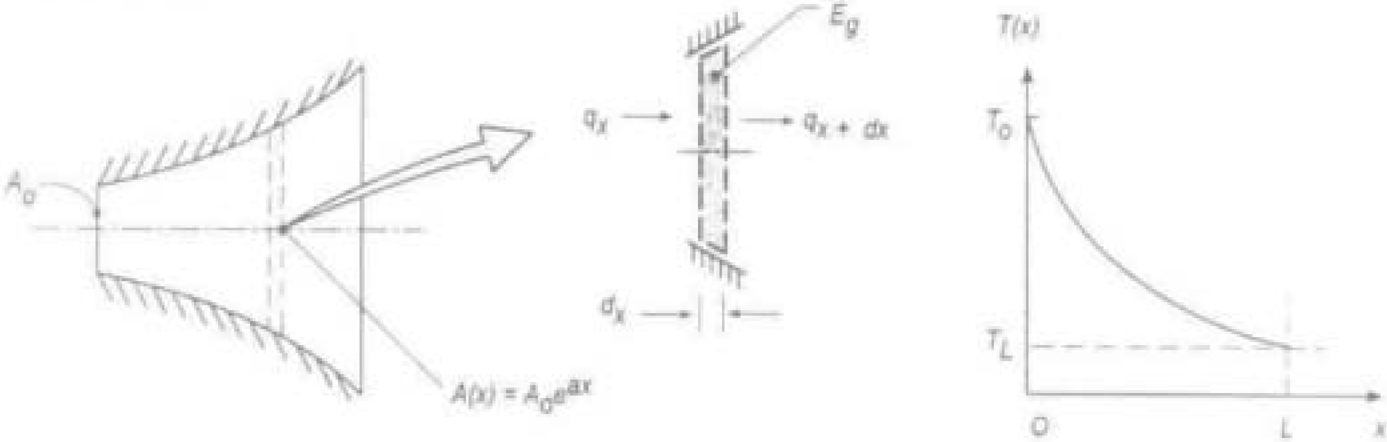
COMMENTS: Heat loss from the side requires the existence of temperature gradients over the shaft cross-section. Hence, specification of T as a function of only x is an approximation.

PROBLEM 2.13

KNOWN: A rod of constant thermal conductivity k and variable cross-sectional area $A_x(x) = A_o e^{ax}$ where A_o and a are constants.

FIND: (a) Expression for the conduction heat rate, $q_x(x)$; use this expression to determine the temperature distribution, T(x); and sketch of the temperature distribution, (b) Considering the presence of volumetric heat generation rate, $\dot{q} = \dot{q}_o \exp(-ax)$, obtain an expression for $q_x(x)$ when the left face, x = 0, is well insulated.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in the rod, (2) Constant properties, (3) Steady-state conditions.

ANALYSIS: Perform an energy balance on the control volume, A(x)·dx,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0$$

$$q_x - q_{x+dx} + \dot{q} \cdot A(x) \cdot dx = 0$$

The conduction heat rate terms can be expressed as a Taylor series and substituting expressions for $\dot{q}$ and A(x),

$$-\frac{d}{dx}(q_x) + \dot{q}_o \exp(-ax) \cdot A_o \exp(ax) = 0 \quad (1)$$

$$q_x = -k \cdot A(x) \frac{dT}{dx} \quad (2)$$

(a) With no internal generation, $\dot{q}_o = 0$, and from Eq. (1) find

$$-\frac{d}{dx}(q_x) = 0 \quad <$$

indicating that the heat rate is constant with x. By combining Eqs. (1) and (2)

$$-\frac{d}{dx}\left(-k \cdot A(x)\frac{dT}{dx}\right) = 0 \qquad \text{or} \qquad A(x) \cdot \frac{dT}{dx} = C_1 \quad (3) <$$

Continued...

PROBLEM 2.13 (Cont.)

That is, the product of the cross-sectional area and the temperature gradient is a constant, independent of x. Hence, with T(0) > T(L), the temperature distribution is exponential, and as shown in the sketch above. Separating variables and integrating Eq. (3), the general form for the temperature distribution can be determined,

$$A_o \exp(ax) \cdot \frac{dT}{dx} = C_1$$

$$dT = C_1 A_o^{-1} \exp(-ax) dx$$

$$T(x) = -C_1 A_o a \exp(-ax) + C_2$$ <

We could use the two temperature boundary conditions, $T_o = T(0)$ and $T_L = T(L)$, to evaluate C_1 and C_2 and, hence, obtain the temperature distribution in terms of T_o and T_L.

(b) With the internal generation, from Eq. (1),

$$-\frac{d}{dx}(q_x) + \dot{q}_o A_o = 0 \qquad \text{or} \qquad q_x = \dot{q}_o A_o x$$ <

That is, the heat rate increases linearly with x.

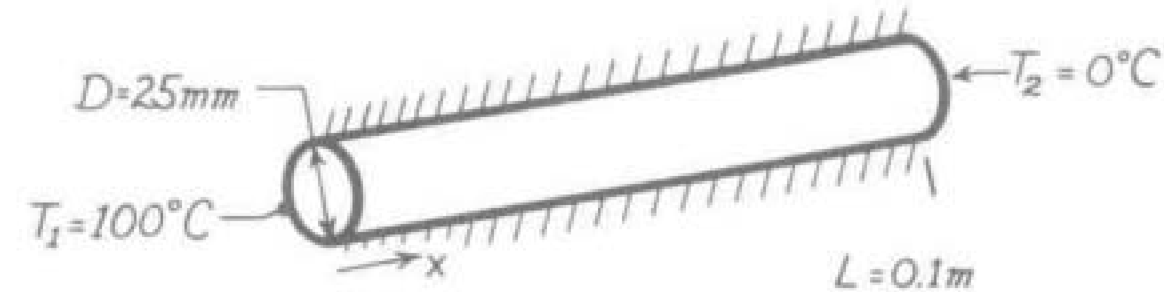
COMMENTS: In part (b), you could determine the temperature distribution using Fourier's law and knowledge of the heat rate dependence upon the x-coordinate. Give it a try!

PROBLEM 2.14

KNOWN: Dimensions and end temperatures of a cylindrical rod which is insulated on its side.

FIND: Rate of heat transfer associated with different rod materials.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction along cylinder axis, (2) Steady-state conditions, (3) Constant properties.

PROPERTIES: The properties may be evaluated from *Tables A-1* to *A-3* at a mean temperature of 50°C = 323K and are summarized below.

ANALYSIS: The heat transfer rate may be obtained from Fourier's law. Since the axial temperature gradient is linear, this expression reduces to

$$q = k\,A\frac{T_1 - T_2}{L} = k\frac{\pi(0.025\text{m})^2}{4}\,\frac{(100-0)°\text{C}}{0.1\text{m}} = 0.491(\text{m}\cdot°\text{C})\cdot k$$

	Cu (pure)	Al (2024)	St.St. (302)	SiN	Oak	Magnesia (85%)	Pyrex	
k(W/m·K)	401	177	16.3	14.9	0.19	0.052	1.4	
q(W)	197	87	8.0	7.3	0.093	0.026	0.69	◁

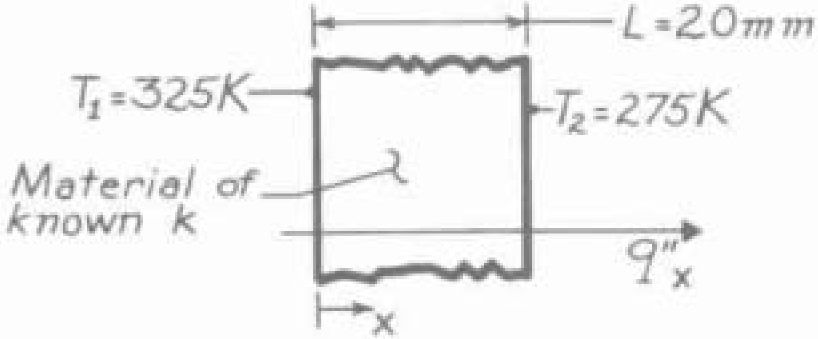
COMMENTS: The k values of Cu and Al were obtained by linear interpolation; the k value of St.St. was obtained by linear extrapolation, as was the value for SiN; the value for magnesia was obtained by linear interpolation; and the values for oak and pyrex are for 300K.

PROBLEM 2.15

KNOWN: One-dimensional system with prescribed surface temperatures and thickness.

FIND: Heat flux through system constructed of these materials: (a) pure aluminum, (b) plain carbon steel, (c) AISI 316, stainless steel, (d) pyroceram, (e) teflon and (f) concrete.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) No heat generation, (4) Constant thermal properties.

PROPERTIES: The thermal conductivity is evaluated at the average temperature of the system, $T = (T_1+T_2)/2 = (325+275)K/2 = 300K$. Property values and table identification are shown below.

ANALYSIS: For this system, Fourier's law can be written as

$$q_x'' = -k\frac{dT}{dx} = -k\frac{T_2-T_1}{L}.$$

Substituting numerical values, the heat flux in terms of the system thermal conductivity is

$$q_x'' = -k\frac{(275-325)K}{20\times10^{-3}m} = +2500\frac{K}{m}\cdot k$$

where q_x'' will have units W/m^2 if k has units W/m·K. The heat fluxes for each system follow.

Material	Thermal conductivity Table	k(W/m·K)	Heat flux q_x'' (kW/m²)	
(a) Pure aluminum	A-1	237	593	◁
(b) Plain carbon steel	A-1	60.5	151	
(c) AISI 316, S.S.	A-1	13.4	33.5	
(d) Pyroceram	A-2	3.98	9.95	
(e) Teflon	A-3	0.35	0.88	
(f) Concrete	A-3	1.4	3.5	

COMMENTS: Recognize the range of thermal conductivity for these solid materials is nearly two decades.

PROBLEM 2.16

KNOWN: Different thicknesses of three materials: rock, 18 ft; wood, 15 in; and fiberglass insulation, 6 in.

FIND: The insulating quality of the materials with given thicknesses as measured by the R-value.

PROPERTIES: *Table A-3* (300K):

Material	Thermal conductivity, W/m·K
Limestone	2.15
Softwood	0.12
Blanket (glass, fiber 10 kg/m^3)	0.048

ANALYSIS: The R-value, a quantity commonly used in the construction industry and building technology, is defined as

$$R \equiv \frac{L(\text{in})}{k(\text{Btu}\cdot\text{in/h}\cdot\text{ft}^2\cdot{}^\circ\text{F})}.$$

The R-value can be interpreted as the thermal resistance of a 1 ft^2 cross section of the material. Using the conversion factor for thermal conductivity between the SI and English systems, the R-values are:

Rock, Limestone, 18 ft:

$$R = \frac{18\ \text{ft} \times 12\dfrac{\text{in}}{\text{ft}}}{2.15\dfrac{\text{W}}{\text{m}\cdot\text{K}} \times 0.5778\ \dfrac{\text{Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F}}{\text{W/m}\cdot\text{K}} \times 12\dfrac{\text{in}}{\text{ft}}} = 15.5\ (\text{Btu/h}\cdot\text{ft}^2\cdot{}^\circ\text{F})^{-1}$$

Wood, Softwood, 15 in:

$$R = \frac{15\ \text{in}}{0.12\dfrac{\text{W}}{\text{m}\cdot\text{K}} \times 0.5778\ \dfrac{\text{Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F}}{\text{W/m}\cdot\text{K}} \times 12\dfrac{\text{in}}{\text{ft}}} = 18\ (\text{Btu/h}\cdot\text{ft}^2\cdot{}^\circ\text{F})^{-1}$$

Insulation, Blanket, 6 in:

$$R = \frac{6\ \text{in}}{0.048\dfrac{\text{W}}{\text{m}\cdot\text{K}} \times 0.5778\ \dfrac{\text{Btu/h}\cdot\text{ft}\cdot{}^\circ\text{F}}{\text{W/m}\cdot\text{K}} \times 12\dfrac{\text{in}}{\text{ft}}} = 18\ (\text{Btu/h}\cdot\text{ft}^2\cdot{}^\circ\text{F})^{-1}$$

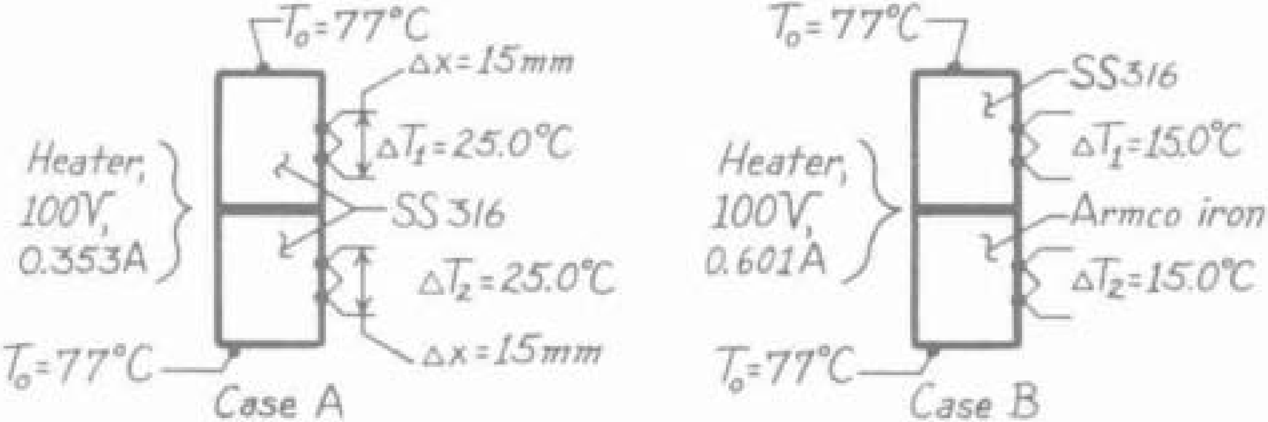
COMMENT: The R-value of 19 given in the advertisement is reasonable.

PROBLEM 2.17

KNOWN: Electrical heater sandwiched between two identical cylindrical (30 mm dia. × 60 mm length) samples whose opposite ends contact plates maintained at T_o.

FIND: (a) Thermal conductivity of SS316 samples for the prescribed conditions (A) and its average temperature, (b) Thermal conductivity of Armco iron sample for the prescribed conditions (B), (c) Comment on advantages of experimental arrangement, lateral heat losses, condition when $\Delta T_1 \neq \Delta T_2$.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional heat transfer in samples, (2) Steady-state conditions, (3) Negligible contact resistance between materials.

PROPERTIES: *Table A.2*, Stainless steel 316 ($\bar{T}$ = 400 K): k_{ss} = 15.2 W/m·K; Armco iron ($\bar{T}$ = 380 K): k_{iron} = 71.6 W/m·K.

ANALYSIS: (a) Recognize that half the heater power will pass through each of the samples which are presumed identical; see Case A above. Apply Fourier's law to a sample

$$q = kA_c \frac{\Delta T}{\Delta x}$$

$$k = \frac{q\Delta x}{A_c \Delta T} = \frac{0.5(100\text{V} \times 0.353\text{A}) \times 0.015\text{ m}}{\pi(0.030\text{ m})^2/4 \times 25.0^\circ\text{C}} = 15.0\text{ W/m·K}. \qquad \triangleleft$$

The total temperature drop across the length of the sample is $\Delta T_1(L/\Delta x)$ = 25°C (60 mm/15 mm) = 100°C. Hence, the heater temperature is T_h = 177°C. Thus, the average temperature of the sample is

$$\bar{T} = (T_o + T_h)/2 = 127^\circ\text{C} = 400\text{ K}. \qquad \triangleleft$$

We compare this result with the tabulated value (see above) at 400 K and note the good agreement.

(b) For the Case B arrangement, we assume that the thermal conductivity of the SS316 is the same as that found in Part (a). The heat rate through the Armco iron sample is

Continued .....

PROBLEM 2.17 (Cont.)

$$q_{iron} = q_{heater} - q_{ss} = 100V\times0.601A - 15.0\ W/m\cdot K\times\frac{\pi(0.030\ m)^2}{4}\times\frac{15.0°C}{0.015\ m}$$

$$q_{iron} = (60.1 - 10.6)W = 49.5\ W$$

where

$$q_{ss} = k_{ss}A_c\Delta T_2/\Delta x_2.$$

Applying Fourier's law to the iron sample,

$$k_{iron} = \frac{q_{iron}\Delta x_2}{A_c\Delta T_2} = \frac{49.5\ W\times0.015\ m}{\pi(0.030\ m)^2/4\times15.0°C} = 70.0\ W/m\cdot K. \qquad \triangleleft$$

The total drop across the iron sample is 15°C(60/15) = 60°C; the heater temperature is (77 + 60)°C = 137°C. Hence the average temperature of the iron sample is

$$\bar{T} = (137 + 77)°C/2 = 107°C = 380\ K. \qquad \triangleleft$$

We compare this result with the tabulated value (see above) at 380 K. Note good agreement.

(c) The principle advantage of having two identical samples is the assurance that all the electrical power dissipated in the heater will appear as heat flow through the samples. With only one sample, heat can flow from the backside of the heater even though insulated.

Heat leakage out the lateral surfaces of the cylindrically shaped samples will become significant when the sample thermal conductivity is only slightly higher than that of the insulating material used on those surfaces. That is, the method is suitable for metallics, but must be used with caution on non-metallic materials.

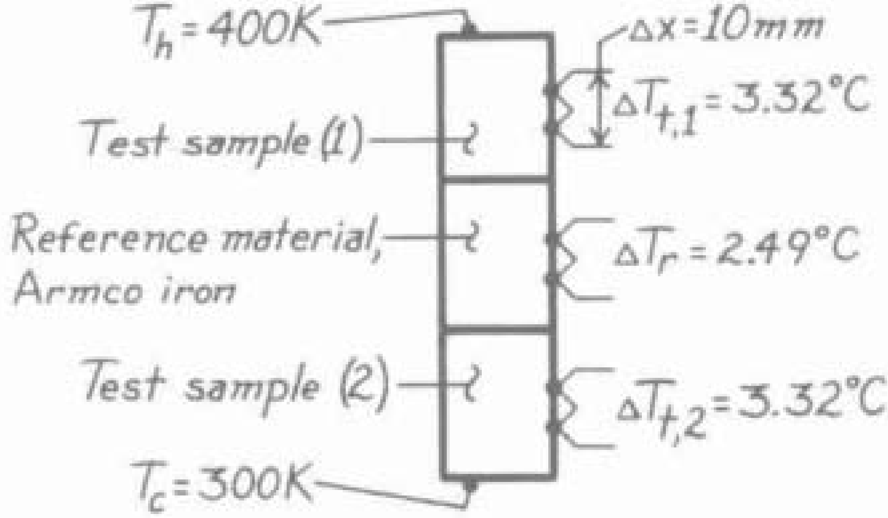
For any combination of materials in the upper and lower position, we expect $\Delta T_1 = \Delta T_2$. However, if the insulation were improperly applied along the lateral surfaces, it is possible that heat leakage will occur. This may cause $\Delta T_1 \neq \Delta T_2$.

PROBLEM 2.18

KNOWN: Comparative method for measuring thermal conductivity involving two identical samples stacked with a reference material.

FIND: (a) Thermal conductivity of test material and average temperatures, (b) Conditions when $\Delta T_1 \neq \Delta T_2$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat transfer through samples and reference material, (3) Negligible thermal contact resistance between materials.

PROPERTIES: *Table A.2*, Armco iron ($\bar{T}$ = 350 K): k_r = 69.2 W/m·K.

ANALYSIS: (a) Recognizing that the heat rate through the samples and reference material, all of the same diameter, is the same, it follows from Fourier's law that

$$k_t \frac{\Delta T_{t,1}}{\Delta x} = k_r \frac{\Delta T_r}{\Delta x} = k_t \frac{\Delta T_{t,2}}{\Delta x}$$

$$k_t = k_r \frac{\Delta T_r}{\Delta T_t} = 69.2\ \text{W/m·K} \frac{2.49°\text{C}}{3.32°\text{C}} = 51.9\ \text{W/m·K}. \quad \triangleleft$$

We should assign this value a temperature of 350 K. ◁

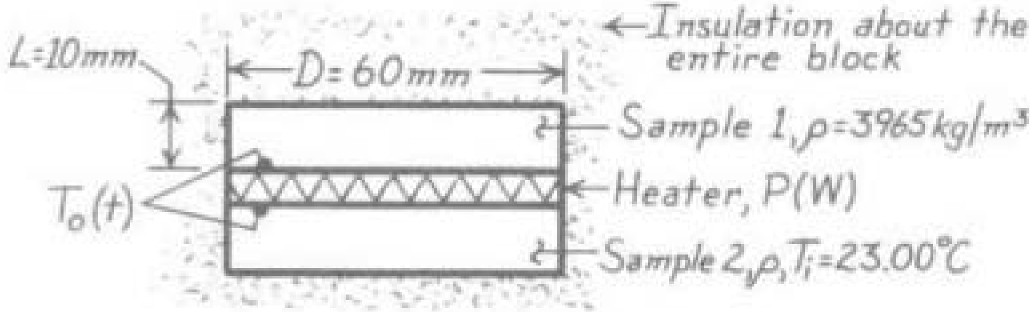
(b) If the test samples are identical in every respect, $\Delta T_1 \neq \Delta T_2$ only when the thermal conductivity is highly dependent upon temperature. Also, if there is heat leakage out the lateral surface, we can expect $\Delta T_2 < \Delta T_1$. This would occur when the thermal conductivity of the test material were only an order of magnitude above that of the insulating material employed.

PROBLEM 2.19

KNOWN: Identical samples of prescribed diameter, length and density initially at a uniform temperatures T_i, sandwich an electric heater which provides a uniform heat flux q''_o for a period of time Δt_o. Conditions shortly after energizing and a long time after de-energizing heater are prescribed.

FIND: Specific heat and thermal conductivity of the test sample material. From these properties, identify type of material using Table A.1 or A.2.

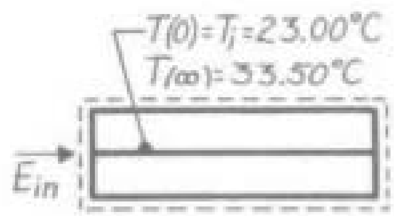
SCHEMATIC:



Case A - for $0 \le t \le \Delta t_o = 120$ s, P = 15 W; $T_o(30\text{ s}) = 25.23°C$
Case B - for $t > \Delta t_o$, P = 0 W; for $t \gg \Delta t_o$, $T_o(\infty) = 33.50°C$

ASSUMPTIONS: (1) One-dimensional heat transfer in samples, (2) Uniform properties, (3) Perfect insulation, no losses of heater power to insulation, (4) Heater has negligible mass.

ANALYSIS: Consider a control volume about the samples and heater for an interval of time t = 0 to ∞ and write the conservation of energy requirement.



$$E_{in} - E_{out} = \Delta E = E_f - E_i$$

$$P\Delta t_o - 0 = Mc_p[T(\infty) - T_i]$$

Solving for c_p, substituting numerical values, and recognizing the energy in is prescribed by Case A power condition and the final temperature T_f by Case B, find

$$c_p = \frac{P\Delta t_o}{M[T(\infty) - T_i]} = \frac{15\text{ W}\times 120\text{ s}}{2\times 3965\text{ kg/m}^3(\pi\times 0.060^2/4)\text{m}^2\times 0.010\text{ m}[33.50 - 23.00]°\text{C}}$$

$$c_p = 765\text{ J/kg}\cdot\text{K} \qquad \triangleleft$$

where $M = \rho V = 2\rho(\pi D^2/4)L$, the mass of both samples. For Case A condition, the temperature rise at the heater surface as a function of time has the form (see Eq. 5.59 with x=0)

$$T_o(t) - T_i = 2q''_o\left[\frac{t}{\pi\rho c_p k}\right]^{1/2}$$

$$k = \frac{t}{\pi\rho c_p}\left[\frac{2q''_o}{T_o(t) - T_i}\right]^2$$

Continued .....

$$k = \frac{30\text{ s}}{\pi \times 3965\text{ kg/m}^3 \times 765\text{ J/kg}\cdot\text{K}}\left[\frac{2\times 2653\text{ W/m}^2}{(24.57-23.00)^\circ\text{C}}\right]^2 = 36.0\text{ W/m}\cdot\text{K} \qquad \triangleleft$$

where

$$q''_o = \frac{P}{2A_s} = \frac{P}{2(\pi D^2/4)} = \frac{15\text{ W}}{2(\pi \times 0.060^2/4)\text{m}^2} = 2653\text{ W/m}^2.$$

With the following properties now known,

$$\rho = 3965\text{ kg/m}^3 \qquad c_p = 765\text{ J/kg}\cdot\text{K} \qquad k = 36\text{ W/m}\cdot\text{K}$$

search now in Table A.2 first to see whether values are typical of metallic or non-metallic materials. Consider the following,

- metallics with low ρ generally have higher thermal conductivities,
- specific heats of both types of materials are of similar magnitude,
- the low k value of the sample is typical of poor metallic conductors which generally have much higher specific heats,
- more than likely, the material is non-metallic.

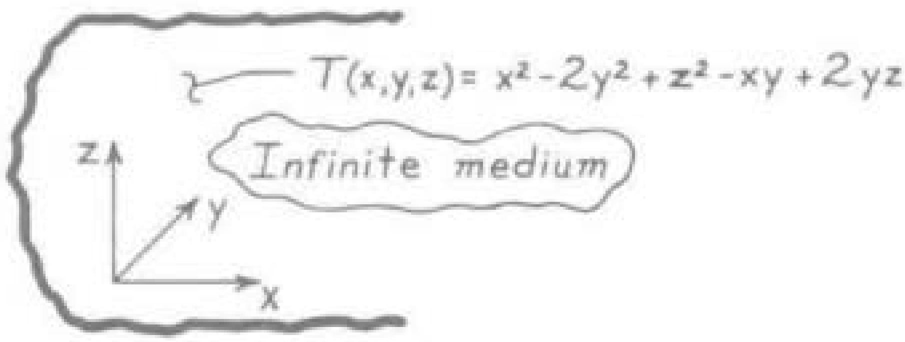
Begin search through Table A.2, and find the second entry, polycrystalline aluminum oxide, has properties at 300 K corresponding to those found for the samples. $\triangleleft$

PROBLEM 2.20

KNOWN: Temperature distribution, T(x,y,z), within an infinite, homogeneous body at a given instant of time.

FIND: Regions where the temperature changes with time.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties of infinite medium and (2) No internal heat generation.

ANALYSIS: The temperature distribution throughout the medium, at any instant of time, must satisfy the heat equation. For the three-dimensional cartesian coordinate system, with constant properties and no internal heat generation, the heat equation, Eq. 2.15, has the form

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}. \quad (1)$$

When T(x,y,z) satisfies this relation, then conservation of energy at every point in the medium is satisfied. Substituting T(x,y,z) into the Eq. (1), first find the gradients, $\partial T/\partial x$, $\partial T/\partial y$, etc.,

$$\frac{\partial}{\partial x}(2x-y) + \frac{\partial}{\partial y}(-4y-x+2z) + \frac{\partial}{\partial z}(2z+2y) = \frac{1}{\alpha}\frac{\partial T}{\partial t}.$$

Performing the differentiation, find

$$2-4+2 = \frac{1}{\alpha}\frac{\partial T}{\partial t}.$$

That is,

$$\frac{\partial T}{\partial t} = 0$$

which implies that, for the given instant of time, the temperature will everywhere not change.

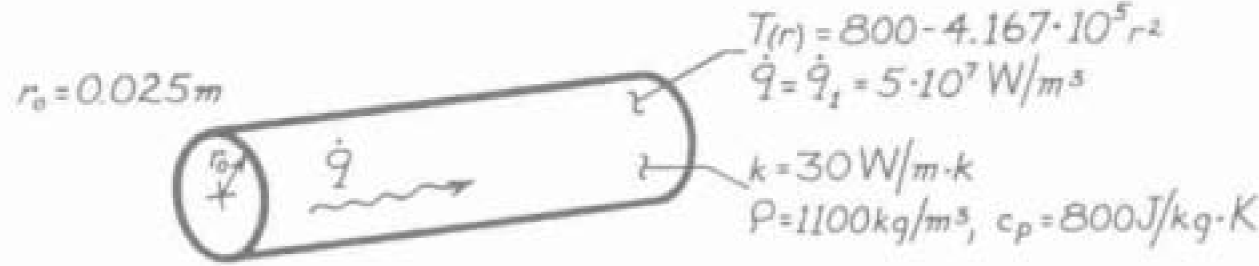
COMMENTS: Since we do not know the initial and boundary conditions, we cannot determine the temperature distribution, T(x,y,z), at any future time. We only can determine that, for this special instant of time, the temperature will not change.

PROBLEM 2.21

KNOWN: Steady-state temperature distribution in a cylindrical rod having uniform heat generation of $\dot{q}_1 = 5\times10^7\ \text{W/m}^3$.

FIND: (a) Steady-state centerline and surface heat transfer rates per unit length, q_r'. (b) Initial time rate of change of the centerline and surface temperatures in response to a change in the generation rate from $\dot{q}_1$ to $\dot{q}_2 = 10^8\ \text{W/m}^3$.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in the r direction, (2) Uniform generation, and (3) Steady-state for $\dot{q}_1 = 5\times10^7\ \text{W/m}^3$.

ANALYSIS: (a) From the rate equations for cylindrical coordinates,

$$q_r'' = -k\frac{\partial T}{\partial r} \qquad\qquad q = -kA_r\frac{\partial T}{\partial r}.$$

Hence,

$$q_r = -k(2\pi rL)\frac{\partial T}{\partial r}$$

or

$$q_r' = -2\pi kr\frac{\partial T}{\partial r} \tag{1}$$

where $\partial T/\partial r$ may be evaluated from the prescribed temperature distribution, T(r).

At r=0, the gradient is $(\partial T/\partial r) = 0$. Hence, from Eq. (1) the heat rate is

$$q_r'(0) = 0. \qquad \triangleleft$$

At $r=r_o$, the temperature gradient is

$$\left.\frac{\partial T}{\partial r}\right]_{r=r_o} = -2\left[4.167\times10^5\frac{\text{K}}{\text{m}^2}\right](r_o) = -2(4.167\times10^5)(0.025\text{m})$$

$$\left.\frac{\partial T}{\partial r}\right]_{r=r_o} = -0.208\times10^5\ \text{K/m}.$$

Continued .....

Hence, the heat rate at the outer surface ($r=r_o$) per unit length is

$$q_r'(r_o) = -2\pi\left[30\ \text{W/m}\cdot\text{K}\right](0.025\text{m})\left[-0.208\times10^5\ \text{K/m}\right]$$

$$q_r'(r_o) = 0.980\times10^5\ \text{W/m}.$$ ◁

(b) Transient (time-dependent) conditions will exist when the generation is changed, and for the prescribed assumptions, the temperature is determined by the following form of the heat equation, Eq. 2.20

$$\frac{1}{r}\frac{\partial}{\partial r}\left[kr\frac{\partial T}{\partial r}\right] + \dot{q}_2 = \rho c_p \frac{\partial T}{\partial t}$$

Hence

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left[kr\frac{\partial T}{\partial r}\right] + \dot{q}_2\right].$$

However, initially (at $t=0$), the temperature distribution is given by the prescribed form, $T(r) = 800 - 4.167\times10^5 r^2$, and

$$\frac{1}{r}\frac{\partial}{\partial r}\left[kr\frac{\partial T}{\partial r}\right] = \frac{k}{r}\frac{\partial}{\partial r}\left[r(-8.334\times10^5\cdot r)\right]$$

$$= \frac{k}{r}(-16.668\times10^5\cdot r)$$

$$= 30\ \text{W/m}\cdot\text{K}\left[-16.668\times10^5\ \text{K/m}^2\right]$$

$$= -5\times10^7\ \text{W/m}^3 \quad (\text{the original } \dot{q} = \dot{q}_1).$$

Hence, everywhere in the wall,

$$\frac{\partial T}{\partial t} = \frac{1}{1100\text{/kg/m}^3 \times 800\ \text{J/kg}\cdot\text{K}}\left[-5\times10^7 + 10^8\right]\text{W/m}^3$$

or

$$\frac{\partial T}{\partial t} = 56.82\ \text{K/s}.$$ ◁

COMMENTS: (1) The value of ($\partial T/\partial t$) will decrease with increasing time beyond $t=0$, until a new steady-state condition is reached and once again ($\partial T/\partial t$) = 0.

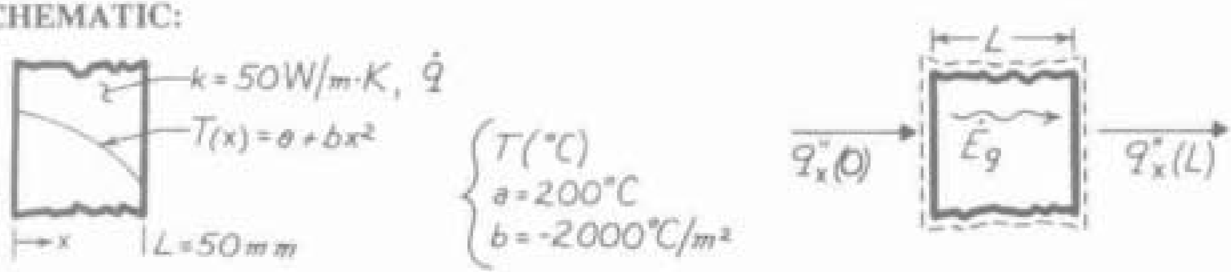
(2) By applying the energy conservation requirement, Eq. 1.11a, to the rod for the steady-state condition, $\dot{E}_{in} + \dot{E}_{gen} - \dot{E}_{out} = 0$. Hence $q_r'(0) - q_r'(r_o) = -\dot{q}_1(\pi r_o^2)$.

PROBLEM 2.22

KNOWN: Temperature distribution in a one-dimensional wall with prescribed thickness and thermal conductivity.

FIND: (a) The heat generation rate, $\dot{q}$, in the wall, (b) Heat fluxes at the wall faces and relation to $\dot{q}$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat flow, (3) Constant properties.

ANALYSIS: (a) The appropriate form of the heat equation for steady-state, one-dimensional conditions with constant properties is Eq. 2.15 re-written as

$$\dot{q} = -k \frac{d}{dx}\left[\frac{dT}{dx}\right]$$

Using the form for the temperature distribution, evaluate the gradient giving,

$$\dot{q} = -k \frac{d}{dx}\left[\frac{d}{dx}(a+bx^2)\right] = -k \frac{d}{dx}[2bx] = -2bk$$

$$\dot{q} = -2(-2000°C/m^2) \times 50\ W/m{\cdot}K = 2.0{\times}10^5\ W/m^3\ . \qquad \triangleleft$$

(b) The heat fluxes at the wall faces can be evaluated from Fourier's law,

$$q''_x(x) = -k \left.\frac{dT}{dx}\right]_x .$$

Using the temperature distribution T(x) to evaluate the gradient, find

$$q''_x(x) = -k \frac{d}{dx}[a+bx^2] = -2kbx\ .$$

The flux at the face x=0, is then

$$q''_x(0) = 0 \qquad \triangleleft$$

and at x = L, $q''_x(L) = -2kbL = -2 \times 50W/m{\cdot}K\ (-2000°C/m^2) \times 0.050m$

$$q''_x(L) = 10{,}000\ W/m^2\ . \qquad \triangleleft$$

COMMENTS: From an overall energy balance on the wall, it follows that

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0 \qquad q''_x(0) - q''_x(L) + \dot{q}L = 0$$

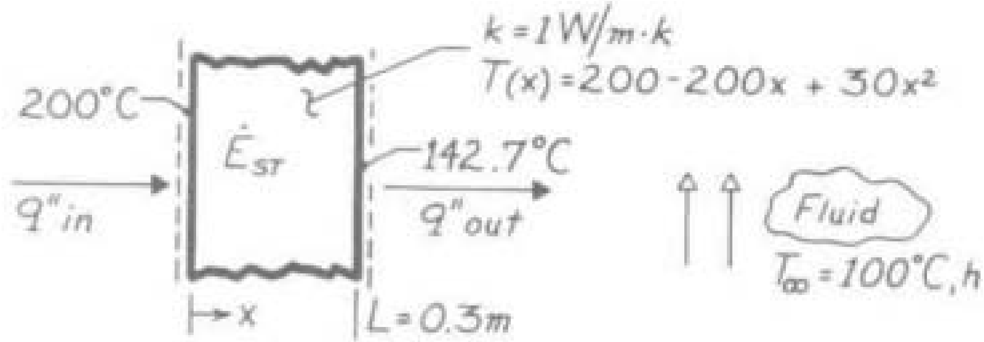
$$\dot{q} = \frac{q''_x(L) - q''_x(0)}{L} = \frac{10{,}000\ W/m^2 - 0}{0.050m} = 2.0 \times 10^5 W/m^3\ .$$

PROBLEM 2.23

KNOWN: Wall thickness, thermal conductivity, temperature distribution, and fluid temperature.

FIND: (a) Surface heat rates and rate of change of wall energy storage per unit area, and (b) Convection coefficient.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in x, (2) Constant k.

ANALYSIS: (a) From Fourier's law,

$$q''_x = -k\,\frac{\partial T}{\partial x} = (200 - 60x)\cdot k$$

$$q''_{in} = q''_{x=0} = 200\frac{°C}{m} \times 1\frac{W}{m\cdot K} = 200\ W/m^2 \qquad \triangleleft$$

$$q''_{out} = q''_{x=L} = (200 - 60 \times 0.3)°C/m \times 1\ W/m\cdot K = 182\ W/m^2\ . \qquad \triangleleft$$

Applying an energy balance to a control volume about the wall, Eq. 1.11a,

$$\dot{E}''_{in} - \dot{E}''_{out} = \dot{E}''_{st}$$

$$\dot{E}''_{st} = q''_{in} - q''_{out} = 18\ W/m^2\ . \qquad \triangleleft$$

(b) Applying a surface energy balance at x=L,

$$q''_{out} = h[T(L) - T_\infty]$$

$$h = \frac{q''_{out}}{T(L) - T_\infty} = \frac{182\ W/m^2}{(142.7 - 100)°C}$$

$$h = 4.3\ W/m^2\cdot K\ . \qquad \triangleleft$$

COMMENTS: (1) From the heat equation,

$$(\partial T/\partial t) = (k/\rho c_p)\,\partial^2 T/\partial x^2 = 60(k/\rho c_p)\ ,$$

it follows that the temperature is increasing with time at every point in the wall.

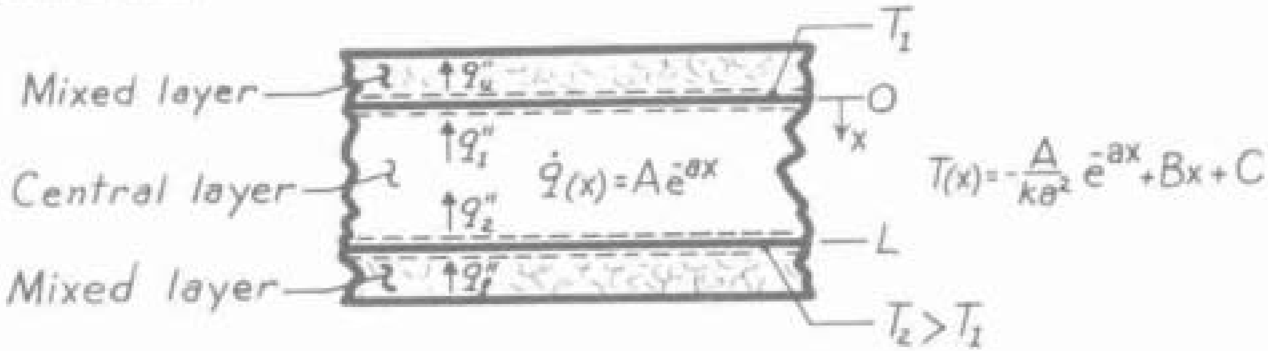
(2) The value of h is small and is typical of free convection in a gas.

PROBLEM 2.24

KNOWN: Temperature distribution and distribution of heat generation in central layer of a solar pond.

FIND: (a) Heat fluxes at lower and upper surfaces of the central layer, (b) Whether conditions are steady or transient, (c) Rate of thermal energy generation for the entire central layer.

SCHEMATIC:



ASSUMPTIONS: (1) Central layer is stagnant, (2) One-dimensional conduction, (3) Constant properties.

ANALYSIS: (a) The desired fluxes correspond to conduction fluxes in the central layer at the lower and upper surfaces. A general form for the conduction flux is

$$q''_{cond} = -k\frac{\partial T}{\partial x} = -k\left[\frac{A}{ka}e^{-ax} + B\right].$$

Hence,

$$q''_\ell = q''_{cond(x=L)} = -k\left[\frac{A}{ka}e^{-aL} + B\right] \quad q''_u = q''_{cond(x=0)} = -k\left[\frac{A}{ka} + B\right]. \quad \triangleleft$$

(b) Conditions are steady if $\partial T/\partial t = 0$. Applying the heat equation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad -\frac{A}{k}e^{-ax} + \frac{A}{k}e^{-ax} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Hence conditions are *steady* since

$$\partial T/\partial t = 0 \qquad \text{(for all } 0 \le x \le L). \quad \triangleleft$$

(c) For the central layer, the energy generation is

$$\dot{E}_g = \int_0^L \dot{q}\,dx = A\int_0^L e^{-ax}\,dx$$

$$\dot{E}_g = -\frac{A}{a}e^{-ax}\Big|_0^L = -\frac{A}{a}(e^{-aL} - 1) = \frac{A}{a}(1 - e^{-aL}). \quad \triangleleft$$

Alternatively, from an overall energy balance,

$$q''_2 - q''_1 + \dot{E}''_g = 0 \qquad \dot{E}''_g = q''_1 - q''_2 = (-q''_{cond(x=0)}) - (-q''_{cond(x=L)})$$

$$\dot{E}_g = k\left[\frac{A}{ka} + B\right] - k\left[\frac{A}{ka}e^{-aL} + B\right] = \frac{A}{a}(1 - e^{-aL}).$$

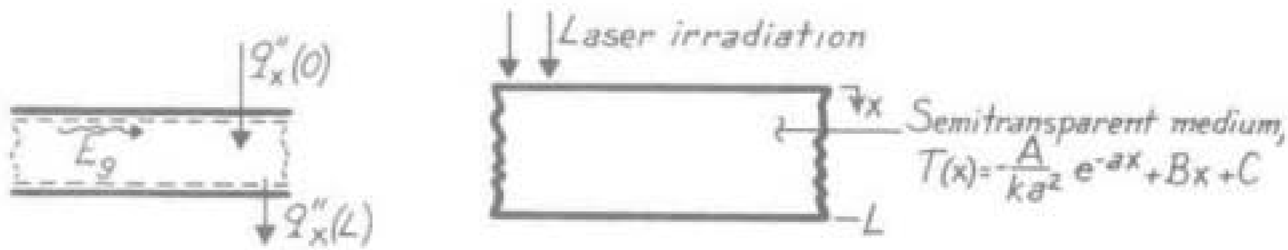
COMMENTS: Conduction is in the negative x-direction, necessitating use of minus signs in the above energy balance.

PROBLEM 2.25

KNOWN: Temperature distribution in a semi-transparent medium subjected to radiative flux.

FIND: (a) Expressions for the heat flux at the front and rear surfaces, (b) Heat generation rate $\dot{q}(x)$, (c) Expression for absorbed radiation per unit surface area in terms of A, a, B, C, L, and k.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in medium, (3) Constant properties, (4) All laser irradiation is absorbed and can be characterized by an internal volumetric heat generation term $\dot{q}(x)$.

ANALYSIS: (a) Knowing the temperature distribution, the surface heat fluxes are found using Fourier's law,

$$q''_x = -k\left[\frac{dT}{dx}\right] = -k\left[-\frac{A}{ka^2}(-a)e^{-ax} + B\right]$$

Front Surface, x = 0: $\quad q''_x(0) = -k\left[+\frac{A}{ka}\cdot 1 + B\right] = -\left[\frac{A}{a} + kB\right]$ ◁

Rear Surface, x = L: $\quad q''_x(L) = -k\left[+\frac{A}{ka}e^{-aL} + B\right] = -\left[\frac{A}{a}e^{-aL} + kB\right].$ ◁

(b) The heat diffusion equation for the medium is

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) + \frac{\dot{q}}{k} = 0 \quad \text{or} \quad \dot{q} = -k\frac{d}{dx}\left(\frac{dT}{dx}\right)$$

$$\dot{q}(x) = -k\frac{d}{dx}\left[+\frac{A}{ka}e^{-ax} + B\right] = Ae^{-ax}.$$ ◁

(c) Performing an energy balance on the medium as shown above,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0$$

recognize that $\dot{E}_{gen}$ represents the absorbed irradiation. On a unit area basis

$$\dot{E}''_g = -\dot{E}''_{in} + \dot{E}''_{out} = -q''_x(0) + q''_x(L) = +\frac{A}{a}(1 - e^{-aL}).$$ ◁

Alternatively, evaluate $\dot{E}''_g$ by integration over the volume of the medium,

$$\dot{E}''_g = \int_0^L \dot{q}(x)dx = \int_0^L Ae^{-ax}dx = -\frac{A}{a}\left[e^{-ax}\right]_0^L = \frac{A}{a}(1 - e^{-aL}).$$

PROBLEM 2.26

KNOWN: Steady-state temperature distribution in a one-dimensional wall of thermal conductivity, $T(x) = Ax^3 + Bx^2 + Cx + D$.

FIND: Expressions for the heat generation rate in the wall and the heat fluxes at the two wall faces (x = 0,L).

ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat flow, (3) Homogeneous medium.

ANALYSIS: The appropriate form of the heat diffusion equation for these conditions is

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad \text{or} \qquad \dot{q} = -k\frac{d^2T}{dx^2}.$$

Hence, the generation rate is

$$\dot{q} = -k\frac{d}{dx}\left[\frac{dT}{dx}\right] = -k\frac{d}{dx}[3Ax^2 + 2Bx + C + 0]$$

$$\dot{q} = -k[6Ax + 2B] \qquad \triangleleft$$

which is linear with the coordinate x. The heat fluxes at the wall faces can be evaluated from Fourier's law,

$$q''_x = -k\frac{dT}{dx} = -k[3Ax^2 + 2Bx + C]$$

using the expression for the temperature gradient derived above. Hence, the heat fluxes are:

Surface x=0:

$$q''_x(0) = -kC \qquad \triangleleft$$

Surface x=L:

$$q''_x(L) = -k[3AL^2 + 2BL + C]. \qquad \triangleleft$$

COMMENTS: (1) From an overall energy balance on the wall, find

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0$$

$$q''_x(0) - q''_x(L) = (-kC) - (-k)[3AL^2 + 2BL + C] + \dot{E}_g = 0$$

$$\dot{E}''_g = -3AkL^2 - 2BkL.$$

From integration of the volumetric heat rate, we can also find $\dot{E}''_g$ as

$$\dot{E}''_g = \int_0^L \dot{q}(x)dx = \int_0^L -k[6Ax + 2B]dx = -k[3Ax^2 + 2Bx]_0^L$$

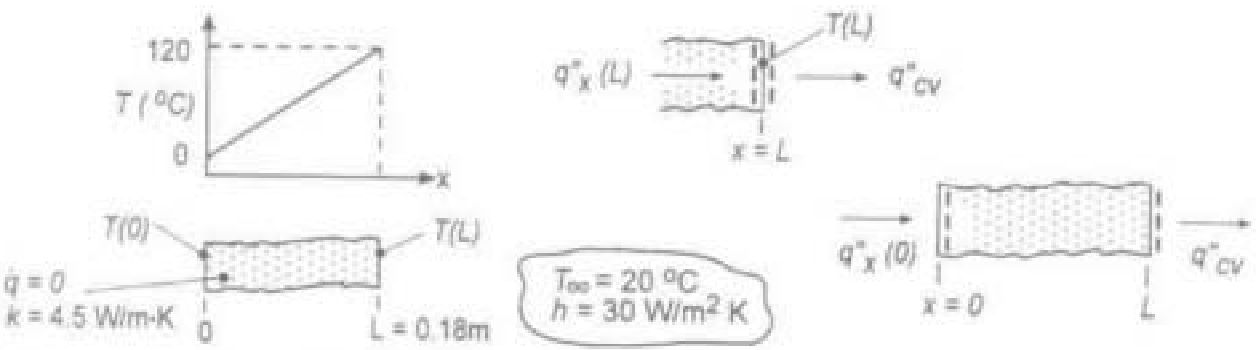
$$\dot{E}''_g = -3AkL^2 - 2BkL.$$

PROBLEM 2.27

KNOWN: Plane wall with no internal energy generation.

FIND: Determine whether the prescribed temperature distribution is possible; explain your reasoning. With the temperatures $T(0) = 0°C$ and $T_\infty = 20°C$ fixed, compute and plot the temperature $T(L)$ as a function of the convection coefficient for the range $10 \le h \le 100\ W/m^2 \cdot K$.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) No internal energy generation, (3) Constant properties, (4) No radiation exchange at the surface x = L, and (5) Steady-state conditions.

ANALYSIS: Is the prescribed temperature distribution possible? If so, the energy balance at the surface x = L as shown above in the Schematic, must be satisfied.

$$\dot{E}_{in} - \dot{E}_{out} \ ?=\ ?0 \qquad\qquad q''_x(L) - q''_{cv} \ ?=\ ?0 \qquad (1,2)$$

where the conduction and convection heat fluxes are, respectively,

$$q''_x(L) = -k\frac{dT}{dx}\bigg)_{x=L} = -k\frac{T(L)-T(0)}{L} = -4.5\,W/m\cdot K \times (120-0)°C/0.18\,m = -3000\,W/m^2$$

$$q''_{cv} = h\left[T(L) - T_\infty\right] = 30\,W/m^2\cdot K \times (120-20)°C = 3000\,W/m^2$$

Substituting the heat flux values into Eq. (2), find (-3000) - (3000) ≠ 0 and therefore, the temperature distribution is not possible.

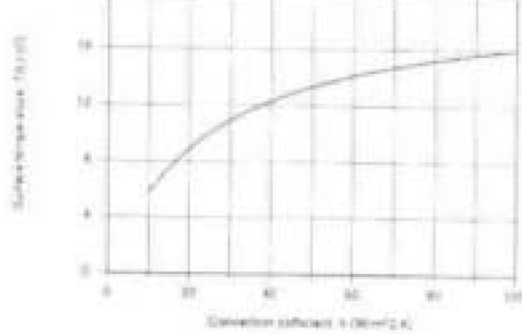
With $T(0) = 0°C$ and $T_\infty = 20°C$, the temperature at the surface x = L, T(L), can be determined from an overall energy balance on the wall as shown above in the Schematic,

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad q''_x(0) - q''_{cv} = 0 \qquad -k\frac{T(L)-T(0)}{L} - h\left[T(L)-T_\infty\right] = 0$$

$$-4.5\,W/m\cdot K\left[T(L) - 0°C\right]/0.18\,m - 30\,W/m^2\cdot K\left[T(L) - 20°C\right] = 0$$

$$T(L) = 10.9°C \qquad <$$

Using this same analysis, T(L) as a function of the convection coefficient can be determined and plotted. We don't expect T(L) to be linearly dependent upon h. Note that as h increases to larger values, T(L) approaches T_∞. To what value will T(L) approach as h decreases?

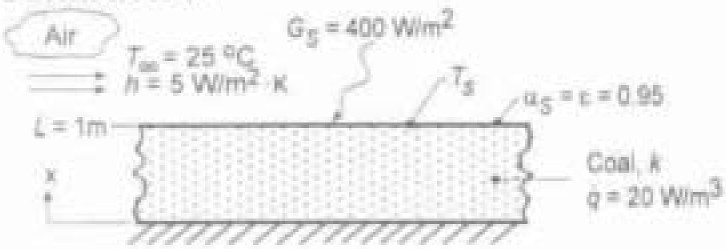


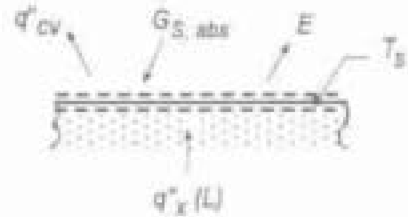
PROBLEM 2.28

KNOWN: Coal pile of prescribed depth experiencing uniform volumetric generation with convection, absorbed irradiation and emission on its upper surface.

FIND: (a) The appropriate form of the heat diffusion equation (HDE) and whether the prescribed temperature distribution satisfies this HDE; conditions at the bottom of the pile, x = 0; sketch of the temperature distribution with labeling of key features; (b) Expression for the conduction heat rate at the location x = L; expression for the surface temperature T_s based upon a surface energy balance at x = L; evaluate T_s and T(0) for the prescribed conditions; (c) Based upon typical daily averages for G_S and h, compute and plot T_s and T(0) for (1) h = 5 $W/m^2 \cdot K$ with $50 \le G_S \le 500$ W/m^2, (2) $G_S = 400$ W/m^2 with $5 \le h \le 50$ $W/m^2 \cdot K$.

SCHEMATIC:





ASSUMPTIONS: (1) One-dimensional conduction, (2) Uniform volumetric heat generation, (3) Constant properties, (4) Negligible irradiation from the surroundings, and (5) Steady-state conditions.

PROPERTIES: *Table A.3*, Coal (300K): k = 0.26 W/m.K

ANALYSIS: (a) For one-dimensional, steady-state conduction with uniform volumetric heat generation and constant properties the heat diffusion equation (HDE) follows from Eq. 2.16,

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) + \frac{\dot{q}}{k} = 0 \qquad (1) <$$

Substituting the temperature distribution into the HDE, Eq. (1),

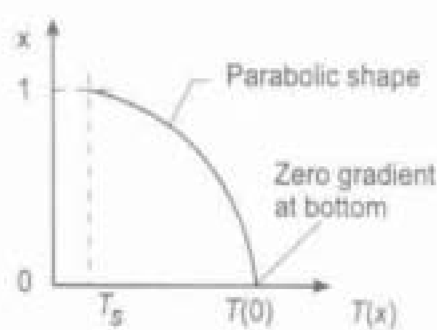
$$T(x) = T_s + \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) \qquad \frac{d}{dx}\left[0 + \frac{\dot{q}L^2}{2k}\left(0 - \frac{2x}{L^2}\right)\right] + \frac{\dot{q}}{k} ? = ?0 \qquad (2,3)$$

we find that it does indeed satisfy the HDE for all values of x. <

From Eq. (2), note that the temperature distribution must be quadratic, with maximum value at x = 0. At x = 0, the heat flux is

$$q_x''(0) = -k\left.\frac{dT}{dx}\right)_{x=0} = -k\left[0 + \frac{\dot{q}L^2}{2k}\left(0 - \frac{2x}{L^2}\right)\right]_{x=0} = 0$$

so that the gradient at x = 0 is zero. Hence, the bottom is insulated.



(b) From an overall energy balance on the pile, the conduction heat flux at the surface must be

$$q_x''(L) = \dot{E}_g'' = \dot{q}L <$$

Continued...

From a surface energy balance per unit area shown in the Schematic above,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0 \qquad q''_x(L) - q''_{cv} + G_{S,abs} - E = 0$$

$$\dot{q}L - h(T_s - T_\infty) + 0.95G_S - \varepsilon\sigma T_s^4 = 0 \qquad (4)$$

$$20\,\mathrm{W/m^3} \times 1\,\mathrm{m} - 5\,\mathrm{W/m^2 \cdot K}(T_s - 298\,\mathrm{K}) + 0.95 \times 400\,\mathrm{W/m^2} - 0.95 \times 5.67 \times 10^{-8}\,\mathrm{W/m^2 \cdot K^4}\,T_s^4 = 0$$

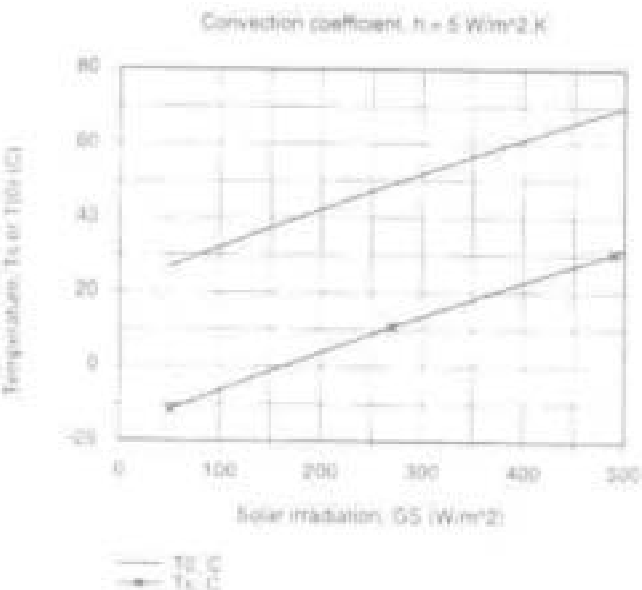
$$T_s = 295.7\ \mathrm{K} = 22.7°\mathrm{C}$$ <

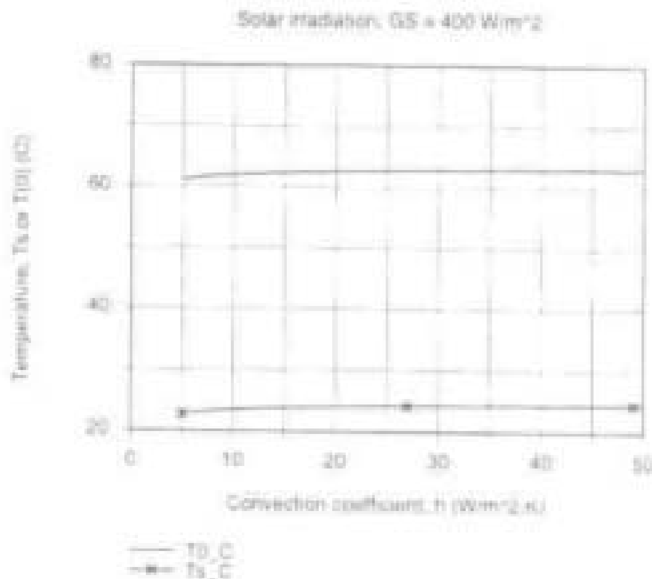
From Eq. (2) with x = 0, find

$$T(0) = T_s + \frac{\dot{q}L^2}{2k} = 22.7°\mathrm{C} + \frac{30\,\mathrm{W/m^3} \times (1\,\mathrm{m})^2}{2 \times 0.26\,\mathrm{W/m \cdot K}} = 61.1°\mathrm{C} \qquad (5)$$ <

where the thermal conductivity for coal was obtained from Table A.3.

(c) Two plots are generated using Eq. (4) and (5) for T_s and T(0), respectively; (1) with h = 5 W/m^2·K for $50 \le G_S \le 500$ W/m^2 and (2) with G_S = 400 W/m^2 for $5 \le h \le 50$ W/m^2·K.





From the T vs. h plot with G_S = 400 W/m^2, note that the convection coefficient does not have a major influence on the surface or bottom coal pile temperatures. From the T vs. G_S plot with h = 5 W/m^2·K, note that the solar irradiation has a very significant effect on the temperatures. The fact that T_s is less than the ambient air temperature, T_∞, and, in the case of very low values of G_S, below freezing, is a consequence of the large magnitude of the emissive power E.

COMMENTS: In our analysis we ignored irradiation from the sky, an environmental radiation effect you'll consider in Chapter 12. Treated as large isothermal surroundings, $G_{sky} = \sigma T_{sky}^4$ where T_{sky} = -30°C for very clear conditions and nearly air temperature for cloudy conditions. For low G_S conditions we should consider G_{sky}, the effect of which will be to predict higher values for T_s and T(0).

PROBLEM 2.29

KNOWN: Cylindrical system with negligible temperature variation in the r,z directions.

FIND: (a) Heat equation beginning with a properly defined control volume, (b) Temperature distribution $T(\phi)$ for steady-state conditions with no internal heat generation and constant properties, (c) Heat rate for Part (b) conditions.

SCHEMATIC:

ASSUMPTIONS: (1) T is independent of r,z, (2) $\Delta r = (r_o - r_i) \ll r_i$.

ANALYSIS: (a) Define the control volume as $V = r_i d\phi \cdot \Delta r \cdot L$ where L is length normal to page. Apply the conservation of energy requirement, Eq. 1.11a,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st} \qquad q_\phi - q_{\phi+d\phi} + \dot{q}V = \rho V c \frac{\partial T}{\partial t} \tag{1,2}$$

where $$q_\phi = -k(\Delta r \cdot L)\frac{\partial T}{r_i \partial \phi} \qquad q_{\phi+d\phi} = q_\phi + \frac{\partial}{\partial \phi}(q_\phi)\, d\phi. \tag{3,4}$$

Eqs. (3) and (4) follow from Fourier's law, Eq. 2.1, and from Eq. 2.7, respectively. Combining Eqs. (3) and (4) with Eq. (2) and canceling like terms, find

$$\frac{1}{r_i^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \dot{q} = \rho c \frac{\partial T}{\partial t}. \tag{5}$$ ◁

Considering that r,z are constant, this form agrees with Eq. 2.20.

(b) For steady-state conditions with $\dot{q} = 0$, the heat equation, (5), becomes

$$\frac{d}{d\phi}\left[k\frac{dT}{d\phi}\right] = 0. \tag{6}$$

With constant properties, it follows that $dT/d\phi$ is constant which implies $T(\phi)$ is linear in ϕ. That is,

$$\frac{dT}{d\phi} = \frac{T_2 - T_1}{\phi_2 - \phi_1} = +\frac{1}{\pi}(T_2 - T_1) \qquad \text{or} \qquad T(\phi) = T_1 + \frac{1}{\pi}(T_2 - T_1)\phi. \tag{7,8}$$ ◁

(c) The heat rate for the conditions of Part (b) follows from Fourier's law, Eq. (3), using the temperature gradient of Eq. (7). That is,

$$q_\phi = -k(\Delta r \cdot L)\frac{1}{r_i}\left[+\frac{1}{\pi}(T_2 - T_1)\right] = -k\left[\frac{r_o - r_i}{\pi r_i}\right] L\,(T_2 - T_1). \tag{9}$$ ◁

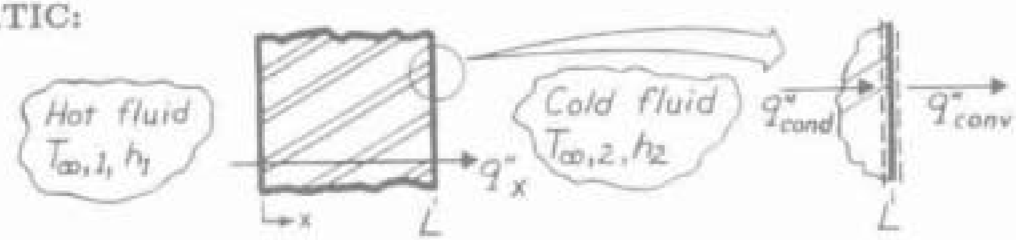
COMMENTS: Note the expression for the temperature gradient in Fourier's law, Eq. (3), is $\partial T / r_i \partial \phi$ not $\partial T / \partial \phi$. For the conditions of Part (b) and (c), note that $q(\phi)$ is independent of ϕ; this is first noted in Eq. (6) and finally confirmed in Eq. (9).

PROBLEM 3.1

KNOWN: One-dimensional, plane wall separating hot and cold fluids at $T_{\infty,1}$ and $T_{\infty,2}$, respectively.

FIND: Temperature distribution, T(x), and heat flux, q''_x, in terms of $T_{\infty,1}$, $T_{\infty,2}$, h_1, h_2, k and L.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Steady-state conditions, (3) Constant properties, (4) Negligible radiation, (5) No generation.

ANALYSIS: For the foregoing conditions, the general solution to the heat diffusion equation is of the form, Equation 3.2,

$$T(x) = C_1 x + C_2 \ . \tag{1}$$

The constants of integration, C_1 and C_2, are determined by using surface energy balance conditions at x=0 and x=L, Equation 2.23, and as illustrated above,

$$-k \left.\frac{dT}{dx}\right)_{x=0} = h_1[T_{\infty,1} - T(0)] \qquad -k \left.\frac{dT}{dx}\right)_{x=L} = h_2[T(L) - T_{\infty,2}] \ . \tag{2,3}$$

For the BC at x=0, Equation (2), use Equation (1) to find

$$-k\ (C_1 + 0) = h_1\ [T_{\infty,1} - (C_1 \cdot 0 + C_2)] \tag{4}$$

and for the BC at x=L to find

$$-k\ (C_1 + 0) = h_2\ [(C_1 L + C_2) - T_{\infty,2}] \ . \tag{5}$$

Multiply Eq. (4) by h_2 and Eq. (5) by h_1, add the equations to obtain C_1. Then substitute C_1 into Eq. (4) to obtain C_2. The results are

$$C_1 = -\frac{(T_{\infty,1} - T_{\infty,2})}{k\left[\dfrac{1}{h_1} + \dfrac{1}{h_2} + \dfrac{L}{k}\right]} \qquad C_2 = -\frac{(T_{\infty,1} - T_{\infty,2})}{h_1\left[\dfrac{1}{h_1} + \dfrac{1}{h_2} + \dfrac{L}{k}\right]} + T_{\infty,1}$$

$$T(x) = -\frac{(T_{\infty,1} - T_{\infty,2})}{\left[\dfrac{1}{h_1} + \dfrac{1}{h_2} + \dfrac{L}{k}\right]}\left[\frac{x}{k} + \frac{1}{h_1}\right] + T_{\infty,1} \ . \qquad \triangleleft$$

From Fourier's law, the heat flux is a constant and of the form

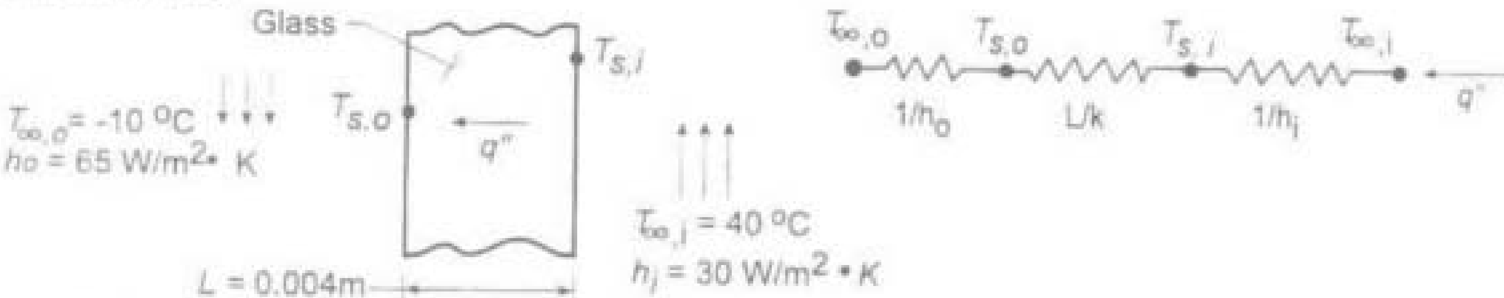
$$q''_x = -k \left.\frac{dT}{dx}\right)_x = -k\ C_1 = +\frac{(T_{\infty,1} - T_{\infty,2})}{\left[\dfrac{1}{h_1} + \dfrac{1}{h_2} + \dfrac{L}{k}\right]} \ . \qquad \triangleleft$$

PROBLEM 3.2

KNOWN: Temperatures and convection coefficients associated with air at the inner and outer surfaces of a rear window.

FIND: (a) Inner and outer window surface temperatures, $T_{s,i}$ and $T_{s,o}$, and (b) $T_{s,i}$ and $T_{s,o}$ as a function of the outside air temperature $T_{\infty,o}$ and for selected values of outer convection coefficient, h_o.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) Negligible radiation effects, (4) Constant properties.

PROPERTIES: *Table A-3*, Glass (300 K): k = 1.4 W/m·K.

ANALYSIS: (a) The heat flux may be obtained from Eqs. 3.11 and 3.12,

$$q'' = \frac{T_{\infty,i} - T_{\infty,o}}{\frac{1}{h_o} + \frac{L}{k} + \frac{1}{h_i}} = \frac{40^\circ C - (-10^\circ C)}{\frac{1}{65\,W/m^2\cdot K} + \frac{0.004\,m}{1.4\,W/m\cdot K} + \frac{1}{30\,W/m^2\cdot K}}$$

$$q'' = \frac{50^\circ C}{(0.0154 + 0.0029 + 0.0333)m^2\cdot K/W} = 968\,W/m^2.$$

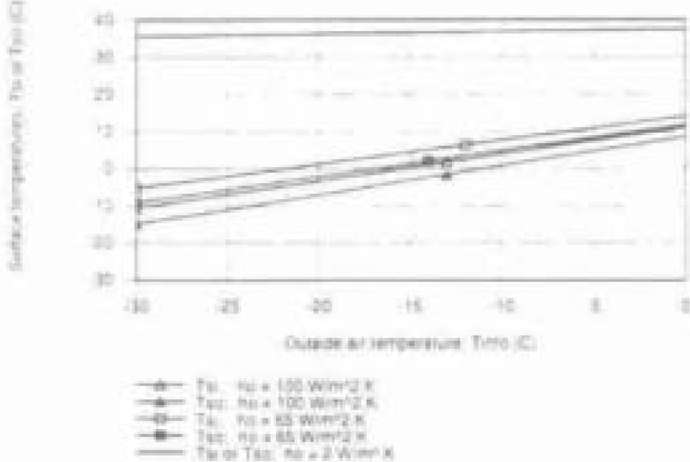
Hence, with $q'' = h_i(T_{\infty,i} - T_{s,i})$, the inner surface temperature is

$$T_{s,i} = T_{\infty,i} - \frac{q''}{h_i} = 40^\circ C - \frac{968\,W/m^2}{30\,W/m^2\cdot K} = 7.7^\circ C$$ <

Similarly for the outer surface temperature with $q'' = h_o(T_{s,o} - T_{\infty,o})$ find

$$T_{s,o} = T_{\infty,o} - \frac{q''}{h_o} = -10^\circ C - \frac{968\,W/m^2}{65\,W/m^2\cdot K} = 4.9^\circ C$$ <

(b) Using the same analysis, $T_{s,i}$ and $T_{s,o}$ have been computed and plotted as a function of the outside air temperature, $T_{\infty,o}$, for outer convection coefficients of h_o = 2, 65, and 100 W/m²·K. As expected, $T_{s,i}$ and $T_{s,o}$ are linear with changes in the outside air temperature. The difference between $T_{s,i}$ and $T_{s,o}$ increases with increasing convection coefficient, since the heat flux through the window likewise increases. This difference is larger at lower outside air temperatures for the same reason. Note that with h_o = 2 W/m²·K, $T_{s,i} - T_{s,o}$ is too small to show on the plot.



COMMENTS: (1) The largest resistance is that associated with convection at the inner surface. The values of $T_{s,i}$ and $T_{s,o}$ could be increased by increasing the value of h_i.

(2) The *IHT Thermal Resistance Network Model* was used to create a model of the window and generate the above plot. The Workspace is shown below.

```
// Thermal Resistance Network Model:
// The Network:
```

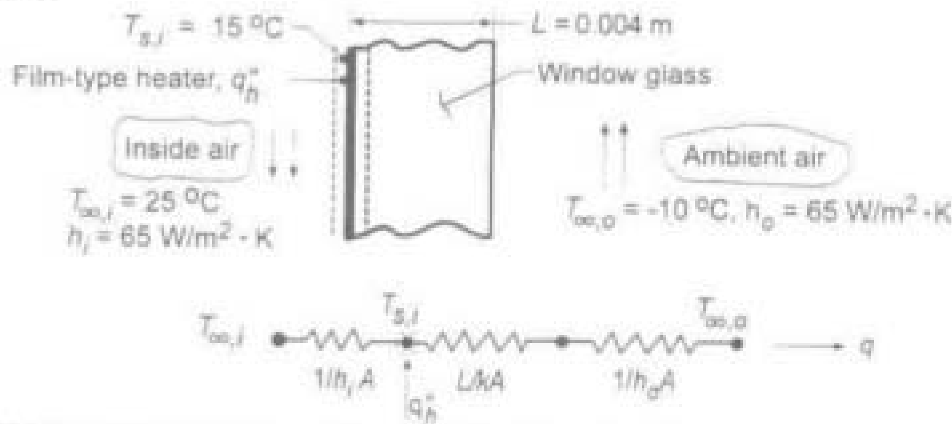

```
// Heat rates into node j,qj, through thermal resistance Rij
q21 = (T2 - T1) / R21
q32 = (T3 - T2) / R32
q43 = (T4 - T3) / R43

// Nodal energy balances
q1 + q21 = 0
q2 - q21 + q32 = 0
q3 - q32 + q43 = 0
q4 - q43 = 0

/* Assigned variables list: deselect the qi, Rij and Ti which are unknowns; set qi = 0 for embedded nodal points
at which there is no external source of heat. */
T1 = Tinfo          // Outside air temperature, C
//q1 =              // Heat rate, W
T2 = Tso            // Outer surface temperature, C
q2 = 0              // Heat rate, W; node 2, no external heat source
T3 = Tsi            // Inner surface temperature, C
q3 = 0              // Heat rate, W; node 2, no external heat source
T4 = Tinfi          // Inside air temperature, C
//q4 =              // Heat rate, W

// Thermal Resistances:
R21 = 1 / ( ho * As )        // Convection thermal resistance, K/W; outer surface
R32 = L / ( k * As )         // Conduction thermal resistance, K/W; glass
R43 = 1 / ( hi * As )        // Convection thermal resistance, K/W; inner surface

// Other Assigned Variables:
Tinfo = -10         // Outside air temperature, C
ho = 65             // Convection coefficient, W/m^2.K; outer surface
L = 0.004           // Thickness, m; glass
k = 1.4             // Thermal conductivity, W/m.K; glass
Tinfi = 40          // Inside air temperature, C
hi = 30             // Convection coefficient, W/m^2.K; inner surface
As = 1              // Cross-sectional area, m^2; unit area
```

PROBLEM 3.3

KNOWN: Desired inner surface temperature of rear window with prescribed inside and outside air conditions.

FIND: (a) Heater power per unit area required to maintain the desired temperature, and (b) Compute and plot the electrical power requirement as a function of $T_{\infty,o}$ for the range $-30 \le T_{\infty,o} \le 0°C$ with h_o of 2, 20, 65 and 100 $W/m^2 \cdot K$. Comment on heater operation needs for low h_o. If $h \sim V^n$, where V is the vehicle speed and n is a positive exponent, how does the vehicle speed affect the need for heater operation?

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat transfer, (3) Uniform heater flux, q''_h, (4) Constant properties, (5) Negligible radiation effects, (6) Negligible film resistance.

PROPERTIES: *Table A-3*, Glass (300 K): k = 1.4 W/m·K.

ANALYSIS: (a) From an energy balance at the inner surface and the thermal circuit, it follows that for a unit surface area,

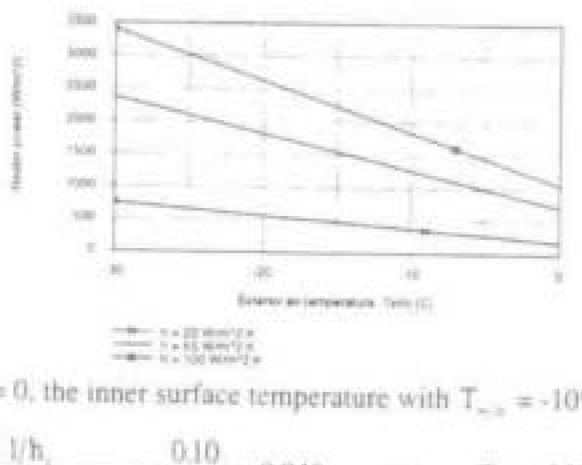
$$\frac{T_{\infty,i} - T_{s,i}}{1/h_i} + q''_h = \frac{T_{s,i} - T_{\infty,o}}{L/k + 1/h_o}$$

$$q''_h = \frac{T_{s,i} - T_{\infty,o}}{L/k + 1/h_o} - \frac{T_{\infty,i} - T_{s,i}}{1/h_i} = \frac{15°C - (-10°C)}{\frac{0.004\,m}{1.4\,W/m \cdot K} + \frac{1}{65\,W/m^2 \cdot K}} - \frac{25°C - 15°C}{\frac{1}{10\,W/m^2 \cdot K}}$$

$$q''_h = (1370 - 100)\,W/m^2 = 1270\,W/m^2$$

<

(b) The heater electrical power requirement as a function of the exterior air temperature for different exterior convection coefficients is shown in the plot. When $h_o = 2\ W/m^2 \cdot K$, the heater is unecessary, since the glass is maintained at 15°C by the interior air. If $h \sim V^n$, we conclude that, with higher vehicle speeds, the exterior convection will increase, requiring increased heat power to maintain the 15°C condition.



COMMENTS: With $q''_h = 0$, the inner surface temperature with $T_{\infty,o} = -10°C$ would be given by

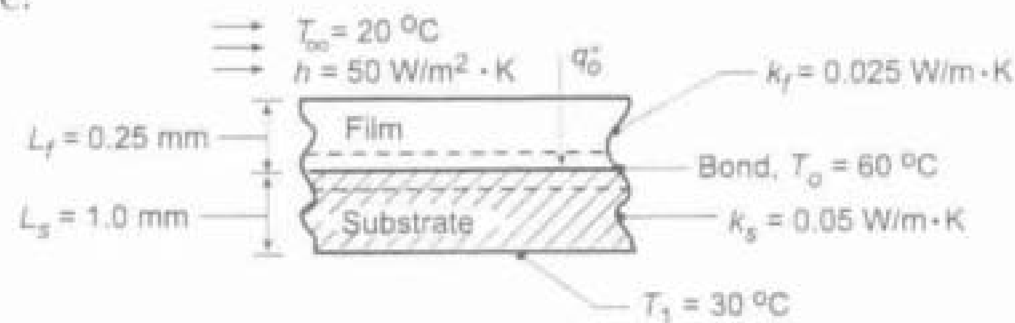
$$\frac{T_{\infty,i} - T_{s,i}}{T_{\infty,i} - T_{\infty,o}} = \frac{1/h_i}{1/h_i + L/k + 1/h_o} = \frac{0.10}{0.118} = 0.846, \qquad \text{or} \qquad T_{s,i} = 25°C - 0.846(35°C) = -4.6°C.$$

PROBLEM 3.4

KNOWN: Curing of a transparent film by radiant heating with substrate and film surface subjected to known thermal conditions.

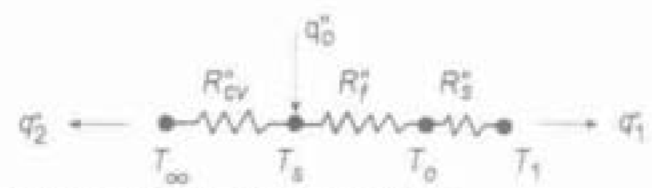
FIND: (a) Thermal circuit for this situation, (b) Radiant heat flux, q''_o (W/m^2), to maintain bond at curing temperature, T_o, (c) Compute and plot q''_o as a function of the film thickness for $0 \le L_f \le 1$ mm, and (d) If the film is not transparent, determine q''_o required to achieve bonding; plot results as a function of L_f.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat flow, (3) All the radiant heat flux q''_o is absorbed at the bond, (4) Negligible contact resistance.

ANALYSIS: (a) The thermal circuit for this situation is shown at the right. Note that terms are written on a per unit area basis.



(b) Using this circuit and performing an energy balance on the film-substrate interface,

$$q''_o = q''_1 + q''_2 \qquad\qquad q''_o = \frac{T_o - T_\infty}{R''_{cv} + R''_f} + \frac{T_o - T_1}{R''_s}$$

where the thermal resistances are

$$R''_{cv} = 1/h = 1/50\,W/m^2 \cdot K = 0.020\,m^2 \cdot K/W$$

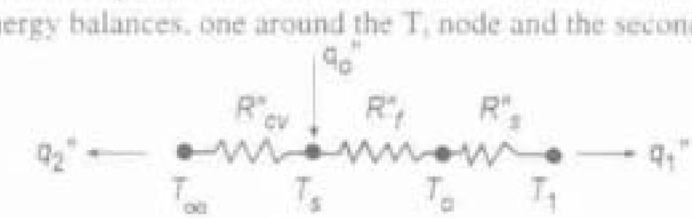
$$R''_f = L_f/k_f = 0.00025\,m/0.025\,W/m \cdot K = 0.010\,m^2 \cdot K/W$$

$$R''_s = L_s/k_s = 0.001\,m/0.05\,W/m \cdot K = 0.020\,m^2 \cdot K/W$$

$$q''_o = \frac{(60-20)^\circ C}{[0.020 + 0.010]\,m^2 \cdot K/W} + \frac{(60-30)^\circ C}{0.020\,m^2 \cdot K/W} = (133 + 1500)\,W/m^2 = 2833\,W/m^2 \qquad <$$

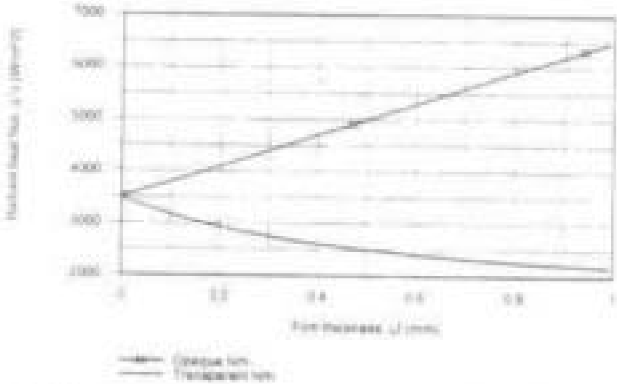
(c) For the transparent film, the radiant flux required to achieve bonding as a function of film thickness L_f is shown in the plot below.

(d) If the film is opaque (not transparent), the thermal circuit is shown below. In order to find q''_o, it is necessary to write two energy balances, one around the T_s node and the second about the T_o node.



The results of the analyses are plotted below.

Continued...



COMMENTS: (1) When the film is transparent, the radiant flux is absorbed on the bond. The flux required decreases with increasing film thickness. Physically, how do you explain this? Why is the relationship not linear?

(2) When the film is opaque, the radiant flux is absorbed on the surface, and the flux required increases with increasing thickness of the film. Physically, how do you explain this? Why is the relationship linear?

(3) The IHT Thermal Resistance Network Model was used to create a model of the film-substrate system and generate the above plot. The Workspace is shown below.

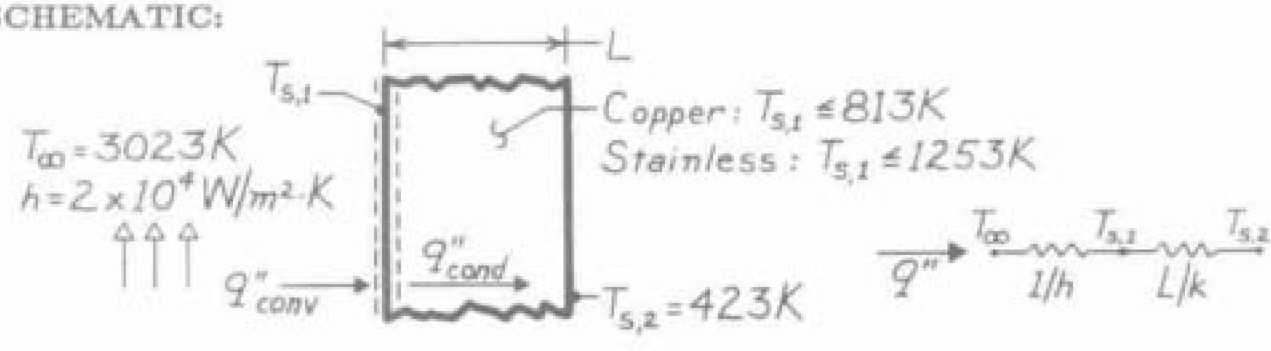

```
// Thermal Resistance Network
Model:
// The Network:

// Heat rates into node j,qij, through thermal resistance Rij
q21 = (T2 - T1) / R21
q32 = (T3 - T2) / R32
q43 = (T4 - T3) / R43

// Nodal energy balances
q1 + q21 = 0
q2 - q21 + q32 = 0
q3 - q32 + q43 = 0
q4 - q43 = 0

/* Assigned variables list: deselect the qi, Rij and Ti which are unknowns; set qi = 0 for embedded nodal points
at which there is no external source of heat. */
T1 = Tinf          // Ambient air temperature, C
//q1 =             // Heat rate, W; film side
T2 = Ts            // Film surface temperature, C
q2 = 0             // Radiant flux, W/m^2; zero for part (a)
T3 = To            // Bond temperature, C
q3 = qo            // Radiant flux, W/m^2; part (a)
T4 = Tsub          // Substrate temperature, C
//q4 =             // Heat rate, W; substrate side

// Thermal Resistances:
R21 = 1 / ( h * As )      // Convection resistance, K/W
R32 = Lf / (kf * As)      // Conduction resistance, K/W; film
R43 = Ls / (ks * As)      // Conduction resistance, K/W; substrate

// Other Assigned Variables:
Tinf = 20          // Ambient air temperature, C
h = 50             // Convection coefficient, W/m^2.K
Lf = 0.00025       // Thickness, m; film
kf = 0.025         // Thermal conductivity, W/m.K; film
To = 60            // Cure temperature, C
Ls = 0.001         // Thickness, m; substrate
ks = 0.05          // Thermal conductivity, W/m.K; substrate
Tsub = 30          // Substrate temperature, C
As = 1             // Cross-sectional area, m^2; unit area
```

PROBLEM 3.5

KNOWN: Maximum allowable temperature and operating conditions of a rocket nozzle wall.

FIND: Preferred material: Cu or 304 Stainless.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction through a plane wall, (2) Steady-state conditions, (3) Constant properties.

PROPERTIES: *Table A-1,* Cu ($\overline{T} = (423+813)/2 = 618K$): $\rho = 8933\ kg/m^3$, $k = 378\ W/m\cdot K$; St.St. (304) ($\overline{T} = 423+1253/2 = 838K$): $\rho = 7900\ kg/m^3$, $k = 23.2\ W/m\cdot K$.

ANALYSIS: The decision concerning which material to use may be made by first computing the thickness L required to insure that $T_{s,1}$ remains within the acceptable limit. The lighter wall (corresponding to the smaller value of $L\cdot\rho$) would then be the obvious choice. Applying an energy balance to the inner surface, $q''_{conv} = q''_{cond}$. Hence,

$$h(T_\infty - T_{s,1}) = k\,\frac{T_{s,1} - T_{s,2}}{L}\ ,$$

$$L = \frac{k}{h}\,\frac{T_{s,1} - T_{s,2}}{T_\infty - T_{s,1}}\ .$$

For the *copper:*

$$L = \frac{378\ W/m\cdot K}{2\times10^4\ W/m^2\cdot K}\,\frac{(813-423)K}{(3023-813)K} = 3.34\ mm$$

$$\rho L = (8933\ kg/m^3 \times 0.00334\ m) = 29.8\ kg/m^2\ .$$

For the *stainless steel:*

$$L = \frac{23.2\ W/m\cdot K}{2\times10^4\ W/m^2\cdot K}\,\frac{(1253-423)K}{(3023-1253)K} = 0.54\ mm$$

$$\rho L = (7900\ kg/m^3 \times 0.000544\ m) = 4.3\ kg/m^2\ .$$

Hence from the standpoint of weight savings the stainless steel is preferred. ◁

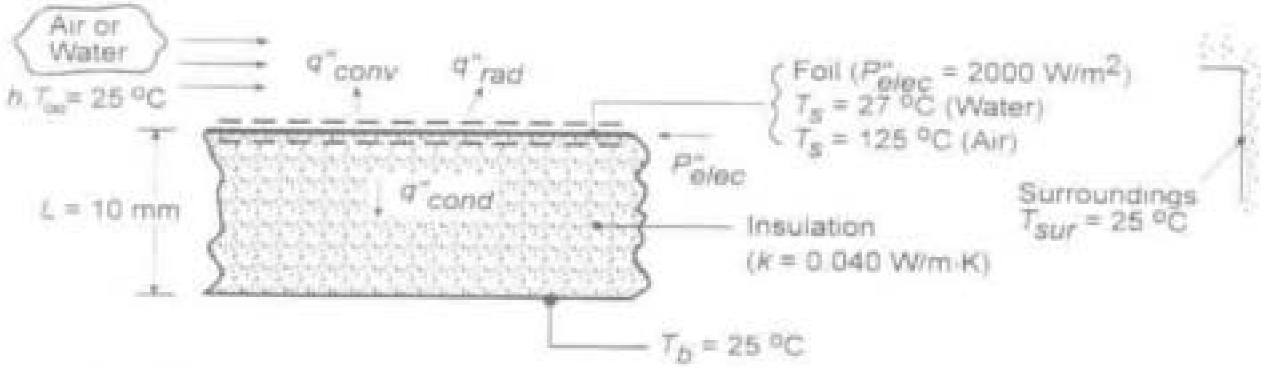
COMMENTS: The above considerations ignore strength requirements, which determine the minimum wall thickness needed to sustain the nozzle loads. Such requirements would have to be considered to complete the design calculations.

PROBLEM 3.6

KNOWN: Design and operating conditions of a heat flux gage.

FIND: (a) Convection coefficient for water flow ($T_s = 27°C$) and error associated with neglecting conduction in the insulation, (b) Convection coefficient for air flow ($T_s = 125°C$) and error associated with neglecting conduction and radiation, (c) Effect of convection coefficient on error associated with neglecting conduction for $T_s = 27°C$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) One-dimensional conduction, (3) Constant k.

ANALYSIS: (a) The electric power dissipation is balanced by convection to the water and conduction through the insulation. An energy balance applied to a control surface about the foil therefore yields

$$P''_{elec} = q''_{conv} + q''_{cond} = h(T_s - T_\infty) + k(T_s - T_b)/L$$

Hence,

$$h = \frac{P''_{elec} - k(T_s - T_b)/L}{T_s - T_\infty} = \frac{2000\,W/m^2 - 0.04\,W/m\cdot K(2\,K)/0.01\,m}{2\,K}$$

$$h = \frac{(2000-8)\,W/m^2}{2\,K} = 996\,W/m^2\cdot K$$ <

If conduction is neglected, a value of $h = 1000\ W/m^2\cdot K$ is obtained, with an attendant error of (1000 - 996)/996 = 0.40%

(b) In air, energy may also be transferred from the foil surface by radiation, and the energy balance yields

$$P''_{elec} = q''_{conv} + q''_{rad} + q''_{cond} = h(T_s - T_\infty) + \varepsilon\sigma(T_s^4 - T_{sur}^4) + k(T_s - T_b)/L$$

Hence,

$$h = \frac{P''_{elec} - \varepsilon\sigma(T_s^4 - T_{sur}^4) - k(T_s - T_\infty)/L}{T_s - T_\infty}$$

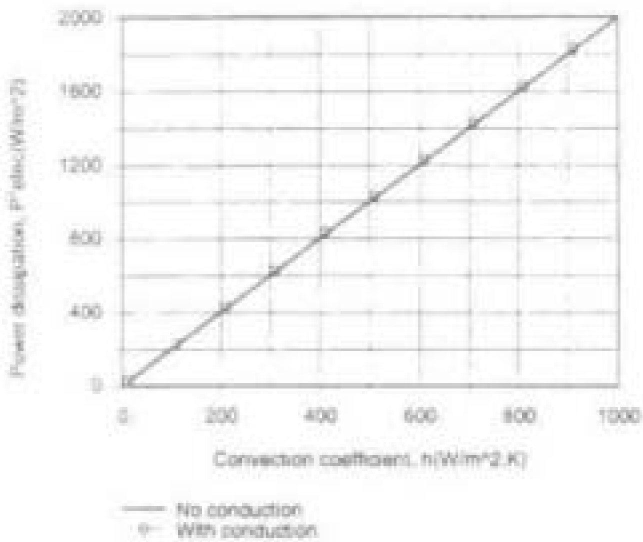
$$= \frac{2000\,W/m^2 - 0.15\times 5.67\times 10^{-8}\,W/m^2\cdot K^4(398^4 - 298^4)K^4 - 0.04\,W/m\cdot K(100\,K)/0.01\,m}{100\,K}$$

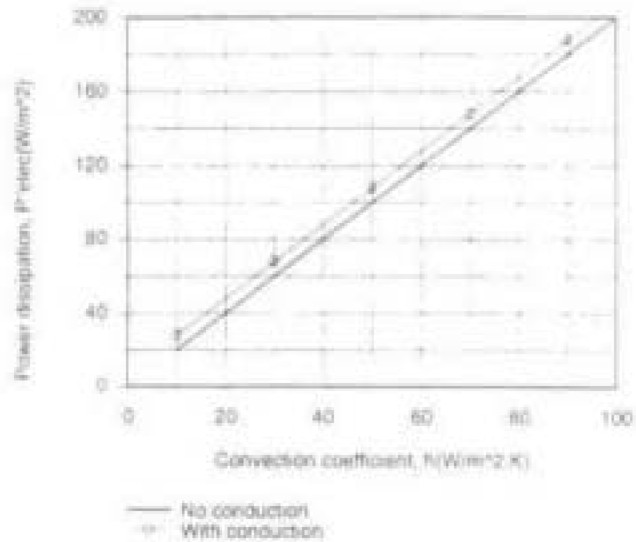
$$= \frac{(2000-146-400)\,W/m^2}{100\,K} = 14.5\,W/m^2\cdot K$$ <

Continued...

If conduction, radiation, or conduction and radiation are neglected, the corresponding values of h and the percentage errors are 18.5 W/m^2·K (27.6%), 16 W/m^2·K (10.3%), and 20 W/m^2·K (37.9%).

(c) For a fixed value of T_s = 27°C, the conduction loss remains at q''_{cond} = 8 W/m^2, which is also the fixed difference between P''_{elec} and q''_{conv}. Although this difference is not clearly shown in the plot for 10 ≤ h ≤ 1000 W/m^2·K, it is revealed in the subplot for 10 ≤ 100 W/m^2·K.





Errors associated with neglecting conduction decrease with increasing h from values which are significant for small h (h < 100 W/m^2·K) to values which are negligible for large h.

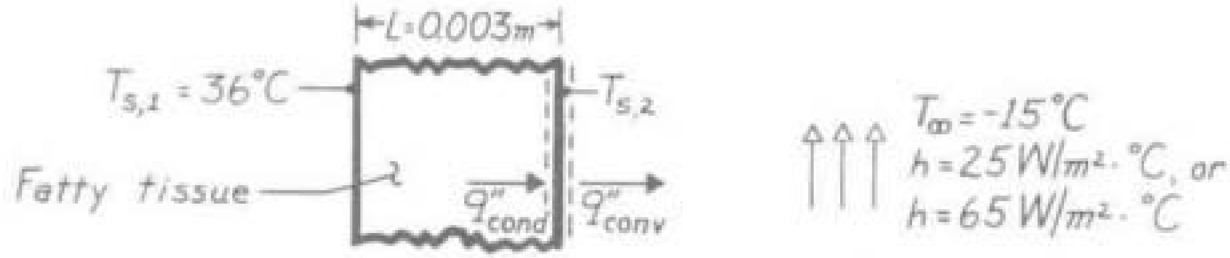
COMMENTS: In liquids (large h), it is an excellent approximation to neglect conduction and assume that all of the dissipated power is transferred to the fluid.

PROBLEM 3.7

KNOWN: A layer of fatty tissue with fixed inside temperature can experience different outside convection conditions.

FIND: (a) Ratio of heat loss for different convection conditions, (b) Outer surface temperature for different convection conditions, and (c) Temperature of still air which achieves same cooling as moving air (*wind chill* effect).

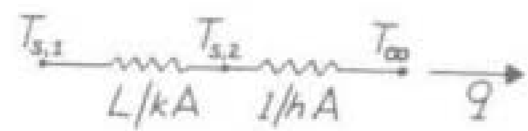
SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction through a plane wall, (2) Steady-state conditions, (3) Homogeneous medium with constant properties, (4) No internal heat generation (metabolic effects are negligible), (5) Negligible radiation effects.

PROPERTIES: *Table A-3*, Tissue, fat layer: $k = 0.2\ W/m\cdot K$.

ANALYSIS: The thermal circuit for this situation is



Hence, the heat rate is

$$q = \frac{T_{s,1} - T_\infty}{R_{tot}} = \frac{T_{s,1} - T_\infty}{L/kA + 1/hA}.$$

Therefore,

$$\frac{q'_c}{q'_w} = \frac{\left[\frac{L}{k} + \frac{1}{h}\right]_{windy}}{\left[\frac{L}{k} + \frac{1}{h}\right]_{calm}}.$$

Applying face energy balance to the outer surface it also follows that

q''_{cc} conv.

Continued .....

Hence,

$$\frac{k}{L}\left(T_{s,1}-T_{s,2}\right)=h\left(T_{s,2}-T_\infty\right)$$

$$T_{s,2}=\frac{T_\infty+\dfrac{k}{hL}T_{s,1}}{1+\dfrac{k}{hL}}.$$

To determine the wind chill effect, we must determine the heat loss for the windy day and use it to evaluate the hypothetical ambient air temperature, T_∞', which would provide the same heat loss on a calm day. Hence,

$$q''=\frac{T_{s,1}-T_\infty}{\left[\dfrac{L}{k}+\dfrac{1}{h}\right]_{windy}}=\frac{T_{s,1}-T_\infty'}{\left[\dfrac{L}{k}+\dfrac{1}{h}\right]_{calm}}$$

From these relations, we can now find the results sought:

(a) $$\frac{q''_{calm}}{q''_{windy}}=\frac{\dfrac{0.003\text{ m}}{0.2\text{ W/m}\cdot\text{K}}+\dfrac{1}{65\text{ W/m}^2\cdot\text{K}}}{\dfrac{0.003\text{ m}}{0.2\text{ W/m}\cdot\text{K}}+\dfrac{1}{25\text{ W/m}^2\cdot\text{K}}}=\frac{0.015+0.0154}{0.015+0.04}$$

$$\frac{q''_{calm}}{q''_{windy}}=0.553 \qquad \triangleleft$$

(b) $$\left.T_{s,2}\right]_{calm}=\frac{-15^\circ\text{C}+\dfrac{0.2\text{ W/m}\cdot\text{K}}{(25\text{ W/m}^2\cdot\text{K})\,(0.003\text{ m})}36^\circ\text{C}}{1+\dfrac{0.2\text{ W/m}\cdot\text{K}}{(25\text{ W/m}^2\cdot\text{K})\,(0.003\text{ m})}}=22.1^\circ\text{C} \qquad \triangleleft$$

$$\left.T_{s,2}\right]_{windy}=\frac{-15^\circ\text{C}+\dfrac{0.2\text{ W/m}\cdot\text{K}}{(65\text{ W/m}^2\cdot\text{K})\,(0.003\text{m})}36^\circ\text{C}}{1+\dfrac{0.2\text{ W/m}\cdot\text{K}}{(65\text{ W/m}^2\cdot\text{K})\,(0.003\text{m})}}=10.8^\circ\text{C} \qquad \triangleleft$$

(c) $$T_\infty'=36^\circ\text{C}-(36+15)^\circ\text{C}\,\frac{(0.003/0.2+1/25)}{(0.003/0.2+1/65)}=-56.3^\circ\text{C} \qquad \triangleleft$$

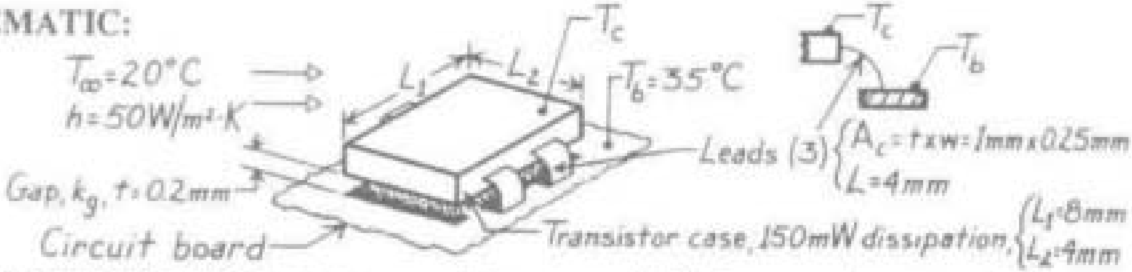
COMMENTS: The wind chill effect is equivalent to a decrease of $T_{s,2}$ by 11.3°C and increase in the heat loss by a factor of $(0.553)^{-1}=1.81$.

PROBLEM 3.8

KNOWN: Surface-mount transistor with prescribed power dissipation and convection cooling conditions.

FIND: Using thermal resistance circuit, an expression for the case temperature T_c; evaluate for stagnant air and conductive paste filled gap.

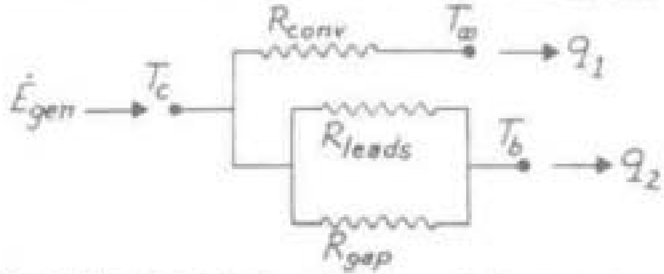
SCHEMATIC:



ASSUMPTIONS: (1) Transistor case is isothermal, (2) Upper surface experiences convection; negligible losses from edges, (3) Leads provide conduction path between case and board, (4) Steady-state conditions.

PROPERTIES: (Given): Air, $k_{g,a} = 0.0263$ W/m·K; Paste, $k_{g,p} = 0.12$ W/m·K; Metal leads, $k_e = 25$ W/m·K.

ANALYSIS: The thermal circuit for the surface mount transistor is



where the thermal resistances are, with $A_s = L_1 \times L_2$ and $A_c = t \times w$,

$$R_{conv} = 1/hA_s = 1/50\ W/m^2\cdot K(8\times4\times10^{-6})m^2 = 625.0\ K/W$$

$$R_{leads} = 1/3(L/k_e A_c) = 1/3(0.004\ m/25\ W/m\cdot K(1\times0.25\times10^{-6})m^2) = 213.3\ K/W$$

$$R_{gap,a} = t/k_{g,a}A_s = 0.0002\ m/0.0263\ W/m\cdot K(8\times4\times10^{-6})m^2 = 237.6\ K/W$$

$$R_{gap,p} = t/k_{g,p}A_s = 0.0002\ m/0.12\ W/m\cdot K(8\times4\times10^{-6})m^2 = 52.1\ K/W.$$

From the thermal circuit and the thermal resistance expressions, find

$$\dot{E}_{gen} = q_1 + q_2 = (T_c - T_\infty)/R_{conv} + (T_c - T_b)[1/R_{leads} + 1/R_{gap}]$$

$$T_c = [\dot{E}_{gen}R_{conv} + T_\infty + T_b \frac{R_{conv}}{[1/R_{leads} + 1/R_{gap}]^{-1}}] / [1 + \frac{R_{conv}}{[1/R_{leads} + 1/R_{gap}]^{-1}}].$$

Substituting values for the stagnant *air-gap* condition, find

$$T_c = \frac{150\times10^{-3}\ W\times625.0\ K/W + 20°C + 35°C\frac{625.0}{[1/213.3 + 1/237.6]^{-1}}}{1 + \frac{625.0}{[1/213.3 + 1/237.6]^{-1}}} = 47.0°C. \quad \triangleleft$$

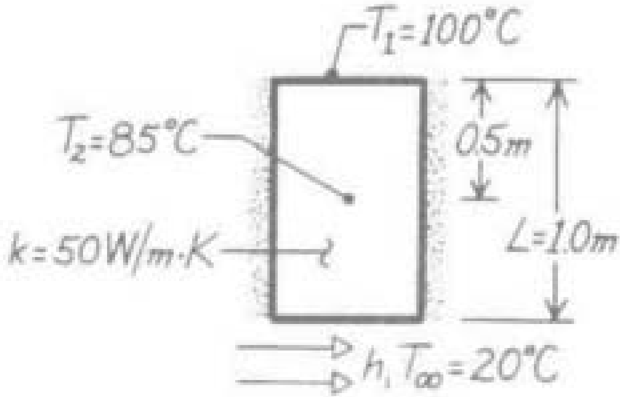
With the *conductive paste* condition, find $T_c = 39.9°C$. ◁

PROBLEM 3.9

KNOWN: Length, surface thermal conditions, and thermal conductivity of a plate. Plate midpoint temperature.

FIND: Surface convection coefficient.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, steady conduction with no generation, (2) Constant properties.

ANALYSIS: For prescribed conditions, q″ is constant. Hence,

$$q''_{cond} = \frac{T_1 - T_2}{(L/2)/k} = \frac{15°C}{0.5\ m/50\ W/m \cdot K} = 1500\ W/m^2$$

$$q'' = \frac{T_1 - T_\infty}{(L/k) + (1/h)} = \frac{30°C}{(0.02 + 1/h) m^2 \cdot K/W} = 1500\ W/m^2$$

$$h = 30\ W/m^2 \cdot K.$$ ◁

COMMENTS: The contributions of conduction and convection to the total thermal resistance are

$$R''_{t,cond} = \frac{L}{k} = 0.02\ m^2 \cdot K/W$$

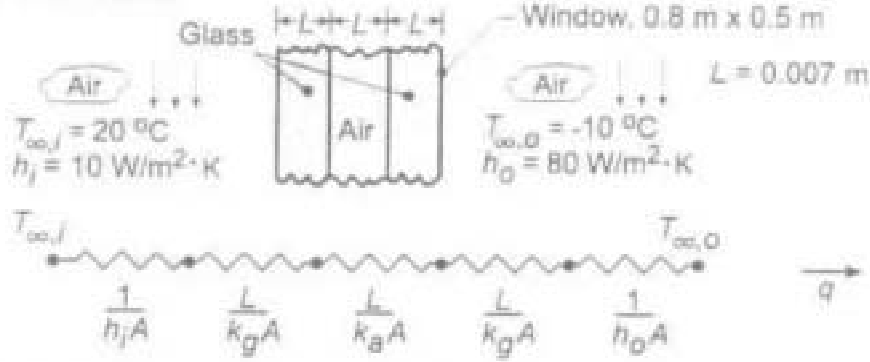
$$R''_{t,cond} = \frac{1}{h} = 0.033\ m^2 \cdot K/W.$$

PROBLEM 3.10

KNOWN: Dimensions of a thermopane window. Room and ambient air conditions.

FIND: (a) Heat loss through window, (b) Effect of variation in outside convection coefficient for double and triple pane construction.

SCHEMATIC (Double Pane):



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat transfer, (3) Constant properties, (4) Negligible radiation effects, (5) Air between glass is stagnant.

PROPERTIES: *Table A-3*, Glass (300 K): k_g = 1.4 W/m·K; *Table A-4*, Air (T = 278 K): k_a = 0.0245 W/m·K.

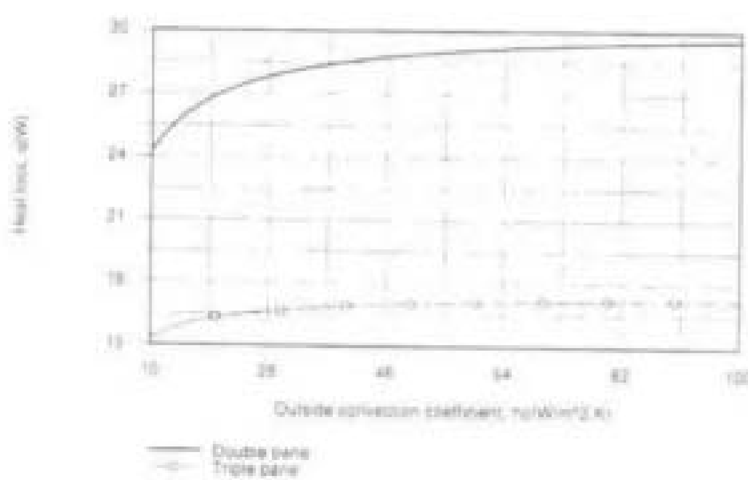
ANALYSIS: (a) From the thermal circuit, the heat loss is

$$q=\frac{T_{\infty,i}-T_{\infty,o}}{\frac{1}{A}\left(\frac{1}{h_i}+\frac{L}{k_g}+\frac{L}{k_a}+\frac{L}{k_g}+\frac{1}{h_o}\right)}$$

$$q=\frac{20^\circ C-(-10^\circ C)}{\left(\frac{1}{0.4\,m^2}\right)\left(\frac{1}{10\,W/m^2\cdot K}+\frac{0.007\,m}{1.4\,W/m\cdot K}+\frac{0.007\,m}{0.0245\,W/m\cdot K}+\frac{0.007\,m}{1.4\,W/m\cdot K}+\frac{1}{80\,W/m^2\cdot K}\right)}$$

$$q=\frac{30^\circ C}{(0.25+0.0125+0.715+0.0125+0.03125)K/W}=\frac{30^\circ C}{1.021\,K/W}=29.4\,W$$ <

(b) For the triple pane window, the additional pane and airspace increase the total resistance from 1.021 K/W to 1.749 K/W, thereby reducing the heat loss from 29.4 to 17.2 W. The effect of h_o on the heat loss is plotted as follows.



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PROBLEM 3.10 (Cont.)

Changes in h_o influence the heat loss at small values of h_o, for which the outside convection resistance is not negligible relative to the total resistance. However, the resistance becomes negligible with increasing h_o, particularly for the triple pane window, and changes in h_o have little effect on the heat loss.

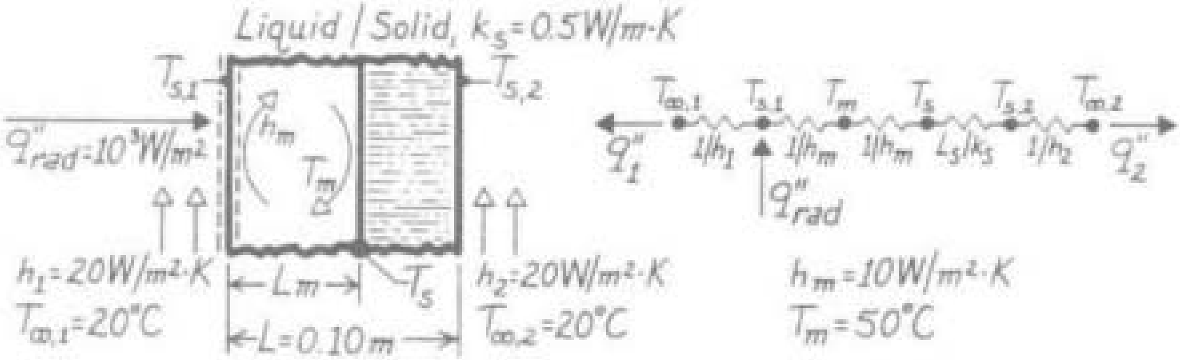
COMMENTS: The largest contribution to the thermal resistance is due to conduction across the enclosed air. Note that this air could be in motion due to free convection currents. If the corresponding convection coefficient exceeded 3.5 $W/m^2 \cdot K$, the thermal resistance would be less than that predicted by assuming conduction across stagnant air.

PROBLEM 3.11

KNOWN: Wall construction for passive solar collector. Net radiation flux to one surface. Ambient temperatures and convection coefficients for opposite surfaces. Melting point, liquid convection coefficient, and solid thermal conductivity of phase change material.

FIND: Melt region thickness and surface temperatures.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, steady-state heat transfer through the wall, (2) Vertical solid-liquid interface in the PCM, (3) Negligible conduction resistance in the supporting surfaces, (4) Constant k_s.

ANALYSIS: An energy balance at surface s,1 yields

$$q''_{rad} = q''_1 + q''_2 = h_1(T_{s,1} - T_{\infty,1}) + h_m(T_{s,1} - T_m)$$

$$T_{s,1} = \frac{q''_{rad} + h_1 T_{\infty,1} + h_m T_m}{(h_1 + h_m)} = \frac{(1000 + 20\times20 + 10\times50)\text{W/m}^2}{(20+10)\text{W/m}^2\cdot\text{K}} = 63.3°\text{C}. \quad \triangleleft$$

Hence, from

$$q''_2 = h_m(T_{s,1} - T_m) = 10\ \text{W/m}^2\cdot\text{K}\times13.3°\text{C} = 133\ \text{W/m}^2$$

$$q''_2 = \frac{T_m - T_{\infty,2}}{(1/h_m + (L_s/k_s) + (1/h_2)} = \frac{(50-20)°\text{C}}{(0.10 + 2.0L_s + 0.05)\text{m}^2\cdot\text{K/W}}$$

and

$$L_s = \frac{(30°\text{C}/133\ \text{W/m}^2) - 0.15\ \text{m}^2\cdot\text{K/W}}{2} = 0.0378\ \text{m}$$

$$L_m = L - L_s = 0.0622\ \text{m}. \quad \triangleleft$$

Also,

$$T_{s,2} = T_{\infty,2} + \frac{q''_2}{h_2} = 20°\text{C} + \frac{133\ \text{W/m}^2}{20\ \text{W/m}^2\cdot\text{K}} = 26.7°\text{C}. \quad \triangleleft$$

COMMENTS: (1) Note the low energy collection efficiency ($\eta = q''_2/q''_{rad} = 0.133$). The efficiency may be increased by increasing h_m and/or decreasing T_m.

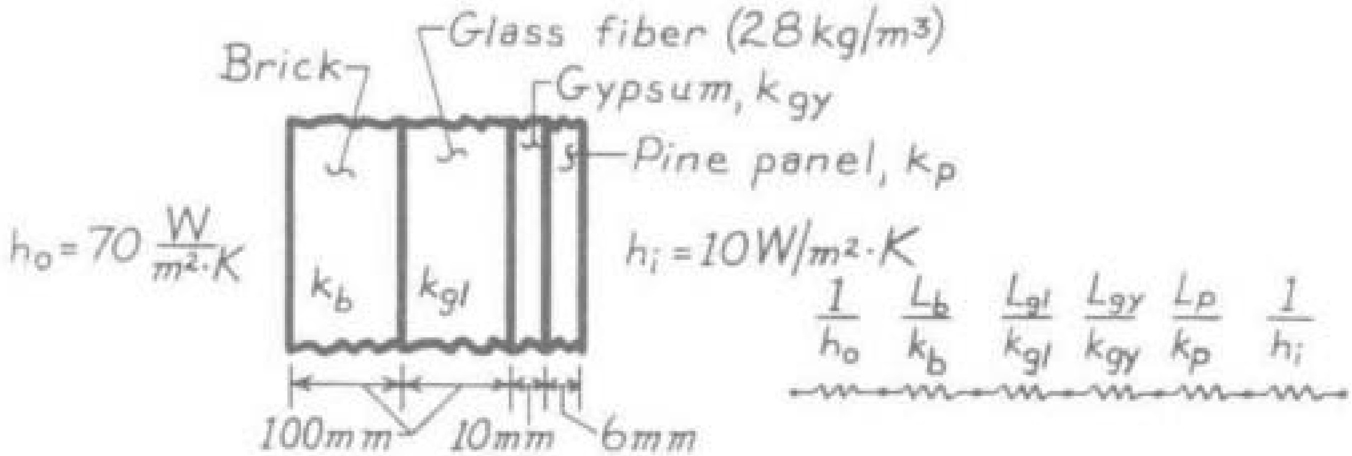
(2) The actual solid-liquid interface will not be vertical, but would slant downward to the left.

PROBLEM 3.12

KNOWN: Material thicknesses in a composite wall consisting of brick, glass fiber, vermiculite and pine panel. Inner and outer convection coefficients.

FIND: Total thermal resistance and overall heat transfer coefficient.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Negligible contact resistance.

PROPERTIES: *Table A-3*, T = 300K: Brick, $k_b = 1.3$ W/m·K; Glass fiber (28 kg/m^3), $k_{g1} = 0.038$ W/m·K; Gypsum, $k_{gy} = 0.17$ W/m·K; Pine panel, $k_p = 0.12$ W/m·K.

ANALYSIS: Considering a unit surface area, the total thermal resistance is

$$R''_{tot} = \frac{1}{h_o} + \frac{L_b}{k_b} + \frac{L_{g1}}{k_{g1}} + \frac{L_{gy}}{k_{gy}} + \frac{L_p}{k_p} + \frac{1}{h_i}$$

$$R''_{tot} = \left[\frac{1}{70} + \frac{0.1}{1.3} + \frac{0.1}{0.038} + \frac{0.01}{0.17} + \frac{0.006}{0.12} + \frac{1}{10}\right] \frac{m^2 \cdot K}{W}$$

$$R''_{tot} = (0.0143 + 0.0769 + 2.6316 + 0.0588 + 0.0500 + 0.1)\ m^2 \cdot K/W$$

$$R''_{tot} = 2.93\ m^2 \cdot K/W . \qquad \triangleleft$$

From Eq. 3.18 the overall heat transfer coefficient is

$$U = \frac{1}{R_{tot} A} = \frac{1}{R''_{tot}} = (2.93\ m^2 \cdot K/W)^{-1}$$

$$U = 0.341\ W/m^2 \cdot K . \qquad \triangleleft$$

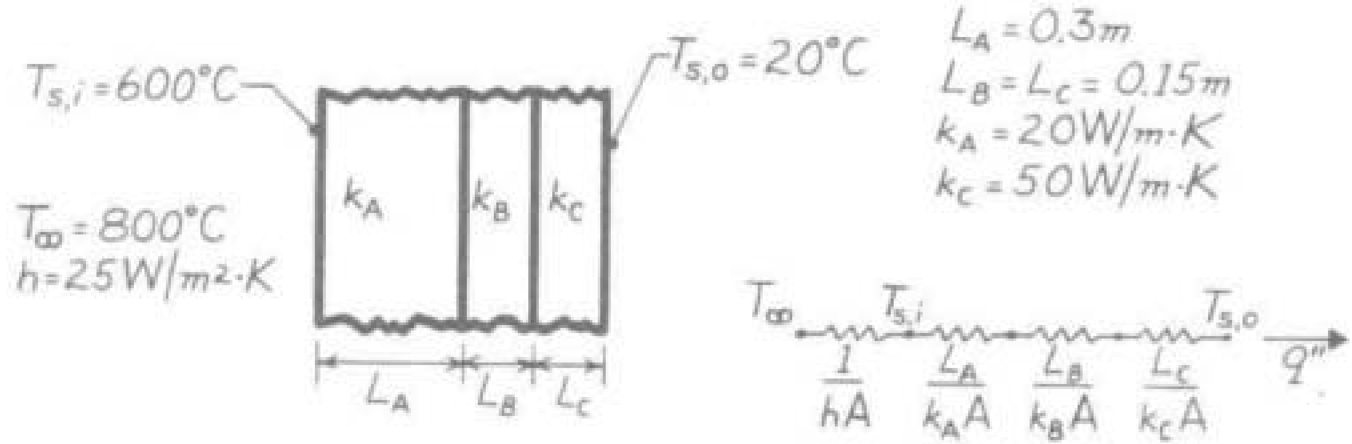
COMMENTS: As anticipated, the dominant contribution to the total resistance is made by the insulation.

PROBLEM 3.13

KNOWN: Thicknesses of three materials which form a composite wall and thermal conductivities of two of the materials. Inner and outer surface temperatures of the composite; also, temperature and convection coefficient associated with adjoining gas.

FIND: Value of unknown thermal conductivity, k_B.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) Constant properties, (4) Negligible contact resistance, (5) Negligible radiation effects.

ANALYSIS: Referring to the thermal circuit, the heat flux may be expressed as

$$q'' = \frac{T_{s,i}-T_{s,o}}{\frac{L_A}{k_A}+\frac{L_B}{k_B}+\frac{L_C}{k_C}} = \frac{(600-20)°C}{\frac{0.3\ m}{20\ W/m\cdot K}+\frac{0.15\ m}{k_B}+\frac{0.15\ m}{50\ W/m\cdot K}}$$

$$q'' = \frac{580}{0.018+0.15/k_B}\ W/m^2 \qquad (1)$$

The heat flux may be obtained from

$$q'' = h(T_\infty - T_{s,i}) = 25\ W/m^2\cdot K(800-600)°C \qquad (2)$$

$$q'' = 5000\ W/m^2 .$$

Substituting for the heat flux from Eq. (2) into Eq. (1), find

$$\frac{0.15}{k_B} = \frac{580}{q''} - 0.018 = \frac{580}{5000} - 0.018 = 0.098$$

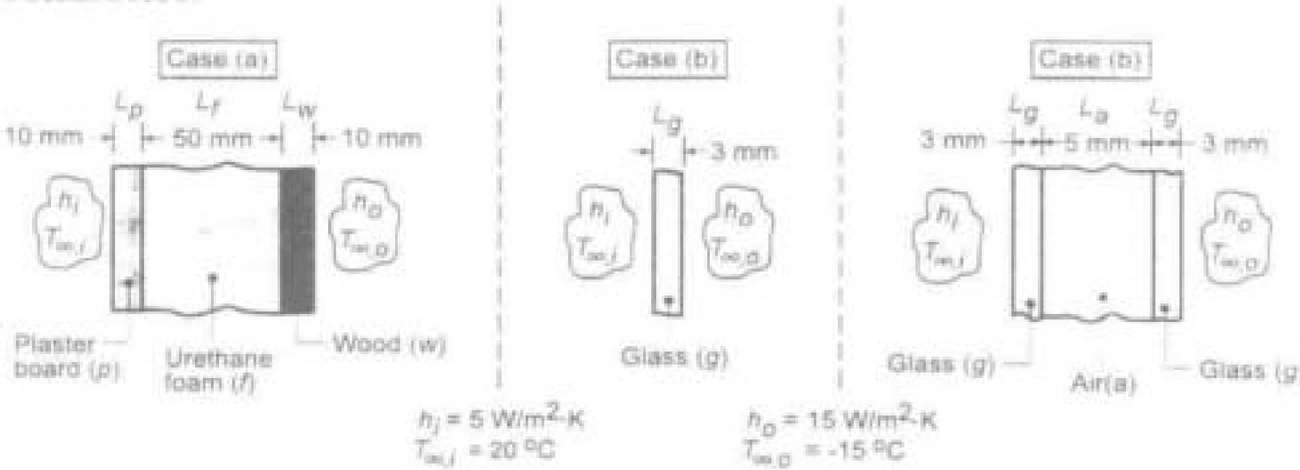
$$k_B = 1.53\ W/m\cdot K . \qquad \triangleleft$$

COMMENTS: Radiation effects are likely to have a significant influence on the net heat flux at the inner surface of the oven.

KNOWN: Configurations of exterior wall. Inner and outer surface conditions.

FIND: Heating load for each of the three cases.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) One-dimensional conduction, (3) Constant properties, (4) Negligible radiation effects.

PROPERTIES: (T = 300 K): *Table A.3*: plaster board, $k_p = 0.17$ W/m·K; urethane, $k_f = 0.026$ W/m·K; wood, $k_w = 0.12$ W/m·K; glass, $k_g = 1.4$ W/m·K. *Table A.4*: air, $k_a = 0.0263$ W/m·K.

ANALYSIS: (a) The heat loss may be obtained by dividing the overall temperature difference by the total thermal resistance. For the composite wall of unit surface area, $A = 1\ m^2$,

$$q = \frac{T_{\infty,i} - T_{\infty,o}}{\left[(1/h_i) + (L_p/k_p) + (L_f/k_f) + (L_w/k_w) + (1/h_o)\right]/A}$$

$$q = \frac{20^\circ C - (-15^\circ C)}{\left[(0.2 + 0.059 + 1.92 + 0.083 + 0.067)\,m^2 \cdot K/W\right]/1\,m^2}$$

$$q = \frac{35^\circ C}{2.33\,K/W} = 15.0\,W \qquad <$$

(b) For the single pane of glass,

$$q = \frac{T_{\infty,i} - T_{\infty,o}}{\left[(1/h_i) + (L_g/k_g) + (1/h_o)\right]/A}$$

$$q = \frac{35^\circ C}{\left[(0.2 + 0.002 + 0.067)\,m^2 \cdot K/W\right]/1\,m^2} = \frac{35^\circ C}{0.269\,K/W} = 130.3\,W \qquad <$$

(c) For the double pane window,

$$q = \frac{T_{\infty,i} - T_{\infty,o}}{\left[(1/h_i) + 2(L_g/k_g) + (L_a/k_a) + (1/h_o)\right]/A}$$

$$q = \frac{35^\circ C}{\left[(0.2 + 0.004 + 0.190 + 0.067)\,m^2 \cdot K/W\right]/1\,m^2} = \frac{35^\circ C}{0.461\,K/W} = 76.4\,W \qquad <$$

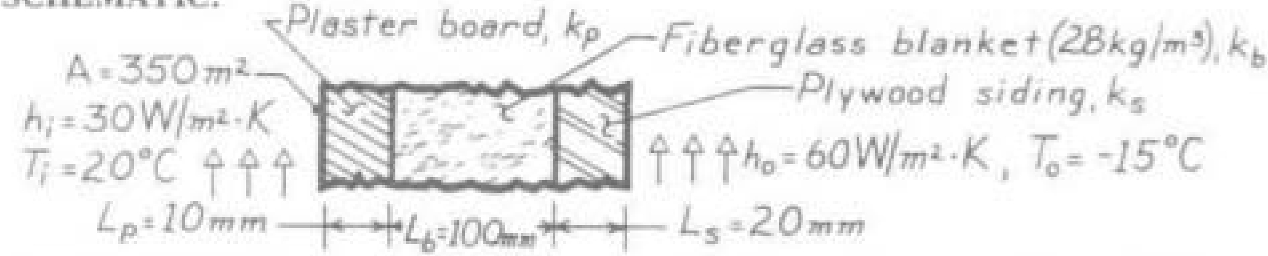
COMMENTS: The composite wall is clearly superior from the standpoint of reducing heat loss, and the dominant contribution to its total thermal resistance (82%) is associated with the foam insulation. Even with double pane construction, heat loss through the window is significantly larger than that for the composite wall.

PROBLEM 3.15

KNOWN: Composite wall of a house with prescribed convection processes at inner and outer surfaces.

FIND: (a) Expression for thermal resistance of house walls, R_{tot}; (b) Total heat loss, q(W); (c) Effect on heat loss due to increase in outside heat transfer convection coefficient, h_o; and (d) Controlling resistance for heat loss from house.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Steady-state conditions, (3) Negligible contact resistance.

PROPERTIES: *Table A-3*, ($\bar{T} = (T_i + T_o)/2 = (20\text{-}15)°C/2 = 2.5°C \approx 300K$): Fiberglass blanket, 28 kg/m³, k_b = 0.038 W/m·K; Plywood siding, k_s = 0.12 W/m·K; Plasterboard, k_p = 0.17 W/m·K.

ANALYSIS: (a) The expression for the total thermal resistance of the house walls follows from Eq. 3.18,

$$R_{tot} = \frac{1}{h_i A} + \frac{L_p}{k_p A} + \frac{L_b}{k_b A} + \frac{L_s}{k_s A} + \frac{1}{h_o A}. \qquad \triangleleft$$

(b) The total heat loss through the house walls is

$$q = \Delta T / R_{tot} = (T_i - T_o)/R_{tot}.$$

Substituting numerical values, find

$$R_{tot} = \frac{1}{30\,\text{W/m}^2\cdot\text{K}\times 350\,\text{m}^2} + \frac{0.01\,\text{m}}{0.17\,\text{W/m}\cdot\text{K}\times 350\,\text{m}^2} + \frac{0.10\,\text{m}}{0.038\,\text{W/m}\cdot\text{K}\times 350\,\text{m}^2}$$

$$+ \frac{0.02\,\text{m}}{0.12\,\text{W/m}\cdot\text{K}\times 350\,\text{m}^2} + \frac{1}{60\,\text{W/m}^2\cdot\text{K}\times 350\,\text{m}^2}$$

$$R_{tot} = [9.52 + 16.8 + 752 + 47.6 + 4.76]\times 10^{-5}\,°\text{C/W} = 831\times 10^{-5}\,°\text{C/W}.$$

The heat loss is then,

$$q = [20 - (-15)]°\text{C}/831\times 10^{-5}\,°\text{C/W} = 4.21\,\text{kW}. \qquad \triangleleft$$

(c) If h_o changes from 60 to 300 W/m²·K, then $R_o = 1/h_o A$ changes from 4.76×10^{-5} °C/W to 0.95×10^{-5} °C/W. This reduces R_{tot} to 826×10^{-5} °C/W which is ≈ 0.5% decrease in R_{tot} or ≈ 0.5% increase in q.

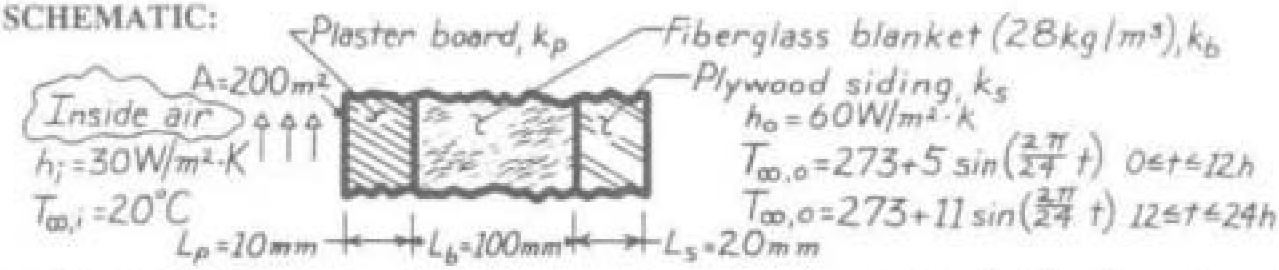
(d) From the R_{tot} numerical expression in part (b), note that the insulation resistance, $L_b/k_b A$, is 752/830 ≈ 90% of the total resistance. Hence, this material layer controls the resistance of the wall. From part (c) note that a 5-fold decrease in resistance due to wind velocity increase has a negligible effect on the heat loss.

PROBLEM 3.16

KNOWN: Composite wall of a house with prescribed convection processes at inner and outer surfaces.

FIND: Daily heat loss for prescribed diurnal variation in ambient air temperature.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, steady-state conduction (negligible change in wall thermal energy storage over 24h period), (2) Negligible contact resistance.

PROPERTIES: *Table A-3*, T = 300K Fiberglass: blanket (28 kg/m^3), k_b = 0.038 W/m·K; Plywood, k_s = 0.12 W/m·K; Plasterboard, k_p = 0.17 W/m·K.

ANALYSIS: The heat loss may be approximated as $Q = \int_0^{24h} \frac{T_{\infty,i} - T_{\infty,o}}{R_{tot}} dt$ where

$$R_{tot} = \frac{1}{A}\left[\frac{1}{h_i} + \frac{L_p}{k_p} + \frac{L_b}{k_b} + \frac{L_s}{k_s} + \frac{1}{h_o}\right]$$

$$R_{tot} = \frac{1}{200\text{m}^2}\left[\frac{1}{30\ \text{W/m}^2\cdot\text{K}} + \frac{0.01\text{m}}{0.17\ \text{W/m}\cdot\text{K}} + \frac{0.1\text{m}}{0.038\ \text{W/m}\cdot\text{K}} + \frac{0.02\text{m}}{0.12\ \text{W/m}\cdot\text{K}} + \frac{1}{60\ \text{W/m}^2\cdot\text{K}}\right]$$

$$R_{tot} = 0.01454\ \text{K/W}.$$

Hence the heat rate is

$$Q = \frac{1}{R_{tot}}\left\{\int_0^{12h}\left[293 - \left[273 + 5\sin\frac{2\pi}{24}t\right]\right]dt + \int_{12}^{24h}\left[293 - \left[273 + 11\sin\frac{2\pi}{24}t\right]\right]dt\right\}$$

$$Q = 68.8\frac{\text{W}}{\text{K}}\left\{\left[20t + 5\left[\frac{24}{2\pi}\right]\cos\frac{2\pi t}{24}\right]\Bigg|_0^{12} + \left[20t + 11\left[\frac{24}{2\pi}\right]\cos\frac{2\pi t}{24}\right]\Bigg|_{12}^{24}\right\}\text{K}\cdot\text{h}$$

$$Q = 68.8\left\{\left[240 + \frac{60}{\pi}(-1-1)\right] + \left[480 - 240 + \frac{132}{\pi}(1+1)\right]\right\}\text{W}\cdot\text{h}$$

$$Q = 68.8\ \{480 - 38.2 + 84.03\}\ \text{W}\cdot\text{h}$$

$$Q = 36.18\ \text{kW}\cdot\text{h} = 1.302\times10^8\ \text{J}.$$ ◁

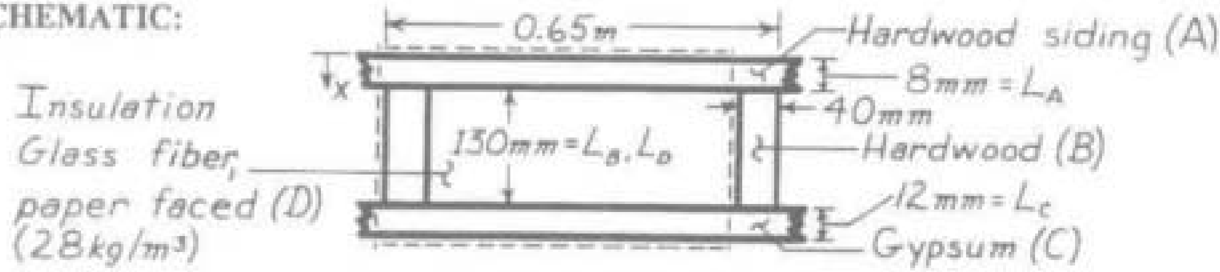
COMMENTS: From knowledge of the fuel cost, the total daily heating bill could be determined. For example, at a cost of 0.10$/kWcdoth, the heating bill would be $3.62/day.

PROBLEM 3.17

KNOWN: Dimensions and materials associated with a composite wall (2.5m × 6.5m, 10 studs each 2.5m high).

FIND: Wall thermal resistance.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Temperature of composite depends only on x (surfaces normal to x are isothermal), (3) Constant properties, (4) Negligible contact resistance.

PROPERTIES: *Table A-3* (T = 300K): Hardwood siding, $k_A = 0.094$ W/m·K; Hardwood, $k_B = 0.16$ W/m·K; Gypsum, $k_C = 0.17$ W/m·K; Insulation (glass fiber paper faced, 28 kg/m³), $k_D = 0.038$ W/m·K.

ANALYSIS: Using the isothermal surface assumption, the thermal circuit associated with a single unit (enclosed by dashed lines) of the wall is

$$(L_A/k_AA_A) = \frac{0.008\text{m}}{0.094\ \text{W/m·K}\ (0.65\text{m}\times 2.5\text{m})} = 0.0524\ \text{K/W}$$

$$(L_B/k_BA_B) = \frac{0.13\text{m}}{0.16\ \text{W/m·K}\ (0.04\text{m}\times 2.5\text{m})} = 8.125\ \text{K/W}$$

$$(L_D/k_DA_D) = \frac{0.13\text{m}}{0.038\ \text{W/m·K}\ (0.61\text{m}\times 2.5\text{m})} = 2.243\ \text{K/W}$$

$$(L_C/k_CA_C) = \frac{0.012\text{m}}{0.17\ \text{W/m·K}\ (0.65\text{m}\times 2.5\text{m})} = 0.0434\ \text{K/W}.$$

The equivalent resistance of the core is

$$R_{eq} = (1/R_B + 1/R_D)^{-1} = (1/8.125 + 1/2.243)^{-1} = 1.758\ \text{K/W}$$

and the total unit resistance is

$$R_{tot,1} = R_A + R_{eq} + R_C = 1.854\ \text{K/W}.$$

With 10 such units in parallel, the total wall resistance is

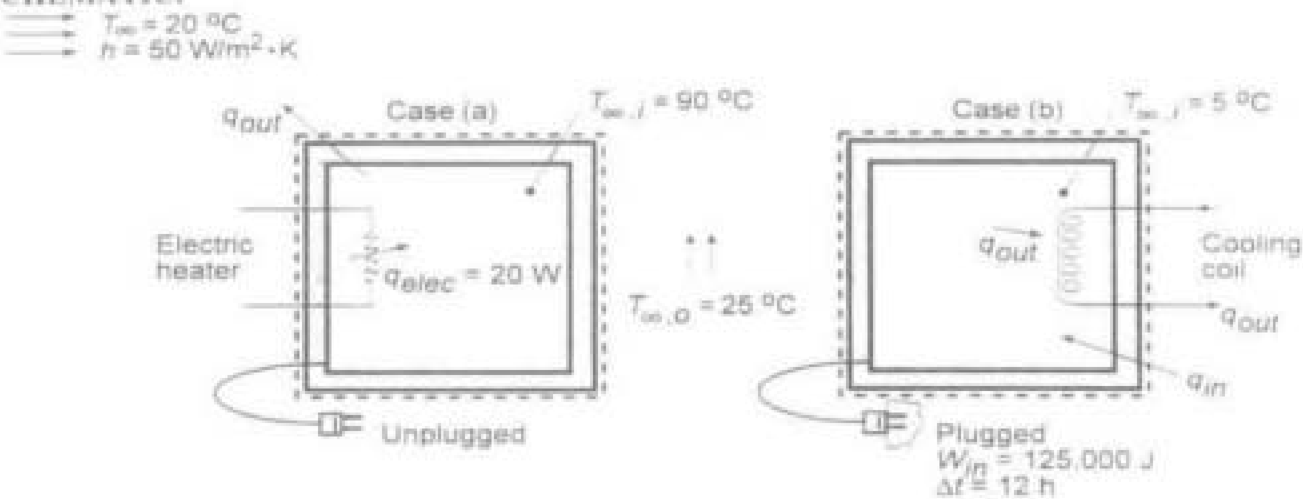
$$R_{tot} = (10\times 1/R_{tot,1})^{-1} = 0.1854\ \text{K/W}.$$ ◁

COMMENTS: If surfaces parallel to the heat flow direction are assumed adiabatic the thermal circuit and the value of R_{tot} will differ.

KNOWN: conditions associated with maintaining heated and cooled conditions within a refrigerator compartment.

FIND: Coefficient of performance (COP).

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state operating conditions. (2) Negligible radiation. (3) Compartment completely sealed from ambient air.

ANALYSIS: The Case (a) experiment is performed to determine the overall thermal resistance to heat transfer between the interior of the refrigerator and the ambient air. Applying an energy balance to a control surface about the refrigerator, it follows from Eq. 1.11a that, at any instant,

$$\dot{E}_g - \dot{E}_{out} = 0$$

Hence,

$$q_{elec} - q_{out} = 0$$

where $q_{out} = (T_{\infty,i} - T_{\infty,o})/R_t$. It follows that

$$R_t = \frac{T_{\infty,i} - T_{\infty,o}}{q_{elec}} = \frac{(90-25)^\circ C}{20\ W} = 3.25^\circ C/W$$

For Case (b), heat transfer from the ambient air to the compartment (the heat load) is balanced by heat transfer to the refrigerant ($q_{in} = q_{out}$). Hence, the thermal energy transferred from the refrigerator over the 12 hour period is

$$Q_{out} = q_{out}\Delta t = q_{in}\Delta t = \frac{T_{\infty,i} - T_{\infty,o}}{R_t}\Delta t$$

$$Q_{out} = \frac{(25-5)^\circ C}{3.25^\circ\ C/W}(12\ h \times 3600\ s/h) = 266{,}000\ J$$

The coefficient of performance (COP) is therefore

$$COP = \frac{Q_{out}}{W_{in}} = \frac{266{,}000}{125{,}000} = 2.13$$ <

COMMENTS: The ideal (Carnot) COP is

$$COP)_{ideal} = \frac{T_c}{T_h - T_c} = \frac{278\ K}{(298-278)K} = 13.9$$

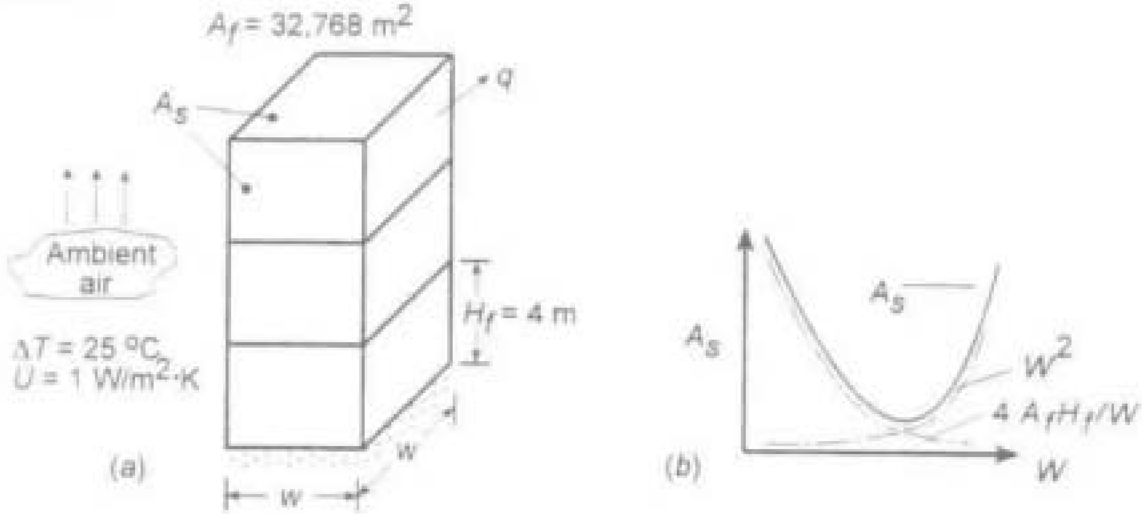
and the system is operating well below its peak possible performance.

PROBLEM 3.19

KNOWN: Total floor space and vertical distance between floors for a square, flat roof building.

FIND: (a) Expression for width of building which minimizes heat loss, (b) Width and number of floors which minimize heat loss for a prescribed floor space and distance between floors. Corresponding heat loss, percent heat loss reduction from 2 floors.

SCHEMATIC:



ASSUMPTIONS: Negligible heat loss to ground.

ANALYSIS: (a) To minimize the heat loss q, the exterior surface area, A_s, must be minimized. From Fig. (a)

$$A_s = W^2 + 4WH = W^2 + 4WN_f H_f$$

where

$$N_f = A_f / W^2$$

Hence,

$$A_s = W^2 + 4WA_f H_f / W^2 = W^2 + 4A_f H_f / W$$

The optimum value of W corresponds to

$$\frac{dA_s}{dW} = 2W - \frac{4A_f H_f}{W^2} = 0$$

or

$$W_{op} = (2A_f H_f)^{1/3} \qquad <$$

The competing effects of W on the areas of the roof and sidewalls, and hence the basis for an optimum, is shown schematically in Fig. (b).

(b) For $A_f = 32,768\ m^2$ and $H_f = 4$ m,

$$W_{op} = (2 \times 32,768\,m^2 \times 4\,m)^{1/3} = 64\,m \qquad <$$

Hence,

$$N_f = \frac{A_f}{W^2} = \frac{32,768\,m^2}{(64\,m)^2} = 8 \qquad <$$

and

$$q = UA_s\Delta T = 1\,W/m^2\cdot K\left[(64\,m)^2 + \frac{4 \times 32,768\,m^2 \times 4\,m}{64\,m}\right]25^\circ C = 307,200\,W \qquad <$$

Continued...

PROBLEM 3.19 (Cont.)

For $N_t = 2$,

$$W = (A_t/N_t)^{1/2} = (32{,}768\ \text{m}^2/2)^{1/2} = 128\ \text{m}$$

$$q = 1\,\text{W/m}^2 \cdot \text{K}\left[(128\,\text{m})^2 + \frac{4 \times 32{,}768\,\text{m}^2 \times 4\,\text{m}}{128\,\text{m}}\right]25^\circ\text{C} = 512{,}000\ \text{W}$$

% reduction in q = (512,000 - 307,200)/512,000 = 40% <

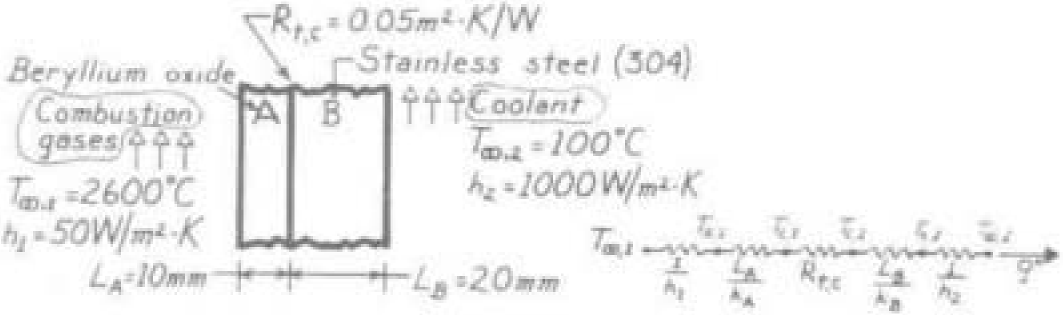
COMMENTS: Even the minimum heat loss is excessive and could be reduced by reducing U.

PROBLEM 3.20

KNOWN: Materials and dimensions of a composite wall separating a combustion gas from a liquid coolant.

FIND: (a) Heat loss per unit area, and (b) Temperature distribution.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional heat transfer, (2) Steady-state conditions, (3) Constant properties, (4) Negligible radiation effects.

PROPERTIES: *Table A-1*, St. St. (304) ($\bar{T}$ = 1000K): k = 25.4 W/m·K; *Table A-2*, Beryllium Oxide (T = 1500K): k = 21.5 W/m·K.

ANALYSIS: (a) The desired heat flux may be expressed as

$$q'' = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1} + \frac{L_A}{k_A} + R_{t,c} + \frac{L_B}{k_B} + \frac{1}{h_2}} = \frac{(2600-100)°C}{\left[\frac{1}{50} + \frac{0.01}{21.5} + 0.05 + \frac{0.02}{25.4} + \frac{1}{1000}\right]\frac{m^2 \cdot K}{W}}$$

$$q'' = 34{,}600\ W/m^2 .$$ ◁

(b) The composite surface temperatures may be obtained by applying appropriate rate equations. From the fact that $q'' = h_1 (T_{\infty,1} - T_{s,1})$, it follows that

$$T_{s,1} = T_{\infty,1} - \frac{q''}{h_1} = 2600°C - \frac{34{,}600\ W/m^2}{50\ W/m^2 \cdot K} = 1908°C .$$

With $q'' = (k_A/L_A)(T_{s,1} - T_{c,1})$, it also follows that

$$T_{c,1} = T_{s,1} - \frac{L_A q''}{k_A} = 1908°C - \frac{0.01m \times 34{,}600\ W/m^2}{21.5\ W/m \cdot K} = 1892°C .$$

Similarly, with $q'' = (T_{c,1} - T_{c,2})/R_{t,c}$,

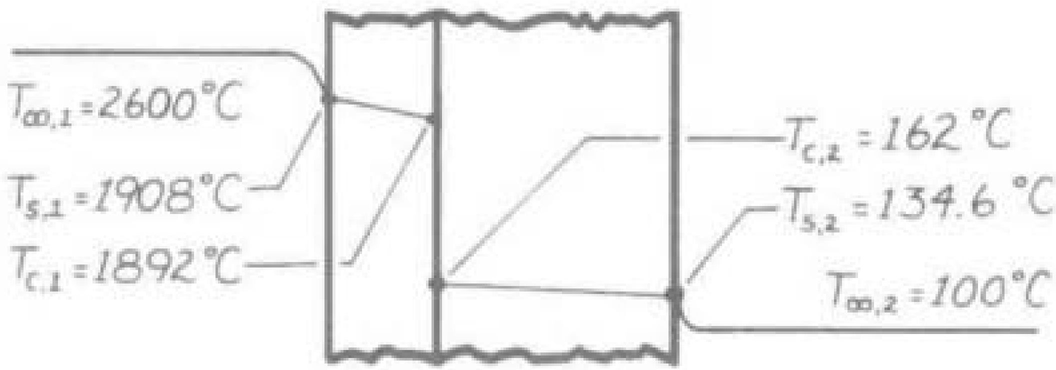
$$T_{c,2} = T_{c,1} - R_{t,c}\, q'' = 1892°C - 0.05\frac{m^2 \cdot K}{W} \times 34{,}600\frac{W}{m^2} = 162°C$$

Continued .....

and with $q'' = (k_B/L_B)(T_{c,2} - T_{s,2})$,

$$T_{s,2} = T_{c,2} - \frac{L_B q''}{k_B} = 162°C - \frac{0.02m \times 34,600 \text{ W/m}^2}{25.4 \text{ W/m·K}} = 134.6°C.$$

The temperature distribution is therefore of the following form:



◁

COMMENTS: (1) The calculations may be checked by recomputing q'' from

$$q'' = h_2(T_{s,2} - T_{\infty,2}) = 1000\text{W/m}^2\cdot\text{K}(134.6-100)°C = 34,600\text{W/m}^2$$

(2) The initial *estimates* of the mean material temperatures are in error, particularly for the stainless steel. For improved accuracy the calculations should be repeated using k values corresponding to T = 1900°C for the oxide and T = 115°C for the steel.

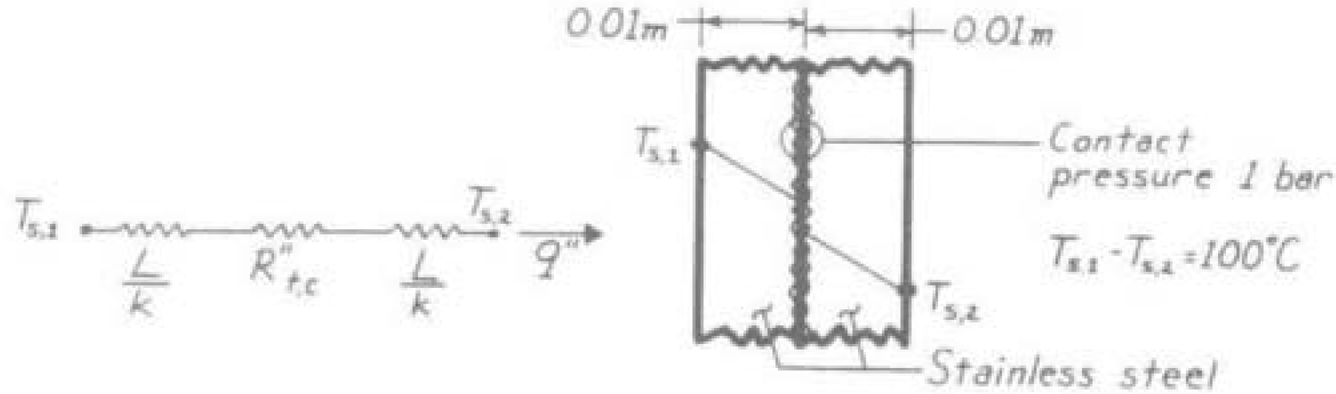
(3) The major contributions to the total resistance are made by the combustion gas boundary layer and the contact, where the temperature drops are largest.

PROBLEM 3.21

KNOWN: Thickness, overall temperature difference, and pressure for two stainless steel plates.

FIND: (a) Heat flux and (b) Contact plane temperature drop.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional heat transfer, (2) Steady-state conditions, (3) Constant properties.

PROPERTIES: *Table A-1*, Stainless Steel (T = 400K): k = 16.6 W/m·K.

ANALYSIS: (a) With $R''_{t,c} = 15\times10^{-4}\ m^2\cdot K/W$ from Table 3.1 and

$$\frac{L}{k} = \frac{0.01m}{16.6\ W/m\cdot K} = 6.02\times10^{-4}\ m^2\cdot K/W\ ,$$

it follows that

$$R''_{tot} = 2(L/k) + R''_{t,c} = 27\times10^{-4}\ m^2\cdot K/W\ ;$$

hence

$$q'' = \frac{\Delta T}{R''_{tot}} = \frac{100°C}{27\times10^{-4}\ m^2\cdot K/W} = 3.70\times10^4\ W/m^2\ . \quad \triangleleft$$

(b) From the thermal circuit,

$$\frac{\Delta T_c}{T_{s,1}-T_{s,2}} = \frac{R''_{t,c}}{R''_{tot}} = \frac{15\times10^{-4}\ m^2\cdot K/W}{27\times10^{-4}\ m^2\cdot K/W} = 0.556.$$

Hence,

$$\Delta T_c = 0.556(T_{s,1}-T_{s,2}) = 0.556(100°C) = 55.6°C\ . \quad \triangleleft$$

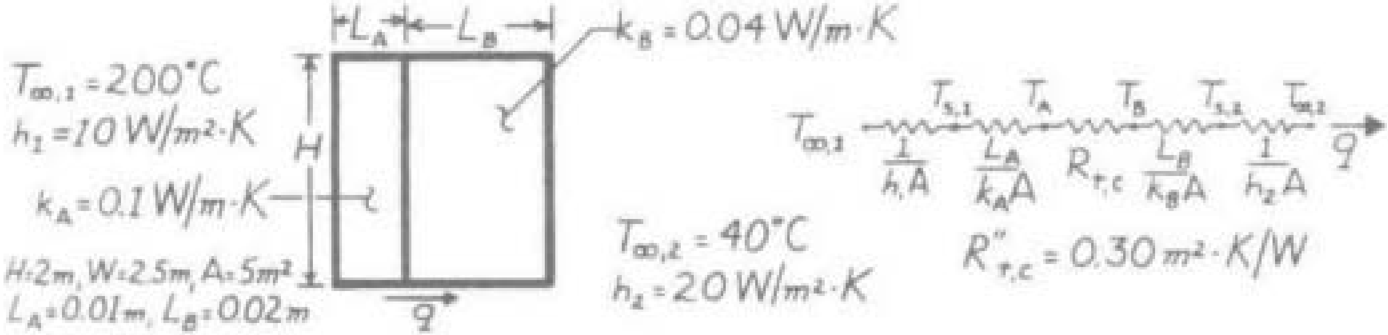
COMMENTS: The contact resistance is significant relative to the conduction resistances. The value of $R''_{t,c}$ would diminish, however, with increasing pressure.

PROBLEM 3.22

KNOWN: Temperatures and convection coefficients associated with fluids at inner and outer surfaces of a composite wall. Contact resistance, dimensions, and thermal conductivities associated with wall materials.

FIND: (a) Rate of heat transfer through the wall, (b) Temperature distribution.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat transfer, (3) Negligible radiation, (4) Constant properties.

ANALYSIS: (a) Calculate the total resistance to find the heat rate,

$$R_{tot} = \frac{1}{h_1A} + \frac{L_A}{k_AA} + R_{t,c} + \frac{L_B}{k_BA} + \frac{1}{h_2A}$$

$$R_{tot} = \left[\frac{1}{10\times5} + \frac{0.01}{0.1\times5} + \frac{0.3}{5} + \frac{0.02}{0.04\times5} + \frac{1}{20\times5}\right]\frac{K}{W}$$

$$R_{tot} = [0.02 + 0.02 + 0.06 + 0.10 + 0.01]\frac{K}{W} = 0.21\frac{K}{W}$$

$$q = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} = \frac{(200-40)°C}{0.21\ K/W} = 762\ W. \qquad \triangleleft$$

(b) It follows that

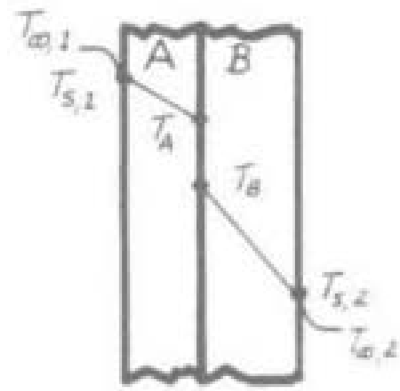
$$T_{s,1} = T_{\infty,1} - \frac{q}{h_1A} = 200°C - \frac{762\ W}{50\ W/K} = 184.8°C$$

$$T_A = T_{s,1} - \frac{qL_A}{k_AA} = 184.8°C - \frac{762W\times0.01m}{0.1\frac{W}{m\cdot K}\times5m^2} = 169.6°C$$

$$T_B = T_A - qR_{t,c} = 169.6°C - 762W\times0.06\frac{K}{W} = 123.8°C$$

$$T_{s,2} = T_B - \frac{qL_B}{k_BA} = 123.8°C - \frac{762W\times0.02m}{0.04\frac{W}{m\cdot K}\times5m^2} = 47.6°C$$

$$T_{\infty,2} = T_{s,2} - \frac{q}{h_2A} = 47.6°C - \frac{762W}{100W/K} = 40°C$$

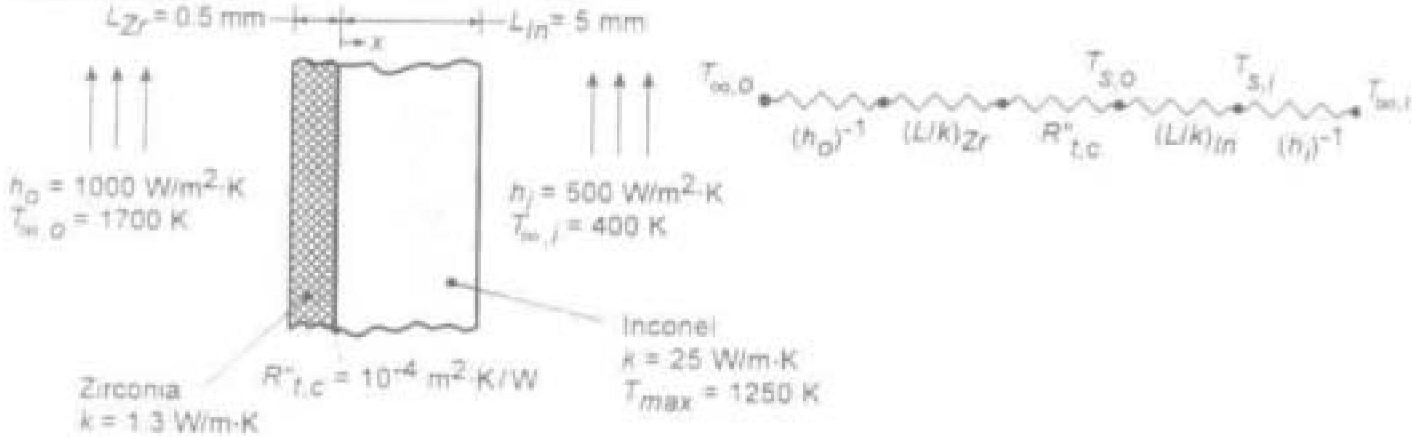


PROBLEM 3.23

KNOWN: Outer and inner surface convection conditions associated with zirconia-coated, Inconel turbine blade. Thicknesses, thermal conductivities, and interfacial resistance of the blade materials. Maximum allowable temperature of Inconel.

FIND: Whether blade operates below maximum temperature. Temperature distribution in blade, with and without the TBC.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, steady-state conduction in a composite plane wall, (2) Constant properties, (3) Negligible radiation.

ANALYSIS: For a unit area, the total thermal resistance with the TBC is

$$R''_{tot,w} = h_o^{-1} + (L/k)_{Zr} + R''_{t,c} + (L/k)_{In} + h_i^{-1}$$

$$R''_{tot,w} = (10^{-3} + 3.85\times10^{-4} + 10^{-4} + 2\times10^{-4} + 2\times10^{-3})\,m^2\cdot K/W = 3.69\times10^{-3}\,m^2\cdot K/W$$

With a heat flux of

$$q''_w = \frac{T_{\infty,o} - T_{\infty,i}}{R''_{tot,w}} = \frac{1300\,K}{3.69\times10^{-3}\,m^2\cdot K/W} = 3.52\times10^5\,W/m^2$$

the inner and outer surface temperatures of the Inconel are

$$T_{s,i(w)} = T_{\infty,i} + (q''_w/h_i) = 400\,K + (3.52\times10^5\,W/m^2/500\,W/m^2\cdot K) = 1104\,K$$

$$T_{s,o(w)} = T_{\infty,i} + \left[(1/h_i) + (L/k)_{In}\right]q''_w = 400\,K + (2\times10^{-3} + 2\times10^{-4})\,m^2\cdot K/W\,(3.52\times10^5\,W/m^2) = 1174\,K$$

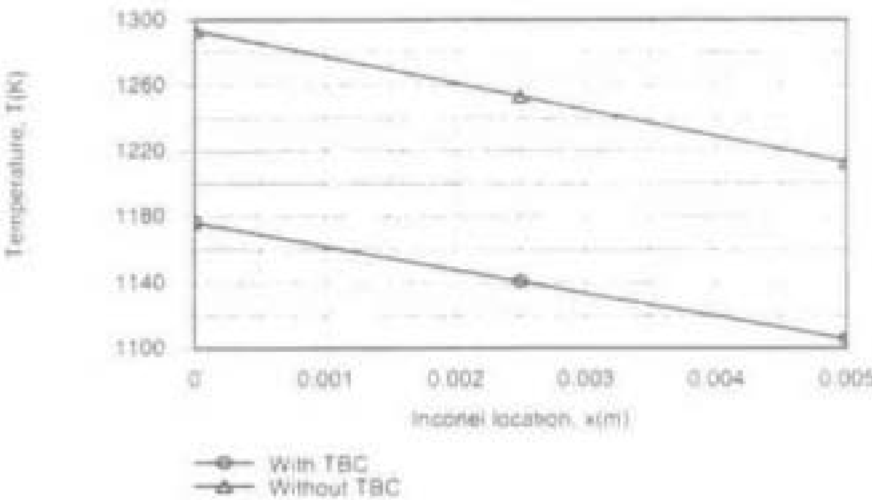
Without the TBC, $R''_{tot,wo} = h_o^{-1} + (L/k)_{In} + h_i^{-1} = 3.20\times10^{-3}\,m^2\cdot K/W$, and $q''_{wo} = (T_{\infty,o} - T_{\infty,i})/R''_{tot,wo}$ = (1300 K)/3.20×10⁻³ m²·K/W = 4.06×10⁵ W/m². The inner and outer surface temperatures of the Inconel are then

$$T_{s,i(wo)} = T_{\infty,i} + (q''_{wo}/h_i) = 400\,K + (4.06\times10^5\,W/m^2/500\,W/m^2\cdot K) = 1212\,K$$

$$T_{s,o(wo)} = T_{\infty,i} + \left[(1/h_i) + (L/k)_{In}\right]q''_{wo} = 400\,K + (2\times10^{-3} + 2\times10^{-4})\,m^2\cdot K/W\,(4.06\times10^5\,W/m^2) = 1293\,K$$

Continued...

PROBLEM 3.23 (Cont.)



Use of the TBC facilitates operation of the Inconel below $T_{max} = 1250$ K.

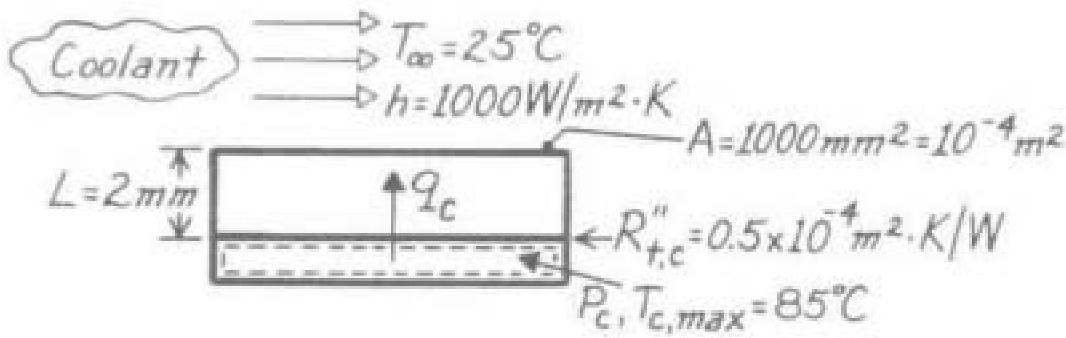
COMMENTS: Since the durability of the TBC decreases with increasing temperature, which increases with increasing thickness, limits to the thickness are associated with reliability considerations.

PROBLEM 3.24

KNOWN: Surface area and maximum temperature of a chip. Thickness of aluminum cover and chip/cover contact resistance. Fluid convection conditions.

FIND: Maximum chip power.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat transfer, (3) Negligible heat loss from sides and bottom, (4) Chip is isothermal.

PROPERTIES: *Table A.1*, Aluminum (T = 325 K): k = 238 W/m·K.

ANALYSIS: For a control surface about the chip, conservation of energy yields

$$\dot{E}_g - \dot{E}_{out} = 0$$

or

$$P_c - \frac{(T_c - T_\infty)A}{[(L/k) + R''_{t,c} + (1/h)]} = 0$$

$$P_{c,max} = \frac{(85-25)°C(10^{-4} m^2)}{[(0.002/238) + 0.5\times10^{-4} + (1/1000)]m^2 \cdot K/W}$$

$$P_{c,max} = \frac{60\times10^{-4} °C \cdot m^2}{(8.4\times10^{-6} + 0.5\times10^{-4} + 10^{-3})m^2 \cdot K/W}$$

$$P_{c,max} = 5.7 \text{ W}. \quad \triangleleft$$

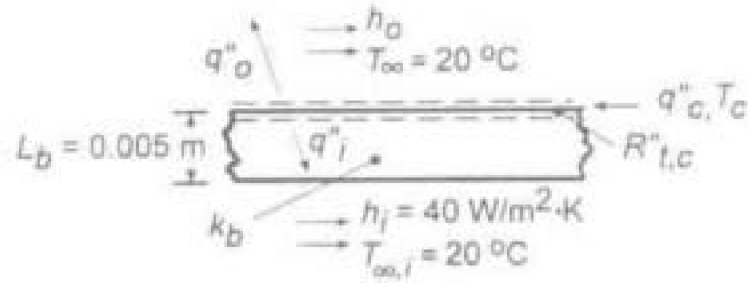
COMMENTS: The dominant resistance is that due to convection ($R_{conv} > R_{t,c} \gg R_{cond}$).

PROBLEM 3.25

KNOWN: Operating conditions for a board mounted chip.

FIND: (a) Equivalent thermal circuit, (b) Chip temperature, (c) Maximum allowable heat dissipation for dielectric liquid ($h_o = 1000$ W/m^2·K) and air ($h_o = 100$ W/m^2·K). Effect of changes in circuit board temperature and contact resistance.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) Negligible chip thermal resistance, (4) Negligible radiation, (5) Constant properties.

PROPERTIES: *Table A-3*, Aluminum oxide (polycrystalline, 358 K): $k_b = 32.4$ W/m·K.

ANALYSIS: (a)

(b) Applying conservation of energy to a control surface about the chip $\left(\dot{E}_{in} - \dot{E}_{out} = 0\right)$,

$$q''_c - q''_i - q''_o = 0$$

$$q''_c = \frac{T_c - T_{\infty,i}}{1/h_i + (L/k)_b + R''_{t,c}} + \frac{T_c - T_{\infty,o}}{1/h_o}$$

With $q''_c = 3\times 10^4$ W/m^2, $h_o = 1000$ W/m^2·K, $k_b = 1$ W/m·K and $R''_{t,c} = 10^{-4}$ m^2·K/W,

$$3\times 10^4\ \text{W/m}^2 = \frac{T_c - 20^\circ\text{C}}{(1/40 + 0.005/1 + 10^{-4})\text{m}^2\cdot\text{K/W}} + \frac{T_c - 20^\circ\text{C}}{(1/1000)\text{m}^2\cdot\text{K/W}}$$

$$3\times 10^4\ \text{W/m}^2 = (33.2T_c - 664 + 1000T_c - 20{,}000)\ \text{W/m}^2\cdot\text{K}$$

$$1003T_c = 50{,}664$$

$$T_c = 49^\circ\text{C}.$$ <

(c) For $T_c = 85^\circ$C and $h_o = 1000$ W/m^2·K, the foregoing energy balance yields

$$q''_c = 67{,}160\ \text{W/m}^2$$ <

with $q''_o = 65{,}000$ W/m^2 and $q''_i = 2160$ W/m^2. Replacing the dielectric with air ($h_o = 100$ W/m^2·K), the following results are obtained for different combinations of k_b and $R''_{t,c}$.

Continued...

k_b (W/m·K)	$R''_{t,c}$ (m²·K/W)	q''_i (W/m²)	q''_o (W/m²)	q''_c (W/m²)
1	10^{-4}	2159	6500	8659
32.4	10^{-4}	2574	6500	9074
1	10^{-5}	2166	6500	8666
32.4	10^{-5}	2583	6500	9083

COMMENTS: 1. For the conditions of part (b), the total internal resistance is 0.0301 m²·K/W, while the outer resistance is 0.001 m²·K/W. Hence

$$\frac{q''_o}{q''_i} = \frac{(T_c - T_{\infty,o})/R''_o}{(T_c - T_{\infty,i})/R''_i} = \frac{0.0301}{0.001} = 30.$$

and only approximately 3% of the heat is dissipated through the board.

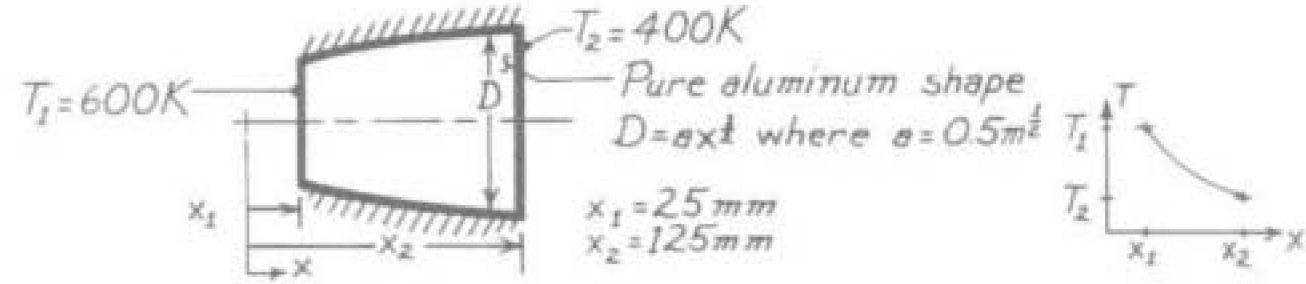
2. With h_o = 100 W/m²·K, the outer resistance increases to 0.01 m²·K/W, in which case $q''_o/q''_i = R''_i/R''_o$ = 0.0301/0.01 = 3.1 and now almost 25% of the heat is dissipated through the board. Hence, although measures to reduce R''_i would have a negligible effect on q''_c for the liquid coolant, some improvement may be gained for air-cooled conditions. As shown in the table of part (b), use of an aluminum oxide board increase q''_i by 19% (from 2159 to 2574 W/m²) by reducing R''_i from 0.0301 to 0.0253 m²·K/W. Because the initial contact resistance ($R''_{t,c} = 10^{-4}\,m^2 \cdot K/W$) is already much less than R''_i, any reduction in its value would have a negligible effect on q''_i. The largest gain would be realized by increasing h_i, since the inside convection resistance makes the dominant contribution to the total internal resistance.

PROBLEM 3.26

KNOWN: Conduction in a conical section with prescribed diameter, D, as a function of x in the form $D = ax^{1/2}$.

FIND: (a) Temperature distribution, T(x), (b) Heat transfer rate, q_x.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in x-direction, (3) No internal heat generation, (4) Constant properties.

PROPERTIES: *Table A-2*, Pure Aluminum (500K): k = 236 W/m·K.

ANALYSIS: (a) Based upon the assumptions, and following the same methodology of Example 3.3, q_x is a constant independent of x. Accordingly,

$$q_x = -kA\frac{dT}{dx} = -k[\pi(ax^{1/2})^2/4]\frac{dT}{dx} \tag{1}$$

using $A = \pi D^2/4$ where $D = ax^{1/2}$. Separating variables and identifying limits,

$$\frac{4q_x}{\pi a^2 k}\int_{x_1}^{x}\frac{dx}{x} = -\int_{T_1}^{T} dT. \tag{2}$$

Integrating and solving for T(x) and then for T_2,

$$T(x) = T_1 - \frac{4q_x}{\pi a^2 k}\ln\frac{x}{x_1} \qquad T_2 = T_1 - \frac{4q_x}{\pi a^2 k}\ln\frac{x_2}{x_1}. \tag{3,4}$$

Solving Eq. (4) for q_x and then substituting into Eq. (3) gives the results,

$$q_x = -\frac{\pi}{4}a^2 k\,(T_1 - T_2)/\ln(x_1/x_2) \tag{5}$$

$$T(x) = T_1 + (T_1 - T_2)\frac{\ln(x/x_1)}{\ln(x_1/x_2)}. \qquad \triangleleft$$

From Eq. (1) note that $(dT/dx)\cdot x$ = Constant. It follows that T(x) has the distribution shown above.

(b) The heat rate follows from Eq. (5),

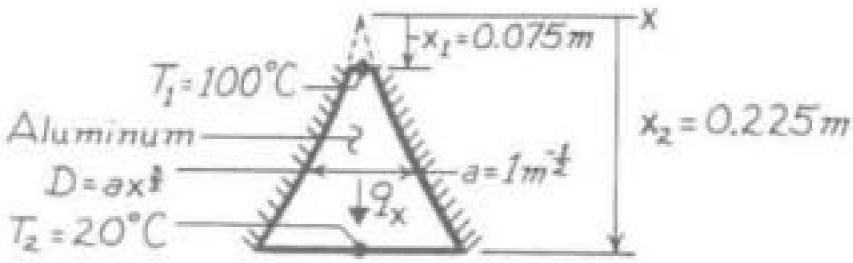
$$q_x = \frac{\pi}{4}\times 0.5^2\,m\times 236\frac{W}{m\cdot K}(600-400)K/\ln\frac{25}{125} = 5.76kW. \qquad \triangleleft$$

PROBLEM 3.27

KNOWN: Geometry and surface conditions of a truncated solid cone.

FIND: (a) Temperature distribution, (b) Rate of heat transfer across the cone.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in x, (3) Constant properties.

PROPERTIES: *Table A-1*, Aluminum (333K): k = 238 W/m·K.

ANALYSIS: (a) From Fourier's law, Eq. (2.1), with $A = \pi D^2/4 = (\pi a^2/4)x^3$, it follows that

$$\frac{4q_x dx}{\pi a^2 x^3} = -kdT.$$

Hence, since q_x is independent of x,

$$\frac{4q_x}{\pi a^2}\int_{x_1}^{x}\frac{dx}{x^3} = -k\int_{T_1}^{T} dT$$

or

$$\frac{4q_x}{\pi a^2}\left[-\frac{1}{2x^2}\right]\Bigg|_{x_1}^{x} = -k(T-T_1).$$

Hence

$$T = T_1 + \frac{2q_x}{\pi a^2 k}\left[\frac{1}{x^2} - \frac{1}{x_1^2}\right]. \qquad \triangleleft$$

(b) From the foregoing expression, it also follows that

$$q_x = \frac{\pi a^2 k}{2}\frac{T_2 - T_1}{[1/x_2^2 - 1/x_1^2]}$$

$$q_x = \frac{\pi(1\,m^{-1})238\ W/m\cdot K}{2} \times \frac{(20-100)°C}{[(0.225)^{-2}-(0.075)^{-2}]m^{-2}}$$

$$q_x = 189\ W. \qquad \triangleleft$$

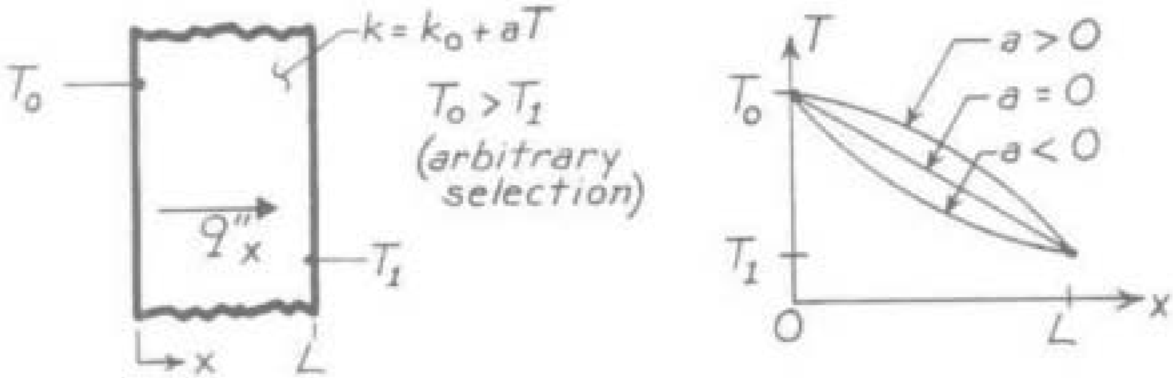
COMMENTS: The foregoing results are approximate due to use of a one-dimensional model in treating a two-dimensional problem.

PROBLEM 3.28

KNOWN: Temperature dependence of the thermal conductivity, k.

FIND: Heat flux and form of temperature distribution for a plane wall.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction through a plane wall, (2) Steady-state conditions, (3) No internal heat generation.

ANALYSIS: For the assumed conditions q_x and A(x) are constant and Eq. 3.21 gives

$$q_x''\int_0^L dx = -\int_{T_o}^{T_1} (k_o+aT)dT$$

$$q_x'' = \frac{1}{L}[k_o(T_1-T_1) + \frac{a}{2}(T_o^2-T_1^2)] .$$

From Fourier's law,

$$q_x'' = -(k_o+aT)\ dT/dx .$$

Hence, since the product of (k_o+aT) and (dT/dx) is constant, decreasing T with increasing x implies,

$a > 0$: decreasing (k_o+aT) and increasing $|dT/dx|$ with increasing x

$a = 0$: $k=k_o$ => constant (dT/dx)

$a < 0$: increasing (k_o+aT) and decreasing $|dT/dx|$ with increasing x.

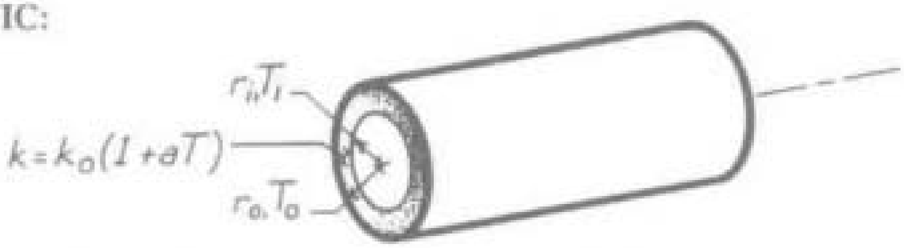
The temperature distributions appear as shown in the above sketch.

PROBLEM 3.29

KNOWN: Temperature dependence of tube wall thermal conductivity.

FIND: Expressions for heat transfer per unit length and tube wall thermal (conduction) resistance.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional radial conduction, (3) No internal heat generation.

ANALYSIS: From Eq. 3.24, the appropriate form of Fourier's law is

$$q_r = -k\,A_r\,\frac{dT}{dr} = -k(2\pi rL)\,\frac{dT}{dr}$$

$$q_r' = -2\pi kr\,\frac{dT}{dr}$$

$$q_r' = -2\pi r k_o(1+aT)\,\frac{dT}{dr}.$$

Separating variables,

$$-\frac{q_r'}{2\pi}\,\frac{dr}{r} = k_o(1+aT)\,dT$$

and integrating across the tube wall, find

$$-\frac{q_r'}{2\pi}\int_{r_i}^{r_o}\frac{dr}{r} = k_o\int_{T_i}^{T_o}(1+aT)\,dT$$

$$-\frac{q_r'}{2\pi}\ln\frac{r_o}{r_i} = k_o\left[T+\frac{aT^2}{2}\right]\Bigg|_{T_i}^{T_o}$$

$$-\frac{q_r'}{2\pi}\ln\frac{r_o}{r_i} = k_o\left[(T_o-T_i)+\frac{a}{2}(T_o^2-T_i^2)\right]$$

$$q_r' = -2\pi k_o\left[1+\frac{a}{2}(T_o+T_i)\right]\frac{(T_o-T_i)}{\ln(r_o/r_i)}. \quad \triangleleft$$

It follows that the overall thermal resistance per unit length is

$$R_t' = \frac{\Delta T}{q_r'} = \frac{\ln(r_o/r_i)}{2\pi k_o\left[1+\frac{a}{2}(T_o+T_i)\right]}. \quad \triangleleft$$

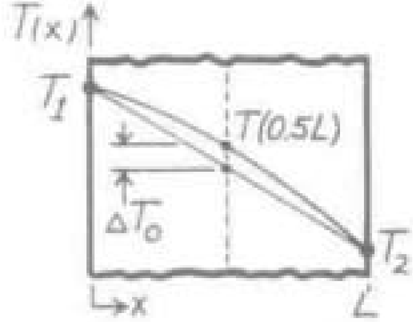
COMMENTS: Note the necessity of the stated assumptions to treating q_r' as independent of r.

PROBLEM 3.30

KNOWN: Steady-state temperature distribution of convex shape for material with $k = k_o(1 + \alpha T)$ where α is a constant and the mid-point temperature is ΔT_o higher than expected for a linear temperature distribution.

FIND: Relationship to evaluate α in terms of ΔT_o and T_1, T_2 the temperatures at the boundaries.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) No internal heat generation, (4) α is positive and constant.

ANALYSIS: At any location in the wall, Fourier's law has the form

$$q''_x = -k_o(1 + \alpha T)\frac{dT}{dx}. \tag{1}$$

Since q''_x is a constant, we can separate Eq. (1), identify appropriate integration limits, and integrate to obtain

$$\int_0^L q''_x dx = -\int_{T_1}^{T_2} k_o(1 + \alpha T)dT \tag{2}$$

$$q''_x = -\frac{k_o}{L}\left[\left(T_2 + \frac{\alpha T_2^2}{2}\right) - \left(T_1 + \frac{\alpha T_1^2}{2}\right)\right]. \tag{3}$$

We could perform the same integration but with the upper limits at $x = L/2$ to obtain

$$q''_x = -\frac{2k_o}{L}\left[\left(T_{L/2} + \frac{\alpha T_{L/2}^2}{2}\right) - \left(T_1 + \frac{\alpha T_1^2}{2}\right)\right] \tag{4}$$

where

$$T_{L/2} = T(L/2) = \frac{T_1 + T_2}{2} + \Delta T_o. \tag{5}$$

Setting Eq. (3) equal to Eq. (4) and substituting from Eq. (5) for $T_{L/2}$ into Eq. (4), and solving for α, eventually find,

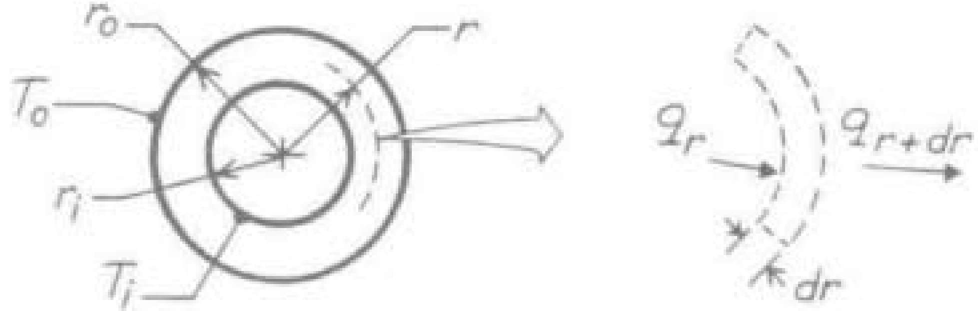
$$\alpha = \frac{2\Delta T_o}{(T_2^2 + T_1^2)/2 - [(T_1 + T_2)/2 + \Delta T_o]^2}. \qquad \triangleleft$$

PROBLEM 3.31

KNOWN: Hollow cylinder of thermal conductivity k, inner and outer radii, r_i and r_o, respectively, and length L.

FIND: Thermal resistance using the alternative conduction analysis method.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional radial conduction, (3) No internal volumetric generation, (4) Constant properties.

ANALYSIS: For the differential control volume, energy conservation requires that $q_r = q_{r+dr}$ for steady-state, one-dimensional conditions with no heat generation. With Fourier's law,

$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr} \qquad (1)$$

where $A = 2\pi rL$ is the area normal to the direction of heat transfer. Since q_r is constant, Eq. (1) may be separated and expressed in integral form,

$$\frac{q_r}{2\pi L}\int_{r_i}^{r_o}\frac{dr}{r} = -\int_{T_i}^{T_o} k(T)dT.$$

Assuming k is constant, the heat rate is

$$q_r = \frac{2\pi Lk(T_i - T_o)}{\ln(r_o/r_i)}.$$

Remembering that the thermal resistance is defined as

$$R_t \equiv \Delta T/q$$

it follows that for the hollow cylinder,

$$R_t = \frac{\ln(r_o/r_i)}{2\pi LK}. \qquad \triangleleft$$

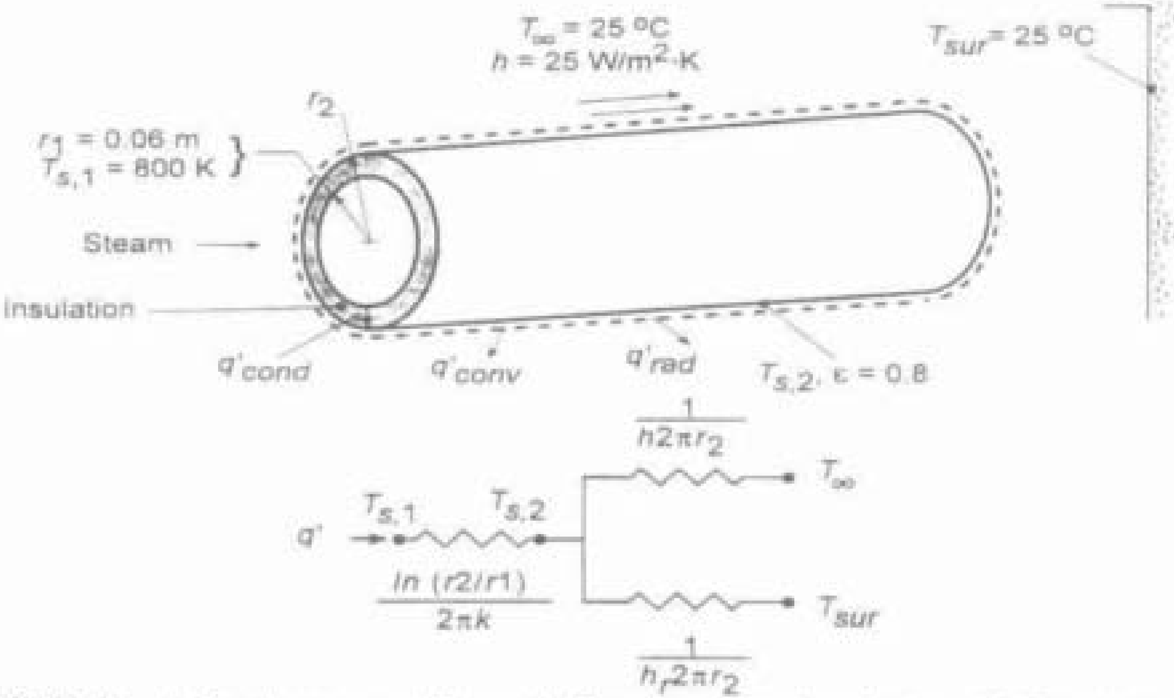
COMMENTS: Compare the *alternative* method used in this analysis with the *standard* method employed in Section 3.3.1 to obtain the same result.

PROBLEM 3.32

KNOWN: Thickness and inner surface temperature of calcium silicate insulation on a steam pipe. Convection and radiation conditions at outer surface.

FIND: (a) Heat loss per unit pipe length for prescribed insulation thickness and outer surface temperature. (b) Heat loss and radial temperature distribution as a function of insulation thickness.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions. (2) One-dimensional conduction. (3) Constant properties.

PROPERTIES: *Table A-3*, Calcium Silicate (T = 645 K): k = 0.089 W/m·K.

ANALYSIS: (a) From Eq. 3.27 with $T_{s,2}$ = 490 K, the heat rate per unit length is

$$q' = q_r/L = \frac{2\pi k(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}$$

$$q' = \frac{2\pi(0.089\,W/m\cdot K)(800-490)K}{\ln(0.08\,m/0.06\,m)}$$

$$q' = 603\,W/m.$$ <

(b) Performing an energy for a control surface around the outer surface of the insulation, it follows that

$$q'_{cond} = q'_{conv} + q'_{rad}$$

$$\frac{T_{s,1}-T_{s,2}}{\ln(r_2/r_1)/2\pi k} = \frac{T_{s,2}-T_\infty}{1/(2\pi r_2 h)} + \frac{T_{s,2}-T_{sur}}{1/(2\pi r_2 h_r)}$$

where $h_r = \varepsilon\sigma(T_{s,2} + T_{sur})(T_{s,2}^2 + T_{sur}^2)$. Solving this equation for $T_{s,2}$, the heat rate may be determined from

$$q' = 2\pi r_2\left[h(T_{s,2} - T_\infty) + h_r(T_{s,2} - T_{sur})\right]$$

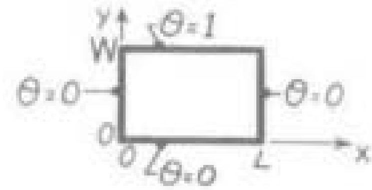
Continued...

PROBLEM 4.1

KNOWN: Method of separation of variables (Section 4.2) for two-dimensional, steady-state conduction.

FIND: Show that negative or zero values of λ^2, the separation constant, result in solutions which cannot satisfy the boundary conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, identification of the separation constant λ leads to the two ordinary differential equations, 4.6 and 4.7, having the forms

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad\qquad \frac{d^2Y}{dy^2} - \lambda^2 Y = 0 \qquad (1,2)$$

and the temperature distribution is $\qquad \theta(x,y) = X(x) \cdot Y(y)\,. \qquad (3)$

Consider now the situation when $\lambda^2 = 0$. From Eqs. (1), (2), and (3), find that

$$X = C_1 + C_2 x\,, \qquad Y = C_3 + C_4 y \qquad \text{and} \qquad \theta(x,y) = (C_1 + C_2 x)\,(C_3 + C_4 y)\,. \qquad (4)$$

Evaluate the constants - C_1, C_2, C_3 and C_4 - by substitution of the boundary conditions:

x=0:	$\theta(0,y) = (C_1 + C_2 \cdot 0)(C_3 + C_4 y) = 0$	$C_1 = 0$
y=0:	$\theta(x,0) = (0 + C_2 X)(C_3 + C_4 \cdot 0) = 0$	$C_3 = 0$
x=L:	$\theta(L,0) = (0 + C_2 L)(0 + C_4 y) = 0$	$C_2 = 0$
y=W:	$\theta(x,W) = (0 + 0 \cdot x)(0 + C_4 W) = 1$	$0 \neq 1$

The last boundary condition evaluation leads to an impossibility ($0 \neq 1$). We therefore conclude that a λ^2 value of zero will not result in a form of the temperature distribution which will satisfy the boundary conditions. Consider now the situation when $\lambda^2 < 0$. The solutions to Eqs. (1) and (2) will be

$$X = C_5 e^{-\lambda x} + C_6 e^{+\lambda x}\,, \qquad\qquad Y = C_7 \cos \lambda y + C_8 \sin \lambda y \qquad (5,6)$$

and $\qquad \theta(x,y) = [C_5 e^{-\lambda x} + C_6 e^{+\lambda x}]\,[C_7 \cos \lambda y + C_8 \sin \lambda y]\,. \qquad (7)$

Evaluate the constants for the boundary conditions identified above.

y=0:	$\theta(x,0) = [C_5 e^{-\lambda x} + C_6 e^{-\lambda x}]\,[C_7 \cos 0 + C_8 \sin 0] = 0$	$C_7 = 0$
x=0:	$\theta(0,y) = [C_5 e^0 + C_6 e^0]\,[0 + C_8 \sin \lambda y] = 0$	$C_8 = 0$

If $C_8 = 0$, a trivial solution results or $C_5 = -C_6$.

x=L: $\qquad \theta(L,y) = C_5 [e^{-xL} - e^{+xL}]\, C_8 \sin \lambda y = 0.$

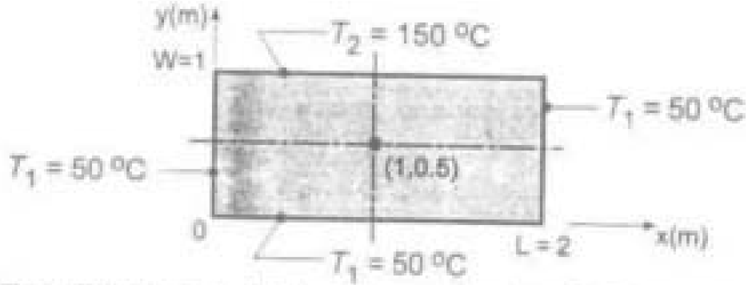
From the last boundary condition evaluation, we require C_5 or C_8 is zero; either case leads to a trivial solution with either no x or y dependence possible.

PROBLEM 4.2

KNOWN: Two-dimensional rectangular plate subjected to prescribed uniform temperature boundary conditions.

FIND: Temperature at the mid-point using the exact solution considering the first five non-zero terms; assess error resulting from using only first three terms. Plot the temperature distributions T(x,0.5) and T(1,y).

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: From Section 4.2, the temperature distribution is

$$\theta(x,y) \equiv \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\cdot\frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)} \qquad (1.4.19)$$

Considering now the point (x,y) = (1.0,0.5) and recognizing x/L = 1/2, y/L = 1/4 and W/L = 1/2,

$$\theta(1,0.5) \equiv \frac{T-T_1}{T_2-T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi}{2}\right)\cdot\frac{\sinh(n\pi/4)}{\sinh(n\pi/2)}.$$

When n is even (2, 4, 6 ...), the corresponding term is zero; hence we need only consider n = 1, 3, 5, 7 and 9 as the first five non-zero terms.

$$\theta(1,0.5) = \frac{2}{\pi}\left\{2\sin\left(\frac{\pi}{2}\right)\frac{\sinh(\pi/4)}{\sinh(\pi/2)} + \frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\frac{\sinh(3\pi/4)}{\sinh(3\pi/2)} + \right.$$

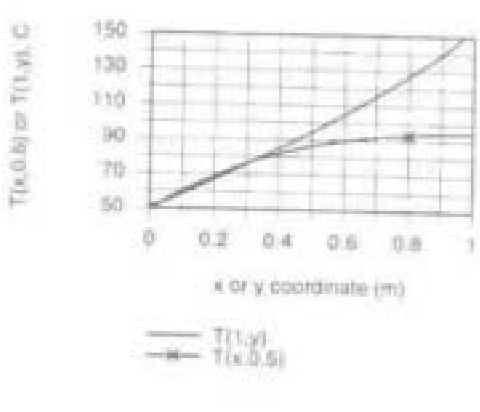
$$\left.\frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\frac{\sinh(5\pi/4)}{\sinh(5\pi/2)} + \frac{2}{7}\sin\left(\frac{7\pi}{2}\right)\frac{\sinh(7\pi/4)}{\sinh(7\pi/2)} + \frac{2}{9}\sin\left(\frac{9\pi}{2}\right)\frac{\sinh(9\pi/4)}{\sinh(9\pi/2)}\right\}$$

$$\theta(1,0.5) = \frac{2}{\pi}[0.755 - 0.063 + 0.008 - 0.001 + 0.000] = 0.445 \qquad (2)$$

$$T(1,0.5) = \theta(1,0.5)(T_2 - T_1) + T_1 = 0.445(150-50) + 50 = 94.5^\circ C. \qquad <$$

If only the first three terms of the series, Eq. (2), are considered, the result will be $\theta(1,0.5) = 0.46$; that is, there is less than a 0.2% effect.

Using Eq. (1), and writing out the first five terms of the series, expressions for θ(x,0.5) or T(x,0.5) and θ(1,y) or T(1,y) were keyboarded into the IHT workspace and evaluated for sweeps over the x or y variable. Note that for T(1,y), that as y → 1, the upper boundary, T(1,1) is greater than 150°C. Upon examination of the magnitudes of terms, it becomes evident that more than 5 terms are required to provide an accurate solution.

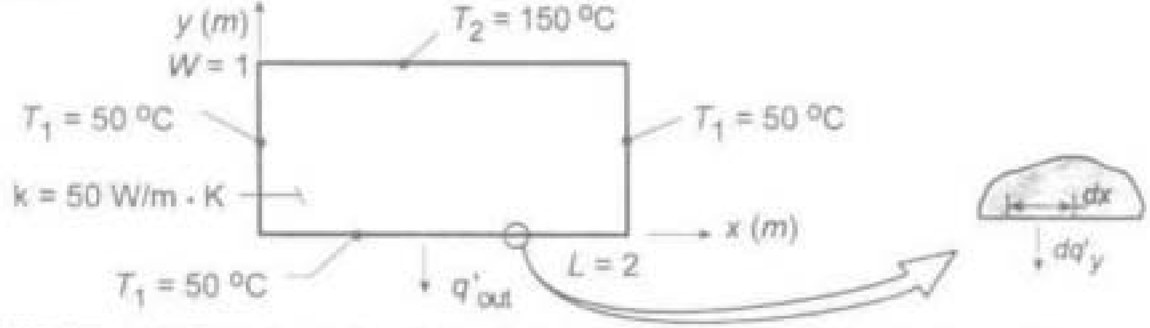


PROBLEM 4.3

KNOWN: Temperature distribution in the two-dimensional rectangular plate of Problem 4.2.

FIND: Expression for the heat rate per unit thickness from the lower surface ($0 \le x \le 2, 0$) and result based on first five non-zero terms of the infinite series.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: The heat rate per unit thickness *from the plate* along the lower surface is

$$q'_{out} = -\int_{x=0}^{x=2} dq'_y(x,0) = -\int_{x=0}^{x=2} -k\left.\frac{\partial T}{\partial y}\right|_{y=0} dx = k(T_2 - T_1)\int_{x=0}^{x=2}\left.\frac{\partial \theta}{\partial y}\right|_{y=0} dx \qquad (1)$$

where from the solution to Problem 4.2,

$$\theta = \frac{T - T_1}{T_2 - T_1} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)}. \qquad (2)$$

Evaluate the gradient of θ from Eq. (2) and substitute into Eq. (1) to obtain

$$q'_{out} = k(T_2 - T_1)\int_{x=0}^{x=2}\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x}{L}\right)\left.\frac{(n\pi/L)\cosh(n\pi y/L)}{\sinh(n\pi W/L)}\right|_{y=0} dx$$

$$q'_{out} = k(T_2 - T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{1}{\sinh(n\pi W/L)}\left[-\cos\left(\frac{n\pi x}{L}\right)\right]_{x=0}^{2}$$

$$q'_{out} = k(T_2 - T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\frac{1}{\sinh(n\pi/L)}[1-\cos(n\pi)] \qquad <$$

To evaluate the first five, non-zero terms, recognize that since $\cos(n\pi) = 1$ for $n = 2, 4, 6, \ldots$, only the n-odd terms will be non-zero. Hence,

$$q'_{out} = 50\,\text{W/m}\cdot\text{K}(150-50)^\circ\text{C}\,\frac{2}{\pi}\left\{\frac{(-1)^2+1}{1}\cdot\frac{1}{\sinh(\pi/2)}(2)+\frac{(-1)^4+1}{3}\cdot\frac{1}{\sinh(3\pi/2)}\cdot(2)\right.$$

$$\left.+\frac{(-1)^6+1}{5}\cdot\frac{1}{\sinh(5\pi/2)}(2)+\frac{(-1)^8+1}{7}\cdot\frac{1}{\sinh(7\pi/2)}(2)+\frac{(-1)^{10}+1}{9}\cdot\frac{1}{\sinh(9\pi/2)}(2)\right\}$$

$$q'_{out} = 3.183\,\text{kW/m}[1.738+0.024+0.00062+(\ldots)] = 5.611\,\text{kW/m} \qquad <$$

Continued...

COMMENTS: If the foregoing procedure were used to evaluate the heat rate into the upper surface, $q'_{in} = -\int_{x=0}^{x=2} dq'_y(x,W)$, it would follow that

$$q'_{in} = k(T_2 - T_1)\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\coth(n\pi/2)\left[1-\cos(n\pi)\right]$$

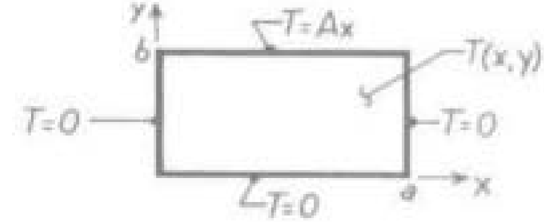
However, with $\coth(n\pi/2) \geq 1$, irrespective of the value of n, and with $\sum_{n=1}^{\infty}\left[(-1)^{n+1}+1\right]/n$ being a divergent series, the complete series does not converge and $q'_{in} \to \infty$. This physically untenable condition results from the temperature discontinuities imposed at the upper left and right corners.

PROBLEM 4.4

KNOWN: Rectangular plate subjected to prescribed boundary conditions.

FIND: Steady-state temperature distribution.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, 2-D conduction, (2) Constant properties.

ANALYSIS: The solution follows the method of Section 4.2. The product solution is

$$T(x,y) = X(x)\cdot Y(y) = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{+\lambda y})$$

and the boundary conditions are: $T(0,y) = 0$, $T(a,y) = 0$, $T(x,0) = 0$, $T(x,b) = Ax$. Applying BC#1, $T(0,y) = 0$, find $C_1 = 0$. Applying BC#2, $T(a,y) = 0$, find that $\lambda = n\pi/a$ with $n = 1,2,\ldots$. Applying BC#3, $T(x,0) = 0$, find that $C_3 = -C_4$. Hence, the product solution is

$$T(x,y) = X(x)\cdot Y(y) = C_2 C_4 \sin\left(\frac{n\pi}{a}x\right)\left(e^{+\lambda y} - e^{-\lambda y}\right).$$

Combine constants and using superposition, find

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right).$$

To evaluate C_n, use orthogonal functions with Eq. 4.16 to find

$$C_n = \int_0^a Ax\cdot\sin\left(\frac{n\pi x}{a}\right)\cdot dx \Big/ \sinh\left(\frac{n\pi b}{a}\right) \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx,$$

noting that $y = b$. The numerator, denominator and C_n, respectively, are:

$$A\int_0^a x\cdot\sin\frac{n\pi x}{a}\cdot dx = A\left[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{a}\right) - \frac{ax}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_0^a = \frac{Aa^2}{n\pi}\left[-\cos(n\pi)\right] = \frac{Aa^2}{n\pi}(-1)^{n+1},$$

$$\sinh\left(\frac{n\pi b}{a}\right)\int_0^a \sin^2\frac{n\pi x}{a}\cdot dx = \sinh\left(\frac{n\pi b}{a}\right)\left[\frac{1}{2}x - \frac{1}{4n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a = \frac{a}{2}\cdot\sinh\left(\frac{n\pi b}{a}\right),$$

$$C_n = \frac{Aa^2}{n\pi}(-1)^{n+1} \Big/ \frac{a}{2}\sinh\left(\frac{n\pi b}{a}\right) = 2Aa\,(-1)^{n+1}/n\pi \sinh\left(\frac{n\pi b}{a}\right).$$

Hence, the temperature distribution is

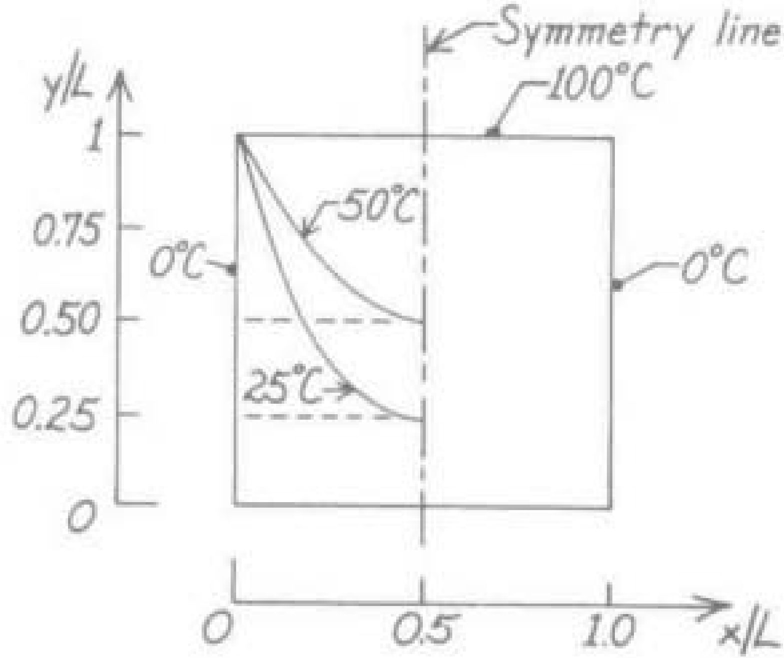
$$T(x,y) = \frac{2\,Aa}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\cdot\sin\left(\frac{n\pi x}{a}\right)\frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}. \quad \triangleleft$$

PROBLEM 4.5

KNOWN: Very long square bar with one side maintained at 100°C while the other three are maintained at 0°C.

FIND: Without performing a flux plot, sketch the 25 and 50°C isotherms; explain how you arrive at their shapes and locations.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction in bar, (3) Constant properties, (4) No internal generation.

ANALYSIS: First recognize that the temperature distribution is symmetrical about the y-axis at x/L = 0.5. Hence, the isotherms must be normal to this symmetry line (0.5, y). Further, the isotherms must converge at the corner (0, 1) on the (x/L, y/L) plane. If we assume, as a *first* approximation, the heat transfer in the direction along the symmetry line is *one-dimensional*, then the 50°C isotherm will intersect the symmetry line at (0.5, 0.5). Likewise, the 25°C isotherm will intersect at (0.5, 0.25). So the isotherms can now be sketched having determined two intersections (in the upper corner and along the symmetry line) and the slope at the symmetry line.

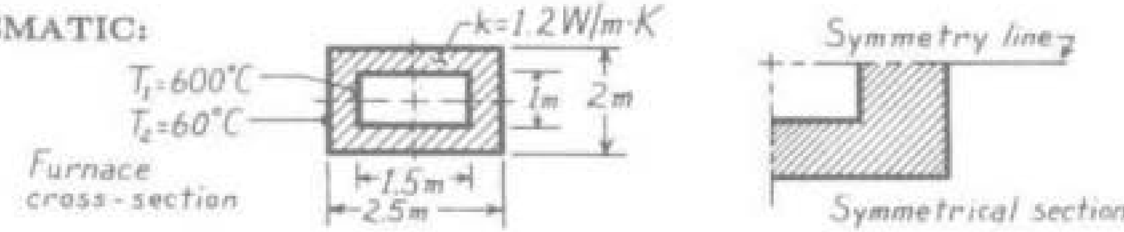
COMMENTS: Using the series solution of Section 4.1, the exact temperatures at (0.5, 0.5) and (0.5, 0.25) are 25.0 and 9.54°C, respectively. Hence, we conclude that our first approximation approach is poor since the heat transfer is *two*-dimensional. If $L_y > L_x$, then our first approximation approach might be more reasonable.

PROBLEM 4.6

KNOWN: Long furnace of refractory brick with prescribed surface temperatures and material thermal conductivity.

FIND: Shape factor and heat transfer rate per unit length using the flux plot method.

SCHEMATIC:

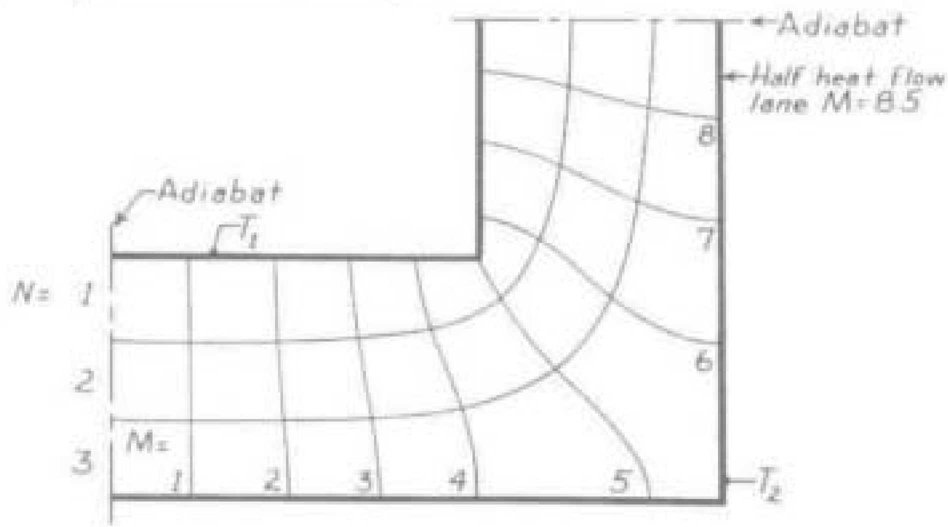


ASSUMPTIONS: (1) Furnace length normal to page, ℓ, >> cross-sectional dimensions, (2) Two-dimensional, steady-state conduction, (3) Constant properties.

ANALYSIS: Considering the cross-section, the cross-hatched area represents a symmetrical element. Hence, the heat rate for the entire furnace per unit length is

$$q' = \frac{q}{\ell} = 4\frac{S}{\ell}k(T_1 - T_2)$$

where S is the shape factor for the symmetrical section. Selecting three temperature increments (N=3), construct the flux plot shown below.



From Eq. 4.26, $S = \frac{M\ell}{N}$ or $\frac{S}{\ell} = \frac{M}{N} = \frac{8.5}{3} = 2.83$ ◁

and from Eq. (1), $q' = 4\times 2.83\times 1.2\frac{W}{m\cdot K}(600-60)^\circ C = 7.34 \text{ kW/m}$. ◁

COMMENTS: The shape factor can also be estimated from the relations of Table 4.1. The symmetrical section consists of two plane walls (horizontal and vertical) with an adjoining edge. Using the appropriate relations, the numerical values are, in the same order,

$$S = \frac{0.75\text{m}}{0.5\text{m}}\ell + 0.54\ell + \frac{0.5\text{m}}{0.5\text{m}}\ell = 3.04\ell$$

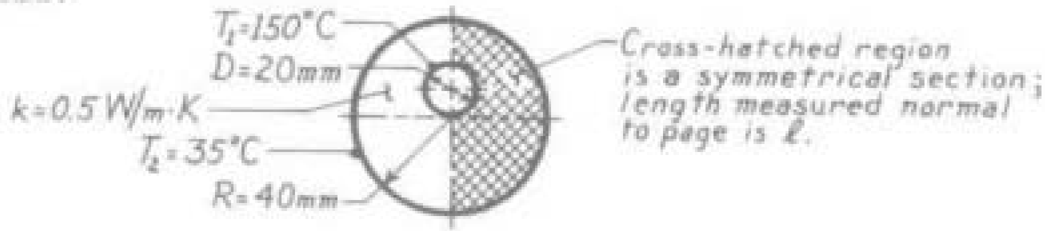
Note that this result compares favorably with the flux plot result of 2.83ℓ.

PROBLEM 4.7

KNOWN: Hot pipe embedded eccentrically in a circular system having a prescribed thermal conductivity.

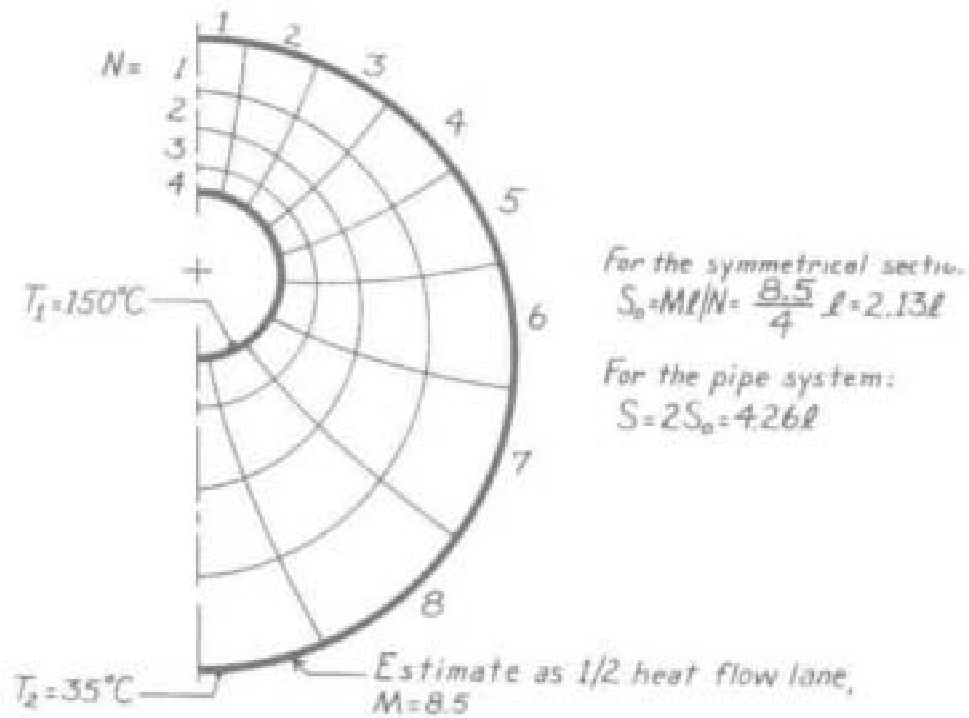
FIND: The shape factor and heat transfer per unit length for the prescribed surface temperatures.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Steady-state conditions, (3) Length $\ell >>$ diametrical dimensions.

ANALYSIS: Considering the cross-sectional view of the pipe system, the symmetrical section shown above is readily identified. Selecting four temperature increments (N=4), construct the flux plot shown below.



For the pipe system, the heat rate per unit length is

$$q' = \frac{q}{\ell} = kS(T_1 - T_2) = 0.5\frac{W}{m \cdot K} \times 4.26(150-35)\,^\circ C = 245\ W/m\,. \qquad \triangleleft$$

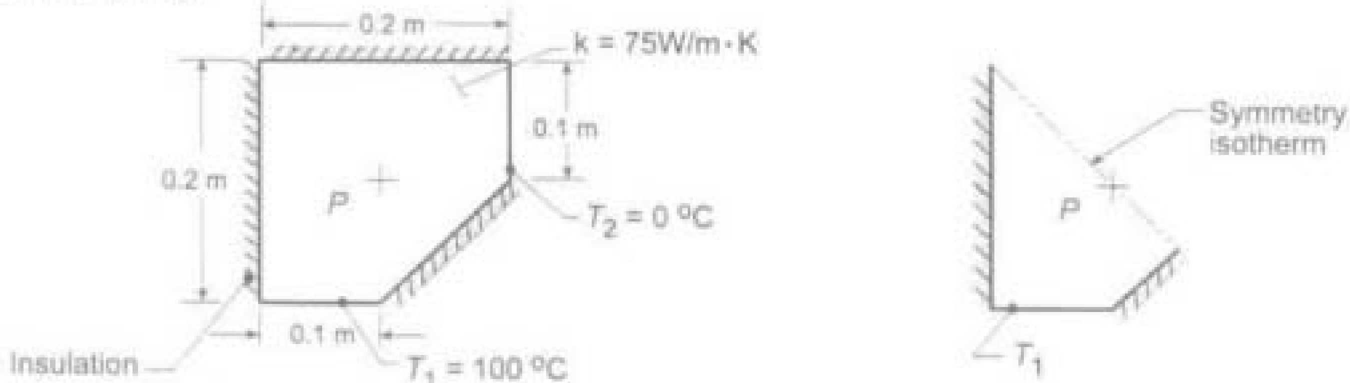
COMMENTS: Note that in the lower, right-hand quadrant of the flux plot, the curvilinear squares are irregular. Further work is required to obtain an improved plot and, hence, obtain a more accurate estimate of the shape factor.

PROBLEM 4.8

KNOWN: Structural member with known thermal conductivity subjected to a temperature difference.

FIND: (a) Temperature at a prescribed point P, (b) Heat transfer per unit length of the strut, (c) Sketch the 25, 50 and 75°C isotherms, and (d) Same analysis on the shape but with adiabatic-isothermal boundary conditions reversed.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Steady-state conditions, (3) Constant properties.

ANALYSIS: (a) Using the methodology of Section 4.3.1, construct a flux plot. Note the line of symmetry which passes through the point P is an isotherm as shown above. It follows that

$$T(P) = (T_1 + T_2)/2 = (100 + 0)^\circ C/2 = 50^\circ C. \quad <$$

(b) The flux plot on the symmetrical section is now constructed to obtain the shape factor from which the heat rate is determined. That is, from Eq. 4.25 and 4.26,

$$q = kS(T_1 - T_2) \quad \text{and} \quad S = M\ell/N. \qquad (1,2)$$

From the plot of the symmetrical section,

$$S_o = 4.2\ell/4 = 1.05\ell.$$

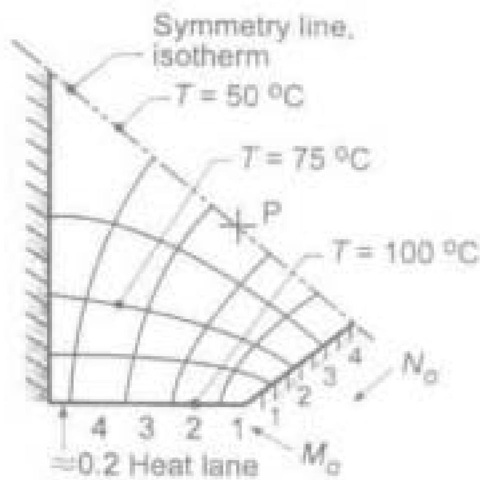
For the full section of the strut,

$$M = M_o = 4.2$$

but $N = 2N_o = 8$. Hence,

$$S = S_o/2 = 0.53\ell$$

and with $q' = q/\ell$, giving

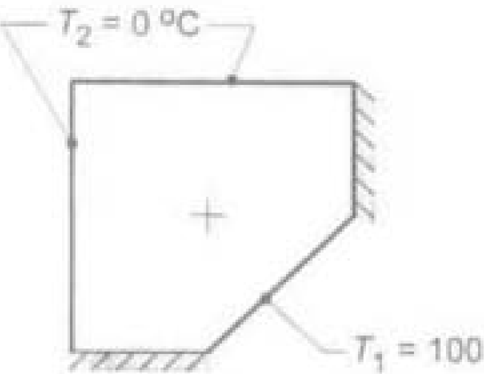


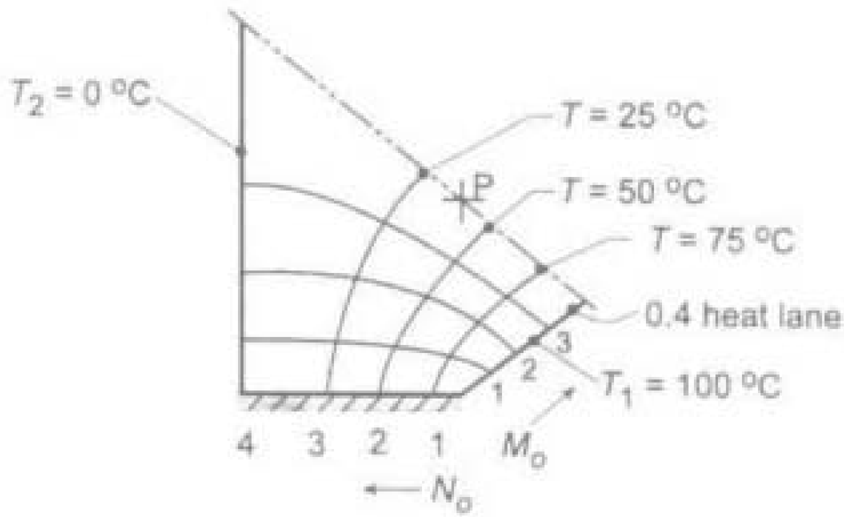
$$q'/\ell = 75\,W/m \cdot K \times 0.53(100 - 0)^\circ C = 3975\,W/m. \quad <$$

(c) The isotherms for T = 50, 75 and 100°C are shown on the flux plot. The T = 25°C isotherm is symmetric with the T = 75°C isotherm.

(d) By reversing the adiabatic and isothermal boundary conditions, the two-dimensional shape appears as shown in the sketch below. The symmetrical element to be flux plotted is the same as for the strut, except the symmetry line is now an adiabat.

Continued...





From the flux plot, find $M_o = 3.4$ and $N_o = 4$, and from Eq. (2)

$$S_o = M_o \ell / N_o = 3.4\ell/4 = 0.85\ell \qquad S = 2S_o = 1.70\ell$$

and the heat rate per unit length from Eq. (1) is

$$q' = 75\,\text{W/m}\cdot\text{K} \times 1.70(100-0)^\circ\text{C} = 12{,}750\,\text{W/m}$$ <

From the flux plot, estimate that

$$T(P) = 40^\circ\text{C}.$$ <

COMMENTS: (1) By inspection of the shapes for parts (a) and (b), it is obvious that the heat rate for the latter will be greater. The calculations show the heat rate is greater by more than a factor of three.

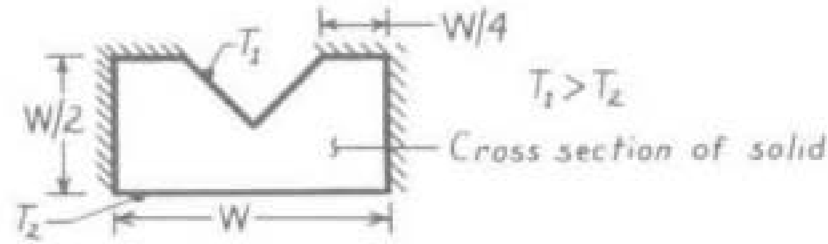
(2) By comparing the flux plots for the two configurations, and corresponding roles of the adiabats and isotherms, would you expect the shape factor for parts (a) to be the reciprocal of part (b)?

PROBLEM 4.9

KNOWN: Relative dimensions and surface thermal conditions of a V-grooved channel.

FIND: Flux plot and shape factor.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Steady-state conditions, (3) Constant properties.

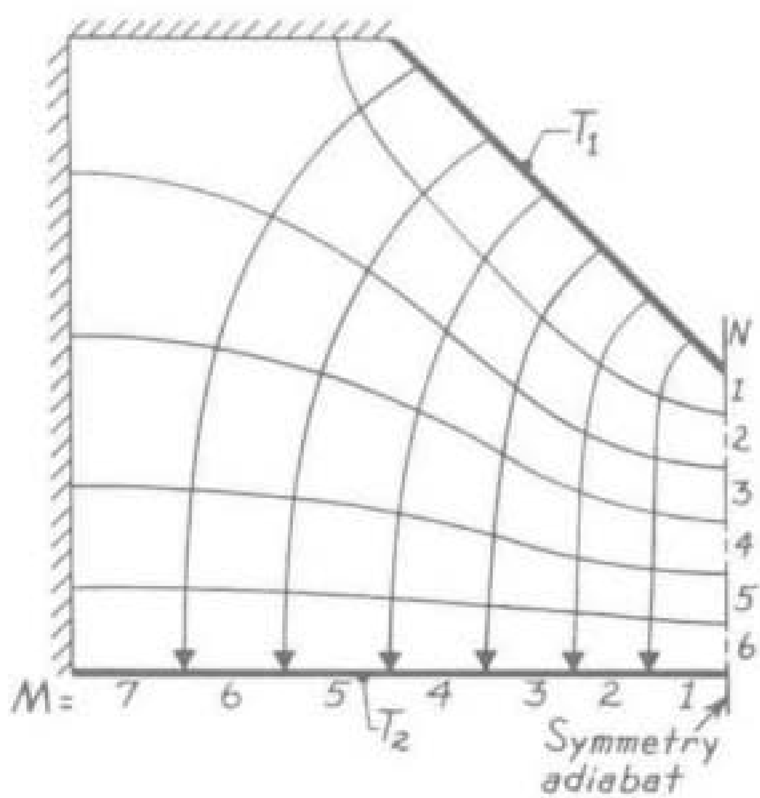
ANALYSIS: With symmetry about the midplane, only one-half of the object need be considered as shown below.

Choosing 6 temperature increments (N=6), it follows from the plot that M≈7. Hence from Eq. 4.26, the shape factor for the half section is

$$S = \frac{M}{N}\ell = \frac{7}{6}\ell = 1.17\ell \ .$$

For the complete system, the shape factor is then

$$S = 2.34\ell \ .$$ ◁

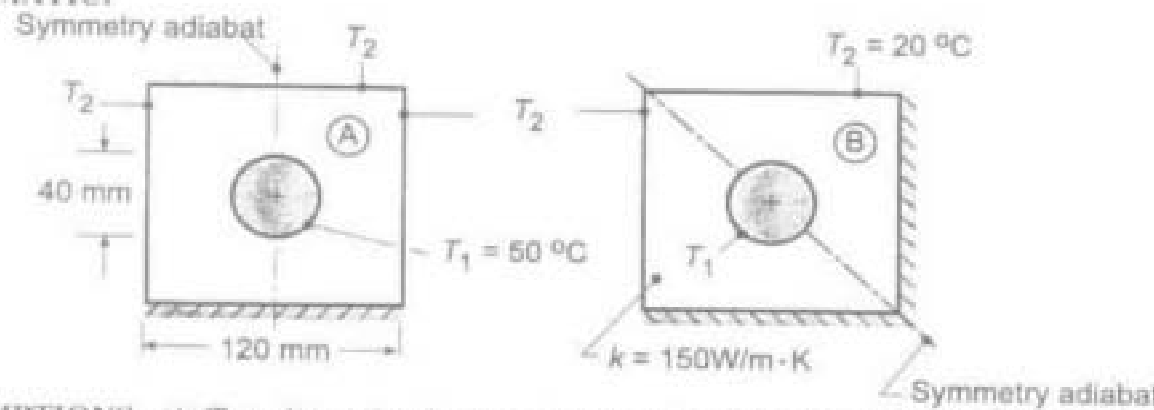


PROBLEM 4.10

KNOWN: Long conduit of inner circular cross section and outer surfaces of square cross section.

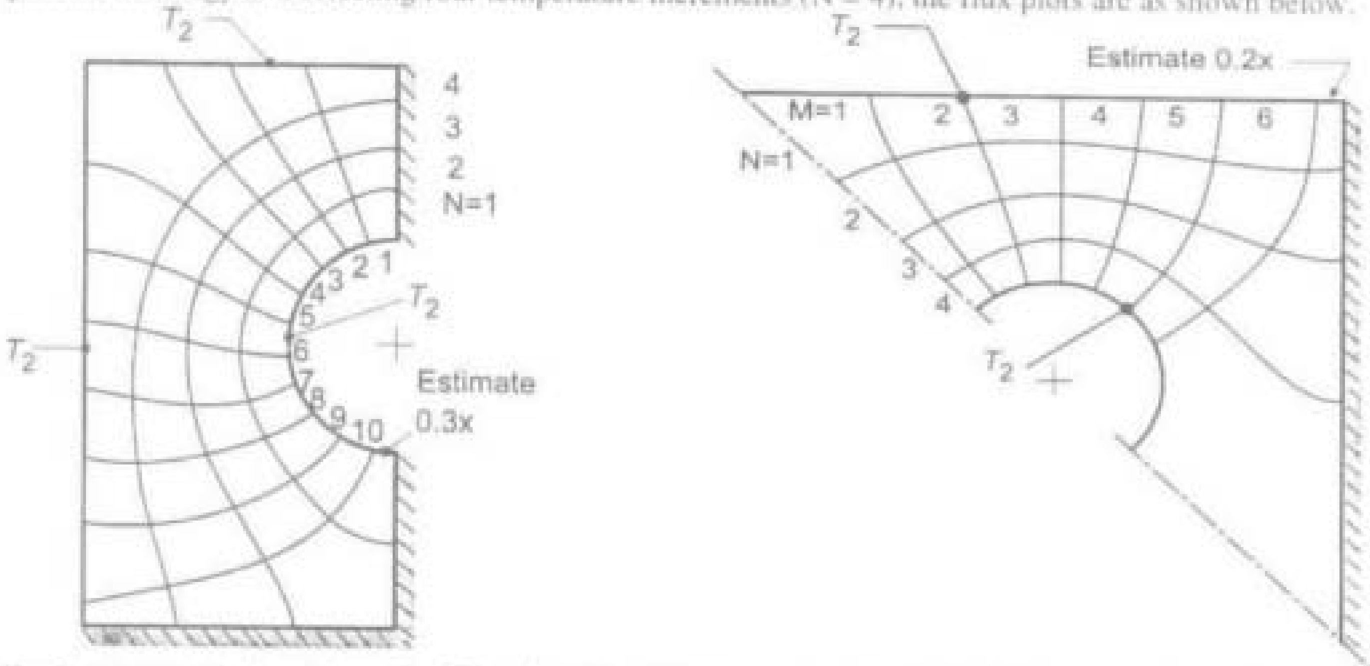
FIND: Shape factor and heat rate for the two applications when outer surfaces are insulated or maintained at a uniform temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties and (3) Conduit is very long.

ANALYSIS: The adiabatic symmetry lines for each of the applications is shown above. Using the flux plot methodology and selecting four temperature increments (N = 4), the flux plots are as shown below.



For the symmetrical sections, $S = 2S_o$, where $S_o = M\ell/N$ and the heat rate for each application is $q = 2(S_o/\ell)k(T_1 - T_2)$.

Application	M	N	S_o/ℓ	q' (W/m)	
A	10.3	4	2.58	11,588	<
B	6.2	4	1.55	6,975	<

COMMENTS: (1) For application A, most of the heat lanes leave the inner surface (T_1) on the upper portion.

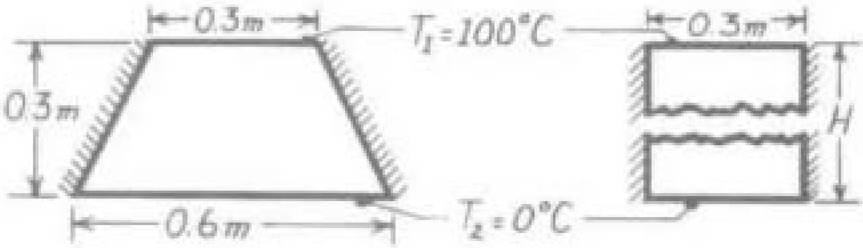
(2) For application B, most of the heat flow lanes leave the inner surface on the upper portion (that is, lanes 1-4). Because the lower, right-hand corner is insulated, the entire section experiences small heat flows (lane 6 + 0.2). Note the shapes of the isotherms near the right-hand, insulated boundary and that they intersect the boundary normally.

PROBLEM 4.11

KNOWN: Shape and surface conditions of a support column.

FIND: (a) Heat transfer rate per unit length. (b) Height of a rectangular bar of equivalent thermal resistance.

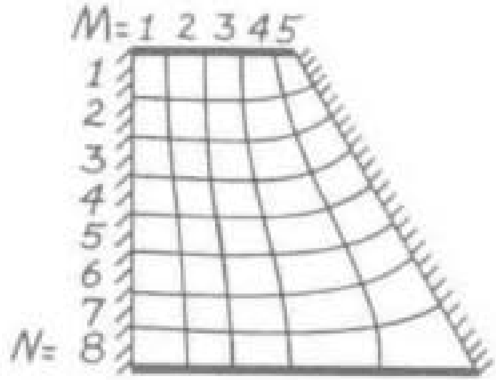
SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Negligible three-dimensional conduction effects, (3) Constant properties, (4) Adiabatic sides.

PROPERTIES: *Table A-1*, Steel, AISI 1010 (323K): k = 62.7 W/m·K.

ANALYSIS: (a) From the flux plot for the half section, M≈5 and N≈8. Hence for the full section



$$S = 2\,\frac{M\ell}{N} \approx 1.25\ell$$

$$q = Sk(T_1 - T_2)$$

$$q' \approx 1.25 \times 62.7\,\frac{W}{m\cdot K}(100-0)\,^\circ C$$

$$q' \approx 7.8\ kW/m\,. \qquad \triangleleft$$

(b) The rectangular bar provides for one-dimensional heat transfer. Hence,

$$q = k\,A\,\frac{(T_1 - T_2)}{H} = k(0.3\ell)\,\frac{(T_1 - T_2)}{H}$$

Hence,

$$H = \frac{0.3k(T_1 - T_2)}{q'} = \frac{0.3m(62.7\ W/m\cdot K)(100\,^\circ C)}{7800\ W/m} = 0.24m\,. \qquad \triangleleft$$

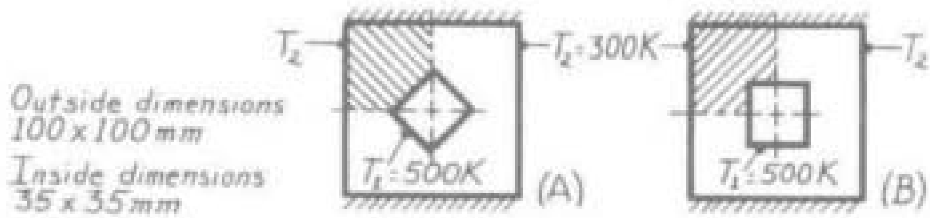
COMMENTS: The fact that H < 0.3m is consistent with the requirement that the thermal resistance of the trapezoidal column must be less than that of a rectangular bar of the same height and top width (because the width of the trapezoidal column increases with increasing distance, x, from the top). Hence, if the rectangular bar is to be of equivalent resistance, it must be of smaller height.

PROBLEM 4.12

KNOWN: Hollow prismatic bars fabricated from plain carbon steel, 1m in length with prescribed temperature difference.

FIND: Shape factors and heat rate per unit length.

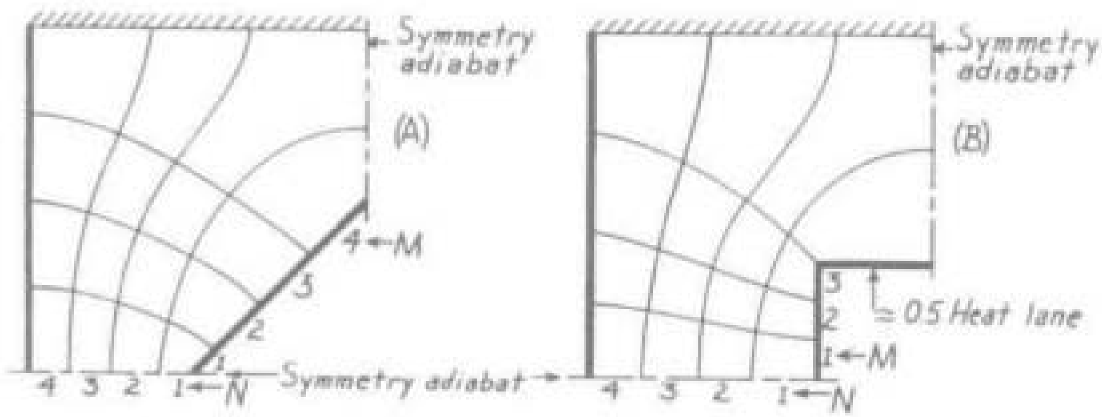
SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

PROPERTIES: *Table A-1*, Steel, Plain Carbon (400K), k = 57 W/m·K.

ANALYSIS: Construct a flux plot on the symmetrical sections (shaded-regions) of each of the bars.



The shape factors for the symmetrical sections are,

$$S_{o,A} = \frac{M\ell}{N} = \frac{4}{4}\ell = 1\ell \qquad S_{o,B} = \frac{M\ell}{N} = \frac{3.5}{4}\ell = 0.88\ell.$$

Since each of these sections is ¼ of the bar cross-section, it follows that

$$S_A = 4\times 1\ell = 4\ell \qquad S_B = 4\times 0.88\ell = 3.5\ell. \qquad \triangleleft$$

The heat rate per unit length is $q' = q/\ell = k(S/\ell)(T_1 - T_2)$,

$$q'_A = 57\frac{W}{m\cdot K}\times 4(500-300)K = 45.6\ kW/m \qquad \triangleleft$$

$$q'_B = 57\frac{W}{m\cdot K}\times 3.5(500-300)K = 39.9\ kW/m. \qquad \triangleleft$$

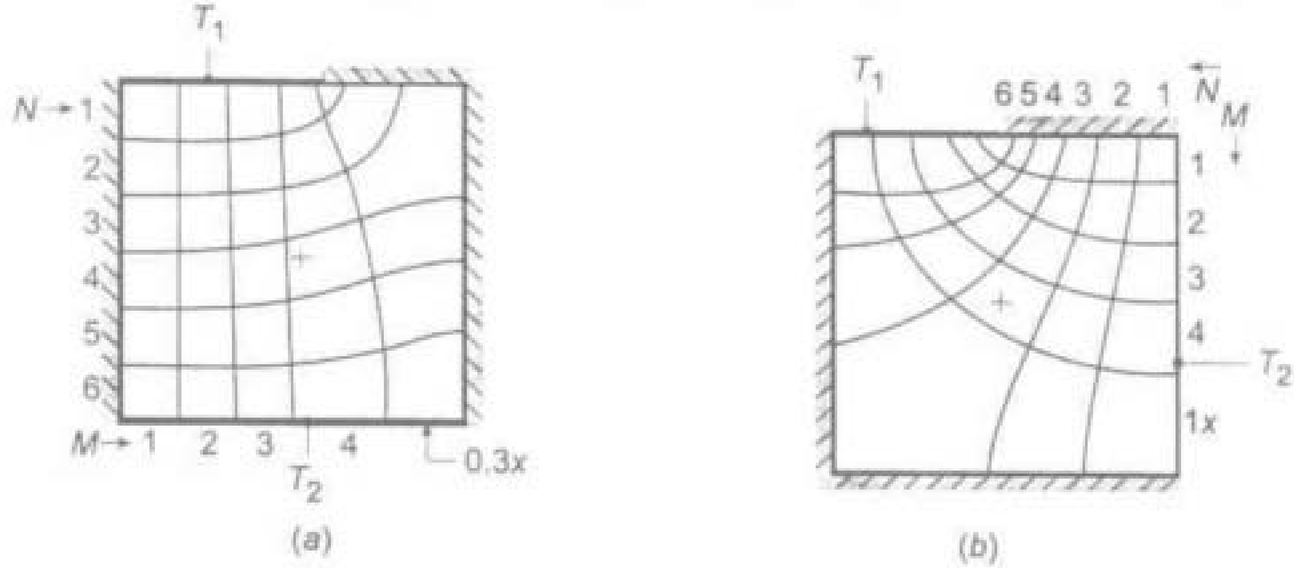
PROBLEM 4.13

KNOWN: Two-dimensional, square shapes maintained at uniform temperatures as prescribed, perfectly insulated elsewhere.

FIND: Using the flux plot method, estimate the shape factors and the center temperatures for the shapes.

ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: Use the methodology of Section 4.3.1 to construct the flux plots. With Figure (a), begin at the left side making the isotherms almost equally spaced since the heat flow will only slightly spread toward the right. Start sketching the adiabats in the vicinity of the T_1 surface. Figure (b) is more difficult to analyze since neither the isotherm or heat flow lanes are regular in any region of the shape. For the present situation, the best approach is to begin in the upper right-hand corner of the shape.



The shape factors are calculated from Eq. 4.26.

$$S' = \frac{M}{N} = \frac{4.3}{6} = 0.72 \qquad\qquad S' = \frac{M}{N} = \frac{5.25}{6} = 0.83$$ <

Assuming that $T_1 = 100°C$ and $T_2 = 0°C$, the center temperatures are estimated as

$$T(0,0) = 53°C \qquad\qquad T(0,0) = 42°C$$

COMMENTS: Using a finite-element package with a fine mesh, we determined shape factors of 0.58 for both Figures (a) and (b). Similarly, the center temperatures are nearly the same, T(0,0) = 43.4 and 43.8°C, respectively. The precision of the flux plots for the heat rates is nearly 50% high, but the center temperature estimates are good ones.

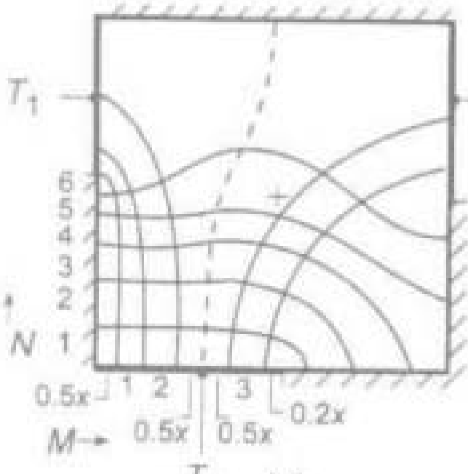
PROBLEM 4.14

KNOWN: Two-dimensional, square shapes, 1 m to a side, maintained at uniform temperatures as prescribed, perfectly insulated elsewhere.

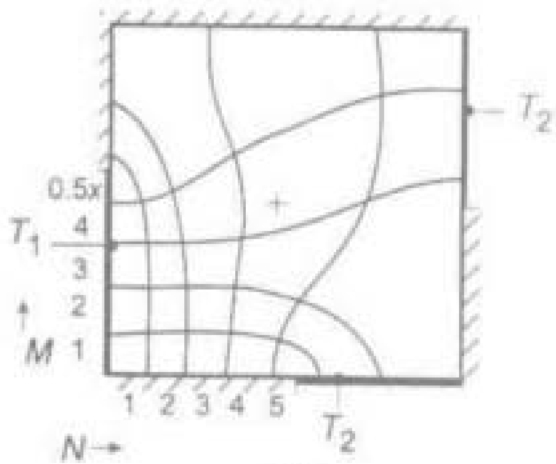
FIND: Using the flux plot method, estimate the heat rate per unit length normal to the page if the thermal conductivity is 50 W/m·K

ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties.

ANALYSIS: Use the methodology of Section 4.3.1 to construct the flux plots to obtain the shape factors from which the heat rates can be calculated. With Figure (a), begin at the lower-left side making the isotherms almost equally spaced, since the heat flow will only slightly spread toward the right. Start sketching the adiabats in the vicinity of the T_2 surface. The dashed line represents the adiabat which separates the shape into two segments. Having recognized this feature, it was convenient to identify partial heat lanes. Figure (b) is less difficult to analyze since the isotherm intervals are nearly regular in the lower left-hand corner.



(a)



(b)

The shape factors are calculated from Eq. 4.26 and the heat rate from Eq. 4.25.

$$S' = \frac{M}{N} = \frac{0.5 + 3 + 0.5 + 0.5 + 0.2}{6} \qquad\qquad S' = \frac{M}{N} = \frac{4.5}{5} = 0.90$$

$$S' = 0.70$$

$$q' = kS'(T_1 - T_2) \qquad\qquad q' = kS'(T_1 - T_2)$$

$$q' = 10\ \text{W/m}\cdot\text{K} \times 0.70(100 - 0)\text{K} = 700\ \text{W/m} \qquad q' = 10\ \text{W/m}\cdot\text{K} \times 0.90(100 - 0)\text{K} = 900\ \text{W/m}$$ <

COMMENTS: Using a finite-element package with a fine mesh, we determined heat rates of 956 and 915 W/m, respectively, for Figures (a) and (b). The estimate for the less difficult Figure (b) is within 2% of the numerical method result. For Figure (a), our flux plot result was 27% low.

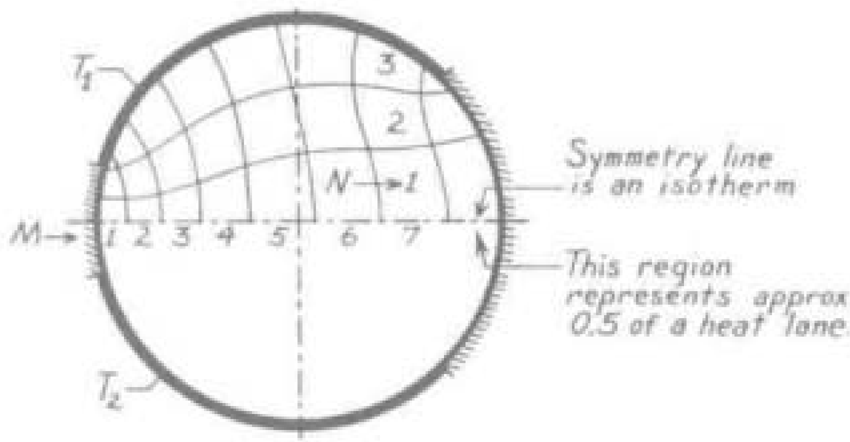
PROBLEM 4.15

KNOWN: Two-dimensional circular shapes maintained at uniform temperatures on portions of their boundaries.

FIND: Shape factors using flux plot method.

ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Constant properties.

ANALYSIS: Use the methodology of Section 4.3.1 to construct the flux plots. With Figure (a), we need only consider the upper half of the shape; hence, the horizontal line of symmetry is an isotherm. We've selected 3 increments of ΔT and then sketched the heat flow lanes, beginning at the left.

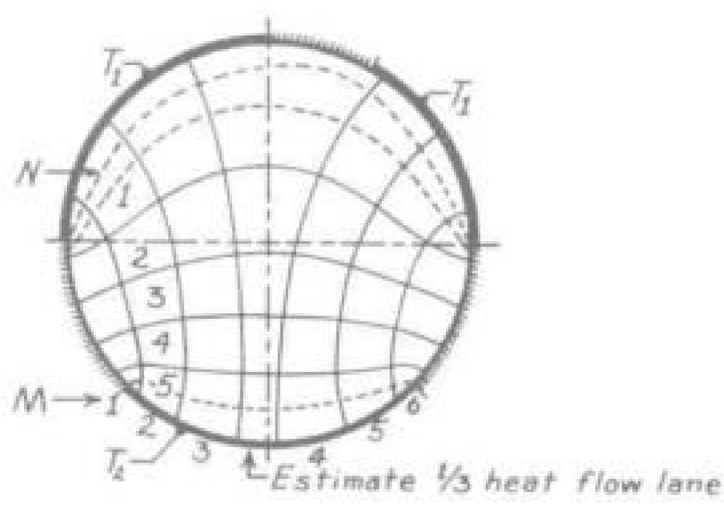


For the full circular shape,

$$S = \frac{M}{2N}\ell \approx \frac{7.5}{2\times 3}\ell \approx 1.25\ell \,. \qquad \triangleleft$$

With Figure (b), there is no symmetry to simplify the flux plotting. A value of N was chosen and isotherms sketched. Then, beginning at the left and right sides, the adiabats were sketched. Note the irregular heat flow lane at the center. The effect of the insulated region at the top has little influence on the isotherms sketched. The dashed lines represent intermediate isotherms.

$$S = \frac{M}{N}\ell \approx \frac{6.3}{5}\ell = 1.25\ell \qquad \triangleleft$$



PROBLEM 4.16

KNOWN: Uniform media of prescribed geometry.

FIND: (a) Shape factor expressions from thermal resistance relations for the plane wall, cylindrical shell and spherical shell, (b) Shape factor expression for the isothermal sphere of diameter D buried in an infinite medium.

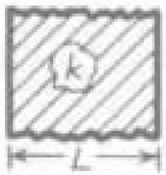
ASSUMPTIONS: (1) Steady-state conditions, (2) Uniform properties.

ANALYSIS: (a) The relationship between the shape factor and thermal resistance of a shape follows from their definitions in terms of heat rates and overall temperature differences.

$$q = kS\Delta T \quad (4.25), \qquad q = \frac{\Delta T}{R_t} \quad (3.18), \qquad S = 1/kR_t \quad (4.27)$$

Using the thermal resistance relations developed in Chapter 3, their corresponding shape factors are:

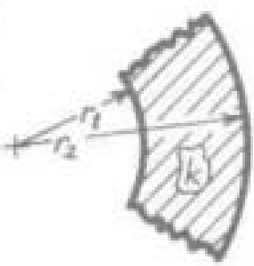
Plane wall:



$$R_t = \frac{L}{kA} \qquad S = \frac{A}{L}. \quad \triangleleft$$

Cylindrical shell:

(L into the page)

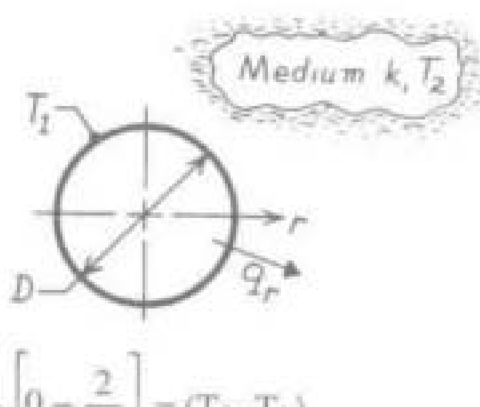


$$R_t = \frac{\ln(r_2/r_1)}{2\pi Lk} \qquad S = \frac{2\pi L}{\ln r_2/r_1}. \quad \triangleleft$$

Spherical shell:

$$R_t = \frac{1}{4\pi k}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \qquad S = \frac{4\pi}{1/r_1 - 1/r_2}. \quad \triangleleft$$

(b) The shape factor for the sphere of diameter D in an infinite medium can be derived easily using the alternative conduction analysis of Section 3.1. For this situation, q_r is a constant and Fourier's law has the form



$$q_r = -k(4\pi r^2)\frac{dT}{dr}.$$

Separate variables, identify limits and integrate.

$$-\frac{q_r}{4\pi k}\int_{D/2}^{\infty}\frac{dr}{r^2} = \int_{T_1}^{T_2} dT \qquad -\frac{q_r}{4\pi k}\left[-\frac{1}{r}\right]_{D/2}^{\infty} = -\frac{q_r}{4\pi k}\left[0 - \frac{2}{D}\right] = (T_2 - T_1)$$

$$q_r = 4\pi k\left[\frac{D}{2}\right](T_1 - T_2) \qquad \text{or} \qquad S = 2\pi D. \quad \triangleleft$$

COMMENTS: Note that the result for the buried sphere, $S = 2\pi D$, can be obtained from the expression for the spherical shell with $r_2 = \infty$. Also, the shape factor expression for the "isothermal sphere buried in a semi-infinite medium" presented in Table 4.1 provides the same result with $z \to \infty$.

PROBLEM 4.17

KNOWN: Heat generation in a buried spherical container.

FIND: (a) Outer surface temperature of the container, (b) Representative isotherms and heat flow lines.

SCHEMATIC:

ASSUMPTIONS: (1) Steady-state conditions, (2) Soil is a homogeneous medium with constant properties.

PROPERTIES: *Table A-3*, Soil (300K): k = 0.52 W/m·K.

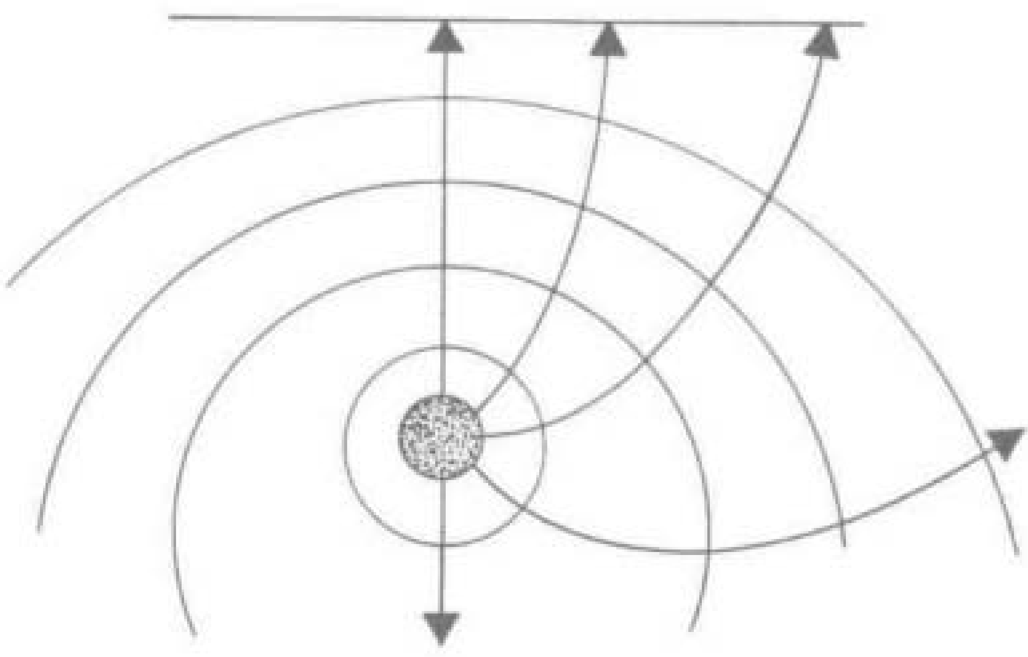
ANALYSIS: (a) From an energy balance on the container, $q = \dot{E}_g$ and from the first entry in Table 4.1,

$$q = \frac{2\pi D}{1-D/4z} k(T_1 - T_2).$$

Hence,

$$T_1 = T_2 + \frac{q}{k}\frac{1-D/4z}{2\pi D} = 20°C + \frac{500W}{0.52\frac{W}{m \cdot K}}\frac{1-2m/40m}{2\pi(2m)} = 92.7°C \quad \triangleleft$$

(b) The isotherms may be viewed as spherical surfaces whose center moves downward with increasing radius. The surface of the soil is an isotherm for which the center is at $z = \infty$.

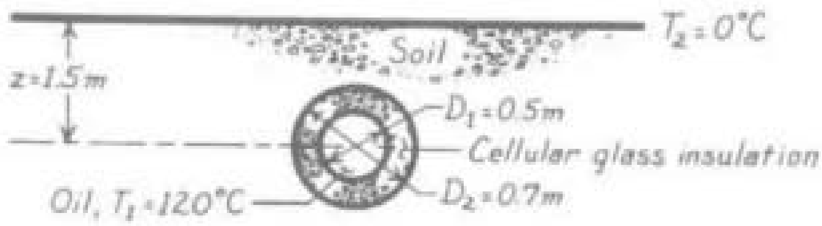


PROBLEM 4.18

KNOWN: Temperature, diameter and burial depth of an insulated pipe.

FIND: Heat loss per unit length of pipe.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction through insulation, two-dimensional through soil, (3) Constant properties, (4) Negligible oil convection and pipe wall conduction resistances.

PROPERTIES: *Table A-3*, Soil (300K): k = 0.52 W/m·K; *Table A-3*, Cellular glass (365K): k = 0.069 W/m·K.

ANALYSIS: The heat rate can be expressed as

$$q = \frac{T_1 - T_2}{R_{tot}}$$

where the thermal resistance is $R_{tot} = R_{ins} + R_{soil}$. From Eq. 3.28,

$$R_{ins} = \frac{\ln(D_2/D_1)}{2\pi L k_{ins}} = \frac{\ln(0.7\text{m}/0.5\text{m})}{2\pi L \times 0.069\ \text{W/m·K}} = \frac{0.776\text{m·K/W}}{L}.$$

From Eq. 4.25 and Table 4.1,

$$R_{soil} = \frac{1}{Sk_{soil}} = \frac{\cosh^{-1}(2z/D_2)}{2\pi L k_{soil}} = \frac{\cosh^{-1}(3/0.7)}{2\pi \times (0.52\ \text{W/m·K})L} = \frac{0.653}{L}\ \text{m·K/W}.$$

Hence,

$$q = \frac{(120-0)\,°\text{C}}{\frac{1}{L}(0.776+0.653)\frac{\text{m·K}}{\text{W}}} = 84\frac{\text{W}}{\text{m}} \times L$$

$$q' = q/L = 84\ \text{W/m}.$$ ◁

COMMENTS: (1) Contributions of the soil and insulation to the total resistance are approximately the same. The heat loss may be reduced by burying the pipe deeper or adding more insulation.

(2) The convection resistance associated with the oil flow through the pipe may be significant, in which case the foregoing result would overestimate the heat loss. A calculation of this resistance may be based on results presented in Chapter 8.

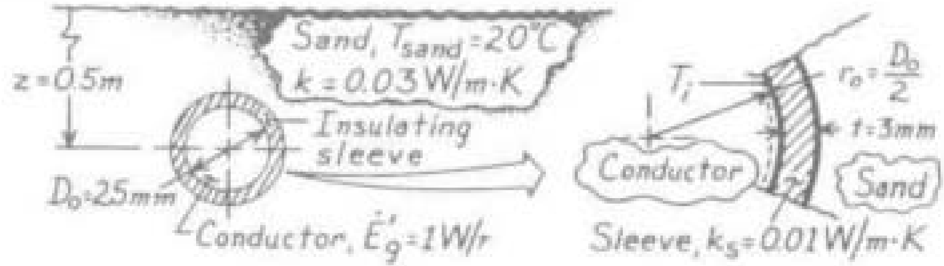
(3) Since $z > 3D/2$, the shape factor for the soil can also be evaluated from $S = 2\pi L/\ln(4z/D)$ of Table 4.1, and an equivalent result is obtained.

PROBLEM 4.19

KNOWN: Electric conductor with insulating sleeve buried in a sand-filled trench.

FIND: Temperature at the conductor-sleeve interface for prescribed dissipation rate.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Conductor approximates horizontal, isothermal cylinder buried in a semi-infinite medium.

ANALYSIS: Perform an energy balance on the conductor to find the radial heat rate per unit length, as $\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$ or $q'_r = \dot{E}'_g$. The insulating sleeve and sand medium may be represented by the thermal circuit,

where the insulating sleeve behaves as a cylindrical shell (Eq. 3.28),

$$R'_{sleeve} = \frac{\ln[(r_o+t)/r_o]}{2\pi k_{sleeve}} = \frac{\ln[(0.0125+0.003)/0.0125]}{2\pi\times 0.01\ \text{W/m}\cdot\text{K}} = 3.42\text{K}\cdot\text{m/W}.$$

The resistance of the sand follows from the appropriate shape factor for a buried cylinder of diameter $D = D_o + 2t$ (see Table 4.1 noting $z > 3D/2$),

$$R'_{sand} = \frac{1}{Sk} = \frac{\ln(4z/D)}{2\pi k_{sand}} = \frac{\ln[4\times 0.5\text{m}/(0.025+0.006)\text{m}]}{2\pi\times 0.03\ \text{W/m}\cdot\text{K}} = 22.11\ \text{K}\cdot\text{m/W}.$$

From the thermal circuit,

$$q'_r = \frac{T_i - T_{sand}}{R'_{sleeve}+R'_{sand}} \qquad \text{or} \qquad T_i = T_{sand} + q'_r(R'_{sleeve}+R'_{sand})$$

$$T_i = 20\,^\circ\text{C} + 1\frac{\text{W}}{\text{m}}(3.42+22.11)\frac{\text{K}\cdot\text{m}}{\text{W}} = 20\,^\circ\text{C} + 25.5\,^\circ\text{C} = 45.5\,^\circ\text{C}. \qquad \triangleleft$$

COMMENTS: (1) The thermal resistance of the insulating sleeve is $3.42/(3.42+22.11) = 13\%$ of the total thermal resistance.

(2) The maximum temperature will occur at the conductor centerline. If $k \approx 400$ W/m·K (pure copper), from Eq. 3.53,

$$T(0) = \frac{\dot{q}r_o^2}{4k} + T_i = \frac{2037\ \text{W/m}^3(0.0125\text{m})^2}{4\times 400\ \text{W/m}\cdot\text{K}} + 20\,^\circ\text{C} \approx 20\,^\circ\text{C}$$

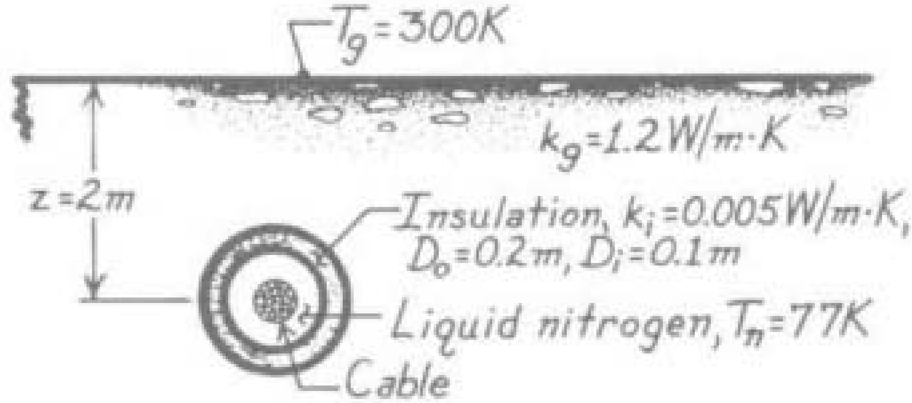
where $\dot{q} = \dot{E}'_g/A_c = (1\ \text{W/m})/(\pi 0.025^2\text{m}^2/4) = 2037\ \text{W/m}^3$. Hence the conductor is nearly isothermal.

PROBLEM 4.20

KNOWN: Operating conditions of a buried superconducting cable.

FIND: Required cooling load.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Two-dimensional conduction in soil, (4) One-dimensional conduction in insulation.

ANALYSIS: The heat rate per unit length is

$$q' = \frac{T_g - T_n}{R'_g + R'_I}$$

$$q' = \frac{T_g - T_n}{[k_g(2\pi/\ln(4z/D_o))]^{-1} + \ln(D_o/D_i)/2\pi k_i}$$

where Tables 3.3 and 4.1 have been used to evaluate the insulation and ground resistances, respectively. Hence,

$$q' = \frac{(300-77)\text{K}}{[(1.2\ \text{W/m}\cdot\text{K})(2\pi/\ln(8/0.2))]^{-1} + \ln(2)/2\pi\times0.005\ \text{W/m}\cdot\text{K}}$$

$$q' = \frac{223\ \text{K}}{(0.489 + 22.064)\text{m}\cdot\text{K/W}}$$

$$q' = 9.9\ \text{W/m}. \qquad \triangleleft$$

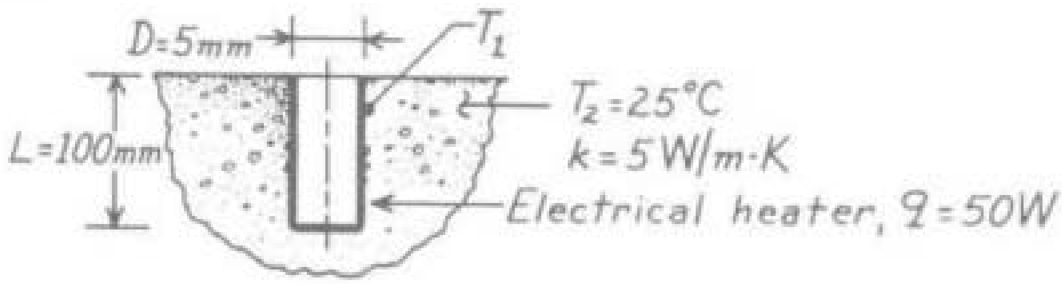
COMMENTS: The heat gain is small and the dominant contribution to the thermal resistance is made by the insulation.

PROBLEM 4.21

KNOWN: Electrical heater of cylindrical shape inserted into a hole drilled normal to the surface of a large block of material with prescribed thermal conductivity.

FIND: Temperature reached when heater dissipates 50 W with the block at 25 °C.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Block approximates semi-infinite medium with constant properties, (3) Negligible heat loss to surroundings above block surface, (4) Heater can be approximated as isothermal at T_1.

ANALYSIS: The temperature of the heater surface follows from the rate equation written as

$$T_1 = T_2 + q/kS$$

where S can be estimated from the conduction shape factor given in Table 4.1 for a "vertical cylinder in a semi-infinite medium,"

$$S = 2\pi L/\ln(4L/D) .$$

Substituting numerical values find

$$S = 2\pi \times 0.1\text{m}/\ln\left(\frac{4\times 0.1\text{m}}{0.005\text{m}}\right) = 0.143\text{m} .$$

The temperature of the heater is then

$$T_1 = 25\,^\circ\text{C} + 50\ \text{W}/(5\ \text{W/m}\cdot\text{K}\times 0.143\text{m}) = 94.9\,^\circ\text{C} .$$ ◁

COMMENTS: (1) Note that the heater has $L \gg D$ which is a requirement of the shape factor expression.

(2) Our calculation presumes there is negligible thermal contact resistance between the heater and the medium. In practice, this would not be the case unless a conducting paste were used.

(3) Since $L \gg D$, the assumptions (3) and (4) are reasonable.

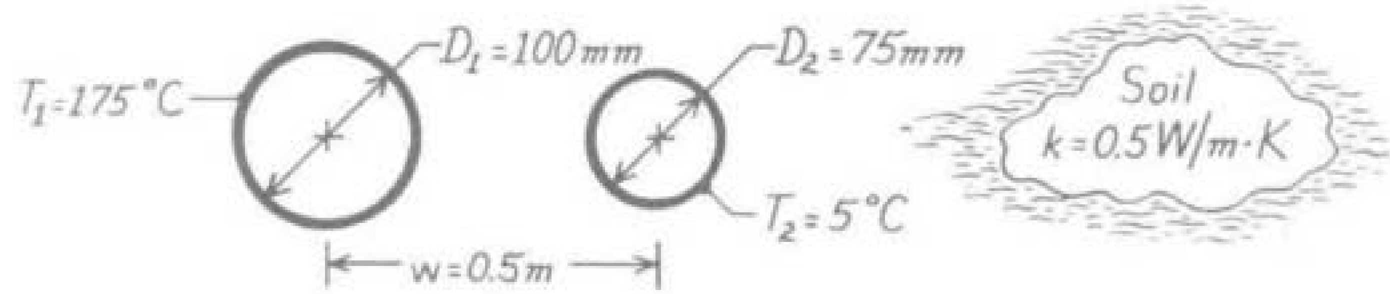
(4) This arrangement, referred to as the line source method, has been used to determine the thermal conductivity of materials from observations of q and T_1.

PROBLEM 4.22

KNOWN: Surface temperatures of two parallel pipe lines buried in soil.

FIND: Heat transfer per unit length between the pipe lines.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties, (4) Pipe lines are buried very deeply approximating burial in an infinite medium, (5) Pipe length $\gg D_1$ or D_2 and $w > D_1$ or D_2.

ANALYSIS: The heat transfer rate per unit length from the hot pipe to the cool pipe is

$$q' = \frac{q}{L} = \frac{S}{L}k(T_1 - T_2).$$

The shape factor S for this configuration is given in Table 4.1 as

$$S = \frac{2\pi L}{\cosh^{-1}\left[\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1 D_2}\right]}.$$

Substituting numerical values,

$$\frac{S}{L} = 2\pi/\cosh^{-1}\left[\frac{4\times(0.5m)^2 - (0.1m)^2 - (0.075m)^2}{2\times 0.1m\times 0.075m}\right] = 2\pi/\cosh^{-1}(65.63)$$

$$\frac{S}{L} = 2\pi/4.88 = 1.29.$$

Hence, the heat rate per unit length is

$$q' = 1.29\times 0.5 W/m\cdot K(175 - 5)°C = 110\ W/m. \qquad \triangleleft$$

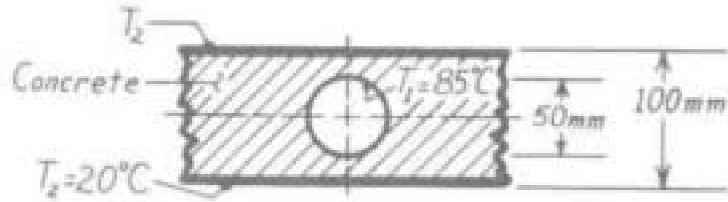
COMMENTS: The heat gain to the cooler pipe line will be larger than 110 W/m if the soil temperature is greater than 5°C. How would you estimate the heat gain if the soil were at 25°C?

PROBLEM 4.23

KNOWN: Tube embedded in the center plane of a concrete slab.

FIND: (a) The shape factor and heat transfer rate per unit length using the appropriate tabulated relation, (b) Shape factor using flux plot method.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional conduction, (2) Steady-state conditions, (3) Constant properties, (4) Concrete slab infinitely long in horizontal plane, $L \gg z$.

PROPERTIES: *Table A-3*, Concrete, stone mix (300K): $k = 1.4\ W/m \cdot K$.

ANALYSIS: (a) The embedded tube-slab system corresponds to the fifth system described in Table 4.1. Recognizing that our tube-slab system meets the restrictive conditions ($z > D/2$, $L \gg z$), the shape factor relation is

$$S = \frac{2\pi L}{\ell n(8z/\pi D)}$$

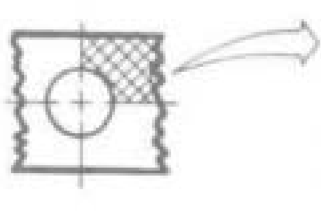
where L is the length of the system normal to the page, z is the half-thickness of the slab and D is the diameter of the tube. Substituting numerical values, find

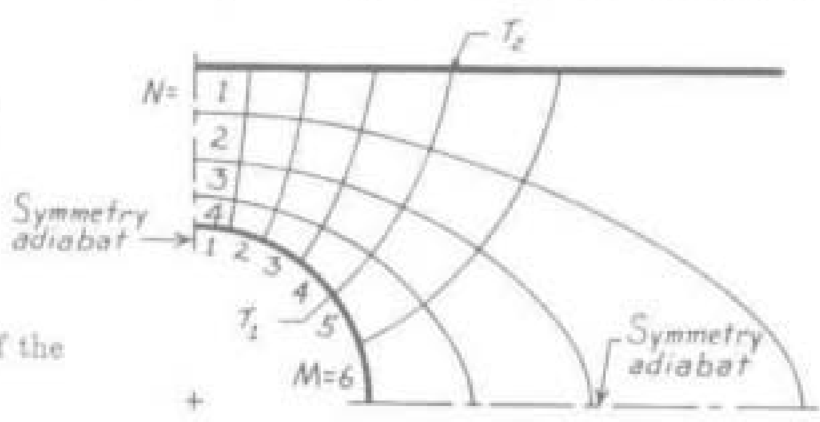
$$S = 2\pi L/\ell n(8\times 50mm/\pi 50mm) = 6.72L.$$

Hence, the heat rate per unit length is

$$q' = \frac{q}{L} = \frac{S}{L}k(T_1 - T_2) = 6.72\times 1.4\frac{W}{m \cdot K}(85 - 20)\,^\circ C = 612\ W.$$

(b) To find the shape factor using the flux plot method, first identify the symmetrical section bounded by the symmetry adiabats formed by the horizontal and vertical center lines. Selecting four temperature increments (N = 4), the flux plot can then be constructed.





From Eq. 4.26, the shape factor of the symmetrical section is

$$S_o = ML/N = 6L/4 = 1.5L.$$

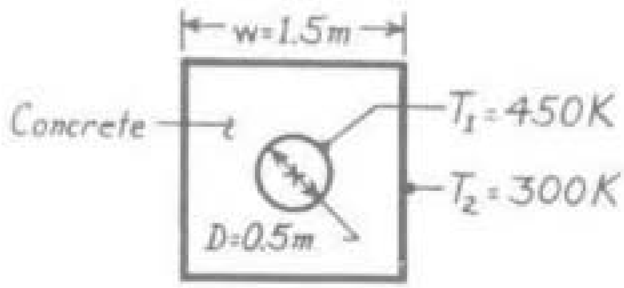
For the tube-slab system, it follows that $S = 4S_o = 6.0L$ which compares favorably with the shape factor relation.

PROBLEM 4.24

KNOWN: Dimensions and boundary temperatures of a steam pipe embedded in a concrete casing.

FIND: Heat loss per unit length.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Negligible steam side convection resistance, pipe wall resistance and contact resistance ($T_1 = 450K$), (3) Constant properties.

PROPERTIES: *Table A-3*, Concrete (300K): $k = 1.4\ W/m \cdot K$.

ANALYSIS: The heat rate can be expressed as

$$q = Sk\Delta T_{1-2} = Sk(T_1 - T_2)$$

From Table 4.1, the shape factor is

$$S = \frac{2\pi L}{\ln\left(\frac{1.08\ w}{D}\right)}.$$

Hence,

$$q' = \left(\frac{q}{L}\right) = \frac{2\pi k(T_1 - T_2)}{\ln\left(\frac{1.08\ w}{D}\right)}$$

$$q' = \frac{2\pi \times 1.4 W/m \cdot K \times (450-300)K}{\ln\left(\frac{1.08 \times 1.5m}{0.5m}\right)} = 1122\ W/m. \qquad \triangleleft$$

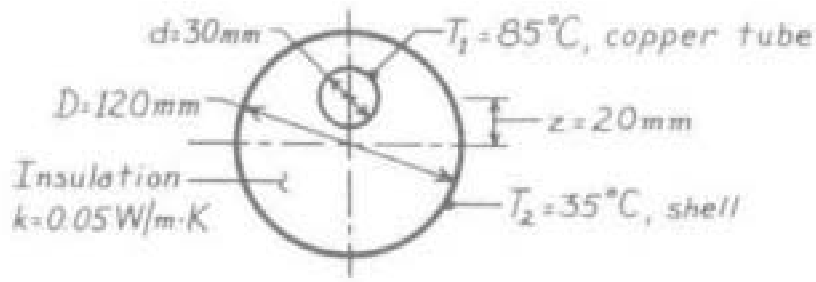
COMMENTS: Having neglected the steam side convection resistance, the pipe wall resistance, and the contact resistance, the foregoing result overestimates the actual heat loss.

PROBLEM 4.25

KNOWN: Thin-walled copper tube enclosed by an eccentric cylindrical shell; intervening space filled with insulation.

FIND: Heat loss per unit length of tube; compare result with that of a concentric tube-shell arrangement.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Thermal resistances of copper tube wall and outer shell wall are negligible, (4) Two-dimensional conduction in insulation.

ANALYSIS: The heat loss per unit length written in terms of the shape factor S is $q' = k(S/\ell)(T_1 - T_2)$ and from Table 4.1 for this geometry,

$$\frac{S}{\ell} = 2\pi/\cosh^{-1}\left(\frac{D^2 + d^2 - 4z^2}{2Dd}\right).$$

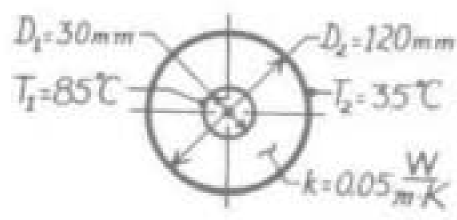
Substituting numerical values, all dimensions in mm,

$$\frac{S}{\ell} = 2\pi/\cosh^{-1}\left(\frac{120^2 + 30^2 - 4(20)^2}{2\times120\times30}\right) = 2\pi/\cosh^{-1}(1.903) = 4.991.$$

Hence, the heat loss is

$$q' = 0.05\,\mathrm{W/m\cdot K}\times 4.991(85-35)\,^\circ\mathrm{C} = 12.5\ \mathrm{W/m}. \quad \triangleleft$$

If the copper tube were concentric with the shell, but all other conditions were the same, the heat loss would be



$$q'_c = \frac{2\pi k(T_1 - T_2)}{\ln(D_2/D_1)}$$

using Eq. 3.27. Substituting numerical values,

$$q'_c = 2\pi\times 0.05\frac{\mathrm{W}}{\mathrm{m\cdot K}}(85-35)\,^\circ\mathrm{C}/\ln(120/30)$$

$$q'_c = 11.3\ \mathrm{W/m}. \quad \triangleleft$$

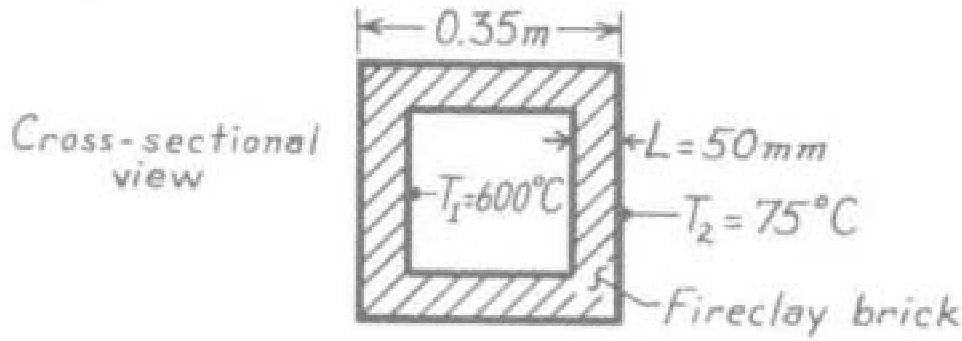
COMMENTS: As expected, the heat loss with eccentric arrangement is larger than that for the concentric arrangement. The effect of the eccentricity is to increase the heat loss by $(12.5-11.3)/11.3 \approx 11\%$.

PROBLEM 4.26

KNOWN: Cubical furnace, 350 mm external dimensions, with 50 mm thick walls.

FIND: The heat loss, q(W).

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

PROPERTIES: *Table A-3*, Fireclay brick ($\bar{T} = (T_1 + T_2)/2 = 610K$): k= 1.1 W/m·K.

ANALYSIS: Using relations for the shape factor from Table 4.1,

Plane Walls (6) $\quad S_W = \dfrac{A}{L} = \dfrac{0.25 \times 0.25 m^2}{0.05m} = 1.25m$

Edges (12) $\quad S_E = 0.54D = 0.54 \times 0.25m = 0.14m$

Corners (8) $\quad S_C = 0.15L = 0.15 \times 0.05m = 0.008m.$

The heat rate in terms of the shape factors is

$$q = kS(T_1 - T_2) = k(6S_W + 12S_E + 8S_C)(T_1 - T_2)$$

$$q = 1.1 \frac{W}{m \cdot K}(6 \times 1.25m + 12 \times 0.14m + 0.15 \times 0.008m)(600 - 75)°C$$

$$q = 5.30\ kW. \qquad \triangleleft$$

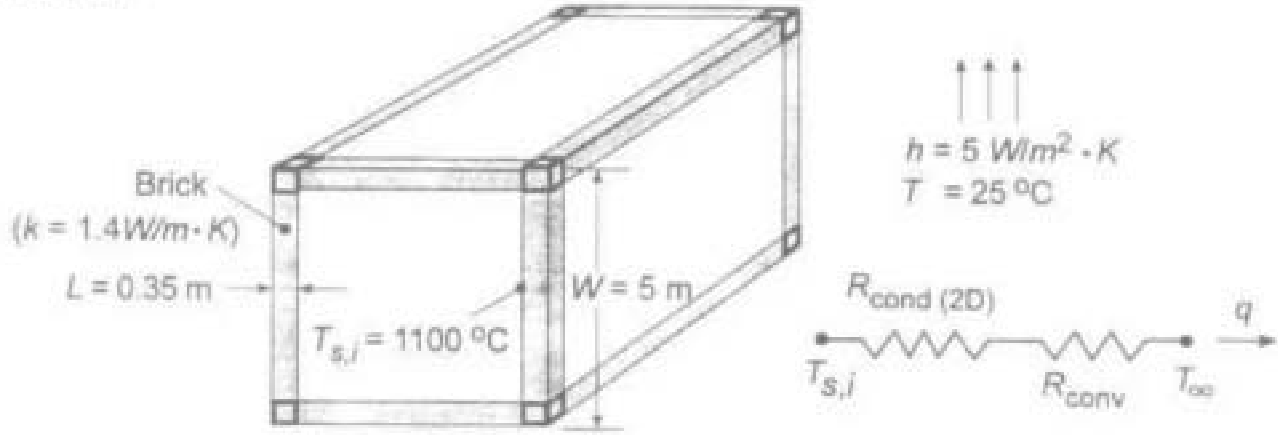
COMMENTS: Be sure to note that the restrictions for S_E and S_C have been met.

PROBLEM 4.27

KNOWN: Dimensions, thermal conductivity and inner surface temperature of furnace wall. Ambient conditions.

FIND: Heat loss.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, (2) Uniform convection coefficient over entire outer surface of container.

ANALYSIS: From the thermal circuit, the heat loss is

$$q = \frac{T_{s,i} - T_{\infty}}{R_{cond(2D)} + R_{conv}}$$

where $R_{conv} = 1/hA_{s,o} = 1/(hW^2) = 1/[5\ W/m^2 \cdot K(5\ m)^2] = 0.008$ K/W. From Eq. (4.27), the two-dimensional conduction resistance is

$$R_{cond(2D)} = \frac{1}{Sk}$$

where the shape factor S must include the effects of conduction through the 8 corners, 12 edges and 6 plane walls. Hence, using the relations for Cases 8 and 9 of Table 4.1,

$$S = 8(0.15L) + 12 \times 0.54(W - L) + 6A_{s,i}/L$$

where $A_{s,i} = (W - L)^2$. Hence,

$$S = [8(0.15 \times 0.35) + 12 \times 0.54(4.65) + 6(21.62)]\text{m}$$

$$S = (0.42 + 30.13 + 129.74)\text{m} = 160.29\text{m}$$

and $R_{cond(2D)} = 1/(160.29\ m \times 1.4\ W/m \cdot K) = 0.00446$ K/W. Hence

$$q = \frac{(1100 - 25)^{\circ}C}{(0.00446 + 0.008)\,K/W} = 86.3\,kW$$ <

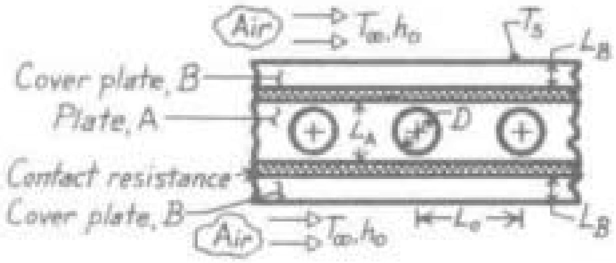
COMMENTS: The heat loss is extremely large and measures should be taken to insulate the furnace.

PROBLEM 4.28

KNOWN: Platen heated by passage of hot fluid in poor thermal contact with cover plates exposed to cooler ambient air.

FIND: (a) Heat rate per unit thickness from each channel, q_i', (b) Surface temperature of cover plate, T_s, (c) q_i' and T_s if lower surface is perfectly insulated, (d) Effect of changing centerline spacing on q_i' and T_s.

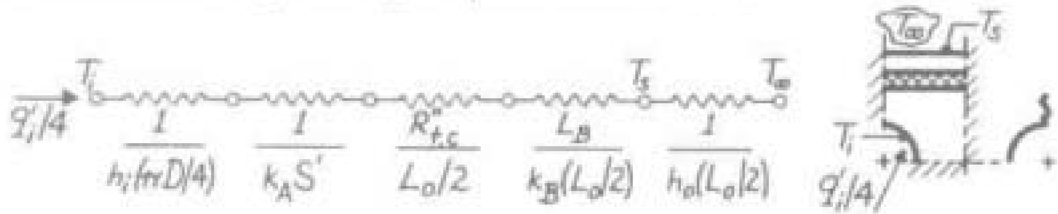
SCHEMATIC:



$D = 15$ mm, $L_o = 60$ mm
$L_A = 30$ mm, $L_B = 7.5$ mm
$T_i = 150°C$, $h_i = 1000$ W/m²·K
$T_\infty = 25°C$, $h_o = 200$ W/m²·K
$k_A = 20$ W/m·K, $k_B = 75$ W/m·K
$R''_{t,c} = 2.0\times10^{-4}$ m²·K/W

ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction in platen, but one-dimensional in coverplate, (3) Temperature of interfaces between A and B is uniform, (4) Constant properties.

ANALYSIS: (a) The heat rate per unit thickness from heat channel can be determined from the following thermal circuit representing the quarter section shown.



The value for the shape factor is $S' = 1.06$ as determined from the flux plot shown on the next page. Hence, the heat rate is

$$q_i' = 4(T_i - T_\infty)/R'_{tot} \qquad (1)$$

$$R'_{tot} = [1/1000\ \text{W/m}^2\cdot\text{K}(\pi 0.015\text{m}/4) + 1/20\ \text{W/m}\cdot\text{K}\times 1.06$$
$$+ 2.0\times10^{-4}\text{m}^2\cdot\text{K/W}/(0.060\text{m}/2) + 0.0075\text{m}/75\ \text{W/m}\cdot\text{K}(0.060\text{m}/2)$$
$$+ 1/200\ \text{W/m}^2\cdot\text{K}(0.060\text{m}/2)]$$

$$R'_{tot} = [0.085 + 0.047 + 0.0067 + 0.0033 + 0.1667]\text{m}\cdot\text{K/W}$$

$$R'_{tot} = 0.309\ \text{m}\cdot\text{K/W}$$

$$q_i' = 4(150 - 25)\text{K}/0.309\ \text{m}\cdot\text{K/W} = 1.62\ \text{kW/m}. \qquad \triangleleft$$

(b) The surface temperature of the cover plate follows also from the thermal circuit as

$$q_i'/4 = \frac{T_s - T_\infty}{1/h_o(L_o/2)} \qquad (2)$$

Continued .....

$$T_s = T_\infty + \frac{q_i'}{4}\frac{1}{h_o(L_o/2)} = 25°C + \frac{1.62\ kW}{4} \times 0.167\ m \cdot K/W$$

$$T_s = 25°C + 67.6°C = 93°C. \quad \triangleleft$$

(c,d) The effect of the centerline spacing on q_i' and T_s can be understood by examining the relative magnitudes of the thermal resistances. The dominant resistance is that due to the ambient air convection process which is inversely related to the spacing L_o. Hence, the heat rate will increase nearly linearly with an increase in L_o, that is from Eq. (1),

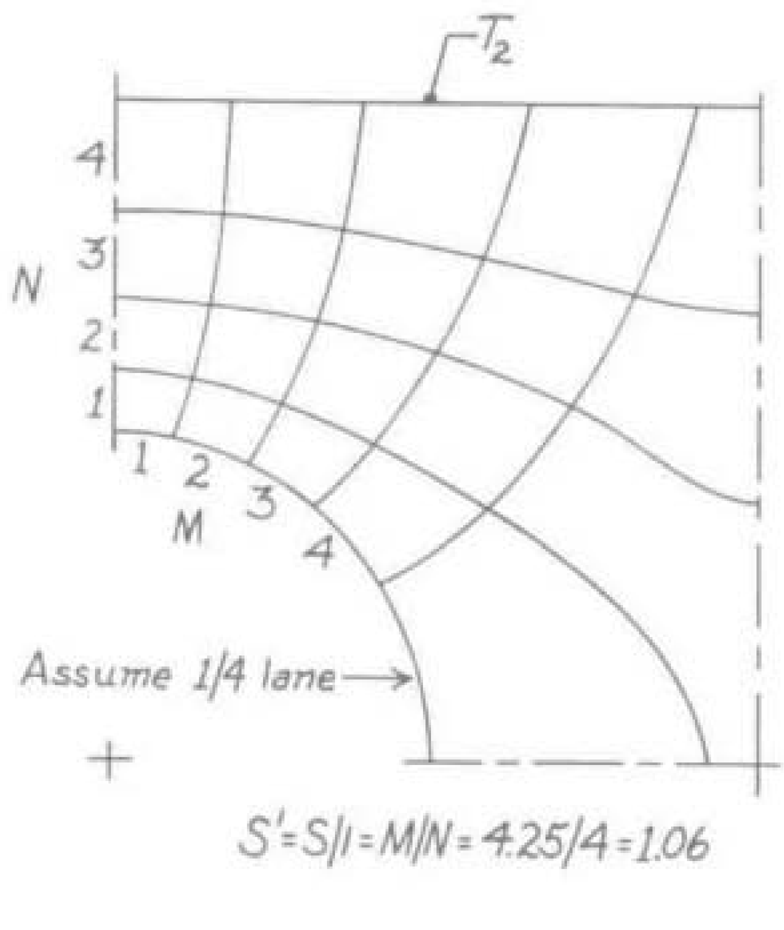
$$q_i' \sim \frac{1}{R_{tot}} \approx \frac{1}{1/h_o(L_o/2)} \sim L_o.$$

From Eq. (2), find

$$\Delta T = T_s - T_\infty = q_i' \frac{1}{h_o(L_o/2)} \sim q_i' \cdot L_o^{-1} \sim L_o \cdot L_o^{-1} = 1.$$

Hence we conclude that ΔT will not increase with a change in L_o. Does this seem reasonable? What effect does the L_o spacing have on Assumptions (2) and (3)?

If the lower surface were insulated, the heat rate would be decreased nearly by half. This follows again from the fact that the overall resistance is dominated by the surface convection process. The temperature difference T_s - T_∞ would only slightly increase.

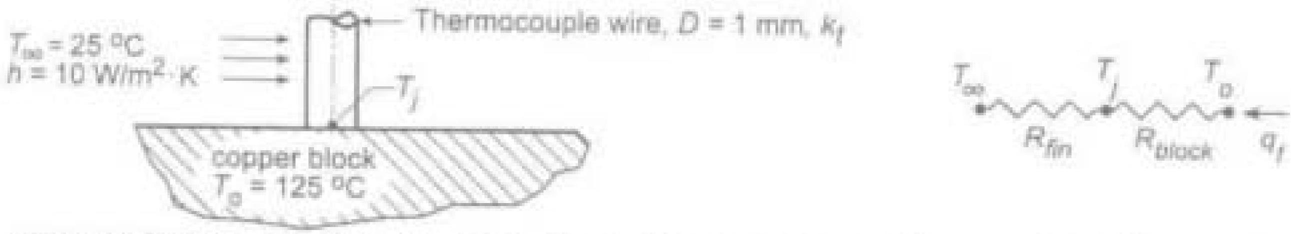


PROBLEM 4.29

KNOWN: Long constantan wire butt-welded to a large copper block forming a thermocouple junction on the surface of the block.

FIND: (a) The measurement error $(T_j - T_o)$ for the thermocouple for prescribed conditions, and (b) Compute and plot $(T_j - T_o)$ for h = 5, 10 and 25 W/m^2·K for block thermal conductivity $15 \le k \le 400$ W/m·K. When is it advantageous to use smaller diameter wire?

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Thermocouple wire behaves as a fin with constant heat transfer coefficient, (3) Copper block has uniform temperature, except in the vicinity of the junction.

PROPERTIES: *Table A-1*, Copper (pure, 400 K), k_b = 393 W/m·K; Constantan (350 K), k_t = 25 W/m·K.

ANALYSIS: The thermocouple wire behaves as a long fin permitting heat to flow from the surface thereby depressing the sensing junction temperature below that of the block T_o. In the block, heat flows into the circular region of the wire-block interface; the thermal resistance to heat flow within the block is approximated as a disk of diameter D on a semi-infinite medium (k_b, T_o). The thermocouple-block combination can be represented by a thermal circuit as shown above. The thermal resistance of the fin follows from the heat rate expression for an infinite fin, $R_{fin} = (hPk_tA_c)^{-1/2}$.

From Table 4.1, the shape factor for the disk-on-a-semi-infinite medium is given as S = 2D and hence $R_{block} = 1/k_bS = 1/2k_bD$. From the thermal circuit,

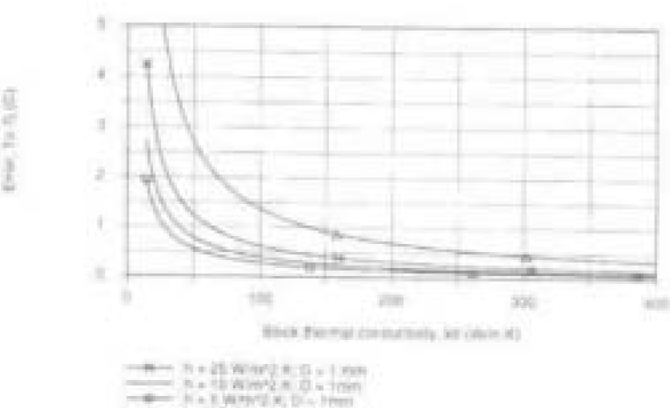
$$T_o - T_j = \frac{R_{block}}{R_{fin} + R_{block}}(T_o - T_\infty) = \frac{1.27}{1273 + 1.27}(125 - 25)^\circ C \approx 0.001(125 - 25)^\circ C = 0.1^\circ C. \quad <$$

with $P = \pi D$ and $A_c = \pi D^2/4$ and the thermal resistances as

$$R_{fin} = \left(10\,W/m^2 \cdot K(\pi/4)25\,W/m \cdot K \times \left(1 \times 10^{-3}\,m\right)^3\right)^{-1/2} = 1273\,K/W$$

$$R_{block} = (1/2) \times 393\,W/m \cdot K \times 10^{-3}\,m = 1.27\,K/W.$$

(b) We keyed the above equations into the IHT workspace, performed a sweep on k_b for selected values of h and created the plot shown. When the block thermal conductivity is low, the error $(T_o - T_j)$ is larger, increasing with increasing convection coefficient. A smaller diameter wire will be advantageous for low values of k_b and higher values of h.

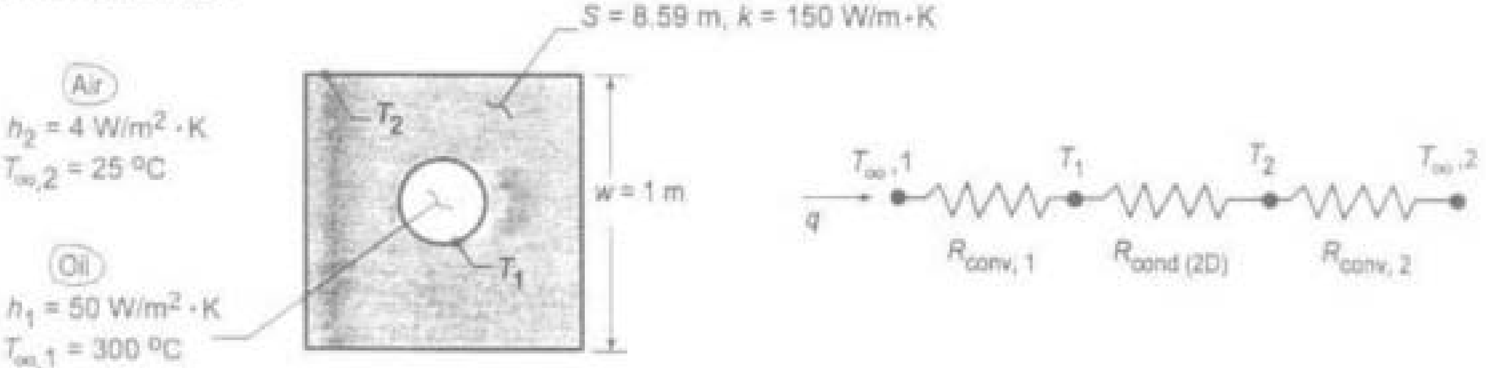


PROBLEM 4.30

KNOWN: Dimensions, shape factor, and thermal conductivity of square rod with drilled interior hole. Interior and exterior convection conditions.

FIND: Heat rate and surface temperatures.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties, (3) Uniform convection coefficients at inner and outer surfaces.

ANALYSIS: The heat loss can be expressed as

$$q = \frac{T_{\infty,1} - T_{\infty,2}}{R_{conv,1} + R_{cond(2D)} + R_{conv,2}}$$

where

$$R_{conv,1} = (h_1 \pi D_1 L)^{-1} = \left(50\,W/m^2\cdot K \times \pi \times 0.25\,m \times 2\,m\right)^{-1} = 0.01273\,K/W$$

$$R_{cond(2D)} = (Sk)^{-1} = (8.59\,m \times 150\,W/m\cdot K)^{-1} = 0.00078\,K/W$$

$$R_{conv,2} = (h_2 \times 4wL)^{-1} = \left(4\,W/m^2\cdot K \times 4\,m \times 1\,m\right)^{-1} = 0.0625\,K/W$$

Hence,

$$q = \frac{(300-25)^\circ C}{0.076\,K/W} = 3.62\,kW$$ <

$$T_1 = T_{\infty,1} - qR_{conv,1} = 300^\circ C - 46^\circ C = 254^\circ C$$ <

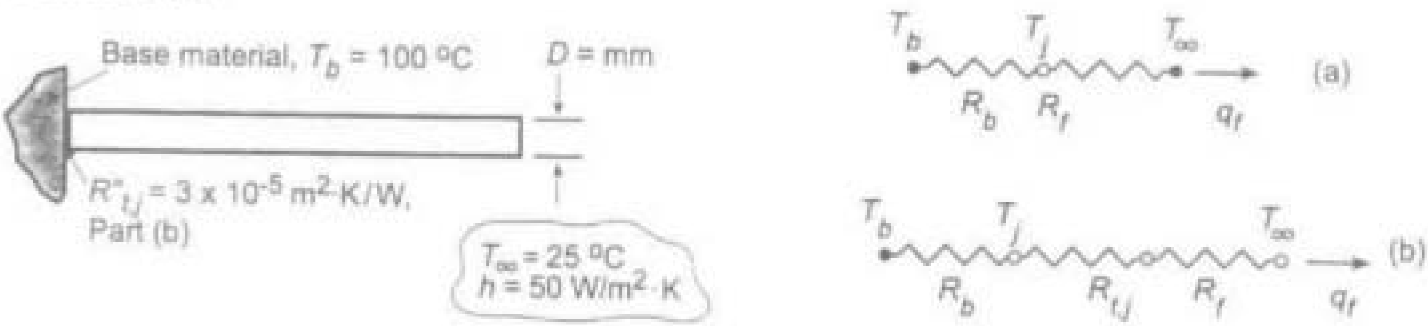
$$T_2 = T_{\infty,2} + qR_{conv,2} = 25^\circ C + 226^\circ C = 251^\circ C$$ <

COMMENTS: The largest resistance is associated with convection at the outer surface, and the conduction resistance is much smaller than both convection resistances. Hence, $(T_2 - T_{\infty,2}) > (T_{\infty,1} - T_1) >> (T_1 - T_2)$.

KNOWN: Long fin of aluminum alloy with prescribed convection coefficient attached to different base materials (aluminum alloy or stainless steel) with and without thermal contact resistance $R''_{t,j}$ at the junction.

FIND: (a) Heat rate q_f and junction temperature T_j for base materials of aluminum and stainless steel, (b) Repeat calculations considering thermal contact resistance, $R''_{t,j}$, and (c) Plot as a function of h for the range $10 \le h \le 1000\ W/m^2 \cdot K$ for each base material.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Infinite fin.

PROPERTIES: (Given) Aluminum alloy, k = 240 W/m·K, Stainless steel, k = 15 W/m·K.

ANALYSIS: (a,b) From the thermal circuits, the heat rate and junction temperature are

$$q_f = \frac{T_b - T_\infty}{R_{tot}} = \frac{T_b - T_\infty}{R_b + R_{t,j} + R_f} \qquad (1)$$

$$T_j = T_\infty + q_f R_f \qquad (2)$$

and, with $P = \pi D$ and $A_c = \pi D^2/4$, from Tables 4.1 and 3.4 find

$$R_b = 1/Sk_b = 1/(2Dk_b) = (2 \times 0.005\,m \times k_b)^{-1}$$

$$R_{t,j} = R''_{t,j}/A_c = 3\times10^{-5}\,m^2 \cdot K/W/\pi(0.005\,m)^2/4 = 1.528\,K/W$$

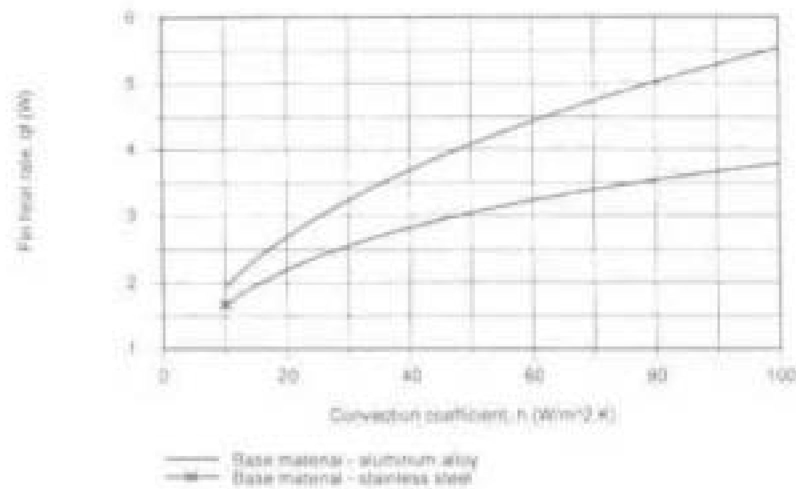
$$R_f = (hPkA_c)^{-1/2} = \left[50\,W/m^2 \cdot K\pi^2(0.005\,m)^3\,240\,W/m \cdot K/4\right]^{-1/2} = 16.4\,K/W$$

		Without $R''_{t,j}$		With $R''_{t,j}$	
Base	R_b (K/W)	q_f (W)	T_j (°C)	q_f (W)	T_j (°C)
Al alloy	0.417	4.46	98.2	4.09	92.1
St. steel	6.667	3.26	78.4	3.05	75.1

(c) We used the *IHT Model for Extended Surfaces, Performance Calculations, Rectangular Pin Fin* to calculate q_f for $10 \le h \le 100\ W/m^2 \cdot K$ by replacing R''_c (thermal resistance at fin base) by the sum of the contact and spreading resistances, $R''_{t,j} + R''_b$.

Continued...

PROBLEM 4.31 (Cont.)



COMMENTS: (1) From part (a), the aluminum alloy base material has negligible effect on the fin heat rate and depresses the base temperature by only 2°C. The effect of the stainless steel base material is substantial, reducing the heat rate by 27% and depressing the junction temperature by 25°C.

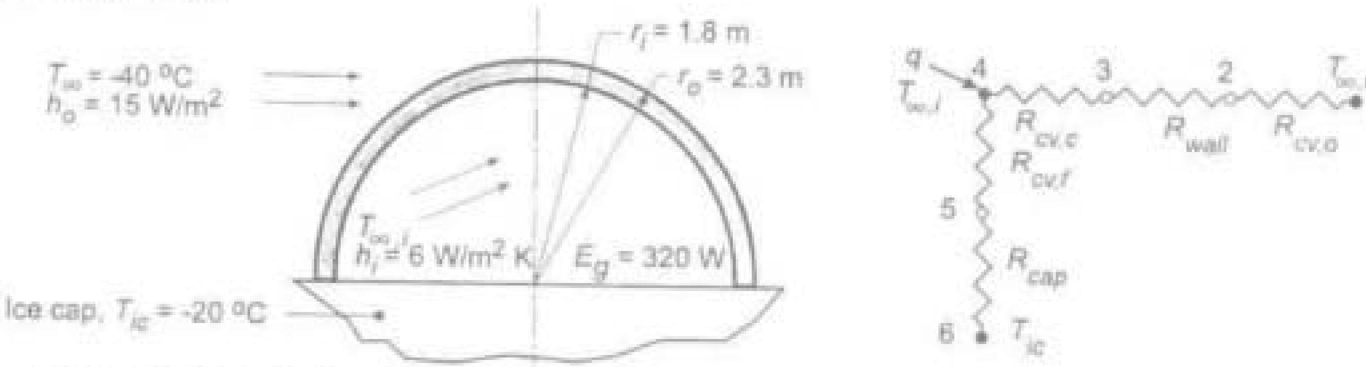
(2) The contact resistance reduces the heat rate and increases the temperature depression relatively more with the aluminum alloy base.

(3) From the plot of q_f vs. h, note that at low values of h, the heat rates are nearly the same for both materials since the fin is the dominant resistance. As h increases, the effect of $R''_{t,c}$ becomes more important.

KNOWN: Igloo constructed in hemispheric shape sits on ice cap; igloo wall thickness and inside/outside convection coefficients (h_i, h_o) are prescribed.

FIND: (a) Inside air temperature $T_{\infty,i}$ when outside air temperature is $T_{\infty,o} = -40°C$ assuming occupants provide 320 W within igloo, (b) Perform parameter sensitivity analysis to determine which variables have significant effect on T_i.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Convection coefficient is the same on floor and ceiling of igloo, (3) Floor and ceiling are at uniform temperature, (4) Floor-ice cap resembles disk on semi-infinite medium, (5) One-dimensional conduction through igloo walls.

PROPERTIES: Ice and compacted snow (given): k = 0.15 W/m·K.

ANALYSIS: (a) The thermal circuit representing the heat loss from the igloo to the outside air and through the floor to the ice cap is shown above. The heat loss is

$$q = \frac{T_{\infty,i} - T_{\infty,o}}{R_{cv,c} + R_{wall} + R_{cv,o}} + \frac{T_{\infty,i} - T_{ic}}{R_{cv,f} + R_{cap}}.$$

Convection, ceiling: $$R_{cv,c} = \frac{2}{h_i(4\pi r_i^2)} = \frac{2}{6\,W/m^2\cdot K \times 4\pi(1.8\,m)^2} = 0.00819\,K/W$$

Convection, outside: $$R_{cv,o} = \frac{2}{h_o(4\pi r_o^2)} = \frac{2}{15\,W/m^2\cdot K \times 4\pi(2.3\,m)^2} = 0.00201\,K/W$$

Convection, floor: $$R_{cv,f} = \frac{1}{h_i(\pi r_i^2)} = \frac{1}{6\,W/m^2\cdot K \times \pi(1.8\,m)^2} = 0.01637\,K/W$$

Conduction, wall: $$R_{wall} = 2\left[\frac{1}{4\pi k}\left(\frac{1}{r_i} - \frac{1}{r_o}\right)\right] = 2\left[\frac{1}{4\pi \times 0.15\,W/m\cdot K}\left(\frac{1}{1.8} - \frac{1}{2.3}\right)m\right] = 0.1281\,K/W$$

Conduction, ice cap: $$R_{cap} = \frac{1}{kS} = \frac{1}{4kr_i} = \frac{1}{4 \times 0.15\,W/m\cdot K \times 1.8\,m} = 0.9259\,K/W$$

where S was determined from the shape factor of Table 4.1. Hence,

$$q = 320\,W = \frac{T_{\infty,i} - (-40)°C}{(0.00818 + 0.1281 + 0.0020)\,K/W} + \frac{T_{\infty,i} - (-20)°C}{(0.01637 + 0.9259)\,K/W}$$

$$320\,W = 7.232(T_{\infty,i} + 40) + 1.06(T_{\infty,i} + 20) \qquad T_{\infty,i} = 1.1°C. \qquad <$$

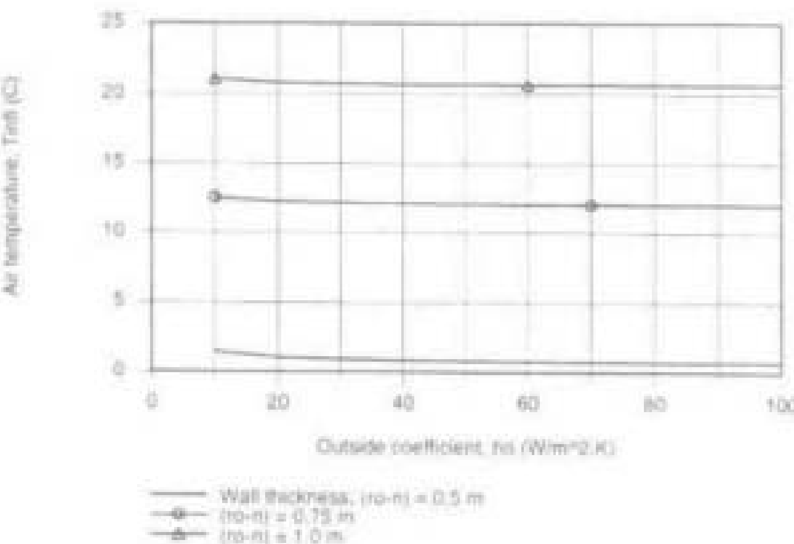
Continued...

(b) Begin the parameter sensitivity analysis to determine important variables which have a significant influence on the inside air temperature by examining the thermal resistances associated with the processes present in the system and represented by the network.

Process	Symbols		Value (K/W)
Convection, outside	$R_{cv,o}$	R21	0.0020
Conduction, wall	R_{wall}	R32	0.1281
Convection, ceiling	$R_{cv,c}$	R43	0.0082
Convection, floor	$R_{cv,f}$	R54	0.0164
Conduction, ice cap	R_{cap}	R65	0.9259

It follows that the convection resistances are negligible relative to the conduction resistance across the igloo wall. As such, only changes to the wall thickness will have an appreciable effect on the inside air temperature relative to the outside ambient air conditions. We don't want to make the igloo walls thinner and thereby allow the air temperature to dip below freezing for the prescribed environmental conditions.

Using the *IHT Thermal Resistance Network Model*, we used the circuit builder to construct the network and perform the energy balances to obtain the inside air temperature as a function of the outside convection coefficient for selected increased thicknesses of the wall.



COMMENTS: (1) From the plot, we can see that the influence of the outside air velocity which controls the outside convection coefficient h_o is negligible.

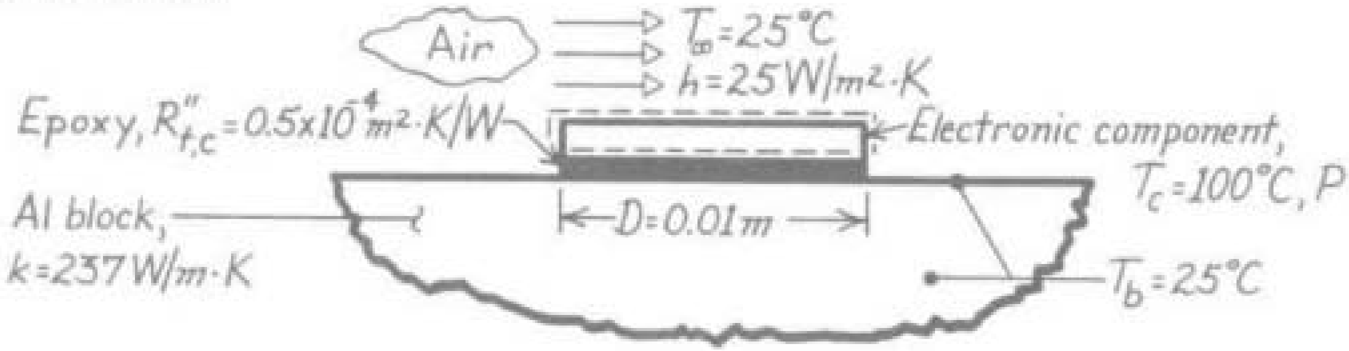
(2) The thickness of the igloo wall is the dominant thermal resistance controlling the inside air temperature.

PROBLEM 4.33

KNOWN: Diameter and maximum allowable temperature of an electronic component. Contact resistance between component and large aluminum heat sink. Temperature of heat sink and convection conditions at exposed component surface.

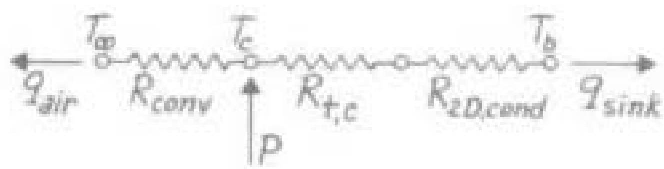
FIND: (a) Thermal circuit, (b) Maximum operating power of component.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Negligible heat loss from sides of chip.

ANALYSIS: (a) The thermal circuit is:



where $R_{2D,cond}$ is evaluated from the shape factor S = 2D of Table 4.1.

(b) Performing an energy balance for a control surface about the component,

$$P = q_{air} + q_{sink} = h(\pi D^2/4)(T_c - T_\infty) + \frac{T_c - T_b}{R''_{t,c}/(\pi D^2/4) + 1/2Dk}$$

$$P = 25\ \text{W/m}^2\cdot\text{K}(\pi/4)(0.01\ \text{m})^2\, 75\,^\circ\text{C} + \frac{75\,^\circ\text{C}}{\{[0.5\times10^{-4}/(\pi/4)(0.01)^2] + (0.02\times237)^{-1}\}\text{K/W}}$$

$$P = 0.15\ \text{W} + \frac{75\,^\circ\text{C}}{(0.64 + 0.21)\text{K/W}} = 0.15\ \text{W} + 88.49\ \text{W} = 88.6\ \text{W}. \qquad \triangleleft$$

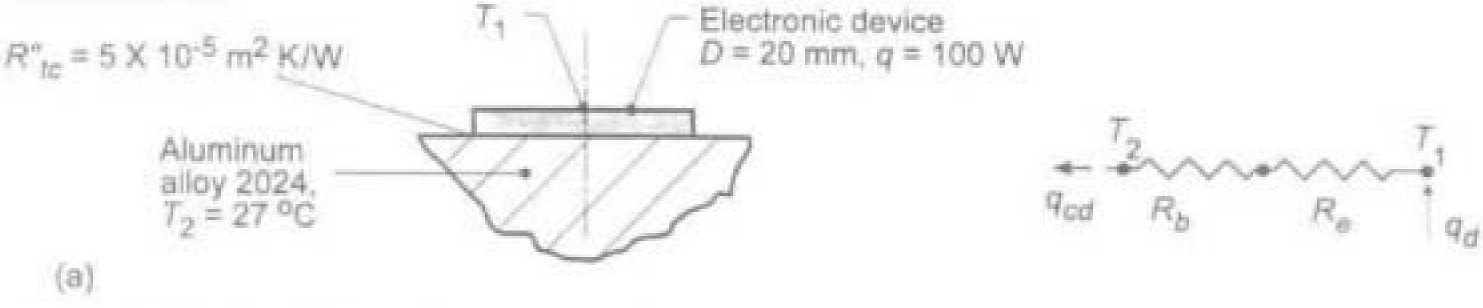
COMMENTS: The convection resistance is much larger than the cumulative contact and conduction resistance. Hence, virtually all of the heat dissipated in the component is transferred through the block. The two-dimensional conduction resistance is significantly underestimated by use of the shape factor S = 2D. Hence, the maximum allowable power is less than 88.6 W.

PROBLEM 4.34

KNOWN: Disc-shaped electronic devices dissipating 100 W mounted to aluminum alloy block with prescribed contact resistance.

FIND: (a) Temperature device will reach when block is at 27°C assuming all the power generated by the device is transferred by conduction to the block and (b) For the operating temperature found in part (a), the permissible operating power with a 30-pin fin heat sink.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional, steady-state conduction, (2) Device is at uniform temperature, T_1, (3) Block behaves as semi-infinite medium.

PROPERTIES: *Table A.1*, Aluminum alloy 2024 (300 K): k = 177 W/m·K.

ANALYSIS: (a) The thermal circuit for the conduction heat flow between the device and the block shown in the above Schematic where R_e is the thermal contact resistance due to the epoxy-filled interface,

$$R_e = R''_{t,c}/A_c = R''_{t,c}/\left(\pi D^2/4\right)$$

$$R_e = 5\times10^{-5}\ \text{K}\cdot\text{m}^2/\text{W}/\left(\pi(0.020\,\text{m})^2\right)/4 = 0.159\,\text{K/W}$$

The thermal resistance between the device and the block is given in terms of the conduction shape factor, Table 4.1, as

$$R_b = 1/Sk = 1/(2Dk)$$

$$R_b = 1/(2\times0.020\,\text{m}\times177\,\text{W/m}\cdot\text{K}) = 0.141\,\text{K/W}$$

From the thermal circuit,

$$T_1 = T_2 + q_d\left(R_b + R_e\right)$$

$$T_1 = 27^\circ\text{C} + 100\,\text{W}(0.141+0.159)\,\text{K/W}$$

$$T_1 = 27^\circ\text{C} + 30^\circ\text{C} = 57^\circ\text{C}$$ <

(b) The schematic below shows the device with the 30-pin fin heat sink with fins and base material of copper (k = 400 W/m·K). The airstream temperature is 27°C and the convection coefficient is 1000 W/m^2·K.

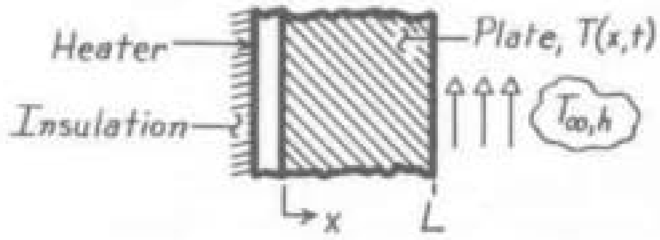
Continued...

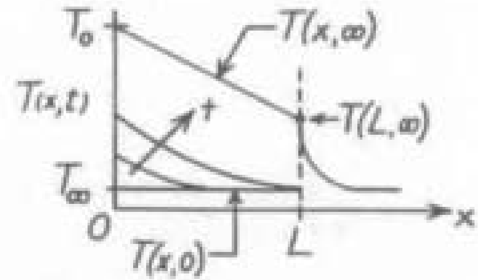
PROBLEM 5.1

KNOWN: Electrical heater attached to backside of plate while front surface is exposed to convection process (T_∞, h); initially plate is at a uniform temperature of the ambient air and suddenly heater power is switched on providing a constant q_o''.

FIND: (a) Sketch temperature distribution, T(x,t), (b) Sketch the heat flux at the outer surface, $q_x''(L,t)$ as a function of time.

SCHEMATIC:





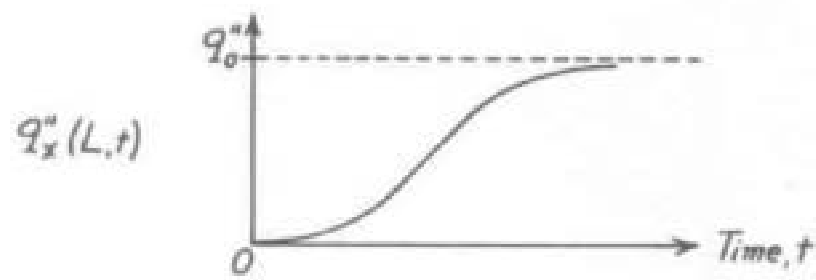
ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Negligible heat loss from heater through insulation.

ANALYSIS: (a) The temperature distributions for four time conditions including the initial distribution, T(x,0), and the steady-state distribution, T(x,∞), are as shown above.

Note that the temperature gradient at x=0, $-dT/dx)_{x=0}$, for t>0 will be a constant since the flux, $q_x''(0)$, is a constant. Noting that $T_o = T(0,\infty)$, the steady-state temperature distribution will be linear such that

$$q_o'' = k\,\frac{T_o - T(L,\infty)}{L} = h[T(L,\infty) - T_\infty]\,.$$

(b) The heat flux at the front surface, x=L, will be given by $q_x''(L,t) = -k\,(dT/dx)_{x=L}$. From the temperature distribution, we can construct the heat flux-time plot.



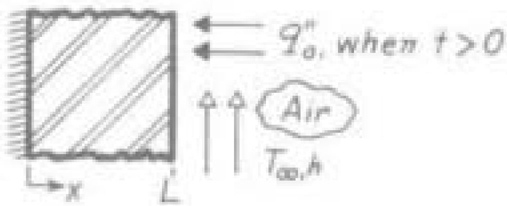
COMMENTS: At early times, the temperature and heat flux at x=L will not change from their initial values. Hence, we show a zero slope for $q_x''(L,t)$ at early times. Eventually, the value of $q_x''(L,t)$ will reach the steady-state value which is q_o''.

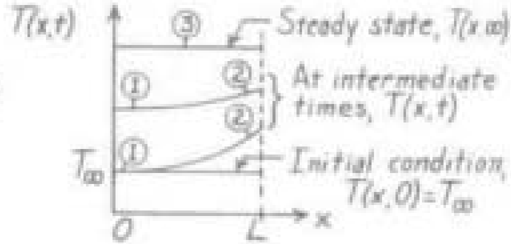
PROBLEM 5.2

KNOWN: Plane wall whose inner surface is insulated and outer surface is exposed to an airstream at T_∞. Initially, the wall is at a uniform temperature equal to that of the airstream. Suddenly, a radiant source is switched on applying a uniform flux, q''_o, to the outer surface.

FIND: (a) Sketch temperature distribution on T-x coordinates for initial, steady-state, and two intermediate times, (b) Sketch heat flux at the outer surface, $q''_x(L,t)$, as a function of time.

SCHEMATIC:





ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) No internal generation, $\dot{E}_g=0$, (4) Surface at x=0 is perfectly insulated, (5) All incident radiant power is absorbed, negligible radiation exchange with surroundings.

ANALYSIS: (a) The temperature distributions are shown on the T-x coordinates and labeled accordingly. Note these special features: ① Gradient at x=0 is always zero, ② gradient is more steep at early times and ③ for steady-state conditions, the radiant flux is equal to the convective heat flux; this follows from an energy balance on the CS at x=L,

$$q''_o = q''_{conv} = h[T(L,\infty) - T_\infty] .$$

(b) The heat flux at the outer surface, $q''_x(L,t)$, as a function of time appears as shown below.

COMMENTS: The sketch must reflect the initial and boundary conditions:

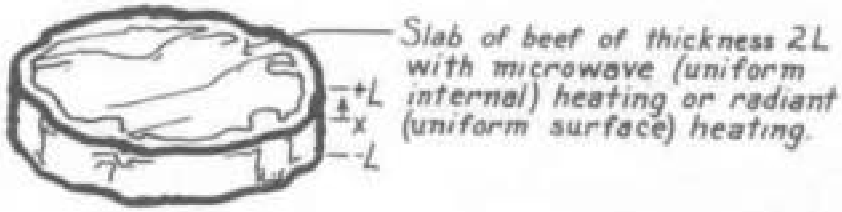
$T(x,0) = T_\infty$	uniform initial temperature.
$-k\dfrac{\partial T}{\partial x}\Big\vert_{x=0} = 0$	insulated at x=0.
$-k\dfrac{\partial T}{\partial x}\Big\vert_{x=L} = h[T(L,t) - T_\infty] - q''_o$	surface energy balance at x=L.

PROBLEM 5.3

KNOWN: Microwave and radiant heating conditions for a slab of beef.

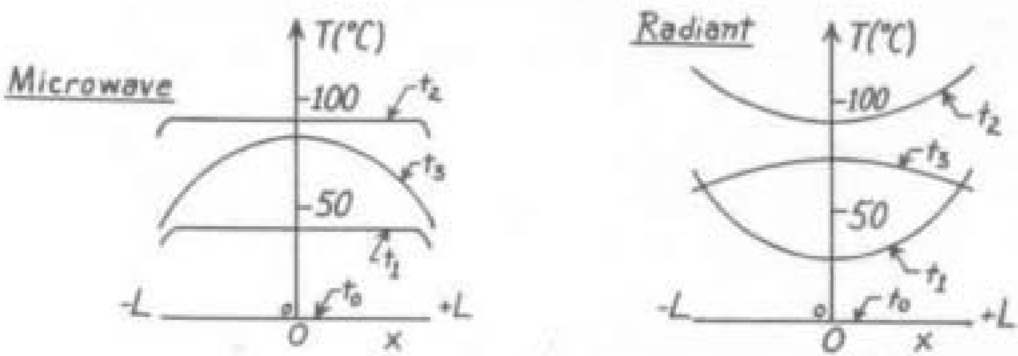
FIND: Sketch temperature distributions at specific times during heating and cooling.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction in x, (2) Uniform internal heat generation for microwave, (3) Uniform surface heating for radiant oven, (4) Heat loss from surface of meat to surroundings is negligible during the heating process, (5) Symmetry about midplane.

ANALYSIS:



COMMENTS: (1) With uniform generation and negligible surface heat loss, the temperature distribution remains nearly uniform during *microwave heating*. During the subsequent surface cooling, the maximum temperature is at the midplane.

(2) The interior of the meat is heated by conduction from the hotter surfaces during *radiant heating*, and the lowest temperature is at the midplane. The situation is reversed shortly after cooling begins, and the maximum temperature is at the midplane.

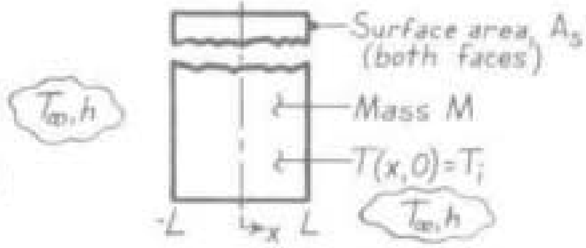
PROBLEM 5.4

KNOWN: Plate initially at a uniform temperature T_i is suddenly subjected to convection process (T_∞, h) on both surfaces. After elapsed time t_o, plate is insulated on both surfaces.

FIND: (a) Assuming $Bi \gg 1$, sketch on $T - x$ coordinates: initial, steady-state $(t \rightarrow \infty)$, $T(x, t_o)$ and temperature distributions for two intermediate times $t_o < t < \infty$, (b) Sketch on $T - t$ coordinates, midplane and surface temperature distributions, (c) Repeat parts (a) and (b) assuming $Bi \ll 1$, and (d) Expression for $T(x,\infty) = T_f$ in terms of plate parameters (M, c_p), thermal conditions (T_i, T_∞, h), surface temperature $T(L,t)$ and heating time t_o.

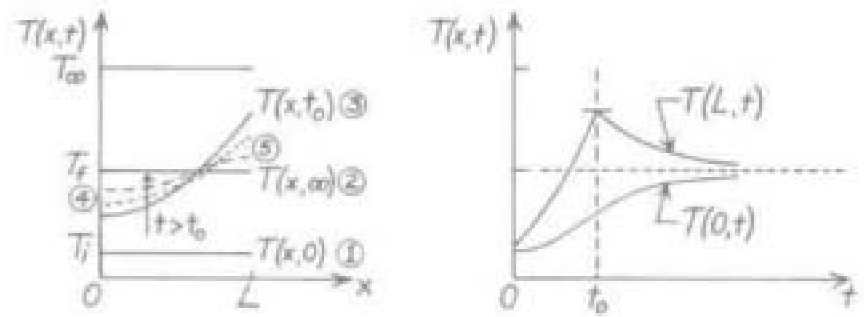
SCHEMATIC:

Time, t	Process
t=0	Uniform T_i
0-t_o	Heating, (T_∞, h)
t>t_o	Insulated



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) No internal generation, (4) Plate is perfectly insulated for $t > t_o$, (5) $T(0, t < t_o) < T_\infty$.

ANALYSIS: (a,b) With $Bi \gg 1$, appreciable temperature gradients exist in the plate following exposure to the heating process.



On T-x coordinates: (1) initial, uniform temperature, (2) steady-state conditions when $t \rightarrow \infty$, (3) distribution at t_o just before plate is covered with insulation, (4) gradients are always zero (symmetry), and (5) when $t > t_o$ (dashed lines) gradients are zero (insulated).

(c) If $Bi \ll 1$, plate behaves space-wise isothermal, hence, no gradients. On T-x coordinates, the temperature distributions are flat; on T-t coordinates, $T(L,t) = T(0,t)$.

(d) The conservation of energy requirement for the interval of time $\Delta t = t_o$ for the entire plate is

$$E_{in} - E_{out} = \Delta E = E_{final} - E_{initial} \qquad 2\int_0^{t_o} hA_s[T_\infty - T(L,t)]dt - 0 = Mc_p(T_f - T_i)$$

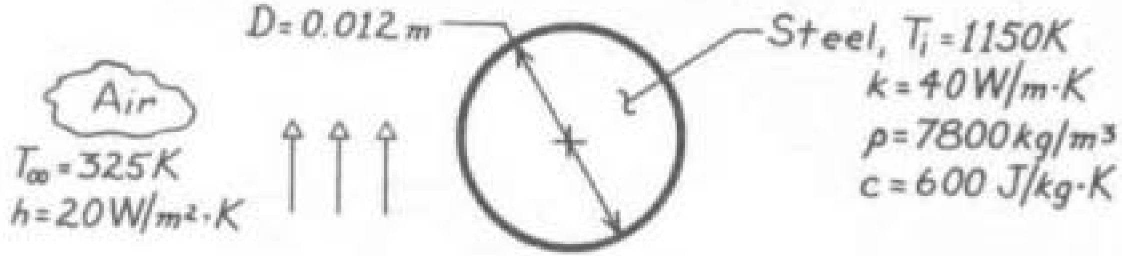
noting that E_{in} is that due to the convection heating process over the period of time $t = 0 \rightarrow t_o$. With knowledge of $T(L,t)$, this expression can then be integrated and a value for T_f determined.

PROBLEM 5.5

KNOWN: Diameter and initial temperature of steel balls cooling in air.

FIND: Time required to cool to a prescribed temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible radiation effects, (2) Constant properties.

ANALYSIS: Applying Eq. 5.10 to a sphere ($L_c = r_o/3$),

$$Bi = \frac{hL_c}{k} = \frac{h(r_o/3)}{k} = \frac{20\ W/m^2{\cdot}K\ (0.002m)}{40\ W/m{\cdot}K} = 0.001\ .$$

Hence the temperature of the steel remains approximately uniform during the cooling process, and the lumped capacitance method may be used. From Eqs. 5.4 and 5.5,

$$t = \frac{\rho V c_p}{hA_s} \ln \frac{T_i - T_\infty}{T - T_\infty} = \frac{\rho(\pi D^3/6)c}{h\pi D^2} \ln \frac{T_i - T_\infty}{T - T_\infty}$$

$$t = \frac{7800kg/m^3(0.012m)600J/kg{\cdot}K}{6{\times}20\ W/m^2{\cdot}K} \ln \frac{1150-325}{400-325}$$

$$t = 1122\ s = 0.312h$$ ◁

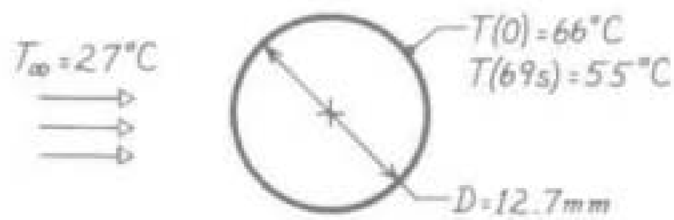
COMMENTS: Due to the large value of T_i, radiation effects are likely to be significant during the early portion of the transient. The effect is to shorten the cooling time.

PROBLEM 5.6

KNOWN: The temperature-time history of a pure copper sphere in an air stream.

FIND: The heat transfer coefficient between the sphere and the air stream.

SCHEMATIC:



ASSUMPTIONS: (1) Temperature of sphere is spatially uniform, (2) Negligible radiation exchange, (3) Constant properties.

PROPERTIES: *Table A-1*, Pure copper (333K): $\rho = 8933\ kg/m^3$, $c_p = 389\ J/kg \cdot K$, $k = 398\ W/m \cdot K$.

ANALYSIS: The time-temperature history is given by Eq. 5.6 with Eq. 5.7.

$$\frac{\theta(t)}{\theta_i} = \exp\left(-\frac{t}{R_t C_t}\right) \qquad \text{where} \qquad R_t = \frac{1}{hA_s} \qquad A_s = \pi D^2$$

$$C_t = \rho V c_p \qquad V = \frac{\pi D^3}{6}$$

$$\theta = T - T_\infty .$$

Recognize that when t = 69s,

$$\frac{\theta(t)}{\theta_i} = \frac{(55-27)°C}{(66-27)°C} = 0.718 = \exp\left(-\frac{t}{\tau_t}\right) = \exp\left(-\frac{69s}{\tau_t}\right)$$

and noting that $\tau_t = R_t C_t$ find

$$\tau_t = 208s .$$

Hence,

$$h = \frac{\rho V c_p}{A_s \tau_t} = \frac{8933\ kg/m^3\ (\pi 0.0127^3\ m^3/6)\ 389 J/kg \cdot K}{\pi 0.0127^2 m^2 \times 208s}$$

$$h = 35.3\ W/m^2 \cdot K. \qquad \triangleleft$$

COMMENTS: Note that with $L_c = D_o/6$,

$$Bi = \frac{hL_c}{k} = 35.3\ W/m^2 \cdot K \times \frac{0.0127}{6}\ m/398\ W/m \cdot K = 1.88 \times 10^{-4} .$$

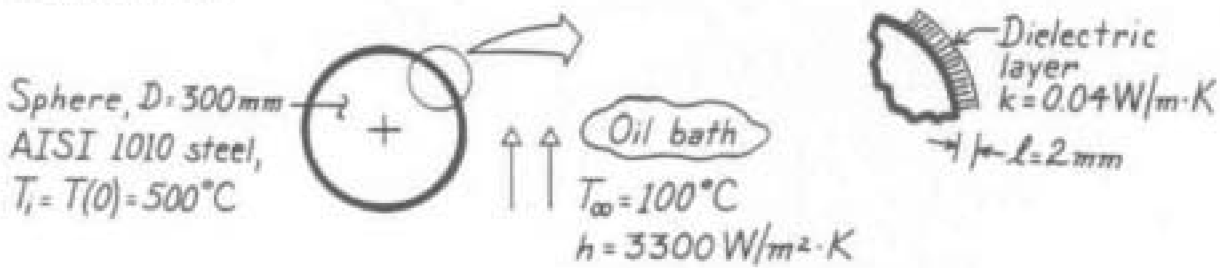
Hence $Bi < 0.1$ and the spatially isothermal assumption is reasonable.

PROBLEM 5.7

KNOWN: Solid steel sphere (AISI 1010), coated with dielectric layer of prescribed thickness and thermal conductivity. Coated sphere, initially at uniform temperature, is suddenly quenched in an oil bath.

FIND: Time required for sphere to reach 140 °C.

SCHEMATIC:



PROPERTIES: *Table A-1*, AISI 1010 Steel ($\overline{T}$ = [500+140] °C/2 = 320 °C≈600K): ρ = 7832 kg/m^3, c = 559 J/kg·K, k = 48.8 W/m·K.

ASSUMPTIONS: (1) Steel sphere is space-wise isothermal, (2) Dielectric layer has negligible thermal capacitance compared to steel sphere, (3) Layer is thin compared to radius of sphere, (4) Constant properties.

ANALYSIS: The thermal resistance to heat transfer from the sphere is due to the dielectric layer and the convection coefficient. That is,

$$R'' = \frac{\ell}{k} + \frac{1}{h} = \frac{0.002\text{m}}{0.04\ \text{W/m·K}} + \frac{1}{3300\ \text{W/m}^2\text{·K}} = (0.050+0.0003) = 0.0503\ \frac{\text{m}^2\text{·K}}{\text{W}},$$

or in terms of an overall coefficient, $U = 1/R'' = 19.88\ \text{W/m}^2\text{·K}$. The effective Biot number is

$$Bi_e = \frac{UL_c}{k} = \frac{U(r_o/3)}{k} = \frac{19.88\ \text{W/m}^2\text{·K}\times(0.300/6)\text{m}}{48.8\ \text{W/m·K}} = 0.0204$$

where the characteristic length is $L_c = r_o/3$ for the sphere. Since $Bi_e < 0.1$, the lumped capacitance approach is applicable. Hence, Eq. 5.5 is appropriate with h replaced by U,

$$t = \frac{\rho c}{U}\left(\frac{V}{A_s}\right)\ln\frac{\theta_i}{\theta_o} = \frac{\rho c}{U}\left(\frac{V}{A_s}\right)\ln\frac{T(0)-T_\infty}{T(t)-T_\infty}.$$

Substituting numerical values with $(V/A_s) = r_o/3 = D/6$,

$$t = \frac{7832\ \text{kg/m}^3\times559\ \text{J/kg·K}}{19.88\ \text{W/m}^2\text{·K}}\left(\frac{0.300\text{m}}{6}\right)\ln\frac{(500-100)\ ^\circ\text{C}}{(140-100)\ ^\circ\text{C}}$$

$$t = 25{,}358\text{s} = 7.04\text{h}.$$ ◁

COMMENTS: (1) Note from calculation of R″ that the resistance of the dielectric layer dominates and therefore nearly all the temperature drop occurs across the layer.

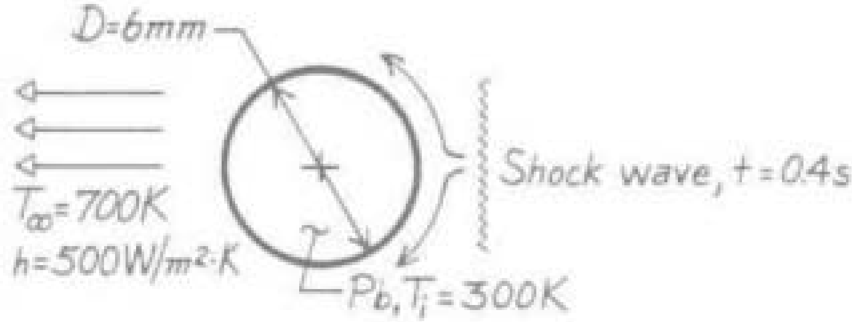
(2) An alternative method of solution would be to use the Heisler chart with $Bi_e^{-1} = 1/0.0611 = 16.4$ and $\theta_o/\theta_i = 0.1$ to find Fo and eventually t.

PROBLEM 5.8

KNOWN: Initial temperature and convection conditions for transient heating of a spherical bullet.

FIND: Surface temperature after a prescribed heating period.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible radiation.

PROPERTIES: *Table A.1* (300 K): $\rho = 11{,}340\ \text{kg/m}^3$, $c_p = 129\ \text{J/kg·K}$, $k = 35.3\ \text{W/m·K}$.

ANALYSIS: Evaluate first the Biot number,

$$\text{Bi} = \frac{h r_o/3}{k} = \frac{hD/6}{k} = \frac{500\ \text{W/m}^2\text{·K}(0.001\ \text{m})}{35.3\ \text{W/m·K}}$$

$$\text{Bi} = 0.0142.$$

Hence, the lumped capacitance method may be used. The transient response of bullet is given by

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp[-(hA_s/\rho Vc)t] = \exp[-(6h/\rho Dc)t].$$

Substituting numerical values, find

$$T(0.4\text{s}) = 700\ \text{K} + (300 - 700)\text{K}\ \exp\left[-\frac{6 \times 500\ \text{W/m}^2\text{·K} \times 0.4\text{s}}{11{,}340\ \text{kg/m}^3 \times 0.006\ \text{m} \times 129\ \text{J/kg·K}}\right]$$

$$T(0.4\text{s}) = 700\ \text{K} - 400\ \text{K}\ \exp(-0.137)$$

$$T(0.4\text{s}) = 351\ \text{K}. \qquad \triangleleft$$

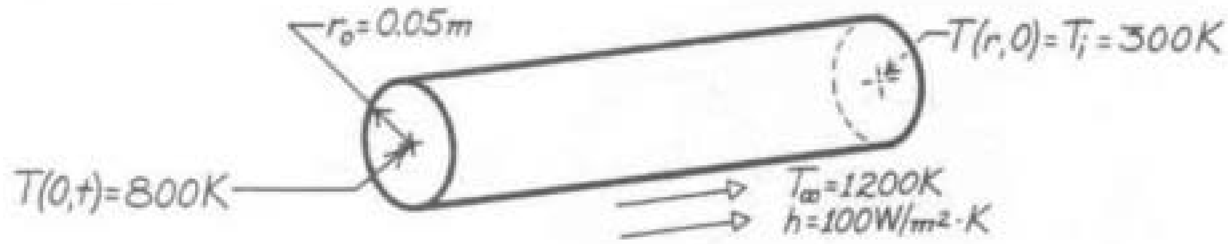
COMMENTS: The heating effect is significant. Note from Fig. 5.15 that, for $\text{Bi}^{-1} = 23.5$, T is approximately independent of r.

PROBLEM 5.9

KNOWN: Diameter and radial temperature of AISI 1010 carbon steel shaft. Convection coefficient and temperature of furnace gases.

FIND: Time required for shaft centerline to reach a prescribed temperature.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional, radial conduction, (2) Constant properties.

PROPERTIES: AISI 1010 carbon steel, *Table A.1* ($\overline{T} = 550$ K): $\rho = 7832$ kg/m^3, k = 51.2 W/m·K, c = 541 J/kg·K, $\alpha = 1.21\times10^{-5}$ m^2/s.

ANALYSIS: The Biot number is

$$Bi = \frac{hr_o/2}{k} = \frac{100\ W/m^2 \cdot K(0.05\ m/2)}{51.2\ W/m \cdot K} = 0.0488.$$

Hence, the lumped capacitance method can be applied. From Equation 5.6,

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hAs}{\rho Vc}\right)t\right] = \exp\left[-\frac{4h}{\rho cD}t\right]$$

$$\ln\left(\frac{800 - 1200}{300 - 1200}\right) = -0.811 = -\frac{4\times100\ W/m^2 \cdot K}{7832\ kg/m^3(541\ J/kg \cdot K)0.1\ m}t$$

$$t = 859\ s. \quad \triangleleft$$

COMMENTS: To check the validity of the foregoing result, use the one-term approximation to the series solution. From Equation 5.49c,

$$\frac{T_o - T_\infty}{T_i - T_\infty} = \frac{-400}{-900} = 0.444 = C_1 \exp(-\varsigma_1^2 Fo)$$

For Bi = $hr_o/k = 0.0976$, Table 5.1 yields $\varsigma_1 = 0.436$ and $C_1 = 1.024$. Hence

$$\frac{-(0.436)^2(1.2\times10^{-5}\ m^2/s)}{(0.05\ m)^2}t = \ln(0.434) = -0.835$$

$$t = 915\ s.$$

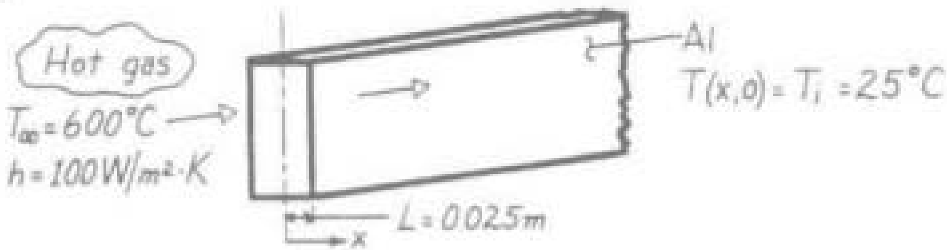
The results agree to within 6%. The lumped capacitance method underestimates the actual time, since the response at the centerline lags that at any other location in the shaft.

PROBLEM 5.10

KNOWN: Configuration, initial temperature and charging conditions of a thermal energy storage unit.

FIND: Time required to achieve 75% of maximum possible energy storage. Temperature of storage medium at this time.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Negligible heat exchange with surroundings.

PROPERTIES: *Table A-1*, Aluminum, pure ($\overline{T} \approx 600K = 327°C$): $k = 231\ W/m \cdot K$, $c = 1033\ J/kg \cdot K$, $\rho = 2702\ kg/m^3$.

ANALYSIS: Recognizing the characteristic length is the half thickness, find

$$Bi = \frac{hL}{k} = \frac{100\ W/m^2 \cdot K \times 0.025m}{231\ W/m \cdot K} = 0.011\ .$$

Hence, the lumped capacitance method may be used. From Eq. 5.8,

$$Q = (\rho Vc)\theta_i\ [1 - \exp(-t/\tau_t)] = -\Delta\ E_{st} \qquad (1)$$

$$-\Delta E_{st,max} = (\rho Vc)\theta_i\ . \qquad (2)$$

Dividing Eq. (1) and (2), the condition sought is for

$$\Delta\ E_{st}/\Delta\ E_{st,max} = 1 - \exp(-t/\tau_{th}) = 0.75\ .$$

Solving for τ_{th} and substituting numerical values, find

$$\tau_{th} = \frac{\rho Vc}{hA_s} = \frac{\rho Lc}{h} = \frac{2702\ kg/m^3 \times 0.025m \times 1033\ J/kg \cdot K}{100\ W/m^2 \cdot K} = 698s\ .$$

Hence, the time required is

$$-\exp(-t/698s) = -0.25 \qquad \text{or} \qquad t = 968s. \qquad \triangleleft$$

From Eq. 5.6,

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-t/\tau_{th})$$

$$T = T_\infty + (T_i - T_\infty)\exp(-t/\tau_{th}) = 600°C - (575°C)\exp(-968/698)$$

$$T = 456°C. \qquad \triangleleft$$

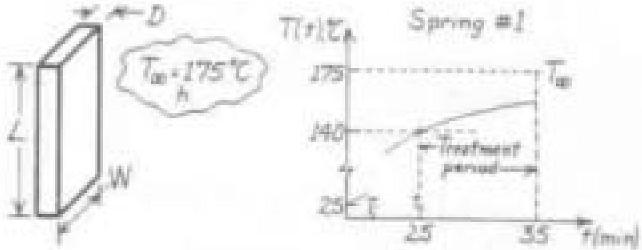
COMMENTS: For the prescribed temperatures, the property temperature dependence is significant and some error is incurred by assuming constant properties. However, selecting properties at 600K was reasonable for this estimate.

PROBLEM 5.11

KNOWN: Conditions for properly treating the coating on a leaf spring vertically suspended in a conveyor oven.

FIND: Time required to treat coating on a different-sized spring.

SCHEMATIC:



ASSUMPTIONS: (1) Springs are space-wise isothermal, (2) Constant properties, (3) Both springs have same convection coefficient.

PROPERTIES: Spring material (given): $\rho = 8131$ kg/m^3, $c_p = 473$ J/kg·K, $k = 42$ W/m·K, $\alpha = k/\rho c_p = 1.092\times10^{-5}$ m^2/s.

ANALYSIS: Consider the smaller spring (#1) for which the coating conditions are known. From Eq. 5.13,

$$\frac{\theta}{\theta_i} = \frac{T(t_c) - T_\infty}{T_i - T_\infty} = \frac{(140-175)°C}{(25-175)°C} = 0.233 = \exp(-Bi\cdot Fo). \tag{1}$$

For the Biot number, $Bi = hL_c/k$, the characteristic length of the spring is

$$L_c = \frac{V}{A_s} = \frac{w\times L\times d}{2(w\times L) + 2(L\times d) + 2(w\times d)} = \frac{0.032\times0.010\times1.1\ m^3}{2(0.032\times1.1)m^2 + 2(0.010\times1.1)m^2 + 2(0.032\times0.01)m^2} = 0.00378\ m$$

With Eq. (1) and $Fo = \alpha t_c/L_c^2$,

$$0.233 = \exp\left(-\frac{h\times0.00378\ m}{42\ W/m\cdot K}\times\frac{1.092\times10^{-5}\ m^2/s(25\times60s)}{(0.0038\ m)^2}\right) \qquad h = 14.1\ W/m^2\cdot K.$$

Consider now the larger spring (#2), the characteristic length is

$$L_c = \frac{0.076\times0.035\times1.6\ m^3}{2(0.076\times1.6)m^2 + 2(0.035\times1.6)m^2 + 2(0.076\times0.035)m^2} = 0.0118\ m.$$

The time required for this spring to reach $T(t_c) = 140°C$,

$$\frac{\theta}{\theta_i} = \frac{140-175}{25-175} = \exp\left(\frac{14.1\ W/m^2\cdot K\times0.0118\ m}{42\ W/m\cdot K}\times\frac{1.092\times10^{-5}\ m^2/s\times t_c}{(0.0120\ m)^2}\right) \qquad t_c = 78\ minutes.$$

Hence, we would leave the larger spring (#2) in the conveyor oven for

$$t = t_c + 10\ min = (78+10)min = 88\ min. \qquad \triangleleft$$

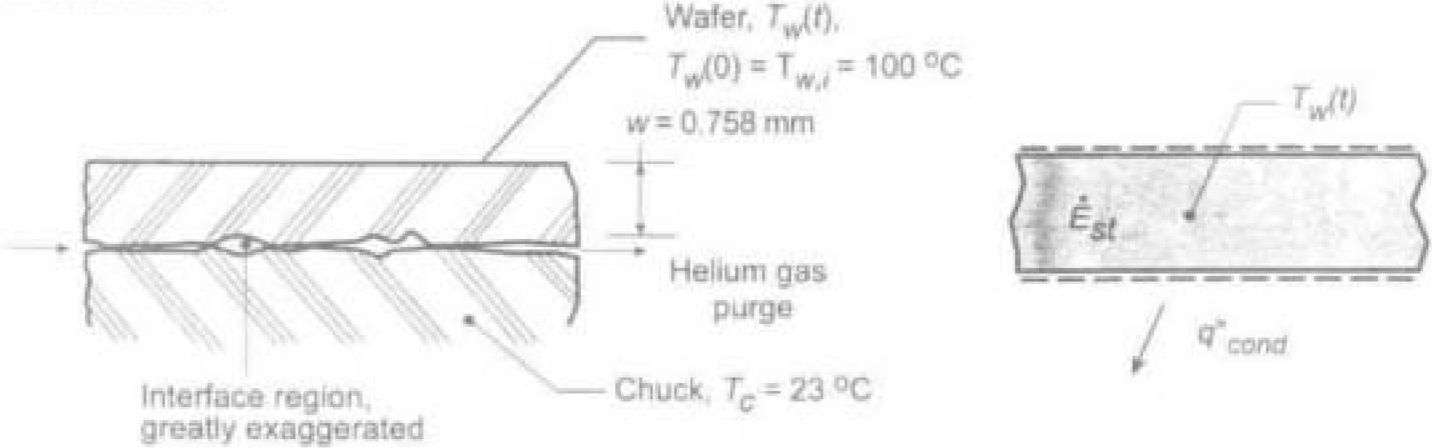
COMMENTS: Using $L_c = d/2$ rather than as V/A_s leads to overpredicting the residence time. Note from the temperature distribution plot on the T–t coordinates how t_c was reasoned to be 25 minutes for spring #1. For the larger spring, $Bi = 0.004$, hence the lumped capacitance method is appropriate.

PROBLEM 5.12

KNOWN: Wafer, initially at 100°C, is suddenly placed on a chuck with uniform and constant temperature, 23°C. Wafer temperature after 15 seconds is observed as 33°C.

FIND: (a) Contact resistance, R''_{tc}, between interface of wafer and chuck through which helium slowly flows, and (b) Whether R''_{tc} will change if air, rather than helium, is the purge gas.

SCHEMATIC:



PROPERTIES: Wafer (silicon, typical values): $\rho = 2700\ \text{kg/m}^3$, $c = 875\ \text{J/kg·K}$, $k = 177\ \text{W/m·K}$.

ASSUMPTIONS: (1) Wafer behaves as a space-wise isothermal object, (2) Negligible heat transfer from wafer top surface, (3) Chuck remains at uniform temperature, (4) Thermal resistance across the interface is due to conduction effects, not convective, (5) Constant properties.

ANALYSIS: (a) Perform an energy balance on the wafer as shown in the Schematic.

$$\dot{E}''_{in} - \dot{E}''_{out} + \dot{E}_g = \dot{E}_{st} \qquad (1)$$

$$-q''_{cond} = \dot{E}''_{st} \qquad (2)$$

$$-\frac{T_w(t) - T_c}{R''_{tc}} = \rho w c \frac{dT_w}{dt} \qquad (3)$$

Separate and integrate Eq. (3)

$$-\int_0^t \frac{dt}{\rho w c R''_{tc}} = \int_{T_{wi}}^{T_w} \frac{dT_w}{T_w - T_c} \qquad (4)$$

$$\frac{T_w(t) - T_c}{T_{wi} - T_c} = \exp\left[-\frac{t}{\rho w c R''_{tc}}\right] \qquad (5)$$

Substituting numerical values for $T_w(15\text{s}) = 33$°C,

$$\frac{(33-23)^\circ\text{C}}{(100-23)^\circ\text{C}} = \exp\left[-\frac{15\,\text{s}}{2700\,\text{kg/m}^3 \times 0.758\times 10^{-3}\,\text{m} \times 875\,\text{J/kg·K} \times R''_{tc}}\right] \qquad (6)$$

$$R''_{tc} = 0.0041\,\text{m}^2\cdot\text{K/W}$$ <

(b) R''_{tc} will increase since $k_{air} < k_{helium}$. See Table A.4.

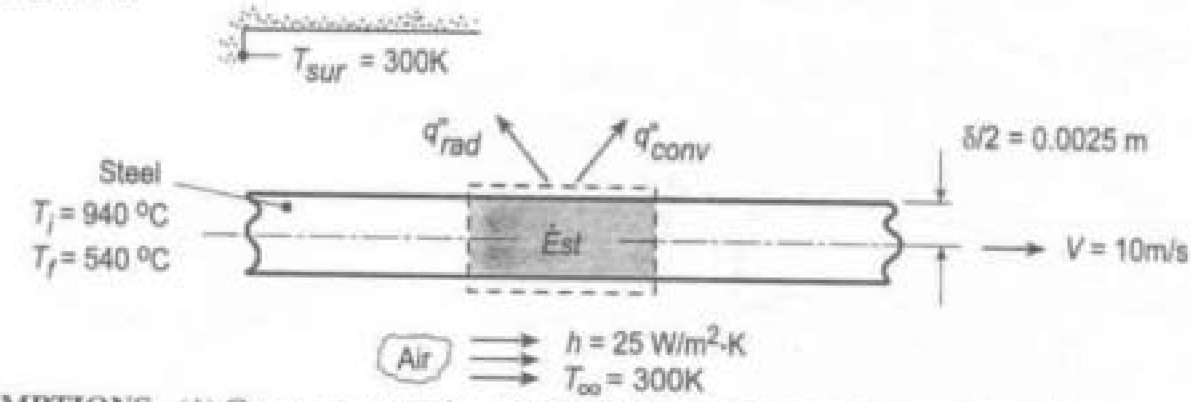
COMMENTS: Note that $Bi = R_{int}/R_{ext} = (w/k)/\ R''_{tc} = 0.001$. Hence the spacewise isothermal assumption is reasonable.

PROBLEM 5.13

KNOWN: Thickness and properties of strip steel cooled by convection and radiation.

FIND: (a) Time required to cool from prescribed initial to final temperature, (b) Relative influence of convection and radiation on heat transfer from the strip.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible strip temperature gradients in transverse direction (across strip thickness), (3) Negligible effect of strip conduction in longitudinal direction.

PROPERTIES: Steel: $\rho = 7900\ \text{kg/m}^3$, $c_p = 640\ \text{J/kg·K}$, $k = 30\ \text{W/m·K}$, $\varepsilon = 0.7$.

ANALYSIS: (a) Considering a fixed (control) mass of the moving strip, its temperature variation with time is governed by an energy balance which equates the change in energy storage to heat transfer by convection and radiation. If the surface area associated with one side of the control mass is designated as A_s, $A_{s,c} = A_{s,r} = 2A_s$ and $V = \delta A_s$ in Equation 5.15, which reduces to

$$\rho c \delta \frac{dT}{dt} = -2\left[h\left(T - T_\infty\right) + \varepsilon\sigma\left(T^4 - T_{sur}^4\right)\right]$$

or, introducing the radiation coefficient from Equations 1.8 and 1.9,

$$T_f - T_i = -\frac{1}{\rho c(\delta/2)}\int_0^{t_f}\left[h\left(T - T_\infty\right) + h_r\left(T - T_{sur}\right)\right]dt$$

Substituting for $T_i = 1213$ K and using the *Lumped Capacitance Model* of IHT to perform the numerical integration, we obtain $T_f = 813\ \text{K} = 540°\text{C}$ at $t_f = 93$ s, in which case the length of the cooling section must be

$$L = Vt_f \approx 10\ \text{m/s} \times 93\text{s} = 930\ \text{m}$$ <

Because this length exceeds practical space limitations within a steel mill, we conclude that the cooling rate is insufficient and should be enhanced.

(b) Setting, first h = 0 and then ε = 0, and repeating the numerical integration, or using Equations 5.18 and 5.5, we obtain for negligible convection and radiation, respectively,

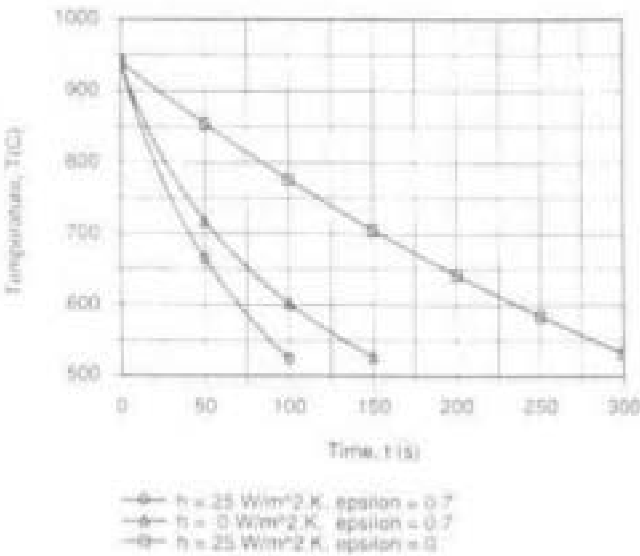
t_f (radiation only) = 140 s <

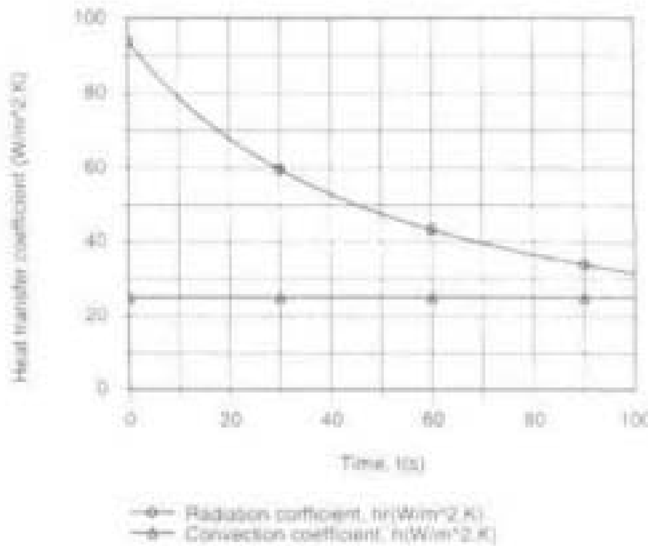
t_f (convection only) = 292 s <

From the foregoing results and the following plots, it is clear that radiation is the dominant heat transfer mode.

Continued...

PROBLEM 5.13 (Cont.)





Initially (t = 0), the radiation coefficient (h_r = 94 W/m^2·K) exceeds the convection coefficient by nearly a factor of 4. Although h_r decreases with decreasing T, it is still larger than h a the conclusion of the cooling process (h_r = 32 W/m^2·K for T = T_f = 813 K).

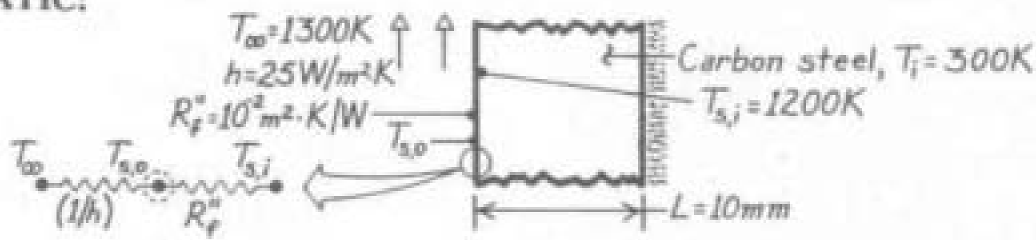
COMMENTS: In a steel mill accelerated cooling is achieved by using water jets, as depicted in Problem 1.30. To check the validity of using a lumped capacitance analysis, we calculate the Biot number using the maximum cumulative coefficient ($h + h_r$) associated with the cooling process. With ($h + h_r$) = (25 + 94) W/m^2·K = 119 W/m^2·K, it follows that $Bi = (h + h_r)(\delta/2)/k = 0.01$ and the approximation is valid.

PROBLEM 5.14

KNOWN: Thickness and properties of furnace wall. Thermal resistance of film on surface of wall exposed to furnace gases. Initial wall temperature.

FIND: (a) Time required for surface of wall to reach a prescribed temperature, (b) Corresponding value of film surface temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible film thermal capacitance, (3) Negligible radiation.

PROPERTIES: Carbon steel (given): $\rho = 7850\ \text{kg/m}^3$, $c = 430\ \text{J/kg·K}$, $k = 60\ \text{W/m·K}$.

ANALYSIS: The overall coefficient for heat transfer from the surface of the steel to the gas is

$$U = (R''_{tot})^{-1} = \left(\frac{1}{h} + R''_f\right)^{-1} = \left(\frac{1}{25\ \text{W/m}^2\cdot\text{K}} + 10^{-2}\text{m}^2\cdot\text{K/W}\right)^{-1} = 20\ \text{W/m}^2\cdot\text{K}.$$

Hence,

$$Bi = \frac{UL}{k} = \frac{20\ \text{W/m}^2\cdot\text{K}\times 0.01\ \text{m}}{60\ \text{W/m}\cdot\text{K}} = 0.0033$$

and the lumped capacitance method can be used.

(a) It follows that

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-t/\tau_t) = \exp(-t/RC) = \exp(-Ut/\rho Lc)$$

$$t = -\frac{\rho Lc}{U}\ln\frac{T - T_\infty}{T_i - T_\infty} = -\frac{7850\ \text{kg/m}^3(0.01\ \text{m})430\ \text{J/kg}\cdot\text{K}}{20\ \text{W/m}^2\cdot\text{K}}\ln\frac{1200 - 1300}{300 - 1300}$$

$$t = 3886\text{s} = 1.08\text{h}.$$ ◁

(b) Performing an energy balance at the outer surface (s,o),

$$h(T_\infty - T_{s,o}) = (T_{s,o} - T_{s,i})/R''_f$$

$$T_{s,o} = \frac{hT_\infty + T_{s,i}/R''_f}{h + (1/R''_f)} = \frac{25\ \text{W/m}^2\cdot\text{K}\times 1300\ \text{K} + 1200\ \text{K}/10^{-2}\text{m}^2\cdot\text{K/W}}{(25 + 100)\text{W/m}^2\cdot\text{K}}$$

$$T_{s,o} = 1220\ \text{K}.$$ ◁

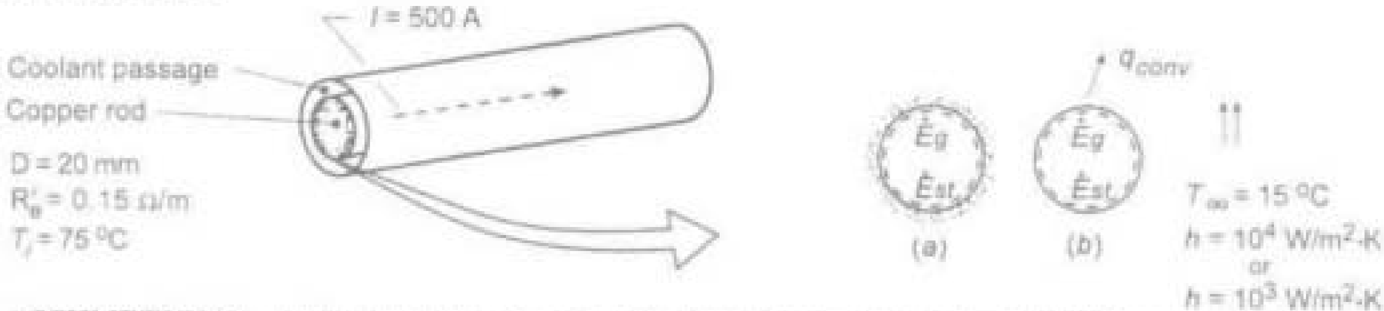
COMMENTS: The film increases τ_t by increasing R_t but not C_t.

PROBLEM 5.15

KNOWN: Copper rod carrying 500 A experiences loss of coolant (LOC).

FIND: (a) Time required for melting to occur following LOC, (b) Performance of back-up cooling systems.

SCHEMATIC:



ASSUMPTIONS: (1) Temperature of rod is spatially uniform at any instant, (2) Without coolant, radiative and convective heat transfer from surface of rod are negligible, (3) Constant properties, (4) Uniform heat generation.

PROPERTIES: *Table A-1*, Copper, $T_{mp} = 1358$ K, ($\overline{T} = (348+1358)/2 \approx 850\,K$): $c_p = 438$ J/kg·K, $\rho = 8933\ kg/m^3$.

ANALYSIS: (a) Since the outer surface of the rod is assumed to lose no energy by heat transfer to its surroundings, the energy balance on the rod is

$$\dot{E}_g = \dot{E}_{st} \qquad \text{or} \qquad I^2 R_e' L = \rho V c_p \frac{dT}{dt}.$$

Separating variables, integrating with definite limits, and substituting numerical values, find

$$\int_0^t dt = \frac{\rho(\pi D^2/4)Lc_p}{I^2 R_e' L}\int_{T_i}^{T_{mp}} dT$$

$$t = \frac{\rho(\pi D^2/4)c_p(T_{mp} - T_i)}{I^2 R_e'}$$

$$t = \frac{(8933\,kg/m^3)(\pi/4)(0.02\,m)^2(438\,J/kg\cdot K)(1358-348)K}{(500)^2(0.15)J/s\cdot m} = 33.1\,s$$ <

(b) The rod temperature at the onset of back-up cooling may be obtained from the expression of part (a). For t = 5s, we obtain

$$T(t = 5s) = 500\ K$$ <

With back-up cooling, the appropriate form of the energy equation is

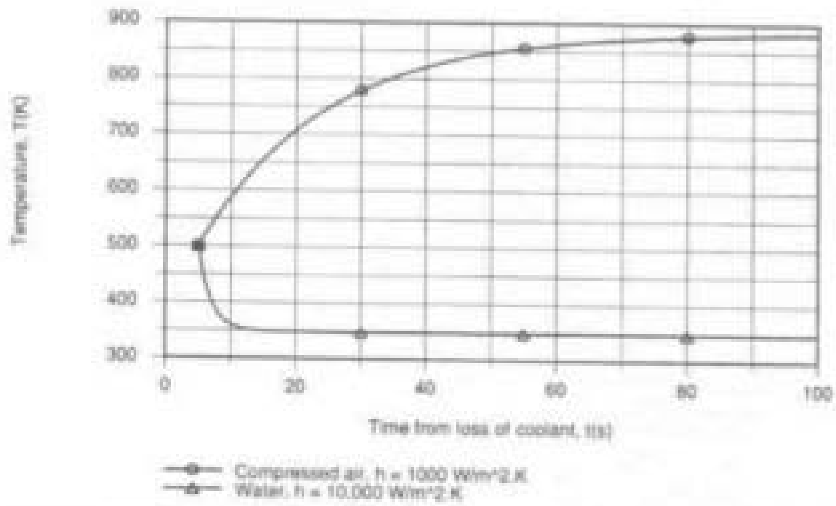
$$-\dot{E}_{out} + \dot{E}_g = \dot{E}_{st} \qquad \text{or} \qquad -hA_s(T - T_\infty) + I^2 R_e' L = \rho V c_p \frac{dT}{dt}$$

where $A_s = \pi DL$. The solution to this linear, first-order nonhomogeneous differential equation is given by Equation (5.25),

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-at) + \frac{b/a}{T - T_\infty}\left[1 - \exp(-at)\right]$$

Continued...

where $a = hA_s/\rho V c_p$ and $b = I^2 R'_e/\rho(\pi D^2/4)c_p$. For the prescribed conditions a(water) = 0.511 s^{-1}, a(air) = 0.051 s^{-1}, and b = 30.5 K/s. With T_i = 500 K at t_i = 5s, the temperature histories associated with the two cooling options are as follows.



With the water back-up system, the rod temperature quickly decays (within 10s) to a steady-state value of 348 K, which corresponds to the normal operating condition. With the air system, the temperature continues to rise and at t = 100s, it is very close (within 4 K) of the steady-state value of 885 K. Although, the melting point is not reached with the air system, the water back-up quickly restores the rod to its original state and is hence much preferred.

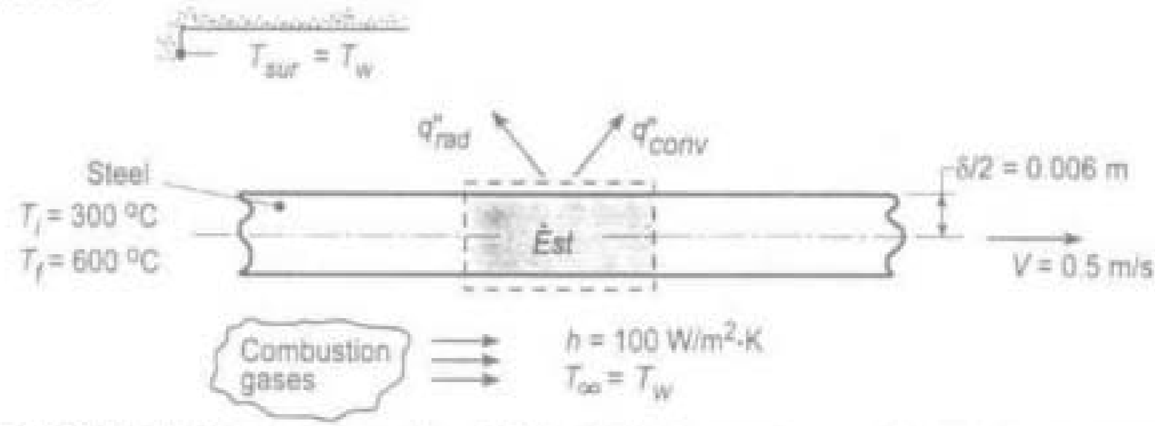
COMMENTS: With k = 400 W/m·K, Bi = h(D/2)/k = 0.25 for water and 0.025 for air. Hence, the lumped capacitance model is appropriate for the air cooling, but only marginally acceptable for the water. The implication is that temperature gradients will develop in the rod during cooling. Nevertheless, a final steady-state of 348 K will be approached in a time span close to that predicted by the model.

PROBLEM 5.16

KNOWN: Thickness and properties of strip steel heated in an annealing process. Furnace operating conditions.

FIND: (a) Time required to heat the strip from 300 to 600°C. Required furnace length for prescribed strip velocity (V = 0.5 m/s), (b) Effect of wall temperature on strip speed, temperature history, and radiation coefficient.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible temperature gradients in transverse direction across strip, (c) Negligible effect of strip conduction in longitudinal direction.

PROPERTIES: Steel: $\rho = 7900$ kg/m^3, $c_p = 640$ J/kg·K, $k = 30$ W/m·K, $\varepsilon = 0.7$.

ANALYSIS: (a) Considering a fixed (control) mass of the moving strip, its temperature variation with time may be obtained from an energy balance which equates the change in energy storage to heat transfer by convection and radiation. If the surface area associated with one side of the control mass is designated as A_s, $A_{s,c} = A_{s,r} = 2A_s$ and $V = \delta A_s$ in Equation 5.15, which reduces to

$$\rho c \delta \frac{dT}{dt} = -2\left[h\left(T - T_\infty\right) + \varepsilon\sigma\left(T^4 - T_{sur}^4\right)\right]$$

or, introducing the radiation coefficient from Equations 1.8 and 1.9 and integrating,

$$T_f - T_i = -\frac{1}{\rho c(\delta/2)}\int_{0}^{t_f}\left[h\left(T - T_\infty\right) + h_r\left(T - T_{sur}\right)\right]dt$$

Using the IHT *Lumped Capacitance Model* to integrate numerically with $T_i = 573$ K, we find that $T_f = 873$ K corresponds to

$$t_f = 209\text{s}$$ <

in which case, the required furnace length is

$$L = Vt_f \approx 0.5\,\text{m/s} \times 209\,\text{s} \approx 105\,\text{m}$$ <

(b) For $T_w = 1123$ K and 1273 K, the numerical integration yields $t_f = 102$s and 62s respectively. Hence, for L = 105 m, V = L/t_f yields

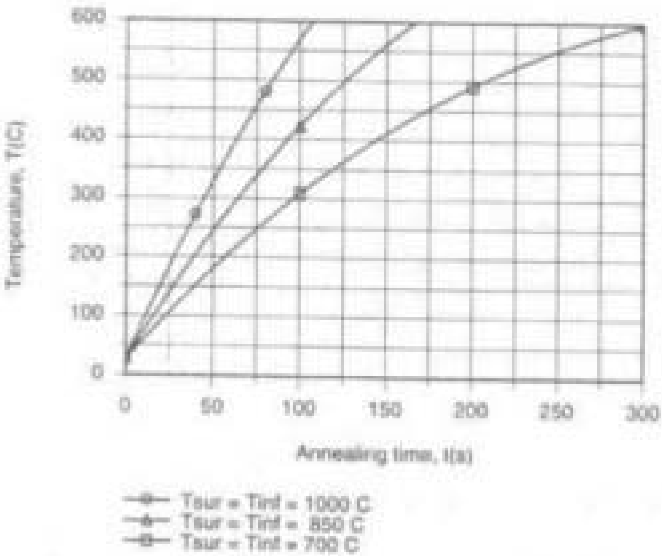
$$V(T_w = 1123\text{K}) = 1.03\,\text{m/s}$$

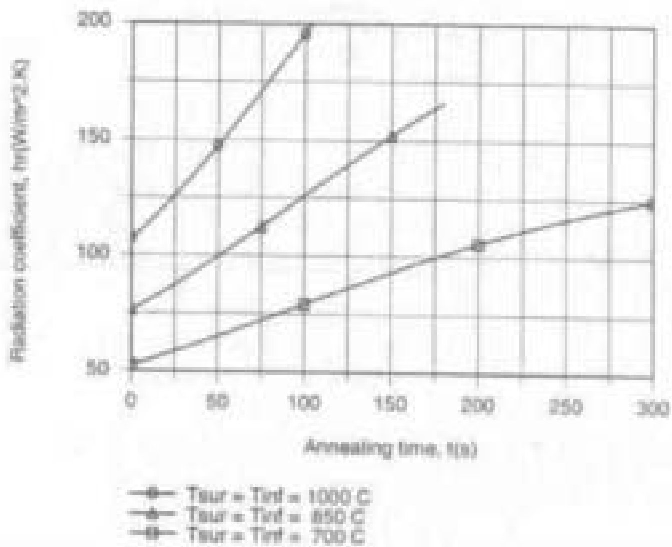
$$V(T_w = 1273\text{K}) = 1.69\,\text{m/s}$$ <

Continued...

which correspond to increased process rates of 106% and 238%, respectively. Clearly, productivity can be enhanced by increasing the furnace environmental temperature, albeit at the expense of increasing energy utilization and operating costs.

If the annealing process extends from 25°C (298 K) to 600°C (873 K), numerical integration yields the following results for the prescribed furnace temperatures.





As expected, the heating rate and time, respectively, increase and decrease significantly with increasing T_∞. Although the radiation heat transfer rate decreases with increasing time, the coefficient h_r increases with t as the strip temperature approaches T_∞.

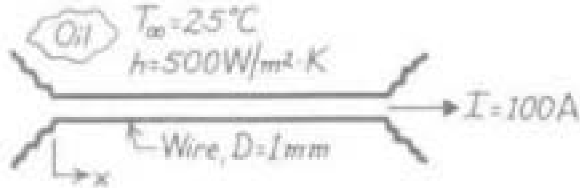
COMMENTS: To check the validity of the lumped capacitance approach, we calculate the Biot number based on a maximum cumulative coefficient of $(h + h_r) = 300\ W/m^2 \cdot K$. It follows that $Bi = (h + h_r)(\delta/2)/k = 0.06$ and the assumption is valid.

PROBLEM 5.17

KNOWN: Diameter, resistance and current flow for a wire. Convection coefficient and temperature of surrounding oil.

FIND: Steady-state temperature of the wire. Time for the wire temperature to come within 1°C of its steady-state value.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Wire temperature is independent of x.

PROPERTIES: Wire (given): $\rho = 8000\ \text{kg/m}^3$, $c_p = 500\ \text{J/kg·K}$, $k = 20\ \text{W/m·K}$, $R_e' = 0.01\ \Omega/\text{m}$.

ANALYSIS: Since

$$Bi = \frac{h(r_o/2)}{k} = \frac{500\ \text{W/m}^2\cdot\text{K}(2.5\times10^{-4}\text{m})}{20\ \text{W/m}\cdot\text{K}} = 0.006 < 0.1$$

the lumped capacitance method can be used. The problem has been analyzed in Example 1.3, and without radiation the steady-state temperature is given by

$$\pi Dh(T - T_\infty) = I^2R_e'.$$

Hence

$$T = T_\infty + \frac{I^2R_e'}{\pi Dh} = 25°\text{C} + \frac{(100\text{A})^2 0.01\Omega/\text{m}}{\pi(0.001\ \text{m})500\ \text{W/m}^2\cdot\text{K}} = 88.7°\text{C}. \qquad \triangleleft$$

With no radiation, the transient thermal response of the wire is governed by the expression (Example 1.3)

$$\frac{dT}{dt} = \frac{I^2R_e'}{\rho c(\pi D^2/4)} - \frac{4h}{\rho cD}(T - T_\infty).$$

With $T = T_i = 25°\text{C}$ at $t = 0$, the solution is

$$\frac{T - T_\infty - (I^2R_e'/\pi Dh)}{T_i - T_\infty - (I^2R_e'/\pi Dh)} = \exp\left(-\frac{4h}{\rho cD}t\right).$$

Substituting numerical values, find

$$\frac{87.7 - 25 - 63.7}{25 - 25 - 63.7} = \exp\left(-\frac{4\times500\ \text{W/m}^2\cdot\text{K}}{8000\ \text{kg/m}^3\times500\ \text{J/kg}\cdot\text{K}\times0.001\ \text{m}}t\right)$$

$$t = 8.31\text{s}. \qquad \triangleleft$$

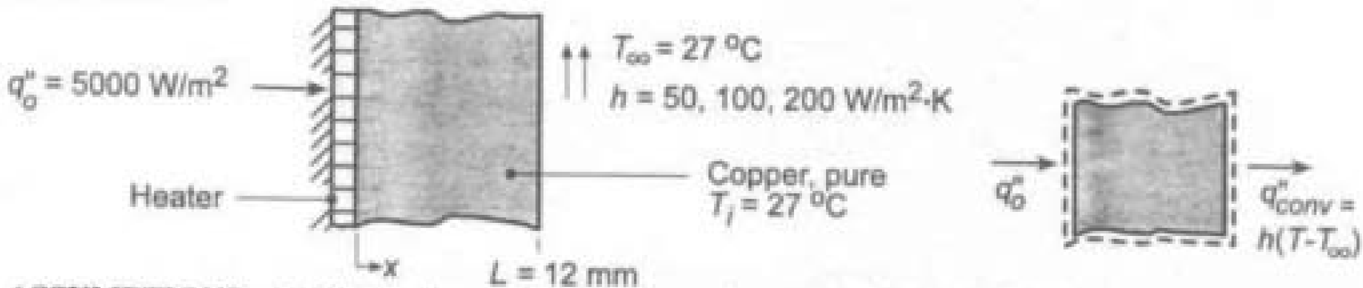
COMMENTS: The time to reach steady state increases with increasing ρ, c and D and with decreasing h.

PROBLEM 5.18

KNOWN: Electrical heater attached to backside of plate while front is exposed to a convection process (T_∞, h); initially plate is at uniform temperature T_∞ before heater power is switched on.

FIND: (a) Expression for temperature of plate as a function of time assuming plate is spacewise isothermal, (b) Approximate time to reach steady-state and $T(\infty)$ for prescribed T_∞, h and q''_o when wall material is pure copper, (c) Effect of h on thermal response.

SCHEMATIC:



ASSUMPTIONS: (1) Plate behaves as lumped capacitance, (2) Negligible loss out backside of heater, (3) Negligible radiation, (4) Constant properties.

PROPERTIES: *Table A-1*, Copper, pure (350 K): k = 397 W/m·K, c_p = 385 J/kg·K, ρ = 8933 kg/m³.

ANALYSIS: (a) Following the analysis of Section 5.3, the energy conservation requirement for the system is $\dot{E}_{in} - \dot{E}_{out} = \dot{E}_{st}$ or $q''_o - h(T - T_\infty) = \rho L c_p \, dT/dt$. Rearranging, and with R''_t = 1/h and C''_t = $\rho L c_p$,

$$T - T_\infty - q''_o/h = -R''_t \cdot C''_t dT/dt \qquad (1)$$

Defining $\theta(t) \equiv T - T_\infty - q''_o/h$ with $d\theta = dT$, the differential equation is

$$\theta = -R''_t C''_t \frac{d\theta}{dt}. \qquad (2)$$

Separating variables and integrating,

$$\int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_0^t \frac{dt}{R''_t C''_t}$$

it follows that

$$\frac{\theta}{\theta_i} = \exp\left(-\frac{t}{R''_t C''_t}\right) \qquad (3) <$$

where $\theta_i = \theta(0) = T_i - T_\infty - (q''_o/h)$ (4)

(b) For h = 50 W/m²·K, the steady-state temperature can be determined from Eq. (3) with $t \to \infty$; that is,

$$\theta(\infty) = 0 = T(\infty) - T_\infty - q''_o/h \qquad \text{or} \qquad T(\infty) = T_\infty + q''_o/h,$$

giving $T(\infty)$ = 27°C + 5000 W/m² /50 W/m²·K = 127°C. To estimate the time to reach steady-state, first determine the thermal time constant of the system,

$$\tau_t = R''_t C''_t = \left(\frac{1}{h}\right)(\rho c_p L) = \left(\frac{1}{50\,W/m^2 \cdot K}\right)(8933\,kg/m^3 \times 385\,J/kg \cdot K \times 12 \times 10^{-3}\,m) = 825\,s$$

Continued...

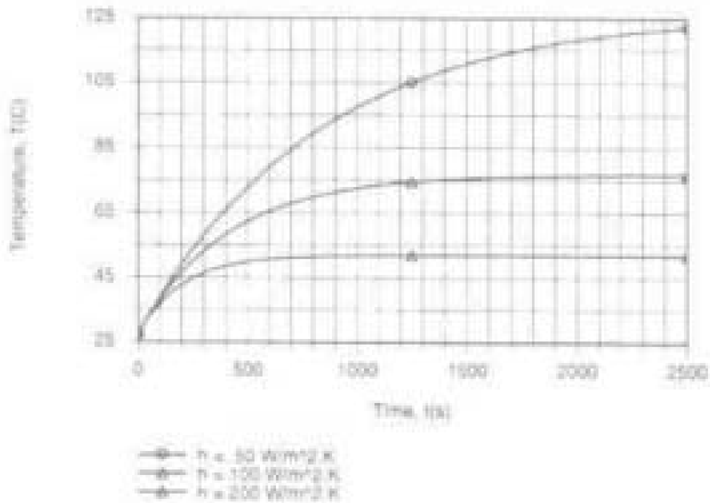
PROBLEM 5.18 (Cont.)

When $t = 3\tau_t = 3\times 825s = 2475s$, Eqs. (3) and (4) yield

$$\theta(3\tau_t) = T(3\tau_t) - 27^\circ C - \frac{5000\,W/m^2}{50\,W/m^2\cdot K} = e^{-3}\left[27^\circ C - 27^\circ C - \frac{5000\,W/m^2}{50\,W/m^2\cdot K}\right]$$

$$T(3\tau_t) = 122^\circ C$$ <

(c) As shown by the following graphical results, which were generated using the IHT *Lumped Capacitance Model*, the steady-state temperature and the time to reach steady-state both decrease with increasing h.



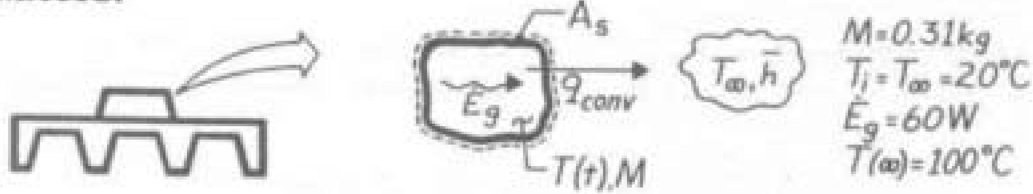
COMMENTS: Note that, even for $h = 200\ W/m^2\cdot K$, $Bi = hL/k << 0.1$ and assumption (1) is reasonable.

PROBLEM 5.19

KNOWN: Electronic device on aluminum, finned heat sink modeled as spatially isothermal object with internal generation and convection from its surface.

FIND: (a) Temperature response after device is energized, (b) Temperature rise for prescribed conditions after 5 min.

SCHEMATIC:



ASSUMPTIONS: (1) Spatially isothermal object, (2) Object is primarily aluminum, (3) Initially, object is in equilibrium with surroundings at T_{sur}.

PROPERTIES: *Table A-1*, Aluminum, pure ($\bar{T} = (20+100)°C/2 = 333K$): $c = 918$ J/kg·K.

ANALYSIS: (a) Following the general analysis of Section 5.3, apply the conservation of energy requirement to the object,

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st} \qquad \dot{E}_g - \bar{h}A_s(T - T_\infty) = Mc\frac{dT}{dt} \tag{1}$$

where $T = T(t)$. Consider now steady-state conditions, in which case the storage term of Eq. (1) is zero. The temperature of the object will be $T(\infty)$ such that

$$\dot{E}_g = \bar{h}A_s\left(T(\infty) - T_\infty\right). \tag{2}$$

Substituting for $\dot{E}_g$ using Eq. (2) into Eq. (1), the differential equation is

$$[T(\infty) - T_\infty] - [T - T_\infty] = \frac{Mc}{\bar{h}A_s}\frac{dT}{dt} \qquad \text{or} \qquad \theta = -\frac{Mc}{\bar{h}A_s}\frac{d\theta}{dt} \tag{3,4}$$

with $\theta \equiv T - T(\infty)$ and noting that $d\theta = dT$. Identifying $R_t = 1/\bar{h}A_s$ and $C_t = Mc$, the differential equation is integrated with proper limits,

$$\frac{1}{R_tC_t}\int_0^t dt = -\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} \qquad \text{or} \qquad \frac{\theta}{\theta_i} = \exp\left[-\frac{t}{R_tC_t}\right] \tag{5}$$ ◁

where $\theta_i = \theta(0) = T_i - T(\infty)$ and T_i is the initial temperature of the object.

(b) Using the information about steady-state conditions and Eq. (2), find first the thermal resistance and capacitance of the system,

$$R_t = \frac{1}{\bar{h}A_s} = \frac{T(\infty) - T_\infty}{\dot{E}_g} = \frac{(100-20)°C}{60\ W} = 1.33\ K/W \qquad C_t = Mc = 0.31\ kg \times 918\ J/kg\cdot K = 285\ J/K.$$

Using Eq. (5), the temperature of the system after 5 minutes is

$$\frac{\theta(5min)}{\theta_i} = \frac{T(5min) - T(\infty)}{T_i - T(\infty)} = \frac{T(5min) - 100°C}{(20-100)°C} = \exp\left[-\frac{5\times 60s}{1.33\ K/W \times 285\ J/K}\right] = 0.453$$

$$T(5min) = 100°C + (20-100)°C \times 0.453 = 63.8°C$$ ◁

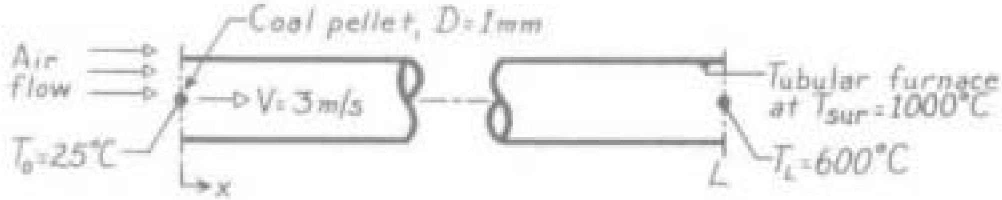
COMMENTS: Eq. 5.24 may be used directly for Part (b) with $a = hA_s/Mc$ and $b = \dot{E}_g/Mc$.

PROBLEM 5.20

KNOWN: Spherical coal pellet at 25°C is heated by radiation while flowing through a furnace maintained at 1000°C.

FIND: Length of tube required to heat pellet to 600°C.

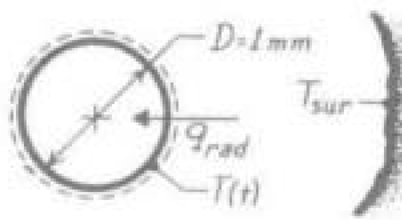
SCHEMATIC:



ASSUMPTIONS: (1) Pellet is suspended in air flow and subjected to only radiative exchange with furnace, (2) Pellet is small compared to furnace surface area, (3) Coal pellet has emissivity, $\varepsilon = 1$.

PROPERTIES: *Table A-3*, Coal ($\bar{T} = (600+25)°C/2 = 585K$, however, only 300K data available): $\rho = 1350\ kg/m^3$, $c_p = 1260\ J/kg\cdot K$, $k = 0.26\ W/m\cdot K$.

ANALYSIS: Considering the pellet as spatially isothermal, use the lumped capacitance method of Section 5.3 to find the time required to heat the pellet from $T_o = 25°C$ to $T_L = 600°C$. From an energy balance on the pellet $\dot{E}_{in} = \dot{E}_{st}$ where



$$\dot{E}_{in} = q_{rad} = \sigma A_s \left(T_{sur}^4 - T_s^4\right) \qquad \dot{E}_{st} = \rho V c_p \frac{dT}{dt}$$

giving $$A_s \sigma \left(T_{sur}^4 - T_s^4\right) = \rho V c_p \frac{dT}{dt}.$$

Separating variables and integrating with limits shown, the temperature-time relation becomes

$$\frac{A_s \sigma}{\rho V c_p} \int_0^t dt = \int_{T_o}^{T_i} \frac{dT}{T_{sur}^4 - T^4}.$$

The integrals are evaluated in Eq. 5.18 giving,

$$t = \frac{\rho V c_p}{4 A_s \sigma T_{sur}^3} \left\{ \ln\left|\frac{T_{sur}+T}{T_{sur}-T}\right| - \ln\left|\frac{T_{sur}+T_i}{T_{sur}-T_i}\right| + 2\left[\tan^{-1}\left[\frac{T}{T_{sur}}\right] - \tan^{-1}\left[\frac{T_i}{T_{sur}}\right]\right]\right\}.$$

Recognizing that $A_s = \pi D^2$ and $V = \pi D^3/6$ or $A_s/V = 6/D$ and substituting values,

$$t = \frac{1350\ kg/m^3 (0.001m)\ 1260\ J/kg\cdot K}{24 \times 5.67 \times 10^{-8}\ W/m^2\cdot K^4 (1273\ K)^3} \left\{ \ln\frac{1273+873}{1273-873} - \ln\frac{1273+298}{1273-298} \right.$$

$$\left. + 2\left[\tan^{-1}\left(\frac{873}{1273}\right) - \tan^{-1}\left(\frac{298}{1273}\right)\right]\right\} = 1.18s.$$

Hence, $L = V \cdot t = 3m/s \times 1.18s = 3.54m.$ ◁

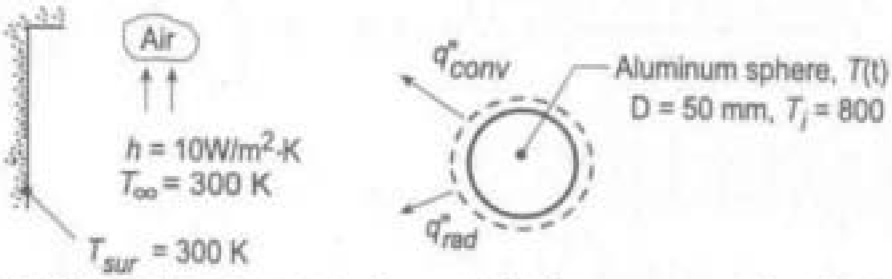
The validity of the lumped capacitance method requires $Bi = h(V/A_s)/k < 0.1$. Using Eq. (1-8) for $h = h_r$ and $V/A_s = D/6$, find that when $T = 600°C$, $Bi = 0.19$; but when $T = 25°C$, $Bi = 0.10$. At early times, when the pellet is cooler, the assumption is reasonable but becomes less appropriate as the pellet heats.

PROBLEM 5.21

KNOWN: Metal sphere, initially at a uniform temperature T_i, is suddenly removed from a furnace and suspended in a large room and subjected to a convection process (T_∞, h) and to radiation exchange with surroundings, T_{sur}.

FIND: (a) Time it takes for sphere to cool to some temperature T, neglecting radiation exchange, (b) Time it takes for sphere to cool to some temperature t, neglecting convection, (c) Procedure to obtain time required if both convection and radiation are considered, (d) Time to cool an anodized aluminum sphere to 400 K using results of Parts (a), (b) and (c).

SCHEMATIC:



ASSUMPTIONS: (1) Sphere is spacewise isothermal, (2) Constant properties, (3) Constant heat transfer convection coefficient, (4) Sphere is small compared to surroundings.

PROPERTIES: *Table A-1*, Aluminum, pure ($\bar{T}$ = [800 + 400] K/2 = 600 K): $\rho = 2702\ kg/m^3$, c = 1033 J/kg·K, k = 231 W/m·K, $\alpha = k/\rho c = 8.276 \times 10^{-5}\ m^2/s$; Aluminum, anodized finish: $\varepsilon = 0.75$, polished surface: $\varepsilon = 0.1$.

ANALYSIS: (a) Neglecting radiation, the time to cool is predicted by Eq. 5.5,

$$t = \frac{\rho Vc}{hA_s}\ln\frac{\theta_i}{\theta} = \frac{\rho Dc}{6h}\ln\frac{T_i - T_\infty}{T - T_\infty} \qquad (1) <$$

where $V/A_s = (\pi D^3/6)/(\pi D^2) = D/6$ for the sphere.

(b) Neglecting convection, the time to cool is predicted by Eq. 5.18,

$$t = \frac{\rho Dc}{24\varepsilon\sigma T_{sur}^3}\left\{\ln\left|\frac{T_{sur}+T}{T_{sur}-T}\right| - \ln\left|\frac{T_{sur}+T_i}{T_{sur}-T_i}\right| + 2\left[\tan^{-1}\left(\frac{T}{T_{sur}}\right) - \tan^{-1}\left(\frac{T_i}{T_{sur}}\right)\right]\right\} \qquad (2)$$

where $V/A_{s,r} = D/6$ for the sphere.

(c) If convection and radiation exchange are considered, the energy balance requirement results in Eq. 5.15 (with $q_s'' = \dot{E}_g = 0$). Hence

$$\frac{dT}{dt} = \frac{6}{\rho Dc}\left[h(T - T_\infty) + \varepsilon\sigma\left(T^4 - T_{sur}^4\right)\right] \qquad (3) <$$

where $A_{s(c,r)} = A_s = \pi D^2$ and $V/A_{s(c,r)} = D/6$. This relation must be solved numerically in order to evaluate the time-to-cool.

(d) For the aluminum (pure) sphere with an anodized finish and the prescribed conditions, the times to cool from T_i = 800 K to T = 400 K are:

Continued...

Convection only, Eq. (1)

$$t = \frac{2702\,\text{kg/m}^3 \times 0.050\,\text{m} \times 1033\,\text{J/kg}\cdot\text{K}}{6 \times 10\,\text{W/m}^2\cdot\text{K}} \ln\frac{800-300}{400-300} = 3743\text{s} = 1.04\text{h}$$ <

Radiation only, Eq. (2)

$$t = \frac{2702\,\text{kg/m}^3 \times 0.050\,\text{m} \times 1033\,\text{J/kg}\cdot\text{K}}{24 \times 0.75 \times 5.67 \times 10^{-8}\,\text{W/m}^2\cdot\text{K}^4 \times (300\,\text{K})^3} \left\{\left(\ln\frac{400+300}{400-300} - \ln\frac{800+300}{800-300}\right) + 2\left[\tan^{-1}\frac{400}{300} - \tan^{-1}\frac{800}{300}\right]\right\}$$

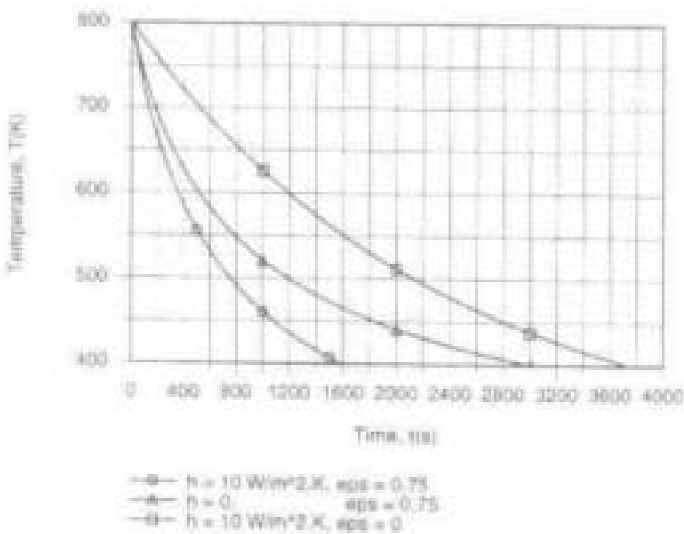
$$t = 5.065 \times 10^3\{1.946 - 0.789 + 2(0.927 - 1.212)\} = 2973\text{s} = 0.826\text{h}$$ <

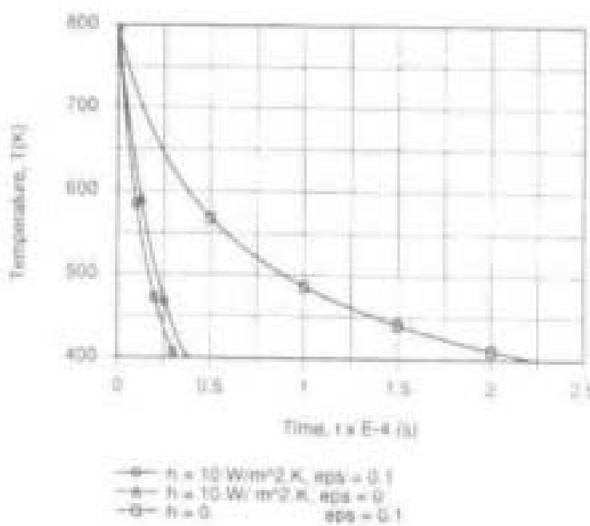
Radiation and convection, Eq. (3)

Using the IHT *Lumped Capacitance Model*, numerical integration yields

$$t \approx 1600\text{s} = 0.444\text{h}$$

In this case, heat loss by radiation exerts the stronger influence, although the effects of convection are by no means negligible. However, if the surface is polished ($\varepsilon = 0.1$), convection clearly dominates. For each surface finish and the three cases, the temperature histories are as follows.





COMMENTS: A summary of the analyses shows the relative importance of the various modes of heat loss:

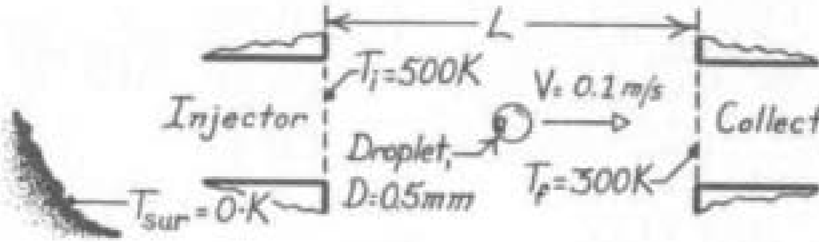
Active Modes	Time required to cool to 400 K (h)	
	$\varepsilon = 0.75$	$\varepsilon = 0.1$
Convection only	1.040	1.040
Radiation only	0.827	6.194
Both modes	0.444	0.889

PROBLEM 5.22

KNOWN: Droplet properties, diameter, velocity and initial and final temperatures.

FIND: Travel distance and rejected thermal energy.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Negligible radiation from space.

PROPERTIES: Droplet (given): $\rho = 885\ \text{kg/m}^3$, $c = 1900\ \text{J/kg·K}$, $k = 0.145\ \text{W/m·K}$, $\varepsilon = 0.95$.

ANALYSIS: To assess the suitability of applying the lumped capacitance method, use Equation 1.9 to obtain the maximum radiation coefficient, which corresponds to $T = T_i$.

$$h_r = \varepsilon\sigma T_i^3 = 0.95\times5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4 (500\ \text{K})^3 = 6.73\ \text{W/m}^2\cdot\text{K}.$$

Hence

$$Bi_r = \frac{h_r(r_o/3)}{k} = \frac{(6.73\ \text{W/m}^2\cdot\text{K})(0.25\times10^{-3}\ \text{m}/3)}{0.145\ \text{W/m}\cdot\text{K}} = 0.0039$$

and the lumped capacitance method can be used. From Equation 5.19,

$$t = \frac{L}{V} = \frac{\rho c(\pi D^3/6)}{3\varepsilon(\pi D^2)\sigma}\left(\frac{1}{T_f^3} - \frac{1}{T_i^3}\right)$$

$$L = \frac{(0.1\ \text{m/s})885\ \text{kg/m}^3(1900\ \text{J/kg}\cdot\text{K})0.5\times10^{-3}\ \text{m}}{18\times0.95\times5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4}\left(\frac{1}{300^3} - \frac{1}{500^3}\right)\frac{1}{\text{K}^3}$$

$$L = 2.52\ \text{m}. \qquad \triangleleft$$

The amount of energy rejected by each droplet is equal to the change in its internal energy.

$$E_i - E_f = \rho V c(T_i - T_f) = 885\ \text{kg/m}^3\,\pi\frac{(5\times10^{-4}\ \text{m})^3}{6}1900\ \text{J/kg}\cdot\text{K}(200\ \text{K})$$

$$E_i - E_f = 0.022\ \text{J}. \qquad \triangleleft$$

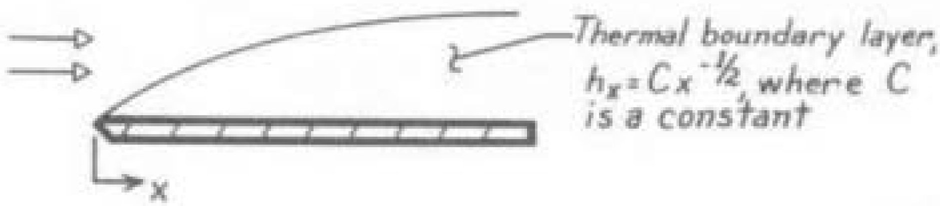
COMMENTS: Because some of the radiation emitted by a droplet will be intercepted by other droplets in the stream, the foregoing analysis overestimates the amount of heat dissipated by radiation to space.

PROBLEM 6.1

KNOWN: Variation of h_x with x for laminar flow over a flat plate.

FIND: Ratio of average coefficient, $\bar{h}_x$, to local coefficient, h_x, at x.

SCHEMATIC:



ANALYSIS: From Eq. 6.5, the average value of h_x between 0 and x is

$$\bar{h}_x = \frac{1}{x}\int_0^x h_x dx = \frac{C}{x}\int_0^x x^{-1/2}\, dx$$

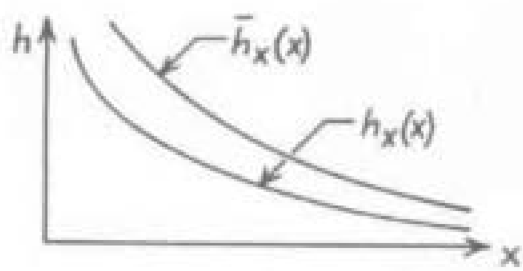
$$\bar{h}_x = \frac{C}{x}\, 2x^{1/2} = 2Cx^{-1/2}$$

$$\bar{h}_x = 2h_x\,.$$

Hence,

$$\frac{\bar{h}_x}{h_x} = 2. \qquad \triangleleft$$

COMMENTS: Both the local and average coefficients decrease with increasing distance x from the leading edge as shown in the sketch below.

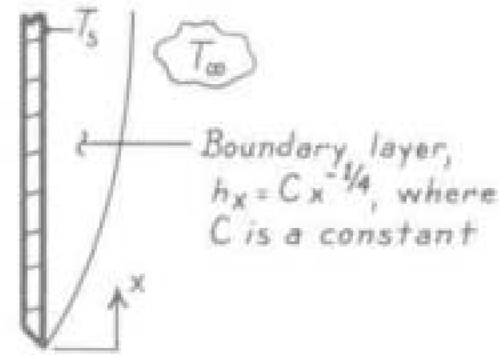


PROBLEM 6.2

KNOWN: Variation of local convection coefficient with x for free convection from a vertical heated plate.

FIND: Ratio of average to local convection coefficient.

SCHEMATIC:



ANALYSIS: From Eq. 6.5, it follows that the average coefficient from 0 to x is

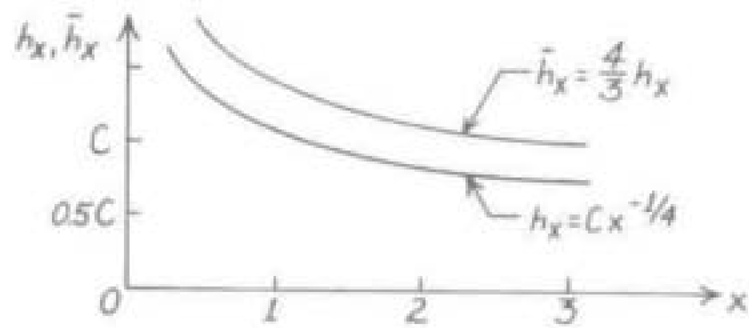
$$\bar{h}_x = \frac{1}{x}\int_0^x h_x\, dx = \frac{C}{x}\int_0^x x^{-1/4}\, dx$$

$$\bar{h}_x = \frac{4}{3}\frac{C}{x}x^{3/4} = \frac{4}{3}C\,x^{-1/4} = \frac{4}{3}h_x\,.$$

Hence,

$$\frac{\bar{h}_x}{h_x} = \frac{4}{3}. \qquad \triangleleft$$

The variations with distance of the local and average convection coefficients are shown in the sketch.



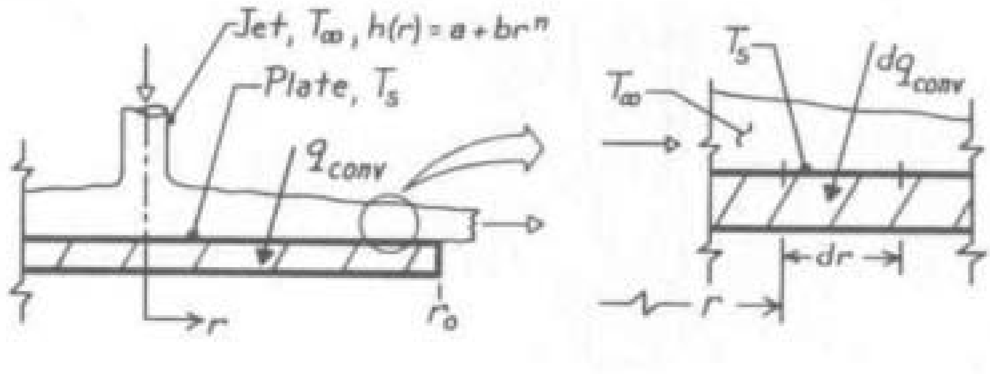
COMMENTS: Note that $\bar{h}_x / h_x = 4/3$, independent of x. Hence the average coefficient for an entire plate of length L is $\bar{h}_L = \frac{4}{3}h_L$, where h_L is the local coefficient at $x = L$. Note also that the average *exceeds* the local. Why?

PROBLEM 6.3

KNOWN: Expression for the local heat transfer coefficient of a circular, hot gas jet at T_∞ directed normal to a circular plate at T_s of radius r_o.

FIND: Heat transfer rate to the plate by convection.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Flow is axisymmetrical about the plate, (3) For h(r), a and b are constants and $n \neq -2$.

ANALYSIS: The convective heat transfer rate to the plate follows from Newton's law of cooling

$$q_{conv} = \int_A dq_{conv} = \int_A h(r) \cdot dA \cdot (T_\infty - T_s) .$$

The local heat transfer coefficient is known to have the form,

$$h(r) = a + br^n$$

and the differential area on the plate surface is

$$dA = 2\pi\, r\, dr .$$

Hence, the heat rate is

$$q_{conv} = \int_0^{r_o} (a + br^n) \cdot 2\pi r\, dr \cdot (T_\infty - T_s)$$

$$q_{conv} = 2\pi\,(T_\infty - T_s) \left[\frac{a}{2} r^2 + \frac{b}{n+2} r^{n+2} \right]_0^{r_o}$$

$$q_{conv} = 2\pi \left[\frac{a}{2} r_o^2 + \frac{b}{n+2} r_o^{n+2} \right] (T_\infty - T_s). \qquad \triangleleft$$

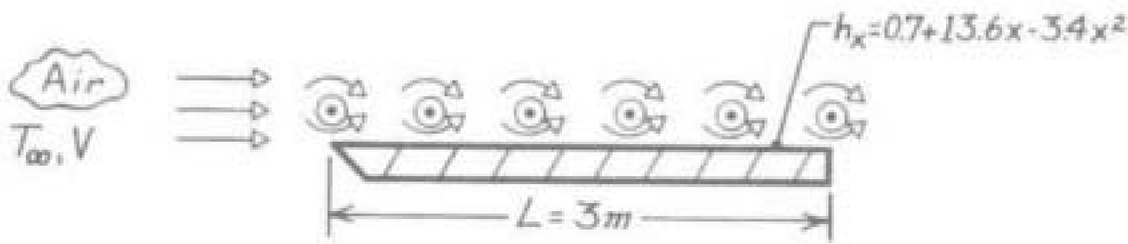
COMMENTS: Note the importance of the assumption requiring $n \neq -2$. Recognize that practically, we would expect the radius of the jet to be much smaller than that of the plate. How does the thickness of the boundary layer vary with plate radius?

PROBLEM 6.4

KNOWN: Distribution of local convection coefficient for obstructed parallel flow over a flat plate.

FIND: Average heat transfer coefficient and ratio of average to local at the trailing edge.

SCHEMATIC:



ANALYSIS: From Equation 6.6, the average convection coefficient (W/m²·K) is

$$\bar{h}_L = \frac{1}{L}\int_0^L h_x dx = \frac{1}{L}\int_0^L (0.7 + 13.6x - 3.4x^2)dx$$

$$\bar{h}_L = \frac{1}{L}(0.7L + 6.8L^2 - 1.13L^3) = 0.7 + 6.8L - 1.13L^2$$

$$\bar{h}_L = 0.7 + 6.8(3) - 1.13(9) = 10.9\ \text{W/m}^2\cdot\text{K}. \quad \triangleleft$$

The local coefficient at x = 3 m is

$$h_L = 0.7 + 13.6(3) - 3.4(9) = 10.9\ \text{W/m}^2\cdot\text{K}.$$

Hence,

$$\bar{h}_L / h_L = 1.0. \quad \triangleleft$$

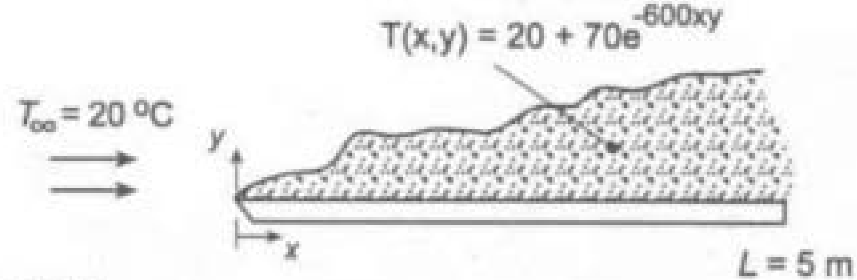
COMMENTS: The result $\bar{h}_L/h_L = 1.0$ is unique to x = 3 m and is a consequence of the existence of a maximum for $h_x(x)$. The maximum occurs at x = 2 m, where $(dh_x/dx) = 0$ and $(d^2h_x/dx^2) < 0$.

PROBLEM 6.5

KNOWN: Temperature distribution in boundary layer for air flow over a flat plate.

FIND: Variation of local convection coefficient along the plate and value of average coefficient.

SCHEMATIC:



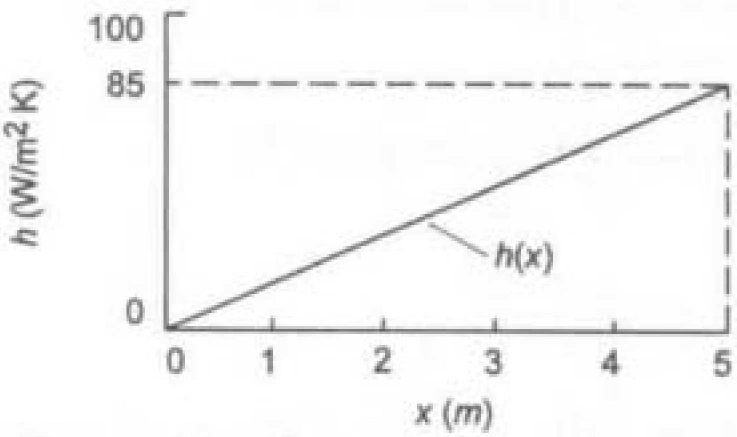
ANALYSIS: From Eq. 6.17,

$$h = -\frac{k\partial T/\partial y|_{y=0}}{(T_s - T_\infty)} = +\frac{k(70 \times 600x)}{(T_s - T_\infty)}$$

where $T_s = T(x,0) = 90°C$. Evaluating k at the arithmetic mean of the freestream and surface temperatures, $\overline{T} = (20 + 90)°C/2 = 55°C = 328$ K, Table A.4 yields $k = 0.0284$ W/m·K. Hence, with $T_s - T_\infty = 70°C = 70$ K,

$$h = \frac{0.0284\,\mathrm{W/m\cdot K}(42{,}000x)\,\mathrm{K/m}}{70\,\mathrm{K}} = 17x\left(\mathrm{W/m^2\cdot K}\right)$$ <

and the convection coefficient increases linearly with x.



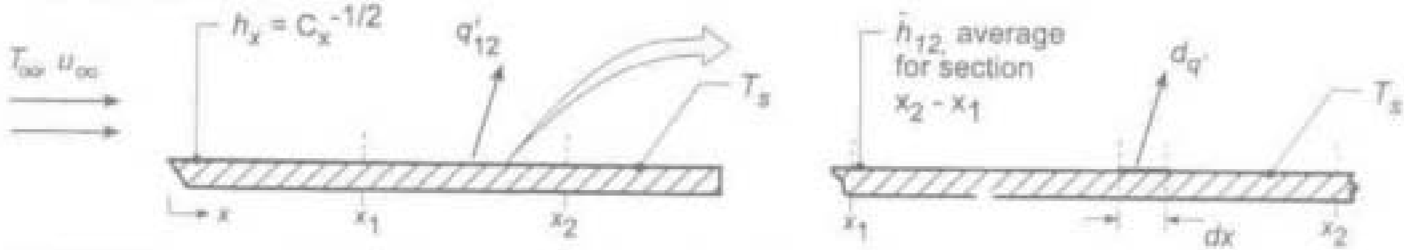
The average coefficient over the range $0 \le x \le 5$ m is

$$\overline{h} = \frac{1}{L}\int_0^L h\,dx = \frac{17}{5}\int_0^5 x\,dx = \frac{17}{5}\frac{x^2}{2}\Big|_0^5 = 42.5\,\mathrm{W/m^2\cdot K}$$ <

KNOWN: Variation of local convection coefficient with distance x from a heated plate with a uniform temperature T_s.

FIND: (a) An expression for the average coefficient $\bar{h}_{12}$ for the section of length $(x_2 - x_1)$ in terms of C, x_1 and x_2, and (b) An expression for $\bar{h}_{12}$ in terms of x_1 and x_2, and the average coefficients $\bar{h}_1$ and $\bar{h}_2$, corresponding to lengths x_1 and x_2, respectively.

SCHEMATIC:



ASSUMPTIONS: (1) Laminar flow over a plate with uniform surface temperature, T_s, and (2) Spatial variation of local coefficient is of the form $h_x = Cx^{-1/2}$, where C is a constant.

ANALYSIS: (a) The heat transfer rate per unit width from a longitudinal section, $x_2 - x_1$, can be expressed as

$$q'_{12} = \bar{h}_{12}(x_2 - x_1)(T_s - T_\infty) \tag{1}$$

where $\bar{h}_{12}$ is the average coefficient for the section of length $(x_2 - x_1)$. The heat rate can also be written in terms of the local coefficient, Eq. (6.3), as

$$q'_{12} = \int_{x_1}^{x_2} h_x dx (T_s - T_\infty) = (T_s - T_\infty)\int_{x_1}^{x_2} h_x dx \tag{2}$$

Combining Eq. (1) and (2),

$$\bar{h}_{12} = \frac{1}{(x_2 - x_1)}\int_{x_1}^{x_2} h_x dx \tag{3}$$

and substituting for the form of the local coefficient, $h_x = Cx^{-1/2}$, find that

$$\bar{h}_{12} = \frac{1}{(x_2 - x_1)}\int_{x_1}^{x_2} Cx^{-1/2} dx = \frac{C}{x_2 - x_1}\left[\frac{x^{1/2}}{1/2}\right]_{x_1}^{x_2} = 2C\frac{x_2^{1/2} - x_1^{1/2}}{x_2 - x_1} \tag{4} <$$

(b) The heat rate, given as Eq. (1), can also be expressed as

$$q'_{12} = \bar{h}_2 x_2 (T_s - T_\infty) - \bar{h}_1 x_1 (T_s - T_\infty) \tag{5}$$

which is the difference between the heat rate for the plate over the section $(0 - x_2)$ and over the section $(0 - x_1)$. Combining Eqs. (1) and (5), find,

$$\bar{h}_{12} = \frac{\bar{h}_2 x_2 - \bar{h}_1 x_1}{x_2 - x_1} \tag{6} <$$

COMMENTS: (1) Note that, from Eq. 6.6,

$$\bar{h}_x = \frac{1}{x}\int_0^x h_x dx = \frac{1}{x}\int_0^x Cx^{-1/2} dx = 2Cx^{-1/2} \tag{7}$$

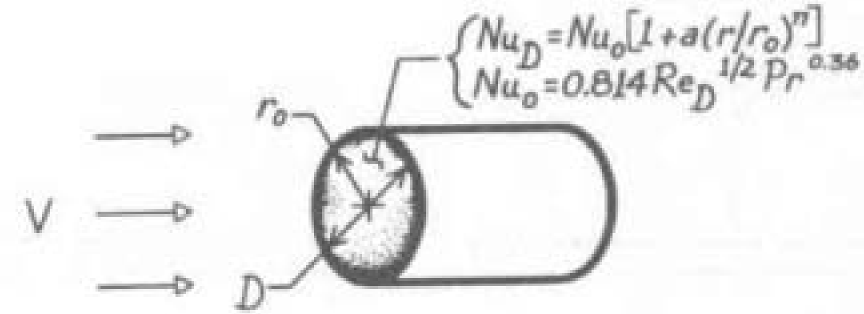
or $\bar{h}_x = 2h_x$. Substituting Eq. (7) into Eq. (6), see that the result is the same as Eq. (4).

PROBLEM 6.7

KNOWN: Radial distribution of local convection coefficient for flow normal to a circular disk.

FIND: Expression for average Nusselt number.

SCHEMATIC:



ASSUMPTIONS: Constant properties.

ANALYSIS: From Equation 6.5, the average convection coefficient is

$$\bar{h} = \frac{1}{A_s}\int_{A_s} h dA_s$$

$$\bar{h} = \frac{1}{\pi r_o^2}\int_0^{r_o} \frac{k}{D} Nu_o[1 + a(r/r_o)^n]2\pi r dr$$

$$\bar{h} = \frac{kNu_o}{r_o^3}\left[\frac{r^2}{2} + \frac{ar^{n+2}}{(n+2)r_o^n}\right]_0^{r_o}$$

where Nu_o is the Nusselt number at the stagnation point ($r = 0$). Hence,

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = 2Nu_o\left[\frac{(r/r_o)^2}{2} + \frac{a}{(n+2)}\left(\frac{r}{r_o}\right)^{n+2}\right]_0^{r_o}$$

$$\overline{Nu}_D = Nu_o[1 + 2a/(n+2)]$$

$$\overline{Nu}_D = [1 + 2a/(n+2)]0.814 Re_D^{1/2} Pr^{0.36}. \qquad \triangleleft$$

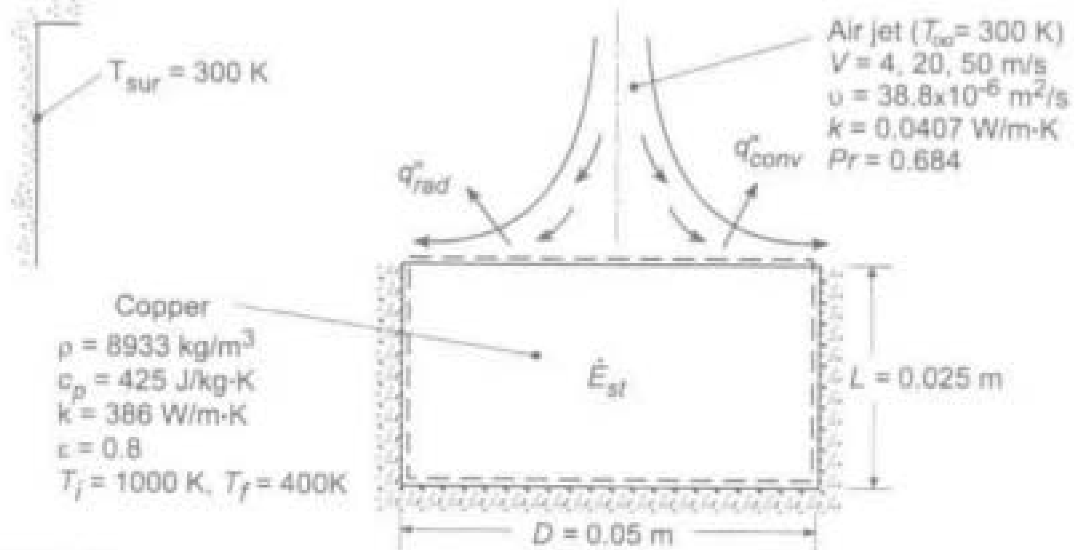
COMMENTS: The increase in h(r) with r may be explained in terms of the sharp turn which the boundary layer flow must make around the edge of the disk. The boundary layer accelerates and its thickness decreases as it makes the turn, causing the local convection coefficient to increase.

PROBLEM 6.8

KNOWN: Convection correlation and temperature of an impinging air jet. Dimensions and initial temperature of a heated copper disk. Properties of the air and copper.

FIND: Effect of jet velocity on temperature decay of disk following jet impingement.

SCHEMATIC:



ASSUMPTIONS: (1) Validity of lumped capacitance analysis, (2) Negligible heat transfer from sides and bottom of disk, (3) Constant properties.

ANALYSIS: Performing an energy balance on the disk, it follows that $\dot{E}_{st} = \rho V c\, dT/dt = -A_s(q''_{conv} + q''_{rad})$. Hence, with $V = A_s L$,

$$\frac{dT}{dt} = -\frac{\bar{h}(T - T_\infty) + h_r(T - T_{sur})}{\rho c L}$$

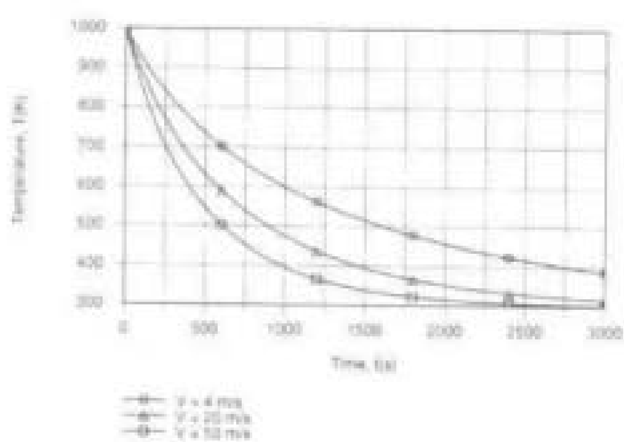
where, $h_r = \varepsilon\sigma(T + T_{sur})(T^2 + T_{sur}^2)$ and, from the solution to Problem 6.7,

$$\bar{h} = \frac{k}{D}\overline{Nu}_D = \frac{k}{D}\left(1 + \frac{2a}{n+2}\right)0.814\, Re_D^{1/2}\, Pr^{0.36}$$

With a = 0.30 and n = 2, it follows that

$$\bar{h} = (k/D)0.936\, Re_D^{1/2}\, Pr^{0.36}$$

where $Re_D = VD/\nu$. Using the *Lumped Capacitance Model* of IHT, the following temperature histories were determined.



Continued...

The temperature decay becomes more pronounced with increasing V, and a final temperature of 400 K is reached at t = 2760, 1455 and 976s for V = 4, 20 and 50 m/s, respectively.

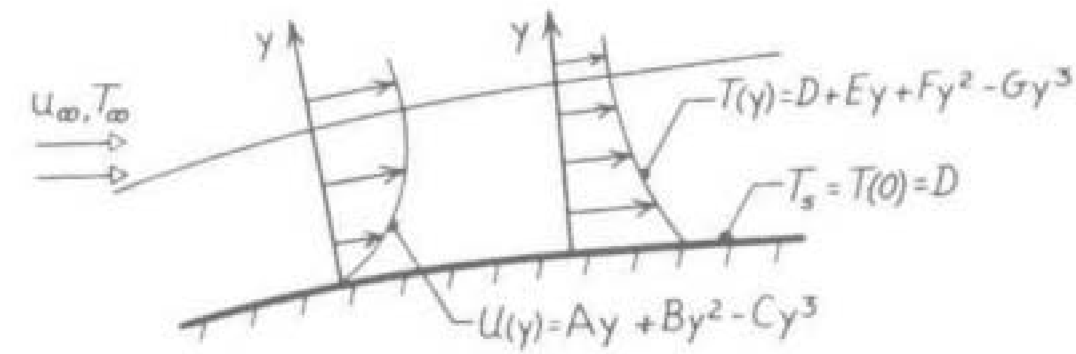
COMMENTS: The maximum Biot number, $Bi = (\overline{h} + h_r)L/k_{Cu}$, is associated with V = 50 m/s (maximum $\overline{h}$ of 169 $W/m^2 \cdot K$) and t = 0 (maximum h_r of 64 $W/m^2 \cdot K$), in which case the maximum Biot number is $Bi = (233\ W/m^2 \cdot K)(0.025\ m)/(386\ W/m \cdot K) = 0.015 < 0.1$. Hence, the lumped capacitance approximation is valid.

PROBLEM 6.9

KNOWN: Form of the velocity and temperature profiles for flow over a surface.

FIND: Expressions for the friction and convection coefficients.

SCHEMATIC:



ANALYSIS: From Section 6.2.1, the shear stress at the wall is

$$\tau_s = \mu \left.\frac{\partial u}{\partial y}\right]_{y=0} = \mu\,[A + 2By - 3Cy^2]_{y=0} = A\mu\,.$$

Hence, the friction coefficient has the form,

$$C_f = \frac{\tau_s}{\rho u_\infty^2/2} = \frac{2A\mu}{\rho u_\infty^2}$$

$$C_f = \frac{2A\nu}{u_\infty^2}. \qquad \triangleleft$$

From Section 6.2.2, the convection coefficient is

$$h = \frac{-k_f(\partial T/\partial y)_{y=0}}{T_s - T_\infty} = \frac{-k_f\,[E + 2Fy - 3Gy^2]_{y=0}}{D - T_\infty}$$

$$h = \frac{-k_f E}{D - T_\infty}. \qquad \triangleleft$$

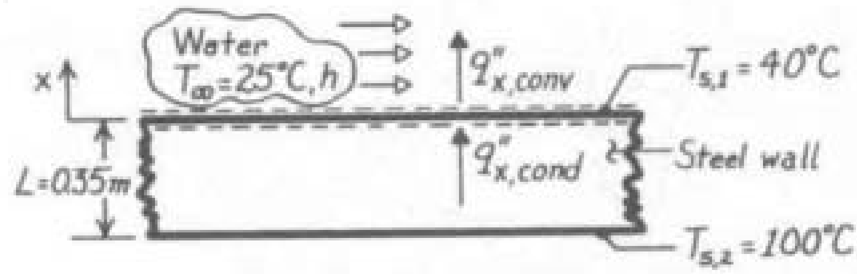
COMMENTS: It is a simple matter to obtain the important surface parameters from knowledge of the corresponding boundary layer profile. However, it is rarely a simple matter to determine the form of the profile.

PROBLEM 6.10

KNOWN: Surface temperatures of a steel wall and temperature of water flowing over the wall.

FIND: (a) Convection coefficient, (b) Temperature gradient in wall and in water at wall surface.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional heat transfer in x, (3) Constant properties.

PROPERTIES: *Table A-1*, Steel Type AISI 1010 (70°C = 343K), $k_s = 61.7$ W/m·K; *Table A-6*, Water (32.5°C = 305K), $k_f = 0.62$ W/m·K.

ANALYSIS: (a) Applying an energy balance Eq. 1.12 to the control surface at x=0, it follows that

$$q''_{x,cond} - q''_{x,conv} = 0$$

and using the appropriate rate equations, Eqs. 1.2 and 1.3a,

$$k_s \frac{T_{s,2} - T_{s,1}}{L} = h(T_{s,1} - T_\infty).$$

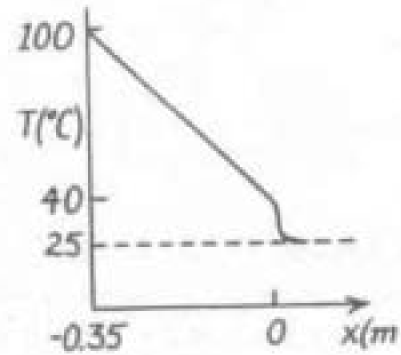
Hence,

$$h = \frac{k_s}{L}\frac{T_{s,2} - T_{s,1}}{T_{s,1} - T_\infty} = \frac{61.7 \text{ W/m·K}}{0.35\text{m}}\frac{60°C}{15°C} = 705 \text{ W/m}^2\text{·K}.$$ ◁

(b) The gradient in the wall at the surface is

$$(dT/dx)_s = -\frac{T_{s,2} - T_{s,1}}{L} = -\frac{60°C}{0.35\text{m}} = -171.4°C/\text{m}.$$

In the water at x=0, the definition of h (Section 6.2.2) gives



$$(dT/dx)_{f,x=0} = -\frac{h}{k_f}(T_{s,1} - T_\infty)$$

$$(dT/dx)_{f,x=0} = -\frac{705 \text{ W/m}^2\text{·K}}{0.62 \text{ W/m·K}}(15°C) = -17{,}056°C/\text{m}.$$ ◁

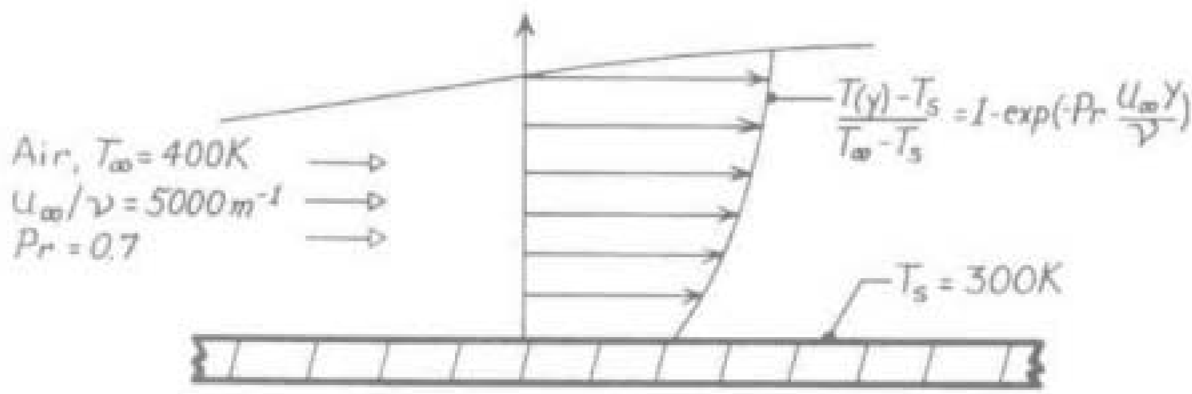
COMMENTS: Note relative magnitudes of the gradients.

PROBLEM 6.11

KNOWN: Boundary layer temperature distribution.

FIND: Surface heat flux.

SCHEMATIC:



PROPERTIES: *Table A-4*, Air (T_s = 300K): k = 0.0263 W/m·K.

ANALYSIS: Applying Fourier's law at y=0, the heat flux is

$$q_s'' = -k\left.\frac{\partial T}{\partial y}\right|_{y=0} = -k(T_\infty - T_s)\left[Pr\frac{u_\infty}{\nu}\right]\exp\left[-Pr\frac{u_\infty y}{\nu}\right]\Bigg|_{y=0}$$

$$q_s'' = -k(T_\infty - T_s)\,Pr\,\frac{u_\infty}{\nu}$$

$$q_s'' = -0.0263\text{ W/m·K }(100\text{K})\ 0.7 \times 5000\text{ 1/m}$$

$$q_s'' = -9205\text{ W/m}^2.$$ ◁

COMMENTS: (1) Negative flux implies convection heat transfer to the surface.

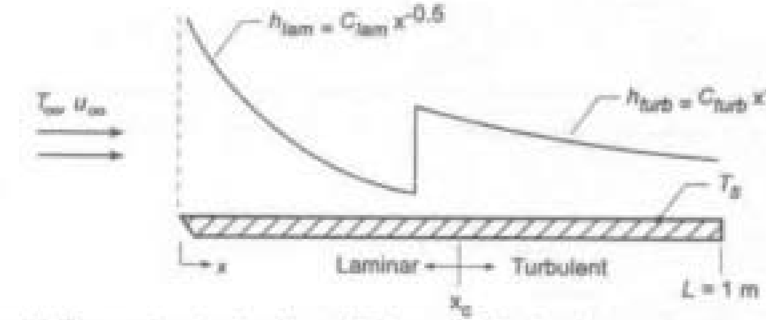
(2) Note use of k at T_s to evaluate q_s'' from Fourier's law.

PROBLEM 6.12

KNOWN: Air flow over a flat plate of length L = 1 m under conditions for which transition from laminar to turbulent flow occurs at $x_c = 0.5$m based upon the critical Reynolds number, $Re_{x,c} = 5\times 10^5$. Forms for the local convection coefficients in the laminar and turbulent regions.

FIND: (a) Velocity of the air flow using thermophysical properties evaluated at 350 K, (b) An expression for the average coefficient $\overline{h}_{lam}(x)$, as a function of distance from the leading edge, x, for the laminar region, $0 \le x \le x_c$, (c) An expression for the average coefficient $\overline{h}_{turb}(x)$, as a function of distance from the leading edge, x, for the turbulent region, $x_c \le x \le L$, and (d) Compute and plot the local and average convection coefficients, h_x and $\overline{h}_x$, respectively, as a function of x for $0 \le x \le L$.

SCHEMATIC:



ASSUMPTIONS: (1) Forms for the local coefficients in the laminar and turbulent regions, $h_{lam} = C_{lam}x^{-0.5}$ and $h_{turb} = C_{turb}x^{-0.2}$ where $C_{lam} = 8.845$ W/m²·K$^{0.5}$, $C_{turb} = 49.75$ W/m²·K$^{0.8}$, and x has units (m).

PROPERTIES: *Table A.4*, Air (T = 350 K): k = 0.030 W/m·K, $\nu = 20.92 \times 10^{-6}$ m²/s, Pr = 0.700.

ANALYSIS: (a) Using air properties evaluated at 350 K with $x_c = 0.5$ m,

$$Re_{x,c} = \frac{u_\infty x_c}{\nu} = 5\times10^5 \qquad u_\infty = 5\times10^5\, \nu/x_c = 5\times10^5 \times 20.92\times10^{-6}\,\text{m}^2/\text{s}/0.5\,\text{m} = 20.9\,\text{m/s} \quad <$$

(b) From Eq. 6.5, the average coefficient in the laminar region, $0 \le x \le x_c$, is

$$\overline{h}_{lam}(x) = \frac{1}{x}\int_0^x h_{lam}(x)dx = \frac{1}{x}C_{lam}\int_0^x x^{-0.5}dx = \frac{1}{x}C_{lam}x^{0.5} = 2C_{lam}x^{-0.5} = 2h_{lam}(x) \qquad (1) <$$

(c) The average coefficient in the turbulent region, $x_c \le x \le L$, is

$$\overline{h}_{turb}(x) = \frac{1}{x}\left[\int_0^{x_c} h_{lam}(x)dx + \int_{x_c}^x h_{turb}(x)dx\right] = \left[C_{lam}\frac{x^{0.5}}{0.5}\bigg|_0^{x_c} + C_{turb}\frac{x^{0.8}}{0.8}\bigg|_{x_c}^x\right]$$

$$\overline{h}_{turb}(x) = \frac{1}{x}\left[2C_{lam}x_c^{0.5} + 1.25C_{turb}\left(x^{0.8} - x_c^{0.8}\right)\right] \qquad (2) <$$

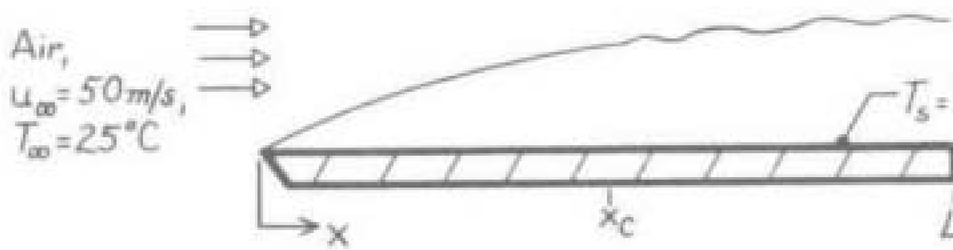
(d) The local and average coefficients, Eqs. (1) and (2) are plotted below as a function of x for the range $0 \le x \le L$.

PROBLEM 6.13

KNOWN: Air speed and temperature in a wind tunnel.

FIND: (a) Minimum plate length to achieve a Reynolds number of 10^8. (b) Distance from leading edge at which transition would occur.

SCHEMATIC:



ASSUMPTIONS: (1) Isothermal conditions, $T_s = T_\infty$.

PROPERTIES: *Table A-4*, Air, (25°C = 298K): $\nu = 15.71\times10^{-6} m^2/s$.

ANALYSIS: (a) From Section 6.3, the Reynolds number is

$$Re_x = \frac{\rho u_\infty x}{\mu} = \frac{u_\infty x}{\nu}.$$

To achieve a Reynolds number of 1×10^8, the minimum plate length is then

$$L_{min} = \frac{Re_x \nu}{u_\infty} = \frac{1\times10^8 \; (15.71\times10^{-6} m^2/s)}{50 \; m/s}$$

$$L_{min} = 31.4 \; m. \quad \triangleleft$$

(b) From Section 6.3, the point of transition corresponds to

$$x_c = \frac{Re_{x,c} \nu}{u_\infty} = \frac{5\times10^5 \; (15.71\times10^{-6} m^2/s)}{50 \; m/s}$$

$$x_c = 0.157 \; m. \quad \triangleleft$$

COMMENTS: Note that

$$\frac{x_c}{L} = \frac{Re_{x,c}}{Re_L}$$

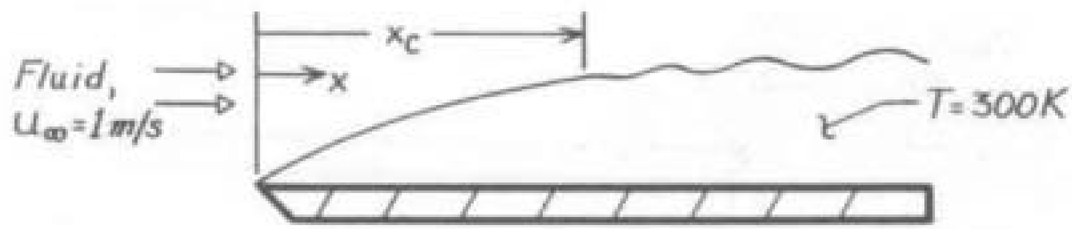
This expression may be used to quickly establish the location of transition from knowledge of $Re_{x,c}$ and Re_L.

PROBLEM 6.14

KNOWN: Transition Reynolds number. Velocity and temperature of atmospheric air, water, engine oil and mercury flow over a flat plate.

FIND: Distance from leading edge at which transition occurs for each fluid.

SCHEMATIC:



ASSUMPTIONS: Transition Reynolds number is $Re_{x,c} = 5\times10^5$.

PROPERTIES: For the fluids at T = 300K;

Fluid	*Table*	$\nu (m^2/s)$
Air (1 atm)	A-4	15.89×10^{-6}
Water	A-6	0.858×10^{-6}
Engine oil	A-5	550×10^{-6}
Mercury	A-5	0.113×10^{-6}

ANALYSIS: From Section 6.3, the point of transition is

$$x_c = Re_{x,c}\,\frac{\nu}{u_\infty} = \frac{5\times10^5}{1\ \text{m/s}}\,\nu .$$

Substituting appropriate viscosity values, find

Fluid	$x_c (m)$
Air	7.95
Water	0.43
Oil	275
Mercury	0.06

◁

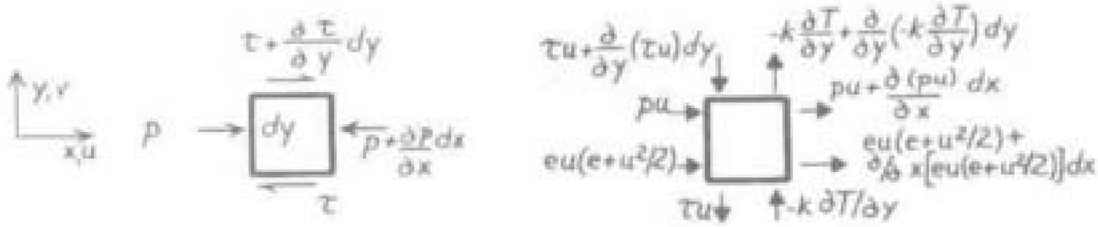
COMMENTS: The distance required to achieve transition increases with increasing ν, due to the effect which viscous forces have on attenuating the instabilities which bring about transition.

PROBLEM 6.15

KNOWN: Two-dimensional flow conditions for which v = 0 and T = T(y).

FIND: (a) Verify that u = u(y), (b) Derive the x-momentum equation, (c) Derive the energy equation.

SCHEMATIC:



Pressure & shear forces Energy fluxes

ASSUMPTIONS: (1) Steady-state conditions, (2) Incompressible fluid with constant properties, (3) Negligible body forces, (4) v = 0, (5) T = T(y) or ∂T/∂x = 0, (6) Thermal energy generation occurs only by viscous dissipation.

ANALYSIS: (a) From the continuity equation, Section 6.4.1, it follows from the prescribed conditions that ∂u/∂x = 0. Hence u = u(y).

(b) From Newton's second law of motion, ΣF_x = (Rate of increase of fluid momentum)$_x$,

$$\left[p - \left[p + \frac{\partial p}{\partial x} dx\right]\right] dy \cdot 1 + \left[-\tau + \left[\tau + \frac{\partial \tau}{\partial y} dy\right]\right] dx \cdot 1 = \left\{(\rho u)u + \frac{\partial}{\partial x}[(\rho u)u]\, dx\right\} dy \cdot 1 - (\rho u)u\, dy \cdot 1$$

Hence, and with $\tau = \mu(\partial u/\partial y)$, it follows that

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} = \frac{\partial}{\partial x}[(\rho u)u] = 0 \qquad \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}. \quad \triangleleft$$

(c) From the conservation of energy requirement and the prescribed conditions, it follows that, $\dot{E}_{in} - \dot{E}_{out} = 0$, or

$$[pu + \rho u(e + u^2/2)]\, dy \cdot 1 + \left[-k\frac{\partial T}{\partial y} + \tau u + \frac{\partial(\tau u)}{\partial y} dy\right] dx \cdot 1$$

$$-\left\{pu + \frac{\partial}{\partial x}(pu)\,dx + \rho u(e + u^2/2) + \frac{\partial}{\partial x}[\rho u(e + u^2/2)]\,dx\right\} dy \cdot 1 - \left[\tau u - k\frac{\partial T}{\partial y} + \frac{\partial}{\partial y}\left[-k\frac{\partial T}{\partial y}\right] dy\right] dx \cdot 1 = 0$$

or,
$$\frac{\partial(\tau u)}{\partial y} - \frac{\partial}{\partial x}(pu) - \frac{\partial}{\partial x}[\rho u(e + u^2/2)] + \frac{\partial}{\partial y}\left[k\frac{\partial T}{\partial y}\right] = 0$$

$$\tau\frac{\partial u}{\partial y} + u\frac{\partial \tau}{\partial y} - u\frac{\partial p}{\partial x} + k\frac{\partial^2 T}{\partial y^2} = 0.$$

Noting that the second and third terms cancel from the momentum equation, hence

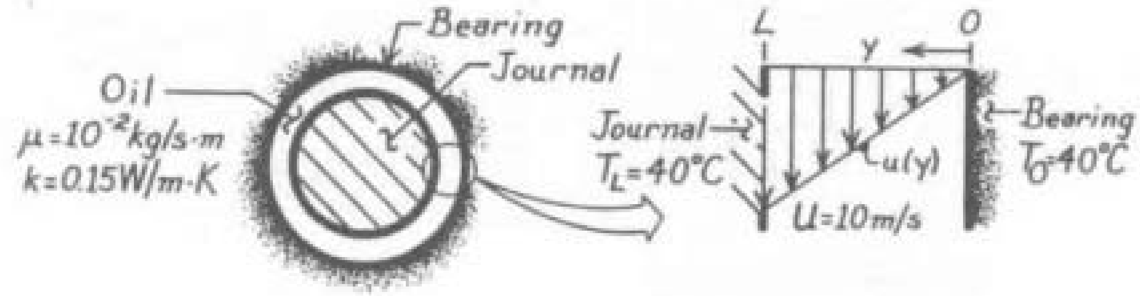
$$\mu\left[\frac{\partial u}{\partial y}\right]^2 + k\left[\frac{\partial^2 T}{\partial y^2}\right] = 0. \quad \triangleleft$$

PROBLEM 6.16

KNOWN: Oil properties, journal and bearing temperatures, and journal speed for a lightly loaded journal bearing.

FIND: Maximum oil temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Incompressible fluid with constant properties, (3) Clearance is much less than journal radius and flow is Couette.

ANALYSIS: The temperature distribution corresponds to the result obtained in the text Example on Couette flow,

$$T(y) = T_0 + \frac{\mu}{2k} U^2 \left[\frac{y}{L} - \left(\frac{y}{L} \right)^2 \right].$$

The position of maximum temperature is obtained from

$$\frac{dT}{dy} = 0 = \frac{\mu}{2k} U^2 \left[\frac{1}{L} - \frac{2y}{L^2} \right]$$

or, $y = L/2.$

The temperature is a maximum at this point since $d^2T/dy^2 < 0$. Hence,

$$T_{max} = T(L/2) = T_0 + \frac{\mu}{2k} U^2 \left[\frac{1}{2} - \frac{1}{4} \right] = T_0 + \frac{\mu U^2}{8k}$$

$$T_{max} = 40°C + \frac{10^{-2} kg/s \cdot m (10 m/s)^2}{8 \times 0.15\ W/m \cdot K}$$

$$T_{max} = 40.83°C. \quad \triangleleft$$

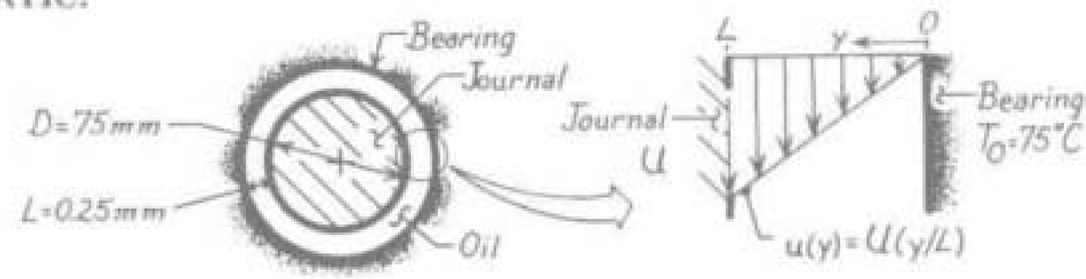
COMMENTS: Note that T_{max} *increases* with *increasing* μ and U, *decreases* with *increasing* k, and is *independent* of L.

PROBLEM 6.17

KNOWN: Diameter, clearance, rotational speed and fluid properties of a lightly loaded journal bearing. Temperature of bearing.

FIND: (a) Temperature distribution in the fluid, (b) Rate of heat transfer from bearing and operating power.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Incompressible fluid with constant properties, (3) Couette flow.

PROPERTIES: Oil (Given): $\rho = 800\ kg/m^3$, $\nu = 10^{-5} m^2/s$, $k = 0.13$ W/m·K; $\mu = \rho\nu = 8\times10^{-3}$ kg/s·m.

ANALYSIS: (a) For Couette flow, the velocity distribution is linear, $u(y) = U(y/L)$, and the energy equation and general form of the temperature distribution are

$$k\frac{d^2T}{dy^2} = -\mu\left[\frac{du}{dy}\right]^2 = -\mu\left[\frac{U}{L}\right]^2 \qquad T = -\frac{\mu}{2k}\left[\frac{U}{L}\right]^2 y^2 + \frac{C_1}{k}y + C_2 .$$

Considering the boundary conditions $dT/dy)_{y=L} = 0$ and $T(0) = T_0$, find $C_2 = T_0$ and $C_1 = \mu U^2/L$. Hence

$$T = T_0 + (\mu U^2)/k\ [(y/L) - 1/2\ (y/L)^2].$$ ◁

(b) Applying Fourier's law at y=0, the rate of heat transfer per unit length to the bearing is

$$q' = -k\,(\pi D)\left.\frac{dT}{dy}\right]_{y=0} = -(\pi D)\frac{\mu U^2}{L} = -(\pi\times75\times10^{-3}m)\frac{8\times10^{-3}\ kg/s{\cdot}m\ (14.14\ m/s)^2}{0.25\times10^{-3}m} = -1507.5\ W/m$$

where the velocity is determined as

$$U = (D/2)\omega = 0.0375m\times3600\ rev/min\ (2\pi\ rad/rev)/(60\ s/min) = 14.14\ m/s.$$

The journal power requirement is

$$P' = F'_{(y=L)}U = \tau_{s(y=L)}\cdot\pi D\cdot U$$

$$P' = 452.5kg/s^2{\cdot}m\ (\pi\times75\times10^{-3}m)\ 14.14m/s = 1507.5kg{\cdot}m/s^3 = 1507.5\ W/m$$ ◁

where the shear stress at y = L is

$$\tau_{s(y=L)} = \mu(\partial u/\partial y)_{y=L} = \mu\frac{U}{L} = 8\times10^{-3}kg/s{\cdot}m\left[\frac{14.14\ m/s}{0.25\times10^{-3}m}\right] = 452.5\ kg/s^2{\cdot}m.$$

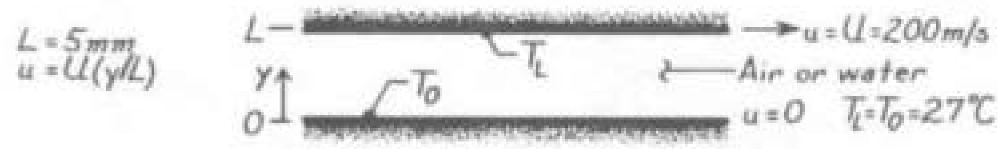
COMMENTS: Note that $q' = P'$, which is consistent with the energy conservation requirement.

PROBLEM 6.18

KNOWN: Conditions associated with the Couette flow of air or water.

FIND: (a) Force and power requirements per unit surface area, (b) Viscous dissipation, (c) Maximum fluid temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Fully-developed Couette flow, (2) Incompressible fluid with constant properties.

PROPERTIES: *Table A-4*, Air (300K): $\mu = 184.6\times10^{-7}\,\text{N·s/m}^2$, $k = 26.3\times10^{-3}\,\text{W/m·K}$; *Table A-6*, Water (300K): $\mu = 855\times10^{-6}\,\text{N·s/m}^2$, $k = 0.613\,\text{W/m·K}$.

ANALYSIS: (a) The force per unit area is associated with the shear stress. Hence, with the linear velocity profile for Couette flow $\tau = \mu(du/dy) = \mu(U/L)$.

Air: $$\tau_{air} = 184.6\times10^{-7}\,\text{N·s/m}^2 \times \frac{200\,\text{m/s}}{0.005\,\text{m}} = 0.738\,\text{N/m}^2 \quad \triangleleft$$

Water: $$\tau_{water} = 855\times10^{-6}\,\text{N·s/m}^2 \times \frac{200\,\text{m/s}}{0.005\,\text{m}} = 34.2\,\text{N/m}^2.$$

With the required power given by $P/A = \tau \cdot U$,

Air: $$(P/A)_{air} = (0.738\,\text{N/m}^2)\,200\,\text{m/s} = 147.6\,\text{W/m}^2 \quad \triangleleft$$

Water: $$(P/A)_{water} = (34.2\,\text{N/m}^2)\,200\,\text{m/s} = 6840\,\text{W/m}^2.$$

(b) The viscous dissipation is $\mu\Phi = \mu(du/dy)^2 = \mu(U/L)^2$. Hence,

Air: $$(\mu\Phi)_{air} = 184.6\times10^{-7}\,\frac{\text{N·s}}{\text{m}^2}\left[\frac{200\,\text{m/s}}{0.005\,\text{m}}\right]^2 = 2.95\times10^4\,\text{W/m}^3 \quad \triangleleft$$

Water: $$(\mu\Phi)_{water} = 855\times10^{-6}\,\frac{\text{N·s}}{\text{m}^2}\left[\frac{200\,\text{m/s}}{0.005\,\text{m}}\right]^2 = 1.37\times10^6\,\text{W/m}^3.$$

(c) From the solution to Part 4 of the text Example, the location of the maximum temperature corresponds to $y_{max} = L/2$. Hence, $T_{max} = T_0 + \mu U^2/8k$ and

Air: $$(T_{max})_{air} = 27°\text{C} + \frac{184.6\times10^{-7}\,\text{N·s/m}^2\,(200\,\text{m/s})^2}{8\times0.0263\,\text{W/m·K}} = 30.5°\text{C} \quad \triangleleft$$

Water: $$(T_{max})_{water} = 27°\text{C} + \frac{855\times10^{-6}\,\text{N·s/m}^2\,(200\,\text{m/s})^2}{8\times0.613\,\text{W/m·K}} = 34.0°\text{C}.$$

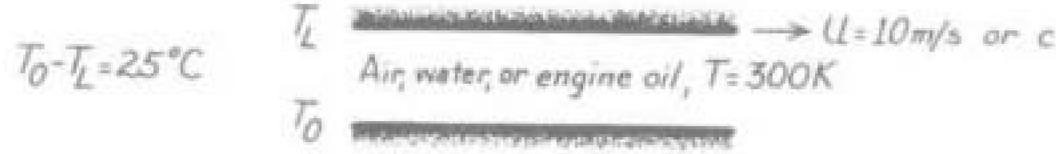
COMMENTS: (1) The viscous dissipation associated with the entire fluid layer, $\mu\Phi(LA)$, must equal the power, P. (2) Although $(\mu\Phi)_{water} \gg (\mu\Phi)_{air}$, $k_{water} \gg k_{air}$. Hence, $T_{max,water} \approx T_{max,air}$.

PROBLEM 6.19

KNOWN: Velocity and temperature difference of plates maintaining Couette flow. Mean temperature of air, water or oil between the plates.

FIND: (a) Pr·Ec product for each fluid, (b) Pr·Ec product for air with plate at sonic velocity.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Couette flow, (3) Air is at 1 atm.

PROPERTIES: *Table A-4*, Air (300K, 1atm), c_p = 1007 J/kg·K, Pr = 0.707, γ = 1.4 R = 287.02 J/kg·K; *Table A-6*, Water (300K): c_p = 4179 J/kg·K, Pr = 5.83; *Table A-5*, Engine oil (300K), c_p = 1909 J/kg·K, Pr = 6400.

ANALYSIS: The product of the Prandtl and Eckert numbers is dimensionless,

$$Pr{\cdot}Ec = Pr\frac{U^2}{c_p \Delta T} \sim \frac{m^2/s^2}{(J/kg{\cdot}K)K} \sim \frac{m^2/s^2}{(kg{\cdot}m^2/s^2)/kg} .$$

Substituting numerical values find ◁

	Air	*Water*	*Oil*
Pr·Ec	0.0028	0.0056	13.41

(b) For an ideal gas, the speed of sound is

$$c = (\gamma R T)^{1/2}$$

where R, the gas constant for air, is $R_u/\mathcal{M}$ = 8.315 kJ/kmol·K/(28.97 kg/kmol) = 287.02 J/kg·K. Hence, at 300K for air,

$$U = c = (1.4 \times 287.02 \text{ J/kg·K} \times 300\text{K})^{1/2} = 347.2 \text{ m/s} .$$

For sonic velocities, it follows that

$$Pr{\cdot}Ec = 0.707 \frac{(347.2 \text{ m/s})^2}{1007 \text{ J/kg·K} \times 25\text{K}} = 3.38.$$ ◁

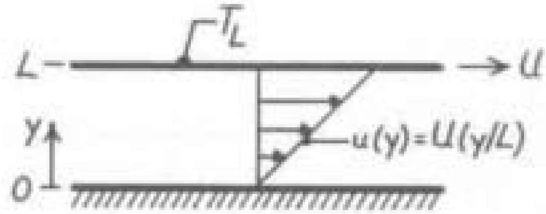
COMMENTS: From the above results it follows that viscous dissipation effects must be considered in the high speed flow of gases and in oil flows at moderate speeds. For Pr·Ec to be less than 0.1 in air with ΔT = 25°C, U should be $\leq$ 60 m/s.

PROBLEM 6.20

KNOWN: Couette flow with moving plate isothermal and stationary plate insulated.

FIND: Temperature of stationary plate and heat flux at the moving plate.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Incompressible fluid with constant properties, (3) Couette flow.

ANALYSIS: The energy equation is given by

$$0 = k\left[\frac{\partial^2 T}{\partial y^2}\right] + \mu\left[\frac{\partial u}{\partial y}\right]^2$$

Integrating twice find the general form of the temperature distribution,

$$\frac{\partial^2 T}{\partial y^2} = -\frac{\mu}{k}\left[\frac{U}{L}\right]^2 \qquad \frac{\partial T}{\partial y} = -\frac{\mu}{k}\left[\frac{U}{L}\right]^2 y + C_1$$

$$T(y) = -\frac{\mu}{2k}\left[\frac{U}{L}\right]^2 y^2 + C_1 y + C_2 .$$

Consider the boundary conditions to evaluate the constants,

$$\partial T/\partial y|_{y=0} = 0 \;\rightarrow\; C_1 = 0 \quad \text{and} \quad T(L) = T_L \;\rightarrow\; C_2 = T_L + \frac{\mu}{2k}U^2 .$$

Hence, the temperature distribution is

$$T(y) = T_L + \left[\frac{\mu U^2}{2k}\right]\left[1 - \left[\frac{y}{L}\right]^2\right].$$

The temperature of the lower plate (y=0) is

$$T(0) = T_L + \left[\frac{\mu U^2}{2k}\right].$$ ◁

The heat flux to the upper plate (y=L) is

$$q''(L) = -k\frac{\partial T}{\partial y}|_{y=L} = \frac{\mu U^2}{L}.$$ ◁

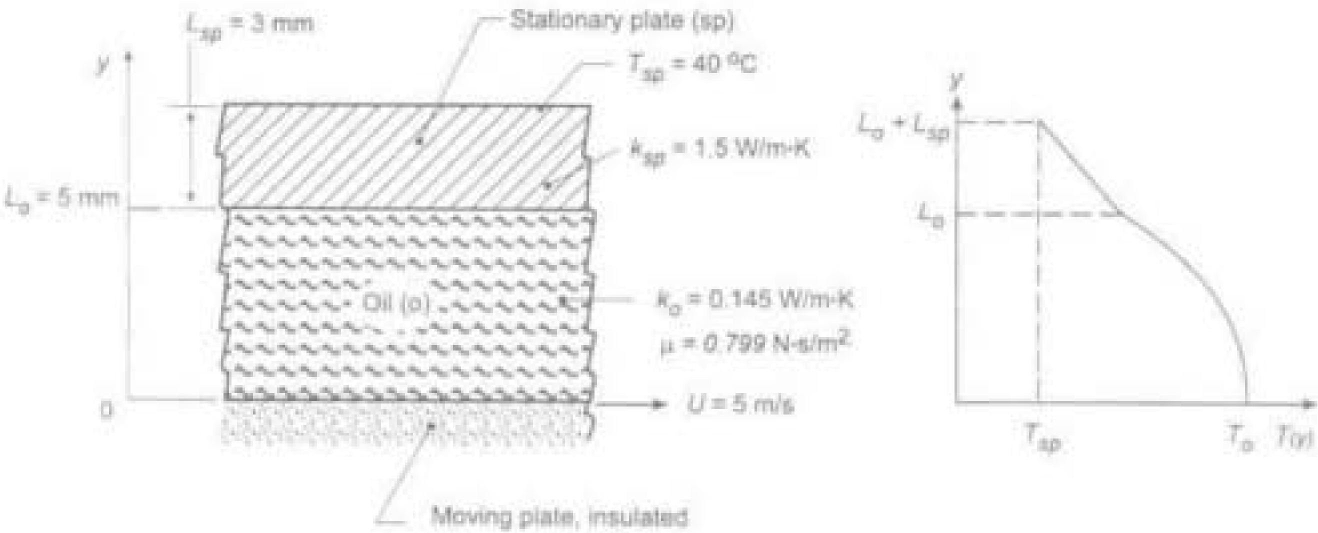
COMMENTS: The heat flux at the top surface may also be obtained by integrating the viscous dissipation over the fluid layer height. For a control volume about a unit area of the fluid layer,

$$\dot{E}_g'' = \dot{E}_{out}'' \qquad \int_0^L \mu\left[\frac{\partial u}{\partial y}\right]^2 dy = q''(L) \qquad q''(L) = \frac{\mu U^2}{L}.$$

KNOWN: Couette flow with heat transfer. Lower (insulated) plate moves with speed U and upper plate is stationary with prescribed thermal conductivity and thickness. Outer surface of upper plate maintained at constant temperature, $T_{sp} = 40°C$.

FIND: (a) On T-y coordinates, sketch the temperature distribution in the oil and the stationary plate, and (b) An expression for the temperature at the lower surface of the oil film, $T(0) = T_o$, in terms of the plate speed U, the stationary plate parameters (T_{sp}, k_{sp}, L_{sp}) and the oil parameters (μ, k_o, L_o). Determine this temperature for the prescribed conditions.

SCHEMATIC:



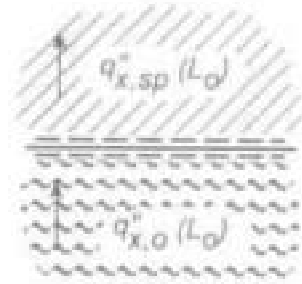
ASSUMPTIONS: (1) Steady-state conditions, (2) Fully developed Couette flow and (3) Incompressible fluid with constant properties.

ANALYSIS: (a) The temperature distribution is shown above with these key features: linear in plate, parabolic in oil film, discontinuity at plate-oil interface, and zero gradient at lower plate surface.

(b) From Example 6.4, the general solution to the conservation equations for the temperature distribution in the oil film is

$$T_o(y) = -Ay^2 + C_3 y + C_4 \qquad \text{where} \qquad A = \frac{\mu}{2k_o}\left(\frac{U}{L_o}\right)^2$$

and the boundary conditions are,



At y = 0, insulated boundary $\quad \left.\frac{dT_o}{dy}\right)_{y=0} = 0; \quad C_3 = 0$

At y = L_o, heat fluxes in oil and plate are equal, $\quad q''_o(L_o) = q''_{sp}(L_o)$

$$-k\left.\frac{dT_o}{dy}\right)_{y=L_o} = \frac{T_o(L_o) - T_{sp}}{R_{sp}} \qquad \begin{cases} \left.\dfrac{dT_o}{dy}\right)_{y=L} = -2AL_o \\ R_{sp} = L_{sp}/k_{sp} \qquad T_o(L) = -AL_o^2 + C_4 \end{cases}$$

$$C_4 = T_{sp} + AL_o^2\left[1 + 2\frac{k_o}{L_o}\frac{L_{sp}}{k_{sp}}\right]$$

Continued...

Hence, the temperature distribution at the lower surface is

$$T_o(0) = -A \cdot 0 + C_4$$

$$T_o(0) = T_{sp} + \frac{\mu}{2k_o} U^2 \left[1 + 2 \frac{k_o}{L_o} \frac{L_{sp}}{k_{sp}} \right]$$ <

Substituting numerical values, find

$$T_o(0) = 40^\circ C + \frac{0.799\, N \cdot s/m^2}{2 \times 0.145\, W/m \cdot K} (5\, m/s)^2 \left[1 + 2 \frac{0.145}{5} \times \frac{3}{1.5} \right] = 116.9^\circ C$$ <

COMMENTS: (1) Give a physical explanation about why the maximum temperature occurs at the lower surface.

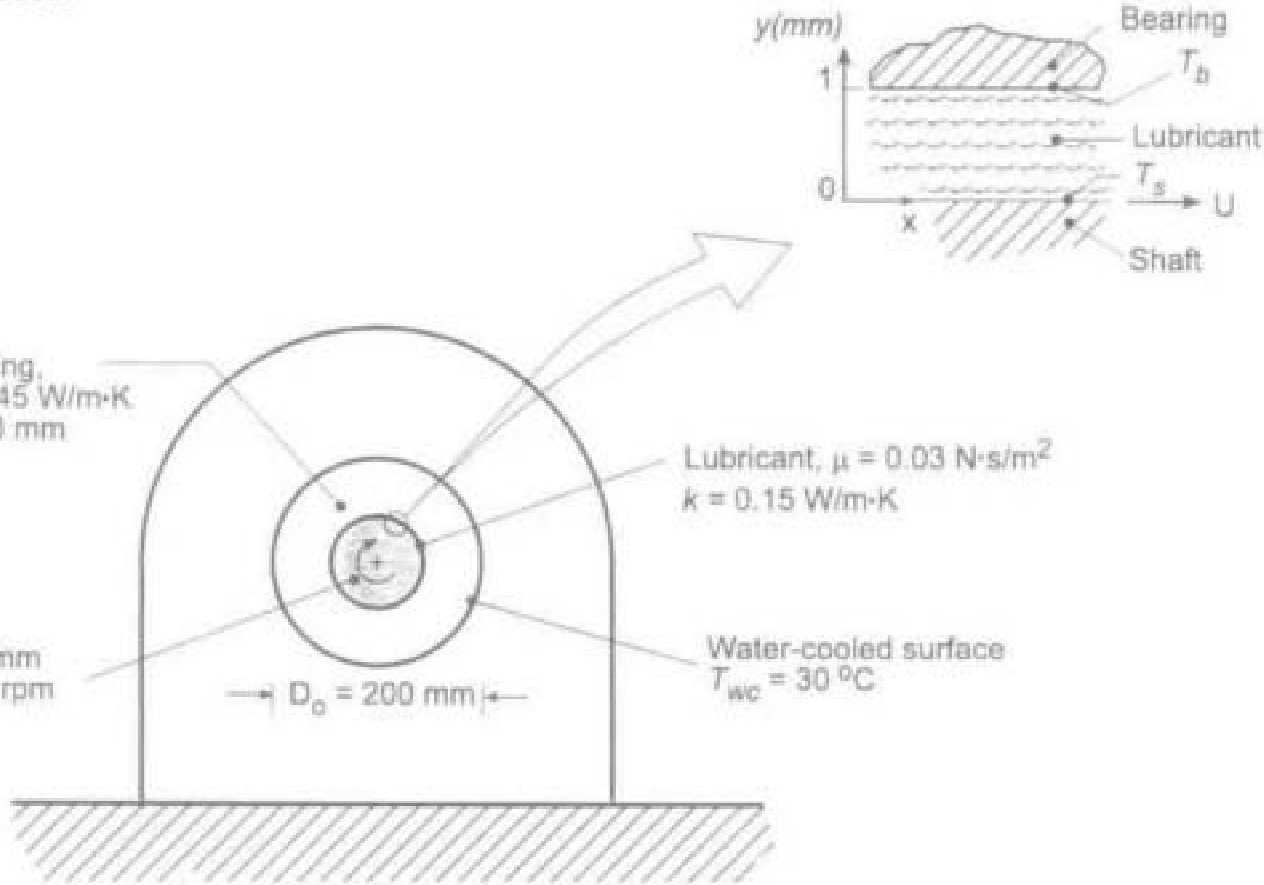
(2) Sketch the temperature distribution if the upper plate moved with a speed U while the lower plate is stationary and all other conditions remain the same.

PROBLEM 6.22

KNOWN: Shaft of diameter 100 mm rotating at 9000 rpm in a journal bearing of 70 mm length. Uniform gap of 1 mm separates the shaft and bearing filled with lubricant. Outer surface of bearing is water-cooled and maintained at T_{wc} = 30°C.

FIND: (a) Viscous dissipation in the lubricant, $\mu\Phi (W/m^3)$, (b) Heat transfer rate from the lubricant, assuming no heat lost through the shaft, and (c) Temperatures of the bearing and shaft, T_b and T_s.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Fully developed Couette flow, (3) Incompressible fluid with constant properties, and (4) Negligible heat lost through the shaft.

ANALYSIS: (a) The viscous dissipation, $\mu\Phi$, Eq. 6.40, for Couette flow from Example 6.4, is

$$\mu\Phi = \mu\left(\frac{du}{dy}\right)^2 = \mu\left(\frac{U}{L}\right)^2 = 0.03\,N\cdot s/m^2\left(\frac{47.1\,m/s}{0.001\,m}\right)^2 = 6.656\times10^7\ W/m^3 \qquad <$$

where the velocity distribution is linear and the tangential velocity of the shaft is

$$U = \pi DN = \pi(0.100\,m)\times 9000\,rpm\times(min/60s) = 47.1\,m/s.$$

(b) The heat transfer rate from the lubricant volume ∀ through the bearing is

$$q = \mu\Phi\cdot\forall = \mu\Phi(\pi D\cdot L\cdot \ell) = 6.65\times10^7\ W/m^3\,(\pi\times0.100\,m\times0.001\,m\times0.070\,m) = 1462\,W \qquad <$$

where ℓ = 70 mm is the length of the bearing normal to the page.

Continued...

PROBLEM 6.22 (Cont.)

(c) From Fourier's law, the heat rate through the bearing material of inner and outer diameters, D_i and D_o, and thermal conductivity k_b is, from Eq. (3.27),

$$q_r = \frac{2\pi\ell k_b(T_b - T_{wc})}{\ln(D_o/D_i)}$$

$$T_b = T_{wc} + \frac{q_r \ln(D_o/D_i)}{2\pi\ell k_b}$$

$$T_b = 30^\circ C + \frac{1462\,W\ln(200/100)}{2\pi \times 0.070\,m \times 45\,W/m\cdot K} = 81.2^\circ C$$ <

To determine the temperature of the shaft, $T(0) = T_s$, first the temperature distribution must be found beginning with the general solution, Example 6.4,

$$T(y) = -\frac{\mu}{2k}\left(\frac{U}{L}\right)^2 y^2 + C_3 y + C_4$$

The boundary conditions are, at y = 0, the surface is adiabatic

$$\left.\frac{dT}{dy}\right)_{y=0} = 0 \qquad C_3 = 0$$

and at y = L, the temperature is that of the bearing, T_b

$$T(L) = T_b = -\frac{\mu}{2k}\left(\frac{U}{L}\right)^2 L^2 + 0 + C_4 \qquad C_4 = T_b + \frac{\mu}{2k}U^2$$

Hence, the temperature distribution is

$$T(y) = T_b + \frac{\mu}{2k}U^2\left(1 - \frac{y^2}{L^2}\right)$$

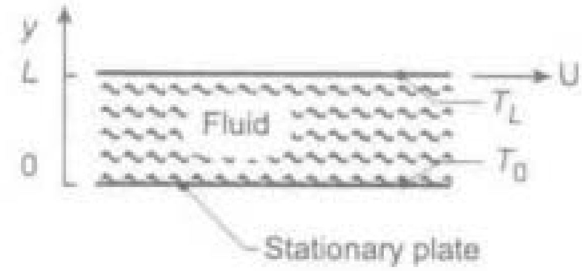
and the temperature at the shaft, y = 0, is

$$T_s = T(0) = T_b + \frac{\mu}{2k}U^2 = 81.3^\circ C + \frac{0.03\,N\cdot s/m^2}{2 \times 0.15\,W/m\cdot K}(47.1\,m/s)^2 = 303^\circ C$$ <

KNOWN: Couette flow with heat transfer.

FIND: (a) Dimensionless form of temperature distribution, (b) Conditions for which top plate is adiabatic, (c) Expression for heat transfer to lower plate when top plate is adiabatic.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) incompressible fluid with constant properties, (3) Negligible body forces, (4) Couette flow.

ANALYSIS: (a) From Example 6.4, the temperature distribution is

$$T = T_0 + \frac{\mu}{2k}U^2\left[\frac{y}{L} - \left(\frac{y}{L}\right)^2\right] + (T_L - T_0)\frac{y}{L}$$

$$\frac{T - T_0}{T_L - T_0} = \frac{\mu U^2}{2k(T_L - T_0)}\left[\frac{y}{L} - \left(\frac{y}{L}\right)^2\right] + \frac{y}{L}$$

or, with

$$\theta \equiv (T - T_0)/T_L - T_0, \qquad \eta \equiv y/L,$$

$$\Pr \equiv c_p\mu/k, \qquad Ec \equiv U^2/c_p(T_L - T_0)$$

$$\theta = \frac{\Pr \cdot Ec}{2}(\eta - \eta^2) + \eta = \eta\left[1 + \frac{1}{2}\Pr \cdot Ec(1-\eta)\right] \quad (1) <$$

(b) For there to be zero heat transfer at the top plate, $dT/dy)_{y=L} = 0$. Hence,

$$\left.\frac{d\theta}{d\eta}\right)_{\eta=1} \cdot \frac{T_L - T_0}{L} = \frac{\Pr \cdot Ec}{2}(1 - 2\eta)\Big|_{\eta=1} + 1 = -\frac{\Pr \cdot Ec}{2} + 1 = 0$$

There is no heat transfer at the top plate if,

$$Ec \cdot \Pr = 2. \quad (2) <$$

(c) The heat transfer rate to the lower plate (per unit area) is

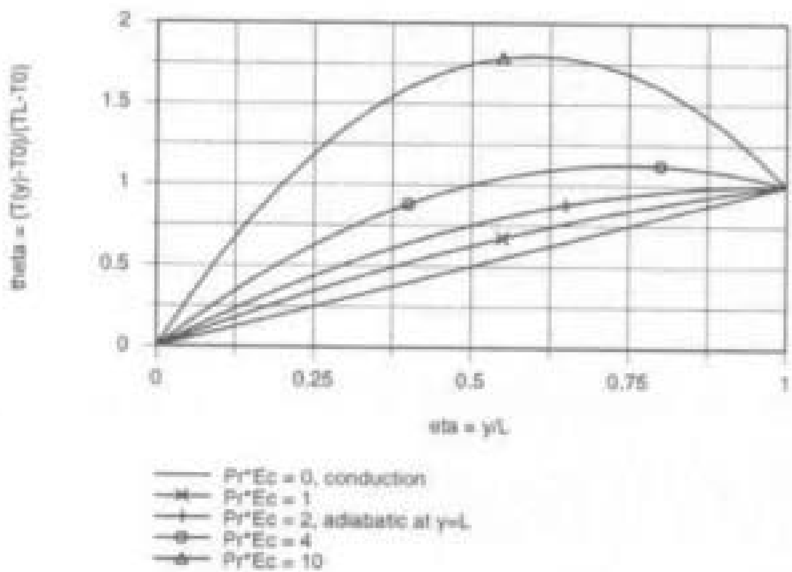
$$q_0'' = -k\left.\frac{dT}{dy}\right|_{y=0} = -k\frac{(T_L - T_0)}{L}\left.\frac{d\theta}{d\eta}\right|_{\eta=0}$$

$$q_0'' = -k\frac{T_L - T_0}{L}\left[\frac{\Pr \cdot Ec}{2}(1 - 2\eta)\Big|_{\eta=0} + 1\right]$$

$$q_0'' = -k\frac{T_L - T_0}{L}\left(\frac{\Pr \cdot Ec}{2} + 1\right) = -2k(T_L - T_0)/L \quad <$$

Continued...

(d) Using Eq. (1), the dimensionless temperature distribution is plotted as a function of dimensionless distance, $\eta = y/L$. When $Pr \cdot Ec = 0$, there is no dissipation and the temperature distribution is linear, so that heat transfer is by conduction only. As $Pr \cdot Ec$ increases, viscous dissipation becomes more important. When $Pr \cdot Ec = 2$, heat transfer to the upper plate is zero. When $Pr \cdot Ec > 2$, the heat rate is out of the oil film at both surfaces.

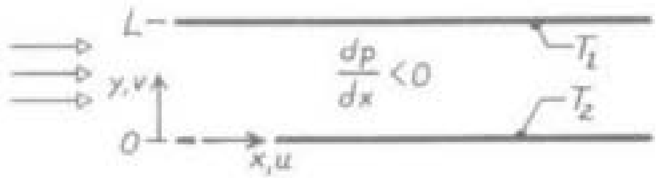


PROBLEM 6.24

KNOWN: Steady, incompressible, laminar flow between infinite parallel plates at different temperatures.

FIND: (a) Form of continuity equation, (b) Form of momentum equations and velocity profile. Relationship of pressure gradient to maximum velocity, (c) Form of energy equation and temperature distribution. Heat flux at top surface.

SCHEMATIC:



ASSUMPTIONS: (1) Two-dimensional flow (no variations in z) between infinite, parallel plates, (2) Negligible body forces, (3) No internal energy generation, (4) Incompressible fluid with constant properties.

ANALYSIS: (a) For two-dimensional, steady conditions, the continuity equation is

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0.$$

Hence for an incompressible fluid (constant ρ) in parallel flow (v=0),

$$\frac{\partial u}{\partial x} = 0. \qquad \triangleleft$$

The flow is fully developed in the sense that, irrespective of y, u is independent of x.

(b) With the above result and the prescribed conditions, the momentum equations of Section 6.4.1 reduce to

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} \qquad 0 = -\frac{\partial p}{\partial y} \qquad \triangleleft$$

Since p is independent of y, $\partial p/\partial x = dp/dx$ is independent of y and

$$\mu\frac{\partial^2 u}{\partial y^2} = \mu\frac{d^2 u}{dy^2} = \frac{dp}{dx}.$$

Since the left-hand side depends only on y and the right-hand side is independent of y, both sides must equal to the same constant C. That is,

$$\mu\frac{d^2 u}{dy^2} = C.$$

Hence, the velocity distribution has the form

$$u(y) = \frac{C}{2\mu}y^2 + C_1 y + C_2. \qquad \triangleleft$$

Using the boundary conditions to evaluate the constants,

$$u(0) = 0 \quad \rightarrow \quad C_2 = 0 \quad \text{and} \quad u(L) = 0 \quad \rightarrow \quad C_1 = -CL/2\mu.$$

Continued .....

The velocity profile is $u(y) = \frac{C}{2\mu}(y^2 - Ly)$.

The profile is symmetric about the midplane, in which case the maximum velocity exists at $y = L/2$. Hence

$$u(L/2) = u_{max} = \frac{C}{2\mu}\left[-\frac{L^2}{4}\right] \qquad \text{or} \qquad u_{max} = -\frac{L^2}{8\mu}\frac{dp}{dx}. \qquad \triangleleft$$

(c) For fully developed thermal conditions, $(\partial T/\partial x) = 0$ and temperature depends only on y. Hence with $v = 0$, $\partial u/\partial x = 0$, and the prescribed assumptions, the energy equation becomes

$$\rho u \frac{\partial i}{\partial x} = k\frac{d^2T}{dy^2} + u\frac{dp}{dx} + \mu\left[\frac{du}{dy}\right]^2.$$

With $i = e + p/\rho$, $\frac{\partial i}{\partial x} = \frac{\partial e}{\partial x} + \frac{1}{\rho}\frac{dp}{dx}$ where $\frac{\partial e}{\partial x} = \frac{\partial e}{\partial T}\frac{\partial T}{\partial x} + \frac{\partial e}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$.

Hence the energy equation becomes $$0 = k\frac{d^2T}{dy^2} + \mu\left[\frac{du}{dy}\right]^2. \qquad \triangleleft$$

This result may be obtained directly by reducing the form of the energy equation in Section 6.4.2 for incompressible fluid and the other prescribed assumptions. With $du/dy = (C/2\mu)(2y - L)$ it follows that

$$\frac{d^2T}{dy^2} = -\frac{C^2}{4k\mu}(4y^2 - 4Ly + L^2).$$

Integrating twice,

$$T(y) = -\frac{C^2}{4k\mu}\left[\frac{y^4}{3} - \frac{2Ly^3}{3} + \frac{L^2y^2}{2}\right] + C_3y + C_4$$

Using the boundary conditions to evaluate the constants,

$$T(0) = T_2 \rightarrow C_4 = T_2 \quad \text{and} \quad T(L) = T_1 \rightarrow C_3 = \frac{C^2L^3}{24k\mu} + \frac{(T_1 - T_2)}{L}.$$

Hence $$T(y) = T_2 + \left[\frac{y}{L}\right](T_1 - T_2) - \frac{C^2}{4k\mu}\left[\frac{y^4}{3} - \frac{2Ly^3}{3} + \frac{L^2y^2}{2} - \frac{L^3y}{6}\right]. \qquad \triangleleft$$

From Fourier's law,

$$q''(L) = -k\left.\frac{\partial T}{\partial y}\right|_{y=L} = \frac{k}{L}(T_2 - T_1) + \frac{C^2}{4\mu}\left[\frac{4}{3}L^3 - 2L^3 + L^3 - \frac{L^3}{6}\right]$$

$$q''(L) = \frac{k}{L}(T_2 - T_1) + \frac{C^2L^3}{24\mu}. \qquad \triangleleft$$

COMMENTS: The third and second terms on the right-hand sides of the temperature distribution and heat flux, respectively, represents the effects of viscous dissipation. If C is large (due to large μ or u_{max}), viscous dissipation is significant. If C is small, conduction effects dominate.

PROBLEM 6.25

KNOWN: The convection conservation equations.

FIND: (a) Identify conservation equations and describe terms, (b) Identify approximations and special conditions used to reduce these equations to the boundary layer equations of Section 6.5, (c) Conditions for which momentum and energy boundary layer equations have the same form and the analogy applies.

ANALYSIS: (a) The *conservation of mass requirement* has the form

$$\underset{①}{\frac{\partial(\rho u)}{\partial x}} + \underset{②}{\frac{\partial(\rho v)}{\partial y}} = 0$$

The terms, as identified, have the following significance:

1. Net change of mass flow in the x-direction, ◁
2. Net change of mass flow in the y-direction.

The expression for *conservation of momentum* in the *x-direction* has the form

$$\rho\left[\underset{①}{u\frac{\partial u}{\partial x}} + \underset{②}{v\frac{\partial u}{\partial y}}\right] = \frac{\partial}{\partial x}(\underset{③}{\sigma_{xx}} - \underset{④}{p}) + \underset{⑤}{\frac{\partial \tau_{yx}}{\partial y}} + \underset{⑥}{X}$$

The terms, as identified, have the following significance:

1. Net rate in x-momentum of fluid leaving control volume in x-direction, ◁
2. Net rate in x-momentum of fluid leaving control volume in y-direction,
3. Change of normal viscous stresses in x-direction,
4. Change of static pressure in x-direction,
5. Change of shear stresses in x-direction,
6. Body force in the x-direction.

The expression for *conservation of energy* has the form

$$\underset{(1a)}{\rho u\frac{\partial i}{\partial x}} + \underset{(1b)}{\rho v\frac{\partial i}{\partial y}} = \underset{(2a)}{\frac{\partial}{\partial x}\left[k\frac{\partial T}{\partial x}\right]} + \underset{(2b)}{\frac{\partial}{\partial y}\left[k\frac{\partial T}{\partial y}\right]}$$

$$+ \underset{③}{\left[u\frac{\partial p}{dx} + v\frac{\partial p}{\partial y}\right]} + \underset{④}{\mu\Phi} + \underset{⑤}{\dot{q}}$$

Continued .....

PROBLEM 6.25 (Cont.)

The terms, as identified, have the following significance:

1. Change of enthalpy (thermal + flow work) advected in x and y directions, ◁
2. Change of conduction rate in x and y directions,
3. Work done by static pressure forces,
4. Work done by viscous dissipation,
5. Rate of energy generation.

(b) The above conservation equations reduce to the boundary layer form when these assumptions are made

constant properties, ◁
incompressible fluid,
negligible body forces,
no energy generation,

and special conditions relating to flow near a surface. The latter are referred to as *boundary layer simplifications*.

(c) Based upon the assumptions and conditions identified above in part (b), the x-momentum and energy equations have the forms:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left[\frac{\partial u}{\partial y}\right]^2$$

The term

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

is zero for a flat plate and the term

$$\frac{\nu}{c_p}\left[\frac{\partial u}{\partial y}\right]^2$$

is negligible for low velocities or a fluid with small viscosity. For such conditions, the *x-momentum* and *energy equations* have the same form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

These equations establish the *analogy* between momentum and heat transfer. ◁

PROBLEM 6.26

KNOWN: Pressure independence of μ, k and c_p .

FIND: Pressure dependence of ν and α for an incompressible liquid and a perfect gas. Values of ν and α for air at 350K and p = 1, 10 atm.

ASSUMPTIONS: Perfect gas behavior for air.

PROPERTIES: *Table A-4*, Air (350K, 1 atm): $\nu=20.92\times10^{-6}m^2/s$, $\alpha=29.9\times10^{-6}m^2/s$.

ANALYSIS: The kinematic viscosity and thermal diffusivity are, respectively,

$$\nu = \mu/\rho \qquad \alpha = k/\rho c_p .$$

Hence, ν and α are inversely proportional to ρ .

For an *incompressible liquid*, ρ is constant.

Hence ν and α are independent of pressure. ◁

For a *perfect gas*, $\rho = p/RT$.

Hence, ρ is directly proportional to p, in which case ν and α vary inversely with pressure. It follows that ν and α are inversely proportional to pressure. ◁

To calculate ν or α for a *perfect* gas at $p \neq 1$ atm,

$$\nu(p) = \nu(1\ \text{atm})\cdot\frac{1}{p}$$

$$\alpha(p) = \alpha(1\ \text{atm})\cdot\frac{1}{p}$$

Hence, for air at 350K,

p(atm)	$\nu(m^2/s)$	$\alpha(m^2/s)$
1	20.92×10^{-6}	29.9×10^{-6}
10	2.09×10^{-6}	2.99×10^{-6}

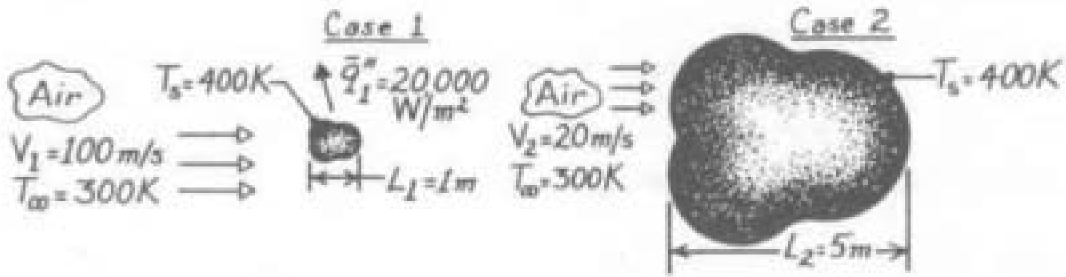
COMMENTS: For the incompressible liquid and the perfect gas, $Pr = \nu/\alpha$ is independent of pressure.

PROBLEM 6.27

KNOWN: Characteristic length, surface temperature and average heat flux for an object placed in an airstream of prescribed temperature and velocity.

FIND: Average convection coefficient if characteristic length of object is increased by a factor of five and air velocity is decreased by a factor of five.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties.

ANALYSIS: From Section 6.6.2, we know that, for a particular geometry,

$$\overline{Nu}_L = f_5(Re_L, Pr).$$

The Reynolds numbers for each case are

Case 1: $$Re_{L,1} = \frac{V_1 L_1}{\nu_1} = \frac{(100\text{m/s})1\text{m}}{\nu_1} = \frac{100\ \text{m}^2/\text{s}}{\nu_1}$$

Case 2: $$Re_{L,2} = \frac{V_2 L_2}{\nu_2} = \frac{(20\text{m/s})5\text{m}}{\nu_2} = \frac{100\ \text{m}^2/\text{s}}{\nu_2}.$$

Hence, with $\nu_1 = \nu_2$, $Re_{L,1} = Re_{L,2}$. Since $Pr_1 = Pr_2$, it therefore follows that

$$\overline{Nu}_{L,2} = \overline{Nu}_{L,1}.$$

Hence,

$$\bar{h}_2 L_2/k_2 = \bar{h}_1 L_1/k_1$$

$$\bar{h}_2 = \bar{h}_1 \frac{L_1}{L_2} = 0.2\,\bar{h}_1.$$

For *Case 1*, using the rate equation, the convection coefficient is

$$q_1 = \bar{h}_1 A_1 (T_s - T_\infty)_1$$

$$\bar{h}_1 = \frac{(q_1/A_1)}{(T_s - T_\infty)_1} = \frac{q_1''}{(T_s - T_\infty)_1} = \frac{20,000\ \text{W/m}^2}{(400-300)\text{K}} = 200\ \text{W/m}^2\cdot\text{K}.$$

Hence, it follows that for *Case 2*

$$\bar{h}_2 = 0.2 \times 200\ \text{W/m}^2\cdot\text{K} = 40\ \text{W/m}^2\cdot\text{K}.$$ ◁

COMMENTS: If $Re_{L,2}$ were *not* equal to $Re_{L,1}$, it would be necessary to know the specific form of $f_5(Re_L, Pr)$ before $\bar{h}_2$ could be determined.

PROBLEM 7.1

KNOWN: Temperature and velocity of fluids in parallel flow over a flat plate.

FIND: (a) Velocity and thermal boundary layer thicknesses at a prescribed distance from the leading edge, and (b) For each fluid plot the boundary layer thicknesses as a function of distance.

SCHEMATIC:

ASSUMPTIONS: (1) Transition Reynolds number is 5×10^5.

PROPERTIES: *Table A.4*, Air (300 K, 1 atm): $\nu = 15.89 \times 10^{-6}$ m²/s, Pr = 0.707; *Table A.6*, Water (300 K): $\nu = \mu/\rho = 855 \times 10^{-6}$ N·s/m²/997 kg/m³ $= 0.858 \times 10^{-6}$ m²/s, Pr = 5.83; *Table A.5*, Engine Oil (300 K): $\nu = 550 \times 10^{-6}$ m²/s, Pr = 6400; *Table A.5*, Mercury (300 K): $\nu = 0.113 \times 10^{-6}$ m²/s, Pr = 0.0248.

ANALYSIS: (a) If the flow is laminar, the following expressions may be used to compute δ and δ_t, respectively,

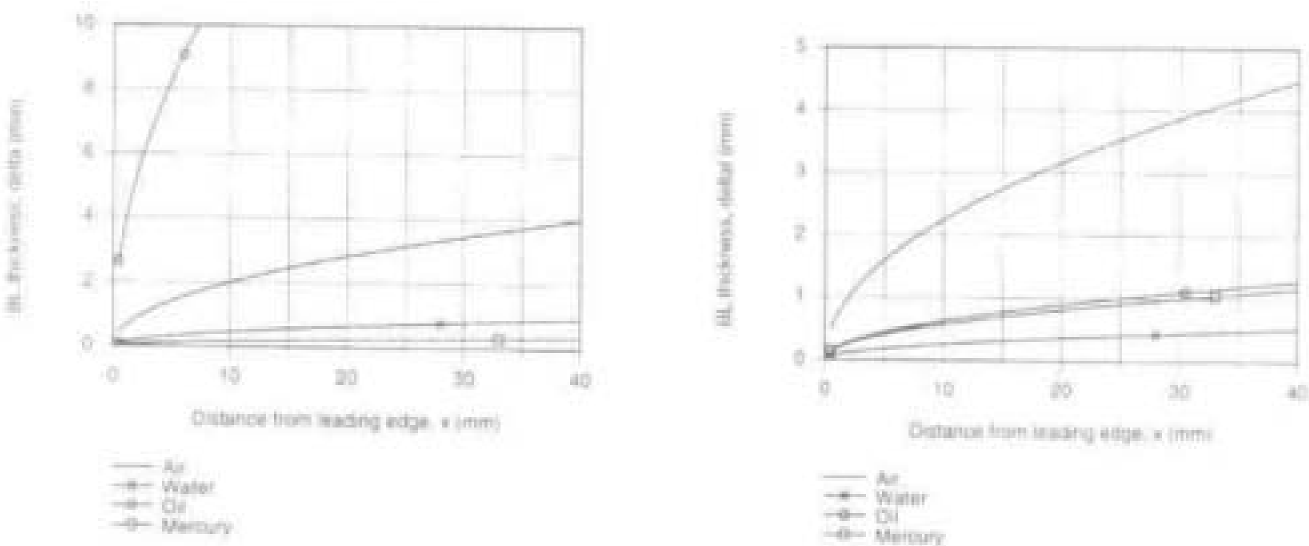
$$\delta = \frac{5x}{Re_x^{1/2}} \qquad \delta_t = \frac{\delta}{Pr^{1/3}}$$

where $Re_x = \frac{u_\infty x}{\nu} = \frac{1\,\text{m/s}(0.04\,\text{m})}{\nu} = \frac{0.04\,\text{m}^2/\text{s}}{\nu}$

Fluid	Re_x	δ (mm)	δ_t (mm)
Air	2517	3.99	4.48
Water	4.66×10^4	0.93	0.52
Oil	72.7	23.5	1.27
Mercury	3.54×10^5	0.34	1.17

<

(b) Using IHT with the foregoing equations, the boundary layer thicknesses are plotted as a function of distance from the leading edge, x.



COMMENTS: (1) Note that $\delta \approx \delta_t$ for air, $\delta > \delta_t$ for water, $\delta >> \delta_t$ for oil, and $\delta < \delta_t$ for mercury. As expected, the boundary layer thicknesses increase with increasing distance from the leading edge.

(2) The value of δ_t for mercury should be viewed as a rough approximation since the expression for δ/δ_t was derived subject to the approximation that Pr > 0.6.

PROBLEM 7.2

KNOWN: Temperature, pressure and velocity of atmospheric air in parallel flow over a plate of prescribed length and temperature.

FIND: (a) Boundary layer thickness, surface shear stress and heat flux at trailing edge, (b) Drag force and total heat transfer per unit width of plate, and (c) Plot the parameters of part (a) as a function of distance from the leading edge.

SCHEMATIC:

ASSUMPTIONS: (1) Critical Reynolds number is 5×10^5, (2) Flow over top and bottom surface.

PROPERTIES: *Table A.4*, Air (T_f = 323 K, 1 atm): $\rho = 1.085\ kg/m^3$, $\nu = 18.2 \times 10^{-6}\ m^2/s$, $k = 0.028$ W/m·K, Pr = 0.707.

ANALYSIS: (a) Calculate the Reynolds number to determine nature of flow,

$$Re_L = \frac{u_\infty L}{\nu} = \frac{5\,m/s \times 1\,m}{18.2 \times 10^{-6}\,m^2/s} = 2.75 \times 10^5.$$

Hence, the flow is laminar, and at x = L, using Eqs. 7.19 and 7.20,

$$\delta = 5L\,Re_L^{-1/2} = 5 \times 1\,m/\left(2.75 \times 10^5\right)^{1/2} = 9.5\,mm \qquad <$$

$$\tau_{s,L} = \left(\rho u_\infty^2/2\right)0.664\,Re_L^{-1/2} = \frac{1.085}{2}\frac{kg}{m^3}(5\,m/s)^2\,0.664/\left(2.75 \times 10^5\right)^{1/2} = 0.0172\,N/m^2 \qquad <$$

Using the appropriate correlation, Eq. 7.23,

$$Nu_L = \frac{h_L L}{k} = 0.332\,Re_L^{1/2}\,Pr^{1/3} = 0.332\left(2.75 \times 10^5\right)^{1/2}(0.707)^{1/3} = 155.1$$

$$h_L = 155.1(0.028\,W/m\cdot K)/1\,m = 4.34\,W/m^2\cdot K$$

Hence, the heat flux is

$$q_s''(L) = h_L\left(T_s - T_\infty\right) = 4.34\,W/m^2\cdot K\left(75^\circ C - 25^\circ C\right) = 217\,W/m^2 \qquad <$$

(b) The drag force per unit plate width is $D' = 2L\bar{\tau}_{s,L}$ where the factor of two is included to account for both sides of the plate. Hence, from Eq. 7.30, with

$$\bar{\tau}_{s,L} = \left(\rho u_\infty^2/2\right)1.328\,Re_L^{-1/2} = \left(1.085\,kg/m^3/2\right)(5\,m/s)^2\,1.328\left(2.75 \times 10^5\right)^{-1/2} = 0.0343\,N/m^2$$

the drag is

$$D' = 2(1\,m)0.0343\,N/m^2 = 0.0686\,N/m \qquad <$$

For laminar flow, the average value $\bar{h}_L$ over the distance 0 to L is twice the local value, h_L,

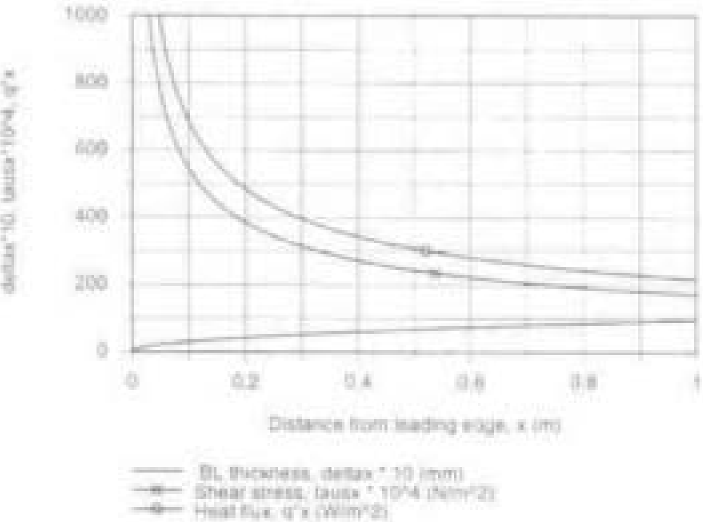
$$\bar{h}_L = 2h_L = 8.68\,W/m^2\cdot K$$

Continued...

The total heat transfer rate per unit width of the plate is

$$q' = 2L\bar{h}_L(T_s - T_\infty) = 2(1\,m)8.68\,W/m^2 \cdot K(75-25)^\circ C = 868\,W/m$$ <

(c) Using IHT with the equations of part (a), the boundary layer thickness, surface shear stress and heat flux as a function of distance form the leading edge were calculated and are plotted below.



COMMENTS: (1) The velocity boundary layer is very thin at the leading edge and increases with increasing distance. The local shear stress and heat flux are very large near the leading edge and decrease with increasing distance. The shapes of the two curves are similar.

(2) A copy of the IHT Workspace used to generate the above plot is shown below.

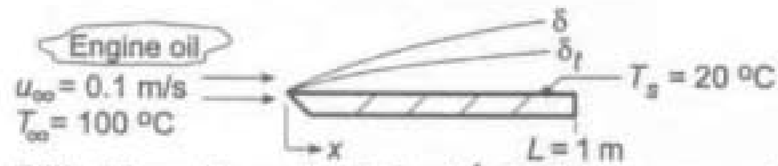
```
// Boundary layer thickness, delta
delta = 5 * x * Rex ^-0.5
delta_mm = delta * 1000
delta_plot = delta_mm * 10          // Scaling parameter for convenience in plotting

// Surface shear stress, tausx
tausx = (rho * uinf^2 / 2) * 0.664 * Rex^-0.5
tausx_plot = tausx * 10000          // Scaling parameter for convenience in plotting

// Heat flux, q''x
q''x = hx * (Ts - Tinf)
Nux = 0.332 * Rex^0.5 * Pr^(1/3)
Nux = hx * x / k

// Reynolds number
Rex = uinf * x / nu

// Properties Tool: Air
// Air property functions : From Table A.4
// Units: T(K); 1 atm pressure
rho = rho_T("Air",Tf)               // Density, kg/m^3
nu = nu_T("Air",Tf)                 // Kinematic viscosity, m^2/s
k = k_T("Air",Tf)                   // Thermal conductivity, W/m.K
Pr = Pr_T("Air",Tf)                 // Prandtl number

// Assigned variables
Tinf = 25 + 273                     // Airstream temperature, K
Ts = 75 + 273                       // Surface temperature, K
uinf = 5                            // Airstream velocity, m/s
Tf = 323                            // Film temperature, K
x = 1                               // Distance from leading edge, m
```

PROBLEM 7.3

KNOWN: Temperature and velocity of engine oil. Temperature and length of flat plate.

FIND: (a) Velocity and thermal boundary layer thickness at trailing edge, (b) Heat flux and surface shear stress at trailing edge, (c) Total drag force and heat transfer per unit plate width, and (d) Plot the boundary layer thickness and local values of the shear stress, convection coefficient, and heat flux as a function of x for $0 \le x \le 1$ m.

SCHEMATIC:



ASSUMPTIONS: (1) Critical Reynolds number is 5×10^5, (2) Flow over top and bottom surfaces.

PROPERTIES: *Table A.5*, Engine Oil ($T_f = 333$ K): $\rho = 864$ kg/m^3, $\nu = 86.1 \times 10^{-6}$ m^2/s, k = 0.140 W/m·K, Pr = 1081.

ANALYSIS: (a) Calculate the Reynolds number to determine nature of the flow,

$$Re_L = \frac{u_\infty L}{\nu} = \frac{0.1\,\text{m/s} \times 1\,\text{m}}{86.1 \times 10^{-6}\,\text{m}^2/\text{s}} = 1161$$

Hence the flow is laminar at x = L, from Eqs. 7.19 and 7.24, and

$$\delta = 5L\,Re_L^{-1/2} = 5(1\,\text{m})(1161)^{-1/2} = 0.147\,\text{m} \qquad <$$

$$\delta_t = \delta\,Pr^{-1/3} = 0.147\,\text{m}(1081)^{-1/3} = 0.0143\,\text{m} \qquad <$$

(b) The local convection coefficient, Eq. 7.23, and heat flux at x = L are

$$h_L = \frac{k}{L}0.332\,Re_L^{1/2}\,Pr^{1/3} = \frac{0.140\,\text{W/m}\cdot\text{K}}{1\,\text{m}}0.332(1161)^{1/2}(1081)^{1/3} = 16.25\,\text{W/m}^2\cdot\text{K}$$

$$q_s'' = h_L(T_s - T_\infty) = 16.25\,\text{W/m}^2\cdot\text{K}(20 - 100)^\circ\text{C} = -1300\,\text{W/m}^2 \qquad <$$

Also, the local shear stress is, from Eq. 7.20,

$$\tau_{s,L} = \frac{\rho u_\infty^2}{2}0.664\,Re_L^{-1/2} = \frac{864\,\text{kg/m}^3}{2}(0.1\,\text{m/s})^2 0.664(1161)^{-1/2}$$

$$\tau_{s,L} = 0.0842\,\text{kg/m}\cdot\text{s}^2 = 0.0842\,\text{N/m}^2 \qquad <$$

(c) With the drag force per unit width given by $D' = 2L\bar{\tau}_{s,L}$ where the factor of 2 is included to account for both sides of the plate, it follows that

$$D' = 2L(\rho u_\infty^2/2)1.328\,Re_L^{-1/2} = 2(1\,\text{m})864\,\text{kg/m}^3(0.1\,\text{m/s})^2 1.328(1161)^{-1/2} = 0.673\,\text{N/m} \qquad <$$

For laminar flow, the average value $\bar{h}_L$ over the distance 0 to L is twice the local value, h_L,

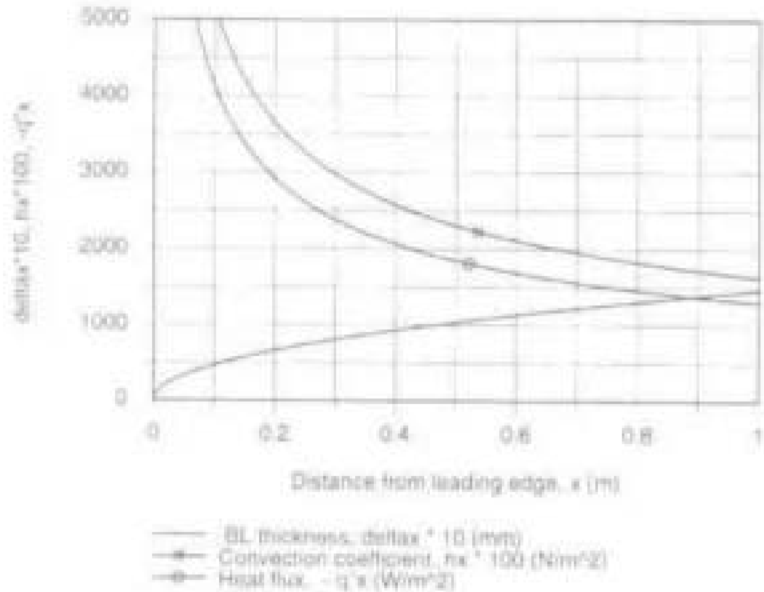
$$\bar{h}_L = 2h_L = 32.5\,\text{W/m}^2\cdot\text{K}$$

The total heat transfer rate per unit width of the plate is

$$q' = 2L\bar{h}_L(T_s - T_\infty) = 2(1\,\text{m})32.5\,\text{W/m}^2\cdot\text{K}(20 - 100)^\circ\text{C} = -5200\,\text{W/m} \qquad <$$

Continued...

(c) Using IHT with the foregoing equations, the boundary layer thickness, and local values of the convection coefficient and heat flux were calculated and plotted as a function of x.



COMMENTS: (1) Note that since Pr >> 1, $\delta >> \delta_t$. That is, for the high Prandtl liquids, the velocity boundary layer will be much thicker than the thermal boundary layer.

(2) A copy of the *IHT Workspace* used to generate the above plot is shown below.

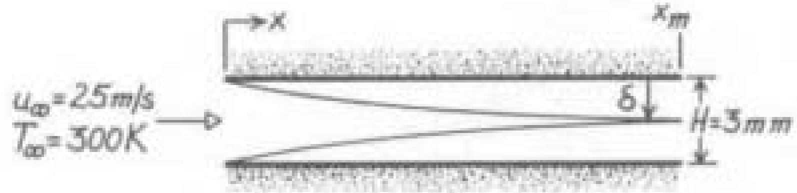
```
// Boundary layer thickness, delta
delta = 5 * x * Rex ^-0.5
delta_mm = delta * 1000
delta_plot = delta_mm * 10          // Scaling parameter for convenience in  plotting

// Convection coefficient and heat flux, q''x
q''x = hx * (Ts - Tinf)
Nux = 0.332 * Rex^0.5 * Pr^(1/3)
Nux = hx * x / k
hx_plot = 100 * hx                  // Scaling parameter for convenience in plotting
q''x_plot = ( -1 ) * q''x           // Scaling parameter for convenience in plotting

// Reynolds number
Rex = uinf * x / nu

// Properties Tool: Engine oil
// Engine Oil property functions : From Table A.5
// Units: T(K)
rho = rho_T("Engine Oil",Tf)        // Density, kg/m^3
cp = cp_T("Engine Oil",Tf)          // Specific heat, J/kg.K
nu = nu_T("Engine Oil",Tf)          // Kinematic viscosity, m^2/s
k = k_T("Engine Oil",Tf)            // Thermal conductivity, W/m.K
Pr = Pr_T("Engine Oil",Tf)          // Prandtl number


// Assigned variables
Tf = (Ts + Tinf) / 2                // Film temperature, K
Tinf = 100 + 273                    // Freestream temperature, K
Ts = 20 + 273                       // Surface temperature, K
uinf = 0.1                          // Freestream velocity, m/s
x = 1                               // Plate length, m
```

PROBLEM 7.4

KNOWN: Velocity and temperature of air in parallel flow over a flat plate.

FIND: (a) Velocity boundary layer thickness at selected stations. Distance at which boundary layers merge for plates separated by H = 3 mm. (b) Surface shear stress and $v(\delta)$ at selected stations.

SCHEMATIC:



ASSUMPTIONS: (1) Steady flow, (2) Boundary layer approximations are valid, (3) Flow is laminar.

PROPERTIES: *Table A.4*, Air (300 K, 1 atm): $\rho = 1.161\ \text{kg/m}^3$, $\nu = 15.89\times10^{-6}\ \text{m}^2/\text{s}$.

ANALYSIS: (a) For laminar flow,

$$\delta = \frac{5x}{Re_x^{1/2}} = \frac{5}{(u_\infty/\nu)^{1/2}}x^{1/2} = \frac{5x^{1/2}}{(25\ \text{m/s}/15.89\times10^{-6}\ \text{m}^2/\text{s})^{1/2}} = 3.99\times10^{-3}\ x^{1/2}.$$

x (m)	0.001	0.01	0.1
δ (mm)	0.126	0.399	1.262

Boundary layer merger occurs at $x = x_m$ when $\delta = 1.5$ mm. Hence

$$x_m^{1/2} = \frac{0.0015\ \text{m}}{3.99\times10^{-3}\ \text{m}^{1/2}} = 0.376\ \text{m}^{1/2} \qquad x_m = 141\ \text{mm}. \qquad \triangleleft$$

(b) The shear stress is

$$\tau_{s,x} = 0.664\frac{\rho u_\infty^2/2}{Re_x^{1/2}} = \frac{\rho u_\infty^2/2}{(u_\infty/\nu)^{1/2}x^{1/2}} = \frac{0.664\times1.161\ \text{kg/m}^3(25\ \text{m/s})^2/2}{(25\ \text{m/s}\big/15.89\times10^{-6}\ \text{m}^2/\text{s})^{1/2}x^{1/2}} = \frac{0.192}{x^{1/2}}(\text{N/m}^2).$$

x(m)	0.001	0.01	0.1
$\tau_{s,x}$(N/m^2)	6.07	1.92	0.61

The velocity distribution in the boundary layer is $v = (1/2)\,(\nu u_\infty/x)^{1/2}(\eta df/d\eta - f)$. At $y = \delta$, $\eta \approx 5.0$, $f \approx 3.24$, $df/d\eta \approx 0.991$.

$$v = \frac{0.5}{x^{1/2}}(15.89\times10^{-6}\ \text{m}^2/\text{s}\times25\ \text{m/s})^{1/2}(5.0\times0.991 - 3.28) = (0.0167/x^{1/2})\text{m/s}.$$

x(m)	0.001	0.01	0.1
v(m/s)	0.528	0.167	0.053

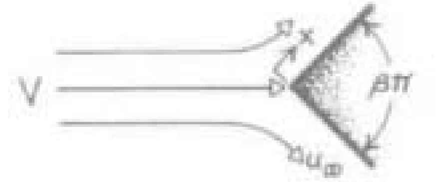
COMMENTS: (1) $v << u_\infty$ and $\delta << x$ are consistent with BL approximations. Note, $v \rightarrow \infty$ as $x \rightarrow 0$ and approximations breakdown very close to the leading edge. (2) Since $Re_{x_m} = 2.22\times10^5$, laminar BL model is valid. (3) Above expressions are approximations for flow between parallel plates, since $du_\infty/dx > 0$ and $dp/dx < 0$.

PROBLEM 7.5

KNOWN: Flow conditions and local Nusselt number-Reynolds number dependence for a wedge.

FIND: (a) Flow conditions at x = 0 for $\beta > 0$ and variation of u_∞ with x for $\beta = 1$, (b) Ratio of average to local convection coefficient for $\beta = 0.5$ and 1.0, (c) Ratio of average convection coefficients associated with wedge flow ($\beta = 0.5$ and 1.0) to convection coefficient associated with parallel flow for air at x = 1m.

SCHEMATIC:



$$u_\infty = V x^m$$
$$m = \beta/(2-\beta)$$
$$Nu_x = C_1(Pr,\beta)\,Re_x^{1/2}$$
$$Nu_x = C_1\left(\frac{V}{\nu}\right)^{1/2} x^{(m+1)/2}$$

ASSUMPTIONS: (1) Laminar, boundary layer flow, (2) Constant properties.

PROPERTIES: *Table A-4*, Air: Pr ≈ 0.70.

ANALYSIS: (a) For $\beta > 0$, $m > 0$. Hence $u_\infty = 0$ at $x = 0$, and we say that a *stagnation point* exists. For $\beta = 1$, $m = 1$ (a flat plate normal to the flow) and u_∞ *increases linearly* with x.

(b) With $Nu_x = C_1\,Re_x^{1/2}$, it follows that

$$\bar{h}_x = \frac{1}{x}\int_0^x h_x\,dx = \frac{C_1}{x}\int_0^x \frac{k}{x}\left(\frac{u_\infty x}{\nu}\right)^{1/2} dx = \frac{C_1}{x}\,\frac{k}{\nu^{1/2}}\,V^{1/2}\int_0^x x^{\frac{m-1}{2}}\,dx$$

$$\bar{h}_x = C_1\left(\frac{V}{\nu}\right)^{1/2} k\left(\frac{2}{m+1}\right) x^{\frac{m-1}{2}} = \frac{2}{m+1}\,h_x\,.$$

Hence,

$$(\bar{h}_x/h_x)_{\beta=0.5} = \frac{2}{1.333} = 1.5 \qquad \triangleleft$$

$$(\bar{h}_x/h_x)_{\beta=1.0} = \frac{2}{2} = 1.0\,. \qquad \triangleleft$$

(c) The ratio of the average coefficients is

$$(\bar{h}_{x,\beta>0}/\bar{h}_{x,\beta=0}) = \frac{C_1(\beta>0)}{C_1(\beta=0)}\;\frac{1}{m(\beta>0)+1}\;\frac{x^{\frac{m(\beta>0)-1}{2}}}{x^{-1/2}}\,.$$

For Pr = 0.7: $C_1 = 0.292$ for $\beta = 0$; $C_1 = 0.384$ for $\beta = 0.5$; $C_1 = 0.496$, $\beta = 1$,

$$\bar{h}_{x,\beta=0.5}/\bar{h}_{x,\beta=0} = \frac{0.384}{0.292}\times\frac{1}{1.333}\,(1)^{0.333} = 0.987 \qquad \triangleleft$$

$$\bar{h}_{x,\beta=1.0}/\bar{h}_{x,\beta=0} = \frac{0.496}{0.292}\times\frac{1}{2}\,(1)^{1} = 0.849\,. \qquad \triangleleft$$

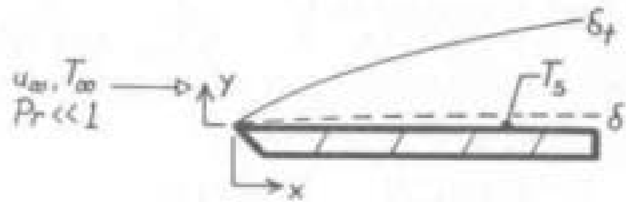
COMMENTS: Flat plate approximation is reasonable, but note dependence on x.

PROBLEM 7.6

KNOWN: Liquid metal in parallel flow over a flat plate.

FIND: An expression for the local Nusselt number.

SCHEMATIC:



ASSUMPTIONS: (1) Steady, incompressible flow, (2) $\delta \gg \delta_t$, hence $u(y) \sim u_\infty$, (3) Boundary layer approximations are valid, (4) Constant properties.

ANALYSIS: The boundary layer energy equation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}.$$

Since $u(y) = u_\infty$, it follows that $v = 0$ and the energy equation becomes

$$u_\infty \frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad \text{or} \qquad \frac{\partial T}{\partial x} = \frac{\alpha}{u_\infty}\frac{\partial^2 T}{\partial y^2}.$$

Boundary Conditions: $T(x,0) = T_s$, $T(x,\infty) = T_\infty$.

Initial Condition: $T(0,y) = T_\infty$.

The differential equation is analogous to that for transient one-dimensional conduction in a plane wall, and the conditions are analogous to those of Fig. 5.17, Case (1). Hence the solution is given by Eqs. 5.55 and 5.56. Substituting y for x, x for t, T_∞ for T_i, and α/u_∞ for α, the boundary layer temperature and the surface heat flux become

$$\frac{T(x,y) - T_s}{T_\infty - T_s} = \text{erf}\left[\frac{y}{2(\alpha x/u_\infty)^{1/2}}\right]$$

$$q_s'' = \frac{k(T_s - T_\infty)}{(\pi\alpha x/u_\infty)^{1/2}}.$$

Hence, with

$$Nu_x \equiv \frac{hx}{k} = \frac{q_s''x}{(T_s - T_\infty)k}$$

find

$$Nu_x = \frac{x}{(\pi\alpha x/u_\infty)^{1/2}} = \frac{(xu_\infty)^{1/2}}{\pi^{1/2}(k/\rho c_p)^{1/2}} = \frac{1}{\pi^{1/2}}\left(\frac{\rho u_\infty x}{\mu}\cdot\frac{c_p\mu}{k}\right)^{1/2}$$

$$Nu_x = 0.564\,(Re_x\,Pr)^{1/2} = 0.564\,Pe^{1/2}$$ ◁

where $Pe = Re \cdot Pr$ is the Peclet number.

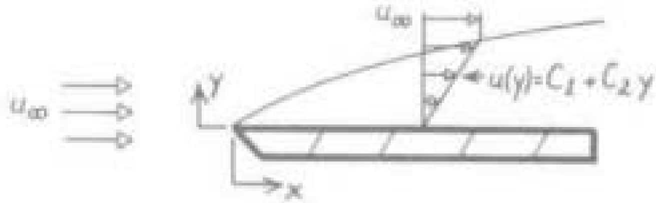
COMMENTS: Because k is very large, axial conduction effects may not be negligible. That is, the $\alpha\,\partial^2 T/\partial x^2$ term of the energy equation may be important.

PROBLEM 7.7

KNOWN: Form of velocity profile for flow over a flat plate.

FIND: (a) Expression for profile in terms of u_∞ and δ, (b) Expression for $\delta(x)$, (c) Expression for $C_{f,x}$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady state conditions, (2) Constant properties, (3) Incompressible flow, (4) Boundary layer approximations are valid.

ANALYSIS: (a) From the boundary conditions

$$u(x,0)=0 \rightarrow C_1=0 \qquad \text{and} \qquad u(x,\delta)=u_\infty \rightarrow C_2=u_\infty/\delta .$$

Hence, $\quad u=u_\infty(y/\delta)$. ◁

(b) From the momentum integral equation for a flat plate

$$\frac{d}{dx}\int_0^\delta (u_\infty-u)u\,dy=\tau_s/\rho$$

$$u_\infty^2\frac{d}{dx}\int_0^\delta\left[1-\frac{u}{u_\infty}\right]\frac{u}{u_\infty}dy=\frac{\mu}{\rho}\left.\frac{\partial u}{\partial y}\right)_{y=0}=\frac{\nu u_\infty}{\delta}$$

$$u_\infty^2\frac{d}{dx}\int_0^\delta\left[1-\frac{y}{\delta}\right]\frac{y}{\delta}dy=\frac{\nu u_\infty}{\delta}$$

$$u_\infty^2\frac{d}{dx}\left[\left(\frac{y^2}{2\delta}-\frac{y^3}{3\delta^2}\right)\Bigg|_0^\delta\right]=\frac{\mu u_\infty}{\delta} \qquad \text{or} \qquad \frac{u_\infty}{6}\frac{d\delta}{dx}=\frac{\nu}{\delta}.$$

Separating and integrating, find

$$\int_0^\delta \delta\,d\delta=\frac{6\nu}{u_\infty}\int_0^x dx \qquad \delta=\left[\frac{12\,\nu x}{u_\infty}\right]^{1/2}=3.46\,x\left[\frac{\nu}{u_\infty x}\right]^{1/2}=3.46\,x\,Re_x^{-1/2}. \quad ◁$$

(c) The shear stress at the wall is

$$\tau_s=\mu\left.\frac{\partial u}{\partial y}\right)_{y=0}=\mu\frac{u_\infty}{\delta}=\frac{\mu u_\infty}{3.46\,x}Re_x^{+1/2}$$

and the friction coefficient is

$$C_{f,x}=\frac{\tau_s}{\rho u_\infty^2/2}=\frac{\mu}{\rho u_\infty x}\,\frac{2}{3.46}Re_x^{+1/2}=0.578\,Re_x^{-1/2}. \quad ◁$$

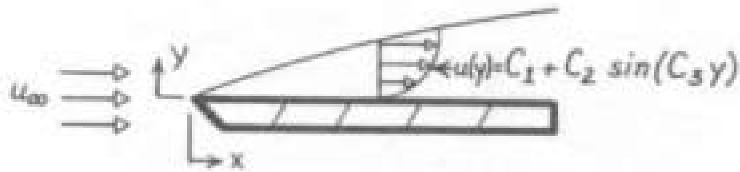
COMMENTS: The foregoing results underpredict those associated with the exact solution ($\delta=4.96\,x\,Re_x^{-1/2}$, $C_{f,x}=0.664\,Re_x^{-1/2}$) and the cubic profile ($\delta=4.64\,x\,Re_x^{-1/2}$, $C_{f,x}=0.646\,Re_x^{-1/1/2}$).

PROBLEM 7.8

KNOWN: Velocity profile for flow over a flat plate.

FIND: (a) Expression for profile in terms of u_∞ and δ, (b) Expression for $\delta(x)$, (c) Expression for $C_{f,x}$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady, incompressible, constant property flow, (2) Boundary layer approximations are valid.

ANALYSIS: (a) The velocity profile with $\eta = y/\delta$ is

$$\left.\begin{array}{l} u(x,0)=0 \quad \rightarrow C_1=0 \\ \left.\dfrac{\partial u}{\partial y}\right|_{y=\delta}=0 \rightarrow C_3=\dfrac{\pi}{2\delta} \\ u(x,\delta)=u_\infty \quad \rightarrow C_2=u_\infty \end{array}\right\} \quad u=u_\infty \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)=u_\infty \sin\left(\frac{\pi}{2}\eta\right). \quad \triangleleft$$

(b) From the momentum integral equation for a flat plate

$$u_\infty^2 \frac{d}{dx}\int_0^\delta\left(1-\frac{u}{u_\infty}\right)\frac{u}{u_\infty}\,dy=\nu\left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$u_\infty^2 \frac{d}{dx}\int_0^1 \delta\left(1-\sin\frac{\pi}{2}\eta\right)\sin\frac{\pi}{2}\eta\,d\eta=\frac{\nu u_\infty}{\delta}\frac{\pi}{2}\cos\left.\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right|_{y=0}$$

$$\frac{d}{dx}\left[\delta\int_0^1\left(\sin\frac{\pi}{2}\eta-\sin^2\frac{\pi}{2}\eta\right)d\eta\right]=\frac{\nu u_\infty}{\delta u_\infty^2}\frac{\pi}{2}=\frac{\pi}{2}\frac{\nu}{\delta u_\infty}$$

$$\frac{d}{dx}\left\{\delta\left[-\frac{2}{\pi}\cos\frac{\pi}{2}\eta\Big|_0^1-\left(\frac{\eta}{2}-\frac{1}{2}\sin\frac{\pi}{2}\eta\cos\frac{\pi}{2}\eta\right)\Big|_0^1\right]\right\}=\frac{\pi}{2}\frac{\nu}{\delta u_\infty}$$

$$\frac{d}{dx}\left\{\delta\left[\frac{2}{\pi}-\frac{1}{2}\right]\right\}=\frac{\pi}{2}\frac{\nu}{\delta u_\infty} \qquad\qquad 0.137\frac{d\delta}{dx}=\frac{\pi}{2}\frac{\nu}{\delta u_\infty}$$

$$\int_0^\delta \delta\, d\delta=11.5\frac{\nu}{u_\infty}\int_0^x dx \qquad \delta^2=23.0\frac{x\nu}{u_\infty}=23.0\,x^2/Re_x \qquad \delta=4.80\,x\,Re_x^{-1/2}. \quad \triangleleft$$

(c) The shear stress and friction coefficient are

$$\tau_s=\mu\left.\frac{\partial u}{\partial y}\right|_{y=0}=\mu\frac{u_\infty}{\delta}\frac{\pi}{2} \qquad C_f=\frac{\tau_s}{\rho u_\infty^2/2}=\frac{\pi\nu}{u_\infty\delta}=\frac{\pi\nu}{4.80\,u_\infty x}Re_x^{1/2}=0.654\,Re_x^{-1/2}. \quad \triangleleft$$

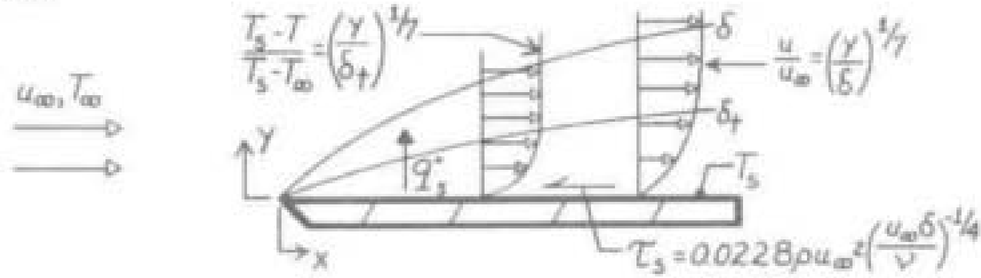
COMMENTS: The foregoing results slightly underpredict those of the exact solution ($\delta = 4.96\,x\,Re_x^{-1/2}$, $C_{f,x} = 0.664\,Re_x^{-1/2}$) and are slightly more accurate than those for the cubic profile ($\delta = 4.64\,x\,Re_x^{-1/2}$, $C_{f,x} = 0.646\,Re_x^{-1/2}$).

PROBLEM 7.9

KNOWN: Velocity and temperature profiles and shear stress-boundary layer thickness relation for turbulent flow over a flat plate.

FIND: (a) Expressions for hydrodynamic boundary layer thickness and average friction coefficient, (b) Expressions for local and average Nusselt numbers.

SCHEMATIC:



ASSUMPTIONS: (1) Steady flow, (2) Constant properties, (3) Fully turbulent boundary layer, (4) Incompressible flow, (5) Isothermal plate, (6) Negligible viscous dissipation, (7) $\delta \approx \delta_t$.

ANALYSIS: (a) The momentum integral equation is

$$\rho u_\infty^2 \frac{d}{dx} \int_0^\delta \left[1 - \frac{u}{u_\infty}\right] \frac{u}{u_\infty}\, dy = \tau_s .$$

Substituting the expression for the wall shear stress

$$\rho u_\infty^2 \frac{d}{dx} \int_0^\delta \left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right] \left(\frac{y}{\delta}\right)^{1/7} dy = 0.0228\, \rho u_\infty^2 \left[\frac{u_\infty \delta}{\nu}\right]^{-1/4}$$

$$\frac{d}{dx} \int_0^\delta \left[\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y}{\delta}\right)^{2/7}\right] dy = \frac{d}{dx} \left[\frac{7}{8} \frac{y^{8/7}}{\delta^{1/7}} - \frac{7}{9} \frac{y^{9/7}}{\delta^{2/7}}\right]\Bigg|_0^\delta$$

$$\frac{d}{dx} \left[\frac{7}{8}\delta - \frac{7}{9}\delta\right] = 0.0228 \left[\frac{u_\infty \delta}{\nu}\right]^{-1/4}$$

$$\frac{7}{72} \frac{d\delta}{dx} = 0.0228 \left[\frac{\nu}{u_\infty}\right]^{1/4} \delta^{-1/4} \qquad \frac{7}{72} \int_0^\delta \delta^{1/4}\, d\delta = 0.0228 \left[\frac{\nu}{u_\infty}\right]^{1/4} \int_0^x dx$$

$$\frac{7}{72} \times \frac{4}{5}\, \delta^{5/4} = 0.0228 \left[\frac{\nu}{u_\infty}\right]^{1/4} x, \qquad \delta = 0.376 \left[\frac{\nu}{u_\infty}\right]^{1/5} x^{4/5}, \quad \frac{\delta}{x} = 0.376 Re_x^{-1/5}. \triangleleft$$

Knowing δ, it follows

$$\tau_s = 0.0228\, \rho u_\infty^2 \left[\frac{u_\infty}{\nu}\right]^{-1/4} [0.376\, x\, Re_x^{-1/5}]^{-1/4}$$

$$C_{f,x} = \frac{\tau_s}{\rho u_\infty^2 / 2} = 0.0456 \left[0.376 \frac{u_\infty}{\nu} \left[\frac{u_\infty}{\nu}\right]^{-1/5} x\, x^{-1/5}\right]^{-1/4} = 0.0592\, Re_x^{-1/5} .$$

Continued .....

The average friction coefficient is then

$$\overline{C}_{f,x} = \frac{1}{x}\int_0^x C_{f,x}\,dx = \frac{1}{x}\,0.0592\left(\frac{u_\infty}{\nu}\right)^{-1/5}\int_0^x x^{-1/5}\,dx$$

$$\overline{C}_{f,x} = \frac{1}{x}\,0.0592\left(\frac{u_\infty}{\nu}\right)^{-1/5} x^{4/5}\left(\frac{5}{4}\right) = 0.074\,Re_x^{-1/5}\,. \qquad \triangleleft$$

(b) The energy integral equation for turbulent flow is

$$\frac{d}{dx}\int_0^{\delta_t} u(T_\infty - T)dy = -\frac{q_s''}{\rho c_p} = -\frac{h}{\rho c_p}(T_s - T_\infty)\,.$$

Hence,

$$u_\infty \frac{d}{dx}\int_0^{\delta_t}\frac{u}{u_\infty}\frac{T-T_\infty}{T_s-T_\infty}dy = u_\infty\frac{d}{dx}\int_0^{\delta_t}(y/\delta)^{1/7}\left[1-(y/\delta_t)^{1/7}\right]dy = \frac{h}{\rho c_p}$$

$$u_\infty \frac{d}{dx}\left[\frac{7}{8}\frac{\delta_t^{8/7}}{\delta^{1/7}} - \frac{7}{9}\frac{\delta_t^{9/7}}{\delta^{1/7}\,\delta_t^{1/7}}\right] = \frac{h}{\rho c_p}$$

or, with $\xi \equiv \delta_t/\delta$,

$$u_\infty \frac{d}{dx}\left[\frac{7}{8}\delta\xi^{8/7} - \frac{7}{9}\delta\xi^{8/7}\right] = \frac{h}{\rho c_p} \qquad u_\infty\frac{d}{dx}\left[\frac{7}{72}\delta\xi^{8/7}\right] = \frac{h}{\rho c_p}\,.$$

Hence, with $\xi \approx 1$ and $\delta/x = 0.376\,Re_x^{-1/5}$,

$$\frac{7}{72}u_\infty(0.376)\left(\frac{u_\infty}{\nu}\right)^{-1/5}\frac{d\,(x^{4/5})}{dx} = \frac{h}{\rho c_p}$$

$$h = 0.0292\,\rho c_p u_\infty\,Re_x^{-1/5} = 0.0292\,\frac{k}{x}\,\frac{\nu}{\alpha}\,\frac{u_\infty x}{\nu}\,Re_x^{-1/5}$$

$$Nu_x = \frac{hx}{k} = 0.0292\,Re_x^{4/5}\,Pr\,. \qquad \triangleleft$$

Hence,

$$\overline{h}_x = \frac{1}{x}\int_0^x h\,dx = \frac{0.0292\,Pr}{x}\,k\left(\frac{u_\infty}{\nu}\right)^{4/5}\int_0^x x^{-1/5}\,dx = 0.0292\,\frac{k}{x}\,Pr\left(\frac{u_\infty x}{\nu}\right)^{4/5}\frac{5}{4}$$

$$\overline{Nu}_x = \frac{\overline{h}_x x}{k} = 0.037\,Re_x^{4/5}\,Pr\,. \qquad \triangleleft$$

COMMENTS: (1) The foregoing results are in excellent agreement with empirical correlations, except that use of $Pr^{1/3}$ instead of Pr, would be more appropriate.

(2) Note that the 1/7 profile breaks down at the surface. For example,

$$\left.\frac{\partial(u/u_\infty)}{\partial y}\right)_{y=0} = \frac{1}{7}\,\delta^{-1/7}\,y^{-6/7} = \infty$$

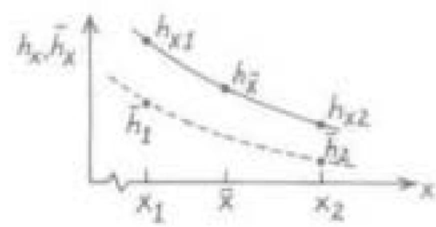
or $\tau_s = \infty$. Despite this unrealistic characteristic of the profile, its use with integral methods provides excellent results.

PROBLEM 7.10

KNOWN: Parallel flow over a flat plate and two locations representing a short span x_1 to x_2 where $(x_2 - x_1) \ll L$.

FIND: Three different expressions for the average heat transfer coefficient over the short span x_1 to x_2, $\bar{h}_{1-2}$.

SCHEMATIC:



ASSUMPTIONS: (1) Parallel flow over a flat plate.

ANALYSIS: The heat rate per unit width for the span can be written as

$$q'_{1-2} = \bar{h}_{1-2}(x_2 - x_1)(T_s - T_\infty) \tag{1}$$

where $\bar{h}_{1-2}$ is the average heat transfer coefficient over the span and can be evaluated in terms of the following three parameters:

(a) *Local coefficient at $\bar{x} = (x_1 + x_2)/2$:* if the span is very short, it may be reasonable to assume that

$$\bar{h}_{1-2} \approx h_{\bar{x}} \tag{2}$$

where $h_{\bar{x}}$ is the local value at the mid-point of the span, $\bar{x} = (x_1 + x_2)/2$.

(b) *Local coefficients at x_1 and x_2:* if the span is very short it may be reasonable to assume $\bar{h}_{1-2}$ is the average of the local values at the ends of the span, that is,

$$\bar{h}_{1-2} \approx [h_{x1} + h_{x2}]/2. \tag{3}$$

(c) *Average coefficients for x_1 and x_2:* the heat rate for the span can also be written as

$$q'_{1-2} = q'_{0-2} - q'_{0-1} \tag{4}$$

where the rate q_{0-x} denotes the heat rate for the plate over the distance 0 to x. In terms of heat transfer coefficients, find

$$\bar{h}_{1-2} \cdot (x_2 - x_1) = \bar{h}_2 \cdot x_2 - \bar{h}_1 \cdot x_1$$

$$\bar{h}_{1-2} = \bar{h}_2 \frac{x_2}{x_2 - x_1} - \bar{h}_1 \frac{x_1}{x_2 - x_1} \tag{5}$$

where $\bar{h}_1$ and $\bar{h}_2$ are the average coefficients for x_1 and x_2, respectively.

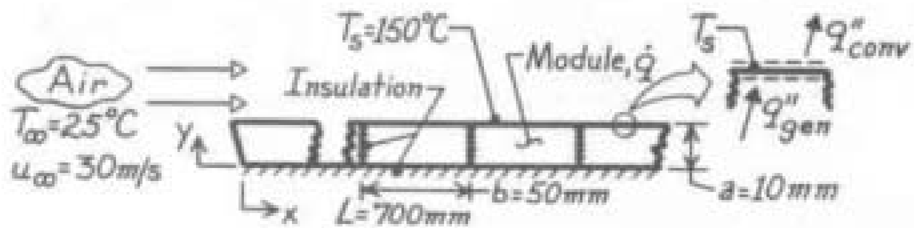
COMMENTS: The expressions, Eqs. (2) and (3), are approximate and work well when the span is small and flow is turbulent rather than laminar ($h_x \approx x^{-0.2}$ vs $h_x \approx x^{-0.5}$). Of course, we require that $x_c < x_1, x_2$ or $x_c > x_1, x_2$; that is, the approximations are inappropriate around the transition region. Eq. (5) is the exact relationship.

PROBLEM 7.11

KNOWN: Flat plate comprised of rectangular modules maintained at surface temperature T_s of thickness a and length b cooled by air at 25°C with velocity 30 m/s. Prescribed thermophysical properties of the module material.

FIND: (a) Required power generation for the module positioned 700 mm from the leading edge of the plate and (b) Maximum temperature in this module.

SCHEMATIC:



ASSUMPTIONS: (1) Laminar flow at leading edge of plate, (2) Transition Reynolds number of 5×10^5, (3) Heat transfer is one-dimensional in y-direction within each module, (4) $\dot{q}$ is uniform within module and (5) Negligible radiation heat transfer.

PROPERTIES: Module material (given): $k = 5.2$ W/m·K, $c_p = 320$ J/kg·K, $\rho = 2300$ kg/m^3; *Table A.4*, Air ($\overline{T}_f = (T_s + T_\infty)/2 = 360$ K, 1 atm): $k = 0.0308$ W/m·K, $\nu = 22.02\times10^{-6}$ m^2/s, Pr = 0.698.

ANALYSIS: (a) The module power generation follows from an energy balance on the module surface,

$$q''_{conv} = q''_{gen}$$

$$\overline{h}A_s(T_s - T_\infty) = \dot{q}(A_s\cdot a) \qquad \text{or} \qquad \dot{q} = \frac{\overline{h}(T_s - T_\infty)}{a}.$$

To select a convection correlation for estimating $\overline{h}$, find first the Reynolds numbers at x = L as

$$Re_L = \frac{u_\infty L}{\nu} = \frac{30\text{ m/s}\times0.70\text{ m}}{22.02\times10^{-6}\text{ m}^2/\text{s}} = 9.537\times10^5.$$

Since the flow is turbulent over the module, the approximation that $\overline{h} = h_x$ (L + b/2) is appropriate with

$$Re_{L+b/2} = \frac{30\text{ m/s}\times(0.700 + 0.050/2)\text{m}}{22.02\times10^{-6}\text{ m}^2/\text{s}} = 9.877\times10^5.$$

Using the turbulent flow correlation, find with x = L + b/2 = 0.725 m

$$Nu_x = \frac{h_x x}{k} = 0.0296 Re_x^{4/5} Pr^{1/3}$$

$$Nu_x = 0.0296(9.877\times10^5)^{4/5}(0.698)^{1/3} = 1640$$

$$\overline{h} = h_x = \frac{Nu_x k}{x} = \frac{1640\times0.0308\text{ W/m}\cdot\text{K}}{0.725} = 69.7\text{ W/m}^2\cdot\text{K}.$$

Continued .....

Hence,

$$\dot{q} = \frac{69.7\ \text{W/m}^2\cdot\text{K}(150 - 25)\text{K}}{0.010\ \text{m}} = 8.713\times10^5\ \text{W/m}^3. \qquad \triangleleft$$

(b) The maximum temperature within the module occurs at the surface next to the insulation ($y = 0$). For one-dimensional conduction with thermal energy generation, use Eq. 3.42, to obtain

$$T(0) = \frac{\dot{q}a^2}{2k} + T_s = \frac{8.713\times10^5\ \text{W/m}^3\times(0.010\ \text{m})^2}{2\times5.2\ \text{W/m}\cdot\text{K}} + 150\ ^\circ\text{C} = 158.4\ ^\circ\text{C}. \qquad \triangleleft$$

COMMENTS: An alternative approach for estimating the average heat transfer coefficient for the module follows from the relation

$$q_{\text{module}} = q_{0\rightarrow L+b} - q_{0\rightarrow L}$$

$$\bar{h}\cdot b = \bar{h}_{L+b}\cdot(L + b) - \bar{h}_L\cdot L \qquad \text{or} \qquad \bar{h} = \bar{h}_{L+b}\frac{L+b}{b} - \bar{h}_L\frac{L}{b}.$$

Recognizing that mixed flow conditions exist, the appropriate correlation is

$$\overline{Nu}_x = (0.037Re_x^{4/5} - 871)Pr^{1/3}$$

and with $x_1 = L + b$ and $x_2 = L$, find

$$\bar{h}_{L+b} = 54.81\ \text{W/m}^2\cdot\text{K} \qquad \text{and} \qquad \bar{h}_L = 53.73\ \text{W/m}^2\cdot\text{K}.$$

Hence,

$$\bar{h} = \left[54.81\frac{0.750}{0.050} - 53.73\frac{0.700}{0.05}\right]\text{W/m}^2\cdot\text{K} = 69.9\ \text{W/m}^2\cdot\text{K}.$$

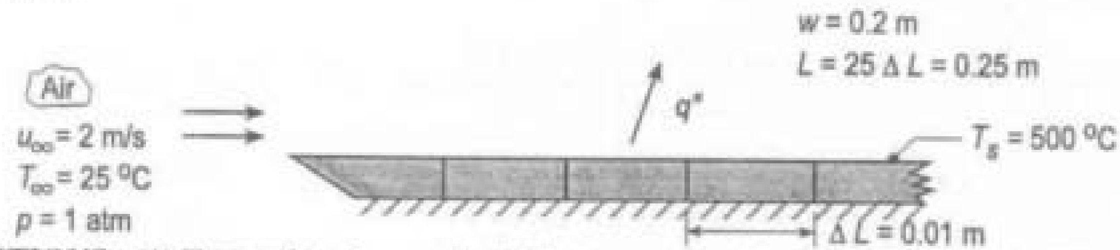
which is in good agreement (10%) with the simpler, but more approximate method employed in part (a).

PROBLEM 7.12

KNOWN: Dimensions and surface temperature of electrically heated strips. Temperature and velocity of air in parallel flow.

FIND: (a) Rate of convection heat transfer from first, fifth and tenth strips as well as from all the strips, (b) For air velocities of 2, 5 and 10 m/s, determine the convection heat rates for all the locations of part (a), and (c) Repeat the calculations of part (b), but under conditions for which the flow is fully turbulent over the entire array of strips.

SCHEMATIC:



ASSUMPTIONS: (1) Top surface is smooth, (2) Bottom surface is adiabatic, (3) Critical Reynolds number is 5×10^5, (4) Negligible radiation.

PROPERTIES: *Table A.4*, Air (T_f = 535 K, 1 atm): $\nu = 43.54 \times 10^{-6}$ m²/s, k = 0.0429 W/m·K, Pr = 0.683.

ANALYSIS: (a) The location of transition is determined from

$$x_c = 5\times 10^5 \frac{\nu}{u_\infty} = 5\times 10^5 \frac{43.54\times 10^{-6}\,\text{m}^2/\text{s}}{2\,\text{m/s}} = 10.9\,\text{m}$$

Since $x_c >> L = 0.25$ m, the air flow is laminar over the entire heater. For the *first* strip, $q_1 = \bar{h}_1(\Delta L \times w)(T_s - T_\infty)$ where $\bar{h}_1$ is obtained from

$$\bar{h}_1 = \frac{k}{\Delta L} 0.664\,\text{Re}_x^{1/2}\,\text{Pr}^{1/3}$$

$$\bar{h}_1 = \frac{0.0429\,\text{W/m}\cdot\text{K}}{0.01\text{m}} \times 0.664\left(\frac{2\,\text{m/s}\times 0.01\,\text{m}}{43.54\times 10^{-6}\,\text{m}^2/\text{s}}\right)^{1/2}(0.683)^{1/3} = 53.8\,\text{W/m}^2\cdot\text{K}$$

$$q_1 = 53.8\,\text{W/m}^2\cdot\text{K}(0.01\,\text{m}\times 0.2\,\text{m})(500-25)^\circ\text{C} = 51.1\,\text{W}$$ <

For the *fifth* strip, $q_5 = q_{0-5} - q_{0-4}$,

$$q_5 = \bar{h}_{0-5}(5\Delta L \times w)(T_s - T_\infty) - \bar{h}_{0-4}(4\Delta L\times w)(T_s - T_\infty)$$

$$q_5 = \left(5\bar{h}_{0-5} - 4\bar{h}_{0-4}\right)(\Delta L \times w)(T_s - T_\infty)$$

Hence, with $x_5 = 5\Delta L = 0.05$ m and $x_4 = 4\Delta L = 0.04$ m, it follows that $\bar{h}_{0-5}$ = 24.1 W/m²·K and $\bar{h}_{0-4}$ = 26.9 W/m²·K and

$$q_5 = (5\times 24.1 - 4\times 26.9)\,\text{W/m}^2\cdot\text{K}(0.01\times 0.2)\text{m}^2(500-25)\text{K} = 12.2\,\text{W}.$$ <

Similarly, where $\bar{h}_{0-10}$ = 17.00 W/m²·K and $\bar{h}_{0-9}$ = 17.92 W/m²·K.

$$q_{10} = \left(10\bar{h}_{0-10} - 9\bar{h}_{0-9}\right)(\Delta L\times w)(T_s - T_\infty)$$

$$q_{10} = (10\times 17.00 - 9\times 17.92)\,\text{W/m}^2\cdot\text{K}(0.01\times 0.2)\text{m}^2(500-25)\text{K} = 8.3\,\text{W}$$ <

Continued...

For the entire heater,

$$\bar{h}_{0-25} = \frac{k}{L}0.664\,Re_L^{1/2}\,Pr^{1/3} = \frac{0.0429}{0.25}\times 0.664\left(\frac{2\times 0.25}{43.54\times 10^{-6}}\right)^{1/2}(0.683)^{1/3} = 10.75\,W/m^2\cdot K$$

and the heat rate over all 25 strips is

$$q_{0-25} = \bar{h}_{0-25}(L\times w)(T_s - T_\infty) = 10.75\,W/m^2\cdot K(0.25\times 0.2)m^2(500-25)^\circ C = 255.3\,W \qquad <$$

(b,c) Using the *IHT Correlations Tool, External Flow*, for *Laminar or Mixed Flow Conditions*, and following the same method of solution as above, the heat rates for the first, fifth, tenth and all the strips were calculated for air velocities of 2, 5 and 10 m/s. To evaluate the heat rates for fully turbulent conditions, the analysis was performed setting $Re_{x,c} = 1\times 10^{-6}$. The results are tabulated below.

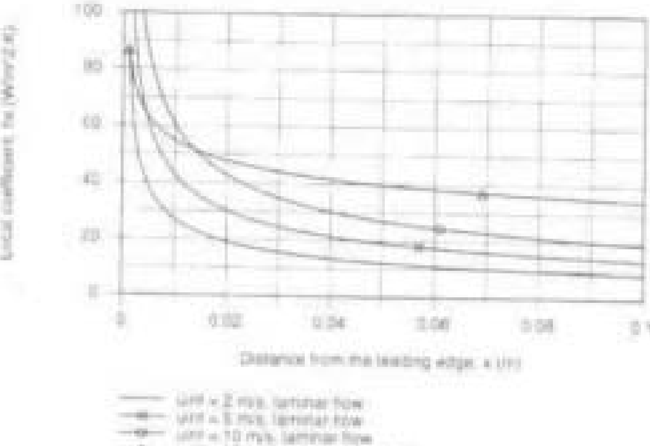
Flow conditions	u_∞ (m/s)	q_1 (W)	q_5 (W)	q_{10} (W)	q_{0-25} (W)
Laminar	2	51.1	12.1	8.3	256
	5	80.9	19.1	13.1	404
	10	114	27.0	18.6	572
Fully turbulent	2	17.9	10.6	9.1	235
	5	37.3	22.1	19.0	490
	10	64.9	38.5	33.1	853

COMMENTS: (1) An alternative approach to evaluating the heat loss from a single strip, for example, strip 5, would take the form $q_5 = \bar{h}_5(\Delta L\times w)(T_s - T_\infty)$, where $h_5 \approx h_{x=4.5\Delta L}$ or $\bar{h}_5 \approx (h_{x=5\Delta L} + h_{x=4\Delta L})/2$.

(2) From the tabulated results, note that for both flow conditions, the heat rate for each strip and the entire heater, increases with increasing air velocity. For both flow conditions and for any specified velocity, the strip heat rates decrease with increasing distance from the leading edge.

(3) The effect of flow conditions, laminar vs. fully turbulent flow, on strip heat rates shows some unexpected behavior. For the u_∞ = 5 m/s condition, the effect of turbulent flow is to increase the heat rates for the entire heater and the tenth and fifth strips. For the u_∞ = 10 m/s, the effect of turbulent flow is to increase the heat rates at all locations. This behavior is a consequence of low Reynolds number ($Re_x = 2.3\times 10^4$) at x = 0.25 m with u_∞ = 10 m/s.

(4) To more fully appreciate the effects due to laminar vs. turbulent flow conditions and air velocity, it is useful to examine the local coefficient as a function of distance from the leading edge. How would you use the results plotted below to explain heat rate behavior evident in the summary table above?

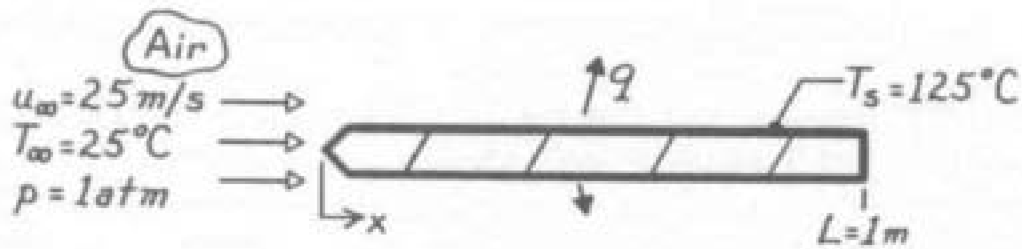


PROBLEM 7.13

KNOWN: Speed and temperature of atmospheric air flowing over a flat plate of prescribed length and temperature.

FIND: Rate of heat transfer corresponding to $Re_{x,c} = 10^5$, 5×10^5 and 10^6.

SCHEMATIC:



ASSUMPTIONS: (1) Flow over top and bottom surfaces.

PROPERTIES: *Table A-4*, Air (T_f = 348K, 1 atm): $\rho = 1.00$ kg/m^3, $\nu = 20.72\times10^{-6}$ m^2/s, k = 0.0299 W/m·K, Pr = 0.700.

ANALYSIS: With

$$Re_L = \frac{u_\infty L}{\nu} = \frac{25 \text{ m/s} \times 1\text{m}}{20.72\times10^{-6} \text{ m}^2/\text{s}} = 1.21\times10^6$$

the flow becomes turbulent for each of the three values of $Re_{x,c}$. Hence,

$$\overline{Nu}_L = (0.037\, Re_L^{4/5} - A)\, Pr^{1/3}$$

$$A = 0.037\, Re_{x,c}^{4/5} - 0.664\, Re_{x,c}^{1/2}$$

$Re_{x,c}$	10^5	5×10^5	10^6
A	160	871	1671
$\overline{Nu}_L$	2272	1641	931
$\bar{h}_L$ (W/m^2)	67.9	49.1	27.8
q′ (W/m)	13,580	9820	5560

◁

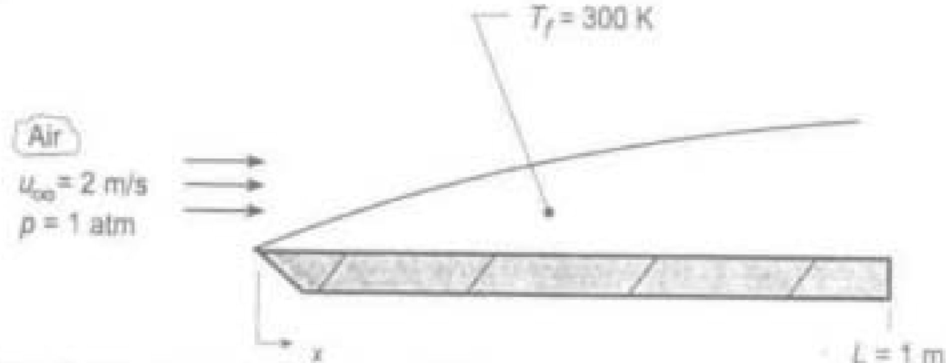
where $q' = 2\,\bar{h}_L L(T_s - T_\infty)$ is the total heat loss per unit width of plate.

COMMENTS: Note that $\bar{h}_L$ decreases with increasing $Re_{x,c}$, as more of the surface becomes covered with a laminar boundary layer.

KNOWN: Velocity and temperature of air in parallel flow over a flat plate of 1-m length.

FIND: (a) Calculate and plot the variation of the local convection coefficient, $h_x(x)$, with distance for flow conditions corresponding to transition Reynolds numbers of 5×10^5, 2.5×10^5 and 0 (fully turbulent), (b) Plot the variation of the average convection coefficient, $\overline{h}_x(x)$, for the three flow conditions of part (a), and (c) Determine the average convection coefficients for the entire plate, $\overline{h}_L$, for the three flow conditions of part (a).

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant surface temperature, and (3) Critical Reynolds depends upon prescribed flow conditions.

PROPERTIES: *Table A.4*, Air ($T_f = 300$ K, 1 atm): $\nu = 15.89 \times 10^{-6}$ m²/s, $k = 0.0263$ W/m·K, Pr = 0.707.

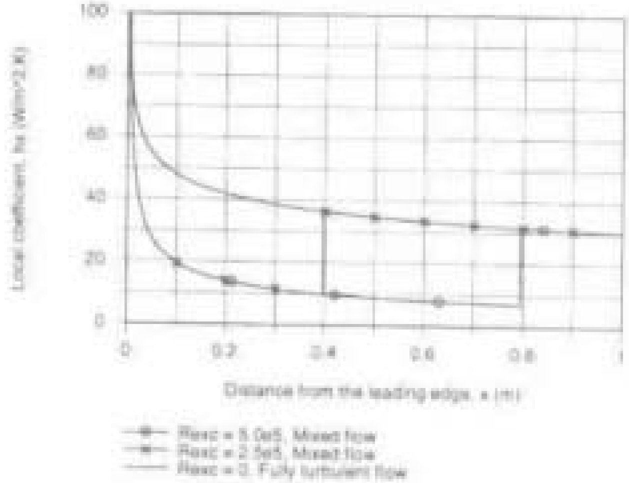
ANALYSIS: (a) The Reynolds number for the plate (L = 1 m) is

$$Re_L = \frac{u_\infty L}{\nu} = \frac{10\,\text{m/s} \times 1\,\text{m}}{15.89 \times 10^{-6}\,\text{m}^2/\text{s}} = 6.29 \times 10^5.$$

Hence, the boundary layer conditions are mixed with $Re_{x,c} = 5 \times 10^5$,

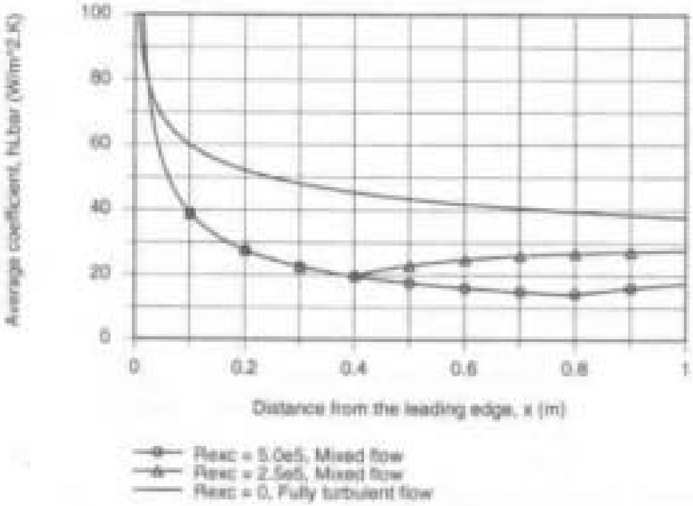
$$x_c = L\left(Re_{x,c}/Re_L\right) = 1\,\text{m}\,\frac{5 \times 10^5}{6.29 \times 10^5} = 0.795\,\text{m}$$

Using the *IHT Correlation Tool, External Flow, Local* coefficients for *Laminar* or *Turbulent Flow*, $h_x(x)$ was evaluated and plotted with critical Reynolds numbers of 5×10^5, 2.5×10^5 and 0 (fully turbulent). Note the location of the laminar-turbulent transition for the first two flow conditions.



Continued...

(b) Using the *IHT Correlation Tool, External Flow, Average* coefficient for *Laminar* or *Mixed Flow*, $\bar{h}_x(x)$ was evaluated and plotted for the three flow conditions. Note that the change in $\bar{h}_x(x)$ at the critical length, x_c, is rather gradual, compared to the abrupt change for the local coefficient, $h_x(x)$.



(c) The average convection coefficients for the plate can be determined from the above plot since $\bar{h}_L = \bar{h}_x(L)$. The values for the three flow conditions are, respectively,

$$\bar{h}_L = 17.4, 27.5 \text{ and } 37.8\,\text{W/m}^2\cdot\text{K} \qquad <$$

COMMENTS: A copy of the *IHT Workspace* used to generate the above plots is shown below.

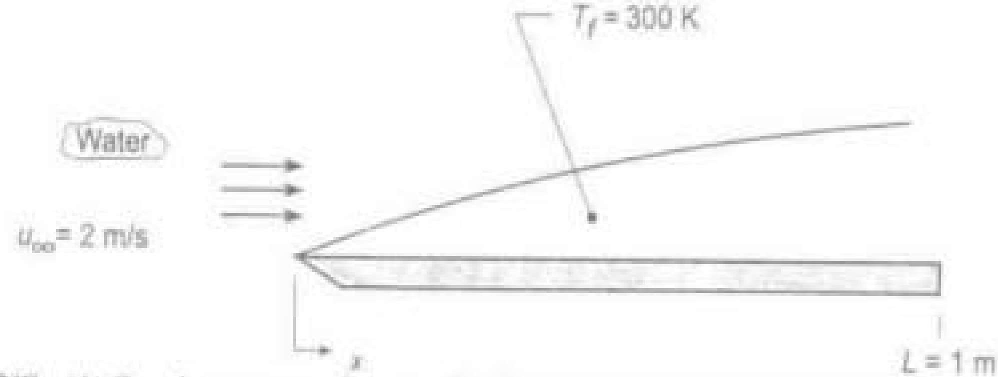
```
// Method of Solution: Use the Correlation Tools, External Flow, Flat Plate, for (i) Local, laminar or turbulent
flow and (ii) Average, laminar or mixed flow, to evaluate the local and average convection coefficients as a
function of position on the plate. In each of these tools, the value of the critical Reynolds number, Rexc, can
be set corresponding to the special flow conditions.

// Correlation Tool: External Flow, Plate Plate, Local, laminar or turbulent flow.
Nux = Nux_EF_FP_LT(Rex,Rexc,Pr)    // Eq 7.23,37
Nux = hx * x / k
Rex = uinf * x / nu
Rexc =1e-10
// Evaluate properties at the film temperature, Tf.
//Tf = (Tinf + Ts) / 2

/* Correlation description: Parallel external flow (EF) over a flat plate (FP), local coefficient; laminar flow (L) for
Rex<Rexc, Eq 7.23; turbulent flow (T) for Rex>Rexc, Eq 7.37; 0.6<=Pr<=60. See Table 7.9. */

// Correlation Tool: External Flow, Plate Plate, Average, laminar or mixed flow.
NuLbar = NuL_bar_EF_FP_LM(Rex,Rexc,Pr)    // Eq 7.31, 7.39, 7.40
NuLbar = hLbar * x / k                    // Changed variable from L to x
//ReL = uinf * x / nu
//Rexc = 5.0E5

/* Correlation description: Parallel external flow (EF) over a flat plate (FP), average coefficient; laminar (L) if
ReL<Rexc, Eq 7.31; mixed (M) if ReL>Rexc, Eq 7.39 and 7.40; 0.6<=Pr<=60. See Table 7.9. */

// Properties Tool - Air:
// Air property functions : From Table A.4
// Units: T(K); 1 atm pressure
nu = nu_T("Air",Tf)          // Kinematic viscosity, m^2/s
k = k_T("Air",Tf)            // Thermal conductivity, W/m-K
Pr = Pr_T("Air",Tf)          // Prandtl number

// Assigned Variables:
x = 1                        // Distance from leading edge; 0 <= x <= 1 m
uinf = 10                    // Freestream velocity, m/s
Tf = 300                     // Film temperature, K
```

PROBLEM 7.15

KNOWN: Velocity and temperature of water in parallel flow over a flat plate of 1-m length.

FIND: (a) Calculate and plot the variation of the local convection coefficient, $h_x(x)$, with distance for flow conditions corresponding to transition Reynolds numbers of 5×10^5, 3×10^5 and 0 (fully turbulent), (b) Plot the variation of the average convection coefficient, $\bar{h}_x(x)$, for the three flow conditions of part (a), and (c) Determine the average convection coefficients for the entire plate, $\bar{h}_L$, for the three flow conditions of part (a).

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant surface temperature, and (3) Critical Reynolds depends upon prescribed flow conditions.

PROPERTIES: *Table A.6*, Water (300 K): $\rho = 997\ \text{kg/m}^3$, $\mu = 855 \times 10^{-6}\ \text{N·s/m}^2$, $\nu = \mu/\rho = 0.858 \times 10^{-6}\ \text{m}^2/\text{s}$, $k = 0.613$ W/m·K, Pr = 5.83.

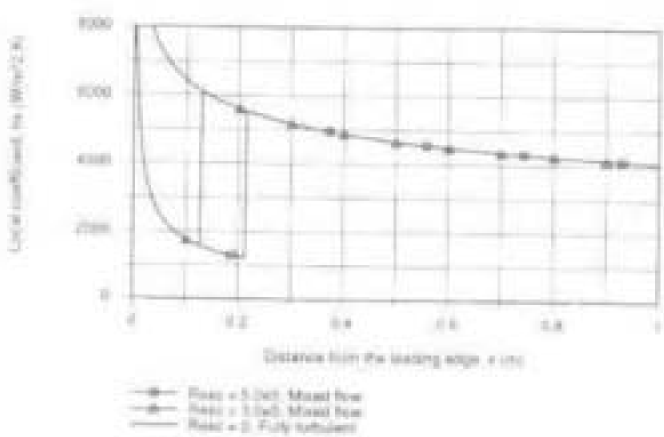
ANALYSIS: (a) The Reynolds number for the plate (L = 1 m) is

$$Re_L = \frac{u_\infty L}{\nu} = \frac{2\ \text{m/s} \times 1\ \text{m}}{0.858 \times 10^{-6}\ \text{m}^2/\text{s}} = 2.33 \times 10^6,$$

and the boundary layer is mixed with $Re_{x,c} = 5 \times 10^5$.

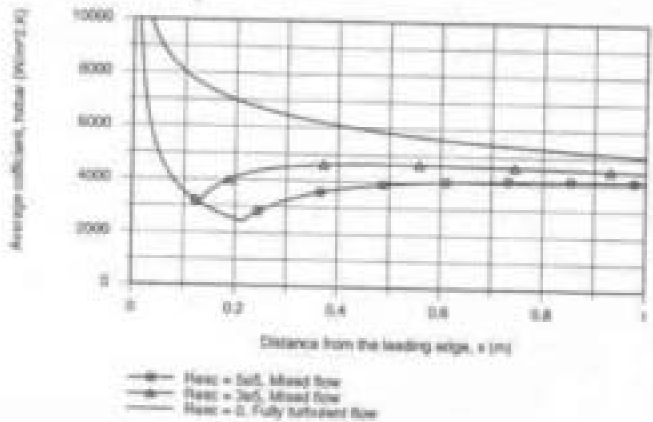
$$x_c = L\left(Re_{x,c}/Re_L\right) = 1\,\text{m}\,\frac{5 \times 10^5}{2.33 \times 10^6} = 0.215\,\text{m}$$

Using the *IHT Correlation Tool, External Flow, Local* coefficients for *Laminar* or *Turbulent Flow*, $h_x(x)$ was evaluated and plotted with critical Reynolds numbers of 5×10^5, 3.0×10^5 and 0 (fully turbulent). Note the location of the laminar-turbulent transition for the first two flow conditions.



Continued...

(b) Using the *IHT Correlation Tool, External Flow, Average* coefficient for *Laminar* or *Mixed Flow*, $\bar{h}_x(x)$ was evaluated and plotted for the three flow conditions. Note that the change in $\bar{h}_x(x)$ at the critical length, x_c, is rather gradual, compared to the abrupt change for the local coefficient, $h_x(x)$.



(c) The average convection coefficients for the plate can be determined from the above plot since $\bar{h}_L = \bar{h}_x(L)$. The values for the three flow conditions are

$$\bar{h}_L = 4110,\ 4490 \text{ and } 5072\ \mathrm{W/m^2 \cdot K}$$ <

COMMENTS: A copy of the *IHT Workspace* used to generate the above plot is shown below.

```
/* Method of Solution: Use the Correlation Tools, External Flow, Flat Plate, for (i) Local, laminar or turbulent
flow and (ii) Average, laminar or mixed flow, to evaluate the local and average convection coefficients as a
function of position on the plate. In each of these tools, the value of the critical Reynolds number, Rexc, can
be set corresponding to the special flow conditions. */

// Correlation Tool: External Flow, Plate Plate, Local, laminar or turbulent flow.
Nux = Nux_EF_FP_LT(Rex,Rexc,Pr)      // Eq 7.23,37
Nux = hx * x / k
Rex = uinf * x / nu
Rexc = 1e-10
// Evaluate properties at the film temperature, Tf.
//Tf = (Tinf + Ts) / 2

/* Correlation description: Parallel external flow (EF) over a flat plate (FP), local coefficient; laminar flow (L) for
Rex<Rexc, Eq 7.23; turbulent flow (T) for Rex>Rexc, Eq 7.37; 0.6<=Pr<=60. See Table 7.9. */

// Correlation Tool: External Flow, Plate Plate, Average, laminar or mixed flow.
NuLbar = NuL_bar_EF_FP_LM(Rex,Rexc,Pr)      // Eq 7.31, 7.39, 7.40
NuLbar = hLbar * x / k                      // Changed variable from L to x
//ReL = uinf * x / nu
//Rexc = 5.0E5

/* Correlation description: Parallel external flow (EF) over a flat plate (FP), average coefficient; laminar (L) if
ReL<Rexc, Eq 7.31; mixed (M) if ReL>Rexc, Eq 7.39 and 7.40; 0.6<=Pr<=60. See Table 7.9. */

// Properties Tool - Water:
// Water property functions :T dependence, From Table A.6
// Units: T(K), p(bars);
xf = 0                          // Quality (0=sat liquid or 1=sat vapor); "x" is used as spatial coordinate
p = psat_T("Water", Tf)         // Saturation pressure, bar
nu = nu_Tx("Water",Tf,x)        // Kinematic viscosity, m^2/s
k = k_Tx("Water",Tf,x)          // Thermal conductivity, W/m-K
Pr = Pr_Tx("Water",Tf,x)        // Prandtl number

// Assigned Variables:
x = 1                           // Distance from leading edge; 0 <= x <= 1 m
uinf = 2                        // Freestream velocity, m/s
Tf = 300                        // Film temperature, K
```

PROBLEM 7.16

KNOWN: Temperature and velocity of atmospheric air in parallel flow over a plate of prescribed length and temperature.

FIND: (a) Average heat transfer coefficient, (b) Local coefficient at the midpoint, (c) Plot the variation of the heat flux with distance.

SCHEMATIC:

ASSUMPTIONS: (1) Uniform surface temperature, (2) $Re_{x,c} = 5 \times 10^5$.

PROPERTIES: *Table A.4*, Air (T_f = 350.5 K ≈ 350 K, 1 atm): $\nu = 20.92 \times 10^{-6}$ m²/s, k = 0.03 W/m K, Pr = 0.700.

ANALYSIS: (a) With

$$Re_L = \frac{u_\infty L}{\nu} = \frac{10\,\text{m/s} \times 3\,\text{m}}{20.92 \times 10^{-6}\,\text{m}^2/\text{s}} = 1.434 \times 10^6,$$

mixed boundary layer conditions exist. Hence, using Eq. 7.41,

$$\overline{h}_L = \frac{k}{L}\left(0.037\,Re_L^{4/5} - 871\right)Pr^{1/3}$$

$$\overline{h}_L = \frac{0.03\,\text{W/m}\cdot\text{K}}{3\,\text{m}}\left[0.037\left(1.434 \times 10^6\right)^{4/5} - 871\right](0.70)^{1/3} = 19.9\,\text{W/m}^2\cdot\text{K}$$ <

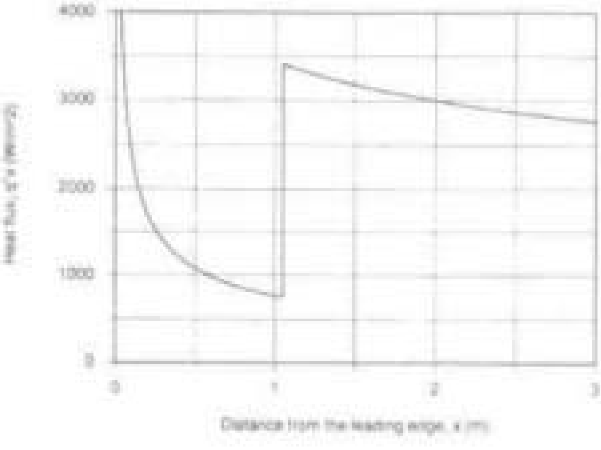
(b) With the transition length as

$$x_c = L\left(Re_{x,c}/Re_L\right) = 3\,\text{m}\left(5 \times 10^5/1.434 \times 10^6\right) = 1.046\,\text{m},$$

the midpoint (x = 1.5 m) is in turbulent flow. Hence, using Eq. 7.37,

$$h_x = \frac{k}{x}\left(0.0296\,Re_x^{4/5}\,Pr^{1/3}\right) = \frac{0.03\,\text{W/m}\cdot\text{K}}{1.5\,\text{m}}\left[0.0296\left(0.717 \times 10^6\right)^{4/5}(0.7)^{1/3}\right] = 25.4\,\text{W/m}^2\cdot\text{K}$$ <

(c) Since $q''_x = h_x\left(T_s - T_\infty\right)$, it follows that the local heat flux will vary with x as the local convection coefficient. From the foregoing results, h_x varies as $x^{-1/2}$ and $x^{-1/5}$ in laminar and turbulent flow, respectively. Using the *IHT Correlation Tool, External Flow, Flat Plate, Local* coefficient for *Laminar* or *Turbulent* flow, h_x (x) was evaluated and q''_x calculated as a function of distance.

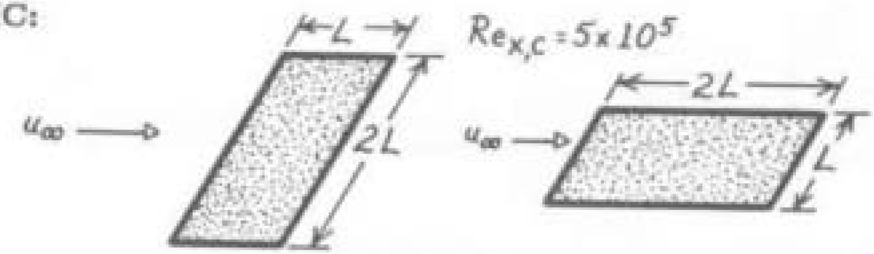


PROBLEM 7.17

KNOWN: Two plates of length L and 2L experience parallel flow with a critical Reynolds number of 5×10^5.

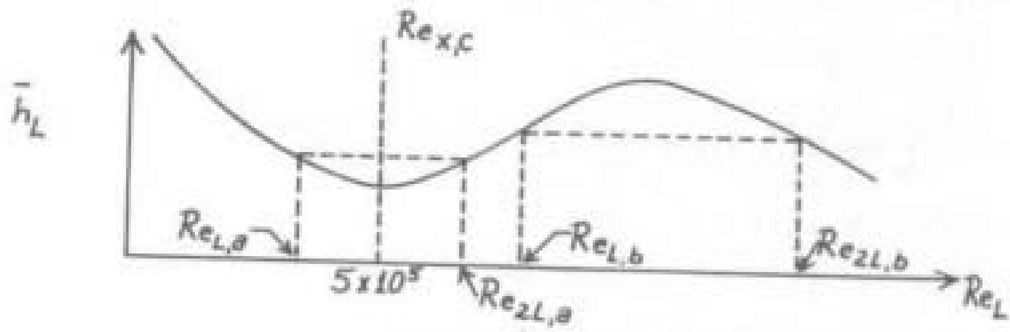
FIND: Reynolds numbers for which the total heat transfer rate is independent of orientation.

SCHEMATIC:



ASSUMPTIONS: (1) Plate temperatures and flow conditions are equivalent.

ANALYSIS: The total heat transfer rate would be the same ($q_L = q_{2L}$), if the convection coefficients were equal, $\bar{h}_L = \bar{h}_{2L}$. Conditions for which such an equality is possible may be inferred from a sketch of $\bar{h}_L$ versus Re_L.



For laminar flow, it follows that $\bar{h}_L \propto L^{-1/2}$ for $Re_L < Re_{x,c}$. Similarly, for mixed laminar and turbulent flow

$$\bar{h}_L = C_1 L^{-1/5} - C_2 L^{-1} \qquad \text{for} \qquad Re_L > Re_{x,c}$$

and $\bar{h}_L$ will vary with Re_L as shown. Hence two possibilities are suggested:

Case (a): Laminar flow exists on the shorter plate, while mixed flow conditions exist on the longer plate.

Case (b): Mixed boundary layer conditions exist on both plates.

In both cases, it is required that

$$\bar{h}_L = \bar{h}_{2L} \qquad \text{and} \qquad Re_{2L} = 2\,Re_L\,.$$

Continued .....

Case (a): From expressions for $\overline{h}_L$ in laminar and mixed flow

$$0.664\,\frac{k}{L}\,Re_L^{1/2}\,Pr^{1/3} = \frac{k}{2L}\left(0.037\,Re_{2L}^{4/5} - 871\right)Pr^{1/3}$$

$$0.664\,Re_L^{1/2} = 0.032\,Re_L^{4/5} - 435\ .$$

Since $Re_L < 5{\times}10^5$ and $Re_{2L} = 2\,Re_L > 5{\times}10^5$, the required value of Re_L may be narrowed to the range

$$2.5{\times}10^5 < Re_L < 5{\times}10^5\ .$$

From a trial-and-error solution, it follows that

$$Re_L \approx 3.2{\times}10^5\ . \qquad \triangleleft$$

Case (b): For mixed flow on both plates

$$\frac{k}{L}\left(0.037\,Re_L^{4/5} - 871\right)Pr^{1/3} = \frac{k}{2L}\left(0.037\,Re_{2L}^{4/5} - 871\right)Pr^{1/3}$$

or

$$0.037\,Re_L^{4/5} - 871 = 0.032\,Re_L^{4/5} - 435$$

$$0.005\,Re_L^{4/5} = 436$$

$$Re_L \approx 1.50{\times}10^6\ . \qquad \triangleleft$$

COMMENTS: (1) Note that it is impossible to satisfy the requirement that $\overline{h}_L = \overline{h}_{2L}$ if $Re_L < 0.25{\times}10^5$ (laminar flow for both plates).

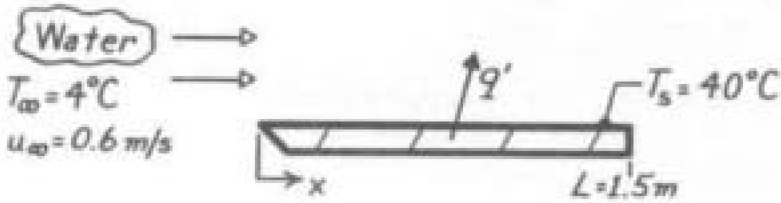
(2) The results are independent of the nature of the fluid.

PROBLEM 7.18

KNOWN: Water flowing over a flat plate under specified conditions.

FIND: (a) Heat transfer rate per unit width, $q'(W/m)$, evaluating properties at $T_f = (T_s + T_\infty)/2$, (b) Error in q' resulting from evaluating properties at T_∞, (c) Heat transfer rate, q', if flow is assumed turbulent at leading edge, $x = 0$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions.

PROPERTIES: *Table A-6,* Water ($T_\infty = 4°C = 277K$): $\rho_f = 1000\,kg/m^3$, $\mu_f = 1560\times10^{-6}\,N\cdot s/m^2$, $\nu_f = \mu_f/\rho_f = 1.560\times10^{-6}\,m^2/s$, $k_f = 0.577\,W/m\cdot K$, $Pr = 11.44$; Water ($T_f = 295K$): $\nu = 0.961\times10^{-6}\,m^2/s$, $k = 0.606\,W/m\cdot K$, $Pr = 6.62$; Water ($T_s = 40°C = 313K$): $\mu = 657\times10^{-6}\,N\cdot s/m^2$.

ANALYSIS: (a) The heat rate is given as $q' = \bar{h}L(T_s - T_\infty)$, and $\bar{h}$ must be estimated by the proper correlation. Calculate first the Reynolds number using properties evaluated at T_f:

$$Re_L = \frac{u_\infty L}{\nu} = \frac{0.6\,m/s\times1.5m}{0.961\times10^{-6}m^2/s} = 9.365\times10^5\ .$$

Hence flow is mixed and the appropriate correlation and convection coefficient are

$$\overline{Nu}_L = [0.037\,Re_L^{4/5} - 871]\,Pr^{1/3} = [0.037(9.365\times10^5)^{4/5} - 871]\,6.62^{1/3} = 2522$$

$$\bar{h}_L \equiv \frac{\overline{Nu}_L\,k}{L} = \frac{2522\times0.606\,W/m\cdot K}{1.5m} = 1019\,W/m^2\cdot K\ .$$

Hence heat rate is

$$q' = 1019\,W/m^2\cdot K\times1.5m(40-4)\,°C = 55.0\,kW/m\ . \qquad \triangleleft$$

(b) Evaluating properties at the free stream temperature, T_∞, find

$$Re_L = \frac{0.6m/s\times1.5m}{1.560\times10^{-6}m^2/s} = 5.769\times10^5$$

and flow is still mixed giving

$$\overline{Nu}_L = [0.037(5.769\times10^5)^{4/5} - 871]\,11.44^{1/3} = 1424$$

$$\bar{h}_L = 1424\times0.577\,W/m\cdot K/1.5m = 575\,W/m\cdot K$$

$$q' = 575\,W/m\cdot K\times1.5m(40-4)\,°C = 31.1\,kW/m\ . \qquad \triangleleft$$

Continued .....

(c) If flow were tripped at the leading edge, the flow would be turbulent over the full length of the plate, in which case,

$$\overline{Nu}_L = 0.037\,Re^{4/5}\,Pr^{1/3} = 0.037(9.365\times10^5)^{4/5}\,6.62^{1/3} = 4157$$

$$\bar{h}_L = \overline{Nu}_L k/L = 4157\times0.606\ W/m{\cdot}K/1.5m = 1679\ W/m^2{\cdot}K$$

$$q' = \bar{h}_L\,L\,(T_s-T_\infty) = 1679\ W/m^2\times1.5m\ (40-4)^\circ C = 90.7\ kW/m\ . \qquad \triangleleft$$

COMMENTS: (1) It is instructive to compare the correlation of Part (a) with the Zhukauskas correlation [9], where all properties are evaluated at T_∞ except for μ_s which is evaluated at T_s.

$$\overline{Nu}_L = 0.036[Re_L^{4/5}-9200]\,Pr^{0.43}\left(\frac{\mu}{\mu_s}\right)^{1/4}$$

$$\overline{Nu}_L = 0.036[(5.769\times10^5)^{4/5}-9200]\,11.44^{0.43}\left(\frac{1560\times10^{-6}\ N{\cdot}s/m^2}{657\times10^{-6}\ N{\cdot}s/m^2}\right)^{1/4} = 4006$$

$$\bar{h}_L = \overline{Nu}_L\,k/L = 4006\times0.577\ W/m{\cdot}K/1.5m = 1541\ W/m^2K$$

$$q' = \bar{h}_L\,L(T_s-T_\infty) = 1541\ W/m^2{\cdot}K\times1.5m\,(40-4)^\circ C = 83.2\ kW/m\ .$$

If the flow is entirely turbulent, the "9200" term for the $\overline{Nu}_L$ disappears and

$$\overline{Nu}_L = 5178 \qquad \bar{h}_L = 1992\ W/m^2{\cdot}K \qquad q' = 108\ kW/m\ .$$

(2) Comparing results:

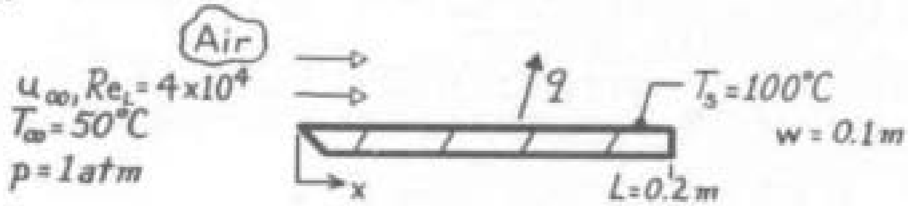
Flow	Part	Property Evaluation	Eqn.	q'(kW/m)	Difference (%)
mixed	(a)	T_f	Text	55.0	--
mixed	--	T_∞, T_s	Zhukauskas	83.2	+51
mixed	(b)	T_∞	Text	31.1	−43
turbulent	(c)	T_f	Text	90.7	--
turbulent	--	T_∞, T_s	Zhukauskas	108	+25

PROBLEM 7.19

KNOWN: Temperature, pressure and Reynolds number for air flow over a flat plate of uniform surface temperature.

FIND: (a) Rate of heat transfer from the plate, (b) Rate of heat transfer if air velocity is doubled and pressure is increased to 10 atm.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Uniform surface temperature, (3) Negligible radiation, (4) $Re_{x,c} = 5\times10^5$.

PROPERTIES: *Table A-4*, Air (T_f = 348K, 1 atm): k = 0.0299 W/m·K, Pr = 0.70.

ANALYSIS: (a) The heat rate is

$$q = \bar{h}_L (w \times L)(T_s - T_\infty).$$

Since the flow is laminar over the entire plate for $Re_L = 4\times10^4$, it follows that

$$\overline{Nu}_L = \frac{\bar{h}_L L}{k} = 0.664\, Re_L^{1/2}\, Pr^{1/3} = 0.664\,(40{,}000)^{1/2}\,(0.70)^{1/3} = 117.9\,.$$

Hence $$\bar{h}_L = 117.9\frac{k}{L} = 117.9\,\frac{0.0299\ W/m\cdot K}{0.2m} = 17.6\ W/m^2\cdot K$$

and $$q = 17.6\frac{W}{m^2\cdot K}(0.1m\times0.2m)\,(100-50)\,°C = 17.6\ W\,. \quad \triangleleft$$

(b) With $p_2 = 10\,p_1$, it follows that $\rho_2 = 10\,\rho_1$ and $\nu_2 = \nu_1/10$. Hence

$$Re_{L,2} = \left(\frac{u_\infty L}{\nu}\right)_2 = 2\times10\left(\frac{u_\infty L}{\nu}\right)_1 = 20\,Re_{L,1} = 8\times10^5$$

and mixed boundary layer conditions exist on the plate. Hence

$$\overline{Nu}_L = \frac{\bar{h}_L L}{k} = (0.037\,Re_L^{4/5} - 871)\,Pr^{1/3} = [0.037\times(8\times10^5)^{4/5} - 871]\,(0.70)^{1/3}$$

$$\overline{Nu}_L = 961\,.$$

Hence, $$\bar{h}_L = 961\,\frac{0.0299\ W/m\cdot K}{0.2m} = 143.6\ W/m^2\cdot K$$

$$q = 143.6\frac{W}{m^2\cdot K}(0.1m\times0.2m)\,(100-50)\,°C = 143.6\ W\,. \quad \triangleleft$$

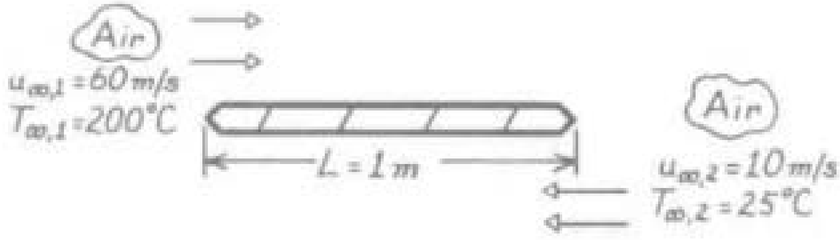
COMMENTS: Note that, in calculating $Re_{L,2}$, ideal gas behavior has been assumed. It has also been assumed that k, μ and Pr are independent of pressure over the range considered.

PROBLEM 7.20

KNOWN: Air flow conditions on opposite sides of a flat plate.

FIND: Heat flux between the two streams at midpoint.

SCHEMATIC:



ASSUMPTIONS: (1) Airstreams are at atmospheric pressure, (2) Transition Reynolds number is 5×10^5, (3) Negligible plate conduction resistance.

PROPERTIES: *Table A-4*, Air (1 atm): Stream 1 – ($T_f \approx 350K$): $\nu = 20.92\times10^{-6}$ m²/s, k = 0.030 W/m·K, Pr = 0.700; Stream 2 – ($T_f \approx 300K$): $\nu = 15.89\times10^{-6}$ m²/s, k = 0.0263 W/m·K, Pr = 0.707.

ANALYSIS: Calculate Re_x at x = L/2.

Stream 1: $$Re_x = \frac{(60\,m/s)\,0.5m}{20.92\times10^{-6}\,m^2/s} = 1.434\times10^6$$

Stream 2: $$Re_x = \frac{(10\,m/s)\,0.5m}{15.89\times10^{-6}\,m^2/s} = 3.147\times10^5\ .$$

Hence, at x = L/2, stream 1 is turbulent and stream 2 is laminar. Hence,

$$h_{L/2,1} = \left(0.0296\,Re_{L/2}^{4/5}\,Pr^{1/3}\right)\frac{k}{L/2} = 0.0296\left(1.434\times10^6\right)^{4/5}(0.7)^{1/3}\times\frac{0.030\,W/m\cdot K}{0.5m}$$

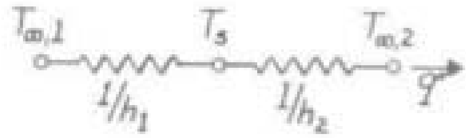
$$h_{L/2,1} = 133\,W/m^2\cdot K\ .$$

Also,

$$h_{L/2,2} = \left(0.332\,Re_{L/2}^{1/2}\,Pr^{1/3}\right)\frac{k}{L/2} = 0.332\left(3.147\times10^5\right)^{1/2}(0.707)^{1/3}\times\frac{0.0263\,W/m\cdot K}{0.5m}$$

$$h_{L/2,2} = 8.73\,W/m^2\cdot K\ .$$

From the thermal circuit,



$$q'' = \frac{T_{\infty,1} - T_{\infty,2}}{h_1^{-1} + h_2^{-1}}$$

$$q'' = \frac{200\,°C - 25\,°C}{(133\,W/m^2\cdot K)^{-1} + (8.73\,W/m^2\cdot K)^{-1}}$$

$$q'' = 1434\,W/m^2\ .$$ ◁

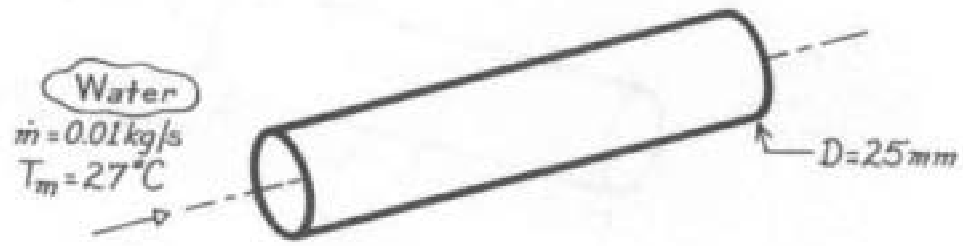
COMMENTS: (1) Neglecting the effects of axial conduction along the plate, its temperature is obtained from $(T_{\infty,1} - T_s)/(T_s - T_{\infty,2}) = h_1^{-1}/h_2^{-1} = 0.0656$ or $T_s = 189\,°C$. (2) Result for midpoint is independent of airstream directions. Not so for any other point.

PROBLEM 8.1

KNOWN: Flowrate and temperature of water in fully developed flow through a tube of prescribed diameter.

FIND: Maximum velocity and pressure gradient.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Isothermal flow.

PROPERTIES: *Table A-6*, Water (300K): $\rho = 998\,kg/m^3$, $\mu = 855\times10^{-6}\,N\cdot s/m^2$.

ANALYSIS: From Eq. 8.6,

$$Re_D = \frac{4\dot{m}}{\pi D \mu} = \frac{4 \times 0.01\,kg/s}{\pi(0.025m)\,855\times10^{-6}\,kg\cdot m/s} = 596\ .$$

Hence the flow is laminar and the velocity profile is given by Eq. 8.15,

$$\frac{u(r)}{u_m} = 2[1 - (r/r_o)^2]\ .$$

The maximum velocity is therefore at $r = 0$, the centerline, where

$$u(0) = 2\,u_m\ .$$

From Eq. 8.5

$$u_m = \frac{\dot{m}}{\rho\pi D^2/4} = \frac{4 \times 0.01\,kg/s}{998\,kg/m^3 \times \pi\,(0.025m)^2} = 0.020\,m/s\ ,$$

hence

$$u(0) = 0.041\,m/s\ . \qquad \triangleleft$$

Combining Eqs. 8.16 and 8.19, the pressure gradient is

$$\frac{dp}{dx} = -\frac{64}{Re_D}\,\frac{\rho u_m^2}{2D}$$

$$\frac{dp}{dx} = -\frac{64}{596} \times \frac{998\,kg/m^3\,(0.020\,m/s)^2}{2 \times 0.025\,m} = -\,0.86\,kg/m^2\cdot s^2$$

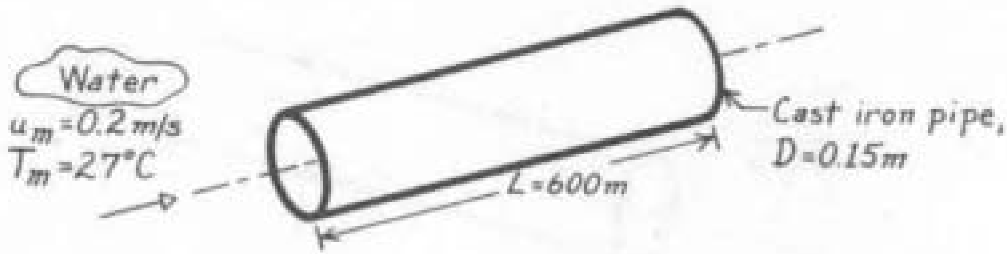
$$\frac{dp}{dx} = -\,0.86 N/m^2\cdot m = -\,0.86\times10^{-5}\,bar/m\ . \qquad \triangleleft$$

PROBLEM 8.2

KNOWN: Temperature and mean velocity of water flow through a cast iron pipe of prescribed length and diameter.

FIND: Pressure drop.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Fully developed flow, (3) Constant properties.

PROPERTIES: *Table A-6*, Water (300K): $\rho = 997\,kg/m^3$, $\mu = 855\times10^{-6}\,N{\cdot}s/m^2$.

ANALYSIS: From Eq. 8.22, the pressure drop is

$$\Delta p = f\,\frac{\rho u_m^2}{2D}\,L\,.$$

With

$$Re_D = \frac{\rho u_m D}{\mu} = \frac{997\,kg/m^3 \times 0.2\,m/s \times 0.15\,m}{855\times10^{-6}\,N{\cdot}s/m^2} = 3.50\times10^4$$

the flow is turbulent and with $e = 2.6\times10^{-4}$ m for cast iron (see Fig. 8.3), it follows that $e/D = 1.73\times10^{-3}$ and

$$f \approx 0.027\,.$$

Hence,

$$\Delta p = 0.027\,\frac{997\,kg/m^3\,(0.2\,m/s)^2}{2\times0.15\,m}\,(600m)$$

$$\Delta p = 2154\,kg/s^2{\cdot}m = 2154\,N/m^2$$

$$\Delta p = 0.0215\,bar\,.$$ ◁

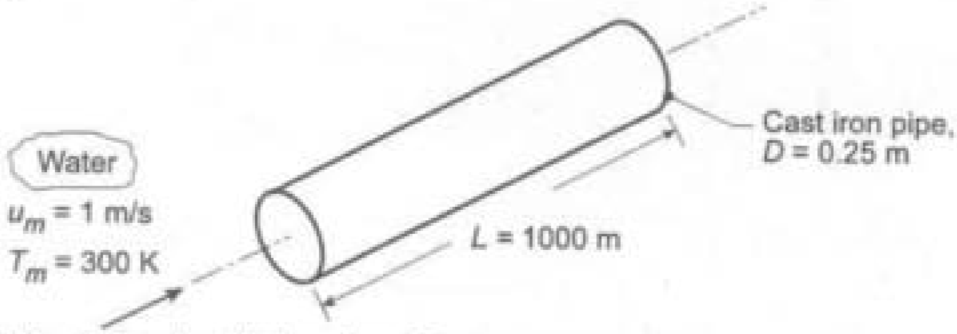
COMMENTS: For the prescribed geometry, $L/D = (600/0.15) = 4000 \gg (x_{fd,h}/D)_{turb} \approx 10$, and the assumption of fully developed flow throughout the pipe is justified.

PROBLEM 8.3

KNOWN: Temperature and velocity of water flow in a pipe of prescribed dimensions.

FIND: Pressure drop and pump power requirement for (a) a smooth pipe, (b) a cast iron pipe with a clean surface, and (c) smooth pipe for a range of mean velocities 0.05 to 1.5 m/s.

SCHEMATIC:



ASSUMPTIONS: (1) Steady, fully developed flow.

PROPERTIES: *Table A.6*, Water (300 K): $\rho = 997\ kg/m^3$, $\mu = 855 \times 10^{-6}\ N{\cdot}s/m^2$, $\nu = \mu/\rho = 8.576 \times 10^{-7}\ m^2/s$.

ANALYSIS: From Eq. 8.22a and 8.22b, the pressure drop and pump power requirement are

$$\Delta p = f\frac{\rho u_m^2}{2D}L \qquad P = \Delta p\dot{V} = \Delta p\left(\pi D^2/4\right)u_m \qquad (1,2)$$

The friction factor, f, may be determined from Figure 8.3 for different relative roughness, e/D, surfaces or from Eq. 8.21 for the smooth condition, $3000 \le Re_D \le 5 \times 10^6$,

$$f = \left(0.790\ln\left(Re_D\right) - 1.64\right)^{-2} \qquad (3)$$

where the Reynolds number is

$$Re_D = \frac{u_m D}{\nu} = \frac{1\,m/s \times 0.25\,m}{8.576 \times 10^{-7}\,m^2/s} = 2.915 \times 10^5 \qquad (4)$$

(a) *Smooth surface:* from Eqs. (3), (1) and (2),

$$f = \left(0.790\ln\left(2.915 \times 10^5\right) - 1.64\right)^{-2} = 0.01451$$

$$\Delta p = 0.01451\left(997\,kg/m^3 \times 1\,m^2/s^2/2 \times 0.25\,m\right)1000\,m = 2.89 \times 10^4\,kg/s^2 \cdot m = 0.289\,bar \quad <$$

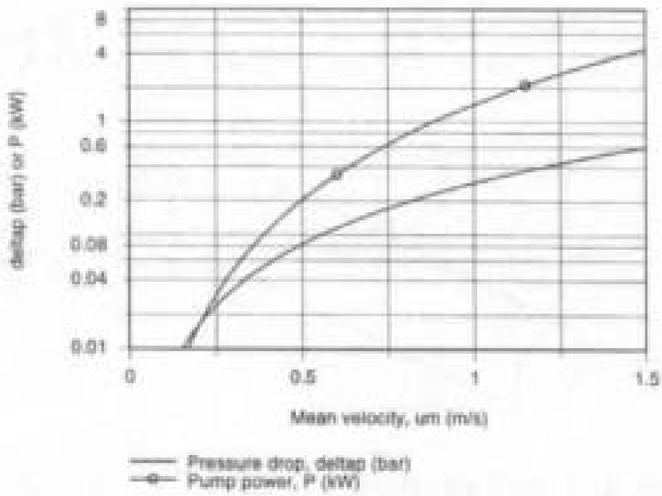
$$P = 2.89 \times 10^4\,N/m^2\left(\pi \times 0.25^2\,m^2/4\right)1\,m/s = 1418\,N \cdot m/s = 1.42\,kW \quad <$$

(b) *Cast iron clean surface:* with e = 260 μm, the relative roughness is $e/D = 260 \times 10^{-6}\ m/0.25\ m = 1.04 \times 10^{-3}$. From Figure 8.3 with $Re_D = 2.92 \times 10^5$, find f = 0.027. Hence,

$$\Delta p = 0.538\ bar \qquad P = 2.64\ kW \quad <$$

(c) *Smooth surface:* Using IHT with the expressions of part (a), the pressure drop and pump power requirement as a function of mean velocity, u_m, for the range $0.05 \le u_m \le 1.5$ m/s are computed and plotted below.

Continued...



The pressure drop is a strong function of the mean velocity. So is the pump power since it is proportional to both Δp and the mean velocity.

COMMENTS: (1) Note that $L/D = 4000 >> (x_{fd,h}/D) = 10$ for turbulent flow and the assumption of fully developed conditions is justified.

(2) Surface fouling results in increased surface roughness and increases operating costs through increasing pump power requirements.

(3) The *IHT Workspace* used to generate the graphical results follows.

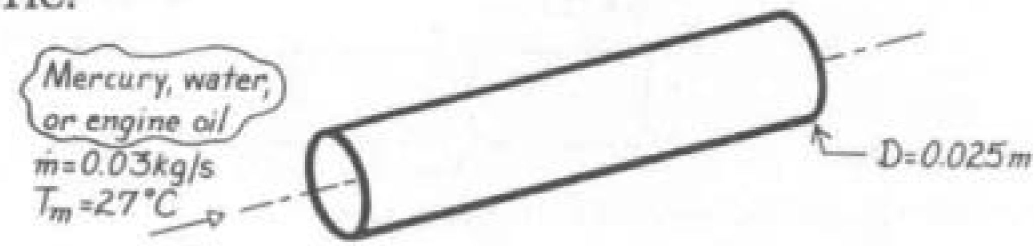
```
// Pressure drop:
deltap = f * rho * um^2 * L / ( 2 * D )          // Eq (1); Eq 8.22a
deltap_bar = deltap / 1.00e5                     // Conversion, Pa to bar units
Power = deltap * ( pi * D^2 / 4 ) * um           // Eq (2); Eq 8.22b
Power_kW = Power / 1000                          // Useful for scaling graphical result

// Reynolds number and friction factor:
ReD = um * D / nu                                // Eq (3)
f = (0.790 * ln (ReD) - 1.64 ) ^ (-2)            // Eq (4); Eq 8.21, smooth surface condition

// Properties Tool - Water:
// Water property functions :T dependence, From Table A.6
// Units: T(K), p(bars);
x = 0                               // Quality (0=sat liquid or 1=sat vapor)
rho = rho_Tx("Water",Tm,x)          // Density, kg/m^3
nu = nu_Tx("Water",Tm,x)            // Kinematic viscosity, m^2/s

// Assigned variables:
um = 1                              // Mean velocity, m/s
Tm = 300                            // Mean temperature, K
D = 0.25                            // Tube diameter, m
L = 1000                            // Tube length, m
```

PROBLEM 8.4

KNOWN: Temperature and mass flow rate of various liquids moving through a tube of prescribed diameter.

FIND: Mean velocity and hydrodynamic and thermal entry lengths.

SCHEMATIC:



ASSUMPTIONS: Constant properties.

PROPERTIES: (T = 300K)

Liquid	*Table*	$\rho(kg/m^3)$	$\mu(N\cdot s/m^2)$	$\nu(m^2/s)$	*Pr*
Engine oil	A-5	884	0.486	550×10^{-6}	6400
Mercury	A-5	13,529	0.152×10^{-2}	0.113×10^{-6}	0.0248
Water	A-6	1000	0.855×10^{-3}	0.855×10^{-6}	5.83

ANALYSIS: The mean velocity is given by

$$u_m = \frac{\dot{m}}{\rho A_c} = \frac{0.03\,kg/s}{\rho\pi(0.025m)^2/4} = \frac{61.1\,kg/s\cdot m^2}{\rho}.$$

The hydrodynamic and thermal entry lengths depend on Re_D,

$$Re_D = \frac{4\dot{m}}{\pi D\mu} = \frac{4\times0.03\,kg/s}{\pi(0.025m)\mu} = \frac{1.53\,kg/s\cdot m}{\mu}.$$

Hence, even for water ($\mu = 0.855\times10^{-3}\,N\cdot s/m^2$), $Re_D < 2300$ and the flow is laminar. From Eqs. 8.3 and 8.23 it follows that

$$x_{fd,h} = 0.05\,D\,Re_D = \frac{1.91\times10^{-3}\,kg/s}{\mu}$$

$$x_{fd,t} = 0.05\,D\,Re_D\,Pr = \frac{(1.91\times10^{-3}\,kg/s)\,Pr}{\mu}.$$

Hence:

Liquid	$u_m(m/s)$	$x_{fd,h}(m)$	$x_{fd,t}(m)$
Oil	0.069	0.0039	25.2
Mercury	0.0045	1.257	0.031
Water	0.061	2.234	13.02

◁

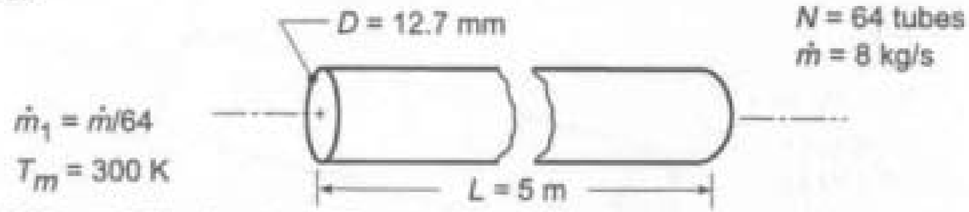
COMMENTS: Note the effect of viscosity on the hydrodynamic entry length and the effect of Pr on the thermal entry length.

PROBLEM 8.5

KNOWN: Number, diameter and length of tubes and flow rate for an engine oil cooler.

FIND: Pressure drop and pump power (a) for flow rate of 8 kg/s and (b) as a function of flow rate for the range $5 \le \dot{m} \le 60$ kg/s.

SCHEMATIC:



ASSUMPTIONS: (1) Fully developed flow throughout the tubes.

PROPERTIES: *Table A.5*, Engine oil (300 K): $\rho = 884$ kg/m^3, $\mu = 0.0486$ kg/s·m.

ANALYSIS: (a) Considering flow through a single tube, find

$$Re_D = \frac{4\dot{m}}{\pi D \mu} = \frac{4(8\,kg/s)}{64\pi(0.0127\,m)0.0486\,kg/s\cdot m} = 258 \quad (1)$$

Hence, the flow is laminar and from Equation 8.19,

$$f = \frac{64}{Re_D} = \frac{64}{258} = 0.248. \quad (2)$$

With

$$u_m = \frac{\dot{m}_1}{\rho(\pi D^2/4)} = \frac{0.125\,kg/s(4)}{(884\,kg/m^3)\pi(0.0127\,m)^2} = 1.116\,m/s \quad (3)$$

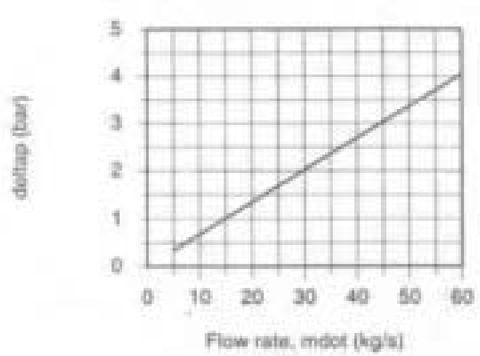
Equation 8.22a yields

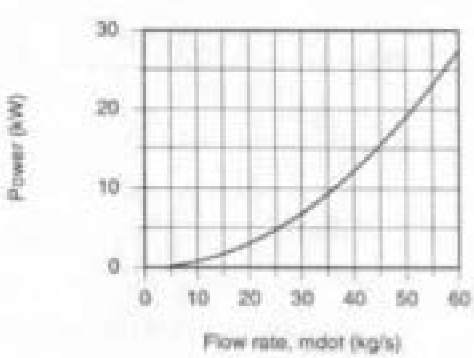
$$\Delta p = f\frac{\rho u_m^2}{2D}L = 0.248\frac{(884\,kg/m^3)(1.116\,m/s)^2}{2(0.0127\,m)}5\,m = 53{,}749\,N/m^2 = 0.537\,bar \quad (4) <$$

The pump power requirement from Equation 8.23b,

$$P = \Delta p \cdot \dot{V} = \Delta p \cdot \frac{\dot{m}}{\rho} = 53{,}749\,N/m^2\frac{8\,kg/s}{884\,kg/m^3} = 486\ N\cdot m/s = 486\ W. \quad (5) <$$

(b) Using IHT with the expressions of part (a), the pressure drop and pump power requirement as a function of flow rate, $\dot{m}$, for the range $5 \le \dot{m} \le 60$ kg/s are computed and plotted below.





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In the plot above, note that the pressure drop is linear with the flow rate since, from Eqs. (2), the friction factor is inversely dependent upon mean velocity. The pump power, however, is quadratic with the flow rate.

COMMENTS: (1) If there is a hydrodynamic entry region, the average friction factor for the entire tube length would exceed the fully developed value, thereby increasing Δp and P.

(2) The *IHT Workspace* used to generate the graphical results follows.

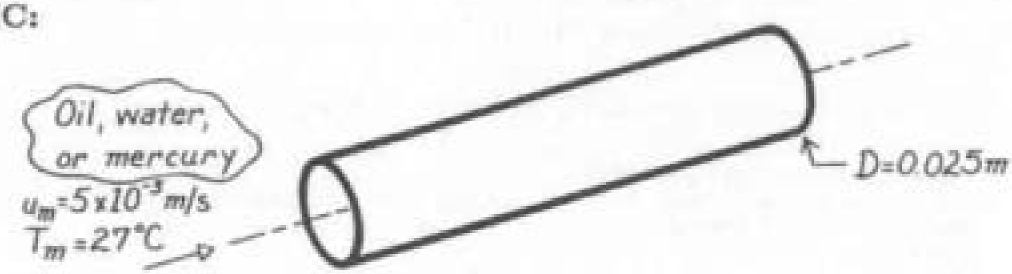
```
// Reynolds number and friction factor:
ReD = 4 * mdot1 / ( pi * D * mu )            // Eq (1); Reynolds number
mdot1= um * rho * pi * D^2 / 4               // Flow rate per tube, kg/s; Eq (3).
mdot = mdot1 * N                             // Total flow rate, N tubes
f = 64 / ReD                                 // Friction factor; laminar flow; Eq (2)

// Pressure drop and power requirement:
deltap = f * rho * um^2 / (2 * D ) * L       // Pressure drop, N/m^2; Eq (4).
deltap_bar = deltap / 1e5                    // Pressure drop, bar
Power = deltap * ( mdot / rho )              // Power, W; Eq (5)
Power_kW = Power / 1000

// Assigned variables:
mdot = 8                  // Flow rate, kg/s; N tubes
N = 64                    // Number of tubes
L = 5                     // Length, m
Tm = 300                  // Mean temperature, K
D = 0.0127                // Tube diameter, m

// Properties Tool - Engine Oil:
rho = 884                 // Density, kg/m^3
mu = 0.0486               // Viscosity, kg/s.m

/* Data Browser results: base case, Part (a)

  Power    Power_kW          ReD     deltap    deltap_bar          f        mdot1    um
  D        L         N       Tm      mdot      mu         rho
  487      0.487             257.9   5.382E4   0.5382              0.2482   0.125    1.116
  0.0127   5         64      300     8         0.0486     884  */
```

PROBLEM 8.6

KNOWN: Mean velocity and temperature of oil, water and mercury flowing through a tube of prescribed diameter.

FIND: Corresponding hydrodynamic and thermal entry lengths.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties.

PROPERTIES: ($T_m = 300K$)

Liquid	*Table*	$\rho(kg/m^3)$	$\mu(N\cdot s/m^2)$	*Pr*
Engine Oil	A-5	884	0.486	6400
Mercury	A-5	13,529	0.152×10^{-2}	0.0248
Water	A-6	997	0.855×10^{-3}	5.83

ANALYSIS: With

$$Re_D = \frac{\rho u_m D}{\mu} = \frac{\rho}{\mu} \times 5\times10^{-3} \text{m/s} \times 0.025\text{m} = 1.25\times10^{-4} \text{m}^2/\text{s} \frac{\rho}{\mu}$$

it follows that

	Oil	*Mercury*	*Water*
Re_D	0.227	1113	146

Hence for each fluid, the flow is laminar and from Eqs. 8.3 and 8.23,

$$x_{fd,h} = 0.05\, D\, Re_D \qquad\qquad x_{fd,t} = 0.5\, D\, Re_D\, Pr.$$

Hence:

Liquid	$x_{fd,h}(m)$	$x_{fd,t}(m)$
Oil	2.84×10^{-4}	1.82
Mercury	1.39	0.0345
Water	0.183	1.06

◁

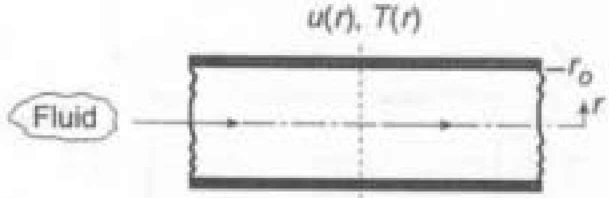
COMMENTS: Note the effect of viscosity on the hydrodynamic entry length and the effect of Prandtl number on the thermal entry length.

PROBLEM 8.7

KNOWN: Velocity and temperature profiles for laminar flow in a tube of radius $r_o = 10$ mm.

FIND: Mean (or bulk) temperature, T_m, at this axial position.

SCHEMATIC:



ASSUMPTIONS: (1) Laminar incompressible flow, (2) Constant properties.

ANALYSIS: The prescribed velocity and temperature profiles, (m/s and K, respectively) are

$$u(r) = 0.1\,[1-(r/r_o)^2] \qquad T(r) = 344.8 + 75.0\,(r/r_o)^2 - 18.8\,(r/r_o)^4 \qquad (1,2)$$

For incompressible flow with constant c_v in a circular tube, from Eq. 8.27, the mean temperature and u_m, the mean velocity, from Eq. 8.8 are, respectively,

$$T_m = \frac{2}{u_m r_o^2}\int_0^{r_o} u(r)\cdot T(r)\cdot r\cdot dr \qquad u_m = \frac{2}{r_o^2}\int_0^{r_o} u(r)\cdot r\cdot dr \qquad (3,4)$$

Substituting the velocity profile, Eq. (1), into Eq. (4) and integrating, find

$$u_m = \frac{2}{r_o^2} r_o^2 \int_0^1 0.1\left[1-(r/r_o)^2\right](r/r_o)d(r/r_o) = 2\left\{0.1\left[\frac{1}{2}(r/r_o)^2 - \frac{1}{4}(r/r_o)^4\right]\right\}_0^1 = 0.05\,\text{m/s}$$

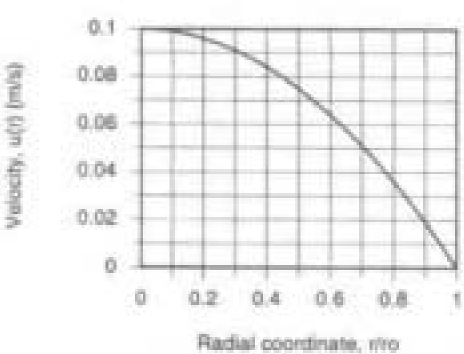
Substituting the profiles and u_m into Eq. (3), find

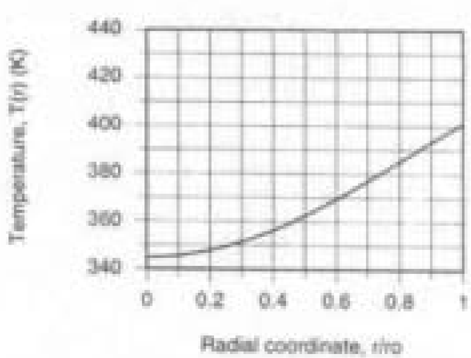
$$T_m = \frac{2}{(0.05\,\text{m/s})r_o^2} r_o^2 \int_0^1 \left\{0.1\left[1-(r/r_o)^2\right]\right\}\left\{344.8 + 75.0(r/r_o)^2 - 18.8(r/r_o)^4\right\}\cdot(r/r_o)\cdot d(r/r_o)$$

$$T_m = 4\int_0^1 \left\{\left[344.8(r/r_o) + 75.0(r/r_o)^3 - 18.8(r/r_o)^5\right] - \left[344.8(r/r_o)^3 + 75.0(r/r_o)^5 - 18.8(r/r_o)^7\right]\right\}d(r/r_o)$$

$$T_m = 4\{[172.40 + 18.75 - 3.13] - [86.20 + 15.00 - 2.35]\} = 356\,\text{K}$$ <

The velocity and temperature profiles appear as shown below. Do the values of u_m and T_m found above compare with their respective profiles as you thought? Is the fluid being heated or cooled?



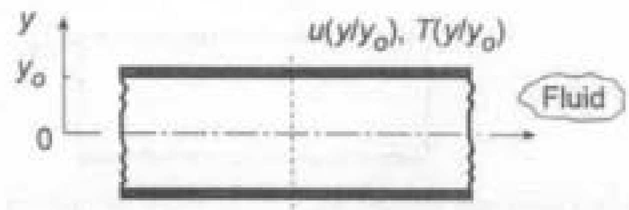


PROBLEM 8.8

KNOWN: Velocity and temperature profiles for laminar flow in a parallel plate channel.

FIND: Mean velocity, u_m, and mean (or bulk) temperature, T_m, at this axial position. Plot the velocity and temperature distributions. Comment on whether values of u_m and T_m appear reasonable.

SCHEMATIC:



ASSUMPTIONS: (1) Laminar incompressible flow, (2) Constant properties.

ANALYSIS: The prescribed velocity and temperature profiles (m/s and °C, respectively) are

$$u(y) = 0.75\left[1-(y/y_o)^2\right] \qquad T(y) = 5.0 + 95.66(y/y_o)^2 - 47.83(y/y_o)^4 \qquad (1,2)$$

The mean velocity, u_m, follows from its definition, Eq. 8.7,

$$\dot{m} = \rho A_c u_m = \rho \int_{A_c} u(y)\cdot dA_c$$

where the flow cross-sectional area is $dA_c = 1\cdot dy$, and $A_c = 2y_o$,

$$u_m = \frac{1}{A_c}\int_{A_c} u(y)\cdot dy = \frac{1}{2y_o}\int_{-y_o}^{+y} u(y)dy \qquad (3)$$

$$u_m = \frac{1}{2y_o}\cdot y_o\int_{-1}^{+1} 0.75\left[1-(y/y_o)^2\right]d(y/y_o)$$

$$u_m = 1/2\left\{0.75\left[(y/y_o) - 1/3(y/y_o)^3\right]\right\}_{-1}^{+1}$$

$$u_m = 1/2\times 0.75\{[1-1/3]-[-1+1/3]\} = 1/2\times 0.75\times 4/3 = 2/3\times 0.75 = 0.50\text{ m/s} \qquad <$$

The mean temperature, T_m, follows from its definition, Eq. 8.25,

$$\dot{E}_t = \dot{m}c_v T_m \qquad \text{where} \qquad \dot{m} = \rho A_c u_m$$

$$\rho A_c u_m c_v T_m = \rho c_v \int_{A_c} u(y)\cdot T(y)dA_c$$

Hence, substituting velocity and temperature profiles,

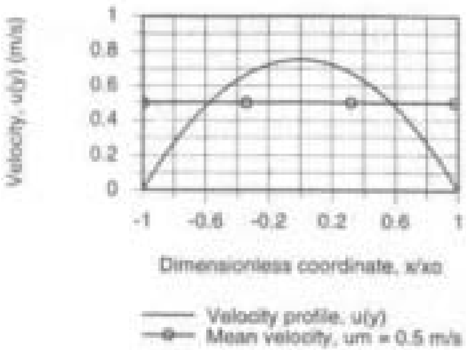
$$T_m = \frac{1}{u_m A_c}\int_{-y_o}^{+y_o} u(y)\cdot T(y)dy \qquad (4)$$

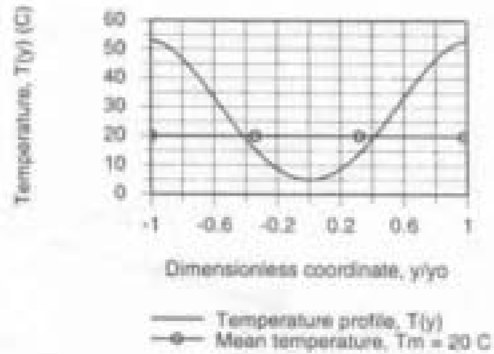
$$T_m = \frac{1}{(0.5\text{ m/s})2y_o}y_o\int_{-1}^{+1}\left\{0.75\left[1-(y/y_o)^2\right]\right\}\left\{5.0+95.66(y/y_o)^2-47.83(y/y_o)^4\right\}d(y/y_o)$$

$$T_m = \frac{0.75}{0.5\times 2}\left\{\left[5(y/y_o)+31.89(y/y_o)^3-9.57(y/y_o)^5\right]-\left[1.67(y/y_o)^3+19.13(y/y_o)^5-6.83(y/y_o)^7\right]\right\}_{-1}^{+1}$$

$$T_m = \frac{0.75}{0.5\times 2}\{[27.32-13.97]-[-27.32-(-13.97)]\} = 20.0^\circ\text{C} \qquad <$$

The velocity and temperature profiles along with the u_m and T_m values are plotted below.





For the velocity profile, the mean velocity is 2/3 that of the centerline velocity, $u_m = 2u(0)/3$. Note that the areas above and below the u_m line appear to be equal. Considering the temperature profile, we'd expect the mean temperature to be closer to the centerline temperature since the velocity profile weights the integral toward the centerline.

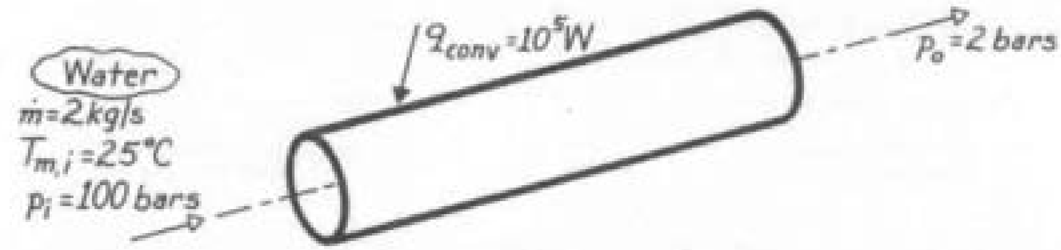
COMMENTS: The integrations required to obtain u_m and T_m, Eqs. (3) and (4), could also be performed using the intrinsic function *INTEGRAL (y,x)* in the *IHT Workspace*.

PROBLEM 8.9

KNOWN: Flow rate, inlet temperature and pressure, and outlet pressure of water flowing through a pipe with a prescribed surface heat rate.

FIND: (a) Outlet temperature, (b) Outlet temperature assuming negligible flow work changes.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible kinetic and potential energy changes, (2) Constant properties, (3) Incompressible liquid.

PROPERTIES: *Table A-6*, Water (T = 300K): $\rho = 997\ kg/m^3$, $c_p = c_v = 4179\ J/kg\cdot K$.

ANALYSIS: (a) Accounting for the flow work effect, Eq. 8.35 may be integrated from inlet to outlet to obtain

$$q_{conv} = \dot{m}\left[c_v(T_{m,o} - T_{m,i}) + (pv)_o - (pv)_i\right]$$

Hence,

$$T_{m,o} = T_{m,i} + \frac{q_{conv}}{\dot{m}c_v} + \frac{1}{\rho c_v}(p_i - p_o)$$

$$T_{m,o} = 25°C + \frac{10^5\ W}{2\ kg/s \times 4179\ J/kg\cdot K} + \frac{(100-2)\ bar(10^5\ N/m^2)/bar}{997\ kg/m^3 \times 4179\ J/kg\cdot K}$$

$$T_{m,o} = 25°C + 12°C + 2.4°C$$

$$T_{m,o} = 39.4°C. \qquad \triangleleft$$

(b) Neglecting the flow work effect, it follows from Eq. 8.37 that,

$$T_{m,o} = T_{m,i} + \frac{q_{conv}}{\dot{m}c_p} = 25°C + 12°C$$

$$T_{m,o} = 37°C. \qquad \triangleleft$$

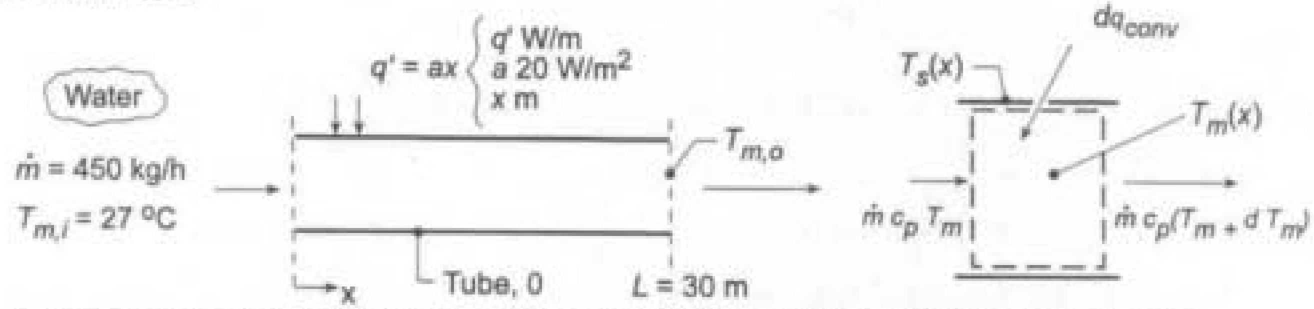
COMMENTS: Even for the large pressure drop of this problem, flow work effects make a small contribution to heating the water. The effects may justifiably be neglected in most practical problems.

PROBLEM 8.10

KNOWN: Internal flow with prescribed wall heat flux as a function of distance.

FIND: (a) Beginning with a properly defined differential control volume, the temperature distribution, $T_m(x)$, (b) Outlet temperature, $T_{m,o}$, (c) Sketch $T_m(x)$, and $T_s(x)$ for fully developed *and* developing flow conditions, and (d) Value of uniform wall flux q_s'' (instead of $q_s' = ax$) providing same outlet temperature as found in part (a); sketch $T_m(x)$ and $T_s(x)$ for this heating condition.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Incompressible flow.

PROPERTIES: *Table A.6*, Water (300 K): $c_p = 4.179$ kJ/kg·K.

ANALYSIS: (a) Applying energy conservation to the control volume above,

$$dq_{conv} = \dot{m}c_p dT_m \tag{1}$$

where $T_m(x)$ is the mean temperature at any cross-section and $dq_{conv} = q' \cdot dx$. Hence,

$$ax = \dot{m}c_p \frac{dT_m}{dx}. \tag{2}$$

Separating and integrating with proper limits gives

$$a\int_{x=0}^{x} x dx = \dot{m}c_p \int_{T_{m,i}}^{T_m(x)} dT_m \qquad\qquad T_m(x) = T_{m,i} + \frac{ax^2}{2\dot{m}c_p} \tag{3,4}$$ <

(b) To find the outlet temperature, let x = L, then

$$T_m(L) = T_{m,o} = T_{m,i} + aL^2/2\dot{m}c_p. \tag{5}$$

Solving for $T_{m,o}$ and substituting numerical values, find

$$T_{m,o} = 27^\circ C + \frac{20\,W/m^2\,(30\,m^2)}{2(450\,kg/h/(3600\,s/h)) \times 4179\,J/kg\cdot K} = 27^\circ C + 17.2^\circ C = 44.2^\circ C.$$ <

(c) For *linear wall heating*, $q_s' = ax$, the fluid temperature distribution along the length of the tube is quadratic as prescribed by Eq. (4). From the convection rate equation,

$$q_s' = h(x) \cdot \pi D(T_s(x) - T_m(x)) \tag{6}$$

For fully developed flow conditions, h(x) = h, a constant; hence, $T_s(x) - T_m(x)$ as a function of distance will be constant. For developing conditions, h(x) will decrease with increasing distance along the tube eventually achieving the fully developed value.

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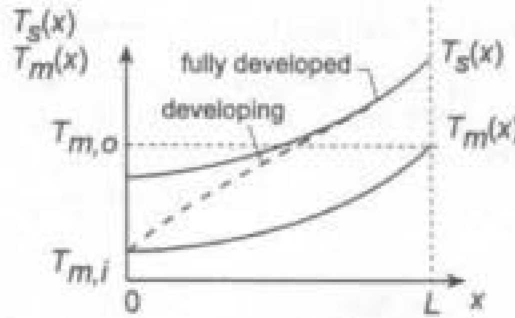


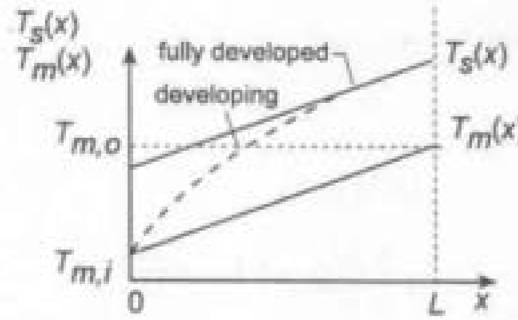
(d) For *uniform wall heat flux heating*, the overall energy balance on the tube yields

$$q = q_s''\pi DL = \dot{m}c_p\left(T_{m,o} - T_{m,i}\right)$$

Requiring that $T_{m,o} = 44.2°C$ from part (a), find

$$q_s'' = \frac{(450/3600)\,kg/s \times 4179\,J/kg\cdot K(44.2-27)K}{\pi D \times 5\,m} = 572/D\,W/m^2$$ <

where D is the diameter (m) of the tube which, when specified, would permit determining the required heat flux, q_s''. For uniform heating, Section 3.3.2, we know that $T_m(x)$ will be linear with distance. $T_s(x)$ will also be linear for fully developed conditions and appear as shown below when the flow is developing.



COMMENTS: (1) Note that c_p should be evaluated at $T_m = (27 + 44)°C/2 = 309$ K.

(2) Why did we show $T_s(0) = T_m(0)$ for both types of history when the flow was developing?

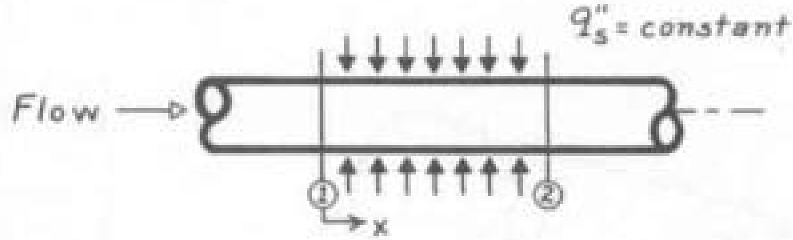
(3) Why must $T_m(x)$ be linear with distance in the case of uniform wall flux heating?

PROBLEM 8.11

KNOWN: Internal flow with constant surface heat flux, q''_s.

FIND: (a) Qualitative temperature distributions, T(x), under developing and fully-developed flow, (b) Exit mean temperature for both situations.

SCHEMATIC:



ASSUMPTIONS: (a) Steady-state conditions, (b) Constant properties, (c) Incompressible flow.

ANALYSIS: Based upon the analysis leading to Eq. 8.40, note for the case of constant surface heat flux conditions,

$$\frac{dT_m}{dx} = \text{constant.}$$

Hence, regardless of whether the hydrodynamic or thermal boundary layer is fully developed, it follows that

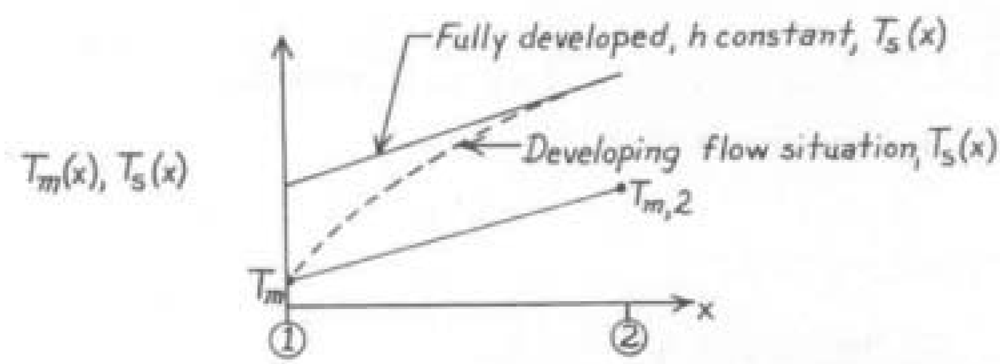
$T_m(x)$ is linear and

$T_{m,2}$ will be the same for all flow conditions. ◁

The surface heat flux can also be written, using Eq. 8.28, as

$$q''_s = h\,[T_s(x) - T_m(x)]\ .$$

Under fully-developed flow and thermal conditions, $h = h_{fd}$ is a constant. When flow is developing $h > h_{fd}$. Hence, the temperature distributions appear as below.



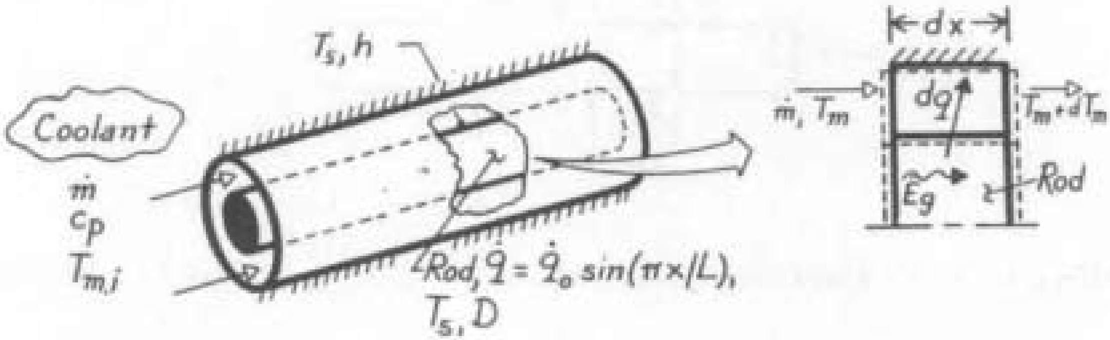
◁

PROBLEM 8.12

KNOWN: Geometry and coolant flow conditions associated with a nuclear fuel rod. Axial variation of heat generation within the rod.

FIND: (a) Axial variation of local heat flux and total heat transfer rate, (b) Axial variation of mean coolant temperature, (c) Axial variation of rod surface temperature and location of maximum temperature.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant fluid properties, (3) Uniform surface convection coefficient, (4) Negligible axial conduction in rod and fluid, (5) Negligible kinetic energy, potential energy and flow work changes, (6) Outer surface is adiabatic.

ANALYSIS: (a) Performing an energy balance for a control volume about the rod,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = 0 \qquad\qquad -dq + \dot{E}_g = 0$$

or

$$-q''(\pi D\, dx) + \dot{q}_o \sin(\pi x/L)(\pi D^2/4)\, dx \qquad\qquad q'' = \dot{q}_o (D/4)\sin(\pi x/L). \qquad \triangleleft$$

The total heat transfer rate is then

$$q = \int_0^L q''\, \pi D\, dx = (\pi D^2/4)\dot{q}_o \int_0^L \sin(\pi x/L)\, dx$$

$$q = \frac{\pi D^2}{4}\dot{q}_o \left(-\frac{L}{\pi}\cos\frac{\pi x}{L}\right)\Big|_0^L = \frac{D^2 \dot{q}_o L}{4}(1+1)$$

$$q = \frac{D^2 L}{2}\dot{q}_o . \qquad (1) \triangleleft$$

(b) Performing an energy balance for a control volume about the coolant,

$$\dot{m} c_p T_m + dq - \dot{m} c_p (T_m + dT_m) = 0 .$$

Hence

$$\dot{m} c_p\, dT_m = dq = (\pi D\, dx)\, q''$$

$$\frac{dT_m}{dx} = \frac{\pi D}{\dot{m} c_p}\frac{\dot{q}_o D}{4}\sin\left(\frac{\pi x}{L}\right).$$

Continued .....

Integrating,

$$T_m(x) - T_{m,i} = \frac{\pi D^2}{4} \frac{\dot{q}_o}{\dot{m} c_p} \int_0^x \sin \frac{\pi x}{L} dx$$

$$T_m(x) = T_{m,i} + \frac{L D^2}{4} \frac{\dot{q}_o}{\dot{m} c_p} \left[1 - \cos \frac{\pi x}{L}\right] \qquad (2) \triangleleft$$

(c) From Newton's law of cooling,

$$q'' = h(T_s - T_m) .$$

Hence

$$T_s = \frac{q''}{h} + T_m$$

$$T_s = \frac{\dot{q}_o D}{4h} \sin \frac{\pi x}{L} + T_{m,i} + \frac{LD^2}{4} \frac{\dot{q}_o}{\dot{m} c_p} \left[1 - \cos \frac{\pi x}{L}\right] . \qquad \triangleleft$$

To determine the location of the maximum surface temperature, evaluate

$$\frac{d T_s}{dx} = 0 = \frac{\dot{q}_o D\pi}{4hL} \cos \frac{\pi x}{L} + \frac{LD^2}{4} \frac{\dot{q}_o}{\dot{m} c_p} \frac{\pi}{L} \sin \frac{\pi x}{L}$$

or

$$\frac{1}{hL} \cos \frac{\pi x}{L} + \frac{D}{\dot{m} c_p} \sin \frac{\pi x}{L} = 0 .$$

Hence

$$\tan \frac{\pi x}{L} = -\frac{\dot{m} c_p}{D h L}$$

$$x = \frac{L}{\pi} \tan^{-1} \left(-\frac{\dot{m} c_p}{D h L} \right) = x_{max} . \qquad \triangleleft$$

COMMENTS: Note from Eq. (2) that

$$T_{m,o} = T_m(L) = T_{m,i} + \frac{L D^2 \dot{q}_o}{2 \dot{m} c_p}$$

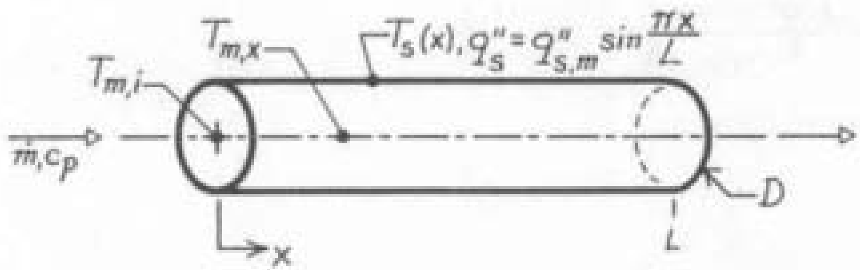
which is equivalent to the result obtained by combining Eq. (1) and Eq. 8.37.

PROBLEM 8.13

KNOWN: Axial variation of surface heat flux for flow through a tube.

FIND: Axial variation of fluid and surface temperatures.

SCHEMATIC:



ASSUMPTIONS: (1) Convection coefficient is independent of x, (2) Negligible axial conduction and kinetic and potential energy changes, (3) Fluid is an ideal gas or a liquid for which $d(pv) << (c_v T_m)$.

ANALYSIS: Since Equation 8.38 is applicable,

$$\frac{dT_m}{dx} = \frac{q_s'' P}{\dot{m} c_p} = \frac{(\pi D) q_{s,m}'' \sin(\pi x/L)}{\dot{m} c_p}.$$

Separating variables and integrating from x = 0

$$\int_{T_{m,i}}^{T_{m,x}} dT_m = \frac{\pi D q_{s,m}''}{\dot{m} c_p} \int_0^x \sin\frac{\pi x}{L} dx$$

$$T_m(x) - T_{m,i} = -\frac{L D q_{s,m}''}{\dot{m} c_p} \cos\frac{\pi x}{L} \Big|_0^x$$

$$T_m(x) = T_{m,i} + \frac{L D q_{s,m}''}{\dot{m} c_p} (1 - \cos \pi x/L). \qquad \triangleleft$$

From Newton's law of cooling, Equation 8.28,

$$T_s(x) = (q_s''/h) + T_m(x)$$

$$T_s(x) = \frac{q_{s,m}''}{h} \sin\frac{\pi x}{L} + T_{m,i} + \frac{L D q_{s,m}''}{\dot{m} c_p} (1 - \cos \pi x/L). \qquad \triangleleft$$

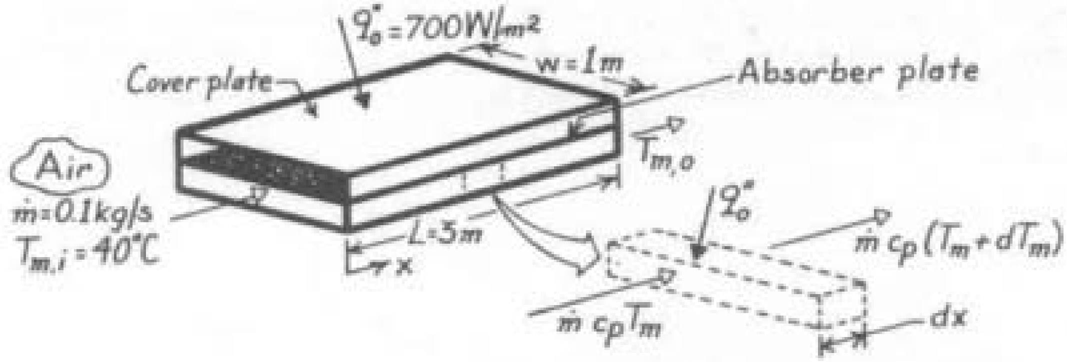
COMMENTS: For the prescribed surface condition, the flow is not fully developed. Hence, the assumption of constant h should be viewed as a first approximation.

PROBLEM 8.14

KNOWN: Surface heat flux for air flow through a rectangular channel.

FIND: (a) Differential equation describing variation in air mean temperature, (b) Air outlet temperature for prescribed conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible change in kinetic and potential energy of air, (2) No heat loss through bottom of channel, (3) Uniform heat flux at top of channel.

PROPERTIES: *Table A-4*, Air (T = 50°C, 1 atm): $c_p = 1008\ \text{J/kg·K}$.

ANALYSIS: (a) For the differential control volume about the air,

$$\dot{E}_{in} = \dot{E}_{out}$$

$$\dot{m} c_p T_m + q''_o (w \cdot dx) = \dot{m} c_p (T_m + d T_m)$$

$$\frac{d T_m}{dx} = \frac{q''_o \cdot w}{\dot{m} c_p}$$

Separating and integrating between the limits of x = 0 and x, find

$$T_m(x) = T_{m,i} + \frac{q''_o (w \cdot x)}{\dot{m} c_p}$$

$$T_{m,o} = T_{m,i} + \frac{q''_o (w \cdot L)}{\dot{m} c_p}. \qquad \triangleleft$$

(b) Substituting numerical values, the air outlet temperature is

$$T_{m,o} = 40°\text{C} + \frac{(700\ \text{W/m}^2)(1 \times 3)\text{m}^2}{0.1\ \text{kg/s}\ (1008\ \text{J/kg·K})}$$

$$T_{m,o} = 60.8°\text{C}. \qquad \triangleleft$$

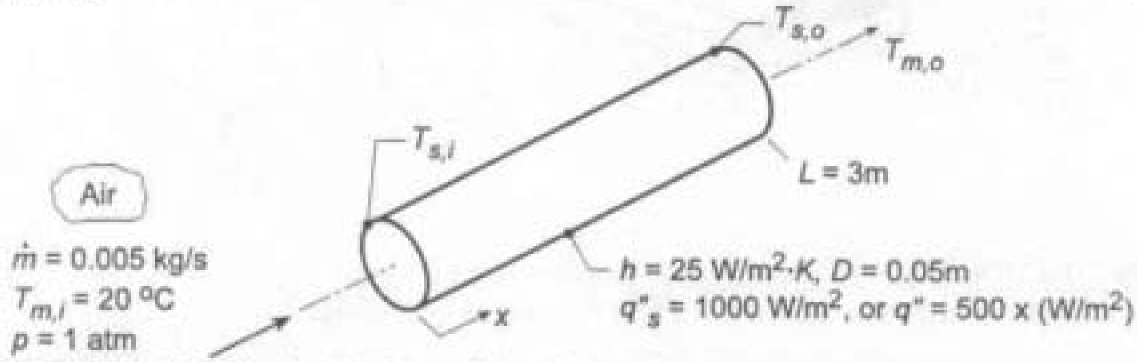
COMMENTS: Due to increasing heat loss with increasing T_m, the net flux q''_o will actually decrease slightly with increasing x.

PROBLEM 8.15

KNOWN: Air inlet conditions and heat transfer coefficient for a circular tube of prescribed geometry. Surface heat flux.

FIND: (a) Tube heat transfer rate, q, air outlet temperature, $T_{m,o}$, and surface inlet and outlet temperatures, $T_{s,i}$ and $T_{s,o}$, for a uniform surface heat flux, q_s''. Air mean and surface temperature distributions. (b) Values of q, $T_{m,o}$, $T_{s,i}$ and $T_{s,o}$ for a linearly varying surface heat flux $q_s'' = 500x$ (m). Air mean and surface temperature distributions, (c) For each type of heating process (a & b), compute and plot the mean fluid and surface temperatures, $T_m(x)$ and $T_s(x)$, respectively, as a function of distance; What is effect of four-fold increase in convection coefficient, and (d) For each type of heating process, heat fluxes required to achieve an outlet temperature of $T_{m,o} = 125°C$; Plot temperatures.

SCHEMATIC:



ASSUMPTIONS: (1) Fully developed conditions in the tube, (2) Applicability of Eq. 8.36, (3) Heat transfer coefficient is the same for both heating conditions.

PROPERTIES: *Table A.4*, Air (for an *assumed* value of $T_{m,o} = 100°C$, $\overline{T}_m = (T_{m,i} + T_{m,o})/2 = 60°C = 333$ K): $c_p = 1.008$ kJ/kg·K.

ANALYSIS: (a) With constant heat flux, from Eq. 8.39,

$$q = q_s''(\pi DL) = 1000\,\text{W/m}^2(\pi \times 0.05\,\text{m} \times 3\,\text{m}) = 471\,\text{W}. \quad (1)$$

From the overall energy balance, Eq. 8.37,

$$T_{m,o} = T_{m,i} + \frac{q}{\dot{m}c_p} = 20°\text{C} + \frac{471\,\text{W}}{0.005\,\text{kg/s} \times 1008\,\text{J/kg}\cdot\text{K}} = 113.5°\text{C} \quad (2) <$$

From the convection rate equation, it follows that

$$T_{s,i} = T_{m,i} + \frac{q_s''}{h} = 20°\text{C} + \frac{1000\,\text{W/m}^2}{25\,\text{W/m}^2\cdot\text{K}} = 60°\text{C} \quad (3) <$$

$$T_{s,o} = T_{m,o} + q_s''/h = 113.5°\text{C} + 40°\text{C} = 153.5°\text{C} \quad <$$

From Eq. 8.40, (dT_m/dx) is a constant, as is (dT_s/dx) for constant h from Eq. 8.31. In the more realistic case for which h decreases with x in the entry region, (dT_m/dx) is still constant but (dT_s/dx) decreases with increasing x. See the plot below.

(b) From Eq. 8.38,

$$\frac{dT_m}{dx} = \frac{500x(\pi D)}{\dot{m}c_p} = \frac{500x\,\text{W/m}^2(\pi \times 0.05\,\text{m})}{0.005\,\text{kg/s} \times 1008\,\text{J/kg}\cdot\text{K}} = 15.6x\,\text{K/m}. \quad (4)$$

Continued...

Integrating from x = 0 to L it follows that

$$T_{m,o} = T_{m,i} + 15.6\int_0^3 x dx = 20^\circ C + 15.6\frac{x^2}{2}\bigg|_0^3 = 20^\circ C + 70.2^\circ C = 90.2^\circ C. \qquad (5) <$$

The heat rate is

$$q = \int q_s'' dA_s = 500(\pi \times 0.05\,m)\int_0^3 x dx = 78.5\frac{x^2}{2}\bigg|_0^3 = 353\,W \qquad <$$

From Eq. 8.28 it then follows that

$$T_s = T_m + q_s''/h = T_{m,i} + 15.6\frac{x^2}{2} + \frac{500}{25}x = 20^\circ C + 7.8x^2 + 20x \qquad (6)$$

Hence, at the inlet (x = 0) and outlet (x = L),

$$T_{s,i} = T_{m,i} = 20^\circ C \qquad \text{and} \qquad T_{s,o} = 150.2^\circ C \qquad <$$

Note that (dT_s/dx) and (dT_m/dx) both increase linearly with x, but $(dT_s/dx) > (dT_m/dx)$.

(c) The foregoing relations can be used to determine $T_m(x)$ and $T_s(x)$ for the two heating conditions:

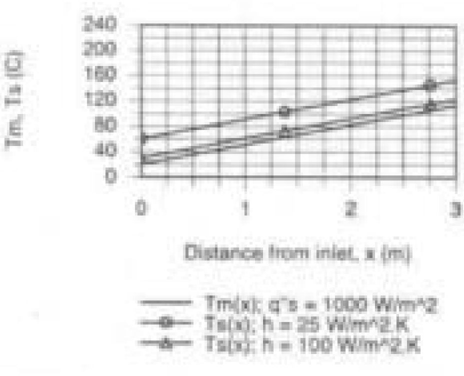
Uniform surface flux, q_s''; Eqs. (1-3),

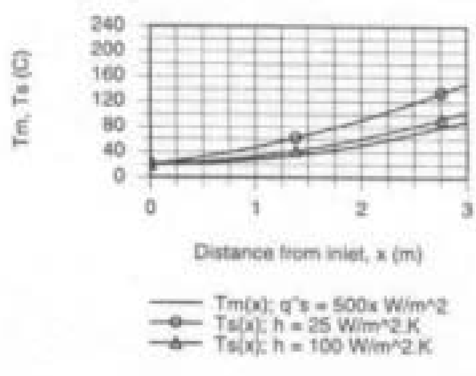
$$T_m(x) = T_{m,i} + q_s''\pi Dx/\dot{m}c_p \qquad T_s(x) = T_m(x) + q_s''/h \qquad (7,8)$$

Linear surface heat flux, $q_s'' = a_o x$, $a_o = 500\ W/m^3$; Eqs. (4-6),

$$T_m(x) = T_{m,i} + \left(a_o\pi D/2\dot{m}c_p\right)x^2 \qquad T_s(x) = T_m(x) + a_o x/h \qquad (9, 10)$$

Using Eqs. (7-10) in IHT, the mean fluid and surface temperatures as a function of distance are evaluated and plotted below. The calculations were repeated with the coefficient increased four-fold, h = 4 × 25 = 100 W/m²·K. As expected, the fluid temperature remained unchanged, but the surface temperatures decreased since the thermal resistance between the surface and fluid decreased.





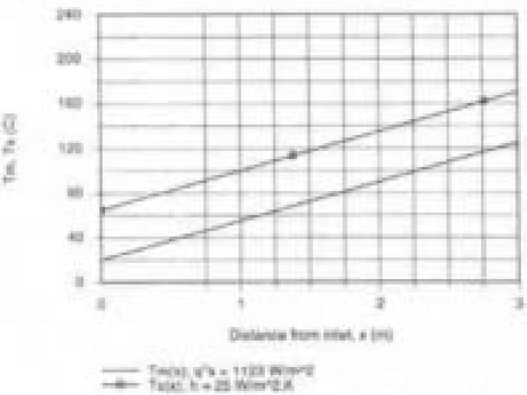
(d) The foregoing set of equations, Eqs. (7-10), in the IHT model can be used to determine the required heat fluxes for the two heating conditions to achieve $T_{m,o} = 125^\circ C$. The results with h = 25 W/m²·K are:

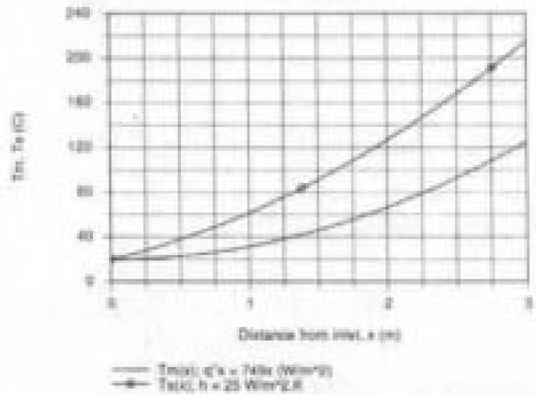
Uniform flux: $q_s'' = 1123\ W/m^2$ *Linear flux:* $q_s'' = 748.7x\ W/m^2$ <

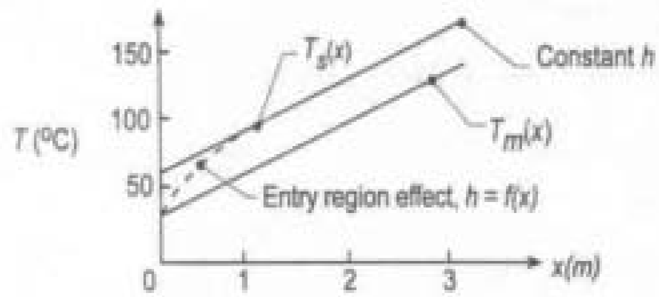
Continued...

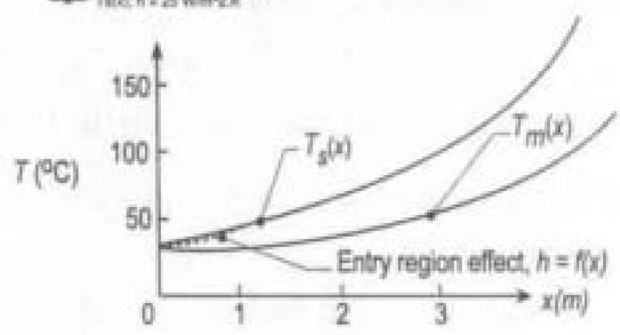
PROBLEM 8.15 (Cont.)

The temperature distributions resulting from these heat fluxes are plotted below. The heat rate for both heating processes is 529 W.









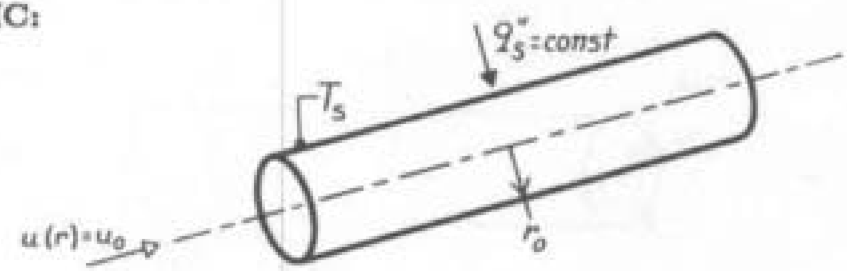
COMMENTS: Note that the assumed value for $T_{m,o}$ (100°C) in determining the specific heat of the air was reasonable.

PROBLEM 8.16

KNOWN: Laminar, slug flow in a circular tube with uniform surface heat flux.

FIND: Temperature distribution and Nusselt number.

SCHEMATIC:



ASSUMPTIONS: (1) Steady, incompressible flow, (2) Constant properties, (3) Fully developed, laminar flow, (4) Uniform surface heat flux.

ANALYSIS: With v = 0 for fully developed flow and $\partial T/\partial x = dT_m/dx$ = const. from Eqs. 8.33 and 8.40, the energy equation, Eq. 8.48, reduces to

$$u_o \frac{dT_m}{dx} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right).$$

Integrating twice, it follows that

$$T(r) = \frac{u_o}{\alpha}\frac{dT_m}{dx}\frac{r^2}{4} + C_1 \ell n(r) + C_2 .$$

Since T(0) must remain finite, $C_1 = 0$. Hence, with $T(r_o) = T_s$

$$C_2 = T_s - \frac{u_o}{\alpha}\frac{dT_m}{dx}\frac{r_o^2}{4} \qquad\qquad T(r) = T_s - \frac{u_o}{4\alpha}\frac{dT_m}{dx}\left(r_o^2 - r^2\right). \qquad \triangleleft$$

From Eq. 8.27, with $u_m = u_o$,

$$T_m = \frac{2}{r_o^2}\int_0^{r_o} Tr\,dr = \frac{2}{r_o^2}\int_0^{r_o}\left[T_s r - \frac{u_o}{4\alpha}\frac{dT_m}{dx}\left(rr_o^2 - r^3\right)\right]dr$$

$$T_m = \frac{2}{r_o^2}\left[T_s\frac{r_o^2}{2} - \frac{u_o}{4\alpha}\frac{dT_m}{dx}\left(\frac{r_o^4}{2} - \frac{r_o^4}{4}\right)\right] = T_s - \frac{u_o r_o^2}{8\alpha}\frac{dT_m}{dx}.$$

From Eq. 8.28 and Fourier's law,

$$h = \frac{q_s''}{T_s - T_m} = \frac{k\left.\frac{\partial T}{\partial r}\right|_{r_o}}{T_s - T_m}$$

Hence,

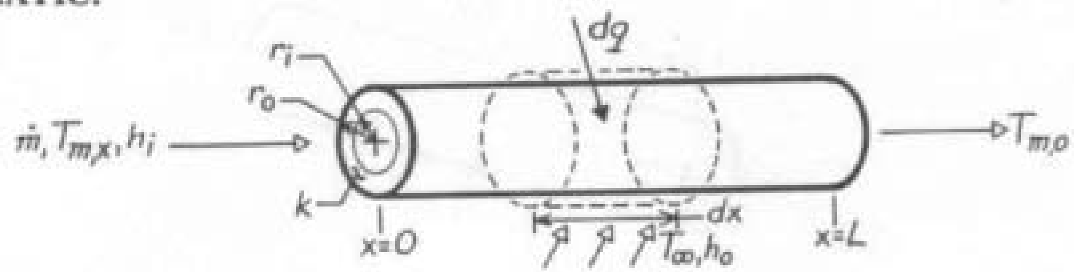
$$h = \frac{k\left(\frac{u_o r_o}{2\alpha}\right)\frac{dT_m}{dx}}{\frac{u_o r_o^2}{8\alpha}\frac{dT_m}{dx}} = \frac{4k}{r_o} = \frac{8k}{D} \qquad\qquad Nu_D = \frac{hD}{k} = 8 . \qquad \triangleleft$$

PROBLEM 8.17

KNOWN: Heat transfer between fluid flow over a tube and flow through the tube.

FIND: Axial variation of mean temperature for inner flow.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible change in kinetic and potential energy, (2) Negligible axial conduction, (3) Constant c_p, (4) Uniform T_∞.

ANALYSIS: From Equation 8.36,

$$dq = \dot{m}c_p dT_m$$

with

$$dq = U dA(T_\infty - T_m) = UP(T_\infty - T_m)dx.$$

The overall heat transfer coefficient may be defined in terms of the inner or outer surface area, with

$$U_i P_i = U_o P_o.$$

For the inner surface, from Equation 3.31,

$$U_i = \left[\frac{1}{h_i} + \frac{r_i}{k}\ln\frac{r_o}{r_i} + \frac{r_i}{r_o}\frac{1}{h_o}\right]^{-1}.$$

Hence,

$$\frac{dT_m}{T_\infty - T_m} = +\frac{UP}{\dot{m}c_p}dx$$

or, with $\Delta T \equiv T_\infty - T_m$,

$$\int_{\Delta T_i}^{\Delta T_o}\frac{d(\Delta T)}{\Delta T} = -\frac{P}{\dot{m}c_p}\int_0^L U dx.$$

Hence,

$$\ln\frac{\Delta T_o}{\Delta T_i} = -\frac{PL}{\dot{m}c_p}\left(\frac{1}{L}\int_0^L U dx\right)$$

$$\frac{T_\infty - T_{m,o}}{T_\infty - T_{m,i}} = \exp\left(-\frac{PL}{\dot{m}c_p}\overline{U}\right). \quad \triangleleft$$

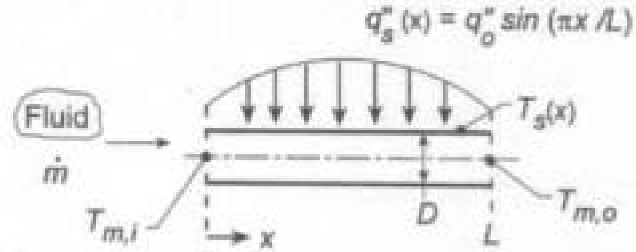
COMMENTS: The development and results parallel those for a constant surface temperature, with $\overline{U}$ and T_∞ replacing $\overline{h}$ and T_s.

PROBLEM 8.18

KNOWN: Thin-walled tube experiences sinusoidal heat flux distribution on the wall.

FIND: (a) Total rate of heat transfer from the tube to the fluid, q, (b) Fluid outlet temperature, $T_{m,o}$, (c) Axial distribution of the wall temperature $T_s(x)$ and (d) Magnitude and position of the highest wall temperature, and (e) For prescribed conditions, calculate and plot the mean fluid and surface temperatures, $T_m(x)$ and $T_s(x)$, respectively, as a function of distance along the tube; identify features of the distributions; explore the effect of ±25% changes in the convection coefficient on the distributions.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Negligible kinetic and potential energy changes, (3) Turbulent, fully developed flow.

ANALYSIS: (a) The total rate heat transfer from the tube to the fluid is

$$q = \int_0^L q_s'' P dx = q_o'' \pi D \int_0^D \sin(\pi x/L) dx = q_o'' \pi D (L/\pi)\left[-\cos(\pi x/L)\right]_0^L = 2DLq_o'' \qquad (1) <$$

(b) The fluid outlet temperature follows from the overall energy balance with knowledge of the total heat rate,

$$q = \dot{m} c_p \left(T_{m,o} - T_{m,i}\right) = 2DLq_o'' \qquad T_{m,o} = T_{m,i} + \left(2DLq_o''/\dot{m} c_p\right) \qquad (2) <$$

(c) The axial distribution of the wall temperature can be determined from the rate equation

$$q_s'' = h\left[T_s(x) - T_m(x)\right] \qquad T_{s,x} = T_{m,x}(x) + q_s''/h \qquad (3)$$

where, by combining expressions of parts (a) and (b), $T_{m,x}(x)$ is

$$\int_0^x q_s'' P dx = \dot{m} c_p \left(T_{m,x} - T_{m,i}\right)$$

$$T_{m,x} = T_{m,i} + \frac{q_o'' \pi D}{\dot{m} c_p} \int_0^x \sin(\pi x/L) dx = T_{m,i} + \frac{DLq_o''}{\dot{m} c_p}\left[1 - \cos(\pi x/L)\right] \qquad (4)$$

Hence, substituting Eq. (4) into (3), find

$$T_s(x) = T_{m,i} + \frac{DLq_o''}{\dot{m} c_p}\left[1 - \cos(\pi x/L)\right] + \frac{q_o''}{h}\sin(\pi x/L) \qquad (5) <$$

(d) To determine the location of the maximum wall temperature x' where $T_s(x') = T_{s,max}$, set

$$\frac{dT_s(x)}{dx} = 0 = \frac{d}{dx}\left\{\frac{DLq_o''}{\dot{m} c_p}\left[1 - \cos(\pi x/L)\right] + \frac{q_o''}{h}\sin(\pi x/L)\right\}$$

$$\frac{DLq_o''}{\dot{m} c_p} \cdot \frac{\pi}{L} \cdot \sin(\pi x'/L) + \frac{q_o''}{h} \cdot \frac{\pi}{L} \cdot \cos(\pi x'/L) = 0 \qquad \tan(\pi x'/L) = -\frac{q_o''/h}{DLq_o''/\dot{m} c_p} = -\frac{\dot{m} c_p}{DLh}$$

Continued...

$$x' = \frac{L}{\pi}\tan^{-1}\left(-\dot{m}c_p/DLh\right) \qquad (6) <$$

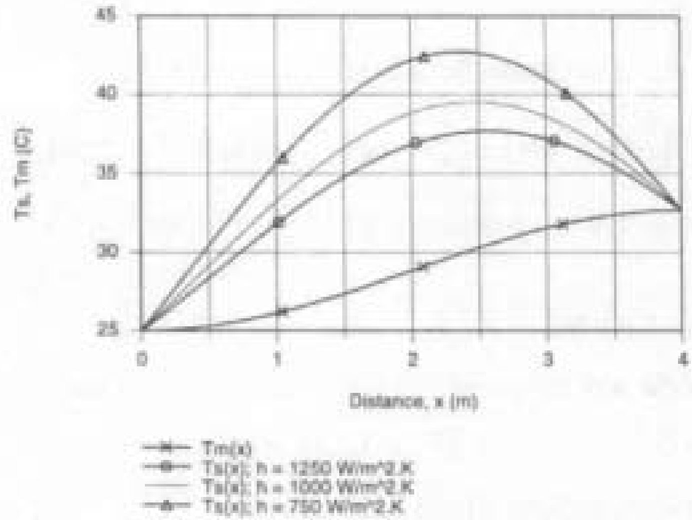
At this location, the wall temperature is

$$T_{s,max} = T_s(x') = T_{m,i} + \frac{DLq''_o}{\dot{m}c_p}\left[1-\cos(\pi x'/L)\right] + \frac{q''_o}{h}\sin(\pi x'/L) \qquad (7) <$$

(e) Consider the prescribed conditions for which to compute and plot $T_m(x)$ and $T_s(x)$.

$D = 40$ mm	$\dot{m} = 0.025$ kg/s	$h = 1000$ W/m^2	$q''_o = 10{,}000$ W/m^2
$L = 4$ m	$c_p = 4180$ J/kg·K	$T_{m,i} = 25°C$	

Using Eqs. (4) and (5) in IHT, the results are plotted below.



The effect of a lower convection coefficient is to increase the wall temperature. The position of the maximum temperature, $T_{s,max}$, moves away from the tube exit with decreasing convection coefficient.

COMMENTS: (1) Because the flow is fully developed and turbulent, assuming h is constant along the entire length of the tube is reasonable.

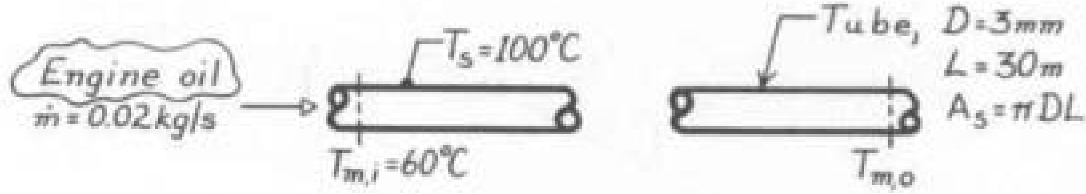
(2) To determine whether the $T_s(x)$ distribution has a maximum (rather than a minimum), you should evaluate $d^2T_s(x)/dx^2$ to show the value is indeed negative.

PROBLEM 8.19

KNOWN: Flow rate of engine oil through a long tube.

FIND: (a) Heat transfer coefficient, $\bar{h}$ (b) Outlet temperature of oil, $T_{m,o}$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Combined entry conditions exist.

PROPERTIES: *Table A-5*, Engine Oil ($T_s = 100°C = 373K$): $\mu_s = 1.73\times10^{-2}\ N\cdot s/m^2$; *Table A-5*, Engine Oil ($\bar{T}_m = 77°C = 350K$): $c_p = 2118\ J/kg\cdot K$, $\mu = 3.56\times10^{-2}\ N\cdot s/m^2$, $k = 0.138\ W/m\cdot K$, $Pr = 546$.

ANALYSIS: (a) The overall energy balance and rate equations have the form

$$q = \dot{m} c_p (T_{m,o} - T_{m,i}) \qquad q = \bar{h} A_s \Delta T_{\ell m} \tag{1,2}$$

Using Eq. 8.42b, with $P = \pi D$, and Eq. 8.6

$$\frac{\Delta T_o}{\Delta T_i} = \frac{T_s - T_{m,o}}{T_s - T_{m,i}} = \exp\left(-\frac{PL}{\dot{m} c_p}\cdot\bar{h}\right). \tag{3}$$

$$Re_D = \frac{4\dot{m}}{\pi D \mu} = \frac{4\times0.02\ kg/s}{\pi\times3\times10^{-3}m\times3.56\times10^{-2}\ N\cdot s/m^2} = 238.$$

For laminar and combined entry conditions, use Eq. 8.57

$$\overline{Nu}_D = 1.86\left(\frac{Re_D\,Pr}{L/D}\right)^{1/3}\left(\frac{\mu}{\mu_s}\right)^{0.14} = \left(\frac{238\times546}{30m/3\times10^{-3}m}\right)^{1/3}\left(\frac{3.56}{1.73}\right)^{0.14} = 4.83$$

$$\bar{h} = \overline{Nu}_D k/D = 4.83\times0.138\ W/m\cdot K/3\times10^{-3}m = 222\ W/m^2\cdot K. \qquad \triangleleft$$

(b) Using Eq. (3) with the foregoing value of $\bar{h}$,

$$\frac{(100 - T_{m,o})°C}{(100-60)°C} = \exp\left(-\frac{\pi\times3\times10^{-3}m\times30m}{0.02\ kg/s\times2118\ J/kg\cdot K}\times222W/m^2\cdot K\right) \quad T_{m,o} = 90.9°C. \qquad \triangleleft$$

COMMENTS: (1) Note that requirements for the correlation, Eq. 8.57, are satisfied.

(2) The assumption of $\bar{T}_m$ for selecting property values was satisfactory.

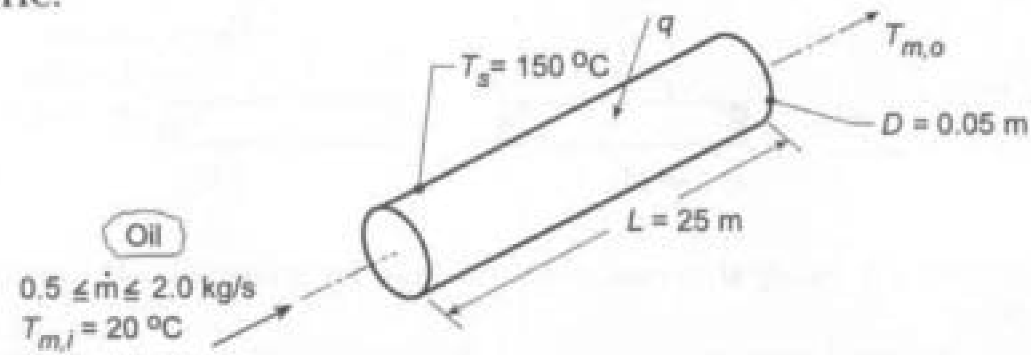
(3) For thermal entry effect only, Eq. 8.56, $\bar{h} = 201\ W/m^2\cdot K$ and $T_{m,o} = 89.5°C$.

PROBLEM 8.20

KNOWN: Inlet temperature and flowrate of oil flowing through a tube of prescribed surface temperature and geometry.

FIND: (a) Oil outlet temperature and total heat transfer rate, and (b) Effect of flowrate.

SCHEMATIC:



ASSUMPTIONS: (1) Negligible temperature drop across tube wall, (2) Negligible kinetic energy, potential energy and flow work effects.

PROPERTIES: *Table A.5*, Engine oil (assume $T_{m,o} = 140°C$, hence $\overline{T}_m = 80°C = 353$ K): $\rho = 852$ kg/m^3, $\nu = 37.5 \times 10^{-6}$ m^2/s, $k = 138 \times 10^{-3}$ W/m·K, Pr = 490, $\mu = \rho \cdot \nu = 0.032$ kg/m·s, $c_p = 2131$ J/kg·K.

ANALYSIS: (a) For constant surface temperature the oil outlet temperature may be obtained from Eq. 8.42b. Hence

$$T_{m,o} = T_s - \left(T_s - T_{m,i}\right)\exp\left(-\frac{\pi DL}{\dot{m}c_p}\overline{h}\right)$$

To determine $\overline{h}$, first calculate Re_D from Eq. 8.6,

$$Re_D = \frac{4\dot{m}}{\pi D\mu} = \frac{4(0.5\,kg/s)}{\pi(0.05m)(0.032\,kg/m\cdot s)} = 398$$

Hence the flow is laminar. Moreover, from Eq. 8.23 the thermal entry length is

$$x_{fd,t} = 0.05D\,Re_D\,Pr = 0.05(0.05\,m)(398)(490) = 486\,m.$$

Since L = 25 m the flow is far from being thermally fully developed. However, from Eq. 8.3, $x_{fd,h} = 0.05DRe_D = 0.05(0.05\text{ m})(398) = 1$ m and it is reasonable to assume fully developed hydrodynamic conditions throughout the tube. Hence $\overline{h}$ may be determined from Eq. 8.56

$$\overline{Nu}_D = 3.66 + \frac{0.0668(D/L)\,Re_D\,Pr}{1+0.04\left[(D/L)\,Re_D\,Pr\right]^{2/3}}.$$

With $(D/L)Re_DPr = (0.05/25)398 \times 490 = 390$, it follows that

$$\overline{Nu}_D = 3.66 + \frac{26}{1+2.14} = 11.95.$$

Hence, $\overline{h} = \overline{Nu}_D \dfrac{k}{D} = 11.95\dfrac{0.138\,W/m\cdot K}{0.05\,m} = 33\,W/m^2\cdot K$ and it follows that

Continued...

$$T_{m,o} = 150^\circ C - (150^\circ C - 20^\circ C)\exp\left[-\frac{\pi(0.05\,m)(25\,m)}{0.5\,kg/s \times 2131\,J/kg\cdot K} \times 33\,W/m^2\cdot K\right]$$

$$T_{m,o} = 35^\circ C.$$ <

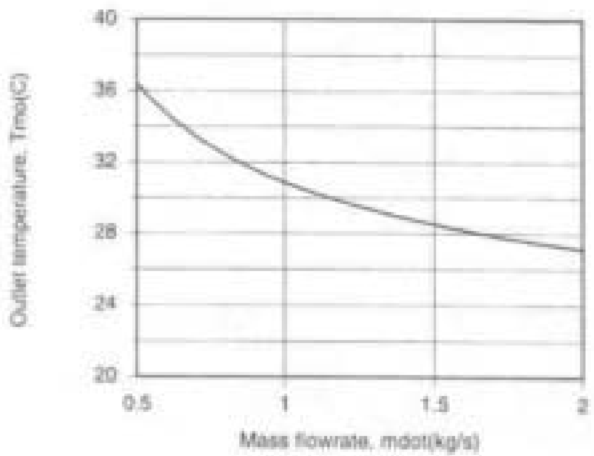
From the overall energy balance, Eq. 8.37, it follows that

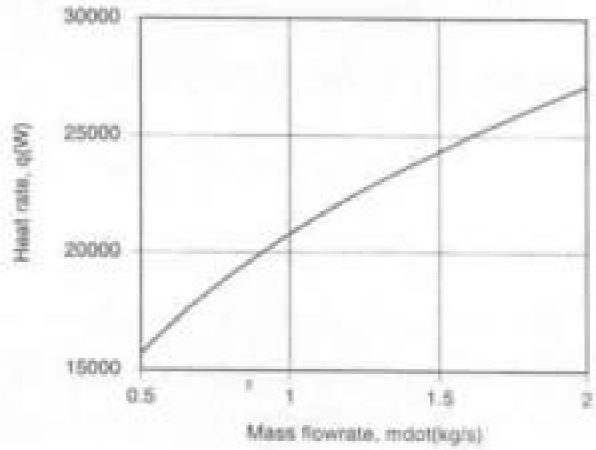
$$q = \dot{m}c_p(T_{m,o} - T_{m,i}) = 0.5\,kg/s \times 2131\,J/kg\cdot K \times (35-20)^\circ C$$

$$q = 15{,}980\ W.$$ <

The value of $T_{m,o}$ has been grossly overestimated in evaluating the properties. The properties should be re-evaluated at $\overline{T} = (20 + 35)/2 = 27^\circ C$ and the calculations repeated. Iteration should continue until satisfactory convergence is achieved between the calculated and assumed values of $T_{m,o}$. Following such a procedure, one would obtain $T_{m,o} = 36.4^\circ C$, $Re_D = 27.8$, $\overline{h} = 32.8\ W/m^2\cdot K$, and $q = 15{,}660$ W. The small effect of reevaluating the properties is attributed to the compensating effects on Re_D (a large decrease) and Pr (a large increase).

(b) The effect of flowrate on $T_{m,o}$ and q was determined by using the appropriate IHT *Correlations* and *Properties* Toolpads.





The heat rate increases with increasing $\dot{m}$ due to the corresponding increase in Re_D and hence $\overline{h}$. However, the increase is not proportional to $\dot{m}$, causing $(T_{m,o} - T_{m,i}) = q/\dot{m}c_p$, and hence $T_{m,o}$, to decrease with increasing $\dot{m}$. The maximum heat rate corresponds to the maximum flowrate ($\dot{m} = 0.20$ kg/s).

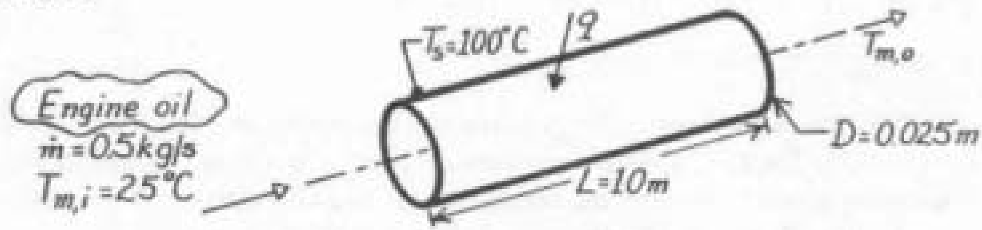
COMMENTS: Note that significant error would be introduced by assuming fully developed thermal conditions and $\overline{Nu}_D = 3.66$. The flow remains well within the laminar region over the entire range of $\dot{m}$.

PROBLEM 8.21

KNOWN: Flow rate and inlet temperature of engine oil in a tube of prescribed length, diameter, and surface temperature.

FIND: Total heat transfer and oil outlet temperature with and without the assumption of fully developed flow.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Negligible kinetic energy, potential energy and flow work changes, (3) Constant properties.

PROPERTIES: *Table A-5*, Engine oil, ($\bar{T}_m$ = 340K): $\rho = 860\ \text{kg/m}^3$, $c_p = 2076\ \text{J/kg·K}$, $\mu = 5.31\times10^{-2}\ \text{kg/s·m}$, $k = 0.139\ \text{W/m·K}$, $Pr = 793$.

ANALYSIS: From Eqs. 8.42b and 8.37,

$$T_{m,o} = T_s - (T_s - T_{m,i})\exp\left(-\frac{\pi D L}{\dot{m} c_p}\bar{h}\right) = 100°C - 75°C\exp\left(-\frac{\pi 0.025\text{m} \times 10\text{m}}{0.5\,\text{kg/s} \times 2076\,\text{J/kg·K}}\bar{h}\right)$$

$$T_{m,o} = 100°C - 75°C\exp(-0.000757\times\bar{h})$$

$$q = \dot{m} c_p (T_{m,o} - T_{m,i}) = 0.5\,\text{kg/s}\times 2076\,\text{J/kg·K}\,(T_{m,o} - 25°C).$$

With $Re_D = 4\dot{m}/\pi D\mu = 4(0.5\ \text{kg/s})/\pi(0.025\ \text{m})0.0531\ \text{kg/s·m} = 480$ the flow is laminar. Considering the *thermal entry (te) region*, it follows from Eq. 8.56 that

$$\bar{h} = \frac{k}{D}\left[3.66 + \frac{0.0668\,(D/L)Re_D Pr}{1+0.04\,[(D/L)Re_D Pr]^{2/3}}\right] = \frac{0.139\,\text{W/m·K}}{0.025\,\text{m}}\left[3.66 + \frac{0.0668\,(940)}{1+0.04\,(940)^{2/3}}\right] = 92.5\,\text{W/m}^2\text{·K}.$$

$$T_{m,o(te)} = 100°C - 75°C\exp(-0.000757\times92.5) = 100°C - 69.9°C = 30.1°C$$ ◁

$$q_{(te)} = 1038\,\text{W/K}\,(30.1-25)°C = 5290\,\text{W}.$$ ◁

If *fully developed* (fd) conditions are assumed for the entire tube,

$$\bar{h} = \frac{k}{D}3.66 = \frac{0.139\,\text{W/m·K}}{0.025\,\text{m}}3.66 = 20.3\,\text{W/m}^2\text{·K}$$

$$T_{m,o(fd)} = 100°C - 73.9°C = 26.1°C$$ ◁

$$q_{(fd)} = 1038\,\text{W/K}\,(26.1-25)°C = 1190\,\text{W}.$$ ◁

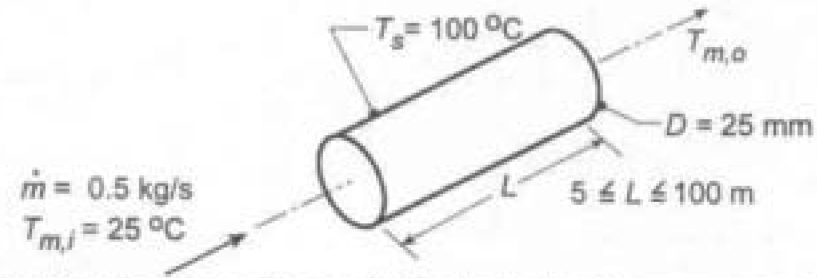
COMMENTS: The assumption of fully developed conditions throughout the tube leads to a large error in the calculation of $\bar{h}$ and hence q. Note that $x_{fd,t} = 0.05\,D\,Re_D\,Pr = 0.05(0.025\,\text{m})480(793) = 476\,\text{m}$, which is much larger than the tube length. The calculations should be repeated with properties evaluated at $\bar{T}_m = 300\text{K}$.

PROBLEM 8.22

KNOWN: Inlet temperature and flowrate of oil moving through a tube of prescribed diameter and surface temperature.

FIND: (a) Oil outlet temperature $T_{m,o}$ for two tube lengths, 5 m and 100 m, and log mean and arithmetic mean temperature differences, (b) Effect of L on $T_{m,o}$ and $\overline{Nu}_D$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Negligible kinetic energy, potential energy and flow work changes, (3) Constant properties.

PROPERTIES: *Table A.4*, Oil (330 K): $c_p = 2035$ J/kg·K, $\mu = 0.0836$ N·s/m², k = 0.141 W/m·K, Pr = 1205.

ANALYSIS: (a) Using Eqs. 8.42b and 8.6

$$T_{m,o} = T_s - (T_s - T_{m,i})\exp\left(-\frac{\pi DL}{\dot{m}c_p}\bar{h}\right) \qquad Re_D = \frac{4\dot{m}}{\pi D\mu} = \frac{4\times 0.5\,kg/s}{\pi\times 0.025\,m\times 0.0836\,N\cdot s/m^2} = 304.6$$

Since entry length effects will be significant, use Eq. 8.56

$$\bar{h} = \frac{k}{D}\left[3.66 + \frac{0.0688(D/L)Re_D\,Pr}{1+0.04[(D/L)Re_D\,Pr]^{2/3}}\right] = \frac{0.141\,W/m\cdot K}{0.025\,m}\left[3.66 + \frac{2.45\times 10^4\,D/L}{1+205(D/L)^{2/3}}\right]$$

For L = 5 m, $\bar{h} = 5.64(3.66 + 17.51) = 119\,W/m^2\cdot K$, hence

$$T_{m,o} = 100^\circ C - (75^\circ C)\exp\left(-\frac{\pi\times 0.025\,m\times 5\,m\times 119\,W/m^2\cdot K}{0.5\,kg/s\times 2035\,J/kg\cdot K}\right) = 28.4^\circ C \qquad <$$

For L = 100 m, $\bar{h} = 5.64(3.66 + 3.38) = 40\,W/m^2\cdot K$, $\quad T_{m,o} = 44.9°C$. $\qquad$ <

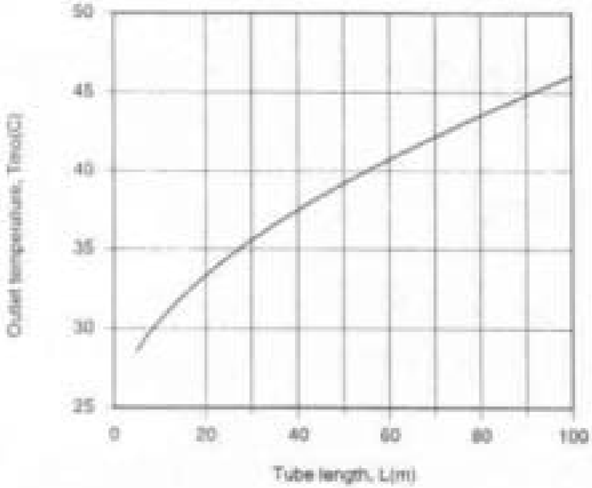
Also, *for* L = 5 m,

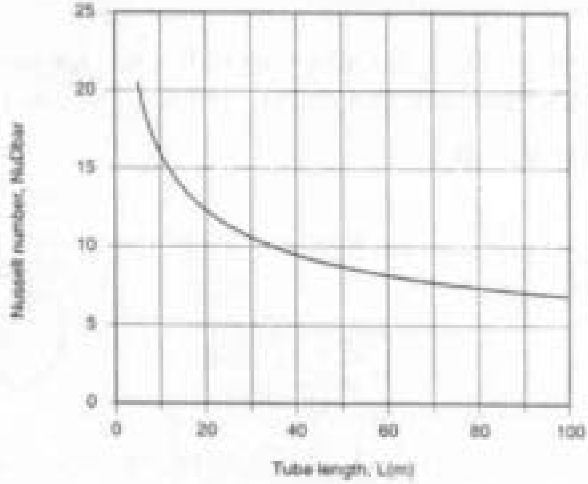
$$\Delta T_{\ell m} = \frac{\Delta T_o - \Delta T_i}{\ell n(\Delta T_o/\Delta T_i)} = \frac{71.6 - 75}{\ell n(71.6/75)} = 73.3^\circ C \qquad \Delta T_{am} = (\Delta T_o + \Delta T_i)/2 = 73.3^\circ C \qquad <$$

For L = 100 m, $\qquad \Delta T_{\ell m} = 64.5^\circ C$, $\qquad \Delta T_{am} = 65.1^\circ C$ $\qquad$ <

(b) The effect of tube length on the outlet temperature and Nusselt number was determined by using the *Correlations* and *Properties* Toolpads of IHT.

Continued...





The outlet temperature approaches the surface temperature with increasing L, but even for L = 100 m, $T_{m,o}$ is well below T_s. Although $\overline{Nu}_D$ decays with increasing L, it is still well above the fully developed value of $Nu_{D,fd} = 3.66$.

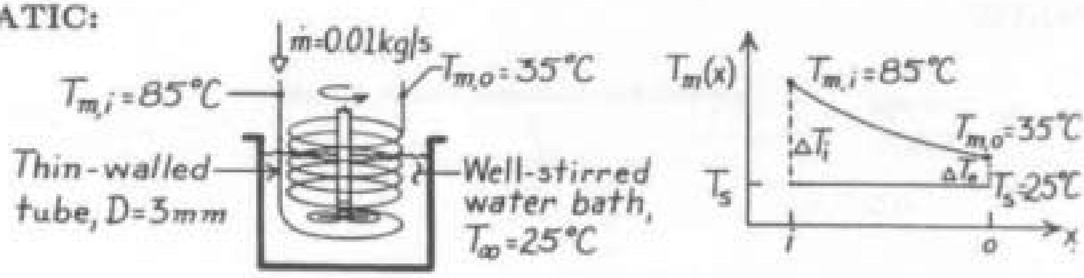
COMMENTS: (1) The average, mean temperature, $\overline{T}_m = 330$ K, was significantly overestimated in part (a). The accuracy may be improved by evaluating the properties at a lower temperature. (2) Use of ΔT_{am} instead of $\Delta T_{\ell m}$ is reasonable for small to moderate values of $(T_{m,i} - T_{m,o})$. For large values of $(T_{m,i} - T_{m,o})$, $\Delta T_{\ell m}$ should be used.

PROBLEM 8.23

KNOWN: Ethylene glycol flowing through a coiled, thin walled tube submerged in a well-stirred water bath maintained at a constant temperature.

FIND: Heat rate and required tube length for prescribed conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Tube wall thermal resistance negligible, (3) Convection coefficient on water side infinite; cooling process approximates constant wall surface temperature distribution, (4) KE, PE and flow work changes negligible, (5) Constant properties, (6) Negligible heat transfer enhancement associated with the coiling.

PROPERTIES: *Table A-5*, Ethylene glycol ($T_m = (85+35)\,^\circ C/2 = 60\,^\circ C = 333K$): $c_p = 2562\ J/kg\cdot K$, $\mu = 0.522\times10^{-2}\ N\cdot s/m^2$, $k = 0.260\ W/m\cdot K$, $Pr = 51.3$.

ANALYSIS: From an overall energy balance on the tube,

$$q_{conv} = \dot{m}\, c_p (T_{m,o} - T_{m,i}) = 0.01\ kg/s \times 2562\ J/kg(35-85)\,^\circ C = -1281\ W\ . \quad (1) \triangleleft$$

For the constant surface temperature condition, from the rate equation,

$$A_s = q_{conv}/\bar{h}\,\Delta T_{\ell m} \quad (2)$$

$$\Delta T_{\ell m} = (\Delta T_o - \Delta T_i)/\ell n \frac{\Delta T_o}{\Delta T_i} = [(35-25)\,^\circ C - (85-25)\,^\circ C]/\ell n \frac{35-25}{85-25} = 27.9\,^\circ C\ . \quad (3)$$

Find the Reynolds number to determine flow conditions,

$$Re_D = \frac{4\dot{m}}{\pi D \mu} = \frac{4 \times 0.01\ kg/s}{\pi \times 0.003m \times 0.522\times10^{-2}\ N\cdot s/m^2} = 813\ . \quad (4)$$

Hence, the flow is laminar and, assuming the flow is fully developed, the appropriate correlation is

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = 3.66\ , \qquad \bar{h} = Nu\frac{k}{D} = 3.66 \times 0.260\frac{W}{m\cdot K}/0.003m = 317\ W/m^2\cdot K\ . \quad (5)$$

From Eq. (2), the required area, A_s, and tube length, L, are

$$A_s = 1281\ W/317\ W/m^2\cdot K \times 27.9\,^\circ C = 0.1448\ m^2$$

$$L = A_s/\pi D = 0.1448m^2/\pi(0.003m) = 15.4m\ . \quad \triangleleft$$

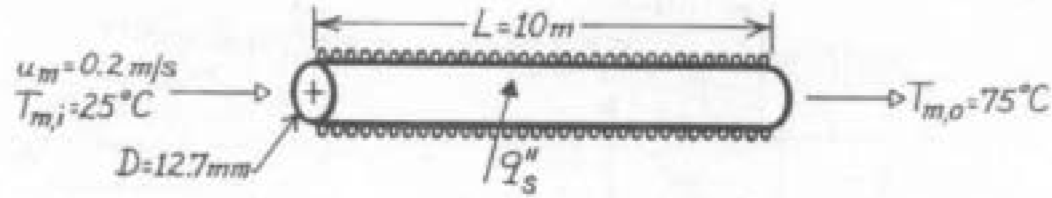
COMMENTS: Note that for fully developed laminar flow conditions, the requirement is satisfied: $Gz^{-1} = (L/D)/Re_D\,Pr = (15.3/0.003)/(813 \times 51.3) = 0.122 > 0.05$. Note also the sign of the heat rate q_{conv} when using Eqs. (1) and (2).

PROBLEM 8.24

KNOWN: Inlet and outlet temperatures and velocity of fluid flow in tube. Tube diameter and length.

FIND: Surface heat flux and temperatures at x = 0.5 and 10 m.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties, (3) Negligible heat loss to surroundings, (4) Negligible potential and kinetic energy changes and axial conduction.

PROPERTIES: Pharmaceutical (given): ρ = 1000 kg/m^3, c_p = 4000 J/kg·K, μ = 2×10^{-3} kg/s·m, k = 0.48 W/m·K, Pr = 10.

ANALYSIS: With

$$\dot{m} = \rho VA = 1000\ \text{kg/m}^3(0.2\ \text{m/s})\pi(0.0127\ \text{m})^2/4 = 0.0253\ \text{kg/s}$$

Equation 8.37 yields

$$q = \dot{m}c_p(T_{m,o} - T_{m,i}) = 0.0253\ \text{kg/s}(4000\ \text{J/kg·K})50\ \text{K} = 5060\ \text{W}.$$

The required heat flux is then

$$q_s'' = q/A_s = 5060\ \text{W}/\pi(0.0127\ \text{m})10\ \text{m} = 12{,}682\ \text{W/m}^2. \qquad \triangleleft$$

With

$$Re_D = \rho VD/\mu = 1000\ \text{kg/m}^3(0.2\ \text{m/s})0.0127\ \text{m}/2\times10^{-3}\ \text{kg/s·m} = 1270$$

the flow is laminar and Equation 8.23 yields

$$x_{fd,t} = 0.05Re_D Pr_D = 0.05(1270)10(0.0127\ \text{m}) = 8.06\ \text{m}.$$

Hence, with fully developed hydrodynamic and thermal conditions at x = 10 m, Equation 8.53 yields

$$h(10\ \text{m}) = Nu_{D,fd}(k/D) = 4.36(0.48\ \text{W/m·K}/0.0127\ \text{m}) = 164.8\ \text{W/m}^2\text{·K}.$$

Hence, from Newton's law of cooling,

$$T_{s,o} = T_{m,o} + (q_s''/h) = 75°\text{C} + (12{,}682\ \text{W/m}^2/164.8\ \text{W/m}^2\text{·K}) = 152°\text{C}. \qquad \triangleleft$$

At x = 0.5 m, $(x/D)/(Re_D Pr) = 0.0031$ and Figure 8.9 yields $Nu_D = 8$ for a thermal entry region with uniform surface heat flux. Hence, h(0.5 m) = 302.4 W/m^2·K and, since T_m increases linearly with x, $T_m(x = 0.5\ \text{m}) = T_{m,i} + (T_{m,o} - T_{m,i})(x/L) = 27.5°\text{C}$. It follows that

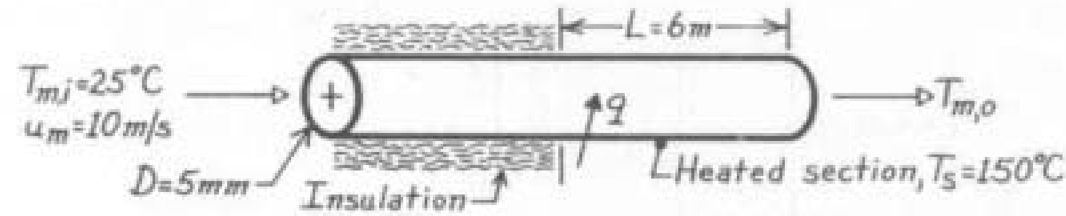
$$T_s(x = 0.5\ \text{m}) = 27.5°\text{C} + (12{,}682\ \text{W/m}^2/302.4\ \text{W/m}^2\text{·K}) = 69.4°\text{C}. \qquad \triangleleft$$

PROBLEM 8.25

KNOWN: Temperature and mean velocity of oil in hydrodynamically fully developed flow through a circular tube. Tube diameter and length and temperature of heated section.

FIND: Oil outlet temperature and heat rate.

SCHEMATIC:



ASSUMPTIONS: (1) Fully developed velocity profile, (2) Negligible kinetic and potential energy and flow work changes.

PROPERTIES: *Table A.5*, Engine oil ($\overline{T}_m$ = 310 K): c_p = 1951 J/kg·K, $\nu = 288\times10^{-6}$ m²/s, k = 0.145 W/m·K, Pr = 3400, ρ = 878 kg/m³.

ANALYSIS: From Equation 8.42b,

$$T_{m,o} = T_s - (T_s - T_{m,i})\exp[-(\pi DL/\dot{m}c_p)\overline{h}]$$

where

$$\dot{m} = \rho(\pi D^2/4)u_m = 878\ \text{kg/m}^3(\pi/4)(0.005\ \text{m})^2 10\ \text{m/s} = 0.172\ \text{kg/s}$$

With $$Re_D = \frac{u_m D}{\nu} = \frac{10\ \text{m/s}(0.005\ \text{m})}{288\times10^{-6}\ \text{m}^2/\text{s}} = 173.6$$

the flow is laminar. For a thermal entry region,

$$(D/L)Re_D Pr = (0.005\ \text{m}/6\ \text{m})(173.6)(3400) = 491.9$$

and Equation 8.56 yields

$$\overline{Nu}_D = 3.66 + \frac{0.0688(491.9)}{1+0.04(491.9)^{2/3}} = 13.4.$$

Hence,

$$\overline{h} = (k/D)\overline{Nu}_D = (0.145\ \text{W/m·K}/0.005\ \text{m})13.4 = 387\ \text{W/m}^2\cdot\text{K}$$

$$T_{m,o} = 150°\text{C} - (125°\text{C})\exp\left[-\frac{\pi(0.005\ \text{m})6\ \text{m}(387\ \text{W/m}^2\cdot\text{k})}{0.172\ \text{kg/s}(1951\ \text{J/kg·K})}\right] = 37.9°\text{C}. \quad \triangleleft$$

Hence, from Equation 8.37

$$q = \dot{m}c_p(T_{m,o} - T_{m,i}) = 0.172\ \text{kg/s}(1951\ \text{J/kg·K})(12.9°\text{C}) = 4330\ \text{W}. \quad \triangleleft$$

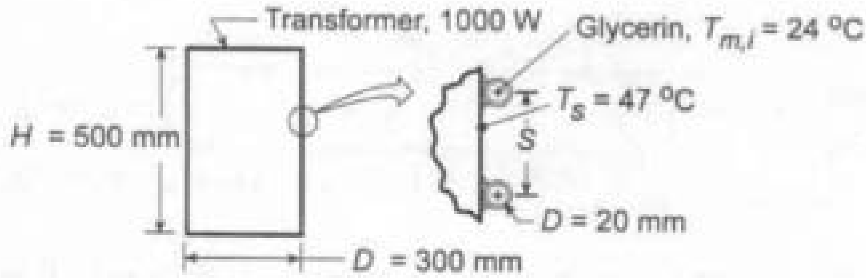
COMMENTS: (1) Although the flow is hydrodynamically fully developed as it enters the heated section, it is not fully developed thermally and $\overline{Nu}_D \neq 3.66$. (2) Equation 8.44 yields q = 4320 W. (3) $\overline{T}_m$ = 304.5 K and the calculations should be repeated with improved oil properties.

PROBLEM 8.26

KNOWN: Tubing with glycerin welded to transformer lateral surface to remove dissipated power. Maximum allowable temperature rise of coolant is 6°C.

FIND: (a) Required coolant rate $\dot{m}$, tube length L and lateral spacing S between turns, and (b) Effect of flowrate on outlet temperature and maximum power.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) All heat dissipated by transformer transferred to glycerin, (3) Fully developed flow (part a), (4) Negligible kinetic and potential energy changes, (5) Negligible tube wall thermal resistance.

PROPERTIES: *Table A.5*, Glycerin ($\overline{T}_m$ = 300 K): $\rho = 1259.9\ kg/m^3$, $c_p = 2427\ J/kg\cdot K$, $\mu = 79.9 \times 10^{-2}\ N\cdot s/m^2$, $k = 286 \times 10^{-3}\ W/m\cdot K$, Pr = 6780.

ANALYSIS: (a) From an overall energy balance assuming the maximum temperature rise of the glycerin coolant is 6°C, find the flow rate as

$$q = \dot{m}c_p\left(T_{m,o} - T_{m,i}\right) \qquad \dot{m} = q/c_p\left(T_{m,o} - T_{m,i}\right) = 1000\,W/2427\,J/kg\cdot K(6\,K) = 6.87\times10^{-2}\ kg/s \quad <$$

From Eq. 8.43, the length of tubing can be determined,

$$\frac{T_s - T_{m,o}}{T_s - T_{m,i}} = \exp\left(-PL\bar{h}/\dot{m}c_p\right)$$

where $P = \pi D$. For the tube flow, find

$$Re_D = \frac{4\dot{m}}{\pi D\mu} = \frac{4\times6.87\times10^{-2}\ kg/s}{\pi\times0.020\,m\times79.9\times10^{-2}\ N\cdot s/m^2} = 5.47$$

which implies laminar flow, and if fully developed,

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = 3.66 \qquad \bar{h} = \frac{3.66\times286\times10^{-3}\ W/m\cdot K}{0.020\,m} = 52.3\,W/m^2\cdot K$$

$$\frac{(47-30)^\circ C}{(47-24)^\circ C} = \exp\left[-\left(\pi(0.020\,m)\times52.3\,W/m^2\cdot K\times L\right)/\left(6.87\times10^{-2}\ kg/s\times2427\,J/kg\cdot K\right)\right]$$

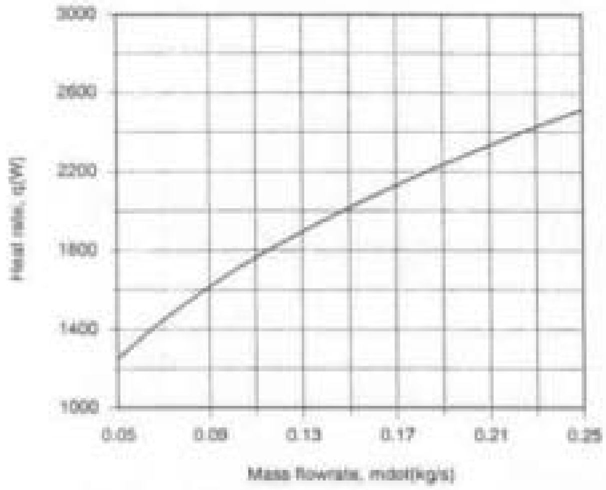
$$L = 15.3\ m. \quad <$$

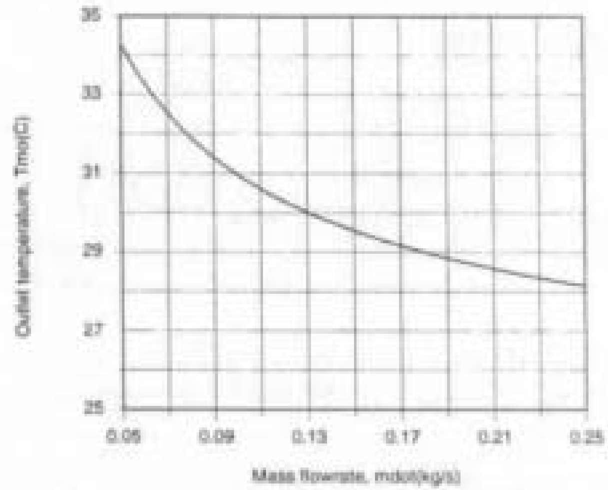
The number of turns of the tubing, N, is $N = L/(\pi D) = (15.3\ m)/\pi(0.3\ m) = 16.2$ and hence the spacing S will be

$$S = H/N = 500\ mm/16.2 = 30.8\ mm. \quad <$$

Continued...

(b) Parametric calculations were performed using the IHT *Correlations* Toolpad based on Eq. 8.56 (a thermal entry length condition), and the following results were obtained.





With T_s maintained at 47°C, the maximum allowable transformer power (heat rate) and glycerin outlet temperature increase and decrease, respectively, with increasing $\dot{m}$. The increase in q is due to an increase in $\overline{Nu}_D$ (and hence $\overline{h}$) with increasing Re_D. The value of $\overline{Nu}_D$ increased from 5.3 to 9.4 with increasing $\dot{m}$ from 0.05 to 0.25 kg/s.

COMMENTS: Since $Gz_D^{-1} = (L/D)/Re_D\,Pr = (15.3\text{ m}/0.02\text{ m})/(5.47 \times 6780) = 0.0206 < 0.05$, entrance length effects are significant, and Eq. 8.56 should be used to determine $\overline{Nu}_D$.

PROBLEM 9.1

KNOWN: Tabulated values of density for water and definition of the volumetric thermal expansion coefficient, β.

FIND: Value of the volumetric expansion coefficient at 300K; compare with tabulated values.

PROPERTIES: *Table A-6,* Water (300K): $\rho = 1/v_f = 1/1.003\times10^{-3}\ \mathrm{m^3/kg} = 997.0\ \mathrm{kg/m^3}$, $\beta = 276.1\times10^{-6}\ \mathrm{K^{-1}}$; (295K): $\rho = 1/v_f = 1/1.002\times10^{-3}\ \mathrm{m^3/kg} = 998.0\ \mathrm{kg/m^3}$; (305K): $\rho = 1/v_f = 1/1.005\times10^{-3}\ \mathrm{m^3/kg} = 995.0\ \mathrm{kg/m^3}$.

ANALYSIS: The volumetric expansion coefficient is defined by Eq. 9.4 as

$$\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p .$$

The density change with temperature at constant pressure can be estimated as

$$\left(\frac{\partial\rho}{\partial T}\right)_p \approx \left(\frac{\rho_1-\rho_2}{T_1-T_2}\right)_p$$

where the subscripts (1,2) denote the property values just above and below, respectively, the condition for T = 300K denoted by the subscript (o). That is,

$$\beta_o \approx -\frac{1}{\rho_o}\left(\frac{\rho_1-\rho_2}{T_1-T_2}\right)_p .$$

Substituting numerical values, find

$$\beta_o \approx \frac{-1}{997.0\,\mathrm{kg/m^3}}\ \frac{(995.0-998.0)\,\mathrm{kg/m^3}}{(305-295)\mathrm{K}} = 300.9\times10^{-6}\ \mathrm{K^{-1}} . \quad \triangleleft$$

Compare this value with the tabulated one, $\beta = 276.1\times10^{-6}\ \mathrm{K^{-1}}$, to find our estimate is 8.7% high. $\triangleleft$

COMMENTS: (1) The poor agreement between our estimate and the tabulated value is due to the poor precision with which the density change with temperature is estimated. The tabulated values of β were determined from very accurate equation of state data.

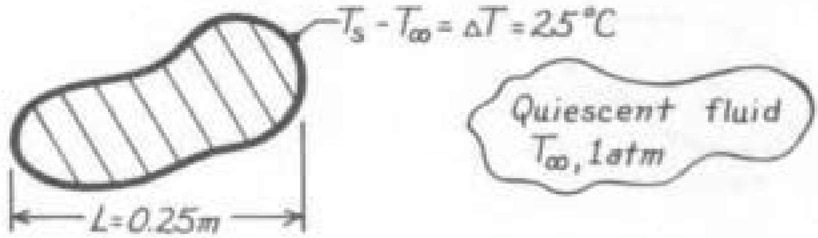
(2) Note that β is negative for $T < 275\mathrm{K}$. Why does this occur? What is the implication of this to free convection?

PROBLEM 9.2

KNOWN: Object with specified characteristic length and temperature difference above ambient fluid.

FIND: Grashof number for air, hydrogen, water, ethylene glycol for a pressure of 1 atm.

SCHEMATIC:



ASSUMPTIONS: (1) Thermophysical properties evaluated at $T_f = 350K$, (2) For gases, $\beta = 1/T_f$.

PROPERTIES: Evaluate at 1 atm, $T_f = 350K$:
Table A-4, Air: $\nu = 20.92\times10^{-6}\ m^2/s$; Hydrogen: $\nu = 143\times10^{-6}\ m^2/s$
Table A-6, Water (Sat. liquid): $\nu = \mu_f v_f = 37.5\times10^{-6}\ m^2/s$, $\beta_f = 0.624\times10^{-3}\ K^{-1}$
Table A-5, Ethylene glycol: $\nu = 3.17\times10^{-6}\ m^2/s$, $\beta = 0.65\times10^{-3}\ K^{-1}$.

ANALYSIS: The Grashof number is given by Eq. 9.12,

$$Gr_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2}.$$

Substituting numerical values for the fluid *air* with $\beta = 1/T_f$, find

$$Gr_{L,air} = \frac{9.8 m/s^2 \times (1/350K)\ (25K)\ (0.25m)^3}{(20.92\times10^{-6}\ m^2/s)^2}$$

$$Gr_{L,air} = 2.50\times10^7 .$$ ◁

Performing similar calculations for the other fluids, find

$$Gr_{L,hyd} = 5.35\times10^5$$ ◁

$$Gr_{L,water} = 1.70\times10^6$$ ◁

$$Gr_{L,eth} = 2.48\times10^8 .$$ ◁

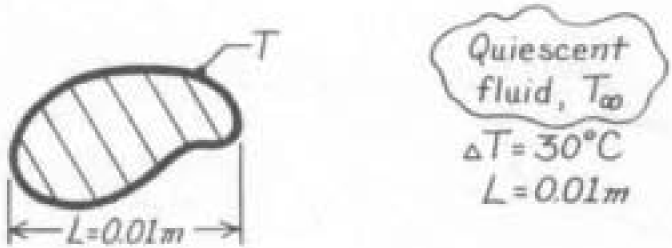
COMMENTS: Higher values of Gr_L imply increased free convection flows. However, other properties affect the value of the heat transfer coefficients. Note that for the gases, $\beta = 1/T_f$, assuming perfect gas behavior.

PROBLEM 9.3

KNOWN: Relation for the Rayleigh number, a dimensionless parameter used in free convection analysis.

FIND: Rayleigh number for four fluids for prescribed conditions.

SCHEMATIC:



ASSUMPTIONS: (1) Perfect gas behavior for specified gases.

PROPERTIES: *Table A-4*, Air (400K, 1 atm): $\nu=26.41\times10^{-6}$ m²/s, $\alpha=38.3\times10^{-6}$ m²/s, $\beta=1/T=1/400K=2.50\times10^{-3}$ K^{-1}; *Table A-4*, Helium (400K, 1 atm): $\nu=199\times10^{-6}$ m²/s, $\alpha=295\times10^{-6}$ m²/s, $\beta=1/T=2.50\times10^{-3}$ K^{-1}; *Table A-5*, Glycerin (12°C=285K): $\nu=2830\times10^{-6}$ m²/s, $\alpha=0.964\times10^{-7}$ m²/s, $\beta=0.475\times10^{-3}$ K^{-1}; *Table A-6*, Water (37°C=310K, sat. liq.): $\nu=\mu_f v_f=695\times10^{-6}$ N·s/m² $\times 1.007\times10^{-3}$ m³/kg $=0.700\times10^{-6}$ m²/s, $\alpha=k_f\, v_f/c_{p,f}=0.628$ W/m·K $\times 1.007\times10^{-3}$ m³/kg/4178 J/kg·K $= 0.151\times10^{-6}$ m²/s, $\beta_f=361.9\times10^{-6}$ K^{-1}.

ANALYSIS: The Rayleigh number, a dimensionless parameter used in free convection analysis, is defined as the product of the Grashof and Prandtl numbers

$$Ra_L \equiv Gr \cdot Pr = \frac{g\beta\Delta T L^3}{\nu^2}\,\frac{\mu c_p}{k} = \frac{g\beta\Delta T L^3}{\nu^2}\cdot\frac{(\nu\rho)\,c_p}{k} = \frac{g\beta\Delta T L^3}{\nu\alpha}$$

where $\alpha=k/\rho c_p$ and $\nu=\mu/\rho$. The numerical values for the four fluids follows:

Air (400K, 1 atm)

$$Ra_L = 9.8\,\text{m/s}^2\,(1/400\text{K})\,30\text{K}(0.01\text{m})^3/26.41\times10^{-6}\,\text{m}^2/\text{s}\times 38.3\times10^{-6}\,\text{m}^2/\text{s} = 727 \qquad \triangleleft$$

Helium (400K, 1 atm)

$$Ra_L = 9.8\,\text{m/s}^2\,(1/400\text{K})\,30\text{K}(0.01\text{m})^3/199\times10^{-6}\,\text{m}^2/\text{s}\times 295\times10^{-6}\,\text{m}^2/\text{s} = 12.5 \qquad \triangleleft$$

Glycerin (285K)

$$Ra_L = 9.8\,\text{m/s}^2\,(0.475\times10^{-3}\text{K}^{-1})\,30\text{K}(0.01\text{m})^3/2830\times10^{-6}\,\text{m}^2/\text{s}\times 0.964\times10^{-7}\,\text{m}^2/\text{s} = 512 \qquad \triangleleft$$

Water (310K)

$$Ra_L = 9.8\,\text{m/s}^2\,(0.362\times10^{-3}\text{K}^{-1})\,30\text{K}(0.01\text{m})^3/0.700\times10^{-6}\,\text{m}^2/\text{s}\times 0.151\times10^{-6}\,\text{m}^2/\text{s} = 9.35\times10^5 \qquad \triangleleft$$

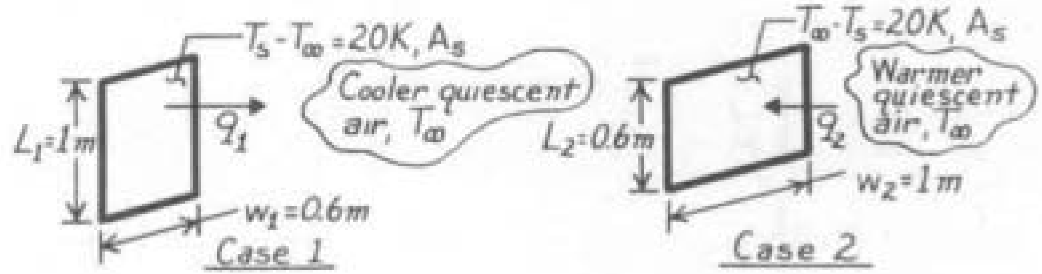
COMMENTS: (1) Note the wide variation in the Ra values for the four fluids. A large value of Ra implies an increased free convection process, however, other properties affect the value of the heat transfer coefficient.

PROBLEM 9.4

KNOWN: Heat transfer rate by convection from a vertical surface, 1m high by 0.6m wide, to quiescent air that is 20K cooler.

FIND: Ratio of the heat transfer rate for the above case to the rate corresponding to a vertical surface that is 0.6m high by 1m wide with quiescent air that is 20K warmer.

SCHEMATIC:



ASSUMPTIONS: (1) Thermophysical properties independent of temperature; evaluate at 300K; (2) Negligible radiation exchange with surroundings, (3) Quiescent ambient air.

PROPERTIES: *Table A-4*, Air (300K, 1 atm): $\nu = 15.89\times10^{-6}\ m^2/s$, $\alpha = 22.5\times10^{-6}\ m^2/s$.

ANALYSIS: The rate equation for convection between the plates and quiescent air is

$$q = \bar{h}_L A_s \Delta T \tag{1}$$

where ΔT is either $(T_s - T_\infty)$ or $(T_\infty - T_s)$; for both cases, $A_s = Lw$. The desired heat transfer ratio is then

$$\frac{q_1}{q_2} = \frac{\bar{h}_{L1}}{\bar{h}_{L2}}. \tag{2}$$

To determine the dependence of $\bar{h}_L$ on geometry, first calculate the Rayleigh number,

$$Ra_L = g\,\beta\,\Delta T L^3/\nu\alpha \tag{3}$$

and substituting property values at 300K find,

Case 1: $Ra_{L1} = 9.8\ m/s^2\ (1/300K)\ 20K\ (1m)^3/15.89\times10^{-6}\ m^2/s \times 22.5\times10^{-6}\ m^2/s = 1.82\times10^9$

Case 2: $Ra_{L2} = Ra_{L1}(L_2/L_1)^3 = 1.82\times10^4\ (0.6m/1.0m)^3 = 3.94\times10^8$.

Hence, Case 1 is turbulent and Case 2 is laminar. Using the correlation of Eq. 9.24,

$$\overline{Nu}_L = \frac{\bar{h}_L L}{k} = C(Ra_L)^n \qquad \bar{h}_L = \frac{k}{L} C\,Ra_L^n \tag{4}$$

where for *Case 1*: $C_1 = 0.10$, $n_1 = 1/3$ and for *Case 2*: $C_2 = 0.59$, $n_2 = 1/4$. Substituting Eq. (4) into the ratio of Eq. (2) with numerical values, find

$$\frac{q_1}{q_2} = \frac{(C_1/L_1)Ra_{L1}^{n_1}}{(C_2/L_2)Ra_{L2}^{n_2}} = \frac{(0.10/1m)(1.82\times10^9)^{1/3}}{(0.59/0.6m)(3.94\times10^8)^{1/4}} = 0.881 \qquad \triangleleft$$

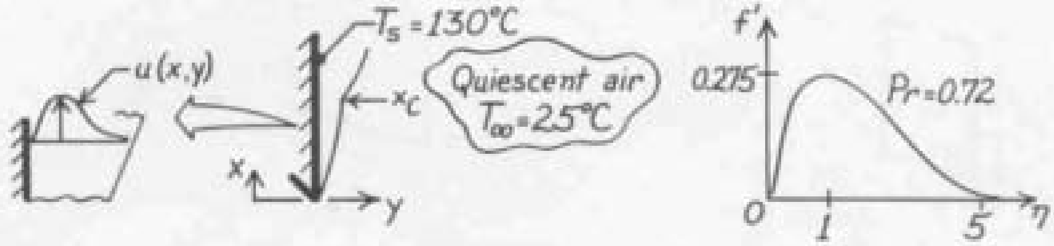
COMMENTS: Is this result to be expected? How do you explain this effect of plate orientation on the heat rates?

PROBLEM 9.5

KNOWN: Large vertical plate with uniform surface temperature of 130°C suspended in quiescent air at 25°C and atmospheric pressure.

FIND: (a) Boundary layer thickness at 0.25m from lower edge, (b) Maximum velocity in boundary layer at this location and position of maximum, (c) Heat transfer coefficient at this location, (d) Location where boundary layer becomes turbulent.

SCHEMATIC:



ASSUMPTIONS: (1) Isothermal, vertical surface in an extensive, quiescent medium, (2) Boundary layer assumptions valid.

PROPERTIES: *Table A-4*, Air ($T_f = (T_s + T_\infty)/2 = 350K$, 1 atm): $\nu = 20.92\times10^{-6}$ m²/s, k = 0.030 W/m·K, Pr = 0.700.

ANALYSIS: (a) From the similarity solution results, Fig. 9.4 (see above right), the boundary layer thickness corresponds to a value η = 5. From Eqs. 9.13 and 9.12,

$$y = \eta x (Gr_x/4)^{-1/4} \tag{1}$$

$$Gr_x = g\beta(T_s - T_\infty)x^3/\nu^2 = 9.8\,\frac{m}{s^2}\times\frac{1}{350K}(130-25)K\,x^3/(20.92\times10^{-6}\,m^2/s)^2 = 6.718\times10^9\,x^3 \tag{2}$$

$$y = 5(0.25m)\left(6.718\times10^9(0.25)^3/4\right)^{-1/4} = 1.746\times10^{-2}m = 17.5\,mm\,. \tag{3}$$ ◁

(b) From the similarity solution shown above, the maximum velocity occurs at η = 1 with $f'(\eta) = 0.275$. From Eq. 9.15, find

$$u = \frac{2\nu}{x}Gr_x^{1/2}f'(\eta) = \frac{2\times20.92\times10^{-6}\,m^2/s}{0.25m}\left(6.718\times10^9(0.25)^3\right)^{1/2}\times0.275 = 0.47\,m/s\,.$$ ◁

The maximum velocity occurs at a value of η = 1; using Eq. (3), it follows that this corresponds to a position in the boundary layer given as

$$y_{max} = 1/5\,(17.5\,mm) = 3.5\,mm.$$ ◁

(c) From Eq. 9.19, the local heat transfer coefficient at x = 0.25 m is

$$Nu_x = h_x x/k = (Gr_x/4)^{1/4}\,g(Pr) = \left(6.718\times10^9(0.25)^3/4\right)^{1/4}0.586 = 41.9$$

$$h_x = Nu_x\,k/x = 41.9\times0.030\,W/m\cdot K/0.25\,m = 5.0\,W/m^2\cdot K\,.$$ ◁

The value for g(Pr) is determined from Eq. 9.20 with Pr = 0.700.

(d) According to Eq. 9.23, the boundary layer becomes turbulent at x_c given as

$$Ra_{x,c} = Gr_{x,c}\,Pr = 10^9 \qquad x_c = \left[10^9/6.718\times10^9(0.700)\right]^{1/3} = 0.60\,m\,.$$ ◁

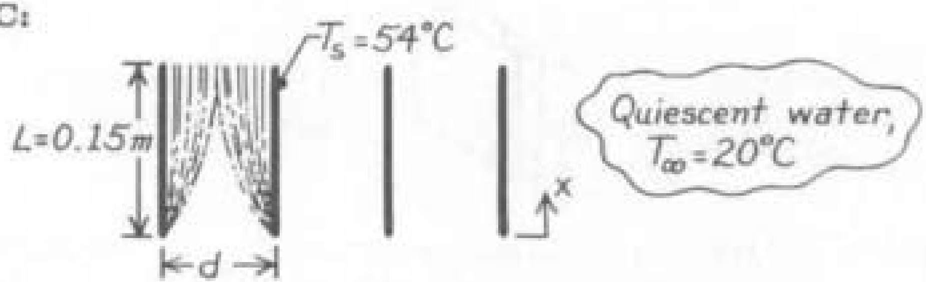
COMMENTS: Note that $\beta = 1/T_f$, a suitable approximation for air.

PROBLEM 9.6

KNOWN: Thin, vertical plates of length 0.15m at 54 °C being cooled in a water bath at 20 °C.

FIND: Minimum spacing between plates such that no interference will occur between free-convection boundary layers.

SCHEMATIC:



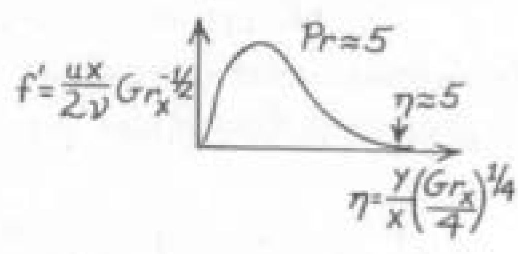
ASSUMPTIONS: (a) Water in bath is quiescent, (b) Plates are at uniform temperature.

PROPERTIES: *Table A-6,* Water ($T_f = (T_s + T_\infty)/2 = (54+20)\,^\circ C/2 = 310K$): $\rho = 1/v_f = 993.05\ kg/m^3$, $\mu = 695\times10^{-6}\ N\cdot s/m^2$, $\nu = \mu/\rho = 6.998\times10^{-7}\ m^2/s$, $Pr = 4.62$, $\beta = 361.9\times10^{-6}\ K^{-1}$.

ANALYSIS: The minimum separation distance will be twice the thickness of the boundary layer at the trailing edge where $x = 0.15m$. Assuming laminar, free convection boundary layer conditions, the similarity parameter, η, given by Eq. 9.13, is

$$\eta = \frac{y}{x}\left(Gr_x/4\right)^{1/4}$$

where y is measured normal to the plate (see Fig. 9.3). According to Fig. 9.4, the boundary layer thickness occurs at a value $\eta \approx 5$. It follows then that,



$$y_{b\ell} = \eta\, x\left(Gr_x/4\right)^{-1/4}$$

where $Gr_x = \dfrac{g\beta(T_s - T_\infty)x^3}{\nu^2}$

$$Gr_x = 9.8\,m/s^2 \times 361.9\times10^{-6}\,K^{-1}(54-20)K \times (0.15m)^3/(6.998\times10^{-7}\,m^2/s)^2 = 8.310\times10^8.$$

Hence,

$$y_{b\ell} = 5 \times 0.15m\left(8.310\times10^8/4\right)^{-1/4} = 6.247\times10^{-3}m = 6.3\,mm$$

and the minimum separation is

$$d = 2\,y_{b\ell} = 2 \times 6.3\,mm = 12.6\,mm.$$ ◁

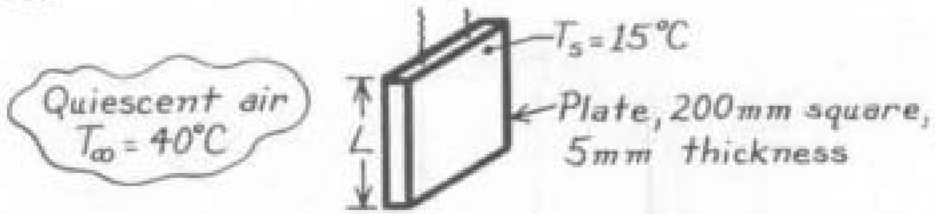
COMMENTS: According to Eq. 9.23, the critical Grashof number for the onset of turbulent conditions in the boundary layer is $Gr_{x,c}\,Pr \approx 10^9$. For the conditions above, $Gr_x\,Pr = 8.31\times10^8 \times 4.62 = 3.8\times10^9$. We conclude that the boundary layer is indeed turbulent at $x = 0.15m$ and our calculation is only an estimate which is likely to be low. Therefore, the plate separation should be greater than 12.6 mm.

PROBLEM 9.7

KNOWN: Square aluminum plate at 15°C suspended in quiescent air at 40°C.

FIND: Average heat transfer coefficient by two methods -- using results of similarity to the boundary layer equations and results from an empirical correlation.

SCHEMATIC:



ASSUMPTIONS: (1) Uniform plate surface temperature, (2) Quiescent room air, (3) Surface radiation exchange with surroundings negligible, (4) Perfect gas behavior for air, $\beta = 1/T_f$.

PROPERTIES: *Table A-4*, Air ($T_f = (T_s + T_\infty)/2 = (40+15)°C/2 = 300K$, 1 atm): $\nu = 15.89\times10^{-6}$ m²/s, k = 0.0263 W/m·K, $\alpha = 22.5\times10^{-6}$ m²/s, Pr = 0.707.

ANALYSIS: Calculate the Rayleigh number to determine the boundary layer flow conditions,

$$Ra_L = g\,\beta\,\Delta T\,L^3/\nu\,\alpha$$

$$Ra_L = 9.8\,m/s^2\,(1/300K)\,(40-15)°C\,(0.2m)^3/(15.89\times10^{-6}\,m^2/s)\,(22.5\times10^{-6}\,m^2/s) = 1.827\times10^7$$

where $\beta = 1/T_f$ and $\Delta T = T_\infty - T_s$. Since $Ra_L < 10^9$, the flow is laminar and the *similarity solution* of Section 9.4 is applicable. From Eqs. 9.21 and 9.20,

$$\overline{Nu}_L = \frac{\bar{h}_L L}{k} = \frac{4}{3}\,(Gr_L/4)^{1/4}\,g(Pr)$$

$$g(Pr) = \frac{0.75\,Pr^{1/2}}{[0.609 + 1.221\,Pr^{1/2} + 1.238\,Pr]^{1/4}}$$

and substituting numerical values with $Gr_L = Ra_L/Pr$, find

$$g(Pr) = 0.75(0.707)^{1/2}/[0.609 + 1.221(0.707)^{1/2} + 1.238\times0.707]^{1/4} = 0.501$$

$$\bar{h}_L = \left(\frac{0.0263\,W/m\cdot K}{0.20\,m}\right)\times\frac{4}{3}\left(\frac{1.827\times10^7/0.707}{4}\right)^{1/4}\times0.501 = 4.42\,W/m^2\cdot K\,. \quad \triangleleft$$

The appropriate empirical correlation for estimating $\bar{h}_L$ is given by Eq. 9.27,

$$\overline{Nu}_L = \frac{\bar{h}_L L}{k} = 0.68 + \frac{0.670\,Ra_L^{1/4}}{[1 + (0.492/Pr)^{9/16}]^{4/9}}$$

$$\bar{h}_L = (0.0263\,W/m\cdot K/0.20\,m)\,[0.68 + 0.670(1.827\times10^7)^{1/4}/[1 + (0.492/0.707)^{9/16}]^{4/9}]$$

$$\bar{h}_L = 4.42\,W/m^2\cdot K\,. \quad \triangleleft$$

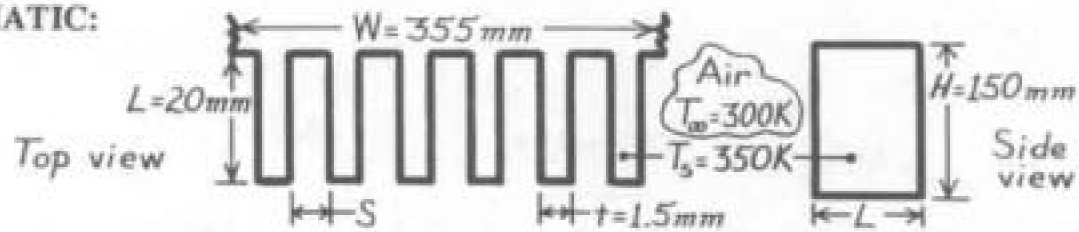
COMMENTS: The agreement of $\bar{h}_L$ calculated by these two methods is excellent. Using the Churchill-Chu correlation, Eq. 9.26, find $\bar{h}_L = 4.87\,W/m^2\cdot K$. This relation is not the most accurate for the laminar regime, but is suitable for both laminar and turbulent regions. As the plate heats up, the average coefficient will decrease.

PROBLEM 9.8

KNOWN: Dimensions of vertical rectangular fins. Temperature of fins and quiescent air.

FIND: (a) Optimum fin spacing, (b) Rate of heat transfer from an array of fins at the optimal spacing.

SCHEMATIC:



ASSUMPTIONS: (1) Fins are isothermal, (2) Radiation effects are negligible, (3) Air is quiescent.

PROPERTIES: *Table A-4*, Air (T_f=325K, 1 atm): $\nu = 18.41\times10^{-6}$ m²/s, k=0.0282 W/m·K, Pr=0.703.

ANALYSIS: (a) If fins are too close, boundary layers on adjoining surfaces will coalesce and heat transfer will decrease. If fins are too far apart, the surface area becomes too small and heat transfer decreases. $S_{op} = \delta_{x=H}$. From Fig. 9.4, the edge of boundary layer corresponds to

$$\eta = (\delta/H)\,(Gr_H/4)^{1/4} = 5.$$

Hence,

$$Gr_H = \frac{g\beta(T_s - T_\infty)H^3}{\nu^2} = \frac{9.8\,\text{m/s}^2(1/325\text{K})\,50\text{K}\,(0.15\text{m})^3}{(18.41\times10^{-6}\,\text{m}^2/\text{s})^2} = 1.5\times10^7$$

$$\delta(H) = 5(0.15\text{m})/(1.5\times10^7/4)^{1/4} = 0.017\text{m} = 17\text{mm} \qquad S_{op} = 34\,\text{mm}. \qquad \triangleleft$$

(b) The number of fins N can be found as

$$N = W/(S_{op}+t) = 355/35.5 = 10$$

and the heat rate is

$$q = 2N\bar{h}(H\cdot L)(T_s - T_\infty).$$

For laminar flow conditions

$$\overline{Nu}_H = 0.68 + 0.67\,Ra_L^{1/4}/[1 + (0.492/Pr)^{9/16}]^{4/9}$$

$$\overline{Nu}_H = 0.68 + 0.67(1.5\times10^7\times0.703)^{1/4}/[1 + (0.492/0.703)^{9/16}]^{4/9} = 30$$

$$\bar{h} = k\,Nu_H/H = 0.0282\,\text{W/m}\cdot\text{K}(30)/0.15\,\text{m} = 5.6\,\text{W/m}^2\cdot\text{K}$$

$$q = 2(10)5.6\,\text{W/m}^2\cdot\text{K}\,(0.15\,\text{m}\times0.02\,\text{m})\,(350-300)\text{K} = 16.8\,\text{W}. \qquad \triangleleft$$

COMMENTS: Part (a) result is a conservative estimate of the optimum spacing. The increase in area resulting from a further reduction in S would more than compensate for the effect of fluid entrapment due to boundary layer merger. From a more rigorous treatment (see Section 9.7.1), S_{op}= 10 mm is obtained for the prescribed conditions.

PROBLEM 9.9

KNOWN: Interior air and wall temperatures; wall height is 2.5m.

FIND: (a) Average heat transfer coefficient when $T_\infty = 20°C$ and $T_s = 10°C$, (b) Average heat transfer coefficient when $T_\infty = 27°C$ and $T_s = 37°C$.

SCHEMATIC:

ASSUMPTIONS: (a) Wall is at a uniform temperature, (b) Room air is quiescent.

PROPERTIES: *Table A-4*, Air ($T_f = 298K$, 1 atm): $\beta = 1/T_f = 3.472\times10^{-3}\ K^{-1}$, $\nu = 14.82\times10^{-6}\ m^2/s$, $k = 0.0253\ W/m\cdot K$, $\alpha = 20.9\times10^{-6}\ m^2/s$, $Pr = 0.710$; ($T_f = 305K$, 1 atm): $\beta = 1/T_f = 3.279\times10^{-3}\ K^{-1}$, $\nu = 16.39\times10^{-6}\ m^2/s$, $k = 0.0267\ W/m\cdot K$, $\alpha = 23.2\times 10^{-6}\ m^2/s$, $Pr = 0.706$.

ANALYSIS: The appropriate correlation for the average heat transfer coefficient for free convection on a vertical wall is Eq. 9.26

$$\overline{Nu}_L = \frac{\bar{h}\,L}{k} = \left\{0.825 + \frac{0.387\ Ra_L^{0.1667}}{[1 + (0.492/Pr)^{0.563}]^{0.296}}\right\}^2$$

where $Ra_L = g\,\beta\,\Delta T\,L^3/\nu\alpha$, Eq. 9.25, with $\Delta T = T_s - T_\infty$ or $T_\infty - T_s$.

(a) Substituting numerical values typical of *winter* conditions gives

$$Ra_L = \frac{9.8\ m/s^2 \times 3.472\times10^{-3}\ K^{-1}\ (20-10)\ K\ (2.5\ m)^3}{14.82\times10^{-6}\ m^2/s \times 20.96\times10^{-6}\ m^2/s} = 1.711\times10^{10}$$

$$\overline{Nu}_L = \left\{0.825 + \frac{0.387(1.711\times10^{10})^{0.1667}}{[1 + (0.492/0.710)^{0.563}]^{0.296}}\right\}^2 = 299.6\ .$$

Hence, $\bar{h} = \overline{Nu}_L\ k/L = 299.6\ (0.0253\ W/m\cdot K)/2.5m = 3.03\ W/m^2\cdot K.$ ◁

(b) Substituting numerical values typical of *summer* conditions gives

$$Ra_L = \frac{9.8\ m/s^2 \times 3.279\times10^{-3}\ K^{-1}(37-27)\ K\ (2.5\ m)^3}{23.2\times10^{-6}\ m^2/s \times 16.39\times10^{-6}\ m^2/s} = 1.320\times10^{10}$$

$$\overline{Nu}_L = \left\{0.825 + \frac{0.387(1.320\times10^{10})^{0.1667}}{[1 + (0.492/0.706)^{0.563}]^{0.296}}\right\}^2 = 275.8\ .$$

Hence, $\bar{h} = \overline{Nu}_L\ k/L = 275.8 \times 0.0267\ W/m\cdot K/2.5\ m = 2.94\ W/m^2\cdot K.$ ◁

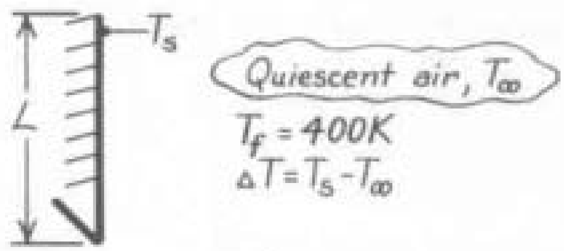
COMMENTS: There is a small influence due to T_f on $\bar{h}$ for these conditions. We should expect radiation effects to be important with such low values of $\bar{h}$.

PROBLEM 9.10

KNOWN: Vertical plate experiencing free convection with quiescent air at atmospheric pressure and film temperature 400 K.

FIND: Form of correlation for average heat transfer coefficient in terms of ΔT and characteristic length.

SCHEMATIC:



ASSUMPTIONS: (1) Air is extensive, quiescent medium, (2) Perfect gas behavior.

PROPERTIES: *Table A.6*, Air (T_f = 400 K, 1 atm): $\nu = 26.41\times10^{-6}$ m²/s, k = 0.0338 W/m·K, $\alpha = 38.3\times10^{-6}$ m²/s.

ANALYSIS: Consider the correlation having the form of Eq. 9.24 with Ra_L defined by Eq. 9.25.

$$\overline{Nu}_L = \bar{h}_L L/k = C Ra_L^n \quad (1)$$

where

$$Ra_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha} = \frac{9.8\ \text{m/s}^2 (1/400\ \text{K})\Delta T \cdot L^3}{26.41\times10^{-6}\ \text{m}^2/\text{s}\times 38.3\times10^{-6}\ \text{m}^2/\text{s}} = 2.422\times10^7 \Delta T \cdot L^3. \quad (2)$$

Combining Eqs. (1) and (2),

$$\bar{h}_L = (k/L) C Ra_L^n = \frac{0.0338\ \text{W/m}\cdot\text{K}}{L} C(2.422\times10^7 \Delta T L^3)^n. \quad (3)$$

From Fig. 9.6, note that for laminar boundary layer conditions, $10^4 < Ra_L < 10^9$, C = 0.59 and n = 1/4. Using Eq. (3),

$$\bar{h}_L = 1.40[L^{-1}(\Delta T\cdot L^3)^{1/4}] = 1.40\left(\frac{\Delta T}{L}\right)^{1/4} \quad \triangleleft$$

For turbulent conditions in the range $10^9 < Ra_L < 10^{13}$, C = 0.10 and n = 1/3. Using Eq. (3),

$$\bar{h}_L = 0.98[L^{-1}(\Delta T\cdot L^3)^{1/3}] = 0.98\Delta T^{1/3}. \quad \triangleleft$$

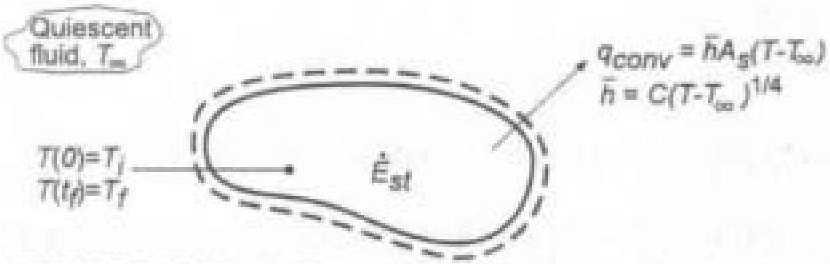
COMMENTS: Note carefully the dependence of ΔT and L on the average heat transfer coefficient for laminar and turbulent conditions. It is important to note that the characteristic length L does not influence $\bar{h}_L$ for turbulent conditions.

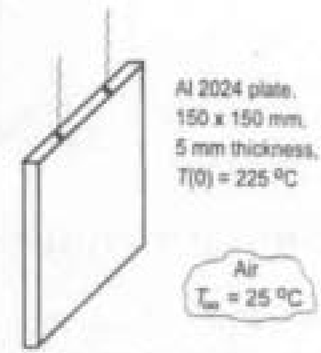
PROBLEM 9.11

KNOWN: Temperature dependence of free convection coefficient, $\bar{h} = C\Delta T^{1/4}$, for a solid suddenly submerged in a quiescent fluid.

FIND: (a) Expression for cooling time, t_f, (b) Considering a plate of prescribed geometry and thermal conditions, the time required to reach 80°C using the appropriate correlation from Problem 9.10 and (c) Plot the temperature-time history obtained from part (b) and compare with results using a constant $\bar{h}_o$ from an appropriate correlation based upon an average surface temperature $\bar{T} = (T_i + T_f)/2$.

SCHEMATIC:





ASSUMPTIONS: (1) Lumped capacitance approximation is valid, (2) Negligible radiation, (3) Constant properties.

PROPERTIES: *Table A.1*, Aluminum alloy 2024 $(\bar{T} = (T_i + T_f)/2 = 400\,K)$: $\rho = 2770$ kg/m^3, $c_p = 925$ J/kg·K, k = 186 W/m·K; *Table A.4*, Air ($\bar{T}_{film} = 362$ K): $\nu = 2.221 \times 10^{-5}$ m^2/s, k = 0.03069 W/m·K, $\alpha = 3.187 \times 10^{-5}$ m^2/s, Pr = 0.6976, $\beta = 1/\bar{T}_{film}$.

ANALYSIS: (a) Apply an energy balance to a control surface about the object, $-\dot{E}_{out} = \dot{E}_{st}$, and substitute the convection rate equation, with $\bar{h} = C\Delta T^{1/4}$, to find

$$-CA_s(T - T_\infty)^{5/4} = d/dt(\rho VcT). \qquad (1)$$

Separating variables and integrating, find

$$dT/dt = -(CA_s/\rho Vc)(T - T_\infty)^{5/4}$$

$$\int_{T_i}^{T_f} \frac{dT}{(T-T_\infty)^{5/4}} = -\left(\frac{CA_s}{\rho Vc}\right)\int_0^{t_f} dt \qquad -4(T-T_\infty)^{-1/4}\Big|_{T_i}^{T_f} = -\frac{CA_s}{\rho Vc}t_f$$

$$t_f = \frac{4\rho Vc}{CA_s}\left[(T_f - T_\infty)^{-1/4} - (T_i - T_\infty)^{-1/4}\right] = \frac{4\rho Vc}{CA_s(T_i - T_\infty)^{1/4}}\left[\left(\frac{T_i - T_\infty}{T_f - T_\infty}\right)^{1/4} - 1\right]. \qquad (2) <$$

(b) Considering the aluminum plate, initially at T(0) = 225°C, and suddenly exposed to ambient air at $T_\infty = 25°C$, from Problem 9.10 the convection coefficient has the form

$$\bar{h}_t = 1.40\left(\frac{\Delta t}{L}\right)^{1/4} \qquad \bar{h}_t = C\Delta T^{1/4}$$

where $C = 1.40/L^{1/4} = 1.40/(0.150)^{1/4} = 2.2496\ W/m^2 \cdot K^{5/4}$. Using Eq. (2), find

Continued...

$$t_f = \frac{4 \times 2770\,\text{kg/m}^3 (0.150^2 \times 0.005)\text{m}^3 \times 925\,\text{J/kg}\cdot\text{K}}{2.2496\,\text{W/m}^2\cdot\text{K}^{5/4} \times 2 \times (0.150\text{m})^2 (225-25)^{1/4}\,\text{K}^{1/4}} \left[\left(\frac{225-25}{80-25} \right)^{1/4} - 1 \right] = 1154\text{s}$$

(c) For the vertical plate, Eq. 9.27 is an appropriate correlation. Evaluating properties at

$$\overline{T}_{film} = \left(\overline{T}_s + T_\infty\right)/2 = \left(\left(T_i + T_f\right)/2 + T_\infty\right)/2 = 362\text{ K}$$

where $\overline{T}_s = 426\text{K}$, the average plate temperature, find

$$Ra_L = \frac{g\beta\left(\overline{T}_s - T_\infty\right)L^3}{\nu\alpha} = \frac{9.8\,\text{m/s}^2 (1/362\text{K})(426-298)\text{K}(0.150\text{m})^3}{2.221\times10^{-5}\,\text{m}^2/\text{s} \times 3.187\times10^{-5}\,\text{m}^2/\text{s}} = 1.652\times10^7$$

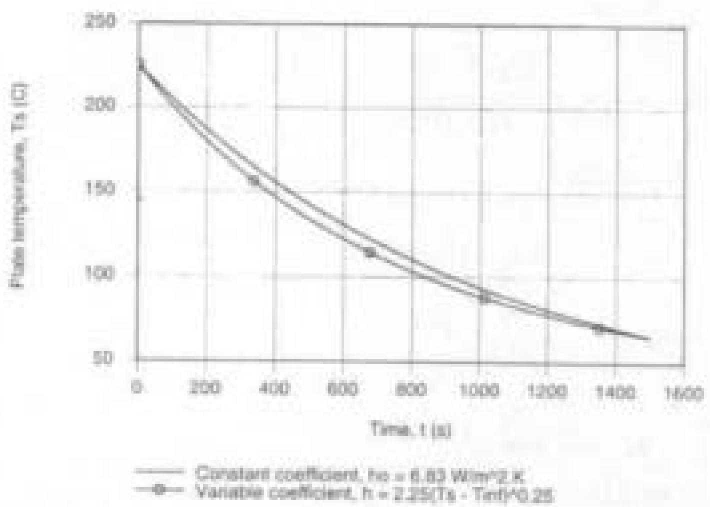
$$\overline{Nu}_L = 0.68 + \frac{0.670 Ra_L^{1/4}}{\left[1 + (0.492/Pr)^{9/16}\right]^{4/9}} = 0.68 + \frac{0.670\left(1.652\times10^7\right)^{1/4}}{\left[1 + (0.492/0.6976)^{9/16}\right]^{4/9}} = 33.4$$

$$\overline{h}_o = \frac{k}{L}\overline{Nu}_L = \frac{0.03069\,\text{W/m}\cdot\text{K}}{0.150\text{m}} \times 33.4 = 6.83\,\text{W/m}^2\cdot\text{K}$$

From Eq. 5.6, the temperature-time history with a constant convection coefficient is

$$T(t) = T_\infty + \left(T_i - T_\infty\right)\exp\left[-\left(\overline{h}_o A_s/\rho V c\right)t\right] \quad (3)$$

where $A_s/V = 2L^2/(L\times L\times w) = 2/w = 400\text{m}^{-1}$. The temperature-time histories for the $h = C\Delta T^{1/4}$ and $\overline{h}_o$ analyses are shown in plot below.



COMMENTS: (1) The times to reach $T(t_o) = 80°C$ were 1154 and 1212s for the variable and constant coefficient analysis, respectively, a difference of 5%. For convenience, it is reasonable to evaluate the convection coefficient as described in part (b).

(2) Note that $Ra_L < 10^9$ so indeed the expression selected from Problem 9.10 was the appropriate one.

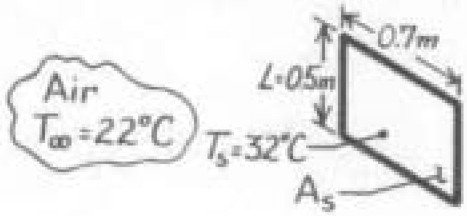
(3) Recognize that if the emissivity of the plate were unity, the average linearized radiation coefficient using Eq. (1.9) is $\overline{h}_{rad} = 11.0\,\text{W/m}^2\cdot\text{K}$ and radiative exchange becomes an important process.

PROBLEM 9.12

KNOWN: Oven door with average surface temperature of 32 °C in a room with ambient air at 22 °C.

FIND: Heat loss to the room. Also, find effect on heat loss if emissivity of door is unity and the surroundings are at 22 °C.

SCHEMATIC:



ASSUMPTIONS: (1) Ambient air is quiescent, (2) Surface radiation effects are negligible.

PROPERTIES: *Table A-4*, Air (T_f=300K, 1 atm): $\nu=15.89\times10^{-6}$ m²/s, k=0.0263 W/m·K, $\alpha=22.5\times10^{-6}$ m²/s, Pr=0.707, $\beta=1/T_f=3.33\times10^{-3}$ K^{-1}.

ANALYSIS: The heat rate from the oven door surface by convection to the ambient air is

$$q=\bar{h}\,A_s\,(T_s-T_\infty) \qquad (1)$$

where $\bar{h}$ can be estimated from the free-convection correlation for a vertical plate, Eq. 9.26,

$$\overline{Nu}_L=\frac{\bar{h}\,L}{k}=\left\{0.825+\frac{0.387\,Ra_L^{1/6}}{[1+(0.492/Pr)^{9/16}]^{8/27}}\right\}^2. \qquad (2)$$

The Rayleigh number, Eq. 9.25, is

$$Ra_L=\frac{g\beta(T_s-T_\infty)L^3}{\nu\alpha}=\frac{9.8\,m/s^2\,(1/300K)(32-22)K\times0.5^3\,m^3}{15.89\times10^{-6}\,m^2/s\times22.5\times10^{-6}\,m^2/s}=1.142\times10^8\,.$$

Substituting numerical values into Eq. (1), find

$$\overline{Nu}_L=\left\{0.825+\frac{0.387\,(1.142\times10^8)^{1/6}}{[1+(0.492/0.707)^{9/16}]^{8/27}}\right\}^2=63.5$$

$$\bar{h}_L=\frac{k}{L}\,\overline{Nu}_L=\frac{0.0263\,W/m\cdot K}{0.5\,m}\times63.5=3.34\,W/m^2\cdot K\,.$$

The heat rate using Eq. (1) is

$$q=3.34\,W/m^2\cdot K\times(0.5\times0.7)\,m^2\,(32-22)\,K=11.7\,W. \qquad \triangleleft$$

Heat loss by radiation, assuming $\epsilon=1$, is

$$q_{rad}=\epsilon\,A_s\,\sigma(T_s^4-T_{sur}^4)$$

$$q_{rad}=1(0.5\times0.7)\,m^2\times5.67\times10^{-8}\,W/m^2\cdot K^4\,[(273+32)^4-(273+22)^4]\,K^4=21.4\,W\,. \qquad \triangleleft$$

Note that heat loss by radiation is nearly double that by free convection.

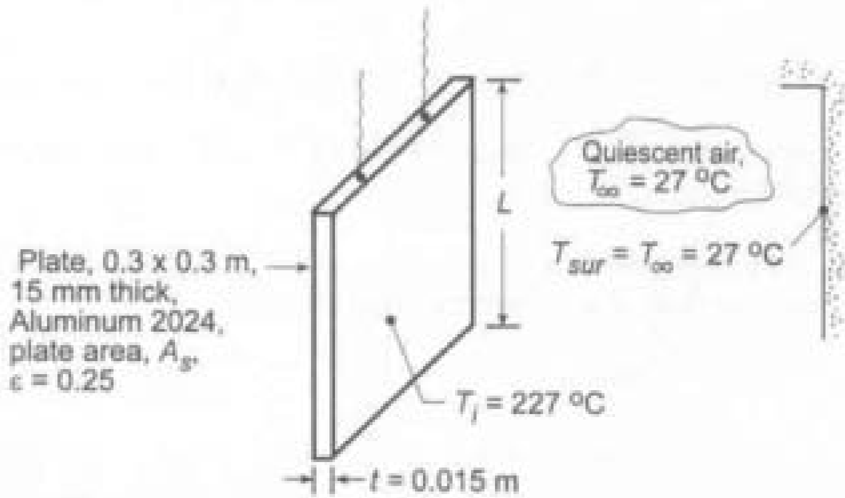
COMMENTS: (1) Note the characteristic length in the Rayleigh number is the height of the vertical plate (door).

PROBLEM 9.13

KNOWN: Aluminum plate (alloy 2024) at an initial uniform temperature of 227°C is suspended in a room where the ambient air and surroundings are at 27°C.

FIND: (a) Expression for time rate of change of the plate, (b) Initial rate of cooling (K/s) when plate temperature is 227°C, (c) Validity of assuming a uniform plate temperature, (d) Decay of plate temperature and the convection and radiation rates during cooldown.

SCHEMATIC:



ASSUMPTIONS: (1) Plate temperature is uniform, (2) Ambient air is quiescent and extensive, (3) Surroundings are large compared to plate.

PROPERTIES: *Table A.1*, Aluminum alloy 2024 (T = 500 K): ρ = 2770 kg/m³, k = 186 W/m·K, c = 983 J/kg·K; *Table A.4*, Air (T_f = 400 K, 1 atm): ν = 26.41 × 10^{-6} m²/s, k = 0.0388 W/m·K, α = 38.3 × 10^{-6} m²/s, Pr = 0.690.

ANALYSIS: (a) From an energy balance on the plate with free convection and radiation exchange, $-\dot{E}_{out} = \dot{E}_{st}$, we obtain

$$-\bar{h}_L 2A_s(T_s - T_\infty) - \varepsilon 2A_s\sigma(T_s^4 - T_{sur}^4) = \rho A_s tc\frac{dT}{dt} \quad \text{or} \quad \frac{dT}{dt} = \frac{-2}{\rho tc}\left[\bar{h}_L(T_s - T_\infty) + \varepsilon\sigma(T_s^4 - T_{sur}^4)\right] \quad <$$

where T_s, the plate temperature, is assumed to be uniform at any time.

(b) To evaluate (dT/dt), estimate $\bar{h}_L$. First, find the Rayleigh number,

$$Ra_L = g\beta(T_s - T_\infty)L^3/\nu\alpha = \frac{9.8\,m/s^2\,(1/400\,K)(227-27)K \times (0.3\,m)^3}{26.41\times10^{-6}\,m^2/s \times 38.3\times10^{-6}\,m^2/s} = 1.308\times10^8.$$

Eq. 9.27 is appropriate; substituting numerical values, find

$$\overline{Nu}_L = 0.68 + \frac{0.670 Ra_L^{1/4}}{\left[1+(0.492/Pr)^{9/16}\right]^{4/9}} = 0.68 + \frac{0.670(1.308\times10^8)^{1/4}}{\left[1+(0.492/0.690)^{9/16}\right]^{4/9}} = 55.5$$

$$\bar{h}_L = \overline{Nu}_L k/L = 55.5\times0.0338\,W/m\cdot K/0.3\,m = 6.25\,W/m^2\cdot K$$

Continued...

$$\frac{dT}{dt} = \frac{-2}{2770\,\text{kg/m}^3 \times 0.015\,\text{m} \times 983\,\text{J/kg}\cdot\text{K}} \times$$

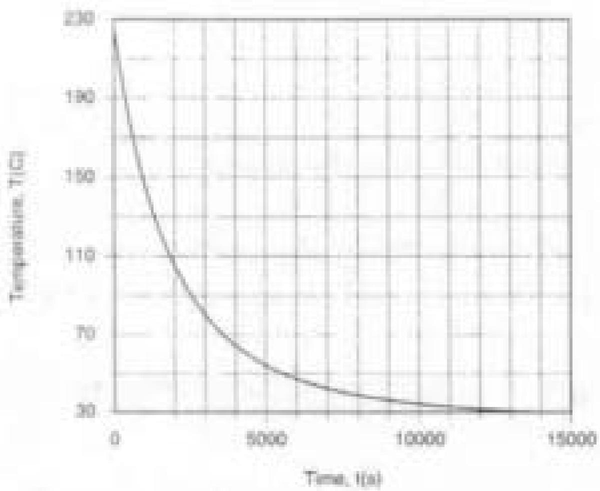
$$\left[6.25\,\text{W/m}^2\cdot\text{K}(227-27)\text{K} + 0.25\left(5.67\times10^{-8}\,\text{W/m}^2\cdot\text{K}^4\right)\left(500^4 - 300^4\right)\text{K}^4\right] = -0.099\,\text{K/s.} <$$

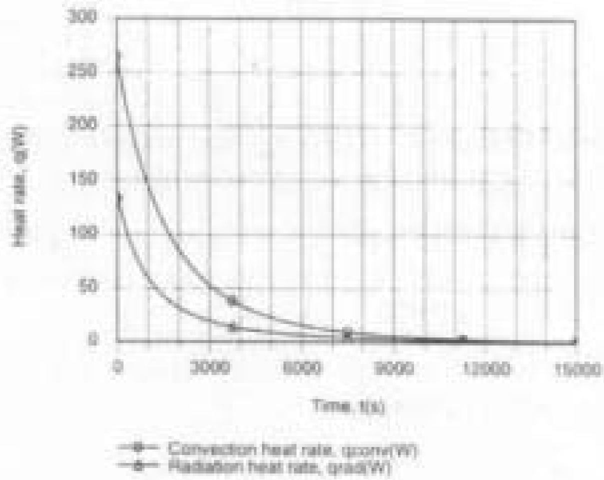
(c) The uniform temperature assumption is justified if the Biot number criterion is satisfied. With $L_c \equiv (V/2A_s) = (A_s \cdot t/2A_s) = (t/2)$ and $\bar{h}_{tot} = \bar{h}_{conv} + \bar{h}_{rad}$, $Bi = \bar{h}_{tot}(t/2)/k \le 0.1$. Using the linearized radiation coefficient relation, find

$$\bar{h}_{rad} = \varepsilon\sigma\left(T_s + T_{sur}\right)\left(T_s^2 + T_{sur}^2\right) = 0.25\left(5.67\times10^{-8}\,\text{W/m}^2\cdot\text{K}^4\right)(500+300)\left(500^2+300^2\right)\text{K}^3 = 3.86\,\text{W/m}^2\cdot\text{K}$$

Hence, $Bi = (6.25 + 3.86)\ \text{W/m}^2\cdot\text{K}(0.015\ \text{m}/2)/186\ \text{W/m}\cdot\text{K} = 4.07 \times 10^{-4}$. Since $Bi << 0.1$, the assumption is appropriate.

(d) The temperature history of the plate was computed by combining the *Lumped Capacitance Model* of IHT with the appropriate *Correlations* and *Properties* Toolpads.





Due to the small values of $\bar{h}_L$ and $\bar{h}_{rad}$, the plate cools slowly and does not reach 30°C until t = 14000s = 3.89h. The convection and radiation rates decrease rapidly with increasing t (decreasing T), thereby decelerating the cooling process.

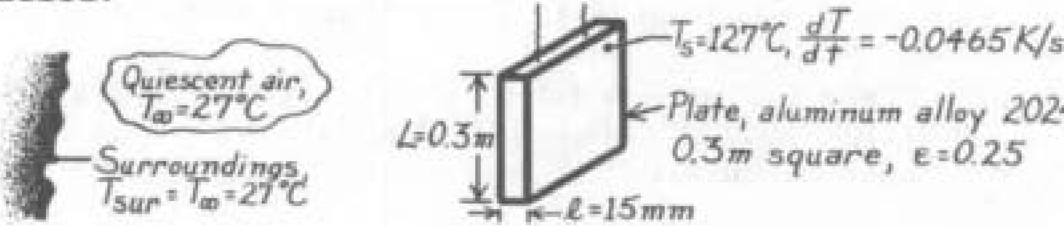
COMMENTS: The reduction in the convection rate with increasing time is due to a reduction in the thermal conductivity of air, as well as the values of $\bar{h}_L$ and T.

PROBLEM 9.14

KNOWN: Instantaneous temperature and time rate of temperature change of a vertical plate as described in Problem 9.13 while cooling in a room.

FIND: Average free convection coefficient for the prescribed conditions; compare with standard empirical correlation.

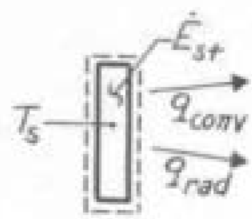
SCHEMATIC:



ASSUMPTIONS: (1) Uniform plate temperature, (2) Quiescent room air, (3) Surroundings large compared to surroundings.

PROPERTIES: *Table A-1*, Aluminum alloy 2024 ($T_s = 127\,^\circ C = 400K$): $\rho = 2770$ kg/m^3, $c_p = 925$ J/kg·K; *Table A-4*, Air ($T_f = (T_s + T_\infty)/2 = 350K$, 1 atm): $\nu = 20.92\times10^{-6}$ m^2/s, k = 0.020 W/m·K, $\alpha = 29.9\times10^{-6}$ m^2/s, Pr = 0.700.

ANALYSIS: From an energy balance on the plate considering free convection *and* radiation exchange,



$$\dot{E}_{in} - \dot{E}_{out} = \dot{E}_{st}$$

$$-\bar{h}_L(2A_s)(T_s - T_\infty) - \epsilon(2A_s)\sigma(T_s^4 - T_{sur}^4) = \rho A_s \ell c_p \frac{dT}{dt}\,.$$

Noting that the plate area is $2A_s$, solving for $\bar{h}_L$, and substituting numerical values, find

$$\bar{h}_L = \left[-\rho \ell c_p \frac{dT}{dt} - 2\epsilon\sigma(T_s^4 - T_{sur}^4)\right]/2(T_s - T_\infty)$$

$$\bar{h}_L = \left[-2770\,\text{kg/m}^3\times0.3\,\text{m}\times925\,\text{J/kg·K}(-0.0465\,\text{K/s}) - 2\times0.25\times5.67\times10^{-8}\,\text{W/m}^2\text{·K}^4(400^4-300^4)\text{K}^4\right]$$
$$/2(127-27)\,^\circ\text{C} = (8.936 - 2.455)\,\text{W/m}^2\text{·K}$$

$$\bar{h}_L = 6.5\,\text{W/m}^2\text{·K}\,. \qquad \triangleleft$$

To select an appropriate empirical correlation, first evaluate the Rayleigh number,

$$Ra_L = g\beta\Delta T L^3/\nu\alpha$$

$$Ra_L = 9.8\,\text{m/s}^2(1/350\text{K})(127-27)\text{K}(0.3\text{m})^3/(20.92\times10^{-6}\,\text{m}^2/\text{s})(29.9\times10^{-6}\,\text{m}^2/\text{s}) = 1.21\times10^8\,.$$

Since $Ra_L < 10^9$, the flow is laminar and Eq. 9.27 is applicable,

$$\overline{Nu}_L = \frac{\bar{h}_L L}{k} = 0.68 + \frac{0.670\,Ra_L^{1/4}}{[1 + (0.492/Pr)^{9/16}]^{4/9}}$$

$$\bar{h}_L = \left(\frac{0.030\,\text{W/m·K}}{0.3\,\text{m}}\right)\{0.68 + 0.670(1.21\times10^8)^{1/4}/[1+(0.492/0.700)^{9/16}]^{4/9}\} = 5.5\,\text{W/m}^2\text{·K}\,. \qquad \triangleleft$$

COMMENTS: (1) The correlation estimate is 15% lower than the experimental result. Using the Churchill-Chu relation, Eq. 9.26, which yields a less accurate estimate, $\bar{h}_L = 6.5\,\text{W/m}^2\text{·K}$. (2) This transient method, useful for obtaining an average free convection coefficient for spacewise isothermal objects, requires $Bi \leq 0.1$.

PROBLEM 9.15

KNOWN: Person, approximated as a cylinder, experiencing heat loss in water or air at 10 °C.

FIND: Whether heat loss from body in water is 30 times that in air.

ASSUMPTIONS: (1) Person can be approximated as a vertical cylinder of diameter D = 0.3 m and length L = 1.8 m, at 25 °C, (2) Loss is only from the lateral surface.

PROPERTIES: *Table A.4*, Air ($\overline{T} = (25 + 10)\,^\circ C/2 = 290$ K, 1 atm): $k = 0.0293$ W/m·K, $\nu = 19.91\times10^{-6}$ m²/s, $\alpha = 28.4\times10^{-6}$ m²/s; *Table A.6*, Water (290 K): $k = 0.598$ W/m·K, $\nu = \mu v_f = 1.081\times10^{-6}$ m²/s, $\alpha = kv_f/c_p = 1.431\times10^{-7}$ m²/s, $\beta_f = 174\times10^{-6}$ K^{-1}.

ANALYSIS: In both water (wa) and air (a), the heat loss from the lateral surface of the cylinder approximating the body is

$$q = \bar{h}\pi DL(T_s - T_\infty)$$

where T_s and T_∞ are the same for both situations. Hence,

$$\frac{q_{wa}}{q_a} = \frac{\bar{h}_{wa}}{\bar{h}_a}.$$

Vertical cylinder in air:

$$Ra_L = \frac{g\beta\Delta TL^3}{\nu\alpha} = \frac{9.8\ \text{m/s}^2\times(1/290\ \text{K})(25-10)\text{K}(1.8\ \text{m})^3}{19.91\times10^{-6}\ \text{m}^2/\text{s}\times28.4\times10^{-6}\ \text{m}^2/\text{s}} = 5.228\times10^9$$

Using Eq. 9.24 with C = 0.1 and n = 1/3,

$$\overline{Nu}_L = \frac{\bar{h}_L L}{k} = CRa_L^n = 0.1(5.228\times10^9)^{1/3} = 173.4 \qquad \bar{h}_L = 2.82\ \text{W/m}^2\cdot\text{K}.$$

Vertical cylinder in water:

$$Ra_L = \frac{9.8\ \text{m/s}^2\times174\times10^{-6}\ \text{K}^{-1}(25-10)\text{K}(1.8\ \text{m})^3}{1.081\times10^{-6}\ \text{m}^2/\text{s}\times1.431\times10^{-7}\ \text{m}^2/\text{s}} = 9.643\times10^{11}$$

Using Eq. 9.24 with C = 0.1 and n = 1/3,

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = CRa_L^n = 0.1(9.643\times10^{11})^{1/3} = 978.9 \qquad \bar{h}_L = 328\ \text{W/m}^2\cdot\text{K}.$$

Hence, from this analysis we find

$$\frac{q_{wa}}{q_a} = \frac{328\ \text{W/m}^2\cdot\text{K}}{2.8\ \text{W/m}^2\cdot\text{K}} = 117$$

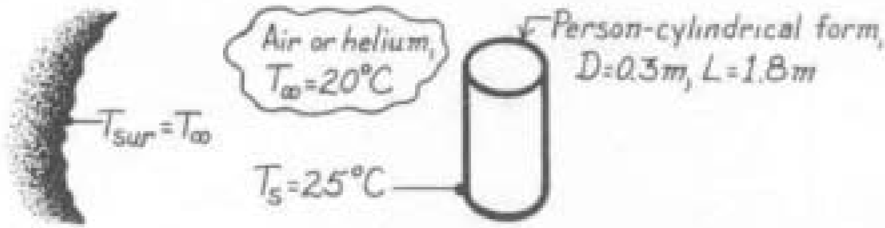
which compares poorly with the claim of 30.

PROBLEM 9.16

KNOWN: Person, approximated as a vertical cylinder (plate), having surface temperature of 25 °C, exposed to surroundings at quiescent ambient conditions at 20 °C.

FIND: Heat loss in ambient air. Compare loss if ambient is helium at 15 atm, the environment for deep sea diver.

SCHEMATIC:



ASSUMPTIONS: (1) Person can be approximated as vertical cylinder (plate), (2) Heat losses occur from lateral surface only, (3) Thermophysical properties k, c_p and μ independent of pressure for helium and air, (4) Surroundings are large compared to person.

PROPERTIES: *Table A.11,* Cloth (300 K): $\epsilon = 0.75$ to 0.90; *Table A.4,* Air (300 K, 1 atm): $k = 0.0263$ W/m·K, $\nu = 15.89\times10^{-6}$ m²/s, $\alpha = 22.5\times10^{-6}$ m²/s, Pr = 0.707; *Table A.4,* Helium (300 K, 15 atm): $k = 0.152$ W/m·K, $\nu = 122\times10^{-6}/15 = 8.13\times10^{-6}$ m²/s, since $\nu = \mu/\rho$ and $\nu \propto \rho^{-1} \propto p^{-1}$, $\alpha = 180\times10^{-6}$ m²/s, Pr = 0.680.

ANALYSIS: The heat loss from the surface due to convection and radiation is

$$q = q_{conv} + q_{rad} = \pi DL[\bar{h} + h_r](T_s - T_\infty) = \pi DL\bar{h}_{tot}(T_s - T_\infty)$$

where $h_r = \epsilon\sigma(T_s + T_\infty)(T_s^2 + T_\infty^2) = \epsilon 5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4(298 + 293)\text{K}(298^2 + 293^2)\text{K}^3 = 5.85\epsilon.$

Since ϵ varies from 0.75 to 0.90, $h_r = 4.4 \rightarrow 5.3$ W/m²·K. Estimate $\bar{h}$ from the vertical plate Churchill-Chu correlation with

$$Ra_L = \frac{g\beta\Delta TL^3}{\nu\alpha} \qquad \overline{Nu}_L = \frac{\bar{h}L}{k} = \left\{0.825 + \frac{0.387Ra_L^{1/6}}{[1 + (0.492/Pr)^{9/16}]^{8/27}}\right\}^2.$$

Substituting numerical values, the following results are obtained

	Air (1 atm)	Helium (15 atm)
Ra_L	2.664×10^9	6.507×10^8
$\overline{Nu}_L$	166.4	107.0
$\bar{h}_L$ (W/m²·K)	2.4	9.0
h_r (W/m²·K)	4.4–5.3	4.4–5.3
$\bar{h}$ (W/m²·K)	6.8–7.7	13.4–14.3
q (W)	57.7–65.3	114–121

◁

The effect of replacing atmospheric air with pressurized helium (15 atm) is to increase the heat loss by nearly 100%.

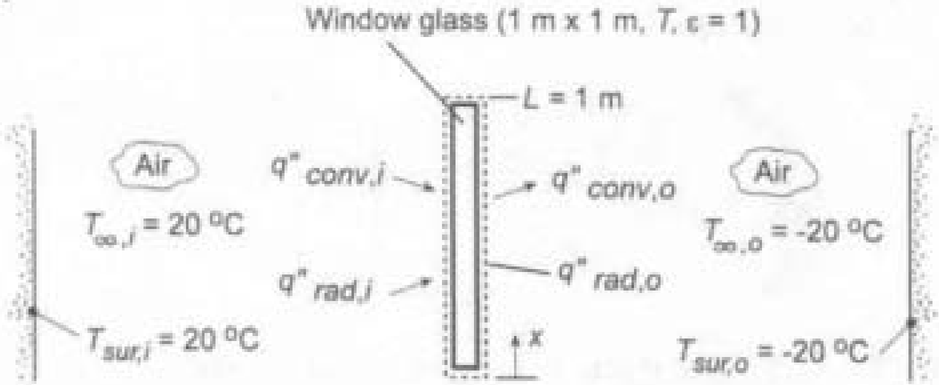
COMMENTS: (1) Note that radiation exchange is twice that of convection with air, but only half with helium. In either situation, using reflective clothing would reduce the heat losses significantly.

PROBLEM 9.17

KNOWN: Room and ambient air conditions for window glass.

FIND: Temperature of the glass and rate of heat loss.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Negligible temperature gradients in the glass, (3) Inner and outer surfaces exposed to large surroundings.

PROPERTIES: *Table A.4*, air ($T_{f,i}$ and $T_{f,o}$): Obtained from the IHT *Properties* Tool Pad.

ANALYSIS: Performing an energy balance on the window pane, it follows that $\dot{E}_{in} = \dot{E}_{out}$, or

$$\varepsilon\sigma\left(T_{sur,i}^4 - T^4\right) + \bar{h}_i\left(T_{\infty,i} - T\right) = \varepsilon\sigma\left(T^4 - T_{sur}^4\right) + \bar{h}_o\left(T - T_{\infty,o}\right)$$

where $\bar{h}_i$ and $\bar{h}_o$ may be evaluated from Eq. 9.26.

$$\overline{Nu}_L = \left\{0.825 + \frac{0.387 Ra_L^{1/6}}{\left[1 + (0.492/Pr)^{9/16}\right]^{8/27}}\right\}^2$$

Using the *First Law* Model for an *Isothermal Plane Wall* and the *Correlations* and *Properties* Tool Pads of IHT, the energy balance equation was formulated and solved to obtain

$$T = 273.8\ K$$ <

The heat rate is then $q_i = q_o$, or

$$q_i = L^2\left[\varepsilon\sigma\left(T_{sur,i}^4 - T^4\right) + \bar{h}_i\left(T_\infty - T\right)\right] = 174.8\ W$$ <

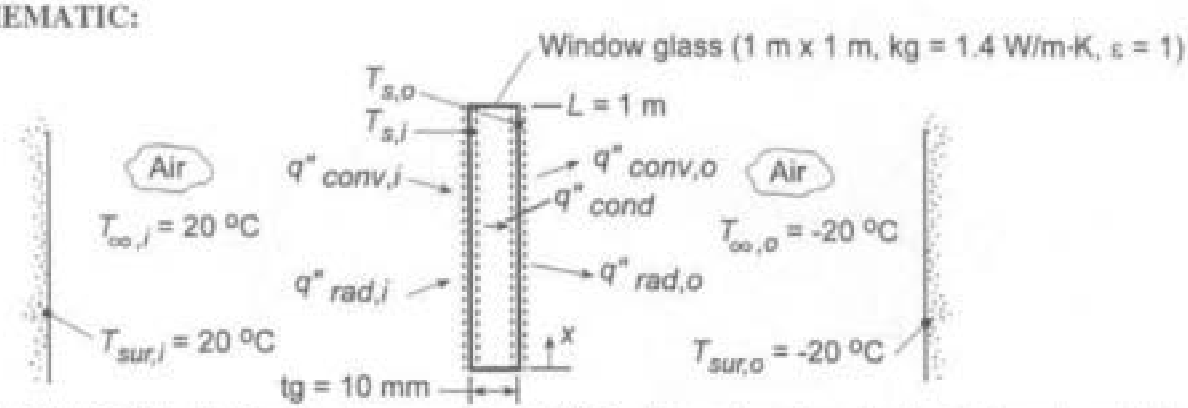
COMMENTS: The radiative and convective contributions to heat transfer at the inner and outer surfaces are $q_{rad,i} = 99.04$ W, $q_{conv,i} = 75.73$ W, $q_{rad,o} = 86.54$ W, and $q_{conv,o} = 88.23$ W, with corresponding convection coefficients of $\bar{h}_i = 3.95\ W/m^2 \cdot K$ and $\bar{h}_o = 4.23\ W/m^2 \cdot K$. The heat loss could be reduced significantly by installing a double pane window.

PROBLEM 9.18

KNOWN: Room and ambient air conditions for window glass. Thickness and thermal conductivity of glass.

FIND: Inner and outer surface temperatures and heat loss.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in the glass, (3) Inner and outer surfaces exposed to large surroundings.

PROPERTIES: *Table A.4*, air ($T_{f,i}$ and $T_{f,o}$): Obtained from the IHT *Properties* Tool Pad.

ANALYSIS: Performing energy balances at the inner and outer surfaces, we obtain, respectively,

$$\varepsilon\sigma\left(T_{sur,i}^4 - T_{s,i}^4\right) + \overline{h}_i\left(T_{\infty,i} - T_{s,i}\right) = (kg/tg)\left(T_{s,i} - T_{s,o}\right) \qquad (1)$$

$$(kg/tg)\left(T_{s,i} - T_{s,o}\right) = \varepsilon\sigma\left(T_{s,o}^4 - T_{sur,o}^4\right) + \overline{h}_o\left(T_{s,o} - T_{\infty,o}\right) \qquad (2)$$

where Eq. 9.26 may be used to evaluate $\overline{h}_i$ and $\overline{h}_o$

$$\overline{Nu}_L = \left\{0.825 + \frac{0.387 Ra_L^{1/6}}{\left[1 + (0.492/Pr)^{9/16}\right]^{8/27}}\right\}^2$$

Using the *First Law* Model for *One-dimensional Conduction* in a *Plane Wall* and the *Correlations* and *Properties* Tool Pads of IHT, the energy balance equations were formulated and solved to obtain

$T_{s,i} = 274.4$ K $\qquad T_{s,o} = 273.2$ K <

from which the heat loss is

$$q = \frac{k_g L^2}{t_g}\left(T_{s,i} - T_{s,o}\right) = 168.8\,W$$ <

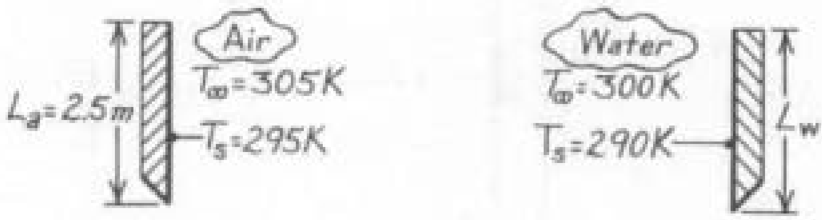
COMMENTS: By accounting for the thermal resistance of the glass, the heat loss is smaller (168.8 W) than that determined in the preceding problem (174.8 W) by assuming an isothermal pane.

PROBLEM 9.19

KNOWN: Air temperature and wall temperature and height for a room. Water temperature and wall temperature for a simulation experiment.

FIND: Required test cell height for similarity. Ratio of average convection coefficients for the two cases.

SCHEMATIC:



ASSUMPTIONS: (1) Air and water are quiescent, (2) Flow conditions correspond to free convection boundary layer development on an isothermal vertical plate, (3) Constant properties.

PROPERTIES: *Table A.4*, Air (T_f = 300 K, 1 atm): $\nu = 15.9\times10^{-6}$ m²/s, $\alpha = 22.5\times10^{-6}$ m²/s, $\beta = 1/T_f = 3.33\times10^{-3}K^{-1}$, k = 0.0263 W/m·K; *Table A.6*, Water (T_f = 295 K): ρ = 998 kg/m³, $\mu = 959\times10^{-6}$ N·s/m², c_p = 4181 J/kg·K, $\beta = 227.5\times10^{-6}$ K^{-1}, k = 0.606 W/m·K; hence, $\nu = \mu/\rho = 9.61\times10^{-7}$ m²/s, $\alpha = k/\rho c_p = 1.45\times10^{-7}$ m²/s.

ANALYSIS: Similarity requires that $Ra_{L,a} = Ra_{L,w}$ where

$$Ra_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha}$$

Hence,

$$\frac{L_w}{L_a} = \left[\frac{(\alpha\nu)_w}{(\alpha\nu)_a}\frac{\beta_{air}}{\beta_{H_2O}}\right]^{1/3} = \left[\frac{9.61\times1.45\times10^{-14}}{15.9\times22.5\times10^{-12}}\frac{3.33\times10^{-3}}{0.228\times10^{-3}}\right]^{1/3}$$

$$L_w = 2.5\text{ m}(0.179) = 0.45\text{ m.} \qquad \triangleleft$$

If $Ra_{L,a} = Ra_{L,w}$, it follows that $\overline{Nu}_{L,a} = \overline{Nu}_{L,w}$. Hence,

$$\frac{\bar{h}_a}{\bar{h}_w} = \frac{L_w}{L_a}\frac{k_a}{k_w} = \frac{0.45}{2.5}\frac{0.0263}{0.606} = 7.81\times10^{-3}. \qquad \triangleleft$$

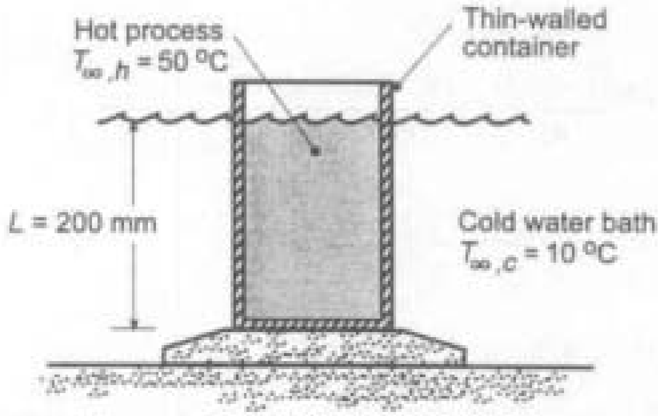
COMMENTS: Similitude allows us to obtain valuable information for one system by performing experiments for a smaller system and a different fluid.

PROBLEM 9.20

KNOWN: Thin-walled container with hot process fluid at 50°C placed in a quiescent, cold water bath at 10°C.

FIND: (a) Overall heat transfer coefficient, U, between the hot and cold fluids, and (b) Compute and plot U as a function of the hot process fluid temperature for the range $20 \le T_{\infty,h} \le 50°C$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) Heat transfer at the surfaces approximated by free convection from a vertical plate, (3) Fluids are extensive and quiescent, (4) Hot process fluid thermophysical properties approximated as those of water, and (5) Negligible container wall thermal resistance.

PROPERTIES: *Table A.6*, Water (assume $T_{f,h}$ = 310 K): $\rho_h = 1/1.007 \times 10^{-3} = 993\ kg/m^3$, $c_{p,h} = 4178\ J/kg\cdot K$, $\nu_h = \mu_h/\rho_h = 695 \times 10^{-6}\ N\cdot s/m^2 / 993\ kg/m^3 = 6.999 \times 10^{-7}\ m^2/s$, $k_h = 0.628\ W/m\cdot K$, $Pr_h = 4.62$, $\alpha_h = k_h/\rho_h c_{p,h} = 1.514 \times 10^{-7}\ m^2/s$, $\beta_h = 361.9 \times 10^{-6}\ K^{-1}$; *Table A.6*, Water (assume $T_{f,c}$ = 295 K): $\rho_c = 1/1.002 \times 10^{-3} = 998\ kg/m^3$, $c_{p,c} = 4181\ J/kg\cdot K$, $\nu_c = \mu_c/\rho_c = 959 \times 10^{-6}\ N\cdot s/m^2 / 998\ kg/m^3 = 9.609 \times 10^{-7}\ m^2/s$, $k_c = 0.606\ W/m\cdot K$, $Pr_c = 6.62$, $\alpha_c = k_c/\rho_c c_{p,c} = 1.452 \times 10^{-7}\ m^2/s$, $\beta_c = 227.5 \times 10^{-6}\ K^{-1}$.

ANALYSIS: (a) The overall heat transfer coefficient between the hot process fluid, $T_{\infty,h}$, and the cold water bath fluid, $T_{\infty,c}$, is

$$U = \left(1/\bar{h}_h + 1/\bar{h}_c\right)^{-1} \quad (1)$$

where the average free convection coeffieicnts can be estimated from the vertical plate correlation Eq. 9.26, with the Rayleigh number, Eq. 9.25,

$$\overline{Nu}_L = \left\{0.825 + \frac{0.387 Ra_L^{1/6}}{\left[1 + (0.492/Pr)^{9/16}\right]^{8/27}}\right\}^2 \qquad Ra_L = \frac{g\beta\Delta T L^3}{\nu\alpha} \quad (2,3)$$

To affect a solution, assume $T_s = \left(T_{\infty,h} - T_{\infty,c}\right)/2 = 30°C = 303\,K$, so that the hot and cold fluid film temperatures are $T_{f,h}$ = 313 K ≈ 310 K and $T_{f,c}$ = 293 K ≈ 295 K. From an energy balance across the container walls,

$$\bar{h}_h\left(T_{\infty,h} - T_s\right) = \bar{h}_c\left(T_s - T_{\infty,c}\right) \quad (4)$$

the surface temperature T_s can be determined. Evaluating the correlation parameters, find:

Hot process fluid:

$$Ra_{L,h} = \frac{9.8\,m/s^2 \times 361.9 \times 10^{-6}\,K^{-1}(50-30)K(0.200\,m)^3}{6.999 \times 10^{-7}\,m^2/s \times 1.514 \times 10^{-7}\,m^2/s} = 5.357 \times 10^9$$

Continued

$$\overline{Nu}_{L,h} = \left\{ 0.825 + \frac{0.387\left(5.357\times10^{9}\right)^{1/6}}{\left[1+(0.492/4.62)^{9/16}\right]^{8/27}} \right\}^2 = 251.5$$

$$\overline{h}_h = \overline{Nu}_{L,h}\frac{h_h}{L} = 251.5\times0.628\,W/m^2\cdot K/0.200m = 790\,W/m^2\cdot K$$

Cold water bath:

$$Ra_{L,c} = \frac{9.8m/s^2\times227.5\times10^{-6}K^{-1}(30-10)K(0.200m)^3}{9.609\times10^{-7}\,m^2/s\times1.452\times10^{-7}\,m^2/s} = 2.557\times10^{9}$$

$$\overline{Nu}_{L,c} = \left\{ 0.825 + \frac{0.387\left(2.557\times10^{9}\right)^{1/6}}{\left[1+(0.492/6.62)^{9/16}\right]^{8/27}} \right\}^2 = 203.9$$

$$\overline{h}_c = 203.9\times0.606\,W/m\,K/0.200\,m = 618\,W/m^2\cdot K$$

From Eq. (1) find

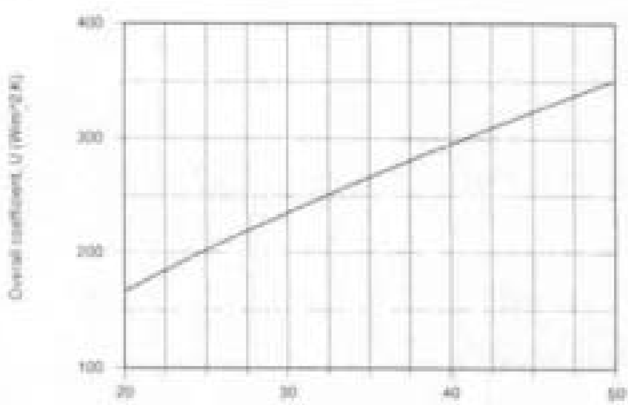
$$U = (1/790 + 1/618)^{-1}\,W/m^2\cdot K = 347\,W/m^2\cdot K$$ <

Using Eq.(4), find the resulting surface temperature

$$790\,W/m^2\cdot K(50-T_s)K = 618\,W/m^2\cdot K(T_s-30)K \qquad T_s = 32.4^\circ C$$

Which compares favorably with our assumed value of 30°C.

(b) Using the *IHT Correlations Tool, Free Convection, Vertical Plate* and following the foregoing approach, the overall coefficient was computed as a function of the hot fluid temperature and is plotted below. Note that U increases almost linearly with $T_{\infty,h}$.



COMMENTS: For the conditions of part (a), using the IHT model of part (b) with thermophysical properties evaluated at the proper film temperatures, find U = 352 W/m·K with T_s = 32.4°C. Our approximate solution was a good one.

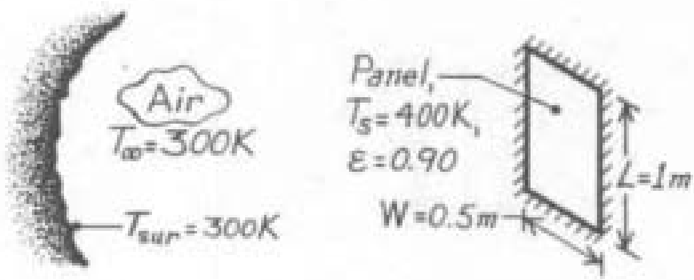
(2) Because the set of equations for part (b) is quite stiff, when using the IHT model you should follow the suggestions in the IHT Example 9.2 including use of the intrinsic function Tfluid_avg (T1,T2).

PROBLEM 9.21

KNOWN: Height, width, emissivity and temperature of heating panel. Room air and wall temperature.

FIND: Net rate of heat transfer from panel to room.

SCHEMATIC:



ASSUMPTIONS: (1) Quiescent air, (2) Walls of room form a large enclosure, (3) Negligible heat loss from back of panel.

PROPERTIES: *Table A.4*, Air (T_f = 350 K, 1 atm): $\nu = 20.9\times10^{-6}$ m²/s, k = 0.03 W/m·K, $\alpha = 29.9\times10^{-6}$ m²/s, Pr = 0.700.

ANALYSIS: The heat loss from the panel by convection and radiation exchange is

$$q = \bar{h}A(T_s - T_\infty) + \varepsilon\sigma A(T_s^4 - T_{sur}^4).$$

With

$$Ra_L = \frac{g\beta(T_s - T_\infty)L^3}{\alpha\nu} = \frac{9.8\ \text{m/s}^2(1/350\ \text{K})(100\ \text{K})(1\ \text{m})^3}{(20.9)(29.9)\times10^{-12}\ \text{m}^4/\text{s}^2} = 4.48\times10^9$$

and using the Churchill and Chu correlation for free convection from a vertical plate,

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = \left\{0.825 + \frac{0.387Ra_L^{1/6}}{[1+(0.492/Pr)^{9/16}]^{8/27}}\right\}^2 = 196$$

$$\bar{h} = 196\,k/L = 196\times0.03\text{W/m·K}/1\ \text{m} = 5.87\ \text{W/m}^2\cdot\text{K}.$$

Hence,

$$q = 5.86\ \text{W/m}^2\cdot\text{K}(0.5\ \text{m}^2)100\ \text{K}$$
$$+ 0.9\times5.67\times10^{-8}\text{W/m}^2\cdot\text{K}^4(0.5\ \text{m}^2)[(400)^4 - (300)^4]\text{K}$$

$$q = 293\ \text{W} + 447\ \text{W} = 740\ \text{W}.$$ ◁

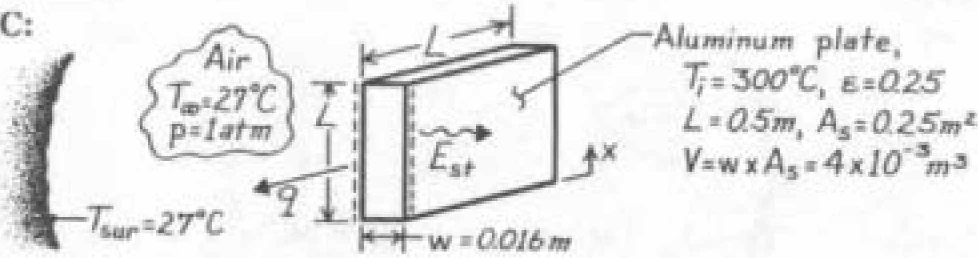
COMMENTS: As is typical of free convection in gases, heat transfer by surface radiation is comparable to, if not larger than, the convection rate. The *relative* contribution of free convection would increase with decreasing L and T_s.

PROBLEM 9.22

KNOWN: Initial temperature and dimensions of an aluminum plate. Condition of the plate surroundings. Plate emissivity.

FIND: (a) Initial cooling rate, (b) Validity of assuming negligible temperature gradients in the plate during the cooling process.

SCHEMATIC:



ASSUMPTIONS: (1) Plate temperature is uniform, (2) Chamber air is quiescent, (3) Plate surface is diffuse-gray, (4) Chamber surface is much larger than that of plate, (5) Negligible heat transfer from edges.

PROPERTIES: *Table A-1*, Aluminum (573 K): k = 232 W/m·K, c_p = 1022 J/kg·K, ρ = 2702 kg/m³; *Table A-4*, Air (T_f = 436 K, 1 atm): ν = 30.72×10⁻⁶ m²/s, α = 44.7×10⁻⁶ m²/s, k = 0.0363 W/m·K, Pr = 0.687, β = 0.00229 K⁻¹.

ANALYSIS: (a) Performing an energy balance on the plate,

$$-q = -2A_s[\bar{h}(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)] = \dot{E}_{st} = \rho V c_p[dT/dt]$$

$$dT/dt = -2[\bar{h}(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)]/\rho w c_p$$

Using the correlation of Eq. 9.27, with

$$Ra_L = \frac{g\beta(T_i - T_\infty)L^3}{\nu\alpha} = \frac{9.8\ \text{m/s}^2 \times 0.00229\text{K}^{-1}(300 - 27)\text{K}(0.5\ \text{m})^3}{30.72\times10^{-6}\ \text{m}^2/\text{s} \times 44.7\times10^{-6}\ \text{m}^2/\text{s}} = 5.58\times10^8$$

$$\bar{h} = \frac{k}{L}\left\{0.68 + \frac{0.670 Ra_L^{1/4}}{[1 + (0.492/Pr)^{9/16}]^{4/9}}\right\} = \frac{0.0363}{0.5}\left\{0.68 + \frac{0.670(5.58\times10^8)^{1/4}}{[1 + (0.492/0.687)^{9/16}]^{4/9}}\right\}$$

$$\bar{h} = 5.8\ \text{W/m}^2\cdot\text{K}.$$

Hence the initial cooling rate is

$$\frac{dT}{dt} = -\frac{2[5.8\ \text{W/m}^2\cdot\text{K}(300 - 27)\text{C}^\circ + 0.25\times5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4[(573\ \text{K})^4 - (300\ \text{K})^4]]}{2702\ \text{kg/m}^3(0.016\ \text{m})1022\ \text{J/kg}\cdot\text{K}}$$

$$\frac{dT}{dt} = -0.136\ \text{K/s}. \qquad \triangleleft$$

(b) To check the validity of neglecting temperature gradients across the plate thickness, calculate Bi = h_{eff} (w/2)/k where $h_{eff} = q''_{tot}/(T_i - T_\infty)$ = (1583 + 1413) W/m²/273 K = 11.0 W/m²·K. Hence

$$Bi = (11\ \text{W/m}^2\cdot\text{K})(0.008\ \text{m})/232\ \text{W/m}\cdot\text{K} = 3.8\times10^{-4} \qquad \triangleleft$$

and the assumption is excellent.

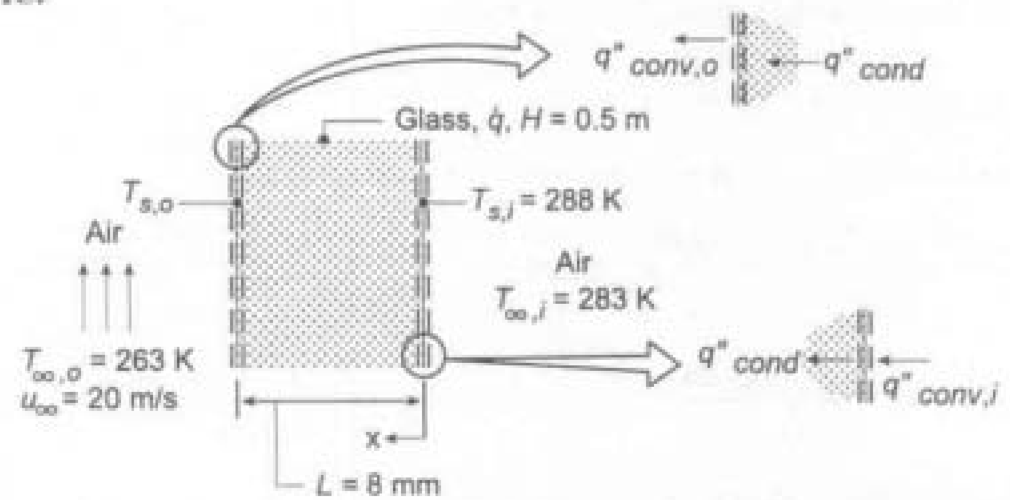
COMMENTS: (1) Longitudinal (x) temperature gradients are likely to be more severe than those associated with the plate thickness due to the variation of h with x. (2) Initially $q''_{conv} \approx q''_{rad}$.

PROBLEM 9.23

KNOWN: Boundary conditions associated with a rear window experiencing uniform volumetric heating.

FIND: (a) Volumetric heating rate $\dot{q}$ needed to maintain inner surface temperature at $T_{s,i} = 15°C$, (b) Effects of $T_{\infty,o}$, u_{∞}, and $T_{\infty,i}$ on $\dot{q}$ and $T_{s,o}$.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, one-dimensional conditions, (2) Constant properties, (3) Uniform volumetric heating in window, (4) Convection heat transfer from interior surface of window to interior air may be approximated as free convection from a vertical plate, (5) Heat transfer from outer surface is due to forced convection over a flat plate in parallel flow.

PROPERTIES: *Table A.3*, Glass (300 K): k = 1.4 W/m·K; *Table A.4*, Air ($T_{f,i}$ = 12.5°C, 1 atm): $\nu = 14.6 \times 10^{-6}$ m²/s, k = 0.0251 W/m·K, $\alpha = 20.59 \times 10^{-6}$ m²/s, $\beta = (1/285.5) = 3.503 \times 10^{-3}$ K⁻¹, Pr = 0.711; ($T_{f,o}$ = 0°C): $\nu = 13.49 \times 10^{-6}$ m²/s, k = 0.0241 W/m·K, Pr = 0.714.

ANALYSIS: (a) The temperature distribution in the glass is governed by the appropriate form of the heat equation, Eq. 3.39, whose general solution is given by Eq. 3.40.

$$T(x) = -(\dot{q}/2k)x^2 + C_1 x + C_2.$$

The constants of integration may be evaluated by applying appropriate boundary conditions at x = 0. In particular, with $T(0) = T_{s,i}$, $C_2 = T_{s,i}$. Applying an energy balance to the inner surface, $q''_{cond} = q''_{conv,i}$

$$-k\left.\frac{dT}{dx}\right|_{x=0} = \bar{h}_i\left(T_{\infty,i} - T_{s,i}\right) \qquad -k\left.\left(-\frac{\dot{q}}{k}x + C_1\right)\right|_{x=0} = \bar{h}_i\left(T_{\infty,i} - T_{s,i}\right)$$

$$C_1 = -\left(\bar{h}_i/k\right)\left(T_{\infty,i} - T_{s,i}\right)$$

$$T(x) = -(\dot{q}/2k)x^2 - \frac{\bar{h}_i\left(T_{\infty,i} - T_{s,i}\right)}{k}x + T_{s,i} \tag{1}$$

The required generation may then be obtained by formulating an energy balance at the outer surface, where $q''_{cond} = q''_{conv,o}$. Using Eq. (1),

$$-k\left.\frac{dT}{dx}\right|_{x=L} = \bar{h}_o\left(T_{s,o} - T_{\infty,o}\right) \tag{2}$$

Continued...

$$-k\frac{dT}{dx}\bigg|_{x=L} = -k\left(-\frac{\dot{q}L}{k}\right) + \bar{h}_i\left(T_{\infty,i} - T_{s,i}\right) = \dot{q}L + \bar{h}_i\left(T_{\infty,i} - T_{s,i}\right) \tag{3}$$

Substituting Eq. (3) into Eq. (2), the energy balance becomes

$$\dot{q}L = \bar{h}_o\left(T_{s,o} - T_{\infty,o}\right) + \bar{h}_i\left(T_{s,i} - T_{\infty,i}\right) \tag{4}$$

where $T_{s,o}$ may be evaluated by applying Eq. (1) at x = L.

$$T_{s,o} = -\frac{\dot{q}L^2}{2k} - \frac{\bar{h}_i\left(T_{\infty,i} - T_{s,i}\right)}{k}L + T_{s,i}. \tag{5}$$

The *inside* convection coefficient may be obtained from Eq. 9.26. With

$$Ra_H = \frac{g\beta(T_{s,i} - T_{\infty,i})H^3}{\nu\alpha} = \frac{9.8\,m/s^2\,(3.503\times10^{-3}\,K^{-1})(15-10)K(0.5\,m)^3}{14.60\times10^{-6}\,m^2/s \times 20.59\times10^{-6}\,m^2/s} = 7.137\times10^7,$$

$$\overline{Nu}_H = \left[0.825 + \frac{0.387Ra_H^{1/6}}{\left[1+(0.492/Pr)^{9/16}\right]^{8/27}}\right]^2 = \left[0.825 + \frac{0.387(7.137\times10^7)^{1/6}}{\left[1+(0.492/0.711)^{9/16}\right]^{8/27}}\right]^2 = 56$$

$$\bar{h}_i = \overline{Nu}_H\frac{k}{H} = \frac{56\times0.0251\,W/m\cdot K}{0.5\,m} = 2.81\,W/m^2\cdot K$$

The *outside* convection coefficient may be obtained by first evaluating the Reynolds number. With

$$Re_H = \frac{u_\infty H}{\nu} = \frac{20\,m/s\times0.5\,m}{13.49\times10^{-6}\,m^2/s} = 7.413\times10^5$$

and with $Re_{x,c} = 5\times10^5$, mixed boundary layer conditions exist. Hence,

$$\overline{Nu}_H = \left(0.037Re_H^{4/5} - 871\right)Pr^{1/3} = \left[0.037\left(7.413\times10^5\right)^{4/5} - 871\right](0.714)^{1/3} = 864$$

$$\bar{h}_o = \overline{Nu}_H(k/H) = (864\times0.0241\,W/m\cdot K)/0.5\,m = 41.6\,W/m^2\cdot K.$$

Eq. (5) may now be expressed as

$$T_{s,o} = -\frac{\dot{q}(0.008\,m)^2}{2(1.4\,W/m\cdot K)} - \frac{2.81\,W/m^2\cdot K(10-15)K}{1.4\,W/m\cdot K}\times0.008\,m + 288\,K = -2.286\times10^{-5}\dot{q} + 288.1\,K$$

or, solving for $\dot{q}$, $\quad \dot{q} = -43{,}745\left(T_{s,o} - 288.1\right) \quad (6)$

and substituting into Eq. (4),

$$-43{,}745\left(T_{s,o} - 288.1\right)(0.008\,m) = 41.6\,W/m^2\cdot K\left(T_{s,o} - 263K\right) + 2.81\,W/m^2\cdot K(288\,K - 283\,K).$$

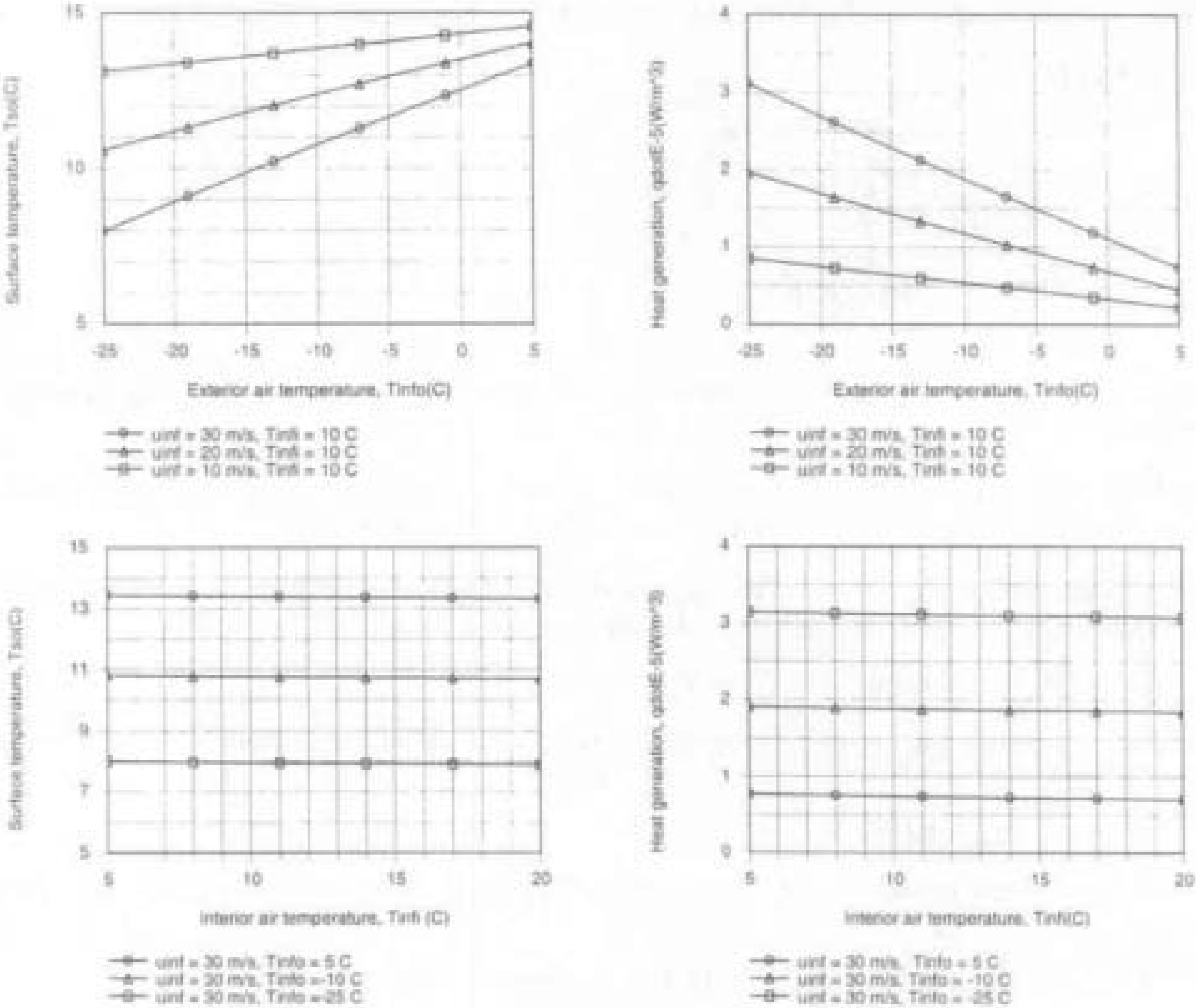
It follows that $T_{s,o} = 285.4$ K in which case, from Eq. (6)

$$\dot{q} = 118\,kW/m^3. \qquad <$$

(b) The parametric calculations were performed using the *One-Dimensional, Steady-state Conduction* Model of IHT with the appropriate *Correlations* and *Properties* Tool Pads, and the results are as follows.

Continued...

PROBLEM 9.23 (Cont.)



For fixed $T_{s,i}$ and $T_{\infty,i}$, $T_{s,o}$ and $\dot{q}$ are strongly influenced by $T_{\infty,o}$ and u_∞, increasing and decreasing, respectively, with increasing $T_{\infty,o}$ and decreasing and increasing, respectively with increasing u_∞. For fixed $T_{s,i}$ and u_∞, $T_{s,o}$ and $\dot{q}$ are independent of $T_{\infty,i}$, but increase and decrease, respectively, with increasing $T_{\infty,o}$.

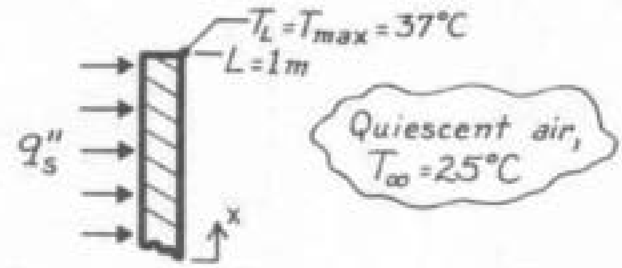
COMMENTS: In lieu of performing a surface energy balance at x = L, Eq. (4) may also be obtained by applying an energy balance to a control volume about the entire window.

PROBLEM 9.24

KNOWN: Vertical panel with uniform heat flux exposed to ambient air.

FIND: Allowable heat flux if maximum temperature is not to exceed a specified value, T_{max}.

SCHEMATIC:



ASSUMPTIONS: (1) Constant properties, (2) Radiative exchange with surroundings negligible.

PROPERTIES: *Table A-4*, Air ($T_f = (T_{L/2} + T_\infty)/2 = (35.4+25)°C/2 = 30.2°C = 303K$, 1 atm): $\nu = 16.19 \times 10^{-6}\ m^2/s$, $k = 26.5 \times 10^{-3}$ W/m·K, $\alpha = 22.9 \times 10^{-6}\ m^2/s$, Pr = 0.707.

ANALYSIS: Following the treatment of Section 9.6.1 for a vertical plate with uniform heat flux (constant q_s''), the heat flux can be evaluated as

$$q_s'' = \bar{h}\,\Delta T_{L/2} \qquad \text{where} \qquad \Delta T_{L/2} = T_s(L/2) - T_\infty \tag{1,2}$$

and $\bar{h}$ is evaluated using an appropriate correlation for a constant temperature vertical plate. From Eq. 9.28,

$$\Delta T_x \equiv T_x - T_\infty = 1.15\,(x/L)^{1/5}\,\Delta T_{L/2} \tag{3}$$

and recognizing that the maximum temperature will occur at the top edge, x = L, use Eq. (3) to find

$$\Delta T_{L/2} = (37-25)°C/1.15\,(1/1)^{1/5} = 10.4°C \qquad \text{or} \qquad T_{L/2} = 35.4°C.$$

Calculate now the Rayleigh number based upon $\Delta T_{L/2}$, with $T_f = (T_{L/2} + T_\infty)/2 = 303K$,

$$Ra_L = \frac{g\beta\Delta T L^3}{\nu\alpha} \qquad \text{where} \qquad \Delta T = \Delta T_{L/2} \tag{4}$$

$$Ra_L = 9.8 m/s^2\,(1/303K) \times 10.4K\,(1m)^3/16.19 \times 10^{-6}\ m^2/s \times 22.9 \times 10^{-6}\ m^2/s = 9.07 \times 10^8.$$

Since $Ra_L < 10^9$, the boundary layer flow is laminar; hence the correlation of Eq. 9.27 is appropriate,

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.68 + \frac{0.670\,Ra_L^{1/4}}{[1+(0.492/Pr)^{9/16}]^{4/9}} \tag{5}$$

$$\bar{h} = \left[\frac{0.0265\ W/m\cdot K}{1m}\right]\{0.68 + 0.670(9.07 \times 10^8)^{1/4}/[1+(0.492/0.707)^{9/16}]^{4/9}\} = 2.38\ W/m^2\cdot K.$$

From Eqs. (1) and (2) with numerical values for $\bar{h}$ and $\Delta T_{L/2}$, find

$$q_s'' = 2.38\ W/m^2\cdot K \times 10.4°C = 24.8\ W/m^2. \qquad \triangleleft$$

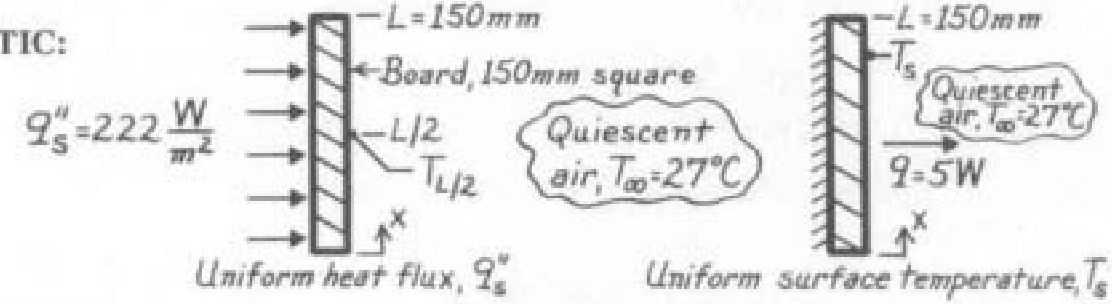
COMMENTS: Recognize that radiation exchange with the environment will be significant. Assuming $\bar{T}_s = T_{L/2}$, $T_{sur} = T_\infty$ and $\varepsilon = 1$, find $q''_{rad} = \sigma(\bar{T}_s^4 - T_{sur}^4) = 66\ W/m^2$.

PROBLEM 9.25

KNOWN: Vertical circuit board dissipating 5W to ambient air.

FIND: (a) Maximum temperature of the board assuming uniform surface heat flux and (b) Temperature of the board for an isothermal surface condition.

SCHEMATIC:



ASSUMPTIONS: (1) Board either uniform q_s'' or constant T_s, (2) Quiescent room air.

PROPERTIES: *Table A-4*, Air ($T_f = (T_{L/2}+T_\infty)/2$ or $(T_s+T_\infty)/2$, 1 atm), values used in iterations:

Iteration	T_f(K)	$\nu \cdot 10^6 (m^2/s)$	$k \cdot 10^3$ (W/m·K)	$\alpha \cdot 10^6 (m^2/s)$	Pr
1	312	17.10	27.2	24.3	0.705
2	323	18.20	28.0	25.9	0.704
3	318	17.70	27.6	25.2	0.704
4	320	17.90	27.8	25.4	0.704

ANALYSIS: (a) For the uniform heat flux case (see Section 9.6.1), the heat flux is

$$q_s'' = \bar{h}\,\Delta T_{L/2} \qquad \text{where} \qquad \Delta T_{L/2} = T_{L/2} - T_\infty \qquad (1,2)$$

and

$$q_s'' = q/A_s = 5W/(0.150m)^2 = 222\ W/m^2.$$

The maximum temperature on the board will occur at x = L and from Eq. 9.28 is

$$\Delta T_x = 1.15 (x/L)^{1/5}\,\Delta T_{L/2} \qquad (3)$$

$$T_L = T_{max} = T_\infty + 1.15\,\Delta T_{L/2}\,.$$

The average heat transfer coefficient $\bar{h}$ is estimated from a vertical (uniform T_s) plate correlation based upon the temperature difference $\Delta T_{L/2}$. Recognize that an iterative procedure is required: (i) assume a value of $T_{L/2}$, use Eq. (2) to find $\Delta T_{L/2}$; (ii) evaluate the Rayleigh number

$$Ra_L = g\,\beta\,\Delta T_{L/2}\,L^3/\nu\alpha \qquad (4)$$

and select the appropriate correlation (either Eq. 9.26 or 9.27) to estimate $\bar{h}$; (iii) use Eq. (1) with values of $\bar{h}$ and $\Delta T_{L/2}$ to find the calculated value of q_s''; and (iv) repeat this procedure until the calculated value for q_s'' is close to $q_s'' = 222\ W/m^2$, the required heat flux.

Continued .....

To evaluate properties for the correlation, use the film temperature,

$$T_f = (T_{L/2} + T_\infty)/2 \,. \qquad (5)$$

Iteration #1: Assume $T_{L/2} = 50°C$ and from Eqs. (2) and (5) find

$$\Delta T_{L/2} = (50-27)°C = 23°C \qquad T_f = (50+27)°C/2 = 312K \,.$$

From Eq. (4), with $\beta = 1/T_f$, the Rayleigh number is

$$Ra_L = 9.8 m/s^2 (1/312K) \times 23°C (0.150m)^3/(17.10 \times 10^{-6} m^2/s) \times (24.3 \times 10^{-6} m^2/s) = 5.868 \times 10^6 \,.$$

Since $Ra_L < 10^9$, the flow is laminar and Eq. 9.27 is appropriate

$$\overline{Nu}_L = \frac{\bar{h} L}{k} = 0.68 + \frac{0.670 \, Ra_L^{1/4}}{[1 + (0.492/Pr)^{9/16}]^{4/9}}$$

$$\bar{h}_L = \frac{0.0272 \, W/m \cdot K}{0.150 m} \{0.68 + 0.670(5.868 \times 10^6)^{1/4}/[1 + (0.492/0.705)^{9/16}]^{4/9}\} = 4.71 \, W/m^2 \cdot K \,.$$

Using Eq. (1), the calculated heat flux is

$$q_s'' = 4.71 \, W/m^2 \cdot K \times 23°C = 108 \, W/m^2 \,.$$

Since $q_s'' < 222 \, W/m^2$, the required value, another iteration with an increased estimate for $T_{L/2}$ is warranted. Further iteration results are tabulated.

Iteration	$T_{L/2}$(°C)	$\Delta T_{L/2}$(°C)	T_f(K)	Ra_L	$\bar{h}(W/m^2 \cdot K)$	$q_s''(W/m^2)$
2	75	48	323	1.044×10^7	5.58	267
3	65	38	318	8.861×10^6	5.28	200
4	68	41	320	9.321×10^6	5.39	221

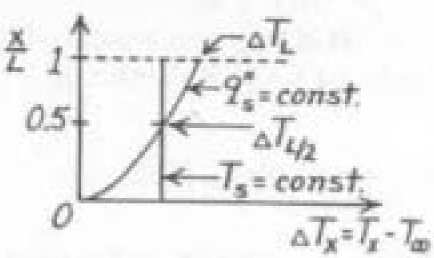
After Iteration 4, close agreement between the calculated and required q_s'' is achieved with $T_{L/2} = 68°C$. From Eq. (3), the maximum board temperature is

$$T_L = T_{max} = 27°C + 1.15(41)°C = 74°C \,. \qquad \triangleleft$$

(b) For the uniform temperature case, the procedure for estimation of the average heat transfer coefficient is the same. Hence,

$$T_s = T_{L/2}|_{q_s''} = 68°C \,. \qquad \triangleleft$$

COMMENTS: In both cases, $q = 5W$ and $\bar{h} = 5.38 \, W/m^2$. However, the temperature distributions for the two cases are quite different as shown on the sketch. For q_s'' = constant, $\Delta T_x \sim x^{1/5}$ according to Eq. 9.28.



CHAPTER 10

EIGENVALUES AND BOUNDARY VALUE PROBLEMS

SECTION 10.1

STURM-LIOUVILLE PROBLEMS AND EIGENFUNCTION EXPANSIONS

1. In the notation of Equation (9) in Section 10.1 of the text we have $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 1$, so Theorem 1 implies that the eigenvalues are all nonnegative. If $\lambda = 0$, then $y'' = 0$ implies that $y(x) = Ax + B$. Then $y'(x) = A$, so the endpoint conditions yield $A = 0$, but B remains arbitrary. Hence $\lambda_0 = 0$ is an eigenvalue with eigenfunction

$$y_0(x) = 1.$$

If $\lambda = \alpha^2 > 0$, then the equation $y'' + \alpha^2 y = 0$ has general solution

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

with

$$y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

Then $y'(0) = 0$ yields $B = 0$ so $A \neq 0$, and then

$$y'(L) = -A\alpha\sin\alpha L = 0,$$

so αL must be an integral multiple of π. Thus the nth positive eigenvalue is

$$\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{L^2},$$

and the associated eigenfunction is

$$y_n(x) = \cos\frac{n\pi x}{L}.$$

2. In the notation of Equation (9) in this section we have $\alpha_1 = \beta_2 = 1$ and $\alpha_2 = \beta_1 = 0$, so Theorem 1 implies that the eigenvalues are all nonnegative. If $\lambda = 0$, then $y'' = 0$ implies $y(x) = Ax + B$. But then $y(0) = B = 0$ and $y'(L) = A = 0$, so it follows that 0 is not an eigenvalue. We may therefore write $\lambda = \alpha^2 > 0$, so our equation is $y'' + \alpha^2 y = 0$ with general solution

$$y(x) = A\cos\alpha x + B\sin\alpha x.$$

Now $y(0) = A = 0$, so $y(x) = B\sin\alpha x$ and

$$y'(x) = B\alpha\cos\alpha x.$$

Hence

$$y'(L) = B\alpha\cos\alpha L = 0,$$

so it follows that αL must be an odd multiple of $\pi/2$. Thus

$$\alpha_n = \frac{(2n-1)\pi}{2L}, \quad \lambda_n = \alpha_n^2, \quad y_n(x) = \sin\alpha_n x.$$

3. If $\lambda = 0$ then $y'' = 0$ yields $y(x) = Ax + B$ as usual. But $y'(0) = A = 0$, and then $hy(L) + y'(L) = h(B) + 0 = 0$, so $B = 0$ also. Thus $\lambda = 0$ is not an eigenvalue. If $\lambda = \alpha^2 > 0$ so our equation is $y'' + \alpha^2 y = 0$, then

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

so

$$y'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

Now $y'(0) = 0$ yields $B = 0$, so we may write

$$y(x) = \cos\alpha x, \qquad y'(x) = -\alpha\sin\alpha x.$$

The equation

$$hy(L) + y'(L) = h\cos\alpha L - \alpha\sin\alpha L = 0$$

then gives

$$\tan\alpha L = \frac{h}{\alpha} = \frac{hL}{\alpha L},$$

so $\beta_n = \alpha_n L$ is the nth positive root of the equation

$$\tan x = \frac{hL}{x}.$$

Thus

$$\lambda_n = \alpha_n^2 = \frac{\beta_n^2}{L^2}, \qquad y_n(x) = \cos\frac{\beta_n x}{L}.$$

Finally, a sketch of the graphs $y = \tan x$ and $y = hL/x$ indicates that $\beta_n \approx (n-1)\pi$ for n large.

4. Here $\alpha_1 = h > 0$, $\alpha_2 = \beta_1 = 1$, and $\beta_2 = 0$, so by Theorem 1 in Section 10.1 there are no negative eigenvalues. If $\lambda = 0$ and $y(x) = Ax + B$, then the equations

$$hy(0) - y'(0) = hB - A = 0, \qquad y(L) = AL + B = 0$$

imply $h = A/B = -1/L < 0$. Thus 0 is not an eigenvalue. If $\lambda = \alpha^2 > 0$ and

$$y(x) = A\cos\alpha x + B\sin\alpha x,$$

then the condition $hy(0) = y'(0)$ yields $B = hA/\alpha$, so

$$y(x) = \frac{A}{\alpha}(\alpha\cos\alpha x + h\sin\alpha x)$$

$$= \frac{A}{\beta}\left(\beta\cos\frac{\beta x}{L} + hL\sin\frac{\beta x}{L}\right)$$

where $\beta = \alpha L$. Then the condition

$$y(L) = \frac{A}{\beta}(\beta\cos\beta + hL\sin\beta) = 0$$

reduces to $\tan\beta = -\dfrac{\beta}{hL}$.

6. $y_n(x) = \sin\dfrac{(2n-1)\pi x}{2L}$ so Equation (25) in Section 10.1 — with $r(x) \equiv 1$ — yields

$$c_n = \frac{\displaystyle\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx}{\displaystyle\int_0^L \sin^2\frac{(2n-1)\pi x}{2L}\,dx} = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx,$$

because the denominator integral here evaluates — by use of the trigonometric identity $\sin^2 A = \frac{1}{2}(1-\cos 2A)$ — to $L/2$.

7. The coefficient c_n in Eq. (23) of this section is given by Formula (25) with $f(x) = r(x) = 1$, $a = 0$, $b = L$, and $y_n(x) = \sin\dfrac{\beta_n x}{L}$. Using the fact that $\tan\beta_n = -\dfrac{\beta_n}{hL}$, so $\dfrac{\sin\beta_n}{\beta_n} = -\dfrac{\cos\beta_n}{hL}$, we find that

$$\int_0^L \sin^2\frac{\beta_n x}{L}\,dx = \int_0^L \frac{1}{2}\left(1-\cos\frac{2\beta_n x}{L}\right)dx = \frac{1}{2}\left[x - \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right]_0^L$$

$$= \frac{1}{2}\left(L - L\frac{\sin\beta_n}{\beta_n}\cos\beta_n\right) = \frac{1}{2}\left(L + L\frac{\cos\beta_n}{hL}\cos\beta_n\right) = \frac{hL+\cos^2\beta_n}{2h}$$

and $\int_0^L \sin\frac{\beta_n x}{L}dx = \frac{L(1-\cos\beta_n)}{\beta_n}$. Hence the desired eigenfunction expansion i

$$1 = 2hL\sum_{n=1}^{\infty}\frac{1-\cos\beta_n}{\beta_n\left(hL+\cos^2\beta_n\right)}\sin\frac{\beta_n x}{L}.$$

for $0 < x < L$.

8. The coefficient c_n in (23) is given by Formula (25) with $f(x) = r(x) = 1$, $a = 0$, $b = L$, and $y_n(x) = \cos\beta_n x/L$:

$$c_n = \frac{\int_0^L \cos\frac{\beta_n x}{L}dx}{\int_0^L \cos^2\frac{\beta_n x}{L}dx} = \frac{\int_0^L \cos\frac{\beta_n x}{L}dx}{\int_0^L \frac{1}{2}\left(1+\cos\frac{2\beta_n x}{L}\right)dx} = \frac{\left[\frac{L}{\beta_n}\sin\frac{\beta_n x}{L}\right]_0^L}{\left[\frac{1}{2}\left(x+\frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right)\right]}$$

$$= \frac{\frac{L}{\beta_n}\sin\beta_n}{\frac{1}{2}\left(L+\frac{L}{2\beta_n}\sin 2\beta_n\right)} = \frac{4\sin\beta_n}{2\beta_n+\sin 2\beta_n}.$$

Hence the desired eigenfunction expansion is

$$1 = \sum_{n=1}^{\infty}\frac{4\sin\beta_n}{2\beta_n+\sin 2\beta_n}\cos\frac{\beta_n x}{L}.$$

9. The coefficient c_n in (23) is given by Formula (25) with $f(x) = r(x) = 1$, $a = 0$, $b = 1$, and $y_n(x) = \sin\beta_n x$. Using the fact that $\tan\beta_n = -\beta_n/h$, so $h\sin\beta_n = -\beta_n\cos\beta_n$, we find that

$$\int_0^1 \sin^2\beta_n x\,dx = \int_0^1\frac{1}{2}(1-\cos 2\beta_n x)\,dx = \frac{1}{2}\left[x-\frac{\sin 2\beta_n x}{2\beta_n}\right]_0^1$$

$$= \frac{1}{2}\left(1-\frac{\sin\beta_n}{\beta_n}\cos\beta_n\right) = \frac{1}{2}\left(1+\frac{\cos^2\beta_n}{h}\right) = \frac{h+\cos^2\beta_n}{2h}$$

and

$$\int_0^1 x\sin\beta_n x\,dx = \frac{1}{\beta_n^2}\int_0^1\beta_n x\sin\beta_n x\cdot\beta_n dx = \frac{1}{\beta_n^2}\int_0^{\beta_n}u\sin u\,du$$

$$= \frac{1}{\beta_n^2}\left[\sin u - u\cos u\right]_0^{\beta_n} = \frac{\sin\beta_n-\beta_n\cos\beta_n}{\beta_n^2}$$

$$= \frac{\sin\beta_n-\beta_n\cos\beta_n}{\beta_n^2} = \frac{(1+h)\sin\beta_n}{\beta_n^2}.$$

It follows that the desired expansion is given by

$$x = 2h(1+h)\sum_{n=1}^{\infty}\frac{\sin\beta_n \sin\beta_n x}{\beta_n^2\left(h+\cos^2\beta_n\right)}$$

for $0 < x < 1$.

10. The coefficient c_n in (23) is given by Formula (25) with $f(x) = x$, $r(x) = 1$, $a = 0$, $b = 1$, and $y_n(x) = \cos\beta_n x$. Integrations similar to those in Problems 8 and 9 give

$$c_n = \frac{\int_0^1 x\cos\beta_n x\,dx}{\int_0^1 \cos^2\beta_n x\,dx} = \frac{4\left(\beta_n\sin\beta_n + \cos\beta_n - 1\right)}{\beta_n\left(2\beta_n + \sin 2\beta_n\right)}.$$

With this value of c_n for $n = 1, 2, 3, \cdots$, the desired eigenfunction expansion is

$$x = \sum_{n=1}^{\infty} c_n \cos\frac{\beta_n x}{L}.$$

11. If $\lambda = 0$ then $y'' = 0$ implies that $y(x) = Ax + B$. Then $y(0) = 0$ gives $B = 0$, so $y(x) = Ax$. Hence

$$hy(L) - y'(L) = h(AL) - A = A(hL - 1) = 0$$

if and only if $hL = 1$, in which case $\lambda_0 = 0$ has associated eigenfunction $y_0(x) = x$.

12. If $\lambda = -\alpha^2 < 0$, then the general solution of $y'' - \alpha^2 y = 0$ is

$$y(x) = A\cosh\alpha x + B\sinh\alpha x.$$

But then $y(0) = A = 0$, so we may take $y(x) = \sinh\alpha x$. Now the condition $hy(L) = y'(L)$ yields

$$h\sinh\alpha L = \alpha\cosh\alpha L.$$

It follows that $\beta = \alpha L$ must be a root of the equation

$$\tanh x = \frac{x}{hL}.$$

The curve $y = \tanh x$ passes through the origin with slope 1, and is concave upward for $x < 0$, concave downward for $x > 0$. Hence this curve and the straight line $y = x/hL$ intersect other than at the origin if and only if the slope of the line is less than 1 — that is, if and only if $hL > 1$. In this case, with β_0 the positive root of $\tanh x = x/hL$, we have $\lambda_0 = -\beta_0^2$ and $y_0(x) = \sinh\beta_0 x$.

13. If $\lambda = +\alpha^2 > 0$, then the general solution of $y'' + \alpha^2 y = 0$ is

$$y(x) = A\cos \alpha x + B\sin \alpha x.$$

But then $y(0) = A = 0$, so we may take $y(x) = \sin \alpha x$. Now the condition $hy(L) = y'(L)$ yields

$$h\sin \alpha L = \alpha \cos \alpha L.$$

It follows that $\beta = \alpha L$ must be a root of the equation

$$\tan x = \frac{x}{hL}.$$

So if β_n is the nth positive root of this equation, then $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and the corresponding eigenfunction is $y_n(x) = \sin \beta_n x / L$.

14. With $\lambda = 0$, $y'' = 0$, and hence $y(x) = Ax + B$, we have $y(0) = B = 0$, so $y(x) = Ax$. Then the condition $hy(L) = y'(L)$ reduces to the equation $hL = A$, which is satisfied because $hL = 1$. Thus $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = x$. Together with the positive eigenvalues and associated eigenfunctions provided by Problem 13, this gives the eigenfunction expansion

$$f(x) = c_0 x + \sum_{n=1}^{\infty} c_n \sin \frac{\beta_n x}{L}$$

where $\tan \beta_n = \beta_n$. The coefficients are given by

$$c_0 = \frac{\int_0^L f(x)\,x\,dx}{\int_0^L x^2\,dx} = \frac{3}{L^3}\int_0^L x f(x)\,dx,$$

$$c_0 = \frac{\int_0^L f(x)\sin \beta_n x/L\,dx}{\int_0^L \sin^2 \beta_n x/L\,dx} = \frac{2}{L\sin^2 \beta_n}\int_0^L f(x)\sin \beta_n x/L\,dx,$$

the latter because

$$\int_0^L \sin^2 \beta_n x/L\,dx = \frac{1}{2}\int_0^L (1-\cos 2\beta_n x/L)\,dx = \frac{1}{2}\left[x - \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right]_0^L$$

$$= \frac{1}{2}\left(L - L\cdot\frac{\sin \beta_n}{\beta_n}\cdot\cos \beta_n\right) = \frac{L}{2}\left(1-\cos^2 \beta_n\right) = \frac{L\sin^2 \beta_n}{2}.$$

15. If $\lambda_0 = 0$, then a general solution of $y'' = 0$ is $y(x) = Ax + B$. The conditions

$$y(0)+y'(0) = B+A = 0, \qquad y(1) = A+B = 0$$

both say that $B=-A$, so we may take $y_0(x)=x-1$ as the eigenfunction associated with $\lambda_0=0$. If $\lambda = +\alpha^2 < 0$, then the general solution of $y''+\alpha^2 y = 0$ is

$$y(x) = A\cos\alpha x + B\sin\alpha x.$$

But $y(0)+y'(0) = A+B\alpha = 0$, so $A = -B\alpha$, and then

$$y(1) = A\cos\alpha + B\sin\alpha = -B(\alpha\cos\alpha - \sin\alpha) = 0.$$

Thus the possible values of α are the positive roots $\{\beta_n\}$ of the equation $\tan x = x$, and the nth eigenfunction is $y_n(x) = \beta_n\cos\beta_n x - \sin\beta_n x$,

17. The Fourier sine series of the constant function $f(x)\equiv w$ for $0<x<L$ is

$$w = \frac{4w}{\pi}\sum_{n\ \text{odd}}\frac{1}{n}\sin\frac{n\pi x}{L}.$$

If $y=\sum b_n \sin n\pi x/L$, then

$$EI\,y^{(4)} = EI\sum_{n=1}^{\infty}\frac{n^4\pi^4 b_n}{L^4}\sin\frac{n\pi x}{L}.$$

Upon equating coefficients in these two series and solving for b_n, we see that

$$y(x) = \frac{4wL^4}{EI\pi^5}\sum_{n\ \text{odd}}\frac{1}{n^5}\sin\frac{n\pi x}{L}.$$

18. By Equation (16) in Section 9.3, the Fourier sine series of $f(x)=bx$ for $0<x<L$ is

$$bx = \frac{2bL}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}.$$

If $y=\sum b_n \sin n\pi x/L$, then

$$EI\,y^{(4)} = EI\sum_{n=1}^{\infty}\frac{n^4\pi^4 b_n}{L^4}\sin\frac{n\pi x}{L}.$$

Upon equating coefficients in these two series and solving for b_n, we see that

$$y(x) = \frac{2bL^5}{EI\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\sin\frac{n\pi x}{L}.$$

19. With $\lambda = \alpha^4$, the general solution of $y^{(4)} - \alpha^4 y = 0$ is

$$y(x) = A\cosh\alpha x + B\sinh\alpha x + C\cos\alpha x + D\sin\alpha x,$$

and then

$$y'(x) = \alpha(A\sinh\alpha x + B\cosh\alpha x - C\sin\alpha x + D\cos\alpha x).$$

The conditions $y(0) = 0$ and $y'(0) = 0$ yield $C = -A$ and $D = -B$, so now

$$y(x) = A(\cosh\alpha x - \cos\alpha x) + B(\sinh\alpha x - \sin\alpha x).$$

The conditions $y(L) = 0$ and $y'(L) = 0$ yield the two linear equations

$$A(\cosh\alpha L - \cos\alpha L) + B(\sinh\alpha L - \sin\alpha L) = 0,$$
$$A(\sinh\alpha L + \sin\alpha L) + B(\cosh\alpha L - \cos\alpha L) = 0.$$

This linear system can have a non-trivial solution for A and B only if its coefficient determinant vanishes,

$$(\cosh\alpha L - \cos\alpha L)^2 - (\sinh^2\alpha L - \sin^2\alpha L) = 0.$$

Using the facts that $\cosh^2 A - \sinh^2 A = 1$ and $\cos^2 A + \sin^2 A = 1$, this equation simplifies to

$$\cosh\alpha L\cos\alpha L - 1 = 0,$$

so $\beta = \alpha L = x$ satisfies the equation

$$\cosh x\cos x = 1.$$

The eigenvalue corresponding to the nth positive root β_n is

$$\lambda_n = \alpha_n^4 = \left(\frac{\beta_n}{L}\right)^4.$$

Finally the first equation in the pair above yields

$$B = -\frac{\cosh\alpha L - \cos\alpha L}{\sinh\alpha L - \sin\alpha L},$$

so we may take

$$\begin{aligned} y_n(x) = {} & (\sinh\beta_n - \sin\beta_n)\left(\cosh\frac{\beta_n x}{L} - \cos\frac{\beta_n x}{L}\right) \\ & - (\cosh\beta_n - \cos\beta_n)\left(\sinh\frac{\beta_n x}{L} - \sin\frac{\beta_n x}{L}\right) \end{aligned}$$

as the eigenfunction associated with the eigenvalue λ_n.

20. As in Problem 19, the solution of $y^{(4)} - \alpha^4 y = 0$ satisfying the left-endpoint conditions $y(0) = 0$ and $y'(0) = 0$ is given by

$$y(x) = A(\cosh \alpha x - \cos \alpha x) + B(\sinh \alpha x - \sin \alpha x).$$

The right-endpoint conditions $y''(L) = 0$ and $y^{(3)}(L) = 0$ now yield the two linear equations

$$A(\cosh \alpha L + \cos \alpha L) + B(\sinh \alpha L + \sin \alpha L) = 0,$$

$$A(\sinh \alpha L - \sin \alpha L) + B(\cosh \alpha L + \cos \alpha L) = 0.$$

This linear system can have a non-trivial solution for A and B only if its coefficient determinant vanishes,

$$(\cosh \alpha L + \cos \alpha L)^2 - (\sinh^2 \alpha L - \sin^2 \alpha L) = 0.$$

This equation simplifies to

$$\cosh \alpha L \cos \alpha L + 1 = 0,$$

so $\beta = \alpha L = x$ satisfies the equation

$$\cosh x \cos x = -1.$$

The eigenvalue corresponding to the nth root β_n is

$$\lambda_n = \alpha_n^4 = \left(\frac{\beta_n}{L}\right)^4.$$

Finally the first equation in the pair above yields

$$B = -\frac{\cosh \alpha L + \cos \alpha L}{\sinh \alpha L + \sin \alpha L},$$

so we may take

$$\begin{aligned} y_n(x) = {} & (\sinh \beta_n + \sin \beta_n)\left(\cosh \frac{\beta_n x}{L} - \cos \frac{\beta_n x}{L}\right) \\ & -(\cosh \beta_n + \cos \beta_n)\left(\sinh \frac{\beta_n x}{L} - \sin \frac{\beta_n x}{L}\right) \end{aligned}$$

as the eigenfunction associated with the eigenvalue λ_n.

21. As in Problem 19, the solution of $y^{(4)} - \alpha^4 y = 0$ satisfying the left-endpoint conditions $y(0) = 0$ and $y'(0) = 0$ is given by

$$y(x) = A(\cosh \alpha x - \cos \alpha x) + B(\sinh \alpha x - \sin \alpha x).$$

The right-endpoint conditions $y(L) = 0$ and $y''(L) = 0$ yield the two linear equations

$$A(\cosh \alpha L - \cos \alpha L) + B(\sinh \alpha L - \sin \alpha L) = 0,$$
$$A(\cosh \alpha L + \cos \alpha L) + B(\sinh \alpha L + \sin \alpha L) = 0.$$

This linear system can have a non-trivial solution for A and B only if its coefficient determinant vanishes,

$$(\cosh \alpha L - \cos \alpha L)(\sinh \alpha L + \sin \alpha L) \\ - (\cosh \alpha L + \cos \alpha L)(\sinh \alpha L - \sin \alpha L) = 0.$$

This equation simplifies to $2\cosh\alpha L \sin\alpha L - 2\cos\alpha L \sinh\alpha L = 0$, which is equivalent to $\tanh\alpha L = \tan\alpha L$. Hence $\beta = \alpha L = x$ satisfies the equation $\tanh x = \tan x$, and the eigenvalue corresponding to the nth positive root β_n is $\lambda_n = \alpha_n^4 = (\beta_n / L)^4$.

SECTION 10.2

APPLICATIONS OF EIGENFUNCTION SERIES

1. The substitution $u(x,t) = X(x)T(t)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad T' = -k\lambda T$$

with separation constant $\lambda = \alpha^2$. In Problem 3 of Section 10.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X'(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \cos\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = hL/x$. The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{\beta_n x}{L}\sinh\frac{\beta_n(L-y)}{L}.$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\sin\frac{\beta_n x}{L}\,dx}{(\sinh\beta_n)\int_0^L \sin^2\frac{\beta_n x}{L}\,dx} = \frac{4\beta_n}{L(\sinh\beta_n)(2\beta_n - \sin 2\beta_n)}\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx,$$

because

$$\int_0^L \sin^2\frac{\beta_n x}{L}\,dx = \int_0^L \frac{1}{2}\left(1-\cos\frac{2\beta_n x}{L}\right)dx = \left[\frac{1}{2}\left(x - \frac{L}{2\beta_n}\sin\frac{2\beta_n x}{L}\right)\right]_0^L$$

$$= \frac{1}{2}\left(L - \frac{L}{2\beta_n}\sin 2\beta_n\right) = \frac{L(2\beta_n - \sin 2\beta_n)}{4\beta_n}.$$

3. The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X'' - \alpha^2 X = 0 \qquad \text{and} \qquad Y'' + \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. Problem 3 of Section 10.1 we saw that the Sturm-Liouville problem

$$Y'' + \alpha^2 Y = 0, \qquad Y'(0) = hY(L) + Y'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$Y_n(y) = \cos\frac{\beta_n y}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = hL/x$. The solution of

$$X_n'' - \frac{\beta_n^2}{L^2}X_n = 0, \qquad X(L) = 0$$

is

$$X_n(x) = \sinh\frac{\beta_n(L-x)}{L},$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{\beta_n(L-x)}{L}\cos\frac{\beta_n y}{L}.$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L g(y)\cos\frac{\beta_n y}{L}\,dy}{(\sinh\beta_n)\int_0^L \cos^2\frac{\beta_n y}{L}\,dy} = \frac{2h}{(\sinh\beta_n)\left(hL+\sin^2\beta_n\right)}\int_0^L g(y)\cos\frac{\beta_n y}{L}\,dy,$$

because

$$\int_0^L \cos^2\frac{\beta_n y}{L}\,dy = \int_0^L \frac{1}{2}\left(1+\cos\frac{2\beta_n y}{L}\right)dy = \left[\frac{1}{2}\left(y+\frac{L}{2\beta_n}\sin\frac{2\beta_n y}{L}\right)\right]_0^L$$

$$= \frac{1}{2}\left(L+\frac{L}{2\beta_n}\sin 2\beta_n\right) = \frac{hL+\sin^2\beta_n}{2h}.$$

The final step here is the same as in Problem 1, using the fact that $(hL\cos\beta_n)/\beta_n = \sin\beta_n$ because $\tan\beta_n = hL/\beta_n$.

4. The substitution $u(x,y) = X(x)Y(y)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \quad\text{and}\quad Y'' - \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. In Example 5 of Section 10.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = -x/hL$. The bounded solution of

$$Y_n'' - \frac{\beta_n^2}{L^2}Y_n = 0$$

is

$$Y_n(y) = \exp\left(-\frac{\beta_n y}{L}\right),$$

so the resulting formal series solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{\beta_n x}{L}\exp\left(-\frac{\beta_n y}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\sin\frac{\beta_n x}{L}\,dx}{\int_0^L \sin^2\frac{\beta_n x}{L}\,dx} = \frac{4\beta_n}{L(2\beta_n - \sin 2\beta_n)}\int_0^L f(x)\cos\frac{\beta_n x}{L}\,dx,$$

the calculation of the denominator integral here being the same as in Problem 2.

5. The substitution $u(x,t) = X(x)T(t)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \quad \text{and} \quad T' = -k\lambda T$$

with separation constant $\lambda = \alpha^2$. In Problem 4 of Section 10.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad hX(0) - X'(0) = X(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation $\tan x = -x/hL$. The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{L^2}\right)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)dx}{\int_0^L \left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)^2 dx}.$$

The evaluation of the denominator integral here is elementary, but there seems little point in carrying it out explicitly.

6. The substitution $u(x,t) = X(x)T(t)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \quad \text{and} \quad T' = -k\lambda T$$

with separation constant $\lambda = \alpha^2$. In Problem 5 of Section 10.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad hX(0) - X'(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation

$$\tan x = \frac{2hLx}{x^2 - h^2L^2}.$$

The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{L^2}\right)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)dx}{\int_0^L \left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)^2 dx}.$$

7. The boundary value problem here is

$$\begin{aligned} &u_{xx} + u_{yy} = 0 \quad (0 < x < 1, \quad y > 0) \\ &u_x(0, y) = u(1, y) + u_x(1, t) = 0, \\ &u(x, 0) = 100. \end{aligned}$$

The substitution $u(x, y) = X(x)Y(y)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \quad \text{and} \quad Y'' - \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. In Problem 3 of Section 10.1 we saw (taking $h = L = 1$) that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X'(0) = X(1) + X'(1) = 0$$

with separation constant $\lambda = \alpha^2$. In Problem 5 of Section 10.1 we saw that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad hX(0) - X'(0) = hX(L) + X'(L) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2 = \beta_n^2 / L^2$ and eigenfunctions

$$X_n(x) = \beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}$$

for $n = 1, 2, 3, \cdots$, with $\{\beta_n\}$ being the positive roots of the equation

$$\tan x = \frac{2hLx}{x^2 - h^2L^2}.$$

The solution of $T_n' = -k\lambda_n T_n$ is then

$$T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{L^2}\right),$$

so the resulting formal series solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{L^2}\right)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L f(x)\left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)dx}{\int_0^L \left(\beta_n \cos\frac{\beta_n x}{L} + hL\sin\frac{\beta_n x}{L}\right)^2 dx}.$$

7. The boundary value problem here is

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad (0 < x < 1, \quad y > 0) \\ u_x(0, y) = u(1, y) + u_x(1, t) &= 0, \\ u(x, 0) &= 100. \end{aligned}$$

The substitution $u(x, y) = X(x)Y(y)$ yields the separated equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad Y'' - \alpha^2 Y = 0$$

with separation constant $\lambda = \alpha^2$. In Problem 3 of Section 10.1 we saw (taking $h = L = 1$) that the Sturm-Liouville problem

$$X'' + \alpha^2 X = 0, \qquad X'(0) = X(1) + X'(1) = 0$$

has eigenvalues $\lambda_n = \alpha_n^2$ and eigenfunctions

$$X_n(x) = \cos\alpha_n x$$

for $n = 1, 2, 3, \cdots$, with $\{\alpha_n\}$ being the positive roots of the equation $\tan x = 1/x$. The bounded solution of $Y_n'' - \alpha_n^2 Y_n = 0$ is then

$$Y_n(y) = \exp(-\alpha_n y),$$

so the resulting formal series solution is

$$u(x, y) = \sum_{n=1}^{\infty} c_n \cos\alpha_n x \exp(-\alpha_n y).$$

The coefficients in the eigenfunction expansion are given by

$$c_n = \frac{\int_0^L 100\cos\alpha_n x\,dx}{\int_0^L \cos^2\alpha_n x\,dx} = \frac{\left[\frac{100}{\alpha_n}\sin\alpha_n x\right]_0^1}{\left[\frac{1}{2}\left(x + \frac{1}{2\alpha_n}\sin 2\alpha_n x\right)\right]_0^1} = \frac{200\sin\alpha_n}{\alpha_n + \sin\alpha_n\cos\alpha_n},$$

so

$$u(x, y) = 200\sum_{n=1}^{\infty} \frac{\sin\alpha_n \cos\alpha_n x \exp(-\alpha_n y)}{\alpha_n + \sin\alpha_n\cos\alpha_n}.$$

The first five positive solutions of $\tan x = 1/x$ are 0.8603, 3.4256, 7.4373, 9.5293, and 12.6453, and we find that

$$u(1,1) \approx 30.8755 + 0.4737 + 0.0074 + 0.0002 + 0.0000 + \cdots \approx 31.4°\text{C}.$$

8. With $m = 0$ the boundary value problem in Example 2 is

$$\begin{aligned} u_{tt} &= a^2 u_{xx} \qquad (0 < x < L, \quad t > 0), \\ u(0,t) &= u_x(L,t) = 0, \\ u_t(x,0) &= 0, \\ u(x,0) &= bx. \end{aligned}$$

The substitution $u(x,t) = X(x)T(t)$ gives the separated equations

$$X'' + \lambda X = T'' + \lambda a^2 T = 0.$$

and the eigenfunctions of the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X(0) = X'(L) = 0$$

are of the form

$$X_n(x) = \sin\frac{n\pi x}{2L}$$

with n odd, with corresponding eigenvalue $\lambda_n = n^2\pi^2/4L^2$. This leads readily to the solution

$$u(x,t) = \sum_{n\ \mathrm{odd}} c_n \sin\frac{n\pi x}{2L}\cos\frac{n\pi at}{2L},$$

where c_n is the odd half-multiple sine coefficient (of Problem 21 in Section 9.3) given by

$$c_{2n-1} = \frac{2}{L}\int_0^L bx\sin\frac{(2n-1)\pi x}{L}\,dx = \frac{8bL\sin\frac{(2n-1)\pi}{2}}{(2n-1)^2\pi^2} = \frac{8bL(-1)^{n+1}}{(2n-1)^2\pi^2}.$$

9. **(a)** With $\lambda = 0$, the endpoint-value problem in (19) is $X'' = 0$, $X(0) = X'(0) = 0$, which has only the trivial solution $X(x) \equiv 0$. Thus $\lambda = 0$ is not an eigenvalue.

(b) With $\lambda = -\alpha^2 < 0$, the endpoint-value problem in (19) is

$$X'' - \alpha^2 X = 0, \qquad X(0) = 0, \quad -m\alpha^2 X(L) = A\delta X'(L).$$

The differential equation and the left-endpoint condition here give $X(x) = \sinh\alpha x$, and substitution in the right-endpoint condition gives

$$-m\alpha^2\sinh\alpha L = A\delta\alpha\cosh\alpha L, \text{ that is, } \tanh\alpha L = -\frac{k}{\alpha L}$$

with $k = A\delta L/m > 0$. But the graph $y = \tanh x$ lies (aside from the origin) in the first and third quadrants, while the graph $y = -k/x$ lies interior to the second and fourth quadrants. Hence the two cannot intersect, and it follows that there cannot be an eigenvalue of the assumed form $\lambda = -\alpha^2 < 0$,.

10. **(a)** With $\delta = 7.75$ gm/cm^3 and $E = 2\cdot 10^{12}$ in Equation (16), the speed of sound in steel is

$$a = \sqrt{\frac{E}{\delta}} \approx 5.08 \times 10^5 \text{ cm/sec} \approx 11364 \text{ mph}.$$

(b) With $\delta = 1$ gm/cm^3 and $K = 2.25\cdot 10^{10}$ in Equation (16), the speed of sound in water is

$$a = \sqrt{\frac{K}{\delta}} \approx 1.50 \times 10^5 \text{ cm/sec} \approx 3355 \text{ mph}.$$

11. **(a)** $$a = \sqrt{\frac{K}{\delta}} = \sqrt{\frac{\lambda p}{m/V}} = \sqrt{\frac{\gamma pV}{m}} = \sqrt{\frac{\gamma nRT_K}{nm_0}} = \sqrt{\frac{\gamma RT_K}{m_0}}$$

(b) $$a = \sqrt{\frac{\gamma RT_K}{m_0}} = \sqrt{\frac{1.4\times 8314(273+T_C)}{29}} = \sqrt{\frac{1.4\times 8314\times 273}{29}\left(1+\frac{T_C}{273}\right)}$$

$$\approx 331.02\sqrt{1+\frac{T_C}{273}}\ \frac{\text{m}}{\text{sec}} \approx 740.47\sqrt{1+\frac{T_C}{273}}\ \frac{\text{miles}}{\text{hour}}$$

$$\approx 740.47\left[1+\frac{1}{2}\left(\frac{T_C}{273}\right)+\cdots\right] \approx 740.47+1.356T_C$$

12. The boundary value problem is

$$u_{tt} = a^2 u_{xx} \quad (0 < x < L,\ t > 0)$$
$$u(0,t) = ku(L,t) + AEu_x(L,t) = 0$$
$$u(x,0) = f(x),$$
$$u_t(x,0) = 0.$$

Starting with the general solution

$$X(x) = A\cos\alpha x + B\sin\alpha x$$

of $X'' + \alpha^2 X = 0$, the condition $X(0) = 0$ gives $A = 0$, so

$$X(x) = \sin\alpha x, \qquad X'(x) = \alpha\cos\alpha x.$$

Then the condition $kX(L) + AEX'(L) = 0$ yields

$$k\sin\alpha L + AE\alpha\cos\alpha L = 0,$$

which is equivalent to the equation

$$\tan x = -\frac{AEx}{kL}$$

with $x = \alpha L$, $\alpha = x/L$. If $\{\beta_n\}$ are the positive roots of this equation, then the nth eigenvalue is $\lambda_n = \alpha_n^2 = (\beta_n/L)^2$ with associated eigenfunction

$$X_n(x) = \sin\frac{\beta_n x}{L}.$$

The associated function of t is

$$T_n(t) = A_n \cos\frac{\beta_n at}{L} + B_n \sin\frac{\beta_n at}{L},$$

but the condition $T'(0) = 0$ yields $B_n = 0$. Hence we obtain a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{\beta_n x}{L}\cos\frac{\beta_n at}{L}.$$

15. $$\int_0^L \sin\frac{\beta_m x}{L}\sin\frac{\beta_n x}{L}dx = \frac{L}{2}\left[\frac{\sin(\beta_m-\beta_n)}{\beta_m-\beta_n} - \frac{\sin(\beta_m+\beta_n)}{\beta_m+\beta_n}\right]$$

$$= \frac{L}{2(\beta_m^2-\beta_n^2)}\left[\begin{array}{r}(\beta_m+\beta_n)(\sin\beta_m\cos\beta_n - \sin\beta_n\cos\beta_m) \\ -(\beta_m-\beta_n)(\sin\beta_m\cos\beta_n + \sin\beta_n\cos\beta_m)\end{array}\right]$$

$$= \frac{L}{\beta_m^2-\beta_n^2}[\beta_n\sin\beta_m\cos\beta_n - \beta_m\sin\beta_n\cos\beta_m]$$

$$= \frac{L}{\beta_m^2-\beta_n^2}\left[\beta_n\cdot\frac{M\cos\beta_m}{m\beta_m}\cdot\cos\beta_n - \beta_m\cdot\frac{M\cos\beta_n}{m\beta_n}\cdot\cos\beta_m\right]$$

$$= \frac{LM}{m(\beta_m^2-\beta_n^2)}\cos\beta_m\cos\beta_n\left(\frac{\beta_n}{\beta_m} - \frac{\beta_m}{\beta_n}\right) = -\frac{LM\cos\beta_m\cos\beta_n}{m\beta_m\beta_n} \neq 0$$

16. When we substitute $v(r,t) = ru_r(r,t)$ we get the boundary value problem

$$v_t = kv_{rr}$$

$$v(0,t) = v(a,t) - av_r(a,t) = 0$$

$$v(r,0) = r f(r).$$

Then $v(r,t) = R(r)T(t)$ yields the equations

$$R'' + \lambda R = 0, \qquad T' = -\lambda kT.$$

If $\lambda_0 = 0$ then $R(r) = Ar + B$. The condition $R(0) = 0$ gives $B = 0$, and $R(r) = Ar$ satisfies the condition $R(a) - aR'(a) = 0$. Thus $\lambda_0 = 0$ is an eigenvalue with eigenfunction

$$R_0(r) = r; \qquad T_0(t) = 1.$$

If $\lambda = \alpha^2 > 0$ then

$$R(r) = A\cos\alpha r + B\sin\alpha r$$

and $R(0) = 0$ gives $A = 0$, so

$$R(r) = \sin\alpha r, \qquad R'(r) = \alpha\cos\alpha r.$$

The condition $R(a) = aR'(a)$ yields $\sin \alpha a = a\alpha \cos \alpha a$, that is,

$$\tan x = x$$

where $x = \alpha a$. If $\{\beta_n\}$ are the roots of this equation, then $\lambda_n = (\beta_n/a)^2$ is an eigenvalue with associated eigenfunction

$$R_n(r) = \sin\frac{\beta_n r}{a}, \qquad \text{and} \qquad T_n(t) = \exp\left(-\frac{\beta_n^2 kt}{a^2}\right).$$

We therefore obtain a solution of the form

$$v(r,t) = c_0 r + \sum_{n=1}^{\infty} c_n \exp\left(-\frac{\beta_n^2 kt}{a^2}\right)\sin\frac{\beta_n r}{a}.$$

The coefficient formulas given in the textbook follow immediately from Problem 14 in Section 10.1, and finally we obtain $u(r, t)$ upon division of $v(r, t)$ by r.

18. The only difference from Example 3 in the text is that the solution of Equation (37) with $T_n'(0) = 0$ is $T_n(t) = \sin\dfrac{n^2\pi^2 a^2 t}{L^2}$.

19. With the given initial velocity function $g(x)$ with constant value $P/2\rho\varepsilon$ concentrated in the interval $L/2-\varepsilon < x < L/2+\varepsilon$, the coefficient formula of Problem 18 gives

$$\begin{aligned} c_n &= \frac{2L}{n^2\pi^2 a^2}\int_{L/2-\varepsilon}^{L/2+\varepsilon}\frac{P}{2\rho\varepsilon}\sin\frac{n\pi x}{L}\,dx \\ &= \frac{L^2 P}{n^3\pi^3 a^2\rho\varepsilon}\left[\cos\left(\frac{n\pi}{2}-\frac{n\pi\varepsilon}{L}\right)-\cos\left(\frac{n\pi}{2}+\frac{n\pi\varepsilon}{L}\right)\right] = \frac{2L^2 P}{n^3\pi^3 a^2\rho\varepsilon}\sin\frac{n\pi}{2}\sin\frac{n\pi\varepsilon}{L}. \end{aligned}$$

This gives the ε-dependent solution

$$y(x,t,\varepsilon) = \frac{2L^2 P}{\pi^3 a^2\rho\varepsilon}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi\varepsilon}{L}\sin\frac{n^2\pi^2 a^2 t}{L^2}\sin\frac{n\pi x}{L}.$$

Because

$$\frac{L}{n\pi\varepsilon}\sin\frac{n\pi\varepsilon}{L} = \frac{\sin(n\pi\varepsilon/L)}{n\pi\varepsilon/L} \to 1 \quad \text{as} \quad \varepsilon \to 0,$$

the limit $y(x,t) = \lim_{\varepsilon\to 0} y(x,t,\varepsilon)$ has the expansion

$$y(x,t) = \frac{2LP}{\pi^2 a^2\rho}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n^2\pi^2 a^2 t}{L^2}\sin\frac{n\pi x}{L}.$$