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MATHEMATICAL AND PHYSICAL

THEORIES

of

HIGHER GEODESY

Preface and Part I

THE MATHEMATICAL THEORIES

by

DR. F. R. HELMERT,

Professor at the School of Technology, Aachen.

LEIPZIG,

PRINTING AND PUBLISHING HOUSE OF B. G. TEUBNER,

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DIE

MATHEMATISCHEN UND PHYSIKALISCHEN

THEORIEEN

DER

HÖHEREN GEODÄSIE.

EINLEITUNG UND I. TEIL:

DIE MATHEMATISCHEN THEORIEEN.

VON

DR. F. R. HELMERT,

PROFESSOR AN DER TECHNISCHEN HOCHSCHULE ZU AACHEN.



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1880.

Preface.

The present book contains the mathematical part of an extensive description of the scientific principles of topography and geodesy, with the objective of practical application. My original intention was to begin with an organic development of the physical part; however, after examining the whole material in an initial review, I came to the conclusion that the tried method of proceeding from the simple to the complicated would be to the advantage of the now chosen arrangement of the parts. At present, the first two chapters of the introduction take care of the general orientation.

In the mathematical development, everything that has no relation to the geodesy of the actual surface of the earth, is omitted; but, witnin this scope, I have tried to combine explicitness and rigor with simplicity.

The desire for explicitness caused me to devote space, among others, to the computation of chords, although it was found that, generally, a chord is less convenient in use than the geodetic line. However, in addition to theoretically interesting results, there also resulted some useful formulae for practical application.

Considering the rigor of the mathematical representation, it appears to me that this rigor in geodetic works is too often looked upon as self-evident. But it is not always a simple case of development of the trigonometrical functions of small angles

in power ries, and there is a special test sometimes necessary, among others, for the validity of the derivation of Legendre's Theorem, in the case of acute triangles which is easily conducted by means of the remainder of Taylor's series.

With regard to simplicity I first confined myself to the geodesy of the sphere and the slightly flattened ellipsoid of revolution, but connected the latter with the geodesy of the real surface of the Earth by means of the concept of deflection of the vertical. This method which Bessel already mentioned, is, by the present

state of our knowledge of the shape of the Earth, not only justified but, in my opinion, the only practical one. I deem it especially useless to choose another surface than the slightly flattened ellipsoid of revolution with its minor axis parallel to the Earth's axis, and I limited myself to basing all theories either on that surface or the simpler spherical surface. Although this knowledge loses every significance for its geodetic application, it is mathematically interesting to learn the range of validity of a theorem in its full sense. This limitation, however, gave me, I hope, the possibility of making myself understood by a larger audience. I have derived especially the fundamental equations of the geodetic line by elementary means, without the calculus of variations; and I have also tried to obtain this theory of geodetic triangles in obvious ways which are sufficient for practice and apparent for general understanding.

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The integral formulae for the geodetic line were, for this purpose, first transformed into differential formulae, and from these the theory of the geodetic circle was derived; whereby only simple integrations were necessary to reach that goal. For better understanding, the theory of geodetic triangles on the sphere as well as those for the ellipsoid were similarily treated. Through this, the validity of the formulae of spherical trigonometry could be proved at the same time without difficulty to such an extent as is important for geodesy.

It is self-evident, of course, that the existing works on geodesy were taken into consideration as is proven by numerous citations; formally, however, I proceeded in most cases independently, corresponding to my intentions.

From the material point of view one could find many a new thing. In this connection I mention, among others, calculations with chords on the ellipsoid (Chapter 4), the differential formulae for the geodetic line (Chapter 6), expansion in a series for the computation of the distance and azimuths from geographic positions (p. 314), the consideration of geodetic lines between points which are almost diametrical (Chapter 7), the examination of the maximal values of the higher forms of Legendre's Theorem (pp. 94, 359, and 384), the examination of the spherical computation of chains of triangulation (p. 400), the relation between rectangular and geographic coordinates (Chapter 9), the adjustment of geodetic, and astronomical measurements, taking into account deflections of the vertical (Chapter 12), and the developments with regard to the conclusiveness of arc measurements for determination of an ellipsoid of revolution for the shape of the

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Although the important problem of astro-geodetic measurements concerning deflections of the vertical is not solved in detail, at least the basic principle is completely developed. According to the present state of the art of observation, I, myself, am convinced that a main result of the study is that, in adjusting large nets of triangulation, one must consider not only the condition equations given by the different base lines, but also the equally valid equations which are derived by means of Laplace's Theorem of azimuth and geographic latitude determinations (p. 537). If this is not taken into consideration, I would deem a simultaneous adjustment of a large complex of triangulation worthless for the purpose of arc measurements, and at least incomplete for the purpose of surveying.

In order to facilitate understanding, abbreviations are introduced to the formulae. The symbolism was obtained from the generally-applied nomenclature by making a series of compromises. Special care was taken in the denotation of different values by identical symbols, so as to avoid any possible error.

Charts are not added to the book, with the exception of the one for the logarithm, of an often occuring function of the meridian-ellipse eccentricity and the geographical latitude; one is rather referred to Albrecht's approved chart collection (p. 34). There are computed numerical examples, sometimes utilizing logarithms with ten decimals, in order to test the usefulness of the

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formulae, especially those for the task of transfer of geographical coordinates and their conversion, which are developed mainly for each of the arbitrary distances as much as 640 km, and for sides of measurable triangles. The accuracy of the calculation arrived at in this manner is indeed in many cases by far larger than necessary in practice, but it was nevertheless desirable because of its special purpose. An operational instruction should cover these examples only insofar as the ready-to-use formulae are intended for practical application.

Aachen, October 1880

The Author

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ERRATA

Page 8 Line 23 from top; before with others insert in pairs.

Page 28 Line 10 and 11 from bottom; in quotients delete π.

Page 65 Line 11 from bottom; for Paragraph 18 read Paragraph 17.

Page 93 Line 3 from bottom; for $\sin : B$ read $\sin B$.

Page 115 Line 7 from top; for y read y_1.

Page 116 Formula (3); for η^3 read η_1^3.

Page 173 Line 2, right; before k in the numerator insert minus sign (-).

Page 191 Line 2, up; the word that is missing at the beginning.

Page 192, in right side of (9); insert $+ Gl_3$.

Page 206 in first square bracket of (8); numerator 5 is missing in fraction $\frac{5}{6}$.

Page 267 center; read: $\mathfrak{p}_1\mathfrak{p}_2' = \sin \Delta\varphi \; \delta a_{1,2}$.

Page 277; (8) delete factor a_1 before the curly bracket and insert before first round parenthesis.

Page 287; (11) lower integration limit should read φ_1.

Page 338 Line 3 from top; index n for P missing.

Page 356; read $6b^2c^2$ instead of $6b_2c_2$ in (14).

Page 378 center; enclose 3,9675889.6 into a square parentheses indicating the logarithm.

Page 383 Line 9 from top; for $\sin^2\beta_1$ read $\sin 2\beta_1$.

Page 386 Line 15 from top; for $b \cos \mathfrak{A}_c$ read $b \cos \mathfrak{A}_b$

Page 387; for $\mp$ read $\pm$ in (21).

Page 437 Line 2 from top; for m^4 read m^2

Page 451; substitute W_0^4 for W_0^2 in (1), (2) and (3).

Page 472; read right sides of lines 6 and 7: 27,24" 0,454' 0,34".

Page 482; (5) to be raised one line higher.

Page 496; for $\cos B$ read $\cos B_4$ in (3)

Page 522; the note provides an even indication of profile $P_1 P_2$.

Page 579 and 581; substitute $l_1 - l_0$, $l_2 - l_0$, .. for $l_1, l_2, \ldots$ in the error equations, and accordingly (adjustment computation p. 143) change the formation of the normal equation, in cases when the Munich plumb line direction also may be assumed to contain an interpolation discrepancy.

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INTRODUCTION

Helmert, Mathematical and Physical Theories of Higher Geodesy

(Translator's Note: Page 2 in original manuscript is blank. -L. D.)

Chapter 1.

Definition of Geodesy.

Par. 1. Geodesy is the science of measuring and mapping the Earth's surface.

The surface of the earth has more or less the shape of a sphere, with a radius of 6370 km. The deviations from the form of a sphere are less than one-half per cent in the radial direction, as the difference between the largest and the smallest distance from the surface to the gravity center of the Earth does not amount to 32 km. In some cases, these deviations are also extremely small where the Earth is taken as a whole, as, for instance, a solid model of the Earth in the form of a globe.

On the other hand, these deviations are an especially important factor of the measurements. It has to be considered here that the immediately visible part of these deviations, like mountains and valleys, has to be taken as irregular because of its complicated law of formation. But, because a determination of the whole by computing single parts is only possible on a regular surface, it is necessary to determine the physical surface of the Earth by means of direct measurements in all those places where it is wanted in detail.

In spite of the fact that the surface of the Earth cannot be reached everywhere, it has been made possible to have a general knowledge of the shape of the Earth through the existence of a simple law of formation of the shape of the whole body of the

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Earth. This is a law of formation of which the factors are the gravitation of mass particles and the centrifugal force, created by the rotation around an axis.

Par. 2. Measurements of small parts of the Earth's surface. Surveying one or the other part of the surface is relatively complete if the space coordinates have been found for as many points of it as are necessary to compile a sufficiently correct picture for a special purpose. The points must be appropriately selected, i.e., that the line and surface elements between them are sufficiently linear, or plane, respectively.

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The direction of the force of gravity which is the resultant of the above-mentioned forces, leads into a coordinate direction. The fact that the center of gravity of a motionless body, suspended in one position, always lies in the direction of the plumb-point below the point of suspension and, that, furthermore, the level of an undisturbed fluid is always perpendicular to the direction of the force of gravity, permits geodetic measuring instruments to be oriented according to this main direction. This is, for short, referred to as plumb line or vertical direction, a normal to this is a horizontal direction, and the geometrical locus of all horizontal lines at a point is the horizontal plane.

The plumb lines of different points converge approximately at the center of gravity of the Earth; therefore, they are not parallel. Nevertheless, one can attribute the property of being

parallel to all plumb lines of sufficiently closely-spaced points, because the convergence is slight, owing to the great distance of the gravity center of the Earth from its surface. It is, of course, necessary to examine whether or not the convergence of the plumb lines may be neglected.

If we consider the plumb lines of the characteristic points of a part of the Earth's surface as parallel, it follows that they project these points as identical figures to all horizontal planes. Regarding the horizontal projection, the choice of the particular horizontal plane, as the imaginary projection plane is, therefore, irrelevant. Contrary to this, the vertical distances, that means the altitudes above the supposed horizon, depend on that choice.

Horizontal projection and altitudes are completely sufficient to answer all the questions about the shape of the considered part of the surface. Not only can the true distances and surfaces can be computed with them - in contradistinction to horizontal distances which are measured in the projection - but one also gets easily the gradients of the line elements of the surface, in addition to the differences in altitude. These figures are so important that a survey which would give the shape of the surface without taking into consideration the plumb line would, therefore, have little importance just because the existence of this fundamental direction facilitates the survey considerably, which is evident when trying to complete the latter without its aid.

Unfortunately, in the horizontal projection, a fixed di-

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rection which can always be reproduced rapidly with sufficient accuracy, is unavailable. However, a fundamental direction exists as north-south direction, but it can generally be obtained with the required precision only through time-consuming astronomical observations. That is why its direct application has been abandoned as a rule. Nevertheless, direct

reference to that direction is convenient, or rather indispensable in some cases, whereby the magnetic needle serves to establish the relation.

It is the object of elementary geodesy (land surveying and leveling) explicitly to describe scientifically the ways and means of carrying out the surveys just discussed in general.

Par. 3. Survey on a Large Scale. If the points of a large part of the Earth's surface, are projected onto a surface considered everywhere perpendicular to the projecting line then, in comparison with Par. 2, two facts are shown:

1. Strictly speaking, following the plumb line of a point, it generally ceases immediately to be a plumb line on leaving the point. The direction of the plumb line rather changes from point to point along the plumb line, and a movable point which at any moment follows that plumb line which corresponds to its momentary position, describes a slightly curved line: the Vertical (line of force). At any point, according to this, the plumb line at the vertical is a tangent line at that point.

2. Instead of the horizontal plane to which the points of a

large part of the Earth's surface have been projected, we obtain a curved surface which intersects the verticals perpendicularly: a level surface.

The level surface, which has to be mentioned specially, is the sea level; but it is here thought of as dependent only on gravity and, therefore, motionless, so that disturbances caused by the tidal movement, winds, and oceanic currents are not considered. This ideal sea-level would form the visible part of a level surface. It is called the mathematical surface of the Earth or, (according to Listing, 1872) the geoid, contrary to the real, the physical surface of the Earth. It would also be possible to imagine the geoid as visible through a system of canals which would be built from the sea coast to the central parts of the continents.*) The motionless levels of ponds, and lakes are, however, as a rule, parts of other level surfaces. Considering the verticals as projecting lines, contrary to Par. 2, the choice of a certain level surface as a projection surface is necessary, because the projections onto various level surfaces are not identical and, especially, differ in their size.

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*)It is assumed, herewith, that the mathematical surface of the Earth is a spherical surface. In fact, we consider as self-evident the possibility of building canals across the continents (not taking into consideration the technical difficulties). But scientific discussion of this problem has to be reserved for the second part of this book, dealing with physics.

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Considering, especially, two level surfaces which belong to two neighboring points of a vertical, then both form closed, approximately parallel surfaces which do not intersect one another but one of which encloses the other entirely. In general, the projection onto the higher located level surface is, according to this, larger.

The projections are forms, similar to each other, if we consider the equipotential surfaces as concentrically spherical surfaces. The sides of the projection forms are, in their relation $\varrho:(\varrho+h)$ or $1:(1+\frac{h}{\varrho})$ if ϱ signifies the radius of the inner sphere and h the distance between both surfaces. The relation of the surfaces is approximately:

$$\varrho^2:(\varrho+h)^2 \text{ or } 1:\left(1+\frac{2h}{\varrho}\right)$$

The horizontal distance between two points is a circular arc whose length depends on the altitude of the level surface.

According to the above-mentioned, a complete survey of the physical surface of the Earth could be effected through the determination of its projection onto a level surface, the shape of which is known, and whose distances between the projected points, and the level surfaces measured in the verticals, are known. This is modified a little in practice.

Higher geodesy treats with the methods to determine the shape of the level surfaces, and to survey large parts of the

Earth's surface by means of horizontal projection and altitudes, taking into consideration the shape of the level surfaces.

The solution of the first-mentioned problem was facilitated by the fact that the level surfaces have, to a close approximation, the shape of an ellipsoid of revolution which is flattened at the poles. Heretofore, it was possible to obtain an approximate knowledge of the Earth's shape from measurements taken at a limited number of points; but the accurate solution requires a study almost as detailed as is necessary for the physical surface of the earth. This study can be conducted only for the smaller part of the Earth's surface, because $\frac{5}{7}$ of the surface are covered by the sea, and only $\frac{2}{7}$ are continental surface. But the study of the latter has only just started.

The mentioned ellipsoid also forms the basis for the second problem not only because the exact shape of the level surface remains, as a rule, unknown, but also because of the simpler computation.

Par. 4. Geographical Conceptions Which Are Independent of the Shape of the Level Surfaces. Although the knowledge of mathematical geography and the elementary geodesy is here considered as a pre-requisite, it is, nevertheless, necessary accurately to establish, and derive some required concepts,

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taking into consideration the true shape of the level surfaces,

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in the following, as far as not mentioned yet. The plumb lines, and the rotational axis of the Earth, the Earth's Axis, are given facts here. As will be discussed later, the axis contains the Earth's center of gravity at which the equatorial plane is perpendicular to it. The axis intersects the surface and every level surface at a North Pole and a South Pole.

At a point where astronomical observations can be made, not only the plumb line but also the direction toward the celestial pole which is above the horizontal plane of a point can be given. The celestial pole, being nothing more than the point of the apparent celestial sphere which is not affected in its position by the daily rotation, is the infinitely distant extention of the Earth's axis. The direction toward this point is parallel to the Earth's axis. The plane cutting this parallel as well as the plumb line is called the meridian plane of the point.

The meridian plane cuts the horizontal plane in a north-south direction to which the east-west direction is perpendicular. Seen from above, the sequence of the four cardinal points is clockwise north, east, south, west (according to agreement of the meteorologists designated as N, E, S, W).

The astronomical azimuth of a vertical plane is its angle with the meridian plane. The astronomical azimuth is measured in the horizontal plane, that is from the meridian plane toward the southwest, or the northeast.

The geographical latitude of a point is the angle between the plumb line of the point and the equatorial plane. The geographical longitude is the angle between the meridian plane of

the point, and another meridian plane at which the counting begins. The latitudes are counted from the equatorial plane to the North, and to the South or, positive and negative, up to 90 degrees. The longitudes are counted to the West or to the East up to 180 degrees or also to 360 degrees, wherein the meridian of a "marked point" of the observatories at Greenwich, near London, or at Paris often serve as starting points of the count in the astronomic and geodetic science.

Par. 5. Geographical Concepts for an Ellipsoid of Revolution. Assuming that the mathematical as well as the physical surface of the Earth form a flattened ellipsoid of revolution, then the plumb line of every point of the surface intersects the Earth's axis. According to this, the Earth's axis becomes the common intersection line of all the meridional planes, and all points of the same geographical longitude are in the same meridional plane and form the curve which crosses both poles, and which is called a geographical meridian.

Also all points of the same geographical latitude are on a

plane curve, the plane of which is parallel to the equatorial plane. This curve is called a geographical parallel, and the one at zero latitude is called especially the <u>geographical equator</u>.

The <u>flattening</u> is the difference between the equatorial and polar diameter compared to the equatorial diameter.

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Par. 6. Geographical Concepts for the True State. In reality a point on the physical surface of the Earth is generally not also on the mathematical surface of the Earth. The physical surface differs a little from the ellipsoid of rotation.

The distance of a point above the geoid measured along the vertical is called elevation above sea level. The elevation, according to this is not the length of a straight line but that of a curved one. However, the curvature is so small that it has significance only in the most exact studies. The curvature effects a deviation of the level surfaces from parallelism.

As a rule, the plumb line of a random point does not intersect the Earth's axis. Therefore, the meridian planes are generally only parallel to the Earth's axis without containing it. But the distance is always small. According to this, points of the same geographical longitude have meridian planes which generally do not coincide but which are parallel to one another with small separations as, generally, all meridian planes intersect with others in pairs whose intersection lines are parallel to the Earth's axis.

The geographical meridian which connects points of the same geographical longitude on the geoidal surface is generally a curve with a double curvature.*) This applies also to the geo-

*)If the geoid were a flattened ellipsoid of revolution with its small axis oblique to the Earth's axis, then the geographical meridians would also be plane curves. (Compare Fergola: "Sulla Posizione del' Asse di Rotazione etc" ("Regarding the Position of the Rotation Axis etc.") or the report on p. 94 of the "Vierteljahrsschrift der Astronomischen Gesellschaft" ("Quarterly of the Astronomical Society") 1876.)

graphical parallel and the geographical equator. It is only approximately possible to pass a plane through the curves. According to accurate concepts, we have to connect the above-mentioned geographical terms with other contents than we are usually accustomed to. But, considering the corresponding concepts of the apparent celestial sphere, there is nothing changed if we understand it as a sphere with an infinite radius which encloses the Earth. The celestial meridians and parallel circles are in any case plane curves, that is circles in the strict sense of the word.

General von Schubert developed the concepts of the true state of the Earth on p. 321, No. 1245 of the "Astronomische Nachrichten" ("Astronomical Reports") Volume 52, as early as 1860.

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Also the expansions, by Laplace, for an almost spherical but otherwise optional sea level, "Mécanique céleste" (Celestial Mechanics) Volume II, Part III, p. 109 etc, published 1799, should be compared.

Par. 7. The Geocentric Latitude, Longitude, and the Radius Vector. If we pass a plane containing the Earth's axis through a connection line between a point on the Earth and its gravity center, i.e., the radius vector of this point, then the geocentric longitude is the angle enclosed by this plane and another plane through the Earth's axis which serves as the beginning of the counting. The geocentric latitude is the incli-

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nation angle of the radius vector with respect to the equatorial plane.

The geocentric latitude is at the maximum, approximately 11½ minutes smaller than the geographical latitude. The difference between geocentric and geographic longitudes which depends only on the deviations of the geoid from a rotational surface is much smaller.*) The difference between geocentric and geographic latitudes originates at the flattened poles.

The radius vector of the geoidal surface is generally the largest at the equator and decreases toward the poles gradually by approximately $\frac{1}{300}$. The radius vector of the physical surface of the Earth differs approximately by the elevation above sea level, from the radius vector of the geoid which has the same direction. The more accurate (but not yet completely exact) value for the difference is equal to the elevation above the sea level, multiplied by the secant of the angle between plumb line and radius vector. It should be noted that, even for altitudes of 10000 m, only centimeters are involved, because the cosine of 11½ minutes differs from unity by only $\frac{1}{150000}$.

Par. 8. Coordinate Systems. According to the preceeding, there are three ways to establish relations between the points on the surface of the Earth.

The geocentric latitude and longitude, and the radius vector form an important system for astronomical purposes. The geographic latitude and longitude, and the elevation above sea level are useful for the description of the mutual position

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*)With the exception of points close to the poles.

of any of two points and are, therefore, the proper geographic coordinates. The elements of the geodetic coordinate system are elevations, and horizontal projections in reference to a level surface with the application of geometrical coordinates, generally known as rectangular coordinates, or polar coordinates for the plane. Higher geodesy applies specifically to direct determinations of geodetic and geographic coordinates for the solution of its problems.

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Chapter 2.

Historical Development of the Knowledge Regarding the Mathematical Surface of the Earth.

Par. 1. Historical Notes up to Newton's Time. Even in antiquity the curvature of the sea's surface could not remain unnoticed by the seafaring nations, where the first step toward the concept of spherical shape was made. About two and a half thousand years ago the Greeks had knowledge of this approximate shape of the surface of the Earth. Aristotle compiled the experiences to prove it. Nevertheless it was not until about 220 B.C. that the Alexandrian scientist Eratosthenes, as far as is known, arrived at a scientifically founded value for the circumference (and with this for the radius of curvature) of the Earth which was considered a sphere. He computed this from the estimated horizontal distance between Alexandria and Syeve and

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the astronomically determined convergence of the plumb lines at the terminal points of this distance. Eratosthenes computed the circumference and radius of the circle by means of the circular arc and the central angle.

For eighteen centuries following no special progress was made, although the accuracy of the values arrived at by Eratosthenes had been surpassed, by far, by some of the following measurements during this period. Not earlier than the seventeenth century, under the influence of the reformatory ideas of Copernicus, Galileo, Kepler, Huygens, and Newton, it became clear that the spherical form was only a crude approximation which were to be replaced by the form of the slightly flattened ellipsoid of revolution for a much better approximation. It should be generally known to the reader what these men accomplished; this is only to remind us of the achievements which concern geodesy especially.

We, therefore, stress the fact that Copernicus also referred to the necessity of the assumption of the Earth's daily revolution in connection with his concept of the planetary system (1543).

Before this time, (about 270 B.C.), the Greek Aristarchus had almost the same concepts, but this was not known to Copernicus. Apparently Aristarchus' teachings remained without any influence because astronomy could get along much better with theories that corresponded more with appearances than with the concepts of Aristarchus. Only after almost two thousand years

was the simplification provided by theoretical astronomy appreciated. Kepler and Galileo accepted the teachings. Kepler developed these teachings further,

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"Astronomia nova de motibus stellae Martis etc., 1609" ("New Astronomy Regarding the Movements of the Star Mars etc., 1609") by applying directly Tycho de Brahe's accurate observations of many years, and by combining this with previously acquired astronomical knowledge which had needed thousands of years to mature. Galileo established the laws of the falling body, and the movements of pendulum, (1602) creating herewith the foundations of mechanics of bodies.

Huygens already studied the physical pendulum and the central movement (1673), and he also recognized the influence of gravity on the Earth's shape. The reform in astronomical teachings was completed with Newton's Theory of Universal Gravitation (established in 1666, and applied, in 1686, to the movement of the celestial bodies etc., in the book: "Philosophiae naturalis principia mathematica" ("Mathematical Principles of the Philosophy of Nature")).

Huygens, the great contemporary of Newton, had his own concepts regarding gravitation, and he hesitated to accept Newton's Theory. He assumed that the Earth's power of attraction did not originate in its single mass-particles but in the Earth's center and he limited the range of its effectiveness to the Earth itself. Newton, on the other hand, did not extend the validity of the law

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of gravitation beyond the range of the solar system only because of caution owing to lack of proof.*) By combining the force of attraction and the centrifugal force (which he discovered first) to a resultant, normal to the Earth's surface, and assuming as $\frac{1}{210}$ the proportion of centrifugal force against gravity at the equator, Huygens obtained the value $\frac{1}{578}$, that is half of the former value (1688) for the flattening.

Newton, however, based his computations on the concept of a homogeneous ellipsoid, and he considered the attraction of all parts, and he found $\frac{1}{230}$ for the flattening, assuming the same relative value for the centrifugal force (1686). He had already noticed that the value for the flattening should decrease in case of an increased density toward the Earth's center. We know now that the truth is, in fact, to be found between the two concepts. The Earth's density is neither constant nor has it such an extension and direction as if the whole mass were concentrated in one single point. On the contrary; the density increases from 2,6 on the surface to approximately 11 in the center (mean value 5,6). However, the theoretical considerations of Huygens and Newton were sufficient to explain the experience of Richers who, in 1672, had to shorten the pendulum of his astronomical clock one and three fourths line during an astronomical expedition from Paris to Cayenne in order to adjust it. This indicated a decrease

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*)cf. Isenkrahe, das Rätsel der Schwerkraft, ("The Problem of Gravity") 1879, pp. 87 ff.

of gravity in comparison with Paris.

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In spite of having to re-establish the former pendulum length on his return to Paris, scientists were inclined to assume errors in observation, or local conditions, as the reason. New, confirming observations, made during the following ten years, were also contradicted by others. Therefore, a confirmation was not to be expected in this way. But the new theories, those of the Earth's rotation not excluded, required more confirmation the more fundamental they proved to be. Even the discovery of the rotation of the planet Mars around its axis by Huygens (1659), and the flattening of the planet Jupiter ($_1{}^1/_4$), discovered by Cassini the Elder (1666, published 1691) did not eliminate the doubts. Direct determinations of the Earth's curvature, according to the same principle already applied by Eratosthenes, seemed to indicate a prolongation of the Earth in the direction of its axis.

Par. 2. From Newton to Laplace. Assuming the Earth to be a sphere, one single determination of the curvature radius is sufficient, at least from the theoretical point of view. Two of such measurements are necessary to determine the shape of the ellipsoid of revolution, because the ellipse depends on two parameters. Considering the meridian arc (as already done by Eratosthenes), it is most adventageous to have one expediently at the equator, and the other expediently close to one of the poles. Today such measurements are called latitude measurements, which

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usage apparently originated in the seventeenth century, the time when the study of the Earth's elliptical shape began. It was impossible to derive the Earth's complete circumference with one measurement. Only the length of the elliptical arc for a latitude difference of one degree could be computed with it.

At the suggestion of the Academy of Sciences, founded in 1666, the Astronomer Picard started a measurement in France, and he already assumed deviations from the spherical shape. His measurement is worth mentioning for that reason that it is the first measurement of those with which the said seventeenth century's numerous and important inventions of the construction of mathematical instruments (we mention pendulum, clocks, and spring clocks, telescope, microscope, tube level, and vernier) won influence. In particular Picard and Azout were the first who applied the telescope furnished with cross hairs during their astronomic and geodetic measurements.

Furthermore, for the first time, the triangulation method was widely applied. The Netherlander, Snellius, had introduced this method in geodetic practice about 1615. This method permitted finding the distance between two points

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more accurately, in an indirect manner, than by the method of direct measurement which was favored until now. The triangulation method was carried out by connecting a chain of triangles with the directly measured ones, determining their angles, and, at one

triangle also measuring one side, in order to compute the unknown distance. (Norwood still measured with chain and compass, an arc of approximately 40 geographical miles in length, from London to York, during the years 1633 to 1635).

The measurement, started by Picard throughout France was, with interruptions, continued by several scientists until 1718. The book, published in 1720 by J. Cassini, showed that, within France, the measured arc length of a degree of the meridian decreased a little toward the North. This indicated an elongated (lemon-shaped)figure of the Earth instead of a flattened (orange-shaped) one.

In the beginning, the contradiction between the theories of Newton and Huygens was used by some scientists to attack Newton and Huygens because those measurements were considered to be very accurate. They were, in fact, more accurate than those of former days, but soon it was discovered that the observation errors nevertheless could be large enough completely to conceal the influence of the flattening for so short a distance of the meridian degree as the measurements within France prove.

The case was decided favorably for the dynamical theories through the latitude arc measurements in Peru (1735-1743), and in Lapland (1736-1737), then carried out by France. The former, and more accurate of the two, provided together with the corrected French arc measurement for the flattening, a value equal to $\frac{1}{301}$.

Since then, and especially since the end of the eighteenth century, numerous latitude arc measurements were carried out,

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with ever increasingly refined equipment.

Longitude arc measurements were also tried, at which each parallel arc was geodetically measured and the longitudinal difference was astronomically determined. But, with exact and convenient astronomical work, difficulties had to be faced in the beginning. Only recently have these difficulties been completely eliminated by electrical telegraphy.

In addition to the purely geometrical method of arc measurements, other methods for the computation of the flattening were found with the development of knowledge of celestial mechanics during the eighteenth century. (In 1743, Clairant showed how to compute the flattening with the help of a simple formula by means of the gravity difference between the equator and the poles of the Earth. D'Alembert studied (1749)

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the relation between the shape and the distribution of mass of the Earth, attracted by the moon and the sun, and also the movement of the Earth's axis in space. On the one hand this axis movement is presented by a continuous movement of the intersection of equator and ecliptic (precession of the equinoxes, discovered by Hipparchus, 150 B.C.,) and on the other hand by small, periodic changes of the celestial poles' position against the fixed stars (nutation, discovered by Bradley, and published 1748). Finally, at the end of the century, by means of the theory of the moon's orbit, Laplace proved those periodic terms which originate

from the Earth's shape and distribution of mass. Those terms were already empirically found.

In the second volume of his celestial mechanics, Laplace derived the flattening of the Earth according to all these methods (the value can only be found by arc measurements, and from the moon's parallax in connection with gravity measurements). He obtained, by using all these methods, a rather good conformity of the same value as a little smaller than $\frac{1}{300}$. And now, after still more numerous observations, all these methods indicate a considerably higher value.

In his computation, Laplace could utilize a greater number of latitude arc measurements as well as pendulum observations. This led him to interesting solutions of the problem of observation-error correction. Shortly afterwards his solutions were, of course, replaced by the more practical methods of least squares. Undoubtedly he noticed in these observation-error computations that the mathematical surface of the Earth does not exactly correspond to the shape of the flattened ellipsoid of revolution. The residual (computed) errors of the measurements - also azimuth measurements showed large deviations - surpassed by far the acceptable limit of observation errors. But, nevertheless, this Earth shape is a good approximation.

A detailed description of the historical development of the theories as well as arc measurements and practically also pendulum measurements from Newton to Laplace is given in the following books:

Todhunter: A History of the Mathematical Theories of At-

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traction and the Figure of the Earth. Two Volumes in 8° of 476 and 508 pages. London 1873.

Furthermore we consulted mainly:

Madler: Geschichte der Himmelskunde von der ältesten bis auf die neueste Zeit. ("History of Astronomy from the Most Ancient Time up to the Modern Time".) Two Volumes in 8° , Braunschweig 1873.

Rud. Wolff: Handbuch der Mathematik, Physik, Geodäsie und Astronomie. ("Manual of Mathematics, Physics, Geodesy and Astronomy".) Two Volumes in 8° , Zurich 1870 and 1872.

Laplace: Traité de Méchanique céleste. ("Manual of Celestial Mechanics".) Volume 1 and 2 in 4° , 1799. Especially compare Volume 3, p. 126 following and the survey of Volume 5, pp. 353-355, as well as Laplace's notes on the origin of the anomalies.

Volume 3, p. 139 - Volume 5 (1825 pp. 2-22 provides a historical survey concerning the studies of dynamics in relation to the figure and rotation of the Earth.

As supplements:

E. Mayer: Über die Gestalt und Grösse der Erde; eine historisch geodätische Studie. ("Regarding the Figure and Size of the Earth; a Historic Geodetic Study".) Fiume, 1876. Seventy-four pages in 8° with Survey Map

(Separate Printing from the Mitteilungen im Gebiete des Seewesens). (Nautical Reports.) Price: Mark 1.80.

Poggendorff: Bibliographisch-literarisches Handwörterbuch. (Bibliographic-literary Lexicon.) Two Volumes. Leipzig 1863.

Par. 3. The Nineteenth Century. Already at the beginning of the nineteenth century a larger number of such operations was carried out at different places of the Earth, because the shortcomings of the determinations of the Earth's flattening from only a few arc measurements were acknowledged. After Laplace it was mainly Bessel who (in critical discussion of the results of the measurements performed up to this date) computed, in 1837, a flattened ellipsoid of revolution. In 1841 he improved the computation by taking into consideration a known error of one of the utilized arc measurements. He found the value $\frac{1}{299}$. for the flattening. But several later computations, consulting the ever growing amount of observational material, made it probable that an increase of this value would better correspond to the shape of the Earth as a whole. The concepts concerning the value of this increase are still discussed. Many scientists are inclined to consider $\frac{1}{298}$, as the most probable value, the same value which expresses the relation of centrifugal force and gravity on the equator, because this value satisfies especially numerous gravity measurements.

The dimensions which Bessel computed in 1841 are, in any case, a very useful approximation. They are still more likely to be applied because of numerous tables for functions of the Earth's

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dimensions which are computed with the aid of them.

Among the more recent computations, we also find some which seem to conform better with the results of the arc measurements by accepting another hypothesis than that of the flattened ellipsoid of revolution. Bowditch, Clarke, Paucker, and Ritter based their computations on a rotational surface with a non-elliptic meridian section. Schubert, and also Clarke, followed up the hypothesis of the triaxial ellipsoid. Lately, Fergola based (his calculations) on the form of a flattened ellipsoid of revolution with its small axis oblique to the Earth's axis. The result of all these attempts is to be considered as negative, because it was neither possible to determine the elements of such figures with sufficient certainty nor to get them to conform better with the observations.

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Furthermore, we are now generally convinced that arc measurements by themselves do not suffice to show with certainty the shape of the mathematical surface of the Earth taken as a whole. That is because the measurements can be carried out only on the continents which cover no more than three-elevenths of the Earth's surface. Fortunately, gravity measurements with the pendulum can be carried out on numerous ocean islands. Even if there is, at present, no satisfying amount available of such measurements to meet all requirements, they suffice already to strengthen considerably the probability of the assumption that the flattened

ellipsoid of revolution*) is a good approximation. It is because they show that the geoid is in conformity with such a figure, up to fractions of the radius vector of the order of the square of the flattening. This form is neither a triaxial ellipsoid, nor does it show considerable differences between the Northern and Southern hemispheres.

The present deviations can be explained through irregularities of the mass distribution within the Earth's crust. Up to the present there is no indication of an irregular mass distribution in the Earth's core. The deviations occur as local ones; as a consequence of the local morphology, different rock densities, and the existence of subterranean cavities; or they have a continental extension as a consequence of the continental elevations above the mean level of the bottom of the sea and the diversity of the ocean's depths.

Assuming the deviations as positive and negative disturbances to a certain ideal mass distribution yielding an ellipsoid of revolution similar to the geoid, then their immediately evident effect is a positive or negative attraction which is combined with the normal gravity, and whose direction, the plumb line of the ideal ellipsoid, leads to the real plumb line of the geoid. The difference between the directions of both lines, the <u>deflection of the vertical</u>, amounts approximately to a few seconds. But,

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*)Some authors call it <u>spheroid</u>, but we keep this term as a name for approximate spherical surfaces in general and, often, where a confusion is not possible, we call the flattened <u>ellipsoid</u> of revolution simply ellipsoid.

close to mountains, mountain ranges, and near the sea coast, this deflection increases slightly up to 10" and more, without, however, surpassing a maximum value of approximately 1½'.

A further geometric consequence of the deflections of the vertical are elevations and depressions of the geoid in comparison to the ellipsoid which, however, in their total amount are better known from gravity measurements

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than from separations of the vertical. Those deflections of a more continental than local character may approach approximately 200 m maximum distance from the geoid. It has not yet been sufficiently studied whether there exists, somewhere, also a difference of 1000 m, as some scientists assume. However, below the Hartz Mountains, the maximum local elevation is not even 2 m (this value is not the locally occuring deviation of the geoid which occurs there, but it is only that part which orginates in the local mass distribution).

The deflections of the vertical have a considerable bearing on the magnitude of the curvature radius because, hereby, the deflection value is not taken into consideration, but its change with the changing position on the geoid. Notwithstanding the small value of the deflection, this change is often reletively considerable, so that deviations of the curvature radius of several per cent are the rule, and even more considerable deviations are not rare. But nowhere was there a curvature change stated and,

according to the shape of the physical surface of the Earth and the geologic properties, this is not to be expected anywhere. The geoid nowhere shows mountains and valleys as the physical surface of the Earth but only an increase and decrease of the curvature which is always concave toward the Earth's core.

The below-mentioned publication contains a survey of the results of most of the more recent computaions for the figure of the Earth.

Listing: Über unsere jetzige Kenntnis der Gestalt und Grösse der Erde (Nachrichten der Kön. Ges. der Wiss, zu Göttingen). (Regarding our present knowledge of the figure and size of the earth (Reports of the Royal Society of Sciences in Göttingen).) 1872. Sixty-six pages in 16°. Separate Copy. Price: Marks 0,80.

Mayer, cited elsewhere, p. 65, serves partially as supplement. Furthermore, for Ritter's spheroid: Wolff, op. cit., Volume 2, p. 141; for Schubert's computations; Listing, op. cit. p. 32 and, as an extract, also "Astronom. Nachrichten" ("Astronomical Reports) Volume 51 and 52. (No. 1201 and 1231). Regarding Fergola's publication which was already mentioned on p. 8, compare "Vierteljahrsschrift der Astronom. Ges". ("Quarterly of the Astronomical Society".) Volume 11, pp. 94 and 280.

Already Legendre gave formulae for the non-elliptic meridian of a rotation-spheroid in a memoir which is a prefix of Delambre's publication "Méthodes analytiques pour la Détermination d'un Arc du Méridien" ("Analytic Methods for the

Determination of a Meridian Arc") 1799. 4° .

Paucker computes the term:

$$3272553{,}2083^t\,(1 - 0{,}003399065\sin^2 - 0{,}000051868\sin^4 + 0{,}000001775\sin^6 - 0{,}000000002\sin^8),$$

as radius vector of the Earth from 11 arc measurements in a publication on the figure of the Earth (Encke cites the same measurements in the "Berl. Astronom. Jahrb." (Astronomical Yearbook of Berlin) 1852, p. 340). The term "sin" means here the sine of the geocentric latitude. Then, the flattening

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becomes $\frac{1}{289{,}9256}$, the equatorial radius becomes 6378325,0 m,*) and the deviation from the ellipse with the same flattening at 45° geocentric latitude is 107,0 m toward the outside. (Bulletin de la Classe physico-mathém. de l'Ac. imp. des sciences de St. Pétersbourg.) (Bulletin of Mathematical Physics, Imperial Academy of Sciences, St. Petersburg.) Volumes 12 and 13. 1854 and 1855, pp. 97-119 and 225 following).

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*)One Toise is equal to 864 Parisian lines and, officially, one meter is accurately equal to 443,296 Parisian lines; Thus:

$$1^t = 1{,}9490363098^m\ldots, \qquad 1^m = 0{,}513074074^t\ldots$$

Regarding the geodetic units of measure, see pp. 34 and 48, of Mayer, op. cit.

The computations by Bowditch and Clarke are cited in the principal publication of the English Survey; "Ordnance Trigonometrical Survey, Principal Triangulation, 1858, (compare Listing, cited elsewhere, p. 29). Bowditch's computations were published as a note to his translation of "Méc. cél." (Celestial Mechanics), Boston 1832. These computers let the curvature radius in the meridian be equal $A+Bs_1^2+Cs_1^4$, where s_1 is the sine of the geographic latitude, and ABC constants to be determined. The deviation at 45^0 latitude is 18 m toward the inside in Bowditch's computation (five arcs), 115 m toward the outside in a first computation by Clarke, without taking into consideration the Russian arc measurement, and 54 m toward the outside in a more complete computation (with a probable error of $\pm$ 22 m). All these computations show only a small deviation from the ellipse in relation to the mean radius of the Earth as well as in comparison to the continental elevations and depressions. The validity of this small deviation is questionable and cannot be stated.

Gauss (p. 73 of the publication regarding the latitude difference between Gottingen and Altona, 1828), and Bessel (in the treatise regarding the influence of the irregularities of the Earth on geodetic measurements, 1837) define very clearly the geoid and its deviations from the ellipsoid. Furthermore, Laplace studied the equipotential surfaces at different altitudes (Celestial

Mechanics, Chapter 7, Volume 3) as well as deflections of the vertical.

The importance of gravity measurements on the determination of the Earth's figure was already known at the beginning of the century. The numerous expeditions for the purpose of gravity measurements and, among others, the article "Erde" (Earth) in Gehler's Physical Dictionary pp. 825-1141, Volume 3, which was written by Muncke, 1827 (according to Fischer) state this fact. Recently, it was repeatedly referred to. See, among others, the following publications:

Ph. Fischer: "Untersuchungen über die Gestalt der Erde" (Studies Regarding the Figure of the Earth), 318 pages, in 8° , Darmstadt 1868. Also in this book, the deviations of the geoid from the spheroid are computed by means of estimated deflections of the vertical. Compare especially pp. 84, 92 and 275. H. J. Klein, in his publication: "Das Sonnensystem" (The Solar System) etc., Braunschweig 1869, computed the size of the Earth's ellipsoid contemporarily and similarly to Fischer (see p. 47, Listing, op. cit.). Based on gravity measurements, he took the value $\frac{1}{289}$ as the flattening and, herewith, found the equatorial radius equal to 3272766,1 toises or 6378739,9 m; that is, 402 m greater than Fischer's computation, solely based on the French-English measurements.

Par. 4. Present and Future Studies.

G. G. Stokes: "On the Variation of Gravity at the Surface of the Earth", Transactions of the Cambridge Phil. Society. VIII, Part V, p. 672-695 (Annual Set 1849). This publication shows the application of gravity measurements for the determination of the differences between geoid and ellipsoid, in addition to other details.

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Par. 4. Present and Future Studies. It is understandable that the present studies are especially dedicated to spheroidal bearings because, at the present time, the measurements concerning the figure of the Earth in general have already yielded a very satisfactory result. But there are still more data gathered to further improve the knowledge of the figure of the Earth in general.

The most extensive studies are at present carried out in Europe. For evident reasons this area has always been the most elaborately worked field of operations of geodesists. Until about the middle of the current century the French latitude measurement had been extended throughout the whole of England, and a latitude difference (amplitude) of $22^{\circ}\,10'$ in extent was determined. To that, the Russian latitude measurement $25^{\circ}\,20'$ amplitude was added in the East. But then, the similarly extended measurement between the two was missing, or only in its beginnings. The fact that this measuring operation is being carried out is due to the intervention of Lieutenant General Baeyer who was, in collaboration with Bessel, working at the East-Prussian arc

measurement (1861). At first, solely Central Europe was considered important enough for this operation but, after five years, it was extended to the European arc measurement.

In this, in addition to the genuine geodetic measurements, not only latitude but also geographic longitude and azimuth measurements are conducted. The latter have gained in accuracy through the use of electrical telegraphy in order that they be equal elements of study beside the latitude measurements. For special studies, the azimuth measurements are a valuable substitute for the longitude determinations. The more accurate the astronomic measurements are being conducted the better one will be able to get the details.

Following that, the gravity and the elevations above sea level (through precise geometric leveling) will be measured in many places. Data are arrived at, herewith, which provide information about the cause of local deflections of the vertical on the one hand, and information about gradual uplifts and subsidences of the continents, as well as other displacements of masses of the Earth's crust on the other hand.

The longitude measurements at the 52nd parallel is an especially outstanding accomplishment in the European arc measurement. This measurement reaches from the Ural Mountains to the westernmost point of Ireland, covering 68° 31′ longitude in all. As early as 1857 it was planned by W. Struve, Director of the Pulkova Observatory near St. Petersburg, and it can be considered completed by now. (Cf. "Petermann's Mitteilungen" (Petermann's

Reports) 1873, p. 332.)

As a rule, the experts for arc measurements

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of the countries involved hold a conference every three years. Furthermore, the Royal Geodetic Institute of Prussia being the Central Office of this arc measurement, publishes a report under the heading "Generalbericht über die europäische Gradmessung" ("Principal Report of the European Arc Measurement") (4°. Publishing House G. Reimer, Berlin.) Usually this report is published together with the proceedings of the conference in general.

The results of the measurements are intended to provide means for the study of the general differences between the ellipsoid of revolution(*) and the geoidal surface in Europe. For a detailed knowledge, operations are necessary which narrow the gap between the astronomically determined points more than usual (that is to only one to two miles distance instead of the former ten miles and more). In the Hartz Mountain area such an operation is undertaken by the above-mentioned Geodetic Institute.

The difficulty and trouble involved to study the character of the geoidal surface is shown, among others, in the English geodetic survey (cf. the main publication "Ordnance Survey,

(*)Translator's Note: At this stage it is obvious that Dr. Helmert considers an "ellipsoid of revolution" to be the appropriate reference surface for mathematical computation. For simplicity, the term "ellipsoid" will be used to indicate this surface, and " - of revolution" will be understood unless otherwise defined. -L.D.

Principal Triangulation", pp. 712 et. seq.). The result of the survey was rather to investigate a number of specifically local deflections of the vertical against a flattened ellipsoid which, with a flattening of $\frac{1}{280,4}$, deviates slightly from the Earth's ellipsoid in general. The information about deflections confirmed the usefulness of the survey to get data regarding the curvature rates in England, too. This knowledge permitted special information regarding the fast changing character of the curvature rates. The same was evident in other places, first probably at the East-Prussian arc measurement (1838; cf. F. W. Bessel, "Abhandlungen" (Proceedings), published by Engelmann, Leipzig, 1876, Volume 3, p. 135).

It is questionable whether a most closely fitting reference ellipsoid will be computed for Europe as was done for England. Bessel's ellipsoid will probably suffice to begin special studies on the geoidal surface. The deviations from the form of the ellipsoid of revolution have, because of their origin, so little in common with the shape of any simple surface that it is almost impossible to achieve any considerable gain in the description of the curvature rates by choosing another form as, for instance, the triaxial ellipsoid.

Outside of Europe, the English part of East India, during this century up to the present, has been covered through extensive geodetic

surveying (cf. "Vierteljahrsschrift der Astronomischen Gesellschaft" ("Quarterly of the Astronomical Society") 1873, Volume 8, pp. 14 et. seq.). Also on the islands in Dutch possession such measurements which will improve our knowledge of the shape of the Earth in general, have been conducted as of late. Further work of this kind was, and is, being conducted at the coasts, and lately also in the interior of the United States of North America as well as Egypt, Algiers, and Chile, for the last few decades. An important longitude and latitude arc measurement is being prepared in Brazil (nine to ten degrees of longitude at a Southern latitude of 23°, and 35 degrees of latitude at 10° longitude west of Rio de Janeiro).

An operation in the Caucasus Mountains, interesting for its results, is mentioned on pp. 49 et. seq., in the general report concerning the European arc measurements, 1871, (published 1872). According to it, there is a difference of 5⅘" in the deflections of the vertical observed between two points less than one degree of latitude distant from one another. (Cf. also "Petermanns Mitteilungen" (Petermann's Reports) 1862, Volume 10, p. 365.)

There were even thoughts of conducting arc measurements during an expedition to the North Pole.

In 1870, Börgen and Copeland measured geodetically an arc of 40 minutes length at the eastern coast of Greenland in order to determine the possibility of angle measurements at higher latitudes. The latitudes of the terminal points were derived

solely from the sun's altitudes as the operation was conducted for reconnaissance purposes only. (Cf. the publication "Zweite deutsche Nordpolfahrt" (Second German North Pole Expedition), 1869-70, Berlin 1871, p. 19.) (The main publication concerning this expedition was not available to the author.)

According to this, the interest in arc measurements is sufficiently intense to expect considerable increase in the knowledge of the Earth's shape from this side. There is also hope for a more systematic resumption of the very important gravity measurements which, lately, have been carried out more frequently after the brilliant operations during the first decades of this century. Being relatively less expensive, these measurements provide, by far, a more general knowledge of the geoid than arc measurements. The gravity measurements cannot even be replaced by arc measurements because they can be carried out geographically to a greater extent.

Gravity measurements also permit estimation of the distances of the geoid from the mean ellipsoid of revolution. But we must say that the computations referred to are not as easy as is often believed, and that they are not always carried out accurately. Stokes, (loc. cit.) provided accurate formulae.

In the future, when the knowledge about the geoid will have increased, it will be of minor importance

to derive such a flattened ellipsoid that, with the same volume, which will correspond to the geoid as close as possible. It is by far more important to study conclusions concerning the mass distribution in the Earth's core. By doing this, we encounter the difficulty that shape and altitude of the equipotential(*) surfaces are, to a certain extent, independent of the mass distribution (for instance, certain homogeneous spherical shells of different thickness have the same equipotential surfaces outside the Earth's sphere). Through this, not all results are rendered useless by any means.

Arc and gravity measurements, which can only be carried out gradually, consequently suggest a satisfactory constancy of the equipotential surfaces. It is now generally accepted that such a constancy exists to a high degree in reality. The masses of the Earth are relatively not completely in repose, but theoretical studies have shown that changes in the position of the Earth's axis are not to be expected. Only local changes of the plumb line directions are produced. It has already been indicated that this circumstance is continuously taken into consideration by the scientists.

Par. 5. Survey Concerning the Development of Theories in

(*)Translator's Note: Heretofore, the literal translation "level surface" has been used. Henceforth, in this theoretical work, the more modern and mathematically definable term "equipotential surface" will be used. -L.D.

the Present Book. A rigorously logical representation of the theories should be started with material which is amenable to precise definition; and to know as a whole, the shape of the Earth, that is with the simplest of propositions of the potential-theory and the gravity measurements. But, in order to settle the purely mathematical part first, we preferred to start with the theory of the arc measurements on the flattened ellipsoid, and to follow with the geometric determination of the geodetic deviations.

The formulae for triangulation nets, and the combination of geodetic and astronomic observations are dir- ctly established for the flattened ellipsoid. On the one hand, the acceptance of a random shape of the Earth is without value because, according to Bessel (1837), it is most convenient to reduce the measurements on the equipotential surfaces to the flattened ellipsoid of revolution. On the other hand, the rapid change of the curvature of the equipotential surfaces, also according to Bessel, would become a great obstacle for the convergence of the expansions of the series which are necessary for those formulae. Strictly speaking, this convergence does not exist because of the discontinuities within the curvatures, as H. Bruns (1876) has shown. Formulae for the triaxial ellipsoid are not mentioned because it has hardly any chances of being accepted as a mean representative of the geoid.

The second volume is intended for filling the gap which the proper

geometrical treatment of the problem of the definition and determination of the Earth's shape leaves through the introduction of potential theory and the following treatment of the gravity measurements. The second volume is further meant to bring some suggestions regarding the utilization of certain astronomical data and the mechanics of the rotation of the bodies, and so on. It is further meant to deal with the method of studying the geodial shape through measurement of zenith distances and, following that, to deal with terrestrial refraction.

Chapter 3.

General Mathematical Notions, Especially Expansion in a Series.

(The "Kompendium der höheren Analysis, Band 1" (Compendium of Higher Analysis, Volume 1) by Schlömilch, and "Algebraische Analysis" ("Algebraic Analysis") by Hattendorff, and also "Höhere Analysis" ("Higher Analysis") by Hattendorff.)

Par. 1. Convergence Conditions. An infinite series with positive terms converges (has a definite sum), if the value of the terms decreases from a given term in such a way that

$$\frac{\text{the following term}}{\text{the preceeding term}} < x;\ x$$ is taken as a proper fraction

The terms are denoted by $u_1, u_2, u_3 \ldots$ when

$$u_{i+1} < x u_i,\ u_{i+2} < x u_{i+1}, \ldots,$$

thus the remainder of the expansion $(u_1 + u_2 + \cdots + u_{i-1})$ is equal to

$$u_i + u_{i+1} + u_{i+2} + \cdots \text{ that means}$$
$$= u_i \left(1 + \frac{u_{i+1}}{u_i} + \frac{u_{i+1}}{u_i}\frac{u_{i+2}}{u_{i+1}} + \frac{u_{i+1}}{u_i}\frac{u_{i+2}}{u_{i+1}}\frac{u_{i+3}}{u_{i+2}} + \cdots\right);$$

therefore the remainder is

$$u_i + u_{i+1} + u_{i+2} + \cdots < u_i\,(1 + x + x^2 + x^3 + \cdots).$$

But the right-hand parenthesis is equal to $1:(1-x)$, and therefore the sum of all terms from u_i to infinity is $< u_i : (1-x)$; The remainder can be kept arbitrarily small, if the subscript i is chosen sufficiently large.

Instead of a given series, it is advantageous to expand a series with an apparently larger sum because all its terms are larger than in the original series, but the larger series has a simpler form than the original.

An infinite series with positive and negative terms converges when it converges also with its terms set positively or, in short, when it converges absolutely.

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The remainder of the series equals $u_i\ \chi$, if the alternating positive and negative terms decrease continuously from u_i on.

Absolutely convergent series permit a different arrangement of the terms; that is, they converge properly. The only condition is that the criterion of convergence given above is always present from one place on.

Par. 2. Rapidly Convergent Series in Geodesy.

It is permitted to sum up or to multiply the terms, and the result is again an absolutely convergent series. This applies as well to the case of subtractions. But special studies are required in the case of divisions.

This is permitted in the case of power series (and replacing the variable by a series) setting instead of the variable a series if the original series and the substitution series converge absolutely, and if the substitution series (taken with positive terms) satisfies the condition which is necessary for the convergence of the original series. Then the result is absolutely convergent.

Special studies are required if the substitution of the variable converges only conditionally and also, in the case wherein a finite number of terms of the substitution, both positive and negative terms appear. Only then is a conditionally convergent result obtained.

Par. 2. Rapidly Convergent Series in Geodesy. In the following we often deal with rapidly convergent series because only they are convenient for practical application. Especially in the case of power series, we limit ourselves to variable values of such a small magnitude that a few series terms are sufficient. We apply such methods that the limitation to small magnitudes sufficies practically.

In all these cases the operation of the series produces only small difficulties because only the significant terms need be treated in the same manner as finite series if we add a remainder (which can be neglected in corresponding practical cases).

Therefore, in most cases we do not need the study of convergence. In order to evade lengthy considerations we consider convergence studies only if it is necessary.

We mostly indicate the remainders by Gl_i, where i is the ordinal number of the remainder, (coordinated to the remainder) because the remainder is essentially a multiple of the $i.$ power of a magnitude of the first order.*)

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We determine the value of approximate 0,1 as the maximum of a magnitude of the First Order. Hereby the flattening becomes a magnitude of the Second Order.

Par. 3. Taylor's Theorem. It is

$$\left.\begin{aligned} f(a+h) = f(a) &+ \frac{h}{1} f'(a) + \frac{h^2}{1.2} f''(a) + \cdots \\ &+ \frac{h^{n-1}}{1.2\ldots(n-1)} f^{(n-1)}(a) \\ &+ \frac{(1-\vartheta)^{n-p} h^n}{1.2\ldots(n-1)\,p} f^{(n)}(a+\vartheta h) \end{aligned}\right\} \quad (1)$$

p is an arbitrary positive magnitude (which we often usefully denominate n). ϑ is a positive and proper fraction which is not known. The last term is called the remainder of the expansion of the series. This expansion at the right hand side holds if $f(x)$ and its differential quotients $f', f'' \ldots f^{(n)}$ within

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*)The expansion is correct with the exception of the terms of the $i.$ order if the remainder is of the $i.$ order.

the interval from $x = a$ to $x = (a + h)$ are finite and continuous. In order not to have to form the next higher differential quotient it is also possible to compute the remainder by means of that one which is in the last formulated term. But then it is necessary to subtract this last term again from the term which is formed in this way. The remainder is then, if we expand till $f^{(n)}$:

$$\frac{\frac{1}{p}(1-\vartheta)^{n-p} f^{(n)}(a+\vartheta h) - \frac{1}{n} f^{(n)}(a)}{1.2\ldots(n-1)} h^n . \qquad (2)$$

Especially for $p = n$:

$$\{f^{(n)}(a+\vartheta h) - f^{(n)}(a)\} \frac{h^n}{1.2\ldots n} . \qquad (3)$$

In some cases it is possible to determine more exactly the remainder of a convergent series which is limited to a finite number of terms by an approximate summation as indicated in Par. 1 p. 23. But this assumes that the law of formation of the series' terms is known. Often we may (see above) set an approximate remainder if u_i is the last term, that means

$$u_i \frac{x}{1-x} . \qquad (4)$$

If we obtain an infinite convergent series for the interval from $x = (a - h)$ to $(a + h)$, in an expansion according to Taylor's Theorem, then by subtraction of the expansions for $f(a + h)$ and $f(a - h)$ we can obtain the following rapidly convergent series

$$\left.\begin{aligned} f(a+h) = f(a-h) + \frac{2h}{1} f'(a) + \frac{2h^3}{1.2.3} f'''(a) + \cdots \\ + \frac{2h^{2n+1}}{1.2.3\ldots(2n+1)} f^{(2n+1)}(a) + \cdots \end{aligned}\right\} (5)$$

If there are only $(n+1)$ terms given in the series (1), which we consider as an infinite convergent series, we can also consider the following in a way that we apply the expansion

$$f^{(n)}\left(a+\frac{h}{n+1}\right)=f^{(n)}(a)+\frac{h}{n+1}f^{(n+1)}(a)+\cdots \qquad (6)$$

in order to sum up the $(n+1)^{\text{th}}$ and the $(n+2)^{\text{th}}$ term, that is:

$$\frac{h^n}{1.2\ldots n}f^{(n)}(a)+\frac{h^{n+1}}{1.2\ldots n\,(n+1)}f^{(n+1)}(a),$$

approximately to one single term:

$$\frac{h^n}{1.2\ldots n}f^{(n)}\left(a+\frac{h}{n+1}\right)$$

We obtain then:

$$\left.\begin{aligned} f(a+h)=f(a)+\frac{h}{1}f'(a)+\frac{h^2}{1.2}f''(a)+\cdots \\ +\frac{h^{n-1}}{1.2\ldots(n-1)}f^{(n-1)}(a) \\ +\frac{h^n}{1.2\ldots n}f^{(n)}\left(a+\frac{h}{n+1}\right) \\ +\text{terms with } h^{n+2} \text{ etc.,} \end{aligned}\right\} \qquad (7)$$

Occasionally this permits a convenient and accurate valuation of the remainder of the series consisting of n terms if the series is rapidly convergent.

By adding up the series for $f(a+h)$ and $f(a-h)$, it follows:

$$\left.\begin{aligned} \frac{f(a+h)+f(a-h)}{2}=f(a) \\ +\text{terms with } h^2, h^4\ldots \end{aligned}\right\} \qquad (8)$$

Par. 4. Binomial Theorem.

By multyplying the same series, it follows:

$$f(a+h)f(a-h) = f(a)^2 + \text{terms with } h^2, h^4 \ldots \quad (9)$$

Par. 4. Binomial Theorem. The relationship: ist.

$$(1 \pm u)^\mu = 1 \pm \frac{\mu}{1} u + \frac{\mu(\mu-1)}{1.2} u^2 \pm \frac{\mu(\mu-1)(\mu-2)}{1.2.3} u^3 + \cdots \quad (1)$$

is valid and absolutely convergent, with any given value of μ, if the absolute value of u is <1 *). The x value always denominates a positive and proper fraction whenever, in the following, some of the often appearing special cases are treated. The remainders are obtained by direct summation.

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(See Illustrative Material on Next Page.)

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*)Here, and in the following, we do not consider the cases $u = 1$. Because of practical reasons we neglect the borderline cases of validity.

$$\left.\begin{aligned}
1:(1+u) &= 1-u+u^2-u^3+u^4-\cdots\pm u^n\cdot x \\
1:(1-u) &= 1+u+u^2+u^3+u^4+\cdots+\frac{u^n}{1-u} \\
1:(1-u)^2 &= 1+2u+3u^2+4u^3+5u^4+\cdots+u^n\left\{\frac{n}{1-u}+\frac{1}{(1-u)^2}\right\}
\end{aligned}\right\}(2)$$

u positive & < 1.

$$\left.\begin{aligned}
\sqrt{1+u} &= 1+\frac{1}{2}u-\frac{1}{8}u^2+\frac{1}{16}u^3-\frac{5}{128}u^4+\cdots\pm\frac{1.3.5\ldots(2n-3)}{1.2.3\ldots n.2^n}u^n x \\
\sqrt{1-u} &= 1-\frac{1}{2}u-\frac{1}{8}u^2-\frac{1}{16}u^3-\frac{5}{128}u^4-\cdots-\frac{1.3.5\ldots(2n-3)}{1.2.3\ldots n.2^n}u^n\frac{x}{1-u} \\
1:\sqrt{1+u} &= 1-\frac{1}{2}u+\frac{3}{8}u^2-\frac{5}{16}u^3+\frac{35}{128}u^4-\cdots\pm\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n.2^n}u^n x \\
1:\sqrt{1-u} &= 1+\frac{1}{2}u+\frac{3}{8}u^2+\frac{5}{16}u^3+\frac{35}{128}u^4+\cdots+\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n.2^n}u^n\frac{x}{1-u} \\
1:\sqrt{1-u^3} &= 1+\frac{3}{2}u+\frac{15}{8}u^2+\frac{35}{16}u^3+\frac{315}{128}u^4+\cdots+\frac{3.5.7\ldots(2n+1)}{1.2.3\ldots n.2^n}u^n\frac{x}{1-\frac{2n+3}{2n+2}u}
\end{aligned}\right\}(3)$$

u positive & < 1

For rapid convergence, small values of u. are required for all these series.

Par. 5. Logarithmic and Power Series. We have

$$\left.\begin{aligned}
\log(1+u) &= M\left\{u-\frac{u^2}{2}+\frac{u^3}{3}-\frac{u^4}{4}+\cdots\pm\frac{u^n}{n}x\right\} \\
\log(1-u) &= -M\left\{u+\frac{u^2}{2}+\frac{u^3}{3}+\frac{u^4}{4}+\cdots+\frac{u^n}{n}\frac{x}{1-u}\right\}
\end{aligned}\right\}(1)$$

u positive & < 1.

$$\log u = 2M\left\{\left(\frac{u-1}{u+1}\right)+\frac{1}{3}\left(\frac{u-1}{u+1}\right)^3+\frac{1}{5}\left(\frac{u-1}{u+1}\right)^5+\cdots+\frac{x}{n}\frac{(u+1)^2}{4u}\left(\frac{u-1}{u+1}\right)^n\right\} \quad (2)$$

$$u = 1+\frac{1}{1}\left(\frac{\log u}{M}\right)+\frac{1}{1.2}\left(\frac{\log u}{M}\right)^2+\frac{1}{1.2.3}\left(\frac{\log u}{M}\right)^3+\cdots \quad (3)$$

u positive

M is the modulus of Brigg's logarithmic system.

The remainder of the last series which is nothing else but the power series $(u=e^x)$, has the values

$$\frac{1}{1.2.3\ldots n}\left(\frac{\log u}{M}\right)^n x \quad \text{for} \quad u < 1,$$

and

$$\frac{1}{1.2.3\ldots n}\left(\frac{\log u}{M}\right)^n \frac{n}{1-\frac{\log u}{(n+1)M}} \quad \text{for} \quad u > 1.$$

Herewith it is assumed that there are set as many terms of the series as are necessary to satisfy the condition $|\log u| < (n+1)M$.(*)

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The above-mentioned series are absolutely convergent. A rapid convergence requires, for equations (1), that the u value be small; and requires for equations (2) and (3), that the u value be almost unity.

Par. 6. Series for Sine, Cosine etc. We have:

$$\left.\begin{aligned}\sin u &= u - \frac{u^3}{6} + \frac{u^5}{120} - \frac{u^7}{5040} + \frac{u^9}{362880} - \cdots \pm \frac{u^{2n+1}}{1.2.3\ldots(2n+1)} \mp \cdots \\ \cos u &= 1 - \frac{u^2}{2} + \frac{u^4}{24} - \frac{u^6}{720} + \frac{u^8}{40320} - \cdots \pm \frac{u^{2n}}{1.2.3\ldots 2n} \mp \cdots\end{aligned}\right\} (1)$$

These series are valid for any value of u, that means in the sense of arc values. They are absolutely convergent. The remainder becomes equal to the first omitted term multiplied by (the value) x. which is the case for $u^2 < 1$ from the start on, if we continue to the place where the terms decrease. Furthermore, the series always converge rapidly for $u^2 < 1$.

(*)Translator's Note: Enclosure between vertical lines e.g. "|log u|" means the absolute value of the quantity enclosed, disregarding algebraic sign. -L.D.

We further obtain

$$\left.\begin{aligned}\tan u &= u + \frac{1}{3}u^3 + \frac{2}{15}u^5 + \frac{17}{315}u^7 + \frac{62}{2835}u^9 + \cdots \\ \sec u &= 1 + \frac{1}{2}u^2 + \frac{5}{24}u^4 + \frac{61}{720}u^6 + \frac{277}{8064}u^8 + \cdots\end{aligned}\right\} \quad (2)$$

$$|u| < \frac{\pi}{2}.$$

$$\left.\begin{aligned}\cot u &= \frac{1}{u}\left\{1 - \frac{1}{3}u^2 - \frac{1}{45}u^4 - \frac{2}{945}u^6 - \frac{1}{4725}u^8 - \cdots\right\} \\ \csc u &= \frac{1}{u}\left\{1 + \frac{1}{6}u^2 + \frac{7}{360}u^4 + \frac{31}{15120}u^6 + \frac{127}{604800}u^8 + \cdots\right\}\end{aligned}\right\} \quad (3)$$

$$|u| < \pi.$$

The coefficients which are present in the absolutely convergent series are not easily expressed in general terms. The terms set up and the fact that the remainder is approximately equal to the last term multiplied by $\frac{4u^2x}{\pi^2 - 4u^2}$ in the series (2) and approximately equal to the last term multiplied by $\frac{u^2x}{\pi^2 - u^2}$ in the series (3) are sufficient for our purposes. This is shown by the remark that the coefficients of the series' terms (2) decrease the more according to the ratio $\pi^2 : 4$, and the coefficients of the series' terms (3) the more according to the ratio $\pi^2 : 1$, the more we continue the series.

The convergence of the series (3) and still more of the series (2) is less rapid than the convergence of the series (1). In order to obtain the value of $\tan u$ or $\sec u$ accurately to 0,00000 00001 by means of the terms set up in the series (2), we have to limit ourselves to absolute values of u which do not appreciably surpass 0,1.

Par. 7. Series for Arc Sine and Arc Tan., and Logarithmic Series Etc.

$$\left.\begin{aligned}\arcsin u &= u + \frac{u^3}{6} + \frac{3u^5}{40} + \frac{5u^7}{112} + \cdots + \frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}\frac{u^{2n-1}}{2n-1}\frac{u}{1-u^2}\\ \arctan u &= u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \frac{u^9}{9} \cdots \pm \frac{u^n}{n}u\end{aligned}\right\}(1)$$

$u < 1.$

If we set $u = \sin v$ respectively $u = \tan v$ and write then instead of v again u, then follows:

$$\left.\begin{aligned}u &= \sin u + \frac{\sin^3 u}{6} + \frac{3\sin^5 u}{40} + \frac{5\sin^7 u}{112} + \cdots + \frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}\frac{\sin^{2n-1}u}{2n-1}\frac{u}{\cos^2 u}\\ &\qquad u < \frac{\pi}{2};\\ u &= \tan u - \frac{1}{3}\tan^3 u + \frac{1}{5}\tan^5 u - \frac{1}{7}\tan^7 u + \frac{1}{9}\tan^9 u \cdots \pm \frac{u}{n}\tan^n u\\ &\qquad u < \frac{\pi}{4}.\end{aligned}\right\}(2)$$

These absolutely convergent series converge only approximately as rapidly as the series (2) of Par. 6. We have further

$$\left.\begin{aligned}\log\sin u &= \log u - M\left\{\frac{u^2}{6} + \frac{u^4}{180} + \frac{u^6}{2835} + \frac{u^8}{37800} + \cdots\right\}\\ &\qquad u \text{ positive } \& < \pi;\\ \log\tan u &= \log u + M\left\{\frac{u^2}{3} + \frac{7u^4}{90} + \frac{62u^6}{2835} + \frac{127u^8}{18900} + \cdots\right\}\\ &\qquad u \text{ positive } \& < \frac{\pi}{2}.\end{aligned}\right\}(3)$$

The first of these two series is obtained by the integration of the series for $\cot u$, the second by integration of the series for $2\csc 2u$ (whereby we have to consider that $\log\frac{\sin u}{u}$ as well as $\log\frac{\tan u}{u}$ are zero for the value $u = 0$). The series converge

definitely with the same interval for u as in the former case (but not more), because the integration decreases the coefficients. But in order to avoid the imaginary because of the log, we have to limit ourselves to positive values of u which is completely sufficient in practice.

If the terms are written as far as u^n, then the remainder of the first series is approximately equal to the last of these terms multiplied by $\frac{n^2\pi}{\pi^2 - n^2}$, and the remainder of the second series is approximately equal to the last of the terms multiplied by $\frac{4n^2\pi}{\pi^2 - 4n^2}$.

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We have further

$$\left.\begin{aligned} \log u &= \log\sin u + M\left\{\frac{1}{6}\sin^2 u + \frac{11}{180}\sin^4 u + \frac{191}{5670}\sin^6 u + \frac{2497}{113400}\sin^8 u + \cdots\right\} \\ &\quad u \text{ positive } \& < \frac{\pi}{2}; \\ \log u &= \log\tan u - M\left\{\frac{1}{3}\tan^2 u - \frac{13}{90}\tan^4 u + \frac{251}{2835}\tan^6 u - \frac{3551}{56700}\tan^8 u + \cdots\right\} \\ &\quad u \text{ positive } \& < \frac{\pi}{4}. \end{aligned}\right\} (4)$$

The first of these series is derived from the first series (3) by substitution of $u = \sin u + \frac{1}{6}\sin^3 u + \frac{3}{40}\sin^5 u + \frac{5}{112}\sin^7 u + \cdots$, that means the first series (2). This latter series is convergent for $u^2 < \left(\frac{\pi}{2}\right)^2$ and as this series has only positive terms, the final result must also converge for $u^2 < \left(\frac{\pi}{2}\right)^2$, taking into consideration the convergence condition.

The remainder of the first series (4) is approximately

equal to the last term multiplied by $x:(1-\sin^2 u)$ that means $x \sec^2 u$.

It is possible to derive the second series (4) from the first one. First we let $\log \sin u = \log \tan u + \log \sqrt{1-\sin^2 u}$, then we expand the last one of these logarithms according to powers of $\sin^2 u$, but substituting $\tan^2 u : (1+\tan^2 u) = \tan^2 u - \tan^4 u + \tan^6 u - \tan^8 u + \cdots$ for $\sin^2 u$. However, this expansion does not completely represent the limits of convergence. In order to study its derivation, compare with "Hohere Analysis" ("Higher Analysis") by Hattendorff. The remainder of the second series (4) is approximately equal to the last term multiplied by $x \tan^2 u$.

The terms which are set up in the series (1) to (4) are sufficient for an accuracy of approximately ten decimal places for $|u| < \frac{1}{10}$.

Par. 8. Determination of the Angle Difference from the Cotangent or Cosine Difference. If we have given:

$$\cot A = \cot A' + h, \qquad (1)$$

letting h be a small magnitude, we can advantageously apply Taylor's Theorem in order to derive $A - A'$. If we set $a = \cot A'$, and $f(a) = \operatorname{arc\,cot} a$, p. 25 (1), because of $A = \operatorname{arc\,cot}(\cot A' + h)$, thus we obtain

$$\begin{aligned} f'(a) &= -\frac{1}{1+a^2} = -\sin^2 A' \\ f''(a) &= 2 \sin^3 A' \cos A' \\ f'''(a) &= 2 \sin^4 A' (1 - 4 \cos^2 A') \\ f^{IV}(a) &= 6 \sin^4 A' \sin 4A' \end{aligned}$$

and thus:

$$A = A' - h\sin^2 A'\{1 - h\sin A'\cos A' - \tfrac{1}{3}h^2\sin^2 A'(1 - 4\cos^2 A')\cdots\} \qquad (2)$$

If we stop at the term with h^3, the remainder of the extension which is taken as $n = p$ in formula (1) p. 25, becomes

$$\tfrac{1}{4}h^4\sin^4 A''\sin 4A'', \qquad (3)$$

wherein A'' corresponds to an angle of which the cot is equal to $\cot A' + \vartheta h$, ϑ is a positive and proper fraction which is not well known. From this it follows that:

$$\sin^2 A'' = \frac{\sin^2 A'}{1 + \vartheta h\sin A'(2\cos A' + \vartheta h\sin A')} \qquad (4)$$

and we recognize that the terms set up in (2) are sufficient if it is possible to neglect $(h\sin A')^4$. Then also $\vartheta h\sin A'$ is certainly so small that $\sin^2 A''$ does not differ much from $\sin^2 A'$, as well as $(h\sin A'')^4$ consequently also does not differ considerably from $(h\sin A')^4$.

If

$$\cos A = \cos A' + h, \qquad (5)$$

then we have, according to Taylor's Theorem, because of $A = \arccos(\cos A' + h)$ for $a = \cos A'$ and $f(a) = \arccos a$ the equations:

(See Illustrative Material on Next Page.)

Par. 8. Determination of the Angle Difference from the Cotangent or Cosine Difference. Par. 9. Arc and Arc Measurements.

$$f'(a) = -(1-a^2)^{-\frac{1}{2}} = -\frac{1}{\sin A'}$$
$$f''(a) = -\frac{\cos A'}{\sin^3 A'}$$
$$f'''(a) = -\frac{1+2\cos^2 A'}{\sin^5 A'}$$
$$f^{IV}(a) = -\frac{9+6\cos^2 A'}{\sin^7 A'}\cos A'$$

and thus

$$A = A' - \frac{h}{\sin A'}\left\{1 + \frac{h\cos A'}{2\sin^2 A'} + \frac{h^2(1+2\cos^2 A')}{6\sin^4 A'} + \cdots\right\}. \qquad (6)$$

Stopping at h^3, then the remainder of the expansion is equal to

$$-\frac{1}{8}h^4\frac{3+2\cos^2 A''}{\sin^7 A''}\cos A'', \qquad (7)$$

whereby the relation $\cos A'' = \cos A' + \vartheta h$ exists for A''. From this follows

$$\sin^2 A'' = \sin^2 A'\left(1 - \frac{\vartheta h}{\sin^2 A'}(2\cos A' + \vartheta h)\right). \qquad (8)$$

The condition for the usefulness of this expansion up to h^3 is that $h : \sin^2 A'$ is sufficiently small to be able to neglect its fourth power.

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Substituting $\frac{\pi}{2} - A$, for the above-mentioned A, then we obtain the expansions for $\tan A$ and $\sin A$.

Par. 9. Arc and Arc Measurements. If u, as mentioned above, denominates an angle measured in fractions of the radius, that

is expressed as an arc, then, in order to obtain u in degrees, we have to take into consideration the proportion:

$$u^0 : 360^0 = u : 2\pi,$$

and thus

$$u^0 = u \cdot \frac{180^0}{\pi} \quad \text{or} \quad u \text{ in degrees} = u\varrho^0, \tag{1}$$

by introducing as an abridgement:

$$\varrho^0 = \frac{180}{\pi}. \tag{2}$$

There follows, in the same way, as a factor for u, for the reduction to u, expressed in minutes or seconds:

$$\varrho' = \frac{180 \cdot 60}{\pi}, \quad \varrho'' = \frac{180 \cdot 60 \cdot 60}{\pi}. \tag{3}$$

Par. 10. Formulae for Very Small Angles. If we neglect u^5, then we obtain

$$\left.\begin{aligned} \sin u &= u\sqrt[3]{\cos u} + \cdots \\ \tan u &= u\sqrt[3]{\sec^2 u} + \cdots \end{aligned}\right\} \tag{1}$$

In order to know the error in these formulae which are named after*) Maskelyne, we form, by means of the binomial theorem:

$$\begin{aligned} \sqrt[3]{\cos u} &= 1 - \frac{u^2}{6} - \frac{u^4}{72} - \cdots \\ \sqrt[3]{\sec^2 u} &= 1 + \frac{u^2}{3} + \frac{u^4}{9} + \cdots \end{aligned}$$

and multiply by u. The comparison with the series for $\sin u$

*)Houel, Five-place Logarithms, p. XXXIX.

and $\tan u$ shows that the neglect of u^5 in the first formula (1) produces an error which is a little larger and that one in the second formula produces an error which is a little smaller than in the abridgement of the series to the first two terms.

For practical computation the formulae (1) and similar ones like $\log \sec \frac{1}{2} u = \frac{1}{4} \log \sec u + \cdots$, often may be preferred to the series, because they require as a reference only the looking up of a cosine etc. instead of the computation of a term.

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The auxiliary logarithms

$$\left.\begin{aligned} S &= \log \frac{\sin u}{u\varrho''} \\ T &= \log \frac{\tan u}{u\varrho''} \end{aligned}\right\} \qquad (2)$$

which are shown in several logarithmic tables serve the same purpose as the formulae (1). The seven-place logarithmic tables by Bruhns and Schrön show S and T values with eight places. The tables by Bremiker also contain the S values but only with seven places. The tables by Schrön have the special advantage that, among others, not only the angle is given in seconds but also the $\log \sin$ is given as an argument to S.

In the same way it is often very advantageous in the evaluation of $\sqrt{1 - v^2}$ and $\sqrt{1 + v^2}$, to set $\sin u$ and $\tan u$ for v, if v is small enough, and we have then:

$$\left.\begin{aligned} \sqrt{1 - v^2} &= \cos u \text{ for } v = \sin u \\ \sqrt{1 + v^2} &= \sec u \text{ for } v = \tan u. \end{aligned}\right\} \qquad (3)$$

Chapter 3. General Mathematical Notions, Especially Expansion in a Series.

Par. 11. Interpolation. If there is given a table of the function values $f(x)$ for values of the variable x, continuing with the same interval, especially for $x = a, (a+\omega), (a+2\omega), \ldots$ and if we want to know $f(a+n\omega)$, $n < 1$, then we form the first, second, third differences etc. according to the scheme:

	1. Differences	2. Differences	Differences
$f(a)$			
$f(a+\omega)$	$\Delta_0 = f(a+\omega) - f(a)$		
$f(a+2\omega)$	$\Delta_1 = f(a+2\omega) - f(a+\omega)$	$\Delta_0'' = \Delta_1' - \Delta_0'$	
$f(a+3\omega)$	$\Delta_2 = f(a+3\omega) - f(a+2\omega)$	$\Delta_1'' = \Delta_2' - \Delta_1'$	$\Delta_0''' = \Delta_1'' - \Delta_0''$
.	.	.	.

Thus, we have following Newton's interpolation formula which is sufficient for our purposes for the computation of:

$$f(a+n\omega) = f(a) + \frac{n}{1}\Delta_0' + \frac{n(n-1)}{1.2}\Delta_0'' + \frac{n(n-1)(n-2)}{1.2.3}\Delta_0''' + \cdots \quad (1)$$

This formula is easy to derive if we base on Taylor's series:

$$f(a+n\omega) = f(a) + \frac{n\omega}{1}f'(a) + \frac{n^2\omega^2}{1.2}f''(a) + \cdots \quad (2)$$

and apply it for $n = 0, 1, 2 \ldots$ but, by means of the formed equations, expressing the differential quotients $f'(a), f''(a) \ldots$ by Δ values. The application of this formula leads, of course, to reliable values only if it is possible to expand the function according to Taylor's Theorem. On another very advantageous formula (by Bessel, Astronomische Nachrichten (Astronomical Review), No. 33, Volume 2),

together with a table of its coefficients, see p. 215 of the book:

Albrecht, Formeln und Hilfstafeln für geographische Ortsbestimmungen u. s. w. 2. Auflage. (Formulae and Auxiliary Tables for Geographical Coordinates of a Place, etc. Second Edition,) Leipzig, 1879.

Often, simple interpolation is not sufficient at the interpolation in logarithmic tables with ten decimal places, because the Δ'' values still have an influence. Then we can take into consideration the higher differences by means of the equations (1). In most cases it is sufficient to include the Δ''_0 values. Then, we set very conveniently:

$$f(a + n\omega) = f(a) + n\left(\Delta_0 - \frac{1-n}{2}\Delta''_0\right) + \cdots \qquad (3)$$

and we notice easily that the n factor is between the values Δ'_0 and Δ'_{-1}. Thus we have a control for the sign of the subtrahend of this factor. Instead of the formulae (3) we can also apply the following formulae (still containing the h^3 terms) which are of the same precision and which are derived from (7) p. 26 for $n = 1$:

$$\left.\begin{aligned} \log(a+h) &= \log a + M\frac{h}{a+\frac{h}{2}} + \cdots \\ \log\sin(a+h) &= \log\sin a + Mh\cot\left(a+\tfrac{h}{2}\right) + \cdots \\ \log\tan(a+h) &= \log\tan a + Mh\csc\left(a+\tfrac{h}{2}\right)\sec\left(a+\tfrac{h}{2}\right) + \cdots \end{aligned}\right\} \qquad (4)$$

Chapter 3. General Mathematical Notions, Especially Expansion in a Series.

If we have to look for the number or the angular value of the logarithm, then the formulae (3) and (4) provide an indirect computation which, in the first formula of (4) can be replaced by the more convenient direct computation according to formula

$$a_1 + h_1 = a_1 + \frac{1}{M}(\log(a_1 + h_1) - \log a_1)\sqrt{(a_1 + h_1)\,a_1} + \cdots \quad (5)$$

This formula is accurate up to terms h_1^3 , (excluded), and it is derived from formulae (7) p. 26, if we let $f(\alpha) = a_1$, $a = \log a_1$ and replace there $f'\left(a + \frac{h}{2}\right)$ by $f'(a)f'(a+h)$ by means of (9) .

Regarding the interpolation which is to apply in the computation of tables, see p. 331 of the "Berliner Astronomisches Jahrbuch" ("Astronomical Annual of Berlin") 1852 by Encke, and p. 279 of Volume 10 of the "Vierteljahrsschrift dre Astronomischen Gesellschaft" ("Quarterly of the Astronomical Society") 1875 by Astrand. The former author provides, in his article for the Annuals of Berlin 1830-1862, numerous hints for practical computing. These articles are compiled and published in 1866 under the title: J. F. Encke's Astronomische Abhandlungen (Astronomical Articles).

Interpolation in tables of logarithms of many decimal-places is treated, among others, in Volume 1 of the "Traité de Géodésie" ("Treatise of Geodesy") by Puissant (partly according to Legendre).

PART I.

The Mathematical Theories of Higher Geodesy.

(Translator's Note: Page 36 is blank in original manuscript. -L. D.)

Chapter 1.

The Flattened Ellipsoid.

Par. 1. Elements of the Meridian Ellipse. Let

a_0 be the semi-major axis (in the equatorial plane),

b_0 be the semi-minor axis (in the rotational axis).

Then:

$\alpha = \dfrac{a_0 - b_0}{a_0} = f$ is the flattening,

$e^2 = \dfrac{a_0^2 - b_0^2}{a_0^2}$ is the numerical value of the eccentricity squared,

$$\left.\begin{aligned} n &= \frac{a_0 - b_0}{a_0 + b_0} \\ \delta &= \frac{a_0^2 - b_0^2}{b_0^2} \\ m &= \frac{a_0^2 - b_0^2}{a_0^2 + b_0^2} \end{aligned}\right\} = e'^2$$ often used auxiliary quantities.

The existing relationships between these five quantities can be easily found by solving for b_0 and equating the resulting expressions for b_0 :

$$b_0 = a_0(1-\alpha) = a_0\frac{1-n}{1+n}$$

$$b_0^2 = a_0^2(1-e^2)$$

$$e^2 = 2\alpha - \alpha^2$$

$$\alpha = 1 - \sqrt{1-e^2} = \frac{e^2}{2} + \frac{e^4}{8} + \frac{e^6}{16} + \frac{5e^8}{128} + \cdots$$

$$\alpha = \frac{2n}{1+n} = 2n - 2n^2 + 2n^3 - 2n^4 + \cdots$$

$$n = \frac{\alpha}{2-\alpha} = \left(\frac{\alpha}{2}\right) + \left(\frac{\alpha}{2}\right)^2 + \left(\frac{\alpha}{2}\right)^3 + \left(\frac{\alpha}{2}\right)^4 + \cdots$$

$$e^2 = \frac{4n}{(1+n)^2} = 4n - 8n^2 + 12n^3 - 16n^4 + \cdots$$

$$n = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} = \frac{e^2}{4} + \frac{e^4}{8} + \frac{5e^6}{64} + \frac{7e^8}{128} + \cdots$$

$$\delta = \frac{e^2}{1-e^2} = e^2 + e^4 + e^6 + e^8 + \cdots$$

$$m = \frac{e^2}{2-e^2} = \frac{e^2}{2} + \frac{e^4}{4} + \frac{e^6}{8} + \frac{e^8}{16} + \cdots$$

$$m = \frac{2n}{1+n^2} = 2n - 2n^3 + 2n^5 - \cdots$$

$$m = \mathfrak{a} + \frac{\mathfrak{a}^2}{2} - \frac{\mathfrak{a}^4}{4} - \frac{\mathfrak{a}^5}{4} \cdots$$

$$1 - e^2 = \frac{1-m}{1+m} = \left(\frac{1-n}{1+n}\right)^2 = (1-\mathfrak{a})^2 = \frac{1}{1+\delta}.$$

All the above derived series are absolutely convergent, which can be seen by applying the usual rules for convergence, since n, m, $\mathfrak{a}$ and $e^2 < 1$ for an ellipsoid, and the series converge also partly for greater values.

According to Bessel (1841; cf. Rud. Engelmann, "Dissertations of F. W. Bessel" Leipzig 1876, Volume 3, p. 62) the following values apply for the ellipsoid

$n = 0{,}0016741848;$	$\mathfrak{a} = 1 : 299{,}1528$
$a_0 = 3272077{,}14'$	[6,5148235.337]
$b_0 = 3261139{,}33$	[6,5133693.539],

where the square brackets indicate common logarithms. Encke, who renders tables for some functions of a_0 and b_0 in the Berliner Astronomisches Jahrbuch of 1852, p. 322 ff., uses

$$n = 0{,}001674184767 \quad [7{,}2238033.861 - 10]. \qquad (1)$$

This value is retained, since it is also used by most computers, and, changing to metric measure (conversion logarithm 0,2898199.2994),

(See Illustrative Material on Next Page.)

$$\begin{aligned} a_0 &= 6377397{,}15500 \quad [6{,}8046434.637] \\ b_0 &= 6356078{,}96325 \quad [6{,}8031892.839]. \end{aligned} \tag{2}$$

The excessive refinement of these values in regard to the real accuracy as representatives of the central Earth ellipsoid is desirable, since they appear as fundamental quantities, and are taken as a basis for all calculations. They have to characterize geometrically a definite ellipsoid. One obtains further:

$$\left.\begin{aligned} e^2 &= 0{,}006674372096 \quad [7{,}8244104.149 - 10] \\ \mathfrak{a} &= 0{,}003342773114 \quad [7{,}5241069.005 - 10] \\ m &= 0{,}003348360149 \quad [7{,}5248321.645 - 10] \\ \delta &= 0{,}006719218662 \quad [7{,}8273187.745 - 10] \end{aligned}\right\} \tag{3}$$

$$\begin{aligned} n^2 &= 0{,}00000280289463 & \mathfrak{a}^2 &= 0{,}000011174132 \\ n^3 &= 0{,}00000000469256 & \mathfrak{a}^3 &= 0{,}000000037352 \\ n^4 &= 0{,}00000000000786 & \mathfrak{a}^4 &= 0{,}000000000125 \\ e^4 &= 0{,}000044547243 & m^2 &= 0{,}000011211516 \\ e^6 &= 0{,}000000297325 & m^3 &= 0{,}000000037540 \\ e^8 &= 0{,}000000001984 & m^4 &= 0{,}000000000126 \\ e^{10} &= 0{,}000000000013 \end{aligned}$$

$$2 \log (1 - \mathfrak{a}) = \log (1 - e^2) = 9{,}9970916.4046 - 10$$

$$\log \frac{2}{2 - e^2} = \log (1 + m) = 0{,}0014517.4521$$

$$\log (1 - m) = 9{,}9985433.8566 - 10$$

$$\log (1 + n) = 0{,}0007264.8124$$

$$\log (1 - n) = 9{,}9992723.0147 - 10.$$

The preceeding values have been evaluated on the one hand with the application of G. Vega's "Thesaurus logarithmorum completus"

(Leipzig 1794), and on the other as a check using direct shortened multiplication and division with the partial use of Thomas' calculating machine: the final eleven-place logarithms have been evaluated directly by the use of series. These values serve as a basis of the following discussion.

Included here are the angles at the center which appertain to a unit arc one radian (p. 32):

$$\begin{aligned} \varrho'' &= 206264{,}806247 \quad [5{,}3144251.332] \\ \varrho' &= 3437{,}74677078 \quad [3{,}5362738.828] \\ \varrho^0 &= 57{,}2957795131 \quad [1{,}7581226.324], \end{aligned}$$

as well as the numerical value for the modulus of the common logarithm system:

$$M = 0{,}4342944819 \quad [9{,}6377843.113 - 10].$$

Par. 2. Coordinates in the Meridian Ellipse. The rectangular coordinates x and z, are measured from the origin M on the major and minor axis, Fig. 1, and the result

$$\frac{x^2}{a_0^2} + \frac{z^2}{b_0^2} = 1.$$

is then obtained.

This equation is satisfied by the substitutions

$$x = a_0 \cos\beta \qquad z = b_0 \sin\beta, \tag{1}$$

The independent variable β is called the reduced latitude.*) They correspond

*)According to Pussiant, Traité de Géodésie, the expression comes from Legendre.

to the known geometric construction of the inscribed and circumscribed circle. The position of a point P is determined completely by β ; although the geographic latitude B is usually used to designate the latter.

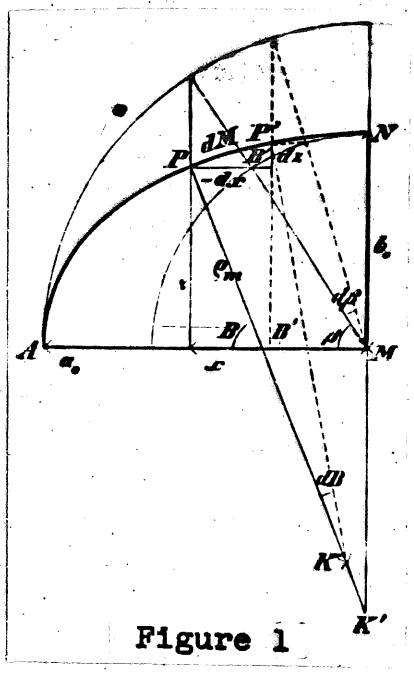


Figure 1

To find the relationship between the latter, β and x, z , P is moved an infinitesimal amount in the direction of increasing B to P'. In the resulting infinitesimal right angle triangle the angle at P' is equal to the angle B except for an infinitesimal error which vanishes towards B . The hypotenuse is the segment dM of the ellipse, which is to be regarded as linear, and the sides have the absolute values $-dx$ and dz . Therefore

$$\tan B = -\frac{dx}{dz}.$$

But from the foregoing formulae for x and z it follows that:

$$-dx = a_0 \sin\beta d\beta \qquad dz = b_0 \cos\beta d\beta;$$

when this is substituted and the value of $a_0 : b_0$ is considered according to Par. 1 p. 37 then

$$\tan\beta = \sqrt{1-e^2}\tan B. \qquad (2)$$

Eliminating $\tan\beta$ from the trigonometric formulae

$$\cos\beta = \frac{1}{\sqrt{1+\tan^2\beta}} \qquad \sin\beta = \frac{\tan\beta}{\sqrt{1+\tan^2\beta}}$$

and multiplying the right hand numerator and denominator by $\cos B$, we obtain

$$\left.\begin{aligned} \cos\beta &= \frac{\cos B}{\sqrt{1-e^2\sin^2 B}} = \frac{\cos B}{W} \\ \sin\beta &= \frac{\sqrt{1-e^2}\sin B}{\sqrt{1-e^2\sin^2 B}} = \frac{\sqrt{1-e^2}\sin B}{W}, \end{aligned}\right\} \qquad (3)$$

where

$$W = \sqrt{1-e^2\sin^2 B}. \qquad (4)$$

Solving this for $\sin B$ and $\cos B$ it follows that

$$\left.\begin{aligned} \cos B &= \frac{\sqrt{1-e^2}\cos\beta}{\sqrt{1-e^2\cos^2\beta}} = \frac{\sqrt{1-e^2}\cos\beta}{w} \\ \sin B &= \frac{\sin\beta}{\sqrt{1-e^2\cos^2\beta}} = \frac{\sin\beta}{w} \end{aligned}\right\} \qquad (5)$$

as well as

$$Ww = \sqrt{1-e^2} \qquad (6)$$

$$w = \sqrt{1-e^2\cos^2\beta}. \qquad (7)$$

Moreover by substitution of the last-found values for $\cos\beta$ and $\sin\beta$, we obtain for x and z , we obtain

$$\left.\begin{aligned} x &= \frac{a_0\cos B}{\sqrt{1-e^2\sin^2 B}} = \frac{a_0\cos B}{W} \\ z &= \frac{a_0(1-e^2)\sin B}{\sqrt{1-e^2\sin^2 B}} = \frac{a_0(1-e^2)\sin B}{W}. \end{aligned}\right\} \qquad (8)$$

According to these formulae one cannot obtain x and z for the complete ellipse but only for one half on one side of the

minor axis, since B is not taken from 0^0 to $\pm 180^0$ as required by the formula $B = -dx : dz$ but only from 0^0 to $\pm 90^0$. But in reality only one half comes into consideration.

The distance MK' of the point of intersection of the normal at P and the axis of rotation equals $(x \tan B - z)$; also

$$MK' = \frac{a_0 e^2 \sin B}{W} = \frac{a_0 e^2}{\sqrt{1-e^2}} \sin \beta. \qquad (9)$$

This distance amounts to approximately 43 km as a maximum.

Par. 3. Conversion of Geographic Latitude into Reduced Latitude and Vice Versa. It can be seen that

$$\frac{\tan \beta}{\tan B} = \sqrt{1-e^2} = \frac{1-n}{1+n}. \qquad (1)$$

Using this, B can be converted to β and vice versa. One obtains, for example, for

$B = 52^0\ 30'\ 16{,}7''$	
$\log \tan B = 0,1150923.1$	$0,1150923.336$
$\log \sqrt{1-e^2} = 9,9985458.2 - 10$	$9,9985458.202 - 10$
$\log \tan \beta = 0,1136381.3$	$0,1136381.538$
$\beta = 52^0\ 24'\ 43{,}01''$	$52^0\ 24'\ 43{,}01144''$.

The first calculation was done with seven-place logarithms where the eighth place of the proportional parts was retained for an increase in certainty. Thereby $\log \tan \beta$ is uncertain at the utmost in a half unit of the seventh decimal place and the error in β does not exceed 0,02". The second calculation with ten decimal places is not correspondingly accurate, since the tenth decimal place is not determined as rigorously as the

seventh decimal in the former calculation.*)

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In order to obtain a greater accuracy with seven-place logarithms than that obtained above formulae for $(B-\beta)$ were set up, the most important of which are here derived. Solving for n: it follows from (1) that

$$n = \frac{\tan B - \tan \beta}{\tan B + \tan \beta} = \frac{\sin (B-\beta)}{\sin (B+\beta)}. \qquad (2)$$

For the moment let $B-\beta = u$ and using the relationship $B+\beta = 2B-u$ it follows that

$$\sin u = n \sin (2B-u). \qquad (3)$$

On expanding the sine on the right hand side and developing $\sin u$ and $\cos u$ as powers of u a series for u can be easily found. We will however use another often used method, the introduction of the imaginary.

Let $i = \sqrt{-1}$ and ε be the base of the natural logarithms, then**)

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*)According to Bremiker's preface of his edition of Vegas seven-place logarithms the tenth decimal place of the logarithms of the numbers in Thes. log. is certain, but not of the trigonometric functions where errors up to four units appear.

**)

$$\sin u = \frac{\varepsilon^{iu} - \varepsilon^{-iu}}{2i},$$

$$\cos u = \frac{\varepsilon^{iu} + \varepsilon^{-iu}}{2};$$

$$\log \text{nat}\,(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots;$$

where $u < 1$.

Par. 3. Conversion of Geographic Latitude into Reduced Latitude and Vice Versa.

$$\varepsilon^{iu} - \varepsilon^{-iu} = n\,(\varepsilon^{2iB-iu} - \varepsilon^{-2iB+iu}).$$

On multiplying by ε^{iu} and reducing to ε^{2iu}, one obtains

$$2iu = \log\text{nat}\,(1 + n\varepsilon^{2iB}) - \log\text{nat}\,(1 + n\varepsilon^{-2iB}).$$

Both logarithms are converted to series and we obtain, since $n < 1$, the convergent expression:

$$u = n\,\frac{\varepsilon^{2iB} - \varepsilon^{-2iB}}{2i} - \frac{n^2}{2}\,\frac{\varepsilon^{4iB} - \varepsilon^{-4iB}}{2i} + \frac{n^3}{3}\,\frac{\varepsilon^{6iB} - \varepsilon^{-6iB}}{2i} - \cdots$$

or

$$B - \beta = n\sin 2B - \frac{n^2}{2}\sin 4B + \frac{n^3}{3}\sin 6B - \frac{n^4}{4}\sin 8B + \cdots \quad (4)$$

The preceeding formulae express the arc in radians; to obtain seconds, the right hand side has to be multiplied by ρ''. The term with n^4 has no bearing on the fifth decimal place of the seconds. Therefore equating:

$$\left.\begin{array}{l} \beta - B \\ \text{in Seconds} \end{array} \right\} = \begin{array}{l} [2{,}5382285.2_n]\sin 2B \\ + [9{,}76203 - 10]\sin 2B\cos 2B \\ + [6{,}51n - 10]\sin 6B + \cdots \end{array} \quad (5)$$

Interchanging B with β, and n with $(-n)$, in equation (1) leaves the equation unaltered. One obtains therefore from (4) with the same interchange:

$$\left.\begin{array}{l} B - \beta \\ \text{in Seconds} \end{array} \right\} = \begin{array}{l} [2{,}5382285.2]\sin 2\beta \\ + [9{,}76203 - 10]\sin 2\beta\cos 2\beta \\ + [6{,}51 - 10]\sin 6\beta + \cdots \end{array} \quad (6)$$

For $B = 52^0\ 30'\ 16{,}7''$, equation (5), yields (where the first term was calculated with eight correct decimal places):

2,5382285.2.	9,76203 − 10	6,51. − 10
$\log \sin 2B =$ 9,9849249.3	9,98492 − 10	$\log \sin 6B =$ 9,85. − 10
2,5231534.5.	$\log \cos 2B =$ 9,41326. − 10	6,36 − 10
− 5′ 33,54425″	9,16021. − 10	+ 0,00023″
	− 0,14461″	

$$\beta - B = -5'\ 33{,}68863'' \qquad \beta = 52^0\ 24'\ 43{,}01137''.$$

If a table for W, (eq. (4) , p. 40) is available, the following formulae can be used to find $B - \beta$

$$\sin (B - \beta) = \frac{n \sin 2B}{2W} = \frac{n \sin 2\beta}{2w} \qquad (7)$$

They follows from $\sin (B - \beta) = n \sin (B + \beta)$, when the right hand side is solved and B or β is eliminated (p. 40). To obtain $(B - \beta)$ in seconds from $\sin (B - \beta)$ the auxiliary logarithm (p. 33) is used.

For $B = 52^0\ 30'\ 16.7''$ is

$\log \sin 2B = 9{,}9849249.3 - 10$
$-\log W = 0{,}0009142.5$
$\log \frac{n}{2} = 7{,}2230769.0 - 10$
Summation 7,2089160.8
S for $334'' = 4{,}6855746.7$
$\log (B - \beta) = 2{,}5233414.1$

$B - \beta = 5'\ 33{,}68863''$ $\beta = 52^0\ 24'\ 43{,}01137''$.

Bremiker in his Studien über höhere Geodäsie 1869, supplied a table for $\log \frac{B - \beta}{\sin 2B}$ with eight decimal places for an easier application of equation (7) . Therefore to obtain $B - \beta$ in seconds one has only to add $\log \sin 2B$. If β is given one has only to take $B = 90 - \beta$ and $\sin 2B$ instead of $\sin 2\beta$. (In the

extract and to seven decimal places by Albrecht p. 198).

A table for $B-\beta$ to two decimal places is included in Albrecht on p. 197.

Par. 4. Radius of Curvature of the Meridian. The radius of curvature is designated by ϱ_m, and therefore (Fig. 1, p. 40) on the one hand:

$$\varrho_m dB = dM; \qquad (1)$$

and on the other hand from the infinitesimal triangle which adjoins dM

$$dM = dz : \cos B;$$

therefore

$$\varrho_m = \frac{dz}{d(\sin B)}. \qquad (2)$$

Introducing z according to equation (8) p. 41:

$$\varrho_m = \frac{a_0(1-e^2)}{W^3} = \frac{[6{,}8017351.042]}{W^3}.$$

This equation renders ϱ_m, with the use of a table for W very easily. The application of the binomial theorem to the numerator in (2) produces on the other hand:

$$\varrho_m = a_0(1-e^2)\left(1+\frac{3}{2}e^2\sin^2 B+\frac{15}{8}e^4\sin^4 B+\frac{35}{16}e^6\sin^6 B+\cdots\right), \qquad (3)$$

and this formula would be well suitable to calculate a single ϱ_m, accurately to nine or ten places with the partial use of di-

rect multiplication and seven-place logarithms. A table for ϱ_m to seven decimal places is included in Albrecht on p. 199. A table for $\frac{\varrho_m}{\varrho''}$ to eight decimal places was also included by Börsch in his Tafeln für geodätische Berechnungen für $B = 35$ to 71^0, (Enclosure to the Program of the Higher Technical School in Kassel).

Par. 5. Calculation of W. In separate cases one would equate $\log(e \sin B) = \log \sin \psi$ where $\log W$ changes to $\log \cos \psi$.

In the calculation of a table the development by series is to be preferred. One obtains immediately:

$$\log W = -\frac{1}{2} M\left(e^2 \sin^2 B + \frac{e^4}{2} \sin^4 B + \frac{e^6}{3} \sin^6 B + \frac{e^8}{4} \sin^8 B + \cdots\right) \quad (1)$$

Here the third term has significance in the seventh decimal place and the fourth term in the tenth.

A more rapidly convergent series is obtained by introducing the auxiliary quantity n and the trigonometric functions of the multiples of B. Instead of introducing this for each power of B separately, we derive it generally by the introduction of $\cos 2B = \frac{1}{2}(\varepsilon^{2iB} + \varepsilon^{-2iB})$. One obtains gradually with consideration of the relation $e^2 = 4n : (1+n)^2$,

$$\sqrt{1 - e^2 \sin^2 B} = \sqrt{1 - \frac{e^2}{2} + \frac{e^2}{2} \cos 2B} = \frac{\sqrt{1 + 2n \cos 2B + n^2}}{1+n}$$

Further

$$W = \frac{\sqrt{(1 + n\varepsilon^{2iB})(1 + n\varepsilon^{-2iB})}}{1+n}. \quad (2)$$

Par. 5. Calculation of $\log W$.

Now the formula on p. 42 is valid for both factors of the radican:

$$\log \text{nat}\,(1 + n\varepsilon^{\pm 2iB}) = n\varepsilon^{\pm 2iB} - \frac{n^2}{2}\varepsilon^{\pm 4iB} + \frac{n^3}{3}\varepsilon^{\pm 6iB} - \cdots .$$

Adding the logarithms of both factors and converting from exponents to cosins one obtains using common logarithms,

$$\log W = \left\{ \begin{array}{l} -\log(1+n) \\ +Mn\cos 2B \\ -M\frac{n^2}{2}\cos 4B \\ +M\frac{n^3}{3}\cos 6B - \cdots \end{array} \right\} = \left\{ \begin{array}{l} [9{,}9992735.1876 - 10] \\ +[3{,}8615877.0]\cos 2B \\ +[0{,}78436_n]\cos 4B \\ +[0{,}83 - 3]\cos 6B + \cdots . \end{array} \right\} \quad (3)$$

The numerical values of the coefficients in the periodic terms are attached in units to the seventh place of the logarithm. The logarithms of the coefficients show that, already in seven-place calculations, the term with n^2 has little bearing.

For $B = 52^\circ\ 30'\ 16{,}7''$

$$\left.\begin{array}{rr} \log W = & 9{,}9992735.188 - 10 \\ - & 1882.983 \\ + & 5.270 \\ + & .005 \end{array}\right\}$$
$$\log W = 9{,}9990857.480 - 10.$$

Formula (3) has not only the advantage of greater convergence than (1), but also that in (1) the interval of the higher members is smaller than in (3) which is quite noticeable in calculations with tables. Therefore one has to evaluate only half as many values of the terms with $\cos 2B$ than the term with $\sin^2 B$. This publication includes a table for $\log W$ to ten decimal places from $B = 47^\circ$ to 57° and a table for $\log W$ and $\log w$ to eight decimal places, B and β ranging from 0° to 90°. Both

tables have been calculated according to equation (3).

Encke computes (loc. cit. p. 328) as stated above: he obtains by means of ten place tables $\log\cos\psi$ directly with $\log\sin\psi$, where the following condensed series is used for interpolation: $\log\cos\psi = \log\cos\psi_0 - \tan^3\frac{\psi+\psi_0}{2}\varkappa(\log\sin\psi - \log\sin\psi_0)$, where ψ_0 is the nearest round of argument of the table. The calculation with equation (3) should be just as convenient, and when the seven-place logarithms of the numbers and

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trigonometric functions are tied up especially one has no need for any notations except the resulting value of the terms of the series themselves.

Par. 6. Rectification of the Meridian Arc. This can be accomplished by the use of the formula $dM = \varrho_m dB$, when ϱ_m is represented as a function of B. The integration is impossible without the use of series. To integrate, the powers of the sin in the series (3) p. 44 have to be converted into trigonometric functions of the multiples of B. The most rapid convergence of the coefficient series thus appearing is obtained by the application of n. It follows in this case with regard to (2) p. 45 when $(1-n)^2:(1+n)^2$, is placed for $1-e^2$ in the expression for ϱ_m without any difficulty that

$$dM = a_0(1-n)(1-n^2)\left\{(1+n\varepsilon^{2iB})(1+n\varepsilon^{-2iB})\right\}^{-\frac{3}{2}}dB. \quad (1)$$

But now

$$(1+n\varepsilon^{\pm 2iB})^{-\frac{3}{2}} = 1 - \frac{3}{2}n\varepsilon^{\pm 2iB} + \frac{15}{8}n^2\varepsilon^{\pm 4iB} - \frac{35}{16}n^3\varepsilon^{\pm 6iB} + \frac{315}{128}n^4\varepsilon^{\pm 8iB} - \ldots \quad (2)$$

When the two obtained series are now multiplied and the cosin is introduced instead of the imaginary it follows that

$$dM = a_0(A_0 - A_2\cos 2B + 2A_4\cos 4B - 3A_6\cos 6B + 4A_8\cos 8B - \cdots)dB \quad (3)$$

$$\left.\begin{aligned} A_0 &= (1-n)(1-n^2)\left\{1 + \tfrac{9}{4}n^2 + \tfrac{225}{64}n^4 + \cdots\right\} \\ A_2 &= (1-n)(1-n^2)\left\{3n + \tfrac{45}{8}n^3 + \cdots\right\} \\ A_4 &= (1-n)(1-n^2)\left\{\tfrac{15}{8}n^2 + \tfrac{105}{32}n^4 + \cdots\right\} \\ A_6 &= (1-n)(1-n^2)\left\{\tfrac{35}{24}n^3 + \cdots\right\} \\ A_8 &= (1-n)(1-n^2)\left\{\tfrac{315}{256}n^4 + \cdots\right\}. \end{aligned}\right\} \quad (4)$$

Integration yields now for the meridian arc from the equator to a latitude B:

$$M = a_0(A_0 B - \tfrac{1}{2}A_2\sin 2B + \tfrac{1}{2}A_4\sin 4B - \tfrac{1}{2}A_6\sin 6B + \tfrac{1}{2}A_8\sin 8B - \cdots). \quad (5)$$

On subtraction of the expression for two values B_1 and B_2 and letting

$$B_2 - B_1 = \Delta B \qquad \tfrac{1}{2}(B_2 + B_1) = B, \quad (6)$$

The following expression for the meridian arc ΔM from B_1 to B_2, $B_2 > B_1$, can be obtained with the application of the relation $\sin\lambda B_2 - \sin\lambda B_1 = 2\cos\lambda B \sin\tfrac{\lambda}{2}\Delta B$:

$$\Delta M = a_0\left\{\begin{aligned} &A_0\Delta B - A_2\cos 2B\sin\Delta B + A_4\cos 4B\sin 2\Delta B \\ &- A_6\cos 6B\sin 3\Delta B + A_8\cos 8B\sin 4\Delta B - \cdots. \end{aligned}\right\} \quad (7)$$

B and ΔB in this arc to be taken as arcs. If ΔB is

given in seconds, then for the arc calculation

$$\Delta B = \frac{1}{\rho''} (\Delta B \text{ in Sek.}). \qquad (8)$$

The convergence of the preceding series is clarified by the fact that (2) is the same as its product absolutely convergent since $n < 1$. The integration strengthens the convergence because of the appearing divisors.

Par. 7. The coefficients **A**. The convergence of the series for A increases when they are multiplied by $(1 - n^2)^2$. It follows that

$$\left.\begin{aligned} A_0 &= \frac{1}{1+n}\left(1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \cdots\right) \\ A_2 &= \frac{3}{1+n}\left(n - \frac{1}{8}n^3 - \cdots\right) \\ A_4 &= \frac{15}{8(1+n)}\left(n^2 - \frac{1}{4}n^4 - \cdots\right) \\ A_6 &= \frac{35}{24(1+n)}\left(n^3 - \cdots\right) \\ A_8 &= \frac{315}{256(1+n)}\left(n^4 - \cdots\right). \end{aligned}\right\} \qquad (1)$$

In comparison the developments for M using e^2 or m produce the following:

$$\left.\begin{aligned} A_0 &= 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16384}e^8 - \cdots \\ A_2 &= \frac{3}{4}\left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6 + \frac{35}{512}e^8 + \cdots\right) \\ A_4 &= \frac{15}{128}\left(e^4 + \frac{3}{4}e^6 + \frac{35}{64}e^8 + \cdots\right) \\ A_6 &= \frac{35}{1536}\left(e^6 + \frac{5}{4}e^8 + \cdots\right) \\ A_8 &= \frac{315}{65536}\left(e^8 + \cdots\right) \end{aligned}\right\} \qquad (2)$$

Par. 8. Introduction of the Mean Length G of a Meridian Degree.

$$\left.\begin{aligned} A_0 &= \left(1 - \tfrac{1}{16} m^2 - \tfrac{15}{1024} m^4 - \cdots\right) : \sqrt{1+m} \\ A_2 &= \tfrac{3}{2}\left(m + \tfrac{3}{32} m^3 + \cdots\right) : \sqrt{1+m} \\ A_4 &= \tfrac{15}{32}\left(m^2 + \tfrac{5}{16} m^4 + \cdots\right) : \sqrt{1+m} \\ A_6 &= \tfrac{35}{192}\left(m^3 + \cdots\right) : \sqrt{1+m} \\ A_8 &= \tfrac{315}{4096}\left(m^4 + \cdots\right) : \sqrt{1+m}. \end{aligned}\right\} \qquad (3)$$

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Equations (2) are most unfavorable; also equations (3) are not as good as equations (1), which can be seen in regard to the relation of m being nearly equal to $2n$.

The derivation using e^2 is already contained in Méthodes analytiques (1799) by Delambre up to e^n . Formulae with m were contained in Traité de Géodésie Volume 1 by Puissant and recently by Ph. Fischer in his Untersuchungen; Bessel used n (Abhandlung, Volume 3, p. 44) 1837. The introduction of n or of powers of sins does not have any advantages of which the authors have convinced themselves.

Par. 8. Introduction of the Mean Length G of a Meridian Degree. We obtain for the meridian quadrant according to equation (5) p. 46 by substituting $B = \frac{\pi}{2}$

$$\text{Mer. Quadr.} = \frac{\pi}{2} a_0 A_0. \qquad (1)$$

On the other hand equating it to $10\,000\,G$, one obtain G di-

rectly the equation for G:

$$G = \frac{1}{\varrho''} a_0 A_0 = \frac{1 + \frac{1}{4} n^2 + \cdots}{1 + n} \frac{a_0}{\varrho''}. \qquad (2)$$

For the meridian arc M from the equator to latitude B it follows that:

$$\begin{aligned} M = G \{ & B \text{ in Degrees} \\ & - \tfrac{3}{2} \varrho^0 \left(n - \tfrac{3}{8} n^3\right) \sin 2B \\ & + \tfrac{15}{16} \varrho^0 \left(n^2 - \tfrac{1}{2} n^4\right) \sin 4B \\ & - \tfrac{35}{48} \varrho^0 n^3 \sin 6B \\ & + \tfrac{315}{512} \varrho^0 n^4 \sin 8B + \cdots \} \\ M = & \,[5{,}0457946.544] B \text{ in Degrees} \\ & + [4{,}2038114.754_n] \sin 2B \\ & + [1{,}2234947] \sin 4B \\ & + [8{,}3382\,n - 10] \sin 6B \\ & + [5{,}49 - 10] \sin 8B + \cdots \\ M = & \,111120{,}6196090\, B \text{ in Degrees} \\ & - 15988{,}63821 \sin 2B \\ & + 16{,}72995 \sin 4B \\ & - 0{,}02178 \sin 6B \\ & + 0{,}00003 \sin 8B + \cdots . \end{aligned} \qquad (3)$$

On the other hand for the meridian arc ΔM from B_1 to B_2:

(See Illustrative Material on Next Page.)

Par. 8. Introduction of the Mean Length G of a Meridian Degree.

$$\left.\begin{aligned} \Delta M = G\,\{&\Delta B \text{ in degrees} \\ &- 3\rho^0\left(n - \tfrac{3}{8}n^3\right)\cos 2B \sin \Delta B \\ &+ \tfrac{15}{8}\rho^0\left(n^2 - \tfrac{1}{2}n^4\right)\cos 4B \sin 2\Delta B \\ &- \tfrac{35}{24}\rho^0 n^3 \cos 6B \sin 3\Delta B \\ &+ \tfrac{315}{256}\rho^0 n^4 \cos 8B \sin 4\Delta B + \cdots\} \\ \Delta M = [5{,}0457946.544]\,\{&\Delta B \text{ in degrees} \\ &+ [9{,}4590468.167_n - 10]\cos 2B \sin \Delta B \\ &+ [6{,}4787300 - 10]\cos 4B \sin 2\Delta B \\ &+ [3{,}5934_n - 10]\cos 6B \sin 3\Delta B \\ &+ [0{,}74_n - 10]\cos 8B \sin 4\Delta B + \cdots\}. \end{aligned}\right\} \quad (4)$$

It is to be noted that in regard to the coefficients of the periodic terms appearing in equations (3) and (4), which are produced by the division of A_0 into $A_2, A_4 \cdots$, the convergence does not remain doubtful on account of the small value of n for the Earth ellipse. Disregarding A_0, the arising common factor of all A terms has the form $1 + u$, where u is such a small quantity that the reciprocal value of $1 + u$, arranged in powers of u, produces in any case an absolutely convergent series.

The computation of a table according to formula (3) can be done with seven-place logarithms if one is also in the possession of ten-place sin tables to form $16000 \sin 2B$ for the second term. Such a table from degree to degree, which suffices, is included among others in Hoüel's table of five-place logarithms. The table of radius of curvature

of meridians progresses first of all from 30' to 30' and is then improved by interpolation. If the table is not interpolated within a very narrow interval it has little practical value.

Encke, in his computation of a table of meridian arcs, does not proceed from a formula which yields them directly. He computes first only the arc lengths for 1" , which appertain to the radii of curvature of the meridians, in intervals of 30'. The table of meridian arcs follows from this through mechanical quadrature. The table by Encke is related to Toises. Börsch gave a table from $B = 35°$ to 71° for meters in the previously mentioned Tafeln für geodätische Berechnungen.

Par. 9. Small Meridian Arc. For the case when the meridian arc has a length of not more than several degrees it is advantageous to introduce the radius ϱ_m which belongs to the mean latitude. Then according to p. 47 (7) and p. 46 (3)

$$\Delta M = a_0 (A_0 \Delta B - A_2 \cos 2B \sin \Delta B + A_4 \cos 4B \sin 2\Delta B - A_6 \cos 6B \sin 3\Delta B + A_8 \cos 8B \sin 4\Delta B - \cdots)$$

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$$\varrho_m \Delta B = a_0 (A_0 \Delta B - A_2 \cos 2B . \Delta B + A_4 \cos 4B . 2\Delta B - A_6 \cos 6B . 3\Delta B + A_8 \cos 8B . 4\Delta B - \cdots).$$

From this follows directly with regard to the sin series p. 28:

$$\left.\begin{aligned} \Delta M = \varrho_m \Delta B + a_0 A_2 \cos 2B \left(\frac{\Delta B^3}{6} - \frac{\Delta B^5}{120} + \cdots\right) \\ - a_0 A_4 \cos 4B \left(\frac{8\,\Delta B^3}{6} - \cdots\right) + \cdots, \end{aligned}\right\} \quad (1)$$

where the omitted terms remain insignificant if one can neglect one to two tenths of a millimeter. Because approximately:

$$\left.\begin{aligned} a_0 A_2 &= 31977 & a_0 A_6 &= 0.04 & A_2 &= \tfrac{1}{200} \\ a_0 A_4 &= 33 & a_0 A_8 &= 0.00006 & A_4 &= \tfrac{1}{1000000} \end{aligned}\right\} \quad (2)$$

and it can easily be shown that for

$$\Delta B = 0{,}1 \text{ d. h. } \Delta B \text{ in Degrees} = 5{,}7$$

the first omitted terms do not exceed 0,2 mm.

Since $\log \Delta M$ is frequently wanted we reduce to it; the calculation for ΔM itself is thereby not more complicated.

We let $\Delta M = \varrho_m \Delta B\,(1 + u)$ and observe the series for $\log(1 + u)$ on p. 27. In the preceeding case the maximum u is nearly $\frac{A_2}{600}$, or $\frac{1}{120000}$; the term with u^2 can therefore be neglected because it will not influence the tenth decimal place of logarithm. Therefore

$$\left.\begin{aligned} \log \Delta M &= \log\left(\frac{\varrho_m \Delta B}{\varrho''}\right) + \beta_4 \Delta B^2 + \beta_6 \Delta B^4 + \cdots \\ &\text{for } \Delta B \text{ in Seconds} \\ \varrho_m &\text{ belonging to mean latitude} = \tfrac{1}{2}(B_1 + B_2) \\ \beta_4 &= \frac{a_0 \text{ Mod.}}{6 \varrho_m \varrho''^2}\left(A_2 \cos 2B - 8 A_4 \cos 4B + \cdots\right) \\ \beta_6 &= -\frac{a_0 \text{ Mod.}}{120 \varrho_m \varrho''^4}\left(A_2 \cos 2B - \cdots\right). \end{aligned}\right\} \quad (3)$$

For the values of $\log \beta_4$ and $\log \beta_6$ a test chart should be made; it is sufficient to calculate the former to five places

of decimals and the latter to two places. In this case it is advantageous to convert the expressions. For this purpose one substitutes into β_4 in place of ϱ_m the value $a_0(A_0 - A_2\cos 2B)$ and into β_6 just $a_0 A_0$. The errors amount then to only $\frac{1}{100000}$ and $\frac{1}{100}$ in the respective terms. Therefore

$$\beta_4 = \frac{\text{Mod.}}{6\varrho''^2}\left(\frac{A_2}{A_0}\cos 2B - 8\frac{A_4}{A_0}\cos 4B + \cdots\right) : \left(1 - \frac{A_2}{A_0}\cos 2B + \cdots\right)$$

$$\beta_6 = -\frac{\text{Mod.}}{120\varrho''^4}\left(\frac{A_2}{A_0}\cos 2B + \cdots\right).$$

One also obtains the following expressions for the quotients of A:

$$\frac{A_2}{A_0} = 3\left(n - \frac{3}{8}n^3 + \cdots\right) \qquad \frac{A_4}{A_0} = \frac{15}{8}\left(n^2 - \frac{1}{2}n^4 + \cdots\right),$$

one can however stop after the first terms without influencing the tenth decimal place of $\log \Delta M$.

At the same time we substitute for the denominator in β_4, which is of the form $1 - u$ the factor $1 + u + u^2 + \cdots$ and omit the squared term, which results in a maximum error of $\frac{1}{100000}$ of the value of β_4, or one unit in the tenth decimal place. Therefore by substituting $\cos 4B = 2\cos^2 2B - 1$

$$\left.\begin{aligned} \beta_4 &= \frac{n\ \text{Mod.}}{2\varrho''^2}\left(\cos 2B - 7n\cos^2 2B + 5n + \cdots\right) \\ \beta_6 &= -\frac{n\ \text{Mod.}\cos 2B}{40\varrho''^4} + \cdots = -\frac{\beta_4}{20\varrho''^2} + \cdots. \end{aligned}\right\} \qquad (4)$$

substituting for n the development of e^2, one obtains

$$\beta_4 = \frac{e^2\ \text{Mod.}}{8\varrho''^2}\left(\cos 2B + 6e^2\sin^2 B - 7e^2\sin^4 B + \cdots\right), \qquad (5)$$

which coefficient form is used by Andrae (Den Danske Gradmaaling, Volume 3, p. 291).

It can be seen from the foregoing development that equation (3) does give the tenth decimal place of $\log \Delta M$ very accurately until $\Delta B = 0{,}1$. Taken analytically, it neglects terms of the ninth order and higher when e and ΔB are regarded as terms of the first order. The same accuracy is obtained with the equation

$$\log \Delta B \text{ in Sec.} = \log\left(\frac{\varrho'' \Delta M}{\varrho_m}\right) - \beta_4 \left(\frac{\varrho'' \Delta M}{\varrho_m}\right)^4 - \beta_6 \left(\frac{\varrho'' \Delta M}{\varrho_m}\right)^6 + \cdots, \quad (6)$$

which follows from equation (3) upon solving for ΔB. First of all the right hand side contains as factor of β_4 and β_6 a power of ΔB; which can be replaced by $\varrho'' \Delta M : \varrho_m$. Since in equation (3) the maximum value of the term with β_4 is 35000 units of the tenth decimal place, ΔB in secs. $= \varrho'' \Delta M : \varrho_m$ except for $\frac{1}{10000000}$ of the value.*)

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The error which arises in the term in β_4 in equation (6) amounts therefore to $\frac{1}{50000}$ of the amount. According to this the accuracy of (6) is essentially that of (3).

The computation of ΔB according to equation (6) is indirect because ϱ_m depends on ΔB inasmuch as it belongs to the independent variable. Therefore one has to start the computations

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*)If the error in a number $z = \frac{1}{u} z$, then the error in $\log z = \frac{\text{Mod.}}{u}$, which is shown by the series development.

with an approximate value of ϱ_m and then to determine ϱ_m and ΔB more accurately with the obtained value for ΔB.

Tabular Statement for β_4.*)

B_m	β_4	$\log \beta_4$ For Units of the 7th Place of Decimals
0°	$+85161 : 10^{19}$	2,93024 — 10
45	+ 715 „	0,85443 „
47	— 5250 „	1,72018. — 10
48	— 8228 „	1,91528. „
49	— 11196 „	2,04907. „
50	— 14153 „	2,15084. „
51	— 17094 „	2,23284. „
52	— 20015 „	2,30136. „
53	— 22914 „	2,36009. „
54	— 25786 „	2,41138. „
55	— 28627 „	2,45677. „
56	— 31435 „	2,49742. „
57	— 34206 „	2,53410. „
90	— 85734 „	2,93315. „

For $B = 45^0$ and $\Delta B = 5^0$ formula (4) p. 49 yields $\log \Delta M = 5{,}7447601.171.$ On the other hand equation (3) of this section yields $\log \Delta M = 5{,}7447600.939 + .232 + .000$ namely the same. $\Delta M = 555597{,}2879^m.$ m.

For $B = 0^0$ and $\Delta B = 5^0$ the respective results are 5,7425852.343 and on the other hand $5{,}7425824.761 + 27.5925 - .0105$ namely the same. $\Delta M = 552821{,}8937^m.$ m.

Neglecting the terms with β_4 and β_6, the error in the

*)Can be easily defined by interpolation.

maximum becomes for $\Delta B = 10'$ in ΔM equals 0,0001m
„ $\Delta B = 1^\circ$ „ ΔM „ 0,03m
„ $\Delta B = 5{,}7^\circ$ „ ΔM „ 5m.

Par. 10. Computation of the Geographic Latitude of the End Point of a Meridian Arc Starting at a Given Point. We establish the following relation for the meridian arc M originating at the equator:

$$\frac{M}{G\varrho^{\circ}} = \sigma. \qquad (1)$$

Then we have accordingto p. 48 (3) the following equation for σ, where B is to be taken as an arc:

$$\sigma = B - \frac{3}{2}\left(n - \frac{3}{8}n^3\right)\sin 2B + \frac{15}{16}n^2 \sin 4B - \frac{35}{48}n^3 \sin 6B + \cdots . \qquad (2)$$

In it the terms with n^4 etc. are neglected since they amount at the most to some hundredths of millimeters. The difference $\sigma - B$, appears now as a function of σ. Since $(\sigma - B)$ is evidently a small magnitude in relation to B and σ no difficulties will be encountered in this calculation.

First we form with equation (2)

$$\sin 2\sigma = \sin 2B \cos\left\{3n \sin 2B - \frac{15}{8}n^2 \sin 4B + \cdots\right\}$$
$$- \cos 2B \sin\left\{3n \sin 2B - \frac{15}{8}n^2 \sin 4B + \cdots\right\}.$$

Regarding the series for cosine and sine and neglecting conse-

quent n^3 etc. we obtain:

$$\sin 2\sigma = \sin 2B - 3n \sin 2B \cos 2B - \frac{9}{2} n^2 \sin^3 2B + \frac{15}{8} n^2 \cos 2B \sin 4B \cdots . \quad (3)$$

Multiplying this on both sides by $\frac{3}{2}\left(n - \frac{3n^3}{8}\right)$ and adding each side to (2) we obtain:

$$\sigma + \frac{3}{2}\left(n - \frac{3n^3}{8}\right) \sin 2\sigma = B - \frac{21}{16} n^2 \sin 4B - \frac{117}{32} n^3 \sin 2B + \frac{227}{96} n^3 \sin 6B - \cdots , \quad (4)$$

where the following relations are used in the reduction of the multiplied terms in n^3

$$\sin^3 2B = \frac{3}{4} \sin 2B - \frac{1}{4} \sin 6B$$
$$\cos 2B \sin 4B = \frac{1}{2} \sin 6B + \frac{1}{2} \sin 2B$$

and the terms in n^4 etc. are neglected. It follows from (2) with further neglect of n^2 etc.

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$$\sin 4\sigma = \sin 4B \cos (6n \sin 2B + \cdots) - \cos 4B \sin (6n \sin 2B + \cdots)$$

also

$$\sin 4\sigma = \sin 4B - 6n \sin 2B \cos 4B + \cdots . \quad (5)$$

Multiplying this on both sides by $\frac{21}{16} n^2$ and adding it to (4) one obtains just as precisely as previously by introducing the value $\sin 6B - \sin 2B$ for $2 \sin 2B \cos 4B$ and simplifying it generally:

(See Illustrative Material on Next Page.)

Par. 10. Computation of the Geographic Latitude of the End Point of a Meridian Arc Starting at a Given Point.

$$\sigma + \frac{3}{2}\left(n - \frac{3n^3}{8}\right)\sin 2\sigma + \frac{21}{16}n^2 \sin 4\sigma = B + \frac{9}{32}n^3 \sin 2B - \frac{151}{96}n^3 \sin 6B + \cdots .$$

Taking all the small terms on the same side as σ and substituting into the ones that contain B one obtains only the insignificant terms with n^4 etc. One obtains finally.

$$\left.\begin{aligned} B &= \sigma + \frac{3}{2}\varrho''\left(n - \frac{9}{16}n^3\right)\sin 2\sigma \\ &\quad + \frac{21}{16}\varrho'' n^2 \sin 4\sigma \\ &\quad + \frac{151}{96}\varrho'' n^3 \sin 6\sigma + \cdots , \\ \sigma &= \frac{3600\,M}{G}. \end{aligned}\right\} \quad (6)$$

This formula can also be derived by the method of undetermined coefficients; the development presented here also shows, however, the admissability of this method clearly.

For the latitude computation of B_2 from B_1 and the length ΔM of the meridian arc P_1P_2 it is easiest to prepare a special formula for ΔB. Applying equation (6) to B_1 with σ_1 and to B_2 with σ_2 and subtracting, it follows by indicating $\sigma_2 - \sigma_1$ by $\Delta\sigma$ and $\frac{1}{2}(\sigma_2 + \sigma_1)$ with σ that:

$$\left.\begin{aligned} \Delta B &= \Delta\sigma + 3\varrho''\left(n - \frac{9}{16}n^3\right)\cos 2\sigma \sin \Delta\sigma \\ &\quad + \frac{21}{8}\varrho'' n^2 \cos 4\sigma \sin 2\Delta\sigma \\ &\quad + \frac{151}{48}\varrho'' n^3 \cos 6\sigma \sin 3\Delta\sigma + \cdots \\ \Delta\sigma &= 3600\,\frac{\Delta M}{G}. \end{aligned}\right\} \quad (7)$$

The following is to be used for the calculation of 2σ, with consideration of the fact that in equation (7) terms with n^4 etc. are neglected.

$$\left.\begin{aligned} 2\sigma_1 &= 2B_1 - 3\varrho'' n \sin 2B_1 + \tfrac{15}{8}\varrho'' n^2 \sin 4B_1 - \cdots \\ 2\sigma &= 2\sigma_1 + \Delta\sigma. \end{aligned}\right\} \qquad (8)$$

These formulae have only an error of about one unit in the fifth decimal place of the seconds for random arc lengths ΔM.

Par. 11. Meridian Arcs by Means of Reduced Latitude. It is desirable to know the meridian arc as a function of the reduced latitude for later derivations. According to p. 40 since $dM^2 = dx^2 + dz^2$

$$dM = \sqrt{a_0^2 \sin^2\beta + b_0^2 \cos^2\beta}\, d\beta,$$

and introducing the equivalent expression $a_0^2(1-e^2)$ for b_0^2 it follows immediately that

$$dM = a_0 \sqrt{1 - e^2\cos^2\beta}\, d\beta. \qquad (1)$$

Since it will become apparent immediately be reference to p. 47 that it is advantageous to introduce n for e^2 we equate $e^2 = 4n:(1+n)^2$ and obtain

$$dM = \frac{a_0}{1+n} \sqrt{1 + n^2 - 2n\cos 2\beta}\, d\beta. \qquad (2)$$

Equating $\cos 2\beta = \frac{1}{2}(\varepsilon^{2i\beta} + \varepsilon^{-2i\beta})$, the radical quantity divides into the two factors

$$(1 - n\varepsilon^{2i\beta})^{\frac{1}{2}} \text{ and } (1 - n\varepsilon^{-2i\beta})^{\frac{1}{2}}$$

and this produces the convergent development since $n<1$ according to the binomial theorem:

$$(1-ne^{\pm 2i\beta})^{\frac{1}{2}}=1-\frac{n}{2}e^{\pm 2i\beta}-\frac{n^2}{8}e^{\pm 4i\beta}-\frac{n^3}{16}e^{\pm 6i\beta}-\frac{5n^4}{128}e^{\pm 8i\beta}-\cdots.$$

The multiplication of the series with the higher signs and the series with the lower signs produces:

$$dM=a_0(A_0-\tfrac{1}{3}A_2\cos 2\beta-\tfrac{2}{15}A_4\cos 4\beta-\tfrac{3}{35}A_6\cos 6\beta-\tfrac{4}{63}A_8\cos 8\beta-\cdots)d\beta, \quad (3)$$

where A_0 to A_8 are the same quantities as on p. 47. This is shown without any effort insofar as the first four powers of n are concerned. To convince oneself that one has the A generally one has to

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compare the so called general terms. There is no value in carrying this out here; the author has convinced himself of the rigorous equality with earlier values of A_0, A_2 and A_4 from which the behavior of the remaining A could be concluded.

We obtain further for the arc ΔM from β_1 to β_2:

$$\Delta M=a_0\left\{\begin{array}{l} A_0\Delta\beta-\frac{1}{3}A_2\cos 2\beta\sin\Delta\beta-\frac{1}{15}A_4\cos 4\beta\sin 2\Delta\beta \\ \qquad -\frac{1}{35}A_6\cos 6\beta\sin 3\Delta\beta-\frac{1}{63}A_8\cos 8\beta\sin 4\Delta\beta-\cdots \\ \Delta\beta=\beta_2-\beta_1 \qquad \beta=\frac{1}{2}(\beta_2+\beta_1). \end{array}\right\} \quad (4)$$

This series converges sooner than that of the geographic latitude and the term with A_8 is completely sufficient; the one with A_6 results only in 1 mm. Although the convergence is

increased by the introduction of β the application of the series in question has no advantages in general when the latitudes of the end points have to be reduced first and are not given directly.

Par. 12. Radius of Transverse Curvature. The normals of two points which belong to the same parallel intersect at one point K' of the axis of rotation. By placing the two normals in one plane and letting the points approach each other until they converge on one point a vertical plane tangent to the parallel of latitude is produced which makes a right angle with the meridian plane at the point of tangency, i.e. is in the azimuth of 90^0 . The radius of curvature of the infinitessimal element of arc of the surface which adjoins P in this direction is the length of the normal PK' since the normals of two coinciding points of the element of arc intersect at K', which can be seen from the previously given presentation. We let

$$PK' = \varrho_n \tag{1}$$

and have with regard to Fig. 1 (p. 40) the following relations for the radius of transverse curvature (Radius of Curvature in the Perpendicular Plane). It follows

$$\varrho_n = x \sec B = \frac{a_0}{W}$$
$$\varrho_n = \frac{\varrho_m W^2}{(1 - e^2)}. \tag{2}$$

The latter of these formulae shows, that ϱ_n is always greater than ϱ_m , i.e.

at every point of the surface the curvature of the meridian is greater than the curvature of the perpendicular.

For computation of a single ϱ_n the following can be used instead of (2)

$$\varrho_n = a_0\left(1+\frac{1}{2}e^2\sin^2 B+\frac{3}{8}e^4\sin^4 B+\frac{5}{16}e^6\sin^6 B+\dots\right) \qquad (3)$$

as a rule $\log\varrho_n$ is evaluated numerically with the above formula and the use of a table for W. A table for $\log\varrho_n$ to seven decimal places is given in Albrecht p. 199, and for $\log\frac{\varrho_n}{\varrho''}$ to eight decimal places in Börsch elsewhere.

Par. 13. Radius of Curvature of a Random Azimuth. To determine the radius of curvature ϱ_α of an element of arc of the line of intersection of the surface adjoining P and a vertical plane through P with azimuth α and which is inclined towards the meridian plane the following theorem by Euler is used:

$$\frac{1}{\varrho_\alpha}=\frac{\cos^2\alpha}{\varrho_m}+\frac{\sin^2\alpha}{\varrho_n}. \qquad (1)$$

This theorem is valid with an appropriate modification for the signification of ϱ_m and ϱ_n for every curved surface, but we restrict ourselves however to prove it here for the ellipsoid of rotation only.

Any point P of the ellipsoid of rotation for which the theorem is to be proven is taken as the origin of the rectangular coordinates ξ, η, ζ. In general equation of the second degree which can also serve as a representation of the ellipsoid of ro-

tation the term without the coordinates disappears because the equation has to remain correct for ξ, η and ζ equal to zero. It may be written therefore as:

$$0 = 2A\xi + 2B\eta + 2C\zeta + D\xi^2 + E\eta^2 + F\zeta^2 + 2G\xi\eta + 2H\xi\zeta + 2I\eta\zeta,$$

where $A, B, C, D, E, F, G, H, I$, represent the given coefficients which depend upon the position ofthe point and the dimensions of the axes. Taking the $\xi\eta$ - plane tangential to the surface and also the vertical of the point as the ζ - axis, A and B, will disappear because the two differential quotients $d\zeta : d\xi$ and $d\zeta : d\eta$ become zero for ξ, η and ζ equal to zero.

Furthermore, placing the ξ- axis into the meridian plane, the latter becomes the $\xi\zeta$ - plane and since it is also the plane of symmetry G and I necessarily have to disappear to make the equation purely quadratic with respect to η. Therefore the equation of the surface is:

$$0 = 2C\zeta + D\xi^2 + E\eta^2 + F\zeta^2 + 2H\xi\zeta.$$

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The polar coordinates ϑ and α are now introduced into the $\xi\eta$ - plane according to the relations

$$\xi = \vartheta \cos\alpha \quad \eta = \vartheta \sin\alpha$$

The equation can therefore be changed by a simple reduction to

$$\vartheta^2 = -\zeta \frac{2C + 2H\vartheta\cos\alpha + F\zeta}{D\cos^2\alpha + E\sin^2\alpha},$$

and this is also the equation of the intersection curve of the

surface with the vertical plane in the azimuth α. An arc of Radius ρ_α which is tangent to the intersection curve at P (Fig. 2) has the equation:

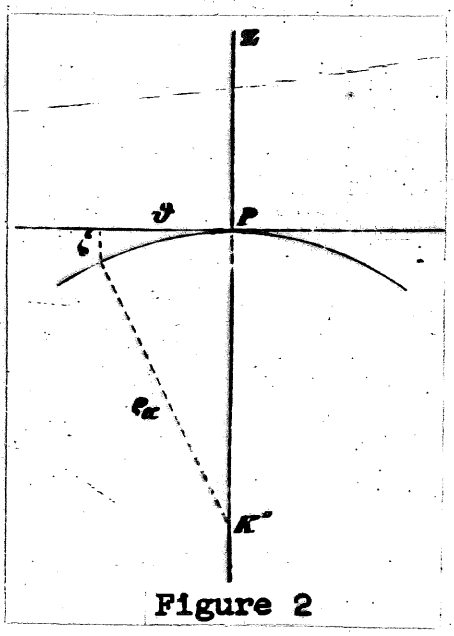


Figure 2

$$\vartheta^2 = \zeta(2\rho_\alpha - \zeta);$$

on choosing ρ_α according to the formula

$$\rho_\alpha = \frac{C}{D\cos^2\alpha + E\sin^2\alpha}, \qquad (2)$$

the equations of both curves correspond except for terms of the order $\vartheta\zeta$ and ζ^2, which are to be neglected for infinitesimal ϑ against the singly multiplied terms in ζ. The arc falls therefore according to the choice of ρ_α closer to the intersection curve at point P than for any other assumption for ρ_α. This value is therefore called the radius of curvature of the surface in the azimuth α.

For $\alpha = 0^0$, $\rho_\alpha = \rho_m$ and ρ_n, respectively and

$$\rho_m = -\frac{C}{D} \text{ und } \rho_n = -\frac{C}{E}. \qquad (3)$$

Eliminating the coefficients C, D and E from (2) by means of these relations one obtains (1).

This shows, that since $\rho_n > \rho_m$, ρ_n is a maximum and ρ_m is a minimum for all ρ_α. ρ_m and ρ_n are called the <u>main radii of curvature</u>.

Par. 14. Computation of ρ_α. From (1) of the previous section it follows upon substitution of the values for ρ_m and ρ_n on p. 44 and 56:

$$\frac{1}{\varrho_\alpha} = \frac{W}{a_0}\left(\frac{W^2}{1-e^2}\cos^2\alpha + \sin^2\alpha\right) = \frac{W}{a_0}(1 + \delta\cos^2 B\cos^2\alpha). \qquad (1)$$

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Introducing $\tan h = \sqrt{\delta}\cos B\cos\alpha$ one obtains the convenient formula:

$$\left.\begin{aligned}\log\varrho_\alpha &= \log a_0 - \log W + 2\log\cos h\\ \tan h &= \sqrt{\delta}\cos B\cos\alpha.\end{aligned}\right\} \qquad (2)$$

To obtain a series development we replace the logarithm of $(1 + \delta\cos^2 B\cos^2\alpha)$ by the known series p. 27. Therefore writing simply on account of this

$$\log\varrho_\alpha = \log a_0 - \log W - M\delta\cos^2 B\cos^2\alpha + \cdots, \qquad (3)$$

than the error is not greater than one unit in the fifth decimal place. For absolute values of $B > 45$ the sixth place of decimals is usually still correct.

A very detailed table for $\log\varrho_\alpha$ to six places and B and α as argument, only in Toise measure, is contained in Bremikers Studien... Albrecht supplies a similar table in metric measure on p. 201 but in the second interval, to simplify interpolation. A table with eight place values, which however requires some calculation is based on the development of

$$\frac{1}{\varrho_\alpha} = \frac{1}{2}\left(\frac{1}{\varrho_m} + \frac{1}{\varrho_n}\right)\left\{1 + \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m}\cos 2\alpha\right\}$$

is contained in Bremiker's tables of six-place logarithms.

Par. 15. The Measure of Curvature. Par. 16. Radius Vector of the Geocentric (Corrected) Latitude.

Par. 15. The Measure of Curvature. According to Gauss the reciprocal value of the product of the main radii of curvature is called the measure of curvature. We substitute for the geographic latitude B

$$\frac{a_0^2}{\varrho_m \varrho_n} = K, \tag{1}$$

And the measure of curvature under latitude B is also

$$K : a_0^2. \tag{2}$$

The substitution of the values for ϱ_m and ϱ_n produces

$$K = \frac{W^4}{1 - e^2} = \frac{1 - e^2}{w^4}. \tag{3}$$

A table for $\log \sqrt{\varrho_m \varrho_n}$ to five decimal places and for K to seven decimal places is given in Albrecht p. 199 and 213; and for K to eight places and in a narrow interval in Gadbeck no. 3 of Rechnungsmethoden des Zentral bureaus (printed as manuscript).

Par. 16. Radius Vector of the Geocentric (Corrected) Latitude. We denote with r, the radius vector MP which is drawn from the origin of the ellipsoid to a point on the surface, its angle of inclination

to the equatorial plane, i.e. the geocentric latitude, by φ. Then, see Fig. 1. p. 40,

$$x = r \cos \varphi \quad z = r \sin \varphi, \tag{1}$$

and according as one eliminates from these two terms r or φ one obtains by substitution of the expressions for x and z, or φ and r as functions of B:

$$\left.\begin{aligned} \tan\varphi &= (1-e^2)\tan B = \frac{1-m}{1+m}\tan B; \\ r &= a_0\sqrt{\frac{1-e^2(2-e^2)\sin^2 B}{1-e^2\sin^2 B}} = a_0\sqrt{\frac{1+m^2}{1+m}}\sqrt{\frac{1+\frac{2m}{1+m^2}\cos 2B}{1+m\cos 2B}} \end{aligned}\right\} \quad (2)$$

Eliminating from both formulae B by use of the relation $\sin^2 B = \tan^2 B : (1+\tan^2 B)$, one obtains:

$$r = \frac{a_0}{\sqrt{1+\delta\sin^2\varphi}} = \frac{a_0\sqrt{1-m}}{\sqrt{1-m\cos 2\varphi}}. \quad (3)$$

Comparing this with p. 41, it can easily be seen that φ depends on B and m in the same manner as β depends on B and n. Therefore among others the following convergent development for arc $(B-\varphi)$ holds:

$$\underset{\text{in Seconds}}{B-\varphi} = \varrho''\left\{m\sin 2B - \frac{m^2}{2}\sin 4B + \frac{m^3}{3}\sin 6B - \cdots\right\} \quad (4)$$

The following development for r can be obtained from formulae (2) and (3):

$$\left.\begin{aligned} \log r = \Big\{\log a_0\sqrt{\frac{1+m^2}{1+m}} - \frac{3Mm^2}{8} + \cdots\Big\} &= 6{,}8039181.997 \\ + \frac{M}{2}\left(m - \frac{m^3}{4} + \cdots\right)\cos 2B &\quad + [3.8615853]\cos 2B \\ - \frac{3Mm^2}{8}\cos 4B + \cdots &\quad + [1.26148_n]\cos 4B \\ + \frac{7Mm^3}{24}\cos 6B + \cdots &\quad + [0.677-2]\cos 6B + \cdots. \end{aligned}\right\} \quad (5)$$

The numerical values of the coefficients are added in units to the seventh place of decimals.

$$r = a_0\left(1 - \frac{1}{2}\delta\sin^2\varphi + \frac{3}{8}\delta^2\sin^4\varphi - \frac{5}{16}\delta^3\sin^6\varphi + \frac{35}{128}\delta^4\sin^8\varphi\cdots\right) \quad (6)$$

$$r = a_0\sqrt{1-m}\left\{\begin{aligned}&\left(1 + \frac{1.3}{1.2}\cdot\frac{2}{1}\left(\frac{m}{4}\right)^2 + \frac{1.3.5.7}{1.2.3.4}\cdot\frac{4.3}{1.2}\left(\frac{m}{4}\right)^4 + \cdots\right)\\ &+\left(\frac{m}{2} + \frac{15m^2}{64} + \cdots\right)\cos 2\varphi\\ &+\left(\frac{3m^2}{16} + \cdots\right)\cos 4\varphi\\ &+\left(\frac{5m^3}{64} + \cdots\right)\cos 6\varphi + \cdots.\end{aligned}\right\} \quad (7)$$

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Tables for r and $B - \varphi$ are given by Albrecht p. 195.

Establishing the polar equation of the meridian for an arbitrary body of rotation generally in the form:

$$r = a_0\,(1 - \alpha_1\sin^2\varphi + \alpha_2\sin^4\varphi - \alpha_3\sin^6\varphi + \alpha_4\sin^8\varphi\cdots),$$

the following relations exist in the case of the ellipse:

$$2\alpha_2 = 3\alpha_1\alpha_1;\ 3\alpha_3 = 5\alpha_1\alpha_2;\ 4\alpha_4 = 7\alpha_1\alpha_3;\ \text{u. s. w.}$$

Par. 17. Computation of Surface Area. The area of an infinitesimal element dZ between two parallel circles a distance dM apart, measured on the meridian, is given by the product of dM and the length of the parallel arcs (which is to be considered equal) between the two adjoining meridian arcs. Their difference in geographic latitude may be designated by L. The radius of the parallel of the geographic latitude B equals according to Fig. 1 (p. 40)

$$x = a_0\cos\beta = a_0\cos B : W.$$

We obtain therefore as the area for the mentioned element:

$$dZ = a_0 L \cos\beta dM = a_0 L \cos B dM : W. \qquad (1)$$

Substituting as the value for dM according to p. 44 and 55:

$$dM = a_0 w d\beta = \frac{a_0 (1 - e^2)\, dB}{W^3},$$

one obtains:

$$dZ = a_0^2 L \sqrt{1 - e^2 \cos^2\beta} \cos\beta d\beta = \frac{a_0^2 (1 - e^2) L \cos B dB}{(1 - e^2 \sin^2 B)^3}. \qquad (2)$$

It follows from this that for the zone from the equator to the geographic latitude B:

$$Z = a_0^2 (1 - e^2) L \int_0^{\sin B} \frac{d(\sin B)}{(1 - e^2 \sin^2 B)^3}.$$

The integration for $\sin B$ can be performed consistently, but for the numerical evaluation a series development is more convenient. Since

$$(1 - e^2 \sin^2 B)^{-2} = 1 + 2e^2 \sin^2 B + 3e^4 \sin^4 B + 4e^6 \sin^6 B + \cdots,$$

therefore:

$$Z = a_0^2 (1 - e^2) L \sin B \left(1 + \frac{2e^2}{3} \sin^2 B + \frac{3e^4}{5} \sin^4 B + \frac{4e^6}{7} \sin^6 B + \cdots\right). \qquad (3)$$

It is now convenient for the rigorous calculation of a zone between two parallels B_1 and B_2 to remove the powers of the sines according to the formulae (which can be derived easily by the introduction of the imaginary):

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$$\left.\begin{aligned}
\sin^3 B &= \tfrac{3}{4}\sin B - \tfrac{1}{4}\sin 3B \\
\sin^5 B &= \tfrac{5}{8}\sin B - \tfrac{5}{16}\sin 3B + \tfrac{1}{16}\sin 5B \\
\sin^7 B &= \tfrac{35}{64}\sin B - \tfrac{21}{64}\sin 3B + \tfrac{7}{64}\sin 5B - \tfrac{1}{64}\sin 7B
\end{aligned}\right\} \quad (4)$$

One obtains also the following expression for Z:

$$Z = L\,(Z_1 \sin B - Z_3 \sin 3B + Z_5 \sin 5B - Z_7 \sin 7B + \cdots) \quad (5)$$

$$\left.\begin{aligned}
Z_1 &= a_0^2\left(1 - \tfrac{e^2}{2} - \tfrac{e^4}{8} - \tfrac{e^6}{16} - \cdots\right) = [13{,}6078327.476] \\
Z_3 &= a_0^2\left(\tfrac{e^2}{6} + \tfrac{e^4}{48} + \text{null} + \cdots\right) = [10{,}6559083] \\
Z_5 &= a_0^2\left(\tfrac{3e^4}{80} + \tfrac{e^6}{40} + \cdots\right) = [7{,}83407] \\
Z_7 &= a_0^2\left(\tfrac{e^6}{112} + \cdots\right) = [5{,}04]
\end{aligned}\right\} \text{ for } z \text{ in square meter} \quad (6)$$

The area for a zone ΔZ with the mean latitude $B = \frac{1}{2}(B_1 + B_2)$ and amplitude $\Delta B = B_2 - B_1$ in geographic latitude between two meridian planes is therefore:

$$\Delta Z = 2L\left\{\begin{aligned}
&Z_1 \cos B \sin \tfrac{1}{2}\Delta B - Z_3 \cos 3B \sin \tfrac{3}{2}\Delta B \\
&+ Z_5 \cos 5B \sin \tfrac{5}{2}\Delta B - Z_7 \cos 7B \sin \tfrac{7}{2}\Delta B + \cdots
\end{aligned}\right\} \quad (7)$$

Where L is the difference in geographic length of the two meridian planes. The following substitution is to be made:

$$\log L = \log (L \text{ in Gr.}) - 1{,}7581226.324. \quad (8)$$

The preceeding formula just yeilds ΔZ in all cases as precisely as is possible with ten-place logarithms which is shown

by equation (6) directly and which can also be seen from equation (3) with regard to the factor $(1-e^2)$. Therefore it did not seem essential to consider e^3.

Substituting $B=90^0$ and $L=\frac{\pi}{2}$ into equation (3) and multiplying by (8) one obtains:

$$\text{surface} = 4a^2\pi\left(1-\frac{e^2}{3}-\frac{e^4}{15}-\frac{e^6}{35}-\ldots\right), \qquad (9)$$

where the progressing of the terms becomes obvious. Using Bessel's dimensions one obtains a value corresponding to one obtained by equation (7):

509950714,1 square kilometers

The integration of dZ by means of β yields according to known integration formulae the following surface without series development:

$$2a^2\pi\left(1+\frac{1-e^2}{2e}\log\text{nat}\left(\frac{1+e}{1-e}\right)\right), \qquad (10)$$

where the calculation verifies the quoted area to all ten places of decimals. Using a series development with powers of $\sin\beta$, before integration, the following strongly convergent series is produced:

$$4a_0b_0\pi\left(1+\frac{1}{6}\delta-\frac{1}{40}\delta^2+\frac{1}{112}\delta^3-\ldots\right). \qquad (11)$$

A table for Z is contained in Jordan, Handbuch der Vermessungskunde, Volume 2, p. 54.

Par. 18. Mean Radius of Curvature at a Point. According to formula (1) p. 57

$$\varrho_\alpha = \frac{\varrho_m \varrho_n}{\varrho_n \cos^2\alpha + \varrho_m \sin^2\alpha}.$$

From this one can obtain the average value by presuming that vertical sections are spread uniformly but very closely around the point. Therefore letting the quantity ϱ_α of the azimuth difference $d\alpha$ be proportional to $d\alpha$, the average equals:

$$\int_0^{2\pi} \frac{\varrho_m \varrho_n d\alpha}{\varrho_n \cos^2\alpha + \varrho_m \sin^2\alpha} : \int_0^{2\pi} d\alpha$$

i.e.

$$\frac{2}{\pi} \varrho_m \varrho_n \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\varrho_n \cos^2\alpha + \varrho_m \sin^2\alpha}.$$

Introducing the variable $\sqrt{\frac{\varrho_m}{\varrho_n}} \tan\alpha = t$, it follows:

$$\frac{2}{\pi} \sqrt{\varrho_m \varrho_n} \int_0^{\infty} \frac{dt}{1+t^2}$$

i.e.

$$\sqrt{\varrho_m \varrho_n} \text{ or } \frac{a_0}{\sqrt{K}}. \qquad (1)$$

The mean radius of curvature at a point of the surface is therefore the reciprocal of the square root of the measure of curvature of that point.

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This statement is valid for every surface. Also the other

equation which is easy to deduce states that the average of all $\frac{1}{\varrho_a}$ at a point is equal.

$$\frac{1}{2}\left(\frac{1}{\varrho_m}+\frac{1}{\varrho_n}\right).$$

Par. 19. Various Mean Radii of Curvature. An average value for the mean radius of curvature at a point in relation to all values of B within the meridian quadrant is obtained by an expression which is formed similarily to that in the last section:

$$\int_0^{\frac{\pi}{2}} \frac{a_0}{\sqrt{K}}\,dB : \int_0^{\frac{\pi}{2}} dB$$

i.e.

$$\int_0^{\frac{\pi}{2}} \frac{a_0\sqrt{1-e^2}\,dB}{1-e^2\sin^2 B} : \int_0^{\frac{\pi}{2}} dB.$$

The substitution $u = \cot B : \sqrt{1-e^2}$ in the numerator of the integral leads to the following value for the mean radius of curvature for all points of a meridian:

$$\frac{2a_0}{\pi}\int_0^{\infty}\frac{du}{1+u^2}$$

i.e.

$$a_0. \qquad (1)$$

Naturally one obtains a different value by taking the uniform distribution according to the reduced latitude β or according to the meridian arcs. To go into detail on this does not seem

appropriate since it is a question of mathematical problems without sufficient practical value.

Considering not the single meridian, but the surface, one had to assume that the values $a_0 : \sqrt{K}$ are distributed uniformly over the surface to obtain in the real sense of the word the mean radius of curvature of the surface. The same is,

$$a_0 \int_0^{\frac{\pi}{2}} \frac{dZ}{\sqrt{K}} : \int_0^{\frac{\pi}{2}} dZ .$$

where we can confine ourselves to the octant for the integration.

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Herein, the differential Zone dZ is to be taken for the difference in geographic longitude $L = \frac{\pi}{2}$. The remaining calculation becomes most convenient by the use of the reduced latitude as variable, p. 40 (2).

Introducing K according to p. 59 (3) and the denominator of the integral according to p. 63 (11) one obtains:

$$\frac{a_0 \sqrt{1-e^2} \int_0^1 \sqrt{1+\delta \sin^2\beta}^3 d(\sin\beta)}{1+\frac{1}{6}\delta - \frac{1}{40}\delta^2 + \frac{1}{112}\delta^3 - \dots} .$$

Developing the numerator according to powers of $\sin\beta$ and integrating one gets the integral $1+\frac{1}{2}\delta+\frac{3}{40}\delta^2-\frac{1}{112}\delta^3+\dots$ and one finally obtains as the value of the mean radius of curvature of the surface:

or

$$\left.\begin{array}{l} b_0\left(1+\frac{1}{3}\delta+\frac{2}{45}\delta^2-\frac{16}{945}\delta^3+\cdots\right) \\ a_0\left(1-\frac{1}{6}e^2+\frac{31}{360}e^4+\frac{1697}{15120}e^6+\cdots\right). \end{array}\right\} \qquad (2)$$

The hereto necessary division of the denominator into the numerator is permissible, since it has the form $(1+u)$, where the series u is surely <1 even if the terms are taken positive as long as we are dealing with the ellipsoid of the Earth.

Evaluating the integrals in closed form, the following expression for the mean radius of curvature is obtained:

$$\frac{b_0\left\{\frac{1}{4(1-e^2)^2}+\frac{3}{8(1-e^2)}+\frac{3}{16e}\log\text{nat}\frac{1+e}{1-e}\right\}}{\frac{1}{2(1-e^2)}+\frac{1}{4e}\log\text{nat}\frac{1+e}{1-e}}.$$

Applying the series development one can see that the second one of the series is valid for $e<1$.

[Vergl. c.f., § 18 (10) and (9)].

An average which is distinguished by its mathematical simplicity can be obtained by multiplying $a_0:\sqrt{K}$ not with dZ but with xdB. This means taking the distribution uniformly according to the geographic latitude over the parallel. This value can replace the former approximately. It follows

i.e. also

$$\left.\begin{array}{l} \frac{a_0 e}{\text{arc sin } e} \\ a_0\left(1-\frac{1}{6}e^2-\frac{17}{360}e^4-\frac{367}{15120}e^6-\cdots\right). \end{array}\right\}. \qquad (3)$$

Par. 20. Mean Radius Vector. Various average values can also be deduced in reference to the radius vector r

according to the assumptions. Only the most important are mentioned here.

Taking the uniform distribution of the r within the meridian plane in relation to the geocentric latitude φ, the average becomes

$$\int_0^{\frac{\pi}{2}} r\,d\varphi : \int_0^{\frac{\pi}{2}} d\varphi \quad \text{or} \quad \frac{2a_0}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1+\delta \sin^2\varphi}}. \tag{1}$$

This expression leads to a complete elliptic integral of the first order by simple transformations. Meanwhile using the series development (7) p. 60 for r according to the cosin of the variable 2φ one obtains, since all integrals with the cosin amount to zero, the mean radius vector of the meridian.

or also

$$\left.\begin{aligned} &a_0\sqrt{1-m}\left(1+\tfrac{3}{16}m^2+\dots\right) \\ &a_0\left(1-\tfrac{1}{4}e^2-\tfrac{7}{64}e^4-\tfrac{17}{256}e^6-\dots\right). \end{aligned}\right\} \tag{2}$$

The mean radius vector of the meridian ellipse differs hereby from the corresponding mean radius of curvature a_0 by magnitudes of the order of the flatness.

Taking the radii vectores r uniformly distributed over all directions of the space around the centrum one obtains their average value from the formula

$$\int r\,d\omega : \int d\omega,$$

where ω represents the surface element of a sphere with radius one and concentric with the ellipsoid, and the integrations are taken over the surface of the latter. Since r retains the same value for a constant geocentric latitude φ one can take for the zone which is situated between the parallels φ and $\varphi + d\varphi$ on the sphere, i.e. one can equate

$$d\omega = 2\pi \cos\varphi d\varphi$$

with it the mean radius vector of the surface in case of uniform distribution over the centrum becomes:

$$\frac{a_0 \int_0^1 \frac{d(\sin\varphi)}{\sqrt{1+\delta\sin^2\varphi}}}{\int_0^{\frac{\pi}{2}} \cos\varphi d\varphi} = b_0 \log\text{nat}\sqrt[2e]{\frac{1+e}{1-e}}. \qquad (3)$$

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Developing on the other hand the series before the integration one obtains immediately

$$\text{or} \quad \left.\begin{array}{l} a_0\left(1 - \frac{1}{6}\delta + \frac{3}{40}\delta^2 - \frac{5}{112}\delta^3 + \cdots\right) \\ a_0\left(1 - \frac{1}{6}e^2 - \frac{11}{120}e^4 - \frac{103}{1680}e^6 - \cdots\right). \end{array}\right\} \qquad (4)$$

The mean radius vector of the surface in the real sense of the word results, when one assumes the r distributed evenly over themselves and one also forms the expression

$$\int_0^{\frac{\pi}{2}} r dZ : \int_0^{\frac{\pi}{2}} dZ$$

p. 61 (2) yields, in that one can limit oneself to the octants:

$$dZ = \frac{a^2\pi}{2}\sqrt{1 - e^2\cos^2\beta}\cos\beta d\beta .$$

The integral in the numerator is the already calculated octant surface. We substitute into the denominator of the integral

$$r = \sqrt{a_0^2\cos^2\beta + b_0^2\sin^2\beta} = a_e\sqrt{1 - e^2\sin^2\beta}.$$

The same can be expressed by complete elliptic integrals in closed form.*) However in the present case the end expression is much simpler in the form of series. We equate in the numerator

$$\frac{a_0^3\pi}{2}\int_0^{\frac{\pi}{2}}\sqrt{(1 - e^2\cos^2\beta)(1 - e^2\sin^2\beta)}\cos\beta d\beta$$
$$= \frac{a_0^2 b_0\pi}{2}\int_0^1\sqrt{1 + \frac{e^4}{1 - e^2}\sin^2\beta(1 - \sin^2\beta)}\,d(\sin\beta).$$

Taking in addition the numerator according to p. 63 (11) we obtain as mean radius vector of the surface:

$$a_0\frac{1 + \frac{1}{15}\left(\frac{e^4}{1 - e^2}\right) - \dots}{1 + \frac{1}{6}\delta - \frac{1}{40}\delta^2 + \frac{1}{112}\delta^3 - \dots} \qquad (5)$$

*)Cf. Drobisch, Berichte über die Verhandlungen der Gesellschaft der Wissenschaft zu Leipzig 1858, p. 159.

or

$$a_0\left(1-\frac{1}{6}e^2-\frac{17}{360}e^4-\frac{83}{3024}e^6-\ldots\right). \qquad (6)$$

Par. 21. The Earth Sphere. If one wants to replace the ellipsoid of the Earth by a sphere of possibly equal properties the following conditions are of use. The various values of the mean radius vector of the surface correspond among themselves and with the values of the mean radius of curvature except for terms of the square of the flattness. In Bessel's dimensions of the Earth the value

$$[3{,}80416] = 6370{,}3 \text{ Kilometer} \qquad (1)$$

is to less than 30 m not only any mean radius vector of the surface, but also a mean radius of curvature.

This value corresponds very closely to the radius of a sphere of the same surface (cf. p. 62 (9)):

$$\sqrt{\frac{\text{Surface}}{4\pi}}$$

i.e.

$$a_0\left(1-\frac{1}{6}e^2-\frac{17}{360}e^4-\frac{67}{3024}e^6-\ldots\right), \qquad (2)$$

further from the sphere of equal volume:

$$\sqrt[3]{a_0^2 b_0}$$

i.e.

$$a_0\left(1-\frac{1}{6}e^2-\frac{5}{72}e^4-\frac{55}{1296}e^6-\ldots\right), \qquad (3)$$

as well as the arithmetic mean of the three semi-axes of the ellipsoid

i.e.

$$\frac{a_0 + a_0 + b_0}{3}$$

$$a_0\left(1 - \tfrac{1}{6}e^2 - \tfrac{1}{24}e^4 - \tfrac{1}{48}e^6 - \ldots\right). \qquad (4)$$

if one wants a closed expression, the following would be most suitable (p. 65 remark):

$$\frac{a_0 e}{\arcsin e}.$$

Chapter 2.

Triangles and Triangular Nets on the Sphere.

Par. 1. Horizontal Distance, Shortest Line, and Geodesic. In this chapter and in the following one, we assume that all level surfaces will be concentric spherical surfaces; therefore, all plumb lines will be straight lines through the common center. *) Two plumb lines have one common vertical plane which intersects each level surface in a great circle. On page 6 already, we called the great-circle arc between the vertical projections of two points onto any level the horizontal distance. We also mentioned that this distance depends on the level's elevation, and we assume the radius of the latter to be unity.

The horizontal distance between two points gives also the shortest distance between the projection of the two points, on the respective level.

To prove this, we divide the great arc P_0P_n (see Fig. 3) between the projections denoted P_0 and P_n, into infinitely small sections $P_0P_1, P_1P_2, P_2P_3,$ etc. Then we describe circles around P_0 using the radii P_0P_1, $P_0P_2, P_0P_3,$ etc. They will be small circles of the sphere which have the same distance between each other. From this, we find that the shortest distance between P_0 and P_n on the sphere's surface will be the great circle itself, and not any other line $P_0P_1'P_2'P_3'\ldots P_n$ since only



Figure 3

*)In later studies, we shall decide how useful the derivations in chapters 2 and 3 will be when applied to existing conditions.

the great circle follows the shortest space between the small circles. On the other hand, we have generally

$$P_0P_1' > P_0P_1;\ P_1'P_2' > P_1P_2;\ P_2'P_3' > P_2P_3; \quad \text{etc.}$$

Hence,

$$P_0P_1'P_2'P_3' \ldots P_n > P_0P_1P_2P_3 \ldots P_n.$$

The one exception is $P_0'P_n > \pi$. In this case, the shortest distance will be the sphere's circumference minus P_0P_n.

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and P_0P_n will not even be a comparative minimum any more since any infinitesimally closer connection between P_0' and P_n on a small circle will be shorter.

The plane of the great-circle arc is the common vertical plane not only for the points at each end but also for any points between them, because all radii are normals to the sphere's surface, and the plane passes through the sphere's center. Hence, this plane is also that one which corresponds to any three infinitesimally close points on the great circle, i.e. its osculating plane in the respective location where it will be a vertical plane. Because of this quality, the great-circle arc may be called the geodesic line on the sphere's surface.

To explain this term, let us assume that the people living on some surface were to run a straight line from a point P_0 along a given direction. The surface on which they are living may be of any curvature, and the line they are supposed to run has to be straight in geodetic terms.

First, they will run a vertical plane from P_0 into the given direction and then assume a point P_1 on it. In P_1 a vertical

plane will be passed through P_0 and on it a point P_1 will be assumed (see Fig. 3). From P_1 another point P_2 will be assumed on the vertical plane through P_1 etc. It is implied that the neighboring points will be located so close to each other that no discernable differences occur between the adjacent vertical planes. Consequently, three neighboring points at a time will be located in one plane which, at this place, will be the vertical plane of the surface.

Hence we see that the geodesist's straight line in the above sense, or, more correctly, the geodesic, has that quality in relation to its osculating plane which was described above.

Par. 2. Horizontal Angles. The direct measurement of horizontal angles is only feasible and to be recommended when profile conditions are favorable. In all other cases, we shall use the triangulation method indicated on page 12, and locate the base line in such a manner that it will be suitable for direct measuring.

The distances which are measured directly in state surveys or degree determinations extend nowadays, as a rule, only to a few kilometers. On the other hand, the directly observed angles often apply to sides which are 50 km long. When the British undertook the survey of India, sights to the Himalayan peaks often spanned a distance of 160 km to 240 km; one length even amounted to 340 km.

Let us have the horizontal projections of three triangulation points and connect two of them by their horizontal distances, namely, their great-circle arcs. We then obtain a spherical triangle in which the angles will be the plane angles formed by the vertical plane of the

respective arcs.

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These plane angles may be observed with suitable instruments, especially the theodolite, which will be set up on the Earth's surface (theoretically: at any point on the plumb line through the vertex.)

The problems which have to be solved further are identical to the problems of spherical trigonometry.

In the following paragraphs of this chapter, we shall first evolve the formulae for solving spherical triangles of any size; to be followed by formulae about triangles having small sides in relation to the sphere's radius. We shall conclude giving formulae which solve problems by utilizing the chord triangle.

Par. 3. The Spherical Triangle. If a great circle is passed through two points on the Earth's surface, the circle will be divided into two parts, one part $> \pi$ and the other $< \pi$. In practice, we are only concerned with the latter case. Although we shall be particularly interested in spherical triangles having sides $< \pi$, other triangles are also to be found in geodesy, especially those in which one side $> \pi$ Triangles in which two sides $> \pi$ are no more true triangles since the vertices and the two sides are intercrossed.

Let the sides of the triangle revolve in an assumed cycle; then we shall count the angles in such a way that they designate the amount of revolution necessary to bring the preceding side in coincidence with the following one. Accordingly, the angle areas will always remain on the same side during the cycle.

Par. 4. Differential Formulae: Sine and Cosine Theorems.

Further, if we count the azimuths in the same sense of revolution initiating from any direction for each vertex, for the time being, then we find

the angle of the triangle equals the azimuth of the following side minus the azimuth of the preceding side. (1)

Let α, β, and γ be the three consecutive sides, expressed in fractions of the sphere's radius; and let A, B, and C denote the opposite vertices. Further, let $\mathfrak{B}_\alpha$ and $\mathfrak{C}_\alpha$ be the azimuths of α in the vertices B and C; and $\mathfrak{C}_\beta$ and $\mathfrak{A}_\gamma$ the azimuths of β in the vertices C and A; and $\mathfrak{A}_\gamma$, and $\mathfrak{B}_\gamma$ the azimuths of γ in the vertices A and B. Then, we have the following relations between the angles and the azimuths

$$\left.\begin{aligned} A &= \mathfrak{A}_\beta - \mathfrak{A}_\gamma \\ B &= \mathfrak{B}_\gamma - \mathfrak{B}_\alpha \\ C &= \mathfrak{C}_\alpha - \mathfrak{C}_\beta. \end{aligned}\right\}$$

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If the angles thus computed are negative we shall understand the addition of 360°.

When the cycle of the sides is reversed the original angles will be replaced by their explements, i. e. A becomes $\mathfrak{A}_\gamma - \mathfrak{A}_\beta$ for example.

Par. 4. Differential Formulae: Sine and Cosine Theorems. To obtain mathematical relations between the sides and the angles, let us employ a method similar to one we shall utilize later to solve geodetic triangles on the ellipsoid of revolution. For this purpose,

Chapter 2. Triangles and Triangular Nets on the Sphere.

we shall first evolve the differential formulae writing the sides as independent variables and the angles as dependent variables.

Let the side a revolve about vertex B by $d\mathfrak{B}_a$, then C will describe the arc differential CC' of a small circle the radius of which will be sin a when the sphere's radius is assumed to be unity. Hence,

$$CC' = \sin a \, d\mathfrak{B}_a .$$

If $a > \pi$, then CC' in Fig. 4, will be located in the opposite place in relation to BC than the one shown, which is also indicated by the negative sign of sin a.

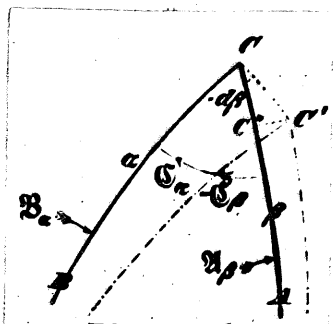


Figure 4

The new location C' of the third vertex will cause changes of β and $\mathfrak{A}_\gamma$ which may be easily deduced from Fig. 4. In that figure, $C'C''$ denotes the differential of the small circle described by

$$C''C' = \sin \beta \, d\mathfrak{A}_\beta .$$

The differential triangle $CC'C''$ forms a right angle at C'' and its small side CC'' indicates the decrease of β:

$$CC'' = -d\beta .$$

Looking further at the differential triangle, we find

$$-d\beta = \sin a \, d\mathfrak{B}_a \, . \sin (\mathfrak{C}_a - \mathfrak{C}_\beta) \qquad (1)$$

$$\sin \beta d\mathfrak{A}_\gamma = \sin a \, d\mathfrak{B}_a \, . \cos (\mathfrak{C}_a - \mathfrak{C}_\beta), \qquad (2)$$

These formulae hold for any size of the azimuths $\mathfrak{C}_a$ and $\mathfrak{C}_\beta$, ; they are also valid for any of their senses of revolution provided that this sense is the same for all azimuths. These formulae may also be applied to the case of side γ turning about B by $d\mathfrak{B}_\gamma$ whereby the positions of vertex A and side β will be moved accordingly.

Replacing γ for α, and $\mathfrak{C}$ for in (1), we now have

$$-d\beta = \sin\gamma\, d\mathfrak{B}_\gamma \cdot \sin(\mathfrak{A}_\gamma - \mathfrak{A}_\beta). \qquad (3)$$

Assuming the size of $d\beta$ to be the same as stated above, we obtain when equating (1) and (3)

$$\sin\alpha \sin(\mathfrak{C}_\alpha - \mathfrak{C}_\beta)\, d\mathfrak{B}_\alpha = \sin\gamma \sin(\mathfrak{A}_\gamma - \mathfrak{A}_\beta)\, d\mathfrak{B}_\gamma . \qquad (4)$$

Both triangles having identical sides $\alpha, \beta - d\beta$ and γ are congruent and may be brought to coincidence by a differential movement. Therefore, the angle B has the same size in both triangles. Referring to (2), P. 71, the variation of B from its value in the original triangle, namely dB equals

in the first movement $-d\mathfrak{B}_\alpha$,

in the second movement $+d\mathfrak{B}_\gamma$.

Hence, both values are equal. Introducing this into (4) we obtain the sin theorem of spherical trigonometry

which is

$$\frac{\sin\alpha}{\sin\gamma} = \frac{\sin(\mathfrak{A}_\beta - \mathfrak{A}_\gamma)}{\sin(\mathfrak{C}_\alpha - \mathfrak{C}_\beta)}$$

$$\frac{\sin\alpha}{\sin\gamma} = \frac{\sin A}{\sin C}. \qquad (5)$$

At the same time, the differential formula (1) becomes

$$d\beta = \sin\alpha \sin C\, dB. \qquad (6)$$

To integrate this differential equation we write (5) referring to β and γ and then have

$$\frac{\sin\beta}{\sin\gamma} = \frac{\sin B}{\sin C}$$

by means of which equation we eliminate $\sin C$, from (6), and thus find

$$-\sin\beta\, d\beta = \sin\alpha \sin\gamma \sin B\, dB,$$

α and γ being constants, we have immediately by integration

$$\cos\beta = \sin\alpha \sin\gamma \cos B + \text{Const.}$$

To determine the constants we make use of the fact that when $B = 180°$ then $\beta = (\alpha + \gamma)$ or $2\pi - (\alpha + \gamma)$. Therefore,

$$\cos(\alpha + \gamma) = -\sin\alpha \sin\gamma + \text{Const.}$$

Subtracting this from the above equation, and rearranging for $\cos\beta$ we have the cosin theorem (for the sides of triangles)

$$\cos\beta = \cos\alpha \cos\gamma + \sin\alpha \sin\gamma \cos B. \qquad (7)$$

The whole spherical trigonometry can be derived from this equation which yields three relations by cyclic substitution of the parts. However, we are going to evolve one more formula from the differential equations.

For this purpose, we divide each member of equation (1) by the corresponding one in (2) and remember that, according to (2), p. 71, $d\mathfrak{A}_\beta = dA$. Then we have

which is

$$\left.\begin{aligned} \frac{d\mathfrak{A}_\beta}{d\beta} &= -\frac{1}{\tan(\mathfrak{C}_\alpha - \mathfrak{C}_\beta)\sin\beta} \\ \frac{dA}{d\beta} &= -\frac{1}{\tan C \sin\beta}. \end{aligned}\right\} \qquad (8)$$

By substituting β and α in (1), and likewise $\mathfrak{B}$ and $\mathfrak{A}$, with regard to (2), p. 71, we find

$$\frac{dA}{d\alpha} = \frac{1}{\sin C \sin \beta}. \qquad (9)$$

However, A is a function of α, β and γ. In (8), α and γ are constants; in (9), β and γ are constants. Now, by using the signs for partial differentiation, we better write equations (8) and (9) as follows

$$\frac{\partial A}{\partial \beta} = -\frac{1}{\tan C \sin \beta},$$
$$\frac{\partial A}{\partial \alpha} = \frac{1}{\sin C \sin \beta}.$$

To form $\frac{\partial^2 A}{\partial \alpha \partial \beta}$, we have a choice of two ways. Using both of them, we get

$$\frac{1}{\sin^2 C \sin \beta} \cdot \frac{\partial C}{\partial \alpha} = -\frac{\cos C}{\sin^2 C \sin \beta} \frac{\partial C}{\partial \beta} - \frac{\cos \beta}{\sin C \sin^2 \beta}. \qquad (10)$$

To find the differential quotients which appear in this equation, we exchange, in the second formula (8) $\alpha\beta\gamma$ in cycle with $\gamma\alpha\beta$, and ABC in cycle with CAB by which the equation retains its validity, of course. Hence,

$$\frac{\partial C}{\partial \alpha} = -\frac{1}{\tan B \sin \alpha}.$$

Further, substituting γ for α and $\mathfrak{C}$ for $\mathfrak{A}$ in the first equation (8), which is certainly permissible, we get

that is

$$\frac{\partial \mathfrak{C}_\beta}{\partial \beta} = -\frac{1}{\tan(\mathfrak{A}_\gamma - \mathfrak{A}_\beta) \sin \beta}$$
$$\frac{\partial C}{\partial \beta} = -\frac{1}{\tan A \sin \beta}.$$

The determined values of the differential quotients are now substituted in (10), and both members multiplied by $\sin A \sin^2 C \sin^2 \beta$ We further note that, in the left-hand member, $\sin A \sin \beta = \sin B \sin \alpha$ and we obtain the cosin theorem (for the angles of the triangle)

$$\cos B = -\cos A \cos C + \sin A \sin C \cos \beta. \qquad (11)$$

At this point, we emphasize again that formulae (5), (7), and (11), being derived from the differential equations (1) and (2), and from set (2), p. 71, are limited only by the condition that, for any cycle of the sides, the angles designate that amount of revolution which is necessary to superpose a preceding side upon the following one, according to equation (2), p. 71. All senses of revolution may be utilized provided they are the same for all the three angles. Negative angle values are not excluded, and the sides may be of any length $\gtrless \pi$.

Par. 5. Cotangent Theorem and Formulae for Five Parts. By cyclic substitution of the parts, formula (7) of the preceding paragraph reads

$$\cos \alpha = \cos \beta \cos \gamma + \sin \beta \sin \gamma \cos A.$$

We eliminate $\cos \beta$ by means of formula (7), and we eliminate $\sin \beta$ by using the relation $\sin \beta = \sin B \sin \alpha : \sin A$, which we obtain from (5), p. 73, by substituting the parts. It follows the cotangent theorem for four consecutive parts

$$\cot \alpha \sin \gamma - \cot A \sin B = \cos B \cos \gamma. \qquad (1)$$

All formulae which we developed so far refer always to four parts. All other formulae referring to four parts result from cyclic substitution of the parts since the only possible cases of combining four parts of the triangle out of six involve one side

and three angles, three sides and one angle, two sides and their opposite angles, and two sides with the included angle and one opposite angle. All these combinations yield only one formula each. However, there are equations for five and six parts which might be valuable.

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Substituting for B in (1) its equivalent $\sin\beta\sin A:\sin\alpha$, it follows for five parts

$$\sin\beta\cos A = \cos\alpha\sin\gamma - \sin\alpha\cos\gamma\cos B. \qquad (2)$$

On the other hand, when replacing $\sin\gamma$ by the expression $\sin\alpha\sin C:\sin A$, we get

$$\sin C\cos\alpha = \cos A\sin B + \sin A\cos B\cos\gamma. \qquad (3)$$

When using formulae (5), (7), and (11) of the preceding paragraph and formulae (1), (2), and (3) of this one we are not bound to employ cyclic substitutions only. We may also exchange two sides if we do likewise with the opposite angles. For the first four equations, this may easily be proven by comparing them with the results of cyclic substitutions, and since the last two equations are derived from the former four, this rule will also apply to them.

Par. 6. Given Three Consecutive Parts. When three consecutive parts of a triangle are known we may solve it completely by utilizing any one of the two following sets which are easily derived from the conclusions of Par. 4 and Par. 5 by cyclic substitution:

$$\left.\begin{aligned}
\cos\alpha &= \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A\\
\sin\alpha\cos B &= \cos\beta\sin\gamma - \sin\beta\cos\gamma\cos A\\
\sin\alpha\sin B &= \sin\beta\sin A\\
\sin\alpha\cos C &= \cos\gamma\sin\beta - \sin\gamma\cos\beta\cos A\\
\sin\alpha\sin C &= \sin\gamma\sin A.
\end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} \cos A &= -\cos B\cos C + \sin B\sin C\cos\alpha \\ \sin A\cos\beta &= \cos B\sin C + \sin B\cos C\cos\alpha \\ \sin A\sin\beta &= \sin B\sin\alpha \\ \sin A\cos\gamma &= \cos C\sin B + \sin C\cos B\cos\alpha \\ \sin A\sin\gamma &= \sin C\sin\alpha. \end{aligned}\right\} \qquad (2)$$

We obtain α, B and C, by solving the first set, and A, β and γ from the second one. They are determined without ambiguity, since we only have to decide whether $\alpha \gtrless \pi$ and $A \gtrless 180^\circ$ respectively. This has to be decided because the first set leaves the sign of $\sin\alpha$ ambiguous, and the second set that of $\sin A$. Otherwise, we obtain two solutions which result in the same position of the three great circles, however; because the values of

α B C in one solution correspond to the values

$2\pi - \alpha$ $180^\circ + B$ $180^\circ + C$ in the other solution of set (1);

and for set (2), the values of

A β γ in one solution correspond to the values of

$360^\circ - A$ $\pi + \beta$ $\pi + \gamma$ in the second solution.

The formulae's numerical accuracy leaves nothing to be desired, since each part may be derived from the tangent or cotangent; thus, by dividing the second and third equations of set (1), we find at first $\cot B$ (this determination relates to the cotangent theorem); then, both equations give us $\sin\alpha$, etc. However, these formulae are not so convenient for logarithmic calculations that we may not desire more suitable ones.

Par. 7. Theorems by Napier and Gauss. From the equations

$$\left.\begin{aligned} \sin\alpha \sin B &= \sin\beta \sin A \\ \sin\alpha \sin C &= \sin\gamma \sin A \end{aligned}\right\} \quad (1)$$

we find by addition, subtraction, and simple reduction

$$\left.\begin{aligned} \sin\alpha \sin\frac{B+C}{2}\cos\frac{B-C}{2} &= \sin A \sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2} \\ \sin\alpha \cos\frac{B+C}{2}\sin\frac{B-C}{2} &= \sin A \cos\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}. \end{aligned}\right\} \quad (2)$$

Using the same method, the equations

$$\left.\begin{aligned} \sin\alpha \cos B &= \cos\beta \sin\gamma - \sin\beta \cos\gamma \cos A \\ \sin\alpha \cos C &= \cos\gamma \sin\beta - \sin\gamma \cos\beta \cos A \end{aligned}\right\} \quad (3)$$

result in set

$$\left.\begin{aligned} \sin\alpha \cos\frac{B+C}{2}\cos\frac{B-C}{2} &= \sin(\beta+\gamma)\sin^2\frac{A}{2} \\ \sin\alpha \sin\frac{B+C}{2}\sin\frac{B-C}{2} &= \sin(\beta-\gamma)\cos^2\frac{A}{2}. \end{aligned}\right\} \quad (4)$$

Dividing the first equation (2) by the first equation (4), and the second equation (2) by the first equation (4), we obtain the following Napier's Analogies for two given sides and the given included angle:

$$\left.\begin{aligned} \tan\frac{B+C}{2} &= \frac{\cos\frac{\beta-\gamma}{2}}{\cos\frac{\beta+\gamma}{2}}\cot\frac{A}{2} \\ \tan\frac{B-C}{2} &= \frac{\sin\frac{\beta-\gamma}{2}}{\sin\frac{\beta+\gamma}{2}}\cot\frac{A}{2}. \end{aligned}\right\} \quad (5)$$

Again, substituting in formulae (1) and (3) α for $180^\circ A$, β for B and γ for C and vice versa, we have again proper formulae.

The same substitution, applied to (5), results in Napier's Analogies for two given angles and the given side between them:

$$\left.\begin{aligned} \tan\frac{\beta+\gamma}{2} &= \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}}\tan\frac{\alpha}{2} \\ \tan\frac{\beta-\gamma}{2} &= \frac{\sin\frac{B-C}{2}}{\sin\frac{B+C}{2}}\tan\frac{\alpha}{2}. \end{aligned}\right\} \tag{6}$$

Using these formulae to solve the problem of Par. 6, we obtain the same ambiguity as mentioned then; however, the two solutions result in the same corresponding positions of the three great arcs.

In order to abbreviate, we write further

$$\left.\begin{aligned} \sin\frac{\beta-\gamma}{2}\cos\frac{A}{2} &= l, & \sin\frac{B-C}{2}\sin\frac{\alpha}{2} &= l' \\ \cos\frac{\beta-\gamma}{2}\cos\frac{A}{2} &= m, & \cos\frac{B-C}{2}\sin\frac{\alpha}{2} &= m' \\ \sin\frac{\beta+\gamma}{2}\sin\frac{A}{2} &= n, & \sin\frac{B+C}{2}\cos\frac{\alpha}{2} &= n' \\ \cos\frac{\beta+\gamma}{2}\sin\frac{A}{2} &= r, & \cos\frac{B+C}{2}\cos\frac{\alpha}{2} &= r', \end{aligned}\right\} \tag{7}$$

Then the (2) and (4) give the relations

$$\left.\begin{aligned} m'n' &= mn, & m'r' &= nr \\ l'r' &= lr, & l'n' &= lm. \end{aligned}\right\} \tag{8}$$

Although we have four equations, we are only able to determine three quantities from the other five since each equation may be derived from the other three equations. However, the substitution which we used above to develop (6) from (5) gives us a new relation. We obtain: l from l', m from m', n from n', r from r' and vice versa.

In addition to the above relations, we have two new ones, of which one is not yet contained in the above:

$$mr = n'r', \quad ln = l'm'. \tag{9}$$

From (8) and (9) follows $m'n' \cdot n'r' = ms \cdot mr$, and since $n'r' = nr$, we get

$$n'^2 = m^2; \quad n' = \pm m.$$

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Hence it follows by means of relations (8) and (9):

if $n' = +m$ then $m' = n, \quad r' = r, \quad l' = l,$

if $n' = -m$ then $m' = -n, \quad r' = -r, \quad l' = -l.$

Substituting the values of $lmnr$ etc. in equation (7), we obtain Gauss's Equations:

$$\left.\begin{aligned}
\sin\frac{B-C}{2}\sin\frac{\alpha}{2} &= \pm\sin\frac{\beta-\gamma}{2}\cos\frac{A}{2}\\
\cos\frac{B-C}{2}\sin\frac{\alpha}{2} &= \pm\sin\frac{\beta+\gamma}{2}\sin\frac{A}{2}\\
\sin\frac{B+C}{2}\cos\frac{\alpha}{2} &= \pm\cos\frac{\beta-\gamma}{2}\cos\frac{A}{2}\\
\cos\frac{B+C}{2}\cos\frac{\alpha}{2} &= \pm\cos\frac{\beta+\gamma}{2}\sin\frac{A}{2},
\end{aligned}\right\} \tag{10}$$

in which the positive and negative signs belong together. The double signs result from the fact that, on the one hand, certain substitutions were permitted in the above formulae (cf. Par. 6, P. 76) of which one resulted in a change of signs, viz. replacing A, β, γ by $360^\circ - A, \pi + \beta, \pi + \gamma$ respectively, in (10). On the other hand, we were also allowed to vary each side by multiples of $\pm 2\pi$ and each angle by multiples of $\pm 360^\circ$; and in particular, to count the angles negatively, from which results an ambiguity of the signs in (10). In spite of the ambiguity, we obtain only two solutions from (10) when three parts are known. Both solutions

belong to the same position of the three great circles.

However, if we limit our triangles to only positive sides and angles, and at least two sides are $<\pi$ whilst the third may be as great as 2π, then only the positive signs hold.

Because, for three positive sides $<\pi$ and three positive angles $<180^\circ$, the second equation (10) proves immediately that the negative sign is not permissible. If, furthermore, each of the three angles $>180^\circ$ and $<360^\circ$, there will be no change since we are allowed to substitute the angles' explements in equations with positive signs of (10). When we finally exchange α for $2\pi-\alpha$ and at the same time B and C for $180^\circ+B$ and $180^\circ+C$; respectively; or replace β by $2\pi-\beta$ and at the same time A and C by $180^\circ+A$ and $180^\circ+C$, respectively, then the positive equations of (10) will still hold. The latter substitutions, however, represent the transition of a triangle having three sides $<\pi$ into one with only two sides $<\pi$.

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Par. 8. Given Two Sides and One Opposite Angle, or Two Angles and One Opposite Side. In this case, the Sine Theorem results either in none or in two values for the remaining opposite part. The two values supplement each other to 180° or π, respectively, and for both values Napier's Analogies furnish a value for the fifth and sixth part. If these parts define a triangle (the conditions for such a case are not of interest here), we obtain generally two solutions. These solutions do not, as was hitherto the case, belong to the same corresponding positions of the three great circles but

to two different ones. This will be proven clearly by constructing the triangles with the help of loci. We may neglect this, however, as we shall also refrain from proving the statement that, if the one given opposite part lies opposite the greater of the two other given parts, we obtain only one solution for triangles with parts $< 180^0$ or π, respectively.

Par. 9. Given Three Sides or Three Angles. When α, β, and γ are known the Cosine Theorem for α gives, after rearranging,

$$\cos A = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}. \qquad (1)$$

This formula is neither convenient for logarithmic use nor is it suitable for precise numerical evaluation in every case. Therefore, we write

$$1 + \cos A = 2\cos^2\frac{A}{2} = \frac{\cos\alpha - \cos(\beta+\gamma)}{\sin\beta\sin\gamma},$$

$$1 - \cos A = 2\sin^2\frac{A}{2} = \frac{-\cos\alpha + \cos(\beta-\gamma)}{\sin\beta\sin\gamma}.$$

Uniting the cosines of the right-hand numerators into products, it follows

$$\left.\begin{aligned}
\cos^2\frac{A}{2} &= \frac{\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{-\alpha+\beta+\gamma}{2}}{\sin\beta\sin\gamma} = \frac{\sin s\sin(s-\alpha)}{\sin\beta\sin\gamma}\\
\sin^2\frac{A}{2} &= \frac{\sin\frac{\alpha-\beta+\gamma}{2}\sin\frac{\alpha+\beta-\gamma}{2}}{\sin\beta\sin\gamma} = \frac{\sin(s-\beta)\sin(s-\gamma)}{\sin\beta\sin\gamma}\\
\tan^2\frac{A}{2} &= \frac{\sin\frac{\alpha-\beta+\gamma}{2}\sin\frac{\alpha+\beta-\gamma}{2}}{\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{-\alpha+\beta+\gamma}{2}} = \frac{\sin(s-\beta)\sin(s-\gamma)}{\sin s\sin(s-\alpha)}
\end{aligned}\right\} \qquad (2)$$

in which $s = \frac{\alpha+\beta+\gamma}{2}$ is used for abbreviation. Multiplying $\cos^2\frac{A}{2}$ and $\sin^2\frac{A}{2}$ gives further

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$$\sin^2 A = \frac{4 \sin s \sin(s-\alpha) \sin(s-\beta) \sin(s-\gamma)}{\sin^2 \beta \sin^2 \gamma}. \qquad (3)$$

Similarly, we obtain the formulae for three given angles A, B and C. We have:

$$\cos \alpha = \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \qquad (4)$$

$$\left.\begin{aligned} \cos^2 \frac{\alpha}{2} &= \frac{\sin\left(B - \frac{\varepsilon}{2}\right) \sin\left(C - \frac{\varepsilon}{2}\right)}{\sin B \sin C} \\ \sin^2 \frac{\alpha}{2} &= \frac{\sin \frac{\varepsilon}{2} \sin\left(A - \frac{\varepsilon}{2}\right)}{\sin B \sin C} \\ \tan^2 \frac{\alpha}{2} &= \frac{\sin \frac{\varepsilon}{2} \sin\left(A - \frac{\varepsilon}{2}\right)}{\sin\left(B - \frac{\varepsilon}{2}\right) \sin\left(C - \frac{\varepsilon}{2}\right)} \end{aligned}\right\} \qquad (5)$$

$$\sin^2 \alpha = \frac{4 \sin \frac{\varepsilon}{2} \sin\left(A - \frac{\varepsilon}{2}\right) \sin\left(B - \frac{\varepsilon}{2}\right) \sin\left(C - \frac{\varepsilon}{2}\right)}{\sin^2 B \sin^2 C}, \qquad (6)$$

in which $\varepsilon = A + B + C - 180^0$ denotes the spherical excess of the triangle.

The above formulae are implicitly applied to B and C, and β and γ, respectively, besides A and α. Then, if the three parts belong to a triangle at all (we shall refrain from stating the conditions of such a case) then, for each unknown part, we have two values which explement each other to 360° or 2π, respectively. These values give, from the Sine Theorem

$$\sin A : \sin B : \sin C = \sin \alpha : \sin \beta : \sin \gamma$$

two solutions for the whole problem, as in Par. 6.

In the preceding proportion, let us substitute for the relations $\sin A : \sin \alpha$, etc, the following values which we obtain from the second equation (2)

$$\frac{\sin A}{\sin \alpha} = \frac{2\sin^2 \frac{A}{2}}{\tan \frac{A}{2} \sin \alpha} = \frac{2 \sin (s-\alpha) \sin (s-\beta) \sin (s-\gamma)}{\tan \frac{A}{2} \sin (s-\alpha) \cdot \sin \alpha \sin \beta \sin \gamma}$$

u. s. f.

etc.

giving

$$\tan \frac{A}{2} : \tan \frac{B}{2} : \tan \frac{C}{2} = \csc (s-\alpha) : \csc (s-\beta) : \csc (s-\gamma), \quad (7)$$

which proportion was first expressed by Mollweide (according to Puissant). Using the second equation (5) leads to the analogous proportion:

$$\tan \frac{\alpha}{2} : \tan \frac{\beta}{2} : \tan \frac{\gamma}{2} = \sin \left(A - \frac{\varepsilon}{2}\right) : \sin \left(B - \frac{\varepsilon}{2}\right) : \sin \left(C - \frac{\varepsilon}{2}\right). \quad (8)$$

Par. 10. Area and Spherical Excess. We may only talk of the triangle's area, in the customary sense of determing an area, if no more than one side $> \pi$ (p. 71). We shall therefore restrict ourselves to this case which is the only improtant one in geodesy; and we define the area to be that section of the two formed on the sphere's surface, which is traversed by the triangle's sides when describing the angles of the triangle.

Figure 5 will be used to represent in a drawing the following discussion, not only in the case of triangles having all three sides $< \pi$ and all three angles $< 180^0$, but also for triangles with two sides $< \pi$, and for any size of angle.

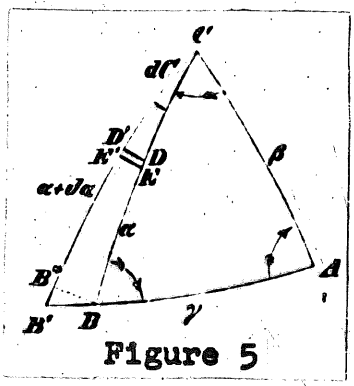


Figure 5

When the side α is turned about vertex C by the amount dC, then the area of the

triangle increases by dF, i.e. by the triangle CBB' in fig. 5 which triangle, in turn differs only by an infinitesimally small amount from triangle CBB'' in which $CB = CB''$. Again, let us consider on CB an arc $CD = \alpha_1$, which is smaller than α, and let it also revolve about C by dC; Then, with the sphere's radius as unity, the infinitely small distance DD' equals $\sin \alpha_1 dC$. Two neighboring points, D for α_1 and E for $\alpha_1 + d\alpha_1$ include with their distances a small quadrangle $DD'EE'$ the area of which equals $\sin \alpha_1 dC . d\alpha_1$ as to size and algebraic signs.

Hence, $$dF = dC \int_0^{\alpha} \sin \alpha_1 d\alpha_1 = dC(1 - \cos \alpha). \qquad (1)$$

The same expression is obtained for the increase of the excess.

First, we have:

$$d\varepsilon = dC + dB. \qquad (2)$$

In the triangle $BB'C$, we now find:

$$\sin (\alpha + d\alpha) \sin (B + dB) = \sin \alpha \sin B,$$

since the angle B' may be denoted $(B + dB)$. From this follow by means of the derivations:

$$\sin (\alpha + d\alpha) = \sin \alpha + \cos \alpha d\alpha$$

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and $$\sin (B + dB) = \sin B + \cos B dB$$

and neglecting the term multiplied by $d\alpha dB$

$$dB = - \cot \alpha \tan B \, d\alpha.$$

From Fig. 5, we have $d\alpha$ i.e. $B'B'' = \sin \alpha \, dC . \cot B$ hence

$$dB = - \cos \alpha \, dC,$$

Finally, substituting this value in (2), and paying attention to (1), we obtain, as already stated above,

$$d\varepsilon = dC(1 - \cos\alpha) = dF. \qquad (3)$$

Integrating this yields $\varepsilon = F +$ const. However, the constant equals zero since ε and F disappear at the same time. Considering this, and designating the sphere's radius ϱ in place of unity, we have

$$\left.\begin{aligned} F &= \varepsilon\varrho^2 \\ \varepsilon &= \frac{F}{\varrho^2}. \end{aligned}\right\} \qquad (4)$$

In this case, ε is expressed in radians. More frequent use is made of the second equation which yields the excess from the area.

The direct determination of the excess from the angles' sum, however, is often impossible in geodesy even when the sum is known. The values of observed angles are subject to observation errors; and to know their influence upon $A + B + C$, we have to determine the excess' quantity at least as accurately as the square of the observation errors. We shall calculate this using at least <u>one</u> side of the triangle.

For this purpose, we first state some remarkably precise formulae for the solution of ε in which we assume to know at least one side.

Par. 11. Precise Spherical Excess Formulae. When α, B, and C are known, i.e. one side and its adjacent angles, we obtain a convenient formula to calculate ε, for small triangles, from the second equation (5), p. 81. When we write:

it follows that:

$$\begin{aligned} A &= 180^0 + \varepsilon - B - C, \\ \sin\frac{\varepsilon}{2} &= \sin^2\frac{\alpha}{2}\,\frac{\sin B \sin C}{\sin\left(B + C - \frac{\varepsilon}{2}\right)}. \end{aligned} \qquad (1)$$

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In general, this equation has two roots

$$\varepsilon_1 \text{ and } \varepsilon_2 = 2(B+C) - \varepsilon_1$$

which results from the ambiguity of the problem to construct a triangle from α, B, and C. However, if all three sides are small then only that solution holds in which ε will be close to zero or to 720^0, the latter in the case of angles $> 180^0$.

In that case, too, it is more convenient to subtract 720^0 from ε and let ε be negative. The above mentioned small triangle being presumed, ε will have a small positive or negative value which may be determined from (1) by successive approximation in which case we start with $\varepsilon =$ zero in the right-hand member.

When two angles A and B and one opposite side α are given, then the ambiguity already mentioned in Par. 8 appears again, which is of no importance, however, if the triangle is known to be a small one. In this case, the second solution pertains to a triangle having one side nearly π and a great excess; however, the approximation which we shall initiate as above does not result in such a triangle. The formula is written

$$\sin\frac{\varepsilon}{2} = \sin^2\frac{\alpha}{2}\,\frac{\sin B \sin(A+B-\varepsilon)}{\sin\left(A-\frac{\varepsilon}{2}\right)}. \qquad (2)$$

To obtain a direct equation for calculating the excess from two sides α and β, and the included angle C, we combine the third equation (5), p. 81,

$$\tan^2\frac{\alpha}{2} = \frac{\sin\frac{\varepsilon}{2}\sin\left(A-\frac{\varepsilon}{2}\right)}{\sin\left(B-\frac{\varepsilon}{2}\right)\sin\left(C-\frac{\varepsilon}{2}\right)}$$

with the following (8), p.82,

$$\frac{\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2}} = \frac{\sin\left(B-\frac{\varepsilon}{2}\right)}{\sin\left(A-\frac{\varepsilon}{2}\right)}$$

by multiplying the left-hand and the right-hand members. We get

$$\left.\begin{aligned} \sin\frac{\varepsilon}{2} &= \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\sin\left(C-\frac{\varepsilon}{2}\right), \\ \tan\frac{\varepsilon}{2} &= \frac{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\sin C}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\cos C} = \frac{\tan C}{1+\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\sec C}. \end{aligned}\right\} \quad (3)$$

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The second equation results from the first one by reducing the right-hand sin, dividing by $\cos\frac{\varepsilon}{2}$, and rearranging for $\tan\frac{\varepsilon}{2}$. The solution yields two values for ε, which differ by 360° and correspond to the two triangles of which α, β, and C are parts. For small triangles, calculating by approximation according to equation 1 of (3) will be most convenient.

When we have to determine ε from two given sides and one opposite angle the most suitable method, for small triangles, will be to calculate the other opposite angle by means of the sin theorem and then proceed with (2).

Par. 12. Continuation: Given Three Sides. From (5), p. 81, we write

(See Illustrative Material on Next Page.)

$$\cos^2\frac{\alpha}{2} = \frac{\sin\left(B-\frac{\varepsilon}{2}\right)\sin\left(C-\frac{\varepsilon}{2}\right)}{\sin B \sin C},$$

$$\cos^2\frac{\beta}{2} = \frac{\sin\left(A-\frac{\varepsilon}{2}\right)\sin\left(C-\frac{\varepsilon}{2}\right)}{\sin A \sin C},$$

$$\cos^2\frac{\gamma}{2} = \frac{\sin\left(A-\frac{\varepsilon}{2}\right)\sin\left(B-\frac{\varepsilon}{2}\right)}{\sin A \sin B},$$

whence

$$\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{\cos\frac{\gamma}{2}} = \pm\frac{\sin\left(C-\frac{\varepsilon}{2}\right)}{\sin C}, \qquad (1)$$

Using this formula (1) to eliminate $\sin\left(C-\frac{\varepsilon}{2}\right)$ from the first equation (3) of the preceding paragraph, it follows

$$\sin\frac{\varepsilon}{2} = \pm\frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}\sin C. \qquad (2)$$

In this we substitute equation (3), p.81, rearranged for C, and we get

$$\sin\frac{\varepsilon}{2} = \pm\frac{\sqrt{\sin s \sin(s-\alpha)\sin(s-\beta)\sin(s-\gamma)}}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}. \qquad (3)$$

Again, eliminating $\cos C$ from the second equation (3) of the preceding paragraph by means of the relation

$$\cos C = \frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta},$$

we obtain, by introducing the functions of half angles

$$\tan\frac{\varepsilon}{2}=\frac{\sin\alpha\sin\beta\sin C}{2\left(\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2}-1\right)}. \qquad (4)$$

This, in connection with the above equation (2), results finally in formula

$$\cos\frac{\varepsilon}{2}=\pm\frac{\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2}-1}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}. \qquad (5)$$

The ambiguity of the signs in the right-hand members of formulae (1), (2), and (5) disappears when we presume that at least two sides $<\pi$ since, in that case, only the positive sign holds.

Namely, assuming that all three sides $<\pi$, and rewriting equation (2) in the following form:

$$\sin\frac{\varepsilon}{2}=\pm 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\frac{\sin C}{\sin\gamma},$$

in which the right-hand term $\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$ will be positive in any case, then certainly only the positive sign is valid since C and $\frac{\varepsilon}{2}$ are simultaneously $\leqslant 180^0$. Exchanging γ for $2\pi-\gamma$, the right-hand member will change its sign; however, the same happens to the left-hand member because the above substitution results in this one: $\varepsilon=360^0+\varepsilon$, which is easily perceived geometrically. This is proof that in (2) only the positive sign holds.

The development shows that the positive and negative signs in (1), (2), and (5) are related to each other; therefore, if at least two sides $<\pi$ the positive signs only will be valid in (1), (2), and (5).

In equation (3), however, we must preserve both signs.

Again, assuming at least two sides $<\pi$, formula (5) gives

$$1-\cos\frac{\varepsilon}{2}=2\sin^2\frac{\varepsilon}{4}=\frac{2\sin\frac{s}{2}\sin\frac{s-\alpha}{2}\sin\frac{s-\beta}{2}\sin\frac{s-\gamma}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}, \quad (6)$$

which is found easily when considering the following identities:

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$$\left.\begin{aligned}\sin\frac{s}{2}\sin\frac{s-\gamma}{2}&=\frac{1}{2}\left(\cos\frac{\gamma}{2}-\cos\frac{\alpha+\beta}{2}\right)\\ \sin\frac{s-\alpha}{2}\sin\frac{s-\beta}{2}&=\frac{1}{2}\left(-\cos\frac{\gamma}{2}+\cos\frac{\alpha-\beta}{2}\right).\end{aligned}\right\} \quad (7)$$

From (6) and (7), we now obtain the following formulae which are most convenient for calculating ε :

$$\left.\begin{aligned}\sin\frac{\varepsilon}{4}&=\sqrt{\frac{\sin\frac{s}{2}\sin\frac{s-\alpha}{2}\sin\frac{s-\beta}{2}\sin\frac{s-\gamma}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}}\\ \cos\frac{\varepsilon}{4}&=\pm\sqrt{\frac{\cos\frac{s}{2}\cos\frac{s-\alpha}{2}\cos\frac{s-\beta}{2}\cos\frac{s-\gamma}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}}\\ \tan\frac{\varepsilon}{4}&=\pm\sqrt{\tan\frac{s}{2}\tan\frac{s-\alpha}{2}\tan\frac{s-\beta}{2}\tan\frac{s-\gamma}{2}}.\end{aligned}\right\} \quad (8)$$

The positive signs of the radicals hold if all angles $<180^0$, the negative ones hold if they are $>180^0$, etc.

When the excess is known, and three sides are given, we shall be able to determine one angle. For example, C may be

found from one of these equations

$$\sin\left(C-\frac{\varepsilon}{2}\right)=\sin\frac{\varepsilon}{2}\cot\frac{\alpha}{2}\cot\frac{\beta}{2}$$

$$\tan\left(\frac{C}{2}-\frac{\varepsilon}{4}\right)=\cot\frac{\varepsilon}{4}\tan\frac{s-\alpha}{2}\tan\frac{s-\beta}{2}.$$

The first formula is the same as the first equation (3) of the preceding paragraph; it holds for any triangle. The second formula holds only for triangles having at least two sides $<\pi$.

The latter may be derived as follows. We first determine $\cos\left(C-\frac{\varepsilon}{2}\right)$, most conveniently by means of the identity

$$\cos\left(C-\frac{\varepsilon}{2}\right)=\sin\left(\frac{A}{2}+\frac{B-C}{2}\right),$$

by reducing the right-hand member and eliminating $\sin\frac{B-C}{2}$ and $\cos\frac{B-C}{2}$ with the help of Gauss's Equations. Retaining the above-mentioned limitation, we get

$$\cos\left(C-\frac{\varepsilon}{2}\right)=\frac{1-\cos^2\frac{\alpha}{2}-\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}}.$$

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We write further

$$1-\cos\left(C-\frac{\varepsilon}{2}\right)=2\sin^2\left(\frac{C}{2}-\frac{\varepsilon}{4}\right)=\frac{2\cos\frac{s}{2}\cos\frac{s-\gamma}{2}\sin\frac{s-\alpha}{2}\sin\frac{s-\beta}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}}$$

in which we have to pay attention, next to the second identity (7), to the following identity

$$\cos\frac{s}{2}\cos\frac{s-\gamma}{2}=\frac{1}{2}\left(\cos\frac{\gamma}{2}+\cos\frac{\alpha+\beta}{2}\right)$$

Now, $\sin\left(\frac{C}{2}-\frac{\varepsilon}{4}\right)$ is known; together with $\sin\left(C-\frac{\varepsilon}{2}\right)$, it becomes $\cos\left(\frac{C}{2}-\frac{\varepsilon}{4}\right)$, and from this results easily the formula which is given for $\tan\left(\frac{C}{2}-\frac{\varepsilon}{4}\right)$. In this case, there will be no ambiguity, since this equation has the same signs, member for member, as its preceding one if we assume at least two sides $<\pi$.

A very subtle development of the formulae involving $\frac{\varepsilon}{4}$ was produced by Werner in Schlömilch's Journal für Mathematik und Physik 1861, p. 146. In general, refer to Baltzer's and Schlömilch's textbooks, that of the former particularly for historical data.

Par. 13. Legendre's Theorem. The application of formulae in spherical trigonometry involves the conversion of radians into degrees, and vice versa. This is inconvenient since the geodesist requires, in most cases, the actual lengths of the sides. It is further troublesome that the logarithmic values of small angles α, β, and γ cannot be found directly from the tables, and the use of the auxiliary logarithms S and T, p. 33, is only partially helpful. Now, Legendre's Theorem states:

A small spherical triangle may be calculated as a plane triangle having the same sides, after each angle has been decreased by one third of the excess.

Let us denote the sides of the auxiliary plane triangle a, b, and c; the angles will be $A-\frac{\varepsilon}{3}, B-\frac{\varepsilon}{3}, C-\frac{\varepsilon}{3}$, and we write further for ε: :

$$\left.\begin{aligned} \varepsilon &= \frac{F'}{\varrho^2}; \\ \underset{\text{in seconds}}{\varepsilon} &= \varrho'' \frac{F'}{\varrho^2}. \end{aligned}\right\} \qquad (1)$$

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In this case, F may be calculated sufficiently accurately from the rules of plane trigonometry. ϱ denotes the sphere's radius.

To evolve the theorem, we proceed from

$$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A, \qquad (2)$$

in which $\alpha, \beta,$ and γ denote the quotients $a:\varrho$, $b:\varrho$, and $c:\varrho$. Since they are small, it will be adequate to substitute for their sines and cosines the first terms of the series (1), p. 28.

Designating $\alpha, \beta,$ and γ as terms of the first order, we find

$$1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24} = \left(1 - \frac{\beta^2}{2} + \frac{\beta^4}{24}\right)\left(1 - \frac{\gamma^2}{2} + \frac{\gamma^4}{24}\right) + \beta\gamma\left(1 - \frac{\beta^2}{6}\right)\left(1 - \frac{\gamma^2}{6}\right)\cos A + Gl_6.$$

By multiplying out, then multiplying both members by $\left(1 + \frac{\beta^2+\gamma^2}{6}\right)$, and rearranging the equation for $\cos A$, we are gradually obtaining

$$\beta\gamma\left(1 - \frac{\beta^2+\gamma^2}{6}\right)\cos A = \frac{-\alpha^2+\beta^2+\gamma^2}{2} + \frac{\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2}{24} + Gl_6,$$

$$\cos A = \frac{-\alpha^2+\beta^2+\gamma^2}{2\beta\gamma} - \frac{2\alpha^2\beta^2+2\alpha^2\gamma^2+2\beta^2\gamma^2-\alpha^4-\beta^4-\gamma^4}{24\beta\gamma} + Gl_4.$$

The terms of the sixth order will always be neglected.

Again, designating the angles of the auxiliary plane triangle A^*, B^*, and C^*, and denoting its area F^*, we have

$$\cos A^* = \frac{-a^2+b^2+c^2}{2bc} = \frac{-\alpha^2+\beta^2+\gamma^2}{2\beta\gamma}, \qquad (3)$$

which applies for measuring the angles in any sense since we may substitute, in (3), $A^* = 360 - A^*$. Furthermore, we find

$$F^{*2} = \frac{1}{16}(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$$

or or

$$F^{*2} = \frac{1}{16}\varrho^4\{2\alpha^2\beta^2 + 2\alpha^2\gamma^2 + 2\beta^2\gamma^2 - \alpha^4 - \beta^4 - \gamma^4\}. \qquad (4)$$

Introducing $\cos A^*$ and F^* into the expression obtained for $\cos A$ above, and observing that we may write $F^* = \frac{1}{2}bc\sin A^*$, it follows

$$\cos A = \cos A^* - \frac{2}{3}\frac{F^{*2}}{bc\varrho^2} + Gl_4$$

or

$$\cos A = \cos A^* - \frac{1}{3}\frac{F^*}{\varrho^2}\sin A^* + Gl_4. \qquad (5)$$

From (5) and (6), p.31, and observing (4), p. 83, we get now

$$A - A^* = \varrho'' \left\{\frac{1}{3}\frac{F^*}{\varrho^2} + Gl_4\right\} = \frac{1}{3}\varepsilon + \cdots. \qquad (6)$$
$$\text{in Sec.}$$

In this, F was substituted for F^* which is sufficiently accurate when the angles $< 180^0$.

In general, F^* has to be introduced negatively for angles $> 180^0$, according to the term $\frac{1}{2}bc\sin A^*$. In the latter case, therefore, ε means the excess with disregard to 720° since, strictly

speaking, F does not equal F^*, but equals $4\pi\varrho^2 + F^*$. By that, s becomes negative.

To determine the error in (6) more accurately, we shall now evolve the higher terms of up to the eighth order.

Par. 14. Higher Terms for Legendre's Theorem. To expand the series more conveniently we write equation (2) of the preceding paragraph as follows

$$\cos A = \frac{2\cos\alpha - \cos(\beta-\gamma) - \cos(\beta+\gamma)}{\cos(\beta-\gamma) - \cos(\beta+\gamma)} \qquad (1)$$

and combine this with formula (3) for $\cos A^*$ to obtain

$$\cos A - \cos A^* = \frac{(s^2-d^2)\cos\alpha + (\alpha^2-s^2)\cos(\beta-\gamma) - (\alpha^2-d^2)\cos(\beta+\gamma)}{2\beta\gamma(\cos(\beta-\gamma) - \cos(\beta+\gamma))}, \qquad (2)$$

$$(\beta+\gamma)^2 = s^2 \quad \text{und} \quad (\beta-\gamma)^2 = d^2 \,, \qquad (3)$$

the latter one used temporarily for abbreviation. We better rearrange the numerator of (2) like this

$$(\alpha^2-s^2)(\cos(\beta-\gamma) - \cos\alpha) - (\alpha^2-d^2)(\cos(\beta+\gamma) - \cos\alpha)$$

and have immediately the well-known series of the cosines

$$\cos A - \cos A^* = \left\{ \begin{array}{l} (\alpha^2-s^2)\left(\frac{\alpha^2-d^2}{2} - \frac{\alpha^4-d^4}{24} + \frac{\alpha^6-d^6}{720} - \frac{\alpha^8-d^8}{40320}\right) \\ -(\alpha^2-d^2)\left(\frac{\alpha^2-s^2}{2} - \frac{\alpha^4-s^4}{24} + \frac{\alpha^6-s^6}{720} - \frac{\alpha^8-s^8}{40320}\right) + Gl_{12} \end{array} \right\} : 2\beta\gamma \left\{ \begin{array}{l} \frac{s^2-d^2}{2} - \frac{s^4-d^4}{24} \\ + \frac{s^6-d^6}{720} + Gl_8 \end{array} \right\}. \qquad (4)$$

Factoring $(\alpha^2-s^2)(\alpha^2-d^2)$ in the numerator of (4), and observing that, with regard to expression (4) for F^* in the preceding paragraph,

$$16F^{*2} = -(\alpha^2 - s^2)(\alpha^2 - d^2)\rho^4 \qquad (5)$$

we find by a simple reduction that the numerator has also the factor $(s^2 - d^2)$; dividing both numerator and denominator of (4) by this factor, we get

$$\cos A^* - \cos A = \frac{2F^{*2}\left(1 - \frac{\alpha^2 + s^2 + d^2}{30} + \frac{\alpha^4 + \alpha^2(s^2 + d^2) + (s^4 + s^2 d^2 + d^4)}{1680} - Gl_6\right)}{3\rho^4\beta\gamma\left(1 - \frac{s^2 + d^2}{12} + \frac{s^4 + s^2 d^2 + d^4}{360} - Gl_6\right)}. \qquad (6)$$

For the parenthesis in the denominator of (6), we introduce into its numerator the factor

$$\left(1 + \frac{s^2 + d^2}{12} + \frac{3s^4 + 8s^2 d^2 + 3d^4}{720} + Gl_6\right)$$

and have, after some reductions,

$$\cos A^* - \cos A = \frac{2}{3}\frac{F^{*2}}{\rho^4\beta\gamma}\left\{1 + \frac{-2\alpha^2 + 3s^2 + 3d^2}{60} + \frac{3\alpha^4 - 11\alpha^2(s^2 + d^2) + 10s^4 + 31s^2 d^2 + 10d^4}{5040} + Gl_6\right\}. \qquad (7)$$

For further development, we shall again apply (5), and (6), p. 31. In them, we substitute A^* for A', and the right-hand member of (7), assumed to be negative, for h.

In substituting h, we must pay attention to the relations

$$\left.\begin{aligned} F^{*2} &= \frac{1}{4}\rho^4\beta^2\gamma^2\sin^2 A^*, \\ 4\beta\gamma &= s^2 - d^2, \\ \cos A^* &= \frac{-2\alpha^2 + s^2 + d^2}{4\beta\gamma} \end{aligned}\right\} \qquad (8)$$

and we obtain, one by one,

$$A - A^* = -\frac{h}{\sin A^*}\left(1 + \frac{2a^2 - s^2 - d^2}{48} + \frac{16a^4 + 8a^2(s^2 + d^2) - 3(s^4 + d^4) - 26s^2d^2}{1440 : 12} + Gl_6\right)$$

and

$$A - A^* = \frac{1}{3}\frac{F^*}{\varrho^2}\left(1 + \frac{2a^2 + 7(s^2 + d^2)}{240} + \frac{16a^4 + 128a^2(s^2 + d^2) + 93(s^4 + d^4) + 310s^2d^2}{120960} + Gl_6\right). \quad (9)$$

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In this, we shall use that sign of F^* which is yielded by the solution from two sides and the included angle.

Regarding the validity of the preceding solution we have to observe the following: To obtain the expression (7) for $\cos A^* - \cos A$, we might also have used the series for cosines, cosecants, and cotangents in the following modified form of (1):

$$\cos A = \cos\alpha \csc\beta \csc\gamma - \cot\beta \cos\gamma$$

However, the former hold generally, the latter only for values $< \pi$. Thus, equation (7) holds as long as β and $\gamma < \pi$. The corresponding developments for B and C require further $\alpha < \pi$.

Naturally, to obtain an accuracy extending to fractions of a second, the terms expressed in (7) satisfy only for small values of $\alpha, \beta,$ and γ. Particularly, we should be able to neglect the eighth power of $\alpha, \beta,$ and γ (cf. p. 28, conclusion of Par. 6).

Also for such values, equation (9) for $A - A^*$ from (7) certainly holds. This development requires, according to p. 31, the possible omission of $(h : \sin^2 A^*)^4$, i.e. approximately $(\frac{1}{2}\beta\gamma)^4$.

Chapter 2. Triangles and Triangular Nets on the Sphere.

Par. 15. Continuation: Excess Portions from the Three Sides.

Substituting in formula (9) of the preceding paragraph

$$a^2 = \frac{a^2}{\varrho^2}, \qquad s^2 + d^2 = 2\,\frac{b^2 + c^2}{\varrho^2}, \qquad s^2 d^2 = \frac{(b^2 - c^2)^2}{\varrho^4},$$

further

$$\left.\begin{aligned} a^2 + b^2 + c^2 &= 3m^2 \\ a^4 + b^4 + c^4 &= 3n^4 \end{aligned}\right\} \qquad (1)$$

and finally

$$s^4 + d^4 = \frac{-12n^4 + 10a^4 + 54m^4 - 36m^2a^2}{\varrho^4},$$

$$s^2 d^2 = \frac{6n^4 - 3a^4 - 9m^4 + 6m^2a^2}{\varrho^4},$$

we obtain after reducing

$$A - A^* = \frac{F^*}{3\varrho^2}\left\{1 + \frac{7m^2 - 2a^2}{40\varrho^2} + \frac{31n^4 - 10a^4 + 93m^4 - 30m^2a^2}{5040\varrho^4} + Gl_6\right\}.$$

By cyclic permutation of A, B, C and a, b, c, we have further

$$B - B^* = \frac{F^*}{3\varrho^2}\left\{1 + \frac{7m^2 - 2b^2}{40\varrho^2} + \frac{31n^4 - 10b^4 + 93m^4 - 30m^2b^2}{5040\varrho^4} + Gl_6\right\},$$

$$C - C^* = \frac{F^*}{3\varrho^2}\left\{1 + \frac{7m^2 - 2c^2}{40\varrho^2} + \frac{31n^4 - 10c^4 + 93m^4 - 30m^2c^2}{5040\varrho^4} + Gl_6\right\}.$$

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The addition of these three equations results in ε in the left-hand if the angles $< 180^0$. If the angles $> 180^0$, then $A^* + B^* + C^* = 900^0$ hence

$$(A + B + C) - (A^* + B^* + C^*) = \text{Excess} - 720^0,$$

the negative value of which will be denoted ε as before. Thus, we have

$$\left.\begin{aligned} \varepsilon_{\text{in Seconds}} &= \varrho'' \frac{F^*}{\varrho^2}\left(1 + \frac{m^2}{8\varrho^2} + \frac{n^4 + 3m^4}{240\varrho^4} + Gl_6\right) \\ \log \varepsilon_{\text{in Seconds}} &= \log\left(\varrho'' \frac{F^*}{\varrho^2}\right) + M\left(\frac{m^2}{8\varrho^2} + \frac{8n^4 + 9m^4}{1920\varrho^4} + Gl_6\right). \end{aligned}\right\} \qquad (2)$$

Subtracting one third of ε from $A - A^*$, etc., we obtain further

Par. 15. Continuation: Excess Portiens from the Three Sides.

$$\left.\begin{aligned} A-A^* &= \underset{\text{in Seconds}}{\frac{\varepsilon}{3}} + \underset{\text{in Seconds}}{\varrho''}\,\frac{F^*}{3\varrho^2}\left(\frac{m^2-a^2}{20\varrho^2} + \frac{(m^4-a^4)+3m^2(m^2-a^2)}{504\varrho^4} + Gl_6\right) \\ B-B^* &= \underset{\text{in Seconds}}{\frac{\varepsilon}{3}} + \underset{\text{in Seconds}}{\varrho''}\,\frac{F^*}{3\varrho^2}\left(\frac{m^2-b^2}{20\varrho^2} + \frac{(m^4-b^4)+3m^2(m^2-b^2)}{504\varrho^4} + Gl_6\right) \\ C-C^* &= \underset{\text{in Seconds}}{\frac{\varepsilon}{3}} + \underset{\text{in Seconds}}{\varrho''}\,\frac{F^*}{3\varrho^2}\left(\frac{m^2-c^2}{20\varrho^2} + \frac{(m^4-c^4)+3m^2(m^2-c^2)}{504\varrho^4} + Gl_6\right). \end{aligned}\right\} \quad (3)$$

In formulae (2) and (3), the sign of F^* must be chosen according to the solution of triangles having two sides and the included angles as known parts. When we utilize (2) to eliminate F^* from (3), the new equations read:

$$\left.\begin{aligned} A-A^* &= \frac{\varepsilon}{3}\left\{1 + \frac{m^2-a^2}{20\varrho^2} + \frac{20(m^4-a^4)-3m^2(m^2-a^2)}{10080\varrho^4} + Gl_6\right\} \\ B-B^* &= \frac{\varepsilon}{3}\left\{1 + \frac{m^2-b^2}{20\varrho^2} + \frac{20(m^4-b^4)-3m^2(m^2-b^2)}{10080\varrho^4} + Gl_6\right\} \\ C-C^* &= \frac{\varepsilon}{3}\left\{1 + \frac{m^2-c^2}{20\varrho^2} + \frac{20(m^4-c^4)-3m^2(m^2-c^2)}{10080\varrho^4} + Gl_6\right\}. \end{aligned}\right\} \quad (4)$$

The necessary divisions of the first parenthesis (2) into the parentheses (3) will be permissible since we are limited to small sides, which are prerequisites for using the preceding derivations.

Legendre published his theorem at first in the Memoires de l'Academis Francaise of the year 1787; 12 years later, he reproduced it in Delambre's book: Méthodes analytiques...; he proceeded from the Sin Theorem by substituting for

$$\sin\alpha : \sin\beta = \sin A : \sin B$$

the expression

$$\alpha : \beta = \sin(A-u) : \sin(B-u)$$

and from this, he determined the unknown u. Since $u = \frac{\varepsilon}{3}$, it was possible to reduce the spherical triangle to a plane

triangle. According to Puissant's Traité de Géodésie, Lagrange applied the Cosin Theorem in his Mémoire sur la Trigonometrie sphérique (Journal de l'École Polytechnique, No. 6).

We utilized, in part, Grunert's treatise in Archiv für Mathematik und Physik, Vol. 9, p. 8 (1847), who wished to present a development which would be suitable for any size of triangles. Grunert also mentioned former works dealing with this subject. In recent years, Nell started from the third equation (2), p. 80, in Schlömilch's Zeitschrift für Mathematik und Physik, Vol. 19, pp. 324-344.

Andrae advanced in quite a different way (Dänische Gradmessung, Vol. 1, 1867, p. 544) by equating two terms for the cosin of the chord angle A which belongs to the spherical angle $\mathfrak{A}$. One of the terms is supplied by plane trigonometry from the chord triangle, the other term is derived by spherical trigonometry from A and the inclination angles between the sides of angle $\mathfrak{A}$ and the tangential plane in A. This method which was reproduced by Zachariae in his book Die geodätischen Hauptpunkte...(German translation by Lamp, 1878, pp. 126-128) has the advantage that it may also be applied directly onto the ellipsoid, in the case of small triangles.

Hansen, too, evolved the theorem, and he states the terms of the sixth order in e and $A-A^*$, etc. (Geodätische Untersuchungen, 1865, pp. 219-221).

Lately, Meissel (Astronomische Nachrichten, Vol. 95, 1879, p. 70, No. 2261) derived formulae by means of elliptic functions which reduce the solution of any spherical triangle to the

solution of a plane triangle. However, in the case of triangles having small sides, these formulae are more complicate than Legendre's theorem, particularly with regard to acute triangles. Meissel's own accounts of the formulae are much too curtailed.

Par. 16. Numerical Value of the Higher Terms. Let the sides $a, b,$ and c of an equilateral triangle have the value of $0{,}1\,\varrho$ each. Then, since $m = n = 0{,}1\,\varrho$, we have

$$\underset{\text{in Seconds}}{\varepsilon} = 2062{,}65\,\frac{\sqrt{3}}{4}\left(1 + \frac{0{,}01}{8} + \frac{0{,}0001}{60}\right)$$

or

$$\varepsilon = 893{,}16'' + 1{,}12'' + 0{,}0015''. \qquad (1)$$

Other triangles, having the same average of m^2 for the sides' squares, will possess smaller areas F^*; namely, by substituting in (5), p. 91, for F^* the term m and the auxiliary term

$$\vartheta^2 = b^2 - c^2$$

-in which we may always regard $b > c$ - we obtain

$$F^{*2} = \frac{1}{16}\,(6a^2m^2 - 3a^4 - \vartheta^4), \qquad (2)$$

This proves that in the case of $\vartheta = 0$ and $a = m$, viz. in an equilateral triangle, F^* will attain a maximum.

We see that the first and second terms of ε decrease when the triangle, with m being constant, departs from the equilateral form.

With regard to the third term, we first have, from the meaning of n in (1), p. 92,

$$n^4 = \frac{1}{6}\,(9m^4 - 6m^2a^2 + 3a^4 + \vartheta^4) \text{ i.e., } \frac{1}{6}\,(9m^4 - 16F^{*2}). \qquad (3)$$

We introduce this into the product $F^*(n^4 + 3m^4)$, on which the third term mostly depends. We then have the expression

$$\varepsilon'' = \frac{F^*(27m^4 - 16F^{*2})}{1440\varrho^6}, \qquad (4)$$

This expression's differential quotient for F^* is always positive since F^{*2} will have a value of only $3m^4 : 16$ in the case of an equilateral triangle, when its area reaches a maximum.

It follows that, in an equilateral triangle, all three terms of the excess are a maximum with regard to all triangles having the same mean square of the three sides.*)

Taking the values of (1) as a basis, we have the following table of the

Maximum Values of the Terms ε:

$m =$	$m =$	2. Term	3. Term
$0{,}2\varrho$	1274^{km}	18″	0,1″
0,1	637	1,12″	0,0015″
0,05	319	0,070	.
0,02	127	0,0018	.
0,01	64	0,0001	.
0,001	6,4	$\varepsilon = 0{,}09''$	

(5)

In order to examine the influence of those terms by which

*) Assuming a constant mean of the sides' first powers would not result in anything essentially different; however, calculating with a constant m^2 is more convenient, and moreover, it does not matter, in practice, which of the two we shall use. On the other hand, a solution referring to different triangles of equal areas would be of little value since, in geodesy, we classify triangles according to the lengths of their sides and not according to their areas.

the reductions $A - A^*$, etc. differ from $\frac{\varepsilon}{3}$, we first take a look at the second term in $\left(A - A^* - \frac{\varepsilon}{3}\right)$, according to equation (3), p. 93, and find

$$\frac{\varrho'' F^* (m^2 - a^2)}{60 \varrho^2} \qquad (6)$$

Introducing (2) shows at once that, in order to obtain the term's maximum, we require ϑ to be zero. The differentiation for a states further, as a condition, $a^2 = m^2 (1 \pm \sqrt{0{,}5})$ with

$$\mp \varrho'' \frac{\sqrt{3}}{480} \frac{m^4}{\varrho^4} \qquad (7)$$

as a maximum of (6). The maximum value will, therefore, be:

if m equals $\begin{matrix} 0{,}2\varrho & 0{,}1\varrho & 0{,}05\varrho & 0{,}02\varrho \\ \mp 1{,}2'' & \mp 0{,}074'' & \mp 0{,}005'' & \mp 0{,}0001''. \end{matrix}$ (8)

It is much less than the maximum of the corresponding term in $\frac{\varepsilon}{3}$.

We may therefore state:

An equal distribution of the excess among the three angles will be sufficient if ε may be calculated from formula $\varrho'' F^* : \varrho^2$.

We presume, of course, that in the reductions $A - A^* - \frac{\varepsilon}{3}$, etc. from (3), p. 93, and also in the reductions from (4), p. 93, the third term will be of lesser importance than the second term since it is of a higher order. However, we can see at once that the term is of necessity smaller than the corresponding one in $\frac{\varepsilon}{3}$, formula (2), p. 93; hence, we may neglect it.

To determine the accuracy of Legendre's theorem, we can also, instead of developing the terms of the fourth and sixth order, look for a limit for the sum of all higher terms which

they may not pass. For this, refer to Mertens, in Schlömilch's Zeitschrift fur Mathematik und Physik, 1875, p. 286.

A different method was chosen by Gauss who stated a precise expression for $(a \sin B^* : b \sin A^*)$ which, apparently, does not vary much from unity if applied to very small triangles. (Crelles Journal, 1841, Vol. 22, pp. 22 ff. or Bauernfeinds Vermessungskunde, Vol. 2, Sixth Edition, p. 259).

Par. 17. Excess Portions from Two Sides and the Included Angle. The case in which two sides and the included angle are known

is, without doubt, more commom than the one in which three sides are given. Therefore, we have to examine it separately using first the formulae of Par. 15 for an indirect calculation.

When b, c, and A, are given, then we must find, by successive approximation,

and

$$\left.\begin{aligned} F^* &= \tfrac{1}{2} bc \sin A^* \\ a^2 &= b^2 + c^2 - 2bc \cos A^* \end{aligned}\right\} \qquad (1)$$

For the first approximation, we state $A^* = A$, and calculate the excess by formula

$$\varepsilon = \frac{\varrho'' bc \sin A}{2\varrho^2} + Gl_4 \qquad (2)$$

In Seconds

For the second approximation, we write $A^* = A - \frac{\varepsilon}{3}$ and, from equations (1), obtain F^* and a^2 accurately enough to calculate the second and third terms for (2), (3), and (4), p. 93. The principal term in ε still shows an error of the sixth order which

may be eliminated by a further approximation.

This approximation is not inconvenient. However, if we wish direct formulae, we shall have to carry out the calculation with the general terms themselves.

In formulae (1), let us put $A^* = A - (A - A^*)$ and form the series for $\sin(A - A^*)$ and $\cos(A - A^*)$; they will now read

$$F^* = \frac{1}{2} bc \left\{ \sin A - (A - A^*) \cos A - \frac{(A - A^*)^2}{2\varrho^2} \sin A + Gl_6 \right\},$$

$$a^2 = b^2 + c^2 - 2bc \cos A - 2bc \{ \sin A (A - A^*) + Gl_4 \}.$$

Further, from p. 92, we get the following expression which is suitable for substitution in the preceding equations:

$$A - A^* = \frac{F^*}{3\varrho^2} \left(1 + \frac{7m^2 - 2a^2}{40\varrho^2} + Gl_4 \right)$$

and this gives, together with the preceding ones, by successive mutual substitution:

$$F^* = \frac{1}{2} bc \sin A \left(1 - \frac{bc \cos A}{6\varrho^2} + Gl_4 \right),$$

$$bc \cos A = \frac{3}{2} m^2 - a^2 + bc\, Gl_2,$$

$$A - A^* = \frac{1}{6} bc \sin A \left(1 + \frac{14a^2 - 9m^2}{120\varrho^2} + Gl_4 \right),$$

$$bc \cos A = \frac{3}{2} m^2 - a^2 - \frac{b^2 c^2 \sin^2 A}{6\varrho^2} + bc\, Gl_4,$$

$$F^* = \frac{1}{2} bc \sin A \left(1 - \frac{3m^2 - 2a^2}{12\varrho^2} - \frac{(3m^2 - 2a^2)(14a^2 - 9m^2)}{1440\varrho^4} + \frac{b^2 c^2 \sin^2 A}{72\varrho^4} + Gl_6 \right),$$

$$\varepsilon = \varrho'' \frac{bc \sin A}{2\varrho^2} \left(1 + \frac{4a^2 - 3m^2}{24\varrho^2} + \frac{14a^4 - 15a^2 m^2 + 3m^4}{720\varrho^4} + \frac{b^2 c^2 \sin^2 A}{72\varrho^4} + Gl_6 \right). \quad (3)$$

in Seconds

In these derivations, it is only doubtful whether we may factor $\sin A$ in F^* and, hence, in ε. Our decision on this is withheld.

Utilizing the well-known series $\log(1+u)=M\left(u-\frac{u^2}{2}+\cdots\right)$ we shall now present ε in logarithmic form and, using (4), p. 93, obtain the following set of formulae:

$$\left.\begin{aligned}
\frac{a^2}{\varrho^2} &= \frac{b^2}{\varrho^2}+\frac{c^2}{\varrho^2}-\frac{2bc\cos A}{\varrho^2}-\frac{b^2c^2\sin^2 A}{3\varrho^4}+Gl_6\\
\log\underset{\text{in Sec.}}{\varepsilon} &= \log\frac{\varrho''bc\sin A}{2\varrho^2}+M\left\{\frac{4a^2-3m^2}{24\varrho^2}+\frac{32a^4-45m^4+24a^2}{5760\varrho^4}+\frac{b^2c^2\sin^2 A}{72\varrho^4}\right\}+Gl_6\\
A-A^* &= \frac{\varepsilon}{3}\left\{1+\frac{m^2-a^2}{20\varrho^2}+\frac{20(m^4-a^4)-3m^2(m^2-a^2)}{10080\varrho^4}+Gl_6\right\}\\
B-B^* &= \frac{\varepsilon}{3}\left\{1+\frac{m^2-b^2}{20\varrho^2}+\frac{20(m^4-b^4)-3m^2(m^2-b^2)}{10080\varrho^4}+Gl_6\right\}\\
C-C^* &= \frac{\varepsilon}{3}\left\{1+\frac{m^2-c^2}{20\varrho^2}+\frac{20(m^4-c^4)-3m^2(m^2-c^2)}{10080\varrho^4}+Gl_6\right\}.
\end{aligned}\right\}\quad(4)$$

As for the validity of the preceding development, we may also interpret (3) as a transformation of the second equation (3), p. 84, which when applied to β, γ, and A, results in

$$\varepsilon = 2\arctan\frac{\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\sin A}{1+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\cos A}$$

Since the series for arc tan u is valid for the $|u|<1$, we are limited to $|\varepsilon|<90^0$. Reducing the denominator $1+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\cos A$ into a series, and a further reduction into series for powers of β and γ require as a further limit β and $\gamma<\frac{\pi}{2}$.;

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and from this, we can see without difficulty that development (3) holds for small values of β and γ; We notice at the same time that ε has the factor $\sin A$

Par. 18. Any Three Parts are Given.

The influence of the second term in ε and $\log\varepsilon$ upon ε will be approximately

$$\varphi'' \frac{F^*}{\varrho^2} \frac{4a^2 - 3m^2}{24\varrho^2}, \qquad (5)$$

in which we may write according to (2), p. 94, for a maximum with $\vartheta = 0$, i.e. $b = c$

$$F^{*2} = \frac{3}{16} a^2(2m^2 - a^2). \qquad (6)$$

The differentiation for a indicates, with m^2 remaining constant, a maximum in the case of a^2 nearly equal to $\frac{3}{2}m^2$, whence (5) will transform into

$$\varphi'' \frac{3}{64} \frac{m^4}{\varrho^4}. \qquad (7)$$

For $m = 0{,}1\varrho$, this will result in 1" which very nearly agrees with the corresponding term in (2), p. 93.

The third and fourth terms in $\log\varepsilon$ influence ε by approximately

$$\varphi'' \frac{F^*}{\varrho^2} \frac{32a^4 - 45m^4 + 24n^4 + 320F^{*2}}{5760\varrho^4}$$

that is: for $\vartheta = 0$ and F^* from (6), further n^4 from (3), p. 95, the influence will also be

$$\varphi'' \sqrt{\frac{3a^2(2m^2 - a^2)}{4\varrho^2}} \frac{96m^2a^2 - 16a^4 - 9m^4}{5760\varrho^4}. \qquad (8)$$

This is, at the maximum, not much smaller than the influence of corresponding terms in (2), p. 93.

Hence, table (5), p. 95, holds, in essence, also for the application of formula (4).

Par. 18. Any Three Parts are Given. In other cases than the two examined so far, we may always proceed by successive approximation according to the formulae in Par. 15, p. 92. Similar to

p. 97, F^* will be calculated repeatedly from improved approximate values of those parts of the auxiliary plane triangle which correspond to the given parts of the spherical triangle.

We shall not involve ourselves further; however, we are going to develop another formula which will be useful to solve observed triangles of unusually great size, and in which one side and the two adjacent angles are known.

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In this case, an equal distribution of ε' among the angles will always be sufficient; however, we may occasionally wish to determine the influence of the fourth order terms upon ε. From (1), p. 83, we have

$$\sin\frac{\varepsilon}{2} = \sin^2\frac{\alpha}{2}\,\frac{\sin B\sin C}{\sin\left(B+C-\frac{\varepsilon}{2}\right)}. \qquad (1)$$

Substituting $\frac{\varepsilon}{2}$ for $\sin\frac{\varepsilon}{2}$, we obtain an error in the order of ε^3, or an error of the sixth order. It is further from (1), p. 32,

$$\sin\frac{\alpha}{2} = \frac{\alpha}{2}\sqrt[3]{\cos\frac{\alpha}{2}} + Gl_5.$$

Hence,

$$\varepsilon = \frac{a^2}{2\varrho^2}\,\frac{\sin B\sin C}{\sin\left(B+C-\frac{\varepsilon}{2}\right)}\sqrt[3]{\cos^2\frac{\alpha}{2}} + Gl_6.$$

Introducing the first equation (5), p. 81, into the right-hand member, we obtain

$$\varepsilon = \frac{a^2}{2\varrho^2}\,\frac{\sqrt[3]{\sin^2 B\sin\left(B-\frac{\varepsilon}{2}\right)\sin^2 C\sin\left(C-\frac{\varepsilon}{2}\right)}}{\sin\left(B+C-\frac{\varepsilon}{2}\right)} + Gl_6. \qquad (2)$$

Hence it follows for logarithmic calculation, when $\Delta \log \sin$ denotes the change of Briggs' logarithm for one second,

$$\log \varepsilon_{\text{in Seconds}} = \log \frac{\varrho'' a^2 \sin B \sin C}{2\varrho^2 \sin (B+C)} \qquad (3)$$

$$+ \frac{\varepsilon \text{ in Sec.}}{2} \left\{ \Delta \log \sin (B+C) - \frac{\Delta \log \sin B + \Delta \log \sin C}{2} \right\} + \ldots .^{*})$$

If all angles are observed, it will be preferable to use A instead of $(B+C)$ for calculation. In this case, $(B+C)$ may simply be replaced by A in the preceding formula.

For example, if $a = 0{,}05\varrho$ with B and $C = 300^0$, then, since, disregarding the preceding sign $\Delta \log \sin (B+C) = + \Delta \log \sin 60^0$ ist, ,

$$\log \varepsilon = 2{,}348866_n - 0{,}000227; \quad \varepsilon = -223{,}172''$$

which agrees with a calculation using the precise initial formula (1).

Par. 19. Numerical Example. $b = c = 0{,}25\varrho$; $A = 270^0$.

Since $\beta = \gamma = 14^0 19' 26{,}202''$, and the positive signs are valid, Gauss's Equations give:

$$\sin \frac{B-C}{2} \sin \frac{\alpha}{2} = 0$$

$$\cos \frac{B-C}{2} \sin \frac{\alpha}{2} = [9{,}2428916.5 - 10]$$

$$\sin \frac{B+C}{2} \cos \frac{\alpha}{2} = [9{,}8494850.0_n - 10]$$

$$\cos \frac{B+C}{2} \cos \frac{\alpha}{2} = [9{,}8357695.2 - 10]$$

*)Using an idea expressed by Krummell in Astronomische Nachrichten, Volume 89, p. 54 (No. 2116).

In this, the eighth decimals are interpolated from the proportionate parts of seven-place logarithms. The third and fourth equations yield, when we ask for $\alpha < \pi$:

$$\frac{B+C}{2} = 360^\circ - 45^\circ 54' 16{,}489'', \text{ the first and second ones add}$$

$$\frac{B-C}{2} = 0$$

$$B = C = 314^\circ 5' 43.511''$$

$$\varepsilon = 718^\circ 11' 27{,}022'' = -1^\circ 48' 32{,}978''.$$

The first and second equations, and the third and fourth ones, respectively, result further in:

$$\sin\frac{\alpha}{2} = [9{,}2428916.5 - 10] \qquad \cos\frac{\alpha}{2} = [9{,}9932505.3 - 10],$$

$$\frac{\alpha}{2} = 10^\circ 4' 30{,}813'' \qquad \alpha = 20^\circ 9' 1{,}626'',$$

$$a = 0{,}35169173\,\varrho.$$

For a check, we may use one of formulae (3), p. 84. The second one gives

$$\tan\frac{\varepsilon}{2} = -\tan^2\frac{\beta}{2} \quad \text{also} \quad \varepsilon = -6512{,}978''.$$

Formulae (4), p. 98, on the other hand, lead to the following figures:

$$b^2 = c^2 = 0{,}0625\varrho^2, \qquad a^2 = 0{,}123698\varrho^2, \quad m^2 = 0{,}082899\varrho^2,$$

$$b^4 = c^4 = 0{,}003906\varrho^4, \quad a^4 = 0{,}015301\varrho^4, \quad n^4 = 0{,}007704\varrho^4,$$

$$m^4 = 0{,}006872\varrho^4,$$

$$\log\varepsilon \text{ in Sec.} = 3{,}8092751.5_n + 44532.4 + 275.5 + 235.7$$

$$= 3{,}8137795.1_n$$

$$\varepsilon = -6512{,}976'' = -1^\circ 48' 32{,}976''.$$

$$A - A^* = -2170{,}992 + 4{,}429 + 0{,}033 - 0{,}002 = -2166{,}532'',$$

$$\left.\begin{matrix} B - B^* = \\ C - C^* = \end{matrix}\right\} - 2170{,}992 - 2{,}215 - 0{,}016 + 0{,}001 = - 2173{,}222''.$$

Hence it follows

$$A^* = 270^0\ 36'\ 6{,}532'' \text{ and since } A^* + B^* + C^* = 900^0$$
$$B^* = C^* = 314^0\ 41'\ 56{,}734'', \text{ also}$$
$$B = C = 314\ \ 5\ 43{,}512.$$

The Sin Theorem gives further:

$$a = 0{,}25\varrho \sin A^* : \sin B^* = 0{,}35169173\varrho.$$

Par. 20. Polar Coordinates. To determine the lengths of degrees, it may be practical to ascertain the distances between astronomically fixed vertices of triangular nets. Thus, we have to solve the problem: From a series of sides of observed triangles, say $P_0P_1, P_1P_2, \ldots P_{n-1}P_n$, which are connected to an astronomically fixed point P_0, to calculate successively the radii vectores $P_0P_2, P_0P_3, \ldots P_0P_n$, and their azimuths with a fixed initial direction through P_0.

It will suffice to consider the calculations for P_0P_2. For the sake of simplicity, we shall only denote the indices of the points P.

In the triangle $0.1.2$, the sides 0.1 and 1.2, are known, also the included angle. Its size will be the difference α of the azimuths of the sides. We shall now introduce the following designation and always observe the cyclic sequence $0.1.2.0$ (cf. p. 71):

(See Illustrative Material on Next Page.)

$$\left.\begin{array}{l} \text{Seite } 0.1 \text{ mit } b, \quad \text{Seite } 0.2 \text{ mit } a, \\ \text{Seite } 1.2 \text{ mit } c, \\ \sphericalangle\binom{2.0}{1} = -\alpha_{1.2} + \alpha_{1.0} \text{ mit } A, \\ \sphericalangle\binom{1.2}{0} = -\alpha_{0.1} + \alpha_{0.2} \quad \text{„} \quad C, \\ \sphericalangle\binom{0.1}{2} = -\alpha_{2.0} + \alpha_{2.1} \quad \text{„} \quad B. \end{array}\right\} \qquad (1)$$

Now we shall apply set (4), p. 98, and reduce the plane triangle by means of the formulae

$$\left.\begin{array}{l} a \sin \dfrac{B^* - C^*}{2} = (b - c) \sin \dfrac{B^* + C^*}{2}, \\ a \cos \dfrac{B^* - C^*}{2} = (b + c) \cos \dfrac{B^* + C^*}{2}, \\ \dfrac{B^* + C^*}{2} \left\{\begin{array}{l} = 90^0 - \dfrac{A^*}{2}; \; A^* < 180, \\ = 450^0 - \dfrac{A^*}{2}; \; A^* > 180. \end{array}\right. \end{array}\right\} \qquad (2)$$

These plane trigonometric formulae are derived from the first two of Gauss's Equations, p. 79, if we form the series for the sines of the sides, multiply both members by ϱ, and then write $\varrho = \infty$. The third and fourth equations (10), p. 79, give the equation for $\frac{B^* + C^*}{2}$.

From B^* and C^*, we now get B and C from (4), p. 98, and it follows that:

$$\left.\begin{array}{l} \alpha_{0.2} = C + \alpha_{0.1}, \\ \alpha_{2.0} = \alpha_{2.1} - B, \end{array}\right\} \qquad (3)$$

which yields everything needed to continue the calculation in the next triangle $0.2.3$.

Par. 21. The Logarithmic Addition Method.

If the logarithms of b and c are given, then the following set will be more convenient than (2):

$$\left.\begin{aligned} &\log\cot\lambda = \log b - \log c,\\ &\tan\frac{B^*-C^*}{2} = \tan\frac{B^*+C^*}{2}\cot(\lambda \mp 45^0),\\ &\frac{B^*+C^*}{2}\begin{cases} = 90^0 - \frac{A^*}{2};\ A^* < 180^0,\\ = 450^0 - \frac{A^*}{2};\ A^* > 180^0,\end{cases}\\ &a = b\frac{\sin A^*}{\sin B^*} = c\frac{\sin A^*}{\sin C^*}, \end{aligned}\right\} \qquad (4)$$

which will be found when forming the expression:

$$\frac{\frac{\sin B^*}{\sin C^*} - 1}{\frac{\sin B^*}{\sin C^*} + 1} = \frac{\cot\lambda - 1}{\cot\lambda + 1} \qquad (5)$$

Set (4) presumes, however, that the triangle is not very obtuse but this happens quite frequently when calculating by means of polar coordinates.

Par. 21. The Logarithmic Addition Method. Legendre's Theorem enables us to calculate all sides of a triangular net, when starting from a fixed base line, by using plane trigonometric formulae. As far as observed triangles are concerned, the theorem's application is further facilitated by the fact that all the excesses have to be figured out in advance to balance the angles by the Method of Least Squares.

To continue the calculation, it will only be necessary to subtract $\frac{\varepsilon}{3}$ from each spherical angle; after that, the plane trigonometry

may be applied at once. But even if ε must be calculated, the procedure is so simple and clear that it is now probably used without exception. However, we are going to scrutinize two further methods which may, under certain conditions, compete with the above. At least they should arouse scientific interest. First of all, let us take a look at the logarithmic addition method.

This method, in general, states for a side s :

$$\log \sin \frac{s}{\varrho} = \log \frac{s}{\varrho} - A_s = \log s - \log \varrho - A_s. \qquad (1)$$

If we need only apply the sin theorem to a, b, A , and B, then we have to substitute in the equation

$$\sin \frac{b}{\varrho} = \sin \frac{a}{\varrho} \frac{\sin B}{\sin A}$$

the following expressions:

$$\log \sin \frac{b}{\varrho} = \log b - \log \varrho - A_b,$$
$$\log \sin \frac{a}{\varrho} = \log a - \log \varrho - A_a.$$

Hence it follows

$$(\log b - A_b) = (\log a - A_a) + \log \sin B - \log \sin A. \qquad (2)$$

If tables for A_s with the argument $\log \sin s$ are on hand, we may now calculate in a consecutive series of triangles by first reducing the logarithm of the base by its additament; (*) then we look up $(\log s - A_s)$ for all sides in every triangle, according to (2), and finally we add A_s to all these values.

(*) Translator's Note: Corresponding value in the adjoining column of the addition-logarithm table. -L.D.

Par. 21. The Logarithmic Addition Method.

The method of logarithmic addition requires the preparing of tables for A_r. For this purpose, we may develop $\log \sin \frac{s}{\varrho}$ into a series. It will be more convenient, however, to use S from the logarithmic tables, p. 33. We have

$$S = \log \sin \frac{s}{\varrho} - \log \frac{\varrho'' s}{\varrho};$$

Comparing this with formula (1), we obtain for determining A_r:

$$A_r = -S - \log \varrho'' = -S - 5{,}3144251.3, \qquad (3)$$

s corresponds to the argument $\frac{\varrho' s}{\varrho}$

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For example, $\log s = 5{,}48000$ and $\log \varrho = 6{,}80416$ yield:

$$\log \frac{\varrho'' s}{\varrho} = 3{,}99027 \text{ with } S = 4{,}6854121.8 - 10$$

and from this follows

$$A_r = 10 - 9{,}9998373.1 = +1626.9.$$

We may also find A_r as the difference between log sin and log arc in Bremiker's six-place logarithmic tables, in which it is found under b in the column next to the log sine. However, these tables are only suitable for calculations involving seven decimals. If eight decimals are required, we may find tables in Bremiker's Studien, p. 80.

Since, according to (8), p. 96, a uniform distribution of the excess among the three angles will not be permissible for triangles with sides greater than $0{,}05\varrho$, the method of calculating the sides by logarithmic addition from the Sin Theorem is more convenient, in this case, than Legendre's Theorem. However, we shall see later

that triangles of those dimensions may no more be solved as spherical ones. We shall also learn (without mentioning it again, though) that the Sin Theorem for triangles on the ellipsoid of rotation points exclusively to the use of Legendre's Theorem (in expanded form).

If the spherical form of the level surfaces is a permissible hypothesis, then the logarithmic addition method may be quite useful.

It was used during the first half of the Nineteeth Century for triangulation in southern Germany, in particular by Soldner when he undertook the survey of Bavaria. Confer Die bayerische Landesvermessung..., Munich 1873, p. 222; further Jordan, Handbuch der Vermessungskunde, Volume 2, p. 132, and also Nell in Zeitschrift fur Mathematik und Physik, Volume 19, 1874, pp. 344-353. In the latter, we also find the logarithmic addition method applied to other cases than the Sin Theorem. For example, when reducing Gauss's Equations, we may write

$$\log\cos\frac{s}{\varrho} = -3A,$$

which will be sufficiently accurate for calculating with seven-place logarithms in the case of sides up to $0{,}05\varrho$ in length. We may see this from the first equation (1), p. 32, Par. 10.

Par. 22. Precise Formulae for Chords and Horizontal Angles. Instead of calculating the sides on the surface of the sphere, we may first derive the corresponding chords from the base chord and the angles; for this reason, let us look at some precise relations between the chords and the horizontal angles.

We do not require the angles of the chord triangles. The chords

will be denoted by Old English letters, and we presume that a, b, and $c < \pi\varrho$*). Then we get immediately from well-known geometric relations

$$\left.\begin{aligned} \mathfrak{a} &= 2\varrho \sin \frac{a}{2\varrho}, \quad \mathfrak{b} = 2\varrho \sin \frac{b}{2\varrho}, \quad \mathfrak{c} = 2\varrho \sin \frac{c}{2\varrho}, \\ \mathfrak{a} &= 2\varrho \sin \frac{\alpha}{2}, \quad \mathfrak{b} = 2\varrho \sin \frac{\beta}{2}, \quad \mathfrak{c} = 2\varrho \sin \frac{\gamma}{2}. \end{aligned}\right\} \quad (1)$$

or

Introducing this in equation (8), p. 82, it follows

$$\mathfrak{a} : \mathfrak{b} : \mathfrak{c} = \cos\frac{\alpha}{2} \sin\left(A - \frac{\varepsilon}{2}\right) : \cos\frac{\beta}{2} \sin\left(B - \frac{\varepsilon}{2}\right) : \cos\frac{\gamma}{2} \sin\left(C - \frac{\varepsilon}{2}\right).$$

Applying the first formula (5), p. 81, for α, β, and γ, we have, on the other hand:

$$\cos^2\frac{\alpha}{2} : \cos^2\frac{\beta}{2} : \cos^2\frac{\gamma}{2} = \frac{\sin A}{\sin\left(A - \frac{\varepsilon}{2}\right)} : \frac{\sin B}{\sin\left(B - \frac{\varepsilon}{2}\right)} : \frac{\sin C}{\sin\left(C - \frac{\varepsilon}{2}\right)};$$

We use this to eliminate $\alpha, \beta,$ and γ from the preceding proportion and obtain a precise formula for calculating the sides $\mathfrak{b}$ and $\mathfrak{c}$ when the third side $\mathfrak{a}$ and the angles $A, B,$ and C: are known:

$$\frac{\mathfrak{a}}{\sqrt{\sin A \sin\left(A - \frac{\varepsilon}{2}\right)}} = \frac{\mathfrak{b}}{\sqrt{\sin B \sin\left(B - \frac{\varepsilon}{2}\right)}} = \frac{\mathfrak{c}}{\sqrt{\sin C \sin\left(C - \frac{\varepsilon}{2}\right)}}. \quad (2)$$

ε may be found from chord $\mathfrak{a}$ and the horizontal angles by the following equation which results from (1), Par. 11, p. 83:

*)This is no limitation, however, because three points on the sphere can be connected by only one chord triangle which always corresponds to a spherical triangle in which the sides satisfy this condition.

$$\sin\frac{\varepsilon}{2}=\frac{\mathfrak{a}^2}{4\varrho^2}\,\frac{\sin B\sin C}{\sin\left(B+C-\frac{\varepsilon}{2}\right)}=\frac{\mathfrak{a}^2}{4\varrho^2}\,\frac{\sin B\sin C}{\sin\left(A-\frac{\varepsilon}{2}\right)}, \qquad (3)$$

This equation is to be solved by approximation as was explained in detail at the place quoted above.

Under certain circumstances, we may also need formulae for two sides and their included angle. We do not obtain precise formulae for this case, however;

they all require successive approximation and are not very convenient. Therefore, we shall not discuss them but proceed to the approximation formulae.

Par. 23. Approximation Formulae. Grunert's Theorem. To obtain relations between chord $\mathfrak{a}$ and side a, we use the first equations of (3) and (4), pp. 29 and 30, for $u=\frac{a}{2\varrho}$ and $\sin u=\frac{\mathfrak{a}}{2\varrho}$ and have the formulae:

$$\left.\begin{aligned}\log\mathfrak{a}&=\log a-\frac{1}{6}M\left\{\left(\frac{a}{2\varrho}\right)^2+\frac{1}{30}\left(\frac{a}{2\varrho}\right)^4+\cdots\right\},\\ \log a&=\log\mathfrak{a}+\frac{1}{6}M\left\{\left(\frac{\mathfrak{a}}{2\varrho}\right)^2+\frac{11}{30}\left(\frac{\mathfrak{a}}{2\varrho}\right)^4+\cdots\right\}.\end{aligned}\right\} \qquad (1)$$

As a rule, however, the auxiliary tables for the difference between sin and arc mentioned on p. 105 will be preferable.

From relation (2) of the preceding paragraph, and omitting the terms of the fourth order (with regard to $\left(\frac{\varepsilon}{\varrho}\right)$:, it follows

$$\mathfrak{a}:\mathfrak{b}:\mathfrak{c}=\sin\left(A-\tfrac{\varepsilon}{4}\right):\sin\left(B-\tfrac{\varepsilon}{4}\right):\sin\left(C-\tfrac{\varepsilon}{4}\right). \qquad (2)$$

This is called Grunert's Theorem. It differs from Legendre's theorem

insofar as the reduced angles do not belong to a triangle.

The following calculation gives us the terms of the fourth order. The proportions

$$b : c = \sqrt{\sin B \sin\left(B - \tfrac{\varepsilon}{2}\right)} : \sqrt{\sin C \sin\left(C - \tfrac{\varepsilon}{2}\right)} \qquad (3)$$

yield, when $\sin\left(B - \frac{\varepsilon}{2}\right)$ and $\sin\left(C - \frac{\varepsilon}{2}\right)$ are reduced to their quantities and both roots divided by $\cos\frac{\varepsilon}{2}$: , the following

$$\frac{b}{c} = \frac{\sin B}{\sin C}\sqrt{\frac{1 - \cot B \tan\frac{\varepsilon}{2}}{1 - \cot C \tan\frac{\varepsilon}{2}}}. \qquad (4)$$

For small triangles, we may develop the numerator and denominator of the root according to the binomial theorem. For example, we get for the denominator from the second formula (3), p. 84

$$\cot C \tan\tfrac{\varepsilon}{2} = 1 : \left(1 + \cot\tfrac{\alpha}{2}\cot\tfrac{\beta}{2}\sec C\right) \qquad (5)$$

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Hence, in the case of small α's and β's, the product $\cot C \tan\frac{\varepsilon}{2}$ will always vary between -1 and +1. This is generally true for all values of C if $\cot\frac{\alpha}{2}\cot\frac{\beta}{2} > 2$, i.e. if the sides do not exceed 70°. When we limit ourselves to small values of α, β, and γ and consider them to be terms of the first order, then the expressions $\cot C \tan\frac{\varepsilon}{2}$, $\cot B \tan\frac{\varepsilon}{2}$, etc. will be terms of the second order, as shown in (5).

We obtain now from (4)

$$\frac{b}{c} = \frac{\sin B}{\sin C} \frac{1 - \frac{1}{2}\cot B \tan\frac{\varepsilon}{2}\left(1 + \frac{1}{4}\cot B \tan\frac{\varepsilon}{2}\right) + Gl_6}{1 - \frac{1}{2}\cot C \tan\frac{\varepsilon}{2}\left(1 + \frac{1}{4}\cot C \tan\frac{\varepsilon}{2}\right) + Gl_6}$$

When we multiply both the numerator and the denominator by

$$1 - \frac{1}{8}\cot B \cot C \tan^2\frac{\varepsilon}{2} - \frac{\varepsilon^2}{32}$$

and consider ε to be negative because the angles are counted outside the triangles, then we have

$$\frac{b}{c} = \frac{\sin B}{\sin C} \frac{1 - \frac{1}{2}\cot B \tan\frac{\varepsilon}{2}\left(1 + \frac{1}{4}\tan\frac{\varepsilon}{2}[\cot B + \cot C]\right) - \frac{\varepsilon^2}{32} + Gl_6}{1 - \frac{1}{2}\cot C \tan\frac{\varepsilon}{2}\left(1 + \frac{1}{4}\tan\frac{\varepsilon}{2}[\cot B + \cot C]\right) - \frac{\varepsilon^2}{32} + Gl_6},$$

Using a simple and permissible transformation in numerator and denominator, we get finally the formula

$$\frac{b}{c} = \frac{\sin\left(B - \frac{\varepsilon}{4} - \frac{\varepsilon^2}{32\varrho''}[\cot B + \cot C]\right) + Gl_6}{\sin\left(C - \frac{\varepsilon}{4} - \frac{\varepsilon^2}{32\varrho''}[\cot B + \cot C]\right) + Gl_6}. \qquad (6)$$

In this formula we shall express ε in seconds. We notice further that the terms of the sixth order, etc. do not always have ε combined with $\cot B$ and $\cot C$. Since they are alone at least once, they disappear like $\sin B$ and $\sin C$, which are the principal terms (as increments of B and C).

To calculate ε by means of logarithmic differences, we shall again use (6) of the preceding paragraph, except for the terms of the sixth order.

it will be sufficient to exchange $\sin\frac{\varepsilon}{2}$ for (ε in seconds): $2\varrho''$ in the left-hand member.

Par. 24. Two Chords b and c and the Horizontal Angle A given. Let us write $(b-c):(b+c)$; then we have from (6), p. 108, to calculate $(B+C)$:

$$\frac{b-c}{b+c}=\frac{\tan\left(\frac{B-C}{2}\right)+Gl_6}{\tan\left(\frac{B+C}{2}-\frac{\varepsilon}{4}-\frac{\varepsilon^2}{32}[\cot B+\cot C]\right)+Gl_6}, \qquad (1)$$

As the above development proves, the terms of the sixth order will disappear in the case of $B=C$. To facilitate logarithmic calculations, we now introduce an auxiliary angle at the left-hand side, as on p. 103, and at the right-hand side, we substitute in the denominator

$$\frac{\varepsilon^2}{32}[\cot B+\cot C]=\frac{\varepsilon a^2}{64\varrho^2}+\cdots. \qquad (2)$$

This substitution is permissible as the following relations show, which were taken from (4), p. 98:

$$\left.\begin{aligned}\frac{2ab\cos C}{\varrho^2}&=\frac{a^2+b^2-c^2}{\varrho^2}+Gl_4=\frac{a^2+b^2-c^2}{\varrho^2}+Gl_4,\\ \frac{ab\sin C}{\varrho^2}&=2\varepsilon+Gl_4.\end{aligned}\right\} \qquad (3)$$

Hence it follows for $\varepsilon\cot C$ and analogously for $\varepsilon\cot B$:

$$\left.\begin{aligned}4\varepsilon\cot C&=\frac{a^2+b^2-c^2}{\varrho^2}+Gl_4,\\ 4\varepsilon\cot B&=\frac{a^2-b^2+c^2}{\varrho^2}+Gl_4.\end{aligned}\right\} \qquad (4)$$

Formula (2), p. 85, yields further, with corresponding substitution of the parts of the triangle:

$$\sin\frac{\varepsilon}{2}=\frac{bc\sin A}{4\varrho^2\cos\frac{\alpha}{2}}. \qquad (5)$$

The series development of $\cos\frac{a}{2}$ gives finally

$$\varepsilon = \frac{bc\sin A}{2\varrho^2}\left(1+\frac{a^2}{8\varrho^2}+Gl_4\right). \qquad (6)$$

Thus, we obtain the following set of formulae to solve a triangle with given b, c, and A:

$$\left.\begin{aligned}
&\frac{a^2}{\varrho^2} = \frac{b^2+c^2-2bc\cos A}{\varrho^2}+Gl_4,\\
&\underset{\text{in Sec.}}{\varepsilon} = \frac{\varrho''bc\sin A}{2\varrho^2}\left(1+\frac{a^2}{8\varrho^2}+Gl_4\right),\\
&\log\cot\lambda = \log b-\log c,\\
&\frac{B+C}{2}\begin{cases} = 90^0-\frac{A}{2}+\frac{\varepsilon}{2}, & A<180^0,\\ = 450^0-\frac{A}{2}+\frac{\varepsilon}{2}, & A>180^0,\end{cases}\\
&\tan\frac{B-C}{2} = \tan\left(\frac{B+C}{2}-\frac{\varepsilon}{4}\left[1+\frac{a^2}{16\varrho^2}\right]\right)\cot(\lambda+45^0)+Gl_6,
\end{aligned}\right\} \qquad (7)$$

a may be determined from (6) of the preceding paragraph, which applies to a and b or a and c, after B and C have been obtained.

The above calculation is only satisfactory, however, when A is not too obtuse. And any set analogous to (2), p. 102, will be complicated even if we omit ε^2 in (6) of the preceding paragraph. Therefore, we have to seek a formula for a^2 similar to the Cosine Theorem of plane trigonometry. We have in a chord triangle

$$a^2 = b^2+c^2-2bc\cos\mathfrak{A}, \qquad (8)$$

in which $\mathfrak{A}$ is the chord angle between b and c. The perpendicular in vertex A and the chords b and c form a trilateral corner with the sides $\left(\frac{\pi}{2}+\frac{\beta}{2}\right), \left(\frac{\pi}{2}+\frac{\gamma}{2}\right)$, and $\mathfrak{A}$; the latter being opposite

A. Hence,

$$\cos \mathfrak{A} = \sin \frac{\beta}{2} \sin \frac{\gamma}{2} + \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \cos A. \qquad (9)$$

In this we introduce formulae (5), p. 81, rearranged for β and γ, and we get, after some reduction,

$$\cos \mathfrak{A} = \cos\left(A - \frac{\varepsilon}{2}\right) \sqrt{\frac{\sin\left(B - \frac{\varepsilon}{2}\right) \sin\left(C - \frac{\varepsilon}{2}\right)}{\sin B \sin C}} = \cos\left(A - \frac{\varepsilon}{2}\right) \cos \frac{\alpha}{2}. \qquad (10)$$

When we further develope $\cos \frac{\alpha}{2}$ for the powers of $\sin \frac{\alpha}{2}$ the resulting equation reads

$$a^2 = b^2 + c^2 - 2bc \cos\left(A - \frac{\varepsilon}{2}\right)\left(1 - \frac{a^2}{8\varrho^2} - \frac{a^4}{128\varrho^4}\right) + \varrho^2 Gl_8. \qquad (11)$$

We conclude from the above that calculations with chords and horizontal angles will be as convenient as the ones with

horizontal distances and angles. This will be particularly so if we refrain from using polar co-ordinates. In chord calculation, however, the fourth-order terms will become more significant than they are when using Legendre's Theorem. This applies especially to the term $\frac{1}{32}\varepsilon^2[\cot B + \cot C] = \frac{1}{64}\varepsilon a^2 : \varrho^2$ in the numerator of (6), p. 108, which will be far greater than its counterpart $\frac{1}{60}\varepsilon(m^2 - a^2) : \varrho^2$ in the first formula (4), p. 93, as may be seen by the substitution $a^2 = m^2$.

Whether it is advantageous to calculate a triangulation by chords and horizontal angles, will depend on the formulae for the ellipsoid of rotation. It matters also how the results of such a

calculation will be used for further evaluation. This will be discussed later on.

Formerly, (e.g. during Delambre's surveys at the end of the Eighteenth Century [cf. Méthodes analytiques, pp. 36-42] and during the Austrian surveys by General Fallon [cf. Nagel's treatise below]) the observed angles were reduced to chord angles and horizontal angles, respectively. This was quite inconvenient; it could not be avoided, however-not even when Legendre's Theorem was used-because the inclined angles were observed by means of Borda's Circle. Nowadays, the horizontal angles are measured with the help of a theodolite. The reduction of directly observed horizontal angles into chord angles is a useless detour, however, as we may see from Faber's and Grunert's formulae. These were developed for this purpose but were only accurate up to the fourth order terms. We may find them, and also Grunert's above-mentioned theorem, in Nagel's comprehensive treatise Über die Reduktion eines sphärischen Dreiecks von geringer Krümmung auf sein Sehnendreieck, in Schlömilch's Zeitschrift fur Mathematik und Physik, Volume 1, 1856, pp. 257-275. Some interesting theorems by Riedl von Leuenstern (1827), which allow a geometric interpretation of the angles' reduced sizes, will also be found there.

The precise relation (2), Par. 22, p. 106, was first seen by the author in Kummell's essay, Astronomische Nachrichten, Volume 89, 1877, No. 2116. Grunert proved theorem (2) of the preceding paragraph in Archiv für Mathematik und Physik, Volume 25, 1855, amongst other developments for the reduction of

horizontal angles to chord angles. Unfortunately, this theorem is not as well known as it deserves to be, probably in the belief that it is only valid in the case when one chord and three horizontal angles are given. Our derivations show, however, that it solves any other case if applied correctly. The actual reduction to chord angles will then be superfluous.

Par. 25. Numerical Example. Given: $a = 0{,}15\varrho$; $B = 350^0$; $C = 330^0$. In this case, formula (3), p. 106, gives $\log \sin \frac{\varepsilon}{2} = 6{,}880696_n - 2.51\,\frac{\varepsilon}{2}$

in which the second member is expressed in units of the sixth decimal, ε being expressed in seconds. Since $\frac{\varepsilon}{2}$ is sufficiently small, we get immediately by adding 5,314425 :

$$\log \frac{\varepsilon}{2}\text{ in Sec.} = 2{,}195121_n - 2.51\,\frac{\varepsilon}{2}\text{ in Sec.}; \quad \frac{\varepsilon}{2} = -\,2'36{,}861''.$$

Further, $A = 90^0 - 5'13{,}722'' - 68^0 = 219^0\,54'\,46{,}278''$; by using eight-place logarithms, we obtain for (2), p. 106:

$A = 219^0 54' 46{,}278''$	$A - \frac{\varepsilon}{2} = 219^0 57' 23{,}139''$	$\log \sqrt{\sin A \sin\left(A - \frac{\varepsilon}{2}\right)} = 9{,}8074763.3_n - 10$
$B = 350\ 0\ 0$	$B - \frac{\varepsilon}{2} = 350\ \ 2\ 36{,}861$	$\log \sqrt{\sin B \sin\left(B - \frac{\varepsilon}{2}\right)} = 9{,}2387316.0_n - 10$
$C = 330\ 0\ 0$	$C - \frac{\varepsilon}{2} = 330\ \ 2\ 36{,}861$	$\log \sqrt{\sin C \sin\left(C - \frac{\varepsilon}{2}\right)} = 9{,}6986837.2_n - 10.$

Since $\log \frac{a}{\varrho} = 9{,}1760912.6 - 10$, the above-mentioned (2) yields

$$\log \frac{b}{\varrho} = 8{,}6073465.3 - 10; \quad \log \frac{c}{\varrho} = 9{,}0672986.5 - 10.$$

If we proceed, in reverse order, from the last two figures and

the angle $A = 219^0\,54'\,46{,}278''$, set (7) of the preceding paragraph results in

$$\mathfrak{a}^2 = (0{,}00164 + 0{,}01363 + 0{,}00725)\,\varrho^2 = 0{,}02252\varrho^2$$

$$\varepsilon = -312{,}838''\left(1 + \frac{0{,}02252}{8}\right) \text{d. i.} - 5'\,13{,}719''.$$

The difference between this and ε above is due to the omission of the sixth-order terms because a precise formula will give us again the first value of ϵ from $\log b$, $\log c$, and A. We obtain further

$$\lambda = 70^0\,52'\,29{,}024''; \quad \log\cot(\lambda + 45^0) = 9{,}6857676.8_n - 10$$

$$\frac{B+C}{2} = 450^0 - 109^0\,57'\,23{,}139'' - 2'\,36{,}860'' = 340^0\,0'\,0{,}001''$$

$$-\frac{\varepsilon}{4}\left(1 + \frac{\mathfrak{a}^2}{16\varrho^2}\right) = +78{,}431''.1{,}00141 = +\,1\,18{,}542$$

$$\text{Summation} = 340\;1\;18{,}543$$

$$\log\tan 340^0\,1'\,18{,}543'' = 9{,}5605510.8_n - 10$$

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$$\log\tan\frac{B-C}{2} = 9{,}2463187.6 - 10; \quad \frac{B-C}{2} = 9^0\,59'\,59{,}999''$$

$$B = 350^0\,0'\,0{,}000'' \qquad C = 330^0\,0'\,0{,}002''.$$

The third side $\mathfrak{a}$ will be determined from (2), p. 106; we have

$$\log\frac{\mathfrak{a}}{\varrho} = 9{,}1760912.5 \text{ und } .6 - 10,$$

the mean of which will be $9{,}1760912.6 - 10$, , as we had stated above.

If we wish to utilize (6), p. 108, we get at first

(See Illustrative Material on Next Page.)

Par. 25. Numerical Example.

$$\frac{\varepsilon}{4} = -1'\,18{,}430''$$

$$-\frac{\varepsilon''^2}{32\varrho''}\cot A = -0{,}0178''$$

$$-\frac{\varepsilon''^2}{32\varrho''}\cot B = +0{,}0848''$$

$$-\frac{\varepsilon''^2}{32\varrho''}\cot C = +0{,}0258''$$

and hence,

$$\mathfrak{a} : \mathfrak{b} = \sin(219^\circ\,56'\,4{,}708'' + 0{,}067'') : \sin(330^\circ\,1'\,18{,}430'' + 0{,}067'')$$
$$\mathfrak{a} : \mathfrak{c} = \sin(219\ \ 56\ \ 4{,}708\ \ + 0{,}008\ ) : \sin(350\ \ 1\ \ 18{,}430\ \ + 0{,}008\ ),$$

from which follows, in accordance with the value determined above,

$$\log \frac{\mathfrak{a}}{\varrho} = 9{,}1760912.5 - 10\,.$$

Now we shall use the second formula (1), p. 107, to obtain the horizontal distances a, b and c. The calculation for a is as follows:

$$\log \frac{a}{\varrho} = 9{,}1760912.6 - 10 + 4071.5 + 8.4 = 9{,}1764992.5 - 10$$

In the same manner, we get for b and c:

$$\log \frac{b}{\varrho} = 8{,}6073762.0 - 10 \qquad \log \frac{c}{\varrho} = 9{,}0675456.7 - 10.$$

To check these figures, we use formulae (4), Par. 15, p. 93. We are not going to calculate ε from (2) of that paragraph, but write at once $\frac{\varepsilon}{3} = -104{,}574''$. Then we have:

$a^2 = 0{,}0225\varrho^2$	$m^2 - a^2 = -0{,}0099\varrho^2$	$\frac{\varepsilon}{3}\frac{m^2 - a^2}{20\varrho^2} = +0{,}052''$
$b^2 = 0{,}0016\varrho^2$	$m^2 - b^2 = +0{,}0110\varrho^2$	$\frac{\varepsilon}{3}\frac{m^2 - b^2}{20\varrho^2} = -0{,}058''$
$c^2 = 0{,}0136\varrho^2$	$m^2 - c^2 = -0{,}0010\varrho^2$	$\frac{\varepsilon}{3}\frac{m^2 - c^2}{20\varrho^2} = +0{,}005$
$m^2 = 0{,}0126\varrho^2$		

$A-A^*=-1'44{,}522''$	$A^*=219°56'30{,}800''$	log sin =	9,8075420.2 − 10
$B-B^*=-1\ 44{,}632$	$B^*=350\ \ 1\ 44{,}632$		9,2384189.7 − 10
$C-C^*=-1\ 44{,}569$	$C^*=330\ \ 1\ 44{,}569$		9,6985884.3 − 10.

These log sines give in combination with the logarithms of the corresponding opposite sides the differences

9,3689472.3; .3; .4 − 10.

Chapter 3.

Rectangular and Geographic Coordinates on the Sphere.*)

Par. 1. Rectangular Spherical Coordinates. It is convenient, to derive rectangular coordinates from the triangle points in order to conveniently use the results of triangulation for a land survey. A great circle is chosen as x axis, and x is computed from the initial point to the terminal point of the great arc, which can be placed normal to the x- axis through the point, the position of which is being determined. The perpendicular determines the ordinate y. Soldner used coordinates of this type at the beginning of the century for the Bavarian land survey, and since that time they have been used almost exclusively (op. cit., p. 271).

An alternate system of rectangular spherical coordinates is obtained by placing a great circle as y axis normal to the x axis and assuming x- and y- to be the length of the perpendicular from the point to the respective axes. Since this technique is less accurate we will not deal with it any further. Disadvantages are found in transferring this system on the ellipsoid, in the computation of geographical coordinates, and due to the fact that the lines of the constants x and y are not normal

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*)We concern ourselves only with the case, where horizontal distances are used for calculation. Chords directly are not very convenient to use for rectangular coordinates. It is better to go first to horizontal distances. On the other hand, chords may be applied directly with advantage to calculate geographic positions. Despite that we have not shown an application of it since the fourth chapter deals with this problem for the ellipsoid of revolution; and the solution for the sphere is not of great interest.

to each other.

Returning to the Soldner system we find as our first problem the following

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To derive the coordinates of a point P_2 from the coordinates of a point P_1, the horizontal distance $P_1 P_2$ and the direction towards P_2. First, it is necessary to state how the direction of a larger arc $P_1 P_2$ is found.

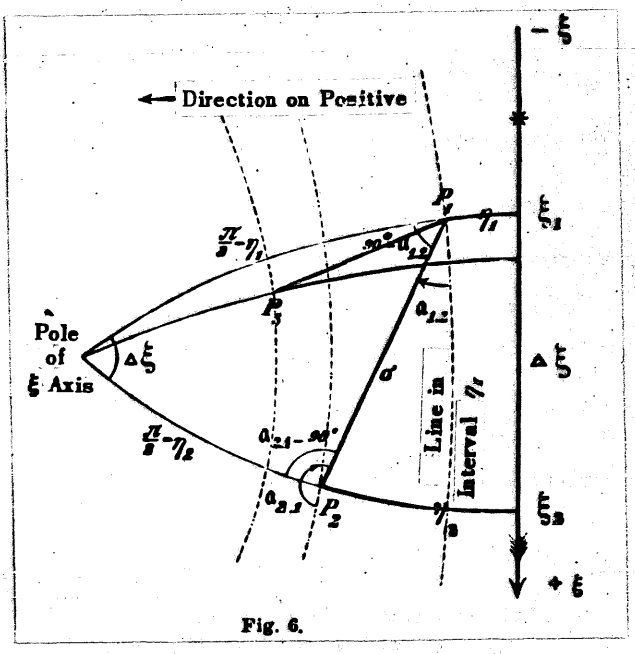


Fig. 6.

Let us assume a line through P_1 running parallel to the x-axis in a distance y. (A small circle, the plane of which is parallel to the plane of the x- axis). From the positive side of this line we count direction angles from 0^0 - 360^0, in such a way that the direction angle with increasing y is 90^0. If we select the direction of positive y in a way that the direction of rotation for direction and measured angles is the

same, we may consider the measured angles to be the differences of the direction angles for both legs, as on page 77 (1). Therefore: To compute the direction angle for $P_1 P_3$ from the one for $P_1 P_2$ and the intermediate angle we have the relation:

$$\mathfrak{a}_{1.3} = \mathfrak{a}_{1.2} + \measuredangle \binom{2.3}{1}, \qquad (1)$$

where $\mathfrak{a}_{1.3}$ and $\mathfrak{a}_{1.2}$ designate the direction angles at P_1 for $P_1 P_3$, $P_1 P_2$ respectively. Fig. 6 shows the spherical surface; and since the radius has been taken as the unit of the length we have that:

$$\left.\begin{aligned} \xi &\text{ is the quotient } \frac{x}{\varrho}, \\ \eta &\text{ is the quotient } \frac{y}{\varrho}, \\ \sigma &\text{ is the quotient } \frac{s}{\varrho}. \end{aligned}\right\} \qquad (2)$$

Par. 2. Differences of Ordinates and Abscissae. For the calculation of η_2 in a spherical triangle with the vertices P_1, P_2, the pole of the ξ- axis, we have according to the Cosine Law

$$\cos\left(\frac{\pi}{2} - \eta_2\right) = \cos\left(\frac{\pi}{2} - \eta_1\right)\cos\sigma + \sin\left(\frac{\pi}{2} - \eta_1\right)\sin\sigma\cos(90^0 - \mathfrak{a}_{1.2})$$

or:

$$\sin\eta_2 = \sin\eta_1\cos\sigma + \cos\eta_1\sin\sigma\sin\mathfrak{a}_{1.2}. \qquad (1)$$

If we say $\eta_2 = \eta_1 + \Delta\eta$, simplify $\mathfrak{a}_{1.2}$ to $\mathfrak{a}$ and reduce, we obtain:

$$\sin \Delta\eta = \sin \sigma \sin \alpha + \tan \eta_1 (\cos \sigma - \cos \Delta\eta). \qquad (2)$$

Into this equation we introduce the following series expansion limiting ourselves to values of σ and η. Since we may consider these to be of first-order magnitude (p. 25) these substitutions and the expansions below are justified.

$$\sin \sigma = \sigma - \frac{1}{6}\sigma^3 + \frac{1}{120}\sigma^5 + Gl_7$$

$$\tan \eta_1 = \eta_1 + \frac{1}{3}\eta_1^3 + Gl_5$$

$$\cos \sigma = 1 - \frac{1}{2}\sigma^2 + \frac{1}{24}\sigma^4 + Gl_6$$

$$\cos \Delta\eta = 1 - \frac{1}{2}\Delta\eta^2 + \frac{1}{24}\Delta\eta^4 + Gl_6.$$

Replacing $\sigma \sin \alpha$, by the abbreviation v we obtain:

$$\sin \Delta\eta = v - \frac{v\sigma^2}{6} + \frac{v\sigma^4}{120} - \eta_1 \frac{\sigma^2 - \Delta\eta^2}{2}\left(1 - \frac{\sigma^2 + \Delta\eta^2}{12}\right) - \eta^3 \frac{\sigma^2 - \Delta\eta^2}{6} + Gl_7. \qquad (3)$$

Introducing this into the first two terms of the arc - sin series on page 29 it follows either:

$$\Delta\eta = v - v\frac{\sigma^2 - v^2}{6} - \eta_1 \frac{\sigma^2 - \Delta\eta^2}{2} + Gl_5,$$

or, if we designate ($\sigma \cos \alpha$) as u and bear in mind that $u^2 + v^2 = \sigma^2$, and $\sigma^2 - v^2 = u^2$, , and $\Delta\eta^2 = v^2 + Gl_4$

$$\Delta\eta = v - \frac{1}{6}u^2 v - \frac{1}{2}u^2\eta_1 + Gl_5.$$

Solving this for $\Delta\eta^2$ and substituting this value in series (3) we obtain:

$$\sin \Delta\eta = v - \frac{1}{6}v\sigma^2 - \frac{1}{2}u^2\eta_1 + \frac{1}{120}v\sigma^4 + \frac{1}{24}u^4\eta_1 - \frac{1}{12}u^2v^2\eta_1 - \frac{1}{2}u^2v\eta_1^2 - \frac{1}{6}u^2\eta_1^3 + Gl_7.$$

Introducing this into the above-mentioned arc - sin series,

bearing in mind its first three terms we obtain the following equation:

$$\left.\begin{aligned} \Delta\eta = v &- \tfrac{1}{6}u^2v - \tfrac{1}{2}u^2\eta_1 \\ &+ \tfrac{1}{24}u^4\eta_1 - \tfrac{1}{3}u^2v^2\eta_1 - \tfrac{1}{2}u^2v\eta_1^2 \\ &- \tfrac{1}{6}u^2\eta_1^3 - \tfrac{1}{15}u^2v^3 + \tfrac{1}{120}u^4v + Gl_7; \end{aligned}\right\} \qquad (4)$$

$$\Delta\eta = \eta_2 - \eta_1.$$

We are able to calculate $\Delta\xi$ from the spherical triangle P_1P_2 pole, applying the Sine Law:

$$\sin\Delta\xi = \sin\sigma\cos\mathfrak{a}_{1,2}\sec\eta_2. \qquad (5)$$

Substituting the series expansion for $\sin\sigma$ and $\sec\eta_2$ we have:

$$\sin\Delta\xi = \left(\sigma - \tfrac{1}{6}\sigma^3 + \tfrac{1}{120}\sigma^5\right)\left(1 + \tfrac{1}{2}\eta_2^2 + \tfrac{5}{24}\eta_2^4\right)\cos\mathfrak{a} + Gl_7,$$

or

$$\begin{aligned} \sin\Delta\xi = u &- \tfrac{1}{6}u^3 - \tfrac{1}{6}uv^2 + \tfrac{1}{2}u\eta_2^2 \\ &+ \tfrac{1}{120}u^5 + \tfrac{1}{60}u^3v^2 + \tfrac{1}{120}uv^4 - \tfrac{1}{12}u^3\eta_2^2 \\ &- \tfrac{1}{12}uv^2\eta_2^2 + \tfrac{5}{24}u\eta_2^4 + Gl_7. \end{aligned}$$

Introducing this expression into the arc - sin series (page 29) for $\sin\Delta\xi$ it follows further that:

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned}\Delta\xi = u &- \frac{1}{6}uv^2 + \frac{1}{2}u\eta_2^2 \\ &- \frac{1}{15}u^3v^2 + \frac{1}{120}uv^4 + \frac{1}{6}u^3\eta_2^2 \\ &- \frac{1}{12}uv^2\eta_2^2 + \frac{5}{24}u\eta_2^4 + Gl_7;\end{aligned}\right\} \qquad (6)$$
$$\Delta\xi = \xi_2 - \xi_1.$$

Using equation (4) we can eliminate η_2 from the above equating $\eta_2 = \eta_1 + \Delta\eta$ and this will give:

$$\left.\begin{aligned}\Delta\xi = u &+ \frac{1}{3}uv^2 + \frac{1}{2}u\eta_1^2 + uv\eta_1 \\ &- \frac{1}{15}u^3v^2 + \frac{2}{15}uv^4 - \frac{1}{3}u^3v\eta_1 \\ &+ \frac{2}{3}uv^3\eta_1 - \frac{1}{3}u^3\eta_1^2 + \frac{7}{6}uv^2\eta_1^2 + \frac{5}{6}uv\eta_1^3 \\ &+ \frac{5}{24}u\eta_1^4 + Gl_7;\end{aligned}\right\} \qquad (7)$$

However, it is quite obvious that this equation is less convenient than (6).

If we substitute now in equations (4) and (6) for ξ and η, $x:\varrho$ and $y:\varrho$ respectively, and change the meaning of u and v as shown further below, we obtain:

$$\left.\begin{aligned}y_2 - y_1 &= v - \left(y_1 + \frac{1}{3}v\right)\frac{u^2}{2\varrho^2} \\ &\quad + \left(y_1 + \frac{1}{5}r\right)\left(\frac{u^4}{24\varrho^4} - \frac{u^2r^2}{3\varrho^4}\right) - (y_1 + 3r)\frac{u^2\eta_1^2}{6\varrho^4} + \varrho\, Gl_7 \\ x_2 - x_1 &= u\left\{1 + \left(\frac{y_2^2}{2\varrho^2} - \frac{v^2}{6\varrho^2}\right)\right. \\ &\quad \left. + \left(\frac{y_2^2}{2\varrho^2} - \frac{r^2}{5\varrho^2}\right)\frac{u^2}{3\varrho^2} + \frac{5}{6}\left(\frac{y_2^2}{2\varrho^2} - \frac{r^2}{10\varrho^2}\right)^2\right\} + \varrho\, Gl_7;\end{aligned}\right\} \qquad (8)$$
$$s\sin\alpha_{1,2} = v \quad s\cos\alpha_{1,2} = u.$$

Among the preceding equations the terms up to the third-

order magnitude (inclusive) should satisfy practical use. However, the higher terms offer the possibility to estimate the error of the approximation. This estimate will be dealt with in Par. 4.

It should be emphasized, that the preceding expansions hold true generally and not only for the special form of Fig. 6 because the derivation of expressions for angles in a triangle for equation (1) and (5) corresponds to the systematic derivation according to (2) page 71.

Par. 3. Difference of Direction Angles. If we wish to continue the calculation for other lines originating in P_2, the coordinate differences and the direction angles in the starting and terminal point of line P_1P_2 are of importance if we go from point P_1 to P_2. It follows from the spherical triangle in Fig. 6 and Napier's First Analogy (5) page 77.

$$\tan\frac{(\alpha_{2.1}-90^0)+(90^0-\alpha_{1.2})}{2}=\frac{\cos\frac{\left(\frac{\pi}{2}-\eta_1\right)-\left(\frac{\pi}{2}-\eta_2\right)}{2}}{\cos\frac{\left(\frac{\pi}{2}-\eta_1\right)+\left(\frac{\pi}{2}-\eta_2\right)}{2}}\cot\frac{\Delta\xi}{2}$$

This equation holds generally for any position of P_1P_2 (as the equations of the preceding paragraph).

Considering that when, $\rho=\infty$, the azimuth $\alpha_{2.1}=\alpha_{1.2}+180^0$ and that this equation has to be valid approximately for small distances and a finite ρ:

$$\alpha_{2.1}=\alpha_{1.2}+180^0+\Delta\alpha \qquad (1)$$

We obtain after simple reduction, taking reciprocal values on the

left and right hand side:

$$\left.\begin{aligned} \tan\frac{\Delta\alpha}{2} &= -\tan\frac{\Delta\xi}{2}\,\frac{\sin\eta}{\cos\frac{\Delta\eta}{2}}, \\ \eta &= \tfrac{1}{2}(\eta_1+\eta_2). \end{aligned}\right\} \qquad (2)$$

We introduce into (2) the following series expansions:

$$\begin{aligned} \sin\eta &= \eta\left(1-\tfrac{1}{6}\eta^2+Gl_4\right) \\ \sec\frac{\Delta\eta}{2} &= 1+\frac{v^2}{8\varrho^2}+Gl_4 \\ \tan\frac{\Delta\xi}{2} &= \frac{1}{2}\Delta\xi\left(1+\frac{u^2}{12\varrho^2}+Gl_4\right) \end{aligned}$$

and obtain the following transformation of (2)

$$\tan\frac{\Delta\alpha}{2} = -\frac{\Delta x}{2\varrho}\,\frac{y}{\varrho}\left(1-\frac{y^2}{6\varrho^2}+\frac{u^2}{12\varrho^2}+\frac{v^2}{8\varrho^2}+Gl_4\right) \qquad (3)$$

and

$$\left.\begin{aligned} \underset{\text{in seconds}}{\Delta\alpha} &= -\varrho''\,\frac{\Delta x}{\varrho}\,\frac{y}{\varrho}\left(1-\tfrac{y^2}{6\varrho^2}+\tfrac{u^2}{12\varrho^2}+\tfrac{v^2}{8\varrho^2}+Gl_4\right) \\ \Delta x &= x_2-x_1 \qquad y=\tfrac{1}{2}(y_1+y_2) \\ \alpha_{2.1} &= \alpha_{1.2}+180^0+\Delta\alpha\,. \end{aligned}\right\} \qquad (4)$$

The above development as well as the one for (8) in the preceding paragraph for small values of σ and η in the first-order magnitude is entirely permissible.

If we eliminate Δx and y_2 (as well if it appears in y) using (4) and (7) of the preceding paragraph we obtain finally:

(See Illustrative Material on Next Page.)

Par. 4. Numerical Value of the Higher Terms.

$$\underset{\text{in seconds}}{\Delta\alpha} = -\varrho'' \frac{u}{\varrho}\left\{\left(\frac{y_1}{\varrho}+\frac{v}{2\varrho}\right) + \left(\frac{y_1}{\varrho}+\frac{v}{4\varrho}\right)\frac{5v^2-u^2}{6\varrho^2} + \left(\frac{y_1}{3\varrho}+\frac{r}{\varrho}\right)\frac{y_1^2}{\varrho^2} + Gl_5\right\} \quad \cdot (5)$$

Par. 4. Numerical Value of the Higher Terms. If we disregard the curvature of the sphere we have:

$$(x_2 - x_1) = u; \quad (y_2 - y_1) = v; \quad \Delta\alpha = \text{zero}.$$

This relation may be used for very small areas. In order to find out to what extent the higher terms may be neglected we estimate the following. The largest neglected term in $(y_2 - y_1)$ is according to (8) p. 118: $\frac{u^2}{2\varrho^2}\left(y_1 + \frac{1}{3}v\right)$, in $x_2 - x_1$, it is $u\left(\frac{y_2^2}{2\varrho^2} - \frac{v^2}{6\varrho^2}\right)$. If we now say that the maximum value of triangle sides and ordinates

equals s, we have both these terms smaller than $\frac{1}{2}\frac{s^3}{\varrho^2}$. If we retain this as limiting value and take the latter to be less than one millimeter, the following results as a condition:

$$\frac{1}{2}\frac{s^3}{\varrho^2} < 0{,}001^m$$

and for $\varrho = 6370000$ and consequently $\frac{s}{\varrho} < \frac{1}{1500}$ approximately or

$$s < 4^{km}. \quad (1)$$

For this value of $(x_2 - x_1)$ and y the term of the second-order magnitude in $\Delta\alpha$ becomes $0{,}1''$ according to equation (4) p. 119. This may be neglected considering the short distance. This leads to the statement:

It is feasible to compute the rectangular coordinates on the

sphere as plane coordinates with millimeter accuracy, as long as the largest lengths of the sides and ordinates do not exceed 4 km significantly.

With respect to the terms of the fifth-order in $(x_2 - x_1)$ and $(y_2 - y_1)$ we find according to (8) p. 118 that the products $u^2v^2 = \frac{1}{4} s^4 \sin 2\alpha_{1.2}$ and $u^2v = s^3 \cos^2 \alpha_{1.2} \sin \alpha_{1.2}$ as shown by the differentiation with respect to $\alpha_{1.2}$, may reach the maximum values $0{,}25s^4$ and $0{,}38s^3$. If we assume the maximum value of the ordinates as s, the same as for the sides, and neglect in $(y_2 - y_1)$ the term with the factor $\frac{1}{24}$, in $(x_2 - x_1)$ the negative terms; then it follows that the maximum value of the fifth-order terms is $0{,}4s^5 : \rho^4$, approximately. If we assume:

$$0{,}4 \frac{s^5}{\rho^4} < 0{,}001^m,$$

then for $\rho = 6370000$ consequently $\frac{s}{\rho} < \frac{1}{75}$ rounded off:

$$s < 100^{km}. \qquad (2)$$

The terms of the fourth-order of $\Delta\alpha$ become $< 0{,}002''$, according to (4) p. 119. For this value this is entirely sufficient and corresponds to the accuracy of $(y_2 - y_1)$ and $(x_2 - x_1)$, since 0,002" will result in a displacement of only 0,001 m in 100 km.

The terms of the third-order in $x_2 - x_1$, $y_2 - y_1$, and the terms of the second order in $\Delta\alpha$ afford an accuracy of 0,001 m as long as the sides of the triangle and the ordinates remain < 100 km.

Looking at table (5) p. 95 we find the following: If a spherical triangle is calculated as a plane triangle with the same sides, the terms of the second- and fourth-order (first and second term)

in the angle reduction up to mean side lengths of the same value may be neglected as stated by (1) and (2). We can conclude as follows: The inclusion of the terms up to the fifth order, inclusive, of the equations for $(x_2 - x_1), (y_2 - y_1)$ and $\Delta\alpha$ will suffice to the same extent as the inclusion of the sixth order, inclusive, (third term) in the angle reductions,(*) i.e. up to dimensions of approximately 600 km. This will become even clearer in the following paragraph.

Par. 5. An Alternate Method of Development. In his work "The Geodetic Principal Points" Zachariae develops the equations for rectangular spherical coordinates. He uses the Legendre law constructing an auxiliary triangle. P_1 and P_2 are two vertices of this auxiliary triangle and the third vertex is given by a point with the coordinates x_2 and y_1. The three sides consequently are $\sigma, \eta_2 - \eta_1$, and one great-circle arc through P_1 and the point (x_2, y_1). The equations follow as:

$$x_2 - x_1 = s\cos\left(\alpha_{1.2} - 2(E+\varepsilon)\right)\cdot\left(1 + \frac{y_1^2}{2\varrho^2}\right) + \varrho Gl_5$$

$$y_2 - y_1 = s\sin\left(\alpha_{1.2} - (E+\varepsilon)\right) + \varrho Gl_5$$

$$E = \underset{\text{in seconds}}{\frac{1}{2}\varrho''\frac{uy_1}{\varrho^2}}$$

$$\varepsilon = \underset{\text{in seconds}}{\frac{1}{6}\varrho''\frac{uv}{\varrho^2}}$$

$$\Delta\alpha = -(2E + 3\varepsilon),$$

(*)Translator's Note: Cf. Table 5, p. 95. -L.D.

which agrees with these derivations. The significance of $2E$ and 3ε as spherical excess of a quadrangle and a triangle becomes quite clear.

Par. 6. Distance and Direction Angle from the Coordinates. By means of equations (8), p. 118, the following is equated in successive approximation:

$$x_2 - x_1 = \Delta x,\ y_2 - y_1 = \Delta y$$

$$\left.\begin{aligned} u &= \Delta x + \varrho Gl_3 \\ v &= \Delta y + \varrho Gl_3 \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} u &= \Delta x\left(1 - \frac{y_1^2}{2\varrho^2} - \frac{y_1\,\Delta y}{\varrho^2} - \frac{\Delta y^2}{3\varrho^2}\right) + \varrho Gl_5 \\ v &= \Delta y + \left(y_1 + \frac{\Delta y}{3}\right)\frac{\Delta x^2}{2\varrho^2} + \varrho Gl_5. \end{aligned}\right\} \qquad (2)$$

Dividing both sides of the second equation (8) p. 118 by the factor of u :

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we have:

$$u = \Delta x\left\{\begin{aligned} &1 - \left(\frac{y_2^2}{2\varrho^2} - \frac{v^2}{6\varrho^2}\right) \\ &+ \left(\frac{y_2^4}{24\varrho^4} - \frac{y_2^2 v^2}{12\varrho^4} + \frac{7v^4}{360\varrho^4}\right) - \left(\frac{y_2^2}{2\varrho^2} - \frac{v^2}{5\varrho^2}\right)\frac{\Delta x^2}{3\varrho^2} \end{aligned}\right\} + \varrho Gl_7.$$

If v is substituted in the terms of third- and fifth-order by the expression (2) or (1) respectively we obtain:

$$\left.\begin{aligned} u = \Delta x &\left\{\begin{aligned} &1 - \left(\frac{y_2^2}{2\varrho^2} - \frac{\Delta y^2}{6\varrho^2}\right) \\ &+ \left(\frac{y_2^4}{24\varrho^4} - \frac{y_2^2\,\Delta y^2}{12\varrho^4} + \frac{7\,\Delta y^4}{360\varrho^4}\right) - \left(\frac{y_1 y_2}{2\varrho^2} + \frac{2\,\Delta y^2}{15\varrho^2}\right)\frac{\Delta x^2}{3\varrho^2} \end{aligned}\right\} + \varrho Gl_7; \\ u &= s\cos\alpha_{1,2}. \end{aligned}\right\} \qquad (3)$$

Par. 6. Distance and Direction Angle from the Coordinates.

If we introduce (2) for u and v into the terms of the third-order of the first equation (8) on p. 118, and introduce (1) into the terms of the fifth-order we have after a simple reduction:

$$\left.\begin{aligned} v = {} & \Delta y + \left(y_1 + \frac{1}{3}\Delta y\right)\frac{\Delta x^2}{2\varrho^2} \\ & - \left\{\left(y_1 + 2\Delta y\right)\frac{y_1^2}{\varrho^2} + \left(y_1 + \frac{2}{15}\Delta y\right)\frac{\Delta y^2}{\varrho^2}\right\}\frac{\Delta x^2}{3\varrho^2} + \left(y_1 + \frac{7}{15}\Delta y\right)\frac{\Delta x^4}{24\varrho^4} + \varrho\, Gl_7; \\ & v = s \sin \alpha_{1,2} \\ & \Delta y = y_2 - y_1 \qquad \Delta x = x_2 - x_1. \end{aligned}\right\} \quad (4)$$

These and the preceding equation offer a means to calculate s and $\alpha_{1,2}$: and formula (4) p. 119 yields $\alpha_{2,1}$, however, this may be determined as well after exchange of both points using the preceding two equations.

The formulas for u and v make it possible to derive equations for s^2 and $\tan \alpha_{1,2}$. However, only the first one can be used conveniently. If we substitute the expressions for u and v in the equation $s^2 = u^2 + v^2$, introducing for further abbreviation $y = \frac{1}{2}(y_1 + y_2)$ and $\Delta y = y_2 - y_1$, instead of y_1 and y_2 we obtain:

$$\left.\begin{aligned} s^2 = {} & \Delta x^2 + \Delta y^2 - \left(\frac{y^2}{\varrho^2} + \frac{\Delta y^2}{12\varrho^2}\right)\Delta x^2 \\ & + \left(\frac{y^4}{3\varrho^4} - \frac{y^2\Delta y^2}{6\varrho^4} - \frac{\Delta y^4}{720\varrho^4}\right)\Delta x^2 - \left(\frac{y^2}{12\varrho^2} + \frac{\Delta y^2}{720\varrho^2}\right)\frac{\Delta x^4}{\varrho^2} + \varrho^2 Gl_8. \\ & y = \frac{1}{2}(y_1 + y_2). \end{aligned}\right\} \quad (5)$$

We could probably effect further simplifications of (3) and (4) by beginning with Gauss'

equations and calculating $s\cos\left(\mathfrak{A}_{1.2}+\frac{1}{2}\Delta\mathfrak{A}\right)$ and $s\sin\left(\mathfrak{A}_{1.2}+\frac{1}{2}\Delta\mathfrak{A}\right)$, which are symmetric magnitudes to both points instead of $\cos\mathfrak{A}_{1.2}$ and $\sin\mathfrak{A}_{1.2}$, which are asymmetric magnitudes to both points. But we prefer the above equations because they permit a control in the above-mentioned sense; and we limit ourselves to produce the expressions accurately only to the fifth-order using the equations already derived.

We have in simple development:

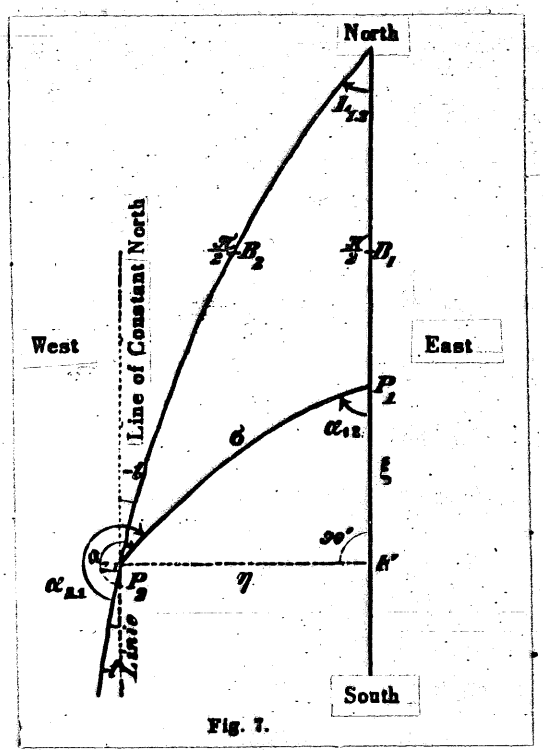
$$\begin{aligned} s\cos\left(\mathfrak{A}_{1.2}+\frac{\Delta\mathfrak{A}}{2}\right) &= u - v\frac{\Delta\mathfrak{A}}{2} + \varrho Gl_5 \\ s\sin\left(\mathfrak{A}_{1.2}+\frac{\Delta\mathfrak{A}}{2}\right) &= v + u\frac{\Delta\mathfrak{A}}{2} + \varrho Gl_5. \end{aligned}$$

Substituting for the expressions of u, v and as well as for $\Delta\mathfrak{A}$, we obtain:

$$\left.\begin{aligned} s\cos\left(\mathfrak{A}_{1.2}+\frac{\Delta\mathfrak{A}}{2}\right) &= \Delta x\left(1-\frac{y^2}{2\varrho^2}+\frac{\Delta y^2}{24\varrho^2}\right)+\varrho Gl_5 \\ s\sin\left(\mathfrak{A}_{1.2}+\frac{\Delta\mathfrak{A}}{2}\right) &= \Delta y\left(1-\frac{\Delta x^2}{12\varrho^2}\right)+\varrho Gl_5 \\ \underset{\text{in seconds}}{\Delta\mathfrak{A}} &= -\varrho''\frac{\Delta x}{\varrho}\frac{y}{\varrho}+Gl_4. \end{aligned}\right\} \qquad (6)$$

Par. 7. Transfer of Geographical Longitude and Latitude Using Horizontal Distance and Azimuth. According to p. 7 we assume a larger circle and its two poles on the sphere surface which correspond to the equator, North and South poles respectively, if the sphere surface represents the mathematical Earth surface. Further we choose a meridian as principal meridian and introduce the coordinate system of geographical latitudes and longitudes.

Par. 7. Transfer of Geographical Longitude and Latitude Using Horizontal Distance and Azimuth.



We take the meridian of a starting point P_1, which is fixed by its geographical latitude, for the moment as the first meridian. From it we count differences in longitude in the same direction of rotation as azimuths from its part, located south of P_1. The difference in longitude $L_{1.2}$ is counted east-west in Fig. 7, and the azimuth $\alpha_{1.2}$ south-west. The radius of the sphere is taken as the unit of length.

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The triangle between the North Pole N, P_1 and P_2 may be solved using Gauss' equations. We will not deal with this any further at the moment, but concern ourselves with a different technique by Gauss, which is convenient for small values of σ :

The larger circle P_2F is placed perpendicular to the first meridian and the triangle P_1P_2F is solved. If the corners F, P_1, P_2, are taken as A, B, C, and we let consequently:

$$\left.\begin{aligned} A &= 90^\circ & \alpha &= \sigma \\ B &= \alpha_{1.2} & \beta &= \eta \\ C &= 270^\circ - \alpha_{2.1} = 90^\circ - \alpha_{1.2} - \Delta\alpha & \gamma &= \xi \end{aligned}\right\} \qquad (1)$$

The equations (1) p. 76 present the following method of calculating η, ξ and $\Delta\alpha$, . Since $\alpha_{1.2}$ is identical with the direction angle $\alpha_{1.2}$, $\Delta\alpha$ has the same meaning here as in the rectangular coordinates.

$$\left.\begin{aligned} \sin\eta &= \sin\sigma \sin\alpha_{1.2} \\ \cos\eta \sin\xi &= \sin\sigma \cos\alpha_{1.2} \\ \cos\eta \cos\xi &= \cos\sigma \end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned} \sin\sigma \sin(\alpha_{1.2} + \Delta\alpha) &= \sin\eta \cos\xi \\ \sin\sigma \cos(\alpha_{1.2} + \Delta\alpha) &= \sin\xi. \end{aligned}\right\} \qquad (3)$$

These equations hold true for every position of P_2 with respect to P_1. See the end of Par. 2 p. 118. Since $|\eta| < \frac{\pi}{2}$ the solution is determined as well.

(2) may be used conveniently to find η and ξ. However, since σ is assumed to have a small value, the equations (3) which will yield $\Delta\alpha$ should be transformed by substituting (3) and the first and second expression of (2) in the right hand side of:

$\Delta\alpha = \sin(\alpha_{1.2} + \Delta\alpha)\cos\alpha_{1.2} - \cos(\alpha_{1.2} + \Delta\alpha)\sin\alpha_{1.2}$. This yields

$$\sin\Delta\alpha = \sin\xi \sin\eta \frac{\cos\xi\cos\eta - 1}{\sin^2\sigma} = \sin\xi \sin\eta \frac{\cos\sigma - 1}{\sin^2\sigma},$$

From this it follows that, if we substitute for $\sin\eta$ its value in (2)

$$\sin\Delta\alpha = -\sin\xi \tan\frac{\sigma}{2} \sin\alpha_{1.2}. \qquad (4)$$

For large values of σ this equation is not as good as (3). However, it permits a more accurate calculation for small values of $\Delta\alpha$ which correspond to small values of σ.

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That this provision for σ makes $\Delta\alpha$ a small quantity as well and not almost 180^0 (as equation (4) alone would permit) is shown by (3), which are reduced by (2) and become:

$$\left.\begin{aligned}\sin(\alpha_{1.2}+\Delta\alpha) &= \sin\alpha_{1.2}\cos\xi\\ \cos(\alpha_{1.2}+\Delta\alpha) &= \cos\alpha_{1.2}\sec\eta.\end{aligned}\right\} \qquad (5)$$

If σ is small, then ξ and η are small positive or negative values; $\cos\xi$ and $\sec\eta$ are, therefore, nearly $+1$, and consequently $\alpha_{1.2}+\Delta\alpha$ is located in the same quadrant with $\alpha_{1.2}$ and differs little from it.

The large right triangle between North Pole N, F and P_2 yields, according to the formula system (1) p. 76, if we assume that:

$$\left.\begin{aligned}A &= 90^0 & \alpha &= \tfrac{\pi}{2}-B_2\\ B &= L_{1.2} & \beta &= \eta\\ C &= 90^0+t & \gamma &= \tfrac{\pi}{2}-F\\ t &= \alpha_{2.1}-\alpha_{2.1} & F &= B_1-\xi,\end{aligned}\right\} \qquad (6)$$

the following equations for the determination of B_2, $L_{1.2}$, and t (t, is called meridian convergence):

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned} \sin B_2 &= \cos\eta \sin F \\ \cos B_2 \sin L_{1.2} &= \sin\eta \\ \cos B_2 \cos L_{1.2} &= \cos\eta \cos F \\ \cos B_2 \sin t &= -\sin\eta \sin F \\ \cos B_2 \cos t &= \cos F. \end{aligned}\right\} \qquad (7)$$

These equations as well as (2) and (3) hold generally, and particularly for negative values of η.

The second and third of the equations in (7) yield $\tan L_{1.2}$, the fourth and fifth, $\tan t$, and since $\cos B_2$ is positive, $L_{1.2}$ is fixed completely with respect to its quadrant.

As far as B_2 is concerned, $F - B_2$ is small for small values of σ and, therefore, it is recommended to derive the difference directly. To do this we eliminate the latitude B_2 from the identity $\sin(F - B_2) = \sin F \cos B_2 - \cos F \sin B_2$ using the first and third equation (7). We obtain: (dropping the subscript of L)

$$\sin(F - B_2) = \sin F \cos F \cos\eta \frac{1 - \cos L}{\cos L}.$$

If the right hand side of this is multiplied by $\cot L \tan\eta \sec F$, which is about 1, we have:

$$\sin(F - B_2) = \sin\eta \tan\frac{L_{1.2}}{2} \sin F. \qquad (8)$$

From this $F - B_2$ is available unambiguously since it has to be a small quantity if dealing with moderate distances.

Combining everything and using a few simple transformations

we finally arrive at the following system of equations for the solution of the problem:

$$\left.\begin{aligned}
\sin\eta &= \sin\sigma \sin\alpha_{1.2}\\
\tan\xi &= \tan\sigma\cos\alpha_{1.2}\\
F &= B_1 - \xi\\
\tan L_{1.2} &= \tan\eta\sec F = \sin\xi\tan\alpha_{1.2}\sec F\\
\tan t &= -\sin\eta\tan F\\
\sin(F-B_2) &= \sin\eta\tan\frac{L_{1.2}}{2}\sin F = -\tan t\tan\frac{L_{1.2}}{2}\cos F\\
\sin\Delta\mathfrak{A} &= -\tan\eta\tan\frac{\sigma}{2}\cos\alpha_{1.2} = -\sin\xi\tan\frac{\sigma}{2}\sin\alpha_{1.2}\\
\alpha_{2.1} &= \alpha_{1.2} + 180^0 + \Delta\mathfrak{A} + t.
\end{aligned}\right\} \quad (9)$$

The second expressions on the right hand side are somewhat more convenient than the first ones; because they do not contain $\tan\eta$, and $\sin F$ but require only $\sin\xi$ which is easy to look up since ξ has to be looked up anyway. The calculation of both systems of expressions would offer some control which is not complete, however, and can be obtained by other means as will be shown later.

In the solution of (9) the S and T of seven-place logarithmic tables will serve very well, as long as these tables guarantee sufficient accuracy. In other cases mostly series expansion will be found advantageous, in order to avoid looking up trigonometric functions of small angles.

Par. 8. Series Expansions for the Preceding Problem. If we assume η, ξ and F known for the time being, formula 4 of (9) yields at once:

$$\tan L_{1.2} = \eta\left(1 + \frac{1}{3}\eta^2 + \frac{2}{15}\eta^4 + Gl_6\right)\sec F. \quad (1)$$

If we develop $L_{1.2}$ from this using arc tangent series (2) p. 29 it follows:

$$L_{1.2} = \varrho'' \eta \sec F \left\{ 1 - \frac{1}{3}\eta^2 \tan^2 F + \frac{1}{15}\eta^4 \tan^2 F (3\tan^2 F + 1) + Gl_6 \right\} \cdot \quad (2)$$

in seconds

If we introduce on the other hand $\tan L_{1.2}$ into the second expression of (4) p. 30 and substitute $L_{1.2}$ for u, we obtain:

$$\log L_{1.2} = \log (\varrho'' \eta \sec F) - \frac{1}{3} M \eta^2 \tan^2 F + \frac{1}{90} M \eta^4 \tan^2 F (13 \tan^2 F + 6) + Gl_6 . \quad (3)$$

in seconds

As far as the applicability of this development is concerned, it is assured for (1) as long as η remains a quantity of first-order magnitude only. On the other hand, since (2) and (3) are derived from $L_{1.2}$ it is necessary that $L_{1.2}$ is a quantity of first-order magnitude. But the absolute value of $L_{1.2}$ is less than $|\eta| \sec F$. The condition for the applicability of (2) and (3) is, therefore: $\eta \sec F$ must be a small quantity.*)

Introducing a series expansion on the right hand side of expression 6 equation (9) (dropping the subscript of L for the time being) we have:

$$\sin (F - B_2) = \frac{1}{2}\eta L \sin F \left(1 - \frac{1}{6}\eta^2 + \frac{1}{120}\eta^4 + Gl_6\right)\left(1 + \frac{1}{12}L^2 + \frac{1}{120}L^4 + Gl_6\right).$$

*)These developments hold true only for differences in latitude up to $45°$, even with indefinite expansion.

Par. 8. Series Expansions for the Preceding Problem.

Applying (2) it follows:

$$\sin(F - B_2) = \frac{1}{2}\eta^2 \tan F \left\{ \begin{array}{l} 1 - \frac{1}{12}\eta^2 [3\tan^2 F + 1] \\ + \frac{1}{360}\eta^4 [45\tan^4 F + 15\tan^2 F + 1] + Gl_6 \end{array} \right\}.$$

If we finally introduce this into the arcsin series, first formula, equation (2) p. 29 we get:

$$F - B_2 = \frac{1}{2}\varrho''\eta^2 \tan F \left\{ \begin{array}{l} 1 - \frac{1}{12}\eta^2 [3\tan^2 F + 1] \\ + \frac{1}{360}\eta^4 [45\tan^4 F + 30\tan^2 F + 1] + Gl_6 \end{array} \right\}. \quad (4)$$

in seconds

On the other hand, the introduction into the first series, equation (4) p. 30 yields:

$$\log(F - B_2) = \log\left(\frac{1}{2}\varrho''\eta^2 \tan F\right) - \frac{1}{12}M\eta^2 [3\tan^2 F + 1] + \frac{1}{1440}M\eta^4 [135\tan^4 F + 90\tan^2 F - 1] + Gl_6. \quad (5)$$

in seconds

If we expand $\sin\eta$ into a series and apply the second series (2) p. 29 to the transition from $\tan t$ to t, we have: from the fifth expression of (5)

$$t = -\varrho''\eta \tan F \left\{ \begin{array}{l} 1 - \frac{1}{6}\eta^2 [2\tan^2 F + 1] \\ + \frac{1}{120}\eta^4 [24\tan^4 F + 20\tan^2 F + 1] + Gl_6 \end{array} \right\}. \quad (6)$$

in seconds

On the other hand the application of the second series (4) p. 30 yields:

$$\log t = \log(-\varrho''\eta\tan F) - \frac{1}{6}M\eta^2[2\tan^2 F + 1] + \frac{1}{180}M\eta^4[26\tan^4 F + 20\tan^2 F - 1] + Gl_6. \quad (7)$$

in seconds

The expansions (4) to (7) as well afford sufficient convergence only as long as $\eta \sec F$ remains a small quantity.

Par. 9. Continuation. It is sufficient to equate $\eta_1 = 0$ in the expressions (5) p. 119 and (4) and (7) p. 117, in order to derive series for η, ξ, and $\Delta\alpha$. We obtain:

$$\left.\begin{aligned} u &= s\cos\alpha_{1.2} \\ v &= s\sin\alpha_{1.2} \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} \eta &= \frac{v}{\varrho}\left(1 - \frac{u^2}{6\varrho^2} - \frac{u^2v^2}{15\varrho^4} + \frac{u^4}{120\varrho^4} + Gl_6\right) \\ \xi &= \varrho''\frac{u}{\varrho}\left(1 + \frac{v^2}{3\varrho^2} - \frac{u^2v^2}{15\varrho^4} + \frac{2v^4}{15\varrho^4} + Gl_6\right) \\ \Delta\alpha &= -\varrho''\frac{uv}{2\varrho^2}\left(1 + \frac{5v^2 - u^2}{12\varrho^2} + Gl_4\right). \end{aligned}\right\} \quad (2)$$

in seconds

in seconds

However it is more convenient now for the use of η and ξ to develop the logarithmic form. The first series (1) Par. 5 p. 27 serves this purpose. We obtain without difficulty the following as well as (2), which may be used for small values of $s:\varrho$:

$$\left.\begin{aligned} \log\eta &= \log\frac{v}{\varrho} - M\left(\frac{u^2}{6\varrho^2} + \frac{u^2v^2}{15\varrho^4} + \frac{u^4}{180\varrho^4}\right) + Gl_6 \\ \log\xi &= \log\varrho''\frac{u}{\varrho} + M\left(\frac{v^2}{3\varrho^2} - \frac{u^2v^2}{15\varrho^4} + \frac{7v^4}{360\varrho^4}\right) + Gl_6 \\ \log\Delta\alpha &= \log\left(-\varrho''\frac{uv}{2\varrho^2}\right) + M\frac{5v^2 - u^2}{12\varrho^2} + Gl_6 \end{aligned}\right\} \quad (3)$$

in seconds

in seconds

$$\alpha_{2.1} = \alpha_{1.2} + 180^0 + \Delta\alpha + \ell.$$

In place of the expressions (3) we could use as well the solution of the right triangle with the sides s, η, and

ξ according to Legendre's Law (taking into consideration the higher terms). In order to compute as accurately as (3) permits, two approximation calculations have to be made.

Par. 10. Given Geographic Latitude and Longitude for Two Points. In order to derive distance and azimuth in this case, we can use again the system of rectangular coordinates as an intermediary link. Considering Fig. 7 p. 123 and the equations (7) and (9) p. 125 and 126 we obtain:

$$\left.\begin{aligned} \sin\eta &= \cos B_2 \sin L_{1.2} \\ \sin(F - B_2) &= \tan\eta \tan\frac{L_{1.2}}{2} \sin B_2 = \sin^2\frac{L_{1.2}}{2} \sec\eta \sin 2B_2 \\ \tan t &= -\sin\eta \sec L_{1.2} \tan B_2 = -\tan L_{1.2} \sin B_2. \end{aligned}\right\} \quad (1)$$

When η, F and t have been found, (2) and (4) on p. 124 yield:

$$\left.\begin{aligned} \sin\sigma \sin\alpha_{1.2} &= \sin\eta \\ \sin\sigma \cos\alpha_{1.2} &= \cos\eta \sin\xi \\ \xi &= B_1 - F \\ \sin\Delta\mathfrak{a} &= -\sin\xi \tan\frac{\sigma}{2} \sin\alpha_{1.2} \\ \alpha_{2.1} &= \alpha_{1.2} + 180^0 + \Delta\mathfrak{a} + t. \end{aligned}\right\} \quad (2)$$

A requirement for the application of these equations (as Par. 7 p. 124) is that σ must be small. They still hold true for arbitrary values of σ but do no longer afford a means to calculate the unknown quantities accurately.

A second solution with interchange of points gives complete control of the calculation.

If an application of the equations seems inconvenient for the

solution, because S and T possibly do not offer enough accuracy, another series expansion may be performed. The equations (1) yield through development of the right hand sides (and by dropping the subscript of $L_{1,2}$):

$$\begin{aligned}
\sin\eta &= L\cos B_2\left(1-\tfrac{1}{6}L^2+\tfrac{1}{120}L^4+Gl_6\right)\\
\sin(F-B_2) &= \tfrac{1}{4}L^2\sin 2B_2\left(1-\tfrac{1}{12}L^2+\tfrac{1}{360}L^4+Gl_6\right)\\
&\quad\times\left(1+\tfrac{1}{2}\eta^2+\tfrac{5}{24}\eta^4+Gl_6\right)\\
\tan t &= -L\sin B_2\left(1+\tfrac{1}{3}L^2+\tfrac{2}{15}L^4+Gl_6\right).
\end{aligned}$$

If this is introduced into the series (2) p. 29 we have further:

$$\left.\begin{aligned}
\eta &= L\cos B_2\left(1-\tfrac{1}{6}L^2\sin^2 B_2-\tfrac{1}{120}L^4\sin^2 B_2[8-9\sin^2 B_2]+Gl_6\right)\\
F-B_2 &= \tfrac{1}{4}\,\varrho''L^2\sin 2B_2\left\{\begin{aligned}&1+\tfrac{1}{12}L^2[5-6\sin^2 B_2]\\&+\tfrac{1}{6}L^4\cos^2 B_2[1-2\sin^2 B_2]+\tfrac{1}{360}L^4+Gl_6\end{aligned}\right\}\\
t &= -\varrho''L\sin B_2\left\{\begin{aligned}&1+\tfrac{1}{3}L^2\cos^2 B_2\\&+\tfrac{1}{15}L^4\cos^2 B_2[2-3\sin^2 B_2]+Gl_6\end{aligned}\right\}
\end{aligned}\right\}(3)$$

$$L=\frac{L_{1,2}\text{ in Sek.}}{\varrho''}.$$

As far as the derivation of the second of these formulas is concerned the following intermediate link is obtained:

$$\begin{aligned}
\sin(F-B_2) &= \tfrac{1}{4}L^2\sin 2B_2\left(1+\tfrac{1}{12}L^2[5-6\sin^2 B_2]\right.\\
&\quad\left.+\tfrac{1}{24}L^4\cos^2 B_2[4-9\sin^2 B_2]+\tfrac{1}{360}L^4+Gl_6\right).
\end{aligned}$$

More convenience for logarithmic calculation is obtained by applying (4) p. 30 to $\sin\eta$, $\sin(F-B_2)$ and $\tan t$:

$$\left.\begin{aligned}
\log \eta &= \log (L \cos B_2) - \frac{1}{6} ML^2 \sin^2 B_2 \\
&\quad - \frac{1}{120} ML^4 \sin^2 B_2 [12 - 11 \sin^2 B_2] + Gl_6 \\
\log \underset{\text{in seconds}}{(F - B_2)} &= \log \left(\frac{1}{4} \varrho'' L^2 \sin 2B_2\right) + \frac{1}{12} ML^2 [5 - 6 \sin^2 B_2] \\
&\quad + \frac{1}{1440} ML^4 [119 - 420 \sin^2 B_2 + 300 \sin^4 B_2] + Gl_6 \\
\log \underset{\text{in seconds}}{t} &= \log (-\varrho'' L \sin B_2) + \frac{1}{3} ML^2 \cos^2 B_2 \\
&\quad + \frac{1}{90} ML^4 \cos^2 B_2 [7 - 13 \sin^2 B_2] + Gl_6 ; \\
L &= \frac{L_{1.2}\ \text{in seconds}}{\varrho''}.
\end{aligned}\right\} \quad (4)$$

The preceding developments (3) and (4) hold absolutely true and suffice as long as L is only a quantity of first-order magnitude, which is quite obvious.

The series corresponding to equations (2) are most conveniently taken from the equations (1) and (2) for v and u on p. 122, by equating $y_1 = 0 = x_1$, and dropping the subscript of y_2 and x_2. It follows and it is quite sufficient, as soon as (3) and (4) are satisfied and $s : \varrho$ is a quantity of first order:

$$\left.\begin{aligned}
s \sin \alpha_{1.2} &= \varrho \eta \left(1 + \frac{1}{6} \xi^2 - \frac{2}{45} \eta^2 \xi^2 + \frac{7}{360} \xi^4 + Gl_6\right) \\
s \cos \alpha_{1.2} &= \varrho \xi \left(1 - \frac{1}{3} \eta^2 - \frac{2}{45} \eta^2 \xi^2 - \frac{1}{45} \eta^4 + Gl_6\right) \\
\xi &= \frac{(B_1 - F)\ \text{in seconds}}{\varrho''} \\
\underset{\text{in seconds}}{\Delta \mathfrak{a}} &= -\frac{1}{2} \varrho'' \xi \eta \left(1 + \frac{1}{12} (\eta^2 + \xi^2) + Gl_4\right) \\
\alpha_{2.1} &= \alpha_{1.2} + 180^0 + \Delta \mathfrak{a} + t,
\end{aligned}\right\} \quad (5)$$

and in logarithmic form we obtain from this:

$$\left.\begin{aligned}
\log(s\sin\alpha_{1.2}) &= \log(\varrho\eta) + \frac{1}{6}M\xi^2 - \frac{2}{45}M\eta^2\xi^2 + \frac{1}{180}M\xi^4 + Gl_6\\
\log(s\cos\alpha_{1.2}) &= \log(\varrho\xi) - \frac{1}{3}M\eta^2 - \frac{2}{45}M\eta^2\xi^2 - \frac{7}{360}M\xi^4 + Gl_6\\
\log\underset{\text{in seconds}}{\Delta\alpha} &= \log\left(-\frac{\varrho''}{2}\varrho''\xi\eta\right) + \frac{1}{12}M(\eta^2+2\xi^2) + Gl_4.
\end{aligned}\right\} \quad (6)$$

Par. 11. Solution of the Preceding Problem Using Gauss' Equations. We again designate the two known points as P_1 and P_2 and apply the equations (10) p. 79, by substituting:

$$\left.\begin{aligned}
L_{1.2} &\text{ for } A & \sigma &\text{ for } \alpha\\
\alpha_{2.1} - 180^0 &\text{ " } B & \frac{\pi}{2} - B_1 &\text{ " } \beta\\
180^0 - \alpha_{1.2} &\text{ " } C & \frac{\pi}{2} - B_2 &\text{ " } \gamma.
\end{aligned}\right\} \quad (1)$$

The following expressions are then found for the determination of $\alpha_{1.2}$, $\alpha_{2.1}$ and σ:

$$\left.\begin{aligned}
\sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\sigma}{2} &= -\sin\frac{B_2-B_1}{2}\cos\frac{L_{1.2}}{2}\\
\cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\sigma}{2} &= -\cos\frac{B_2+B_1}{2}\sin\frac{L_{1.2}}{2}\\
\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\sigma}{2} &= +\cos\frac{B_2-B_1}{2}\cos\frac{L_{1.2}}{2}\\
\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\sigma}{2} &= +\sin\frac{B_2+B_1}{2}\sin\frac{L_{1.2}}{2}.
\end{aligned}\right\} \quad (2)$$

Only the higher symbols are used from the above equations, since the sides of a triangle, except for σ, are always less than π. σ itself may be taken as equal to or less than π; as a rule, however, only the shorter connection (less than π) is of interest, and, consequently, the solution of (2) is well defined.

The application of (2), provided the angles of the triangle are

positive and located between 0° and 360°. If, however $L_{1.2}$ is greater than 180°, the expressions (1) for B and C become at the same time negative in the case of $\sigma < \pi$. But if we assume, that we add 360° to B and C we have again equation (2). Therefore (2) hold true generally for every location of $P_1 P_2$.

If we introduce (2) into the preceding equations for abbreviation we have:

$$\left.\begin{aligned} B_2 - B_1 &= \Delta B \qquad \frac{B_2 + B_1}{2} = B \\ \frac{\alpha_{1.2} + \alpha_{2.1} - 180^\circ}{2} &= \alpha \\ \alpha_{2.1} &= \alpha_{1.2} + 180^\circ + \Delta\alpha, \end{aligned}\right\} \qquad (3)$$

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where $\Delta\alpha$ is nothing else but $\Delta\alpha + t$ using the symbols of the preceding paragraphs and we obtain:

$$\left.\begin{aligned} \cos\alpha \sin\frac{\sigma}{2} &= -\sin\frac{\Delta B}{2}\cos\frac{L_{1.2}}{2} \\ \sin\alpha \sin\frac{\sigma}{2} &= +\cos B \sin\frac{L_{1.2}}{2} \\ \cos\frac{\Delta\alpha}{2}\cos\frac{\sigma}{2} &= +\cos\frac{\Delta B}{2}\cos\frac{L_{1.2}}{2} \\ \sin\frac{\Delta\alpha}{2}\cos\frac{\sigma}{2} &= -\sin B \sin\frac{L_{1.2}}{2}; \end{aligned}\right\} \qquad (4)$$

$$\left.\begin{aligned} \alpha_{1\,2} &= \alpha - \frac{\Delta\alpha}{2} \\ \alpha_{2.1} &= \alpha + \frac{\Delta\alpha}{2} + 180^\circ. \end{aligned}\right\} \qquad (5)$$

These expressions afford a solution as accurate as possible,

a solution which is equally convenient for small mutual distances of the points if the auxiliary terms S and T are applied.

Gauss uses (in: "Investigations of the Subjects of Higher Geodesy" part 1 (1844) p. 33) these equations to solve indirectly the problem of Par. 7 of this chapter on p. 123; very rapidly convergent series expansions are obtained this way. In the following, we will deal only with those series expansions which concern the problem of this paragraph.

Par. 12. Series Development for the Preceding Problem. (4) yields Napier's Analogy:

$$\tan\alpha = -\tan\frac{L_{1.2}}{2}\csc\frac{\Delta B}{2}\cos B.$$

We convert this expression to logarithmic form and substitute for $\log\tan\frac{L_{1.2}}{2}$ and $\log\sin\frac{\Delta B}{2}$. the series (3) p. 29. This yields the expression:

$$\left.\begin{aligned}\log\tan\alpha = \log\left(-\frac{L\cos B}{\Delta B}\right) &+ \frac{1}{24}M[2L^2+\Delta B^2]\\ &+ \frac{1}{2880}M[14L^4+\Delta B^4] + Gl_6\end{aligned}\right\} \quad (1)$$

$$L = \frac{L_{1.2}\text{ in seconds}}{\rho''} \qquad \Delta B = \frac{\Delta B\text{ in seconds}}{\rho''}.$$

$L_{1.2}$ is assumed here to be a small positive or negative quantity. To determine the quadrant for α, it must be $\alpha \lesssim 180^\circ$ for $+L_{1.2}$. a condition which follows from Fig. 7 p. 23, assuming a small value for $\Delta\alpha$.

Further, we derive from (4)

Par 12. Series Development for the Preceding Problem.

$$\tan\frac{\Delta\alpha}{2} = -\tan\frac{L_{1.2}}{2}\sin B\sec\frac{\Delta B}{2}.$$

On the right hand side we substitute for $\tan\frac{L_{1.2}}{2}$ the series expansion, and introduce the expression for $\frac{\Delta\alpha}{2}$, into the second series (2) p. 29:

$$\Delta\alpha = -Lb\left(1+\frac{1-b^2}{12}L^2+\frac{(1-b^2)(2-3b^2)}{240}L^4+Gl_6\right),$$

If we let $b = \underset{\sin}{B}\sec\frac{\Delta B}{2}$ for the time being. Converting to logarithmic form and rearranging, we get:

$$\log\underset{\text{in seconds}}{\Delta\alpha} = \log\left(-\underset{\text{in seconds}}{L_{1.2}}\sin B\sec\frac{\Delta B}{2}\right)$$
$$+\frac{1}{12}ML^2\cos B_1\cos B_2\sec^2\frac{\Delta B}{2}\left(1-\frac{1}{120}L^2(13\sin^2 B-7)\right)+Gl_6; \quad (2)$$

We use here the identity $1-b^2=\cos B_1\cos B_2\sec^2\frac{\Delta B}{2}$, which can be verified easily, and in the last term on the right hand side we equate $b^2=\sin^2 B+Gl_2$.

The developments (1) and (2) are useful, if L and ΔB may be regarded as quantities of first-order magnitude. (The condition that σ must be small, is, by itself, not sufficient either here or in the preceding problems.)

For the calculation of σ and s the first two equations (4) of the preceding paragraph render as a choice the expressions:

$$\sin\frac{\sigma}{2} = \left(-\cos\frac{L_{1.2}}{2}\sec\alpha\right)\sin\frac{\Delta B}{2} = (\cos B\csc\alpha)\sin\frac{L_{1.2}}{2}.$$

Both have the form $\sin\frac{\sigma}{2}=p\sin\frac{q}{2}$. Considering the equation $s=\sigma\varrho$:, the application of the sin series to $\sin\frac{q}{2}$ and the arcsin series (2) p. 29 or the series for its logarithm p. 30 (4)

with respect to $\sin\frac{\sigma}{2}$ yields:

$$\left.\begin{aligned} s &= pq\varrho\left(1-\frac{1-p^2}{24}q^2+\frac{(1-p^2)(1-9p^2)}{1920}q^4+Gl_6\right)\\ \log s &= \log(pq\varrho)-\frac{1}{24}Mq^2[1-p^2]\left(1+\frac{1}{120}q^2[1+11p^2]\right)+Gl_6. \end{aligned}\right\}(3)$$

We equate here:

$$\left.\begin{aligned} p &= -\cos\frac{L_{1.2}}{2}\sec\alpha & q &= \frac{\Delta B\text{ in sec.}}{\varrho''}\\ \text{or}\quad & & & \\ p &= \cos B\csc\alpha. & q &= \frac{L_{1.2}\text{ in sec.}}{\varrho''} \end{aligned}\right\}\qquad(4)$$

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The most accurate calculation is obtained (as in the case of calculating a hypotenuse from the two short sides) using the smallest p-value.

If $\log\sin\frac{\sigma}{2}$ is desired, the substitution (4) must be introduced into the equation:

$$\log\sin\frac{\sigma}{2}=\log\frac{pq}{2}-\frac{1}{24}Mq^2-\frac{1}{2880}Mq^4+Gl_6\qquad(5)$$

The expressions (3) and (5) are useful only under the same conditions as (1) and (2).

Chapter 4.

The Vertical Section and Chord Triangle for the Flattened Ellipsoid of Revolution.

Par. 1. Difference Between Reciprocal Vertical Sections. Turning now more particularly to the geodesy of a reference surface having the form of a flattened ellipsoid of revolution, it must at once be pointed out that we shall disregard the curvature of the plumb lines; that is, in the cases where we have to deal with points not lying on the chosen reference surface, the plumb lines of these points also will be assumed normal to that surface.*)

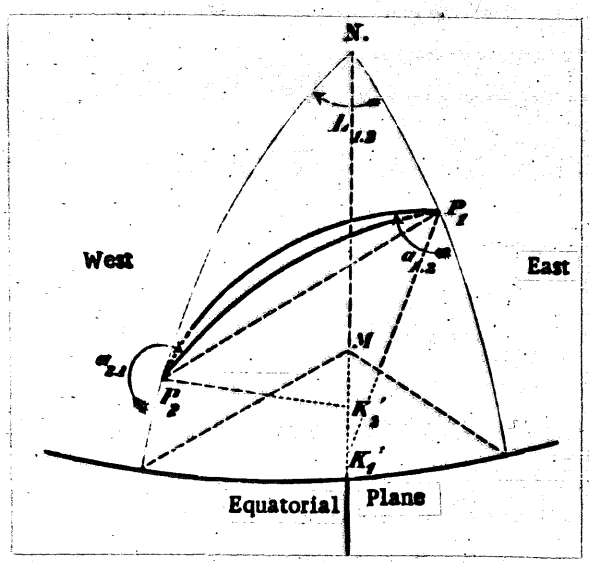


Figure 8

If from a point P_1 we pass a vertical plane, that is, a plane containing the plumb line P_1K_1', through a second point P_2 of

*)This assumption will be adhered to not only in Chapter 4 but in all following chapters dealing with the geodesy of the flattened ellipsoid of revolution.

the surface (Fig. 8), the plumb line from the latter point will not in general lie in the vertical plane just mentioned, since the points of intersection K_1' and K_2' of the plumb lines with the axis of rotation do not in general coincide. In accord

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with p. 41, equation (9),

$$K_1'K_2' = a_0c^2\left(\frac{\sin B_2}{\sqrt{1-e^2\sin^2 B_2}} - \frac{\sin B_1}{\sqrt{1-e^2\sin^2 B_1}}\right) = a_0c^2\,\frac{\sin\beta_2 - \sin\beta_1}{\sqrt{1-e^2}},$$

whence, assuming P_1P_2 to be a short distance, and assuming ΔB to be a first-order magnitude and e^2 to be a second-order magnitude, we have approximately:

$$K_1'K_2' = a_0c^2(\cos B \sin \Delta B + Gl_3) \qquad (1)$$

where

$$\Delta B = B_2 - B_1 \qquad B = \tfrac{1}{2}(B_2 + B_1). \qquad (2)$$

If both points have the same geographical latitude B, the two points K_1' and K_2' will coincide; in this case, P_1 and P_2 will have a common vertical plane. This is also the case if they lie on the same meridian. In all other cases, however, the differing vertical planes of P_1 through P_2 and of P_2 through P_1 generate upon the surface two vertical sections which intersect at small angles at points P_1 and P_2.

Here we have already found an important difference from the case of the sphere; and we immediately observe another one if we

imagine the points P_1 and P_2 moved upward along their plumb lines. In such a case P_2 will leave the vertical plane passed from P_1 through the original position of P_2, and P_1 will leave the vertical plane passed from P_2 through the original position of P_1. In other words:

The height of an object above the chosen reference surface has an effect upon the astronomical azimuths of the vertical planes (p. 7) and thus also upon the horizontal angles.

Par. 2. The equation of the Ellipsoid and of the Vertical Section. In rectangular coordinates, with the center as origin and the axis of rotation as z axis, this equation is:

$$\frac{x^2}{a_0^2} + \frac{y^2}{a_0^2} + \frac{z^2}{a_0^2(1-e^2)} = 1. \qquad (1)$$

Let us now take the plane of the meridian through P_1 as the xz plane, and a plane $W'P$ of longitude westward of this as the yz plane, with positive values of z toward the north. Then, in accordance with the formulae of p. 39 Par. 2, and with reference to Fig. 8, it follows that

$$\left.\begin{aligned} x_1 &= a_0 \cos\beta_1 = \frac{a_0 \cos B_1}{\sqrt{1 - e^2 \sin^2 B_1}} \\ y_1 &= 0 \\ z_1 &= b_0 \sin\beta_1 = \frac{a_0(1-e^2)\sin B_1}{\sqrt{1 - e^2 \sin^2 B_1}}; \end{aligned}\right\} \qquad (2)$$

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned} x_2 &= a_0 \cos\beta_2 \cos L_{1.2} = \frac{a_0 \cos B_2 \cos L_{1.2}}{\sqrt{1 - e^2 \sin^2 B_2}} \\ y_2 &= a_0 \cos\beta_2 \sin L_{1.2} = \frac{a_0 \cos B_2 \sin L_{1.2}}{\sqrt{1 - e^2 \sin^2 B_2}} \\ z_2 &= b_0 \sin\beta_2 \qquad\quad = \frac{a_0 (1 - e^2) \sin B_2}{\sqrt{1 - e^2 \sin^2 B_2}} \end{aligned}\right\} \qquad (3)$$

In the equations for x_2 and y_2 it should be noted that $L_{1.2}$ is the inclination in the equatorial plane of $a_0 \cos\beta_2$ (i.e. the x referred to on p. 40) relative to the x axis.

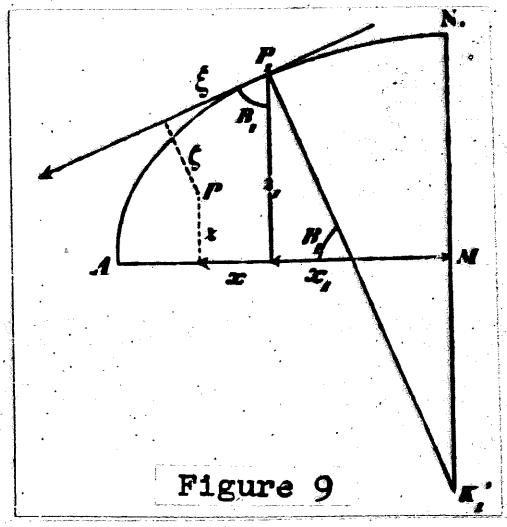


Figure 9

Projection Upon the xz Plane.

In the xz plane (see Fig. 9) we now define a ξ axis as the tangent to the meridian curve, the positive direction being toward the south. Normal to this, that is to say in the direction of the plumb line, we define a ζ axis, positive toward the interior of the ellipsoid. Finally, we define an η -axis thru P_1 parallel to the y -axis. Then for any point P of our space,

there result the following relationships between the xyz and the $\xi\eta\zeta$ coordinates:

$$\left.\begin{aligned}\xi &= (x - x_1)\sin B_1 - (z - z_1)\cos B_1\\ \eta &= y\\ \zeta &= -(x - x_1)\cos B_1 - (z - z_1)\sin B_1\end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned}x &= x_1 + \xi\sin B_1 - \zeta\cos B_1\\ y &= \eta\\ z &= z_1 - \xi\cos B_1 - \zeta\sin B_1,\end{aligned}\right\} \quad (5)$$

as can readily be ascertained by projection upon the various axes.

Substituting from (5) in (1), the latter becomes

$$\begin{aligned}&\xi^2(1-e^2\sin^2 B_1) + \eta^2(1-e^2) + \zeta^2(1-e^2\cos^2 B_1) + 2\xi\zeta e^2\sin B_1\cos B_1\\ &+ 2\xi\left(x_1(1-e^2)\sin B_1 - z_1\cos B_1\right) - 2\zeta\left(x_1(1-e^2)\cos B_1 + z_1\sin B_1\right)\\ &= a_0^2(1-e^2) - x_1^2(1-e^2) - z_1^2,\end{aligned}$$

which with the aid of (2) can be considerably simplified, as follows:

$$\xi^2(1-e^2\sin^2 B_1) + \eta^2(1-e^2) + \zeta^2(1-e^2\cos^2 B_1) + 2\xi\zeta e^2\sin B_1\cos B_1 - 2\zeta\frac{a_0(1-e^2)}{\sqrt{1-e^2\sin^2 B_1}} = 0. \quad (6)$$

If (as on p. 58) we substitute herein for ξ and η the polar coordinate expressions

$$\xi = \vartheta\cos\alpha \qquad \eta = \vartheta\sin\alpha, \quad (7)$$

we obtain, if α remains constant, an equation between ϑ and ζ alone, the equation of the vertical section, which, in passing

from P_1 on azimuth α becomes:

$$\left.\begin{aligned}\vartheta^2(1-e^2+e^2\cos^2 B_1\cos^2\alpha)+\zeta^2(1-e^2\cos^2 B_1)\\ +2\vartheta\zeta e^2\sin B_1\cos B_1\cos\alpha-2\zeta\frac{a_0(1-e^2)}{\sqrt{1-e^2\sin^2 B_1}}\end{aligned}\right\}=0. \qquad (8)$$

This curve of intersection is an ellipse, since the equation is of second degree and the curve is closed. The minor axis lies in that meridian plane which is normal to the plane of the vertical section; the major axis lies in the plane of the equator. This follows from the fact that the said meridian plane of the ellipsoid of rotation, as well as the equatorial plane, are planes of symmetry, while the major axis can in any case lie only on the equatorial plane.

The meridian plane normal to the plane of the vertical section is also the plane in which lies the angle of inclination of the axis of rotation to the plane of the vertical section, which we denote by 90^0-U.

In order to compute this angle, let us construct a sphere of radius 1 about K_1' as center and draw thru K_1' parallels to all the lines concerned. Then the great-circle arcs between the points of intersection of the lines thru K_1' with the sphere give the angles of inclination of these lines. Fig. 10 shows the resulting figure upon the spherical surface. N corresponds to the northern part of the axis of rotation, P_1 to the plumbline from point P_1, NP_1 to the meridian plane thru P_1, $P_0P_1P_2$ to

(See Illustrative Material on Next Page.)

Par. 2. The Equation of the Ellipsoid and of the Vertical Section.

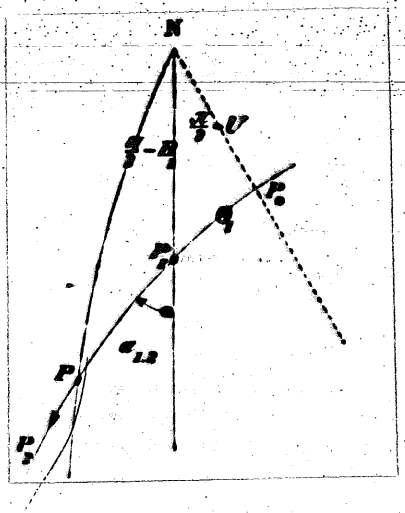


Figure 10

the vertical section plane P_1P_2 and NP_0 to the meridian plane normal thereto.

The spherical triangle NP_0P_1 has a right angle at P_0. The angle at P_1, which is the astronomical azimuth of the vertical plane from P_1 thru P_2 will now be denoted by $a_{1.2}$. By the sin law we then have

$$\cos U = \cos B_1 \sin a_{1.2}. \qquad (9)$$

We shall not deal with the correction of the arc lengths here, as it will be more appropriate to treat this later

in connection with the chord. (On this correction see also Hansen, Geodät. Untersuchungen (Geodetic Investigations), p. 66).

Here we shall merely note the form of the equation of the vertical section expressed in the polar coordinates

Chapter 4. The Vertical Section and Chord Triangle for the Flattened Ellipsoid of Revolution.

$K_1' P = r$ and $\sphericalangle\left(\begin{smallmatrix} P_0 P' \\ K_1' \end{smallmatrix}\right) = \Theta$; namely:

$$r^2(1 - e^2 + e^2 \cos^2 U \cos^2 \Theta) - r \frac{2 a_0 e^2 \cos U \sin B_1}{\sqrt{1 - e^2 \sin^2 B_1}} \cos \Theta = \frac{a_0^2 (1 - e^2 - e^2 \sin^2 B_1)}{1 - e^2 \sin^2 B_1},$$

which can be derived from (8) by the following substitutions:

$$\vartheta = r \sin (\Theta - \Theta_1) \text{ und } \zeta = \left(\frac{a_0}{\sqrt{1 - e^2 \sin^2 B_1}} - r \cos (\Theta - \Theta_1) \right).$$

Par. 3. The Astronomical Azimuth as a Function of the Geographic Positions. The astronomical azimuth $a_{1.2}$ of the vertical plane of P_1 thru P_2 is given by the following formulae (cf. (7) and (4) of the preceding section):

$$\left.\begin{aligned} \cot a_{1.2} &= \frac{\xi_2}{\eta_2} \\ \xi_2 &= (x_2 - x_1) \sin B_1 - (z_2 - z_1) \cos B_1 \\ \eta_2 &= y_2, \end{aligned}\right\} \qquad (1)$$

in connection with which x_1, z_1, x_2, y_2 and z_2 must be obtained from formulae (2) and (3) of the preceding section.

By making use of the expressions containing the reduced latitudes we obtain

$$\cot a_{1.2} = \frac{(\cos \beta_2 \cos L_{1.2} - \cos \beta_1) \sin B_1 - (\sin \beta_2 - \sin \beta_1) \cos B_1 \sqrt{1 - e^2}}{\cos \beta_2 \sin L_{1.2}}.$$

If in this equation we set

$$\sin B_1 = \sin \beta_1 : \sqrt{1 - e^2 \cos^2 \beta_1}$$

and

$$\cos B_1 = \cos \beta_1 \sqrt{1 - e^2} : \sqrt{1 - e^2 \cos^2 \beta_1}$$

(see (5) on p. 40), the equation becomes

$$\cot a_{1\,2} = \frac{(\cos\beta_2 \sin\beta_1 \cos L_{1.2} - \sin\beta_2 \cos\beta_1) + 2e^2 \sin\frac{\Delta\beta}{2} \cos\beta \cos\beta_1}{\cos\beta_2 \sin L_{1.2} \sqrt{1 - e^2\cos^2\beta_1}}, \quad (2)$$

$$\Delta\beta = \beta_2 - \beta_1; \qquad \beta = \frac{1}{2}(\beta_2 + \beta_1). \quad (3)$$

If for $L_{1.2}$ we write $\left(1 - 2\sin^2\frac{L_{1.2}}{2}\right)$ and denote, as before, $\sqrt{1 - e^2\cos^2\beta_1}$ by w_1 we obtain the following transformation suitable for precise computation:

$$\cot a_{1.2} = -\frac{1}{w_1}\left\{\frac{\sin\Delta\beta}{\cos\beta_2 \sin L_{1.2}}\left(1 - e^2\cos\beta_1\cos\beta\sec\frac{\Delta\beta}{2}\right) + \sin\beta_1 \tan\frac{L_{1.2}}{2}\right\}. \quad (4)$$

No special formula is required for $a_{2.1}$ the azimuth of the vertical section through P_2 and containing P_1, since $\cot a_{2.1}$ is obtained from $\cot a_{1.2}$ if, corresponding to an exchange of function of the points, the subscripts 1 and 2 are exchanged and $\Delta\beta$ is taken as negative in order that it may continue to denote $\beta_2 - \beta_1$. It may also be pointed out that

$$L_{2.1} = -L_{1.2}.$$

We shall now give the expression for $\cot a_{1.2}$ in terms of geographical latitude. This is most readily formulated by direct substitution of the appropriate values for x, y and z in the equation (1). We have:

$$\cot a_{1.2} = \frac{(\cos B_2 \sin B_1 \cos L_{1.2} - \sin B_2 \cos B_1) + e^2 \cos B_1 \left(\sin B_2 - \sin B_1 \frac{\sqrt{1 - e^2\sin^2 B_2}}{\sqrt{1 - e^2\sin^2 B_1}}\right)}{\cos B_2 \sin L_{1.2}}, \quad (5)$$

Par. 4. Series Expansions for the Preceding Problem.

and from this the form

$$\cot a_{1.2} = -\left\{\frac{\sin \Delta B}{\sin L_{1.2} \cos B_2} + \sin B_1 \tan \frac{L_{1.2}}{2} + \frac{e^2 \cos B_1}{\sin L_{1.2} \cos B_2}\left(\sin B_1 \frac{W_2}{W_1} - \sin B_2\right)\right\} \cdot \quad (6)$$

more appropriate for precise computation, in which W denotes $\sqrt{1 - e^2 \sin^2 B}$.

The formulae (4) and (6) leave $a_{1.2}$ undetermined in the vicinity of 180^0 which is however of no importance, as both values of $a_{1.2}$ characterize the same vertical plane. It is also evident at once that the chord P_1P_2 will always be west of the meridian through P_1 if the longitude of P_2 is less than 180^0 farther west than that of P_1 ; and east of that meridian if P_2 is more than 180^0 farther west than P_1 .

Accordingly we put

$$\left.\begin{array}{l} \text{The southwestward azimuth } a_{1.2} < 180^0 \text{ for a W longitude } L_{1.2} < 180^0 \\ \text{The southwestward azimuth } a_{1.2} > 180^0 \text{ for a W longitude } L_{1.2} > 180^0. \end{array}\right\} \quad (7)$$

Par. 4. Series Expansions for the Preceding Problem. Formula (6) can be further expanded by transforming the term having e^2 as a factor in the following manner. To begin with:

$$\sqrt{1 - e^2 \sin^2 B_2} = \sqrt{1 - \frac{e^2}{2} + \frac{e^2}{2} \cos (2B_1 + 2\Delta B)}.$$

Eliminating the cosine, a simple reduction gives:

$$\sqrt{1 - e^2 \sin^2 B_2} = \sqrt{1 - e^2 \sin^2 B_1} \sqrt{1 - \left\{\frac{e^2 \sin 2B \sin \Delta B}{1 - e^2 \sin^2 B_1}\right\}}.$$

The series expansion in accordance with the binomial theorem gives (see second formula (3), p. 27, Par. 4) in finite form:

$$\frac{\sqrt{1-e^2\sin^2 B_2}}{\sqrt{1-e^2\sin^2 B_1}} = 1 - \frac{1}{2}\left\{\frac{e^2 \sin 2B \sin \Delta B}{1-e^2\sin^2 B_1}\right\}\left(1+\frac{1}{4}\left\{\div\right\}+\frac{x}{8(1-(\div))}\left\{\div\right\}^2\right), \quad (1)$$

wherein x denotes a proper fraction. If the forgoing is inserted into the formula for $\cot \alpha_{1.2}$, e^2 being again regarded as a second-order magnitude, we obtain:

$$\cot\alpha_{1.2} = -\left\{\frac{\sin \Delta B}{\sin L_{1.2}\cos B_2}\left[\begin{array}{l} 1-e^2\cos B_1\cos B\sec\frac{\Delta B}{2}+Gl_8 \\ -\frac{e^4\sin 2B_1\sin 2B}{4(1-e^2\sin^2 B_1)}\left(1+\frac{e^2\sin 2B\sin\Delta B}{4(1-e^2\sin^2 B_1)}\right)\end{array}\right]+\sin B_1\tan\frac{L_{1.2}}{2}\right\}. \quad (2)$$

This formula will always give a precision adequate for the use of ten-place logarithms. Namely, neglecting certain factors in the denominator which are always approximately equal to 1, the terms Gl_8 (eight-order terms) in the square brackets amount only to

$$+\frac{1}{32}\varkappa e^8 \sin 2B_1 \sin^3 2B \sin^2 \Delta B,$$

that is, less than 1 : 16000 000000. The effect upon the logarithm of the quantity in the square brackets is thus $< M:16000\,000000$, that is, only two units in the eleventh-place decimal.

If ΔB is a small first-order magnitude, the term shown in very small type within the square brackets also becomes negligible, since then, because of the short distance involved, $\alpha_{1.2}$ need not be so precisely computed as for great distances.

Formula (2) can be still further transformed to advantage if

small values for ΔB and $L_{1,2}$ are assumed.

Within the square brackets of (2) we introduce in place of B_1 and B a latitude

$$B' = \frac{3B_1 + B_2}{4} = B_1 + \frac{\Delta B}{4} \qquad (3)$$

so that the substitutions to be made are $B_1 = B' - \frac{\Delta B}{4}$ and $B = B' + \frac{\Delta B}{4}$. Eliminating then the relevant cosine and sine, we obtain for the quantity in the square brackets:

$$[\div] = 1 - e^2 \sec\frac{\Delta B}{2}\left(\cos^2 B' - \sin^2\frac{\Delta B}{4}\right) - \frac{e^4}{4}\frac{\sin^2 2B' - \sin^2\frac{\Delta B}{2}}{1 - e^2\sin^2 B_1} + Gl_7.$$

In the last denominator at the right B_1 may be replaced simply by B', since the error thus introduced is of seventh-order. With this substitution and some simple reductions we obtain:

$$[\div] = \left(1 - e^2\cos^2 B' - \frac{e^4}{4}\frac{\sin^2 2B'}{1 - e^2\sin^2 B'}\right) - \frac{e^2}{2}\cos 2B'\left(\sec\frac{\Delta B}{2} - 1\right) + \frac{e^4}{4}\frac{\sin^2\frac{\Delta B}{2}}{1 - e^2\sin^2 B'} + Gl_7.$$

Strictly speaking, the first of these terms is $(1 - e^2) : W^2$, with W referred to B', since

$$\frac{1 - e^2}{1 - e^2\sin^2 B'} = (1 - e^2)\left(1 + e^2\sin^2 B' + \frac{e^4\sin^4 B'}{1 - e^2\sin^2 B'}\right),$$

from which this form of the first term immediately follows. Introducing this and the series for secant and sine, we obtain:

$$\left[\frac{\cdot}{\cdot}\right] = \frac{1-e^2}{1-e^2\sin^2 B'} - \frac{e^2}{2}\cos 2B'\left(\frac{\Delta B^2}{8} + \frac{5\Delta B^4}{384}\right) + \frac{e^4\Delta B^2}{16} + Gl_7$$

or

$$\left[\frac{\cdot}{\cdot}\right] = \frac{1-e^2}{W'^2}\left\{1 - \frac{1}{16}e^2\Delta B^2\left(\cos 2B'\left[1 + \frac{5}{48}\Delta B^2\right] - e^2\sin^2 B'\,[2 + \cos 2B']\right) + Gl_7\right\}, \quad (4)$$

wherein now only eight-order terms have been neglected. Furthermore, for small values of ΔB and $L_{1.2}$ we can write, suppressing the subscript of L,

$$\frac{\sin\Delta B}{\sin L} = \frac{\Delta B}{L}\left(1 - \frac{1}{6}\Delta B^2 + \frac{1}{120}\Delta B^4 + Gl_6\right)\left(1 + \frac{1}{6}L^2 + \frac{7}{360}L^4 + Gl_6\right).$$

Carrying out the multiplication on the right and introducing the resulting expression, along with (4), into (2), we finally obtain

$$\left.\begin{aligned} \cot\alpha_{1.2} &= -\frac{(1-e^2)\,\Delta B}{W'^2 L\cos B_2}F - \frac{1}{2}L\sin B_1\left(1 + \frac{1}{12}L^2 + \frac{1}{120}L^4 + Gl_6\right), \\ &\text{whereby} \\ F &= \left\{1 - \frac{1}{6}[\Delta B^2 - L^2]\left(1 - \frac{1}{60}[3\Delta B^2 - 7L^2]\right) - \frac{1}{16}e^2\Delta B^2\cos 2B' + Gl_6\right\} \\ \Delta B &= \frac{\Delta B \text{ in seconds}}{\varrho''} \qquad L = \frac{L_{1.2} \text{ in seconds}}{\varrho''} \end{aligned}\right\} \quad (5)$$

The largest terms here neglected are the following sixth-order terms of the expression in braces F:

$$+\frac{1}{16}e^4\Delta B^2\sin^2 B'\,[2 + \cos 2B'] + \frac{1}{256}e^2\Delta B^4\cos 2B' - \frac{1}{96}e^2\Delta B^2 L^2\cos 2B' + \frac{31}{15120}L^6. \quad (6)$$

Par. 5. Chord and Azimuths Derived from the Geographic Position of Two Points Using the Reduced Latitudes.

Multiplying these by M, gives us their effect upon the logarithm of the expression in braces. The first and third terms have the greatest effect. In units in the seventh decimal place of the logarithm, this amounts at most to approximately

$$13\,\Delta B^2 \text{ and } 300\,\Delta B^2 L^2, \tag{7}$$

so that even for $\Delta B = 0{,}1$ and $L = 0{,}1$ an error of 1 unit will appear only in the eight place of the logarithm of the expression in braces (the error being due solely to the first term).

Applying (5) to the case of $\Delta B = 0{,}02$ and $L = 0{,}08$, the error of the logarithm of the expression in braces in (5) on account of all neglected terms is at the most eight units in the tenth decimal place.

Par. 5. Chord and Azimuths Derived from the Geographic Position of Two Points Using the Reduced Latitudes. So far it has been assumed that only one azimuth had to be computed. If, however, both azimuths and also the chord are to be found, the formulae for computing the azimuth must be suitably modified. Let us first consider the chord $P_1P_2 = k$. For this we have immediately the equation:

$$k^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Substituting the expressions for the coordinates in terms of the reduced latitudes in accordance with equations (2) and (3), pp. 135 and 136, the foregoing becomes

$$k^2 = a_0^2 \left\{ (\cos\beta_2 \cos L_{1,2} - \cos\beta_1)^2 + \cos^2\beta_2 \sin^2 L_{1,2} + (1 - c^2)(\sin\beta_2 - \sin\beta_1)^2 \right\},$$

which after some transformation, and in particular after the introduction of half angles, becomes

$$k^2 = 4a_0^2\left(\sin^2\frac{\Delta\beta}{2}\cos^2\frac{L_{1.2}}{2} + \cos^2\beta\sin^2\frac{L_{1.2}}{2} - e^2\sin^2\frac{\Delta\beta}{2}\cos^2\beta\right) \tag{1}$$

To avoid dealing with numbers having so many digits, this formula can be advantageously transformed by setting

$$\left.\begin{aligned} -\sin\frac{\Delta\beta}{2}\cos\frac{L_{1.2}}{2} &= \sin\frac{\sigma'}{2}\cos a' \\ +\cos\beta\sin\frac{L_{1.2}}{2} &= \sin\frac{\sigma'}{2}\sin a' \end{aligned}\right\} \tag{2}$$

$$\sin q' = \frac{e}{\sin\frac{\sigma'}{2}}\sin\frac{\Delta\beta}{2}\cos\beta. \tag{3}$$

Therewith, formula (1) becomes the much simpler

$$k = 2a_0\sin\frac{\sigma'}{2}\cos q'. \tag{4}$$

The meaning of the auxiliary magnitudes a' and σ' will be seen by considering a spherical triangle with the following sides and angles:

σ'	$\frac{\pi}{2} - \beta_1$	$\frac{\pi}{2} - \beta_2$
$L_{1.2}$	$a'_{2.1} - 180^0$	$180^0 - a'_{1.2}$.

(5)

Namely, if we define at the same time

$$\left.\begin{aligned} a'_{2.1} &= a'_{1.2} + 180^0 + \Delta a' \\ \frac{a'_{1.2} + a'_{2.1} - 180^0}{2} &= a' \end{aligned}\right\} \tag{6}$$

then the Gaussian Equations give for this triangle (cf. p. 131):

$$\left.\begin{aligned}
\cos\alpha' \sin\frac{\sigma'}{2} &= -\sin\frac{\Delta\beta}{2}\cos\frac{L_{1.2}}{2} \\
\sin\alpha' \sin\frac{\sigma'}{2} &= +\cos\beta \sin\frac{L_{1.2}}{2} \\
\cos\frac{\Delta\alpha'}{2}\cos\frac{\sigma'}{2} &= +\cos\frac{\Delta\beta}{2}\cos\frac{L_{1.2}}{2} \\
\sin\frac{\Delta\alpha'}{2}\cos\frac{\sigma'}{2} &= -\sin\beta \sin\frac{L_{1.2}}{2};
\end{aligned}\right\} \quad (7)$$

$$\Delta\beta = \beta_2 - \beta_1 \qquad \beta = \tfrac{1}{2}(\beta_2 + \beta_1)$$

$$\alpha'_{1.2} = \alpha' - \frac{\Delta\alpha'}{2} \qquad \alpha'_{2.1} = \alpha' + \frac{\Delta\alpha'}{2} + 180^0.$$

If we find, σ', from the above in accord with (2), and also compute $\alpha'_{1.2}$ and $\alpha'_{2.1}$ we can thereupon find the astronomical azimuths $\alpha_{1.2}$ and $\alpha_{2.1}$ also. For it is obvious that formula (2) on p. 138 is transformed into a formula for $\cot\alpha'_{1.2}$ if e is set equal to zero. This formula can thereupon be reduced as follows:

$$\cot\alpha_{1.2} = \frac{\cot\alpha'_{1.2} + 2e^2 \dfrac{\sin\frac{\Delta\beta}{2}\cos\beta\cos\beta_1}{\sin L_{1.2}\cos\beta_2}}{\sqrt{1 - e^2\cos^2\beta_1}}.$$

If we set the second part of the numerator equal to $\tan\varepsilon_{1.2}$ and introduce the notation $w_1 = \sqrt{1 - e^2\cos^2\beta_1}$, we have:

$$\left.\begin{aligned}
\tan\alpha_{1.2} &= \frac{w_1 \sin\alpha'_{1.2}\cos\varepsilon_{1.2}}{\cos(\alpha'_{1.2} - \varepsilon_{1.2})} \\
\tan\varepsilon_{1.2} &= 2e^2 \frac{\sin\frac{\Delta\beta}{2}\cos\beta\cos\beta_1}{\sin L_{1.2}\cos\beta_2}.
\end{aligned}\right\} \quad (8)$$

Since $L_{2.1}=-L_{1.2}$ with the interchange of the subscripts 1 and 2, the foregoing becomes:

$$\left.\begin{aligned}\tan \alpha_{2.1} &= \frac{w_2 \sin \alpha'_{2.1} \cos \varepsilon_{2.1}}{\cos (\alpha'_{2.1} - \varepsilon_{2.1})} \\ \tan \varepsilon_{2.1} &= 2e^2 \frac{\sin \frac{\Delta\beta}{2} \cos\beta \cos\beta_2}{\sin L_{1.2} \cos \beta_1}.\end{aligned}\right\} \qquad (9)$$

The whole solution is now comprised in the formulae (7), (3), (4), (8) and (9). With regard to (7) it has further to be noted that σ' must always be taken $<\pi$ in order that the azimuths shall correspond to the position of the chord (cf. p. 139, Par. 3). This having been done, the solution of (7) is entirely determined.

If the formulae of this section are applied to two points a short distance apart, under certain circumstances it will be found appropriate to develop series expansions for formulae (7). Since formulae (7) fully correspond to (4) on p. 132, the formulae of Par. 12, p. 132 can be applied to the present case with very slight changes in the notations; and formula (5) can then also be applied here.

Seven-place logarithms (possibly with the use of the auxiliary logarithms T) should always be adequate for the determination of the ε, values, which, however, can be computed as above; but then in the expressions for $\tan \varepsilon$, using the first two equations of (7) it will be necessary to replace:

$$\frac{\sin \frac{\Delta\beta}{2}}{\sin L_{1.2}} \quad \text{by} \quad \left(-\frac{1}{2} \sec^2 \frac{L_{1.2}}{2} \cos\beta \cot \alpha'\right).$$

Par. 6. Chord and Azimuths Derived from the Geographical Latitudes and the Difference of Longitude of Two Points. Let us first consider the spherical auxiliary triangle having the following sides and angles:

σ'	$\frac{\pi}{2} - B_1$	$\frac{\pi}{2} - B_2$
$L_{1.2}$	$a'_{2.1} - 180^0$	$180^0 - a'_{1.2}$

(1)

As in the similar case in the preceding section, we have the following formulae appropriate to the determination of σ', $a'_{2.1}$ and $a'_{1.2}$:

$$\cos a' \sin \frac{\sigma'}{2} = -\sin \frac{\Delta B}{2} \cos \frac{L_{1.2}}{2}$$

$$\sin a' \sin \frac{\sigma'}{2} = +\cos B \sin \frac{L_{1.2}}{2}$$

$$\cos \frac{\Delta a'}{2} \cos \frac{\sigma'}{2} = +\cos \frac{\Delta B}{2} \cos \frac{L_{1.2}}{2}$$

$$\sin \frac{\Delta a'}{2} \cos \frac{\sigma'}{2} = -\sin B \sin \frac{L_{1.2}}{2}$$

$$\Delta B = B_2 - B_1 \qquad B = \frac{1}{2}(B_2 + B_1)$$

$$a'_{1.2} = a' - \frac{\Delta a'}{2} \qquad a'_{2.1} = a' + \frac{\Delta a'}{2} + 180^0.$$

(2)

In these formulae $a'_{1.2}$ etc. do not denote the same magnitudes as in the foregoing section. On the contrary, $a'_{1.2}$ is identical with that value of $a_{1.2}$, given by formula (6) p. 139 for $e^2 = 0$. Hence we have

$$\cot a_{1.2} = \cot a'_{1.2} + \frac{e^2 \cos B_1}{\sin L_{1.2} \cos B_2} \left(\sin B_2 - \frac{W_2}{W_1} \sin B_1\right). \qquad (3)$$

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If the term to be added to $\cot \alpha'_{1.2}$ is set equal to $\tan \varepsilon'_{1.2}$, it follows at once, for both $\alpha_{1.2}$ and by interchange of the subscripts 1 and 2 for $\alpha_{2.1}$ also, that

$$\left.\begin{aligned} \tan \alpha_{1.2} &= \frac{\sin \alpha'_{1.2} \cos \varepsilon'_{1.2}}{\cos (\alpha'_{1.2} - \varepsilon'_{1.2})} \qquad \tan \alpha_{2.1} = \frac{\sin \alpha'_{2.1} \cos \varepsilon'_{2.1}}{\cos (\alpha'_{2.1} - \varepsilon'_{2.1})} \\ \tan \varepsilon'_{1.2} &= \frac{e^2 \cos B_1}{\sin L_{1.2} \cos B_2} \left(\sin B_2 - \frac{W_2}{W_1} \sin B_1 \right) \\ \tan \varepsilon'_{2.1} &= \frac{e^2 \cos B_2}{\sin L_{1.2} \cos B_1} \left(\sin B_2 - \frac{W_2}{W_1} \sin B_1 \right) \frac{W_1}{W_2}. \end{aligned}\right\} \qquad (4)$$

As in the first part of the foregoing section, for the chord k we obtain the equation:

$$k^2 = a_0^2 \left\{ \left(\frac{\cos B_2 \cos L_{1.2}}{W_2} - \frac{\cos B_1}{W_1} \right)^2 + \left(\frac{\cos B_2 \sin L_{1.2}}{W_2} \right)^2 + (1 - e^2)^2 \left(\frac{\sin B_2}{W_2} - \frac{\sin B_1}{W_1} \right)^2 \right\}.$$

Since for $e = 0$, k^2 must become $4 a_0^2 \sin^2 \frac{\sigma'}{2}$, it follows from the above that

$$k = 2 a_0 \sin \frac{\sigma'}{2} \cos q' : \sqrt{W_1 W_2} \qquad (5)$$

$$2 \sin \frac{\sigma'}{2} \sin q' = \sqrt{\frac{W_1}{W_2}} \sqrt{e^2 (2 - e^2) \left(\sin B_2 - \frac{W_2}{W_1} \sin B_1 \right)^2 - \left(1 - \frac{W_2}{W_1} \right)^2} \qquad (6)$$

In view of the fact that the second part of the radicand under the larger square-root sign always has e^4 as a factor, while the first part has only e^2, whereas with respect to ΔB the order is the same, the last equation above, for $\sin q'$, is better

written as follows:

$$\left.\begin{aligned} \sin q' &= \frac{e\sqrt{2-e^2}\left(\sin B_2 - \frac{W_2}{W_1}\sin B_1\right)}{2\sin\frac{\varepsilon'}{2}}\sqrt{\frac{W_1}{W_2}}\cos q'' \\ \sin q'' &= \left(1-\frac{W_2}{W_1}\right) : e\sqrt{2-e^2}\left(\sin B_2 - \frac{W_2}{W_1}\sin B_1\right). \end{aligned}\right\} \quad (7)$$

It may be advantageous to compute some of the expressions occurring in the above formula by means of series. However, in accord with equation (1) on p. 140 we have the following expression which is of sufficient accuracy for all cases in which ten-place logarithms are satisfactory for the computation of k:

$$1-\frac{W_2}{W_1} = \left\{\frac{e^2\sin 2B\sin \Delta B}{2W_1^2}\right\}\left(1+\frac{1}{2}\left\{\frac{e^2\sin 2B\sin\Delta B}{2W_1^2}\right\}+Gl_4\right).$$

Thence:

$$\left.\begin{aligned} \log\left(1-\frac{W_2}{W_1}\right) &= \log E + \frac{M}{2}E + Gl_4 \\ E &= \frac{e^2\sin 2B\sin\Delta B}{2W_1^2}. \end{aligned}\right\} \quad (8)$$

The terms Gl_4 are of fourth-order because of the factor e^4. As the factor $\sin^2\Delta B$ is still present, they become of sixth-order for small ΔB. values. Now from equation (1), p. 140 it also follows that

$$\sin B_2 - \frac{W_2}{W_1}\sin B_1 = 2\sin\frac{\Delta B}{2}\cos B + E\sin B_1\left(1+\frac{1}{2}E+Gl_4\right)$$

and thence using logarithms:

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned} &\log\left(\sin B_2 - \frac{W_2}{W_1}\sin B_1\right) \\ &= \log D + \frac{ME}{D}\sin B_1\left(1 - \frac{E}{2D}\sin B_1 + \frac{1}{2}E + Gl_4\right), \\ &D = 2\sin\frac{\Delta B}{2}\cos B, \end{aligned}\right\} \qquad (9)$$

here even for small ΔB values, some of the Gl_4 terms do not change their order, since they lack the factor $\sin^2\Delta B$.

Par. 7. Series Expansions for the Preceding Problem. From now on we shall regard h as a first-order magnitude, so that σ and ΔB are also

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first-order magnitudes, while L can be as large as desired unless the requirement as to h demands a small L value.

Our point of departure is formula (3), p. 145; in accord with this we may write:

$$\left.\begin{aligned} &\cot a_{1.2} = \cot a'_{1.2} + h \\ &h = \frac{e^2\cos B_1}{\sin L_{1.2}\cos B_2}\left(\sin B_2 - \frac{W_2}{W_1}\sin B_1\right). \end{aligned}\right\} \qquad (1)$$

But in accord with equation (1), p. 140 and ultimately in accord with p. 146 we also have:

$$h = \frac{2e^2\sin\frac{\Delta B}{2}\cos B\cos B_1}{\sin L_{1.2}\cos B_2}\left\{1 + \frac{e^2\sin B\sin B_1\cos\frac{\Delta B}{2}}{1-e^2\sin^2 B_1}\left(1 + \frac{e^2\sin 2B\sin\Delta B}{4(1-e^2\sin^2 B_1)} + Gl_4\right)\right\}, \qquad (2)$$

wherein as indicated on p. 140 the residual term Gl_4 is negli-

gible for any value of ΔB, and all the more certainly so for ΔB values amounting to first-order magnitudes since this term is then of eight-order with respect to the expression in braces in equation (2), because of the presence in it of the factor $\sin^2 \Delta B$.

If in the above expression (2) we now set

$$\frac{1}{1 - e^2 \sin^2 B_1} = 1 + e^2 \sin^2 B_1 + \frac{e^4 \sin^4 B_1}{1 - e^2 \sin^2 B_1},$$

$$\frac{1}{(1 - e^2 \sin^2 B_1)^2} = 1 + \frac{e^2 \sin^2 B_1}{1 - e^2 \sin^2 B_1} + \frac{e^2 \sin^2 B_1}{(1 - e^2 \sin^2 B_1)^2},$$

and introduce the abbreviations

$$\left.\begin{aligned} 2 \cos \frac{\Delta B}{2} \sin B &= \sin B_2 + \sin B_1 = S \\ 2 \sin \frac{\Delta B}{2} \cos B &= \sin B_2 - \sin B_1 = D \end{aligned}\right\} \qquad (3)$$

and note that $\sin \Delta B \sin 2B = SD$, we find:

$$h = \frac{e^2 D \cos B_1}{\sin J_{1,2} \cos B_2} \left\{ \begin{aligned} &1 + \tfrac{1}{2} e^2 S \sin B_1 \\ &+ \tfrac{1}{8} e^4 S \sin B_1 (3 \sin^2 B_1 + \sin^2 B_2) + \varkappa_1 e^6 \sin^6 B \end{aligned} \right\}. \qquad (4)$$

Here it is clearly evident that the remainder comprised in $\varkappa_1 e^6 \sin^6 B$ is a small term whose largest component:

$$\tfrac{1}{2} e^6 S \sin^5 B_1 : (1 - e^2 \sin^2 B_1)$$

differs only negligibly from $e^6 \sin^6 B$ so that $\varkappa_1$ is very nearly equal to 1.

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Chapter 4. The Vertical Section and Chord Triangle for the Flattened Ellipsoid of Revolution.

In order to use the foregoing to derive the difference of $a'_{1.2}$ and $a_{1.2}$, we use the expansion given on p. 30 for

$$\cot a_{1.2} = \cot a'_{1.2} + h$$

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and obtain:

$$\left.\begin{aligned} a_{1.2} - a'_{1.2} = -h\sin^2 a'_{1.2}\Big\{1 - h\sin a'_{1.2}\cos a'_{1.2} - \tfrac{1}{3}h^2\sin^2 a'_{1.2}(1-4\cos^2 a'_{1.2})\Big\} \\ + \tfrac{1}{4}h^4\sin^4 a''_{1.2}\sin 4a''_{1.2}, \end{aligned}\right\} \quad (5)$$

In the last term $a''_{1.2}$ denotes an angle whose cotangent equals $\cos a'_{1.2} + \vartheta h$, where ϑ is a positive real fraction. The spherical triangle (1) on p. 144 gives, however,

$$\sin a'_{1.2}\sin\sigma' = \sin L_{1.2}\cos B_2.$$

Using this relationship we obtain:

$$h\sin^2 a'_{1.2} = \frac{e^2 D\cos B_1\cos B_2\sin L_{1.2}}{\sin^2\sigma'} \times\left(1 + \tfrac{1}{2}e^2 S\sin B_1 + \tfrac{1}{8}e^4 S\sin B_1(3\sin^2 B_1 + \sin^2 B_2) + \varkappa_1 e^6\sin^6 B\right)$$

and

$$h\sin a'_{1.2} = \frac{e^2 D\cos B_1}{\sin\sigma'}\left(1 + \frac{e^2 S}{2}\sin B_1 + \varkappa_2 e^4\sin^4 B\right),$$

wherein $\varkappa_2$ denotes a number which, like $\varkappa_1$, is very nearly equal to 1. Introducing the above into (5) yields

$$\left.\begin{aligned} a_{1.2} = a'_{1.2} - \frac{e^2 D\cos B_1\cos B_2\sin L_{1.2}}{\sin^2\sigma'}\cdot F \\ + \tfrac{1}{4}h^4\sin^4 a''_{1.2}\sin 4a''_{1.2} - \varkappa_{1.2}e^8\frac{D\cos B_1\sin a'_{1.2}}{\sin\sigma'}, \end{aligned}\right\} \quad (6)$$

$$F = \left\{\begin{aligned} 1 + e^2\left(\frac{S}{2}\sin B_1 - \frac{D\cos B_1\cos a'_{1.2}}{\sin\sigma'}\right) + e^4\left(\frac{S}{8}\sin B_1(3\sin^2 B_1 + \sin^2 B_2)\right) \\ - e^4\left[\frac{DS\sin B_1\cos B_1\cos a'_{1.2}}{\sin\sigma'} + \frac{D^2\cos^2 B_1(1-4\cos^2 a'_{1.2})}{3\sin^2\sigma'}\right] \end{aligned}\right\}$$

Par. 7. Series Expansions for the Preceding Problem.

Here the terms with e^3 and so on, which arise from the prsence of h, h^2 and h^3, are brought together in the term which contains $x_{1.2}$ as a factor. At its maximum this coefficient is approximately equal to 1. Now for substitution in the remaining terms it is sufficiently accurate to take $D:\sin\sigma' = \cos B\cos a'$, whence the maximum value of the term with $x_{1.2}$ as a factor is about $\frac{1}{2}e^3$.

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In order to evaluate the term containing h^4 as a factor, we have first to note that, as indicated on p. 31, we may, to a sufficient approximation, set $a''_{1.2}$ equal to $a'_{1.2}$ or to a'. If we also take the above-mentioned value for $D:\sin\sigma'$, the term in question becomes

$$\frac{1}{4}e^3\cos^6 B\cos^4 a'\sin 4a'$$

the value of which at its maximum will remain considerably less than that of the residual term just dealt with.

Accordingly, the residual terms of the eight-order will amount at a maximium, in seconds of arc, to about $\frac{1}{2}\rho'' e^3$, that is, only 0,0002".

If the subscripts 1 and 2 are interchanged in formula (6), D and $L_{1.2}$ must also be replaced by $-D$ and $-L_{1.2}$. Subtracting then one formula from the other term by term, we obtain:

$$a_{2.1} - a_{1.2} = a'_{2.1} - a'_{1.2} - \frac{e^4 D\cos B_1\cos B_2\sin L_{1.2}}{\sin^2\sigma'}(A_1 + A_2 + A_3), \quad (7)$$

$$A_1 = D\left(\frac{S}{2} + \frac{\cos B_2\cos a'_{2.1} + \cos B_1\cos a'_{1.2}}{\sin\sigma'}\right)$$

$$A_2 = e^2\left\{\begin{array}{c}\frac{1}{8}S(3\sin^3 B_2 - 3\sin^3 B_1 + \sin^2 B_1\sin B_2 - \sin^2 B_2\sin B_1)\\ + DS\,\frac{\sin B_2\cos B_2\cos a'_{2.1} + \sin B_1\cos B_1\cos a'_{1.2}}{\sin\sigma'}\\ - D^2\,\frac{\cos^2 B_2 - \cos^2 B_1 - 4\cos^2 B_2\cos^2 a'_{2.1} + 4\cos^2 B_1\cos^2 a'_{1.2}}{8\sin^3\sigma'}\end{array}\right\},$$

We simplify A_1 and A_2 by eliminating σ' through the relationships:

$$\left.\begin{aligned}\cos B_1 \cos a'_{1.2} \sin\sigma' &= \cos\sigma' \sin B_1 - \sin B_2 = -D - 2\sin B_1 \sin^2\frac{\sigma'}{2}\\ \cos B_2 \cos a'_{2.1} \sin\sigma' &= \cos\sigma' \sin B_2 - \sin B_1 = +D - 2\sin B_2 \sin^2\frac{\sigma'}{2},\end{aligned}\right\} \quad (8)$$

which follow from the cosine law in the spherical triangle. A rather long but not difficult computation finally yields:

$$\left.\begin{aligned}A_1 &= -\frac{1}{2} DS \tan^2\frac{\sigma'}{2},\\ A_2 &= -\frac{1}{2} c^2 DS \left((\sin^2 B_1 + \sin^2 B_2)\tan^2\frac{\sigma'}{2} + \frac{1}{4} D^2\right).\end{aligned}\right\} \quad (9)$$

As for the quantity A_3, arising from the residual terms in (6), its effect upon $a_{2.1} - a_{1.2}$ is unquestionably far smaller than 0,0002", the maximum value of those residual terms. For whenever ΔB is a small quantity, i.e. when B_1 and B_2 differ little in value, it is to be expected that the residual terms

occurring in formula (6) for $a_{1.2}$ and $a_{2.1}$ must in part cancel each other out. Substituting from (9) into (7) now gives:

$$\left.\begin{aligned}&\underset{\text{in seconds}}{a_{2.1} - a_{1.2}} = \underset{\text{in seconds}}{a'_{2.1} - a'_{1.2}}\\ &+\varrho'' \frac{c^4 D^2 S \cos B_1 \cos B_2 \sin L_{1.2}}{2\sin^2\sigma'}\left(\tan^2\frac{\sigma'}{2} + c^2\left[(\sin^2 B_1 + \sin^2 B_2)\tan^2\frac{\sigma'}{2} + \frac{1}{4} D^2\right] + \ldots\right)\\ &D = \sin B_2 - \sin B_1 = 2\sin\frac{\Delta B}{2}\cos B\\ &S = \sin B_2 + \sin B_1 = 2\cos\frac{\Delta B}{2}\sin B.\end{aligned}\right\} \quad (10)$$

Par. 8. Dalby's Theorem. Formula (10) shows that, as Dalby assumed, for small distances the azimuthal difference may be computed on a purely spherical basis, since the most significant term ignored in so doing is found after some simplification to be equivalent to (in seconds of arc):

$$+\frac{1}{8}\rho'' e^4 \sin L_{1.2} \cos B \sin^2 \Delta B \sec \frac{\Delta B}{2} \sec^4 \frac{\sigma'}{2} \sin 2B \cos B_1 \cos B_2, \qquad (1)$$

wherein the value of $\sin 2B \cos B_1 \cos B_2$, assuming all B equal, is at most 0,65. Thus at the maximum term (1) will be approximately

$$\tfrac{3}{4} \sin^2 \Delta B \cdot \sin L_{1.2} \cos B \text{ seconds}$$

Setting $\sin L_{1.2} \cos B = \sin \sigma' \sin a'$ and $\sin \Delta B = \sin \sigma' \cos a'$, which is perfectly adequate, and noting that the maximum value of $\cos^2 a' \sin a'$, is about 0,4, we have as the maximum value of the term (1)

$$0{,}3'' \sin^3 \sigma'. \qquad (2)$$

Thus even if $\sin \sigma' = 0{,}2$ the error in the formula

$$a_{2.1} - a_{1.2} = a'_{2.1} - a'_{1.2} + \cdots \qquad (3)$$

is only a few thousandths of a second. (However, expression (2) is valid only as long as σ' is small. Hence it cannot be applied to values of $\sin \sigma'$ significantly larger than 0,2.)

The terms in (10) involving e^6 account for only about one percent of the value of (1). If we omit them, while taking account of the terms having e^4 as a factor in accordance with formula (1) and setting $\sec \frac{\Delta B}{2} = 1$, $\sec^4 \frac{\sigma'}{2} = 1$ and $\cos B_1 \cos B_2 = \cos^2 B$, therein, which involves neglecting only second-order fractions of the value

of (1), we obtain:

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$$a_{2.1} - a_{1.2} = a'_{2.1} - a'_{1.2} + \frac{1}{4}\varrho''e^2 \sin L_{1.2} \sin^2 \Delta B \sin B \cos^2 B + \cdots \qquad (4)$$

(in seconds) (in seconds)

With this formula a small uncertainty in the ten thousandths of a second begins to appear only when ΔB and $\sin L_{1.2} \cos B$ equal 0,2.

The Dalby problem is investigated by geometrical means on p. 235 of the classical work on English surveying, the Ordnance Survey, Principal Triangulation. It had already been discussed by Ivory and Tiark in the Philosophical Magazine of 1828 (Vol. 4, p. 241, 364, and 432) and 1829 (Vol. 5, p. 24, 52, and 106), as well as by Anger in Astronom. Nachrichtungen (No. 212, Vol. 9, p. 249). It has also been studied recently by Andrae in the third volume of the Dänische Gradmessung, p. 324.

Already in 1790 and 1795 in the Philosophical Transactions, Dalby applied formula (3) to the problem of finding the difference of longitude from the geographical latitudes and azimuths of a geodetic line by purely spherical computation (cf. the third and fourth formulae under (2), p. 145). In so doing the differences between the astronomical azimuths and the azimuths of the geodetic line are also neglected, as they are small for short distances and nearly equal for both azimuths, so that they cancel each other.

Par. 9. Continuation of the Expansions for Short Distances.

It should also be mentioned there exists no theorem similar to Dalby's applicable to the use of the reduced latitudes; in this case the difference

$$\{(a_{2.1} - a_{1.2}) - (a'_{2.1} - a'_{1.2})\}$$

is of the order of $e^2 s'$.

Par. 9. Continuation of the Expansions for Short Distances.

Having established in the foregoing a formula for $a_{2.1} - a_{1.2}$, it is appropriate to derive a formula for $a_{2.1} + a_{1.2}$. For this purpose we again take formula (6), p. 148, apply it as before to $a'_{2.1}$ also, by interchange of the subscripts 1 and 2, and add the two formulae. It then follows that

$$a_{2.1} + a_{1.2} = a'_{2.1} + a'_{1.2} - \frac{e^2 D \cos B_1 \cos B_2 \sin L_{1.2}}{\sin^2 s'} (2 + e^2 A' + e^4 A'' + Gl_6) \qquad (1)$$

wherein the coefficients A' and A'' assume the following values:

$$A' = \frac{1}{2} S^2 + D \frac{\cos B_2 \cos a'_{2.1} - \cos B_1 \cos a'_{1.2}}{\sin s'},$$

$$A'' = \left\{ \begin{array}{c} \frac{1}{8} S \left(3 (\sin^2 B_1 + \sin^2 B_2) + S \sin B_1 \sin B_2\right) \\ + DS \frac{\sin B_2 \cos B_2 \cos a'_{2.1} - \sin B_1 \cos B_1 \cos a'_{1.2}}{\sin s'} \\ - D^2 \frac{\cos^2 B_1 + \cos^2 B_2 - 4 (\cos^2 B_1 \cos^2 a'_{1.2} + \cos^2 B_2 \cos^2 a'_{2.1})}{3 \sin^2 s'} \end{array} \right\}.$$

We now make use of the relationships (8), p. 149, to simplify the two A by the elimination of $\cos a'$. In so doing we have to recall that by (10), p. 150

$$S = 2\cos\frac{\Delta B}{2}\sin B \quad \text{and} \quad D = 2\sin\frac{\Delta B}{2}\cos B \qquad (2)$$

but that in view of the first formula (2), p. 145, we may also write

$$\sin\frac{\Delta B}{2} = -\sin\frac{\sigma'}{2}\cos a'\sec\frac{L_{1.2}}{2}. \qquad (3)$$

The expression for A' first takes the form:

$$A' = 2\left(\sin^2 B\cos^2\frac{\Delta B}{2} + \cos^2 B\sec^2\frac{L_{1.2}}{2}\cos^2 a'\right).$$

The two first equations (2), p. 145, however, give us also

$$\cos B\cos a'\tan\frac{L_{1.2}}{2} = -\sin\frac{\Delta B}{2}\sin a' \qquad (4)$$

and thus we obtain from the preceding equation, after setting $\sec^2 = 1 + \tan^2$, the form:

$$A' = 2\left(\sin^2 B\cos^2\frac{\Delta B}{2} + \cos^2 B\cos^2 a' + \sin^2\frac{\Delta B}{2}\sin^2 a'\right). \qquad (5)$$

In simplifying the expression for A'' we first substitute for $\sin B_1$ and $\sin B_2$ their equivalents in terms of S and D, using for this in accordance with (10), p. 150, the relationships:

$$\sin B_1 = \frac{S-D}{2} \qquad \sin B_2 = \frac{S+D}{2}.$$

We then have:

(See Illustrative Material on Next Page.)

Par. 9. Continuation of the Expansions for Short Distances.

$$A'' = \left\{ \begin{array}{l} \left(\frac{S^4}{8} + \frac{7D^2S^2}{6\sin^2\sigma'} - \frac{2D^3}{3\sin^2\sigma'} + \frac{D^4}{6\sin^4\frac{\sigma'}{2}} \right) \\ + \left(\frac{D^2S^2}{4} - \frac{D^2S^2}{2\cos^2\frac{\sigma'}{2}} + \frac{D^2S^2}{6\cos^4\frac{\sigma'}{2}} + \frac{D^4}{6\sin^2\sigma'} \right) \end{array} \right\}, \qquad (6)$$

in which, in agreement with (2), (3) and (4), we now substitute for S, DS and D the expressions:

$$\left. \begin{array}{c} S = 2\sin B\cos\frac{\Delta B}{2} \qquad DS = \sin\Delta B\sin 2B \\ \frac{D^2}{\sin^2\sigma'} = \cos^2 B\cos^2 a'\sec^2\frac{\sigma'}{2} + \sin^2\frac{\Delta B}{2}\sin^2 a'\sec^2\frac{\sigma'}{2} \\ \frac{D^2}{\sin^2\frac{\sigma'}{2}} = 4\cos^2 B\cos^2 a' + 4\sin^2\frac{\Delta B}{2}\sin^2 a'. \end{array} \right\} \qquad (7)$$

It is evident, however, that the second parenthesis in (6)

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is a second-order term which (with some small permissible omissions) can be simplified to:

$$\frac{1}{12}\sin^2\Delta B\,(-\sin^2 2B + 2\cos^4 B\cos^2 a') \qquad (8)$$

Considering that the factor of A'' will be at most $\frac{1}{2}c^6\rho''$ or $\frac{1}{34}$ second, and expression (8) will at its maximum equal $\frac{1}{12}\sin^2\Delta B$, its influence upon $a_{2.1} + a_{1.2}$, for $\Delta B = 0{,}1$, is at most 0,000025: and can be neglected.

Taking this into account, upon substituting from (7) we obtain for A'':

$$A'' = 2(1 - \cos^2 B\sin^2 a')^2 - \frac{1}{6}\cos^4 B\sin^2 2a' + Gl_3. \qquad (9)$$

Disregarding (8), the second-order residual terms take the following approximate form:

$$\Delta B''\left(\frac{7}{6}\sin^2 B\sin^2 a' - \sin^4 B + \frac{7}{24}\sin^2 2B\sin^2 a' - \frac{1}{6}\sin^2 a' - \frac{1}{6}\cos^2 B\cos 4a'\right), \quad (10)$$

wherein σ' is eliminated by use of the relationship

$$\Delta B^2\cos^2 B = \sigma'^2\cos^2 a'\cos^2 B + Gl_4 \quad (11)$$

which is easily derived from the third formula of (7). Comparison of (8) and (10) shows that the influence of (10) is significantly larger than that of (8), as is evident from inspection of its first two terms, which are the most important. Nevertheless, formula (9) can be retained for A'' without danger of incurring more than about 0,0002" of error in $a_{2.1} + a_{1.2}$.

We now substitute A' and A'' in (1) above, simplify the coefficient of the parenthesis in (1) by means of the relationship

$$D\sin L_{1.2} = 2\sin^2\frac{\sigma'}{2}\sin 2a' \quad (12)$$

derived from the first two equations (2), p. 145, and obtain

$$\underset{\text{in seconds}}{a_{2.1} + a_{1.2}} = \underset{\text{in seconds}}{a'_{2.1} + a'_{1.2}} + \varrho'' c^2\cos B_1\cos B_2\sin 2a'\sec^2\frac{\sigma'}{2}\cdot F, \quad (13)$$

$$F = \left\{\begin{array}{l} 1 + c^2\left[1 - \cos^2 B\sin^2 a' + \sin^2\frac{\Delta B}{2}(\sin^2 a' - \sin^2 B)\right] \\ + c^4\left[(1 - \cos^2 B\sin^2 a')^2 - \frac{1}{12}\cos^4 B\sin^2 2a'\right] + Gl_6 \end{array}\right\},$$

a formula which for $\Delta B = 0{,}1$ gives an error of only a few ten thousandths of a second.

If we write the formula more simply as

$$\underset{\text{in seconds}}{a_{2.1} + a_{1.2}} = \underset{\text{in seconds}}{a'_{2.1} + a'_{1.2}} + \frac{\varrho'' e^2 \cos B_1 \cos B_2 \sin 2a' \sec^2 \frac{a'}{2}}{1 - e^2 (1 - \cos^2 B \sin^2 a')} + Gl_6, \quad (14)$$

the error derived from the sixth-order terms is approximately

$$\varrho'' e^4 \cos^2 B \left\{ \frac{1}{4} \Delta B^2 (\sin^2 a' - \sin^2 B) \sin 2a' - \frac{1}{12} e^2 \cos^4 B \sin^3 2a' \right\}. \quad (15)$$

Both parts of this expression are a maximum for $B = 0$. Differentiation with respect to a' shows that for this value of B and for $\Delta B = 0{,}1$ the first part has a maximum value of 0,015" while the second, independent of ΔB, has a maximum value 0,005".

Since the values of the two parts in some degree compensate for each other, the maximum error of (14) can be set at about 0,01". If $B = 45^0$, for $\Delta B = 0{,}1$ the maximum error of the first part is only 0,003" and of the second less than 0,001".

It appears from all this that formula (14) can be regarded as adequate in many cases. It is more convenient than (13). In particular, its numerator can readily be dealt with logarithmically by means of the substitution

$$\cos B \sin a' = \cos u, \quad (16).$$

whereby it is transformed into W^2 with u as the argument of W.

Par. 10. Continuation. The Chord. It will no doubt be most convenient to compute the chord by the general formulae of Par. 6, p. 145, which afford a not inconsiderable simplification for short distances. However, we shall here briefly cite another method of

computation, namely, that using the radius of curvature in the mean azimuth and at the mean geographical latitude.

For this purpose we write

$$k^2 = 4a^2 \sin^2 \frac{e''}{2} \left\{ \frac{1}{\sqrt{(1 - e^2 \sin^2 B_1)(1 - e^2 \sin^2 B_2)}} - \frac{e^2(2 - e^2)(\sin B_2 \sqrt{1 - e^2 \sin^2 B_1} - \sin B_1 \sqrt{1 - e^2 \sin^2 B_2})^2 - (\sqrt{1 - e^2 \sin^2 B_2} - \sqrt{1 - e^2 \sin^2 B_1})^2}{4 \sin^2 \frac{e''}{2} (1 - e^2 \sin^2 B_1)(1 - e^2 \sin^2 B_2)} \right\}$$

and expand the terms of the expression in brackets in series, neglecting all terms of higher than sixth-order with respect to e or ΔB. The minuend gives:

$$1 + e^2 \sin^2 B + \left(\frac{1}{4} e^2 \Delta B^2 - \frac{1}{48} e^2 \Delta B^4\right) \cos 2B + e^4 \sin^4 B + e^6 \sin^6 B + e^4 \Delta B^2 \sin^2 B \left(1 - \frac{3}{2} \sin^2 B\right) + Gl_8.$$

The subtrahend gives:

$$\frac{2e^2 \sin^2 \frac{\Delta B}{2} \cos^2 B}{\sin^2 \frac{e''}{2}} \left\{ 1 + e^2 \left(\frac{5}{2} \sin^2 B - \frac{1}{2}\right) - \frac{1}{8} e^2 \Delta B^2 (7 \sin^2 B - 2) + \frac{3}{2} e^4 \sin^2 B (3 \sin^2 B - 1) + Gl_6 \right\},$$

in which, using equations (3) and (4), p. 152, we have further to substitute:

$$\frac{\sin^2 \frac{\Delta B}{2}}{\sin^2 \frac{e''}{2}} = \cos^2 a' \left(1 + \tan^2 \frac{L}{2}\right)$$

$$= \cos^2 a' + \sin^2 a' \sec^2 B \left(\frac{1}{4} \Delta B^2 - \frac{1}{48} \Delta B^4 + Gl_6\right).$$

With this substitution, the expression in brackets becomes:

$$\left.\begin{aligned}&1+e^2(\sin^2 B-2\cos^2 B\cos^2 a')+e^4\left(\sin^4 B-\cos^2 B\cos^2 a'(5\sin^2 B-1)\right)\\&+e^6\left(\sin^6 B-3\cos^2 B\cos^2 a'(3\sin^4 B-\sin^2 B)\right)+\tfrac{1}{4}e^2\,\Delta B^2(\cos 2B-2\sin^2 a')\\&-\tfrac{1}{48}e^2\,\Delta B^4(\cos 2B-2\sin^2 a')+e^4\,\Delta B^2\left\{-\tfrac{1}{4}\sin^2 B(1+6\sin^2 B)\right\}\\&+e^4\,\Delta B^2\left\{\tfrac{7}{4}\sin^2 B\cos^2 a'(1+\cos^2 B)+\tfrac{1}{4}(1-3\cos^2 a')\right\}+Gl_6.\end{aligned}\right\}$$

This expression is now multiplied by the right side of the following equation derived from formula (1), p. 58:

$$\frac{1}{\varrho^2}=\frac{(1-e^2\sin^2 B)\,(1-e^2+e^2\cos^2 B\cos^2 a')^2}{a_0^2\,(1-e^2)^2},$$

which corresponds to the radius of curvature in azimuth a' at latitude B. Thus we obtain finally:

$$k=2\varrho\sin\frac{\sigma'}{2}\left\{1+\frac{3}{8}e^4\cos^4 B\sin^2 2a'+\frac{1}{8}e^2\,\Delta B^2(\cos 2B-2\sin^2 a')+R\right\},\quad(1)$$

wherein

$$\left.\begin{aligned}R=&\tfrac{1}{4}e^4\cos^4 B\sin^2 2a'(3-2\cos^2 B+\cos^2 B\cos^2 a')-\tfrac{1}{96}e^2\,\Delta B^4(\cos 2B-2\sin^2 a')\\&+\tfrac{1}{8}e^4\,\Delta B^2(1-4\sin^2 B-2\cos 2B\cos^2 a'-3\cos^4 B\cos^2 a'+4\cos^2 B\cos^4 a')+Gl_8.\end{aligned}\right\}\quad(2)$$

thus R includes the terms of sixth and higher order with respect to e and ΔB.

If, however, the whole transformation is to afford any added ease of computation we must be able to neglect R.

The influence of R upon $\log k$ is equal to R times the modulus M. Accordingly, the first term in R gives for $B=0^g$

a maximum of four units in the eighth decimal place of the logarithm. However, with increasing latitude this value decreases, and at $B = 45^0$ no longer amounts to two units. For $\Delta B = 0{,}1$ the second term in R has no more than a slight effect upon the ninth decimal place. The third term has a maximum effect of about seven units in the eight place; but this maximum occurs with a zero value of the first term, and in general these two terms combine with each other not unfavorably.

The neglect of R thus corresponds to the accuracy afforded by the use of seven-place logarithmic tables and the determination of the eight place by interpolation. Equally accurate is the following logarithmic form:

$$\log k = \log\left(2\varrho \sin\frac{\sigma'}{2}\right) + \frac{1}{8} M \left\{3 e^4 \cos^4 B \sin^2 2a' + e^2 \Delta B^2 (\cos 2B - 2\sin^2 a')\right\} + Gl_6 \quad (3)$$

Since formula (3) requires the computation of ϱ with the spherical azimuth a' as argument, it is appropriate to introduce the corresponding astronomical azimuth also.

But by formula (14), p. 154, and letting a denote:

$$\frac{a_{1.2} + a_{2.1} - 180^0}{2} \quad (4)$$

we can write:

$$a = a' + \frac{1}{2} e^2 \cos B_1 \cos B_2 \sin 2a' \sec^2\frac{\sigma'}{2} \left\{1 + e^2(1 - \cos^2 B \sin^2 a') + Gl_4\right\}.$$

Whence it follows that:

$$a' = a - \frac{1}{2} e^2 \sin 2a \left\{\cos^2 B \left[1 + e^2(1 - \cos^2 B \cos^2 a) + \frac{\sigma'^2}{4}\right] - \frac{1}{4}\Delta B^2\right\} + Gl_6.$$

Inserting this into the expression:

$$\log(\rho \text{ for } a') = \log \frac{a_0(1-e^2)}{\sqrt{1-e^2\sin^2 B}} - \log(1 - e^2 + e^2\cos^2 B\cos^2 a')$$

in accordance with (11), p. 153, we have:

$$\log(\rho \text{ for } a') = \log(\rho \text{ for } a)$$
$$-\tfrac{1}{2}e^4\cos^4 B\sin^2 2a\left(1+e^2\left(2-3\cos^2 B\cos^2 a+\tfrac{1}{2}\cos^2 B\right)\right)$$
$$-\tfrac{1}{2}e^4\,\Delta B^2\cos^2 B\sin^2 a\,(\cos^2 B-\cos^2 a)+Gl_6. \qquad (5)$$

If we now retain only fourth-order terms; in accordance with formula (3), the substitution of this last expression for $\log(\rho \text{ for } a')$ in (3) gives:

$$\log k = \log\left(2\rho\sin\frac{\sigma'}{2}\right)$$
$$+\tfrac{1}{8}M\{-e^4\cos^4 B\sin^2 2a+e^2\,\Delta B^2(\cos 2B-2\sin^2 a)\}+Gl_6, \qquad (6)$$

This formula has the same accuracy as (3). We can verify this if in the substitution of $\log(\rho \text{ for } a)$ we retain the sixth-order terms, and in the fourth-order terms of (3) we set $a' = a - \frac{1}{2}e^2\sin 2a\cos^2 B + Gl_4$. We then obtain, disregarding the coefficient M, the following residual terms of (6):

$$\left\{\begin{array}{l} -\frac{1}{4}e^4\cos^4 B\sin^2 2a\,(1-\cos^2 B\cos^2 a)-\frac{1}{96}e^2\,\Delta B^2(\cos 2B-2\sin^2 a) \\ +\frac{1}{8}e^4\,\Delta B^2\{-3+2\sin^2 2B+\cos^2 a\,(2+\cos^4 B+4\cos^2 B\sin^2 a)\}. \end{array}\right\} \qquad (7)$$

These terms have approximately the same effect as that noted earlier for the corresponding terms in (2). However, there is

always this advantage in using the astronomical azimuth, that the terms independent of the distance are smaller than before (by about half), so that for ΔB values significantly smaller than 0,1 formula (6) becomes more accurate than (3).

Par. 11. Summary of Formulae for Distances Less than $0{,}1\ a_0$.

Given: B_1, B_2 and $L_{1.2}$.

$$\Delta B = B_2 - B_1 \qquad B = \frac{1}{2}(B_2 + B_1)$$

$$\left.\begin{aligned} \cos\alpha' \sin\frac{\sigma'}{2} &= -\sin\frac{\Delta B}{2}\cos\frac{L_{1.2}}{2} \\ \sin\alpha' \sin\frac{\sigma'}{2} &= +\cos B \sin\frac{L_{1.2}}{2} \\ \cos\frac{\Delta\alpha'}{2}\cos\frac{\sigma'}{2} &= +\cos\frac{\Delta B}{2}\cos\frac{L_{1.2}}{2} \\ \sin\frac{\Delta\alpha'}{2}\cos\frac{\sigma'}{2} &= -\sin B \sin\frac{L_{1.2}}{2} \end{aligned}\right\} \quad (1)$$

$$\frac{\alpha_{2.1} - \alpha_{1.2} - 180^\circ}{2} = \frac{\Delta\alpha'}{2} + Gl_7. \quad (2)$$

(in seconds) (in seconds)

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$$\left.\begin{aligned} \frac{\alpha_{2.1} + \alpha_{1.2} - 180^\circ}{2} &= \alpha' + \frac{1}{2}\varrho'' e^2 \frac{\cos B_1 \cos B_2 \sin 2\alpha' \sec^2\frac{\sigma'}{2}}{W_u^2} + Gl_6 \\ \cos u &= \cos B \sin\alpha', \quad u \text{ argument for } W_u. \end{aligned}\right\} \quad (3)$$

(in seconds) (in seconds)

$$\left.\begin{aligned} \log k = \log\left(2\varrho \sin\frac{\sigma'}{2}\right) + \frac{1}{8} M \left\{ 3e^4 \cos^4 B \sin^2 2\alpha' + e^2 \Delta B^2 (\cos 2B - 2\sin^2\alpha') \right\} + Gl_6 \\ \Delta B = \frac{\Delta B \text{ in seconds}}{\varrho''} \end{aligned}\right\} \quad (4)$$

ϱ for the arguments of α' and B.

For the computation of ϱ we have, according to p. 59,

$$\left.\begin{aligned} \log\varrho &= \log a_0 - \log W + 2\log\cos h \\ \tan h &= \sqrt{\delta}\cos B \cos\alpha' \end{aligned}\right\} \quad (5)$$

w for the argument B.

Par. 12. Numerical Example I.

With Bessel's dimensions of the Earth ellipsoid, the logarithms of the coefficients in the foregoing expression are:

$$\log\left(\tfrac{1}{2}\,\varrho''\,e^2\right) = 2{,}8378056 \qquad \log\varrho'' = 5{,}31443$$

$$\log\left(\tfrac{1}{8}\,M\,.\,3e^4\right) = 1{,}86064$$
$$\log\left(\tfrac{1}{8}\,M\,.\,e^2\right) = 3{,}55910$$

For units of the seventh-decimal of the logarithm k.

$$\log a_0 = 6{,}8046434.637 \qquad \log\sqrt{\delta} = 8{,}9136593.9 - 10$$

For $\Delta B = 0{,}1$ and $k = 0{,}1$ the maximum error of the azimuths is $< 0{,}01''$, while that of $\log k$ is < 1 unit in the seventh decimal place. In regard to the accuracy, see particularly Pars. 8 - 10.

Par. 12. Numerical Example I. For B_1, B_2 and $L_{1.2}$ we take the following numerical values (see Bremiker, Studien über höhere Geodäsie, and Förster and Tietjen, Berliner Astronomisches Jahrbuch 1880):

Berlin $B_1 = 52^\circ\ 30'\ 16{,}7''$
Königsberg $B_2 = 54\ \ 42\ \ 50{,}6$
$L_{1.2} = 7^\circ\ 6'\ 0{,}0''$ East

Spherical Computation, using the series expansion given on p. 132 instead of (2) Par. 6, p. 145, yields the following numerical values:

$L_{1.2} = -25560{,}00000$ in seconds $\quad \log = 4{,}4075608.495_n \quad \log L = 9{,}0931357.163_n - 10$

$\Delta B = +\ 7953{,}90000$ in seconds $\quad 3{,}9005801.265 \quad \log \Delta B = 8{,}5861549.933 - 10$

$B = 53^\circ 36' 33{,}65'' \quad \log\cos B = 9{,}7732652.484 - 10 \quad \log\sin B = 9{,}9057908.074 - 10;$

$\log\frac{M}{24} = 5{,}2575731$ and $\log\frac{M}{2880} = 3{,}1784$ for units of seventh decimal.

Formula (1), p. 132, gives in particular:

$$\begin{cases} \log\left(-\frac{L\cos B}{\Delta B}\right) = 0{,}2802459.714 & \\ +\frac{1}{24}M.2L^2 = +\ 5557.436 & \log = 3{,}7448745 \\ +\frac{1}{24}M.\Delta B^2 = +\ 269.081 & 2{,}4298831 \\ +\frac{1}{2880}M.14L^4 = +\ 4.977 & 0{,}6970 \\ +\frac{1}{2880}M.\Delta B^4 = +\ .003 & 0{,}52-3 \end{cases}$$

$$\log\tan\alpha' = 0{,}2808291.211 \qquad \alpha' = 242^0\ 21'\ 14{,}9790''.$$

Since L^6 may be expected to have a slight effect upon the tenth decimal place of the logarithm, α' is uncertain by about one unit in the fourth decimal place of the seconds.

Corresponding to formula (2), p. 133 we have the numerical values

$$\log\sec\frac{\Delta B}{2} = 0{,}0000807.294 \qquad \begin{matrix}\log\cos B_1 = 9{,}7844013-10\\ \log\cos B_2 = 9{,}7616703-10\end{matrix}$$

$$\begin{cases} \log\left(-L_{1,2}\sin B\sec\frac{\Delta B}{2}\right) = 4{,}3134323.863-10 & \log M = 6{,}6377843 \\ \text{in seconds} & \text{For units of the seventh decimal.} \\ \left\{\frac{1}{12}M.L^2\cos B_1\cos B_2\sec^2\frac{\Delta B}{2}\right\} = +\ 1954.824 & \log = 3{,}2911076 \\ -\frac{13}{120}\sin^2 B\,L^2.\{\div\} = -\ 2.107 & 0{,}3237 \\ +\frac{7}{120}L^2.\{\div\} = +\ 1.751 & 0{,}2433 \end{cases}$$

$$\log\Delta\alpha' = 4{,}3136278.331 \qquad \frac{\Delta\alpha'}{2} = 2^0\ 51'\ 34{,}3241''.$$

$$\text{in seconds}$$

We now obtain:

$$\alpha'_{1,2} = 239^0\ 29'\ 40{,}6549'' \qquad \alpha'_{2,1} = 65^0\ 12'\ 49{,}3031''.$$

Par. 12. Numerical Example I.

Taking $q = \Delta B$ and $p = -\cos\frac{L_{1.2}}{2}\sec a'$, formula (5), p. 134, gives:

$$\log\cos\frac{L_{1.2}}{2} = 9{,}9991658.506 - 10 \qquad \log\cos a' = 9{,}6665225.564_n - 10$$

$$\left\{\begin{array}{rl} \log\frac{pq}{2} = & 8{,}6177682.918 - 10 \\ -\frac{1}{24}M.\Delta B^2 = - & 269.081 \\ -\frac{1}{2880}M.\Delta B^4 = - & .003 \end{array}\right\}$$ Already computed above.

$$\log\sin\frac{\sigma'}{2} = -8{,}6177413.834 - 10.$$

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If, however, we set $q = L$ and $p = \cos B \csc a'$, we have, since $\log\sin a' = 9{,}9473516.775_n - 10$:

$$\left\{\begin{array}{rl} \log\frac{pq}{2} = & 8{,}6180192.915 - 10 \\ -\frac{1}{24}M.L^2 = - & 2778.718 \\ -\frac{1}{2880}M.L^4 = - & .355 \end{array}\right\}$$ Already computed above

$$\log\sin\frac{\sigma'}{2} = 8{,}6177413.842 - 10.$$

The assumption of an increase of not quite 0,0002" in a' is sufficient to explain the difference of the two values of $\log\sin\frac{\sigma'}{2}$. The second value should be retained as being the more exact; or better yet, it should, making the assumption of this increase in a', be written

Result of the spherical computation.

$$\left\{\begin{array}{l} \log\sin\frac{\sigma'}{2} = 8{,}6177413.840 - 10 \\ a'_{1.2} = 239^0\ 29'\ 40{,}6549'' \\ a'_{2.1} = 65^0\ 12'\ 49.3031''. \end{array}\right.$$

Chapter 4. The Vertical Section and Chord Triangle for the Flattened Ellipsoid of Revolution.

In regard to the foregoing, it must be added that in the present case it would have been more convenient to apply formulae (2), p. 145, without series expansion. If, however, ΔB and L are even smaller than in the above example, in conjunction with ten-place logarithms the use of the series expansions may in fact be preferable. In any case the present purpose was simply to illustrate the numerical values of the series terms.

For formula (8), p. 146, we have:

$$\log c^2 = 7{,}8244104 - 10 \qquad \log\left(\tfrac{1}{2}\sin 2B\right) = 7{,}6790560 - 10,$$
$$-\log W_1 = 0{,}0009143 \qquad \log\sin\Delta B = 8{,}5860473 - 10,$$

the latter, derived from $\log \Delta B$ in seconds using the auxiliary logarithm S.

$$\begin{array}{r} \log E = 6{,}09134 - 10 \\ +\tfrac{1}{2} M.E = \quad 3 \\ \hline \log\left(1 - \frac{W_2}{W_1}\right) = 6{,}09137 - \overline{10}. \end{array}$$

For formula (9), p. 146, the values are:

(See Illustrative Material on Next Page.)

Par. 12. Numerical Example I.

$$\log\left(2\sin\frac{\Delta B}{2}\right) = \log\sin\Delta B + \log\sec\frac{\Delta B}{2} = 8{,}5861280-10$$

$\log M = 6{,}63778$ for units of the seventh decimal place. $\log\sin B_1 = 9{,}89949-10$

$$\log D = 8{,}3593932-10$$

$+\left\{\frac{ME}{D}\sin B_1\right\} = +$	18588	$\log = 4{,}26922$
$-\frac{E\sin B_1}{2D}\left\{\frac{ME}{D}\sin B_1\right\} = -$	40	1,600
$+\frac{1}{2}E\left\{\frac{ME}{D}\sin B_1\right\} = +$	1	0,060

$$\log\left(\sin B_2 - \frac{W_2}{W_1}\sin B_1\right) = 8{,}3612481-10.$$

For formulae (7), p. 146, we find:

$$\log e\sqrt{2-e^2} = 9{,}0619943-10$$

$\log\sin q'' = 8{,}66813 \quad -10$	$\log\cos q'' = 9{,}9995285 \quad -10$
$\log W_2 = 9{,}9990321-10$	$\log\sqrt{\frac{W_1}{W_2}} = 0{,}0000268$
$\log\sin q' = 8{,}5040263-10$	$\log\cos q' = 9{,}9997786.761-10.$

The formulae (5) and (6), p. 145, yield

$$\left.\begin{array}{l}\log 2a_0 = 7{,}1056734.594 \text{ und}\\ \log W_1 = 9{,}9990857.480-10,\\ \log W_2 = 9{,}9990321.461-10:\end{array}\right\} \log k = 5{,}7241345.725.$$

The formulae (4), p. 145, lead to the following numerical values:

(See Illustrative Material on Next Page.)

$\log \sin L_{1.2} = 9{,}0920237_n - 10$ $\qquad \log \frac{\cos B_1}{\cos B_2} = 0{,}0227310$

$\log \tan \varepsilon'_{1.2} = 7{,}1163658_n - 10$ $\qquad \log \tan \varepsilon'_{2.1} = 7{,}0709574_n - 10$

$T = 4{,}6855751 - 10$ $\qquad T = 4{,}6855751 - 10$

$\log \varepsilon'_{1.2} = 2{,}4307907_n$ $\qquad \log \varepsilon'_{2.1} = 2{,}3853823_n$

in seconds $\qquad$ in seconds

$\varepsilon'_{1.2} = -\ 4'\ 29{,}6440''$ $\qquad \varepsilon'_{2.2} = -\ 4'\ 2{,}8747''$

$a'_{1.2} - \varepsilon'_{1.2} = 239^0\ 34'\ 10{,}2989''$ $\qquad a'_{2.1} - \varepsilon'_{2.1} = 65^0\ 16'\ 52{,}1778''$

$\log \sin a'_{1.2} = 9{,}9352963.933_n - 10$

$\log \cos \varepsilon'_{1.2} = 9{,}9999996.289 - 10$

$\log \cos (a'_{1.2} - \varepsilon'_{1.2}) = 9{,}7045729.476_n - 10$

$\log \tan a_{1.2} = 0{,}2307230.746$

$\log \sin a'_{2.1} = 9{,}9580273.552 - 10$

$\log \cos \varepsilon'_{2.1} = 9{,}9999996.989 - 10$

$\log \cos (a'_{2.1} - \varepsilon'_{2.1}) = 9{,}6213484.891 - 10$

$\log \tan a_{2.1} = 0{,}3366785.660.$

With this we reach the following

result: $\log k = 5{,}7241345.725$

$a_{1.2} = 239^0\ 33'\ 0{,}9324''$

$a_{2.1} = 65^0\ 16'\ 9{,}5806''.$

Bremiker gives in his Studien, etc., on p. 37 the values 0,9325" and 9,5809" respectively, but on p. 28 the values 0,9326" and 9,5800" as the seconds in the azimuths. His value (p. 28) for the last three places of $\log k$ is .730. The agreement of these two sets of results may be considered satisfactory.

Par. 13. Continuation of Numerical Example I. In formula (3), p. 158,

(See Illustrative Material on Next Page.)

Par. 13. Continuation of Numerical Example I.

$$\log \cos B_1 \cos B_2 = 9{,}5460716 - 10 \qquad \log \sin 2a' = 9{,}9149042 - 10$$

$$\log \cos u = 9{,}72062_n - 10 \qquad \log \sec^2 \frac{a'}{2} = 0{,}0007475$$

$$u = 58^0\ 17',7 \qquad 2 \log W_u = 9{,}9978969 - 10$$

$$\log \left\{ \frac{1}{2} \varrho'' e^2 \frac{\cos B_1 \cos B_2 \sin 2a' \sec^2 \frac{a'}{2}}{W_u^2} \right\} = 2{,}3016320.$$

Consequently, formulae (2) and (3), p. 157 and 158, yield the values:

$$\frac{a_{2.1} - a_{1.2}}{2} = 92^0\ 51'\ 34{,}3241''$$

$$\frac{a_{2.1} + a_{1.2}}{2} = 332^0\ 21'\ 14{,}9790'' + 3'\ 20{,}2774''$$

$$= 332^0\ 24'\ 35{,}2564''.$$

From this follows the

Result: $\begin{cases} a_{1.2} = 239^0\ 33'\ 0{,}9323'' \\ a_{2.1} = 65^0\ 16'\ 9{,}5805''. \end{cases}$

These values differ from the more rigorous results of the preceding section by only about 0,0001", since the neglected terms, which could make themselves felt in the ten-thousandths of a second in the sum $a_{2.1} + a_{1.2}$, happen to balance each other.

For formula (5), p. 158, we obtain:

$$\begin{cases} \log \sqrt{\delta} = 8{,}9136593.9 - 10 \\ \log \cos B = 9{,}7732652.5 - 10 \\ \log \cos a' = 9{,}6665225.6_n - 10 \\ \log \tan h = 8{,}3534472.0_n - 10 \end{cases} \qquad \begin{cases} \log a_0 = 6{,}8046434.637 \\ \log W = 9{,}9990587.885 - 10 \\ 2 \log \cos h = 9{,}9997789.106 - 10 \\ \log \varrho = 6{,}8053635.858. \end{cases}$$

Bremiker's table in his six-place logarithms gives 6,8053635.9, a value which would be perfectly adequate if we wished to use only formula (4), p. 158, and accordingly computed to seven or eight places. However, the higher terms which were neglected should now be computed.

Since $\log \cos 2B = 9{,}47132_n - 10$, we have for formula (4), p. 158,

$$\begin{aligned}
&\log\left(2\varrho \sin\frac{\sigma'}{2}\right) = 5{,}7241349.655\\
&+\frac{M}{8}\cdot 3e^4 \cos^4 B \sin^2 2\alpha' = \quad +6.074\\
&+\frac{M}{8}\cdot e^2 \Delta B^2 \cos 2B = \quad -1.594\\
&-\frac{M}{8}\cdot 2e^2 \Delta B^2 \sin^2 \alpha' = \quad -8.456\\
&\log k = 5{,}7241345.679.
\end{aligned}$$

Formula (2), p. 155, gives for the three terms of the remainder R the values

$$+146, +3, -47$$

in units of the tenth decimal place. For $\log k$ we have further to multiply by M , which yields a correction of 44 units in the tenth decimal place of $\log k$. Hence the result is:

Result: $\log k = 5{,}7241345.723,$

a value which agrees with that found on p. 162 as closely as the uncertainty of computation of the last place allows.

For formula (6), p. 157, we obtain:

(See Illustrative Material on Next Page.)

Par. 13. Continuation of Numerical Example I.

$$a = 242^0\,24'\,35{,}2564'' \qquad \log \varrho = 6{,}8053644.048$$

$$\begin{cases} \log\left(2\varrho \sin \frac{\sigma'}{2}\right) = 5{,}7241357.845 \\ -\frac{M}{8} \cdot e^4 \cos^4 B \sin^2 2a = \quad -2.019 \\ +\frac{M}{8} \cdot e^2 \Delta B^2 \cos 2B = \quad -1.594 \\ -\frac{M}{8} \cdot 2e^2 \Delta B^2 \sin^2 a = \quad -8.464 \end{cases}$$

$$\log k = 5{,}7241345.768.$$

The neglected sixth-order terms of (7), p. 157, amount to (in units of the tenth decimal place M), $(-56+3-41)$

that is,..... -41;

hence the result is: $\log k = 5{,}7241345.727$,

which again is in good agreement with the precise computation.

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Formula (5), p. 141, for the direct computation of a single azimuth gives for $a_{1,2}$:

(See Illustrative Material on Next Page.)

$B' = 53^0\,3'\,25{,}175''$ $\quad \log\cos 2B' = 9{,}4433_n - 10$ $\quad \log\sin^2 B' = 9{,}80 - 10$

$$\begin{cases} \log\dfrac{\Delta B}{L} = 9{,}4930192.770_n - 10 \\ \log(1-e^2) = 9{,}9970916.405 \end{cases} \qquad \begin{cases} \log\cos B_2 = 9{,}7616703.379 - 10 \\ 2\log W' = 9{,}9981444.626 - 10 \end{cases}$$

$$\text{Sum} = 9{,}4901109.175_n - 10 \qquad \text{Sum} = 9{,}7598148.005 - 10$$

$$\begin{cases} \log\left(-\dfrac{(1-e^2)\Delta B}{W'^2 L\cos B_2}\right) = 9{,}7302961.17 - 10; & \text{num.} = 0{,}537398088 \\ \log\left(-\dfrac{(1-e^2)\Delta B}{W'^2 L\cos B_2}\right) + \log\left(-\dfrac{\Delta B^2}{6}\right) = 6{,}124455_n - 10; & \text{num.} = - \quad 133185 \\ \log\left(-\dfrac{(1-e^2)\Delta B}{W'^2 L\cos B_2}\right) + \log\left(\dfrac{L^2}{6}\right) = 7{,}1384163 - 10; & \text{num.} = + \quad 1375360 \\ \log\left(-\dfrac{(1-e^2)\Delta B}{W'^2 L\cos B_2}\right) + \log\left(\dfrac{\Delta B^2 - L^2}{6}\,\dfrac{3\Delta B^2 - 7L^2}{60}\right) = 4{,}32900 - 10; & \\ & \text{num.} = + \quad 2133 \\ \log\left(-\dfrac{(1-e^2)\Delta B}{W'^2 L\cos B_2}\right) + \log\left(-\dfrac{1}{16}e^2\Delta B^2\cos 2B'\right) = 2{,}966 - 10; & \\ & \text{num.} = + \quad 93 \\ \log\left(-\dfrac{(1-e^2)\Delta B}{W'^2 L\cos B_2}\right) + \log\left(\dfrac{1}{16}e^4\Delta B^2\sin^2 B'(2+\cos 2B')\right) = 1{,}39 - 10; & \\ & \text{num.} = + \quad 2 \\ \log\left(-\dfrac{(1-e^2)\Delta B}{W'^2 L\cos B_2}\right) + \log\dfrac{31L^6}{15120} = 1{,}60 - 10; & \text{num.} = + \quad 4 \end{cases}$$

$$\begin{cases} \log\left(-\dfrac{L}{2}\sin B_1\right) = 8{,}6915993.5 - 10; & \text{num.} = + \; 49158582 \\ \log\left(-\dfrac{L}{2}\sin B_1\right) + \log\dfrac{L^2}{12} = 5{,}79869 - 10; & \text{num.} = + \quad 62906 \\ \log\left(-\dfrac{L}{2}\sin B_1\right) + \log\dfrac{L^4}{120} = 2{,}986 - 10; & \text{num.} = + \quad 97 \end{cases}$$

$$\log\cot a_{1.2} = 9{,}7692769.24 - 10 \qquad \cot a_{1.2} = 0{,}587864080.$$

The final result is thus:

$$a_{1.2} = 239^0\,33'\,0{,}9326''.$$

In this computation the two largest sixth-order terms are taken into account, since they alter $a_{1.2}$ by some 0,0009". In cases for which this formula is designed, and in which it alone is advantageous, the inclusion of these terms is of course unnecessary.

Par. 14. Numerical Example II. Given:

$$B_1 = 57^\circ \quad B_2 = 56^\circ\,13'\,49{,}02186''$$
$$L_{1,2} = 1^\circ\,22'\,6{,}03270'' \text{ Easterly}$$

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We apply the formulae of Par. 11, p. 157, to these data and make use of eight-place logarithms. To begin with, making partial use of the auxiliary logarithm S from the seven-place tables we have:

$$\frac{\Delta B}{2} = -1385{,}48907'' \quad \log\sin = 7{,}8271747.0_n - 10 \quad \log\cos = 9{,}9999902.0 - 10$$
$$\frac{L_{1,2}}{2} = 180^\circ - 2463{,}01635'' \quad \log\sin = 8{,}0770318.4 - 10 \quad \log\cos = 9{,}9999690.4_n - 10$$
$$B = 56^\circ\,36'\,54{,}51093'' \quad \log\sin = 9{,}9216830.0 - 10 \quad \log\cos = 9{,}7405680.1 - 10.$$

The formulae (1), p. 157, now give:

$$\cos a' \sin\frac{\sigma'}{2} = [7{,}8271437.4_n - 10]$$
$$\sin a' \sin\frac{\sigma'}{2} = [7{,}8175998.5 - 10]$$
$$\cos\frac{\Delta a'}{2}\cos\frac{\sigma'}{2} = [9{,}9999592.4_n - 10]$$
$$\sin\frac{\Delta a'}{2}\cos\frac{\sigma'}{2} = [7{,}9987148.4_n - 10]$$
$$a' = 135^\circ\,37'\,46{,}216'' \quad \log\sin = 9{,}8446606.3 - 10 \quad \log\cos = 9{,}8542045.2_n - 10$$
$$\log\sin\frac{\sigma'}{2} = 7{,}9729392.2$$
$$\frac{\Delta a'}{2} = 180^\circ\,34'\,16{,}678'' \quad \log\sin = 7{,}9987340.1_n - 10 \quad \log\cos = 9{,}9999784.1_n - 10$$
$$\log\cos\frac{\sigma'}{2} = 9{,}9999808.3.$$

For the formulae (2) and (3) of p. 157 we have:

(See Illustrative Material on Next Page.)

$$\begin{cases} \log\left(\frac{1}{2}\varrho''e^2\right) = 2{,}837806 \\ \log\cos B_1 = 9{,}736109 - 10 \\ \log\cos B_2 = 9{,}744963 - 10 \\ \log\sin 2a' = 9{,}999895_n - 10 \\ \log\sec^2\frac{\sigma'}{2} = 0{,}000038 \\ -2\log W_m = 0{,}002476 \end{cases} \qquad \begin{cases} \log\cos B = 9{,}74057 - 10 \\ \log\sin a' = 9{,}84466 - 10 \end{cases}$$

$$\text{Sum} = 2{,}321287_n \quad \log\cos u = 9{,}58523 - 10$$

$$u = 67^0\,22',2 \qquad \text{num.} = -209{,}550''$$

$$\log W_m = 9{,}998762 - 10$$

$$\begin{cases} \frac{\alpha_{2.1} - \alpha_{1.2}}{2} = 270^0\,34'\,16{,}678'' \\ \frac{\alpha_{2.1} + \alpha_{1.2}}{2} = 225^0\,34'\,16{,}666'' \end{cases}$$

$$\alpha_{2.1} = 136^0\ 8'\ 33{,}344''$$

$$\alpha_{1.2} = 314\ 59\ 59{,}988.$$

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For the formulae (5) and (4) of p. 158 we find:

$$\begin{cases} \log\sqrt{\delta} = 8{,}913659 - 10 \\ \log\cos B = 9{,}740568 - 10 \\ \log\cos a' = 9{,}854205_n - 10 \end{cases} \qquad \begin{cases} \log a_0 = 6{,}8046434.6 \\ -\log W = 0{,}0010128.5 \\ 2\log\cos h = 9{,}9995487.4 - 10 \end{cases}$$

$$\log\tan h = 8{,}508432_n - 10 \qquad \log\varrho = 6{,}8052050.5$$

$$\begin{cases} \log\left(\frac{1}{8}M.3e^4\right) = 1{,}861 \\ \log\cos^4 B = 8{,}962 - 10 \\ \log\sin^2 2a' = 0{,}000 \end{cases} \quad \begin{cases} \log\left(\frac{1}{8}Me^2\right) = 3{,}56 \\ \log\Delta B^2 = 6{,}26 \\ \log\cos 2B = 9{,}60_n - 10 \end{cases} \quad \begin{cases} \log\left(\frac{1}{8}Me^2.2\right) = 3{,}86 \\ \ldots\ldots\ 6{,}26 \\ \log\sin^2 a' = 9{,}69 - 10 \end{cases}$$

$$\text{Sum} = 0{,}823 \qquad \text{Sum} = 9{,}42_n \qquad \text{Sum} = 9{,}81 - 10$$

$$\text{num.} = +6.65 \qquad \text{num.} = -0.26 \qquad \text{num.} = +0.64$$

$$\log k = 5{,}0791742.7 + 6.65 - 0.26 - 0.64.$$

$$= 5{,}0791748.5.$$

Accordingly, the following values emerge as the

result

$$\begin{cases} \log k = 5{,}0791748.5 \\ \alpha_{1.2} = 314^0\,59'\,59{,}988'' \\ \alpha_{2.1} = 136\ \ 8\ 33{,}344. \end{cases}$$

Par. 15. Transfer of the Geographic Position by Means of Chord and Astronomical Azimuth.

Let there be given for P_1 the geographical latitude B_1 the chord k to the point P_2, and the azimuth $\alpha_{1.2}$ of the vertical section from P_1 passing through P_2. The values required are B_2, $L_{1.2}$ and $\alpha_{2.1}$. We denote the angle of inclination of k relative to the horizontal plane at P_1 (the plane $\xi\eta$, p. 136), which is called the angle of depression, by $\mu_{1.2}$. In accord with Par. 2, p. 136, the projection of k will now give us at once:

$$\begin{aligned} k\cos\mu_{1.2}\cos\alpha_{1.2} &= \xi_2 = (x_2 - x_1)\sin B_1 - (z_2 - z_1)\cos B_1 \\ k\cos\mu_{1.2}\sin\alpha_{1.2} &= \eta_2 = y_2 \\ k\sin\mu_{1.2} &= \zeta_2 = -(x_2 - x_1)\cos B_1 - (z_2 - z_1)\sin B_1. \end{aligned}$$

This is readily reduced to:

$$\left.\begin{aligned} \frac{\cos B_2\cos L_{1.2}\sin B_2}{\sqrt{1-e^2\sin^2 B_2}} - (1-e^2)\frac{\sin B_2\cos B_1}{\sqrt{1-e^2\sin^2 B_2}} &= \frac{k}{a_0}\cos\mu_{1.2}\cos\alpha_{1.2} + \frac{e^2\sin B_1\cos B_1}{\sqrt{1-e^2\sin^2 B_1}} \\ \frac{\cos B_2\sin L_{1.2}}{\sqrt{1-e^2\sin^2 B_2}} &= \frac{k}{a_0}\cos\mu_{1.2}\sin\alpha_{1.2} \\ \frac{\cos B_2\cos L_{1.2}\cos B_1}{\sqrt{1-e^2\sin^2 B_2}} + (1-e^2)\frac{\sin B_2\sin B_1}{\sqrt{1-e^2\sin^2 B_2}} &= -\frac{k}{a_0}\sin\mu_{1.2} + \sqrt{1-e^2\sin^2 B_1}. \end{aligned}\right\} (1)$$

These three equations serve to determine the unknowns B_2, $L_{1.2}$ and $\mu_{1.2}$. The attempt to derive the first two of these unknowns with the elimination of $\mu_{1.2}$ leads to very complicated formulae. Hence, we take the opposite course, eliminating B_2 and $L_{1.2}$ and deriving $\mu_{1.2}$.

Eliminating $\cos B_2\cos L_{1.2}$ and $\sin B_2$ respectively from the

first and third of the equations designated by (1), we obtain:

$$\left.\begin{aligned} \frac{\sin B_2}{\sqrt{1-e^2\sin^2 B_2}} &= \frac{\sin B_1}{\sqrt{1-e^2\sin^2 B_1}} \\ &\quad - \frac{k}{a_0(1-e^2)}(\cos\mu_{1.2}\cos B_1\cos\alpha_{1.2} + \sin\mu_{1.2}\sin B_1) \\ \text{and} \\ \frac{\cos B_2\cos L_{1.2}}{\sqrt{1-e^2\sin^2 B_2}} &= \frac{\cos B_1}{\sqrt{1-e^2\sin^2 B_1}} \\ &\quad + \frac{k}{a_0}(\cos\mu_{1.2}\sin B_1\cos\alpha_{1.2} - \sin\mu_{1.2}\cos B_1). \end{aligned}\right\} \quad (2)$$

Squaring the second of the equations denoted by (2), adding this square to the square of the second equation of (1), multiplying the sum by $(1-e^2)$ and adding the product obtained by multiplying the square of the first equation of (1) by $(1-e^2)^2$ in each case, side for side, we obtain:

$$0 = -\frac{2k(1-e^2)\sin\mu_{1.2}}{a_0\sqrt{1-e^2\sin^2 B_1}} + \frac{k^2}{a_0^2}(1-e^2+e^2\cos^2\chi),$$

wherein we set

$$\cos\chi = \cos\mu_{1.2}\cos B_1\cos\alpha_{1.2} + \sin\mu_{1.2}\sin B_1. \quad (3)$$

$180^\circ - \chi$ is the angle between the axis of the Earth and the chord P_1P_2, as will be readily seen by imagining a parallel to the Earth's axis drawn thru P_1 and considering the angle between this parallel, the chord and the plumb line.

If we regard $\cos\chi$ as known, we have for the determination of $\mu_{1.2}$ the formula

$$\begin{aligned} \sin\mu_{1.2} &= \frac{k\sqrt{1-e^2\sin^2 B_1}}{2a_0}\left(1+\frac{e^2\cos^2\chi}{1-e^2}\right) \\ &= \frac{kW_1}{2a_0}(1+\vartheta\cos^2\chi) = \frac{kW_1}{2a_0}\sec^2 u, \end{aligned} \quad (4)$$

setting $\sqrt{\vartheta}\cos\chi = \tan u$.

Now since χ is not known, but requires a knowledge of $\mu_{1.2}$ in order to compute it, we must make use of successive approximations in order to find $\mu_{1.2}$. We set:

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$$\left.\begin{aligned}\cot h &= \cot B_1 \cos a_{1.2}\\ \cos\chi &= \cos(\mu_{1.2} - h)\,\frac{\sin B_1}{\sin h}\end{aligned}\right\} \qquad (5)$$

and, if a table of values of ϱ for the arguments B and a is available, we take as a first approximation:

$$\sin\mu_{1.2} = \frac{k}{2\varrho};\quad \varrho \text{ for } B_1 \text{ and } a_{1.2}, \qquad (6)$$

otherwise, based on the expression (1) for ϱ on p. 58, we write

$$\log\sin\mu_{1.2} = \log\frac{k W_1}{2a_0} + M\vartheta\cos^2 B_1\cos^2 a_{1.2} + \cdots. \qquad (6^*)$$

In order to find B_2 we now start from the first equation of (2) preferably using the reduced latitude as intermediate term (cf. p. 40) by setting:

$$\left.\begin{aligned}\sin\beta_1 &= \frac{\sqrt{1-e^2}\sin B_1}{W_1}\\ \sin\beta_2 &= \sin\beta_1 - \frac{k\cos\chi}{a_0\sqrt{1-e^2}}\end{aligned}\right\} \qquad (7)$$

We can further derive from this

$$\left.\begin{aligned}\sin\frac{\Delta\beta}{2} &= -\frac{k\cos\chi}{2a_0\sqrt{1-e^2}}\sec\beta\\ \Delta\beta &= \beta_2 - \beta_1 \qquad \beta = \tfrac{1}{2}(\beta_2 + \beta_1)\end{aligned}\right\} \qquad (8)$$

and finally

$$\sin \Delta B = \frac{2 W_1}{w_2} \sin \frac{\Delta\beta}{2} \cos \frac{\Delta\beta}{2}, \qquad (9)$$

this last formula being obtained by introducing the reduced latitudes into the right side of the identity:

$$\sin \Delta B = \sin B_2 \cos B_1 - \cos B_2 \sin B_1$$

using formulae (5), (6), (7) of pp. 40 and 41.

The computation with these formulae begins with an approximate determination of β_2 from (7), insofar as this is conveniently possible by the use of Gaussian logarithms. Formula (8) gives $\sin \frac{\Delta\beta}{2}$ by successive approximations, and (9) thereupon gives ΔB.

For the computation of $L_{1.2}$ we have next, from the second formula of (1):

$$\sin L_{1.2} = \frac{k}{a_0} \cos \mu_{1.2} \sec \beta_2 \sin a_{1.2} = \frac{k\sqrt{1-e^2}}{a_0 w_2} \cos \mu_{1.2} \sec B_2 \sin a_{1.2}. \qquad (10)$$

Finally, formula (4) of p. 138 is, in general, the appropriate means of finding $a_{2.1}$; however, it will usually be sufficient to compute:

$$\left.\begin{aligned} \tan \frac{\Delta a'}{2} &= -\tan \frac{L_{1.2}}{2} \sin B \sec \frac{\Delta B}{2} \\ \text{and} \\ a_{2.1} &= a_{1.2} + 180^0 + \Delta a' + \tfrac{1}{4} \psi'' e^4 \sin L_{1.2} \sin^2 \Delta B \sin B \cos^4 B + \cdots \end{aligned}\right\} \qquad (11)$$

in accord with Par. 6 (2), p. 145 and Par. 8 (4), p. 151. By this procedure values of $\alpha_{2.1}$ will be obtained which are uncertain by no more than 0,0001" even for values of $k = 0{,}2\,a_0$.

The above formulae do not permit a satisfactory solution in all cases, in particular because $\mu_{1.2}$, B_2 or ΔB and $L_{1.2}$ are derived from the sine. However, this inadequacy at least in particular cases, will extend only to moderately large values of $\mu_{1.2}$ and ΔB, which are capable of determination by the sine. On the other hand, even for small distances, $L_{1.2}$ can be large if the points lie near the poles. Then, however, the determination of B_2 from (7), (8), (9) is also unsatisfactory, and it is better to derive B_2 and $L_{1.2}$ from the second formula of (1) and the second formula of (2).

The question of series expansions of the foregoing may arise, chiefly in connection with the indirect derivation of $\mu_{1.2}$ and ΔB. The series for $\mu_{1.2}$ will be given below; it is poorly suited for computation because of its considerable complexity. Since this is even more true of the series for ΔB, the latter expansion will be entirely omitted.

If ΔB and $L_{1.2}$ are small, the application of the first series under (4), p. 30, to $\sin \Delta B$ and $\sin L_{1.2}$ may in some cases be useful, while for the first formula of (11) the applicable expansion is that of (2), p. 133.

If in formula (10) the subscripts 1 and 2 are interchanged and the two expressions for $\sin L_{1.2}$ are then set equal to each other, we obtain the following interesting equation which will be important

hereafter:

$$\sin \alpha_{1.2} \cos \beta_1 \cos \mu_{1.2} = \sin (\alpha_{2.1} - 180^0) \cos \beta_2 \cos \mu_{2.1}, \qquad (12)$$

This equation can also serve for the computation of $\alpha_{2.1}$ after the derivation of $\mu_{2.1}$ from formula (4) (wherein only B_1 and B_2 are to be interchanged, χ remaining unaltered). However, we shall not go into detail concerning this, nor concerning the transformations necessary in order that $\alpha_{2.1}$ should always be obtained with the necessary precision (cf. Bremiker, Studies....etc. p. 34).

Par. 16. Numerical Example I. (Inverted form of the numerical example on p. 158 et. seq.) Given: $B_1 = 52^0\ 30'\ 16{,}7''$

$$\log k = 5{,}7241345.725 \qquad \alpha_{1.2} = 239^0\ 33'\ 0{,}9324''.$$

According to formulae (4), (5) and (6*) of the preceding section we have:

$$\log 2a_0 = 7{,}1056734.594$$

$$\left\{\begin{array}{l} \log \frac{k}{2a_0} = 8{,}6184611.131 - 10 \\ \log W_1 = 9{,}9990857.480 - 10 \end{array}\right.$$

$$\left\{\begin{array}{l} \log \frac{k W_1}{2a_0} = 8{,}6175468.611 - 10 \\ [6{,}45 - 10] = 0{,}00028 \quad \ldots\ldots \end{array}\right.$$

$$\left\{\begin{array}{l} \log M\delta = 7{,}47 - 10 \\ \log \cos^2 \alpha_{1.2} = 9{,}41 - 10 \\ \log \cos^2 B_1 = 9{,}57 - 10 \\ \text{Sum} = 6{,}45 - 10 \end{array}\right.$$

First approximation: $\log \sin \mu_{1.2} = 8{,}61783 - 10$; $\mu_{1.2} = 2^0\ 22'\ 38''$.

(See Illustrative Material on Next Page.)

Par. 16. Numerical Example I.

$$\begin{cases} \log\cos a_{1.2} = & 9{,}7048214.896_n - 10 \\ \log\cot B_1 = & 9{,}8849076.665 - 10 \\ \hline \log\cot h = & 9{,}5897291.561_n - 10 \end{cases}$$

$$h = -68^\circ\,45'\,13{,}6416''$$

$$\mu_{1.2} - h = 71^\circ\,7'\,52''$$

First Approximation

$$\begin{cases} \log\sin a_{1.2} = 9{,}9355445.641_n - 10 \\ \log\sin B_1 = 9{,}8994936.333 - 10 \\ \log\sin h = 9{,}9694307.477_n - 10 \end{cases}$$

$$\begin{cases} \log\dfrac{\sin B_1}{\sin h} = 9{,}9300628.856_n - 10 \\ \log\cos(\mu_{1.2} - h) = 9{,}509745 - 10 \end{cases}$$

Difference for 1": 6,1 units of the 6th decimal place.

$$\begin{cases} \log\cos\chi = 9{,}439808_n - 10 \\ \log\sqrt{\delta} = 8{,}913659 - 10 \end{cases}$$

$$\log\tan u = 8{,}353467_n - 10.$$

To this there corresponds: $\log\sec u = 0{,}0001105.55$ with an increase of 0,51 unit in the ninth place for each unit in the sixth place of $\log\tan u$.

We now have for the second approximation:

$$\log\sin\mu_{1.2} = \log\frac{k W_1}{2a_0} + 2 \times 0{,}0001105.55 = 8{,}6177679.71 - 10,$$

$$\mu_{1.2} = 2^\circ\,22'\,36{,}9242''$$

with an increase of 0,000020" for each unit in the ninth decimal place of $\log\sin\mu_{1.2}$.

The value $(\mu_{1.2} - h) = 71^\circ\,7'\,50{,}5658''$ now obtained is more precise than the value shown above. Corresponding to the decrease of 1,43" in this angle we may note an increase of nine units in the sixth place of $\cos(\mu_{1.2} - h)$ as well as $2 \times 9 \times 0{,}51 = 9$ units in the ninth place of $\sin\mu_{1.2}$ and finally 0,0002" increase in $\mu_{1.2}$. Hence we have definitively

$$\mu_{1.2} = 2^\circ\,22'\,36{,}9244'' \qquad \mu_{1.2} - h = 71^\circ\,7'\,50{,}5660''$$

$$\log\cos(\mu_{1.2} - h) = 9{,}5097537.572 - 10 \qquad \log\cos\chi = 9{,}4398166.428_n - 10.$$

Furthermore, we have for the formulae (7) and (8):

$$\left\{\begin{array}{ll} \log \sin \beta_1 = 9{,}8989537.055 - 10 & \log \sqrt{1-e^2} = 9{,}9985458.202 - 10 \\ \log \frac{k \cos z}{a_0 \sqrt{1-e^2}} = 8{,}360762_n - 10 & \log \frac{k \cos z}{2 a_0 \sqrt{1-e^2}} = 8{,}0597319.357_n - 10. \end{array}\right.$$

The two logarithms on the left connected by a brace give with the use of six-place Gaussian logarithms (seven-place tables being unavailable to the author)

$$\log \sin \beta_2 = 9{,}911353 - 10 \text{ und } \beta_2 = 54^0\ 37'\ 25''.$$

On the other hand, from the $\log \sin \beta_1$, we find the value $\beta_1 = 52^0\ 24'\ 43{,}0115''$ which is in sufficiently close agreement with the more rigorous value derived on p. 43:

$$\beta_1 = 52^0\ 24'\ 43{,}01137''$$

when formula (9) is used. However, we cannot conveniently use this formula since a table of $\log w$ to ten places is lacking; hence, we pass from β_2 as found by equation (8) to B_2. We must then, however, use the precise value for β_1.

In first approximation we now obtain $\beta = 53^0\ 31'\ 4''$ and $\log \sec \beta = 0{,}2257945.702$ with an increase of 28.472 for each 1" in β; furthermore,

$$\log \sin \frac{\Delta\beta}{2} = 8{,}2855265 - 10; \quad \frac{\Delta\beta}{2} = 1^0\ 6'\ 20{,}875''.$$

This leads in second approximation to $\beta = 53^0\ 31'\ 3{,}886''$, $\log \sec \beta = 0{,}2257942.456$ and

$$\log \sin \frac{\Delta\beta}{2} = 8{,}2855261.813 - 10; \quad \frac{\Delta\beta}{2} = 1^0\ 6'\ 20{,}87257''.$$

A change of 0,00001" in $\frac{\Delta\beta}{2}$ changes the log sin by 11 units in the tenth decimal place; and since in the foregoing computation β was taken 0,00206" too large, and thus $\log\sec\beta$ and $\log\sin\frac{\Delta\beta}{2}$ have to be diminished by 59 units, $\frac{\Delta\beta}{2}$ has also to be diminished by a further 0,00005". Hence, we have definitively:

$$\log\sin\frac{\Delta\beta}{2} = 8{,}2855261.754 - 10; \quad \frac{\Delta\beta}{2} = 1^\circ\,6'\,20{,}87252''$$

$$\beta_2 = 54^\circ\,37'\,24{,}75641''.$$

Passing from β_2 to B_2 in accordance with the formulae of Par. 3, p. 41 et. seq., we obtain $B_2 - \beta_2 = 5'\,25{,}84361''$; thence

$$B_2 = 54^\circ\,42'\,50{,}60002''.$$

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Finally, with respect to formula (10) we have $\log\cos\beta_2 = 9{,}7626381.919 - 10$

$$\log\cos\mu_{1.2} = 9{,}9996261.786 - 10 \qquad \log\sin L_{1.2} = 9{,}0920236.596_n - 10$$

$$L_{1.2} = 7^\circ\,6'\,0{,}00000''\ \text{eastward}$$

There now remains $a_{2.1}$ to be computed, which we shall here omit since the formulae (11) have already been dealt with on pp. 159 and 162.

Par. 17. Numerical Example II. (Inverted form of the numerical example on p. 164). Given:

$$B_1 = 57^\circ$$

$$\log k = 5{,}0791748.5 \qquad a_{1.2} = 314^\circ\,59'\,59{,}988''.$$

From the formulae (4), (5) and (6*) of pp. 167 and 168 we have:

(See Illustrative Material on Next Page.)

$\log 2a_0 = 7{,}1056734.6 - 10$

$\log \frac{k}{2a_0} = 7{,}9735013.9 - 10$

$\log W_1 = 9{,}9989781.9 - 10$

$\log \frac{k W_1}{2a_0} = 7{,}9724795.8 - 10$

$[6{,}6363 - 10] = 0{,}0004328$ . . . .

$\log M\delta = 7{,}4651 - 10$

$\log \cos^2 B_1 = 9{,}4722 - 10$

$\log \cos^2 a_{1.2} = 9{,}6990 - 10$

Sum $= 6{,}6363 - 10$

First Approximation: $\log \sin \mu_{1.2} = 7{,}9729124 - 10$; $\mu_{1.2} = 0^0\ 32'\ 17{,}96''$.

$\log \cos a_{1.2} = 9{,}8494849.8 - 10$

$\log \cot B_1 = 9{,}8125173.6 - 10$

$\log \cot h = 9{,}6620023.4 - 10$

$h = 65^0\ 20'\ 7{,}403''$

$h - \mu_{1.2} = 64\ 47\ 49{,}443$ . . .

$\log \sin a_{1.2} = 9{,}8494850.3 - 10$

$\log \sin B_1 = 9{,}9235914.0 - 10$

$\log \sin h = 9{,}9584521.3 - 10$

$\log \frac{\sin B_1}{\sin h} = 9{,}9651392.7 - 10$

$\log \cos(\mu_{1.2} - h) = 9{,}629232 - 10$

First Approximation

$\log \frac{k W_1}{2a_0} = 7{,}9724795.8 - 10$

$2 \log \sec u = 0{,}0004504.2$ . . . . . .

$\log \cos \chi = 9{,}594371 - 10$

$\log \sqrt{\delta} = 8{,}913659 - 10$

$\log \tan u = 8{,}508030 - 10$

$\log \cos u = 9{,}9997747.9 - 10$

$\log \sin \mu_{1.2} = 7{,}9729300.0 - 10$

$S = 4{,}6855684.7 - 10$

$\log \mu_{1.2} = 3{,}2873615.3$ . . . . . .

in secs.

Second Approximation (Gives definitive value.)

$\mu_{1.2} = 0^0\ 32'\ 18{,}035''$

$h - \mu_{1.2} = 64^0\ 47'\ 49{,}368''$ . .

$\log \cos(\mu_{1.2} - h) = 9{,}6292321.0 - 10$

$\log \cos \chi = 9{,}5943713.7 - 10.$

Further, with respect to formulae (7) and (8), p. 168, we now have:

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(See Illustrative Material on Next Page.)

By Gaussian Logarithms	$\log \sin \beta_1 = 9{,}9231590.3 - 10$	$\log \sqrt{1-e^2} = 9{,}9985458.2 - 10$	
	$\log \frac{k \cos \chi}{a_0 \sqrt{1-e^2}} = 7{,}870357 - 10$	$\log \frac{k \cos \chi}{2a_0 \sqrt{1-e^2}} = 7{,}5693269.4_n - 10$	
	$\log \sin \beta_2 = 9{,}919296 - 10$	$\log \sec \beta = 0{,}2584192.3$	+31.8 for 1"
		First Approximation	
	$\beta_1 = 56^0\ 54'\ 44{,}315''$	$\log \sin \frac{\Delta\beta}{2} = 7{,}8277461.7_n - 10$	
	$\beta_2 = 56^0\ 8'\ 29{,}6''$	$S = 4{,}6855715.9 - 10$	
First Approximation	$\beta = 56^0\ 31'\ 37{,}9''$	$\log \frac{\Delta\beta}{2} = 3{,}1421745.8_n$	
		in seconds	
Second Approximation	$\beta = 56^0\ 31'\ 37{,}002''$	$\frac{\Delta\beta}{2} = -23'\ 7{,}313''$.	

Since β has increased by 0,002", there is a change of $+0.1$ in $\log \sec \beta$ and $\log \frac{\Delta\beta}{2}$. The formulae (9), (10) and (11), pp. 168 and 169, now give

(See Illustrative Material on Next Page.)

$$\begin{cases} \log \sin \frac{\Delta\beta}{2} = & 7{,}8277461.8_n - 10 \\ \log \cos \frac{\Delta\beta}{2} = & 9{,}9999901.8 - 10 \\ \log 2 W_1 = & 0{,}3000081.9 \\ -\log w_2 = & 0{,}0004503.4 \end{cases} \qquad \begin{cases} \log \frac{k\sqrt{1-e^2}}{a_0} = & 8{,}2730772.1 - 10 \\ -\log w_2 = & 0{,}0004503.4 \\ \log \cos \mu_{1.2} = & 9{,}9999808.3 - 10 \\ \log \sec B_2 = & 0{,}2550374.5 \\ \log \sin a_{1.2} = & 9{,}8494850.3_n - 10 \end{cases}$$

$$\begin{cases} \log \sin \Delta B = & 8{,}1281948.9_n - 10 \\ S = & 4{,}6855618.0 - 10 \end{cases} \qquad \begin{cases} \log \sin L_{1.2} = & 8{,}3780308.6_n - 10 \\ S = & 4{,}6855335.8 - 10 \end{cases}$$

$\log \Delta B =$ 3,4426330.9$_n$ (in seconds)

$\Delta B = -$ 0° 46′ 10,9781″

$B_2 =$ 56° 13′ 49,0219″

$\log L_{1.2} =$ 3,6924972.8$_n$ (in seconds)

$L_{1.2} = -$ 1° 22′ 6,0326″.

$-\frac{1}{2} L_{1.2} = +$ 2463,0163″ . . .

$B =$ 56° 36′ 54,511″ . . .

$\frac{\Delta B}{2} = -$ 23′ 5,49″ . . .

$$\begin{cases} \log\left(-\frac{L_{1.2}}{2}\right) = & 3{,}3914672.9 \\ T = & 4{,}6855955.1 - 10 \\ \log \sin B = & 9{,}9216830.0 - 10 \\ \log \sec \frac{\Delta B}{2} = & 0{,}0000098.0 \end{cases}$$

Δa can be set equal to $\Delta a'$, hence:

$$\begin{cases} \log \tan \frac{\Delta a'}{2} = & 7{,}9987556.0 - 10 \\ T = & 4{,}6855892.6 - 10 \end{cases}$$

$\frac{\Delta a}{2} = +$ 0° 34′ 16,6782″ . . . . $\log \frac{\Delta a'}{2} =$ 3,3131663.4 (in seconds)

$\Delta a =$ 1° 8′ 33,356″ . . . . . $a_{2.1} =$ 136° 8′ 33,344″.

Results: $\begin{cases} B_2 = & 56° 13' 49{,}0219'' \\ L_{1.2} = & 1\ 22\ 6{,}0326 \text{ East} \\ a_{2.1} = & 136\ 8\ 33{,}344. \end{cases}$

Par. 18. Series Expansion for the Angle of Depression μ.

According to p. 167,

$$\sin \mu_{1.2} = \frac{k W_1}{2 a_0}(1 + \delta \cos^2 \chi), \qquad (1)$$

wherein

$$\cos\chi = \cos\mu_{1.2}\cos B_1\cos a_{1.2} + \sin\mu_{1.2}\sin B_1. \qquad (2)$$

Furthermore, from p. 58, equation (1),

$$\frac{1}{\varrho} = \frac{W_1}{a_0}(1 + \delta\cos^2 B_1\cos^2 a_{1.2}), \qquad (3)$$

where ϱ denotes the radius of curvature of the vertical section in the azimuth $a_{1.2}$ for the point P_1 having the geographical latitude B_1 . It is obviously advantageous to expand $\mu_{1.2}$ using ϱ since for infinitely small k it must always become true that $\sin\mu_{1.2} = \mu_{1.2} = k:2\varrho$. In accordance with (1) and (3) we now have

$$\sin\mu_{1.2} = \frac{k}{2\varrho}\,\frac{1+\delta\cos^2\chi}{1+\delta\cos^2 B_1\cos^2 a_{1.2}}$$

or

$$\sin\mu_{1.2} = \frac{k}{2\varrho}\left\{1 + (\cos^2\chi - \cos^2 B_1\cos^2 a_{1.2})\,\frac{\delta}{1+\delta\cos^2 B_1\cos^2 a_{1.2}}\right\}. \qquad (4)$$

But by transforming (2) we readily obtain

$$\cos^2\chi - \cos^2 B_1\cos^2 a_{1.2} = \sin^2\mu_{1.2}(\sin^2 B_1 - \cos^2 B_1\cos^2 a_{1.2}) + 2\sin\mu_{1.2}\cos\mu_{1.2}\cos B_1\sin B_1\cos a_{1.2}. \qquad (5)$$

If in this expression we expand $\cos\mu_{1.2}$ in powers of $\sin\mu_{1.2}$ and for brevity temporarily set

$$\left.\begin{aligned} \frac{\delta\frac{k}{2\varrho}}{1+\delta\cos^2 B_1\cos^2 a_{1.2}}(\sin^2 B_1 - \cos^2 B_1\cos^2 a_{1.2}) &= c_1 \\ \frac{\delta\frac{k}{2\varrho}}{1+\delta\cos^2 B_1\cos^2 a_{1.2}}\sin 2B_1\cos a_{1.2} &= c_2, \end{aligned}\right\} \qquad (6)$$

it follows after some readily-accomplished simplification by introducing (5) into (4) that

$$(1-c_2)\sin\mu_{1.2} = \frac{k}{2\varrho} + c_1\sin^2\mu_{1.2} - \frac{1}{2}c_2\sin^3\mu_{1.2} + \cdots \quad (7)$$

If we suppose k and thus also $\sin\mu_{1.2}$

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be small magnitudes, the neglected remainder is, at a maximum, about $\frac{1}{8}c_2\sin^5\mu_{1.2}$ that is to say, an eight-order magnitude. Multiplied by ϱ'' this gives only 0,000003" as the maximum effect upon $\mu_{1.2}$ for $k = 0{,}1\,a_0$.

Dividing equation (7) by $(1-c_2)$ yields

$$\sin\mu_{1.2} = \frac{k}{2\varrho\,(1-c_2)} + c_1\sin^2\mu_{1.2} - \frac{1}{2}c_2\sin^3\mu_{1.2} + Gl_8,$$

a formula which differs from (7) chiefly in neglecting a further term $c_1c_2\sin^2\mu_{1.2}$ which is also of eighth-order and would at most make itself felt in the hundred-thousandths of a second. If for $\sin\mu_{1.2}$ on the right side we now also insert the principal term on the left side and write

$$\sin\mu_{1.2} = \frac{k}{2\varrho\,(1-c_2)}\left\{1 + c_1\frac{k}{2\varrho} - \frac{1}{2}c_2\left(\frac{k}{2\varrho}\right)^2 + Gl_7\right\}$$

or

$$\log\sin\mu_{1.2} = \log\frac{k}{2\varrho\,(1-c_2)} + M\frac{k}{2\varrho}\left\{c_1 - \frac{1}{2}c_2\frac{k}{2\varrho}\right\} + Gl_7, \quad (8)$$

the accuracy remains unaffected, since the new errors thus introduced are of an order higher by one than those already admitted. The expressions (6) for c_1 and c_2, necessary for a computation in accord with (8), are transformed by means of formula (3) into:

$$\left.\begin{aligned} c_1 &= \delta \frac{k W_1}{2 a_0} (\sin^2 B_1 - \cos^2 B_1 \cos^2 a_{1.2}) \\ c_2 &= \delta \frac{k W_1}{2 a_0} \sin 2 B_1 \cos a_{1.2}. \end{aligned}\right\} \qquad (9)$$

By applying to the expression found above for $\sin \mu_{1.2}$ the first series of (4), p. 30, with $u = \mu_{1.2}$ we now find immediately this expression for $\mu_{1.2}$ itself:

$$\left.\begin{aligned} \log \mu_{1.2} = \log \frac{k}{2\varrho (1 - c_2)} + M \frac{k}{2\varrho} \left(c_1 - \frac{1}{6} c_2 \frac{k}{2\varrho} + \frac{1}{3} c_1 \left(\frac{k}{2\varrho}\right)^2\right) \\ + \frac{M}{6} \left(\frac{k}{2\varrho}\right)^2 \left(1 + \frac{11}{30} \left(\frac{k}{2\varrho}\right)^2 + \frac{191}{945} \left(\frac{k}{2\varrho}\right)^4\right) + Gl_7. \end{aligned}\right\} (10)$$

The seventh-order terms herein neglected have, on $\mu_{1.2}$, as before, an effect only upon the hundred-thousandths of a second. If we further neglect the sixth-order terms

$$M \left(\frac{1}{3} c_1 \left(\frac{k}{2\varrho}\right)^3 + \frac{191}{5670} \left(\frac{k}{2\varrho}\right)^6\right),$$

the influence of which on $\mu_{1.2}$ in seconds is equal to the above expression multiplied by $\frac{\varrho''}{M} \frac{k}{2\varrho}$, the maximum error, for $k = 0{,}1\, a_0$ becomes 0,00014", which however is usually inconsequential.

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Expanding likewise the denominator $(1 - c_2)$ of the first term on the right of formula (10) in a series, we next obtain

$$\log \mu_{1.2} = \log \frac{\varrho'' k}{2\varrho} + M \left\{\frac{1 - c_2}{6} \left(\frac{k}{2\varrho}\right)^2 + c_2 + c_1 \left(\frac{k}{2\varrho}\right) + \frac{11}{180} \left(\frac{k}{2\varrho}\right)^4 + \frac{1}{2} c_2^2\right\} + Gl_6 \quad (11)$$

in seconds

ϱ for the arguments B_1 and $a_{1.2}$

when $k = 0{,}1 a_0$, this formula is not more accurate in ten-thousandths of a second of $\mu_{1.2}$.

Finally, if we set

$$\left.\begin{array}{l}\log \mu_{1.2} = \log \dfrac{\varrho'' k}{2\varrho} + M\left\{\dfrac{1}{6}\left(\dfrac{k}{2\varrho}\right)^2 + c_2 + c_1 \dfrac{k}{2\varrho}\right\} + Gl_4, \\ \text{in seconds} \qquad\qquad \varrho \text{ for the arguments } B_1 \text{ and } a_{1.2},\end{array}\right\} \quad (12)$$

in further simplification, the error in the value of $\mu_{1.2}$ thus computed is indeed some few thousandths of a second for $k = 0{,}1 a_0$ but it diminishes to less than 0,00001" for $k = 0{,}02 a_0$ as may be seen by consideration of those terms neglected in (12) as compared to (11).

Par. 19. Numerical Example I, using the given data on p. 170 (Par. 16), with formulae (3), (9) and (11)

$$\left\{\begin{array}{l}\log \cos a_{1.2} = 9{,}7048215_n - 10 \\ \log \cos B_1 = 9{,}7844013 - 10 \\ \log \sqrt{\delta} = 8{,}9136594 - 10 \\ \log \tan h' = 8{,}4028822_n - 10 \text{*)}\end{array}\right. \qquad \left\{\begin{array}{l}\log \dfrac{k W_1}{2a_0}'' = 8{,}6175468.611 - 10 \\ \log \sec^2 h' = 0{,}0002775.938' \\ \log \varrho'' = 5{,}3144251.332\end{array}\right.$$

$$\left\{\begin{array}{l}\log \sin^2 B_1 = 9{,}798987 - 10 \\ \log \cos^2 B_1 \cos^2 a_{1.2} = 8{,}978446 - 10\end{array}\right. \qquad \begin{array}{l}\log \dfrac{\varrho'' k}{2\varrho} = 3{,}9322495.88 - 10 \\ \log \delta = 7{,}827319 - 10\end{array}$$

By Gaussian Logarithms

$$\left\{\begin{array}{l}\log \delta \dfrac{k W_1}{2a_0} = 6{,}444865 - 10 \\ \log(\sin^2 B_1 - \cos^2 B_1 \cos^2 a_{1.2}) = 9{,}727809 - 10\end{array}\right.$$

$$\log c_1 = 6{,}172674 - 10.$$

At the same time $\log \sin 2B_1 \cos a_{1.2} = 9{,}689747_n - 10$

$$\log c_2 = 6{,}134612_n - 10.$$

Using $\log M = 6{,}637784$ reduced to units of the seventh decimal place, we further obtain

---

*) h' here denotes the auxiliary angle h introduced in equation (2) p. 59, for computation of ϱ.

$$\log \frac{\varrho'' k}{2\varrho} = 3{,}9322495.88 - 10$$

$$+\frac{M}{6}\left(\frac{k}{2\varrho}\right)^2 = +1245.32 \qquad \log = 3{,}095282$$

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$-\frac{M}{6}\left(\frac{k}{2\varrho}\right)^2 c_2 =$	+	.17	0,230−1
$+ Mc_2 =$	−	592.09	2,77239.
$+ M\left(\frac{k}{2\varrho}\right) c_1 =$	+	26.81	1,4283
$+\frac{11M}{180}\left(\frac{k}{2\varrho}\right)^4 =$	+	.79	0,895−1
$+ M\frac{c_2^2}{2} =$	+	4	0,60−2

$\log \mu_{1.2} = 3{,}9323176.92$ (in seconds) $\qquad \mu_{1.2} = 8556{,}9243''$,

which agrees sufficiently well with the computation given on p. 170.

Numerical Example II as set forth on p. 172 yields by formula (12) of p. 176:

$$\begin{cases} \log \frac{1}{6} M = 5{,}8596 \\ \log \left(\frac{k}{2\varrho}\right)^2 = 5{,}9458 - 10 \end{cases}$$ For units of seventh decimal place

$$\log \left(\frac{1}{6} M \left(\frac{k}{2\varrho}\right)^2\right) = 1{,}8054$$

$$\log \left(M\delta \frac{k W_1}{2 a_0}\right) = 2{,}4376$$

$$\begin{cases} \log \sin^2 B_1 = 9{,}8472 - 10 \\ \log(\cos^2 B_1 \cos^2 a_{1.2}) = 9{,}1712 - 10 \end{cases}$$

By Gaussian Logarithms: $\log(\sin^2 B_1 - \cos^2 B_1 \cos^2 a_{1.2}) = 9{,}7444 - 10$

$$\log \sin 2B_1 \cos a_{1.2} = 9{,}8102 - 10$$

$$\log Mc_1 = 2{,}1820$$

$$\log Mc_2 = 2{,}2478$$

$$\log \left(Mc_1 \frac{k}{2\varrho}\right) = 0{,}1549$$

$$\begin{cases} \log \frac{k}{2\varrho} = 7{,}9729124 - 10 \\ \log \varrho'' = 5{,}3144251 \end{cases}$$

$$\begin{cases} \log \frac{\varrho'' k}{2\varrho} = 3{,}2873375 \\ [1{,}8054] = + \quad 63.9 \\ [2{,}2478] = + \quad 176.9 \\ [0{,}1549] = + \quad 1.4 \end{cases}$$

$\log \mu_{1.2} = 3{,}2873617$ (in seconds)

$\mu_{1.2} = 32' 18{,}035''$.

Chapter 4. The Vertical Section and Chord Triangle for the Flattened Ellipsoid of Revolution.

Par. 20. Rectification of the Vertical Section. For isolated applications it is necessary to ascertain the relationship between the lengths of the chord and of the vertical section for small values of the distance. This relationship can however be formulated with the aid of the formula derived on p. 175 for $\sin \mu_{1.2}$ which reads

$$\sin \mu_{1.2} = \frac{k}{2\varrho(1-c_2)} \left\{ 1 + c_1 \frac{k}{2\varrho} - \frac{1}{2} c_2 \left(\frac{k}{2\varrho}\right)^2 + Gl_7 \right\}. \qquad (1)$$

Here $\mu_{1.2}$ appears as a function of k, B_1 and $a_{1.2}$.

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For the purpose of rectification let us now regard k as a radius vector and $\mu_{1.2}$ as an anomaly, considering both of these quantities to be variables and all the other quantities to be constant. Included among the latter is ϱ (arguments: B_1 and $a_{1.2}$), while c_1 and c_2 contain the variable k as a factor. We therefore separate from c_1 and c_2 (see the expressions (6), p. 174) the constant components

$$\left.\begin{aligned} c_1' &= \frac{\delta(\sin^2 B_1 - \cos^2 B_1 \cos^2 a_{1.2})}{1 + \delta \cos^2 B_1 \cos^2 a_{1.2}} \\ c_2' &= \frac{\delta \sin 2B_1 \cos a_{1.2}}{1 + \delta \cos^2 B_1 \cos^2 a_{1.2}} \end{aligned}\right\} \qquad (2)$$

and then obtain from (1) by expanding the denominator $(1-c_2)$ on the right side in a series and neglecting a seventh-order term dependent upon $\delta^2 k^3$, the expression

$$\sin \mu_{1.2} = \frac{k}{2\varrho} \left\{ 1 + c_2' \left(\frac{k}{2\varrho}\right) + c_1' \left(\frac{k}{2\varrho}\right)^2 - \frac{1}{2} c_2' \left(\frac{k}{2\varrho}\right)^3 + Gl_6 \right\}.$$

From this it follows, by the first formula of (2), p. 29, that

$$\mu_{1.2}=\left(\frac{k}{2\varrho}\right)+\frac{1}{6}\left(\frac{k}{2\varrho}\right)^3+\frac{3}{40}\left(\frac{k}{2\varrho}\right)^5+c_2'\left(\frac{k}{2\varrho}\right)^2+c_1'\left(\frac{k}{2\varrho}\right)^3+Gl_7. \quad (3)$$

If however we denote by ds the increment of arc of the vertical section, by the familiar formula for polar coordinates we have

$$ds^2=dk^2+k^2d\mu_{1.2}^2.$$

By differentiation of expression (3) for $\mu_{1.2}$ and multiplication by k we now obtain

$$kd\mu_{1.2}=\left\{\left(\frac{k}{2\varrho}\right)+\frac{1}{2}\left(\frac{k}{2\varrho}\right)^3+\frac{3}{8}\left(\frac{k}{2\varrho}\right)^5+2c_2'\left(\frac{k}{2\varrho}\right)^2+3c_1'\left(\frac{k}{2\varrho}\right)^3+Gl_7\right\}dk.$$

From this there is easily derived:

$$ds^2=dk^2\left\{1+\left(\frac{k}{2\varrho}\right)^2+\left(\frac{k}{2\varrho}\right)^4+\left(\frac{k}{2\varrho}\right)^6+4c_2'\left(\frac{k}{2\varrho}\right)^3+6c_1'\left(\frac{k}{2\varrho}\right)^4+Gl_7\right\}$$

and:

$$ds=dk\left\{1+\frac{1}{2}\left(\frac{k}{2\varrho}\right)^2+\frac{3}{8}\left(\frac{k}{2\varrho}\right)^4+\frac{5}{16}\left(\frac{k}{2\varrho}\right)^6+2c_2'\left(\frac{k}{2\varrho}\right)^3+3c_1'\left(\frac{k}{2\varrho}\right)^4+Gl_7\right\}.$$

Integrating the above from $k=0$ to k, we obtain for the arc $s_{1.2}$ lying above the chord k, that is to say, for the vertical section from P_1 through P_2, the expression:

$$s_{1.2}=k\left\{1+\frac{1}{6}\left(\frac{k}{2\varrho}\right)^2+\frac{3}{40}\left(\frac{k}{2\varrho}\right)^4+\frac{5}{112}\left(\frac{k}{2\varrho}\right)^6+\frac{c_2'}{2}\left(\frac{k}{2\varrho}\right)^3+\frac{3c_1'}{5}\left(\frac{k}{2\varrho}\right)^4+Gl_7\right\}; \quad (4)$$

ϱ for the arguments s_1 and $s_{1.2}$.

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In order to be able to compute $s_{1.2}$ as an arc from k and ϱ, one must be able to neglect the terms having c_2' and c_1' as

factors. The first of these, the larger, when at its maximum has an effect upon $\log s_{1.2}$ equal to

$$\frac{1}{2} M\delta\left(\frac{k}{2\varrho}\right)^3,$$

which for $k = 0{,}04\ a_0$ amounts to one unit in the eighth decimal place of $\log s_{1.2}$.

This formula can be somewhat simplified while retaining the same accuracy by substituting for ϱ the geometric mean of the two ϱ for the arguments B_1, $a_{1.2}$ and B_2, $a_{2.1}$ respectively.

Denoting these two ϱ values by ϱ_1 and ϱ_2 for the present, we have in accordance with equation (1) p. 58,

$$\left.\begin{aligned}\frac{1}{\varrho_1} &= \frac{\sqrt{1-e^2\sin^2 B_1}}{a_0}\,(1+\delta\cos^2 B_1\cos^2 a_{1.2})\\ \frac{1}{\varrho_2} &= \frac{\sqrt{1-e^2\sin^2 B_2}}{a_0}\,(1+\delta\cos^2 B_2\cos^2 a_{2.1}).\end{aligned}\right\} \qquad (5)$$

Taking into account the relationship $\delta = e^2 : (1-e^2)$ and neglecting the terms with δ^2, it follows from the above, first:

$$\begin{aligned}\frac{1}{\varrho_1} &= \frac{1}{a_0}\left(1+\delta\left(1-\frac{3}{2}\sin^2 B_1-\cos^2 B_1\sin^2 a_{1.2}\right)+\cdots\right)\\ \frac{1}{\varrho_2} &= \frac{1}{a_0}\left(1+\delta\left(1-\frac{3}{2}\sin^2 B_2-\cos^2 B_2\sin^2 a_{2.1}\right)+\cdots\right),\end{aligned}$$

and then:

$$\frac{1}{\varrho_1^2}-\frac{1}{\varrho_1\varrho_2} = \frac{\delta}{a_0^2}\left(\frac{3}{2}(\sin^2 B_2-\sin^2 B_1)+(\cos^2 B_2\sin^2 a_{2.1}-\cos^2 B_1\sin^2 a_{1.2})\right)+\cdots.$$

But according to equation (6), p. 148, the azimuths $a_{1.2}$ and

$a_{2.1}$ differ from the azimuths $a'_{1.2}$ and $a'_{2.1}$ there introduced only by amounts of the order of magnitude of e^2 or δ. Now, by the Sine Law, in the spherical triangle (1), p. 144

$$\sin^2 a'_{2.1} \cos^2 B_2 = \sin^2 a'_{1.2} \cos^2 B_1,$$

consequently, $\sin^2 a_{2.1} \cos^2 B_2 - \sin^2 a_{1.2} \cos^2 B_1$ is only a magnitude of the order of δ. Thus, neglecting δ^2 as before, we have

$$a_0^2 \left(\frac{1}{\varrho_1^2} - \frac{1}{\varrho_1 \varrho_2}\right) = \frac{3}{2} \delta (\sin^2 B_2 - \sin^2 B_1) + Gl_5. \qquad (6)$$

The remainder of the expansion on the right side is taken as a fifth-order term, since in addition to δ^2 it must also contain the factor k, so that it disappears when $k = 0$.

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The factor k is also contained in $\sin^2 B_2 - \sin^2 B_1$; namely, in accordance with equations (2) and (4), p. 167, it can be shown without difficulty that we may write

$$\sin^2 B_2 - \sin^2 B_1 = -\frac{2k}{a_0} \sin B \cos \chi + Gl_3$$

and accordingly, from (6)

$$\frac{k^2}{\varrho_1^2} - \frac{k^2}{\varrho_1 \varrho_2} = -\frac{3\delta k^2}{a_0^3} \sin B \cos \chi + Gl_7. \qquad (7)$$

Abbreviating $\varrho_1 \varrho_2$ now as ϱ^2, the introduction of (7) into formula (4) with some further easily-understood omissions of terms, gives us

$$s_{1.2} = k \left\{1 + \frac{1}{6}\left(\frac{k}{2\varrho}\right)^2 + \frac{3}{40}\left(\frac{k}{2\varrho}\right)^4 + \frac{5}{112}\left(\frac{k}{2\varrho}\right)^6 - \delta \sin B \cos \chi \left(\frac{k}{2\varrho}\right)^3 + \frac{c'_2}{2}\left(\frac{k}{2\varrho}\right)^3 + \frac{3c'_1}{5}\left(\frac{k}{2\varrho}\right)^4 + Gl_7\right\}. \qquad (8)$$

Chapter 4. The Vertical Section and Chord Triangle for the Flattened Ellipsoid of Revolution.

The neglected seventh-order terms in the parenthesis are functions of $\delta\left(\frac{k}{2\varrho}\right)^5$ and $\delta^2\left(\frac{k}{2\varrho}\right)^3$, and ought scarcely to be perceptible in the ninth place of $\log s_{1.2}$ for $k = 0{,}1\, a_e$.

The terms containing δ^2 in e_2' and e_1' can in fact also be neglected and the same accuracy retained. In that case, if we substitute for $\cos\chi$:

$$\cos\chi = \cos B_1 \cos a_{1.2} + \frac{k}{2\varrho}\sin B_1 + Gl_2, \qquad (9)$$

the last three terms on the right side of (8) are transformed, except for the factor k, into the expression

$$\delta\left(\frac{k}{2\varrho}\right)^3\Big\{-\sin B\cos B_1\cos a_{1.2} + \sin B_1\cos B_1\cos a_{1.2} - \frac{k}{2\varrho}\left(\sin B\sin B_1 - \frac{3}{5}\sin^2 B_1 + \frac{3}{5}\cos^2 B_1\cos^2 a_{1.2}\right)\Big\} + Gl_1.$$

But according to equation (2), p. 167,

$$\sin B_2 = \sin B_1 - \frac{k}{\varrho}\cos\chi + Gl_2;$$

furthermore, since $\sin B_2 + \sin B_1 = 2\sin B\cos\frac{\Delta B}{2}$ we obtain in accord with equation (9)

$$\sin B = \sin B_1 - \frac{k}{2\varrho}\cos B_1\cos a_{1.2} + Gl_2,$$

whereby, with the disappearance of the fifth-order terms, the above expression for the last three terms of (8) reduces to

$$\frac{2}{5}\delta\left(\frac{k}{2\varrho}\right)^4(\cos^2 B\cos^2 a - \sin^2 B) + \cdots. \qquad (10)$$

Here a is taken as in (4) p. 156.

The influence of (10) on $\log s_{1.2}$ is M times as great and, for $k = 0{,}1\, a_0$ and $0{,}05\, a_0$, is still not one unit of the eighth or ninth decimal place. Accordingly, we have by use of some further reductions and insignificant omissions, provided that at the same time the index 1.2 is omitted from s:

$$\left.\begin{aligned} s &= \begin{cases} = k\left\{1 + \frac{1}{6}\left(\frac{k}{2\varrho}\right)^2 + \frac{3}{40}\left(\frac{k}{2\varrho}\right)^4 + Gl_6\right\} \\ = 2\varrho\left\{\arcsin\frac{k}{2\varrho} + Gl_7\right\}; \end{cases} \\ \log s &= \log k + \frac{1}{6} M\left\{\left(\frac{k}{2\varrho}\right)^2 + \frac{11}{30}\left(\frac{k}{2\varrho}\right)^4\right\} + Gl_6 \\ \log\frac{M}{6} &= 5{,}8596331 \text{ u. } \log\frac{11\,M}{180} = 5{,}4239 \text{ for units of the seventh decimal} \end{aligned}\right\} \quad (11)$$

These formulae do not give at first the tenth decimal place of the logarithm accurately with $k = 0{,}05\, a_0$ but with $k = 0{,}1\, a_0$ at least eight decimals are certain.

For computation of the ϱ appearing in (11) we obtain from (5):

$$\left.\begin{aligned} \frac{1}{\varrho} &= \frac{W}{a_0}(1 + \delta\cos^2 B\cos^2 a + Gl_4) \\ \log\varrho &= \log a_0 - \log W - M\delta\cos^2 B\cos^2 a + Gl_4 \\ W \text{ with argument } B &= \frac{1}{2}(B_1 + B_2) \\ a &= \frac{1}{2}(a_{1.2} + a_{2.1} - 180^0). \end{aligned}\right\} \quad (12)$$

The development (9), p. 26 shows the accuracy of the first formula (12), from which it is evident that

$$W_1 W_2 = W^2 + Gl_4$$

$$(1 + \delta\cos^2 B_1\cos^2 a_{1.2})(1 + \delta\cos^2 B_2\cos^2 a_{2.1}) = 1 + \delta\cos^2 B\cos^2 a + Gl_4.$$

The fourth-order terms are dependent on $e^2\frac{k^2}{a_0^2}$ and small terms for

the vicinity of the pole too, as is to be expected and is confirmed by detailed observation.

The formula (11) is interesting not only in that it shows the possibility of computing with considerable approximation the length of the vertical section from the chord with suitable choice of ρ as that of an arc of a circle; but it shows that the length

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of the vertical sections from P_1 to P_2 and P_2 to P_1 deviate from one another only by terms of very high order. We cannot yet recognize from (11) which order the difference will show up since for that we must carry the developments further. But there is no reason to do so now, particularly since that difference is yielded later on a different occasion.

If we write (11) in the form of $\log k = \log 2\rho + \log \sin \frac{s}{2\rho}$ and consider formula (6), p. 157, then we have

$$\log \sin \frac{s}{2\rho} = \log \sin \frac{\sigma'}{2} - \frac{1}{8} M e^4 \cos^4 B \sin^2 2\alpha + \frac{1}{8} M e^2 \Delta B^2 (\cos 2B - 2\sin^2 \alpha) + Gl_6$$

and from this

$$\log s = \log (\sigma' \rho) - \frac{1}{8} M e^4 \cos^4 B \sin^2 2\alpha + \frac{1}{8} M e^2 \Delta B^2 (\cos 2B - 2\sin^2 \alpha) + Gl_6, \quad (13)$$

where ρ is to be computed according to (12) and σ' refers to the spherical triangle (1), p. 144. If we compute ρ with α' (instead of α), then the small terms in (13) are to be interchanged with those of (3), p. 156.

This formula (13) is like (6), p. 157, independent of the distance in the eighth decimal place, but not quite precise.

Par. 21. Numerical Example I, cf. pp. 170 and 176, gives with formula (11), p. 181 with rigorous computation of the radii of curvature:

$\log \varrho = 6{,}8053644$ for $B = 53^\circ 36' 34''$ and $a = 242^\circ 24' 35''$ by S. 163

$\left\{\begin{array}{l} 6{,}8052801 \text{ for } B_1 = 52\ 30\ 17 \text{ and } a_{1.2} = 239\ 33\ 1 \text{ by S. 176} \\ 6{,}8054409 \text{ for } B_2 = 54\ 42\ 51 \text{ and } a_{2.1} = 65\ 16\ 10; \text{ whence it follows} \end{array}\right.$

$\log \varrho = 6{,}8053605$, precisely corresponding to $\sqrt{\varrho_1 \varrho_2}$ S. 180

The application of formulae (12), p. 181 and (3), p. 59 respectively to the computation of the three $\log \varrho$ leads to values that deviate by one unit in the seventh decimal place.

We now obtain for the third formula (11), p. 181:

with $\log \varrho = 6{,}8053644$	with 6,8053605
$\log \frac{k}{2\varrho} = 8{,}6177402 - 10$	$8{,}6177441 - 10$
$\log k = 5{,}7241345.725$	$5{,}7241345.725$
$\frac{1}{6} M \left(\frac{k}{2\varrho}\right)^2 = + 1244.840$	$+ 1244.862$
$\frac{11}{180} M \left(\frac{k}{2\varrho}\right)^4 = + 0.785$	$+ 0.785$
$\log s = 5{,}7242591.350$	$5{,}7242591.372.$

If we consider, for the second of these values, the sixth-order terms according to (8), p. 180 and (10), p. 181 respectively, then there is yielded...5,7242591.353.

It will later appear that this value is correct to the tenth place inclusive. Assuming this, we find from comparison with the

first value of $\log s$ (obtained from $\log \varrho$ with the arguments B and a) that it is also extremely precise without consideration of the sixth-order terms.

Numerical Example II, cf. p. 172, gives $\log \varrho = 6{,}805$ and

$$\log s = 5{,}0791748.5 + 64.0 + 0.0 = 5{,}07918125,$$

which value will likewise later prove true.

Par. 22. Azimuthal Difference of the Vertical Sections. Referring to the developments of Pars. 1 and 2 of this chapter pp. 134 et. seq. we set up the equation of the vertical plane of P_1 to P_2. As this latter goes through the origin of coordinates, its equation is of the form:

$$A\xi + B\eta + C\zeta = 0. \qquad (1)$$

If we assume the coefficients A, B and C to be already determined, then the function

$$\cot a'_{1,2} = -\frac{B}{A} \qquad (2)$$

serves for the determination of the azimuth $a'_{1,2}$ of the vertical section of P_2 to P_1.

The equation (1) becomes known in that it is applied to P_2 and K'_2 (Fig. 8, p. 134). According to (4), p. 136, for P_2 we have:

$$\left.\begin{aligned} \xi_2 &= (x_2 - x_1)\sin B_1 - (z_2 - z_1)\cos B_1 \\ \eta_2 &= y_2 \\ \zeta_2 &= -(x_2 - x_1)\cos B_1 - (z_2 - z_1)\sin B_1. \end{aligned}\right\} \qquad (3)$$

And noting the circumstance that for K'_2 the coordinates x'_2 and y'_2

are equal to zero:

$$\left.\begin{aligned} \xi'_2 &= -x_1 \sin B_1 - (z'_2 - z_1)\cos B_1 \\ \eta'_2 &= 0 \\ \zeta'_2 &= +x_1 \cos B_1 - (z'_2 - z_1)\sin B_1 \end{aligned}\right\} \qquad (4)$$

The values x_2, y_2 and z_2 are stated in (3), p. 136; further, according to (9), p. 41:

$$z'_2 = -\frac{a_0 e^2 \sin B_2}{W_2}. \qquad (5)$$

By application of the equation (1) to the points P_2 and K'_2 we now have:

$$\begin{aligned} A\xi_2 + B\eta_2 + C\zeta_2 &= 0 \\ A\xi'_2 + B\eta'_2 + C\zeta'_2 &= 0 \end{aligned}$$

and from this can determine the ratio $(B:A)$ and $(C:A)$. In particular

$$\cot a'_{1,2} = -\frac{B}{A} = \frac{\xi_2\zeta'_2 - \xi'_2\zeta_2}{\eta_2\zeta'_2 - \eta'_2\zeta_2}$$

and from this by means of (3) and (4):

$$\cot a'_{1,2} = \frac{(z'_2 - z_2)\,x_1 - (z'_2 - z_1)\,x_2}{y_2\,(x_1 \cos B_1 - (z'_2 - z_1)\sin B_1)}. \qquad (6)$$

If here we replace the value of z'_2 according to (5) and furthermore eliminate x_1, z_1, x_2, y_2 and z_2 by means of the following relations:

$$\left.\begin{aligned} x_1 &= a_0 \cos B_1 : W_1 \\ z_1 &= a_0 (1 - e^2) \sin B_1 : W_1 \end{aligned}\right\} \qquad (7)$$

$$\left.\begin{aligned} x_2 &= a_0 \cos B_2 \cos L_{1.2} : W_2 \\ y_2 &= a_0 \cos B_2 \sin L_{1.2} : W_2 \\ z_2 &= a_0 (1 - e^2) \sin B_2 : W_2 \end{aligned}\right\} \qquad (8)$$

then, through simple reductions, there is obtained the expression:

$$\cot a'_{1.2} = \frac{\dfrac{\cos B_2 \sin B_1 \cos L_{1.2} - \sin B_2 \cos B_1}{\cos B_2 \sin L_{1.2}} + e^2 \cot L_{1.2} \left(\dfrac{W_1}{W_2} \sin B_2 - \sin B_1 \right)}{1 + e^2 \sin B_1 \left(\dfrac{W_1}{W_2} \sin B_2 - \sin B_1 \right)} . \qquad (9)$$

From this formula we can determine $a'_{1.2}$ at a given geographic position of the points. Similarly, we can recognize for $\cot a_{1.2}$ according to formula (6), p. 139, the reciprocal behaviour of the vertical sections in general and still more simply geometrically from observation of the relative position of the points K'_1 and K'_2 at the chord $P_1 P_2$ in Fig. 8, p. 134. In either way we easily notice that the difference of $a'_{1.2}$ and $a_{1.2}$ for nearly diametrically-opposite points can approach $\pm 180^0$. The difference is small generally only with small distances and for such the difference may still be represented.

If we correspondingly designate the spherical triangle (1), p. 144

$$\frac{\cos B_2 \sin B_1 \cos L_{1.2} - \sin B_2 \cos B_1}{\cos B_2 \sin L_{1.2}} \text{ with } \cot a'_{1.2}$$

and in passing, further

$$\left.\begin{aligned} &\sin B_2 - \frac{W_2}{W_1} \sin B_1 \text{ with } p_1 \\ &\frac{W_1}{W_2} \text{ with } q_1 , \end{aligned}\right\} \qquad (10)$$

then according to (9) and (3) respectively, p. 145:

$$\cot a'_{1.2} = \frac{\cot a'_{1.2} + e^2 \cot L_{1.2} \cdot p_1 q_1}{1 + e^2 \sin B_1 \cdot p_1 q_1}, \tag{11}$$

$$\cot a_{1.2} = \cot a'_{1.2} + \frac{e^2 \cos B_2}{\cos B_2 \sin L_{1.2}} p_1. \tag{12}$$

When we subtract equation (12) from (11), as well as undertaking some simplifications by application of the relations:

$$\cot L_{1.2} \sin a'_{1.2} - \cot \sigma' \cos B_1 = \sin B_1 \cos a'_{1.2} \tag{13}$$

and

$$\cos B_2 \sin L_{1.2} = \sin \sigma' \sin a'_{1.2}, \tag{14}$$

which refer to the spherical auxiliary triangle already referred to above, we obtain:

$$\cot a'_{1.2} - \cot a_{1.2} = \frac{e^2 p_1 \cos B_1}{\sin \sigma' \sin a'_{1.2}} \left(-1 + \frac{q_1 \cos \sigma'}{1 + e^2 p_1 q_1 \sin B_1} \right). \tag{15}$$

For further simplification of the right side of this equation we remember the formulae (1), p. 140 and (9), p. 146, according to which:

$$\left.\begin{aligned} q_1 &= 1 + \frac{1}{2} e^2 \sin 2B \sin \Delta B + Gl_5 \\ &\text{and} \\ p_1 &= 2 \sin \frac{\Delta B}{2} \cos B (1 + e^2 \sin B_1 \cos B + Gl_4), \end{aligned}\right\} \tag{16}$$

if ΔB is regarded as being of first-order magnitude. From this we have:

$$\frac{p_1}{\sin \sigma'} = \frac{\sin B_2 - \sin B_1}{\sin \sigma'} (1 + e^2 \sin B_1 \cos B + Gl_4);$$

but we have $\sin B_2 = \sin B_1 \cos \sigma' - \cos B_1 \sin \sigma' \cos a'_{1.2}$, with which the previous expression for $p_1 : \sin \sigma'$ develops into:

$$\frac{p_1}{\sin \sigma'} = -\left(\cos B_1 \cos a'_{1.2} + \sin B_1 \tan \frac{\sigma'}{2}\right)(1 + e^2 \sin B_1 \cos B + Gl_4). \qquad (17)$$

Further, by means of (16), if σ' too is regarded as being of first-order magnitude, there is yielded:

$$1 - \frac{q_1 \cos \sigma'}{1 + e^2 p_1 q_1 \sin B_1} = 2\sin^2 \frac{\sigma'}{2} - 2e^2 \cos^2 B \sin^2 \frac{\Delta B}{2} + Gl_5. \qquad (18)$$

Equation (15) hence takes for small values of σ' the form:

$$\cot a'_{1.2} - \cot a_{1.2} = \frac{e^2 \sigma'^2 \cos B_1}{2 \sin a'_{1.2}} \left(\cos B_1 \cos a'_{1.2} + \frac{\sigma'}{2} \sin B_1\right) + Gl_6. \qquad (19)$$

For $\sin a'_{1.2}$ as denominator one can now also put $\sin a_{1.2}$ for with consideration of (12) and (14):

$$\csc a_{1.2} = \sqrt{1 + \cot^2 a_{1.2}}$$
$$= \csc a'_{1.2} \sqrt{1 + 2e^2 \cos B_1 \cos a'_{1.2} \frac{p_1}{\sin \sigma'} + e^4 \cos^2 B_1 \frac{p_1^2}{\sin^2 \sigma'}}$$

and hence in every case $\csc a'_{1.2} = \csc a_{1.2}(1 + Gl_2)$, as long as σ' is small. Likewise we can, and still with exactitude from (19) put for $\cos a'_{1.2}$ and σ', respectively $\cos a_{1.2}$ and $\frac{s}{a_0}$. Thus we have:

$$\cot a'_{1.2} - \cot a_{1.2} = \frac{e^2 s^2 \cos B_1}{2 a_0^2 \sin a_{1.2}} \left(\cos B_1 \cos a_{1.2} + \frac{s}{2 a_0} \sin B_1\right) + Gl_6. \qquad (20)$$

And from this, according to (2), p. 31, we immediately have:

Par. 22. Azimuthal Difference of the Vertical Sections.

$$\underset{\text{in seconds}}{a'_{1.2} - a_{1.2}} = -\frac{1}{4}\varrho''e^2\frac{s^2}{a_0^2}\left(\cos^2 B_1 \sin 2a_{1.2} + \frac{1}{2}\frac{s}{a_0}\sin 2B_1 \sin a_{1.2}\right) + Gl_6. \qquad (21)$$

By interchange of the indices 1 and 2 we have further:

$$\underset{\text{in seconds}}{a'_{2.1} - a_{2.1}} = -\frac{1}{4}\varrho''e^2\frac{s^2}{a_0^2}\left(\cos^2 B_2 \sin 2a_{2.1} + \frac{1}{2}\frac{s}{a_0}\sin 2B_2 \sin a_{2.1}\right) + Gl_6. \qquad (22)$$

These formulae can be written otherwise and indeed more simply by introducing a mean azimuth and a mean latitude. Namely,

$$\left.\begin{aligned} \cos B_1 \sin a'_{1.2} &= -\cos B_2 \sin a'_{2.1}, \\ -\cos B_1 \cos a'_{1.2} &= \sin B_2 \sin\sigma' + \cos B_2 \cos a'_{2.1} \cos\sigma' \end{aligned}\right\} \qquad (23)$$

and identical by reason of $a' = \frac{1}{2}(a'_{1.2} + a'_{2.1}) - 90^0$:

$$\begin{aligned} &-\cos B_1 \cos B_2 \sin 2a' \\ &= \cos B_1 \sin a'_{1.2} \cdot \cos B_2 \cos a'_{2.1} + \cos B_1 \cos a'_{1.2} \cdot \cos B_2 \sin a'_{2.1}. \end{aligned}$$

If from this we eliminate $a'_{2.1}$ and $\cos B_2$ on the right side, and note that $\cos B_1 \cos B_2$ is equal to $\cos^2 B + Gl_2$, then we have:

$$\cos^2 B \sin 2a' = \cos^2 B_1 \sin 2a'_{1.2} + \frac{1}{2}\sigma' \sin 2B_1 \sin a'_{1.2} + Gl_2. \qquad (24)$$

Now we can easily see that (21) and (22) can be brought into the following form:*)

$$\left.\begin{aligned} \underset{\text{in seconds}}{a'_{1.2} - a_{1.2}} = \underset{\text{in seconds}}{a'_{2.1} - a_{2.1}} &= -\frac{1}{4}\varrho''e^2\frac{s^2}{a_0^2}\cos^2 B \sin 2a + Gl_6 \\ B &= \frac{1}{2}(B_1 + B_2) \\ a &= \frac{1}{2}(a_{1.2} + a_{2.1} - 180^0). \end{aligned}\right\} \qquad (25)$$

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*)Development (8) p. 26 fully confirms this.

According to this we have

$$s = 0{,}1\,a_0 = 640^{km} \quad a'_{1.2} - a_{1.2} = 3{,}4'' \text{ maximum}$$
$$s = 0{,}01\,a_0 = 64^{km} \quad a'_{1.2} - a_{1.2} = 0{,}034 \text{ maximum}$$

Par. 23. Surface Angle of both Vertical Planes and Distance of the Vertical Sections.

If we imagine K'_2 (Fig. 8, p. 134) projected on the one side on the vertical plane from P_1 to P_2 i.e. on the plane $P_1 P_2 K'_1$, as well as on the other side on the chord, then a right triangle is formed in which the surface angle ν appears as the angle opposite the line of projection from K'_2 onto $P_1 P_2 K'_1$. The length of this line is $K'_1 K'_2 \cos U$, since $90^0 - U$ is the angle of inclination of the axis of the Earth to the vertical plane $P_1 P_2 K'_1$ (Fig. 10, p. 137); against this the distance of the point K'_2 from the chord is equal to $\varrho_m \cos \mu_{2.1}$, when $\varrho_m = P_2 K'_2$ designates the radius of the cross section at P_2, and $\mu_{2.1}$ the angle of depression of the chord at P_2. Since according to (9), p. 137 $\cos U = \cos B_1 \sin a_{1.2}$ we thus have, precisely.

$$\sin \nu = \frac{K'_1 K'_2 \cdot \cos B_1 \sin a_{1.2}}{\varrho_m \cos \mu_{2.1}}. \qquad (1)$$

p. 41 (9) and p. 168 (7), as well as p. 167 (4) give, however;

$$K'_1 K'_2 = \frac{a_0 e^2}{\sqrt{1 - e^2}} (\sin \beta_1 - \sin \beta_2) = \delta\, k \cos \chi$$
$$\cos \chi = \cos \mu_{1.2} \cos B_1 \cos a_{1.2} + \sin \mu_{1.2} \sin B_1. \qquad (2)$$

Whilst we insert this, at the same time we cause an inaccuracy to enter because of the assumption that the distances are small:

we put as denominator of (1) for $\cos \mu_{2.1}$ the almost equal value $\cos \mu_{1.2}$. According to p. 167 (4), however, $\sin \mu_{1.2}$ and $\sin \mu_{2.1}$ differ only in the factor W, thus by a fourth-order term. Hence with reference to the expansion in series $\cos = 1 - \frac{1}{2} \sin^2 + \cdots$ we have

$$\cos \mu_{2.1} = \cos \mu_{1.2} (1 + Gl_3).$$

We now obtain from (1):

$$\sin v = \frac{\delta k}{2\varrho_n} (\cos^2 B_1 \sin 2a_{1.2} + \tan \mu_{1.2} \sin 2B_1 \sin a_{1.2} + Gl_3). \qquad (3)$$

If we again put $k = s(1 + Gl_2)$ and $\tan \mu_{1.2} = \frac{s}{2\varrho_a}(1 + Gl_2)$ and take note that according to p. 56 (2) ϱ_n for P_2 is equal to ϱ_n for P_1 times $(1 + Gl_3)$ then we have:

$$v = \frac{1}{2} \varrho'' \frac{\delta s}{\varrho_n} \left(\cos^2 B_1 \sin 2a_{1.2} + \frac{s}{2\varrho_a} \sin 2B_1 \sin a_{1.2} + Gl_2\right) \qquad (4)$$

ϱ_n radius of curvature in the prime vertical
ϱ_a radius of curvature in azimuth $a_{1.2}$, both for point P_1.

If we desire a formula symmetrical to P_1 and P_2, it can be easily gained for short distances by consideration of the development at the end of the previous paragraph. According to that:

$$v \underset{\text{in Sek.}}{=} \frac{1}{2} \varrho'' \frac{\delta s}{\varrho_n} (\cos^2 B \sin 2a + Gl_2)$$
$$B = \frac{1}{2}(B_1 + B_2); \quad a = \frac{1}{2}(a_{1.2} + a_{2.1} - 180^0) \qquad (5)$$

ϱ_n radius of curvature in the prime vertical for argument B.

According to this:

$$s = 0{,}1\,a_0 = 640^{km} \quad \nu = 69''$$
$$s = 0{,}01\,a_0 = 64^{km} \quad \nu = 7''.$$

In order to arrive at $a'_{1.2} - a_{1.2}$ from ν, it is sufficient to imagine P_2 as the center of a sphere of radius 1, and to observe the spherical triangle which is formed on its surface by the sections of both vertical planes and the horizontal plane at P_1. This triangle is right-angled; the cathodes are $(a_{1.2} - a'_{1.2})$ in the horizontal plane and $\mu_{1.2}$, the angle of depression in the vertical plane from P_1 to P_2. The side $(a_{1.2} - a'_{1.2})$ lies opposite to angle ν. The cotangent law for four successive parts now immediately gives the rigorous formula:

$$\tan (a_{1.2} - a'_{1.2}) = \tan \nu \sin \mu_{1.2}. \qquad (6)$$

This formula also gives in connection with (1) and (2) the sign of the difference $a_{1.2} - a'_{1.2}$ (cf. Fig. 8, p. 134), as long as one may assume that $|\nu| < 90^0$.

Without now going further with the derivation of the already known expression for $a_{1.2} - a'_{1.2}$ at short distances, we use the formula again for an evaluation of the maximum distances of the two vertical sections from each other.

If we pass a plane perpendicular to the chord then it cuts the surface area of the ellipsoid in an ellipse from which a linear segment falls between the vertical sections.

Par. 24. Variation of the Astronomical Azimuth and the Horizontal Angle with the Height of the Objects.

Since this segment of arc lies nearly normal to the vertical sections as long as we consider only small distances P_1P_2, one can interpret it as the separation of the vertical sections and compute it as an arc of a circle, with the central angle ν and the radius of which is the distance of the vertical sections from the chord at the relevant position. The distance is thus a maximum, anywhere where this radius is a maximum, namely very nearly at the mid-point between P_1 and P_2. Similarly the radius is approximately equal to $\frac{s^2}{8\varrho_a}$ and thus the

$$\text{Maximum distance} = \frac{\delta s^3}{16\,\varrho_m\varrho_a}\cos^2 B \sin 2\alpha + \cdots. \qquad (7)$$

Accordingly we have for

$$s = 0{,}1\,a_0 = 640^{km} \text{ the maximum distance} = 2{,}7^m$$
$$s = 0{,}01\,a_0 = 64^{km} \text{ the maximum distance} = 0{,}003^m.$$

Par. 24. Variation of the Astronomical Azimuth and the Horizontal Angle with the Height of the Objects. It has already been indicated at the close of Par. 1, p. 135 that the vertical plane from a point P_1, the station, to a point P_2, the object, rotates somewhat in the event of the object being displaced in the perpendicular from the equipotential surface (the ellipsoid of revolution) as a base. Let the displacement be $+H_2$ outwards. Then there enter in Par. 2 (3), p. 136 to the expression:

$$\begin{array}{lll} \text{for } x_2: & \text{the increment} & +H_2\cos B_2\cos L_{1,2} \\ \text{for } y_2: & \text{the increment} & +H_2\cos B_2\sin L_{1,2} \\ \text{for } z_2: & \text{the increment} & +H_2\sin B_2, \end{array} \qquad (1)$$

for the projection of H_2 onto the equatorial plane is $H_2 \cos B_2$ and onto the axis of the Earth $H_2 \sin B_2$. In Par. 3 (1), p. 138 is now to be added

$$\left.\begin{aligned} \text{for } \xi_2 &: \quad + H_2 (\cos B_2 \sin B_1 \cos L_{1.2} - \sin B_2 \cos B_1) \\ \text{for } \eta_2 &: \quad + H_2 \cos B_2 \sin L_{1.2}, \end{aligned}\right\} \tag{2}$$

and the expression (5), p. 139 thus becomes the following one, which hence no longer gives the cotangent of the azimuth $a_{1.2}$ according to an object lying in the equipotential surface, but that of the azimuth $a^{(h)}_{1.2}$ for the object displaced outward along the perpendicular:

$$\cot a^{(h)}_{1.2} = \frac{(\cos B_2 \sin B_1 \cos L_{1.2} - \sin B_2 \cos B_1)(a_0 + H_2 W_2) + a_0 e^2 \cos B_1 \left(\sin B_2 - \frac{W_2}{W_1} \sin B_1\right)}{\cos B_2 \sin L_{1.2} (a_0 + H_2 W_2)} \tag{3}$$

If one compares this with $\cot a_{1.2}$ according to (5), p. 139, we find that:

$$\cot a^{(h)}_{1.2} = \cot a_{1.2} - e^2 \frac{H_2 W_2 \cos B_1}{a_0 + H_2 W_2} \frac{\sin B_2 - \frac{W_2}{W_1} \sin B_1}{\cos B_2 \sin L_{1.2}}. \tag{4}$$

According to p. 184 (10) p. 185 (14) and p. 185 (17), and with some obvious insignificant omissions, it follows that:

$$\cot a^{(h)}_{1.2} = \cot a_{1.2} + e^2 \frac{H_2 \cos B_1}{a_0 \sin a_{1.2}} \left(\cos B_1 \cos a_{1.2} + \frac{1}{2} \frac{s}{a_0} \sin B_1 + G l_2\right), \tag{5}$$

and p. 31 (2) gives:

$$\left.\begin{aligned} a^{(h)}_{1.2} - a_{1.2} &= -\frac{1}{2} \varrho'' e^2 \frac{H_2}{a_0} \left(\cos^2 B_1 \sin 2a^{(h)}_{1.2} + \frac{1}{2} \frac{s}{a_0} \sin 2B_1 \sin a^{(h)}_{1.2} + G l_2\right), \\ &\text{in seconds} \\ &H_2 \text{ the height of the object above the ellipsoid of rotation.} \end{aligned}\right\} \tag{6}$$

Consequently we have

$H_2 = 640^m$	for	$a_{1.2}^{(h)} - a_{1.2} = 0{,}069''$	maximum
$H_2 = 1280$	for	$a_{1.2}^{(h)} - a_{1.2} = 0{,}14$	maximum
$H_2 = 1920$	for	$a_{1.2}^{(h)} - a_{1.2} = 0{,}21$	maximum

While (cf. p. 135) the height of the station above the ellipsoid of rotation does not influence the azimuth and with it the horizontal angle, the foregoing shows that the height of the object can be an influence not to be disregarded. The amount of s, insofar as (in practice) mutually intervisible points along are concerned, is, besides, largely without influence.

Par. 25. The Sine Series for Chords and Horizontal Angles. To be able to compute a triangulation net from a directly measured base line out to its individual chords, a reduction, as with the sphere, of the measured horizontal angles to chord angles is not needed; one can compute much more directly with the horizontal angles.

We name the sides in the chord triangle $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and A, B, C or 1, 2, 3, those lying opposite angles $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$. The latter correspond to the horizontal angles A, B, C. Our task is now to eliminate $\mathfrak{A}$ and $\mathfrak{B}$ with the help of the horizontal angles from the equation

$$\mathfrak{a} : \mathfrak{b} = \sin \mathfrak{A} : \sin \mathfrak{B} \qquad (1)$$

For simplicity we presume that the angles $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are counted

inside the chord triangle, and correspondingly to A, B, C, these

angles are thus $< 180°$.

We take the triangle sides a, b, c as small and, in comparison to a_0, as first-order magnitudes, corresponding to the intended application to an observed triangulation net.

If we now observe the spherical triangle formed by the vertical planes from A to B and C and the plane triangle on a sphere having A as center (Fig. 11), then

$$\frac{\sin \alpha}{\sin A} = \frac{\cos \mu_{1.3}}{\sin \chi_{1.2}} = \frac{\cos \mu_{1.2}}{\sin \chi_{1.3}}. \qquad (2)$$

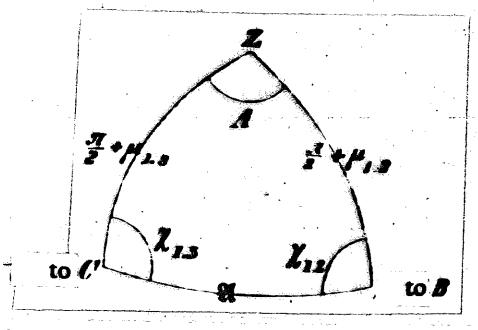


Figure 11. Sphere with Side A as Center.

Here the χ are the angles of inclination of the vertical planes to the plane triangle in the sense given in the figure. Through cyclic exchange of the indices there is found from (2):

$$\frac{\sin \beta}{\sin B} = \frac{\cos \mu_{2.1}}{\sin \chi_{2.3}} = \frac{\cos \mu_{2.3}}{\sin \chi_{2.1}} \qquad (3)$$

$$\frac{\sin \gamma}{\sin C} = \frac{\cos \mu_{3.2}}{\sin \chi_{3.1}} = \frac{\cos \mu_{3.1}}{\sin \chi_{3.2}}. \qquad (4)$$

The equation (1) thus gives:

$$a \cos \mu_{2.1} \frac{\sin B}{\sin \chi_{2.3}} = b \cos \mu_{1.2} \frac{\sin A}{\sin \chi_{1.3}}.$$

If this is side for side multiplied by the following relations from equation (4)

$$\cos \mu_{3.2} \sin \chi_{3.2} = \cos \mu_{3.1} \sin \chi_{3.1} \qquad (5)$$

then we have:

$$\mathfrak{a} \cos \mu_{2.1} \cos \mu_{3.2} \sin B \frac{\sin z_{3.2}}{\sin z_{2.3}} = \mathfrak{b} \cos \mu_{1.2} \cos \mu_{3.1} \sin A \frac{\sin z_{3.1}}{\sin z_{1.3}}. \qquad (6)$$

With the further conversion of this equation it is sufficient next to consider the right side alone, while both sides can be transformed by exchange of $\mathfrak{a}$ and $\mathfrak{b}$, Index 1 and 2 etc. into each other.

We now put

$$z_{3.1} = z_{1.3} - \nu_b, \qquad (7)$$

whereby ν_b indicates the surface angle of both of the vertical sections lying on the chord $\mathfrak{b}$; then we have:

$$\mathfrak{b} \sin A \frac{\sin z_{3.1}}{\sin z_{1.3}} = \mathfrak{b} \sin A \cos \nu_b - \mathfrak{b} \sin A \sin \nu_b \cot z_{1.3}. \qquad (8)$$

In order now to obtain a value for $z_{1.3}$, we pass a plane from the midpoint M_1 of the circle circumscribed around the chord triangle (Fig. 12) and normal to the chord $\mathfrak{b}$, and construct in the same a right triangle at M_1. The sides of this are $M_1 Q$, further a normal $M_1 M_2$ to the plane of the chord triangle and a normal $Q M_2$ to the chord $\mathfrak{b}$, which last normal lies at the same time in the vertical plane from A to C. Then the angle at Q is equal to $180^0 - z_{1.3}$ and:

(See Illustrative Material on Next Page.)

Chapter 4. The Vertical Section and Chord Triangle for the Flattened Ellipsoid of Revolution.

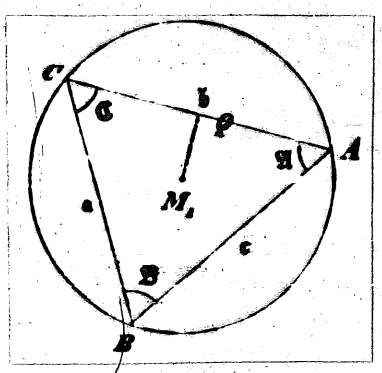


Figure 12

$$\cos \chi_{1.3} = - \frac{M_1 Q}{Q M_2}.$$

$\frac{1}{2} b \cot \mathfrak{B}$ is to be put for $M_1 Q$, but for $Q M_2$, if in place of the ellipsoid we imagine a sphere of radius $a_0 (1 + Gl_2)$ passed through ABC to be put simply as a_0. Thus $\cos \chi_{1.3}$ is indeed not obtained exactly, but precisely to one term of the order $e^2 \frac{h}{a_0}$, where h designates a mean chord length. For the normals of the ellipsoid and of the sphere chosen at random will only at the most deviate from each other by terms of this order (as observation of the expression for the radius of curvature shows), and only by just as much the angles χ and their cosines for ellipsoid and sphere too. Thus we have

$$\cos \chi_{1.3} = - \frac{b \cot \mathfrak{B}}{2 a_0} (1 + Gl_2). \qquad (9)$$

If we introduce for $\cos \chi_{1.3}$ in equation (8) the value obtained in (9), and put at the same time according to (2) $\sin A = \sin \mathfrak{A} \sin \chi_{1.3} \sec \mu_{1.2}$, as well as for $b \sin \mathfrak{A}$ the equivalent expression $a \sin \mathfrak{B}$, then we shall have:

$$b \sin A \frac{\sin \chi_{3.1}}{\sin \chi_{1.3}} = b \left\{ \sin A \cos \nu_b + \frac{a}{2 a_0} \sin \nu_b \cos \mathfrak{B} \sec \mu_{1.2} + Gl_6 \right\}. \qquad (10)$$

The exchange of a and b, the indices 1 and 2, etc. gives for the left side of the equation (6):

$$a \sin B \frac{\sin z_{3.2}}{\sin z_{2.3}} = a\left\{\sin B \cos \nu_a + \frac{b}{2a_0} \sin \nu_a \cos \mathfrak{A} \sec \mu_{2.1} + Gl_6\right\}, \qquad (11)$$

$$z_{3.2} = z_{2.3} - \nu_a.$$

Equation (6), by use of the previous conversions, can now be brought into the form:

$$a\left\{\cos \mu_{2.1} \cos \mu_{3.2} \cos \nu_a \sin B - \frac{b}{2a_0} \cos \mu_{3.1} \sin \nu_b \cos \mathfrak{B} + Gl_6\right\}$$
$$= b\left\{\cos \mu_{1.2} \cos \mu_{3.1} \cos \nu_b \sin A - \frac{a}{2a_0} \cos \mu_{3.2} \sin \nu_a \cos \mathfrak{A} + Gl_6\right\} \qquad (12)$$

According to (4), p. 188 and with the determination of signs according to the terms of (7) and (11), whereby it can be noted that the interior of the chord triangle (Fig. 12) lies on the right hand, there is here to be inserted, if one runs through the periphery abc:

$$\left.\begin{aligned} \nu_a &= -\varrho'' \delta \left(\frac{a}{2a_0} \cos^2 B_3 \sin 2a_{3.2} + \frac{a^2}{4a_0^2} \sin 2B_3 \sin a_{3.2}\right) + Gl_b \\ \text{in seconds} \\ \nu_b &= +\varrho'' \delta \left(\frac{b}{2a_0} \cos^2 B_3 \sin 2a_{3.1} + \frac{b^2}{4a_0^2} \sin 2B_3 \sin a_{3.1}\right) + Gl_b. \end{aligned}\right\} \qquad (13)$$

If we now write the formula (12) as follows:

$$\frac{a}{b} = \frac{\sin A - \frac{a}{2a_0} \sin \nu_a \cos \mu_{3.2} \sec \mu_{1.2} \sec \mu_{3.1} \sec \nu_b \cos \mathfrak{A} + Gl_a}{\sin B - \frac{b}{2a_0} \sin \nu_b \cos \mu_{3.1} \sec \mu_{2.1} \sec \mu_{3.2} \sec \nu_a \cos \mathfrak{B} + Gl_a} \cdot \frac{\cos \mu_{3.1} \cos \mu_{1.2} \cos \nu_b}{\cos \mu_{3.2} \cos \mu_{2.1} \cos \nu_a}, \qquad (14)$$

we can simply insert $\cos A$, with consideration of the fact that

in numerators and denominators sixth-order terms are already disregarded and, also, ν_a and ν_b are only developed to fifth-order terms, for $\cos \mathfrak{A}$, which equals

$$\sin \mu_{1.2} \sin \mu_{1.3} + \cos \mu_{1.2} \cos \mu_{1.3} \cos A$$

and simply $\cos \mathfrak{B}$ correspondingly for $\cos B$, as this produces only sixth-order errors in numerators and denominators. Further, we can with equal reason take as equal to 1 the cosine and secant factors of $\cos \mathfrak{A}$ and $\cos \mathfrak{B}$, and equally $\cos \nu_b = 1 = \cos \nu_a$ in the last part of (14). Thus from the expression (14) we obtain:

$$\frac{a}{b} = \frac{\sin A - \cos A \cdot \frac{a}{2a_0} \nu_a + Gl_a}{\sin B - \cos B \cdot \frac{b}{2a_0} \nu_b + Gl_b} \frac{\cos \mu_{3.1}}{\cos \mu_{3.2}} \frac{\cos \mu_{1.2}}{\cos \mu_{2.1}}. \qquad (15)$$

Par. 26. Continuation. Sine Series for the Computation of Triangles. According to p. 174 (1) and (2) we have:

$$\sin \mu_{3.1} = \frac{b}{2a_0} W_3 \left(1 + e^2 \cos^2 B_3 \cos^2 a_{3.1} + \frac{e^2 b}{2a_0} \sin 2B_3 \cos a_{3.1}\right) + Gl_3$$

or on introduction of the curvature for point C with the geographic latitude B_3 according to p. 59, Par. 15:

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$$\sin \mu_{3.1} = \frac{b}{2a_0} \sqrt{K_3} \left(1 + \frac{1}{2} e^2 \cos^2 B_3 \cos 2a_{3.1} + \frac{e^2 b}{2a_0} \sin 2B_3 \cos a_{3.1}\right) + Gl_3.$$

If we now put the previous expression for $\sin \mu_{3.1}$ in the series $\cos = 1 - \frac{1}{2} \sin^2 - \frac{1}{8} \sin^4 \ldots$, we obtain for $\cos \mu_{3.1}$, after some reduction:

Par. 26. Continuation. Sine Series for the Computation of Triangles.

$$\cos\mu_{3.1} = \left(1 - \frac{b^2}{8a_0^2} K_3 - \frac{b^4}{128a_0^4} K_3^2\right) \times \left(1 - \frac{e^2b^2}{8a_0^2}\cos^2 B_3 \cos 2a_{3.1} - \frac{e^2b^3}{8a_0^3}\sin 2B_3 \cos a_{3.1}\right) + Gl_6. \tag{1}$$

The interchange of the indices 1 and 2 etc. gives:

$$\cos\mu_{3.2} = \left(1 - \frac{a^2}{8a_0^2} K_3 - \frac{a^4}{128a_0^4} K_3^2\right) \times \left(1 - \frac{e^2a^2}{8a_0^2}\cos^2 B_3 \cos 2a_{3.2} - \frac{e^2a^3}{8a_0^3}\sin 2B_3 \cos a_{3.2}\right) + Gl_6. \tag{2}$$

If in the same way one defines $\cos\mu_{2.1}$ and $\cos\mu_{1.2}$, then one easily sees that

$$\frac{\cos\mu_{1.2}}{\cos\mu_{2.1}} = 1 - \frac{e^2r^2}{8a_0^2}\left[\cos^2 B_1 \cos 2a_{1.2} - \cos^2 B_2 \cos 2a_{2.1}\right] + Gl_6. \tag{3}$$

In order to simplify this formula we use the relations (23), p. 186:

$$\begin{aligned} \cos B_1 \sin a'_{1.2} &= -\cos B_2 \sin a'_{2.1} \\ -\cos B_1 \cos a'_{1.2} &= \sin B_2 \sin\sigma' + \cos B_2 \cos a'_{2.1} \cos\sigma'. \end{aligned}$$

From this, with consideration to p. 148 (6) we have:

$$\left.\begin{aligned} \cos B_1 \sin a_{1.2} &= -\cos B_2 \sin a_{2.1} + Gl_3, \\ \cos B_1 \cos a_{1.2} &= -\cos B_2 \cos a_{2.1} - \frac{r}{a_0}\sin B_2 + Gl_3 \end{aligned}\right\} \tag{4}$$

and

$$\cos^2 B_1 \cos 2a_{1.2} = \cos^2 B_2 \cos 2a_{2.1} + \frac{r}{a_0}\sin 2B_2 \cos a_{2.1} + Gl_3, \tag{5}$$

with which (3) becomes:

$$\frac{\cos\mu_{1.2}}{\cos\mu_{2.1}} = 1 - \frac{e^2r^2}{8a_0^2}\sin 2B_2 \cos a_{2.1} + Gl_6. \tag{6}$$

Chapter 4. The Vertical Section and Chord Triangle for the Flattened Ellipsoid of Revolution.

For the conversion of the quotients to be formed from (1) and (2) we next consider the quotients:

$$\frac{1-\frac{1}{8}b_1^2-\frac{1}{128}b_1^4-\cdots}{1-\frac{1}{8}a_1^2-\frac{1}{128}a_1^4-\cdots}, \qquad (7)$$

where b_1 and a_1 designate $b\sqrt{K_3}:a_0$ and $a\sqrt{K_3}:a_0$ respectively.

Multiplication by $\left(1+\frac{a_1^2+b_1^2-c_1^2}{16}\right)$ in the numerator and denominator produces with equal precision, namely, to sixth-order terms:

$$\frac{1+\frac{a_1^2-b_1^2-c_1^2}{16}-\frac{a_1^2+2b_1^2-c_1^2}{128}b_1^2+\cdots}{1+\frac{-a_1^2+b_1^2-c_1^2}{16}-\frac{2a_1^2+b_1^2-c_1^2}{128}a_1^2+\cdots}. \qquad (8)$$

Now $a_1^2-b_1^2-c_1^2=-2b_1c_1\cos\mathfrak{A}$, or if we remember the relation of $\cos\mathfrak{A}$ to $\cos A$ (cf. p. 193).

$$a_1^2-b_1^2-c_1^2=-2b_1c_1\left(\cos A\left[1-\frac{b_1^2+c_1^2}{8}\right]+\frac{b_1c_1}{4}\right)+Gl_6.$$

Here we also insert for abbreviation:

$$\frac{1}{2}b_1c_1\sin A\left(1+\frac{1}{8}a_1^2\right)=\varepsilon_a. \qquad (9)$$

Which thus (without doubtful factors appearing in the sixth-order terms) becomes:

$$a_1^2-b_1^2-c_1^2=-4\varepsilon_a\cot A\left(1-\frac{a_1^2+b_1^2+c_1^2}{8}\right)-\frac{1}{2}b_1^2c_1^2+Gl_6$$

or

$$a_1^2 - b_1^2 - c_1^2 = -4\varepsilon_a \cot A - \frac{1}{2} b_1^2 c_1^2 - \frac{(a_1^2 - b_1^2 - c_1^2)(a_1^2 + b_1^2 + c_1^2)}{8} + Gl_6.$$

Thus there is yielded for the numerator of (8):

$$1 - \frac{1}{4}\varepsilon_a \cot A + \frac{1}{128}(-a_1^4 - b_1^4 + c_1^4 - a_1^2 b_1^2 - c_1^2 b_1^2) + Gl_6. \qquad (10)$$

By interchange of a_1 and b_1, there follows from this expression the denominator:

$$1 - \frac{1}{4}\varepsilon_b \cot B + \frac{1}{128}(-a_1^4 - b_1^4 + c_1^4 - a_1^2 b_1^2 - c_1^2 a_1^2) + Gl_6, \qquad (11)$$

where we put:

$$\frac{1}{2} a_1 c_1 \sin B \left(1 + \frac{1}{8} b_1^2\right) = \varepsilon_b. \qquad (12)$$

We easily note that the identical fourth-order terms in the numerator and denominator can be omitted and in return $c_1^2(a_1^2 + b_1^2 - c_1^2):256$ may be added, because this gives only a small sixth-order error. The expression (7) or (8) respectively then becomes:

$$\frac{1 - \frac{1}{4}\varepsilon_a \cot A \left(1 + \frac{c^2}{16 a_0^2}\right)}{1 - \frac{1}{4}\varepsilon_b \cot B \left(1 + \frac{c^2}{16 a_0^2}\right)} + Gl_6. \qquad (13)$$

It may be shown that ε_a and ε_b agree up to sixth-order terms. According to (2), p. 191, we next have:

$$\sin A = \sin \mathfrak{A} \sin \chi_{1.2} \sec \mu_{1.3} = \sin \mathfrak{A} \sin \chi_{1.2} \left(1 + \frac{1}{8} b_1^2\right) + Gl_4.$$

Thus for ε_a and then through due interchange we obtain for ε_b :

$$\varepsilon_a = \frac{1}{2} b_1 r_1 \sin \mathfrak{A} \left(1 + \frac{a_1^2 + b_1^2}{8}\right) \sin \chi_{1.2} + Gl_4$$
$$\varepsilon_b = \frac{1}{2} a_1 r_1 \sin \mathfrak{B} \left(1 + \frac{a_1^2 + b_1^2}{8}\right) \sin \chi_{2.1} + Gl_4.$$

As $b_1 \sin \mathfrak{A} = a_1 \sin \mathfrak{B}$, both expressions agree up to $\sin \chi$. Now, with $\chi_{2.1} = \chi_{1.2} - \nu_c$:

$$\sin \chi_{1.2} = \sin \chi_{2.1} \cos \nu_c + \cos \chi_{2.1} \sin \nu_c;$$

the difference of the χ hence produces mainly the term apparently having fifth-order:

$$\frac{1}{2} b_1 r_1 \nu_c \sin \mathfrak{A} \cos \chi_{2.1}.$$

But if we insert the equal value $r_1 \sin \mathfrak{A}$ for $a_1 \sin \mathfrak{C}$, then it is easy to gather from (9), p. 192 that $\sin \mathfrak{C} \cos \chi_{2.1}$ becomes almost equal to $\left(-\frac{1}{2} r_1 \cos \mathfrak{C}\right)$ and as ν_c has third-order, there is accordingly that difference too from the sixth-order.

Thus to (13), whilst we now designate ε_a and ε_b as ε_3 :

$$\varepsilon_3 \underset{\text{in seconds}}{=} \begin{cases} = \frac{1}{2} \varrho'' \frac{b r}{a_0^2} K_3 \sin A \left(1 + \frac{1}{8} \frac{a^2}{a_0^2}\right) + Gl_6 \\ = \frac{1}{2} \varrho'' \frac{a r}{a_0^2} K_3 \sin B \left(1 + \frac{1}{8} \frac{b^2}{a_0^2}\right) + Gl_6. \end{cases} \qquad (14)$$

If we now insert everything into the equation (15) of the preceding paragraph, we have:

$$\frac{a}{b} = \frac{\sin A - \cos A \left\{\frac{\varepsilon_3}{4}\left(1 + \frac{r^2}{16 a_0^2}\right) + \frac{a}{2 a_0} \nu_a\right\} + Gl_6}{\sin B - \cos B \left\{\frac{\varepsilon_3}{4}\left(1 + \frac{r^2}{16 a_0^2}\right) + \frac{b}{2 a_0} \nu_b\right\} + Gl_6} \cdot F \qquad (15)$$

$$\log F = M\left\{\begin{array}{l}\frac{a^2}{8a_0^2}e^2\cos^2 B_1\cos 2a_{1.2} - \frac{b^2}{8a_0^2}e^2\cos^2 B_2\cos 2a_{2.1} \\ +\frac{a^2}{8a_0^2}e^2\sin 2B_1\cos a_{1.2} - \frac{b^2}{8a_0^2}e^2\sin 2B_2\cos a_{2.1} + \frac{c^2}{8a_0^2}e^2\sin 2B_c\cos a_c + Gl_3\end{array}\right\} \quad (16)$$

$$B_c = \frac{1}{2}(B_1 + B_2) \quad a_c = \frac{1}{2}(a_{1.2} + a_{2.1} \pm 180^0)$$

Designation of $\cos a_c$ same as of $\cos a_{1.2}$.

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If we multiply the right side of (15) in numerator and denominator by $\left(1 - \frac{1}{32}\varepsilon_3^2\right)$, we finally obtain:

$$\frac{a}{b} = \frac{\sin\left\{A - \frac{\varepsilon_3}{4}\left(1 + \frac{c^2}{16a_0^2}\right) - \frac{a}{2a_0}\nu_a\right\} + Gl_a}{\sin\left\{B - \frac{\varepsilon_3}{4}\left(1 + \frac{c^2}{16a_0^2}\right) - \frac{b}{2a_0}\nu_b\right\} + Gl_b} \cdot F. \quad (17)$$

Par. 27. The Sine Series runs according to the developments of both the last paragraphs if we still neglect fifth-order terms:

$$\frac{a}{b} = \frac{\sin\left\{A - \frac{\varepsilon}{4}\left(1 + \frac{c^2}{16a_0^2}\right) + \varrho'\frac{e^2a^2}{4a_0^2}\cos^2 B_0\sin 2a_{3.2}\right\} + Gl_a}{\sin\left\{B - \frac{\varepsilon}{4}\left(1 + \frac{c^2}{16a_0^2}\right) - \varrho''\frac{e^2b^2}{4a_0^2}\cos^2 B_0\sin 2a_{3.1}\right\} + Gl_b} \cdot F$$

$$\log F = M\left(\frac{e^2a^2}{8a_0^2}\cos^2 B_0\cos 2a_{3.2} - \frac{e^2b^2}{8a_0^2}\cos^2 B_0\cos 2a_{3.1} + Gl_3\right) \quad (1)$$

$$\varepsilon = \text{in seconds} \begin{cases} = -\frac{1}{2}\varrho''\frac{bc}{a_0^2}K_0\sin A\left(1 + \frac{1}{8}\frac{a^2}{a_0^2}\right) + Gl_a \\ = \frac{1}{2}\varrho''\frac{ac}{a_0^2}K_0\sin B\left(1 + \frac{1}{8}\frac{b^2}{a_0^2}\right) + Gl_b. \end{cases}$$

Here B_0 is a mean latitude of the triangle and $K_0 : a_0^2$ the curvature measure. A mean azimuth of the relevant sides can be introduced for $a_{3.2}$ and $a_{3.1}$.

We must note besides that in the first formula (1) the terms multiplied in e^2 have the sign $+$ for the side a following from left to right in the triangle outline, but (-) for the pre-

ceding side b . On the other hand it is not necessary, as formerly accepted, to be confined to an angle less than 180^0 ; it suffices (much better) to determine the angles in general according to (1), p. 71. This is yielded by (1) above, giving still the same value for the ratio $a:b$, if we take a as the side preceding b and interchange accordingly A and B with their supplements at 360^0 . In $\log F$ the signs correspond with $\log a - \log b$. .

The formulae for $b:c$ and $c:a$ proceed out of (1) through cyclic interchange of the parts of the triangle, with B_0, K_0 and e (as one easily sees) also retaining their values.

If we put $e^2 = 0$ then the first formula (1) (relating to b and c) becomes the formula (6), p. 108. At the same time ε then indicates the spherical excess (possibly $- 720^0$). The question whether $\varepsilon = A + B + C - 180^0$ (possibly $- 720^0$) is still up to fifth-order terms, remains temporarily open.

We will now go over to discuss the influence of e^2 on the expression for $a:b$.

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Next it turns out that the fourth-order terms independent of e^2 even for triangle sides equal to $0{,}1a_0$ markedly diminish against the terms dependent on e^2 .

The terms combined with A and B are the most influential of the terms multiplied by e^2 , as soon as $\cot A$ and $\cot B$ respectively are greater than $\frac{1}{2}$, for then the influence of one of

these terms amounts to more than $M \cdot \frac{k^3}{8a_0^2} e^2$ on the $\log \frac{a}{b}$, when k designates the chords a and b, and this is evidently the maximum influence of one of the terms with e^2 in $\log F$.

For well-formed triangles, that is to say those that do not deviate considerably from equilateral or rectangular form, one can put the cotangents of the relevant angles as approximately equal to I. Then the influence is that of a fourth-order term which holds e^2, at maximum with $k = 0{,}02 a_0$ as equal to 3 units of the seventh-place decimal of the logarithm and with $k = 0{,}01 a_0$ as equal to 7 units of the eighth decimal place. It greatly diminishes for the case of the triangle lying at a distance from the equator.

The influence of these terms is thus only considerable if we ascertain the sides of a chain of triangles of average 64 km chord length simply according to the Grunert series by means of eight-figure logarithms. The inaccuracy causes the geographic positions computed from one point of origin to differ by some units in the fourth decimal of the seconds when they are obtained in different ways through the chord polygon.

Thus the circumstance is favorable that no accumulation of the terms should arise at the successive computation of a', a'', a''' etc. in a chain of the form of Fig. 13, which is formed approxi-

(See Illustrative Material on Next Page.)

Chapter 4. The Vertical Section and Chord Triangle for the Flattened Ellipsoid of Revolution.

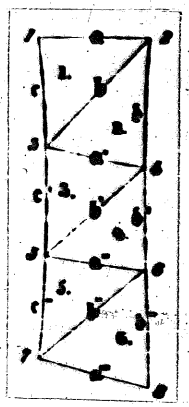


Figure 13

mately from parallelograms, - because the influence for each bracket in the expression $\left(\frac{a}{b}\cdot\frac{b}{a'}\right)\left(\frac{a'}{b'}\cdot\frac{b'}{a''}\right)\cdots$ becomes nearly zero. This one can easily see when one observes the first bracket which belongs to the rectangle 1. 2. 3. 4, and when one applies the formulae (1) to it, but carefully considers that not only a and b, but also a' and b are sides following from left to right, the one after the other, and finally assumes a and a', and c and b respectively are nearly parallel.

(The fifth-order terms contained in the ν and F do not substantially compensate one another in the computation of a chain of triangles in general; but they are extremely small for measurable triangles, as is easily seen, so that an accumulation does not become soon noticeable.)

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The length of the whole chain is accordingly influenced in its logarithm only as much as the c or b.

If we now examine how the influence of the terms under discussion for very acute or obtuse triangles affects $\log\frac{a}{b}$,

then it is immediately evident that it can expand to far more than for well-formed triangles. It is anyhow taken in seconds, and if we disregard unusually large measurable triangles, and put $k < 0{,}01 a_0$, the amount of the fourth-order terms supervene upon A and B and dependent on e^2 is only some hundredths of a second, and thus far smaller than the errors of observation. But as one applies similar triangles only at the setting up of the side equations*) in connection with well-formed triangles for the purpose of adjustment of the angle measurement, so here presumably too the final influence of the terms on the amount computed above is reduced.

A detailed rigorous investigation on how the influence of the terms forms after ensuing adjustment we must deny ourselves. It may be noted only that in rectangles which approach parallelograms in shape as 1. 2. 3. 4 of Fig. 13, the side equations of the terms with e^2 do not become influenced significantly, for the side equation amounts in each form to the adjustment of both the values for the ratio $a:a'$, which is computed with help of both the diagonals. But for each of these values, the influence, as noted above, is almost zero in the event that a is equal and parallel to a'.

We do not want to leave unmentioned that the form of the expression for $a:b$ indicates the impossibility of computing a chord triangle of an ellipsoid of rotation in general as a chord triangle of a sphere with relevant reduction of the

*)The side equations are better called chord equations. They form in the same manner as in spherical figures. Cf. to this the author's Ausgleichungsrechnung (English: Computation of Adjustment) p. 323 ff.

horizontal angle. This is confirmed by the experiment of first representing $\cos A$ according to the powers of e^2 by means of the $\cos A = -\tan \mu_{1.2} \tan \mu_{1.3} + \sec \mu_{1.2} \sec \mu_{1.3} \cos \Delta$ formula, and introduce as assumed values for $\mu_{1.2}$ and $\mu_{1.3}$ values that correspond to a sphere with the radius $(a_0 : \sqrt{K_0})$. If the horizontal angle A', belongs to these assumed values then it is just not possible to derive from $\cos A - \cos A'$ a formula valid for every A to give $A - A'$. The reason is that for very obtuse triangles the radius of the circumscribed circle can become greater than $(a_0 : \sqrt{K_0})$. The triangle then cannot be placed in the selected sphere.

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Par. 28. Numerical Example III. Bremiker derives in his Studies... pp. 17 and 28 the following horizontal angles and chords of a triangle from the geographic positions of three places:

		Azimuth a.	Horizontal Angle	log k in Meters
Berlin	K	239° 33′ 0,9324″	95° 19′ 4,6513″	$\log KW = 5{,}8905264.742$
	W	334 52 5,5837		
Königsberg	W	23 11 41,9353	42 4 27,6453	$\log WB = 5{,}7180944.366$
	B	65 16 9,5806		
Wien	B	157 10 6,9817	42 48 6,9048	$\log BK = 5{,}7241345.725$
	K	199 58 13,8865		
		Sum = 180° 11′ 39,2014″		

Par. 28. Numerical Example III.

	Geographical Longitude	Length
Berlin	52° 30′ 16,7″	0° 0′ 0″
Königsberg	54 42 50,6	7 6 0 East
Wien	48 12 35,5	2 59 6 East

These figures do not correspond to the Bremiker data for the line BK, for which the data of the numerical example 1, p. 158 were introduced. Moreover, $\log WB$ is diminished from Bremiker's statement by 28 units of the tenth place. Recomputed and found agreeing are $\log KW$ and the differences of the final azimuth of the Earth for each one of both the lines WB and KW.

While we introduce the designations of our formulae, we put:

$\log \mathfrak{a} = 5{,}7180944.4$	$A = 42^0\,4'\,27{,}645''$	$B_1 = 54^0\,43'$	Königsb. $= A$
$\log \mathfrak{b} = 5{,}7241345.7$	$B = 42\,48\,6{,}905$	$B_2 = 48\,13$	Wien $= B$
$\log \mathfrak{c} = 5{,}8905264.7$	$C = 95\,19\,4{,}651$	$B_3 = 52\,30$	Berlin $= C$

$a_{3.2} = 334^0\,52'$		$\log \frac{M}{8a_0^2} = 2{,}125-10$	For units of the seventh decimal place.
$a_{3.1} = 239\,33$	$\log \frac{\varrho'' e^2}{4a_0^2} = 8{,}927-20$	$\log \frac{Me^2}{8a_0^2} = 9{,}950-20$	
$a_{2.1} = 199\,58$	$\log \frac{\varrho'' e^2}{8a_0^2} = 1{,}821-30$	$\log \frac{Me^3}{8a_0^3} = 3{,}145-20$	

At the computation of $\log \frac{\mathfrak{a}}{\mathfrak{b}}$ we consider also the fifth-order terms, very noticeable here because of the unusual magnitude of the triangle.

Next ε is to be computed according to formula (14), p. 196.

For this by means of $\log W$ we have for the argument B_3 according to (3), p. 59, Par. 14:

$$\log K_3 = 9{,}999252-10$$

and from this by means of the first formula (14):

$$\begin{aligned} \log\left(\tfrac{1}{2}\varrho''\,\tfrac{\mathfrak{br}}{a_0^2}\,K_3\sin A\right) &= 2{,}844159 \\ +M.\tfrac{1}{8}\tfrac{\mathfrak{s}^3}{a_0^2} &= +\ 365 \\ \log \varepsilon_3 = 2{,}844524 \quad & \varepsilon_3 = 699{,}076''. \end{aligned}$$

in seconds

Computation with $\mathfrak{ar}$ according to the second formula (14) gives the same. With this value of ε_3 ,

$$\begin{aligned} \tfrac{\varepsilon_3}{4} &= 174{,}769'' \\ +\tfrac{\varepsilon_3\mathfrak{r}^2}{64a_0^2} &= +0{,}163 \\ \tfrac{\varepsilon_3}{4}\left(1+\tfrac{\mathfrak{r}^2}{16a_0^2}\right) &= 174{,}932''. \end{aligned}$$

We have further for the computation of the terms dependent on v_a and v_b according to formula (13), p. 193:

$\log\sin 2a_{3.2} = 9{,}886_n-10$	$\log\sin a_{3.2} = 9.628_n-10$	$\log\cos^2 B_3 = 9{,}569-10$
$\log\sin 2a_{3.1} = 9{,}941\ -10$	$\log\sin a_{3.1} = 9.936_n-10$	$\log\sin 2B_3 = 9{,}985-10$

and from this to formula (17), p. 197:

$$\begin{aligned} -\tfrac{\mathfrak{a}}{2a_0}v_a &= +\left(\tfrac{\varrho''e^2}{4a_0^2}\cos^2 B_3\right)\mathfrak{a}^2\sin 2a_{3.2} \\ &\quad +\left(\tfrac{\varrho''e^2}{8a_0^2}\sin 2B_3\right)\mathfrak{a}^3\sin a_{3.2} = -0{,}658''-0{,}039'' \\ -\tfrac{\mathfrak{b}}{2a_0}v_b &= -\left(\tfrac{\varrho''e^2}{4a_0^2}\cos^2 B_3\right)\mathfrak{b}^2\sin 2a_{3.1} \\ &\quad -\left(\tfrac{\varrho''e^2}{8a_0^2}\sin 2B_3\right)\mathfrak{b}^3\sin a_{3.1} = -0{,}768''+0{,}083''. \end{aligned}$$

Hence we have

$$\frac{\mathfrak{a}}{\mathfrak{b}} = \frac{\sin(42^\circ\,4'\,27{,}645'' - 2'\,54{,}932'' - 0{,}697'')}{\sin(42^\circ\,48'\,6{,}905'' - 2'\,54{,}932'' - 0{,}685'')} \cdot F$$

and thus

$$\log \mathfrak{a} - \log \mathfrak{b} = 9{,}8257259.7 - 9{,}8317680.2 + \log F.$$

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According to formula (16), p. 196 we have for $\log F$:

$\log\cos 2a_{3.2} = 9{,}806 - 10$	$\log\cos a_{3.2} = 9{,}957 - 10$	$\log\cos a_c = 9{,}969 - 10$
$\log\cos 2a_{3.1} = 9{,}687_n - 10$	$\log\cos a_{3.1} = 9{,}705_n - 10$	$\log\sin 2B_c = 9{,}989 - 10$

$$\left\{\begin{array}{ll}
+\left(\frac{Me^2}{8a_0^2}\cos^2 B_3\right)\mathfrak{a}^2\cos 2a_{3.2} = + & 5.77 \text{ units of the 7th decimal place,}\\
-\left(\frac{Me^2}{8a_0^2}\cos^2 B_3\right)\mathfrak{b}^2\cos 2a_{3.1} = + & 4.51 \quad " \quad "\\
+\left(\frac{Me^2}{8a_0^3}\sin 2B_3\right)\mathfrak{a}^3\cos a_{3.2} = + & 1.74 \quad " \quad "\\
-\left(\frac{Me^2}{8a_0^3}\sin 2B_3\right)\mathfrak{b}^3\cos a_{3.1} = + & 1.02 \quad " \quad "\\
+\left(\frac{Me^2}{8a_0^3}\sin 2B_c\right)\mathfrak{c}^3\cos a_c = + & 5.95 \quad " \quad "
\end{array}\right.$$

$$\log F = + 19.0;$$

with this

$$\log \mathfrak{a} - \log \mathfrak{b} = 9{,}9939598.5 - 10,$$

agreeing to within 1 unit with the data of Bremiker.

If in the formulae we interchange $\mathfrak{abc}$ with $\mathfrak{bca}$, 1.2.3 with 2.3.1 etc., then we obtain the formulae for the computation of $\mathfrak{b}:\mathfrak{c}$. The computation gives here:

(See Illustrative Material on Next Page.)

$$\frac{b}{c} = \frac{\sin(42^\circ\,48'\,6{,}905'' - [2'\,54{,}683'' + 0{,}073''] + [0{,}604'' + 0{,}085''])}{\sin(95^\circ\,19'\,4{,}651'' - [2'\,54{,}683'' + 0{,}073''] - [1{,}239'' + 0{,}115'')} F$$

$$\log b - \log c = 9{,}8317715.5 - 9{,}9981609.9 + \log F$$

$$\log F = (-5.43 - 12.42) + (0.78 - 5.69 - 1.79) = -24.5 \text{ units of the 7th decimal place,}$$

consequently:

$$\log b - \log c = 9{,}8336081.1 - 10,$$

this is by only 1 unit of the eighth decimal place too large against Bremiker's data; an agreement (in fact) sufficient because the disregarded sixth-order terms can most certainly influence the eighth decimal place slightly.

Par. 29. The Sum of the Horizontal Angles above a Chord-Triangle.

We imagine a concentric sphere laid around the mid-point of the ellipsoid of rotation and all verticals of the ellipsoid displaced parallel to the mid-point.

Then three points A', B', C' of the sphere with the same geographic latitudes and lengths correspond to the three points A, B, C of the ellipsoid.

If we designate the azimuths of the vertical sections on the sphere by "prime", then according to p. 151 (4) we have for a chord triangle the sides of which are first-order magnitudes in relation to a_0

(See Illustrative Material on Next Page.)

Par. 29. The Sum of the Horizontal Angles above a Chord-Triangle.

$$\left.\begin{aligned} a_{2.1} - a_{1.2} &= a'_{2.1} - a'_{1.2} + Gl_7 \\ a_{1.3} - a_{3.1} &= a'_{1.3} - a'_{3.1} + Gl_7 \\ a_{3.2} - a_{2.3} &= a'_{3.2} - a'_{2.3} + Gl_7 . \end{aligned}\right\} \qquad (1)$$

The seventh-order terms depend on the fourth power of the eccentricity e and the third power of the distance and give for side lengths equal to $0{,}2a_0$ a few thousandths of a second, but for side lengths equal to $0{,}1a_0$ only a few ten thousandths of a second.

If we add the equations (1), side for side, and consider that the horizontal angles become defined on the ellipsoid and the sphere by the equations:

$$\left.\begin{aligned} A &= a_{1.3} - a_{1.2} & A' &= a'_{1.3} - a'_{1.2} \\ B &= a_{2.1} - a_{2.3} & B' &= a'_{2.1} - a'_{2.3} \\ C &= a_{3.2} - a_{3.1} & C' &= a'_{3.2} - a'_{3.1} , \end{aligned}\right\} \qquad (2)$$

then the remarkable equation immediately yielded is:

$$A + B + C = A' + B' + C' + Gl_7 . \qquad (3)$$

If we designate $(A + B + C - 180^0)$ as the excess of the horizontal angle sum of the chord triangle then, consequently, within the indicated limits and the indicated precision:

the excess of the horizontal angle sum of the chord triangle is equal to the excess of the spherical triangle.

The radius of the sphere does not matter here; but it would on other grounds be best to take $r_0 : \sqrt{K_0}$ for it, where for K_0

$$B_0 = \tfrac{1}{3}(B_1 + B_2 + B_3) \qquad (4)$$

and according to p. 59 there follows from the formula:

$$K_0 = \frac{(1 - e^2 \sin^2 B_0)^2}{1 - e^2} = \frac{W_0^4}{1 - e^2}. \qquad (5)$$

With the help of this radius and of the formulae stated in Pars. 7 to 11 of this chapter we can ascertain from the elements of the chord triangle on the ellipsoid those of the triangle on the sphere, i.e. a', b', c', A', B', C' . It suffices perfectly to consider only the terms up to fifth-order finally to obtain the excess up to sixth-order terms.

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According to Par. 10, p. 155 we have especially for the chord between B and C :

$$a = 2 \sin \frac{\sigma'_{2.3}}{2} \left\{ \frac{a_0 (1 - e^2)}{\sqrt{1 - e^2 \sin^2 B_a}} : (1 - e^2 + e^2 \cos^2 B_a \cos^2 a_a + G l_4) \right\}, \qquad (6)$$

$$B_a = \frac{1}{2}(B_2 + B_3), \quad a_a = \frac{1}{2}(a_{2.3} + a_{3.2} \pm 180^0), \qquad (7)$$

where it is immaterial whether one computes on the right hand side with the azimuths on the sphere or with the ellipsoid, because these deviate from one another only by small second-order terms.

On the other side one has for the chord between B' and C' :

$$a' = 2 \sin \frac{\sigma'_{2.3}}{2} \frac{a_0}{\sqrt{K_0}} = 2 \sin \frac{\sigma'_{2.3}}{2} \frac{a_0 \sqrt{1 - e^2}}{1 - e^2 \sin^2 B_0},$$

and hence we have:

$$\frac{a'}{a} = \frac{\sqrt{1 - e^2 \sin^2 B_a}}{1 - e^2 \sin^2 B_0} \left(1 - \frac{1}{2} e^2 + e^2 \cos^2 B_a \cos^2 a_a + G l_4\right)$$

or developed further:

$$\frac{s'}{s} = 1 + \frac{1}{2} e^2 \cos^2 B_a \cos 2a_a + e^2 (\sin^2 B_0 - \sin^2 B_a) + Gl_4. \qquad (8)$$

For the horizontal angles, and in particular for the angles B from the relations (2) there is yielded further:

$$B' - B = (a'_{2.1} - a_{2.1}) - (a'_{2.3} - a_{2.3}), \qquad (9)$$

and for this according to (2) and (3), pp. 157 and 158 we have:

$$B' - B = \frac{1}{2} e^2 \cos B_2 \left\{ \cos B_3 \sin 2a_a - \cos B_1 \sin 2a_c \right\} + Gl_4, \qquad (10)$$

if we put:

$$a_c = \frac{1}{2} (a_{1.2} + a_{2.1} \pm 180^0). \qquad (11)$$

We can now find the sides and the horizontal angles of the spherical triangle (where naturally the interchanges belonging to b and c in (8) and to A and C in (10) have to take place). With reference to the determination of the excess we now consider the most important case of the determination from two chords and the contained angle.

Par. 30. Excess from Two Chords and the Included Horizontal Angle. We take as chords s' and t', and as angle B' and according to(6), p. 109 for the excess, where it is computed negatively in case the horizontal angles are $> 180^0$, we obtain:

$$\varepsilon = \frac{s' t'}{2a_0^2} K_0 \sin B' \left(1 + \frac{y'^2}{8a_0^2} K_0 + Gl_4\right). \qquad (1)$$

In addition, according to (8) of the previous paragraph if at the same time $e^2 \cos B_2 \cos B_3$ is substituted for $e^2 \cos^2 B_a$ which gives only a fourth-order error - we have:

$$\frac{s'}{s} = 1 + \frac{1}{2} e^2 \cos B_2 \cos B_3 \cos 2a_a + e^2 (\cos^2 B_a - \cos^2 B_0) + Gl_4. \qquad (2)$$

In order to develop this further, let us consider next the first term with e^2 on the right hand. It is identical:

$$\cos 2a_a = -\cos (a_{2.3} + a_{3.2}) = (-\cos a_{2.3} \cos a_{3.2} + \sin a_{2.3} \sin a_{3.2}).$$

According to (4), p. 194 after appropriate interchange of the indices etc. we have:

$$\left.\begin{aligned} \cos B_3 \sin a_{3.2} &= -\cos B_2 \sin a_{2.3} + Gl_3 \\ \cos B_3 \cos a_{3.2} &= -\cos B_2 \cos a_{2.3} - \frac{s}{a_0} \sin B_2 + Gl_3; \end{aligned}\right\} \qquad (3)$$

and if we substitute this in the foregoing identity, multiplied by $\cos B_2 \cos B_3$ then there follows:

$$\cos B_2 \cos B_3 \cos 2a_a = \cos^2 B_2 \cos 2a_{2.3} + \frac{s}{2a_0} \sin 2B_2 \cos a_{2.3} + Gl_2.$$

This we insert in the second term on the right hand of (2); but at the same time in the third term:

$$\left.\begin{aligned} \cos^2 B_2 &= \cos^2 B_0 - \sin 2B_0 (B_2 - B_0) + Gl_2 \\ \sin 2B_2 &= \sin 2B_0 + Gl_1 \\ \cos^2 B_a = \cos^2 \frac{B_2 + B_3}{2} &= \cos^2 B_0 - \sin 2B_0 \left(\frac{B_2 + B_3}{2} - B_0\right) + Gl_2, \end{aligned}\right\} \qquad (4)$$

with which is yielded:

(See Illustrative Material on Next Page.)

Par. 30. Excess from Two Chords and the Included Horizontal Angle.

$$\frac{s'}{s} = 1 + \frac{1}{2} e^2 \cos^2 B_0 \cos 2a_{2.3}$$
$$+ \frac{1}{2} e^2 \sin 2B_0 \left\{ \frac{s}{2a_0} \cos a_{2.3} - \frac{B_2 - B_1}{3} (2 + \cos 2a_{2.3}) + \frac{B_3 - B_2}{3} (1 - \cos 2a_{2.3}) \right\} + Gl_4. \quad (5)$$

One gathers from formula (2), p. 167 easily now that it is permissible to put:

$$(B_2 - B_1) \cos B_0 = \frac{r}{a_0} \cos B_0 \cos a_{2.1} + Gl_2$$
$$(B_2 - B_3) \cos B_0 = \frac{s}{a_0} \cos B_0 \cos a_{2.3} + Gl_2, \quad (6)$$

where the second-order terms for each geographic position of the points have small values. Thus we have finally:

$$\frac{s'}{s} = 1 + \frac{1}{2} e^2 \cos^2 B_0 \cos 2a_{2.3}$$
$$+ \frac{1}{2} e^2 \sin 2B_0 \left\{ \frac{s}{a_0} \cos a_{2.3} \left[\frac{5}{6} - \frac{1}{3} \cos 2a_{2.3} \right] - \frac{r}{a_0} \cos a_{2.1} \left[\frac{2}{3} + \frac{1}{3} \cos 2a_{2.3} \right] \right\} + Gl_4 \quad (7)$$

and from this by interchange of index 3 and 1 etc.:

$$\frac{r'}{r} = 1 + \frac{1}{2} e^2 \cos^2 B_0 \cos 2a_{2.1}$$
$$+ \frac{1}{2} e^2 \sin 2B_0 \left\{ \frac{r}{a_0} \cos a_{2.1} \left[\frac{5}{6} - \frac{1}{3} \cos 2a_{2.1} \right] - \frac{s}{a_0} \cos a_{2.3} \left[\frac{2}{3} + \frac{1}{3} \cos 2a_{2.1} \right] \right\} + Gl_4. \quad (8)$$

We go further by introduction of (10), p. 204 into the identity $\sin B' = \sin B \cos (B' - B) + \cos B \sin (B' - B)$, and have:

$$\sin B' = \sin B + \cos B (B' - B) + Gl_4$$
$$B = a_{2.1} - a_{2.3}, \quad (9)$$

$$B' - B = \frac{1}{2} e^2 \cos B_2 \{ \cos B_3 \sin 2a_c - \cos B_1 \sin 2a_c \} + Gl_4.$$

The formulae (3) show, however, that one can put:

$$\cos B_2 \cos B_3 \sin 2a_m = \cos^2 B_2 \sin 2a_{2.3} + \frac{s}{2a_0} \sin 2B_0 \sin a_{2.3} + Gl_3,$$

for which one may, with due consideration to the first equation (4) and that of (6), write:

$$\cos B_2 \cos B_3 \sin 2a_m$$
$$= \cos^2 B_0 \sin 2a_{2.3} - \sin 2B_0 \left\{ \sin 2a_{2.3} \left[\frac{s}{3a_0} \cos a_{2.3} + \frac{r}{3a_0} \cos a_{2.1} \right] - \frac{s}{2a_0} \sin a_{2.3} \right\} + Gl_3.$$

The interchange of the indices 1 and 3 etc. next yields from this:

$$\cos B_2 \cos B_1 \sin 2a_m$$
$$= \cos^2 B_0 \sin 2a_{2.1} - \sin 2B_0 \left\{ \sin 2a_{2.1} \left[\frac{s}{3a_0} \cos a_{2.3} + \frac{r}{3a_0} \cos a_{2.1} \right] - \frac{r}{2a_0} \sin a_{2.1} \right\} + Gl_4.$$

By use of both of the last formulae we finally obtain after some reduction:

$$B' - B = \frac{1}{2} e^2 \cos^2 B_0 (\sin 2a_{2.3} - \sin 2a_{2.1})$$
$$+ \frac{1}{2} e^2 \sin 2B_0 \left\{ \begin{array}{c} \left[\frac{s}{3a_0} \cos a_{2.3} + \frac{r}{3a_0} \cos a_{2.1} \right] (\sin 2a_{2.1} - \sin 2a_{2.3}) \\ + \left[\frac{s}{2a_0} \sin a_{2.3} - \frac{r}{2a_0} \sin a_{2.1} \right] \end{array} \right\} + Gl_4. \qquad (10)$$

The formulae (7) to (10) now give:

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned}\frac{s's'}{ss}\sin B' = \sin B + \frac{1}{2}e^2\cos^2 B_0 \left\{ \begin{matrix} \sin(a_{2.1}-a_{2.3})(\cos 2a_{2.3}+\cos 2a_{2.1}) \\ +\cos(a_{2.1}-a_{2.3})(\sin 2a_{2.3}-\sin 2a_{2.1}) \end{matrix} \right\} \\ -\frac{1}{2}e^2\sin 2B_0 \left\{ \begin{matrix} \text{as} \\ \text{before} \end{matrix} \right\} \left(\frac{s}{3a_0}\cos a_{2.3}+\frac{s'}{3a_0}\cos a_{2.1}\right) \\ +\frac{1}{2}e^2\sin 2B_0 \left\{ \begin{matrix} \left[\frac{s}{6a_0}\cos a_{2.3}+\frac{s'}{6a_0}\cos a_{2.1}\right]\sin(a_{2.1}-a_{2.3}) \\ +\left[\frac{s}{2a_0}\sin a_{2.3}-\frac{s'}{2a_0}\sin a_{2.1}\right]\cos(a_{2.1}-a_{2.3}) \end{matrix} \right\} + Gl_4. \end{aligned}\right\} \quad (11)$$

But as identically

$$(\cos 2a_{2.3}+\cos 2a_{2.1}) = 2\cos(a_{2.3}+a_{2.1})\cos(a_{2.3}-a_{2.1})$$

and furthermore

$$(\sin 2a_{2.3}-\sin 2a_{2.1}) = 2\cos(a_{2.3}+a_{2.1})\sin(a_{2.3}-a_{2.1}),$$

the grouped brackets appearing twice here disappear. In the last part left by itself we put for the large bracket:

$$\begin{aligned}&\frac{s}{6a_0}\left(\cos a_{2.3}\sin(a_{2.1}-a_{2.3})+\sin a_{2.3}\cos(a_{2.1}-a_{2.3})\right)+\frac{s}{3a_0}\sin a_{2.3}\cos B \\ &+\frac{s'}{6a_0}\left(\cos a_{2.1}\sin(a_{2.1}-a_{2.3})-\sin a_{2.1}\cos(a_{2.1}-a_{2.3})\right)-\frac{s'}{3a_0}\sin a_{2.1}\cos B,\end{aligned}$$

which expression permits grouping as follows:

$$\frac{s}{6a_0}\sin a_{2.1}-\frac{s'}{6a_0}\sin a_{2.3}+\frac{1}{3}\cos B\left(\frac{s}{a_0}\sin a_{2.3}-\frac{s'}{a_0}\sin a_{2.1}\right).$$

With this (1) proceeds into the formula:

$$\left.\begin{aligned}\varepsilon = \underset{\text{in seconds.}}{\rho''}\frac{ss'}{2a_0^2}K_0\left\{\sin B\left(1+\frac{b^2}{8a_0^2}K_0\right)+E\right\} \\ E = +\frac{1}{12}e^2\sin 2B_0\left(\frac{s}{a_0}\sin a_{2.1}-\frac{s'}{a_0}\sin a_{2.3}+2\cos B\left[\frac{s}{a_0}\sin a_{2.3}-\frac{s'}{a_0}\sin a_{2.1}\right]\right)+Gl_6.\end{aligned}\right\} \quad (12)$$

For this formula it must be remembered that B designates

the horizontal angle between $\mathfrak{a}$ and $\mathfrak{r}$ but B_0 the mean geographic latitude.

This formula may be put in another form; in which $\cos \mathfrak{S} = \sin \mu_{2.1} \sin \mu_{2.3} + \cos \mu_{2.1} \cos \mu_{2.3} \cos B$ hence

$$\cos B = \cos \mathfrak{S} + Gl_2 = \frac{\mathfrak{a}^2 - \mathfrak{b}^2 + \mathfrak{r}^2}{2\mathfrak{a}\mathfrak{r}} + Gl_2. \qquad (13)$$

With introduction of this value we obtain for the round bracket of E in (12), multiplied into $\mathfrak{a}\mathfrak{r} : a_0^2$, the relation:

$$\frac{\mathfrak{a}\mathfrak{r}}{a_0^2}\left(\cdot\right) = \frac{1}{6a_0^3}\left(\mathfrak{a}^3 \sin a_{2.3} - \mathfrak{r}^3 \sin a_{2.1} + [\mathfrak{r} \sin a_{2.1} - \mathfrak{a} \sin a_{2.3}]\,\mathfrak{b}^2\right) + Gl_5. \qquad (14)$$

If furthermore we now apply the second equation (1), p. 166, then we have:

$$\left.\begin{aligned} \cos B_1 \sin L_{2.1} &= \frac{\mathfrak{r}}{a_0} \sin a_{2.1} + Gl_3; & \cos B_3 \sin L_{2.3} &= \frac{\mathfrak{a}}{a_0} \sin a_{2.3} + Gl_3; \\ \cos B_3 \sin L_{1.3} &= \frac{\mathfrak{b}}{a_0} \sin a_{1.3} + Gl_3; & \cos B_1 \sin L_{3.1} &= \frac{\mathfrak{b}}{a_0} \sin a_{3.1} + Gl_3. \end{aligned}\right\} \qquad (15)$$

With this the identity gives:

$$\sin L_{1.3} = \sin (L_{2.3} - L_{2.1}) = \sin L_{2.3} \cos L_{2.1} - \sin L_{2.1} \cos L_{2.3}$$

with previously carried out multiplication by $\cos B_1 \cos B_3$:

$$\begin{aligned} &\frac{\mathfrak{a}}{a_0} \sin a_{2.3} \cos B_1 \cos L_{2.1} - \frac{\mathfrak{r}}{a_0} \sin a_{2.1} \cos B_3 \cos L_{2.3} \\ &= \frac{\mathfrak{b}}{a_0} \sin a_{1.3} \cos B_1 + Gl_3 = -\frac{\mathfrak{b}}{a_0} \sin a_{3.1} \cos B_3 + Gl_3. \end{aligned}$$

One can now, with consideration of the second equation (2), p. 167,

write for this:

$$\left(\frac{a}{a_0}\sin a_{2.3}-\frac{c}{a_0}\sin a_{2.1}\right)\cos B_0=\frac{b}{a_0}\cos B_0\sin\frac{a_{1.3}+a_{3.1}\pm 180^0}{2}+Gl_2, \qquad (16)$$

where the sign of the sine on the right-hand side is to be taken like that of $\sin a_{1.3}$. This we introduce into (14) and with it reduce (12). We then finally have:

$$\left.\begin{aligned}\varepsilon &= \varrho''\frac{ca}{2a_0^2}K_0\sin B\left(1+\frac{b^2}{8a_0^2}K_0\right)\\ &+\frac{1}{24}\varrho''e^2\sin 2B_0\left\{\frac{a^2}{a_0^2}\sin a_{3.1}-\frac{c^2}{a_0^2}\sin a_{3.1}+\frac{b^2}{a_0^2}\sin a_b\right\}+Gl_6,\\ a_b &= \frac{1}{2}(a_{3.1}+a_{1.3}\pm 180^0);\ \left\{\begin{array}{l}\sin a_b \text{ has the}\\ \text{same sign}\\ \text{as } \sin a_{3.1}\end{array}\right.\end{aligned}\right\} \qquad (17)$$

By cyclic interchange we have the following from this:

$$\left.\begin{aligned}\varepsilon &= \varrho''\frac{ab}{2a_0^2}K_0\sin C\left(1+\frac{c^2}{8a_0^2}K_0\right)\\ &+\frac{1}{24}\varrho''e^2\sin 2B_0\left\{\frac{b^2}{a_0^2}\sin a_{3.1}-\frac{a^2}{a_0^2}\sin a_{3.2}+\frac{c^2}{a_0^2}\sin a_c\right\}+Gl_6,\\ a_c &= \frac{1}{2}(a_{1.2}+a_{2.1}\pm 180)^0;\ \left\{\begin{array}{l}\sin a_c \text{ has the}\\ \text{same sign}\\ \text{as } \sin a_{1.2}\end{array}\right.\end{aligned}\right\} \qquad (18)$$

$$\left.\begin{aligned}\varepsilon &= \varrho''\frac{bc}{2a_0^2}K_0\sin A\left(1+\frac{a}{8a_0^2}K_0\right)\\ &+\frac{1}{24}\varrho''e^2\sin 2B_0\left\{\frac{b^2}{a_0^2}\sin a_{1.2}-\frac{b^2}{a_0^2}\sin a_{1.3}+\frac{a^2}{a_0^2}\sin a_a\right\}+Gl_6\\ a_a &= \frac{1}{2}(a_{2.3}+a_{3.2}\pm 180^0);\left\{\begin{array}{l}\sin a_a \text{ has the}\\ \text{same sign}\\ \text{as } \sin a_{2.3}\end{array}\right.\end{aligned}\right\} \qquad (19)$$

Par. 31. Numerical Example III, cf. p. 200

(See Illustrative Material on Next Page.)

$\log \frac{e''}{2a_0^2} = 1{,}4041082-10$ $\log K_0 = 9{,}9993197-10$

$B_0 = 51^\circ 48',6$ $\log \sin 2B_0 = 9{,}988-10$

$\log \frac{M}{8a_0^2} = 2{,}1255-10$ $\log \frac{e^2 \sin 2B_0}{12a_0} = 9{,}929-20$

units of the 7th decimal place

$\alpha_{1.2}$ =	23° 12′	log sin = 9,595 −10	α_a =	156° 1′	log sin = 9,609 −10
$\alpha_{1.3}$ =	65 16	9,958 −10	α_b =	242 25	9,947$_n$ −10
$\alpha_{2.1}$ =	199 58	9,533$_n$ −10	α_c =	21 35	9,566 −10
$\alpha_{2.3}$ =	157 10	9,589 −10	A =	42 4	log cos = 9,871 −10
$\alpha_{3.1}$ =	239 33	9,936$_n$ −10	B =	42 48	9,866 −10
$\alpha_{3.2}$ =	334 52	9,628$_n$ −10	C =	95 19	8,967$_n$ −10

The formula (12), p. 207 gives for the computation with respect to ra, ab, br :

$\varepsilon = 699{,}1825'' - 0{,}0156'' - 0{,}0264'' + 0{,}0260'' + 0{,}0341'' = 699{,}2006''$

$699{,}1798 - 0{,}0134 + 0{,}0269 + 0{,}0051 - 0{,}0025 = 699{,}1959$

$699{,}1797 + 0{,}0625 - 0{,}0184 + 0{,}0403 - 0{,}0633 = 699{,}2008$

Mean $= 699{,}1991''$.

On the other hand the formulae (17) to (19), p. 208 and 209 respectively give:

$\varepsilon = 699{,}1825'' + 0{,}0119'' + 0{,}0344'' - 0{,}0283'' = 699{,}2005''$

$699{,}1798 - 0{,}0275 + 0{,}0130 + 0{,}0371 = 699{,}2022$

$699{,}1797 + 0{,}0397 - 0{,}0290 + 0{,}0124 = 699{,}2028$

Mean $= 699{,}2019''$.

According to p. 200 the exact value is equal to 699,2014". The mean values from the computations of both types, namely 699,2006..., 1992..., 2018 agree in general with this better than the results of the individual computations.

Par. 32. Summary. The sine series developed in the foregoing

and the formula for ε form the basis for the adjustment and computation of a chain of triangles. For completion

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of the collective formula, particularly in consideration of the computation of polar-coordinates, such formulae as of p. 109, Par. 24 for spherical computation are necessary.

(The tangent series would permit derivation from the sin series as before, while the derivation of a formula giving $\mathfrak{r}^2$ directly from $\mathfrak{a}, \mathfrak{b}$ and C would have to start from the formulae"

$$\begin{pmatrix} \mathfrak{r}^2 = \mathfrak{a}^2 + \mathfrak{b}^2 - 2\mathfrak{a}\mathfrak{b}\cos\mathfrak{C}, \\ \cos\mathfrak{C} = \sin\mu_{3.1}\sin\mu_{3.2} + \cos\mu_{3.1}\cos\mu_{3.2}\cos C. \end{pmatrix}$$

This disposition may be withheld meanwhile because no application of the formulae is in view. For the computation, (already troublesome for purely spherical calculations) of chord polar coordinates (cf. p. 111) becomes evidently still further complicated for the ellipsoid.

With the working of a triangulation net by means of chords and horizontal angles, one will hence compute latitude and length differences most suitably, directly by means of the chords that appear in the observed net. As with this the Grunert series (with application of a special mean K_0 for each triangle) just suffices, and as, further, the computation of geographic latitude and length differences by means of the chords in the event that a table for $\log \Pi'$ is at hand is not inconvenient, the course of the calculation for this method is very simple. Hence it becomes

worthy of attention in all cases where the geographic coordinates of all triangulation points form a suitable aim for the computations, as with large surveys.

Bremiker in his Studien has likewise carried out the computation with chords, yet in the way (cf. Par. 9) that each horizontal angle is reduced to its chord angle according to one of the formulae:

$$\sin^2\frac{A-\mathfrak{A}}{2} = \frac{n^2\cos 2N}{\sin\frac{A+\mathfrak{A}}{2}} \text{ mit } \begin{cases} n\cos N = \sin\frac{A}{2}\sin\frac{\mu_{1.2}+\mu_{1.3}}{2} \\ n\sin N = \cos\frac{A}{2}\sin\frac{\mu_{1.2}-\mu_{1.3}}{2}, \end{cases}$$

$$A-\mathfrak{A} = \varrho''\tan\frac{A}{2}\left(\frac{\mu_{1.2}+\mu_{1.3}}{2}\right)^2 - \varrho''\cot\frac{A}{2}\left(\frac{\mu_{1.2}-\mu_{1.3}}{2}\right)^2+\cdots,$$

in seconds

the second of which is a formula of approximation which, although it may not be applied to values of A near 0° or 180°, will suffice for well-formed measurable triangles as a rule. These are formulae that can be easily derived from the equation:

$$\cos\mathfrak{A} = \sin\mu_{1.2}\sin\mu_{1.3} + \cos\mu_{1.2}\cos\mu_{1.3}\cos A$$

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This method is much less suited for the computation of polar coordinates than the one stated above, as it requires a continual reduction back and forth between chord-and-horizontal angles. Since chord angles lack the advantage of horizontal angles of lying in one plane if they have the same vertex, and consequently permitting

direct addition and subtraction.

Moreover, the adjustment and computation of an observed triangulation net by means of chords and horizontal angles is, as is easily recognized, much more simple than those with chords and chord angles. If there is ever a question of whether one should compute with chords, the further decision whether horizontal angles or chord angles should be used, will be in favor of the former.

As to the formulae for the transfer of geographic coordinates and the computation of chord and azimuth from geographic positions, our developments are not essentially different from those of Bremiker. In his Studien however, the important formulae of approximation for small distances in reference to the last named task is missing.

These formulae of approximation have also a value, apart from the application of the chord, as they are quite suitable as they later show for the ascertaining of the geodetic line between two points from their geographic positions. A transition is already there, insofar as the chord is brought into contact with the length of the vertical sections.

They show the simple connections stated on p. 203 which take place between the horizontal angles on the ellipsoid and the corresponding ones on the sphere at equal geographic positions. The general use of the representation of the ellipsoid on a sphere without alteration of the geographic positions is, however, opposed to the size of the azimuth reductions according to p. 157, Par. 11 (2) and (3).

This last goes together with those peculiarities of our solution (of the task) of finding the azimuths of the vertical sections from the geographical positions, from which the correction of the spherical calculation rising out of the ellipticity is almost independent of the distance. Nevertheless the calculation for short distances is greatly simplified, and as the formulae are generally simple, that circumstance is no defect insofar as this task only comes under consideration.

If however we wanted to have formulae that make that correction dependent on the distance, then such could easily be gather from observations which Hansen makes for other objectives in his Geodätischen Untersuchungen (Geodetic Investigations) (especially from those of the second Section). If we are particularly concerned with two points P_1 and P_2 then Hansen reduces the geographic latitude of one of the points to roughly that of P_2 on the line P_1K_1 (Fig. 8, p. 134), so that it is possible to take K_1 as the center of an auxiliary sphere.

We must also mention here that Delambre in 1798 in his Méthodes analytiques pour la Détermination d'un Arc du Méridien likewise (as Bremiker did later) recommended the calculation using chord triangles and formed formulae for the transfer of geographic coordinates by chords. First he gives spherical formulae and series developments which still consider k^2. He then applies these to

the sphere with the center K, (as Hansen later, and Legendre already earlier in 1787), and ascertains the corrections due to the elliptic form. In the final formulae he considers only e^2 and k^2 (loc. cit., p. 83; in the formula for δ the sign of e^2 is to be reversed). These final formulae give sufficient precision for very limited chord lengths only.

In the main work of the English Survey Ordnance Trigonometrical Survey, Principal Triangulation pp. 228 - 253 there are formulae for a geodesy with horizontal angles and vertical sections. We are, however, of the opinion that the application of the latter instead of the chord is not to be recommended, in which opinion the formulae of the place quoted have not confused us.

Sonderhof in the Zeitschr. f. Math. u. Physik (Journal for Mathematics and Physics) Vol. 17, 1872, pp. 89 and 177 ff. has given an independent contribution, interesting in many regards, to geodesy without application of the geodetic line.

Chapter 5

Basic Formulae for the Geodetic Line.

Par. 1. Horizontal Distance and Geodetic Line on the Ellipsoid of Rotation. In the last chapter it has been shown, that, in general, the opposite vertical sections of two points deviate from each other. According to page 189, however, the amount of deviation for distances up to 64 km is so small, that as a rule, it will be, for all practical applications, without any significance. If therefore the measurements of vertical sections are carried out by intervisibility of the points, there will be no noticeable deviation between the sections. Therefore it will be not difficult for a geodesist to determine, what he has to consider in the sense of geodetic practice, as a straight line, or as a horizontal distance.

For small distances, however, a vagueness exists in the mathematical sense, also for larger distances in a practical sense. Referring first to the practical application, we can eliminate the vagueness by following the procedure outlined in the note Par. 1 p. 70. For the sake of simplicity we disregard the connection of two points, and consider only the measurement of a line from one point P_0 in a given direction:

We start (fig. 14) from P_0 in a vertical plane to P_1, corresponding to a given azimuth, assuming P_1 is so close to P_0, that there will be no noticeable deviation between points P_0 and P_1 of one vertical plane from P_1 to P_0, and another plane which we can lay from P_1 to P_0. From this last plane we proceed to P_2

and follow from here the reversed vertical plane, extended from P_2 to P_1 , to P_3, etc. If the distances P_0P_1, P_1P_2 are very small the line measured according to the procedure described will be for all practical purposes a continuous line, and in practice the geodesist will consider it as a straight line between any two points P .

In order to eliminate entirely the small breaks in the line, we assume the elements P_0P_1, P_1P_2, P_2P_3 are infinitely small. In accordance with (25) p. 187 the deflection angles therefore are in this case infinitely small and of second order. We therefore obtain a continuous line, which, according to p. 70, can be considered a <u>geodetic line</u>, in accordance with the strict requirements of mathematics. For this line, it is insignificant in which direction we start with measurements, because it is evident that we obtain the same line (fig. 14) if we envision the measurements repeated backward from P_n . As a clear junction of two points, and a junction corresponding to the procedure of the geodetic practice for the measurement and elongation of straight lines, the geodetic line is highly suitable to define the <u>horizontal distance</u> between two points.

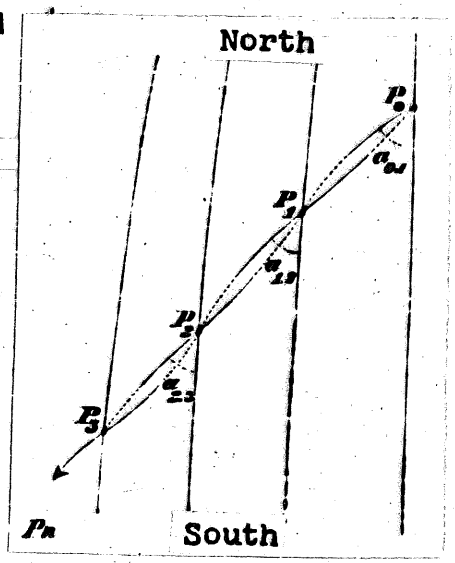


Figure 14

Further we will see, that the geodetic line represents, as a rule, the shortest distance and leads to a comparatively simple computation.

The next task is now to deduce from the geometrical derivation of the geodetic line an equation which defines this line.

Par. 2. Basic Equation of the Geodetic Line on the Ellipsoid of Rotation. On p. 169 of the preceding chapter we found in formula (12) the following relation for two points P_1 and P_2, connected by their opposite vertical sections:

$$\sin \alpha_{1.2} \cos \beta_1 \cos \mu_{1.2} = \sin (\alpha_{2.1} - 180^0) \cos \beta_2 \cos \mu_{2.1}. \quad (1)$$

Herein $\alpha_{1.2}$ and $\alpha_{2.1}$ are the azimuths of the vertical sections from P_1 to P_2 and from P_2 to P_1, $\mu_{1.2}$ and $\mu_{2.1}$ angles of depression of chord P_1P_2 at P_1 and P_2; and β_1 and β_2 the reduced latitudes of points P_1 and P_2.

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Applying this formula to points P_0 and P_1, P_1 and P_2, P_2 and P_3 etc., we obtain:

$$\left.\begin{aligned} \sin \alpha_{0.1} \cos \beta_0 &= \sin (\alpha_{1.0} - 180^0) \cos \beta_1 \frac{\cos \mu_{1.0}}{\cos \mu_{0.1}} \\ \sin \alpha_{1.2} \cos \beta_1 &= \sin (\alpha_{2.1} - 180^0) \cos \beta_2 \frac{\cos \mu_{2.1}}{\cos \mu_{1.2}} \\ \sin \alpha_{2.3} \cos \beta_2 &= \sin (\alpha_{3.2} - 180^0) \cos \beta_3 \frac{\cos \mu_{3.2}}{\cos \mu_{2.3}} \end{aligned}\right\} \quad (2)$$

etc.

Because simultaneously $\alpha_{1.2} = \alpha_{1.0} - 180^0$, $\alpha_{2.3} = \alpha_{2.1} - 180^0$ etc; so also, $\sin \alpha_{1.2} = \sin (\alpha_{1.0} - 180^0)$, $\sin \alpha_{2.3} = \sin (\alpha_{2.1} - 180^0)$ etc. In multiplying equations (2) side after side and continuing to P_n, we obtain:

$$\sin \alpha_{0.1} \cos \beta_0 = \sin (\alpha_{n.n-1} - 180^0) \cos \beta_n \left(\frac{\cos \mu_{1.0}}{\cos \mu_{0.1}} \frac{\cos \mu_{2.1}}{\cos \mu_{1.2}} \cdots \frac{\cos \mu_{n.n-1}}{\cos \mu_{n-1.n}} \right). \quad (3)$$

This is now the basic equation for the geodetic line; it remains only to verify that the expression in parentheses, for infinetly small intervals between adjacent points $P_0, P_1, P_2 \ldots P_n$ etc., for $n = \infty$ equals exactly 1.

According to formulae (3), p. 29, we can set $-\frac{1}{2}\varkappa\mu^2$, for the natural logarithm of cos μ ; where $\varkappa$ for small values of μ nearly equals 1. Denoting the expression in parentheses in (3) by P and taking all cosines in it in the numerator or denominator, we get:

$$-\frac{1}{2}\sum\varkappa\mu^2 < \log P < +\frac{1}{2}\sum\varkappa\mu^2. \qquad (4)$$

Σ designates the summation of all values of $\varkappa\mu^2$ with various indices. Setting now on the left and on the right for each $\varkappa$ the largest one, and for each factor μ the largest of all μ, we obtain:

$$-\frac{1}{2}\varkappa_m\mu_m\sum\mu < \log P < +\frac{1}{2}\varkappa_m\mu_m\sum\mu. \qquad (5)$$

Assuming the elements infinitely small, $\Sigma\mu$ is still a finite angle, namely the one, which will be described by the normal in sliding along the geodetic line from P_0 to P_n. On the other hand μ_m is at the same time infinitely small, i. e., when log P will equal zero, $P = 1$. We then have the relation:

$$\sin\alpha_0\cos\beta_0 = \sin\alpha_n\cos\beta_n, \qquad (6)$$

whereby the azimuth $a_{0,1}$ of the line beginning at P_0 is designated α_0, and

the azimuth $(a_{n,n-1} - 180^0)$ at P_n for the further extending line is designated α_n (fig. 15).

Since P_n may designate any point on the geodetic line, which starts from P_0 and has azimuth α_0, equation (6) means that:

The product of $\sin\alpha\cos\beta$ for all points of a geodetic line is constant.

Considering the precise expression for P and μ, it is possible to state the deviation of P from 1 by a finite n. We will carry out this computation.

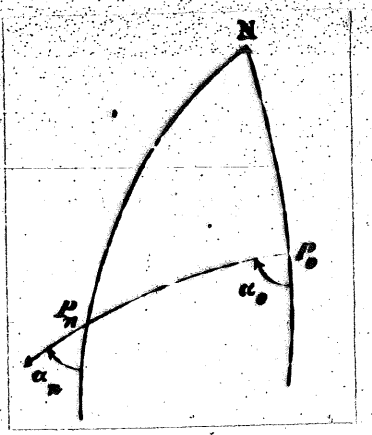


Figure 15

According to formula (1), p. 174, it follows that:

$$\left.\begin{aligned}\sin\mu_{1.2} &= \frac{k_{1.2}\sqrt{1-e^2\sin^2 B_1}}{2a_0}\left(1+\delta\cos^2\chi_{1.2}\right)\\ \sin\mu_{2.1} &= \frac{k_{1.2}\sqrt{1-e^2\sin^2 B_2}}{2a_0}\left(1+\delta\cos^2\chi_{1.2}\right),\end{aligned}\right\} \qquad (7)$$

where $k_{1.2}$ denotes the chord P_1P_2, and $\chi_{1.2}$ denotes the chord's angle of inclination to the axis of rotation. The quotient $(\cos\mu_{2.1}:\cos\mu_{1.2})$ can easily be derived from the difference of the squares of these expressions:

$$\sin^2\mu_{2.1}-\sin^2\mu_{1.2}=\frac{e^2k_{1.2}^2}{4a_0^2}\left(1+\delta\cos^2\chi_{1.2}\right)\left(\sin^2 B_1-\sin^2 B_2\right) \qquad (8)$$

log stands for the natural logarithm:

$$\log\frac{\cos\mu_{2.1}}{\cos\mu_{1.2}}=\frac{1}{2}\log\left(1-\sin^2\mu_{2.1}\right)-\frac{1}{2}\log\left(1-\sin^2\mu_{1.2}\right)$$

$$=-\frac{1}{2}\left\{\left(\sin^2\mu_{2.1}-\sin^2\mu_{1.2}\right)+\frac{1}{2}\left(\sin^4\mu_{2.1}-\sin^4\mu_{1.2}\right)+\frac{1}{3}\left(\sin^6\mu_{2.1}-\sin^6\mu_{1.2}\right)+\cdots\right\}.$$

The expression $(\sin^2\mu_{2.1}-\sin^2\mu_{1.2})$ can be removed from the parentheses and we obtain inside of the brace:

the 2 term: $\frac{1}{2}\left(\sin^2\mu_{2.1}+\sin^2\mu_{1.2}\right)$,

the 3 term: $\frac{1}{3}\left(\sin^4\mu_{2.1}+\sin^2\mu_{2.1}\sin^2\mu_{1.2}+\sin^4\mu_{1.2}\right)$,

etc.

The nth term gives $\frac{1}{n}$ times an aggregate of n sine powers of

$\varkappa$ th degree. When μ denotes a value which is larger than $\mu_{2.1}$ and $\mu_{1.2}$ but which, when $k_{1.2} = 0$, is also zero, and when we set μ inside of the brace for $\mu_{2.1}$ and $\mu_{1.2}$, it follows that:

$$\log \frac{\cos \mu_{2.1}}{\cos \mu_{1.2}} = -\frac{1}{2}(\sin^2 \mu_{2.1} - \sin^2 \mu_{1.2})(1 + \sin^2 \mu + \sin^4 \mu + \cdots)\varkappa$$

$$= -\frac{1}{2}\frac{\sin^2 \mu_{2.1} - \sin^2 \mu_{1.2}}{1 - \sin^2 \mu}\varkappa,$$

where $\varkappa$ denotes a positive, real fraction. Using formula (8) we obtain:

$$\log \frac{\cos \mu_{2.1}}{\cos \mu_{1.2}} = \frac{k_{1.2}^2 (\sin^2 B_2 - \sin^2 B_1)}{8 a_0^2}(1 + \delta \cos^2 \chi_{1.2})\frac{e^2 \varkappa}{1 - \sin^2 \mu}. \qquad (9)$$

We can apply this formula to $(\cos \mu_{1.0} : \cos \mu_{0.1})$, $(\cos \mu_{2.1} : \cos \mu_{1.2})$, $(\cos \mu_{3.2} : \cos \mu_{2.3})$ etc., and add the logarithms. We take the right side as always positive, and set further for

$$(\sin^2 B_i - \sin^2 B_{i-1}),\ i = 1 \text{ to } n,$$

the largest absolute value among these differences, denoting it as $\Delta \sin^2 B$, and introduce finally for $\cos^2 \chi_{1.2}$ and $\varkappa$ simply 1, and for μ in $(1 - \sin^2 \mu)$ the largest of all possible μ, and obtain:

$$\log P < \frac{\delta \cdot \Delta \sin^2 B}{8 a_0^2 (1 - \sin^2 \mu)} \Sigma k^2. \qquad (10)$$

The summation of k^2 has to be carried out for all distances:

$$\Sigma k^2 = k_{0.1}^2 + k_{1.2}^2 + \cdots k_{n.n-1}^2.$$

Substituting in each part for one of the k, one maximum value k_m occurring among all k, we obtain:

$$\Sigma k^2 < k_m \Sigma k. \qquad (11)$$

Par. 3. The Course of the Geodetic Line.

If we now assume the number of distances to the infinitely large, but each single distance to be infinitely small, Σk will have a finite value, namely the length of the geodetic line, whereas k_m will be infinitely small as will $\Delta \sin^2 B$. We obtain therefore for $n = \infty$, as stated above, log P = zero.

If however, the line $P_0 P_n$ consists of a finite number of nearly equally long distances, and if we confine ourselves to the area where the variations of azimuth α are not very large, we can set for $\Sigma(\sin^2 B_i - \sin^2 B_{i-1})$, n times the average value of $\sin^2 B_i - \sin^2 B_{i-1}$, i. e. approximately

$$n\left(-\frac{s}{a_0 n}\cos\alpha \sin 2B\right) \quad \text{or simpler} \quad -\frac{s}{a_0}\cos\alpha \sin 2B,$$

wherein α is a mean azimuth, B a mean geographical latitude, and s the length of the entire line $P_0 P_n$. By means of some other approximations which are self-evident, we obtain:

$$\log(\sin\alpha_0 \cos\beta_0) = \log(\sin\alpha_n \cos\beta_n) - \frac{\delta s^3}{8 a_0^3 n^3}\cos\alpha \sin 2B + \cdots \qquad (12)$$

This represents the approximate relation for the starting and terminal points of a line of the length s, which has been measured in n equally long, small but finite distances, and the mean azimuth α and mean geographical latitude B.

Par. 3. The Course of the Geodetic Line. If we consider the product $\sin\alpha \cos\beta$ which for a given geodetic line has an equal value in all points of the line, according to the relative magnitude of

its factors, it is clear that for $\alpha = 90^0$, β will reach its maximum value. If a geodetic line proceeds from one point P_1 with northern latitude in an azimuth α_1 between 90^0 and 180^0, then β is increasing and $\cos\beta$ is decreasing. When $\cos\beta$ is decreasing, $\sin\alpha$ is increasing, so that simultaneously in the beginning, β is increasing and α decreasing, until finally α equals 90^0.

If this value of α is reached, and with it the highest latitude, then β is in the further course decreasing again since, because of the constancy of the product $\sin\alpha\cos\beta$, β is neither able to increase any more, nor can it remain constant, because in that case the geodetic line would coincide with the parallel circle. (This circle is certainly not a geodetic line, because its adjoining plane stands oblique to the surface.) Because $\sin\alpha$ decreases as $\cos\beta$ increases, but in doing so $\alpha < 90^0$, α must be getting smaller and smaller. When $\cos\beta = 1$, i. e. at the equator, a minimum is reached. This minimum of α supplements the maximum of β to 90^0, because the sine must be equal to the cosine:

$$\alpha_{\text{Min.}} + \beta_{\text{Max.}} = 90^0. \qquad (1)$$

After crossing the equator, $\cos\beta$ is decreasing and $\sin\alpha$ is increasing and this continues until $\alpha = 90^0$ and β reaches its negative maximum. As soon as α enters the second quadrant, $\sin\alpha$ is again decreasing and $\cos\beta$ is increasing, until at a further equator crossing α reaches a maximum in the second quadrant, and the geodetic line re-enters the northern half of the ellipsoid. Between $\alpha_{\text{Min.}}$ and $\alpha_{\text{Max.}}$, we have the following relation:

$$\alpha_{\text{Min.}} + \alpha_{\text{Max.}} = 180^0. \qquad (2)$$

Accordingly, the advance of a geodetic line, with an azimuth between 0^0 and 180^0 at any point along the line, will always be carried forward in azimuths situated between the limits $\alpha_{Min.}$ and $\alpha_{Max.}$ These limit values represent the azimuths at equator crossing.

If the azimuth is situated, even only in one point along the line, between 180^0 and 360^0, then all azimuths $> 180^0$ and fluctuate between two limits, located in the third, and fourth quadrants. In that case (up to an error of 360^0) we can still use the relation (2).

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In the assumed manner of counting the azimuths, the geodetic line will have in one case a western course, in the other an eastern course. However, there will be no basic difference between the two kinds of lines. In drawing a geodetic line first in a western direction, and proceeding in a backward motion, we notice an increase of the azimuth in the direction of the movement by 180^0, i. e. one described the previous course in an eastern direction.

Each geodetic line is confined in its course to a certain area, that is situated between two parallel circles which are equi-distant from the equator to the north and to the south. The amplitude of the latitude depends on the constants $\sin \alpha \cos \beta$.

If the amplitude equals zero, either α at any time equals zero, or β at any time equals 90^0. The former represents the meridians, the latter produces no line.

If the constant equals ± 1, then $\alpha = 90^0$ or 270^0 and $\beta = 0$. This represents the equator.

Par. 4. Length of Geodetic Line, Proceeding from a Given Point in a Definite Direction. If β_1 represents the reduced latitude of the given point P_1, and α_1 the azimuth of the geodetic line, then we have the following relation for any point P whose reduced latitude is β:

$$\sin\alpha\cos\beta = \sin\alpha_1\cos\beta_1, \qquad (1)$$

α denoting the azimuth of an extending geodetic line.

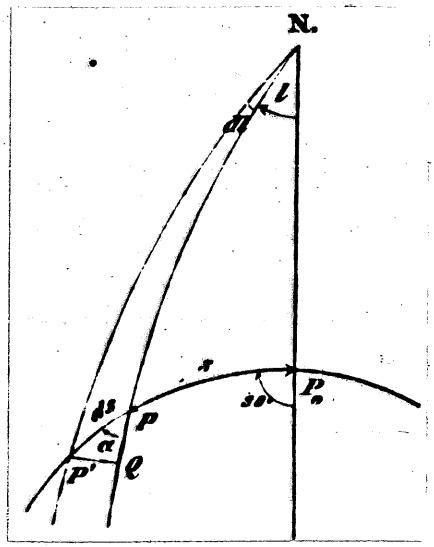


Figure 16

Of the various points we especially emphasize one of those where β is reaching its positive maximum. In fig. 16 it is denoted P_0. If we denominate the maximum reduced latitude β_0, we obtain:

$$\sin\alpha\cos\beta = \cos\beta_0. \qquad (2)$$

We denote s as the length of the geodetic line from P_0 to any other point P and consider it temporarily as being positive in a western direction, negative in an eastern direction.

If by shifting P to P', s increases by ds, with the reduced latitude β changing to $\beta + d\beta$, and if we assume that P shifts again along the meridian to Q with the reduced latitude $\beta + d\beta$,

then P' and Q are points of the same parallel circle. Considering $P'Q$ as being perpendicular to the meridian, we obtain: (fig. 16)

$$ds = PQ\sec\alpha.$$

Par. 4. Length of Geodetic Line, Proceeding from a Given Point

According to p. 55 (1), the element dM of the meridian, belonging to $d\beta$, is equal to:

$$a_0\sqrt{1-e^2\cos^2\beta}\,d\beta;$$

Introducing this for PQ and taking into account, that, when $\alpha<90^0$, $d\beta$ is negative, but ds has to be taken as positive, it follows that:

$$ds=-a_0\sqrt{1-e^2\cos^2\beta}\sec\alpha\,d\beta. \qquad (3)$$

Insofar as the sign is concerned, this formula is also applicable to those parts of the line, where α is located between 90 and 180^0; in that case $d\beta$ will be positive, but $\sec\alpha$ negative, and consequently, ds positive.

In order to find s by means of integration, we introduce tentatively, according to (2), for $\sec\alpha$ the value $\pm 1:\sqrt{1-\cos^2\beta_0\sec^2\beta}$ and obtain:

$$ds=\pm a_0\sqrt{\frac{1-e^2\cos^2\beta}{\cos^2\beta-\cos^2\beta_0}}\cos\beta\,d\beta, \qquad (4)$$

of which the top sign applies to increasing values of β, and the lower sign to decreasing values of β.

However, before proceeding with integration, we recall the the result obtained with $e^2=$ zero, i. e. in case of a sphere. Fig. 17 shows the corresponding spherical figure on a sphere of the radius 1.

The arc φ is replacing s, and on the sphere of the radius a_0, $s=a_0\varphi$ and $ds=a_0\,d\varphi$. Formula (4) yields for e^2 equal to zero:

$$ds=\pm a_0\frac{\cos\beta\,d\beta}{\sqrt{\cos^2\beta-\cos^2\beta_0}};$$

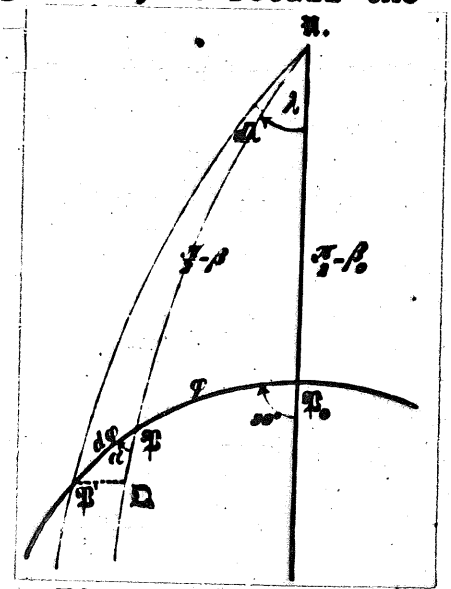


Figure 17

It is obvious, that the introduction of the variable φ is simplifying the computation considerably since φ can be computed from the spherical figure, without integration, by applying known formulae.

A favorable circumstance allows us to retain the variable φ, as well as the basic auxiliary spherical figure (fig. 17) for the ellipsoid. If we select on an auxiliary sphere

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of the radius 1 a point $\mathfrak{N}$ as North Pole and assume a point $\mathfrak{P}_0$ at the latitude β_0, and then draw perpendicular to the meridian the great circle $\mathfrak{P}\mathfrak{P}_0$, we obtain again fig. 17. In the spherical triangle, which is formed by the meridians of $\mathfrak{P}_0$ and $\mathfrak{P}$ and the great circle $\mathfrak{P}_0\mathfrak{P}$ we find according to the sine law:

$$\sin\alpha\cos\beta = \cos\beta_0.$$

This is equation (2), i. e. if a point $\mathfrak{P}$ has as latitude the reduced latitude β of point P of the geodetic line fig. 16, the great circle $\mathfrak{P}\mathfrak{P}_0$ in $\mathfrak{P}$ fig. 17 has the same azimuth as the geodetic line has at P.

However, this does not yet establish what other relation still exists between figs. 16 and 17. For the time being, we denote the meridian angles in both figures differently, and will in fact find that the geographic longitudinal differences l and λ in figs. 16 and 17 are dissimilar.*)

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*)The auxiliary sphere introduced here is different from the one used in the previous chapter, p. 114, insofar as it does not present an image of the entire ellipsoid, like the other one. Now only the points of the geodetic line and the great circle have a relation to each other.

As φ increases, a differential right triangle $PP'Q$ (fig. 17) will be formed. From this triangle we derive the relation

$$d\varphi = -\sec\alpha\, d\beta \tag{5}$$

and by eliminating $\sec\alpha$, formula (3) demonstrates that:

$$ds = a_0\sqrt{1 - e^2\cos^2\beta}\, d\varphi. \tag{6}$$

Compared with (4), this formula is much simpler and eliminates the ambiguity of the signs, because ds and $d\varphi$ increase simultaneously. Integrating and using the cosine law for β, we can derive from the spherical triangle the following relation to φ:

$$\sin\beta = \sin\beta_0\cos\varphi.$$

Introducing it in (6) and simultaneously eliminating from the root the factor $\sqrt{1 - e^2\cos^2\beta_0}$, it follows that:

$$ds = a_0\sqrt{1 - e^2\cos^2\beta_0}\sqrt{1 - \frac{e^2\sin^2\beta_0}{1 - e^2\cos^2\beta_0}\sin^2\varphi}\, d\varphi. \tag{7}$$

This formula will be more elegant, when we introduce the geographical latitude B_0, belonging to β_0. We obtain, according to p. 40 (3) and (5)

$$\sin^2 B_0 = \frac{\sin^2\beta_0}{1 - e^2\cos^2\beta_0}, \qquad \cos^2\beta_0 = \frac{\cos^2 B_0}{1 - e^2\sin^2 B_0},$$

and setting for abbreviation

$$k = e\sin B_0, \tag{8}$$

formula (7) will change to:

$$ds = a_0\sqrt{\frac{1 - e^2}{1 - k^2}}\sqrt{1 - k^2\sin^2\varphi}\, d\varphi. \tag{9}$$

Par. 5. Continuation. The Length of the Geodetic Line. Comparing formula (9) for the differential of the arc of the geodetic line with the formula for the differential of the arc dM for the arc of the meridian, p. 55 (1), we notice a considerable similarity. It is insignificant that there the sine, here, the cosine appears under the radical, because the sine will change to the cosine by introducing in place of the variable its complement to 90°. According to p. 55, the integral for the arc of the meridian depended on the coefficients A_0, A_2, A_4 etc., which were represented on p. 47 in three forms:

1. as power series of e^2,
2. as power series of n, where $n = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}$,
3. as power series of m, where $m = \frac{e^2}{2-e^2}$.

The application of n has been found to be the most advantageous. Therefore, we now perform the corresponding conversion, introducing in place of k^2 a new auxiliary quantity k_1. The relations of k_1 to k^2 are as follows:

$$\left.\begin{aligned} k_1 &= \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} = \frac{k^2}{4} + \frac{k^4}{8} + \frac{5k^6}{64} + \frac{7k^8}{128} + \cdots, \\ k^2 &= \frac{4k_1}{(1+k_1)^2} = 4k_1 - 8k_1^2 + 12k_1^3 - 16k_1^4 + \cdots. \end{aligned}\right\} \qquad (1)$$

The series outlined are derived from p. 37 by interchanging e and k, n and k_1. Their convergence is beyond doubt, because $k \leq e$, and regarding e there never existed any doubt. In formula (9) of the preceeding paragraph we set for $\sin^2\varphi$ the cosine of the double angle and, introducing simultaneously k_1, obtain:

$$\sqrt{1-k^2\sin^2\varphi} = \frac{\sqrt{1+k_1^2+2k_1\cos 2\varphi}}{1+k_1},$$

$$\sqrt{1-k^2} = \frac{1-k_1}{1+k_1}.$$

and it follows in place of (9) that:

$$ds = a_0 \frac{\sqrt{1-e^2}}{1-k_1} \sqrt{1 + k_1^2 + 2k_1 \cos 2\varphi}\, d\varphi. \qquad (2)$$

The further evolution is the same as on p. 55. In setting

$$\cos 2\varphi = \frac{1}{2}(\varepsilon^{2i\varphi} + \varepsilon^{-2i\varphi}),$$

and denoting the base of natural logarithms as ε, and the $\sqrt{-1}$ as i (p. 42), the following formula is obtained for the root in (2):

$$\sqrt{1 + k_1^2 + 2k_1 \cos 2\varphi} = (1 + k_1 \varepsilon^{2i\varphi})^{\frac{1}{2}} (1 + k_1 \varepsilon^{-2i\varphi})^{\frac{1}{2}}, \qquad (3)$$

$$(1 + k_1 \varepsilon^{\pm 2i\varphi})^{\frac{1}{2}} = 1 + \frac{k_1}{2}\varepsilon^{\pm 2i\varphi} - \frac{k_1^2}{8}\varepsilon^{\pm 4i\varphi} + \frac{k_1^3}{16}\varepsilon^{\pm 6i\varphi} - \frac{5k_1^4}{128}\varepsilon^{\pm 8i\varphi} + \cdots \qquad (4)$$

We multiply the two factors of the root quantity and eliminate the imaginary unit by using the relation $\cos u = \frac{1}{2}(\varepsilon^{iu} + \varepsilon^{-iu})$. According to p. 56, it is unnecessary to take k_1^4 into consideration, at least it is possible to omit the term $\sin 8\varphi$ at once. For better control of the strong convergence in the coefficient series, however, we shall continue with other terms up to k_1^4. It follows that:

$$\sqrt{1 + k_1^2 + 2k_1 \cos 2\varphi} = \left(1 + \frac{1}{4}k_1^2 + \frac{1}{64}k_1^4 + \cdots\right) + \left(k_1 - \frac{1}{8}k_1^3 + \cdots\right)\cos 2\varphi$$

$$- \left(\frac{1}{4}k_1^2 - \frac{1}{16}k_1^4 + \cdots\right)\cos 4\varphi + \left(\frac{1}{8}k_1^3 - \cdots\right)\cos 6\varphi + \cdots. \qquad (5)$$

Introducing this in (2) and integrating, and keeping in mind that with $\varphi = 0$, s will also be zero, we obtain:

$$s = a_0 \frac{\sqrt{1-e^2}}{1-k_1}\left\{\left(1 + \frac{1}{4}k_1^2 + \frac{1}{64}k_1^4 + \cdots\right)\varphi + \left(\frac{1}{2}k_1 - \frac{1}{16}k_1^3 + \cdots\right)\sin 2\varphi\right.$$

$$\left. - \left(\frac{1}{16}k_1^2 - \frac{1}{64}k_1^4 + \cdots\right)\sin 4\varphi + \left(\frac{1}{48}k_1^3 - \cdots\right)\sin 6\varphi \cdots\right\}. \qquad (6)$$

Here we set for $a_0\sqrt{1-e^2}$ the small semi-axis b_0 of the meridian

ellipse. We put the factor φ in front of the large parenthesis, and obtain in the same way as before:

$$s = b_0 \frac{1+\frac{1}{4}k_1^2+\frac{1}{64}k_1^4+\cdots}{1-k_1}\left\{\varphi+\left(\frac{1}{2}k_1-\frac{3}{16}k_1^3+\cdots\right)\sin 2\varphi - \left(\frac{1}{16}k_1^2-\frac{1}{32}k_1^4+\cdots\right)\sin 4\varphi+\left(\frac{1}{48}k_1^3-\cdots\right)\sin 6\varphi+\cdots\right\}. \quad (7)$$

The greatest value of k_1 is n, i. e. almost $\frac{1}{600}$, therefore

$\frac{k_1^4}{64} \leq 1:8\,000\,000\,000\,000$ approximately,

and this term in the factor in front of the parenthesis, may be omitted, because even for a geodetic line encircling the entire Earth, with φ approaching nearly 2π, this term would give only $\frac{1}{200}$ of a millimeter. Even more so, we may disregard the term with k_1^4 in the factor of $\sin 4\varphi$, and now obtain for the length of the line P_0P the following formula:

$$s = b_0 \frac{1+\frac{1}{4}k_1^2}{1-k_1}\left\{\varphi+\left(\frac{1}{2}k_1-\frac{3}{16}k_1^3\right)\sin 2\varphi-\frac{1}{16}k_1^2\sin 4\varphi+\frac{1}{48}k_1^3\sin 6\varphi\right\}+\cdots, \quad (8)$$

which is exact in the hundredths of a millimeter for a line encircling the whole Earth.

If an inaccuracy of 5 mm is permissible, k_1^3 in the parenthesis may also be omitted.

Applying formula (8) to two points P_1 and P_2, for which φ has the values φ_1 and φ_2, and assuming $\varphi_2 > \varphi_1$, it follows from (8) by subtraction:

$$s_2 - s_1 = b_0 \frac{1+\frac{1}{4}k_1^2}{1-k_1}\left\{(\varphi_2-\varphi_1)+\left(\frac{1}{2}k_1-\frac{3}{16}k_1^3\right)(\sin 2\varphi_2-\sin 2\varphi_1) - \frac{1}{16}k_1^2(\sin 4\varphi_2-\sin 4\varphi_1)+\frac{1}{48}k_1^3(\sin 6\varphi_2-\sin 6\varphi_1)\right\}+\cdots$$

or in simpler notation:

$$s = b_0 \frac{1+\frac{1}{4}k_1^2}{1-k_1}\left\{\Delta\varphi + \left(k_1 - \frac{3}{8}k_1^3\right)\cos 2\varphi \sin \Delta\varphi - \frac{1}{8}k_1^2 \cos 4\varphi \sin 2\Delta\varphi + \frac{1}{24}k_1^3 \cos 6\varphi \sin 3\Delta\varphi\right\} + \cdots \quad (9)$$

$$s = \text{horizontal distance } P_1P_2,$$

$$\varphi = \frac{1}{2}(\varphi_1 + \varphi_2),\ \Delta\varphi = \varphi_2 - \varphi_1.$$

Par. 6. Given the Length s of a Geodetic Line, the Location of One of the End Points, and the Azimuth; Desired: the Reduced Latitude and the Azimuth at the Other End Point. We denote the given point P_1 its reduced latitude β_1 and the azimuth of the line α_1. With reference to the azimuth we assume for the time being that $\alpha_1 \leq 180^0$, so that in accordance with the earlier remarks, especially fig. 16, we obtain a positive value for the distance of the desired point P_2 from the given point P_1. Compare fig. 18 and 19 on p. 225.

From the right triangle $\mathfrak{N}\mathfrak{P}_0\mathfrak{P}_1$ we obtain first β_0, from which follow

k and k_1; then we also obtain φ_1, and therefore from formula (8) of the previous paragraph s_1.

Adding to s_1 the given length s, we obtain s_2. Formula (8), applied to φ_2 and s_2, has now to be solved for φ_2.

We proceed exactly as on p. 53 Par. 10. Denoting first for abbreviation:

$$\frac{s_1(1-k_1)}{b_0\left(1+\frac{1}{4}k_1^2\right)} \text{ by } \sigma_1 \text{ and } \frac{s_2(1-k_1)}{b_0\left(1+\frac{1}{4}k_1^2\right)} \text{ by } \sigma_2 \quad (1)$$

we set correspondingly:

$$\frac{s(1-k_1)}{b_0\left(1+\frac{1}{4}k_1^2\right)} = \sigma_2 - \sigma_1 = \Delta\sigma. \qquad (2)$$

Dividing the factor of φ_2 to the left, we obtain from formula (8):

$$\sigma_2 = \varphi_2 + \left(\frac{1}{2}k_1 - \frac{3}{16}k_1^3\right)\sin 2\varphi_2 - \frac{1}{16}k_1^2 \sin 4\varphi_2 + \frac{1}{48}k_1^3 \sin 6\varphi_2 - \cdots. \qquad (3)$$

The apparent difference between σ_2 and φ_2 in this formula is always a small quantity, at maximum it equals only $\frac{1}{1200}$, or is equal in seconds to 206265 : 1200, i. e., not even 200". Therefore, the following evolution will be uncontestable.

Neglecting small terms in the third and higher powers of k_1 it follows from (3):

$$\sin 2\sigma_2 = \sin 2\varphi_2 \cos\left\{k_1 \sin 2\varphi_2 - \frac{1}{8}k_1^2 \sin 4\varphi_2 + \cdots\right\} + \cos 2\varphi_2 \sin\left\{k_1 \sin 2\varphi_2 - \frac{1}{8}k_1^2 \sin 4\varphi_2 + \cdots\right\}.$$

Considering the cosine and sine series, and consequently omitting the small terms appearing as factors of k_1^3 etc., we obtain:

$$\sin 2\sigma_2 = \sin 2\varphi_2 + k_1 \sin 2\varphi_2 \cos 2\varphi_2 - \frac{1}{2}k_1^2 \sin^3 2\varphi_2 - \frac{1}{8}k_1^2 \cos 2\varphi_2 \sin 4\varphi_2 + \cdots.$$

We set as follows

$$\sin^3 2\varphi_2 = \frac{3}{4}\sin 2\varphi_2 - \frac{1}{4}\sin 6\varphi_2,$$

$$\sin 2\varphi_2 \cos 2\varphi_2 = \frac{1}{2}\sin 4\varphi_2,$$

$$\cos 2\varphi_2 \sin 4\varphi_2 = \frac{1}{2}\sin 6\varphi_2 + \frac{1}{2}\sin 2\varphi_2$$

and obtain:

$$\sin 2\sigma_2 = \left(1 - \frac{7}{16} k_1^2\right) \sin 2\varphi_2 + \frac{1}{2} k_1 \sin 4\varphi_2 + \frac{1}{16} k_1^2 \sin 6\varphi_2 + \cdots \quad (4)$$

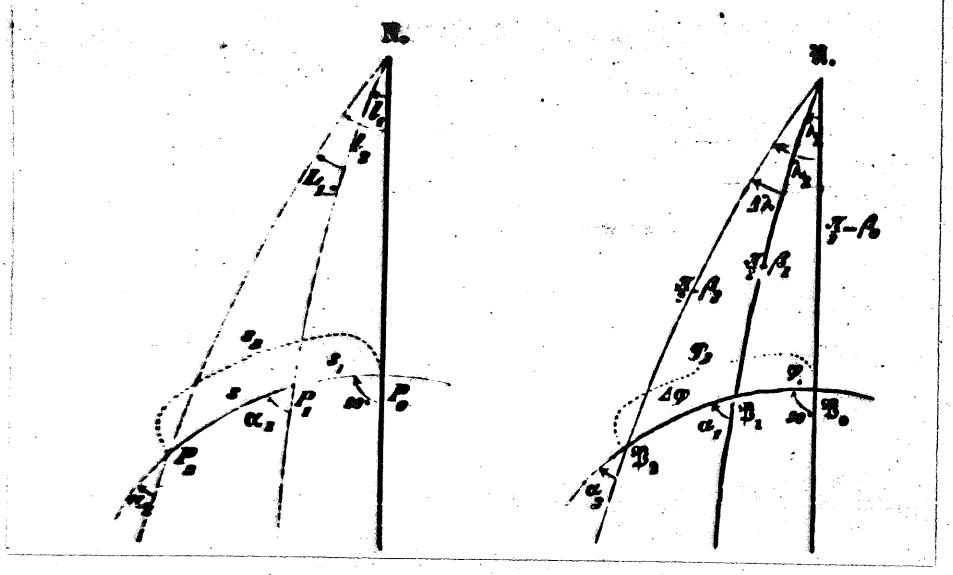


Figure 18. Ellipsoid

Figure 19. Sphere

Using this equation we eliminate sin $2\varphi_2$ from (3) by multiplying it by $\left(\frac{1}{2} k_1 - \frac{3}{16} k_1^3\right) : \left(1 - \frac{7}{16} k_1^2\right)$ and subtracting it from (3); we obtain:

$$\sigma_2 - \left(\frac{1}{2} k_1 + \frac{1}{32} k_1^3\right) \sin 2\sigma_2 = \varphi_2 - \frac{5}{16} k_1^2 \sin 4\varphi_2 - \frac{1}{96} k_1^3 \sin 6\varphi_2 + \cdots \quad (5)$$

All terms with higher powers than k_1^3 are omitted in this formula, in the same manner as in (3). Considering the higher terms, it is obvious that formula (5) is numerically just as exact as formula (3).

Using formula (3) we set:

$$\sin 4\sigma_2 = \sin 4\varphi_2 \cos(2k_1 \sin 2\varphi_2 + \cdots) + \cos 4\varphi_2 \sin(2k_1 \sin 2\varphi_2 + \cdots)$$

or neglecting small terms, appearing as factors of k_1^2 etc.:

$$\sin 4\sigma_2 = \sin 4\varphi_2 + 2k_1 \sin 2\varphi_2 \cos 4\varphi_2 + \cdots,$$

or

$$\sin 4\sigma_2 = \sin 4\varphi_2 + k_1(\sin 6\varphi_2 - \sin 2\varphi_2) + \cdots . \tag{6}$$

Multiplying (6) by $\frac{5}{16}k_1^2$ and adding to (5), it follows that:

$$\sigma_2 - \left(\frac{1}{2}k_1 + \frac{1}{32}k_1^3\right)\sin 2\sigma_2 + \frac{5}{16}k_1^2 \sin 4\sigma_2 = \varphi_2 + \frac{29}{96}k_1^3 \sin 6\varphi_2 - \frac{5}{16}k_1^3 \sin 2\varphi_2 + \cdots . \tag{7}$$

Taking the factors of k_1^3 to the left side and setting for φ_2 simply σ_2, will produce only small errors of the order of k_1^4. Hence we obtain finally:

$$\varphi_2 = \sigma_2 - \left(\frac{1}{2}k_1 - \frac{9}{32}k_1^3\right)\sin 2\sigma_2 + \frac{5}{16}k_1^2 \sin 4\sigma_2 - \frac{29}{96}k_1^3 \sin 6\sigma_2 + \cdots . \tag{8}$$

Comparing this formula with (3), it seems that formula (8) is converging less succinctly than formula (3). Although formula (8) is indeed a little less exact than (3), it still can be used in all instances. This becomes evident from the fact that the maximum value of k_1^4 is approximately 1 : 130 000 000 000, whereas the neglected terms will never approach that figure. The maximum value of k_1^4 would have an influence on s_2 of only a_0 : 130 000 000 000, or 0,05 mm. If an inaccuracy of 10 mm is permissible, k_1^3 in formula (8) may also be omitted.

Having determined φ_2 from (8), we obtain β_2 and α_2 from the relation of two sides and enclosed right angle in the spherical triangle $\mathfrak{U}\mathfrak{P}_0\mathfrak{P}_2$. For the time being we do not take into consideration the applicable formulae, which offer nothing new. Later on, we shall correlate all pertinent formulae in a computation plan. It must be emphasized at this point that because of the comparatively small

Par. 7. Computation of k_1 and of the Coefficients Depending on k_1.

magnitude of the distance $P_1P_2 = s$, it will be as a rule advantageous not to compute directly φ_{21} but $\Delta\varphi$. Interchanging in (8) φ_2 with φ_1 and σ_2 with σ_1 and subtracting the new formula side after side from (8) we obtain:

$$\left.\begin{aligned} \Delta\varphi = \Delta\sigma &- \left(k_1 - \tfrac{9}{16}k_1^3\right)\cos 2\sigma \sin \Delta\sigma + \tfrac{5}{8}k_1^2 \cos 4\sigma \sin 2\Delta\sigma \\ &- \tfrac{29}{48}k_1^3 \cos 6\sigma \sin 3\Delta\sigma + \cdots; \\ 2\sigma &= \sigma_2 + \sigma_1 = 2\sigma_1 + \Delta\sigma, \\ \varphi_2 &= \varphi_1 + \Delta\varphi. \end{aligned}\right\} \quad (9)$$

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Formula (2) yields $\Delta\sigma$ directly from s, but formula (3) produces, after interchanging σ_2 with σ_1 and φ_2 with φ_1:

$$2\sigma_1 = 2\varphi_1 + k_1 \sin 2\varphi_1 - \tfrac{1}{2}k_1^2 \sin 4\varphi_1 + \cdots, \quad (10)$$

As assumed, it is sufficient to consider only k_1 and k_1^2, because for the computation of the trigonometric functions of σ, only σ_1 is used. These functions appear in (9) multiplied only by k_1, k_1^2 and k_1^3, hence the neglected terms of formula (10) will produce in formula (9) only small errors of k_1^4 degree.

Par. 7. Computation of k_1 and of the Coefficients Depending on k_1. According to p. 221 (8) $k = e \sin B_0 = e \sin \beta_0 : \sqrt{1 - e^2\cos^2\beta_0}$ and hence:

$$k_1 = \frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}} = \frac{\sqrt{1 - e^2\cos^2\beta_0} - \sqrt{1-e^2}}{\sqrt{1 - e^2\cos^2\beta_0} + \sqrt{1-e^2}},$$

and because $\delta = \dfrac{e^2}{1-e^2}$:

$$k_1 = (\sqrt{1 + \delta \sin^2\beta_0} - 1) : (\sqrt{1 + \delta \sin^2\beta_0} + 1).$$

If we set:

$$\tan\{Rz\} = \tan E = \sqrt{\delta}\sin\beta_0, = \tan\{Oz\}\times\sin\{\beta_0\} \qquad (1)$$

error (should be "$\sin\beta_0^2$" or "$\sqrt{\delta}\sin\beta_0$")

it follows that

$$\tan\{\tfrac{1}{2}Rz\}^2 = k_1 = \tan^2\frac{E}{2}. \qquad (2)$$

Hence the computation of k_1 is very simplified. Further we require the expression:

$$\frac{1+\frac{1}{4}k_1^2}{1-k_1} \text{ or more exactly } \frac{1+\frac{1}{4}k_1^2+\frac{1}{64}k_1^4+\cdots}{1-k_1},$$

where it has been shown that the omission of k_1^4 in the numerator, is permissible. We have a relation:

$$\log\left(1+\tfrac{1}{4}k_1^2+\tfrac{1}{64}k_1^4+\cdots\right) = M\left(\tfrac{1}{4}k_1^2-\tfrac{1}{64}k_1^4+\cdots\right)$$

and note that it is permissible to use in its place simply $\frac{1}{4}Mk_1^2$. Further it follows that:

$$\cos\{Rz\} = 1-k_1 = 1-\tan^2\frac{E}{2} = \cos E\sec^2\frac{E}{2};$$

and we obtain finally with a sufficient accuracy:

$$\log\frac{1+\frac{1}{4}k_1^2+\cdots}{1-k_1} = 2\log\cos\frac{E}{2} - \log\cos E + \tfrac{1}{4}Mk_1^2+\cdots. \qquad (3)$$

However, this expression is still not perfect. It is possible to reduce the right side to two terms. According to p. 29 (3) we have:

$$\log\cos E = -M\left(\tfrac{1}{2}E^2+\tfrac{1}{12}E^4+\tfrac{1}{45}E^6+\tfrac{17}{2520}E^8+\cdots\right)$$

and substituting for E simply $\frac{1}{2}E$:

$$4\log\cos\frac{E}{2} = -M\left(\frac{1}{2}E^2 + \frac{1}{48}E^4 + \frac{1}{16\cdot 45}E^6 + \frac{17}{64\cdot 2520}E^8 + \cdots\right)$$

It follows by subtraction:

$$\log\cos E = 4\log\cos\frac{E}{2} - M\left(\frac{1}{16}E^4 + \frac{1}{48}E^6 + \frac{17}{64\cdot 40}E^8 + \cdots\right).$$

But according to (2):

$$k_1 = \frac{1}{4}E^2\left(1 + \frac{1}{12}E^2 + \frac{1}{120}E^4 + \cdots\right)^2$$
$$k_1^2 = \frac{1}{16}E^4\left(1 + \frac{1}{3}E^2 + \frac{3}{40}E^4 + \cdots\right).$$

Therefore, it is easy to see that:

$$\log\cos E = 4\log\cos\frac{E}{2} - Mk_1^2 - \frac{1}{2}Mk_1^4 - \cdots. \qquad (4)$$

Hence (3) will change to

$$\log\frac{1 + \frac{1}{4}k_1^2 + \frac{1}{64}k_1^4 + \cdots}{1 - k_1} = -2\log\cos\frac{E}{2} + \frac{5}{4}Mk_1^2 + \frac{31}{64}Mk_1^4 + \cdots \qquad (5)$$

During the computation with ten-place logarithms the factor of k_1^4 will be completely undiscernible. Also, its influence in formula (9), p. 223, results only in 0,2 mm, even for a line encircling the entire Earth. Hence we can set for all instances:

$$\log\frac{1 + \frac{1}{4}k_1^2 + \cdots}{1 - k_1} = -2\log\cos\frac{E}{2} + \frac{5}{4}Mk_1^2 + \cdots \qquad (5^*)$$

According to (9) of the previous paragraph it follows that:

$$\log\left(k_1 - \frac{9}{16}k_1^3\right) = \log k_1 - \frac{9}{16}Mk_1^2 + \cdots, \qquad (6)$$

again with a completely sufficient accuracy, because

the neglected terms of the logarithmic evolution in the mentioned formula will produce only terms with k_1^5 as an error.

Par. 8. Determination of the Geographical Longitude Difference. If x denotes the normal distance of a point P from the axis of rotation, the differential triangle in fig. 16, p. 218, which adjoins P, yields:

$$x\,dl = ds \cdot \sin\alpha .$$

According to p. 39 (1), $x = a_0 \cos\beta$, therefore $x\,dl$ equals

$$a_0 \cos\beta\, dl = \sin\alpha\, ds . \qquad (1)$$

Substituting for ds the value $a_0\sqrt{1 - e^2\cos^2\beta}\, d\varphi$, and for $\sin\alpha$, according to p. 218 (2), the value $\cos\beta_0 : \cos\beta$, it follows further:

$$dl = \frac{\cos\beta_0}{\cos^2\beta}\sqrt{1 - e^2\cos^2\beta}\, d\varphi . \qquad (2)$$

Integrating this equation, we recall the usefulness of introducing the elements of the spherical fig. 17, which is associated with fig. 16 on the ellipsoid. Fig. 17 shows that the variation of the geographical longitude difference on the sphere, belonging to $d\varphi$, equals:

$$d\lambda = \frac{d\varphi \cdot \sin\alpha}{\cos\beta} = \frac{\cos\beta_0}{\cos^2\beta}\, d\varphi . \qquad (3)$$

Subtracting this equation from (2) side after side results in:

$$dl = d\lambda - \cos\beta_0 \left(\frac{1 - \sqrt{1 - e^2\cos^2\beta}}{\cos^2\beta}\right) d\varphi . \qquad (4)$$

The factor of $d\varphi$ in the parenthesis must be expressed by φ. But the transition from (6) to (9) on p. 220 showed that:

Par. 8. Determination of the Geographical Longitude Difference.

$$\sqrt{1-e^2\cos^2\beta}=\sqrt{1-e^2\cos^2\beta_0}\sqrt{1-k^2\sin^2\varphi},$$
$$k^2=\frac{e^2\sin^2\beta_0}{1-e^2\cos^2\beta_0}. \tag{5}$$

Hence:

$$(1-\sqrt{1-e^2\cos^2\beta})=\sqrt{1-e^2\cos^2\beta_0}\left\{\frac{1}{\sqrt{1-e^2\cos^2\beta_0}}-\sqrt{1-k^2\sin^2\varphi}\right\} \tag{6}$$

It follows further from the equation for k^2 in solving for e^2, that $e^2=k^2:(\sin^2\beta_0+k^2\cos^2\beta_0)$ and we find:

$$\frac{1}{\sqrt{1-e^2\cos^2\beta_0}}=\sqrt{1+k^2\cot^2\beta_0}. \tag{7}$$

We introduce this in equation (6), inside of the brace, and in this manner change (4) to the following form:

$$dl=d\lambda-\cos\beta_0\sqrt{1-e^2\cos^2\beta_0}\,\frac{\sqrt{1+k^2\cot^2\beta_0}-\sqrt{1-k^2\sin^2\varphi}}{1-\sin^2\beta_0\cos^2\varphi}\,d\varphi. \tag{8}$$

It is immediately evident that by developing the root quantities on the right side, the differences of the single terms belonging to k^2, k^4 etc. are divisible by $(\cot^2\beta_0+\sin^2\varphi)$. The denominator $(1-\sin^2\beta_0\cos^2\varphi)$ can be written $(\cot^2\beta_0+\sin^2\varphi)\sin^2\beta_0$, hence the quotient will change to:

$$\frac{k^2}{2\sin^2\beta_0}\left(1+\frac{k^2}{4}(\sin^2\varphi-\cot^2\beta_0)+\frac{k^4}{8}(\sin^4\varphi-\sin^2\varphi\cot^2\beta_0+\cot^4\beta_0)+\cdots\right).$$

Since $k\cot\beta$, the same as k, is a small term of the same power as e, the neglected terms will produce also small terms of power k^3. In setting further

$$\sin^2\varphi=\frac{1}{2}-\frac{1}{2}\cos 2\varphi \text{ and } \sin^4\varphi=\frac{3}{8}-\frac{1}{2}\cos 2\varphi+\frac{1}{8}\cos 4\varphi,$$

we finally obtain for equation (8):

$$dl = d\lambda - \frac{k^2 \cos\beta_0}{2\sin^2\beta_0}\sqrt{1-e^2\cos^2\beta_0}\left\{\left(1-\frac{k^2}{4}\cot^2\beta_0+\frac{k^4}{8}+\frac{k^4}{8}\cot^4\beta_0-\frac{k^4}{16}\cot^2\beta_0+\frac{3k^4}{64}+\cdots\right)\right.$$
$$\left.-\left(\frac{k^2}{8}-\frac{k^4}{16}\cot^2\beta_0+\frac{k^4}{16}\right)\cos 2\varphi+\frac{k^4}{64}\cos 4\varphi+\cdots\right\}d\varphi. \qquad (9)$$

Before proceeding with the integration of this equation, we simplify the coefficients by introducing k_1 and n. Setting $(\sin^2\beta_0 + \cos^2\beta_0)$ in the denominator for I, it follows from $k^2 = e^2 \sin^2\beta_0 : (1 - e^2 \cos^2\beta_0)$ that:

$$k^2 \cot^2\beta_0 = (e^2 - k^2) : (1 - e^2). \qquad (10)$$

Since according to p. 221 (1) for k^2 we can set $4k_1 : (1+k_1)^2$, and because $e^2 = 4n : (1+n)^2$, it follows that:

$$k^2 \cot^2\beta_0 = 4\frac{(n-k_1)(1-nk_1)}{(1+k_1)^2(1-n)^2} = 4(n-k_1)+8(n-k_1)^2+\cdots,$$
$$k^2 = 4k_1 - 8k_1^2 + \cdots. \qquad (11)$$

From this relation for k^2 we obtain at once the relation $k^2(1 - e^2\cos^2\beta_0) = e^2\sin^2\beta_0$, hence:

$$\frac{k^2\sqrt{1-e^2\cos^2\beta_0}}{\sin^2\beta_0} = \frac{e^2}{\sqrt{1-e^2\cos^2\beta_0}} = e^2\sqrt{1+k^2\cot^2\beta_0},$$

and finally:

$$\frac{k^2\sqrt{1-e^2\cos^2\beta_0}}{\sin^2\beta_0} = e^2\left(1+2(n-k_1)+2(n-k_1)^2+\cdots\right). \qquad (12)$$

We set equation (12) in equation (9) as a factor in front of the parenthesis, and equation (11) inside the parenthesis, and obtain:

$$dl = d\lambda - \frac{1}{2}e^2\cos\beta_0\left\{\left(1+n-\frac{1}{2}k_1-\frac{1}{4}k_1^2+\cdots\right)\right.$$
$$\left.-\frac{1}{2}k_1\cos 2\varphi+\frac{1}{4}k_1^2\cos 4\varphi+\cdots\right\}d\varphi. \qquad (13)$$

Formula (13) covers all terms within the parenthesis up to but excluding the order n^3. This fact must be stressed, because, in the factor of cos 2φ the term k_1^2 does not appear. The largest neglected terms of the power n^4, or k_1^4, are also completely insignificant. Integration, however, will intensify the convergence of the series.

Integrating from l_1 to l_2 and denoting $L_{1,2}$ as the longitudinal difference from P_1 to P_2, and after contracting the terms which appear as factors in sin $2\varphi_1$ and sin $2\varphi_2$ as well as in sin $4\varphi_1$ and sin $4\varphi_2$ (compare also fig. 18 and 19, p. 225), we obtain the following relation:

$$L_{1,2} = \Delta\lambda - \frac{1}{2}e^2\cos\beta_0\left\{\begin{array}{l}\left(1+n-\frac{1}{2}k_1-\frac{1}{4}k_1^2+\cdot\cdot\right)\Delta\varphi \\ -\frac{1}{2}k_1\cos 2\varphi\sin\Delta\varphi+\frac{1}{8}k_1^2\cos 4\varphi\sin 2\Delta\varphi+\cdot\cdot\end{array}\right\} \qquad (14)$$

$$L_{1,2}=l_2-l_1 \qquad \Delta\lambda=\lambda_2-\lambda_1$$

$$\varphi=\frac{1}{2}(\varphi_1+\varphi_2) \qquad \Delta\varphi=\varphi_2-\varphi_1 .$$

Introducing the oblateness α makes it possible to transform the coefficient of $\Delta\varphi$ in such a manner that it is more convenient for small distances. According to p. 37 we have:

$$e^2 = 4n:(1+n)^2 = 2\alpha:(1+n)$$

and hence we obtain for the coefficient of $\Delta\varphi$ in (14) exactly as before, the formula:

$$\begin{array}{l}\frac{1}{2}e^2\cos\beta_0\left(1+n-\frac{1}{2}k_1-\frac{1}{4}k_1^2+\cdot\cdot\right) \\ =\alpha\cos\beta_0\left(1-\frac{1}{2}k_1-\frac{1}{4}k_1^2+\frac{1}{4}\alpha k_1+\cdot\cdot\right).\end{array} \qquad (15)$$

When, however, it is permissible to neglect the terms of power k_1^2 in the parenthesis to the right,

the coefficient will change to the simpler form of $a\cos\beta_0(1-\frac{1}{2}k_1)$.

Even for a line encircling the entire Earth, formula (14) leaves only a small uncertainty in the hundred thousandths of a second of $L_{1,2}$, as can be seen from the approximately correct assumption that the coefficient of $\Delta\varphi$ is erroneous by less than n^4.

Par. 9. Compilation of Formulae for Transfer of Geographical Latitude and Longitude by Means of a Geodetic Line of a Known Length and With a Known Initial Azimuth.

The horizontal distance $P_1P_2 = s$ may be of arbitrary length.

If the dimensions of the Earth ellipsoid, according to Bessel, are not applied, we have to compute first:

$$e^2 = 2a - a^2 \qquad \sqrt{1-e^2} = 1 - a$$
$$\sqrt{\delta} = \frac{e}{\sqrt{1-e^2}} \qquad b_0 = a_0\sqrt{1-e^2}.$$

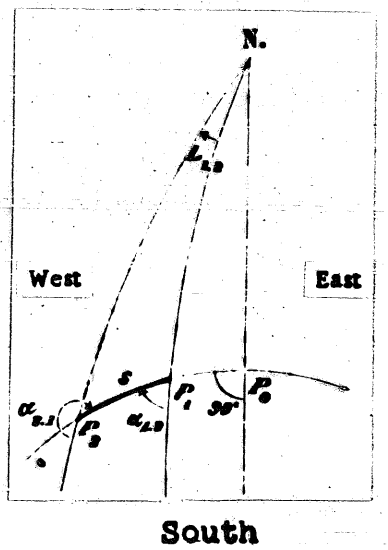


Figure 20. Ellipsoid

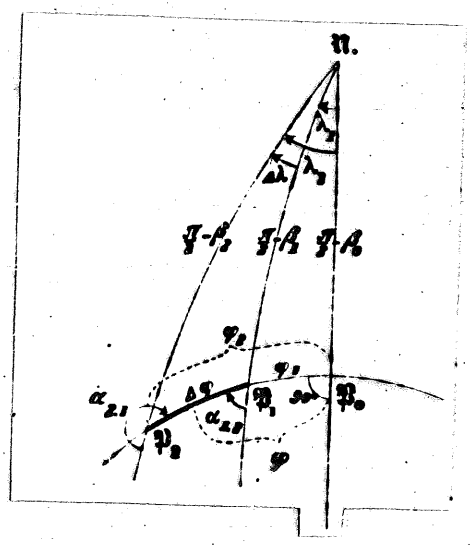


Figure 21. Sphere

If B_1 is the geographical latitude of P_1, then the reduced latitude

β_1 is to be determined by means of the formula:

$$\tan \beta_1 = \sqrt{1-e^2} \tan B_1; \quad \log \sqrt{1-e^2} = 9{,}9985458.202 - 10. \tag{1}$$

Application of this formula from p. 40 seems to be the most suitable way, because $\tan \beta_1$ will be needed immediately. The next step is to resolve the spherical triangle $\mathfrak{N}\mathfrak{P}_0\mathfrak{P}_1$ (fig. 21), for which purpose we apply the formulae (1) p. 76 (for $A = 90^0$ etc.):

$$\left.\begin{aligned} \sin \beta_0 \cos \varphi_1 &= \sin \beta_1 \\ \sin \beta_0 \sin \varphi_1 &= \cos \beta_1 \cos \alpha_{1.2} \\ \cos \beta_0 \quad . \quad &= \cos \beta_1 \sin \alpha_{1.2} \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} \cos \beta_1 \cos \lambda_1 &= \cos \varphi_1 \cos \beta_0 \\ \cos \beta_1 \sin \lambda_1 &= \sin \varphi_1 . \end{aligned}\right\} \tag{3}$$

The reduction of formulae (2) and (3) is highly exact and precise as long as we continue to take β_0 as positive.

Now we have to compute according to p. 227 (1) and (2), p. 224 (2) and p. 228 (5):

$$\left.\begin{aligned} \tan E = \sqrt{\delta} \sin \beta_0 \qquad & \log \sqrt{\delta} = 8{,}9136593.9 - 10 \\ \log k_1 = 2 \log \tan \tfrac{1}{2} E; & \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} \underset{\text{in sec.}}{\Delta\sigma} &= s \frac{\varphi''}{b_0} \frac{1-k_1}{1+\frac{1}{4}k_1^2} \\ \log \frac{1-k_1}{1+\frac{1}{4}k_1^2} &= 2 \log \cos \frac{E}{2} - \frac{5}{4} M k_1^2 + \cdots \\ \log \left(\tfrac{5}{4} M\right) &= 6{,}73469 \quad \text{for units of the 7th decimal} \\ \log \varphi'' &= 5{,}3144251.332 \\ \log \frac{\varphi''}{b_0} &= 8{,}5112358.493 - 10. \end{aligned}\right\} \tag{5}$$

L.D. 4

Chapter 5. Basic Formulae for the Geodetic Line.

According to formulae (9) and (10), p. 226 and 227, it is necessary to compute further:

$$\left.\begin{aligned} 2\sigma_1 &= 2\varphi_1 + \varrho'' k_1 \sin 2\varphi_1 - \tfrac{1}{8}\varrho'' k_1^2 \sin 4\varphi_1 \\ &\text{in sec.} \qquad \text{in Sek.} \\ \log \varrho'' &= 5{,}31443 \\ \log\left(-\tfrac{1}{8}\varrho''\right) &= 4{,}4114_n \end{aligned}\right\} \qquad (6)$$

it is sufficient to determine $2\sigma_1$ in hundredths of a second, if one is satisfied with hundred thousandths of a second; also with regard to (6), p. 228:

$$\left.\begin{aligned} \Delta\varphi &= \Delta\sigma + \mathfrak{A}\cos 2\sigma \sin\Delta\sigma + \mathfrak{B}\cos 4\sigma \sin 2\Delta\sigma + \mathfrak{C}\cos 6\sigma \sin 3\Delta\sigma + \cdots \\ &\text{in sec.} \qquad \text{in sec.} \\ 2\sigma &= 2\sigma_1 + \Delta\sigma \\ \log\mathfrak{A} &= \log(-\varrho'') + \log k_1 - \tfrac{9}{16} k_1^2 + \cdots \\ &= 5{,}3144251.3_n + \log k_1 - [6{,}388]\, k_1^2 + \cdots \quad \text{in units of the 7th decimal} \\ \log\mathfrak{B} &= \log\left(\tfrac{5}{8}\varrho'' k_1^2 + \cdots\right) = 5{,}11031 + 2\log k_1 + \cdots \\ \log\mathfrak{C} &= \log\left(-\tfrac{25}{48}\varrho'' k_1^3 + \cdots\right) = 5{,}096_n + 3\log k_1 + \cdots . \end{aligned}\right\} \qquad (7)$$

Hence we obtain $\varphi_2 = \varphi_1 + \Delta\varphi$, and the reduction of the right triangle $\mathfrak{N}\mathfrak{P}_0\mathfrak{P}_2$ according to formulae

$$\left.\begin{aligned} \sin\beta_2 &= +\sin\beta_0 \cos\varphi_2 \\ \cos\beta_2 \cos\alpha_{2.1} &= -\sin\beta_0 \sin\varphi_2 \\ \cos\beta_2 \sin\alpha_{2.1} &= -\cos\beta_0 \end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned} \cos\beta_2 \cos\lambda_2 &= \cos\varphi_2 \cos\beta_0 \\ \cos\beta_2 \sin\lambda_2 &= \sin\varphi_2 \end{aligned}\right\} \qquad (9)$$

results in β_2, $\alpha_{2.1}$ and λ_2. We derive these formulae from (2) and (3)

by interchanging $\beta_1, \alpha_{1.2}$ and λ_1 with β_2, or $\alpha_{2.1} - 180^0$ and λ_2. ($\alpha_{2.1}$ is the azimuth of the line P_2P_1 in P_2, see fig. 20 and 21). Because cos β_2 has to be positive, the reduction of (8) and (9) will be very definite.

The following formula enables us to obtain from β_2 the geographical latitude for point P_2:

$$\tan B_2 = \frac{1}{\sqrt{1-e^2}} \tan \beta_2, \qquad \log \frac{1}{\sqrt{1-e^2}} = 0{,}0014541.798, \tag{10}$$

with tan β_2 already known from (8).

For computation of the geographical longitudinal difference $L_{1.2}$ we have determined already λ_1 and λ_2 by formulae (3) and (9). In accordance with (14) and (15), p. 231 it follows that:

$$\left.\begin{aligned} L_{1.2} &= \lambda_2 - \lambda_1 - \cos\beta_0 \{\mathfrak{A}' \Delta\varphi + \mathfrak{B}' \cos 2\varphi \sin \Delta\varphi + \mathfrak{C}' \cos 4\varphi \sin 2\Delta\varphi + \cdots\} \\ &\text{in sec.} \qquad \text{in sec.} \qquad \text{in sec.} \\ 2\varphi &= \varphi_1 + \varphi_2 \\ \log \mathfrak{A}' &= \log \mathfrak{a} - \tfrac{1}{2} M \left(1 - \tfrac{1}{2}\mathfrak{a}\right) k_1 - \tfrac{3}{8} M k_1^2 + \cdots \\ &= 7{,}5241069.0 - 10 - [6{,}33603] k_1 - [6{,}212] k_1^2 + \cdots \\ &\qquad\qquad \text{for units of the 7th decimal} \\ \log \mathfrak{B}' &= \log\left(-\tfrac{1}{4}\rho'' e^2 k_1 + \cdots\right) = 2{,}53678_n + \log k_1 + \cdots \\ \log \mathfrak{C}' &= \left(\tfrac{1}{16}\rho'' e^2 k_1^2 + \cdots\right) = 1{,}935 + 2\log k_1 + \cdots .\ ^*) \end{aligned}\right\} \tag{11}$$

In case we do not operate with Bessel's $\mathfrak{a}$, there will be a slight change in the figures in formula (11) for log $\mathfrak{B}'$ and log $\mathfrak{C}'$, and strictly speaking, also in the coefficient $\frac{1}{2} M \left(1 - \frac{1}{2}\mathfrak{a}\right)$ of k_1 in log $\mathfrak{A}'$. However, in most cases it will be possible to retain this coefficient. The change for log $\mathfrak{B}'$ and log $\mathfrak{C}'$ will be determined by the fact that we now set log $e^2 = 7{,}82441 - 10$.

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*)For partial use of the computation of (7) we set in the first formula (11):

$$\{\div\} = \mathfrak{A}' \Delta\varphi + \mathfrak{B}' \cos 2\varphi \sin \Delta s + 3\mathfrak{C}' \cos 4\varphi \sin 2\Delta s + \cdots$$

In evolving the preceding formulae it was assumed that $\alpha_{1.2} < 180^0$ in order to maintain the horizontal distance s positive. We may, however, apply these formulae without question to any value of $\alpha_{1.2}$ and still set s always as positive. Since for $\alpha_{1.2} > 180^0$ cos β_0 becomes negative, which according to the initial geometrical meaning of β_0 as geographical latitude is impossible, we have to disregard the initial meaning of β_0, and prove that these formulae also are valid for $\alpha_{1.2} > 180^0$.

These formulae are valid in any case when we compute simultaneously an eastern λ as positive, a south-eastern $\alpha_{1.2}$ as positive, and φ as positive in an eastern direction. This corresponds to the figures which are symmetrical to the previously used fig. 16 to 21, but are situated on the other side of the meridian of P_0. Denoting those quantities and the corresponding β with primes, we obtain, moreover, from (2) and (3):

$$\begin{aligned}
\sin \beta'_0 \cos \varphi'_1 &= \sin \beta'_1 \\
\sin \beta'_0 \sin \varphi'_1 &= \cos \beta'_1 \cos \alpha'_{1.2} \\
\cos \beta'_0 &= \cos \beta'_1 \sin \alpha'_{1.2} \\
\cos \beta'_1 \cos \lambda'_1 &= \cos \varphi'_1 \cos \beta'_0 \\
\cos \beta'_1 \sin \lambda'_1 &= \sin \varphi'_1.
\end{aligned}$$

Because $\beta'_1 = \beta_1$ and $\alpha'_{1.2} = 360^0 - \alpha_{1.2}$ it follows:

$$\begin{aligned}
\sin \beta'_0 \cos \varphi'_1 &= \sin \beta_1 \\
\sin \beta'_0 \sin \varphi'_1 &= \cos \beta_1 \cos \alpha_{1.2} \\
\cos \beta'_0 &= -\cos \beta_1 \sin \alpha_{1.2} \\
\cos \beta_1 \cos \lambda'_1 &= \cos \varphi'_1 \cos \beta'_0 \\
\cos \beta_1 \sin \lambda'_1 &= \sin \varphi'_1.
\end{aligned}$$

Par. 10. Abbreviation of Formulae.

Setting instead of β_0 a new auxiliary quantity $\pi - \beta_0$, and in place of λ_1' the quantity $180^0 - \lambda_1$, and writing further for φ_1' simply φ_1, it follows that:

$$\begin{aligned}
&\sin\beta_0 \cos\varphi_1 = \sin\beta_1 \\
&\sin\beta_0 \sin\varphi_1 = \cos\beta_1 \cos\alpha_{1.2} \\
&\cos\beta_0 \quad\quad\quad = \cos\beta_1 \sin\alpha_{1.2} \\
&\cos\beta_1 \cos\lambda_1 = \cos\varphi_1 \cos\beta_0 \\
&\cos\beta_1 \sin\lambda_1 = \sin\varphi_1 .
\end{aligned}$$

These formulae agree with (2) and (3). Formulae (8) and (9), too, are valid again. Also, formulae (4) to (7) remain valid, because $\sin\beta_0$ is not changing and φ' is equal to φ.

Regarding the expression for $L_{1.2}$, in accordance with (11), we notice that the right side, written with primes on λ, β_0 and φ, has the opposite value, because $\lambda' = 180^0 - \lambda$ and $\cos\beta_0' = -\cos\beta_0$.

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However the formula produces the eastern longitudinal difference $L_{1.2}'$, which is equal to the negative western longitudinal difference $L_{1.2}$. Hence we retain also this formula for the western longitudinal difference.

Par. 10. Abbreviation of Formulae. In most cases it will be sufficient to omit in the mentioned formulae the terms in small print. In the position of P_2, this would only produce errors corresponding to a displacement of 0,001 to 0,002 equator seconds, or 0,03 to 0,06 m. For small distances the error is considerably smaller.

Looking in this respect at the formulae (7) for $\Delta\varphi$ and

considering also the term $2\sigma_1$ omitted in formula (6), we find that the terms in small print have the following effect:

$$\varrho'' \left\{ \begin{array}{c} -\frac{1}{8} k_1^3 \sin 2\sigma \sin \Delta\sigma \sin 4\varphi_1 + \frac{9}{16} k_1^3 \cos 2\sigma \sin \Delta\sigma \\ -\frac{29}{48} k_1^3 \cos 6\sigma \sin 3\Delta\sigma . \end{array} \right\} \qquad (1)$$

The most unfavorable assumption is that the three parts of that formula accumulate absolutely. Setting $k_1 = n$ this produces as a maximum:

$$\varrho'' \frac{62}{48} n^3 \text{ i.e., } 0{,}0012''.$$

The true maximum value is slightly smaller. If $\Delta\sigma$ is small, we may interchange φ_1 with σ also sin $\Delta\sigma$ and sin 3 $\Delta\sigma$ with $\Delta\sigma$, or 3 $\Delta\sigma$, and obtain instead of (1):

$$-\varrho'' k_1^3 \Delta\sigma \cos 2\sigma \left(7 \cos^2 2\sigma - \frac{23}{4}\right), \qquad (2)$$

the maximal value of which is nearly equal to

$$\pm 2\varrho'' n^3 \Delta\sigma \quad \text{i.e.,} \quad \pm 0{,}002'' \Delta\sigma.$$

For $\Delta\sigma = 0{,}2$, this amounts to only $0{,}0004''$. This figure represents an error in s of approximately 0,013 mm.

The minuteness of (2) becomes even more evident when we seek to determine the effect on log $\Delta\varphi$. For the maximum value of $0{,}002''$ $\Delta\sigma$, this amounts to only four units of the 9th decimal place, because 0,002 M : $\varrho'' = 0{,}0000000.04$.

With regard to the terms in small print in formula (11), their value is equal to:

$$\frac{1}{8} \varrho'' n k_1^2 \cos\beta_0 (3\Delta\varphi - \cos 4\varphi \sin 2\Delta\varphi). \qquad (3)$$

Setting for k_1 the approximate value $n \sin^2\beta_0$ and taking into consideration, that the maximum of $\sin^4\beta_0 \cos\beta_0$ equals nearly 0,3, we obtain 0,0013" as the greatest value of (3) for a line encircling the Earth. Assuming $\Delta\varphi$ to be small, however, formula (3) will change to

$$\frac{1}{4}\varrho'' n^3 \sin^4\beta_0 \cos\beta_0 \Delta\varphi\,(3-2\cos 4\varphi). \qquad (4)$$

The maximum value is nearly equal to:

$$0{,}4\varrho'' n^3 \Delta\varphi \quad \text{i.e.,} \quad 0{,}0004'' \Delta\varphi$$

and this results for $\Delta\varphi = 0{,}2$ in only 0,0001". The effect of the terms in small print in (11) is also smaller than in (7).

$L_{1,2}$ is also affected by the omissions in φ and $\Delta\varphi$, however, noticeably only in the term λ_2, and in such a degree as a small error in the distance carries over to the geographical longitude as a result of spherical relationships.

It is possible to modify the spherical computation for moderate values of distances by determining φ_2 and λ_2 directly by reduction of the oblique triangle $\mathfrak{N}\mathfrak{P}_1\mathfrak{P}_2$, instead of seeking them by reduction of the right triangle $\mathfrak{N}\mathfrak{P}_0\mathfrak{P}_2$ and deriving from it $\Delta\varphi$ and $\Delta\lambda$. This way we save a decimal of the logarithms. The necessary formulae may be obtained from (9), p. 126, in setting β or φ instead of B and σ. To place them there seems unnecessary, because we are going to develop further special formulae for small s values. The above-mentioned formulae yield geographical latitude and longitude when we use eight-place logariths with an accuracy of 0,002". Using seven-place logarithms and adding the eighth place from the proportional parts, an

accuracy of hundredths of a second may be achieved.

We may add here another formula for $\Delta\lambda = \lambda_2 - \lambda_1$, which usually can be used for the checking of small s values.

The oblique triangle $\mathfrak{N}\mathfrak{P}_1\mathfrak{P}_2$, according to the sine law, is expressed as follows:

$$\sin\Delta\lambda = \sin\Delta\varphi\frac{\sin\alpha_{1.2}}{\cos\beta_2} = -\sin\Delta\varphi\frac{\sin\alpha_{2.1}}{\cos\beta_1}, \qquad (5)$$

This formula becomes useless only for locations of s close to the Pole.

For ease of computation, the coefficients $\mathfrak{A}$, $\mathfrak{A}'$ etc., which appear as functions of k_1 may be arranged in tables. In that case, log tan E has to be taken as an argument. Such tables would be valid not only for Bessel's dimensions of the Earth ellipsoid,

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but for the values of the same, when log a and log e^2 are separated from the logarithms of $\mathfrak{A}'$, $\mathfrak{B}'$ and $\mathfrak{C}'$. For the small term ak_1, present in log $\mathfrak{A}'$, a constant value of a would be sufficient, because its maximum value is only 0,0004" $\Delta\varphi$ ($\Delta\varphi$ taken as an arc).

However, also without tables, the computation is comparatively easy.

Very convenient tables are found in Albrecht, p. 207 etc. They give the coefficients of the formulae established by Bessel in 1825, (see Engelmann, treatises by Bessel, Volume 3, p. 6 etc), partly in a narrower interval of the argument, than the original tables by Bessel. These tables are valid for any given oblateness.

Par. 10. Different Formulae.

Bessel's formulae differ from the above mentioned mainly in two respects. First, in that $\Delta\varphi$ is derived by successive approximation from formula (9), p. 223, and second, in the way the coefficients of the series for $L_{1,2}$ are formed. For the latter we have indicated a less artificial derivation.

Dealing with great distances and with great accuracy in computation, we would prefer the procedures of our own formulae, instead of Bessel's, because his tables are not sufficient. For moderate distances and an accuracy of 0,001", the application of his tables for computation is very convenient. The indirect determination of $\Delta\varphi$ is here not only more convenient than the direct one by means of Δs (because already the second approximation leads to the objective), but has the advantage of revealing some logarithms (namely for sin $\Delta\varphi$ etc.) for the computation of $L_{1,2}$.

Hansen, in 1865, in his Geodetical Investigations (Proceedings of the mathematical-physical group of the royal society of science in Saxony, Leipzig, Volume 8), dealt in great detail with this problem; yet for great given distances an indirect computation of $\Delta\varphi$ is required, and his choice for the development of $L_{1,2}$ is perhaps not very fortunate.

In the first instance, it is important not to start out with a formula for $P_1P_2 = s$, but one for P_0P_1, i. e. to proceed from the distance beginning at the vertex of a geodetic line. In recognizing this fact, we have been able to obtain a direct formula for $\Delta\varphi$.

Our formulae agree with those of Winterberg, which are contained in Volume 89 of Astronomical News No. 2119 and 2120,

in regard to the introduction of s in the expression for $L_{1,2}$. However, Winterberg is omitting all terms which correspond to those in small print. The author bases his procedure on the development of series by Jacobi with the aid of elliptical functions (Crelle's Journal, Volume 53; Astronomical News, Volume 41, No. 974 and 1006).

Apart from the interesting solution of the problem as furnished by Jacobi, with aid of theta functions in closed form, it is unnecessary (as our evolutions show) to go beyond the elements of the analysis in order to obtain the results in a series form. The final results contain only series progressing according to multiples of φ and σ. Those functions, may be determined by elementary methods. The only

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reversion to elliptical functions is found in Winterberg in a modulus q appearing in the coefficients for which, apart from the terms of power q^5 and higher, the following relation exists:

$$q = \frac{1}{2}\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} = \frac{1}{4}\left(\frac{k^2}{4}+\frac{k^4}{8}+\frac{21k^6}{256}+\cdots\right).$$

The comparison with p. 221 (1) shows that $4q$ and k_1 deviate from each other only by $\frac{21}{256}k^6$, i. e. by a value which is insignificant. Setting in our formulae:

$$k_1 = 4q - 16q^3 + \cdots,$$

we will not notice any magnification of the convergence of the coefficient series, whereas by neglecting q^3 the formulae become

identical with those of Winterberg (apart from insignificant modifications which otherwise may be desirable). Compare our detailed remarks in Astronomical News, Volume 94, No. 2252, p. 314, 1879.

Hansen also introduced q in his development of series. Bessel used k (and called it ε).

We mention also the treatment of this problem by Baeyer (see The Measurement on the Spheroid Earth Surface, Berlin 1862, p. 52 ff., especially the third paper, printed as manuscript, on the Scientific Foundation of Computing Methods of the Central Office of European Measurements of Degrees, with tables). Baeyer's expressions for s and $L_{1,2}$ represent the difference of two series each, which are progressing for s, in accordance with the powers of $\sin^2\beta_1$ respectively $\sin^2\beta_2$, and for $L_{1,2}$ according to the powers of $\cos^2\beta_1$ or $\cos^2\beta_2$. The tables do not give the coefficients in logarithmic form, and are only applicable to Bessel's eccentricity of the Earth ellipsoid. A direct computation of the coefficients (or their logarithms) may be less convenient than the other solutions mentioned. Regardless of the magnitude of s, the solution is valid the same way as shown by other authors.

A paper, Die Kuerzeste auf dem Erdsphaeroid (The Shortest Line on the Earth Spheroid) by Bachoven von Echt, 1865, is interesting in many respects. The computation in this case is carried out directly with the geographical latitude, instead of the reduced one. We do not see any advantage in this, because the convergence of the series is decreasing, as has been shown

for the linear length in the special case of the arc of the meridian, and neither does it simplify the computation. The author studies mainly the course of the geodetic line, with which we shall deal in one of the following chapters.

Legendre probably was the first to use the geodetic line for the transfer of geographical coordinates. In 1806, he published in a paper in the Memoirs of the French Academy, formulae of general validity as well as series for short distances (according to Trepied, also see Chapter 6). He still used in 1799, p. 14 ff. of the work by Delambre, Méthodes analytique pour la Détermination d'un arc du Méridien, (Analytical Methods for the Determination of an Arc of Meridian), using for the latter purpose spherical distances. The introduction of the auxiliary quantity φ is generally attributed to Legendre with reference to his Théorie des Fonctions Elliptiques,(Theory of Elliptic Functions), 1825.

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According to Soldner, Die Bayerische Landesvermessung (Bavarian Land Surveying, 1873) p. 533, it was suggested by Euler (Mémoirs de l'Academie de Berlin, 1753) and du Séjour to consider terrestrial distances at shortest lines, and they performed accordingly the transfer of geographical coordinates.

According to Todhunter, History of Attraction etc, Volume 1; pp. 83 and 118, Clairaut, in the Memoirs of the Paris Academy for 1733 (published in 1735), treated the geodetic line on the rotation surfaces, based on its characteristic as the shortest

line. He found that for each point from the rotation axis times sin α (for the ellipsoid, cos β sin α, p. 215) the distance is constant, and remarks that only on a sphere, the line will be a plane curve. The reason for this investigation was Cassini's approach, (cf. those Memoirs, 1734), to refer the points of a triangle, by means of a perpendicular, to the principal meridian. Clairaut proved that strictly speaking, these perpendiculars are the shortest lines.

Par. 11. Numerical Example IV. We select an example, which has been computed exactly by Baeyer, Rechnungsmethoden (Computing methods, third bulletin, p. 22), and has been discussed by Winterberg, Hansen and Albrecht.

Given:

$$B_1 = 55^0\,45' \quad \log s = 7{,}1495432.083 \quad \alpha_{1.2} = 83^0\,23'\,51{,}200''.$$

The formulae of Par. 9, p. 232 et seq., yield successively*):

$$\log \tan B_1 = 0{,}1669321.238$$
$$9{,}9985458.202 - 10$$
$$\log \tan \beta_1 = 0{,}1654779.440$$
$$\log \sin \beta_1 = 9{,}9168283.498 - 10 \qquad \log \sin \alpha_{1.2} = 9{,}9971101.048 - 10$$
$$\log \cos \beta_1 = 9{,}7513504.058 - 10 \qquad \log \cos \alpha_{1.2} = 9{,}0606205.339 - 10$$

$$\left.\begin{aligned} \sin \beta_0 \cos \varphi_1 &= [9{,}9168283.498 - 10] \\ \sin \beta_0 \sin \varphi_1 &= [8{,}8119709.397 - 10] \\ \cos \beta_0 \qquad\quad &= [9{,}7484605.106 - 10] \end{aligned}\right\}$$

$$\left.\begin{aligned} \cos \beta_1 \cos \lambda_1 &= [9{,}7471248.284 - 10] \\ \cos \beta_1 \sin \lambda_1 &= [8{,}8938069.077 - 10] \end{aligned}\right\}$$

*)For ease of comparison with the computation according to different formulae, all figures resulting from the computation are retained, except those for interpolation of logarithms, and those which can be remembered.

Hence:

$$\log \tan \varphi_1 = 8{,}8951425.899 - 10 \quad \text{and} \quad \varphi_1 = 4^\circ\,29'\,28{,}76702''$$
$$\left.\begin{aligned}\log \sin \varphi_1 &= 8{,}8938069.077 - 10\\ \log \cos \varphi_1 &= 9{,}9986643.178 - 10\end{aligned}\right\} \log \sin \beta_0 = 9{,}9181640.320 - 10,$$
$$\left.\begin{aligned}\log \cos \lambda_1 &= 9{,}9957744.226 - 10\\ \log \sin \lambda_1 &= 9{,}1424565.019 - 10\end{aligned}\right\} \lambda_1 = 7^\circ\,58'\,46{,}75083''.$$

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log sin β_0 and log cos β_0, as well as log cos λ_1, and log sin λ_1 belong to the same angle within the accuracy of the tables.

For the computation of k_1 and $\Delta\sigma$ we obtain:

$$\log \sqrt{\delta} = 8{,}9136593.9 - 10$$
$$\log \sin \beta_0 = 9{,}9181640.3 - 10$$
$$\log \tan E = 8{,}8318234.2 - 10 \qquad E = 3^\circ\,53'\,2{,}429''$$
$$\log \tan \frac{E}{2} = 8{,}5302938.5 - 10 \qquad \frac{E}{2} = 1\ 56\ 31{,}2145$$
$$\log k_1 = 7{,}0605877.0 - 10;$$

$$\left.\begin{aligned}\log s &= 7{,}1495432.083\\ \log \frac{\varrho''}{b_0} &= 8{,}5112358.493 - 10\\ 2\log \cos \frac{E}{2} &= 9{,}9995009.748 - 10\end{aligned}\right.$$
$$\text{Sum} = 5{,}6602800.324$$

$$\left.\begin{aligned}\log\left(\frac{5}{4}M\right) &= 6{,}73469\\ 2\log k_1 &= 4{,}12118 - 10\end{aligned}\right.$$
$$\text{Sum} = 0{,}85587$$
$$\text{number} = 7.176$$

$$\log \Delta\sigma = 5{,}6602793.148 \quad \text{in sec.}$$
$$\Delta\sigma = 457382{,}25862'' = 127^\circ\,3'\,2{,}25862''.$$

The computation of $2\sigma_1$ produces:

$$\left.\begin{aligned}\log \varrho'' &= 5{,}31443\\ \log k_1 &= 7{,}06059 - 10\\ \log \sin 2\varphi_1 &= 9{,}19350 - 10\end{aligned}\right.$$
$$\text{Sum} = 1{,}56852$$

$$\left.\begin{aligned}\log\left(-\frac{1}{8}\varrho''\right) &= 4{,}411_n\\ 2\log k_1 &= 4{,}121 - 10\\ \log \sin 4\varphi_1 &= 9{,}489 - 10\end{aligned}\right.$$
$$\text{Sum} = 8{,}021_n - 10$$

$$2\varphi_1 = 8^\circ\,58'\,57{,}534''$$
$$[1{,}56852] = +\,37{,}027$$
$$[8{,}021_n - 10] = -\,0{,}010$$
$$2\sigma_1 = 8^\circ\,59'\,34{,}551''$$
$$2\sigma = 136\ 2\ 36{,}810.$$

Then follows the computation for $\Delta\varphi$ and φ_2:

$$\begin{array}{l|l} \log\left(\frac{2}{16}M\right) = 6{,}388 & \log(-\varrho'') = 5{,}3144251.3_n \\ 2\log k_1 = 4{,}121 - 10 & \log k_1 = 7{,}0605877.0 - 10 \\ \hline \text{Sum} = 0{,}509 & -[0.509] = \quad -3.2 \\ & \log \mathfrak{A} = 2{,}3750125.1_n \end{array}$$

$$\begin{array}{l|l} \log\left(\frac{5}{8}\varrho''\right) = 5{,}110 & \log\left(-\frac{29}{48}\varrho''\right) = 5{,}096_n \\ 2\log k_1 = 4{,}121 - 10 & 3\log k_1 = 1{,}182 - 10 \\ \hline \log \mathfrak{B} = 9{,}231 - 10 & \log \mathfrak{C} = 6{,}278_n - 10 \end{array}$$

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$$\begin{array}{l|l} \log \mathfrak{A} = 2{,}3750125.1_n & \log \mathfrak{B} = 9{,}231 - 10 \\ \log\cos 2\sigma = 9{,}8572526.9_n - 10 & \log\cos 4\sigma = 8{,}561 - 10 \\ \log\sin \Delta\sigma = 9{,}9020591.7 - 10 & \log\sin 2\Delta\sigma = 9{,}983_n - 10 \\ \hline \text{Sum} = 2{,}1343243.7 & \text{Sum} = 7{,}775_n - 10 \end{array}$$

$$\begin{array}{l} \log \mathfrak{C} = 6{,}278_n - 10 \\ \log\cos 6\sigma = 9{,}825 - 10 \\ \log\sin 3\Delta\sigma = 9{,}557 - 10 \\ \hline \text{Sum} = 5{,}660_n - 10 \end{array}$$

$$\begin{array}{rl} \Delta\sigma = & 127^\circ\,3'\,2{,}25862'' \\ [2{,}1343243.7] = + & 2\,16{,}24620 \\ [7{,}775_n - 10] = - & 596 \\ [5{,}660_n - 10] = - & 5 \\ \hline \Delta\varphi = & 127^\circ\,5'\,18{,}49881'' \\ = & 457518{,}49881''. \end{array}$$

To check the formulae numerically, $(\Delta\varphi - \Delta\sigma)$ has been computed according to formula (9), p. 223. The same value as before has been found:

$$2'\,16{,}24140'' - 0{,}00122'' + 0{,}00000'' = 2'\,16{,}24018''.$$

It follows that:

$$\varphi_2 = 131^\circ\,34'\,47{,}26583''$$

and

$$\log \sin \varphi_2 = 9{,}8739202.704 - 10, \quad \log \cos \varphi_2 = 9{,}8219472.649_n - 10.$$

Further:

$$\begin{aligned} \sin \beta_2 &= [9{,}7401112.969_n - 10] \\ \cos \beta_2 \cos \alpha_{2.1} &= [9{,}7920843.024_n - 10] \\ \cos \beta_2 \sin \alpha_{2.1} &= [9{,}7484605.106_n - 10] \\ \cos \beta_2 \cos \lambda_2 &= [9{,}5704077.755_n - 10] \\ \cos \beta_2 \sin \lambda_2 &= [9{,}8739202.704 - 10] \end{aligned}$$

Hence we obtain:

$$\begin{aligned} \log \tan \alpha_{2.1} &= 9{,}9563762.082 - 10 & \alpha_{2.1} &= 222^0\ 7'\ 37{,}98543'' \\ \log \tan \lambda_2 &= 0{,}3035124.949_n & \lambda_2 &= 116^0 26'\ 3{,}37617'' \\ \log \sin \alpha_{2.1} &= 9{,}8265794.683_n - 10 & \log \sin \lambda_2 &= 9{,}9520392.280 - 10 \\ \log \cos \alpha_{2.1} &= 9{,}8702032.601_n - 10 & \log \cos \lambda_2 &= 9{,}6485267.332_n - 10. \end{aligned}$$

and it follows from the above equations, that:

$$\log \cos \beta_2 = 9{,}9218810.423 - 10,$$

coinciding three times, and once with 4 instead of 3 in the 10th decimal place. The value for log cos β_2 belongs, within the accuracy of the tables, to the same β_2 as the above mentioned log sin β_2.

Further:

$$\begin{aligned} &\log \tan \beta_2 = 9{,}8182302.546_n - 10 \\ &-\log \sqrt{1-e^2} = 0{,}0014541.798 \\ \hline &\log \tan B_2 = 9{,}8196844.344_n - 10; \quad B_2 = -33^0\ 26'\ 0{,}00002''. \end{aligned}$$

For the computation of $L_{1.2}$ we use:

$$\Delta\lambda = \lambda_2 - \lambda_1 = 108^0\ 27'\ 16{,}62534''.$$

Then:

$$\left\{\begin{array}{r}\log\left\{\frac{1}{2}M\left(1-\frac{1}{2}\mathfrak{n}\right)\right\}=6{,}33603\\ \log k_1=7{,}06059-10\end{array}\right.\qquad \text{Sum}=3{,}39662$$

$$\left\{\begin{array}{r}\log\left(\frac{3}{8}M\right)=6{,}212\\ 2\log k_1=4{,}121-10\end{array}\right.\qquad \text{Sum}=0{,}333$$

$$\left\{\begin{array}{r}\log\mathfrak{n}=7{,}5241069.0-10\\ -[3{,}39662]=\quad -2492.4\\ -[0{,}333]=\quad -\quad 2.2\end{array}\right.$$

$$\log\mathfrak{A}'=7{,}5238574.4-10$$

$$\left\{\begin{array}{r}\log\left(-\frac{1}{4}\varrho''e^2\right)=2{,}53678_n\\ \log k_1=7{,}06059-10\end{array}\right.\qquad \log\mathfrak{B}'=9{,}59737_n-10$$

$$\left\{\begin{array}{r}\log\left(\frac{1}{16}\varrho''e^2\right)=1{,}935\\ 2\log k_1=4{,}121-10\end{array}\right.\qquad \log\mathfrak{C}'=6{,}056-10$$

$$\left\{\begin{array}{r}\log\cos\beta_0=9{,}7484605.1-10\\ \log\mathfrak{A}'=7{,}5238574.4-10\\ \log\Delta\varphi=5{,}6604086.6\end{array}\right.$$

$$\text{Sum}=2{,}9327266.1$$

$$\left\{\begin{array}{r}\log\cos\beta_0=9{,}74846-10\\ \log\mathfrak{B}'=9{,}59737_n-10\\ \log\cos 2\varphi=9{,}85745_n-10\\ \log\sin\Delta\varphi=9{,}90184-10\end{array}\right.\qquad \text{Sum}=9{,}10512-10$$

$$\left\{\begin{array}{r}\log\cos\beta_0=9{,}748-10\\ \log\mathfrak{C}'=6{,}056-10\\ \log\cos 4\varphi=8{,}572-10\\ \log\sin 2\Delta\varphi=9{,}983_n-10\end{array}\right.\qquad \text{Sum}=4{,}36_n-10$$

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Therefore we obtain for $L_{1.2}$:

$$\left\{\begin{array}{r}\lambda_2-\lambda_1=108^0\,27'\,16{,}62534''\\ -[2{,}9327266.1]=-\quad 14\;16{,}49848\\ -[9{,}10512-10]=-\quad\quad 0{,}12739\\ +[4{,}36-10]=\quad\quad\quad 0\end{array}\right.$$

$$L_{1.2}=108^0\,12'\,59{,}99947''.$$

The computation of $\Delta\lambda$ from the first equation (5), p. 237, results in the following:

$$\log \sin \Delta\lambda = \left\{ \begin{array}{l} 9{,}9018424.667 - 10 \\ 9{,}9971101.048 - 10 \\ 0{,}0781189.577 \end{array} \right\} = 9{,}9770715.292 - 10$$

with $\Delta\lambda = 108^\circ\, 27'\, 16{,}6254''$. For a readily understandable reason, the determination is not very precise, and is uncertain up to 0,0001". The value otherwise corresponds closely to the one found above. The

Results:
$$\left\{ \begin{array}{rl|l} \alpha_{2.1} = & 222^\circ\ 7'\ 37{,}98543'' & 37{,}986'' \\ B_2 = & -33^\circ 26'\ 0{,}00002'' & 0{,}000'' \\ L_{1.2} = & 108^\circ 12'\ 59{,}99947'' & 60{,}000'' \end{array} \right.$$

fully agree, as far as possible, with the indicated findings of Baeyer who used ten-place logarithms but in the results went only as far as thousandths of a second. Because of the magnitude of s, in the present case, Baeyer's as well as Bessel's tables are not sufficient to achieve an accuracy of hundred thousandths of a second. But the formulae of both permit achievement of any desired accuracy, although, as stated, the direct computation of the coefficients is more troublesome.

Par. 12. Numerical Example I. Given:

$$B_1 = 52^\circ\, 30'\, 16{,}7'' \qquad s = 529979{,}5784^m \qquad \alpha_{1.2} = 239^\circ\, 33'\, 0{,}68921''.$$

β_1 was already computed on p. 43, so there is no need to compute it again from the tangent:

$$\beta_1 = 52^\circ\, 24'\, 43{,}01137''.$$

$$\log \sin \beta_1 = 9{,}8989537.053 - 10 \qquad \log \sin \alpha_{1.2} = 9{,}9355442.631_n - 10$$
$$\log \cos \beta_1 = 9{,}7853155.518 - 10 \qquad \log \cos \alpha_{1.2} = 9{,}7048223.603_n - 10.$$

It follows from the formulae on p. 232 et seq.:

$$\sin\beta_0\cos\varphi_1 = [9{,}8989537.053 - 10]$$
$$\sin\beta_0\sin\varphi_1 = [9{,}4901379.121_n - 10]$$
$$\cos\beta_0 = [9{,}7208598.149_n - 10].$$

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Hence:

$$\log\tan\varphi_1 = 9{,}5911842.068_n - 10 \qquad \varphi_1 = -21^\circ\,18'\,40{,}05148''$$
$$\log\sin\varphi_1 = 9{,}5604233.251_n - 10 \qquad \log\cos\varphi_1 = 9{,}9692391.183 - 10$$
$$\log\sin\beta_0 = 9{,}9297145.870 - 10.$$

And further:

$$\left.\begin{aligned}\cos\beta_1\cos\lambda_1 &= [9{,}6900989.332_n - 10]\\ \cos\beta_1\sin\lambda_1 &= [9{,}5604233.251_n - 10]\end{aligned}\right\}$$

and then:

$$\left.\begin{aligned}\log\cos\lambda_1 &= 9{,}9047833.814_n - 10\\ \log\sin\lambda_1 &= 9{,}7751077.733_n - 10\end{aligned}\right\} \lambda_1 = 216^\circ\,34'\,13{,}42302''.$$

Furthermore:

$$\left.\begin{aligned}\log\sqrt{\delta} &= 8{,}9136593.9 - 10\\ \log\sin\beta_0 &= 9{,}9297145.9 - 10\end{aligned}\right\}$$
$$\log\tan E = 8{,}8433739.8 - 10 \qquad E = 3^\circ\,59'\,18{,}097''$$
$$\log\tan\frac{E}{2} = 8{,}5418171 - 10 \qquad \frac{E}{2} = 1^\circ\,59'\,39{,}0485''$$
$$\log k_1 = 7{,}0836342 - 10$$

$$\left.\begin{aligned}\log s &= 5{,}7242591.353\\ \log\frac{\varrho''}{b_0} &= 8{,}5112358.493 - 10\\ 2\log\cos\frac{E}{2} &= 9{,}9994737.942 - 10\end{aligned}\right\} \text{Sum} = 4{,}2349687.788$$

$$\left.\begin{aligned}\log\left(\frac{5}{4}M\right) &= 6{,}73469\\ 2\log k_1 &= 4{,}16727 - 10\end{aligned}\right\} \text{Sum} = 0{,}90196$$

number = 7.979

$$\log\Delta\sigma = 4{,}2349679.809$$
in sec.
$$\Delta\sigma = 17177{,}81736'' = 4^\circ\,46'\,17{,}81736''.$$

The computation of $2\sigma_1$ produces:

$$\log \sin 2\varphi_1 = 9{,}83069_n - 10 \qquad \log \sin 4\varphi_1 = 9{,}999_n - 10$$

$$\begin{aligned} 2\varphi_1 &= -\ 42^\circ\ 37'\ 20{,}10'' \\ \varrho'' k_1 \sin 2\varphi_1 &= -\ \ 2\ 49{,}34 \\ -\tfrac{1}{8}\varrho'' k_1^2 \sin 4\varphi_1 &= +\ \ 0{,}04 \\ \hline 2\sigma_1 &= -\ 42^\circ\ 40'\ \ 9{,}40'' \\ 2\sigma &= -\ 37\ 53\ 51{,}58. \end{aligned}$$

The computation of $\Delta\varphi$ yields:

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$$\log \sin \Delta\sigma = 8{,}9200407 - 10 \qquad \log \sin 2\Delta\sigma = 9{,}21959 - 10$$
$$\log \cos 2\sigma = 9{,}8971371 - 10 \qquad \log \cos 4\sigma = 9{,}38985 - 10$$
$$\log \sin 3\Delta\sigma = 9{,}393 - 10 \qquad \log \cos 6\sigma = 9{,}604_n - 10$$

$$\log \mathfrak{A} = 2{,}3980589_n \qquad \log \mathfrak{B} = 9{,}27758 - 10 \qquad \log \mathfrak{C} = 6{,}35_n - 10$$

$$\begin{aligned} \Delta\sigma &= \quad 4^\circ\ 46'\ 17{,}81736'' \\ \mathfrak{A} \cos 2\sigma \sin \Delta\sigma &= -\ \ 16{,}41484 \\ \mathfrak{B} \cos 4\sigma \sin 2\Delta\sigma &= +\ \ 0{,}00771 \\ \mathfrak{C} \cos 6\sigma \sin 3\Delta\sigma &= +\ \ 2 \\ \hline 17161{,}41025'' = \Delta\varphi &= \quad 4^\circ\ 46'\ \ 1{,}41025'' \end{aligned}$$

$$\varphi = -\ 18^\circ\ 55'\ 39''; \quad \begin{aligned} \varphi_1 &= -\ 21\ 18\ 40{,}05148 \\ \varphi_2 &= -\ 16^\circ\ 32'\ 38{,}64123'' \end{aligned}$$

$$\log \sin \varphi_2 = 9{,}4544678.545_n - 10 \qquad \log \cos \varphi_2 = 9{,}9816378.827 - 10.$$

Thus we obtain further:

$$\begin{aligned} \sin \beta_2 &= [9{,}9113524.697 - 10] \\ \cos \beta_2 \cos \alpha_{2.1} &= [9{,}3841824.415 - 10] \\ \cos \beta_2 \sin \alpha_{2.1} &= [9{,}7208598.149 - 10] \\ \cos \beta_2 \cos \lambda_2 &= [9{,}7024976.976_n - 10] \\ \cos \beta_2 \sin \lambda_2 &= [9{,}4544678.545_n - 10]. \end{aligned}$$

Therefore:

$\log \tan \alpha_{2.1} = 0{,}3366773.734$	$\alpha_{2.1} = 65^\circ\ 16'\ 9{,}36534''$
$\log \tan \lambda_2 = 9{,}7519701.569 - 10$	$\lambda_2 = 209^\circ\ 27'\ 43{,}28944''$
$\log \sin \alpha_{2.1} = 9{,}9582216.230 - 10$	$\log \sin \lambda_2 = 9{,}6918296.627_n - 10$
$\log \cos \alpha_{2.1} = 9{,}6215442.496 - 10$	$\log \cos \lambda_2 = 9{,}9398595.058_n - 10$,

and thence:

$\log \cos \beta_2 = 9{,}7626381.919 - 10$ or - 918 every 2 times
$\log \tan \beta_2 = 0{,}1487142.779$
$\log \tan B_2 = 0{,}1501684.577$; $B_2 = 54^\circ\ 42'\ 50{,}60000''$.

Further $\Delta\lambda = \lambda_2 - \lambda_1 = -\ 7^\circ\ 6'\ 30{,}13358''$. Using the

sine law, we can compute this from $\Delta\varphi$ for verification (cf. p. 237 (5)), and obtain with $\log \sin \Delta\varphi = 8{,}9196266.655 - 10$ from $\alpha_{1.2}$ and $\alpha_{2.1}$:

$\log \sin \Delta\lambda = 9{,}0925327.367_n - 10$; $\Delta\lambda = -\ 7^\circ\ 6'\ 30{,}13350''$.

This value will be retained as the more exact one. Computing $L_{1.2}$, we proceed as follows:

$\log \Delta\varphi = 4{,}2345530$	$\log \mathfrak{A}' = 7{,}5238439 - 10$	
$\log \cos 2\varphi = 9{,}8974 - 10$	$\log \mathfrak{B}' = 9{,}6204_n - 10$	
$\log \sin \Delta\varphi = 8{,}9196 - 10$	$\log \mathfrak{C}' = 6{,}10 - 10$	
	$\Delta\lambda = -\ 7^\circ\ 6'\ 30{,}13350''$	The calculation of the 3rd and 4th term with σ produces the same value.
$-\cos\beta_0 . \mathfrak{A}' \Delta\varphi =$	$+\ 30{,}14788$	
$-\cos\beta_0 . \mathfrak{B}' \cos 2\varphi \sin \Delta\varphi =$	$-\ 0{,}01440$	
$-\cos\beta_0 . \mathfrak{C}' \cos 4\varphi \sin 2\Delta\varphi =$	0	
	$L_{1.2} = -\ 7^\circ\ 6'\ 0{,}00002''$.	

Hence we obtain

Results:

$B_2 = 54^\circ\ 42'\ 50{,}60000''$
$L_{1.2} = 7^\circ\ 6'\ 0{,}00002''$ easterly
$\alpha_{2.1} = 65^\circ\ 16'\ 9{,}36534''$.

Chapter 5. Basic Formulae for the Geodetic Line.

Par. 13. Determination of the Geodetic Line from the Geographic Position of Two Points. So far, no direct solution has been found for this problem, with the exception of the case of a short distance between the two points (which will be treated in a later chapter).

But the problem can be solved indirectly by determining $\alpha_{1.2}$, $\alpha_{2.1}$ and $\Delta\varphi$ from the oblique spherical triangle $\mathfrak{N}\mathfrak{P}_1\mathfrak{P}_2$, fig. 21, p. 248, using sides $\frac{\pi}{2}-\beta_1$, $\frac{\pi}{2}-\beta_2$, and the interior angle $\Delta\lambda=\lambda_2-\lambda_1$, and setting for it $L_{1.2}$ in the first approximation. Tentative values for $\varphi_1, \varphi_2, \beta_0, k_1$ can be found, and the equation for $L_{1.2}$ and $\Delta\lambda$ will indicate the difference between the two.

This way, we obtain a closer value for $\Delta\lambda$ and proceed with a new reduction of the oblique triangle, etc., until the difference $L_{1.2}-\Delta\lambda$ remains constant.

We apply the Gauss equations for the reduction of the oblique triangle $\mathfrak{N}\mathfrak{P}_1\mathfrak{P}_2$ (cf. p. 131):

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$$\left.\begin{aligned}
\sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= -\sin\frac{\beta_2-\beta_1}{2}\cos\frac{\Delta\lambda}{2}\\
\cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= -\cos\frac{\beta_2+\beta_1}{2}\sin\frac{\Delta\lambda}{2}\\
\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= +\cos\frac{\beta_2-\beta_1}{2}\cos\frac{\Delta\lambda}{2}\\
\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= +\sin\frac{\beta_2+\beta_1}{2}\sin\frac{\Delta\lambda}{2}
\end{aligned}\right\} \qquad (1)$$

proceeding with the reduction, we restrict ourselves to the assumption $\Delta\varphi<\pi$.

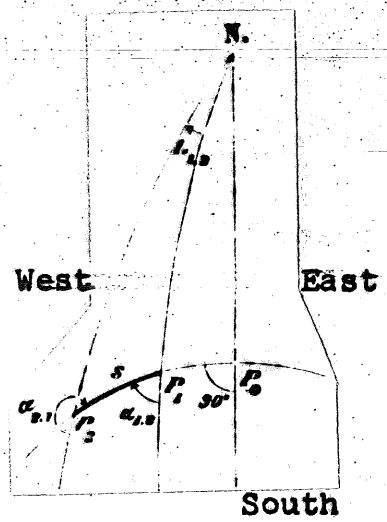


Figure 20. Ellipsoid

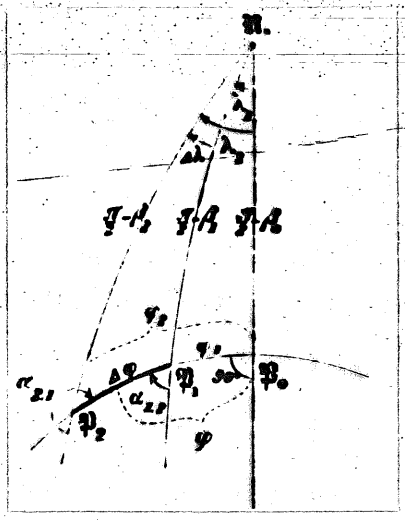


Figure 21. Sphere

For the computation of β_1 and β_2 from B_1 and B_2, the following formula may be applied:

$$\tan\beta = \sqrt{1-e^2}\tan B;\quad \log\sqrt{1-e^2} = 9{,}9985458.202 - 10; \tag{2}$$

and the following formulae may be used for the determination of $\Delta\lambda$ from $L_{1,2}$:

$$\left.\begin{aligned} \Delta\lambda &= L_{1,2} + \cos\beta_0\{\mathfrak{A}'\Delta\varphi + \mathfrak{B}'\cos 2\varphi\sin\Delta\varphi + \mathfrak{C}'\cos 4\varphi\sin 2\Delta\varphi + \cdots\}\\ &\quad\text{in sec.}\qquad\text{in sec.}\qquad\qquad\text{in sec.}\\ \log\mathfrak{A}' &= \log\mathfrak{a} - \frac{1}{2}M\left(1-\frac{1}{2}\mathfrak{a}\right)k_1 - \frac{3}{8}Mk_1^2 + \cdots\\ &= 7{,}5241069.0 - 10 - [6{,}33603]\,k_1 - [6{,}212]\,k_1^2 + \cdots\\ &\qquad\qquad\text{for units of the 7th decimal}\\ \log\mathfrak{B}' &= \log\left(-\frac{1}{4}\varrho''e^2k_1 + \cdots\right) = 2{,}53678_n + \log k_1 + \cdots\\ \log\mathfrak{C}' &= \log\left(\frac{1}{16}\varrho''e^2k_1^2 + \cdots\right) = 1{,}935 + 2\log k_1 + \cdots; \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} \sin\beta_0\cos\varphi_1 &= \sin\beta_1\\ \sin\beta_0\sin\varphi_1 &= \cos\beta_1\cos\alpha_{1.2}\\ \cos\beta_0 &= \cos\beta_1\sin\alpha_{1.2} \end{aligned}\right| \text{ or also } \left|\begin{aligned} \sin\beta_0\cos\varphi_2 &= +\sin\beta_2\\ \sin\beta_0\sin\varphi_2 &= -\cos\beta_2\cos\alpha_{2.1}\\ \cos\beta_0 &= -\cos\beta_2\sin\alpha_{2.1} \end{aligned}\right\} \tag{4}$$

$$\left.\begin{array}{c} 2\varphi = 2\varphi_1 + \Delta\varphi = 2\varphi_2 - \Delta\varphi \\ \tan E = \sqrt{\delta}\sin\beta_0; \qquad \log\sqrt{\delta} = 8{,}9136593.9-10 \\ \log k_1 = 2\log\tan\frac{E}{2}. \end{array}\right\} \qquad (5)$$

In the first computation of $\Delta\lambda$ the terms with k_1 may be omitted. The process of computation is elucidated in the following examples.

First, let us emphasize that for small distances a slightly different procedure is preferrable which not merely substitutes $L_{1.2}$ for $\Delta\lambda$ in the first approximation in (1), but uses a value (as stated by Albrecht, p. 80), which will be obtained by applying the two following formulae:

$$\left.\begin{array}{l} \underset{\text{in sec.}}{\Delta\lambda_1} = \underset{\text{in sec.}}{L_{1.2}} + \varrho''\mathfrak{a}\cos\beta_1\cos\beta_2\sin\underset{\text{in sec.}}{L_{1.2}} \\ \underset{\text{in sec.}}{\Delta\lambda} = \underset{\text{in sec.}}{L_{1.2}} + \varrho''\mathfrak{a}\cos\beta_1\cos\beta_2\sin\Delta\lambda_1 + \cdots \end{array}\right\} \qquad (6)$$

We obtain these formulae by omitting in the first equation (3) all terms with k_1, i. e. setting $\mathfrak{A}=\mathfrak{a}$, and writing sin $\Delta\varphi$ for $\Delta\varphi$. The precise result (cf. the third equation (4) above and p. 237 (5)) is:

$$\cos\beta_0\sin\Delta\varphi = \cos\beta_1\sin\alpha_{1.2}\sin\Delta\varphi = \cos\beta_2\cos\beta_1\sin\Delta\lambda.$$

Further we will see that equation (6) delivers the value of $\Delta\lambda$ for small distances with a precision of thousandths of a second, and if this accuracy is considered sufficient, the computation may be considered as a direct one. (cf. p. 264 (3)).

After determining an unchangeable value for $\Delta\lambda$ by the method of approximation, we may derive s from $\Delta\varphi$, by applying the following formulae (cf. p. 223 (9) and p. 233 (4) and (5)):

$$s = b_0 \frac{1+\frac{1}{4}k_1^2}{1-k_1}\left\{\begin{array}{l}\frac{1}{\varrho''}\Delta\varphi + \left(k_1 - \frac{3}{8}k_1^3\right)\cos 2\varphi \sin \Delta\varphi \\ \qquad -\frac{1}{8}k_1^2\cos 4\varphi \sin 2\Delta\varphi \\ \qquad +\frac{1}{24}k_1^3\cos 6\varphi \sin 3\Delta\varphi + \cdots\end{array}\right\} \text{ in sec.}$$

$$\log\frac{b_0}{\varrho''} = 1{,}4887641.507 \qquad \log\left(-\frac{k_0}{8}\right) = 5{,}90010_n$$

$$\log b_0 = 6{,}8031892.8 \qquad \log\frac{k_0}{24} = 5{,}423$$

$$\log\frac{1+\frac{1}{4}k_1^2}{1-k_1} = 2\log\sec\frac{E}{2} + \frac{5}{4}Mk_1^2 + \cdots$$

$$\log\left(\frac{5}{4}M\right) = 6{,}73469 \text{ und } \log\left(-\frac{3}{8}M\right) = 6{,}212_n \text{ for units of the 7th decimal}$$

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The formulae of paragraphs 9 and 13 of this chapter allow the solution of all problems related to the positions of two points. (cf. the detailed discussion by Winterberg, Astronomische Nachrichten (Astronomical News, Volume 95, No. 2271 et seq., p. 223 et seq., and Hansen, Geodaetische Untersuchungen (Geodetic Research). Applying the formulae in the mentioned paragraphs, we are able to solve the most important problems for geodetic triangles of any size, especially the determination from two sides and the interior angle from a given geographic position and azimuths at the vertex of the triangle.

Par. 14. Numerical Example IV. Given:

$$B_1 = 55^\circ 45', \quad B_2 = -33^\circ 26', \quad L_{1,2} = 108^\circ 13'.$$

We carry out the computation with seven-place logarithms (tables by Bruhns), and add the eighth decimal place from proportional parts.

According to formula (2) p. 248:

$$\begin{array}{ll}
\left\{\begin{array}{l}\log\sqrt{1-e^2}=9{,}9985458.2-10\\ \log\tan B_1=0{,}1669321\end{array}\right. & \left\{\begin{array}{l}\log\sqrt{1-e^2}=9{,}9985458.2-10\\ \log\tan B_2=9{,}8196844_n\end{array}\right.\\
\log\tan\beta_1=0{,}1654779.2-10 & \log\tan\beta_2=9{,}8182302.2_n-10\\
\beta_1=55^\circ 39' 38{,}500'' & \beta_2=-33^\circ 20' 42{,}642''\\
\log\sin\beta_1=9{,}9168283.4-10 & \log\sin\beta_2=9{,}7401112.6_n-10\\
\log\cos\beta_1=9{,}7513504.2-10 & \log\cos\beta_2=9{,}9218810.5-10.
\end{array}$$

Hence the formulae (1) p. 248 produce:

$$\left.\begin{array}{l}
\sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2}=\cos\frac{\Delta\lambda}{2}[9{,}8456844.2-10]=[9{,}6138\ -10]\\
\cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2}=\sin\frac{\Delta\lambda}{2}[9{,}9917124.5_n-10]=[9{,}9003_n-10]\\
\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2}=\cos\frac{\Delta\lambda}{2}[9{,}8532202.1-10]=[9{,}6213-10]\\
\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2}=\sin\frac{\Delta\lambda}{2}[9{,}2867058.0-10]=[9{,}1953-10].
\end{array}\right\}$$

The extreme right parts refer to the assumption that

$$\Delta\lambda=108^\circ 13'.$$

It follows that:

$$\Delta\varphi=126^\circ 57',\quad \Delta\varphi=457020'',\quad \log\Delta\varphi=5{,}6599;$$
$$\alpha_{1.2}=83^\circ 13',\quad \alpha_{2.1}=222^\circ 7'$$

and according to the third equations (4), p. 248:

$$\log\cos\beta_0=9{,}7484-10.$$

Thus we obtain the second approximation from formula (3), p. 248:

$$\Delta\lambda=L_{1.2}+[9{,}7484-10+7{,}5241-10+5{,}6599]=108^\circ 27' 16''.$$

With this value for $\Delta\lambda$ we obtain from the above system of equations:

$$\left.\begin{aligned}
\sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [9{,}612522-10]\\
\cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [9{,}900916.-10]\\
\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}620058-10]\\
\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}195910-10];
\end{aligned}\right\}$$

$\Delta\varphi = 127^\circ 5' 18{,}1''$, $\alpha_{1.2} = 83^\circ 23' 51{,}0''$, $\alpha_{2.1} = 222^\circ 7' 37{,}8''$.

The first system (4) p. 248 yields:

$$\left.\begin{aligned}
\sin\beta_0\cos\varphi_1 &= [9{,}916828-10]\\
\sin\beta_0\sin\varphi_1 &= [8{,}811974-10]\\
\cos\beta_0 &= [9{,}748460-10],
\end{aligned}\right\}$$

$\varphi_1 = 4^\circ 29' 28{,}9''$, $\log\sin\beta_0 = 9{,}918164-10$;

both values of β_0, from $\cos\beta_0$ and $\sin\beta_0$, agree with sufficient accuracy.

We now compute $\Delta\lambda$ and $L_{1.2}$ in the third approximation, according to formulae (5), p. 249, and (3) p. 248:

$2\varphi_1 = 8^\circ 58' 57{,}8''$, $2\varphi = 136^\circ 4' 16''$, $\Delta\varphi = 457518{,}1''$;

$$\left\{\begin{aligned}
\log\sqrt{\delta} &= 8{,}913659-10\\
\log\sin\beta_0 &= 9{,}918164-10
\end{aligned}\right.$$

$\log\tan E = 8{,}831823-10$ $\qquad E = 3^\circ 53'\ 2{,}4''$

$\log\tan\frac{E}{2} = 8{,}530293-10$ $\qquad \frac{E}{2} = 1^\circ 56' 31{,}2''$

$\log k_1 = 7{,}060586-10$ $\qquad \log\mathfrak{G} = 9{,}597.\ -10$

$\log\mathfrak{A}' = 7{,}5241069-10-2492 = 7{,}523858-10$,

(2492 = units of the 7th decimal)

in $\log\mathfrak{A}'$, terms with k_1^2 are omitted. Further:

$$\left\{\begin{aligned}
\log\cos\beta_0 &= 9{,}748460-10\\
\log\mathfrak{A}' &= 7{,}523858-10\\
\log\Delta\varphi &= 5{,}660408
\end{aligned}\right.$$

Sum $= 2{,}932726$

$$\left\{\begin{aligned}
\log\cos\beta_0 &= 9{,}748-10\\
\log\mathfrak{G} &= 9{,}597.-10\\
\log\cos 2\varphi &= 9{,}857.-10\\
\log\sin\Delta\varphi &= 9{,}902-10
\end{aligned}\right.$$

Sum $= 9{,}104-10$

$$\begin{aligned} L_{1.2} &= 108^\circ 13' \ 0'' \\ [2{,}932726] &= + \ 14'16{,}497'' \\ [9{,}104-10] &= + \ 0{,}127'' \\ \hline \Delta\lambda &= 108^\circ 27' 16{,}624''. \end{aligned}$$

Hence we obtain from the system of equations mentioned above:

$$\left.\begin{aligned} \sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [9{,}6125217.6-10] \\ \cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [9{,}9009166.8-10] \\ \sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}6200575.5-10] \\ \cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}1959100.3-10]; \end{aligned}\right\}$$

$$\Delta\varphi = 127^\circ 5' 18{,}47'', \quad \alpha_{1.2} = 83^\circ 23' 51{,}18'', \quad \alpha_{2.1} = 222^\circ 7' 38{,}00''.$$

The deviation from the values previously obtained is so small, that a new computation would change the value of $\Delta\lambda$ only in the thousandths of a second. We therefore accept the obtained results and determine s.

For this purpose, the formulae (7), p. 249 yield:

$$\Delta\varphi = 457518{,}47''$$

and

$$2\varphi = 136^\circ 4' 16{,}3'',$$

and successively:

$$\begin{aligned} \log \Delta\varphi &= 5{,}6604086.5 \\ \log \frac{b_0}{\varrho''} &= 1{,}4887641.5 \\ \hline \text{Sum} &= 7{,}1491728.0 \end{aligned}$$

$$\begin{cases} \log b_0 = 6{,}803189 \\ \log k_1 = 7{,}060586 - 10 \\ \log\cos 2\varphi = 9{,}857455_n - 10 \\ \log\sin \Delta\varphi = 9{,}901842 - 10 \end{cases}$$

$$\text{Sum} = 3{,}623072_n$$

$$\begin{cases} \log\left(-\frac{b_2}{8}\right) = 5{,}900_n \\ 2\log k_1 = 4{,}121 - 10 \\ \log\cos 4\varphi = 8{,}572 - 10 \\ \log\sin 2\Delta\varphi = 9{,}983_n - 10 \end{cases}$$

$$\text{Sum} = 8{,}576 - 10$$

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$$\begin{cases} \log\left(\frac{5}{4}M\right) = 6{,}735 \\ 2\log k_1 = 4{,}121 - 10 \end{cases}$$

$$\log\left(\frac{5}{4}Mk_1^2\right) = 0{,}856$$

$$\begin{cases} 2\log\sec\frac{E}{2} = 0{,}0004990.4 \\ [0{,}856] = + 7.2 \end{cases}$$

$$\log\frac{1+\frac{1}{4}k_1^2}{1-k_1} = 0{,}0004997.6.$$

This will be added to the previously-found three sums of logarithms, and we get:

$$s = 14114729{,}3 - 4203{,}2 + 0{,}0 = 14110526{,}1.$$

The results:

$\log s = 7{,}1495432.0$	...432.1	
$\alpha_{1.2} = 83^\circ 23' 51{,}18''$	51,20	
$\alpha_{2.1} = 222\ 7\ 38{,}00$	37,99	

agree with the data on the right part of pp. 240 and 241 with a sufficient degree of accuracy.

Par. 15. Numerical Example V. Given:

$$B_1 = 51^\circ 12', \quad B_2 = 51^\circ 55', \quad L_{1.2} = 69^\circ 3'.$$

We are computing again with seven-digit logarithms (tables by Bruhns), as in the previous example.

Formula (2), p. 248, yields first:

$\log\sqrt{1-e^2} = 9{,}9985458.2-10$	$\log\sqrt{1-e^2} = 9{,}9985458.2-10$
$\log\tan B_1 = 0{,}0947328$	$\log\tan B_2 = 0{,}1058886$
$\log\tan\beta_1 = 0{,}0932786.2$	$\log\tan\beta_2 = 0{,}1044344.2$
$\beta_1 = 51^\circ 6' 22{,}603''$	$\beta_2 = 51^\circ 49' 24{,}545''$
$\log\sin\beta_1 = 9{,}8911537.2-10$	$\log\sin\beta_2 = 9{,}8954835.0-10$
$\log\cos\beta_1 = 9{,}7978751.1-10$	$\log\cos\beta_2 = 9{,}7910490.7-10.$

Equations (1), p. 248, lead to:

$$\sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} = \cos\frac{\Delta\lambda}{2}[7{,}7964889_n-10] = [7{,}71235_n-10]$$

$$\cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} = \sin\frac{\Delta\lambda}{2}[9{,}7944840_n-10] = [9{,}54789_n-10]$$

$$\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} = \cos\frac{\Delta\lambda}{2}[9{,}9999915.0-10] = [9{,}91585-10]$$

$$\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} = \sin\frac{\Delta\lambda}{2}[9{,}8933325.1-10] = [9{,}64674-10].$$

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The extreme right parts refer to $\Delta\lambda = 69^\circ 3'$. It follows that:

$$\Delta\varphi = 41^\circ 21' 30'' = 148890''; \quad \log = 5{,}17287;$$
$$\alpha_{2.1} = 242^\circ 33' 4'', \quad \alpha_{1.2} = 119^\circ 7' 20''.$$

The third equations (4), p. 248, deliver coincidently:

$$\log\cos\beta_0 = 9{,}73918 - 10.$$

Hence we obtain the second approximation from formula (3), p. 248:

$$\Delta\lambda = L_{1.2} + [9{,}73918 - 10 + 7{,}52411 - 10 + 5{,}17287] = 69^\circ 7' 33{,}00''.$$

With this value, the above system of equations produces:

Par. 15. Numerical Example V.

$$\left.\begin{aligned}
\sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [7{,}7121545_n - 10]\\
\cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [9{,}5483051_n - 10]\\
\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}9156570.7 - 10]\\
\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}6471536.0 - 10];
\end{aligned}\right\}$$

$$\Delta\varphi = 41^\circ 23' 57{,}2'', \quad \alpha_{2.1} = 242^\circ 30' 57{,}26'', \quad \alpha_{1.2} = 119^\circ 9' 18{,}26''.$$

The first system of (4), p. 248 yields:

$$\left.\begin{aligned}
\sin\beta_0\cos\varphi_1 &= [9{,}891154 - 10]\\
\sin\beta_0\sin\varphi_1 &= [9{,}485560_n - 10]\\
\cos\beta_0 &= [9{,}739040 - 10];
\end{aligned}\right\}$$

$$\varphi_1 = -21^\circ 27' 19{,}5'', \qquad \log\sin\beta_0 = 9{,}922344.$$

and the values of β_0 from $\cos\beta_0$ and $\sin\beta_0$ agree with sufficient accuracy.

We now compute $\Delta\lambda$ and $L_{1.2}$ in a third approximation, applying formulae (5), p. 249, and (3), p. 248:

$$2\varphi_1 = -42^\circ 54' 39{,}0'', \quad 2\varphi = -1^\circ 30' 41{,}8'', \quad \Delta\varphi = 149037{,}2'';$$

$$\log\tan E = 8{,}836003 - 10 \qquad E = 3^\circ 55' 17{,}2''$$

$$\log\tan\frac{E}{2} = 8{,}534464 - 10 \qquad \frac{E}{2} = 1^\circ 57' 38{,}6''$$

$$\log k_1 = 7{,}068928 - 10 \qquad \log\mathfrak{C} = 9{,}606_n - 10$$

$$\log\mathfrak{A}' = 7{,}5241069 - 10 - 2541 \text{ units of the 7th decimal} = 7{,}523853 - 10,$$

In $\log\mathfrak{A}'$, terms with k_1^2 are omitted. Further:

$$\begin{cases} \log\cos\beta_0 = 9{,}739040-10 \\ \log \mathfrak{A}' = 7{,}523853-10 \\ \log \Delta\varphi = 5{,}173295 \end{cases} \qquad \text{Sum} = 2{,}436188$$

$$\begin{cases} \log\cos\beta_0 = 9{,}739 -10 \\ \log \mathfrak{S}' = 9{,}606_n-10 \\ \log\cos 2\varphi = 0{,}000 \\ \log\sin\Delta\varphi = 9{,}820 -10 \end{cases} \qquad \text{Sum} = 9{,}165_n-10$$

$$\begin{cases} L_{1.2} = 69^\circ\ 3'\ 0'' \\ [2{,}436188] = +\ 4'\ 33{,}016'' \\ [9{,}165_n-10] = -\ \ 0{,}146'' \end{cases}$$

$$\Delta\lambda = 69^\circ\ 7'\ 32{,}870''.$$

The change in the second approximation is so small that in the third approximation $\Delta\varphi$, $\alpha_{2.1}$ and $\alpha_{1.2}$ will not change by more than a tenth of a second, and for a seven-digit computation, the following values can be considered as definitive.

It follows that:

$$\frac{\Delta\lambda}{2} = 34^\circ\ 33'\ 46{,}435''; \begin{cases} \log\sin\frac{\Delta\lambda}{2} = 9{,}7538208.9-10 \\ \log\cos\frac{\Delta\lambda}{2} = 9{,}9156656.7-10; \end{cases}$$

$$\left.\begin{aligned} \sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [7{,}7121546_n-10] \\ \cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [9{,}5483049_n-10] \\ \sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}9156571.7-10] \\ \cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}6471534.0-10]; \end{aligned}\right\}$$

$$\Delta\varphi = 41^\circ\ 23'\ 57{,}34'',\quad \alpha_{2.1} = 242^\circ\ 30'\ 57{,}32'',\quad \alpha_{1.2} = 119^\circ\ 9'\ 18{,}20''.$$

For the computation of s the formulae (7), p. 249 come forth with

$$\Delta\varphi = 149037{,}34'' \quad \text{and} \quad 2\varphi = -\ 1^\circ\ 30'\ 41{,}8''$$

and thence:

$$\begin{cases} \log \Delta\varphi = 5{,}17329506 \\ \log \frac{b_0}{\varrho''} = 1{,}48876415 \end{cases}$$
$$\text{Sum} = 6{,}6620592.1$$

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$$\begin{cases} \log b_0 = 6{,}80319 \\ \log k_1 = 7{,}06893-10 \\ \log \cos 2\varphi = 9{,}99985-10 \\ \log \sin \Delta\varphi = 9{,}82040-10 \end{cases} \qquad \text{Sum} = 3{,}69237$$

$$\begin{cases} \log\left(-\frac{b_0}{8}\right) = 5{,}90_n \\ 2\log k_1 = 4{,}14 - 10 \\ \log \cos 4\varphi = 0{,}00 \\ \log \sin 2\Delta\varphi = 0{,}00 \end{cases} \qquad \text{Sum} = 0{,}04_n$$

$$\begin{cases} \log\left(\frac{5}{4} M\right) = 6{,}735 \\ 2\log k_1 = 4{,}138-10 \end{cases} \qquad \log\left(\frac{5}{4} M k_1^2\right) = 0{,}873-10$$

$$\begin{cases} 2\log \sec \frac{E}{2} = 0{,}0005087.2 \\ [0{,}873] = + 7.5 \end{cases} \qquad \log \frac{1+\frac{1}{4}k_1^2}{1-k_1} = 0{,}0005094.7$$

Adding this to the previously-found three sums of logarithms, we receive:

$$s = 4597996{,}6 + 4930{,}4 - 1{,}1 = 4602925{,}9^m.$$

Result: $\left\{ \begin{array}{l} \log s = 6{,}6630340 \\ \alpha_{1.2} = 119^\circ\ 9'\ 18{,}20'' \\ \alpha_{2.1} = 242\ 30\ 57{,}32 \end{array} \right\}$.

Hansen, treating the same example in his Geodetic Research, obtained (loc. cit., pp. 47 and 88) for $\alpha_{2.1}$ 57,30" and 57,27"; $\alpha_{1.2}$ and log s (reduced to meters) agree with our results.

Par. 16. Numerical Example I. Given:

$$\left.\begin{array}{ll} \text{Königsberg} & B_1 = 54^\circ\ 42'\ 50{,}6'' \\ \text{Berlin} & B_2 = 52\ 30\ 16{,}7 \end{array}\right\} L_{1.2} = + 7^\circ\ 6'.$$

Chapter 5. Basic Formulae for the Geodetic Line.

According to previous computations:

$$\beta_1 = 54^\circ\,37'\,24{,}75639''\ (S.171) \qquad \beta_2 = 52^\circ\,24'\,43{,}01137''\ (S.43).$$

Using seven-place logarithms (tables by Bruhns), and retaining the eighth decimal-place from proportional parts, we obtain from equations (1), p. 248:

$$\left.\begin{aligned}
\sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [8{,}2855261.9 - 10]\cos\frac{\Delta\lambda}{2} = [8{,}28470 - 10]\\
\cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [9{,}7742057.3_n - 10]\sin\frac{\Delta\lambda}{2} = [8{,}56604_n - 10]\\
\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}9999191.0 - 10]\cos\frac{\Delta\lambda}{2} = [9.99909 - 10]\\
\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}9052782.2 - 10]\sin\frac{\Delta\lambda}{2} = [8{,}69711 - 10]
\end{aligned}\right\}.$$

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The extreme right parts refer to $\Delta\lambda = 7^\circ\,6'\,0''$. It follows that:

$$\alpha_{1.2} = 65^\circ\,14'\,17'' \qquad \alpha_{2.1} = 239^\circ\,31'\,33''$$
$$\Delta\varphi = 4^\circ\,45'\,46'' = 17146'' \qquad \log\Delta\varphi = 4{,}23416.$$

The third equations (4), p. 248 agree sufficiently:

$$\log\cos\beta_0 = 9{,}72075 - 10.$$

It follows in the second approximation from formula (3), p. 248 that:

$$\Delta\lambda = L_{1.2} + [9{,}72075 - 10 + 7{,}52411 - 10 + 4{,}23416]$$
$$\Delta\lambda = 7^\circ\,6'\,30{,}131'' \qquad \frac{\Delta\lambda}{2} = 3^\circ\,33'\,15{,}066''$$
$$\log\sin\frac{\Delta\lambda}{2} = 8{,}7923388.3 - 10 \qquad \log\cos\frac{\Delta\lambda}{2} = 9{,}9991638.9 - 10.$$

In this case we obtain this value more rapidly from the formulae (6), p. 249, which yield:

Par. 16. Numerical Example I.

$$\begin{aligned}
\log(\varrho'' a) &= 2{,}83853\\
\log\cos\beta_1 &= 9{,}78532-10\\
\log\cos\beta_2 &= 9{,}76264-10\\
\log\sin 7^\circ 6' &= 9{,}09202-10\\
\text{Sum} &= 1{,}47851\\
\Delta\lambda &= 7^\circ 6' 30{,}10''.
\end{aligned}$$

With $\log\sin 7^\circ 6' 30{,}10'' = 9{,}09253-10$ instead of the value shown on the fourth line, a total of 1,47902 results and hence:

$$\Delta\lambda = 7^\circ 6' 30{,}131''.$$

We obtain, with the second approximate value for $\Delta\lambda$ (which may be determined in different ways):

$$\begin{aligned}
\sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [8{,}2846900.8-10]\\
\cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [8{,}5665445.6_8-10]\\
\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}9990829.9-10]\\
\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [8{,}6976170.5-10]
\end{aligned}$$

and hence:

$$\Delta\varphi = 4^\circ 46' 1{,}410'' \quad \alpha_{2.1} = 239^\circ 33' 0{,}67'' \quad \alpha_{1.2} = 65^\circ 16' 9{,}34''.$$

The first system of equations (4), p. 248, leads to:

$$\begin{aligned}
\sin\beta_0\cos\varphi_1 &= [9{,}9113525-10]\\
\sin\beta_0\sin\varphi_1 &= [9{,}3841825-10]\\
\cos\beta_0 &= [9{,}7208598-10]
\end{aligned}$$

$$\varphi_1 = 16^\circ 32' 38{,}64'' \quad \log\sin\beta_0 = 9{,}9297146-10,$$

with the latter value corresponding to the same angle, together with $\cos \beta_0$.

Using seven-place logarithms, instead of six-place (the latter will be sufficient here), we obtain in third approximation for formulae (5) and (3), p. 249 and 248:

$$2\varphi_1 = 33^\circ\,5'\,17{,}28'' \qquad 2\varphi = 37^\circ\,51'\,18{,}69'' \qquad \Delta\varphi = 17161{,}410''$$

$$\log \tan E = 8{,}8433740-10 \qquad E = 3^\circ\,59'\,18{,}097''$$

$$\log \tan \frac{E}{2} = 8{,}5418171-10 \qquad \frac{E}{2} = 1\;\;59\;\;39{,}048$$

$$\log k_1 = 7{,}0836342-10 \qquad \log \mathfrak{B}' = 9{,}6204_n-10$$

$$\log \mathfrak{A}' = 7{,}5241069-10 - (2628+2)^{\text{units of the 7th decimal}} = 7{,}5238439-10$$

$\log \cos \beta_0$	$= 9{,}7208598-10$	$\log \cos \beta_0$	$= 9{,}7209\;\;-10$
$\log \mathfrak{A}'$	$= 7{,}5238439-10$	$\log \mathfrak{B}'$	$= 9{,}6204_n-10$
$\log \Delta\varphi$	$= 4{,}2345530$	$\log \cos 2\varphi$	$= 9{,}8974\;\;-10$
		$\log \sin \Delta\varphi$	$= 8{,}9196\;\;-10$
Sum	$= 1{,}4792567$	Sum	$= 8{,}1583_n-10$

$$\begin{aligned} L_{1.2} &= 7^\circ\,6'\;\;0{,}00000'' \\ [1{,}4792567] &= +\;\;30{,}14788'' \\ [8{,}1583_n-10] &= -\;\;\;0{,}01440'' \end{aligned}$$

$$\Delta\lambda = 7^\circ\,6'\,30{,}13348''.$$

This value deviates from the one obtained in the second approximation only by two units in the third decimal-place of seconds. Therefore we can retain the results for $\alpha_{1.2}$, $\alpha_{2.1}$ and $\Delta\varphi$ for the computation with seven-place logarithms. The latter computation of $\Delta\lambda$ could have been omitted, had it been known in advance that in the second approximation is accurate up to 0,002". The following paragraph will give some explanation in this regard.

For computation of s we obtain from formulae (7), p. 249, with

$$\Delta\varphi = 17161{,}410'' \text{ and } 2\varphi = 37^\circ\,51'\,18{,}69''$$

thence:

$$\left.\begin{aligned}\log \Delta\varphi &= 4{,}2345529.7\\ \log \frac{b_0}{\varrho''} &= 1{,}4887641.5\end{aligned}\right\}$$

$$\text{Sum} = 5{,}7233171.2$$

$$\left.\begin{aligned}\log b_0 &= 6{,}80319\\ \log k_1 &= 7{,}08363-10\\ \log\cos 2\varphi &= 9{,}89739-10\\ \log\sin \Delta\varphi &= 8{,}91963-10\end{aligned}\right\}$$

$$\text{Sum} = 2{,}70384$$

$$\left.\begin{aligned}\log\left(-\frac{b_0}{8}\right) &= 5{,}90.\\ 2\log k_1 &= 4{,}17-10\\ \log\cos 4\varphi &= 9{,}39-10\\ \log\sin 2\Delta\varphi &= 9{,}22-10\end{aligned}\right\}$$

$$\text{Sum} = 8{,}68.-10$$

$$\left.\begin{aligned}\log\left(\frac{5}{4}M\right) &= 6{,}735\\ 2\log k_1 &= 4{,}167-10\end{aligned}\right\}$$

$$\log\left(\frac{5}{4}Mk_1^2\right) = 0{,}902$$

$$\left.\begin{aligned}2\log\sec\frac{E}{2} &= 0{,}0005262.0\\ [0{,}902] &= +8.0\end{aligned}\right\}$$

$$\log\frac{1+\frac{1}{4}k_1^2}{1-k_1} = 0{,}0005270.0\,.$$

Adding this to the previously found three sums of logarithms, we get:

$$s = 529473{,}32 + 506{,}26 - 0{,}04 = 529979{,}54^m\,.$$

The

$$\text{Results:}\quad \left\{\begin{aligned}s &= 529979{,}54^m\\ \alpha_{1.2} &= 65^\circ\,16'\,9{,}34''\\ \alpha_{2.1} &= 239^\circ\,33'\,0{,}67''\end{aligned}\right\}$$

agree satisfactorily with the computation on pp. 244 et seq.

In order to achieve an even greater accuracy, we now shall use ten-place logarithms.

Using for $\Delta\lambda$: the value found above:

$$\frac{\Delta\lambda}{2} = 3^\circ\,33'\,15{,}06674''\quad \left\{\begin{aligned}\log\sin\frac{\Delta\lambda}{2} &= 8{,}7923388.593-10\\ \log\cos\frac{\Delta\lambda}{2} &= 9{,}9991638.814-10\end{aligned}\right.$$

and setting above:

$$\beta_1 = 54^\circ\,37'\,24{,}75639'' \qquad \beta_2 = 52^\circ\,24'\,43{,}01137''\,,$$

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we obtain according to equations (1), p. 248:

$$\left.\begin{aligned}
\sin\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [8{,}2846900.557 - 10]\\
\cos\frac{\alpha_{2.1}+\alpha_{1.2}}{2}\sin\frac{\Delta\varphi}{2} &= [8{,}5665446.196_n - 10]\\
\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [9{,}9990829.928 - 10]\\
\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}\cos\frac{\Delta\varphi}{2} &= [8{,}6976170.810 - 10]
\end{aligned}\right\}$$

$$\alpha_{2.1} = 239^\circ\,33'\,0{,}68891'' \qquad \alpha_{1.2} = 65^\circ\,16'\,9{,}36499''$$

$$\left.\begin{aligned}
\log\sin\frac{\Delta\varphi}{2} &= 8{,}6189725.722 - 10\\
\log\cos\frac{\Delta\varphi}{2} &= 9{,}9996240.975 - 10
\end{aligned}\right\} \quad \frac{\Delta\varphi}{2} = 2^\circ\,23'\,0{,}70512''$$

$$\Delta\varphi = 4^\circ\,46'\,1{,}41024'' = 17161{,}41024''\,.$$

For E and k_1, as well as 2φ, the values obtained from the previous computation can be used.

The computation of s is:

$$\frac{3}{8}\,Mk_1^2 = 2 \text{ units of the 7th decimal}$$

$$\left.\begin{aligned}
\log\Delta\varphi &= 4{,}2345529.731\\
\log\frac{b_0}{\varrho''} &= 1{,}4887641.507
\end{aligned}\right.$$

Sum $= 5{,}7233171.238$

$$\left.\begin{aligned}
\log b_0 &= 6{,}8031893\\
\log\left(k_1 - \frac{3}{8}k_1^3\right) &= 7{,}0836340 - 10\\
\log\cos 2\varphi &= 9{,}8973875 - 10\\
\log\sin 2\Delta\varphi &= 8{,}9196266 - 10
\end{aligned}\right.$$

Sum $= 2{,}7038374$

$$\left.\begin{aligned}
\log\left(-\frac{b_0}{8}\right) &= 5{,}9001_n\\
2\log k_1 &= 4{,}1673 - 10\\
\log\cos 4\varphi &= 9{,}3924 - 10\\
\log\sin 2\Delta\varphi &= 9{,}2191 - 10
\end{aligned}\right.$$

Sum $= 8{,}6789_n - 10$

Par. 16. Numerical Example I.

$$\begin{cases} \log\left(\frac{5}{4}M\right) = 6{,}73469 \\ 2\log k_1 \quad = 4{,}16727-10 \end{cases} \qquad \begin{cases} 2\log\sec\frac{E}{2} = \quad 0{,}0005262.058 \\ [0{,}90196] \quad = + \qquad 7.979 \end{cases}$$

$$\log\left(\frac{5}{4}Mk_1^2\right) = 0{,}90196 \qquad \log\frac{1+\frac{1}{4}k_1^2}{1-k_1} = \quad 0{,}0005270.037$$

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$$\begin{cases} [5{,}7238441.275] \quad = \quad 529473{,}3769 \\ [2{,}7043644] \quad = + \quad 506{,}2493 \\ [8{,}6794_n-10] \quad = - \quad 0{,}0478 \end{cases}$$

The term with $3\Delta\varphi$ = 0

$$s = 529979{,}5784$$

Result:
$$\begin{cases} s = 529979{,}5784 \\ \log s = 5{,}7242591.353 \\ \alpha_{1.2} = 65^\circ\,16'\,9{,}36499'' \\ \alpha_{2.1} = 239^\circ\,33'\,0{,}68891''. \end{cases}$$

These figures are not in complete agreement with the computation on pp. 244 et seq., because $\alpha_{1.2}$ had been assumed 0,0003" too big. Correspondingly, $\alpha_{2.1}$ also is too big by 0,0003". Further, because of the greater azimuth, B_2 should be decreased by 0,00002" and the longitude difference increased by 0,00002". The latter exists indeed, according to p. 247.

Par. 17. Convergence of Approximation Methods in Solving the Problem of Par. 13. We consider first the method of exclusive application of Gauss equations. This method sets for a first approximation $L_{1.2}$ for $\Delta\lambda$, and then in a second approximation $\Delta\lambda = L_{1.2} + a\cos\beta_0'\,\Delta\varphi'$, where $\cos\beta_0'\,\Delta\varphi'$ denotes the result of the first approximation for $\cos\beta_0\Delta\varphi$. Subtracting the second approximate value from the precise value:

$$L_{1.2} + \mathfrak{a} \cos \beta_0 \left\{ \Delta\varphi \left(1 - \frac{1}{2} k_1\right) - \frac{1}{2} k_1 \cos 2\varphi \sin \Delta\varphi + \cdots \right\},$$

where k_1^2 in the parenthesis has been neglected, we obtain for the error of the second approximate value as a correction:

$$\mathfrak{a} \left\{ (\cos\beta_0 \Delta\varphi - \cos\beta_0' \Delta\varphi') - \frac{1}{2} k_1 \cos\beta_0 (\Delta\varphi + \sin\Delta\varphi \cos 2\varphi) + \cdots \right\}. \quad (1)$$

Limiting ourselves to short distances, we may set in the first part of (1) for $\Delta\varphi$ and $\Delta\varphi'$ the sine of these angles, but in the second part for sin $\Delta\varphi$ simply $\Delta\varphi$. Because (cf. the development for (6) p. 249):

$$\cos\beta_0 \sin\Delta\varphi - \cos\beta_0' \sin\Delta\varphi' = \cos\beta_1 \cos\beta_2 (\sin\Delta\lambda - \sin L_{1.2})$$

and further because $L_{1.2} = \Delta\lambda - \mathfrak{a} \cos\beta_0 \Delta\varphi + \cdots$ we may set for short distances, with sufficient approximation:

$$\sin\Delta\lambda - \sin L_{1.2} = \mathfrak{a} \cos\beta_0 \cos\Delta\lambda \, \Delta\varphi + \cdots$$

Considering the first of the equations (4), p. 248, and the relation

$$1 + \cos 2\varphi = 2\cos^2\varphi,$$

$$\frac{1}{2} k_1 \cos\beta_0 \Delta\varphi (1 + \cos 2\varphi) = \frac{1}{2} \mathfrak{a} \cos\beta_0 \sin\beta_1 \sin\beta_2 \Delta\varphi + \cdots$$

will be sufficiently accurate, and we may write (1), for short distances, in the following form:

$$\mathfrak{a}^2 \cos\beta_0 \Delta\varphi \left(\cos\beta_1 \cos\beta_2 \cos\Delta\lambda - \frac{1}{2} \sin\beta_1 \sin\beta_2\right) + \cdots. \quad (2)$$

Considering that $\cos\beta_1 \cos\beta_2 \cos\Delta\lambda + \sin\beta_1 \sin\beta_2 = \cos\Delta\varphi$, it follows in simpler form:

$$\left.\begin{array}{l} \mathfrak{a}^2 \cos\beta_0 \Delta\varphi \left(\cos\Delta\varphi - \frac{3}{2} \sin\beta_1 \sin\beta_2\right) + \cdots \\ \log \mathfrak{a}^2 = 5{,}048 - 10. \end{array}\right\} \quad (3)$$

In the evolution of this formula, terms with s^3, especially such with $s^3\Delta\varphi^3$, have been neglected, and it would be logical to set simply 1 for $\cos\Delta\varphi$. Retaining $\cos\Delta\varphi$, however, improves the accuracy of the results, particularly in higher latitudes, because the neglected terms of order $s^3\Delta\varphi^3$ are decreasing*) considerably.

The form of expression (3) shows that the convergence of the approximation method depends mainly on the magnitude of the reduced latitude. Setting $\cos\beta_0 = \cos\beta\sin\alpha + \cdots$, letting β and α denote the arithmetic means of β_1 and β_2, and of $\alpha_{1.2}$ and $\alpha_{2.1} - 180^0$, and assuming further that $\cos\Delta\varphi = 1$; then equation (3) will change to

$$s^2\,\Delta\varphi\sin\alpha\,.\cos\beta\left(1-\tfrac{3}{2}\sin^2\beta\right)+\cdots. \qquad (4)$$

The factor $\cos\beta\left(1-\frac{3}{2}\sin^2\beta\right)$ is equal to 1 for $\beta = 0$, for $\sin\beta = \pm\sqrt{\frac{2}{3}}$, i. e. for $\beta = \pm 55^0$ and also for $\beta = \pm 90^0$, it equals zero, and has maximum for $\beta = 0$, and for $\sin\beta = \pm\frac{2}{3}\sqrt{2}$, i. e. for $\beta = \pm 70^0$.

The latter smaller maximum amounts to $-\frac{1}{9}$. For $\beta = 48^0$ the factor is $+\frac{1}{9}$.

The convergence of the approximation method is therefore very sharp only for regions north of $\beta = 48^0$, and south of $\beta = -48^0$ up to the Poles, and the error for the second approximate value of $\Delta\lambda$ remains smaller than $\Delta\varphi$: 800000.

*)Taking $\Delta\varphi$ as an arc, it is possible to contract these terms in the following expression:

$$\frac{1}{12}\varrho''s^2\cos\beta_0\Delta\varphi^3\left\{\cos^2\beta\,(1-6\sin\alpha_{1.2}\sin\alpha_{2.1})-1\right\}.$$

The error in general increases up to $\Delta\varphi : 90000$, and can not be neglected, even for very short distances.

Expression (3) may be helpful in converting the second approximate value for $\Delta\lambda$ to a definite value.

Numerical Example V, p. 253, yields for expression (3):

$\cos \Delta\varphi = 0{,}750$	$\frac{s}{2} \sin \beta_1 \sin \beta_2 = 0{,}918$
$\log \Delta\varphi = 5{,}173$	$\log \cos \beta_0 = 9{,}739 - 10$

Correction of $(\Delta\lambda = 69^0\ 7'\ 33{,}00'')$ equals $-\ 0{,}15''$.
The exact value is $-\ 0{,}13''$; despite the magnitude of the distance, formula (3) is still sufficiently accurate.

Numerical Example I, p. 256, yields for expression (3):

$\cos \Delta\varphi = 0{,}997$	$\frac{s}{2} \sin \beta_1 \sin \beta_2 = 0{,}969$
$\log \Delta\varphi = 4{,}234$	$\log \cos \beta_0 = 9{,}721 - 10$

Correction of $(\Delta\lambda = 7^0\ 6'\ 30{,}131'')$ equals $+\ 0{,}0029''$.
Performing the computation on p. 256 and 257 in the first approximation with seven decimal-places instead of five decimal-places, we obtain as the second approximate value for $\Delta\lambda$ $7^0\ 6'\ 30{,}1307''$; adding 0,0029", produces the exact value from p. 258, with an accuracy of 0,0001".

Par. 18. Continuation: Method 2. If from the beginning of the approximation, we apply formulae (6) p. 249, the second approximate value for $\Delta\lambda$ with the exception of an insignificant error, corresponds to the equation:

$$\Delta\lambda = L_{1,2} + \mathfrak{k} \cos \beta_1 \cos \beta_2 \sin \Delta\lambda.$$

Comparing this with the exact value stated above, we obtain the following correction in the error of this approximate value:

$$\mathfrak{a}\cos\beta_0\left\{\Delta\varphi-\tfrac{1}{2}k_1(\Delta\varphi+\sin\Delta\varphi\cos 2\varphi)\right\}-\mathfrak{a}\cos\beta_1\cos\beta_2\sin\Delta\lambda\Big]+\cdots(1)$$

Setting herein: $\cos\beta_0\Delta\varphi=\cos\beta_0\sin\Delta\varphi+\frac{1}{6}\cos\beta_0\sin^3\Delta\varphi+\cdots$

$=\cos\beta_1\cos\beta_2\sin\Delta\lambda+\frac{1}{6}\cos\beta_0\Delta\varphi\sin^2\Delta\varphi+\cdots$

and substituting in the second part of the brace $k_1\sin\Delta\varphi$, for $k_1\Delta\varphi$; equation (1) will, after further reduction, change to:

$$\mathfrak{a}\cos\beta_0\Delta\varphi\left(\tfrac{1}{6}\sin^2\Delta\varphi-\tfrac{1}{2}\mathfrak{a}\sin\beta_1\sin\beta_2\right)+\cdots. \qquad (2)$$

A comparison of this expression with (3), p. 262, for $\Delta\varphi<\frac{1}{7}\cos\beta$ is more favorable to the second method. If $\Delta\varphi$ exceeds this amount, expression (2) increases much more rapidly than (3).

For long distances, the first method (application of Gauss' equations) is preferable to the second method.

Example V, p. 253, shows, in fact, that the second approximate value of $\Delta\lambda$, determined according to the second method, is erroneous by 23", i. e. equal to $69^0\,7'\,10'',0$, instead of 32",9.

For the sides of measurable triangles, where $\frac{1}{2}\mathfrak{a}\sin\beta_1\sin\beta_2$ has precedence over $\frac{1}{6}\sin^2\Delta\varphi$ rather than using (6), p. 249, a closer approximation may be achieved by determining $\Delta\lambda$ from the equation:

$$\underset{\text{in sec.}}{\Delta\lambda}=\underset{\text{in sec.}}{L_{1.2}}+\varrho''\mathfrak{a}\cos\beta_1\cos\beta_2\sin\Delta\lambda\left(1-\tfrac{1}{2}\mathfrak{a}\sin\beta_1\sin\beta_2\right) \qquad (3)$$

This value has to be introduced in formulae (1), p. 248.

Albrecht described his method in detail and determined the degree of approximation (cf. No. 2294 of Astronomische Nachrichten (Astronomical News,) Volume 96, p. 211, 1880).

Chapter 5. Basic Formulae for the Geodetic Line.

Par. 19. The Problem of Par. 13 in General. It was assumed in Par. 13, p. 248, that $\Delta\varphi < \pi$, and that the approximation method will definitely produce a solution. However, the former is a restriction, and the latter does not always happen.

It is easy to see that equations (1), p. 248, will be satisfied, not only by a value $\Delta\varphi < \pi$, which we denote $\Delta\varphi'$, but by an infinite number of other values, between π and 2π, 2π and 3π, etc. To each one, there corresponds a particular geodetic line. That the shortest connection belongs to $\Delta\varphi'$, is evident for moderate distances, which will also be proved to be generally valid in Chapter 7, where we will study the courses of geodetic lines.

At this point, we can also settle the case, where the methods of Par. 13 are more or less insufficient. This occurrs when considering nearly diametrical positions of points. If we set here $L_{1,2}$ for $\Delta\lambda$ as a first approximation, the error is close to $180^0 \cdot \cos\beta_0$, i. e. $0{,}6^0 \cos\beta_0$, and $\Delta\varphi$ is close to 180 .

Geometrically, it is easy to see that in nearly diametral positions of the auxiliary points $\mathfrak{P}_1$ and $\mathfrak{P}_2$ on the sphere ($\beta_1 + \beta_2$ being of the order $0{,}6^0 \cos\beta_0$), a small displacement of one of them in geographic longitude will strongly affect the position of the great circle connecting these points, i. e., the value of cos β_0 obtained from the approximation method. In case of extremes, as for instance $\beta_1 = \beta_2 =$ zero, we can no longer consider such a series of computations as successive approximations.

Par. 19. The Problem of Par. 13 in General.

The determination of a true first approximate value $\Delta\lambda$ has been shown elsewhere. Applying this value, we can proceed with the computation according to Par. 13.

Hansen solved the problem of Par. 13, p. 247 (as many others) by using the vertical section which can be extended from one of the two points to the other. However, this method does not strike us as particularly useful.

The solution is not a direct one, because with long distances the results obtained by reduction of data for vertical sections to geodetic lines, have to be corrected by means of differential formulae.

Besides, these formulae will be useless for nearly diametral points, as shown in (52), p. 60, on the assumption of z being close to 180°. (Hansen maintains, erroneously, the general validity of the formulae derived from a development of the series).

Finally, Hansen's formulae are less simple than the ones stated in Par. 13 which have the additional advantage of not differing from formulae used for the problem in Par. 9, p. 232.

For comparison, see examples IV and V, p. 250 and 253, with which Hansen deals on p. 87.

Chapter 6.

Differential Formulae and Series Expansions for the Geodetic Line.

Par. 1. Rotation of a Geodetic Line P_1P_2 about One of its End Points. We think of the line P_1P_2 of constant length s rotated around the point P_1. According to p. 220 (6) one obtains.

$$s = a_0 \int_{\varphi_1}^{\varphi_2} \sqrt{1 - e^2 \cos^2 \beta}\, d\varphi, \qquad (1)$$

where β is given as a function of φ with respect to the fixed point P_1 by the formulae:

$$\sin \beta = \sin \beta_0 \cos \varphi \qquad \cos \beta_0 = \cos \beta_1 \sin \alpha_{1.2}. \qquad (2)$$

As $\alpha_{1.2}$ varies by the differential quantity $\delta\alpha_{1.2}$, so also does β_0 as equations (2) indicate; and as a consequence of this, for a constant value of φ, β varies similarly, and even φ_1 does, because β_1 remains constant. These variations finally produce, in accordance with (1) a change of φ_2.

(See Illustrative Material on Next Page.)

Par. 1. Rotation of a Geodetic Line P_1P_2 about One of its End Points.

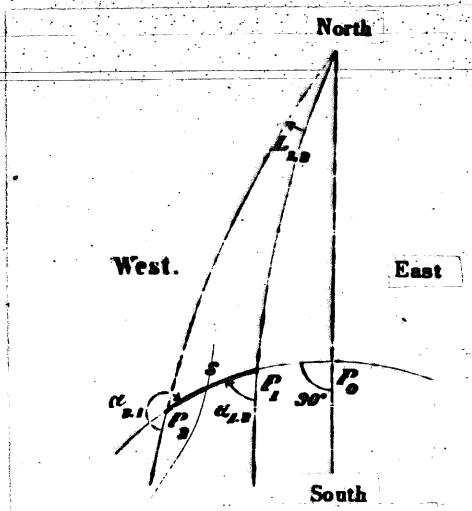


Figure 20 Ellipsoid

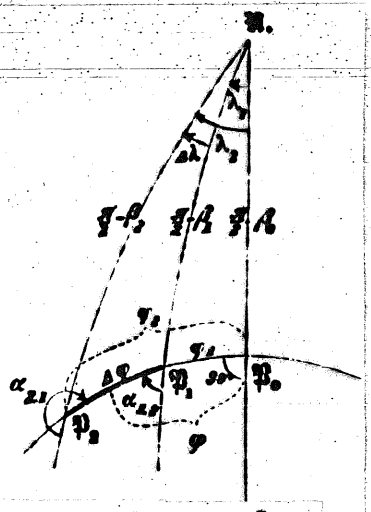


Figure 21 Circle

Denote variations by prefixing δ to the respective variables and observe that if $\delta s = \text{zero}$, (1) gives

$$0 = w_2 \delta\varphi_2 - w_1 \delta\varphi_1 + e^2 \int_{\varphi_1}^{\varphi_2} \frac{\cos\beta \sin\beta \, \delta\beta}{w} \, d\varphi, \qquad (3)$$

where, as earlier, w serves as an abbreviation for $\sqrt{1 - e^2 \cos^2\beta}$. $\delta\beta$ is to be understood as determined while φ is constant. Equations (2) give

$$\begin{aligned} \delta\beta &= \sec\beta \cos\beta_0 \cos\varphi \, \delta\beta_0 \\ \delta\beta_0 &= -\csc\beta_0 \cos\beta_1 \cos\alpha_{1.2} \, \delta\alpha_{1.2}. \end{aligned}$$

If one observes the relations on the sphere in Figure 21 (cf. p. 248 (4)), he finds from the preceding

$$\left.\begin{aligned} \delta\beta_0 &= -\sin\varphi_1 \delta\alpha_{1.2} \\ \delta\beta &= -\sec\beta \cos\beta_0 \cos\varphi \sin\varphi_1 \delta\alpha_{1.2}. \end{aligned}\right\} \qquad (4)$$

This we substitute in (3) and obtain, since $\beta = \sin\beta_0 \cos\varphi$:

$$0 = w_2 \delta\Delta\varphi + (w_2 - w_1)\delta\varphi_1 - e^2 \sin\beta_0 \cos\beta_0 \sin\varphi_1 J \delta\alpha_{1.2} \tag{5}$$

for $\delta\Delta\varphi = \delta\varphi_2 - \delta\varphi_1$

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and

$$J = \int_{\varphi_1}^{\varphi_2} \frac{\cos^2\varphi}{w} d\varphi. \tag{6}$$

If one expresses $\delta\varphi_1$ in (5) by $\delta\alpha_{1.2}$, then this equation provides a means for determining $\delta\Delta\varphi$, and one can, further, by means of the spherical triangle $\mathfrak{N}\,\mathfrak{P}_1\,\mathfrak{P}_2$ Figure 21, find the variations of β_2, $L_{1.2}$ and $\alpha_{2.1}$ which belong to $\delta\Delta\varphi$ and $\delta\alpha_{1.2}$. It then follows from $\sin\beta_1 = \sin\beta_0 \cos\varphi_1$ by differentiating and referring to the first equation of (4):

or

$$\sin\beta_0 \sin\varphi_1 \delta\varphi_1 = \cos\beta_0 \cos\varphi_1 \delta\beta_0,$$

$$\delta\varphi_1 = -\cot\beta_0 \cos\varphi_1 \delta\alpha_{1.2}.$$

Par. 1. Rotation of a Geodetic Line $P_1 P_2$ about One of its End Points.

With this the first equation of (5) gives:

$$\delta \Delta\varphi = \left\{(w_2 - w_1)\cot\beta_0 \cos\varphi_1 + e^2 \sin\beta_0 \cos\beta_0 \sin\varphi_1 J\right\} \frac{\delta\alpha_{1.2}}{w_2}. \quad (7)$$

For the determination of $\delta\beta_2$ and $\delta\Delta\lambda$, one immediately has from the spherical Figure 22, in which $\mathfrak{P}_2\mathfrak{P}_2' = \sin\Delta\varphi\,\alpha\,\delta\alpha_{1.2}$ and $\mathfrak{P}_2'\mathfrak{P}_3' = \delta\Delta\varphi$, through the projection of $\mathfrak{P}_3'$ on $\mathfrak{N}\mathfrak{P}_2$:

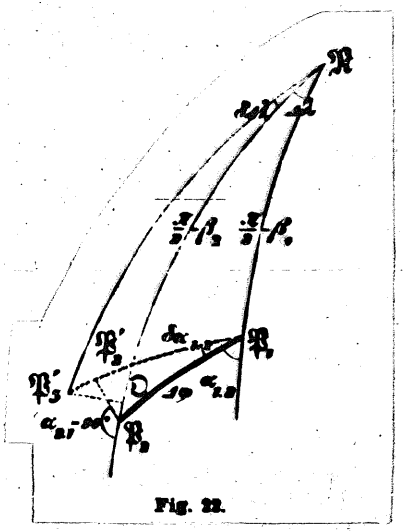


Fig. 22.

$$\begin{aligned}\delta\beta_2 &= \mathfrak{P}_2\mathfrak{Q} \\ &= \cos\alpha_{2.1}\delta\Delta\varphi - \sin\Delta\varphi \sin\alpha_{2.1}\delta\alpha_{1.2}\end{aligned} \quad (8)$$

and

$$\begin{aligned}\cos\beta_2\delta\Delta\lambda &= \mathfrak{P}_3'\mathfrak{Q} \\ &= -\sin\alpha_{2.1}\delta\Delta\varphi - \sin\Delta\varphi\cos\alpha_{2.1}\delta\alpha_{1.2}.\end{aligned} \quad (9)$$

We substitute (7) in (8) and (9), which gives

$$\delta\beta_2 = \left\{\begin{array}{c} w_2(\cot\beta_0\cos\varphi_1\cos\alpha_{2.1} - \sin\Delta\varphi\sin\alpha_{2.1}) \\ -w_1\cot\beta_0\cos\varphi_1\cos\alpha_{2.1} + e^2\sin\beta_0\cos\beta_0\sin\varphi_1\cos\alpha_{2.1}J \end{array}\right\} \frac{\delta\alpha_{1.2}}{w_2}$$

and

$$\delta\Delta\lambda = \left\{\begin{array}{c} -w_2(\cot\beta_0\cos\varphi_1\sin\alpha_{2.1} + \sin\Delta\varphi\cos\alpha_{2.1}) \\ +w_1\cot\beta_0\cos\varphi_1\sin\alpha_{2.1} - e^2\sin\beta_0\cos\beta_0\sin\varphi_1\sin\alpha_{2.1}J \end{array}\right\} \frac{\delta\alpha_{1.2}}{w_2\cos\beta_2}. \quad (10)$$

If one multiplies the formula for $\delta\beta_2$ by $\tan\beta_0 \sin\varphi_2 \tan\alpha_{2.1}$, i.e., since $\tan\beta_0 \sin\varphi_2 = \cot\alpha_{2.1}$ by about 1, it is converted to:

in which:

$$\delta\beta_2 = -\frac{\mathfrak{M}\sin\alpha_{2.1}}{a_0 w_2}\,\delta\alpha_{1.2}, \tag{11}$$

$$\mathfrak{M} = a_0(w_1\cos\varphi_1\sin\varphi_2 - w_2\sin\varphi_1\cos\varphi_2 - e^2\sin^2\beta_0\sin\varphi_1\sin\varphi_2 J). \tag{12}$$

For the determination of $\delta L_{1.2}$, consider the following equation derived from (2) and (3) of p. 229:

$$L_{1.2} = \int_{\lambda_1}^{\lambda_2}\sqrt{1 - e^2\cos^2\beta}\,d\lambda. \tag{13}$$

Here β is given as a function of λ with respect to fixed point P_1 by the relations:

$$\tan\beta = \tan\beta_0\cos\lambda \qquad \cos\beta_0 = \cos\beta_1\sin\alpha_{1.2}. \tag{14}$$

Differentiation of (13) leads to

$$\delta L_{1.2} = w_2\delta\lambda_2 - w_1\delta\lambda_1 + e^2\int_{\lambda_1}^{\lambda_2}\frac{\cos\beta\sin\beta\,\delta\beta}{w}\,d\lambda, \tag{15}$$

where $\delta\beta$ is determined while λ remains constant. It may be expressed by (14) and reference to the first equation of (4) as:

$$\delta\beta = \cos^2\beta\sec^2\beta_0\cos\lambda\,\delta\beta_0 = -\cos^2\beta\sec^2\beta_0\cos\lambda\sin\varphi_1\,\delta\alpha_{1.2}.$$

We substitute this in (15) and at the same time transform the integral by means of the relations:

$$d\lambda = \cos\beta_0\sec^2\beta\,d\varphi \qquad \cos\lambda = \tan\beta\cot\beta_0 \qquad \sin\beta = \sin\beta_0\cos\varphi,$$

so that it again is a function of the variable φ. With this we obtain:

$$\delta L_{1.2} = w_2 \delta \Delta\lambda + (w_2 - w_1)\,\delta\lambda_1 - e^2 \sin\beta_0 \sin\varphi_1 J \delta\alpha_{1.2}. \qquad (16)$$

For $\delta\lambda_1$ it follows eventually from $\tan\beta_1 = \tan\beta_0 \cos\lambda_1$ without difficulty:

$$\tan\beta_0 \sin\lambda_1 \delta\lambda_1 = \sec^2\beta_0 \cos\lambda_1 \delta\beta_0 = -\sec^2\beta_0 \cos\lambda_1 \sin\varphi_1 \delta\alpha_{1.2},$$
$$\delta\lambda_1 = -\cos\varphi_1 \csc\beta_0 \delta\alpha_{1.2}.$$

We substitute this, as with $\delta\Delta\lambda$ of (10) in (16) and after some reduction using the formulae:

$$\cot\beta_0 = \tan\alpha_{2.1} \sin\varphi_2 \quad \text{und} \quad \cos\beta_2 \cos\alpha_{2.1} = -\sin\beta_0 \sin\varphi_2$$

we finally obtain:

$$\delta L_{1.2} = -\frac{m}{a_0} \sec\beta_2 \cos\alpha_{2.1} \delta\alpha_{1.2}. \qquad (17)$$

Par. 2. The Reduced Length and the Geodetic Circle: If for the moment one calls the differential-arc u (that arc traced out by P_2 in the rotation of the line P_1P_2) and its azimuth χ, then

$$u\cos\chi = -a_0 w_2 \delta\beta_2 \text{ und } u\sin\chi = a_0 \cos\beta_2\, \delta L_{1.2}, \qquad (1)$$

because the element of the meridian arc is in general equal to $a_0 w \delta\beta$, according to (1) p. 55 and the radius of the parallel-circle for P_2 will equal $a_0\cos\beta_2$, by p. 40.

By substitution of (11) and (17) of the previous section, (1) gives:

$$\varkappa \cos \chi = + \mathfrak{m} \sin \alpha_{2.1} \delta \alpha_{1.2} = \mathfrak{m} \delta \alpha_{1.2} \cdot \cos (\alpha_{2.1} - 90^0)$$
$$\varkappa \sin \chi = - \mathfrak{m} \cos \alpha_{2.1} \delta \alpha_{1.2} = \mathfrak{m} \delta \alpha_{1.2} \cdot \sin (\alpha_{2.1} - 90^0),$$

from which one sees that

$$\left.\begin{aligned} \varkappa &= \mathfrak{m} \delta \alpha_{1.2} \\ \chi &= \alpha_{2.1} - 90^0. \end{aligned}\right\} \qquad (2)$$

Here we see that differential-arc $\mathfrak{m}\delta\alpha_{1.2}$ determined by P_2 is normal to the geodetic line P_1P_2.

The curve determined by P_2 is called a geodetic circle and $\mathfrak{m}$ is the reduced length of the geodetic line P_1P_2.

These terms were introduced by Christoffel on pp. 130 and 131 of his book: Allgemeine Theorie der geodätschen Dreiecke (General Theory of the Geodetic Triangle, published by the Royal Academy of Science in Berlin, 1868). The Theorem itself, however, which is the basis for the designation geodetic circle, (which can be used to refer to any curved surface) appeared first in Diquisitiones generales circa superficies curvas by Gauss in 1828.

On the basis of the preceding, a system of polar coordinates may be constructed by means of geodetic lines and circles. If A is the center of such a system, upon which a curve is based, and ds an arc element PP' of the curve, one has, as in the case of plane polar coordinates,

(See Illustrative Material on Next Page.)

Par. 3. Geometric Illustration of the Geodetic Circle Theorem.

$$ds^2 = dr^2 + \mathfrak{m}_r^2 da_0^2, \qquad (3)$$

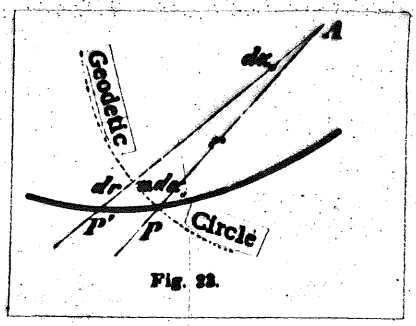


Fig. 23.

where r is the length of the geodetic radius vector AP, $\mathfrak{m}_r$ its reduced length and da_0 the angle between r and $r + dr$.

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Par. 3. Geometric Illustration of the Geodetic Circle Theorem. In Figure 24 P_1P_2 denotes a curve constructed according to an arbitrary law, and P_1P_2' a curve infinitely close, which

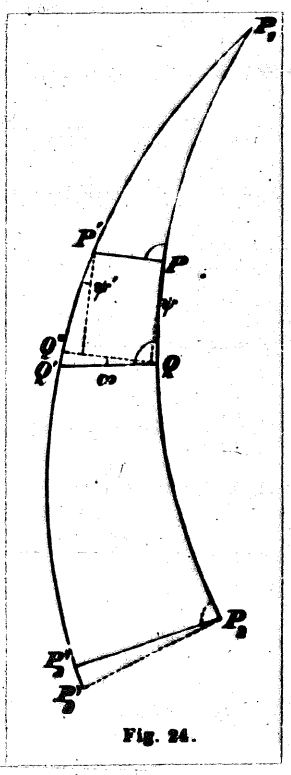


Fig. 24.

is gotten from a rotation of P_1P_2 around the point P_1. At two infinitely close points P and Q we construct line elements

normal to P_1P_2 extending to the curve P_1P_2'. Thus arises an infinitely small rectangle $PQP'Q'$, which we may treat as a plane figure because its projection on a medial plane tangential to the surface $PQP'Q'$ differs from the rectangle itself only by infinitely small amounts of second- or third-order, as the magnitude of the slope of the rectangle with respect to the tangent plane shows.

It is now important to observe that PP' and QQ' are in general no closer to being parallel than PQ and $P'Q'$, but, as in this last case, they have an infinitely small slope with respect to one another. The infinitesimal curvature of the line element PQ certainly has no noticeable influence on its length, but this implies that at P and Q the directions of the curve P_1P_2 differ only infinitesimally; consequently, PP' and QQ' will also, in general, deviate from parallel.

If now, QQ' is parallel to PP' and one considers normals from P and P' to QQ', reference to the figure shows that

$$P'Q' = P'Q'' + Q''Q' = PQ\cos\psi\sec\psi' + QQ'.\omega.$$

If one applies this formula to all line elements from P_1 to P_2, which we consider to be all the same size, then it follows, if P_2' denotes the point on P_1P_2' on the line from P_2 normal to P_1P_2, that:

$$P_1P_2' = P_1P_2.\cos\psi_m\sec\psi_m' + \Sigma(QQ'.\omega).$$

Here ψ_m is the average value for all ψ and ψ_m' an average of all ψ'. Now, since the ψ are infinitesimal, $\cos\psi_m$ and $\sec\psi_m'$ differ from unity only by infinitely small values of the second-order, and

$P_1P_2 \cdot \cos\psi_m \sec\psi'_m = P_1P_2$ to within a small second-order magnitude (in proportion to P_1P_2).

As far as $\Sigma(QQ' \cdot \omega)$ is concerned, it is certainly a sum of infinitesimals of second-order in general,

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and so is itself infinitesimal of first-order. The osculating plane of a curve P_1P_2 in general is oblique to the surface; whence it follows, as is easily recognizable, that a finite part of the curvature of the line element PQ is in divergence from PP' and QQ', so that ω is infinitely small, of first-order, and only when the osculating plane is normal to the surface does ω become zero or infinitesimal of order greater than one.

Hence, only for the geodetic line is that sum always less than a first-order infinitesimal and only for the geodetic line can it in general be asserted that $P_1P_2 = P_1P_2'$ except for infinitesimal values of greater than first-order. For other lines, that sum will only be zero in special instances, those in which the product $QQ' \cdot \omega$ occasionally cancels.

Reasoning in the same way, one may conclude that in the rotation of a geodetic line P_1P_2 about the endpoint P_1 the other endpoint P_2 describes a path normal to P_1P_2. Then if in the infinitesimal rotation, P_2 goes to P_2', P_2' deviates from P_2' only by an infinitesimal value of higher order, which, compared to the first-order infinitesimal P_2P_2' and P_2P_2' is inconsequential--or in other words, in the infinitesimal straight line triangle $P_2P_2''P_2'$, the angle at P_2 is infinitesimal.

To be sure, the case may arise where P_2P_2' itself is of higher than first-order, although the angle of rotation at P_1 and the transformations of the point P from P_1 to P_2 are in general of the first-order. Then $m = 0$ and the geodetic circle at the desired point is no longer necessarily normal to the geodetic line. However, the extension of such a point is infinitely small, and so in the finite case it creates no exception.

Par. 4. The Geodetic Line is the Shortest. Just as we earlier made a projection onto the sphere fig. 3 p. 69, we can now make one onto the ellipsoid of rotation in such a way that we rotate a geodetic line P_0P_n about one of its end points and thereby through intermediate points trace out infinitely close geodetic circles. As before, one sees immediately that a particular line $P_0P_1'P_2'\ldots P_n$ must be longer than the geodetic line $P_0P_1P_2\ldots P_n$.

The pattern of the proof is the same as on p. 69 for the sphere in figure 3. However, the small spherical circles here denote geodetic circles.

This is valid only under the condition, that the geodetic circle of the endpoint P_n cuts no geodetic circle which is at a smaller distance from its center P_0. Then and only then can one be sure that no line can be drawn shorter than the geodetic line.

Such cross sections arise from the ellipsoid of rotation as soon as the length of the geodetic line is significant enough that it stretches more than half way around the surface. These

circumstances will be further investigated in Chapter 7.

That the geodetic line is not necessarily the shortest connection between endpoints, follows from the assertions of p. 264, according to which there are infinitely many geodetic connections between two points. Not all can be the shortest line; as a rule only one shortest line exists. In many cases there are available several shortest lines of the same length.

On the other hand every shortest line is a geodetic line; since one can divide the shortest line in different ways into such small pieces, that each of them doubtless coincides with the shortest geodetic line between the endpoints of the particular segment - for an appropriate reduction of length, the proof accompanying Figure 3 leaves no doubt that the geodetic line is at the same time the shortest line, and the converse of this is necessarily valid. If one now fits the parts together it becomes evident that all the geodetic line segments must belong to the same geodetic line and the shortest is therefore a geodetic line (compare journal for measurement Zeitschift für Vermessungswesen 1880, August edition).

In order to show that the shortest line and the shortest geodetic line between two points coincide, one may use the computational rule in which one produces the equation of the shortest line by means of the calculus of variations and compares it with the equation of the geodetic line, which one can derive from the condition that the osculating plane is normal to the surface.

In considering the shortest line, the calculus of variations

at the same time demonstrates the existence of the geodetic circle (Christoffel, loc. cit.). However, one can show by pure geometry that a rotating shortest line describes a course normal to the surface. This was the course taken by Gauss (loc. cit., art. 15).

The following would also suffice in the last case: Consider a point around which are drawn infinitely close lines equal to the shortest distance which cover the whole closed surface area without intersection. Then one obtains the shortest path from the center to the exterior of the surface area as the normal to those lines.

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Par. 5. The Reduced Length. By p. 268 (12) and p. 267 (6) the reduced length $\mathfrak{m}$ is given by the formulae:

$$\left.\begin{aligned} \mathfrak{m} &= a_0(w_1 \cos\varphi_1 \sin\varphi_2 - w_2 \sin\varphi_1 \cos\varphi_2 - e^2 \sin^2\beta_0 \sin\varphi_1 \sin\varphi_2 J), \\ J &= \int_{\varphi_1}^{\varphi_2} \frac{\cos^2\varphi}{w}\, d\varphi, \quad w = \sqrt{1 - e^2\cos^2\beta}. \end{aligned}\right\} \quad (1)$$

According to the development valid for $\mathfrak{m}$ for a rotation about P_1; it is easy to see that it is also valid for a rotation about P_2. In particular, if the points are interchanged, there results an interchange of the indices in the first equation of (1) and temporarily a change of sign for $\mathfrak{m}$. But equations (4) p. 248 show that in the interchange φ_1 and φ_2 go over into $-\varphi_1$ and $-\varphi_2$, not φ_2 and φ_1. With this observation $\mathfrak{m}$ obtains its

value:

The reduced length $\mathfrak{m}$ has the same value for rotation around each of the two endpoints of the geodetic line.

We now develop the expression (1) for $\mathfrak{m}$ in a way similar to our earlier development for s, without however using k_1 but getting simplicity from the retention of k^2.

According to Par. 4, pp. 220 and 221:

$$\left.\begin{aligned} \sqrt{1-e^2\cos^2\beta} &= \sqrt{\frac{1-e^2}{1-k^2}}\sqrt{1-k^2\sin^2\varphi}, \\ e^2\sin^2\beta_0 &= k^2\frac{1-e^2}{1-k^2}. \end{aligned}\right\} \qquad (2)$$

One obtains then

$$e^2\sin^2\beta_0\, J = k^2\sqrt{\frac{1-e^2}{1-k^2}}\int\frac{\cos^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,d\varphi$$
$$= k^2\sqrt{\frac{1-e^2}{1-k^2}}\int\left(\cos^2\varphi+\frac{1}{2}k^2\cos^2\varphi\sin^2\varphi+\cdots\right)d\varphi.$$

If one sets $\cos^2\varphi=\frac{1}{2}+\frac{1}{2}\cos 2\varphi$ and $\cos^2\varphi\sin^2\varphi=\frac{1}{8}-\frac{1}{8}\cos 4\varphi$ and integrates, it follows that:

$$e^2\sin^2\beta_0 J=k^2\sqrt{\frac{1-e^2}{1-k^2}}\left\{\frac{1}{2}\Delta\varphi\left(1+\frac{1}{8}k^2+\cdots\right)+\frac{1}{4}(\sin 2\varphi_2-\sin 2\varphi_1)\right.$$
$$\left.-\frac{1}{64}k^2(\sin 4\varphi_2-\sin 4\varphi_1)+\cdots\right\}. \qquad (3)$$

On the other hand considering the first formula of (2) and by means of the series expansion of $\sqrt{1-k^2\sin^2\varphi}$ and a simple reduction:

(See Illustrative Material on Next Page.)

$$w_1 \cos\varphi_1 \sin\varphi_2 - w_2 \sin\varphi_1 \cos\varphi_2$$
$$= \sqrt{\frac{1-e^2}{1-k^2}} \left\{ \sin \Delta\varphi + k^2 \sin\varphi_1 \sin\varphi_2 \left[\frac{1}{4}(\sin 2\varphi_2 - \sin 2\varphi_1) \right.\right.$$
$$\left.\left. + \frac{1}{8} k^2 (\cos\varphi_2 \sin^3\varphi_2 - \cos\varphi_1 \sin^3\varphi_1) + \cdots \right] \right\}. \quad (4)$$

If one substitutes (3) and (4) in (1) one obtains for every value of the length s the valid formula:

$$\left.\begin{aligned} \mathfrak{m} &= \frac{b_0}{\sqrt{1-k^2}} \left\{ \sin \Delta\varphi - \frac{1}{2} k^2 \sin\varphi_1 \sin\varphi_2 [F] \right\}, \\ [F] &= \left[\Delta\varphi \left(1 + \frac{1}{8} k^2 + \cdots\right) - \frac{1}{8} k^2 \cos 2\varphi \sin \Delta\varphi + \cdots \right]; \\ \Delta\varphi &= \varphi_2 - \varphi_1 \qquad 2\varphi = \varphi_2 + \varphi_1. \end{aligned}\right\} \quad (5)$$

This formula shows, that $\mathfrak{m}$ increases at first as the length of the line increases, but then decreases until $\Delta\varphi$ is almost π when it vanishes (it does not however, vanish for $\Delta\varphi = \pi$, but only for a somewhat longer value). The reason is that $\mathfrak{m}$ contains a term involving $\Delta\varphi$. When $\mathfrak{m}$ goes through zero it becomes negative, reaches a negative maxium, then decreases in absolute value etc.

It is of special interest for geodesy to be able to develop $\mathfrak{m}$ in terms of powers of s. With this as our goal, we seek the differential quotients of $\mathfrak{m}$ with respect to s, in order eventually to use Taylor's Theorem. This procedure, if not simpler, is yet of more interest than the development by means of the substitution of s as a function of φ in (5).

Par. 6. The Differential Quotients of $\mathfrak{m}$ by s. In all that follows we consider P_1 a fixed point. This convention is important, because the differential quotients of $\mathfrak{m}$ have other values when P_2 is the center of rotation.

Differentiation of $\mathfrak{m}$ (cf. Formula (1) on the previous page) with respect to φ_2 gives:

$$\frac{1}{a_0}\frac{d\mathfrak{m}}{d\varphi_2} = w_1\cos\varphi_1\cos\varphi_2 + w_2\sin\varphi_1\sin\varphi_2 - e^2\sin\varphi_1\cos\varphi_2\cos\beta_2\sin\beta_2\frac{d\beta_2}{w_2\,d\varphi_2}$$
$$- e^2\sin^2\beta_0\sin\varphi_1\cos\varphi_2 J - \frac{1}{w_2}e^2\sin^2\beta_0\sin\varphi_1\sin\varphi_2\cos^2\varphi_2 .$$

From a spherical figure analogous to Figure 22, p. 267 one infers without difficulty, that

$$d\beta_2 = \cos\alpha_{2,1}\,d\varphi_2; \qquad (1)$$

if one inserts this in the previous equation and eliminates J by means of the formula for $\mathfrak{m}$, it follows from noticing (4), p. 248:

$$\frac{d\mathfrak{m}}{d\varphi_2} = \mathfrak{m}\cot\varphi_2 + a_0 w_2\frac{\sin\varphi_1}{\sin\varphi_2}. \qquad (2)$$

Now we have, if one thinks of s extended to P_2

$$\frac{d\mathfrak{m}}{ds} = \frac{d\mathfrak{m}}{d\varphi_2}\frac{d\varphi_2}{ds},$$

according to (6), p. 220 we have further:

$$\frac{d\varphi_2}{ds} = \frac{1}{a_0 w_2}. \qquad (3)$$

One therefore has

$$\frac{d\mathfrak{m}}{ds} = \frac{\mathfrak{m}\cot\varphi_2}{a_0 w_2} + \frac{\sin\varphi_1}{\sin\varphi_2}, \tag{4}$$

or, with regard to equations (4), p. 248:

$$\frac{d\mathfrak{m}}{ds} = -\frac{\cos\beta_1\cos\alpha_{1\cdot 2}}{\cos\beta_2\cos\alpha_{2\cdot 1}} - \frac{\mathfrak{m}\tan\beta_2}{a_0 w_2\cos\alpha_{2\cdot 1}}. \tag{5}$$

From (4) by further differentiation one obtains:

$$\frac{d^2\mathfrak{m}}{ds^2} = \frac{\cot\varphi_2}{a_0 w_2}\frac{d\mathfrak{m}}{ds} - \frac{\mathfrak{m}\csc^2\varphi_2}{a_0 w_2}\frac{d\varphi_2}{ds} - \frac{\sin\varphi_1\cos\varphi_2}{\sin^2\varphi_2}\frac{d\varphi_2}{ds} - e^2\frac{\mathfrak{m}\cot\varphi_2}{a_0 w_2^3}\cos\beta_2\sin\beta_2\frac{d\beta_2}{d\varphi_2}\frac{d\varphi_2}{ds}.$$

If we reduce this by means of formulae (1), (3) and (4) and refer to (4), p. 248, it becomes:

$$\frac{d^2\mathfrak{m}}{ds^2} = -\frac{\mathfrak{m}}{a_0^2 w_2^2}\left(1 - \frac{e^2\sin^2\beta_2}{w_2^2}\right).$$

It follows from this however

$$\frac{d^2\mathfrak{m}}{ds^2} = -\frac{\mathfrak{m}}{a_0^2}\frac{1-e^2}{w_2^4} = -\mathfrak{m}\frac{K_2}{a_0^2}, \tag{6}$$

wherein $K_2 : a_0^2$ according to p. 59, (3) denotes the measure of curvature at P_2 (the moving point).

The equation $\frac{d^2\mathfrak{m}}{ds^2} = -\mathfrak{m}$. (measure of curvature) is valid for the geodetic line of every curved surface. It was first derived by Gauss in 1828 in Disquisitiones generales circa superficies curvas, art. 19 in the Gesammelte Werke (Collected Works, Vol. 4, p. 244), and later by Christoffel in a different way from the integrability condition for the differential

Par. 7. Development of K_2 as a Function of s.

formulae of a geodetic triangle (1868, Allgemeine Theorie der geodätischen Dreiecke (General Theory of the Geodetic Triangle), p. 141).

Par. 7. Development of K_2 as a Function of s. It is with use of the first of formulae (2), p. 273:

$$K = \frac{1 - e^2}{(1 - e^2 \cos^2\beta)^2} = \frac{(1 - k^2)^2}{(1 - e^2)(1 - k^2 \sin^2\varphi)^2}. \qquad (1)$$

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Then one has:

$$K_2 = K_1 \left(\frac{1 - k^2 \sin^2\varphi_1}{1 - k^2 \sin^2\varphi_2}\right)^2. \qquad (2)$$

It follows easily from this, neglecting k^4 etc.

$$K_2 = K_1 \left(1 + k^2 [\cos 2\varphi_1 - \cos(2\varphi_1 + 2\Delta\varphi)] + \cdots\right). \qquad (3)$$

According to (9), p. 223 however, $\Delta\varphi$ deviates from $s : a_0$ only by terms of order $e^2 s : a_0$. Moreover $k^2 = e^2 \sin^2\beta_0 + Gl_4$. One notices here, that according to (4), p. 248

$$\text{and} \quad \left.\begin{aligned} \sin^2\beta_0 \cos 2\varphi_1 &= \sin^2\beta_1 - \cos^2\beta_1 \cos^2\alpha_{1.2} \\ \sin^2\beta_0 \sin 2\varphi_1 &= \sin 2\beta_1 \cos\alpha_{1.2} \end{aligned}\right\} \qquad (4)$$

and so (3) gives us:

$$K_2 = K_1 \left\{1 + e^2 \left[\sin\frac{2s}{a_0} \sin 2\beta_1 \cos\alpha_{1.2} + 2\sin^2\frac{s}{a_0}(\sin^2\beta_1 - \cos^2\beta_1 \cos^2\alpha_{1.2})\right] + \cdots\right\}. \qquad (5)$$

This formula gives K_2 to within terms of order e^4 as long as it is used for distances, for which

$$\sin 2\Delta\varphi - \sin\frac{2s}{a_0} \text{ and } \sin\Delta\varphi_1 - \sin\frac{s}{a_0}$$

are values of order e^2, which is the case for particular distances, insofar as one limits oneself to the shortest line.

If one takes $s:a_0$ as a magnitude of first-order like e, it follows:

$$K_2 = K_1\left(1 + 2e^2 \sin 2\beta_1 \cos\alpha_{1.2}\frac{s}{a_0} + Gl_4\right). \qquad (6)$$

Par. 8. The Reduced Length as a Function of s. Since the second differential quotient of m with respect to s is so easily formed, one suspects that instead of using Taylor's Theorem, one obtains m as a function of s after obtaining expressions for the third and higher differential quotients directly by integration of equation (6) of the previous page. Set:

$$K_2 = K_1\left(1 + c_1 \sin\frac{2s}{a_0} + c_2 \sin^2\frac{s}{a_0} + \cdots\right), \qquad (1)$$

$$\left.\begin{aligned} c_1 &= e^2 \sin 2\beta_1 \cos\alpha_{1.2}, \\ c_2 &= 2e^2(\sin^2\beta_1 - \cos^2\beta_1 \cos^2\alpha_{1.2}), \end{aligned}\right\} \qquad (2)$$

where c_1 and c_2 may be computed from the geographic instead of the reduced latitude, and it follows from (6):

$$\frac{d^2m}{ds^2} + m\frac{K_1^2}{a_0^2}\left(1 + c_1 \sin\frac{2s}{a_0} + c_2 \sin^2\frac{s}{a_0} + \cdots\right) = 0. \qquad (3)$$

This equation neglects all higher powers of e^2, but contains all terms with e^2. It may be integrated precisely, insofar as the

question is solely the conservation of terms involving e^2.

Equating c_1 and c_2 to zero it follows that $\mathfrak{m} = \frac{a_0}{\sqrt{K_1}} \sin \frac{s\sqrt{K_1}}{a_0}$, as one realizes that in this case $\mathfrak{m}$ becomes the reduced longitude on the sphere of radius $a_0 : \sqrt{K_1}$.

In general we set

$$\mathfrak{m} = \frac{a_0}{\sqrt{K_1}} \sin \frac{s\sqrt{K_1}}{a_0} + a_0\mu. \qquad (4)$$

Differentiating twice gives:

$$\frac{d^2\mathfrak{m}}{ds^2} = -\frac{\sqrt{K_1}}{a_0} \sin \frac{s\sqrt{K_1}}{a_0} + a_0 \frac{d^2\mu}{ds^2}.$$

This we substitute in (3) and neglect the products μc_1 and μc_2, assuming that each has a factor e^4 while μ contains probably the factor e^2. It follows that:

$$\frac{d^2\mu}{ds^2} + \frac{K_1}{a_0^2}\mu + \frac{\sqrt{K_1}}{a_0^2} \sin \frac{s\sqrt{K_1}}{a_0} \left(c_1 \sin \frac{2s}{a_0} + c_2 \sin^2 \frac{s}{a_0} \right) + \cdots = 0,$$

or, neglecting e^4:

$$a_0^2 \frac{d^2\mu}{ds^2} + \mu + c_1 \sin \frac{s}{a_0} \sin \frac{2s}{a_0} + c_2 \sin^3 \frac{s}{a_0} + \cdots = 0. \qquad (5)$$

One still has for μ the conditions

$$\mu = 0 \text{ and } \frac{d\mu}{ds} = 0 \text{ for } s = 0. \qquad (6)$$

This results from the condition, that for $s = 0$, $\mathfrak{m}$ becomes the reduced length for the plane, and therefore, since here $\mathfrak{m} = s$, it must be:

$$\mathfrak{m} = 0, \quad \frac{d\mathfrak{m}}{ds} = 1 \text{ for } s = 0$$

If one now introduces expression (4), one obtains (6).

Our goal being integration, we write (5) as follows:

$$a_0^2\frac{d\mu^2}{ds^2}+\mu+\tfrac{1}{2}c_1\left(\cos\frac{s}{a_0}-\cos\frac{3s}{a_0}\right)+\tfrac{1}{4}c_2\left(3\sin\frac{s}{a_0}-\sin\frac{3s}{a_0}\right)+\cdots=0 \qquad (7)$$

and now tentatively set (which is in part pointed to by series expansions):

$$\mu=a_1\left\{\begin{array}{l}\left(\frac{s}{a_0}\cos\frac{s}{a_0}-\sin\frac{s}{a_0}\right)+b_1\frac{s}{a_0}\sin\frac{s}{a_0}\\ +a_3\left(\cos\frac{3s}{a_0}-\cos\frac{s}{a_0}\right)+b_3\left(\sin\frac{3s}{a_0}-3\sin\frac{s}{a_0}\right).\end{array}\right\} \qquad (8)$$

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Repeated differentiation of (8) results in

$$a_0^2\frac{d^2\mu}{ds^2}=\left\{\begin{array}{l}-a_1\left(\sin\frac{s}{a_0}+\frac{s}{a_0}\cos\frac{s}{a_0}\right)+b_1\left(2\cos\frac{s}{a_0}-\frac{s}{a_0}\sin\frac{s}{a_0}\right)\\ -a_3\left(9\cos\frac{3s}{a_0}-\cos\frac{s}{a_0}\right)-b_3\left(9\sin\frac{3s}{a_0}-3\sin\frac{s}{a_0}\right).\end{array}\right\}$$

We substitute this as well as (8) in (7) and this equation becomes

$$\left.\begin{array}{l}\left(2b_1+\frac{1}{2}c_1\right)\cos\frac{s}{a_0}-\left(2a_1-\frac{3}{4}c_2\right)\sin\frac{s}{a_0}\\ -\left(8a_3+\frac{1}{2}c_1\right)\cos\frac{3s}{a_0}-\left(8b_3+\frac{1}{4}c_2\right)\sin\frac{3s}{a_0}\end{array}\right\}=0,$$

which must vanish indentically. It therefore can be assumed that:

$$\left.\begin{array}{ll}a_1=+\frac{3}{8}c_2 & b_1=-\frac{1}{4}c_1\\ a_3=-\frac{1}{16}c_1 & b_3=-\frac{1}{32}c_2.\end{array}\right\} \qquad (9)$$

If one substitutes these coefficient values in (8) and the value obtained for μ in (4), then one finds for m the following expression, from which no e^2 terms have been omitted:

Par. 8. The Reduced Length as a Function of s.

$$\mathfrak{m} = \left\{ \begin{aligned} &\frac{a_0}{\sqrt{K_1}} \sin \frac{s\sqrt{K_1}}{a_0} - \frac{1}{4} c_1 s \sin \frac{s}{a_0} - \frac{1}{16} c_1 a_0 \left(\cos \frac{3s}{a_0} - \cos \frac{s}{a_0} \right) \\ &\quad + \frac{3}{8} c_2 \left(s \cos \frac{s}{a_0} - a_0 \sin \frac{s}{a_0} \right) \\ &\quad - \frac{1}{32} c_2 a_0 \left(\sin \frac{3s}{a_0} - 3 \sin \frac{s}{a_0} \right) + \cdots \end{aligned} \right\}. \quad (10)$$

For c_1 and c_2 the values given under (2) are valid. Expression (10) gives $\mathfrak{m}$ up to values of the order $a_0 e^4$ for arbitrary shortest distances s (cf. the close of Par. 7, p. 276).

If one uses the series expansions in which $s:a_0$ is considered as first-order magnitude, it follows that:

$$\mathfrak{m} = \frac{a_0}{\sqrt{K_1}} \sin \frac{s\sqrt{K_1}}{a_0} - \frac{1}{6} c_1 a_0 \left(\frac{s^4}{a_0^4} - \frac{11}{30} \frac{s^4}{a_0^4} + Gl_6 \right) - \frac{1}{20} c_2 a_0 \left(\frac{s^5}{a_0^5} + Gl_7 \right) + a_0 Gl_6. \quad (11)$$

That the remainder is a fractional part of the eight-order of a_0, is shown by entering the previous expression in the strict equation (6), p. 275 which is satisfied up to terms of $e^4 s^3$.

If e^4 is also considered in the development of K_2, one obtains for small distances:

$$K_2 = K_1 \left\{ 1 + c_1' \frac{2s}{a_0} + c_2' \frac{s^2}{a_0^2} - \frac{4}{3} c_1' \frac{s^3}{a_0^3} + Gl_6 \right\} \quad (12)$$

$$\left. \begin{aligned} c_1' &= e^2 \sin 2\beta_1 \cos \alpha_{1.2} \left(1 + \frac{3}{2} e^2 \cos^2 \beta_1 \right) \\ c_2' &= 2e^2 \left(\sin^2 \beta_1 - \cos^2 \beta_1 \cos^2 \alpha_{1.2} \right) \\ &\quad + e^4 \left(\sin^2 2\beta_1 \left[1 + \frac{9}{2} \cos^2 \alpha_{1.2} \right] - 4 \cos^2 \beta_1 \cos^2 \alpha_{1.2} \right) \end{aligned} \right\} \quad (13)$$

$$\mathfrak{m} = \frac{a_0}{\sqrt{K_1}} \sin \frac{s\sqrt{K_1}}{a_0} - a_0 K_1 \left\{ \frac{1}{6} c_1' \frac{s^4}{a_0^4} + \frac{1}{20} c_2' \frac{s^5}{a_0^5} - \frac{11}{180} c_1' \frac{s^6}{a_0^6} + Gl_6 \right\}. \quad (14)$$

Chapter 6. Differential Formulae and Series Expansions for the Geodetic Line.

Par. 9. Differential Formulae for the Geodetic Line with a Displacement of an Endpoint. We refer to the line P_1P_2, of Figure 20, p. 226 and consider a displacement of P_2 in the sense of increasing length s by δs and then a rotation $\delta\alpha_{1.2}$ in amount, about P_1.

1. Displacement δs of P_2. According to (3), p. 219 and (1), p. 229, one has, first, setting for α, β and dl or $\alpha_{2.1} - 180^\circ, \beta_2$ and $dL_{1.2}$ respectively:

$$\left.\begin{aligned} \delta\beta_2 &= \frac{\delta s}{a_0 w_2}\cos\alpha_{2.1} \\ \delta L_{1.2} &= -\frac{\delta s}{a_0}\sec\beta_2\sin\alpha_{2.1}. \end{aligned}\right\} \qquad (1)$$

In order to find $\delta\alpha_{2.1}$ notice that the product $\cos\beta_2\sin\alpha_{2.1}$ is constant during a displacement of P_2, from which it follows that:

$$\cos\beta_2\sin\alpha_{2.1} = \cos(\beta_2+\delta\beta_2)\sin(\alpha_{2.1}+\delta\alpha_{2.1}).$$

hence

$$\cos\beta_2\cos\alpha_{2.1}\delta\alpha_{2.1} = \sin\beta_2\sin\alpha_{2.1}\delta\beta_2$$

and referring to the first of formulae (1):

$$\delta\alpha_{2.1} = \frac{\delta s}{a_0 w_2}\tan\beta_2\sin\alpha_{2.1}. \qquad (2)$$

2. Rotation of P_1P_2 about P_1. By (11), p. 268 and (17), p. 269 we have:

$$\left.\begin{aligned} \delta\beta_2 &= -\frac{m}{a_0 w_2}\sin\alpha_{2.1}\delta\alpha_{1.2} \\ \delta L_{1.2} &= -\frac{m}{a_0}\sec\beta_2\cos\alpha_{2.1}\delta\alpha_{1.2}. \end{aligned}\right\} \qquad (3)$$

The fundamental equation applied to P_1 and P_2 <u>after</u> the rotation, provides:

$$\cos\beta_1\sin(\alpha_{1.2}+\delta\alpha_{1.2}) + \cos(\beta_2+\delta\beta_2)\sin(\alpha_{2.1}+\delta\alpha_{2.1}) = 0.$$

Since:

$$\cos\beta_1 \sin\alpha_{1.2} + \cos\beta_2 \sin\alpha_{2.1} = 0$$

this becomes:

$$\cos\beta_2 \cos\alpha_{2.1}\delta\alpha_{2.1} = \sin\beta_2 \sin\alpha_{2.1}\delta\beta_2 - \cos\beta_1 \cos\alpha_{1.2}\delta\alpha_{1.2},$$

where $\delta\beta_2$ is gotten from the first formula of (3). From this it follows that:

$$\delta\alpha_{2.1} = \left\{ -\frac{\cos\beta_1 \cos\alpha_{1.2}}{\cos\beta_2 \cos\alpha_{2.1}} - \frac{m}{a_0 w_2} \tan\beta_2 \sin\alpha_{2.1} \tan\alpha_{2.1} \right\} \delta\alpha_{1.2}.$$

From here we eliminate the first term of the right member, with the help of equation (5) of p. 275, and obtain:

$$\delta\alpha_{2.1} = \left(\frac{dm}{ds} + \frac{m}{a_0 w_2} \tan\beta_2 \cos\alpha_{2.1} \right) \delta\alpha_{1.2}. \qquad (4)$$

For the differential quotient $\frac{dm}{ds}$ it should be remembered that it refers to a displacement of P_2 in the sense of a lengthening of s along a constant azimuth $\alpha_{1.2}$.

3. Displacement and Rotation. When the geodetic line P_1P_2 undergoes a rotation about P_1 of magnitude $\delta\alpha_{1.2}$ and at the same time increases by ds, the total changes of β_2, $L_{1.2}$ and $\alpha_{2.1}$ are equal to the sums of the related expressions (1) to (4).

While performing the summation we substitute according to pp. 40 and 41, (2), (3) and (6):

$$\left.\begin{aligned} &\frac{W_2}{\sqrt{1-e^2}} \text{ for } \frac{1}{w_2}, \quad W_2 \sec B_2 \text{ for } \sec\beta_2, \\ &\sqrt{1-e^2}\tan B_2 \text{ for } \tan\beta_2, \quad \delta B_2 \frac{\sqrt{1-e^2}}{W_2^2} \text{ for } \delta\beta_2 \end{aligned}\right\} \quad (5)$$

and arrive thereby at the formulae for the total changes:

$$\left.\begin{aligned} \delta B_2 &= \frac{\delta s}{a_0}\frac{W_2^3}{1-e^2}\cos\alpha_{2.1} - \frac{m}{a_0}\frac{W_2^3}{1-e^2}\sin\alpha_{2.1}\delta\alpha_{1.2} \\ \delta L_{1.2} &= -\frac{\delta s}{a_0}W_2\sec B_2\sin\alpha_{2.1} - \frac{m}{a_0}W_2\sec B_2\cos\alpha_{2.1}\delta\alpha_{1.2} \\ \delta\alpha_{2.1} &= \frac{\delta s}{a_0}W_2\tan B_2\sin\alpha_{2.1} + \left\{\left(\frac{dm}{ds}\right)_{1.2} + \frac{m}{a_0}W_2\tan B_2\cos\alpha_{2.1}\right\}\delta\alpha_{1.2}. \end{aligned}\right\} \quad (6)$$

Here $W_2 = \sqrt{1-e^2\sin B_2}$ and the index 1.2 applied to $\frac{dm}{ds}$ denotes the direction from the fixed to the moving point, in the sense of the above remark on (4) at the close of the second paragraph.

4. Variation of the Geographic Coordinates of P_2. The values

δs and $\delta\alpha_{1.2}$, which belong to δB_2 and $\delta L_{1.2}$, result from the first two equations of (6) by elimination of $\delta\alpha_{1.2}$ and δs respectively. If one inserts the obtained results in the expressions for $\delta\alpha_{2.1}$ and reduces it by parts by means of formula (5), p. 275, which gives by introduction of the substitutions (5):

$$\left(\frac{dm}{ds}\right)_{1.2} = -\frac{W_2\cos B_1\cos\alpha_{1.2}}{W_1\cos B_2\cos\alpha_{2.1}} - \frac{W_2 m\tan B_2}{a_0\cos\alpha_{2.1}}, \quad (7)$$

then, also this $\delta\alpha_{2.1}$ is a function of δB_2 and $\delta L_{1.2}$. The results are:

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned}
\delta s &= a_0 \frac{1-e^2}{W_2^3}\cos\alpha_{2.1}\delta B_2 - \frac{a_0}{W_2}\cos B_2 \sin\alpha_{2.1}\delta L_{1.2} \\
\delta\alpha_{1.2} &= -\frac{a_0}{m}\frac{1-e^2}{W_2^3}\sin\alpha_{2.1}\delta B_2 - \frac{a_0}{mW_2}\cos B_2\cos\alpha_{2.1}\delta L_{1.2} \\
\delta\alpha_{2.1} &= -\frac{a_0}{m}\frac{1-e^2}{W_2^3}\left(\frac{dm}{ds}\right)_{1.2}\sin\alpha_{2.1}\delta B_2 + \frac{a_0}{mW_1}\cos B_1\cos\alpha_{1.2}\delta L_{1.2}.
\end{aligned}\right\} \quad (8)$$

It should be observed, that in the preceding paragraphs δB, δL and $\delta\alpha$ are to be understood as arcs.

Par. 10. Displacement of Both Endpoints of the Geodetic Line. In all formulae of the preceding paragraphs we may interchange the indices, if we accept P_2 as fixed and P_1 as moving. Hence (8) give:

$$\left.\begin{aligned}
\delta s &= a_0 \frac{1-e^2}{W_1^3}\cos\alpha_{1.2}\delta B_1 + \frac{a_0}{W_1}\cos B_1 \sin\alpha_{1.2}\delta L_{1.2} \\
\delta\alpha_{2.1} &= -\frac{a_0}{m}\frac{1-e^2}{W_1^3}\sin\alpha_{1.2}\delta B_1 + \frac{a_0}{mW_1}\cos B_1\cos\alpha_{1.2}\delta L_{1.2} \\
\delta\alpha_{1.2} &= -\frac{a_0}{m}\frac{1-e^2}{W_1^3}\left(\frac{dm}{ds}\right)_{2.1}\sin\alpha_{1.2}\delta B_1 - \frac{a_0}{mW_2}\cos B_2\cos\alpha_{2.1}\delta L_{1.2}.
\end{aligned}\right\} \quad (1)$$

These variations in $s, \alpha_{1.2}$ and $\alpha_{2.1}$ correspond therefore to a displacement of P_1 with respect to fixed P_2. Then one can set

$$L_{1.2} = L_2 - L_1, \quad (2)$$

where L in general, is the west longitude in relation to any prime meridian, and so we have:

$$\left.\begin{aligned}
\delta L_{1.2} &= +\delta L_2 \text{ for a fixed } P_1, \text{ for (8)} && \text{of the previous paragraph} \\
\delta L_{1.2} &= -\delta L_1 \text{ for a fixed } P_2, \text{ for (1)} && \text{of this paragraph}
\end{aligned}\right\} \quad (3)$$

Through addition of the relevant values according to (8) of the previous paragraph and (1) of this paragraph we obtain

variations of the length and azimuth, which correspond to particular differential displacements of both endpoints:

Given: $\delta B_1, \delta B_2, \delta L_1, \delta L_2$;

$$\left.\begin{aligned}
\rho''\delta s = {} & a_0\frac{1-e^2}{W_1^2}\cos\alpha_{1.2}\,\delta B_1 + a_0\frac{1-e^2}{W_2^2}\cos\alpha_{2.1}\,\delta B_2 \\
& + \frac{a_0}{W_2}\cos B_2\sin\alpha_{2.1}(\delta L_1 - \delta L_2) \quad \text{in seconds} \\
\delta\alpha_{1.2} = {} & -\frac{a_0}{\mathfrak{M}}\frac{1-e^2}{W_1^2}\left(\frac{d\mathfrak{M}}{ds}\right)_{2.1}\sin\alpha_{1.2}\,\delta B_1 - \frac{a_0}{\mathfrak{M}}\frac{1-e^2}{W_2^2}\sin\alpha_{2.1}\,\delta B_2 \\
& + \frac{a_0}{\mathfrak{M}W_2}\cos B_2\cos\alpha_{2.1}(\delta L_1 - \delta L_2) \quad \text{in seconds} \\
\delta\alpha_{2.1} = {} & -\frac{a_0}{\mathfrak{M}}\frac{1-e^2}{W_1^2}\sin\alpha_{1.2}\,\delta B_1 - \frac{a_0}{\mathfrak{M}}\frac{1-e^2}{W_2^2}\left(\frac{d\mathfrak{M}}{ds}\right)_{1.2}\sin\alpha_{2.1}\,\delta B_2 \\
& - \frac{a_0}{\mathfrak{M}W_1}\cos B_1\cos\alpha_{1.2}(\delta L_1 - \delta L_2) \quad \text{in seconds}
\end{aligned}\right\} \tag{4}$$

For the reduction of the first of these equations the relation $\cos\beta_2\sin\alpha_{2.1} + \cos\beta_1\sin\alpha_{1.2} = 0$, that is

$$\frac{1}{W_2}\cos B_2\sin\alpha_{2.1} + \frac{1}{W_1}\cos B_1\sin\alpha_{1.2} = 0 \tag{5}$$

was used. With reference to this last, it can be seen that (4) is symmetric in P_1 and P_2. We now assume that the changes of the geographic coordinates of both points are not given, but only those of P_1, besides δs and $\delta\alpha_{1.2}$; then from the first two of (4) δL_2, and then δB_2 may be eliminated and reduced to δB_2 and δL_2 respectively. At the same time one sets

$$\alpha_{2.1} - \alpha_{1.2} - 180^0 = \Delta\alpha, \tag{6}$$

and finds

Given: $\delta B_1, \delta L_1, \delta s, \delta\alpha_{1.2}$;

$$\delta B_2 = \left\{\begin{aligned}
& \frac{W_2^2}{1-e^2}\left\{\rho''\cos\alpha_{2.1}\frac{\delta s}{a_0} - \frac{\mathfrak{M}}{a_0}\sin\alpha_{2.1}\,\delta\alpha_{1.2}\right\} \\
& + \frac{W_2^2}{W_1^2}\left\{\cos\Delta\alpha + \left[1 - \left(\frac{d\mathfrak{M}}{ds}\right)_{2.1}\right]\sin\alpha_{1.2}\sin\alpha_{2.1}\right\}\delta B_1
\end{aligned}\right\} \quad \text{in sec.} \tag{7}$$

(See Illustrative Material on Next Page.)

Par. 11. Calculation of the Coefficients of the Differential Formulas.

$$\delta L_2 \underset{\text{in sec.}}{=} \left\{ \begin{array}{c} \delta L_1 - W_2 \sec B_2 \left\{ \varrho'' \sin \alpha_{2.1} \frac{\delta s}{a_0} + \frac{m}{a_0} \cos \alpha_{2.1} \delta \alpha_{1.2} \right\} \\ - \frac{(1-e^2) W_2}{W_1^3} \sec B_2 \left\{ \sin \Delta\alpha - \left[1 - \left(\frac{dm}{ds} \right)_{2.1} \right] \sin \alpha_{1.2} \cos \alpha_{2.1} \right\} \delta B_1 \end{array} \right\} \quad (8)$$

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These expressions we finally insert in the third equation of (4) and after some reduction using (7), p. 281, we obtain:

$$\delta \alpha_{2.1} \underset{\text{in sec.}}{=} \left\{ \begin{array}{l} \varrho'' W_2 \tan B_2 \sin \alpha_{2.1} \frac{\delta s}{a_0} + \left\{ \left(\frac{dm}{ds} \right)_{1.2} + \frac{m}{a_0} W_2 \tan B_2 \cos \alpha_{2.1} \right\} \delta \alpha_{1.2} \\ + \frac{a_0}{m} \frac{1-e^2}{W_1^3} \left\{ \left[1 - \left(\frac{dm}{ds} \right)_{1.2} + 1 - \left(\frac{dm}{ds} \right)_{2.1} \right] \cos \Delta\alpha \sin \alpha_{2.1} - \frac{\sin^2 \Delta\alpha}{\sin \alpha_{1.2}} \right\} \delta B_1 . \\ \quad + \left[1 - \left(\frac{dm}{ds} \right)_{1.2} \right] \left[1 - \left(\frac{dm}{ds} \right)_{2.1} \right] \sin^2 \alpha_{2.1} \sin \alpha_{1.2} \end{array} \right\} \quad (9)$$

In formulae (7) to (9) δB, δL and $\delta \alpha$ are understood to be expressed in seconds, as is indicated in part.

Par. 11. Calculation of the Coefficients of the Differential Formulae. For particular large shortest distances s in the previous differential formulae for (10) equation (10), p. 278 is to be used; to be sure, it neglects e^4, but is most acceptable.

For smaller distances the more convenient expression (11), p. 278, is preferred. But in most cases, it is completely satisfactory to set

$$m = \frac{a_0}{\sqrt{K}} \sin \frac{s\sqrt{K}}{a_0} + \cdots = s \left(1 - \frac{s^2}{6 a_0^2} K + G l_4 \right), \quad (1)$$

in which K is some intermediate value for the magnitude K, in relation to the endpoints of the line s (cf. (3), p. 59). One has further

$$\left(\frac{d\mathfrak{m}}{ds}\right)_{1,2} - \left(\frac{d\mathfrak{m}}{ds}\right)_{2,1} + \cdots = 1 - \frac{s^2}{2a_0^2} K + Gl_4, \qquad (2)$$

and in logarithmic form:

$$\left.\begin{aligned} &\log \mathfrak{m} = \log s - \frac{M}{6a_0^2} s^2 K + Gl_4 \\ &\log \frac{d\mathfrak{m}}{ds} = -\frac{M}{2a_0^2} s^2 K + Gl_4; \\ \text{moreover} \quad &\left[1 - \frac{d\mathfrak{m}}{ds}\right] = \frac{s^2}{2a_0^2} K + Gl_4. \end{aligned}\right\} \qquad (3)$$

The introduction of these values considerably simplifies formulae (7) to (9) of the previous paragraph, while (4) are generally already simple.

For sides of a directly measureable triangle, as a rule it will be permissible to neglect the small parantheses in (7) and (8), and in (9) at least the third term of the coefficients of δB_1.

Spherical Computation of Coefficients. If it does not depend upon fractional parts of the order e^2 in the coefficients of the differential formulae, it can be calculated

as if the triangle between the North Pole, P_1, and P_2 were spherical.

Granting this supposition, one has at first:

$$\mathfrak{m} = a_0 \sin \frac{s}{a_0}, \quad \frac{d\mathfrak{m}}{ds} = \cos \frac{s}{a_0} \qquad (4)$$

and then, by an easily evident reduction, especially by a solution of $\sin \Delta\alpha$ and $\cos \Delta\alpha$, for some of the complicated coefficients in

(7) to (9) of the previous paragraphs:

$$\left.\begin{aligned}&\cos \Delta\alpha + \left[1 - \left(\frac{dm}{ds}\right)_{2.1}\right] \sin \alpha_{1.2} \sin \alpha_{2.1} \\ &= -\cos \alpha_{1.2} \cos \alpha_{2.1} - \cos \frac{s}{a_0} \sin \alpha_{1.2} \sin \alpha_{2.1},\end{aligned}\right\} \quad (5)$$

$$\left.\begin{aligned}&\sin \Delta\alpha - \left[1 - \left(\frac{dm}{ds}\right)_{2.1}\right] \sin \alpha_{1.2} \cos \alpha_{2.1} \\ &= -\cos \alpha_{1.2} \sin \alpha_{2.1} + \cos \frac{s}{a_0} \sin \alpha_{1.2} \cos \alpha_{2.1},\end{aligned}\right\} \quad (6)$$

$$\left(\frac{dm}{ds}\right)_{1.2} + \frac{m}{a_0} W_2 \tan B_2 \cos \alpha_{2.1} = \cos \frac{s}{a_0} + \sin \frac{s}{a_0} \tan B_2 \cos \alpha_{2.1}, \quad (7)$$

$$\left.\begin{aligned}&\left\{\begin{aligned}&\left[1 - \left(\frac{dm}{ds}\right)_{1.2} + 1 - \left(\frac{dm}{ds}\right)_{2.1}\right] \cos \Delta\alpha \sin \alpha_{2.1} \\ &\qquad - \frac{\sin^2 \Delta\alpha}{\sin \alpha_{1.2}}\end{aligned}\right\} \\ &+ \left[1 - \left(\frac{dm}{ds}\right)_{1.2}\right]\left[1 - \left(\frac{dm}{ds}\right)_{2.1}\right] \sin^2 \alpha_{2.1} \sin \alpha_{1.2}\end{aligned}\right\} \quad (8)$$

$$= \left\{\begin{aligned}&2 \cos \frac{s}{a_0} \cos \alpha_{1.2} \cos \alpha_{2.1} \sin \alpha_{2.1} \\ &- \sin \alpha_{1.2} \cos^2 \alpha_{2.1} - \cos^2 \alpha_{1.2} \frac{\sin^2 \alpha_{2.1}}{\sin \alpha_{1.2}} \\ &- \sin^2 \frac{s}{a_0} \sin^2 \alpha_{2.1} \sin \alpha_{1.2}.\end{aligned}\right.$$

But according to formulae of spherical trigonometry:

$$-\cos \alpha_{1.2} \cos \alpha_{2.1} - \cos \frac{s}{a_0} \sin \alpha_{1.2} \sin \alpha_{2.1} = \cos L_{1.2} \quad (5^*)$$

$$\cos \alpha_{1.2} \sin \alpha_{2.1} - \cos \frac{s}{a_0} \sin \alpha_{1.2} \cos \alpha_{2.1} = \sin L_{1.2} \sin B_2, \quad (6^*)$$

from which (5) and (6) are immediately simplified. Further,

$$\cos \frac{s}{a_0} = \sin B_1 \sin B_2 + \cos B_1 \cos B_2 \cos L_{1.2}$$

$$\sin \frac{s}{a_0} \cos \alpha_{2.1} = -\sin B_1 \cos B_2 + \cos B_1 \sin B_2 \cos L_{1.2}$$

and from there, for (7)

$$\cos\frac{s}{a_0} + \sin\frac{s}{a_0}\tan B_2 \cos\alpha_{2.1} = \cos L_{1.2}\cos B_1 \sec B_2. \qquad (7^*)$$

For reduction of (8), according to the cotangent theorem (or by Formula (5), p. 275):

$$\tan B_1 \sin\frac{s}{a_0} = \cot\alpha_{2.1}\sin\alpha_{1.2} - \cos\frac{s}{a_0}\cos\alpha_{1.2}.$$

Squaring this and multiplying by $\sin^2\alpha_{2.1} : \sin\alpha_{1.2}$, one arrives at the relation:

$$2\cos\frac{s}{a_0}\cos\alpha_{1.2}\cos\alpha_{2.1}\sin\alpha_{2.1} - \sin\alpha_{1.2}\cos^2\alpha_{2.1} - \cos^2\alpha_{1.2}\frac{\sin^2\alpha_{2.1}}{\sin\alpha_{1.2}} = \sin^2\frac{s}{a_0}(\sin^2\alpha_{1.2} - \sec^2 B_1)\frac{\sin^2\alpha_{2.1}}{\sin\alpha_{1.2}},$$

with the help of which the expression for the right side of (8) may be simplified:

$$-\sin^2\frac{s}{a_0}\sec^2 B_1 \frac{\sin^2\alpha_{2.1}}{\sin\alpha_{1.2}} \text{ i.e., } -\sin\frac{s}{a_0}\sin L_{1.2}\sec B_2. \qquad (8^*)$$

The differential formulae (4) and (7) to (9) of Par. 10 may now be transformed without difficulty into the following formulae, in which δB, δL and $\delta\alpha$ are expressed in seconds:

(See Illustrative Material on Next Page.)

Par. 11. Calculation of the Coefficients of the Differential Formulae.

$$\left.\begin{aligned}
\varrho''\frac{\delta s}{a_0} &= \cos\alpha_{1.2}\,\delta B_1 + \cos\alpha_{2.1}\,\delta B_2 + \begin{pmatrix}\cos B_2 \sin\alpha_{2.1}\\ \text{or even}\\ -\cos B_1\sin\alpha_{1.2}\end{pmatrix}(\delta L_1 - \delta L_2)\\
\delta\alpha_{1.2} &= -\frac{\sin\alpha_{1.2}}{\tan\frac{s}{a_0}}\delta B_1 - \frac{\sin\alpha_{2.1}}{\sin\frac{s}{a_0}}\delta B_2 + \frac{\cos B_2\cos\alpha_{2.1}}{\sin\frac{s}{a_0}}(\delta L_1 - \delta L_2)\\
\delta\alpha_{2.1} &= -\frac{\sin\alpha_{1.2}}{\sin\frac{s}{a_0}}\delta B_1 - \frac{\sin\alpha_{2.1}}{\tan\frac{s}{a_0}}\delta B_2 - \frac{\cos B_1\cos\alpha_{1.2}}{\sin\frac{s}{a_0}}(\delta L_1 - \delta L_2)
\end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned}
\delta B_2 = {} & \varrho''\cos\alpha_{2.1}\frac{\delta s}{a_0} + \sin L_{1:2}\cos B_1\,\delta\alpha_{1.2}\\
& + \cos L_{1.2}\,\delta B_1\\
\delta L_2 = {} & \delta L_1 - \varrho''\sec B_2\sin\alpha_{2.1}\frac{\delta s}{a_0} - \sin L_{1.2}\csc\alpha_{1.2}\cos\alpha_{2.1}\,\delta\alpha_{1.2}\\
& + \sin L_{1:2}\tan B_2\,\delta B_1\\
\delta\alpha_{2.1} = {} & \varrho''\tan B_2\sin\alpha_{2.1}\frac{\delta s}{a_0} + \cos L_{1.2}\cos B_1\sec B_2\,\delta\alpha_{1.2}\\
& - \sin L_{1.2}\sec B_2\,\delta B_1 .
\end{aligned}\right\} \quad (10)$$

It is obvious, that these formulae can be obtained directly and far more simply through differentiation of the formulae of spherical trigonometry. The course chosen here clearly shows that the omitted terms are not always fractional parts of the coefficients of order e^2, but in the first and third formulae of (10) there are only fractional parts of the order $e^2 s : a_0$ in the coefficients of δB_1 and $\delta\alpha_{1.2}$ respectively. Other methods of obtaining the formulae do not do this. Since both these coefficients are almost equal to 1, the above-mentioned circumstance deserves mention.

The coefficient of $\delta\alpha_{1.2}$ in the second formula of (10) can also be writted as follows:

$$-\sin\frac{s}{a_0}\sec B_2\cos\alpha_{2.1}, \quad (10^a)$$

which expression however, is less convenient than that used above.

For the formulae (9) $\sin\frac{s}{a_0}$ can eventually be calculated by one of the relations

$$\sin\frac{s}{a_0} = \frac{\sin L_{1.2}\cos B_2}{\sin\alpha_{1.2}} = -\frac{\sin L_{1.2}\cos B_1}{\sin\alpha_{2.1}} \qquad (9^*)$$

and immediately $\log\sin\frac{s}{a_0}$ as used for $\log\tan\frac{s}{a_0}$.

Par. 12. Differential Formulae for s, $\alpha_{1.2}$ and $\alpha_{2.1}$ for Given Geographic Positions in Relation to Changes of a_0 and e^2. We consider first the spherical auxiliary triangle $\mathfrak{N}\mathfrak{P}_1\mathfrak{P}_2$. We can immediately apply (9) of the previous paragraph to this, if we write for $\frac{s}{a_0}$, B and $L_{1.2}$ respectively $\Delta\varphi$, β and $\Delta\lambda$. With that, it results, taking $\delta\beta$ and $\delta\Delta\lambda$ as arcs and since $\cos\beta_1\sin\alpha_{1.2} = \cos\beta_0$:

$$\delta\Delta\varphi = \cos\alpha_{1.2}\,\delta\beta_1 + \cos\alpha_{2.1}\,\delta\beta_2 + \cos\beta_0\,\delta\Delta\lambda \qquad (1)$$

and

$$\sin\Delta\varphi\,\delta\alpha_{1.2} = -\cos\Delta\varphi\sin\alpha_{1.2}\delta\beta_1 - \sin\alpha_{2.1}\delta\beta_2 - \cos\beta_2\cos\alpha_{2.1}\delta\Delta\lambda. \qquad (2)$$

The formula for $\delta\alpha_{2.1}$ can be ignored here, since it will be obtained eventually most conveniently by interchange of indices in the formula for $\delta\alpha_{1.2}$.

For $\delta\beta_1$ and $\delta\beta_2$ one has by differentation and the relation $\tan\beta = \sqrt{1-e^2}\tan B$:

$$\delta\beta = -\tan B\cos^2\beta\,\frac{\delta e^2}{2\sqrt{1-e^2}} = -\frac{1}{4}\sin 2\beta\,\frac{\delta e^2}{-e^2}, \qquad (3)$$

and with this (1) and (2), referring to (4), p. 248, are transformed:

$$\delta \Delta\varphi = \frac{1}{2}\frac{\delta e^2}{1-e^2}\sin^2\beta_0\,(\sin\varphi_2\cos\varphi_2 - \sin\varphi_1\cos\varphi_1) + \cos\beta_0\,\delta\Delta\lambda \tag{4}$$

$$\delta\alpha_{1.2} = \frac{1}{2}\frac{\delta e^2}{1-e^2}\sin\beta_0\cos\beta_0\sin\varphi_1 + \frac{\sin\beta_0\sin\varphi_2}{\sin\Delta\varphi}\,\delta\Delta\lambda. \tag{5}$$

The determination of $\delta\Delta\lambda$ is connected to equation (13), p. 268:

$$L_{1.2} = \int_{\lambda_1}^{\lambda_2} \sqrt{1 - e^2\cos^2\beta}\; d\lambda, \tag{6}$$

where

$$\tan\beta = \tan\beta_0\cos\lambda \qquad \cos\beta_0 = \cos\beta_1\sin\alpha_{1.2}. \tag{7}$$

Through differentiation, since $\delta L_{1.2}$ is zero (cf. p. 268, (15)):

$$0 = (w_2 - w_1)\,\delta\lambda_1 + w_2\,\delta\Delta\lambda - \frac{1}{2}\,\delta e^2 \int_{\lambda_1}^{\lambda_2} \frac{\cos^2\beta}{w}\,d\lambda + e^2\int_{\lambda_1}^{\lambda_2}\frac{\cos\beta\sin\beta\,\delta\beta}{w}\,d\lambda, \tag{8}$$

$$\delta\beta = \cos^2\beta\sec^2\beta_0\cos\lambda\,\delta\beta_0, \tag{9}$$

$$\sin\beta_0\,\delta\beta_0 = \sin\beta_1\sin\alpha_{1.2}\,\delta\beta_1 - \cos\beta_1\cos\alpha_{1.2}\,\delta\alpha_{1.2}.$$

$\delta\beta$ is put in the preceding integral according to the second formula of (8) and reduced by means of the formula:

$$d\lambda = \cos\beta_0\sec^2\beta\,d\varphi, \quad \cos\lambda = \tan\beta\cot\beta_0, \quad \sin\beta = \sin\beta_0\cos\varphi,$$

as on p. 268, the first equation of (8) becomes

$$0 = (w_2 - w_1)\,\delta\lambda_1 + w_2\,\delta\Delta\lambda - \frac{1}{2}\cos\beta_0 H\,\delta e^2 + e^2\sin\beta_0 J\,\delta\beta_0, \tag{10}$$

where J has the meaning defined in (6), p. 267 and H is given by

the equation

$$H = \int_{\varphi}^{\varphi_2} \frac{d\varphi}{w}. \tag{11}$$

It remains to determine $\delta\lambda_1$. The differentiation of equation $\cot\lambda_1 = \sin\beta_1 \tan\alpha_{1.2}$, serves for this, by which one finds

$$\delta\lambda_1 = -\sin^2\lambda_1 \cos\beta_1 \tan\alpha_{1.2}\,\delta\beta_1 - \sin^2\lambda_1 \sin\beta_1 \sec^2\alpha_{1.2}\,\delta\alpha_{1.2}.$$

If we set here, as in (9), the values of $\delta\beta_1$ and $\delta\alpha_{1.2}$ in (3) and (5) we obtain by observing (2) and (3), pp. 232 and 233:

$$\left.\begin{aligned} \delta\lambda_1 &= -\frac{\cos\varphi_1 \sin\varphi_2}{\sin\Delta\varphi}\,\delta\Delta\lambda \\ \delta\beta_0 &= -\frac{1}{2}\frac{\delta e^2}{1-e^2}\sin\beta_0\cos\beta_0 - \sin\beta_0 \frac{\sin\varphi_1 \sin\varphi_2}{\sin\Delta\varphi}\,\delta\Delta\lambda. \end{aligned}\right\} \tag{12}$$

Inserting this in (10) and solving for $\delta\Delta\lambda$ upon the introduction of $\mathfrak{m}$, p. 273 (1):

$$\delta\Delta\lambda = \frac{1}{2}\frac{\delta e^2}{1-e^2}\left\{(1-e^2)H + e^2\sin^2\beta_0 J\right\}\frac{a_0\cos\beta_0\sin\Delta\varphi}{\mathfrak{m}}. \tag{13}$$

But,

$$s = a_0\int_{\varphi_1}^{\varphi_2}\sqrt{1-e^2\cos^2\beta}\,d\varphi$$

and identically

$$\sqrt{1-e^2\cos^2\beta} = \frac{1-e^2+e^2\sin^2\beta}{\sqrt{1-e^2\cos^2\beta}}.$$

Setting here $\sin\beta = \sin\beta_0 \cos\varphi$ and considering the values of H and J according to (11), p. 287 and (6), p. 267, it follows that:

$$s = a_0 \left((1 - e^2) H + e^2 \sin^2\beta_0 J \right), \tag{14}$$

from which (13) is transformed to:

$$\delta \Delta\lambda = \frac{1}{2} \frac{\delta e^2}{1 - e^2} \frac{s}{w} \cos\beta_0 \sin \Delta\varphi. \tag{15}$$

Before we substitute these formulae in (4) and (5) we differentiate first the equation for s, to which (1) and (2), p. 265 are to be compared. It becomes:

$$\frac{\delta s}{a_0} = \frac{s}{a_0} \frac{\delta a_0}{a_0} + w_2 \delta \Delta\varphi + (w_2 - w_1)\, \delta\varphi_1 - \frac{1}{2} \delta e^2 \int_{\varphi_1}^{\varphi_2} \frac{\cos^2\beta}{w}\, d\varphi + e^2 \int_{\varphi_1}^{\varphi_2} \frac{\cos\beta \sin\beta\, \delta\beta}{w}\, d\varphi,$$

$$\delta\beta = \sec\beta \cos\beta_0 \cos\varphi\, \delta\beta_0,$$

with $\delta\beta_0$ corresponding to the second formula of (12). One uses this form hereafter and sets $\cos^2\beta = 1 - \sin^2\beta_0 \cos^2\varphi$, in the first integral, obtaining:

$$\frac{\delta s}{a_0} = \frac{s}{a_0} \frac{\delta a_0}{a_0} + w_2 \delta \Delta\varphi + (w_2 - w_1) \delta\varphi_1 - \frac{1}{2} \frac{\delta e^2}{1 - e^2} \left(\frac{s}{a_0} - J \sin^2\beta_0 (1 - e^2 \cos^2\beta_0) \right) - e^2 J \sin^2\beta_0 \cos\beta_0 \frac{\sin\varphi_1 \sin\varphi_2}{\sin \Delta\varphi} \delta \Delta\lambda. \tag{16}$$

In order to simplify this further $\delta\varphi_1$ must still be determined. For this purpose differentiation of the equation $\tan\varphi_1 = \cot\beta_1 \cos\alpha_{1.2}$ yields:

$$\delta\varphi_1 = -\frac{\cos^2\varphi_1}{\sin^2\beta_1} \cos\alpha_{1.2}\, \delta\beta_1 - \cot\beta_1 \cos^2\varphi_1 \sin\alpha_{1.2} \delta\alpha_{1.2}.$$

Here $\delta\beta_1$ is introduced according to (3) and $\delta\alpha_{1,2}$ according to (5), with which is obtained:

$$\delta\varphi_1 = \frac{1}{2}\frac{\delta e^2}{1-e^2}\sin^2\beta_0 \sin\varphi_1\cos\varphi_1 - \frac{\cos\varphi_1 \sin\varphi_2}{\sin J\varphi}\cos\beta_0\,\delta\Delta\lambda. \qquad (17)$$

The introduction of (4) and (17) in (16) gives without difficulty, referring to (1), p. 273:

$$\frac{\delta s}{a_0} = \frac{s}{a_0}\frac{\delta a_0}{a_0} + \frac{\mathfrak{M}\cos\beta_0}{a_0 \sin J\varphi}\delta\Delta\lambda$$
$$+ \frac{1}{2}\frac{\delta e^2}{1-e^2}\left\{-w_1\sin^2\beta_0\sin\varphi_1\cos\varphi_1 + w_2\sin^2\beta_0\sin\varphi_2\cos\varphi_2 - \frac{s}{a_0} + J\sin^2\beta_0(1-e^2\cos^2\beta_0)\right\}.$$

The substitution of the value of $\delta\Delta\lambda$ according to (15) and restitution of the value of s according to (14) lead finally to the formula

$$\delta s = \frac{\delta a_0}{a_0}s - \frac{1}{2}\frac{\delta e^2}{1-e^2}\mathfrak{S}, \qquad (18)$$

$$\mathfrak{S} = a_0\sin^2\beta_0\left(\frac{1}{2}w_1\sin 2\varphi_1 - \frac{1}{2}w_2\sin 2\varphi_2 + (1-e^2)\int_{\varphi_1}^{\varphi_2}\frac{\sin^2\varphi}{w}\,d\varphi\right). \qquad (19)$$

Moreover, according to (5) and (15)

$$\underset{\text{in seconds}}{\delta\alpha_{1,2}} = \varrho'' \frac{1}{2}\frac{\delta e^2}{1-e^2}\sin\beta_0\cos\beta_0\left(\sin\varphi_1 + \frac{s}{\mathfrak{M}}\sin\varphi_2\right) \qquad (20)$$

and hereafter by interchanging indices and referring to (4), p. 248:

$$\underset{\text{in seconds}}{\delta\alpha_{2,1}} = \varrho'' \frac{1}{2}\frac{\delta e^2}{1-e^2}\sin\beta_0\cos\beta_0\left(\sin\varphi_2 + \frac{s}{\mathfrak{M}}\sin\varphi_1\right). \qquad (21)$$

Par. 13. Computation of $\mathfrak{S}$. If our concern is with deriving a formula useable in all cases, then we would naturally want to develop the integral for formula (19) above, as we had done earlier

for s. We will be satisfied to undertake a simpler procedure which neglects all terms of order e^4.

Referring to (2), p. 273, (19) is first transformed as follows:

$$\mathfrak{S}=a_0\sin^2\beta_0\sqrt{\frac{1-e^2}{1-k^2}}\left\{\begin{array}{c}\frac{1}{2}(\sin 2\varphi_1-\sin 2\varphi_2)-\frac{1}{4}k^2(\sin^2\varphi_1\sin 2\varphi_1-\sin^2\varphi_2\sin 2\varphi_2)\\ +(1-k^2)\int_{\varphi_1}^{\varphi_2}(\sin^2\varphi+\frac{1}{2}k^2\sin^4\varphi)\,d\varphi+\cdots\end{array}\right\}.$$

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Since now $\sin^2\varphi=\frac{1}{2}-\frac{1}{2}\cos 2\varphi$, $\sin^4\varphi=\sin^2\varphi-\frac{1}{8}+\frac{1}{8}\cos 4\varphi$ and

$$\sin^2\varphi\,\sin 2\varphi=\frac{1}{2}\sin 2\varphi-\frac{1}{4}\sin 4\varphi,$$

so one easily arrives at the formula:

$$\mathfrak{S}=a_0\sin^2\beta_0\sqrt{\frac{1-e^2}{1-k^2}}\left\{\begin{array}{c}\frac{1}{2}\left(1-\frac{5}{8}k^2\right)\Delta\varphi-\frac{1}{4}(3-k^2)(\sin 2\varphi_2-\sin 2\varphi_1)\\ -\frac{3}{64}k^2(\sin 4\varphi_2-\sin 4\varphi_1)+\cdots\end{array}\right\}$$

and from this reaching a sufficient development as a rule:

$$\mathfrak{S}=b_0\sin^2\beta_0\left\{\begin{array}{c}\frac{1}{2}\left(1-\frac{1}{8}k^2\right)\Delta\varphi-\frac{3}{2}\left(1+\frac{1}{6}k^2\right)\cos 2\varphi\sin\Delta\varphi\\ -\frac{3}{32}k^2\cos 4\varphi\sin 2\Delta\varphi+\cdots.\end{array}\right\}\qquad(1)$$

For small distances it is expedient to express $\mathfrak{S}$ directly as a function of s. In this event we consider s as a value of first-order. Then, according to (9), p. 223:

$$s=b_0\left(1+\frac{1}{4}k^2+Gl_4\right)\left(\Delta\varphi+\frac{1}{4}k^2\cos 2\varphi\sin\Delta\varphi+Gl_5\right)$$

and from this, since $\sin \Delta\varphi$ and $\Delta\varphi$ may be interchanged in the last term of the right member:

$$\Delta\varphi = \frac{s}{b_0}\left(1 - \frac{1}{2}k^2\cos^2\varphi + Gl_4\right). \qquad (2)$$

Inserting this in (1) and at the same time $\Delta\varphi\left(1 - \frac{1}{6}\Delta\varphi^2\right)$ for $\sin\Delta\varphi$, and $2\Delta\varphi$ for $\sin 2\Delta\varphi$, it results that, since the terms multiplied by k^2 disappear:

$$\mathfrak{S} = \frac{1}{2}\, s \sin^2\beta_0 \left\{1 - 3\cos 2\varphi\left[1 - \frac{1}{6}\frac{s^2}{a_0^2}\right] + Gl_4\right\}. \qquad (3)$$

If one considers that $2\varphi = \varphi_1 + \varphi_2$ and introduces the latitudes of the endpoints by means of (4), p. 248, it follows that:

$$\mathfrak{S} = \frac{1}{2}\, s \left\{\begin{array}{l} 1 - \cos^2\beta_1 \sin^2\alpha_{1.2} - 3\sin\beta_1 \sin\beta_2\left[1 - \frac{1}{6}\frac{s^2}{a_0^2}\right] \\ -3\cos\beta_1\cos\beta_2\cos\alpha_{1.2}\cos\alpha_{2.1}\left[1 - \frac{1}{6}\frac{s^2}{a_0^2}\right] + Gl_4 \end{array}\right\}. \qquad (4)$$

One can write here $\cos^2\beta_2 \sin^2\alpha_{2.1}$ for $\cos^2\beta_1 \sin^2\alpha_{1.2}$.

Par. 14. Formulae for Short Distances. We introduce here the geographic latitude obtained from the formulae (19), (20) and (21) of Par. 12 and (4) of Par. 13:

$$\delta s = \frac{\delta a_0}{a_0}\, s - \frac{1}{2}\,\frac{\delta e^2}{1 - e^2}\,\mathfrak{S}, \qquad (1)$$

$$\mathfrak{S} = \frac{1}{2}\, s \left\{\begin{array}{l} 1 - \left(\frac{1}{W_1^2}\cos^2 B_1 \sin^2\alpha_{1.2} \text{ oder } \frac{1}{W_2^2}\cos^2 B_2 \sin^2\alpha_{2.1}\right) \\ -\frac{3}{W_1 W_2}\left\{\begin{array}{c} \sin B_1 \sin B_2 [1 - e^2] \\ +\cos B_1 \cos B_2 \cos\alpha_{1.2}\cos\alpha_{2.1} \end{array}\right\}\left[1 - \frac{1}{6}\frac{s^2}{a_0^2}\right] + Gl_4 \end{array}\right\}; \qquad (2)$$

$$\delta\alpha_{1.2} = \varrho'' \frac{1}{2} \frac{\delta e^2}{1-e^2} \left\{ \begin{array}{l} \frac{1}{W_1^2} \cos^2 B_1 \sin\alpha_{1.2} \cos\alpha_{1.2} \\ + \frac{1}{W_2^2} \frac{s}{m} \cos^2 B_2 \sin\alpha_{2.1} \cos\alpha_{2.1} \end{array} \right\}; \quad (3)$$

$$\delta\alpha_{2.1} = \varrho'' \frac{1}{2} \frac{\delta e^2}{1-e^2} \left\{ \begin{array}{l} \frac{1}{W_2^2} \frac{s}{m} \cos^2 B_1 \sin\alpha_{1.2} \cos\alpha_{1.2} \\ + \frac{1}{W_2^2} \cos^2 B_2 \sin\alpha_{2.1} \cos\alpha_{2.1} \end{array} \right\}; \quad (4)$$

$$\log\left[1 - \frac{1}{6}\frac{s^2}{a_0^2}\right] = -\frac{M}{6a_0^2} s^2 + Gl_4; \quad \log\frac{s}{m} = \frac{M}{6a_0^2} s^2 + Gl_4. \quad (5)$$

One can also introduce the mean latitude and mean azimuth. We find this to be of no advantage however, since the terms with s^2 must be considered. If, however, one may omit these, since as a rule they are sides of a measurable triangle, and then can more easily write:

$$\left. \begin{array}{c} \delta s = \frac{\delta a_0}{a_0} s - \frac{1}{2} \frac{\delta e^2}{1-e^2} \mathbb{S}, \\ \mathbb{S} = \frac{s}{W^2} (2\cos^2 B \cos^2\alpha - [1-e^2]\sin^2 B + \cdots); \\ \delta\alpha_{1.2} = \delta\alpha_{2.1} = \frac{1}{2} \frac{\delta e^2}{1-e^2} \frac{\varrho''}{W^2} (2\cos^2 B \sin\alpha\cos\alpha + \cdots); \\ B = \frac{B_1 + B_2}{2}, \qquad \alpha = \frac{\alpha_{1.2} + \alpha_{2.1} - 180^\circ}{2}, \end{array} \right\} \quad (6)$$

W with argument B

In the parentheses the terms with $s^2 : a_0^2$ are neglected, but e^2 is retained.

Par. 15. Differential Formulae for the Endpoint P_2 of a Geodetic Line in Relation to Changes of a_0 and e^2. These

can be deduced from Par. 12, where, after the modification of a_1 and e^2 under the assumption of constant geographic position s and $\alpha_{1.2}$

are thought of as being changed again, because of the negative values of the previous changes, and by p. 280, (5) and (6) the subsequent variations of β_2, $L_{1,2}$ and $\alpha_{2,1}$ may be determined. We therefore reverse the signs of the last named formulae (6), keep the reduced latitude and so it results in:

$$\delta\beta_2 = -\frac{\delta s}{a_0 w_2}\cos\alpha_{2,1} + \frac{\mathfrak{m}}{a_0 w_2}\sin\alpha_{2,1}\,\delta\alpha_{1,2}$$

$$\delta L_{1,2} = +\frac{\delta s}{a_0}\sec\beta_2\sin\alpha_{2,1} + \frac{\mathfrak{m}}{a_0}\sec\beta_2\cos\alpha_{2,1}\,\delta\alpha_{1,2}$$

$$\delta\alpha_{2,1} = -\frac{\delta s}{a_0 w_2}\tan\beta_2\sin\alpha_{2,1} - \left\{\left(\frac{d\mathfrak{m}}{ds}\right)_{1,2} + \frac{\mathfrak{m}}{a_0 w_2}\tan\beta_2\cos\alpha_{2,1}\right\}\delta\alpha_{1,2}.$$

Here we insert for δs and $\delta\alpha_{1,2}$ the values (18) and (20) of p. 289 and combine $\delta\alpha_{2,1}$ and the expression according to (21), p. 289.

This procedure gives:

$$\left.\begin{aligned}\delta\beta_2 &= -\varrho''\frac{\delta a_0}{a_0}\frac{s}{a_0 w_2}\cos\alpha_{2,1}\\ &+\varrho''\frac{1}{2}\frac{\delta e^2}{1-e^2}\frac{1}{w_2}\left\{\frac{\mathfrak{S}}{a_0}\cos\alpha_{2,1}+\frac{\mathfrak{T}}{a_0}\sin\alpha_{2,1}\right\}\end{aligned}\right\}\quad\text{in sec.}\qquad(1)$$

$$\left.\begin{aligned}\delta L_{1,2} &= +\varrho''\frac{\delta a_0}{a_0}\frac{s}{a_0}\sec\beta_2\sin\alpha_{2,1}\\ &+\varrho''\frac{1}{2}\frac{\delta e^2}{1-e^2}\left\{-\frac{\mathfrak{S}}{a_0}\sec\beta_2\sin\alpha_{2,1}+\frac{\mathfrak{T}}{a_0}\sec\beta_2\cos\alpha_{2,1}\right\}\end{aligned}\right\}\quad\text{in sec.}\qquad(2)$$

$$\left.\begin{aligned}\delta\alpha_{2,1} &= -\varrho''\frac{\delta a_0}{a_0}\frac{s}{a_0 w_2}\tan\beta_2\sin\alpha_{2,1}\\ &+\varrho''\frac{1}{2}\frac{\delta e^2}{1-e^2}\frac{1}{w_2}\left\{\frac{\mathfrak{S}}{a_0}\tan\beta_2\sin\alpha_{2,1}-\frac{\mathfrak{T}}{a_0}\tan\beta_2\cos\alpha_{2,1}+\mathfrak{U}w_2\right\}\end{aligned}\right\}\quad\text{in sec.}\qquad(3)$$

$$\mathfrak{T} = (\mathfrak{m}\sin\varphi_1 + s\sin\varphi_2)\sin\beta_0\cos\beta_0$$

$$\mathfrak{U} = \left\{\begin{aligned}&\left(\sin\varphi_1+\frac{s}{\mathfrak{m}}\sin\varphi_2\right)\left[1-\left(\frac{d\mathfrak{m}}{ds}\right)_{1,2}\right]\\ &-(\sin\varphi_2-\sin\varphi_1)\left[\frac{s}{\mathfrak{m}}-1\right]\end{aligned}\right\}\sin\beta_0\cos\beta_0.\qquad(4)$$

Par. 15. Differential Formulae for the Endpoint P_2 of a Geodetic Line in Relation to Changes of e^2 etc.

Here expressions (5), p. 274 and (1), p. 290 are to be considered for $\mathfrak{m}$ and $\mathbb{S}$, if any large distances are involved.

If $s : a_0$ is a value of first-order, then the formulae (4), p. 290 suffice for $\mathbb{S}$ and the expression $s\left(1 - \frac{1}{6}\frac{s^2}{a_0^2} + Gl_4\right)$ for $\mathfrak{m}$, (cf. Par. 11, p. 283).

The preceding formulas may be put into essentially different form, in which are substituted the complete expressions for $\mathbb{S}$, $\mathfrak{m}$ and s according to (19), p. 289, (1), p. 273 and (14), p. 288.

One arrives at the same modifications through direct development of the differential formulae as yielded in a computation employed as a check. The formulae are:

$$\left.\begin{array}{l}\delta\beta_2 = -\varrho''\dfrac{\delta a_0}{a_0}\dfrac{s}{a_0 w_2}\cos\alpha_{2.1} \\ \text{in sec.} \\ +\varrho''\dfrac{1}{2}\dfrac{\delta e^2}{1-e^2}\dfrac{1}{w_2}\left\{\begin{array}{l}-\mathfrak{A}(\sin\beta_1\sin\beta_2\cos\alpha_{2.1}+\cos\beta_1\cos\beta_2\cos\alpha_{1.2}) \\ +(\mathfrak{B}-[1-\delta\cos^2\beta_0\cos^2\varphi_1]\mathfrak{C})\cos\alpha_{2.1}\end{array}\right.\end{array}\right\} \quad (5)$$

$$\left.\begin{array}{l}\delta L_{1.2} = +\varrho''\dfrac{\delta a_0}{a_0}\dfrac{s}{a_0}\sec\beta_2\sin\alpha_{2.1} \\ \text{in sec.} \\ +\varrho''\dfrac{1}{2}\dfrac{\delta e^2}{1-e^2}\sec\beta_2\sin\alpha_{2.1}\left\{\begin{array}{l}\mathfrak{A}\sin\beta_1\sin\beta_2-\mathfrak{B}\sin^2\beta_2 \\ +(1+\delta\sin^2\beta_1\sin^2\varphi_2)\mathfrak{C}\end{array}\right.\end{array}\right\} \quad (6)$$

$$\left.\begin{array}{l}\delta\alpha_{2.1} = -\varrho''\dfrac{\delta a_0}{a_0}\dfrac{s}{a_0 w_2}\tan\beta_2\sin\alpha_{2.1} \\ \text{in sec.} \\ +\varrho''\dfrac{1}{2}\dfrac{\delta e^2}{1-e^2}\sec\beta_2\sin\alpha_{2.1}\dfrac{1}{w_2}\left\{\begin{array}{l}-\mathfrak{A}\sin\beta_1+\mathfrak{B}\sin\beta_2 \\ -[1-\delta\cos^2\beta_0\cos^2\varphi_1]\mathfrak{C}\sin\beta_2\end{array}\right.\end{array}\right\} \quad (7)$$

$$\mathfrak{A} = w_2\sin\Delta\varphi \quad (8)$$

$$\mathfrak{B} = (1-e^2)H + (w_1 - w_2)\sin\varphi_1\cos\varphi_1 \quad (9)$$

$$\mathfrak{C} = (1-e^2)J\sin^2\beta_0\,. \quad (10)$$

δ in the parentheses on the right has its earlier meaning $e^2:(1-e^2)$, but otherwise denotes differentiation. J and H are the integrals introduced earlier:

$$J=\int_{\varphi_1}^{\varphi_2}\frac{\cos^2\varphi}{w}\,d\varphi, \qquad H=\int_{\varphi_1}^{\varphi_2}\frac{1}{w}\,d\varphi.$$

The first of these was already developed in (3), p. 273; the latter comes from J together with the integral developed for $\mathfrak{S}$ p. 290. One generally deals with $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{S}$ and obtains:

$$\mathfrak{B}=\sqrt{1-e^2}\left\{\begin{array}{l}\left(1-\frac{1}{4}k^2\right)J\varphi-\frac{1}{4}k^2\cos 2\varphi\sin J\varphi\\ +\frac{1}{4}k^2\sin 2\varphi_1\sin 2\varphi\sin J\varphi+\cdots\end{array}\right\} \qquad (9^*)$$

$$\mathfrak{C}=\sqrt{1-e^2\sin^2\beta_0}\left\{\begin{array}{c}\frac{1}{2}\left(1-\frac{3}{8}k^2\right)J\varphi+\frac{1}{2}\left(1-\frac{1}{2}k^2\right)\cos 2\varphi\sin J\varphi\\ -\frac{1}{32}k^2\cos 4\varphi\sin 2J\varphi+\cdots.\end{array}\right\} \qquad (10^*)$$

Par. 16. The Preceding Formulae for Short Distances: We introduce here the geographic latitude and obtain from formulae (1) through (3) of the preceding paragraphs, observing relations (5), p. 280:

(See Illustrative Material on Next Page.)

Par. 16. The Preceding Formulae for Short Distances:

$$\left.\begin{aligned}
\delta B_2 &= -\varrho'' \frac{\delta a_0}{a_0}\frac{s}{a_0}\frac{W_2^2}{1-e^2}\cos\alpha_{2.1} \\
&\quad +\varrho''\frac{1}{2}\frac{\delta e^2}{1-e^2}\frac{W_2^2}{1-e^2}\left\{\frac{\mathfrak{S}}{a_0}\cos\alpha_{2.1}+\frac{\mathfrak{T}}{a_0}\sin\alpha_{2.1}\right\}, \\
\delta L_{1.2} &= +\varrho''\frac{\delta a_0}{a_0}\frac{s}{a_0}W_2\sec B_2\sin\alpha_{2.1} \\
&\quad +\varrho''\frac{1}{2}\frac{\delta e^2}{1-e^2}W_2\left\{-\frac{\mathfrak{S}}{a_0}\sec B_2\sin\alpha_{2.1}+\frac{\mathfrak{T}}{a_0}\sec B_2\cos\alpha_{2.1}\right\}, \\
\delta\alpha_{2.1} &= -\varrho''\frac{\delta a_0}{a_0}\frac{s}{a_0}W_2\tan B_2\sin\alpha_{2.1} \\
&\quad +\varrho''\frac{1}{2}\frac{\delta e^2}{1-e^2}W_2\left\{\frac{\mathfrak{S}}{a_0}\tan B_2\sin\alpha_{2.1}-\frac{\mathfrak{T}}{a_0}\tan B_2\cos\alpha_{2.1}+\frac{1}{W_2}\mathfrak{U}\right\}.
\end{aligned}\right\} \quad (1)$$

According to (2), p. 291, when $s:a_0$ is a magnitude of first-order, then:

$$\mathfrak{S}=\frac{1}{2}s\left\{\begin{aligned}
&1-\left(\frac{1}{W_1^2}\cos^2 B_1\sin^2\alpha_{1.2}\ \text{oder}\ \frac{1}{W_2^2}\cos^2 B_2\sin^2\alpha_{2.1}\right) \\
&-\frac{3}{W_1W_2}\left\{\begin{aligned}&\sin B_1\sin B_2[1-e^2]\\&+\cos B_1\cos B_2\cos\alpha_{1.2}\cos\alpha_{2.1}\end{aligned}\right\}\left[1-\frac{1}{6}\frac{s^2}{a_0^2}\right]+Gl_4.
\end{aligned}\right\} \quad (2)$$

With $\mathfrak{m}=s\left(1-\frac{1}{6}\frac{s^2}{a_0^2}+Gl_4\right)$ one further obtains:

$$\mathfrak{T}=s\left\{\begin{aligned}&\frac{1}{W_1^2}\cos^2 B_1\cos\alpha_{1.2}\sin\alpha_{1.2}\left[1-\frac{1}{6}\frac{s^2}{a_0^2}\right]\\&+\frac{1}{W_2^2}\cos^2 B_2\cos\alpha_{2.1}\sin\alpha_{2.1}+Gl_4\end{aligned}\right\}, \quad (3)$$

$$\mathfrak{U}=\frac{s}{2a_0^2}\mathfrak{T}-\frac{s^3}{6a_0^3}\sin B_1\cos B_1\sin\alpha_{1.2}+Gl_4. \quad (4)$$

If one is concerned with the sides of measurable triangles, then one will be able to neglect the terms with $s^3:a_0^3$ in the curly brackets of the first two formulae of 1, and terms with $s^2:a_0^2$ in the third formula of 1. Then it becomes simpler:

$$\left.\begin{aligned}
&\delta B_2 \underset{\text{in sec.}}{=} -\varrho'' \frac{\delta a_0}{a_0}\frac{s}{a_0}\frac{W_2^3}{1-e^2}\cos\alpha_{2.1} \\
&-\varrho''\frac{1}{2}\frac{\delta e^2}{1-e^2}\frac{s}{a_0}\frac{W_2}{1-e^2}\frac{W_2^2}{W^2}\left\{\begin{matrix}\cos^2 B\cos\alpha_{1.2}\\ -(1-[2-e^2]\sin^2 B)\cos\alpha_{2.1}+\cdots\end{matrix}\right\}, \\
&\delta L_{1.2} \underset{\text{in sec.}}{=} +\varrho''\frac{\delta a_0}{a_0}\frac{s}{a_0} W_2 \sec B_2 \sin\alpha_{2.1} \\
&-\varrho''\frac{1}{2}\frac{\delta e^2}{1-e^2}\frac{s}{a_0}\frac{\sec B_2}{W_2}\frac{W_2^2}{W^2}\left\{\begin{matrix}\cos^2 B\sin\alpha_{1.2}\\ +(1-[2-e^2]\sin^2 B)\sin\alpha_{2.1}+\cdots\end{matrix}\right\}, \\
&\delta\alpha_{2.1} \underset{\text{in sec.}}{=} -\varrho''\frac{\delta a_0}{a_0}\frac{s}{a_0} W_2\tan B_2\sin\alpha_{2.1} \\
&+\varrho''\frac{1}{2}\frac{\delta e^2}{1-e^2}\frac{s}{a_0}\frac{\tan B_2}{W_2}\frac{W_2^2}{W^2}\left\{\begin{matrix}\cos^2 B\sin\alpha_{1.2}\\ +(1-[2-e^2]\sin^2 B)\sin\alpha_{2.1}+\cdots\end{matrix}\right\},
\end{aligned}\right\} \quad (5)$$

$$B=\frac{1}{2}(B_1+B_2),\quad W \text{ with argument } B.$$

In the curly brackets of the first two formulae, the terms with $s^3:a_0^3$ are neglected and in the formula for $\delta\alpha_{2.1}$ already the terms with $s:a_0$.

Differential forms which correspond to (5), (6) and (7) of the previous paragraph were developed by Bessel in the 1837 Astronomische Nachrichten (Astronomical Proceedings), Vol. 14, pp. 269 ff. (Proceedings, Vol. 3, p.34 (27)). He proceeded in part differently than we (it is apparent that we generally did not show direct derivations) in that he considered $\Delta\varphi$ and Δl, instead of φ and l, as the variables in the integrals. This is not however an advantageous procedure.

Our exact formulae for $\delta\beta_2$ and $\delta\alpha_{1.2}$ do not agree with $\delta L_{1.2}$ after

the appropriate transformation. Bessel is mistaken here. Specifically if one differentiates the integrals for both s and $L_{1,2}$ $\delta\beta$ arises in both cases (compare, for example, pp. 266 and 268). But these are different, for one involves a constant φ (for Bessel, $\Delta\varphi$) the other, a constant λ, (for Bessel, $\Delta\lambda$) but in both cases Bessel takes the first value. As a result the coefficient of δc^2 in his $\delta L_{1,2}$ is deficient by a term containing e^2 as a factor. (In fact, the second part of his P, in order to be made correct must be multiplied under the integral sign by $\cos u \sec u' \cos s \sec \omega$.) This coefficient assumes a very complicated form in the series development due to this oversight, but the results of the numerical computation are not greatly influenced.

Moreover, for Bessel in the first term of the right member of the second formula of (27), p. 34, one of the two r's in the numerator is to be cancelled.

It may be observed, that in the given place, p. 35, Column 1 and in the

formula serving for a different development for $d\sigma$, the term depending on $\delta\alpha'$ is absent, which only affects terms with e^2.

Jordan, in the second volume, p. 441 of his Handbuch der Vermessungskunde (Handbook of Surveying) gives shortened differential formulae, which assume s as given and, instead of $\alpha_{1,2}$ and B_1, the arithmetic mean of the latitude

and azimuth.

Par. 17. Series Development for the Transformation of Expressions of Geographic Coordinates with Powers of s. In general, according to Taylor's Theorem, one has:

$$\left.\begin{aligned} B_2 &= B_1 + \left(\frac{dB}{ds}\right)_1 \cdot \frac{s}{1} + \left(\frac{d^2B}{ds^2}\right)_1 \cdot \frac{s^2}{1.2} + \left(\frac{d^3B}{ds^3}\right)_1 \cdot \frac{s^3}{1.2.3} + \cdots \\ L_2 &= L_1 + \left(\frac{dL}{ds}\right)_1 \cdot \frac{s}{1} + \left(\frac{d^2L}{ds^2}\right)_1 \cdot \frac{s^2}{1.2} + \left(\frac{d^3L}{ds^3}\right)_1 \cdot \frac{s^3}{1.2.3} + \cdots \\ \alpha_{2.1} &= \alpha_{1.2} + 180^0 + \left(\frac{d\alpha}{ds}\right)_1 \cdot \frac{s}{1} + \left(\frac{d^2\alpha}{ds^2}\right)_1 \cdot \frac{s^2}{1.2} + \left(\frac{d^3\alpha}{ds^3}\right)_1 \cdot \frac{s^3}{1.2.3} + \cdots, \end{aligned}\right\} \quad (1)$$

where the index 1, attached to the differential quotients means that the values of B, L and α for the point P_1 are to be inserted after differentiation. α is to be understood as indicating the azimuth of the line from P_1 increasing in the direction of P_2.

If in formula (6), p. 280 $\delta\alpha_{1.2} =$ zero one obtains without difficulty, when d is written in place of δ as the sign of differentiation and the index 2 is neglected:

$$\left.\begin{aligned} \frac{dB}{ds} &= -\frac{W^3 \cos\alpha}{a_0(1-e^2)} = -\frac{W^3(1+\delta)\cos\alpha}{a_0} \\ \frac{dL}{ds} &= +\frac{W}{a_0} \sec B \sin\alpha \\ \frac{d\alpha}{ds} &= -\frac{W}{a_0} \tan B \sin\alpha. \end{aligned}\right\} \quad (2)$$

The δ appearing here is the already used value $e^2 : (1-e^2)$, the introduction of which is advantageous at this point. One has further

$$W = \sqrt{1 - e^2 \sin^2 B}, \quad (3)$$

$$\frac{dW}{ds} = \frac{dW}{dB}\frac{dB}{ds} = -\frac{e^2 \sin 2B}{2W}\frac{dB}{ds} = \delta \frac{W^3}{2a_0} \sin 2B \cos\alpha \quad (4)$$

and

$$\frac{d^2B}{ds^2} = -\frac{W^2}{a_0}(1+\delta)\left\{3\cos\alpha\,\frac{dW}{ds} - W\sin\alpha\,\frac{d\alpha}{ds}\right\}$$

$$\frac{d^2L}{ds^2} = \frac{1}{a_0}\sec B\left\{\sin\alpha\,\frac{dW}{ds} + W\tan B\sin\alpha\,\frac{dB}{ds} + W\cos\alpha\,\frac{d\alpha}{ds}\right\}$$

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$$\frac{d^2\alpha}{ds^2} = -\frac{1}{a_0}\left\{\begin{array}{c}\tan B\sin\alpha\,\dfrac{dW}{ds} + W\sec^2 B\sin\alpha\,\dfrac{dB}{ds} \\ + W\tan B\cos\alpha\,\dfrac{d\alpha}{ds}\end{array}\right\}.$$

substituting here (2) and (4) and observing the relation

$$W^2(1+\delta) = 1 + \delta\cos^2 B. \qquad (5)$$

it follows that:

$$\left.\begin{aligned}\frac{d^2B}{ds^2} &= -\frac{W^4}{a_0^2}(1+\delta)\left\{\tan B\sin^2\alpha + \frac{3}{2}\delta\sin 2B\cos^2\alpha\right\} \\ \frac{d^2L}{ds^2} &= -\frac{W^2}{a_0^2}\tan B\sec B\sin 2\alpha \\ \frac{d^2\alpha}{ds^2} &= +\frac{W^2}{2a_0^2}\sin 2\alpha\left\{1 + 2\tan^2 B + \delta\cos^2 B\right\}.\end{aligned}\right\} \qquad (6)$$

Repeating the differentiation gives:

$$\left.\begin{aligned}\frac{d^3B}{ds^3} &= \frac{W^5}{a_0^3}(1+\delta)\cos\alpha\left\{[1+3\tan^2 B]\sin^2\alpha + \delta(3\cos 2B\cos^2\alpha\right. \\ &\qquad \left. + [1-10\sin^2 B]\sin^2\alpha) - 3\delta^2[6\sin^2 B - 1]\cos^2 B\cos^2\alpha\right\} \\ \frac{d^3L}{ds^3} &= \frac{2W^3}{a_0^3}\sec B\sin\alpha\left\{[1+3\tan^2 B]\cos^2\alpha\right. \\ &\qquad \left. - \tan^2 B\sin^2\alpha + \delta\cos^2 B\cos^2\alpha\right\} \\ \frac{d^3\alpha}{ds^3} &= \frac{W^3}{a_0^3}\tan B\sin\alpha\left\{-[5+6\tan^2 B]\cos^2\alpha\right. \\ &\qquad \left. + [1+2\tan^2 B]\sin^2\alpha - \delta\cos^2 B\cos 2\alpha + 4\delta^2\cos^4 B\cos^2\alpha\right\}.\end{aligned}\right\} \qquad (7)$$

Chapter 6. Differential Formulae and Series Expansions for the Geodetic Line.

In the formation of yet higher differential quotients we neglect e^2 and ϑ and obtain

$$\left.\begin{aligned}
\frac{d^4B}{ds^4} &= \frac{1}{a_0^4}\tan B\sin^2\alpha\,\{-4\,[2+3\tan^2B]\cos^2\alpha \\
&\qquad +[1+3\tan^2B]\sin^2\alpha+\cdots\} \\
\frac{d^4L}{ds^4} &= \frac{8}{a_0^4}\tan B\sec B\cos\alpha\sin\alpha\,\{-[2+3\tan^2B]\cos^2\alpha \\
&\qquad +[1+3\tan^2B]\sin^2\alpha+\cdots\} \\
\frac{d^4\alpha}{ds^4} &= \frac{1}{a_0^4}\cos\alpha\sin\alpha\,\{[5+28\tan^2B+24\tan^4B]\cos^2\alpha \\
&\qquad -[1+20\tan^2B+24\tan^4B]\sin^2\alpha+\cdots\}
\end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned}
\frac{d^5B}{ds^5} &= \frac{1}{a_0^5}\cos\alpha\sin^2\alpha\,\{[8+60\tan^2B+60\tan^4B]\cos^2\alpha \\
&\qquad -[1+30\tan^2B+45\tan^4B]\sin^2\alpha+\cdots\} \\
\frac{d^5L}{ds^5} &= \frac{8}{a_0^5}\sec B\sin\alpha\,\{[2+15\tan^2B+15\tan^4B]\cos^4\alpha \\
&\qquad -[1+20\tan^2B+30\tan^4B]\sin^2\alpha\cos^2\alpha \\
&\qquad +[1+3\tan^2B]\tan^2B\sin^4\alpha+\cdots\}
\end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned}
\frac{d^5\alpha}{ds^5} &= \frac{1}{a_0^5}\tan B\sin\alpha\,\{-[61+180\tan^2B+120\tan^4B]\cos^4\alpha \\
&\qquad +[58+280\tan^2B+240\tan^4B]\sin^2\cos^2\alpha \\
&\qquad -[1+20\tan^2B+24\tan^4B]\sin^4\alpha+\cdots\}.
\end{aligned}\right\} \quad (9)$$

We designate:

$$\left.\begin{aligned}
&\frac{W_1}{a_0}\,s\cos\alpha_{1,2} \quad \text{with} \quad u \\
&\frac{W_1}{a_0}\,s\sin\alpha_{1,2} \quad \text{with} \quad v \\
&\tan B_1 \quad \text{with} \quad t_1
\end{aligned}\right\} \quad (10)$$

and find from the developed differential quotients of the series expansion (1):

(See Illustrative Material on Next Page.)

$$B_2 = B_1 - \varrho'' W_1^2 (1+\delta) \left\{ \begin{array}{l} u + \frac{1}{2} t v^2 - \frac{1}{6}[1+3t^2] u v^2 + \frac{1}{6}[2+3t^2] t u^2 v^2 \\ - \frac{1}{24}[1+3t^2] t v^4 - \frac{1}{30}[2+15t^2+15t^4] u^3 v^2 \\ + \frac{1}{120}[1+30t^2+45t^4] u v^4 + \frac{3}{4}\delta \sin 2B_1 u^2 \\ - \delta\left(\frac{1}{2}\cos 2B_1 u^3 + \frac{1}{6}[5\cos 2B_1 - 4] u v^2\right) + Gl_6 \end{array} \right\} \quad (11)$$

in sec. in sec.

$$L_2 = L_1 + \varrho'' \sec B_1 \left\{ \begin{array}{l} v - t u v + \frac{1}{3}[1+3t^2] u^2 v - \frac{1}{3} t^2 v^3 \\ - \frac{1}{3}[2+3t^2] t u^3 v + \frac{1}{3}[1+3t^2] t u v^3 \\ + \frac{1}{15}[1+3t^2] t^2 v^5 + \frac{1}{15}[2+15t^2+15t^4] u^4 v \\ - \frac{1}{15}[1+20t^2+30t^4] u^2 v^3 + \frac{1}{3}\delta \cos^2 B_1 u^2 v + Gl_6 \end{array} \right\} \quad (12)$$

in sec. in sec.

$$\alpha_{2,1} = \alpha_{1,2} + 180^0 - \varrho'' \left\{ \begin{array}{l} t v - \frac{1}{2}[1+2t^2] u v - \frac{1}{6}[1+2t^2] t v^3 \\ + \frac{1}{6}[5+6t^2] t u^2 v - \frac{1}{24}[5+28t^2+24t^4] u^3 v \\ + \frac{1}{24}[1+20t^2+24t^4] u v^3 \\ + \frac{1}{120}[61+180t^2+120t^4] t u^4 v \\ - \frac{1}{120}[58+280t^2+240t^4] t u^2 v^3 \\ + \frac{1}{120}[1+20t^2+24t^4] t v^5 - \frac{1}{2}\delta \cos^2 B_1 u v \\ + \frac{1}{12}\delta \sin 2B_1 (u^2 - v^2) v + Gl_6. \end{array} \right\} \quad (13)$$

in sec. in sec.

If one equates e^2 and δ to zero, one arrives at the formulae, which may be derived from the developments on pp. 126 et. seq.

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For $\alpha_{1,2} = 0^0$ and 90^0 respectively one obtains formulae from (11), (12) and (13) which can be used to transform longitude, latitude and azimuth as soon as the rectangular geodetic coordinates x and y of P_2 with respect to P_1 and its meridian are given. In particu-

lar s is transformed to x for $\alpha_{1,2} = 0^0$ and y for $\alpha_{1,2} = 90^0$. The point of the foot of the ordinate y serves as intermediate point for the transformation of the geographic latitude, as is easy to see.

In the statement about the order of the neglected terms in (11) to (13) it is assumed that the value $t = \tan B_1$, which appears in general in any term with the same power as s, does not exceed unity by so much, that the character of the order of magnitude is essentially changed. The rapid convergence not only implies a small value of $\frac{s}{a_0}$, but also of $\frac{s}{a_0} \tan B_1$.

In order to obtain an idea of the precision in every case, one may form the remainder according to p. 25, (1) by means of the first neglected differential quotient, or on the basis of (2), by means of the highest of the retained differential quotients. In the last case it will be shown for the highest retained term that the maximum of its variation arises by use of the values of B and α for a point between P_1 and P_2 (instead of for P_1).

It is easily recognized, that for $s = 0{,}1 a_0$ the formulas for some distance from the equator B_2, L_2 and $\alpha_{2,1}$ can be short by tenths of a second, but on the other hand, for decreasing values of s the accuracy rapidly increases. For $s = 0{,}05 a_0$ in medium latitudes, accuracy to within hundredths of a second will surely be obtained.

Formulae (11), (12) and (13) can only be used, when not <u>one</u>, but <u>several</u> lines radiate from a point for which the transformation of coordinates is carried out. This applies to polar coordinates with P_1 as center, and in some measure for stations of a dense triangulation net. In this case the tedious work in the compu-

tation of the many functions of B_1 is reduced, because it may be divided to some extent among more lines.

If s is a line whose length equals the side of a measurable triangle, whence we have approximately $\leq 0{,}01 a_0$, then in the above formula one may omit, as rule, the highest of the given terms without affecting the fourth decimal place of B_2, L_2 and $\alpha_{2.1}$ expressed in seconds.

Closer investigation involves lengthy computation.

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Since in a particular case it is easy enough to perform and should not be dispensed with , we have ample cause not to go into it here.

The author noticed the significance of formulae (11) to (13) for the case when P_1 is a center for polar coordinates in the work: Die bayerische Landesvermessung (The Bavarian Land Measurement), etc., 1873, pp. 517 ff. In it there is an 1810 treatise of Soldner's (cf. loc. cit., p. 262), which begins with a summary of generally valid formulae on the transformation of geographic coordinates, in which only the first power of the flattening is retained. Soldner employs series expansions, but only up to s^3. There is also a treatment by C. V. Orff, which avoids the general formulae and proceeds directly to the differential formulae, retaining s^4 (excluded in $\alpha_{2.1}$). Our formulae agree with Orff's

up to a deviation in the fifth differential quotient of B_2 with respect to s. In the development we have the simplification δ instead of e^2 and the two values u and v instead of the three: $s_{12}\cos\alpha_{1,2}$ and $\sin\alpha_{1,2}$. (Orff even introduced rectangular coordinates, the change of which by transformations of geographic coordinates will be considered further in this book.)

In the 80th volume of Comptes Rendus of the French Academy of Science for the year 1875 on pp. 36 ff. are the Trepied formulae, to which our formulae (11) to (13) correspond, without however containing terms with s^5, and without being brought into simplest form by the introduction of δ, u and v. These formulae come from formulae given by Legendre in an 1806 supplement of the Memoirs of the Academy, which formulae retain s^3 and e^2s^3. They are not completely correct however. In the formula for B_2 in the term with s^2, the denominator ϱN must be inserted instead of ϱ^2. Moreover, the terms in e^4 are erroneous. This shows a development of Levret in Volume 76 of the Comptes Rendus pp. 410 ff. which gives terms to s^3 completely and with which we agree (after adjustment of a misprint in the term with s^2 in the azimuth form).

Par. 18. Numerical Example I. Given: $B_1 = 52^\circ\,30'\,16{,}7''$ (Berlin)

$$s = 529979{,}58^m \qquad \alpha_{1,2} = 239^\circ\,33'\,0{,}689''.$$

We compute now with seven-place logarithms, determining the

eighth decimal place by proportional parts. One has next, as is except for special values for s and $\alpha_{1,s}$:

$$\log W_1 = 9{,}9990857.5 - 10$$
$$\log \frac{W_1}{a_0} = 3{,}1944422.9 - 10$$
$$\log (1+\delta) = 0{,}0029083.6$$
$$\log \varrho'' = 5{,}3144251.3$$
$$\log \delta = 7{,}82732 - 10$$

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$$\log t = 0{,}1150923.1$$
$$t^2 = 1{,}69897.$$
$$t^4 = 2{,}886$$
$$1 + 2t^2 = 4{,}39794 = [0{,}643249]$$
$$1 + 3t^2 = 6{,}09691 = [0{,}785110]$$
$$2 + 3t^2 = 7{,}0969 = [0{,}85107]$$
$$5 + 6t^2 = 15{,}1938 = [1{,}18167]$$
$$2 + 15t^2 + 15t^4 = 70{,}78 = [1{,}8499]$$
$$5 + 28t^2 + 24t^4 = 121{,}83 = [2{,}0858]$$
$$1 + 30t^2 + 45t^4 = 181{,}85 = [2{,}2597]$$
$$1 + 20t^2 + 30t^4 = 121{,}56 = [2{,}0848]$$
$$1 + 20t^2 + 24t^4 = 104{,}24 = [2{,}0180]$$
$$58 + 280t^2 + 240t^4 = 1226{,}4 = [3{,}0886]$$
$$61 + 180t^2 + 120t^4 = 713{,}1 = [2{,}8532]$$
$$\log \cos B_1 = 9{,}7844012.8 - 10$$
$$\log \cos^2 B_1 = 9{,}5688 - 10$$
$$\log \sin 2B_1 = 9{,}9849 - 10$$
$$\log \cos 2B_1 = 9{,}4133_n - 10$$
$$\log [5 \cos 2B_1 - 4] = 0{,}7239_n.$$

with this it is found from formulae (11), (12) and (13), p. 298:

(See Illustrative Material on Next Page.)

$$B_2 = 52^\circ\,30'\,16{,}7'' + [5{,}3155049.9_n]\,u + [5{,}1295673_n]\,v^2$$
in sec.
$$+ [5{,}322464]\,uv^2 + [5{,}50352_n]\,u^2v^2 + [4{,}8355]\,v^4$$
$$+ [5{,}6883]\,u^3v^2 + [5{,}4960_n]\,uv^4 + [3{,}0028_n]\,u^2$$
$$+ [2{,}2551_n]\,u^3 + [3{,}0885_n]\,uv^2 + \cdots$$

$$L_2 = L_1 + [5{,}5300238.5]\,v + [5{,}6451162_n]\,uv$$
in sec. in sec.
$$+ [5{,}838013]\,u^2v + [5{,}283088_n]\,v^3 + [6{,}01906_n]\,u^3v$$
$$+ [5{,}95310]\,uv^3 + [5{,}3692]\,v^5 + [6{,}2038]\,u^4v$$
$$+ [6{,}4387_n]\,u^2v^3 + [2{,}4490]\,u^2v + \cdots$$

$$\alpha_{2,1} = \alpha_{1,2} + 180^\circ + [5{,}4295174.4_n]\,v + [5{,}656644]\,uv$$
in sec. in sec.
$$+ [5{,}294615]\,v^3 + [5{,}83304_n]\,u^2v + [6{,}0200]\,u^3v$$
$$+ [5{,}9522_n]\,uv^3 + [6{,}2035_n]\,u^4v + [6{,}4389]\,u^2v^3$$
$$+ [5{,}3683_n]\,v^5 + [2{,}4095]\,uv + [2{,}4074_n]\,(u^2 - v^2)\,v + \cdots$$

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With the special values of s and $\alpha_{1,2}$ one now has

(See Illustrative Material on Next Page.)

Par. 19. Numerical Example II.

$\log s = 5{,}7242591.6$
$\log \cos \alpha_{1.2} = 9{,}7048223.3_n - 10$
$\log \sin \alpha_{1.2} = 9{,}9355442.5_n - 10$

$\log u = 8{,}6235237.8_n - 10$
$\log v = 8{,}8542457.0_n - 10.$

$B_2 =$	$L_2 - L_1 =$	$\alpha_{2.1} =$
52° 30′ 16,7″	− 6° 43′ 45,32″	59° 33′ 0,69″
+ 2′ 24 50,18	− 22 7,05	+ 5 20 20,43
− 11 28,74	− 1 26,96	+ 22 42,74
− 45,13	+ 1 10,12	− 1 12,00
− 2,88	− 5,54	+ 1 25,97
+ 1,79	+ 13,78	+ 5,56
− 0,19	− 0,44	− 13,75
+ 0,34	− 0,36	+ 0,36
− 1,78	+ 1,77	− 1,77
+ 0,01	− 0,04	+ 0,44
+ 0,26		+ 0,77
		− 0,03
54° 42′ 50,56″	− 7° 6′ 0,04″	65° 16′ 9,41″
instead 50,60	instead 0,00	instead 9,37 .

For the exact statements compare p. 247. It should be mentioned that terms below the first horizontal line of the above three columns are dependent upon δ.

Par. 19. Numerical Example II. Given: $B_1 = 57°$

$$s = 120000^m \quad \alpha_{1.2} = 315°.$$

We compute here with logarithms correct to nine decimals. Then:

$\log W_1 = 9{,}9989781.93 - 10$
$\log \frac{W_1}{a_e} = 3{,}1943347.29 - 10$
$\log (1 + \delta) = 0{,}0029083.60$
$\log \varrho'' = 5{,}3144251.33$
$\log \delta = 7{,}82732 - 10$
$\log t = 0{,}1874826.38$
$t^2 = 2{,}371184$
$t^4 = 5{,}6225$

$$1 + 2t^2 = 5{,}742368 = [0{,}7590910]$$
$$1 + 3t^2 = 8{,}113552 = [0{,}909211]$$
$$2 + 3t^2 = 9{,}11355 = [0{,}9597]$$
$$5 + 6t^2 = 19{,}22710 = [1{,}283914]$$
$$2 + 15t^2 + 15t^4 = 121{,}91 = [2{,}086]$$
$$5 + 28t^2 + 24t^4 = 206{,}33 = [2{,}3146]$$
$$1 + 30t^2 + 45t^4 = 325{,}2 = [2{,}512]$$
$$1 + 20t^2 + 30t^4 = 217{,}1 = [2{,}337]$$
$$1 + 20t^2 + 24t^4 = 183{,}36 = [2{,}2633]$$
$$58 + 280t^2 + 240t^4 = 2071 = [3{,}316]$$
$$61 + 180t^2 + 120t^4 = 1163 = [3{,}065]$$
$$\log\cos B_1 = 9{,}7361087.65 - 10$$
$$\log\cos^2 B_1 = 9{,}47222 - 10$$
$$\log\sin 2B_1 = 9{,}96073 - 10$$
$$\log\cos 2B_1 = 9{,}609_n - 10$$
$$\log[5\cos 2B_1 - 4] = 0{,}7806_n.$$

With this, formulae (11) to (13) of p. 298 give:

$$B_2 = 57^\circ\,0'\,0'' + [5{,}3152898.79_n]\,u + [5{,}2017425_n]\,v^2$$
$$\text{in sec.}$$
$$+ [5{,}446350]\,uv^2 + [5{,}6843_n]\,u^2v^2 + [5{,}0318]\,v^4$$
$$+ [5{,}924]\,u^3v^2 + [5{,}748_n]\,uv^4 + [2{,}97840_n]\,u^2$$
$$+ [2{,}450_n]\,u^3 + [3{,}1450_n]\,uv^2 + \cdots$$

$$L_2 = L_1 + [5{,}5783163.68]\,v + [5{,}7657990.1_n]\,uv$$
$$\text{in sec.} \quad \text{in sec.}$$
$$+ [6{,}010406]\,u^2v + [5{,}47616_n]\,v^3 + [6{,}2484_n]\,u^2v$$
$$+ [6{,}1979]\,uv^3 + [5{,}686]\,v^5 + [6{,}488]\,u^4v + [6{,}739_n]\,u^2v^3$$
$$+ [2{,}40074]\,u^2v + \cdots$$

$$\alpha_{2.1} = \alpha_{1.2} + 180^\circ + [5{,}5019077.71_n]\,v + [5{,}7724861]\,uv$$
$$\text{in sec.} \quad \text{in sec.}$$
$$+ [5{,}482848]\,v^3 + [6{,}007670_n]\,u^2v + [6{,}2488]\,u^3v$$
$$+ [6{,}1975_n]\,uv^3 + [6{,}487_n]\,u^4v + [6{,}738]\,u^2v^3$$
$$+ [5{,}685_n]\,v^5 + [2{,}31294]\,uv + [2{,}023_n]\,(u^2 - v^2)\,v + \cdots.$$

Par. 20. Formulae with Mean Values of the Geographic Latitude and Azimuth.

With special values of s and $\alpha_{1.2}$ one now has

$$\left.\begin{array}{l}\log s = 5{,}0791812.46 \\ \log\cos\alpha_{1.2} = 9{,}8494850.02 - 10 \\ \log\sin\alpha_{1.2} = 9{,}8494850.02_n - 10\end{array}\right\} \quad \begin{array}{l} u = 8{,}1230009.77 - 10 \\ v = 8{,}1230009.77_n - 10\end{array}$$

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$B_2 =$	$L_2 - L_1 =$	$\alpha_{2.1} =$
57° 0′ 0″	− 1° 23′ 47,09792″	135° 0′ 0″
− 45 43,41088	+ 1 42,75453	+ 1 10 16,07908
− 28,03783	− 2,39557	− 1 44,34892
+ 0,65366	+ 0,70010	− 0,71097
− 0,01501	+ 0,05500	+ 2,38053
+ 0,00334	− 0,04897	− 0,05506
+ 0,00035	− 0,00020	+ 0,04892
− 0,00023	− 0,00127	+ 0,00127
− 0,16765	+ 0,00226	− 0,00225
− 0,00066	− 0,00059	+ 0,00020
− 0,00327		− 0,03622
		− 0,00000
56° 13′ 49,02182″	− 1° 22′ 6,03263″	135° 8′ 33,35658″

These values must be correct to four decimals in seconds, which will in fact be confirmed later by the use of other formulae.

Par. 20. Formulae with Mean Values of the Geographic Latitude and Azimuth. The formulae of Par. 17 contain B_1 and $\alpha_{1.2}$ as initial values. But for many purposes formulae with mean values of B and α for both endpoints P_1 and P_2 are desirable. To accomplish this we begin with the general formulae of the previous chapter.

According to p. 248, (1):

$$\tan\frac{\Delta\beta}{2} = -\tan\frac{\Delta\varphi}{2}\,\frac{\cos\alpha}{\cos\frac{\Delta\alpha}{2}}, \qquad (1)$$

in which use is made of the abbreviations

$$\left.\begin{aligned} \beta_2-\beta_1=\Delta\beta \qquad \frac{\beta_2+\beta_1}{2}=\beta \\ \frac{\alpha_{1,2}+\alpha_{2,1}}{2}=180^\circ+\alpha \\ \alpha_{2,1}=\alpha_{1,2}+180^\circ+\Delta\alpha \end{aligned}\right\} \qquad (2)$$

In formula (1) we introduce s and M in place of $\Delta\varphi$ and $\Delta\beta$, the latter to denote the meridian-arc for the latitude-difference B_2-B_1.

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According to p. 223, (9) for small distances, in which we view $\frac{s}{a_0}$, like e, as of first-order magnitude:

$$s=b_0\frac{1+\frac{1}{4}k_1^2}{1-k_1}\left\{\Delta\varphi+k_1\cos 2\varphi\sin\Delta\varphi-\frac{1}{8}k_1^2\cos 4\varphi\sin 2\Delta\varphi+Gl_7\right\} \qquad (3)$$

and from this by the development of $\sin\Delta\varphi$ and $\sin 2\Delta\varphi$ in series:

$$s=b_0\frac{1+\frac{1}{4}k_1^2}{1-k_1}\left\{1+k_1\cos 2\varphi\left(1-\frac{1}{6}\Delta\varphi^2\right)-\frac{1}{4}k_1^2\cos 4\varphi+Gl_6\right\}\Delta\varphi,$$

in which we now substitute according to p. 220 and 221:

$$k_1=\frac{1}{4}k^2+\frac{1}{8}k^4+\cdots=\frac{1}{4}e^2\sin^2\beta_0\left(1+e^2-\frac{1}{2}e^2\sin^2\beta_0\right)+\cdots.$$

At the same time we set $\cos 4\varphi=2\cos^2 2\varphi-1$ and reduce to $\Delta\varphi$ which gives after a short computation:

$$\Delta\varphi=\frac{s}{b_0}\left\{\begin{aligned} &1-\tfrac{1}{4}e^2(1+e^2)\sin^2\beta_0(1+\cos 2\varphi)+\tfrac{1}{24}e^2\Delta\varphi^2\sin^2\beta_0\cos 2\varphi \\ &+\tfrac{3}{32}e^4\sin^4\beta_0(1+\cos 2\varphi)^2+Gl_6. \end{aligned}\right\} \qquad (4)$$

In the right member, β_0 and φ are still to be eliminated and expressed by β_1, β_2 and the azimuths $\alpha_{1.2}$ and $\alpha_{2.1}$.

However, one has since $2\varphi = \varphi_1 + \varphi_2$ and with reference to (4), p. 248:

$$\begin{aligned}\sin^2\beta_0(1+\cos 2\varphi) &= \sin^2\beta_0 + \sin^2\beta_0\cos(\varphi_1+\varphi_2)\\ &= 1 - \tfrac{1}{2}(\cos^2\beta_1\sin^2\alpha_{1.2} + \cos^2\beta_2\sin^2\alpha_{2.1})\\ &\quad + \sin\beta_1\sin\beta_2 + \cos\beta_1\cos\beta_2\cos\alpha_{1.2}\cos\alpha_{2.1}.\end{aligned}$$

If one substitutes for $\sin^2\alpha$ the equal value $1-\cos^2\alpha$, then after some reduction:

$$\sin^2\beta_0(1+\cos 2\varphi) = \left\{\begin{aligned}&1 - \cos(\beta_1+\beta_2) - \tfrac{1}{2}(\cos\beta_1 - \cos\beta_2)^2\\ &+ \tfrac{1}{2}(\cos\beta_1\cos\alpha_{1.2} + \cos\beta_2\cos\alpha_{2.1})^2.\end{aligned}\right\} \qquad (5)$$

Now, according to well-known formulae and with reference to the spherical auxiliary triangle, Figure 21, p. 248

$$\begin{aligned}\cos\beta_1 - \cos\beta_2 &= 2\sin\frac{\Delta\beta}{2}\sin\frac{\beta_1+\beta_2}{2}\\ &= -2\sin\frac{\Delta\varphi}{2}\sin\frac{\beta_1+\beta_2}{2}\sin\frac{\alpha_{1.2}+\alpha_{2.1}}{2}\sec\frac{\Delta l}{2}\end{aligned}$$

and if we introduce the abbreviations (2), and $\sin\frac{\Delta\varphi}{2}$ and $\sec\frac{\Delta l}{2}$ expressed in series:

$$\cos\beta_1 - \cos\beta_2 = -\Delta\varphi\sin\beta\cos\alpha + Gl_2. \qquad (6)$$

One has further, referring to (4), p. 248:

$$\begin{aligned}\cos\beta_1\cos\alpha_{1.2} + \cos\beta_2\cos\alpha_{2.1} &= \sin\beta_0(\sin\varphi_1 - \sin\varphi_2)\\ &= -2\sin\frac{\Delta\varphi}{2}\sin\beta_0\cos\frac{\varphi_1+\varphi_2}{2}.\end{aligned}$$

From which

$$\cos\beta_1\cos\alpha_{1.2}+\cos\beta_2\cos\alpha_{2.1}=-\Delta\varphi\sin\beta+Gl_3; \qquad (7)$$

since one has from the addition of the first two equations of (4), p. 248 the relation:

$$\sin\beta_0\cos\frac{\varphi_1+\varphi_2}{2}=\sin\beta\cos\frac{\beta_1-\beta_2}{2}\sec\frac{\varphi_1-\varphi_2}{2}$$
$$=\sin\beta+Gl_3.$$

In the same way one finds the relation

$$\sin^2\beta_0\cos 2\varphi=\sin\beta_1\sin\beta_2+\cos\beta_1\cos\beta_2\cos\alpha_{1.2}\cos\alpha_{2.1}$$

or

$$\sin^2\beta_0\cos 2\varphi=\sin^2\beta-\cos^2\beta\cos^2\alpha+Gl_3. \qquad (8)$$

If (7) and (6) are substituted in (5), and also (5) and (8) in (4) it is found without difficulty that:

$$\Delta\varphi=\frac{s}{b_0}\left\{\begin{array}{c}1-\frac{1}{2}e^2(1+e^2)\sin^2\beta+\frac{3}{8}e^4\sin^4\beta\\+\frac{1}{24}e^2\Delta\varphi^2(\cos^2\alpha[4\sin^2\beta-1]-2\sin^2\beta)+Gl_6.\end{array}\right\} \qquad (9)$$

Setting $\alpha=180^0$, this formula becomes that for the meridian-arc M from β_1 to β_2, and $\Delta\varphi$ becomes $\Delta\beta$. One has then

$$\Delta\beta=\frac{M}{b_0}\left\{\begin{array}{c}1-\frac{1}{2}e^2(1+e^2)\sin^2\beta+\frac{3}{8}e^4\sin^4\beta\\+\frac{1}{24}e^2\Delta\beta^2(2\sin^2\beta-1)+Gl_6.\end{array}\right\} \qquad (10)$$

If we divide this side for side into equation (9), and observe that from (1):

$$\Delta\beta=-\Delta\varphi\cos\alpha+Gl_3,$$

in order to eliminate $\Delta\beta$ from the parenthesis of (10) we find:

$$\frac{\Delta\varphi}{\Delta\beta} = \frac{s}{M}\left(1 - \frac{1}{12}e^2\Delta\varphi^2\sin^2\beta\sin^2\alpha + Gl_6\right). \qquad (11)$$

We now return to formula (1), change the right member $\tan\frac{\Delta\varphi}{2}$ into a series and transform $\tan\frac{\Delta\beta}{2}$ to $\frac{\Delta\beta}{2}$ by means of the second series (2) of p. 29. For the moment we write the abbreviation

$$\tan\frac{\Delta\beta}{2} = Q\tan\frac{\Delta\varphi}{2},$$

then it becomes:

$$\tan\frac{\Delta\beta}{2} = \frac{1}{2}Q\Delta\varphi\left(1 + \frac{1}{12}\Delta\varphi^2 + \frac{1}{120}\Delta\varphi^4 + Gl_6\right)$$

and

$$\Delta\beta = Q\Delta\varphi\left\{1 + \frac{1}{12}\Delta\varphi^2(1-Q^2) + \frac{1}{240}\Delta\varphi^4(1-Q^2)(2-3Q^2) + Gl_6\right\}. \qquad (12)$$

Here Q^2 is equated to $\cos^2\alpha : \cos^2\frac{\Delta\alpha}{2}$, whence:

$$1 - Q^2 = \frac{\sin^2\alpha}{\cos^2\frac{\Delta\alpha}{2}} - \tan^2\frac{\Delta\alpha}{2}. \qquad (13)$$

Introducing this in (12) and connecting this equation by multiplication of corresponding sides by (11), this formula arises:

$$M = -s\frac{\cos\alpha}{\cos\frac{\Delta\alpha}{2}}\left\{1 + \frac{1}{12}\Delta\varphi^2\left(\frac{\sin^2\alpha}{\cos^2\frac{\Delta\alpha}{2}} - \tan^2\frac{\Delta\alpha}{2}\right) - \frac{1}{12}e^2\Delta\varphi^2\sin^2\beta\,\mathrm{sin}^2\alpha + \frac{1}{240}\Delta\varphi^4\sin^2\alpha(2-3\cos^2\alpha) + Gl_6.\right\} \qquad (14)$$

In these developments for (12) and (14) there is the condition, that not only $s:a_0$, but also $\Delta\alpha$ is a small magnitude of first-order. Referring to the fourth equation of (1), p. 248 we see the same condition holds for $L_{1.2}$.

The second and fourth equations of (1), p. 248 now give without difficulty:

$$\sin\frac{\Delta\alpha}{2} = -\tan\frac{\Delta\varphi}{2}\tan\beta\sin\alpha,$$

from which one obtains immediately:

$$\Delta\alpha = -\Delta\varphi\tan\beta\sin\alpha + Gl_3. \qquad (15)$$

We introduce this into (14) and also on the basis of (9) the expression for $\Delta\varphi$:

$$\Delta\varphi = \frac{s}{a_0}\left(1+\frac{1}{2}e^2\cos^2\beta + Gl_4\right), \qquad \text{and further,}$$

we insert in the second term of the curly brackets

$$1:\cos^2\frac{\Delta\alpha}{2} = 1+\sin^2\frac{\Delta\alpha}{2} + Gl_4$$

and eventually interchange in the smallest term β with B. Then, it is found that:

$$M = -s\,\frac{\cos\alpha}{\cos\frac{\Delta\alpha}{2}}\left\{\begin{array}{c} 1+\frac{1}{12}\frac{s^2}{a_0^2}\sin^2\alpha(1+e^2\cos 2B) \\ -\frac{1}{240}\frac{s^4}{a_0^4}\sin^2\alpha(-2+[5\tan^2 B+3]\cos^2\alpha) + Gl_6 \end{array}\right\}; \qquad (16)$$

$$\left.\begin{array}{c} B = \frac{1}{2}(B_1+B_2), \\ \alpha = \frac{\alpha_{1.2}+\alpha_{2.1}-180^\circ}{2}, \quad \Delta\alpha = \alpha_{2.1}-\alpha_{1.2}-180^\circ. \end{array}\right\} \qquad (17)$$

Par. 21. Continuation: Longitude Difference.

This formula agrees, after an appropriate reduction, to that given by Bessel in 1837 in the 14th volume of Astronomische Nachrichten (Astronomical Proceedings), No. 331, p. 310 (Proceedings, Volume 3, p. 40).

Par. 21. Continuation: Longitude Difference. By p. 231 we have

$$L_{1.2}=\Delta\lambda-\frac{1}{2}e^2\cos\beta_0\left\{\left(1+n-\frac{1}{2}k_1\right)\Delta\varphi-\frac{1}{2}k_1\cos 2\varphi\sin\Delta\varphi+Gl_5\right\}. \quad (1)$$

From this, by development of $\sin\Delta\varphi$ in a series and substitution of the values

$$k_1=\frac{1}{4}e^2\sin^2\beta_0+Gl_4, \qquad n=\frac{1}{4}e^2+Gl_4$$

without difficulty we obtain:

$$L_{1.2}=\Delta\lambda-\frac{1}{2}e^2\cos\beta_0\left(1+\frac{1}{4}e^2[1-\sin^2\beta_0\cos^2\varphi]+Gl_4\right)\Delta\varphi. \quad (2)$$

One now has by the equations (4) of p. 248:

$$\cos\beta_0=\frac{1}{2}(\cos\beta_1\sin\alpha_{1.2}-\cos\beta_2\sin\alpha_{2.1}).$$

Herein we set

$$\beta_1=\frac{1}{2}(\beta_2+\beta_1)-\frac{1}{2}(\beta_2-\beta_1) \text{ und } \beta_2=\frac{1}{2}(\beta_2+\beta_1)+\frac{1}{2}(\beta_2-\beta_1)$$

and similarly

$$\alpha_{1.2}=\frac{1}{2}(\alpha_{2.1}+\alpha_{1.2})-\frac{1}{2}(\alpha_{2.1}-\alpha_{1.2})$$

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and

$$\alpha_{2.1}=\frac{1}{2}(\alpha_{2.1}+\alpha_{1.2})+\frac{1}{2}(\alpha_{2.1}-\alpha_{1.2}).$$

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Chapter 6. Differential Formulae and Series Expansions for the Geodetic Line.

Referring then to abbreviations (2) of p. 304 we obtain:

$$\cos\beta_0 = \cos\beta \sin\alpha \cos\frac{\beta_2-\beta_1}{2} \sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2} + \sin\beta\cos\alpha \sin\frac{\beta_2-\beta_1}{2}\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2}. \quad (3)$$

For the further expansion of this expression, (1) of p. 248 give immediately:

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$$\sin\frac{\beta_2-\beta_1}{2} = -\frac{1}{2}\Delta\varphi\cos\alpha + Gl_3$$

$$\cos\frac{\alpha_{2.1}-\alpha_{1.2}}{2} = \frac{1}{2}\Delta\varphi\tan\beta\sin\alpha + Gl_3,$$

from which it is found:

$$\cos\frac{\beta_2-\beta_1}{2} = 1 - \frac{1}{8}\Delta\varphi^2\cos^2\alpha + Gl_4$$

$$\sin\frac{\alpha_{2.1}-\alpha_{1.2}}{2} = 1 - \frac{1}{8}\Delta\varphi^2\tan^2\beta\sin^2\alpha + Gl_4.$$

By means of these formulae expression (3) for $\cos\beta_0$ is formed as follows:

$$\cos\beta_0 = \cos\beta\sin\alpha\left(1 - \frac{1}{8}\Delta\varphi^2\sec^2\beta\,[\sin^2\beta + \cos^2\alpha] + Gl_4\right). \quad (4)$$

This value is introduced into (2) and at the same time referring back to the first two equations of (4), p. 248 we set:

$$\sin\beta_0\cos\varphi = \sin\beta + Gl_2.$$

One then obtains

$$L_{1.2} = \Delta\lambda - \frac{1}{2}e^2\cos\beta\sin\alpha\left\{1 + \frac{1}{4}e^2\cos^2\beta - \frac{1}{8}\Delta\varphi^2\sec^2\beta\,[\sin^2\beta + \cos^2\alpha] + Gl_4\right\}\Delta\varphi. \quad (5)$$

Par. 21. Continuation: Longitude Difference.

For $\Delta\lambda$ the second equation of (1), p. 248 gives the relation

$$\sin\tfrac{1}{2}\Delta\lambda = \sin\tfrac{1}{2}\Delta\varphi \sec\beta \sin\alpha.$$

We convert $\sin\frac{1}{2}\Delta\varphi$ into a series and derive from $\sin\frac{1}{2}\Delta\lambda$ by means of the first series of (2), p. 29, $\frac{1}{2}\Delta\lambda$. It follows that:

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$$\Delta\lambda = \Delta\varphi \sec\beta \sin\alpha \left\{1 - \tfrac{1}{24}\Delta\varphi^2(1 - \sec^2\beta \sin^2\alpha) + \tfrac{1}{1920}\Delta\varphi^4(1 - \sec^2\beta \sin^2\alpha)(1 - 9\sec^2\beta \sin^2\alpha) + Gl_6\right\} \quad (6)$$

and specifically under the assumption that not only $s : a_0$, but even $L_{1.2}$ is of first-order. (5) and (6) together give:

$$L_{1.2} = \Delta\varphi \sec\beta \sin\alpha \left\{\begin{array}{r} 1 - \tfrac{1}{2}e^2\cos^2\beta\left(1 + \tfrac{1}{4}e^2\cos^2\beta\right) \\ + \tfrac{1}{16}e^2\Delta\varphi^2(\sin^2\beta + \cos^2\alpha) \\ - \tfrac{1}{24}\Delta\varphi^2(1 - \sec^2\beta \sin^2\alpha) \\ + \tfrac{1}{1920}\Delta\varphi^4(1 - \sec^2\beta\sin^2\alpha)(1 - 9\sec^2\beta\sin^2\alpha) + Gl_6. \end{array}\right\} \quad (7)$$

For $\Delta\varphi$ we introduce here the series (9), p. 306, after we have substituted $a_0\sqrt{1-e^2}$ for b_0 in it, and thus $\Delta\varphi$ is transformed into

$$\Delta\varphi = \frac{s}{a_0}\left\{1 + \tfrac{1}{2}e^2\cos^2\beta + \tfrac{3}{8}e^4\cos^4\beta + \tfrac{1}{24}e^2\Delta\varphi^2(\cos^2\alpha[4\sin^2\beta - 1] - 2\sin^2\beta) + Gl_6\right\} \quad (8)$$

For (7) we now find:

(See Illustrative Material on Next Page.)

$$L_{1,2}=\frac{s}{a_0}\sec\beta\sin\alpha\left\{1-\frac{1}{24}\frac{s^2}{a_0^2}(1-\sec^2\beta\sin^2\alpha-e^2[\sin^2\beta+(4\sin^2\beta-1)\cos^2\alpha])\right.$$
$$\left.+\frac{1}{1920}\frac{s^4}{a_0^4}(1-\sec^2\beta\sin^2\alpha)(1-9\sec^2\beta\sin^2\alpha)+Gl_6.\right\} \quad (9)$$

Here we introduce the mean geographic latitude B. But according to p. 42:

$$B_1=\beta_1+n\sin 2\beta_1+\frac{1}{2}n^2\sin 4\beta_1+Gl_6$$
$$B_2=\beta_2+n\sin 2\beta_2+\frac{1}{2}n^2\sin 4\beta_2+Gl_6,$$

from which

$$B=\beta+n\sin 2\beta\cos\Delta\beta+\frac{1}{2}n^2\sin 4\beta+Gl_6. \quad (10)$$

From which, the following computation (in which) $n=\frac{1}{4}e^2+\frac{1}{8}e^4+Gl_6$ gradually yields:

$$\cos B=\cos\beta\cos(n\sin 2\beta\cos\Delta\beta+Gl_4)$$
$$-\sin\beta\sin(n\sin 2\beta\cos\Delta\beta+\tfrac{1}{2}n^2\sin 4\beta+Gl_6)$$

or

$$\cos B=\cos\beta\left\{1-\frac{1}{2}e^2\sin^2\beta\cos\Delta\beta-\frac{1}{2}e^4\sin^2\beta\right.$$
$$\left.+\frac{3}{8}e^4\sin^4\beta+Gl_6\right\};$$

on the other hand:

$$1:\sqrt{1-e^2\sin^2 B}=1+\frac{1}{2}e^2\sin^2\beta+\frac{1}{2}e^4\sin^2\beta-\frac{1}{8}e^4\sin^4\beta+Gl_6, \quad (11)$$

from which:

$$\frac{\cos B}{\sqrt{1-e^2\sin^2 B}} = \cos\beta\left(1+\frac{1}{4}e^2\sin^2\beta\sin^2\Delta\beta + Gl_6\right).$$

According to (9) and (12), p. 306 and 307 $\Delta\beta = -\frac{s}{a_0}\cos\alpha + Gl_3$, from which this formula can finally be brought into the following form:

$$\frac{\cos B}{\sqrt{1-e^2\sin^2 B}} = \cos\beta\left(1+\frac{1}{4}e^2\frac{s^2}{a_0^2}\sin^2\beta\cos^2\alpha + Gl_6\right).^{*} \quad (12)$$

With this (9) gives:

$$L_{1\cdot 2} = \varrho''\frac{s}{a_0}W\sec B\sin\alpha\left\{1-\frac{1}{24}\frac{s^2}{a_0^2}\left(W^2[1-\sec^2 B\sin^2\alpha]-e^2[10\sin^2 B-1]\cos^2\alpha\right) + \frac{1}{1920}\frac{s^4}{a_0^4}(1-\sec^2 B\sin^2\alpha)(1-9\sec^2 B\sin^2\alpha) + Gl_6.\right\} \quad (13)$$

in sec.

The meaning of B and α is defined in (17), p. 308. Otherwise, as earlier $W = \sqrt{1-e^2\sin^2 B}$.

Par. 22. Continuation: Azimuth Difference. The second and fourth equations of (1), p. 248 give:

$$\sin\frac{\Delta\alpha}{2} = -\tan\frac{\Delta\varphi}{2}\tan\beta\sin\alpha. \quad (1)$$

We express $\tan\frac{\Delta\varphi}{2}$ in a series and then, by means of the first series of (2), p. 29 we determine $\frac{\Delta\alpha}{2}$. It is:

$$\Delta\alpha = -\Delta\varphi\tan\beta\sin\alpha\left\{1+\frac{1}{24}\Delta\varphi^2(2+\tan^2\beta\sin^2\alpha) + \frac{1}{1920}\Delta\varphi^4\left(\begin{matrix}16+20\tan^2\beta\sin^2\alpha\\ +9\tan^4\beta\sin^4\alpha\end{matrix}\right)+Gl_6.\right\}$$

For $\Delta\varphi$ we now insert expression (8), p. 310 and obtain:

(See Illustrative Material on Next Page.)

$$\Delta\alpha = -\frac{s}{a_0}\frac{\tan\beta\sin\alpha}{\sqrt{1-e^2\cos^2\beta}}\left\{1+\frac{1}{24}\frac{s^2}{a_0^2}\begin{pmatrix}2+\tan^2\beta\sin^2\alpha\\+e^2[1-3\sin^2\beta\sin^2\alpha+\sin^2\alpha]\end{pmatrix}\right.$$
$$\left.+\frac{1}{1920}\frac{s^4}{a_0^4}\begin{pmatrix}16+20\tan^2\beta\sin^2\alpha\\+9\tan^4\beta\sin^4\alpha\end{pmatrix}+Gl_6.\right\} \quad (2)$$

From (12) p. 311 we easily derive

$$\cos\beta = \frac{\cos B}{\sqrt{1-e^2\sin^2 B}}\left(1-\frac{1}{4}e^2\frac{s^2}{a_0^2}\sin^2 B\cos^2\alpha+Gl_6\right)$$

from which

$$\sin\beta = \frac{\sqrt{1-e^2}\sin B}{\sqrt{1-e^2\sin^2 B}}\left(1+\frac{1}{4}e^2\frac{s^2}{a_0^2}\cos^2 B\cos^2\alpha+Gl_6\right),$$

which yields

$$\sqrt{1-e^2}\tan B = \tan\beta\left(1-\frac{1}{4}e^2\frac{s^2}{a_0^2}\cos^2\alpha+Gl_6\right), \quad (3)$$

while from (11), p. 311

$$\frac{\sqrt{1-e^2}}{\sqrt{1-e^2\sin^2 B}} = \sqrt{1-e^2\cos^2\beta}+Gl_6. \quad (4)$$

Substituting this in (2) we obtain the formula

$$\Delta\alpha = -\varrho''\frac{s}{a_0}W\tan B\sin\alpha\left\{1+\frac{1}{24}\frac{s^2}{a_0^2}\begin{pmatrix}2+\tan^2 B\sin^2\alpha\\+e^2[7-(5+\tan^2 B+3\sin^2 B)\sin^2\alpha]\end{pmatrix}\right.$$
$$\left.+\frac{1}{1920}\frac{s^4}{a_0^4}\begin{pmatrix}16+20\tan^2 B\sin^2\alpha\\+9\tan^4 B\sin^4\alpha\end{pmatrix}+Gl_6.\right\} \quad (5)$$

in sec.

This formula assumes as does (16), p. 308 and (13), p. 311, that not only $s:a_0$, but also $L_{1,2}$ are small magnitudes of first-order.

If in (16), p. 308 the meridian-arc M is expressed by B and ΔB and reduced to ΔB, then one obtains in the two formulae, (13), p. 311 and (5), above, a means for the indirect computation of $B_2, L_{1,2}$ and $\alpha_{2,1}$ from B_1, s and $\alpha_{1,2}$, which is very conducive to rapid computation, in the case where the desired values are already known.

Gauss presented such a formula in 1847 in his Untersuchungen über Gegenstände der höhere Geodäsie (Investigation of Topics of Higher Geodesy), part 2, pp. 26 ff., in which he developed only terms up to those with s^3, but those completely. The course taken by him is different from ours, since he, in particular, begins directly with the differential forms (11) to (13), p. 298 and uses a point lying half the distance between P_1 and P_2 as a starting point and P_1 and P_2 as endpoints. Out of the six equations thus arising he eliminates the values of B, L and α of the mentioned midpoint. For

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our purposes however, the course chosen here seemed best, though, it necessitated the use of the general formulae since it brings formula (16), p. 308 directly into the most unified form--this for the sake of later applications.

Par. 23. Distance and Azimuth from Geographic Positions.

The formulae (16), p. 308 and (13), p. 311 give without difficulty

(See Illustrative Material on Next Page.)

$$s\cos\alpha = -M\cos\frac{\Delta\alpha}{2}\left\{1-\frac{1}{12}\frac{s^2}{a_0^2}\sin^2\alpha(1+e^2\cos 2B) + \frac{1}{720}\frac{s^4}{a_0^4}\sin^2\alpha\left([15\tan^2 B+3]\cos^2\alpha-\sin^2\alpha\right)+Gl_6\right\}, \quad (1)$$

$$s\sin\alpha = P\left\{1+\frac{1}{24}\frac{s^2}{a_0^2}\left(W^2[\cos^2\alpha-\tan^2 B\sin^2\alpha]-e^2[10\sin^2 B-1]\cos^2\alpha\right) + \frac{1}{5760}\frac{s^4}{a_0^4}(\cos^2\alpha-\tan^2 B\sin^2\alpha)(7+17\sec^2 B\sin^2\alpha)+Gl_6\right\}, \quad (2)$$

in which P denotes the arc of the parallel for the longitude difference $L_{1,2}$ at the geographic latitude B:

$$P = a_0\frac{L_{1,2}\text{ in Sek.}}{\varrho''}\frac{\cos B}{W}, \qquad W \text{ with argument } B = \frac{1}{2}(B_1+B_2), \quad (3)$$

and M according to p. 50, (3) and p. 44, (2) is computed by means of the formula:

$$\log M = \log\left(a_0\frac{\Delta B\text{ in Sek.}}{\varrho''}\frac{1-e^2}{W^3}\right)+\frac{1}{8}\,\text{Mod.}\,e^2\cos 2B\left(\frac{\Delta B\text{ in Sek.}}{\varrho''}\right)^2+Gl_6. \quad (4)$$

Moreover, by the introduction of (2) and (3) in formula (5) of the preceding paragraph:

$$\Delta\alpha = -L_{1,2}\sin B\left\{1+\frac{1}{24}\frac{s^2}{a_0^2}\left\{3\cos^2\alpha+2\sin^2\alpha - e^2[(11\sin^2 B-8)\cos^2\alpha+(4\sin^2 B-2)\sin^2\alpha]\right\} + \frac{1}{5760}\frac{s^4}{a_0^4}\left\{75\cos^4\alpha+140\cos^2\alpha\sin^2\alpha+48\sin^4\alpha + [60\cos^2\alpha\sin^2\alpha+16\sin^4\alpha]\tan^2 B\right\}+Gl_6.\right\} \quad (5)$$

Using these formulae computation becomes in part indirect. First P and M are definitely determined by (3) and (4). At the same time these are the values of $s\cos\alpha$ and $s\sin\alpha$ neglecting terms of the third-order. From this (5) gives the value of $\Delta\alpha$ up to terms of the fifth-order, however

one already has justification for retaining terms in the parenthesis dependent on e^2s^2. Now, by means of (1) and (2), $s\cos\alpha$ and $s\sin\alpha$ may be more exactly computed, even up to terms of fifth-order, but with retention of the e^2s^2 terms of the parenthesis. A subsequent approximation gives the fifth-order terms completely.

For practical use the logarithmic form for (1), (2) and (5) recommends itself. We will derive this form. Moreover we produce those forms, which result from the stated operations with the general symbols and especially for $s\cos\alpha$ by use of the series

$$\log\cos\frac{\Delta\alpha}{2} = -\operatorname{Mod.}\left(\frac{1}{8}\Delta\alpha^2 + \frac{1}{192}\Delta\alpha^4 + Gl_6\right)$$

according to p. 29, (3).

Finally, we write both formulae (2) and (4) somewhat differently, as seems best from looking back on the whole exposition.

In this way we obtain the following formulae for direct computation:

$$\left.\begin{array}{ll} P' = \dfrac{L_{1,2}\text{ in sec.}}{\varrho''}\dfrac{\cos B}{W} & M' = \dfrac{\Delta B\text{ in sec.}}{\varrho''}\dfrac{1-e^2}{W^3} \\ \Delta B = B_2 - B_1 & B = \frac{1}{2}(B_2 + B_1) \\ & W \text{ with Argument } B \\ P = a_0 P' & \log M = \log(a_0 M') + \frac{1}{24}\operatorname{Mod.} M'^2 e^2 (3 - 6\sin^2 B) + Gl_6 \end{array}\right\} (6)$$

$$\left.\begin{array}{l} \log\Delta\alpha = \log(-L_{1,2}\sin B) \\ + \frac{1}{24}\operatorname{Mod.}\left\{M'^2(3 + e^2(3 - 11\sin^2 B)) + P'^2(2 + e^2(2 - 4\sin^2 B))\right\} + Gl_4. \end{array}\right\} (7)$$

$$\log(s\cos\alpha) = \log(-M) - \frac{1}{24}\operatorname{Mod}\left\{\begin{array}{c} P'^2(2 + e^2(2 - 4\sin^2 B)) \\ + 3P'^2 W^2 \tan^2 B \end{array}\right\} + Gl_4 \quad (8)$$

$$\log(s\sin\alpha) = \log P + \frac{1}{24}\operatorname{Mod}\left\{\begin{array}{c} M'^2(1 + e^2(1 - 11\sin^2 B)) \\ - P'^2 W^2 \tan^2 B \end{array}\right\} + Gl_4. \quad (9)$$

Here, the terms of fourth-order independent of e^2 are ignored.

If a still closer approximation is desirable it may be obtained by adding pre-determined values for the logarithms of $\Delta\alpha$, and $s\cos\alpha$ and $s\sin\alpha$ respectively, to the following expressions:

$$+\frac{1}{2880}\,\mathrm{Mod}\left\{15M'^4+14F'^4-[60M'^2F'^2+12F'^4]\tan^2B\right\} \qquad (7^*)$$

$$+\frac{1}{2880}\,\mathrm{Mod}\left\{-8M'^2F'^2-14F'^4-[30M'^2F'^2+40F'^4]\tan^2B-15F'^4\tan^4B\right\} \qquad (8^*)$$

$$+\frac{1}{2880}\,\mathrm{Mod}\left\{M'^4-8M'^2F'^2-[30M'^2F'^2+12F'^4]\tan^2B-F'^4\tan^4B\right\}, \qquad (9^*)$$

by which the omissions in the logarithm diminish to Gl_6. The formulae are easily arrived at, since in (5), (1) and (2) in the right member we set $s\cos\alpha = -M(1-\frac{1}{8}I'^2\tan^2B-\frac{1}{12}P'^2+Gl_4)$ and $s\sin\alpha = P(1+\frac{1}{24}M'^2-\frac{1}{24}I'^2\tan^2B+Gl_4)$ and take logarithms.

Eventually one has

$$\left.\begin{aligned}\alpha_{1.2} &= \alpha-\frac{\Delta\alpha}{2}\\ \alpha_{2.1} &= \alpha+\frac{\Delta\alpha}{2}+180^\circ.\end{aligned}\right\} \qquad (10)$$

For the previous formulae exactly as for those of the preceding paragraphs, from which they have come, we have the restriction that not only $s:a_0$, but also $L_{1.2}$ is a magnitude of first-order.

This may be expressed in still another form, toward which the purely spherical formulae of pp. 132 ff. point the way. We have, among other things the difference of formulae (8) and (9) equalling formula (1) of the given page up to terms multiplied by e^2

and likewise (7) may be put in the form of (2). We are content however with the very convenient, succinct formulae for the above simultaneous computation of s and α. The small terms in them are computed much more easily since the terms appear in different formulae.

For sides of measurable triangles it is often sufficient to omit terms of the fourth-order. But then one has purely spherical forms, with the modification that different curvature radii are used for $L_{1,2}$ and for ΔB. This difference disappears upon introduction of the reduced latitude difference $\Delta B \frac{1-e^2}{W^2}$, and it is easily demonstrated within the given accuracy, by a spherical treatment of $L_{1,2}$ and the reduced latitude difference by means of the curvature radius $a_0 : W$ (i.e., the normal at mean latitude) brings one back to the above formulae. From this one can carry out, in general, all computations on a sufficiently small geodetic line purely spherically. This is also immediately clear geometrically, lacking only the specification of omissions. (Compare also Jordan, Handbuch (Handbook) Volume 2, p. 236.)

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Par. 24. Numerical Example I: Given:

$$\left.\begin{array}{l} B_1 = 54^\circ\,42'\,50{,}6'' \\ B_2 = 52\;\;30\;\;16{,}7 \end{array}\right\} \; L_{1,2} = 7^\circ\,6'\,0''.$$

One then has

(See Illustrative Material on Next Page.)

$B = 53^\circ\,36'\,33{,}65''$ | $\log W = 9{,}9990587.89 - 10$

$\Delta B = -\,7953{,}9''$ | $\log \Delta B = 3{,}9005801.27_n$

$L_{1.2} = 25560{,}0''$ | $\log L_{1.2} = 4{,}4075608.50$

$\log \frac{\text{Mod}}{24} = 5{,}2575731$ | $\log a_0 = 6{,}8046434.64$

$\log \frac{\text{Mod}}{2880} = 3{,}17839$ (for units of the 7th decimal) | $\log \tan^2 B = 0{,}2650511$

$\log e^2 = 7{,}82441 - 10$ | $\log \sin^2 B = 9{,}81158 - 10$

$\log \varrho'' = 5{,}3144251.33$

$\log \frac{\Delta B}{\varrho''} = 8{,}5861549.94_n - 10$

$\log \frac{L_{1.2}}{\varrho''} = 9{,}0931357.17 - 10$

$\log \sin B = 9{,}9057908.07 - 10$

$\log \cos B = 9{,}7732652.48 - 10$

$\log (1 - e^2) = 9{,}9970916.40 - 10$

$\log M' = 8{,}5860702.67_n - 10$

$\log (a_0 M') = 5{,}3907137.31_n$

$\log P' = 8{,}8673421.76 - 10$

$\log P = 5{,}6719856.40$

$\frac{1}{24}\,\text{Mod} \cdot M'^2 = 268.976$

„ $P'^2 = 982.330$

„ $P'^2 W^2 \tan^2 B = 1800.637$

(Units of 7th decimal)

$\frac{1}{24}\,\text{Mod} \cdot e^2 M'^2 = 1.7952$

„ $e^2 M'^2 \sin^2 B = 1.1634$

„ $e^2 P'^2 = 6.5564$

„ $e^2 P'^2 \sin^2 B = 4.2486$

„

$\frac{1}{2880}\,\text{Mod} \cdot M'^4 = 0.0033$

„ $P'^4 = 0.0444$

„ $M'^2 P'^2 = 0.0122$

„

(See Illustrative Material on Next Page.)

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$$\left.\begin{aligned} \tfrac{1}{2880}\,\text{Mod}\,.\,M'^2P'^2\tan^2B &= 0.02240 \\ \text{„}\quad P'^4\tan^2B &= 0.08183 \\ \text{„}\quad P'^4\tan^4B &= 0.15065 \end{aligned}\right\}\ \text{Units of the 7th decimal}$$

From which results the following summary:

(See Illustrative Material on Next Page.)

	$\log(-L_{1.2}\sin B)$	=	4,3133516.57.	
+ $\frac{1}{24}$	Mod. $3M'^2$	= +	806.928	
+ $\frac{1}{24}$	Mod. $e^2M'^2.8$	= +	14.362	
−	„ „ $11\sin^2B$	= −	12.797	
+ $\frac{1}{24}$	Mod. $2P'^2$	= +	1964.660	used
+ $\frac{1}{24}$	Mod. $e^2P'^2.2$	= +	13.113	below
−	„ „ $4\sin^2B$	= −	16.994	
+ $\frac{1}{2880}$	Mod. $15M'^4$	= +	0.050	
+	„ $14P'^4$	= +	0.622	
−	„ $60M'^2P'^2\tan^2B$	= −	1.344	
−	„ $12P'^4\tan^2B$	= −	0.982	
		$\log \Delta\alpha$ =	4,3136284.19.	

$$\Delta\alpha = -20588{,}6760'' = -5^0\,43'\,8{,}6760''.$$

	$\log(-a_0M')$	=	5,3907137.31	
+ $\frac{1}{24}$	Mod. $e^2M'^2.3$	= +	5.386	is log M
−	„ „ $6\sin^2B$	= −	6.980	
− $\frac{1}{24}$	Mod. $I'^2(2+e^2[2-4\sin^2B])$	= −	1960.779	see above
− $\frac{1}{24}$	Mod. $3P'^2W'^2\tan^2B$	= −	5401.911	
− $\frac{1}{2880}$	Mod. $8M'^2P'^2$	= −	0.098	
−	„ $14P'^4$	= −	0.622	see above
−	„ $30M'^2P'^2\tan^2B$	= −	0.672	see above
−	„ $40P'^4\tan^2B$	= −	3.273	
−	„ $15P'^4\tan^4B$	= −	2.260	
		$\log(s\cos\alpha)$ =	5,3899766.10	

(See Illustrative Material on Next Page.)

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	$\log P$	=	5,6719856.40
+	$\frac{1}{24}$ Mod. M'^2	= +	268.976
+	$\frac{1}{24}$ Mod. $e^2 M'^2 . 1$	= +	1.795
−	„ „ $11 \sin^2 B$	= −	12.797 see above
−	$\frac{1}{24}$ Mod. $P'^2 W'^2 \tan^2 B$	= −	1800.637
+	$\frac{1}{2880}$ Mod. M'^4	= +	0.003
−	„ $8 M'^2 P'^2$	= −	0.098 see above
−	„ $30 M'^2 P'^2 \tan^2 B$	= −	0.672 see above
−	„ $12 P'^4 \tan^2 B$	= −	0.982 see above
−	„ $P'^4 \tan^4 B$	= −	0.151
	$\log (s \sin \alpha)$	=	5,6718311.84

From $\log (s \cos \alpha)$ and $\log (s \sin \alpha)$ one obtains:

$$\alpha = 62^0\ 24'\ 35{,}029'',$$

from which the following results for s, $\alpha_{1.2}$ and $\alpha_{2.1}$ are given:

Results:

	instead	
$\log s = 5{,}7242591.34$	.35	acc. to p. 261
$\alpha_{1.2} = 65^0\ 16'\ 9{,}367''$	,365	
$\alpha_{2.1} = 239\ 33\ 0{,}691$	,689	

The accuracy of the results is satisfactory, in comparison with the stricter computation p. 261 shows $\Delta\alpha$ even being to within 0,0001".

Par. 25. Numerical Example II. Given:

$$\left.\begin{aligned} B_1 &= 57^0 \\ B_2 &= 56^0\ 13'\ 49{,}02186'' \end{aligned}\right\} L_{1.2} = 1^0\ 22'\ 6{,}03270'' \text{ eastern}$$

One then has:

$B =$	$56^\circ\ 36'\ 54{,}51093''$	$\log W' =$	$9{,}9989871.55 - 10$
$\Delta B =$	$- 2770{,}97814''$	$\log \Delta B =$	$3{,}4426331.00_n$
$L_{1.2} =$	$- 4926{,}03270''$	$\log L_{1.2} =$	$3{,}6924972.90_n$
$\log \frac{\text{Mod}}{24} =$	$5{,}25757$	$\log a_0 =$	$6{,}8046434.64$
$\log \frac{\text{Mod}}{2880} =$	$3{,}17839$	$\log \tan^2 B =$	$0{,}36223$
$\log e^2 =$	$7{,}82441 - 10$	$\log \sin^2 B =$	$9{,}8434 - 10$

units of the 7th decimal place.

$$\log \varrho'' = 5{,}314251.33$$
$$\log (1 - e^2) = 9{,}9970916.40 - 10$$

$$\log \frac{\Delta B}{\varrho''} = 8{,}1282079.67_n - 10$$
$$\log \frac{L_{1.2}}{\varrho''} = 8{,}3780721.57_n - 10$$
$$\log \sin B = 9{,}9216829.98 - 10$$
$$\log \cos B = 9{,}7405680.15 - 10$$
$$\log M' = 8{,}1283381.42_n - 10$$
$$\log (a_0 M') = 4{,}9329816.06_n$$
$$\log P' = 8{,}1196530.17_n - 10$$
$$\log P = 4{,}9242964.81_n$$

$\frac{1}{24}$ Mod. M'^2	$= 32.678$	units of the 7th decimal place.
„ P'^2	$= 31.396$	
„ $P'^2 W^2 \tan^2 B$	$= 71.958$	
$\frac{1}{24}$ Mod. $e^2 M'^2$	$= 0.2181$	„
„ $e^2 M'^2 \sin^2 B$	$= 0.1521$	
„ $e^2 P'^2$	$= 0.2096$	
„ $e^2 P'^2 \sin^2 B$	$= 0.1461$	
$\frac{1}{2880}$ Mod. M'^4	$= 0.0000$	„
„ P'^4	$= 0.0000$	
„ $M'^2 P'^2$	$= 0.0000$	
$\frac{1}{2880}$ Mod. $M'^2 P'^2 \tan^2 B$	$= 0.00011$	„
„ $P'^4 \tan^2 B$	$= 0.00011$	
„ $P'^4 \tan^4 B$	$= 0.00024$	

From which the following summary results:

$\log(-L_{1,2}\sin B)$	=	3,6141802.88
$+\frac{1}{24}$ Mod. $3M'^2$	= +	98.034
$+\frac{1}{24}$ Mod. $e^2M'^2.8$	= +	1.745
$-$ „ „ $11\sin^2 B$	= −	1.673
$+\frac{1}{24}$ Mod. $2P'^2$	= +	62.792
$+\frac{1}{24}$ Mod. $e^2P'^2.2$	= +	0.419
$-$ „ „ $4\sin^2 B$	= −	0.584
the terms (7*) p. 314	= −	0.008
$\log \Delta\alpha$ =		3,6141963.61

is used again later

$\Delta\alpha = 4113{,}35659'' = 1^0\ 8'\ 33{,}35659''.$

$\log(-a_0 M')$	=	4,9329816.06
$+\frac{1}{24}$ Mod. $e^2M'^2.3$	= +	0.654
$-$ „ „ $6\sin^2 B$	= −	0.913
$-\frac{1}{24}$ Mod. $P'^2(2+e^2[2-4\sin^2 B])$	= −	62.627 see above
$-\frac{1}{24}$ Mod. $3P'W'^2\tan^2 B$	= −	215.874
the terms (8*) p. 315	= −	0.011
$\log(s\cos\alpha)$ =		4,9329537.29

$\log P$	=	4,9242964.81.
$+\frac{1}{24}$ Mod. M'^2	= +	32.678
$+\frac{1}{24}$ Mod. $e^2M'^2.1$	= +	0.218
$-$ „ „ $11\sin^2 B$	= −	1.673 see above
$-\frac{1}{24}$ Mod. $P'^2W'^2\tan^2 B$	= −	71.958
the terms (9*) p. 315	= −	7
$\log(s\sin\alpha)$ =		4,9242924.07.

From $\log(s\cos\alpha)$ and $\log(s\sin\alpha)$ one obtains:

$$\alpha = 315^0\ 34'\ 16{,}6775'',$$

from which we have the following result for s, $\alpha_{1.2}$ and $\alpha_{2.1}$.

$$\text{Result:} \begin{cases} \log s = 5{,}0791812.47 \\ \alpha_{1.2} = 314^\circ\ 59'\ 59{,}9992'' \\ \alpha_{2.1} = 136\ \ 8\ 33{,}3558'', \end{cases}$$

which just as $\Delta\alpha$ must be as accurate in the last digit as the numerical computation's uncertainty permits.

Completely ignoring terms of fourth-order of the logarithms, one obtains:

$$\begin{array}{l|l} \log \Delta\alpha = 3{,}6141963.7 & \text{from which with 7 digit log it follows:} \\ \log (s\cos\alpha) = 4{,}9329536.4 & \Delta\alpha = \ 1^\circ\ 8'\ 33{,}3568'' \\ \log (s\sin\alpha) = 4{,}9242925.2., & \alpha = 315^\circ\ 34'\ 16{,}63'' \end{array}$$

$$\begin{cases} \log s = 5{,}0791812.8 \\ \alpha_{1.2} = 314''\ 59'\ 59{,}95'' \\ \alpha_{2.1} = 135\ \ 8\ 33{,}31. \end{cases}$$

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One recognizes, that for sides of measurable triangles, using seven digit logs, terms of fourth-order as a rule are without influence.

CHAPTER 7

The Course of the Geodetic Line.

Par. 1. The Form of Geodetic Circles in the Vicinity of Points Situated Diametrically Opposite the Pivot Point of a Geodetic Line on the Ellipsoid of Rotation. In addition to our remarks on p. 217 regarding the course of the geodetic line, we turn our attention to the behavior of the geodetic circles referred to in the heading of this paragraph. In our analysis of this matter we shall progress only as far as terms with e^2, neglecting e^4 and higher powers of the eccentricity e of the meridian ellipse. According to p. 223, the formula for the length of the geodetic line between points P_1 and P_2, will therefore be as follows (regarding e as a magnitude of the first-order):

$$s = b_0\left\{(1 + k_1)\,\Delta\varphi + \frac{1}{2}k_1(\sin 2\varphi_2 - \sin 2\varphi_1) + Gl_4\right\}, \tag{1}$$

where retaining the stipulated accuracy

$$k_1 = \frac{1}{4}k^2 + \cdots = \frac{e^2}{4}\sin^2\beta_0 + \cdots = \frac{1}{2}\alpha\sin^2\beta_0 + \cdots \tag{2}$$

may be accepted. β_0 is the greatest reduced latitude which covers the whole course, and may be attained by the geodetic line determined by points P_1 and P_2.

For geodetic circles s is constant. If we assume further that $\Delta\varphi$ equals π within a fraction of the second-order, so that in the equation

$$\Delta\varphi = \varphi_2 - \varphi_1 = \pi + \chi \tag{3}$$

χ is of the second-order, the factor of b_0 in equation (1) will be transformed into

$$\pi + \tfrac{1}{2}\,\varrho\pi \sin^2\beta_0 + \chi + Gl_4.$$

Therefore, if χ_0 designates a constant, s is constant, assuming that

$$\chi = \chi_0 - \tfrac{1}{2}\,\varrho\pi \sin^2\beta_0 + Gl_4 \qquad (4)$$

This conditional equation for the geodetic circles may now be applied to the auxiliary spherical triangle, fig. 21, p. 232. Bearing in mind equation (3), we obtain:

$$\sin\beta_2 = -\sin\beta_1 \cos\chi + \cos\beta_1 \sin\chi \cos\alpha_{1.2}$$

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and hence:

$$\beta_2 = -\beta_1 + \chi \cos\alpha_{1.2} + Gl_4.$$

This relation neglects, as indicated earlier, only terms of the fourth-order, if P_1 and P_2 are not situated in the proximity of the poles. We disregard this special case, and substituting the value of χ from (4) obtain:

$$\beta_2 = -\beta_1 - \tfrac{1}{2}\,\varrho\pi \sin^2\beta_0 \cos\alpha_{1.2} + \chi_0 \cos\alpha_{1.2} + Gl_4. \qquad (5)$$

The longitude difference $L_{1.2}$ of the points P_1 and P_2, is obtained according to p. 231 (14) and retaining the same accuracy as before, will be expressed as follows:

$$L_{1.2} = \Delta\lambda - \varrho \cos\beta_0 \Delta\varphi + Gl_4. \qquad (6)$$

The auxiliary spherical triangle, fig. 21, supplies

$$\sin\Delta\lambda = \sin\Delta\varphi \sin\alpha_{1.2} \sec\beta_2,$$

and, as a result of expression (3) for $\Delta\varphi$, and disregarding the

proximity of the poles, we obtain:

$$\Delta\lambda = \pi + \chi \sec\beta_2 \sin\alpha_{1.2} + Gl_4$$

If this is substituted in (6), deriving the value of χ from (4), but setting for $\Delta\varphi$ in the last term simply π, it follows:

$$L_{2.2} = \pi - \frac{1}{2}\mathfrak{A}\pi(2\cos\beta_0 + \sin^2\beta_0 \sec\beta_2 \sin\alpha_{1.2}) + \chi_0 \sec\beta_2 \sin\alpha_{1.2} + Gl_4. \qquad (7)$$

Since we ignore the proximity of the pole, we may further substitute $-\beta_1$ for β_2. If we also substitute for cos β_0, according to p. 232 (2), the value cos $\beta_1 \sin\alpha_{1.2}$ and for $\sin^2\beta_0$ accordingly, $1 - \cos^2\beta_1 \sin^2\alpha_{1.2}$, we obtain from (5) and (7) by a simple reduction:

$$\left.\begin{aligned} \beta_2 &= -\beta_1 - \frac{1}{2}\mathfrak{A}\pi\cos\alpha_{1.2}(1 - \cos^2\beta_1\sin^2\alpha_{1.2}) + \chi_0\cos\alpha_{1.2} + Gl_4 \\ L_{1.2} &= \pi - \frac{1}{2}\mathfrak{A}\pi\cos\beta_1\sin\alpha_{1.2}(1 + \sec^2\beta_1 + \cos^2\alpha_{1.2}) \\ &\quad + \chi_0 \sec\beta_1 \sin\alpha_{1.2} + Gl_4. \end{aligned}\right\} \qquad (8)$$

The variation of $\alpha_{1.2}$ corresponds to a motion of P_2 in a geodetic circle, whose configuration (except for members of the fourth-order) may be derived from (8).

A comparison of point C, which is situated directly opposite P_1 on the surface and which possesses the geographic latitude $-\beta_1$, with

P_2, shows that the latitudinal difference is $\beta_2 + \beta_1$, and their geographic difference of longitude, $L_{1.2} - \pi$.

If we relate, on a plane, a point B to an origin C of rectangular coordinates ξ and η, assuming that

$$\left.\begin{aligned} \xi &= -(\beta_2 + \beta_1) \\ \eta &= (L_{1.2} - \pi)\cos\beta_1 \end{aligned}\right\}, \qquad (9)$$

the curve B, corresponding to a constant χ_0 and variable $\alpha_{1.2}$, shows a very accurate picture of the geodetic circle traced by P_2, namely, in the reduction $\frac{1}{a_0}$.

If, however, χ_0 is made variable while $\alpha_{1.2}$ remains constant, B will trace a straight line in the azimuth $180^\circ - \alpha_{1.2}$ with respect to the axis of ξ. This agrees with the azimuth of the ascending geodetic line P_2 up to magnitudes of the second-order, because, according to (9), we have:

$$d\xi : d\eta = -d\beta_2 : \cos\beta_1 dL_{1.2},$$

and the latter relationship up to magnitudes of the second-order equals the cotangent of that azimuth, i.e., according to p. 279 (1).

$$-\nu_2 d\beta_2 : \cos\beta_2 dL_{1.2}.$$

Taking into account (8) and disregarding henceforth the residual members, we arrive at the following formulae for ξ and η:

$$\left.\begin{aligned} \xi &= \frac{1}{2}\mathfrak{A}\pi\cos\alpha_{1.2}(1 - \cos^2\beta_1\sin^2\alpha_{1.2}) - \chi_0\cos\alpha_{1.2} \\ \eta &= -\frac{1}{2}\mathfrak{A}\pi\cos^2\beta_1\sin\alpha_{1.2}(1 + \sec^2\beta_1 + \cos^2\alpha_{1.2}) + \chi_0\sin\alpha_{1.2}. \end{aligned}\right\} \qquad (10)$$

The positive directions of ξ and η are considered to be from C, i.e. toward south and west.

Differentiating with respect to $\alpha_{1.2}$, we obtain

$$\left.\begin{aligned} \frac{d\xi}{d\alpha_{1.2}} &= -\sin\alpha_{1.2}\left\{\frac{1}{2}\mathfrak{A}\pi(\sin^2\beta_1 + 3\cos^2\beta_1\cos^2\alpha_{1.2}) - \chi_0\right\} \\ \frac{d\eta}{d\alpha_{1.2}} &= -\cos\alpha_{1.2}\left\{\frac{1}{2}\mathfrak{A}\pi(\sin^2\beta_1 + 3\cos^2\beta_1\cos^2\alpha_{1.2}) - \chi_0\right\}. \end{aligned}\right\} \qquad (11)$$

From this follows, first, that $\frac{d\eta}{d\xi} = \cot\alpha_{1.2}$, with χ_0 being constant; therefore, in the illustration, the azimuth of the geodetic circle equals $90^0 - \alpha_{1.2}$, so that the circle is in a normal position in relation to the geodetic line -- a fact which has already been conclusively proved for the ellipsoid itself.

We see further that the differential quotients of ξ and η according to $\alpha_{1.2}$ become equal to zero for

$$\chi_0 = \frac{1}{2}\,\varrho\pi\,(\sin^2\beta_1 + 3\cos^2\beta_1\cos^2\alpha_{1.2}). \tag{12}$$

If, with χ_0 being constant, there exists a $\alpha_{1.2}$, corresponding to the above equation, the pertinent geodetic circle in the illustration will have cusps, whose coordinates, according to (10), will be as follows:

$$\xi_0 = -\varrho\pi\cos^2\beta_1\cdot\cos^3\alpha_{1.2} \qquad \eta_0 = -\varrho\pi\cos^2\beta_1\cdot\sin^3\alpha_{1.2} \tag{13}$$

To this, (12) supplies:

$$\cos^2\alpha_{1.2} = \frac{\chi_0 - \frac{1}{2}\varrho\pi\sin^2\beta_1}{\frac{3}{2}\varrho\pi\cos^2\beta_1}. \tag{14}$$

It is also possible to arrive at the conditional equation for the appearance of a cusp by assuming that $m = 0$.

With this, equation (5), p. 274, yields:

$$\sin\Delta\varphi = \varrho\sin^2\beta_0\sin\varphi_1\sin\varphi_2\,\Delta\varphi + Gl_4, \tag{15}$$

from which it is evident that cusps are possible only for such geodetic circles which are situated almost diametrically opposite

their pivot centers. From (15), as a result of (3) and (4) as well as (2), p. 232, we come back to (12), which substantiates the correctness of the conditional equation, up to the terms of the fourth-order. Since $\cos^2\alpha_{1,2}$ is confined to the limits of zero and 1, it follows from (12) that, except for terms of the fourth-order, cusps in geodetic circles will occur only for such values of χ_0, which conform to the requirement

$$\frac{1}{2}\mathfrak{a}\pi\sin^2\beta_1 \leq \chi_0 \leq \frac{1}{2}\mathfrak{a}\pi(1+2\cos^2\beta_1). \qquad (16)$$

Par. 2. Continuation: Shape of Geodetic Circles With Cusps. In order to gain an overall picture, a figure has been designed (fig. 25) which, with an equatorial point as the center of rotation, gives a plane presentation of the course of geodetic circles with cusps. The figure was designed with the aid of formulae (10) which assume that $\cos\beta_1 = 1$.

The figure gives geodetic circles for values of χ_0 in the interval $\frac{1}{2}\mathfrak{a}\pi$. The geodetic circles I, II, III, and IV, in particular, correspond to values of χ_0 equals zero, $\frac{1}{2}\mathfrak{a}\pi$, $\mathfrak{a}\pi$ and $\frac{3}{2}\mathfrak{a}\pi$, and therefore increasing distances from the pivot center.

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In addition to the geodetic circles and the geodetic lines from 10° to 10° azimuth, fig. 25 also indicates the connecting curve of the cusps, which, in the plane presentation, is generally represented by the equation

$$\xi_0^{\frac{2}{3}} + \eta_0^{\frac{2}{3}} = (\mathfrak{a}\pi\cos^2\beta_1)^{\frac{2}{3}}, \qquad (1)$$

into which, for the special case of fig. 25, it is necessary to introduce $\cos \beta_1 = 1$

Equatorial Point as Center.

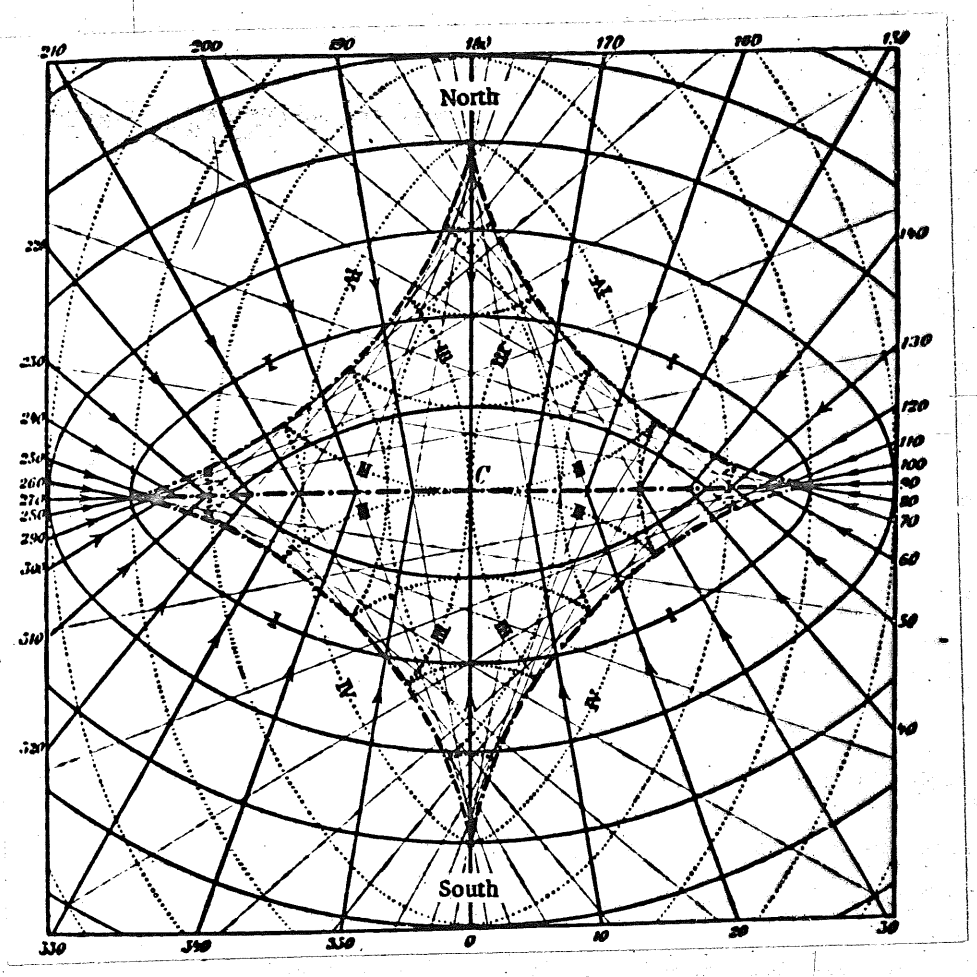


Figure 25. The Numbers Give the Azimuth $\alpha_{1,2}$ in Degrees.

According to established theorems of analytical geometry, this is the equation for the evolute of all straight lines from the constant length $a\pi \cos^2 \beta_1$ between the axes of ξ and η.

These straight lines are identical with those which, in the plane presentation, correspond to the geodetic lines; for if we form $\frac{d\eta_0}{d\xi_0}$, it is evident that the tangent of the evolute runs in the same direction at the point (ξ_0, η_0) as the straight line cor-

responding to the geodetic line which traverses that point.

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It can easily be seen geometrically, however, that the locus of all cusps must be an evolute of the geodetic lines, because when $m = 0$ two infinitesimally close lines intersect in each cusp.

Fig. 25 also retains substantially its validity for any point as the pivot center; however, as a result of equation (1) the evolute contracts ever more, the closer the center comes to one of the poles. The half-lengths of its axis, except for fractions of the second-order, equal

$$a_0 \mathfrak{A} \pi \cos^2 B_1, \qquad (2)$$

if B_1 designates the geographic latitude of the center.

As a result of this contraction the area, in which the geodetic circles depart from the simple circular form, grows ever smaller the closer the center comes to one of the poles, and it disappears entirely, when one of the poles itself becomes the center. Since our developments have, in part, been conducted with the exclusion of the proximity of the poles, the latter conclusion still requires confirmation, which, however, may be derived quite simply from the fact that, with a pole as center, the geodetic lines are the parallel circles.*)

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*)An investigation of the evolute whose center lies on the equator is given by Cayley in the Philosophical Magazine, Volume 40, 1870, second half, p. 10 ff. He calls this line the geodetic evolute. Cayley's investigation, valid for any flattening and employing elliptic functions, came to the author's knowledge only while this work was being printed.

Par. 3. Shortest Lines Between Two Nearly Diametrical Points.

The geodetic circles in fig. 25 also show directly how the shortest lines run forth from the point of departure. We only have to note that the circles I and IV, as indicated, lie at increasing distances from the pivot center, and we shall easily verify that the heavy straight lines belong to the shortest lines emanating in the direction of the arrows from the pivot center. The geodetic circles in fig. 25, insofar as they are lines of equally shortest distance, have been drawn in heavily.

To establish precisely how the shortest lines run, we need an exact knowledge of the middle lines (axes) of the area bounded on the ellipsoid by the evolute; fig. 25 shows at a glance that the course of a shortest line is determined by the position of its vertex P_2 relative to these middle lines or, in short, by the quadrant containing P_2 in the evoluted area.

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One of the middle lines falls into the meridian of point C lying opposite P_1. This requires no proof.

The other one possesses the peculiar quality that its points are connected by two equal shortest lines with the point of departure P_1 and that it falls into the parallel circle of point C This remains to be proved.

Par. 3. Shortest Lines Between Two Nearly Diametrical Points. If, on the auxiliary sphere (fig. 21, p. 232) we connect two diametrically lying points $\mathfrak{P}_1$ and $\mathfrak{P}_2$ by means of a great circle, we get $\Delta\lambda = \Delta\varphi = \pi$, and according to p. 231 (14)

$$L_{1.2} = \pi - a\pi \cos\beta_1 \sin\alpha_{1.2}\left(1 - \frac{1}{2}k_1 + \cdots\right) \qquad (1)$$

with all periodical members disappearing. At the same time we obtain, for the lineal length of the geodetic line P_1P_2, according to p. 223:

$$s = \pi b_0 \frac{1 + \frac{1}{4}k_1^2 + \cdots}{1 - k_1}, \tag{2}$$

also with all periodical members disappearing. We have, according to p. 221

$$k_1 = \frac{1}{4}k^2 + \frac{1}{8}k^4 + \cdots \qquad k^2 = \frac{e^2 \sin^2\beta_0}{1 - e^2 + e^2 \sin^2\beta_0}, \tag{3}$$

with the following value for $\sin^2\beta_0$ as a result of p. 232 (2):

$$\sin^2\beta_0 = 1 - \cos^2\beta_1 \sin^2\alpha_{1.2}. \tag{4}$$

These formulae show that $L_{1.2}$ and s in respect to $\alpha_{1.2}$ depend only on the sine, and that therefore for two points P_1 and P_2 with the geographic latitudes B_1 and $B_2 = -B_1$ there may exist two equally long geodetic connecting lines, whose azimuths $\alpha_{1.2}$ will then add to 180°. These two connections, of course, need not exist; they will occur only when $L_{1.2}$ fulfills equation (1). Since the marginal values of $\sin\alpha_{1.2}$ equal ±1, $L_{1.2}$ must satisfy the condition:

$$\left.\begin{aligned} \pi - \text{ß}\pi\cos\beta_1\left(1 - \tfrac{1}{8}e^2\sin^2\beta_1 + \cdots\right) &\leq L_{1.2} \\ L_{1.2} &\leq \pi + \text{ß}\pi\cos\beta_1\left(1 - \tfrac{1}{8}e^2\sin^2\beta_1 + \cdots\right). \end{aligned}\right\} \tag{5}$$

If $L_{1.2} = \pi \mp \text{ß}\pi\cos\beta_1\left(1 - \frac{1}{8}e^2\sin^2\beta_1 + \cdots\right)$, the

two equally long connections will fall into a single line which

emanates in the azimuth $\alpha_{1.2} = 90°$, or 270° from P_1, and enters in the azimuth $\alpha_{2.1} = 270°$, or 90° in P_2.

In this case, P_2 lies in the western or eastern cusp of the evolute. The geometric locus of all P_2 with two equally long connections is the east-west axis of the evolute (fig. 25) which, according to the postulate $B_2 = -B_1$, is a part of the parallel circle (q.e.d.) passing through point P_1 which is diametrical to C.

On the basis of fig. 25 we also understand now that the shortest connection between two nearly diametrical points P_1 and P_2 can never intersect the east-west axis -- running through point P_1, diametrical to C -- of the evolute of the geodetic lines which emanate from P_1, no more than it can cross the meridian of C. The quadrant of $\alpha_{1.2}$ is determined hereby.

Since, with sufficient extension, all geodetic lines emanating from P_1 will intersect the east-west axis of the evolute, and, in this case, $\Delta\varphi = \pi$, we recognize further that for the shortest line we must always get

$$\Delta\varphi \leqq \pi \tag{6}$$

which is being noted to complement p. 264.

In order to gain an approximation formula for the azimuth of the shortest line between two given, nearly diametrical points, we eliminate from the formulae (8), p. 322, the unknown χ_0, as follows: we multiply the first by $\sin\alpha_{1.2}$, and the second by $\cos\alpha_{1.2}\cos\beta_1$, after which we subtract both. There follows:

$$\begin{aligned}(\beta_1 + \beta_2)\sin\alpha_{1.2} + (\pi - L_{1.2})\cos\alpha_{1.2}\cos\beta_1 \\ = \mathfrak{A}\pi\sin\alpha_{1.2}\cos\alpha_{1.2}\cos^2\beta_1 + Gl_4.\end{aligned} \tag{7}$$

From this, we can derive an equation of the fourth degree for

$\sin \alpha_{1.2}$; however, it would seem to be more convenient to determine the only relevant root by trial from equation (7) or from the following, usually adequate, equation:

$$(\beta_1 + \beta_2) \tan \alpha_{1.2} + (\pi - L_{1.2}) \cos \beta_1 = a\pi \cos^2 \beta_1 \sin \alpha_{1.2} + \cdots \qquad (8)$$

Here we have to bear in mind that, according to the theory developed above, the following relationship exists:

$$\left.\begin{array}{ll} \alpha_{1.2} < 180^0 & \text{for} \quad L_{1.2} < 180^0 \\ \alpha_{1.2} > 180^0 & \text{"} \quad L_{1.2} > 180^0 \end{array}\right\}. \qquad (9)$$

If the point P_2 lies inside of the space of the evolute, i.e., according to p. 323 (9) and 325 (1), when

$$(\beta_1 + \beta_2)^{\frac{2}{3}} + (\pi - L_{1.2})^{\frac{2}{3}} \cos \beta_1^{\frac{2}{3}} \leq (a\pi \cos^2 \beta_1)^{\frac{2}{3}} + \cdots,$$

the above equations (7) and (8) have two roots smaller or larger than 180°, where corresponding to the shortest line (cf. fig. 25):

$$\left.\begin{array}{lll} \alpha_{1.2} & \text{is in the first or fourth quadr. for} & (\beta_2 + \beta_1) < 0 \\ \alpha_{1.2} & \text{" " " second " third " " } & (\beta_2 + \beta_1) > 0 \end{array}\right\}. \qquad (10)$$

With the aid of these formulae, it will be possible to obtain, in all cases in which in the procedure of p. 247, Par. 13, the first approximation $\Delta\lambda = L_{1.2}$ should be insufficient, a serviceable first approximation value for $\alpha_{1.2}$, from which there follows further in first approximation:

$$\Delta\lambda = L_{1.2} + a\pi \cos \beta_1 \sin \alpha_{1.2} + \cdots. \qquad (11)$$

It goes without saying that even this first approximation is

relatively disadvantageous for the positions of P_2 near the eastern or western cusp of the evolute of the geodetic lines emanating from P_1 (or conversely). This is so because a finite revolution of the geodetic line results in a change of locus for P_2 which may be regarded as more or less infinitesimal.

Par. 4. Difference Between the Astronomic and Geodetic Azimuth.

For geodesy, the difference between the vertical section and the geodetic line connecting the same two points is especially important in view of the fact that the observed horizontal angles and azimuths are related to vertical sections, whereas the introduction of the geodetic line demands of course that we take into account the azimuths of that line. For this reason, we must reduce the azimuths of vertical sections, i.e., from astronomic azimuths to geodetic azimuths.

Here we are dealing only with small distances of two points, and for these the directional difference may be found easily by means of the earlier developed formulae. For points in any mutual position, however, we can no more establish a direct formula, than for the azimuthal difference of the two connecting vertical sections (p. 183) because the pertinent series do not generally converge.

The difference between astronomic and geodetic azimuth, even for moderate distances of points is only of minor magnitude; but it may lose these characteristics entirely and gain values of up to 180°, as we shall see later.

Turning now to the formulae, we find first, with the aid of the spherical triangle, fig. 21, p. 232, that for the geodetic line running from P_1 to P_2 the following is true:

$$\cot \alpha_{1.2} = (\cos \Delta\lambda \sin \beta_1 - \tan \beta_2 \cos \beta_1) : \sin \Delta\lambda, \tag{1}$$

and, according to p. 231 and because of $\cos \beta_0 = \cos \beta_1 \sin \alpha_{1.2}$ there exists the following relationship of $\Delta\lambda$ to $L_{1.2}$:

$$L_{1.2} = \Delta\lambda - \frac{1}{2} e^2 \varkappa \Delta\varphi \cos \beta_1 \sin \alpha_{1.2}. \tag{2}$$

The factor $\varkappa$, introduced here, differs from 1 only to the extent of terms which have been multiplied by e^2 and higher powers thereof, and which yield, at any time, only a minor fraction of the same order as e^2 .

On the other hand, we have for the vertical section, placed from P_1 to P_2 and as a result of p. 138 (2):

$$\cot \alpha_{1.2} = \frac{(\cos L_{1.2} \sin \beta_1 - \tan \beta_2 \cos \beta_1) + e^2 (\sin \beta_2 - \sin \beta_1) \cos \beta_1 \sec \beta_2}{\sin L_{1.2} \sqrt{1 - e^2 \cos^2 \beta_1}}, \tag{3}$$

into which we now introduce $\Delta\lambda$. Assuming, for the time being, that $L_{1.2} = \Delta\lambda - \zeta$, we get, first:

$$\left.\begin{aligned} \cos L_{1.2} &= \cos \Delta\lambda \cos \zeta + \sin \Delta\lambda \sin \zeta, \\ \sin L_{1.2} &= \sin \Delta\lambda \cos \zeta - \cos \Delta\lambda \sin \zeta. \end{aligned}\right\} \tag{4}$$

Since sin $L_{1.2}$ is in the denominator of (3), the second of these equations may be better expressed thus:

$$\sin L_{1.2} = \sin \Delta\lambda \cos \zeta \left(1 - \cos \Delta\lambda \frac{\tan \zeta}{\sin \Delta\lambda}\right).$$

From this it follows -- $\zeta = \frac{1}{2} e^2 x \Delta\varphi \cos\beta_1 \sin\alpha_{1.2}$ being small even for lines of half a circumference -- that:

$$\sin L_{1.2} = \sin\Delta\lambda \cos\zeta \left(1 - \frac{1}{2} e^2 x' \Delta\varphi \cot\Delta\lambda \cos\beta_1 \sin\alpha_{1.2}\right),$$

with x' differing from I only by small magnitudes dependent on e^2, whereas $\cos\zeta$ actually coincides with 1, except for one term with $e^4\Delta\varphi^2$.

If we eliminate in the right-hand parenthesis $\sin\Delta\lambda$ by means of the relation of the spherical triangle

$$\sin\Delta\varphi : \sin\Delta\lambda = \cos\beta_2 : \sin\alpha_{1.2},$$

we shall then have:

$$\sin L_{1.2} = \sin\Delta\lambda \cos\zeta \left(1 - \frac{1}{2} e^2 x' \frac{\Delta\varphi}{\sin\Delta\varphi} \cos\beta_1 \cos\beta_2 \cos\Delta\lambda\right). \qquad (5)$$

We now introduce this expression for $\sin L_{1.2}$ into (3), and obviously can undertake, for moderate distances, a series development of the terms in the parenthesis entering into the denominator of (3). If we also develop the value of the root in the denominator of (3), substitute the value of $\cos L_{1.2}$ according to (4), and take into account (1), we obtain:

$$\cot\alpha_{1.2} = \left\{\begin{array}{l} \cot\alpha_{1.2}\left(1 + \frac{1}{2} e^2 \cos^2\beta_1 + \frac{1}{2} e^2 \frac{\Delta\varphi}{\sin\Delta\varphi} \cos\beta_1 \cos\beta_2 \cos\Delta\lambda + \cdots\right) \\ + e^2 \cos\beta_1 \sec\beta_2 \frac{\sin\beta_2 - \sin\beta_1}{\sin\Delta\lambda} + \frac{1}{4} e^2 \Delta\varphi \sin 2\beta_1 \sin\alpha_{1.2} + \cdots. \end{array}\right\} \qquad (6)$$

This equation neglects e^4, ect., but contains all of the terms depending on e^2.

We further eliminate β_2 and $\Delta\lambda$, in order to have everything

represented by $\alpha_{1.2}$, β_1 and $\Delta\varphi$. To this end, the following relations will be employed. First, we have, in the spherical triangle, the equation: $\cos\Delta\varphi = \sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos\Delta\lambda$. If we assume that $\sin\beta_2 = \sin\beta_1 \cos\Delta\varphi - \cos\beta_1 \sin\Delta\varphi \cos\alpha_{1.2}$, we find:

$$\cos\beta_1\cos\beta_2\cos\Delta\lambda = \cos\Delta\varphi - \sin^2\beta_1\cos\Delta\varphi + \frac{1}{2}\sin 2\beta_1 \sin\Delta\varphi\cos\alpha_{1.2}. \tag{7}$$

It follows further from this equation that for $\sin\beta_2$ we get the difference $\sin\beta_2 - \sin\beta_1 = -2\sin\beta_1 \sin^2\frac{\Delta\varphi}{2} - \cos\beta_1 \sin\Delta\varphi\cos\alpha_{1.2}$. Since, however, $\sin\Delta\lambda = \sin\Delta\varphi \sec\beta_2 \sin\alpha_{1.2}$, we have further:

$$\frac{\sin\beta_2 - \sin\beta_1}{\sin\Delta\lambda} = -\tan\frac{\Delta\varphi}{2}\frac{\sin\beta_1}{\sin\alpha_{1.2}}\cos\beta_2 - \cot\alpha_{1.2}\cos\beta_1\cos\beta_2. \tag{8}$$

If we insert (7) and (8) in (6), we come, through a slight reduction, to the equation

$$\cot\alpha_{1.2} = \cot\alpha_{1.2} - \frac{1}{2}e^2\left(1 - \frac{\Delta\varphi}{\tan\Delta\varphi}\right)\cos^2\beta_1\cot\alpha_{1.2} - \frac{1}{2}e^2\left(\tan\frac{\Delta\varphi}{2} - \frac{\Delta\varphi}{2}\right)\frac{\sin 2\beta_1}{\sin\alpha_{1.2}} + \cdots. \tag{9}$$

If in this equation we substitute $s:a_0$, for $\Delta\varphi$, the degree of precision will not change, because the terms multiplied by e^2 are still completely present.

This introduction of s we undertake simultaneously with the transition to the azimuthal difference $\alpha_{1.2} - \alpha_{1.2}$. According to Par. 8, p. 30,

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we may, as long as the development (9) is at all valid, assume:

(See Illustrated Material on Next Page)

Par. 5. Difference Between the Astronomic and Geodetic Azimuth.

$$\alpha_{1.2} - \alpha_{1.2} = \left\{ \begin{array}{l} \frac{1}{4} e^2 \left(1 - \frac{\frac{s}{a_0}}{\tan \frac{s}{a_0}} \right) \cos^2 \beta_1 \sin 2\alpha_{1.2} \\ + \frac{1}{2} e^2 \left(\tan \frac{s}{2a_0} - \frac{s}{2a_0} \right) \sin 2\beta_1 \sin \alpha_{1.2} + \cdots . \end{array} \right\} \qquad (10)$$

This development contains all of the terms multiplied by e^2 -- even, if instead of β_1 we write B_1, i.e., if we introduce the geographic latitude.

If we apply, finally, the series developments for $\tan \frac{s}{a_0}$ and $\tan \frac{s}{2a_0}$, and neglect s^5, we get:

$$\underset{\text{in Sec.}}{\alpha_{1.2} - \alpha_{1.2}} = \frac{1}{12} \rho'' e^2 \frac{s^2}{a_0^2} \left(\cos^2 B_1 \sin 2\alpha_{1.2} + \frac{1}{4} \frac{s}{a_0} \sin 2B_1 \sin \alpha_{1.2} \right) + Gl_6,$$

in which formula, just as in (10), with unaltered right-hand exactitude, $\alpha_{1.2}$ may also be replaced by $a_{1.2}$.

If it is stated in formula (11) that terms of the sixth-order have been neglected, e and $s:a_0$ being, as before, magnitudes of the first-order, it is assumed that the neglected multiplied terms by e^4 possess at least the factors s^2. This factor is contained in all members of the development for $\alpha_{1.2} - \alpha_{1.2}$ in any case. For, if $\frac{s}{a_0}$ is infinitesimal of the first-order, the vertical section, as presented on p. 212 ff., coincides with the geodetic line, i.e., its directional difference is in the infinitesimal of a higher order than $s:a_0$. In the development of the former according to powers of s (of whose possibility for sufficiently small s there can be no doubt) s^2 must everywhere by present as a factor.

Par. 5. Continuation: Difference Between the Astronomic and Geodetic Azimuth. If we compare formula (11) with formula (21), p. 186, we find complete correspondence in the form of the main members

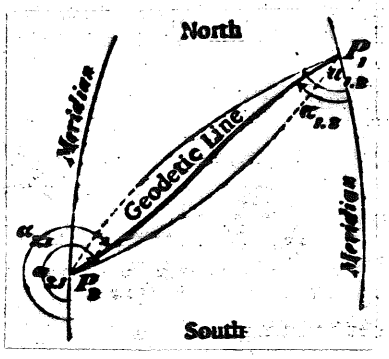


Figure 26.

for $\alpha_{1.2} - \alpha_{1.2}$ and $\alpha_{1.2} - \alpha'_{1.2}$, but the former is only a third of the latter.

For a short geodetic line, the course in respect of the vertical sections in the vicinity of the vertices P_1 and P_2 is therefore generally determined by the fact that the line divides the angular space between the vertical sections into two parts in the proportion 1 : 2, lying closest to that vertical section which takes its departure from the referenced

vertex.

The difference $\alpha_{1.2} - \alpha_{1.2}$ is for $s = 64^{km}$ at most 0,01", and therefore so small that we may relate the observed horizontal angles and azimuths directly to the geodetic lines, without committing notable errors in comparison with the observational errors.

It is different, however, when larger distances are involved than those generally corresponding to the sides of measurable triangles. For $s = a_0$ the difference $\alpha_{1.2} - \alpha_{1.2}$ at the maximum already amounts to about 2', as shown in formula (10) on the preceding page. With s further increasing, this difference always increases more rapidly than with the square of s, as may be seen from the formula cited. This formula, however, loses its usefulness completely for nearly diametrical points P_1 and P_2, as was stated during its development.

For such points formulae, (1) and (3) of the preceding para-

graph show the relation of $a_{1.2}$ to $\alpha_{1.2}$. For the sake of simplicity, we confine ourselves to examining only the special case, where with any $\alpha_{1.2}$ $\beta_2 = -\beta_1$, and Δl as well as $\Delta\varphi$ equals π, where, therefore, P_1 and P_2 are situated on opposite parallel circles, and where P_2 with fixed P_1 may assume all positions within the eastern and western cusp of that evolute which surrounds point C diametrical to P_1 (fig. 25, p. 325). Under these circumstances, formula (2) for a fixed value of $\alpha_{1.2}$ gives $L_{1.2}$ immediately, by means of the equation

$$L_{1.2} = \pi - \frac{1}{2} e^2 \pi \varkappa \cos\beta_1 \sin\alpha_{1.2}. \qquad (1)$$

The introduction of this expression into equation (3) already transformed by the condition $\beta_2 = -\beta_1$:

$$\cot a_{1.2} = \frac{-\sin\beta_1 \cot\frac{L_{1.2}}{2} - 2e^2 \frac{\sin\beta_1}{\sin L_{1.2}}}{\sqrt{1 - e^2\cos^2\beta_1}},$$

yields:

$$\cot a_{1.2} = -\frac{\sin\beta_1}{\sqrt{1-e^2\cos^2\beta_1}}\left(\frac{4\varkappa'}{\pi\cos\beta_1\sin\alpha_{1.2}} + \frac{1}{4}e^2\pi\varkappa''\cos\beta_1\sin\alpha_{1.2}\right), \qquad (2)$$

wherein $\varkappa'$ and $\varkappa''$ are coefficients deviating from 1 only by small magnitudes in the order of e^2. For the sake of brevity, we assume β_1 as positive only.

In this case, $a_{1.2}$ equals 180° for $\alpha_{1.2}$ = zero; with increasing $\alpha_{1.2}$ it decreases to a minimum value which pertains to a $\alpha_{1.2}$ approximating 90°; however, if $\alpha_{1.2}$ grows further, $a_{1.2}$ again increases and for $\alpha_{1.2} = 180°$ also equals 180°, etc.

A good insight into the mutual relations of vertical sections and shortest line may be obtained from fig. 25 by sketching in those vertical sections which emanate from P_1 and converge in a point C'. C' is the point where the extended normal line of the point of departure P_1 contacts the ellipsoid for the second time (fig. 27).

A Point at 45° Latitude as Center.

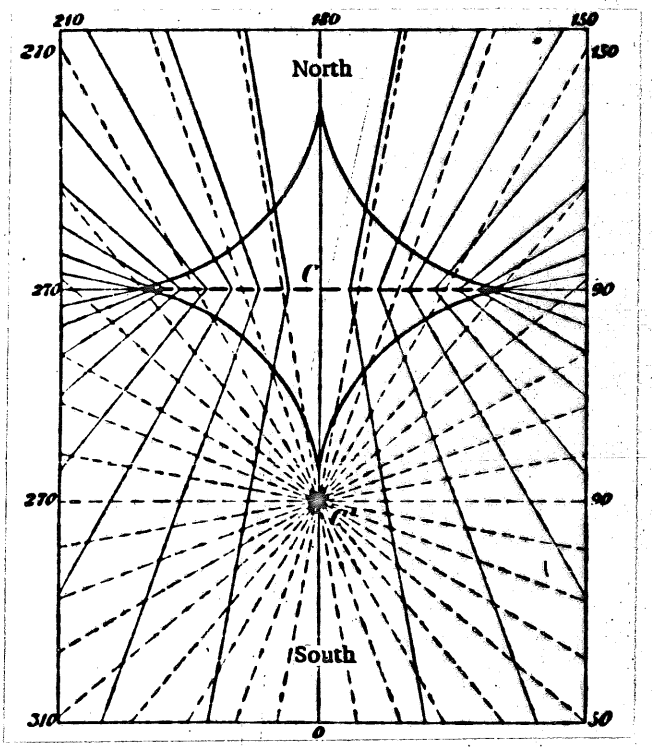


Figure 27. The numbers give the azimuths $\alpha_{1,2}$ and/or $\alpha_{2,1}$ in Degrees.

In order to find the reduced latitude β' of the point C', we deduce from fig. 1 on p. 40 as equation for the normal line through P_1:

$$z + MK_1' = x \tan B_1,$$

in which we substitute

$$x = -a_0 \cos\beta', \quad z = a_0\sqrt{1-e^2}\sin\beta', \quad \tan B_1 = \tan\beta_1 : \sqrt{1-e^2}$$

and for MK_1 the value given on p. 41 (9). A slight reduction gives:

$$\sin(\beta' + \beta_1) = e^2 \cos\beta_1 (\sin\beta' - \sin\beta_1).$$

Since β' undoubtedly nearly equals $-\beta_1$, we assume that $\beta' = -\beta_1 - \zeta$ and obtain:

$$\sin\zeta = e^2 \cos\beta_1 (\sin(\beta_1 + \zeta) + \sin\beta_1).$$

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Approximately, therefore, $\zeta = e^2 \sin 2\beta_1 + Gl_4$, so that we arrive at the following formula for the geographic latitude of C':

$$B' = -B_1 - e^2 \sin 2B_1 + Gl_4. \qquad (3)$$

The position of C' relative to C is determined by the distance CC', which equals $a_0(B' + B_1)$, except for terms with e^4. If, instead of e^2, we simply write 2α, and reckoning CC' from C directly southward, as positive we obtain:

$$CC' = 2a_0\alpha \sin 2B_1 + \cdots, \qquad (4)$$

except for fractions of the second-order of its value.

Fig. 27 sketches for a point P_1 at 45° geographic latitude a picture of the shortest lines and of the vertical sections on that portion of the surface which is diametrical to P_1, on the same scale as in fig. 25.

Par. 6. Another Procedure for Determining the Difference of the Astronomic and Geodetic Azimuth for Short Distances. Between

points P_0 and P_n we imagine, as in fig. 14, p. 213, a geodetic line composed of an infinity of infinitesimal vertical sections P_0P_1, P_1P_2 , etc. For the time being, we designate the number of these parts by n and the length of each part as $s:n$.

If we consider in particular the first two vertical sections, and imagine their length up to the value of s, that is, extended to the vicinity of point P_n , it is clear that their distance from one another will be proportional, first, to the dihedral angle ν between the pertinent planes, and, second, to the distance of the chord P_0P_1 from the surface at P_n . This distance approximates $s^2:2\varrho_\alpha$, insofar as we may regard the vertical section P_0P_1 and its extension as an arc with a radius ϱ_α , if ϱ_α is the radius of curvature of the surface at P_0 in the azimuth $\alpha_{0.1}$. According to p. 188, we must further assume (if ϱ_n stands for the transverse radius of curvature at P_0 and B_0 for the geographic latitude of P_0) that:

$$\nu = \frac{1}{2}\delta\frac{s}{n\varrho_n}\cos^2 B_0 \sin 2\alpha_{0.1}$$

accompanied by the loss of the higher-order terms which disappear because $n = \infty$ in relation to the component term. The separation of the extended vertical sections P_0P_1 and P_1P_2 is therefore:

$$\frac{1}{4}\delta\frac{s^2}{\varrho_n\varrho_\alpha}\frac{n^2}{n^2}\cos^2 B_0 \sin 2\alpha_{0.1}. \qquad (1)$$

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If we consider in the same manner the sufficiently extended vertical sections P_1P_2 and P_2P_3, P_2P_3 and P_3P_4, etc., we obtain analogous expressions for the separations at P_n, however, n^2 will be

replaced in the numerator in turn by $(n-1)^2$, $(n-2)^2$, etc., while $B_0, \alpha_{0.1}, \varrho_n$ and ϱ_a are related successively to P_1, P_2, etc.

With the aid of the differential formulae (1) and (2), p. 279, in which we assume that $\delta s = is : n$, $i = 0, 1, 2 \ldots n-1$, we find easily, while neglecting terms multiplied by e^2:

$$\cos^2 B_i \sin 2\alpha_{i.i+1} = \cos^2 B_0 \sin 2\alpha_{0.1} + \frac{i}{n}\frac{s}{a_0} \sin 2B_0 \sin \alpha_{0.1} + \cdots . \tag{2}$$

If we take no account of changes of $\varrho_n \varrho_a$, amounting only to fractions of the order $e^2 s : a_0$, we obtain as sum of the separations between any two neighboring extended vertical sections at P_n, i.e. for the separation of this point from the extended vertical section $P_0 P_1$:

$$\frac{1}{4}\delta \frac{s^2}{\varrho_n \varrho_a} \left\{ \begin{array}{l} \cos^2 B_0 \sin 2\alpha_{0.1} \dfrac{\Sigma (n-i)^2}{n^2} \\ + \dfrac{s}{a_0} \sin 2B_0 \sin \alpha_{0.1} \dfrac{\Sigma i (n-i)^2}{n^4} \end{array} \right\} + \cdots , \tag{3}$$

where Σ suggests the summation of $i = 0$ to $n-1$. If we write the quotients dependent on n and i in this form:

$$\Sigma \left(1 - \frac{i}{n}\right)^2 \frac{1}{n} \quad \text{and} \quad \Sigma \frac{i}{n} \left(1 - \frac{i}{n}\right)^2 \frac{1}{n}$$

and note that we must take $n = \infty$, we can compute them as integrals with the variable $\frac{i}{n} = x$ from 0 to 1, and with $\frac{1}{n} = dx$. Finally, we get, instead of (3), for the separation:

$$\frac{1}{12}\delta \frac{s^2}{\varrho_n \varrho_a} \left(\cos^2 B_0 \sin 2\alpha_{0.1} + \frac{s}{4a_0} \sin 2B_0 \sin \alpha_{0.1} \right) + \cdots . \tag{4}$$

Insofar as we have assumed $n = \infty$, that is an approximation

for the separation between a geodetic line and a vertical section, both of which emanate in the same azimuth $\alpha_{0.1}$ from P_0, at a distance s from the point of departure.

If we revolve the geodetic line around P_0, until P_n attains a position in the unchanged vertical section, the angle of rotation will evidently be equal to that magnitude which, in the preceding

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paragraph was designated as $\alpha_{1.2} - \alpha_{1.2}$. Since, however, the product of the angle into the length m belonging to s closely approximates the distance computed above, and since m agrees with s except for terms of the third-order, we obtain for the angle of rotation:

$$\alpha_{1.2} - \alpha_{1.2} = \frac{1}{12}\varrho''\delta\frac{s^2}{\varrho_a \varrho_n}\left(\cos^2 B_1 \sin 2\alpha_{1.2} + \frac{1}{4}\frac{s}{a_0}\sin 2B_1 \sin\alpha_{1.2}\right) + Gl_0. \qquad (5)$$

Here, as before the symbols P_1 and P_2 have been used to designate the vertices P_0 and P_n.

This formula agrees substantially with (11), p. 332. It is, however, more exact in the first term than (11) since, in regard to powers of e^2, the first term is now complete.

The above approximate development also shows clearly in what manner the difference between the astronomic and the geodetic azimuth comes about.

The difference between the astronomic and the geodetic azimuth was given without development in 1821 by Bessel, in Astronom. Nachr. Astronomical News, Volume 1, No. 3, p. 33; (cf. Engelmann, Abhandl. von Bessel, Volume 3, p. 1). A development was given by Bessel later (1837) Astronom. Nachr. Volume 14,

No. 330, p. 289, equation (19) (Abhandl, Volume 3, p. 29); it does not differ substantially from the procedure followed here on p. 329 ff., and leads to formula (10), p. 332.

A substantially different procedure -- which, by the way, is applicable for any curved surface -- was adopted by Weingarten (cf. Baeyer, Das Messen auf der sphäroidischen Erdoberfläche (Measurements on the Spheroid Earth Surface), 1862, p. 88, or a short notice in Astronom. Nachr., Volume 60, No. 1425, p. 135).

This development has been reproduced by Jordan in the second volume of his Handbook of Surveying, 1878, p. 334. With Weingarten's procedure, we obtain the considered directional difference developed directly according to powers of s, namely, the first member of formula (11), p. 332; this, of course, presupposes that the differential equations of the geodetic line have been derived for any surface, and, in addition, it fails to provide for nearly diametrical points of the ellipsoid of rotation. An analogous development, continued, however, up to and including s^3 and higher powers of e^2, was given by Andrae in 1867 in Volume 1 of the Danske Gradmaaling p. 181 (cf. also Vierteljahrsschrift der Astronom. Ges. (Quarterly of the Astronomical Society), Volume 13, p. 19). His expression agrees with our formula (5) except for the denominator of the member multiplied by s^3, which he gives as ϱ^2. This difference, however, is insignificant.

The method of our development in this paragraph corresponds to the procedure of Sonderhof, in 1869, in the 51st

part of Grunerts Archiv (2. Abhandlung). This method also permits an estimate of the error committed when the geodetic line is composed of n finite sections, according to fig. 14, p. 213.

For this purpose, we have to determine the distance of the chords P_0P_1, P_1P_2,

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etc. from the surface more exactly than was done above; however, we consider only terms multiplied by e^2. If we consider, first, the arc of the vertical section P_0P_1, extended into the vicinity of P, in the expansion s as an arc of the radius a_0, there belongs to it an angle at the center $s:a_0$. A simple figure will show that the arc tangent which is parallel to the chord P_0P_1 diverges from the vertex of the arc by

$$a_0\left(1-\cos\left\{\frac{s}{a_0}\left[1-\frac{1}{2n}\right]\right\}\right)$$

but from the chord by

$$a_0\left(1-\cos\frac{s}{2a_0n}\right),$$

therefore, the chord diverges from the relevant vertex of the arc by

$$a_0\left(\cos\frac{s}{2a_0n}-\cos\left\{\frac{s}{a_0}\left[1-\frac{1}{2n}\right]\right\}\right)$$

i.e., very nearly:

$$\frac{s^3}{2a_0n^2}(n^2-n).$$

For P_1P_2, P_2P_3 etc. we obtain similar expressions, except that n in the parenthesis is replaced in turn by $n-1$, $n-2$ etc.,

up to 1. Each of these expressions is multiplied by $\frac{1}{2} e^2 \frac{s}{a_0 n} \cos B_0 \sin 2\alpha_{0.1}$, and their addition gives the following separation of the point P_n from the first vertical section $P_0 P_1$:

$$\frac{1}{4} e^2 \frac{s^2}{a_0^2} \cos B_0 \cos 2\alpha_{0.1} \cdot \frac{\Sigma(n-i)^2 - \Sigma(n-i)}{n^3}.$$

Since i runs from 0 to n, we may also write the numerator as: $\Sigma i^2 - \Sigma i$.

Since we have $\Sigma i^2 = \frac{1}{6} n(n+1)(2n+1)$, $\Sigma i = \frac{1}{2} n(n+1)$, and consequently $\Sigma(i^2 - i) = \frac{1}{3} n(n^2-1)$, the spacing therefore approximates

$$\frac{1}{12} e^2 \frac{s^2}{a_0^2} \cos^2 B_0 \sin 2\alpha_{0.1} \left(1 - \frac{1}{n^2}\right). \qquad (6)$$

This expression, for the time being, refers to the separation of the vertical section $P_0 P_1$ from the broken line composed of n vertical sections in the distance s from P_0. If, however, we deduct from (6) the expression (4), which gives the corresponding distance of the vertical section $P_0 P_1$ from the geodetic line touching the section at P_0, we obtain the following for the separation of the geodetic line from a line beginning in the same azimuth and composed of n vertical sections of finite length (fig. 14, p. 213), at the distance s:

$$\frac{1}{12} e^2 \frac{s^2}{a_0^2} \cos^2 B_1 \sin 2\alpha_{1.2} \cdot \frac{1}{n^2} + \cdots. \qquad (7)$$

Here the point of departure and the vertex, instead of

being designated by P_0 and P_n, are rendered as P_1 and P_2, just as in formula (5).

For example, if we have in an equatorial area a chain of triangles in a southwesterly direction, if ten triangle sides are joined with 180° angles, and if the length of a side equals $s:10$ with $s:10 = 0{,}01\,a_0$, i.e., $s = 640000^m$, we obtain, through (7), as the separation of an exact geodetic line having the same initial azimuth as the first triangle side, from the vertex of the tenth triangle side, $0{,}036^m$.

Par. 7. Numerical Example I. For the line Koenigsberg-Berlin we have, according to pp. 158 and 162, as well as 256 and 261:

$$B_1 = 54^\circ\,42'\,50{,}6'' \qquad B_2 = 52^\circ\,30'\,16{,}7'' \qquad \log s = 5{,}72426$$

$$\left.\begin{aligned} \alpha_{1.2} &= 65^\circ\,16'\,9{,}5806'' \\ \alpha_{1.2} &= 65\;\;16\;\;9{,}3650 \end{aligned}\right\} \alpha_{1.2} - \alpha_{1.2} = +\,0{,}2156''$$

$$\left.\begin{aligned} \alpha_{2.1} &= 239\;33\;0{,}9324 \\ \alpha_{2.1} &= 239\;33\;0{,}6889 \end{aligned}\right\} \alpha_{2.1} - \alpha_{2.1} = +\,0{,}2435''.$$

Since we also have

$$\log \frac{1}{12}\rho''\delta = 2{,}0626 \quad \text{and} \quad \log \frac{1}{48}\rho''\delta = 1{,}461$$

we obtain, according to formula (5), p. 337:

$$\alpha_{1.2} - \alpha_{1.2} = [13{,}5111]\frac{\cos^2 B_1 \sin 2\alpha^0_{1.2}}{\varrho_{\alpha_1}\varrho_{\alpha_2}} + [18{,}634]\frac{\sin 2B_1 \sin \alpha_{1.2}}{a_0 \varrho_{\alpha_1}\varrho_{\alpha_2}}$$

$$\alpha_{2.1} - \alpha_{2.1} = [13{,}5111]\frac{\cos^2 B_2 \sin 2\alpha_{2.1}}{\varrho_{\alpha_2}\varrho_{\alpha_2}} + [18{,}634]\frac{\sin 2B_2 \sin \alpha_{2.1}}{a_0 \varrho_{\alpha_2}\varrho_{\alpha_2}}$$

According to Albrecht, pp. 200 and 201, we find for the above values of

B_1 and $\alpha_{1.2}$: $\log \varrho_{\alpha_1} = 6{,}8056$ $\log \varrho_{\alpha_2} = 6{,}8054$

and for

B_2 and $\alpha_{2.1}$: $\log \varrho_{\alpha_2} = 6{,}8056$ $\log \varrho_{\alpha_2} = 6{,}8053$.

Par. 8. The Difference of the Lineal Lengths of Geodetic Line and Vertical Section Between the Same Two Points.

We have further

$\log \cos^2 B_1 = 9{,}5233 - 10$	$\log \sin 2B_1 = 9{,}975 - 10$
$\log \sin 2\alpha_{1.2} = 9{,}8808 - 10$	$\log \sin \alpha_{1.2} = 9{,}958 - 10$
$\log \cos^2 B_2 = 9{,}5689 - 10$	$\log \sin 2B_2 = 9{,}985 - 10$
$\log \sin 2\alpha_{2.1} = 9{,}9414 - 10$	$\log \sin \alpha_{2.1} = 9{,}936 - 10$

and thus:

$$a_{1.2} - \alpha_{1.2} = + 0{,}2015 + 0{,}0141 = + 0{,}2156''$$
$$a_{2.1} - \alpha_{2.1} = + 0{,}2573 - 0{,}0138 = + 0{,}2435''.$$

This fully agrees with the values computed according to the exact formulae.

Numerical Example II. Here we obtain, as stipulated on pp. 166 and 320, the following differences:

$$a_{1.2} - \alpha_{1.2} = - 0{,}012'' \quad \text{and} \quad a_{2.1} - \alpha_{2.1} = - 0{,}012''.$$

Par. 8. The Difference of the Lineal Lengths of Geodetic Line and Vertical Section between the Same Two Points. This difference can easily be specified by the use of the preceding method. If we take P_1, from which the vertical section may emanate, as the center A of geodetic polar coordinates (fig. 23, p. 269), we get according to equation (3), loc. cit., for the lineal element ds of the vertical section contacting any point P of that section:

$$ds^2 = dr^2 + \mathfrak{m}_r^2 d\alpha_0^2, \qquad (1)$$

with r as the geodetic radius vector AP, $\mathfrak{m}_r$ its reduced length,

and α_0 its azimuth. If we designate the azimuth of the vertical section at A as $a_{1.2}$, we get, according to formula (11), p. 332, in the first approximation:

$$\alpha_0 = a_{1.2} - \frac{1}{12} e^2 \frac{r^2}{a_0^2} \cos^2 B_1 \sin 2a_{1.2} + \cdots,$$

in which B_1 represents the geographic latitude of A. From this it follows:

$$d\alpha_0 = -\left(\frac{1}{6} e^2 \frac{r}{a_0^2} \cos^2 B_1 \sin 2a_{1.2} + \cdots\right) dr. \qquad (2)$$

If we consider that it is evidently sufficient to replace, in the second term of formula (1) of the right hand in the first approximation, m, by r, for moderate distances, we obtain from (1) through this substitution, and because of (2):

$$ds = \sqrt{1 + \frac{1}{36} e^4 \frac{r^4}{a_0^4} \cos^4 B_1 \sin^2 2a_{1.2} + \cdots}\, dr.$$

From this, it follows that:

$$ds = \left(1 + \frac{1}{72} e^4 \frac{r^4}{a_0^4} \cos^4 B_1 \sin^2 2a_{1.2} + \cdots\right) dr.$$

If we integrate this from the initial point A, i.e., P_1, to the vertex P_2, we get:

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$$s = r_{1.2}\left\{1 + \frac{1}{360} e^4 \frac{r_{1.2}^4}{a_0^4} \cos^4 B_1 \sin^2 2a_{1.2} + \cdots\right\}, \qquad (3)$$

in which s stands for the lineal length of the vertical section P_1P_2, whereas $r_{1.2}$ stands for the lineal length of the geodetic line P_1P_2.

Par. 8. The Difference of the Lineal Lengths of Geodetic Line and Vertical Section Between the Same Two Points.

This formula still suffices for distances equaling $0{,}2a_0$. If, however, the distance increases to a_0 and beyond, the development will, at first, still be convergent, but the component term will yield but a rough approximation.

If we take $r_{1.2} = 0{,}2a_0$ the maximum will be

$$s = r_{1.2}\left(1 + \tfrac{1}{300000000000}\right),$$

so that, with a ten-figure computation, the logarithms of s and $r_{1.2}$ do not yet differ from each other by a unit of the last decimal; therefore, in most practical cases the vertical section, as far as lineal length is concerned, does not differ appreciably from the geodetic line. (For the line Berlin-Koenigsberg the difference amounts to $0{,}0002^{mm}$. Cf. also examples I and II, pp. 182, 183, 261, and 320).

If we interchange in (3) indices 1 and 2, we find that, e and $r_{1.2}:a_0$ being, as always, magnitudes of the first-order, both vertical sections are lengthened, compared with the geodetic line, by a term of the eighth-order in fractions of the length. The differences among the lengths of the vertical sections because of the inequality of these terms of the eighth-order and higher-order terms is, therefore, in fractions of the length, at most of the ninth-order, as may be remarked in addition to p. 182.

As a result of the preceding remarks, we may regard the relations of the length of the vertical section to the chord and to σ' (pp. 181 and 182) directly, if we so desire, as relations of the geodetic line to the chord, or to σ'.

Formula (3) was given by Bessel in 1837 without derivation in Astronom. Nachr., Volume 14, No. 330, p. 285 (Abhand-

lungen, Volume 3, p. 26). Weingarten arrived at this formula in the course of the mentioned (p. 337) development for any curved surface (cf. Baeyer, op. cit., p. 93; Jordan, Volume 2, p. 339). An estimate of the order of the lineal difference of geodetic line and vertical section in a simple geometric manner was given by Andrae in Volume I of Daenische Gradmessung (Danish Arc Measurement), p. 179 (cf. Quarterly of the Astronomical Society, Volume 13, p. 18).

Par. 9. Separation of the Geodetic Line from Both Vertical Sections, With Small Distances. In fig. 26, p. 332, we assume a point P on the vertical section P_1P_2, and a geodetic line r, connecting P_1 and P. If α_0 is its azimuth in P_1, we obtain in the first approximation:

$$\alpha_0 = \alpha_{1.2} - \frac{1}{12} e^2 \frac{r^2}{a_0^2} \cos^2 B_1 \sin 2\alpha_{1.2} + \cdots.$$

For the geodetic line P_1P_2, however, whose length we now designate by s, the azimuth in P_1 equals

$$\alpha_{1.2} = \alpha_{1.2} - \frac{1}{12} e^2 \frac{s^2}{a_0^2} \cos^2 B_1 \sin 2\alpha_{1.2} + \cdots.$$

The azimuthal difference of the geodetic lines P_1P and P_1P_2 is now $\alpha_0 - \alpha_{1.2}$. If we multiply the difference by the reduced length (on the permissible assumption that $\alpha_0 - \alpha_{1.2}$ may be considered an infinitesimal rotation) and replace this length, in approximation, by r, we obtain as the separation of the vertical section P_1P_2 from the geodetic line P_1P_2 at a distance r from P_1 in the first approximation:

Par. 9. Separation of the Geodetic Line from Both Vertical Sections, With Small Distances.

$$\frac{1}{12} e^2 r \frac{s^2 - r^2}{a_0^2} \cos^2 B_1 \sin 2a_{1.2} + \cdots. \qquad (1)$$

If, in this formula, we interchange points P_1 and P_2, replacing r with $s - r$, we obtain for the separation of the vertical section $P_2 P_1$ from the geodetic line in the same place as before, and after a simple reduction:

$$\frac{1}{12} e^2 r \frac{(s-r)(2s-r)}{a_0^2} \cos^2 B_2 \sin 2a_{2.1} + \cdots. \qquad (2)$$

Since we do not consider higher terms, we can introduce, in this formula and in (1), for B_1 and B_2 on the one hand, and for $2a_{1.2}$ and $2a_{2.1}$ on the other hand, the same mean value, or B_1, and $2a_{1.2}$ respectively. The addition of (1) and (2) yields the separation of the vertical sections. This separation is therefore, at the distance r from P_1, in the first approximation, equal to

$$\frac{1}{4} e^2 \frac{sr(s-r)}{a_0^2} \cos^2 B_1 \sin 2a_{1.2} + \cdots. \qquad (3)$$

The comparison of (1) and (3) shows that expression (3) will always be greater than (1), in view of the proportion:

$$(1) : (3) = s + r : 3s. \qquad (4)$$

Therefore, the geodetic line, in the first approximation for small distances, lies always between the two vertical sections which connect its two vertices.

The differentiation of (3) or (1) with respect to r also shows that in the first approximation the maximum separation of the vertical sections will lie midway between P_1 and P_2; whereas the

maximum separation of the geodetic line,

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and the vertical section will lie at a distance $s:\sqrt{3}$ from that of the two points which serves as the beginning of the vertical section. Therein the distances (1) and (3) are in the following proportion:

$$\left(1+\frac{1}{\sqrt{3}}\right):3 \quad \text{i.e., as } 1 : 1{,}902.$$

The above relationships, just as the one concerning the trisection of the angle between the vertical sections by the geodetic line, mentioned on p. 332, cannot of course be accepted as fully valid and applicable to lines of any length. If we want to establish the relations with greater exactitude, we have to find a more exact expression of the astronomic-geodetic azimuthal difference; as to the rest, however, the above procedure would have to be followed.

In this manner, we shall now consider the case where vertical section and geodetic line have nearly the azimuth 90°, in particular when $B_1 = B_2$. According to formulae (1) and (3), the vertical sections would, for $\alpha_{1,2} = 90''$ coincide with one another as well as with the geodetic line. This, must be modified however more exactly for the two vertical sections to the extent that they will coincide only if both points P_1 and P_2 are situated on the same parallel circle -- in every other case (except for the position of both points on the same meridian) they will not coincide. The geodetic line has, in any case, for values of $\alpha_{1,2}$ at approximately 90° a somewhat deviate position in respect to the two vertical sections

Par. 9. Separation of the Geodetic Line from Both Vertical Sections, With Small Distances.

(the line can coincide exactly with the vertical sections only for two points P_1 and P_2 situated on the same meridian, because only in such a case is the osculating plane of each of the two connecting vertical sections simultaneously the vertical plane for all intermediate points).

From formula (11), p. 332, we see that for values of $a_{1.2}$ and $\alpha_{1.2}$ near 90° the azimuthal difference is reduced to a magnitude of the fifth-order; the lineal separation between geodetic line and vertical section is, therefore, for such positions one-order higher than usual. It is obvious, however, that it does not become zero.

If as a result of $B_1 = B_2$, the vertical sections coincide, we get according to p. 144 in the auxiliary spherical triangle (1)

$$\tan\frac{\sigma'}{2} = -\cot B_1 \cos a_{1.2}.$$

Omitting fractions of the second-order (cf. p. 182 (13)), it follows:

$$\cos a_{1.2} = -\frac{1}{2}\frac{s}{a_0}\tan B_1 (1 + Gl_2). \qquad (5)$$

According to p. 332 (11) we have further:

$$\alpha_{1.2} = a_{1.2} - \frac{1}{12}e^2\frac{s^2}{a_0^2}\left(\cos^2 B_1 \sin 2a_{1.2} + \frac{1}{4}\frac{s}{a_0}\sin 2B_1 \sin a_{1.2}\right) + Gl_6. \qquad (6)$$

If we replace in the first term $\cos a_{1.2}$ by the expression (5), we obtain, due to the fact that $\sin a_{1.2}$ deviates from 1 only by a magnitude of the second-order:

$$\alpha_{1.2} = a_{1.2} + \frac{1}{48}e^2\frac{s^2}{a_0^2}\sin 2B_1 + \cdots. \qquad (7)$$

Chapter 7. The Course of the Geodetic Line.

The frequently cited formula (11), p. 332, with $s:2$ as distance in relation to the geodetic line which may be laid from P_1 to the point of the vertical sections situated midway between P_1 and P_2, yields the following azimuth:

$$\alpha_0 = \alpha_{1.2} - \frac{1}{48} e^2 \frac{s^2}{a_0^2} \left(\cos^2 B_1 \sin 2\alpha_{1.2} + \frac{1}{8} \frac{s}{a_0} \sin 2B_1 \sin \alpha_{1.2} \right) + Gl_6. \tag{8}$$

If we replace in the second term of the right hand $\cos\alpha_{1.2}$ with the value given in (5), we get:

$$\alpha_0 = \alpha_{1.2} + \frac{1}{128} e^2 \frac{s^2}{a_0^2} \sin 2B_1 + \cdots. \tag{9}$$

Comparing α_0 with $\alpha_{1.2}$, we see that for positive B_1 the geodetic line lies north of the coincident vertical sections. The distance is greatest in the middle and equals $\frac{1}{2}(\alpha_{1.2} - \alpha_0)$ s i.e.,

$$\frac{5}{768} e^2 \frac{s^4}{a_0^2} \sin 2B_1 + \cdots.^*) \tag{10}$$

For $s = 0{,}1 a_0$ this gives at the maximum roughly 1/4 meter and, according to (7), $\alpha_{1.2} = \alpha_{1.2} + 0{,}03''$.

For $s \leqq 0{,}02 a_0$ the relative values are so small that, for practical purposes, there ceases to be a difference between the coin-

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*)According to (10), our statement, without development, made casually in a book review in the Zeitschr. f. Math. u. Phys. von Schlömilch, Litteraturzeitung (Journal for Mathematics and Physics of Schloeemilch, Literary Review, 1873, p. 39) will have to be corrected. The correct expression (10) derives, in fact, quite simply from an investigation conducted as early as 1870 by A. R. Clarke in the Philosophical Magazine 1870, Volume 39, p. 361 Art. 8.

The rest of (10) is in the form $a_0 Gl_6$, since (10) must be a straight function of s; because it is possible to regard (10) as a function of each of the two separations $\pm s:2$, which P_1 and P_2 have from the apex of the geodetic line. The remainder of (7) also is of the seventh-order.

cident vertical sections and the geodetic line.

Par. 10. Survey of the Position of the Geodetic Line Relative to the Two Vertical Sections for Small Distances. If we examine, with the aid of the two-term expression for azimuthal differences,

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the position of a geodetic line P_1P_2 relative to the two vertical sections in general, we get the following results.

Let us imagine P_2 at a distance s from P_1, having a northern latitude, with describing a geodetic circle around that point. Let the southwestern azimuth $\alpha_{1.2}$ initially be equal to zero, thus the geodetic and the two vertical sections will coincide. If $\alpha_{1.2}$ increases, the geodetic line, for values of this azimuth in the first quadrant, will always lie between the two vertical sections, of which P_1P_2 is the one farther north.

If, however, with further increase, P_1 has become

$$\cos \alpha_{1.2} = -\frac{1}{4}\frac{s}{a_0}\tan B_1 + \cdots \qquad (1)$$

the geodetic line will touch the vertical section P_1P_2 at P_1; with continued rotation that part of the geodetic line which borders on P_1 moves northward out of the space enclosed by the two vertical sections, and for

$$\cos \alpha_{1.2} = -\frac{3}{8}\frac{s}{a_0}\tan B_1 + \cdots \qquad (2)$$

the line lies entirely outside, touching the vertical section P_1P_2 at P_2. With further rotation, the vertical sections for

$$\cos \alpha_{1.2} = -\frac{1}{2}\frac{s}{a_0}\tan B_1 + \cdots \qquad (3)$$

become identical; from then on the geodetic line lies to the north of them, up to that moment when the northern most of the two vertical sections, that is P_2P_1, touches the geodetic line at P_1 for

$$\cos \alpha_{1.2} = -\frac{5}{8}\frac{s}{a_0}\tan B_1 + \cdots; \qquad (4)$$

From then on, the geodetic line again comes to lie, in part, between the two vertical sections, until it touches, for

$$\cos \alpha_{1.2} = -\frac{3}{4}\frac{s}{a_0}\tan B_1 + \cdots \qquad (5)$$

the vertical section P_2P_1 at P_2; being contained entirely, for further values of $\alpha_{1.2}$, in the second quadrant between the two vertical sections. And so forth.

The above is derived from formulae (21) and (25), pp. 186 and 187, and from formula (11), p. 332. Hence we have, for the azimuthal differences of the two vertical sections:

$$\left.\begin{aligned} a_{1.2} - a'_{1.2} &= \\ a_{2.1} - a'_{2.1} &= \end{aligned}\right\} \frac{1}{4}e^2\frac{s^3}{a_0^3}\sin 2B_1 \sin\alpha_{1.2}\left(\cot B_1 \cos\alpha_{1.2} + \frac{s}{2a_0}\right) + Gl_6, \qquad (6)$$

and for the azimuthal difference of the geodetic line and the vertical section P_1P_2 at P_1:

$$a_{1.2} - \alpha_{1.2} = \frac{1}{12}e^2\frac{s^3}{a_0^3}\sin 2B_1 \sin\alpha_{1.2}\left(\cot B_1 \cos\alpha_{1.2} + \frac{s}{4a_0}\right) + Gl_6. \qquad (7)$$

Therefore,

$$a_{1.2} - \alpha_{1.2} = \frac{1}{3}(a_{1.2} - a'_{1.2}) - \frac{1}{48}e^2\frac{s^3}{a_0^3}\sin 2B_1 \sin\alpha_{1.2} + Gl_6.$$

If we interchange indices 1 and 2, it follows, since with the

omission of terms of the sixth-order in the second part on the right B_2 may be exchanged for B_1, and $\alpha_{2.1}$ may be exchanged for $180^\circ + \alpha_{1.2}$, that

$$\alpha_{2.1} - \alpha_{2.1} = \frac{1}{3}(\alpha_{2.1} - \alpha'_{2.1}) + \frac{1}{48} e^2 \frac{s^3}{a_0^3} \sin 2B_1 \sin \alpha_{1.2} + Gl_6.$$

The substitution of the value of $\alpha_{2.2} - \alpha'_{2.1}$ from (6) finally leads to

$$\alpha_{2.1} - \alpha_{2.1} = \frac{1}{12} e^2 \frac{s^2}{a_0^2} \sin 2B_1 \sin \alpha_{1.2} \left(\cot B_1 \cos \alpha_{1.2} + \frac{3s}{4a_0}\right) + Gl_6. \qquad (8)$$

Finally, we obtain by the combination of (7), or (8), with (6):

$$\alpha_{1.2} - \alpha'_{1.2} = \frac{1}{6} e^2 \frac{s^2}{a_0^2} \sin 2B_1 \sin \alpha_{1.2} \left(\cot B_1 \cos \alpha_{1.2} + \frac{5s}{8a_0}\right) + Gl_6 \qquad (9)$$

$$\alpha_{2.1} - \alpha'_{2.1} = \frac{1}{6} e^2 \frac{s^2}{a_0^2} \sin 2B_1 \sin \alpha_{1.2} \left(\cot B_1 \cos \alpha_{1.2} + \frac{3s}{8a_0}\right) + Gl_6. \qquad (10)$$

With the aid of these formulae, it is possible to complete the picture of the mutual position of the three lines; this, however, is left to the reader.

Chapter 8

The Geodetic Triangle; Triangulation Nets.

Par. 1. Basic Theorem for Geodetic Polar Coordinates. Some relations for geodetic polar coordinates have been stated on p. 269, and so that we can start discussing the theory of the geodetic triangle, one further relation must be added, showing how, along a curve, its angle of inclination Θ changes in relation to the geodetic radius vector r, particularly if this curve is again a geodetic line (cf. fig. 28). Θ refers to the directions of the increasing radius vector and the increasing curve, from the former in the sense of increasing azimuths to the latter. Thus

$$\alpha' - \alpha = \Theta, \tag{1}$$

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where α' is the azimuth of the curve and α that of the radius vector. It follows that

$$\frac{d\alpha'}{ds} - \frac{d\alpha}{ds} = \frac{d\Theta}{ds}, \tag{2}$$

where ds defines the arc element PP' of the curve. It is self-evident that, in order to compute $\frac{d\Theta}{ds}$, the characteristics of the curve must be known so that $\frac{d\alpha'}{ds}$ can be determined.

As to $\frac{d\alpha}{ds}$, it can be found by the addition of the expressions (2), p. 279 and (4), p. 280. It is to be noted that α_1 becomes $\alpha + 180^0$ and s becomes r.

Par. 1. Basic Theorem for Geodetic Polar Coordinates.

First we obtain:

$$d\alpha = \left(\frac{d\mathfrak{m}_r}{dr} - \frac{\mathfrak{m}_r}{a_0 w}\tan\beta\cos\alpha\right)d\alpha_0 - \frac{dr}{a_0 w}\tan\beta\sin\alpha,$$

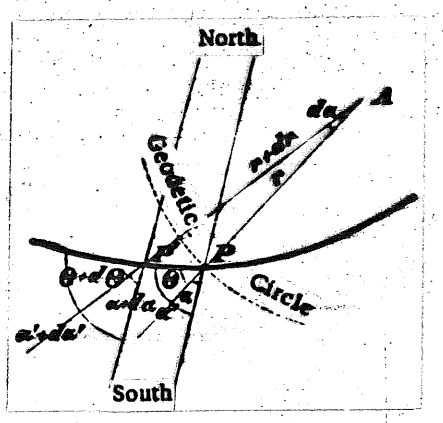


Figure 28

where β is the reduced latitude of P, and $w = \sqrt{1 - e^2\cos^2\beta}$. Since $\mathfrak{m}_r d\alpha_0 = ds.\sin\Theta$ and $dr = ds.\cos\Theta$, it follows that:

$$\frac{d\Theta}{ds} = \frac{d\alpha'}{ds} - \frac{\sin\Theta}{\mathfrak{m}_r}\frac{d\mathfrak{m}_r}{dr} + \frac{\tan\beta\sin\alpha'}{a_0 w}. \qquad (3)$$

since $\alpha + \Theta = \alpha'$.

But if PP' is itself a geodetic curve, then according to p. 279 (2)

$$\frac{d\alpha'}{ds} = -\frac{\tan\beta\sin\alpha'}{a_0 w}$$

since $\alpha_{2,1}$ corresponds to $\alpha' + 180^0$. Thus we obtain for the inclination angle of a <u>geodetic</u> line PP' relative to the geodetic radius vector AP:

$$\frac{d\Theta}{ds} = -\frac{\sin\Theta}{\mathfrak{m}_r}\frac{\partial\mathfrak{m}_r}{\partial r}. \qquad (4)$$

It is to be remembered (p. 280) that $\frac{\partial\mathfrak{m}_r}{\partial r}$ refers to a displacement of P in the direction of the radius vector $AP = r$.

To indicate this, the sign of the partial differentiation

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was chosen so there would be no confusion with that differential quotient which refers to the total variation of r due to the displacement of the point P to P'.

It is evident that this equation can be written in the form:

$$\frac{d\Theta}{ds} = -\sin\Theta \frac{\partial \log \text{nat } m_r}{\partial r}. \qquad (5)$$

Setting $dr \cdot \sec\Theta$ for ds, we obtain:

$$\frac{d \log \text{nat} \sin\Theta}{dr} = -\frac{\partial \log \text{nat } m_r}{\partial r}. \qquad (6)$$

Formulae (3) to (6) like (6) p. 275, are generally valid for geodetic lines on all surfaces, as Gauss, (loc. cit., Art. 19) has proven (by Christoffel loc. cit., p. 132).

Par. 2. Sine Theorem for the Geodetic Triangle. It is now possible to formulate the differential equations for a triangle with shortest lines as sides, i. e., for a geodetic triangle, and to integrate with the aid of the expressions found for m.

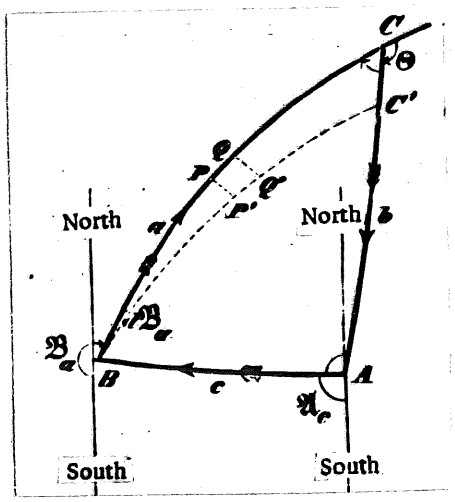


Figure 29

When the differential formulae for the spherical triangle were formulated on pp. 72 et seq. we could use the fact that each triangle on the sphere can be displaced without change of form for the derivation of finite formulae. Since this cannot be done with the ellipsoid of revolution, the equations (5) or (6) must serve as substitutes. This characteristic, just as that of the sphere, leads us to the sine law.

In the triangle ABC, fig. 29, with the sides abc following one another in any order, we form the angles of the triangle (as before, p. 71) according to the formulae

$$\left.\begin{aligned} A &= \mathfrak{A}_b - \mathfrak{A}_c \\ B &= \mathfrak{B}_c - \mathfrak{B}_a \\ C &= \mathfrak{C}_a - \mathfrak{C}_b \end{aligned}\right\} \qquad (1)$$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are the azimuths in the corners A, B, C for the sides appearing as subscripts. The azimuths are now read off with reference to the meridians of the respective corners,

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as is indicated, in part, in fig. 29.

The definition of the angles according to formulae (1) makes it possible, as before, to satisfy the two conditions that the angles be measured within or outside the triangle; cf. p. 71.

If the side a is rotated by $d\mathfrak{B}_a$ in the direction of increasing azimuths, then the corner C moves along the fixed side b to C', and from formula (5) it can be indicated by how much the angle Θ increases at C, taken from the increasing radius vector a to the direction from C to C'. Θ is equal to $\mathfrak{C}_b - (\mathfrak{C}_a + 180^0)$ i. e., $180^0 - C$.

Therefore we have:

$$d \log \text{nat} \sin (180^0 - C) = d \log \text{nat} \sin C = - \frac{\partial \log \text{nat} \sin \mathfrak{m}_a}{\partial a} da. \qquad (2)$$

Considering the triangle sides as magnitudes of the first order in relation to a_0, we have according to formula (11), p. 278, for the side a with B as the center of rotation:

$$\mathfrak{m}_a = \frac{a_0}{\sqrt{K_2}} \sin \frac{a\sqrt{K_2}}{a_0} - \frac{1}{6} e^2 \sin 2\beta_2 \cos \mathfrak{B}_a \frac{a^4}{a_0^3} + a_0 G l_7, \qquad (3)$$

where a_0 is the equatorial radius, $K_2 : a_0^2$ the curvature in the corner B, and β_2 the reduced latitude.

For the sake of simplification, we set:

$$a_0 = 1,$$

so that, temporarily, the linear dimensions abc are assumed to be fractions of the equatorial radius. The return to any unit of measurement is easily accomplished later. If we put

$$\frac{1}{6} e^2 \sin 2\beta_2 = E_2 \tag{4}$$

we obtain from (3):

$$m_a = \frac{\sin [a\sqrt{K_2}]}{\sqrt{K_2}} - E_2 a^4 \cos \mathfrak{B}_a + Gl_7$$

or

$$m_a = \frac{\sin [a\sqrt{K_2}]}{\sqrt{K_2}} (1 - E_2 a^3 \cos \mathfrak{B}_a + Gl_6). \tag{5}$$

It follows for the natural logarithm of m_a with the same accuracy as before:

$$\log \text{nat}\, m_a = \log \text{nat} \sin [a\sqrt{K_2}] - \log \text{nat} \sqrt{K_2} - E_2 a^3 \cos \mathfrak{B}_a + Gl_6,$$

If this is differentiated partially with respect to a and the differential quotient of (2) is introduced, we obtain:

$$d \log \text{nat} \sin C = (-\sqrt{K_2} \cot [a\sqrt{K_2}] + 3E_2 a^2 \cos \mathfrak{B}_a + Gl_5)\, da. \tag{6}$$

In the integration now to be performed, we must consider the variability of $\mathfrak{B}_a$ with a. If the side $a = BC$, fig. 29, is rotated about B, then E_2 remains constant, but $\mathfrak{B}_a$ varies. $\mathfrak{B}_a$ appears to be a dependent variable of a. The integration leads to the equation:

$$\log \text{nat} \sin C = -\log \text{nat} \sin [a\sqrt{K_2}] + 3E_2 \int a^2 \cos \mathfrak{B}_a da + \int Gl_5 da + \text{Const.} \tag{7}$$

In evaluating the first integral, we consider the side $CA = b$ as an independent variable. Regardless of the nature of fig. 29, we get:

$$da = \cos C \, db. \tag{8}$$

If we apply formula (11), p. 298, to 2 of the 3 vertices of the triangle it follows that:

$$\begin{aligned}(B_2 - B_1)\cos\beta &= -c\cos\mathfrak{A}_c\cos\beta + Gl_2\\ (B_3 - B_2)\cos\beta &= -a\cos\mathfrak{B}_a\cos\beta + Gl_2\\ (B_1 - B_3)\cos\beta &= -b\cos\mathfrak{C}_b\cos\beta + Gl_2,\end{aligned}$$

where B_1, B_2, B_3 are the geographic latitudes of A, B, C and some mean reduced latitude (geographic or otherwise) of the triangle is taken for β. Addition of the three equations gives:

$$\cos\beta\,(a\cos\mathfrak{B}_a + b\cos\mathfrak{C}_b + c\cos\mathfrak{A}_c) + Gl_2 = 0. \tag{9}$$

Similarly, by use of formula (12) p. 298, we find:

$$\cos\beta\,(a\sin\mathfrak{B}_a + b\sin\mathfrak{C}_b + c\sin\mathfrak{A}_c) + Gl_2 = 0. \tag{10}$$

These formulae are still valid in the vicinity of the Poles, for if formulae (11) and (12) are reduced to their first term and multiplied by $\cos\beta$ or $\cos^2\beta$, the remaining terms at the Poles are small second-order terms, as the first two differential quotients (6), p. 297, show with respect to the remainder of the general development (1), p. 25.

Moreover, the derivation shows that in formulae (9) and (10) the azimuths of the sides, increased by 180°, may always be substituted at the other end point by the azimuths of the same sides. Thus, we obtain from (9):

$$\cos\beta\,.\,a\cos\mathfrak{A}_a = \cos\beta\,(b\cos\mathfrak{A}_b + c\cos\mathfrak{A}_c) + Gl_2. \tag{11}$$

Formulae (9) and (10) show the possibility of applying the theorems of plane polygonometry to geodetic triangles at a first approximation.

From such formulae, we can derive theorems concerning the angles and sides of a triangle. However, it is preferable to trace them directly in order to show that the validity of these formulae at the Poles is independent of the coefficient $\cos\beta$. If we consider the first and fourth equation (1) p. 76, the series expressions for the cosine and sine of the sides yield the generally valid result for spherical triangles:

$$a^2 = b^2 + c^2 - 2bc\cos A + Gl_4, \qquad (12)$$

$$a\cos C = b - c\cos A + Gl_3. \qquad (13)$$

These relations are obtained for the ellipsoid from (9) and (10), which still contain the coefficient $\cos\beta$, and are apparently one order less precise which, however, is immaterial here. That the coefficient can be omitted, follows from a geometric consideration. Let us imagine a sphere defined by the three points ABC, its radius corresponding as nearly as possible to the mean curvature of the area covered. For the parts of the spherical triangle, (12) and (13) are directly valid. Those parts differ from the parts of the geodetic triangle in the terms which are multiplied by e^2, which however do not influence equations (12) and (13) at sufficient distance from the pole. The divergences are obviously more insignificant in the vicinity of the poles than in general. Therefore, (12) and (13) must be generally valid.

We obtain by multiplication of (11) and (13):

$$\cos\beta \cdot a^2 \cos\mathfrak{B}_a \cos C = \cos\beta\,(b - c\cos A)\,(b\cos\mathfrak{A}_b + c\cos\mathfrak{B}_c) + Gl_5.$$

If we introduce this into (7) and note that the first integral is further multiplied by E_2, a quantity of the second order that contains cos β as a factor, then we get through simple reduction:

$$\log \text{nat}\left(\sin C \sin [a\sqrt{K_2}]\right) = 3E_2\int(b - c\cos A)\,(b\cos \mathfrak{A}_b + c\cos \mathfrak{B}_c)\,db + \int Gl_5\,da + \text{Const.} \tag{14}$$

The first integral on the right-hand side still contains only the variable b; through multiplication and integration we get:

$$\tfrac{1}{2}(b^2 - 2bc\cos A)\,c\cos \mathfrak{B}_c + \left(\tfrac{1}{3}b^3 - \tfrac{1}{2}b^2 c\cos A\right)\cos \mathfrak{A}_b,$$

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or by elimination of $bc\cos A$ by means of (12):

$$\tfrac{1}{4}(a^2 - c^2)\,(b\cos \mathfrak{A}_b + 2c\cos \mathfrak{B}_c) + \tfrac{1}{12}b^3\cos \mathfrak{A}_b + Gl_5,$$

or, finally, by substitution of the value of $b\cos \mathfrak{A}_b$ from the first term of (11):

$$\tfrac{1}{4}(a^2 - c^2)\,(a\cos \mathfrak{B}_a + c\cos \mathfrak{B}_c) + \tfrac{1}{12}b^3\cos \mathfrak{A}_b + Gl_4. \tag{15}$$

If we multiply this by E_2, the Gl_4 give sixth-order terms. The same holds true for the second integral of formula (14), and we need not fear that the Gl_5 terms are large in the vicinity of the pole. For, the development of K, p. 275, leaves no doubt that in the higher terms of K, and with it of m, as well as of (14), the azimuth α only appears in the form cos β cos α, so that the strong variations of α in the vicinity of the pole are reduced by the coefficient cos β to ordinary proportions.

The result of (14) and (15) is now:

$$\log \text{nat}\left(\sin C \sin [a\sqrt{K_2}]\right) = \tfrac{3}{4}E_2(a^2 - c^2)\,(a\cos \mathfrak{B}_a + c\cos \mathfrak{B}_c) + \tfrac{1}{4}E_2 b^3\cos \mathfrak{A}_b + Gl_6 + \text{Constant} \tag{16}$$

To determine the constant, we assume $b = 0$. Thus a becomes identical with c, and $\mathfrak{C}_a = \mathfrak{A}_c$, $\mathfrak{C}_b = \mathfrak{A}_b - 180^0$, therefore:

$$C \text{ i.e., } \mathfrak{C}_a - \mathfrak{C}_b = \mathfrak{A}_c - \mathfrak{A}_b + 180^0 \text{ i.e., } 180^0 - A$$

Consequently, $\log \text{nat}\left(\sin A \sin [c\sqrt{K_2}]\right) = \text{Const.}$ and (16) becomes:

$$\log \text{nat}\left(\frac{\sin C \sin [a\sqrt{K_2}]}{\sin A \sin [c\sqrt{K_2}]}\right) = \frac{3}{4} E_2 (a^2 - c^2)(a \cos \mathfrak{B}_a + c \cos \mathfrak{B}_c) + \frac{1}{4} E_2 b^3 \cos \mathfrak{A}_b + Gl_6.$$

From this we obtain without any substantial change in accuracy:

$$\frac{\sin C}{\sin A} = \frac{\sin [c\sqrt{K_2}]}{\sin [a\sqrt{K_2}]} \left\{ 1 + \frac{3}{4} E_2 (a^2 - c^2)(a \cos \mathfrak{B}_a + c \cos \mathfrak{B}_c) + \frac{1}{4} E_2 b^3 \cos \mathfrak{A}_b + Gl_6. \right\} \qquad (17)$$

This formula can still be simplified; however, in the above arrangement it is sufficient for the following, so that we can retain it for the time being.

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Par. 3. The Cosine Theorem in the Geodetic Triangle. To obtain a as a function of b, c and A we consider c in fig. 30 as being fixed and b as constant, but movable about A. If AC changes to AC', and CC'' is drawn at right angles to BC', the resultant differential orthogonal triangle $CC'C''$ gives:

$$da = C'C'' = CC' \sin C = m_b \sin C \, d\mathfrak{A}_b, \qquad (1)$$

which is generally valid for $C = \mathfrak{C}_a - \mathfrak{C}_b$.

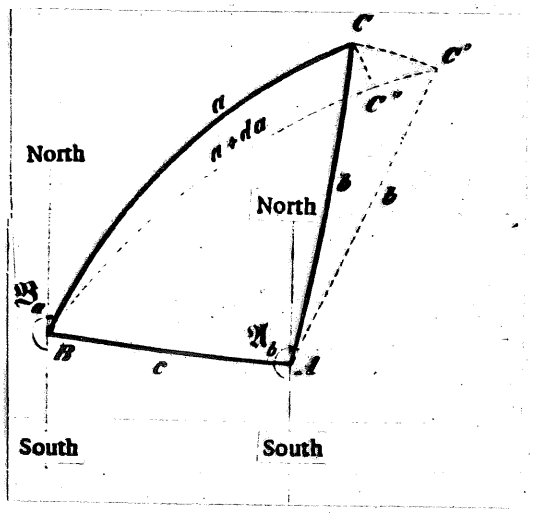


Figure 30

This differential formula for a is easily integrated, since the constancy of b makes $\mathfrak{m}_b$ only a function of $\mathfrak{A}_b$, whereas sin C may be expressed as a function of A and a by means of the Sine Theorem.

First, according to formula (17) of the previous paragraph, we have:

$$\sin C = \sin A \frac{\sin [c\sqrt{K_2}]}{\sin [a\sqrt{K_2}]} \Big(1 + \frac{3}{4} E_2 (a^2 - c^2)(a \cos \mathfrak{B}_a + c \cos \mathfrak{B}_c) + \frac{1}{4} E_2 b^3 \cos \mathfrak{A}_b + Gl_6\Big). \tag{2}$$

Moreover, according to (11), p. 278, and as a result of the introduction of the equatorial radius as the unit of length, we get:

$$\mathfrak{m}_b = \frac{\sin [b\sqrt{K_1}]}{\sqrt{K_1}} (1 - E_1 b^3 \cos \mathfrak{A}_b + Gl_6). \tag{3}$$

Before substituting these expressions in (1), we also introduce K_1 instead of K_2 into (2) so that we may have the same K in the sine expression in all sides.

But generally we have for one side u:

$$\sin [u\sqrt{K_2}] = u\sqrt{K_2}\Big(1 - \frac{1}{6} u^2 K_2 + \frac{1}{120} u^4 K_2^2 + Gl_6\Big),$$

as well as

$$\sin [u\sqrt{K_1}] = u\sqrt{K_1}\Big(1 - \frac{1}{6} u^2 K_1 + \frac{1}{120} u^4 K_1^2 + Gl_6\Big)$$

since $K_2 - K_1$ is of the third order according to formula (5), p. 276, and since

$$K_1 = K_2 + 12 E_2 c \cos \mathfrak{B}^c + Gl_4$$

the division of both equations gives:

$$\frac{\sin[u\sqrt{K_2}]}{\sin[u\sqrt{K_1}]} = \frac{\sqrt{K_2}}{\sqrt{K_1}}\left(1 - \frac{1}{6}[K_2 - K_1]\,u^2 + Gl_7\right).$$

If this is used in $u = c$ and $u = a$ and divided, we obtain:

$$\frac{\sin[c\sqrt{K_2}]}{\sin[a\sqrt{K_2}]} = \frac{\sin[c\sqrt{K_1}]}{\sin[a\sqrt{K_1}]}\left(1 + 2E_2\,(c^2 - a^2)\,c\cos\mathfrak{B}_c + Gl_7\right),$$

whereupon (2) becomes:

$$\sin C = \sin A\,\frac{\sin[c\sqrt{K_1}]}{\sin[a\sqrt{K_1}]}\left(1 + E_2\,(a^2 - c^2)\left[\frac{3}{4}a\cos\mathfrak{B}_a - \frac{5}{4}c\cos\mathfrak{B}_c\right] + \frac{1}{4}E_2 b^3\cos\mathfrak{A}_b + Gl_6\right). \tag{4}$$

If we insert this and (3) in (1) and write E_1 for E_2, which results only in a small error of the third order in E and thus a sixth-order error in the brackets of (4), it follows that:

$$\sqrt{K_1}\sin[a\sqrt{K_1}]\,da = \sin[b\sqrt{K_1}]\sin[c\sqrt{K_1}]\left(1 + E_1(a^2 - c^2)\left[\frac{3}{4}a\cos\mathfrak{B}_a - \frac{5}{4}c\cos\mathfrak{B}_c\right] - \frac{3}{4}E_1 b^3\cos\mathfrak{A}_b + Gl_6\right)\sin A\,d\mathfrak{A}_b. \tag{5}$$

On the right side, we eliminate $a\cos\mathfrak{B}_a$ and a^2 by means of the relations (11) and (12), p. 350 and p. 351, and obtain:

$$\sin[a\sqrt{K_1}]\,d(a\sqrt{K_1}) = \sin[b\sqrt{K_1}]\sin[c\sqrt{K_1}]\left(1 - bcE_1\left[\frac{1}{2}b\cos\mathfrak{B}_c + \frac{3}{2}b\cos A\cos\mathfrak{A}_c - c\cos\mathfrak{B}_c\cos A\right] + Gl_6\right)\sin A\,d\mathfrak{A}_b. \tag{6}$$

For A we set $\mathfrak{A}_b - \mathfrak{A}_c$ and apply the formulae to integrate:

$$\cos A\sin A = \frac{1}{2}\sin 2(\mathfrak{A}_b - \mathfrak{A}_c)$$

$$\cos A\cos\mathfrak{A}_b\sin A = \frac{1}{4}\left(\sin(3\mathfrak{A}_b - 2\mathfrak{A}_c) + \sin(\mathfrak{A}_b - 2\mathfrak{A}_c)\right).$$

If we introduce this into (6) integration yields without difficulty:

$$\cos[a\sqrt{K_1}]+\text{Constant}=\sin[b\sqrt{K_1}]\sin[c\sqrt{K_1}]\{\cos(\mathfrak{A}_b-\mathfrak{A}_c) -bcE_1(g_1+g_2+g_3)+Gl_6\}, \quad (7)$$

where temprorarily the following symbols serve as abbreviations:

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$$g_1 = \frac{1}{2}b\cos\mathfrak{B}_c\cos(\mathfrak{A}_b-\mathfrak{A}_c)=\frac{1}{2}b\cos\mathfrak{B}_c\cos A$$

$$g_2 = \frac{3}{8}b\left(\frac{1}{3}\cos(3\mathfrak{A}_b-2\mathfrak{A}_c)+\cos(\mathfrak{A}_b-2\mathfrak{A}_c)\right)$$

$$= \frac{1}{4}b\left(\cos 2A\cos\mathfrak{A}_b+\cos(A-\mathfrak{A}_c)\right)$$

$$g_3 = -\frac{1}{4}c\cos\mathfrak{B}_c\cos 2(\mathfrak{A}_b-\mathfrak{A}_c)=-\frac{1}{4}c\cos\mathfrak{B}_c\cos 2A.$$

The constant of integration is determined by the fact that for $\mathfrak{A}_b=\mathfrak{A}_c+180^0$ the side a must become $=b+c$. At the same time $A=180^0$ and correspondingly $g_1=-\frac{1}{2}b\cos\mathfrak{B}_c$; $g_2=-\frac{1}{2}b\cos\mathfrak{A}_c$; and $g_3=-\frac{1}{4}c\cos\mathfrak{B}_c$. The sum $g_1+g_2=-\frac{1}{2}b(\cos\mathfrak{B}_c+\cos\mathfrak{A}_c)$ is reduced to zero, except for first-order terms, because $\mathfrak{B}_c=\mathfrak{A}_c+180^0+Gl_1$. This is not true at first in the vicinity of the poles; however, the coefficient $\cos\beta$ in E_1 extends the area of validity into the poles.

Accordingly, we obtain:

$$\cos[(b+c)\sqrt{K_1}]+\text{Constant}=-\sin[b\sqrt{K_1}]\sin[c\sqrt{K_1}]\left(1-\frac{1}{4}bc^2E_1\cos\mathfrak{B}_c+Gl_6\right).$$

If we subtract this from (7) and observe during the consolidation of the g that $\mathfrak{B}_c=\mathfrak{A}_c+180^0+Gl_1$, it follows after some reduction:

$$\cos[a\sqrt{K_1}]=\cos[b\sqrt{K_1}]\cos[c\sqrt{K_1}] +\sin[b\sqrt{K_1}]\sin[c\sqrt{K_1}]\left(\cos A+\frac{1}{2}E_1bc(b\cos\mathfrak{A}_b+c\cos\mathfrak{A}_c)\sin^2A+Gl_6\right). \quad (8)$$

But we can easily write the terms multiplied in E_1 by use of the differences of the curvatures. According to (5), p. 276, we have:

$$\left.\begin{aligned} K_2 &= K_1 + 12E_1 c\cos\mathfrak{A}_c + Gl_4 \\ K_3 &= K_1 + 12E_1 b\cos\mathfrak{A}_b + Gl_4; \end{aligned}\right\} \tag{9}$$

therefore:

$$E_1(b\cos\mathfrak{A}_b + c\cos\mathfrak{A}_c) = \tfrac{1}{4}[K - K_1] + Gl_4, \tag{10}$$

where $K = \frac{1}{3}(K_1 + K_2 + K_3)$ is the arithmetic mean of the K of the three angles. The large bracket in (8) yields:

$$\cos A + \tfrac{1}{8}[K - K_1]\,bc\sin^2 A + Gl_6, \tag{11}$$

for which we can write with the same accuracy:

$$\cos\left(A - \tfrac{1}{4}[K - K_1]\,F_A\right) + Gl_6. \tag{12}$$

when $\frac{1}{2}bc\sin A$ is temporarily designated as F_A. Thus we get:

$$\begin{aligned} &\cos[a\sqrt{K_1}] = \cos[b\sqrt{K_1}]\cos[c\sqrt{K_1}] \\ &+ \sin[b\sqrt{K_1}]\sin[c\sqrt{K_1}]\left\{\cos\left(A - \tfrac{1}{4}[K - K_1]\,F_A\right) + Gl_6\right\}. \end{aligned} \tag{13}$$

This formula becomes even more convenient if we substitute $\sqrt{K}$ for $\sqrt{K_1}$ in the argument of the cosine and sine. The following relations serve this purpose:

$$\begin{aligned} \cos[u\sqrt{K_1}] &= 1 - \tfrac{1}{2}u^2K_1 + \tfrac{1}{24}u^4K_1^2 - \tfrac{1}{720}u^6K_1^3 + Gl_8 \\ \sin[u\sqrt{K_1}] &= u\sqrt{K_1}\left(1 - \tfrac{1}{6}u^2K_1 + \tfrac{1}{120}u^4K_1^2 - Gl_6\right) \\ \cos[u\sqrt{K}] &= 1 - \tfrac{1}{2}u^2K + \tfrac{1}{24}u^4K^2 - \tfrac{1}{720}u^6K^3 + Gl_8 \\ \sin[u\sqrt{K}] &= u\sqrt{K}\left(1 - \tfrac{1}{6}u^2K + \tfrac{1}{120}u^4K^2 - Gl_6\right). \end{aligned}$$

Par. 3. The Cosine Theorem in the Geodetic Triangle.

If we introduce in (13) the first two of these series for $u = a, b, c$ and subtract 1 from both sides, then all terms contain K_1 as coefficient. This is divided out, and the series multiplied by K. K_1 will continue to appear in fourth-and sixth-order terms. If we replace it by K, there will result only an error of the ninth order in the fourth- and sixth-order terms. This is negligible. However, a correction must be added for the fourth-order terms. These fourth-order terms run:

left hand	right hand
$+\frac{a^4}{24} K K_1$	$+\frac{b^4 + c^4 + 6b^2c^2 - 4(b^2+c^2)bc\cos A}{24} K K_1.$

If we write K for K_1, and add 1 to both sides in the reshaped equation (13), we obtain through consolidation an equation that differs from (13) only in the cosine and sine terms where $\sqrt{K}$ replaces $\sqrt{K_1}$, and the added correction required by the substitution of K for K_1 in the mentioned fourth-order terms. If we transpose all terms to the right-hand side, we have:

$$\frac{1}{24}\left\{-a^4 + b^4 + c^4 + 6b_2c_2 - 4(b^2+c^2)bc\cos A\right\} K[K_1 - K]. \qquad (14)$$

If we eliminate $2bc\cos A$ by means of (12), p. 351, and apply the existing relations for the area of the plane triangle with sides abc:

$$16F^{*2} = 2(a^2b^2 + a^2c^2 + b^2c^2) - a^4 - b^4 - c^4,$$

we obtain from (14):

$$+\frac{2}{3} F^{*2} K[K_1 - K] + Gl_9. \qquad (15)$$

But if we square equation (12), p. 351, we get as a result of some

reduction:

$$F^{*2} = F_A^2 + Gl_6. \tag{16}$$

This enables us to write expression (15) as follows:

$$\sin[b\sqrt{K}]\sin[c\sqrt{K}]\left(-\tfrac{1}{6}[K-K_1]\,bc\sin^2 A\right) + Gl_9. \tag{17}$$

This is to be added to (13), when $\sqrt{K}$ is introduced. A glance at (11) and (12) shows how (17) may be combined with (13). Thus:

$$\begin{aligned}\cos[a\sqrt{K}] &= \cos[b\sqrt{K}]\cos[c\sqrt{K}] \\ &+ \sin[b\sqrt{K}]\sin[c\sqrt{K}]\cos\left(A + \tfrac{1}{12}[K-K_1]\,F_A\right) + Gl_8.\end{aligned} \tag{18}$$

The Gl_8 of this formula, which can be considered as functions of b, c and A according to the powers of b and c and the cosine and sine of A and its multiples, must contain however the factor $bc\sin^2 A$; for they disappear for b and c, and for $A = 0^0$ and 180^0; and must finally remain valid if A is replaced by $360^0 - A$.

(The existence of the factor $\sin^2 A$ can, moreover, be easily recognized from the analysis of the transition from formula (4) to (8). The further transformations which only mean transitions from one sphere to another, make no difference because we can always take the plane as the connecting link. The existence of the factor $\sin^2 A$ for the plane has been proved earlier.)

The Gl_8 of formula (18) can subsequently be written in the form $\sin b\sqrt{K}\sin c\sqrt{K}\sin^2 A \cdot Gl_6$ and formula (18) can be written as follows:

$$\begin{aligned}\cos[a\sqrt{K}] &= \cos[b\sqrt{K}]\cos[c\sqrt{K}] \\ &+ \sin[b\sqrt{K}]\sin[c\sqrt{K}]\cos\left(A + \tfrac{1}{12}[K-K_1]\,F_A + Gl_6\right).\end{aligned} \tag{19}$$

In conclusion, it must be repeated that the equatorial radius a_0 has been selected as the unit of length.

The comparison of (13) and (19) shows again that with purely spherical computation it is more practical to apply the mean curvature than that of an angle. But as far as formula (16) is concerned, it is still more practical to apply instead of K the curvature

$$K' = \frac{1}{4}(2K_1 + K_2 + K_3)$$

from which follows:

$$\cos[a\sqrt{K'}] = \cos[b\sqrt{K'}]\cos[c\sqrt{K'}] + \sin[b\sqrt{K'}]\sin[c\sqrt{K'}]\cos(A + Gl_6).$$

Par. 4. Reduction of the Geodetic Triangle to a Spherical or Plane Triangle with the Same Sides. A comparison of formula (19) in the previous paragraph with the first formula (1), p. 76, shows that the opposite angle A' of the side a in a spherical triangle with the sides a, b and c and $1:\sqrt{K}$ as spherical radius, is determined by the relation:

$$A' = A + \frac{1}{12}[K - K_1]F_A + Gl_6. \qquad (1)$$

$F_A = \frac{1}{2}ab\sin A$ can be used instead of $\frac{1}{2}ab\sin A'$, which produces only seventh-order errors in F_A. According to p. 98, F^* equals $\frac{1}{2}ab\sin A'$. up to Gl_4. Thus, if we set

$$A' = A + \frac{1}{12}[K - K_1]F^* + Gl_6, \qquad (2)$$

then F^* can, without substantial effect upon the result, be computed from the 3 sides abc or from any 2 sides and the included angle, according to the formulae of plane geometry.

If we assume a_0 to be different from 1, express the differences $A' - A$ in seconds, and finally apply formula (2) to B and C, we

obtain, for the reduction of the geodetic triangle to a spherical triangle, these formulae:

$$\left.\begin{aligned} A' - A &= \frac{1}{12}\varrho''\frac{F^*}{a_0^2}[K - K_1] + Gl_6 \\ B' - B &= \frac{1}{12}\varrho''\frac{F^*}{a_0^2}[K - K_2] + Gl_6 \\ C' - C &= \frac{1}{12}\varrho''\frac{F^*}{a_0^2}[K - K_3] + Gl_6. \end{aligned}\right\} \quad \text{in seconds} \qquad (3)$$

A', B' and C' refer to a spherical triangle whose spherical radius corresponds to the arithmetical mean of the curvature of the angles, i.e., to the center of gravity of the triangle;

the radius of curvature thus equals

$$\frac{a_0}{\sqrt{K}}; \quad K = \frac{K_1 + K_2 + K_3}{3}. \qquad (4)$$

F^*, the area of the plane triangle with the same sides, is positive or negative, depending on whether the angles are taken inside or outside the triangle. The expressions $\frac{1}{2}ab\sin C$, $\frac{1}{2}ac\sin B$ or $\frac{1}{2}bc\sin A$ are the easiest way to describe the area with the proper sign and sufficient accuracy.

If one adds the expressions (3), then we have:

$$A + B + C = A' + B' + C' + Gl_6, \qquad (5)$$

i.e., the excess of the geodetic triangle, up to sixth-order terms, is equal to that of a spherical triangle with the same sides, if the spherical radius corresponds to the mean curvature of the triangle.

If we reduce the spherical triangle to a plane one with the same sides (p. 93), it follows through combination with (3) for the reduction of the geodetic triangle to a plane triangle that:

Par. 5. Fifth-Order Terms Dependent Upon e^2.

$$\left.\begin{aligned} A - A^* &= \frac{\varepsilon}{3} + \varrho'' \frac{F^*}{3a_0^2} K \left\{ \frac{m^2 - a^2}{20a_0^2} K + \frac{K_1 - K}{4K} + Gl_4 \right\} \\ B - B^* &= \frac{\varepsilon}{3} + \varrho'' \frac{F^*}{3a_0^2} K \left\{ \frac{m^2 - b^2}{20a_0^2} K + \frac{K_2 - K}{4K} + Gl_4 \right\} \\ C - C^* &= \frac{\varepsilon}{3} + \varrho'' \frac{F^*}{3a_0^2} K \left\{ \frac{m^2 - c^2}{20a_0^2} K + \frac{K_3 - K}{4K} + Gl_4 \right\} \\ \varepsilon &= \varrho'' \frac{F^*}{a_0^2} K \left(1 + \frac{m^2}{2a_0^2} K + Gl_4 \right). \end{aligned}\right\} \qquad (6)$$

in seconds

In these formulae the remaining terms could be included in the brackets, since they disappear together with F^* (cf. p. 357).

Par. 5. Fifth-Order Terms Dependent Upon e^2. In order to determine the influence of these terms in the mentioned formulae which represent the error which occurs if the geodetic triangle is treated like a spherical triangle without any correction, we consider the expression

$$\frac{1}{12} \varrho'' \frac{F^*}{a_0^2} [K - K_1], \qquad (1)$$

which refers to angle A. Here, according to p. 349 (4) and p. 355 (10) we set:

$$K - K_1 = \frac{2e^2 K_1}{3a_0} \sin 2\beta_1 (b \cos \mathfrak{A}_b + c \cos \mathfrak{A}_c) + \cdots \qquad (2)$$

and further, according to p. 94 (2), assuming $3m^2 = a^2 + b^2 + c^2$:

$$F^{*2} = \frac{1}{16} \left(6a^2 m^2 - 3a^4 - (b^2 - c^2)^2 \right). \qquad (3)$$

Then, instead of (1), we have the equally valid expression:

$$\varrho'' \frac{e^2 K_1}{72 a_0^2} \sin 2\beta_1 (b \cos \mathfrak{A}_b + c \cos \mathfrak{A}_c) \sqrt{3a^2 (2m^2 - a^2) - (b^2 - c^2)^2}. \qquad (4)$$

If we now assume the angle A to be fixed, but let the direction and length of the sides be variable on condition that m^2 remains constant,

(4) becomes a maximum when

$$(b\cos\mathfrak{A}_b + c\cos\mathfrak{A}_c)\sqrt{3a^2(2m^2-a^2)-(b^2-c^2)^2} \qquad (5)$$

is a maximum under the following conditions:

$$\left.\begin{aligned} 3m^2 &= a^2+b^2+c^2 \\ a^2 &= b^2+c^2-2bc\cos(\mathfrak{A}_b-\mathfrak{A}_c)+a_0^2\cdot Gl_4. \end{aligned}\right\} \qquad (6)$$

These expressions are symmetric with $b\cos\mathfrak{A}_b$ and $c\cos\mathfrak{A}_c$, and $b\sin\mathfrak{A}_b$ and $\pm c\sin\mathfrak{A}_c$. Therefore, excluding the case $\mathfrak{A}_b=\mathfrak{A}_c$, we have for the maximum:

$$b=c \qquad \mathfrak{A}_b = 360^\circ - \mathfrak{A}_c, \qquad (7)$$

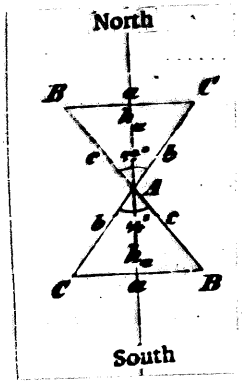


Figure 31

the values of which change (4) into the expression:

$$\varrho''\frac{e^2K_1}{36a_0^3}\sin 2\beta_1 \cdot b\cos\mathfrak{A}_b\sqrt{3a^2(2m^2-a^2)}. \qquad (8)$$

Except for the sign, $b\cos\mathfrak{A}_b$ is equal to the height h_a of the triangle, fig. 31 (considered as plane), from which we obtain $b^2=c^2=\frac{1}{4}a^2+h_a^2$; we also have $3m^2=2h_a^2+\frac{3}{2}a^2$.

From this it follows that $2m^2-a^2=\frac{4}{3}h_a^2$ and (8) becomes:

$$\pm\varrho''\frac{e^2K_1}{18a_0^3}\sin 2\beta_1 \cdot ah_a^2. \qquad (9)$$

The variable ah_a^2 that is, $\frac{3}{4}(2m^2a-a^3)$ becomes a maximum for

$$\left.\begin{aligned} &a=m\sqrt{\tfrac{2}{3}};\ h_a=m;\ A=44^\circ \text{ approx. with } \\ &\mathfrak{A}_b=22^\circ \text{ and } \mathfrak{A}_c=338^\circ \text{ or } \mathfrak{A}_b=202^\circ \text{ and } \mathfrak{A}_c=158^\circ. \end{aligned}\right\} \qquad (10)$$

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*In fig. 31 the base CB is assumed somewhat too wide for the maximum.

Thus, in the maximum, the term under discussion, i. e., $A'-A$ equals

$$\pm \varrho'' \frac{e^2 K_1}{18} \sqrt{\frac{s}{s}} \sin 2\beta_1 \left(\frac{m}{a_0}\right)^3, \tag{11}$$

which for $m = 0{,}1 a_0$ and $\beta_1 = 45^\circ$ yields this small amount:

$$\pm 0{,}062''. \tag{12}$$

This amount is reduced still further for measurable triangles, because it is:

$$\left.\begin{aligned} &\pm 0{,}008'' \text{ for } m = 0{,}05 a_0 \text{ i. e. } 319^{km} \\ &\pm 0{,}0005'' \text{ for } m = 0{,}02 a_0 \text{ i. e. } 127^{km} \end{aligned}\right\}. \tag{13}$$

Even if in special cases the latter figure should be exceeded it will still be permissible, considering the lower accuracy of the observation at very great distances, to treat the directly observed geodetic triangles as spherical, if the radius of curvature is assumed to correspond to the mean curvature of the three angles. If we compare (12) and (13) with p. 96 (8), we see that in the formulae (6), p. 359, about the same maximal errors arise for $A - A^*$ through neglect of the terms multiplied by $m^2 - a^2$ and $K_1 - K$, and correspondingly for $B - B^*$ and $C - C^*$. In other words:

If we consider the dissimilarity of the sides in distributing the excess over the 3 angles, we must also consider the dissimilarity of the curvature.

As long as we consider a triangle as spherical it is only logical to distribute the excess uniformly. This does not exclude that for the computation of ε the formula given under (6), p. 359, with fourth-order

terms is applicable. For observed triangles it might in general be superfluous to consider the latter cf. p. 95 (5).

If we now calculate the excess simply according to the formula:

$$\varepsilon'' \frac{F''}{a_0^2} \frac{K_1 + K_2 + K_3}{3} \qquad (14)$$

and distribute it uniformly among the 3 angles, then the question can arise whether in a series of triangles an unfavorable accumulation of the errors thus caused is possible. With the limited magnitude of these errors for observed triangles in particular this however need not be feared for the lengths of the sides nor for the azimuths. If we investigate especially the influence of the errors with the application of the Sine Theorem,

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then it becomes evident that they become significant only in the ninth decimal-place of the logarithm of the sides, with individual well-formed triangles, and that the seventh decimal-place always remains unaffected even with twenty to thirty connected triangles.

Par. 6. Excess of the Geodetic Triangles from Two Sides and Included Angle. If we first remember the formula for the spherical triangle then we have, according to p. 98, if A' is the included angle, b and c the sides:

$$\varepsilon = \frac{bc \sin A'}{2a_0^2} K \left(1 + \frac{4a^2 - 3m^2}{24 a_0^2} K + Gl_4\right).$$

This formula is also valid for the geodetic triangle because up to sixth-order terms it has an excess equal to that of the spherical. We have only to put:

Par. 6. Excess of Geodetic Triangle from Two Sides and Included Angle.

$$A' = A - \frac{1}{12}\frac{F^*}{a_0^2}[K_1 - K] + Gl_6,$$

where we may take F^* equals $\frac{1}{2} bc \sin A$. If we introduce this, then:

$$\varepsilon = \frac{bc \sin A}{2a_0^2} K \left\{1 + \frac{4a^2 - 3m^2}{24a_0^2} K - \frac{K_1 - K}{24} \frac{bc \cos A}{a_0^2} + Gl_4\right\}.$$

We see here immediately that on the right-hand side, the term dependent on $bc \cos A$ can be omitted, because it is an order higher than the neglected fourth-order terms. Considering p. 98 we have now:

$$\left.\begin{aligned} \frac{a^2}{a_0^2} &= \frac{b^2}{a_0^2} + \frac{c^2}{a_0^2} - \frac{2bc \cos A}{a_0^2} + Gl_4 \\ \log \varepsilon = \log \underset{\text{in seconds}}{\varrho''} \frac{bc \sin A}{2a_0^2} K &+ M \left\{\frac{4a^2 - 3m^2}{24a_0^2} K + Gl_4\right\} \\ A - A^* &= \frac{\varepsilon}{3}\left\{1 + \frac{m^2 - a^2}{20a_0^2} K + \frac{K_1 - K}{4K} + Gl_4\right\} \\ B - B^* &= \frac{\varepsilon}{3}\left\{1 + \frac{m^2 - b^2}{20a_0^2} K + \frac{K_2 - K}{4K} + Gl_4\right\} \\ C - C^* &= \frac{\varepsilon}{3}\left\{1 + \frac{m^2 - c^2}{20a_0^2} K + \frac{K_3 - K}{4K} + Gl_4\right\}. \end{aligned}\right\} \quad (1)$$

If we shorten these formulae to their first terms, then the errors for the angle reductions are just as large as in the corresponding case of the previous paragraph; for as already shown on p. 99 the error for $\log \varepsilon$ is even smaller, if not negligible. This simple computation will therefore in general appear permissible for directly observed triangles.

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Par. 7 Area of the Geodetic Triangle. In order to find the area, we determine first, fig. 29, p. 348, the area of the strip BCC. If P is a point between B and C, which gives the position P' at the rotation of BP, and Q and Q' are further two infinitesimally adjacent

points and we put:

$$BP = BP' = r \qquad PQ = P'Q' = dr,$$

then the area of the infinitely small rectangle is $PQP'Q' = \mathfrak{m}_r d\mathfrak{B}_a dr$, within triangle BCC' or

$$-dF = d\mathfrak{B}_a \int_c^a \mathfrak{m}_r dr.$$

Now considering p. 349 (5) we have, if a_0 is temporarily $= 1$:

$$\mathfrak{m}_r = \frac{\sin [r\sqrt{K_2}]}{\sqrt{K_2}} - E_2 r^4 \cos \mathfrak{B}_a + Gl_7.$$

After integration and simplification

$$dF = -\frac{2 \sin^2 \left[\frac{a}{2}\sqrt{K_2}\right]}{K_2} \left(1 - \frac{2}{5} E_2 a^5 \cos \mathfrak{B}_a + Gl_6\right) d\mathfrak{B}_a. \qquad (1)$$

In order to integrate again, we introduce b as an independent variable and, according to fig. 29 p. 348 put:

$$\mathfrak{m}_a d\mathfrak{B}_a = -db \sin C,$$

further, according to p. 352 (17):

$$\sin C = \sin A \frac{\sin [c\sqrt{K_2}]}{\sin [a\sqrt{K_2}]} \left(1 + \frac{3}{4} E_2 (a^2 - c^2)(a \cos \mathfrak{B}_a + c \cos \mathfrak{B}_c) + \frac{1}{4} E_2 b^3 \cos \mathfrak{A}_b + Gl_6\right)$$

and according to p. 349 (5) as for above $\mathfrak{m}_r$:

$$\mathfrak{m}_a = \frac{\sin [a\sqrt{K_2}]}{\sqrt{K_2}} \left(1 - E_2 a^3 \cos \mathfrak{B}_a + Gl_6\right).$$

Introducing these expressions into (1) we obtain after some reduction:

$$dF = \frac{\sin [c\sqrt{K_2}]}{2 \cos^2 \left[\frac{a}{2}\sqrt{K_2}\right]} \frac{\sin A \, db}{\sqrt{K_2}} \left\{ \begin{array}{l} 1 + \frac{3}{4} E_2 (a^2 - c^2)(a \cos \mathfrak{B}_a + c \cos \mathfrak{B}_c) \\ + \frac{3}{5} E_2 a^3 \cos \mathfrak{B}_a + \frac{1}{4} E_2 b^3 \cos \mathfrak{A}_b + Gl_6 \end{array} \right\}. \qquad (2)$$

From this formula it follows immediately that for the differential of the area F_2' of a spherical triangle with sides b and c and the included angle A, if $1:\sqrt{K_2}$ is the spherical radius:

$$dF_2' = \frac{\sin[c\sqrt{K_2}]}{2\cos^2\left[\frac{a'}{2}\sqrt{K_2}\right]} \frac{\sin A\, db}{\sqrt{K_2}} \tag{3}$$

The side a' is to be calculated spherically from b, c and A according to the formula:

$$\cos[a'\sqrt{K_2}] = \cos[b\sqrt{K_2}]\cos[c\sqrt{K_2}] + \sin[b\sqrt{K_2}]\sin[c\sqrt{K_2}]\cos A.$$

On the other hand, according to pp. 355 and 356 it is evident that:

$$\cos[a\sqrt{K_2}] = \cos[b\sqrt{K_2}]\cos[c\sqrt{K_2}] + \sin[b\sqrt{K_2}]\sin[c\sqrt{K_2}]\cos A + Gl_7.$$

Therefore $\cos[a'\sqrt{K_2}]$ and $\cos[a\sqrt{K_2}]$ diverge from each other only by seventh-order terms, and considering the equation $2\cos^2\frac{u}{2} = 1 + \cos u$; we have

$$2\cos^2\left[\frac{a'}{2}\sqrt{K_2}\right] = 2\cos^2\left[\frac{a}{2}\sqrt{K_2}\right] + Gl_7 = 2\cos^2\left[\frac{a}{2}\sqrt{K_2}\right](1 + Gl_7).$$

Now considering the above we have:

$$dF_2' = \frac{\sin[c\sqrt{K_2}]}{2\cos^2\left[\frac{a}{2}\sqrt{K_2}\right]} \frac{\sin A\, db}{\sqrt{K_2}}(1 + Gl_7). \tag{4}$$

From this it follows after some permissible simplification with (2):

$$dF = dF_2' + \left\{\left[\frac{3}{4}(a^2 - c^2)(a\cos\mathfrak{B}_a + c\cos\mathfrak{B}_c) + \frac{3}{5}a^3\cos\mathfrak{B}_a + \frac{1}{4}b^3\cos\mathfrak{A}_b\right]E_2 + Gl_7\right\}\frac{c\sin A\, db}{2}. \tag{5}$$

The dependent variables a and $\mathfrak{B}_a$, which appear still on the right-hand side in the small terms we eliminate by means of the known relations (11) and (12) pp. 350 and 351, whereby the square bracket

of the subsequent value gives:

$$\left(\frac{8}{5}b^2+\frac{3}{5}c^2-\frac{27}{10}bc\cos A\right)b\cos\mathfrak{A}_b+\left(\frac{21}{10}b^2+\frac{3}{5}c^2-\frac{21}{5}bc\cos A\right)c\cos\mathfrak{B}_c,$$

after its introducing (5) can be integrated immediately. If we consider that for $\delta=0$ the general integral vanishes

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and F as well as F_2 are zero, then we obtain without difficulty the equation:

$$F=F_2''+\left\{\left[\begin{array}{l}\dfrac{4b^2+3c^2-9bc\cos A}{10}\,b\cos\mathfrak{A}_b\\ +\dfrac{7b^2+6c^2-21bc\cos A}{10}\,c\cos\mathfrak{B}_c\end{array}\right]E_2+Gl_2\right\}\frac{bc\sin A}{2}. \qquad (6)$$

Here $F_2''=\varepsilon\varrho^2$ with $\varrho=1:\sqrt{K_2}$ i. e according to p. 98 (3):

$$F_2''=\frac{1}{2}bc\sin A\left\{1+\frac{4a^2-3m^2}{24}K_2+\frac{14a^4-15a^2m^2+3m^4}{720}K_2^2+\frac{b^2c^2\sin^2A}{72}K_2^2+Gl_6\right\}, \qquad (7)$$

where actually a' is to be applied instead of a ; which is negligible according to what has been said above about the difference of the pertinent cosines.

If we put K_2 for the mean curvature K in F_2'' , then the area F'' of a spherical triangle with the parts b, c, A on the sphere of the radius $1:\sqrt{K}$ results. We see that

$$F_2''=F''+\frac{1}{2}bc\sin A\left(\frac{4a^2-3m^2}{24}[K_2-K]+Gl_4\right),$$

where because of

$$a^2 = b^2 + c^2 - 2bc\cos A + Gl_4$$

one can also put

$$4a^2 - 3m^2 = 2b^2 + 2c^2 - 6bc\cos A + Gl_4$$

if in the right-hand side of (6) we use, according to p. 276 (5), the relations

$$E_2 c\cos\mathfrak{B}_c = \frac{K_1 - K_2}{12} + Gl_4 = \frac{[K_1 - K] - [K_2 - K]}{12} + Gl_4$$

$$E_2 b\cos\mathfrak{A}_b = \frac{K_3 - K_1}{12} + Gl_4 = \frac{[K_3 - K] - [K_1 - K]}{12} + Gl_4$$

we have, after simplification, for the area F of the geodetic triangle (a_0 is no longer the unit of length):

$$F = F^{\sphericalangle}\left\{1 + \frac{3b^2+3c^2-12bc\cos A}{120a_0^2}[K_1 - K] + \frac{3b^2+4c^2-9bc\cos A}{120a_0^2}[K_2 - K] + \frac{4b^2+3c^2-9bc\cos A}{120a_0^2}[K_3 - K] + Gl_6\right\} \quad (8)$$

in connection with the area $F^{\sphericalangle}$ of the spherical triangle, that has

the parts b, c and A in common and lies on the sphere of radius $a_0 : \sqrt{K}$. For $F^{\sphericalangle}$ we have:

$$\log F^{\sphericalangle} = \log\left(\frac{1}{2}bc\sin A\right) + M\left\{\frac{4a^2-3m^2}{24a_0^2}K + \frac{32a^4-45m^4+24n^4}{5760a_0^4}K^2 + \frac{b^2c^2\sin^2 A}{72a_0^4}K^2 + Gl_6\right\}, \quad (9)$$

where according to p. 98 we have to compute:

$$\frac{a^2}{a_0^2} = \frac{b^2}{a_0^2} + \frac{c^2}{a_0^2} - \frac{2bc\cos A}{a_0^2} - \frac{b^2c^2\sin^2 A}{3a_0^4} + Gl_6 \quad (10)$$

$$m^2 = \frac{a^2+b^2+c^2}{3} \qquad n^4 = \frac{a^4+b^4+c^4}{3}.$$

Formula (8) permits further simplification in that zero is added in the brackets in the form:

$$-\frac{3b^2+3c^2-12bc\cos A}{120a_0^2}\{[K_1-K]+[K_2-K]+[K_3-K]\}.$$

Then we have instead of (8):

$$F=F''\left\{1+\frac{c^2+3bc\cos A}{120a_0^2}[K_2-K]+\frac{b^2+3bc\cos A}{120a_0^2}[K_3-K]+Gl_6\right\}. \qquad (11)$$

Par. 8. Continuation. Geodetic and Spherical Triangle with the Same Sides. In order to compare the area F of the geodetic triangle with the area F'' of a spherical triangle of the same sides, we note that in the spherical triangle with the sides a, b and c on the sphere with the radius $1:\sqrt{K}$ the angle A', which lies opposite a is, according to (3) p. 358, given by the equation:

$$A'=A-\frac{K_1-K}{12}F^*+Gl_6, \qquad (12)$$

if we again let a_0 equals 1. Now we have

$$F''=\frac{1}{2}bc\sin A'\left\{1+\frac{4a^2-3m^2}{24}K+\frac{14a^2-15a^2m^2+3m^4}{720}K^2+\frac{b^2c^2\sin^2 A}{72}K^2+Gl_6\right\} \qquad (13)$$

and according to (12):

$$\sin A'=\sin A-\frac{K_1-K}{12}F^*\cos A+Gl_6.$$

If we put in the latter formula $F^*=\frac{1}{2}bc\sin A+Gl_4$, then we obtain from (9) and (13) without difficulty:

$$F''=F'\left(1+\frac{K_1-K}{24}bc\cos A+Gl_7\right). \qquad (14)$$

The introduction of this expression for F'' in (11) now immediately gives

$$F = F'\left\{1 + \frac{bc\cos A}{24}[K_1 - K] + \frac{c^2 + 3bc\cos A}{120}[K_2 - K] + \frac{b^2 + 3bc\cos A}{120}[K_3 - K] + Gl_6\right\},$$

which becomes more elegant when we add zero in the brackets:

$$-\frac{b^2 + c^2 + 11bc\cos A}{360}\cdot\left([K_1 - K] + [K_2 - K] + [K_3 - K]\right).$$

If we then eliminate $bc\cos A$ by means of the relation $a^2 = b^2 + c^2 - 2bc\cos A + Gl_4$ and introduce m^2, then we obtain finally (a_0 is no longer the length unit)

$$F = F''\left\{1 + \frac{m^2 - a^2}{120a_0^2}[K_1 - K] + \frac{m^2 - b^2}{120a_0^2}[K_2 - K] + \frac{m^2 - c^2}{120a_0^2}[K_3 - K] + Gl_6\right\}. \qquad (15)$$

Here F and F'' are the areas of the <u>geodetic</u> and <u>spherical</u> triangles with the same sides a, b, c; the radius of the sphere is assumed equivalent to $a_0 : \sqrt{K}$ corresponding to the arithmetic mean of the curvatures of the three angles.*)

For the calculation of F'' we have according to p. 93 (2), since $F'' = \varepsilon\varrho^2$ for $\varrho^2 = a_0^2 : K$, the formula:

$$\log F'' = \log F^* + M\left(\frac{m^2}{8a_0^2}K + \frac{8n^4 + 9m^4}{1920a_0^4}K^2 + Gl_6\right), \qquad (16)$$

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*)This formula agrees with one given by Hansen in his <u>Geodätischen Untersuchungen</u> (<u>Geodetic Investigations</u>), p. 191, If the curvature is introduced when considering the difference which occurs between F'' and the Hansen spherical triangle F''' which lies on the sphere of radius a_0 . This difference is easily derived by means of our formula (16).

where F^* is the area of the plane triangle with sides a, b and c computed as before.

For triangles with sides greater than $0{,}2a_0$ in length, it will be preferable to derive F'' from the exact value of the spherical excess which is computed according to p. 87 (8); just as F' is computed from the formula (3) p. 84.

Thereby the inconvenience is met, that the eighth-order terms,

which originate from m^6 etc. are not disregarded relative to the numerically smaller terms that are produced by $K_1 - K$ etc.

With reference to the validity of previous developments, the permissibility of which in particular is not everywhere emphasized for the sake of brevity, there remains no doubt for moderately large values of the lengths of the sides. The effect of the fifth-order terms appearing in the brackets of (11) and (15), which to a first approximation indicate the difference between exact and spherical area calculation, accounts for $m = 0{,}1a_0$, for hardly one unit of the seventh decimal-place of $\log F$.

The area of geodetic triangles may therefore be generally computed purely spherically from arbitrary parts only if the radius of the sphere is chosen according to the mean curvature.

Par. 9. The Theory of the Geodetic Triangle on arbitrarily curved surfaces is the main aim of Gauss' treatise: Disquisitiones generales circa superficies curves (General discussion about surface

curves). Gottingae 1828.*) If we have proved the formulae pp. 358 and 359 for the reduction of the geodetic triangle to a spherical or plane triangle with same sides only for the ellipsoid of rotation, then Gauss shows the general validity of these formulae (except for unessential differences of the presentation) for each surface.

Gauss does not connect onto the oblique triangle, but he splits it into two right triangles and formulates differential formulae for a right triangle, both small sides of which are assumed variable in that they both form a system of rectangular geodetic coordinates. The development of formulae of this type and the development of an auxiliary variable $\mathfrak{n}$, which replaces our $\mathfrak{m}$ is presented in the next chapter in the treatment of the problem of rectangular geodetic coordinates.

Should we wish to apply the method of Gauss to the tracing of the problem to the integration of a partial differential equation with two independent variables in connection to our developments for the oblique triangle, then we would have to form according to fig. 29, p. 348:

$$\frac{\partial a}{\partial b} = \cos C \qquad (1)$$

as partial differential quotient from a to b with constant A and $\mathfrak{A}$. Further from fig. 30 p. 353:

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*)Cf. also in Gauss works volume 4, pp. 341 et seq., the reading of this treatise made by Gauss himself, with a comprehensive explanation of contents.

$$\frac{\partial a}{m_b \partial \mathfrak{A}_b} = \sin C \qquad (2)$$

as partial differential quotient from a to $\mathfrak{A}_b$ with constant b. Together both equations give by elimination of the variables C with the relation $\cos^2 C + \sin^2 C = 1$ the equation

$$\left(\frac{\partial a}{\partial b}\right)^2 + \left(\frac{\partial a}{m_b \partial \mathfrak{A}_b}\right)^2 = 1 \qquad (3)$$

or, as $\partial(a^2) = 2a\partial a$ we find:

$$\left(\frac{\partial (a^2)}{\partial b}\right)^2 + \left(\frac{\partial (a^2)}{m_b \partial \mathfrak{A}_b}\right)^2 = 4a^2. \qquad (4)$$

Concerning integration, if we introduce a^2 as a function of b and $\mathfrak{A}_b$, and also as a power series of b and the sine and cosine of the multiples of $\mathfrak{A}_b$, then the coefficients of this series would be determined according to the method of undetermined coefficients. The expression for a^2 then replaces the Cosine Theorem in the further computation and permits comparison of $\cos A$ and $\cos A^*$.

For a plane triangle we now have:

$$a^2 = b^2 + c^2 - 2bc\cos A = b^2 + c^2 - 2bc(\cos \mathfrak{A}_b \cos \mathfrak{A}_c + \sin \mathfrak{A}_b \sin \mathfrak{A}_c)$$

and we recognize that in general a^2 becomes a long expression that involves a fairly extensive computation yielding no data for the convergence of the results--at least this is only immediately clear for infinitely small triangles.

With Gauss' method the computation becomes more convenient in that the azimuth variable drops out and we are concerned with only two variable short sides, thus with only simple power series.

The computation also allows simplification in that, as in fig. 29,

we introduce only b as an independent variable, but put angle A as equal to 90°. This very commendable process requires the integration of a system of three ordinary differential equations which can also occur according to the method of undetermined coefficients. Hansen has applied it and we will outline it more thoroughly in the next paragraph.

The method of successive approximations followed by us might perhaps, although in no way ought it to replace the method by Gauss, be not unwelcome to many readers, as the developments become relatively simple and the zone of validity of the omissions is perhaps somewhat easier to survey than with the methods of Gauss and Hansen.

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We can, in this way, increase the exactness of the series development further. It would be most useful to apply the expression (11) p. 278 for m, which is more accurate in second-order terms than the one used earlier. To go only one order further would offer little success because for distances $> 0{,}1 a_0$ with which it would be concerned, the convergence of the series for K is limited.

We by-pass this development, as geodetic triangles of more than $0{,}1 a_0$ average side length can always be avoided, and in any case will be avoided on account of the involved computation. These formulae are however conveyed further according to Weingarten and Hansen.

Gauss went only one order further in his formula development, which, however, offers little profit as was just shown, and as Hansen too has discussed in his geodetic examinations by observation of final formulae.

Chapter 8. The Geodetic Triangle; Triangulation Nets.

The treatise of Christoffel: Allgemeine Theorie der geodätischen Dreiecke (General Theory of Geodetic Triangles). Berlin 1868 (Abh. der Kön. Akad. der Wiss.) is still to be remembered as that type of work which deals with the theory of the oblique triangle in a direct manner. Like Gauss, Christoffel establishes the theory of the geodetic line by means of the calculus of variation. He first proves that for a revolution the shortest line is always normal to the line joining the moved end point, and that for geodetic polar coordinates formula (5), p. 348 exists.

Now that the differential formulae become established in the triangle, the conditions of integrability which (cf. p. 74, the development of formula (10)) subsequently lead to the basic qualities of the reduced length, among others: uniformity for revolution around both end points, as well as satisfaction of the differential equation (6) p. 275.

An integration of the differential formula will not be carried out, whereas further conclusions are drawn in the field of the classification of the areas.

Par. 10. Higher Terms in the Formulae for the Computation of Geodetic Triangles. Agreeing most directly with Gauss' developments are those of Weingarten (Astronom. Nachr. 1869 volume 73 No. 1733, p. 65 and 1870 volume 75 No. 1782, p. 91; also in the second issue of the publication: Wissenschaftliche Begründung der Rechnungsmethoden des Zentralbureaus der Europäischen Gradmessung--Scientific Foundation of the Methods of Computation of the Central Bureau of the European Arc Measurement). He finds:

$$\varepsilon \underset{\text{in seconds}}{=} \varrho'' \frac{F}{a_0^2} \left\{ K + \left[(\sin\beta_1 - \sin\beta_2)^2 + (\sin\beta_2 - \sin\beta_3)^2 + (\sin\beta_3 - \sin\beta_1)^2 \right] \frac{e^2}{6} - \left[\frac{b^2+c^2}{a_0^2} \sin^2\beta_1 + \frac{a^2+c^2}{a_0^2} \sin^2\beta_2 + \frac{a^2+b^2}{a_0^2} \sin^2\beta_3 \right] \frac{e^2}{12} + Gl_6 \right\}. \quad (1)$$

Here the area F of the geodetic triangle is to be computed according to Par. 7 (8) to (11) p. 365 or (15) and (16) p. 367, possibly by partial use of the formulae (8) p. 87 and (3) p. 84 for the spherical part of the computation.

According to Weingarten's developments, we find further for the reduction onto the plane triangle (with complement of the Weingarten expression around the term dependent on n^4 and m^4, cf. p. 93):

$$A - A^* \underset{\text{in seconds}}{=} \varrho'' \frac{F}{a_0^2} \left\{ \frac{3K+K_1}{12} + \frac{m^2-a^2}{60a_0^2}(1+2e^2) + \frac{20(n^4-a^4) - 3m^2(m^2-a^2)}{30240a_0^4} + \left[\begin{matrix} 2(\sin\beta_1 - \sin\beta_2)^2 + 2(\sin\beta_3 - \sin\beta_1)^2 \\ + (\sin\beta_2 - \sin\beta_3)^2 \end{matrix} \right] \frac{e^2}{30} - \left[\begin{matrix} \frac{-10a^2+34b^2+34c^2}{a_0^2} \sin^2\beta_1 + \frac{a^2+5b^2+25c^2}{a_0^2} \sin^2\beta_2 \\ + \frac{a^2+25b^2+5c^2}{a_0^2} \sin^2\beta_3 \end{matrix} \right] \frac{e^2}{720} + Gl_6 \right\}. \quad (2)$$

If we deduct $\frac{\varepsilon}{3}$ from both sides of this, with the right-hand written according to the above formula, then we have:

$$A - A^* \underset{\text{in seconds}}{=} \frac{\varepsilon}{3} + \varrho'' \frac{F}{a_0^2} \left\{ \frac{K_1 - K}{12} + \frac{m^2-a^2}{60a_0^2}(1+2e^2) + \frac{20(n^4-a^4) - 3m^2(m^2-a^2)}{30240a_0^4} + \left[\begin{matrix} (\sin\beta_1 - \sin\beta_2)^2 + (\sin\beta_3 - \sin\beta_1)^2 \\ - 2(\sin\beta_2 - \sin\beta_3)^2 \end{matrix} \right] \frac{e^2}{90} - \left[\begin{matrix} \frac{14m^2-8a^2}{a_0^2} \sin^2\beta_1 \\ + \frac{5m^2-8a^2}{a_0^2} (\sin^2\beta_2 + \sin^2\beta_3) \end{matrix} \right] \frac{e^2}{240} + Gl_6 \right\}. \quad (3)$$

From this the expressions for $B-B^*$ and $C-C^*$ follow through cyclical interchange of the symbols relative to the three corners.

We may also apply in these formulae the geographic latitude B instead of the reduced latitude β.

In his Geodätische Untersuchungen (Geodetic Investigations) of 1865, Hansen did not depart from the Gauss partial differential equation with

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two independent variables (both the small sides of the right-angled triangle), but proceeded in the following manner, if we consider fig. 29 p. 348 (cf. Hansen op. cit. pp. 137, 153 and 155, et seq.; there the designations σ, φ, ψ are used, which here are a, B, C).

Thus:

$$\left.\begin{aligned} da &= db\cos C \\ \mathfrak{m}_a dB &= db\sin C \\ dC &= -\frac{\partial \mathfrak{m}_a}{\partial a} dB. \end{aligned}\right\} \qquad (4)$$

This system, with the help of the identities $d(a\sin C) = \sin C da + a\cos C dC$ and $d(a\cos C) = \cos C da - a\sin C dC$ is introduced into the new and more convenient:

$$\left.\begin{aligned} \frac{d(a\sin C)}{db} &= \left(\mathfrak{m}_a - a\frac{\partial \mathfrak{m}_a}{\partial a}\right)\frac{dB}{db}\cos C \\ \frac{d(a\cos C)}{db} &= 1 - \left(\mathfrak{m}_a - a\frac{\partial \mathfrak{m}_a}{\partial a}\right)\frac{dB}{db}\sin C \\ \frac{d(B+C)}{db} &= \left(1 - \frac{\partial \mathfrak{m}_a}{\partial a}\right)\frac{dB}{db}. \end{aligned}\right\} \qquad (5)$$

As we will see immediately, it is now a question of finding $a\sin B$

and $a\cos B$ (Gauss works toward this unknown for the same reason). Hansen therefore sets series with undetermined coefficients for $a\sin B$, $a\cos B$ and $B+C$, arranged according to powers of b. He then uses the third series to express $a\sin C$ and $a\cos C$ from the first two expressions. This calculation becomes simplest when angle A equals 90°, i. e. $B+C=90°$ in the first approximation.

Further functions of a and C, which appear in m_a and $\frac{\partial m_a}{\partial a}$ follow then from $a\sin C$ and $a\cos C$. If all these expressions are then substituted into the above system, then an identity must result, i. e., the coefficients of the various powers of b have to disappear singularily. From these conditions the undetermined coefficients in the assumed expressions for $a\sin B$, $a\cos B$ and $B+C$ finally result. (Hansen op. cit. 159.)

If they are applied to the two rectangular triangles ABC_1 and ABC_2, which have A and B common, then the following results for the obtuse triangle BC_1C_2 : $a_1a_2\cos(B_1-B_2)$ from $a_1\cos B_1 . a_2\cos B_2$ $+ a_1\sin B_1 . a_2\sin B_2$, in powers of a_1, a_2 and b_1-b_2, i. e., one obtains $\cos(B_1-B_2)$, the cosine of an angle of the obtuse triangle, represented by the three sides and one can

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compare it directly with the cosine of the corresponding angle in the plane triangle.

According to pp. 188 and 189, Hansen, ibid, finds:

$$\varepsilon = \varrho'' \frac{F}{a_0^2} \left\{ \begin{array}{l} 1 + (\cos 2\beta_1 + \cos 2\beta_2 + \cos 2\beta_3) \dfrac{e^2 + \frac{1}{2} e^4}{3} \\ + (\cos^2 2\beta_1 + \cos^2 2\beta_2 + \cos^2 2\beta_3 - 1) \dfrac{e^4}{4} \\ + \left[\begin{array}{l} \dfrac{-14a^2 + 2b^2 + 2c^2}{a_0^2} \cos 2\beta_1 + \dfrac{7a^2 + b^2 - 3c^2}{a_0^2} \cos 2\beta_2 \\ + \dfrac{7a^2 - 3b^2 + c^2}{a_0^2} \cos 2\beta_3 \end{array} \right] \dfrac{e^2}{360} \\ - \left[\dfrac{3m^2}{a_0^2} \sin^2 \beta_1 - 2 \dfrac{b^2 \cos^2 \mathfrak{A}_b + c^2 \cos^2 \mathfrak{A}_c - bc \cos \mathfrak{A}_b \cos \mathfrak{A}_c}{a_0^2} \cos^2 \beta_1 \right] \dfrac{e^2}{6} \\ + \left[\dfrac{a^2 + 6b^2 - 2c^2}{a_0^2} b \cos \mathfrak{A}_b + \dfrac{a^2 - 2b^2 + 6c^2}{a_0^2} c \cos \mathfrak{A}_c \right] \dfrac{2e^2}{45} \sin 2\beta_1 + Gl_6 \end{array} \right\} \quad (6)$$

in seconds

$$A - A^* = \varrho'' \frac{F}{a_0^2} \left\{ \begin{array}{l} (2\cos 2\beta_1 + \cos 2\beta_2 + \cos 2\beta_3) \dfrac{2e^2 + e^4}{24} \\ + \left(2\cos^2 2\beta_1 + \cos^2 2\beta_2 + \cos^2 2\beta_3 - \dfrac{4}{3} \right) \dfrac{e^4}{16} \\ + \left[\begin{array}{l} \dfrac{-58a^2 + 54b^2 + 54c^2}{a_0^2} \cos 2\beta_1 \\ + \dfrac{11a^2 + 23b^2 + 31c^2}{a_0^2} \cos 2\beta_2 \\ + \dfrac{11a^2 + 31b^2 + 23c^2}{a_0^2} \cos 2\beta_3 \end{array} \right] \dfrac{e^2}{4320} \\ - \left[\begin{array}{l} \dfrac{a^2 + 2b^2 + 2c^2}{a_0^2} \sin^2 \beta_1 \\ - \dfrac{3b^2 \cos^2 \mathfrak{A}_b + 3c^2 \cos^2 \mathfrak{A}_c - 2bc \cos \mathfrak{A}_b \cos \mathfrak{A}_c}{a_0^2} \cos^2 \beta_1 \end{array} \right] \dfrac{e^2}{30} \\ + \left[\begin{array}{l} \dfrac{a^2 + 11b^2 - 2c^2}{a_0^2} b \cos \mathfrak{A}_b \\ + \dfrac{a^2 - 2b^2 + 11c^2}{a_0^2} c \cos \mathfrak{A}_c \end{array} \right] \dfrac{e^2}{135} \sin 2\beta_1 + Gl_6 \end{array} \right\} \quad (7)$$

in seconds

$$+ \varrho'' \frac{F}{a_0^2} \left\{ \begin{array}{l} \dfrac{1}{3} + \dfrac{m^2 - a^2}{60a_0^2} + \dfrac{20(n^4 - a^4) - 3m^2(m^2 - a^2)}{30240a_0^4} \\ - \left[\begin{array}{l} \dfrac{a^2 + 2b^2 + 2c^2}{a_0^2} \cos 2\beta_1 + \dfrac{2a^2 + b^2 + 2c^2}{a_0^2} \cos 2\beta_2 \\ + \dfrac{2a^2 + 2b^2 + c^2}{a_0^2} \cos 2\beta_3 \end{array} \right] \dfrac{e^2}{360} \end{array} \right\}$$

Hansen gives only the first part of the expression for $A - A^*$

as the reduction of A onto the angle of a spherical triangle with equal sides on a sphere of radius a_0. We add to this the reduction on the plane triangle by use of p. 93 (4), where ε was to be exchanged with $F'':a_0^2$, if F'' is the area of the spherical triangle the relationship of which to F, Hansen, ibid, states on p. 192. (cf. the note to p. 367.)

One finds the formulae for $B-B^*$ and $C-C^*$ out of the above for $A-A^*$ by interchanging the angles; moreover Hansen gives further expressions in order to be able to apply the same two azimuths for all three reductions.

We will rearrange the above formulae some more. According to p. 59 one easily obtains:

$$K_1 = 1 + \left(e^2 + \frac{1}{2}e^4\right)\cos 2\beta_1 + \frac{1}{4}e^4(3\cos^2 2\beta_1 - 1) + Gl_6 \qquad (8)$$

and correspondingly for K_2 and K_3. From this the magnitude of K is immediately found from the value of the first line in ε, i. e., the arithmetic mean of K_1, K_2 and K_3.

One therefore gets:

$$K_2 - K_1 = e^2(\cos 2\beta_2 - \cos 2\beta_1) + Gl_4,$$

but which according to p. 355 (9) is also equal to:

$$K_2 - K_1 = 2e^2 \sin 2\beta_1 \frac{c}{a_0}\cos \mathfrak{A}_c + Gl_4.$$

By equating the two values we obtain:

$$e^2(\cos 2\beta_2 - \cos 2\beta_1) = 2e^2 \sin 2\beta_1 \frac{c}{a_0}\cos \mathfrak{A}_c + Gl_4. \qquad (9)$$

Also:

$$e^2(\cos 2\beta_3 - \cos 2\beta_1) = 2e^2 \sin 2\beta_1 \frac{b}{a_0}\cos \mathfrak{A}_b + Gl_4. \qquad (10)$$

With the help of both these formulae the second line of the

above expression for ε can be brought to the form of the last line and one obtains:

$$\varepsilon = \rho'' \frac{F}{a_0^2} \text{ in seconds} \left\{ K - \left[\frac{3m^2}{a_0^2} \sin^2\beta_1 - 2 \frac{b^2\cos^2\mathfrak{A}_b + c^2\cos^2\mathfrak{A}_c - bc\cos\mathfrak{A}_b\cos\mathfrak{A}_c}{a_0^2} \cos^2\beta_1 \right] \frac{e^2}{6} \right\} + \left[\frac{a^2+3b^2-c^2}{a_0^2} b\cos\mathfrak{A}_b + \frac{a^2-b^2+3c^2}{a_0^2} c\cos\mathfrak{A}_c \right] \frac{e^2}{12} \sin 2\beta_1 + Gl_6 . \quad (11)$$

In the same manner the following can be found without difficulty:

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$$A - A^* = \rho'' \frac{F}{a_0^2} \text{ in seconds} \left\{ \frac{3K+K_1}{12} + \frac{m^2-a^2}{60a_0^2}(1+2e^2) + \frac{20(n^4-a^4)-3m^2(m^2-a^2)}{30240a_0^4} - \left[\frac{8m^2-3a^2}{a_0^2} \sin^2\beta_1 - \frac{3b^2\cos^2\mathfrak{A}_b + 3c^2\cos^2\mathfrak{A}_c - 2bc\cos\mathfrak{A}_b\cos\mathfrak{A}_c}{a_0^2} \cos^2\beta_1 \right] \frac{e^2}{30} \right\} + \left[\frac{a^2+61b^2-7c^2}{a_0^2} b\cos\mathfrak{A}_b + \frac{a^2-7b^2+61c^2}{a_0^2} c\cos\mathfrak{A}_c \right] \frac{e^2}{720} \sin 2\beta_1 + Gl_6 . \quad (12)$$

If we introduce here $\frac{\varepsilon}{3}$ with help of the previous expression (11) for ε, then we finally obtain:

$$A - A^* = \frac{\varepsilon}{3} \text{ in seconds} + \rho'' \frac{F}{a_0^2} \text{ in seconds} \left\{ \frac{K_1-K}{12} + \frac{m^2-a^2}{60a_0^2}(1+2e^2-6e^2\sin^2\beta_1) + \frac{20(n^4-a^4)-3m^2(m^2-a^2)}{30240a_0^2} - \frac{b^2\cos^2\mathfrak{A}_b + c^2\cos^2\mathfrak{A}_c - 4bc\cos\mathfrak{A}_b\cos\mathfrak{A}_c}{90a_0^2} e^2\cos^2\beta_1 \right\} + \left[\frac{-19a^2+b^2+13c^2}{a_0^2} b\cos\mathfrak{A}_b + \frac{-19a^2+13b^2+c^2}{a_0^2} c\cos\mathfrak{A}_c \right] \frac{e^2}{720} \sin 2\beta_1 + Gl_6 . \quad (13)$$

In these formulae (11) and (13) we may likewise apply the geographic latitude B in the place of the reduced latitude β .

Par. 11. Continuation. Comparison of the Higher Terms According to Hansen and Weingarten. In order to compare the formulae expressed by both authors, we set with reference to fig. 19 p. 225 by means of the spherical auxiliary triangle which is subordinate to the geodetic line $AB = c$.

$$\sin\beta_2 = \sin\beta_1 \cos\Delta\varphi - \cos\beta_1 \sin\Delta\varphi \cos\mathfrak{A}_c .$$

But according to p. 223 (9) $\Delta\varphi = \frac{c}{a_0} + Gl_3$; therefore we have immediately from this:

$$\sin\beta_2 = \sin\beta_1 - \frac{c}{a_0}\cos\mathfrak{A}_c\cos\beta_1 - \frac{c^2}{2a_0^2}\sin\beta_1 + Gl_3 . \qquad (1)$$

Correspondingly:

$$\sin\beta_3 = \sin\beta_1 - \frac{b}{a_0}\cos\mathfrak{A}_b\cos\beta_1 - \frac{b^2}{2a_0^2}\sin\beta_1 + Gl_3 \qquad (2)$$

and by subtraction of both equations:

$$\sin\beta_2 - \sin\beta_3 = \left\{\begin{array}{l}\left(\frac{b}{a_0}\cos\mathfrak{A}_b - \frac{c}{a_0}\cos\mathfrak{A}_c\right)\cos\beta_1 \\ + \left(\frac{b^2}{2a_0^2} - \frac{c^2}{2a_0^2}\right)\sin\beta_1 + Gl_3 .\end{array}\right\} \qquad (3)$$

From this it follows that:

$$\left[(\sin\beta_1 - \sin\beta_2)^2 + (\sin\beta_2 - \sin\beta_3)^2 + (\sin\beta_3 - \sin\beta_1)^2\right]\frac{e^2}{6}$$
$$= \left\{\begin{array}{l}\frac{b^2\cos^2\mathfrak{A}_b + c^2\cos^2\mathfrak{A}_c - bc\cos\mathfrak{A}_b\cos\mathfrak{A}_c}{3a_0^2}\, e^2\cos^2\beta_1 \\ + \left[\frac{2b^2 - c^2}{a_0^3}\, b\cos\mathfrak{A}_b + \frac{2c^2 - b^2}{a_0^3}\, c\cos\mathfrak{A}_c\right]\frac{e^2}{12}\sin 2\beta_1 + Gl_6\end{array}\right\},$$

and:

$$\left[\frac{b^2+c^2}{a_0^2}\sin^2\beta_1 + \frac{a^2+c^2}{a_0^2}\sin^2\beta_2 + \frac{a^2+b^2}{a_0^2}\sin^2\beta_3\right]\frac{e^2}{12}$$
$$= \frac{m^2}{2a_0^2}\, e^2\sin^2\beta_1 - \left[\frac{a^2+b^2}{a_0^3}\, b\cos\mathfrak{A}_b + \frac{a^2+c^2}{a_0^3}\, c\cos\mathfrak{A}_c\right]\frac{e^2}{12}\sin 2\beta_1 + Gl_6 .$$

If we subtract both equations side for side then we soon see that

the expressions for ε are, according to (1) p. 371 and (11) p. 374, identical up to the disregarded eighth-order terms.

We obtain also from the formulae (1) to (3):

$$\left[(\sin\beta_1-\sin\beta_2)^2-2(\sin\beta_2-\sin\beta_3)^2+(\sin\beta_3-\sin\beta_1)^2\right]\frac{e^2}{90}$$

$$=\left\{\begin{array}{l}-\dfrac{b^2\cos^2\mathfrak{A}_b+c^2\cos^2\mathfrak{A}_c-4bc\cos\mathfrak{A}_b\cos\mathfrak{A}_c}{90a_0^2}e^2\cos^2\beta_1\\ +\left[\dfrac{-4b^2+8c^2}{a_0^3}b\cos\mathfrak{A}_b+\dfrac{8b^2-4c^2}{a_0^3}c\cos\mathfrak{A}_c\right]\dfrac{e^2}{720}\sin 2\beta_1+Gl_6\end{array}\right\}$$

and:

$$\left[\frac{14m^2-8a^2}{a_0^2}\sin^2\beta_1+\frac{5m^2-8a^2}{a_0^2}(\sin^2\beta_2+\sin^2\beta_3)\right]\frac{e^2}{240}$$

$$=\frac{m^2-a^2}{10a_0^2}e^2\sin^2\beta_1+\frac{19a^2-5b^2-5c^2}{a_0^3}(b\cos\mathfrak{A}_b+c\cos\mathfrak{A}_c)\frac{e^2}{720}\sin 2\beta_1+Gl_6.$$

Here it is easily seen that the formulae (3) p. 371 and (13) p. 375 are likewise identical up to the eighth-order terms.

For the application, the formulae (1) and (3) p. 371 are just as convenient as (11) and (13) pp. 374 and 375.

It is evident though that previous formulae at the limit of application can give noticeably different results to the different modifications. This is, in fact, the case for a

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total mean side length of $m=0{,}3a_0$; differently modified formulae for the excess show here differences of up to approximately 0,2".

Hansen succeeded with equal analytical precision in finding such a modification, that for $m=0{,}3a_0$ is still certain in the hundredth of a second. He utilizes for this the curvature of certain auxiliary points outside those of the angles; a process which Schering in 1868, still restricted to more limited analytical precision, worked out.

Par. 12. Numerical Example of Hansen's and Weingarten's Formulae.

To this is to be compared the information of Hansen: Reflexionen über the Reduktion der Winkel eines sphärischen Dreiecks (Reflections on the Reduction of the Angles of a Spherical Triangle) and following in the reports on the Proceedings of the Society of Science at Leipzig, math-physic class, 1869, p. 139.

More explicitly we have the examples (which connect up with those given in the Geodätischen Untersuchungen, pp. 191 ff. in the Supplement zu der "Geodätische Untersuchungen" benannten Abhandlung (Supplement to the treatise named "Geodetic Investigations") etc. in the treatises of the same Society of 1869, volume 9, No. III.

Compare also a comprehensive discussion in the quarterly of the Astronomical Society 1870, volume 5, p. 222, as well as some remarks in volume 13, 1878, p. 79.

We may confine ourselves here to the application of one of the examples computed by Hansen for very large triangles by the formulae stated above. We will at the same time sufficiently recognize that with $m \leq 0{,}2a_0$ these formulae are adequate in each modification and in each case sufficient to assess the errors of the more simple formulae developed earlier by us for $m < 0{,}1a_0$.

Par. 12. Numerical Example of Hansen's and Weingarten's Formulae. In his Geodätischen Untersuchungen p. 199 and particularly in the supplement p. 348, Hansen gives the following triangle, computed according to the strict formulae for geodetic lines (which formulae correspond to those of Chapter 5):

A	$\beta_1 = 15^0\,14'\,30{,}0318''$	$\varrho'' \frac{a}{a_0} = 70351{,}4184$	$\mathfrak{A}_b = 32^0\,18'\,53{,}2155'' = -\mathfrak{A}_c$
B	$\beta_2 = 0$	$\varrho'' \frac{b}{a_0} = 64800$	$\mathfrak{B}_c = 148^0$ $\quad \mathfrak{B}_a = 90^0$
C	$\beta_3 = 0$	$\varrho'' \frac{c}{a_0} = 64800$	$\mathfrak{C}_a = 270^0$ $\quad \mathfrak{C}_b = 212^0$.

The earlier computation of Hansen in the Geodätischen Untersuchungen agrees essentially with these data taken from the Supplement; and it is therefore not checked. The excess is, according to those figures, equal to

$$\varepsilon = 9466{,}4310''.$$

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If we compute further the pertinent angles of the plane triangle to the linear lengths 70351, 4184, 64800 and 64800 of the three sides according to known formulae of plane trigonometry, then we have:

$$\begin{aligned} A^* &= 65^\circ\,45'\,13{,}2493'' \\ B^* &= 57\;\;7\;\;23{,}3754 \\ C^* &= 57\;\;7\;\;23{,}3754 \\ \text{Sum} &= 180\;\;0\;\;0{,}0001\,. \end{aligned}$$

As we now have for the geodetic triangle:

$$\begin{aligned} A &= 66^\circ\,37'\,46{,}4310'' \\ B &= 58^\circ \\ C &= 58^\circ, \end{aligned}$$

then:

$$\begin{aligned} A - A^* &= 3153{,}1817'' = \tfrac{\varepsilon}{3} - 2{,}2953'' \\ B - B^* &= 3156{,}6246 \;\; = \tfrac{\varepsilon}{3} + 1{,}1476 \\ C - C^* &= 3156{,}6246 \;\; = \tfrac{\varepsilon}{3} + 1{,}1476 \\ \text{Sum} &= 9466{,}4309. \end{aligned}$$

At the same time we have

$$\rho'' \frac{F^*}{a_0^2} = 3{,}9675889.6\,.$$

Concerning the application of the formulae of Par. 10 we next

have, according to p. 59:

$$\log K_1 = 9{,}9970916.4 - 10 - 4 \log w_1 = 0{,}0025051.0$$
$$K_1 = 1{,}00578487.$$
$$K_2 = K_3 = 1 : (1 - e^2) = 1 + \delta.$$

thus according to p. 38:

$$K_2 = K_3 = 1{,}00671922$$
$$K = \frac{K_1 + K_2 + K_3}{3} = 1{,}00640777.$$

We further obtain:

$\frac{a^2}{a_0^2} = 0{,}1163311$	$\frac{m^2 - a^2}{a_0^2} = -0{,}0117567$	$\frac{a^4}{a_0^4} = 0{,}013533$	$\frac{n^4 - a^4}{a_0^4} = -0{,}002528$
$\frac{b^2}{a_0^2} = 0{,}0986961$	$\frac{m^2 - b^2}{a_0^2} = +0{,}0058783$	$\frac{b^4}{a_0^4} = 0{,}009741$	$\frac{n^4 - b^4}{a_0^4} = +0{,}001264$
$\frac{c^2}{a_0^2} = 0{,}0986960$	$\frac{m^2 - c^2}{a_0^2} = +0{,}0058783$	$\frac{c^4}{a_0^4} = 0{,}009741$	$\frac{n^4 - c^4}{a_0^4} = +0{,}001264$
$\frac{m^2}{a_0^2} = 0{,}1045744$		$\frac{n^4}{a_0^4} = 0{,}011005$	

According to formula (16) p. 367 we now have:

$$\log F'' = \log F^* + 0{,}0057133.9 + 0{,}0000427.2,$$

thus:

$$\log \varrho'' \frac{F''}{a_0^2} = 3{,}9733450.7.$$

Because of the significant size of the triangle this amount is somewhat too small, as the subsequent rigorous computation shows.

Next we have $\sqrt{K} = 1{,}00319878$ and as sides of the spherical triangle F'':

$$\varrho'' \frac{a}{a_0} \sqrt{K} = 70576{,}4569'' = 19^\circ\ 36'\ 16{,}4569''$$
$$\varrho'' \frac{b}{a_0} \sqrt{K} = 65007{,}2809 = 18\ \ 3\ 27{,}2809$$
$$\varrho'' \frac{c}{a_0} \sqrt{K} = 65007{,}2809 = 18\ \ 3\ 27{,}2809.$$

Thus, according to (8) p. 87, there follows:

$$\log \tan \frac{\varepsilon'}{4} = 8{,}0596534.9-10 \qquad \log \frac{\varepsilon'}{4} = 8{,}0596344.3-10$$

$$\log \varepsilon' = 8{,}6616944.3-10,$$

where ε' is to be understood as the arc. If we subtract $\log K$ and add $\log \varrho''$, then we have: $\log \varrho'' \frac{F'}{a_0^2} = 3{,}9733455.8$.

In order to obtain F, we compute:

$$\left.\begin{aligned} \frac{m^2-a^2}{a_0^2}(K_1-K) &= -0{,}0118.(-0{,}00062) = +0{,}0000073 \\ \frac{m^2-b^2}{a_0^2}(K_2-K) &= +0{,}0059.(+0{,}00031) = +\quad 18 \\ \frac{m^2-c^2}{a_0^2}(K_3-K) &= +0{,}0059.(+0{,}00031) = +\quad 18 \end{aligned}\right\}$$

$$+0{,}0000109.$$

This gives an increase in the logarithm, according to p. 367 (15):

$$+M\frac{0{,}0000109}{120} = +0{,}0000000.4;$$

hence:

$$\log \varrho'' \frac{F}{a_0^2} = 3{,}9733456.2-10.$$

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Formula (1) p. 371 now gives for the computation of ε :

(See Illustrative Material on the Next Page)

$$\log \varrho'' \frac{F}{a_0^2} K = 3{,}9761196.0 \qquad \varrho'' \frac{F}{a_0^2} K = 9464{,}978''$$

But $\sin\beta_2 = \sin\beta_3 = 0$, $\log c^2 \frac{\sin^2\beta_1}{6} = 5{,}88582 - 10$,

hence

$$\varrho'' \frac{F}{a_0^2}\left[(\sin\beta_1 - \sin\beta_2)^2 + (\sin\beta_2 - \sin\beta_3)^2 + (\sin\beta_3 - \sin\beta_1)^2\right]\frac{c^2}{6} = + 1{,}446''$$

Further:

$$-\varrho'' \frac{F}{a_0^2}\left[\frac{b^2+c^2}{a_0^2}\sin^2\beta_1 + \frac{a^2+c^2}{a_0^2}\sin^2\beta_2 + \frac{a^2+b^2}{a_0^2}\sin^2\beta_3\right]\frac{c^2}{12} = - 0{,}071''$$

These three terms give together $\varepsilon = 9466{,}353''$

erroneous by $+ 0{,}078$.

If we compute with the geographic latitude $B_1 = 15^0\,17'\,25{,}46''$, we will have for the second and third term respectively $+ 1{,}455 - 0{,}072$ and

$\varepsilon = 9466{,}361''$

erroneous by $+ 0{,}070$.

Formula (6) p. 359 gives $\varepsilon = 9340{,}346 + 122{,}877 = 9463{,}223''$

erroneous by $+ 3{,}208$.

The error diminishes somewhat through rigorous spherical calculation, whence is found:

$$\log \varepsilon' = \log \varrho'' \frac{F''}{a_0^2} K = 3{,}9761195.6$$

and with $\varepsilon' = 9464{,}977''$

erroneous by $+ 1{,}458$.

Formula (3) p. 371 gives for $A - A^*$:

$$\begin{cases} \frac{K_1 - K}{12} = - 0{,}00005191 \\ + \frac{m^2 - a^2}{60 a_0^2}(1 + 2e^2) = - \quad 19856 \\ + \frac{m^4 - a^4}{1512 a_0^4} = - \quad 17 \\ - \frac{m^2(m^2 - a^2)}{10080 a_0^4} = + \quad 12 \\ + \frac{e^2 \sin^2\beta_1}{45} = + \quad 1025 \\ - \frac{14m^2 - 8a^2}{240 a_0^2} e^2 \sin^2\beta_1 = - \quad 103 \end{cases}$$

Sum $= - 0{,}00024130$

for $B - B^*$ and $C - C^*$:

$$\begin{aligned}
&\frac{K_2 - K}{12} && = +\ 0{,}00002596\\
+\ &\frac{m^2 - b^2}{60 a_0^2}(1 + 2e^2) && = +\ 9928\\
+\ &\frac{m^4 - b^4}{1512 a_0^4} && = +\ 8\\
-\ &\frac{m^2(m^2 - b^2)}{10080 a_0^4} && = -\ 6\\
-\ &\frac{e^2 \sin^2 \beta_1}{90} && = -\ 513\\
&\frac{-5m + 8b^2}{240 a_0^2} e^2 \sin^2 \beta_1 && = +\ 52\\
& && \text{Sum} = +\ 0{,}00012065.
\end{aligned}$$

These numbers are to be multiplied by $\varrho'' \frac{F}{a_0^2}$. It then follows:

$A - A^* = \frac{s}{3} - 2{,}269''$ erroneous by $-\ 0{,}026$ $\quad B - B^* = C - C^* = \frac{s}{3} + 1{,}134''$ erroneous by $+\ 0{,}014$.

If we introduce $\frac{s}{3}$ according to formula (1), as obtained above, as equal to $3155{,}451''$, then

$A - A^* = 3153{,}182''$ erroneous by $0{,}000$ $\quad B - B^* = C - C^* = 3156{,}585''$ erroneous by $+\ 0{,}040$.

The formulae (6) p. 359 lead to following figures:

$$\begin{aligned}
&\frac{K_1 - K}{12} && = -\ 0{,}00005191\\
&\frac{m^2 - a^2}{60 a_0^2} K^2 && = -\ 0{,}00019846\\
& && \text{Sum} = -\ 0{,}00025037
\end{aligned}
\qquad
\begin{aligned}
&\frac{K_2 - K}{12} && = +\ 0{,}00002596\\
&\frac{m^2 - b^2}{60 a_0^2} K^2 && = +\ 0{,}00009923\\
& && \text{Sum} = +\ 0{,}00012519.
\end{aligned}$$

This multiplied by $\varrho'' \frac{F^*}{a_0^2}$, yields:

$A - A^* = \frac{s}{3} - 2{,}323''$ erroneous by $+\ 0{,}028$ $\quad B - B^* = C - C^* = \frac{s}{3} + 1{,}162''$ erroneous by $-\ 0{,}014$;

and the results here are just as exact as with the rigorous computation. We shall see later on that the formulae (6) p. 359 are still sufficient in fact for very large triangles, only if ε is computed according to a more accurate formula.

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Formula (11) p. 374 gives for the excess:

$$\varrho'' \frac{F}{a_0^2} K = \quad 9464{,}978''$$

$$\log\left(\frac{e^2 m^2}{2a_0^2}\sin^2\beta_1\right) = 5{,}38236 - 10 \text{ and } -\varrho'' \frac{F}{a_0^2}\cdot\frac{e^2 m^2}{2a_0^2}\sin^2\beta_1 = - \quad 0{,}227$$

$$+\varrho'' \frac{F}{a_0^2}\cdot\frac{b^2}{a_0^2}\cos^2 A_b \cdot \frac{e^2}{3}\cos^2\beta_1 = [0{,}13768] \quad = + \quad 1{,}373$$

$$+\varrho'' \frac{F}{a_0^2}\cdot\frac{3m^2}{a_0^2} b\cos A_b \cdot \frac{e^2}{6}\sin 2\beta_1 = [0{,}9645 - 10] \quad = + \quad 0{,}442$$

$$\varepsilon = \quad 9466{,}566''$$

erroneous by $- \quad 0{,}135.$

The introduction of the geographic latitude B_1 does not noticeably alter the result, whereas another value appears if the angles are interchanged cyclically in formula (11) i. e., everything that refers to angle A and its adjacent sides, refers to angle B or C and their adjacent sides. For example, we obtain for angle C, since, except for the main term, all of the other terms but one disappear:

$$\varrho'' \frac{F}{a_0^2} K = 9464{,}978''$$

$$\varrho'' \frac{F}{a_0^2}\cdot\frac{b^2\cos^2 212^\circ + a^2\cos^2 270^\circ - ab\cos 212^\circ\cos 270^\circ}{a_0^2}\cdot\frac{e^2}{3} = \quad 1{,}485$$

$$\varepsilon = 9466{,}463''$$

erroneous by $- 0{,}032.$

This value results also if we start from angle B.

The arithmetic mean of the three values of ε with angle A, B

and C as origin leads to the value:

$$\varepsilon = 9466{,}497''$$
erroneous by $-0{,}066$.

The mean of this last value and of the value from formula (1) p. 371 finally give:

$\varepsilon = 9466{,}425''$	respectively	$429''$
erroneous by $+0{,}006$		$+0{,}002$

according to whether one computes with reduced or geographic latitude.

According to Hansen's remarks in the treatise Reflexionen etc. pp. 142 and 143, we must assume that the mean just formed is especially exact in all cases,

since presumably it includes the numerically largest of the higher terms.

Formula (13) p. 375 gives:

for $A - A^*$:

$$\begin{cases} \dfrac{K_1 - K}{12} = -0{,}00005191 \\ +\dfrac{m^2 - a^2}{60 a_0^2}(1 + 2e^2 - 6e^2 \sin^2\beta_1) = - \quad 19803 \\ +\dfrac{20(m^4 - a^4) - 3m^2(m^2 - a^2)}{30240 a_0^4} = - \quad 5 \\ +\dfrac{b^2 \cos^2 \mathfrak{A}_0}{45 a_0^2} e^2 \cos^2\beta_1 = + \quad 973 \\ -\dfrac{19a^2 + 14b^2}{360 a_0^2} b \cos \mathfrak{A}_0 e^2 \sin^2\beta_1 = - \quad 207 \end{cases}$$

Sum $= -0{,}00024233$

Par. 12. Numerical Example of Hansen's and Weingarten's Formulae.

for $B - B^*$ and $C - C^*$:

$$\begin{cases} \frac{K_2 - K}{12} = + 0{,}00002596 \\ + \frac{m^2 - b^2}{60 a_0^2}(1 + 2e^2) = + \quad 9928 \\ + \frac{20(n^4 - b^4) - 3m^2(m^2 - b^2)}{30240 a_0^4} = + \quad 2 \\ - \frac{b^2 \cos^2 212^\circ}{90 a_0^2} e^2 = - \quad 526 \\ \text{last term} = \quad 0 \end{cases}$$

$$\text{Sum} = + 0{,}00012000.$$

These coefficients multiplied by $\varrho'' \frac{F}{a_0^2}$ give:

$$A - A^* = \frac{\varepsilon}{3} - 2{,}279'' \qquad B - B^* = C - C^* = \frac{\varepsilon}{3} + 1{,}129''.$$

Due to the influence of higher terms the sum of

$$- 2{,}279 + 1{,}129 + 1{,}129$$

is not zero, but -0,021. If we correct these figures by ±0,007, then

$$A - A^* = \frac{\varepsilon}{3} - 2{,}272'' \qquad B - B^* = C - C^* = \frac{\varepsilon}{3} + 1{,}136''$$

erroneous by $- 0{,}023$ erroneous by $+ 0{,}012$.

If we take $\frac{\varepsilon}{3} = 3155{,}499''$, corresponding to the mean value of the three computations according to formula (11) p. 374, then we obtain:

$$A - A^* = 3153{,}227'' \qquad B - B^* = C - C^* = 3156{,}635''$$

erroneous by $- 0{,}045$ erroneous by $- 0{,}010$.

If we combine the mean values of the results from (3) and (13):

$$A - A^* = \frac{\varepsilon}{3} - 2{,}271'' \qquad B - B^* = C - C^* = \frac{\varepsilon}{3} + 1{,}135''$$

with the mean value of the results from (1) and (11), i. e. with $\varepsilon = 9466{,}425''$, then:

$$A - A^* = 3153{,}204'' \qquad B - B^* = C - C^* = 3156{,}610''$$

erroneous by $- 0{,}022$ erroneous by $+ 0{,}015$.

Par. 13. Maximum Magnitudes of the Sixth- and Seventh-Order Terms. The formulae of Par. 10 answer the question of how long it is permissible to disregard the sixth- and seventh-order terms in computing the reductions $A - A^*$ etc., using the formulae (6), p. 359.

If we next turn to the excess ε, then the following formulae are considered. On one hand, according to p. 359 (6):

$$\underset{\text{in seconds}}{\varepsilon = \varrho'' \frac{F^*}{a_0^2}} \left(K + \frac{m^2}{8a_0^2} K^2 + Gl_4\right). \tag{1}$$

On the other, according to p. 374 (11) and p. 367 (15) and (16), where we replace the logarithmic formula in the latter case by the direct formula according to p. 93 (2):

$$\underset{\text{in seconds}}{\varepsilon = \varrho'' \frac{F}{a_0^2}} \left\{ K - \left[\frac{3m^2}{a_0^2}\sin^2\beta_1 - 2\frac{b^2\cos^2\mathfrak{A}_b + c^2\cos^2\mathfrak{A}_c - bc\cos\mathfrak{A}_b\cos\mathfrak{A}_c}{a_0^2}\cos^2\beta_1\right]\frac{e^2}{6} \right. \tag{2}$$
$$\left. + \left[\frac{a^2+3b^2-c^2}{a_0^2} b\cos\mathfrak{A}_b + \frac{a^2-b^2+3c^2}{a_0^2} c\cos\mathfrak{A}_c\right]\frac{e^2}{12}\sin^2\beta_1 + Gl_6 \right\}$$

$$\varrho'' \frac{F'}{a_0^2} = \varrho'' \frac{F^*}{a_0^2} \left\{ 1 + \frac{m^2}{8a_0^2} K + \frac{3m^4+n^4}{240a_0^4} K^2 \right. \tag{3}$$
$$\left. + \frac{(m^2-a^2)(K_1-K) + (m^2-b^2)(K_2-K) + (m^2-c^2)(K_3-K)}{120a_0^2} + Gl_6 \right\}$$

The formula (2) becomes (1), if the third and fourth term of the brackets in (3) and complete terms in (2) up to the first i. e. K, are neglected.

The third term in (3):

$$\varrho'' \frac{F^*}{a_0^2} \cdot \frac{3m^4+n^4}{240a_0^4} K^2 \tag{4}$$

has its maximum value in the equilateral triangle. As already computed on p. 95, we have for $m = n = 0{,}1a_0$:

$$0{,}0015''. \tag{5}$$

We can therefore without hesitation omit the third term of formula (3) for triangles with $m \leqq 0{,}1a_0$.

This is valid in still higher degree of the fourth term of this formula:

$$\varrho'' \frac{F^*}{a_0^2} \cdot \frac{(m^2-a^2)(K_1-K)+(m^2-b^2)(K_2-K)+(m^2-c^2)(K_3-K)}{120a_0^2} \tag{6}$$

We examine its maximum for constant m, remembering the conditions (3) and (6) p. 360, l. e.:

$$\left.\begin{aligned} F^{*2} &= \frac{1}{16}\left(6a^2m^2-3a^4-(b^2-c^2)^2\right) \\ 3m^2 &= a^2+b^2+c^2 \\ a^2 &= b^2+c^2-2bc\cos(\mathfrak{A}_b-\mathfrak{A}_c)+a_0^2Gl_4 \end{aligned}\right\} \tag{7}$$

Here, according to p. 360 (2) and p. 355 (9) conditional equations occur for $K-K_1$ etc.:

$$\left.\begin{aligned} K-K_1 &= \frac{2}{3}e^2\sin 2\beta\,\frac{b\cos\mathfrak{A}_b+c\cos\mathfrak{A}_c}{a_0}+Gl_4 \\ K-K_2 &= \frac{2}{3}e^2\sin 2\beta\,\frac{b\cos\mathfrak{A}_b-2c\cos\mathfrak{A}_c}{a_0}+Gl_4 \\ K-K_3 &= \frac{2}{3}e^2\sin 2\beta\,\frac{-2b\cos\mathfrak{A}_b+c\cos\mathfrak{A}_c}{a_0}+Gl_4 \end{aligned}\right\} \tag{8}$$

By β a mean reduced latitude of the three angles can be understood.

We now notice immediately that term (6) to be examined is symmetrical to $b\cos\mathfrak{A}_b$ and $c\cos\mathfrak{A}_c$, $b\sin\mathfrak{A}_b$ and $\pm c\sin\mathfrak{A}_c$.

$$b=c \qquad \mathfrak{A}_c = 360^0-\mathfrak{A}_b. \tag{9}$$

therefore corresponds to the maximum. If we substitute this, then the expression (6), remembering that $b\cos\mathfrak{A}_b$ is the positive or negative height h_a of the triangle, becomes:

$$\pm\varrho''\frac{e^2\sin 2\beta}{180}\,\frac{(b^2-a^2)ah_a^2}{a_0^5}$$

or

$$\pm\varrho''\frac{e^2\sin 2\beta}{160}\,\frac{(2m^2-a^2)(m^2-a^2)a}{a_0^5}. \tag{10}$$

This has its maximum for $a^2 = \frac{9 \pm \sqrt{41}}{10} m^2$. To the smaller a

which is very nearly equal to $\frac{m}{2}$, belongs the greater maximum; for $\beta = 45^0$ and $m = 0{,}1 a_0$: it amounts to:

$$\pm 0{,}00006''. \tag{11}$$

The second term of formula (2) is approximately equal to

$$-\varrho'' \frac{F^*}{a_0^2} \cdot \frac{m^2}{2 a_0^2} e^2 \sin^2 \beta_1. \tag{12}$$

The maximum is obtained for the equilateral triangle and with $\beta_1 = 90^0$ for $m = 0{,}1 a_0$: , amounts to:

$$-0{,}030''. \tag{13}$$

The third term of formula (2) is approximately equal to

$$+\varrho'' \frac{F^*}{a_0^2} \frac{b^2 \cos^2 \mathfrak{A}_b + c^2 \cos^2 \mathfrak{A}_c - bc \cos \mathfrak{A}_b \cos \mathfrak{A}_c}{3 a_0^2} e^2 \cos^2 \beta_1. \tag{14}$$

If it becomes a maximum with constant value m , then the conditions (7) to which attention should be given are that $b = c$ and either $\mathfrak{A}_c$ must equal $360^0 - \mathfrak{A}_b$ or $180^0 - \mathfrak{A}_b$. For terms (14) and (7) are symmetrical to $b \cos \mathfrak{A}_c$ and $\pm c \cos \mathfrak{A}_c$, $b \sin \mathfrak{A}_b$ and $\pm c \sin \mathfrak{A}_c$.

For $b = c$, $\mathfrak{A}_c = 360^0 - \mathfrak{A}_b$ the triangle lies symmetrically to the meridian of A, $b \cos \mathfrak{A}_b$, the height is again $\pm h_a$ and the term becomes equal to



Fig. 32

$$+\varrho'' \frac{a h_a^3}{6 a_0^4} e^2 \cos^2 \beta_1 \tag{15}$$

with $h_a = \sqrt{\frac{3(2m^2 - a^2)}{4}}$. The maximum occurs at $a^2 = \frac{1}{2} m^2$, $\tan \mathfrak{A}_b = \frac{1}{3}$ with $\mathfrak{A}_b = 18{,}4^0$ or $198{,}4^0$, cf. fig. 32. It amounts to:

$$+\varrho'' \frac{9 m^4}{64 a_0^4} e^2 \cos^2 \beta_1. \tag{16}$$

Of particular interest is its value for $\beta_1 = 0^0$ and $m = 0{,}1 a_0$:

$$+0{,}019''. \tag{17}$$

For $b = c$, $\mathfrak{A}_c = 180^0 - \mathfrak{A}_b$ the triangle lies to the east or west of the meridian A, symmetrical to its perpendicular. The term (14), since $b \cos \mathfrak{A}_b$ is now $\pm \frac{a}{2}$ here becomes equal to

$$+\varrho'' \frac{e^2 \cos^2 \beta_1}{16 a_0^4} a^2 \sqrt{6a^2 m^2 - 3a^4} \tag{18}$$

and a maximum for $a^2 = \frac{3}{2} m^2$. This amounts to as much as (16), and is true for $\beta_1 = 0^0$ and $m = 0{,}1 a_0$ as large as (17). There, however $\mathfrak{A}_c = 45^0$ or 225^0; cf. fig. 33.

The fourth term of formula (2) is approximately equal to

$$\varrho'' \frac{F^*}{a_0^2} \left(\frac{a^2 + 3b^2 - c^2}{a_0^3} b \cos \mathfrak{A}_b + \frac{a^2 - b^2 + 3c^2}{a_0^3} c \cos \mathfrak{A}_c \right) \frac{e^2}{12} \sin 2\beta_1. \tag{19}$$

Remembering the conditions (7) we easily see that with constant m, $b = c$ and $\mathfrak{A}_c = 360^0 - \mathfrak{A}_b$ is a requisite for a maximum, whereupon the expression (19) becomes

$$\varrho'' \frac{F^*}{a_0^2} \frac{m^2 b \cos \mathfrak{A}_b}{2 a_0^3} e^2 \sin 2\beta_1$$

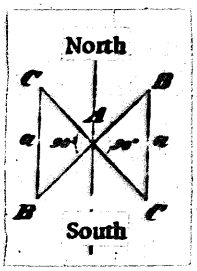


Fig. 33

Now $b \cos \mathfrak{A}_b$ is again $\pm h_a$, hence the expression (19) is also equal to

$$\pm \varrho'' \frac{m^2 a h_a^2}{4 a_0^5} e^2 \sin 2\beta_1. \tag{20}$$

Because h_a^2 equals $\frac{3}{4}(2m^2 - a^2)$ this becomes a maximum for $a = m\sqrt{\frac{2}{3}}$ where $h_a = m$. The maximum is given by

$$\mp \varrho'' \frac{m^5 \sqrt{6}}{12 a_0^5} e^2 \sin 2\beta_1. \tag{21}$$

which for $\beta_1 = 45^0$ and $m = 0{,}1 a_0$ amounts to:

$$\pm 0{,}003''. \tag{22}$$

If we now survey the maximum values of the particular terms, it is shown that for $m \leq 0{,}1a_0$ only two terms will be considered, namely, the second and third term in ε. In order to establish the combined effect of them we consider their sum for $\beta_1 = 45^0$. With respect to the above we easily recognize that two maxima occur.

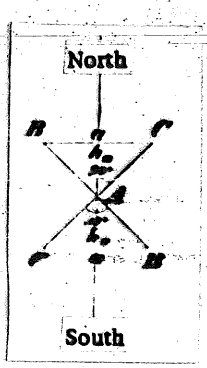


Fig. 34

We first obtain for $b = c$ and $\mathfrak{A}_c = 360^0 - \mathfrak{A}_b$ where the triangle lies symmetrical to the meridian of A, fig. 34:

$$-\varrho'' \frac{e^2}{24} \frac{a h_a}{a_0^4} (3m^2 - 2h_a^2); \qquad (23)$$

and secondly for $b = c$ and $\mathfrak{A}_c = 180^0 - \mathfrak{A}_b$, where the triangle lies symmetrical to the perpendicular of A, fig. 35:

$$-\varrho'' \frac{e^2}{32a_0^4} (2m^2 - a^2) \sqrt{3(2m^2 - a^2)a^2}. \qquad (24)$$

(23) becomes a maximum for $a = m\sqrt{\frac{3}{2}}$ and $\mathfrak{A}_b = 45$ or 225^0; whereas (24) does so for $a = \frac{m}{\sqrt{2}}$, $\tan \mathfrak{A}_c = 3$, $\mathfrak{A}_c = 71{,}6^0$ or $251{,}6^0$. Both are equal to

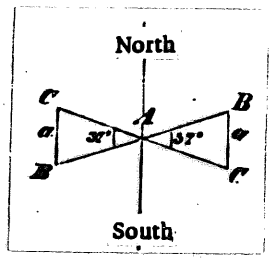


Fig. 35

$$-\varrho'' \frac{9e^2}{128} \frac{m^4}{a_0^4}; \qquad (25)$$

That is for $m = 0{,}1a_0$:

$$-0{,}010''. \qquad (26)$$

With reference to the two most considered terms of formula (2), we now have the following compilation which would be only insignificantly modified by retaining the small disregarded terms:

Par. 14. Continuation. Maxima of Sixth- and Seventh-Order Terms.

Maximum Error of Formula (1) in Comparison to (2)

for $m = 0{,}1a_0$.

$B_1 = 0°$	$+0{,}019''$	Fig. 32 and 33.
45	$-0{,}010$	Fig. 34 and 35.
90	$-0{,}030$	Arbitrarily lying equilateral triangle.

(27)

Par. 14. Continuation. Maxima of the Sixth- and Seventh-Order Terms in $A - A^*$, etc. In order to compare formula (6) p. 359 for $A - A^*$, i.e.

$$A - A^* = \frac{\varepsilon}{3} + \varrho'' \frac{F^*}{a_0^2}\left(\frac{m^2 - a^2}{60a_0^2}K^2 + \frac{K_1 - K}{12}\right) + Gl_6 \qquad (1)$$

in seconds in seconds

with the exact formula (13) p. 375, we introduce in this the area F^* for F by means of the relation (3) p. 384. It then becomes:

$$A - A^* = \frac{\varepsilon}{3} + \varrho'' \frac{F^*}{a_0^2}\left\{\begin{aligned}&\frac{m^2 - a^2}{60a_0^2}(1 + 2e^2 - 6e^2\sin^2\beta_1) + \frac{K_1 - K}{12}\left(1 + \frac{m^2}{8a_0^2}\right)\\ &+ \frac{(n^4 - a^4) + 3m^2(m^2 - a^2)}{1512a_0^4}\\ &- \frac{b^2\cos^2\mathfrak{A}_b + c^2\cos^2\mathfrak{A}_c - 4bc\cos\mathfrak{A}_b\cos\mathfrak{A}_c}{90a_0^2}e^2\cos^2\beta_1\\ &+ \left[\begin{aligned}&\frac{-19a^2 + b^2 + 13c^2}{a_0^2}b\cos\mathfrak{A}_b\\ &+ \frac{-19a^2 + 13b^2 + c^2}{a_0^2}c\cos\mathfrak{A}_c\end{aligned}\right]\frac{e^2}{720}\sin 2\beta_1 + Gl_6\end{aligned}\right\}.$$

in seconds in seconds

Here, according to p. 360 (2), we put:

$$\frac{K_1 - K}{12}\frac{m^2}{8a_0^2} = -\frac{(5b\cos\mathfrak{A}_b + 5c\cos\mathfrak{A}_c)m^2}{a_0^3}\frac{e^2}{720}\sin 2\beta_1 + \cdots$$

as well as:

$$\begin{aligned}1 + 2e^2 - 6e^2\sin^2\beta_1 &= K_1^2 - 2e^2\sin^2\beta_1 + \cdots\\ &= K^2 + (K_1^2 - K^2) - 2e^2\sin^2\beta_1 + \cdots\\ &= K^2 - \frac{4}{3}\frac{b\cos\mathfrak{A}_b + c\cos\mathfrak{A}_c}{a_0}e^2\sin 2\beta_1 - 2e^2\sin^2\beta_1 + \cdots\end{aligned}$$

With this we have:

$$A-A^*=\underset{\text{in seconds}}{\frac{\varepsilon}{3}}+\underset{\text{in seconds}}{\varrho''\frac{F^*}{a_0^2}}\left\{\begin{array}{l}\dfrac{m^2-a^2}{60a_0^2}K^2+\dfrac{K_1-K}{12}+\dfrac{(m^4-a^4)+3m^2(m^2-a^2)}{1512a_0^4}\\ \qquad-\dfrac{m^2-a^2}{30a_0^2}e^2\sin^2\beta_1\\ -\dfrac{b^2\cos^2\mathfrak{A}_b+c^2\cos^2\mathfrak{A}_c-4bc\cos\mathfrak{A}_b\cos\mathfrak{A}_c}{90a_0^2}e^2\cos^2\beta_1\\ -\left[\begin{array}{l}\dfrac{5a^2+3b^2-3c^2}{a_0^2}b\cos\mathfrak{A}_b\\ +\dfrac{5a^2-3b^2+3c^2}{a_0^2}c\cos\mathfrak{A}_c\end{array}\right]\dfrac{e^2}{360}\sin 2\beta_1+Gl_6.\end{array}\right\} \qquad (2)$$

The third term in the brackets of (2), as comparison with (4) p.384, shows, amounts to only a few ten thousandths of a second for $m=0{,}1a_0$.

The fourth term is a maximum for $a^2=m^2(1\pm\sqrt{0{,}5})$, because for this value of a^2, according to p. 96 (6) and (7), the product $F^*(m^2-a^2)$ becomes a maximum. For $m=0{,}1a_0$ in the maximum, it amounts to:

$$\pm 0{,}0010''\sin^2\beta_1. \qquad (3)$$

For the fifth term we immediately obtain with consideration of the previous paragraph, $\frac{1}{15}$ as the maximum of the expression (16) p. 386 i. e., with $m=0{,}1a_0$, the amount

$$\pm 0{,}0013''\cos^2\beta_1. \qquad (4)$$

The sixth term becomes a maximum for $b=c$ and $\mathfrak{A}_c=360^0-\mathfrak{A}_b$ with $a=\sqrt{\frac{6}{5}}m$, $b\cos\mathfrak{A}_b=\pm h_a=\pm\sqrt{\frac{3}{5}}m$. For $m=0{,}1a_0$ it becomes equal to

$$\pm 0{,}00015''\sin 2\beta_1. \qquad (5)$$

From the preceding it is evident that formula (1) is fully sufficient, as long as $m\leq 0{,}1a_0$. With regard to the results of the previous paragraph we can therefore say:

The formulae (6) p. 359 are generally sufficient for the calculation of geodetic triangles, as long as

$$m < 0{,}1 a_0 \quad \text{or} \quad 637^{km}. \tag{6}$$

The formulae (3) are valid there, just as long, since the corresponding formulae in the system (3) and in (6) differ from one another only in the reduction from spherical triangle to plane, for which reduction the formula applied on p. 359 is nevertheless sufficient according to p. 96.

Formulae (1) p. 362 too are just as long sufficient. The development itself shows that it matters considerably whether it is permissible to interchange A with A' or not. Except for this it deals only with the computation of the spherical excess from two sides and the included angle, and for this the formula is perfectly sufficient if applied according to p. 99. Now, as we easily infer from Par. 6, p. 362, the error in ε caused by interchanging A with A' is approximately equal to

$$-\varrho'' \frac{F^*}{a_0^2} \frac{[K_1 - K]\, bc \cos A}{24 a_0^2}. \tag{7}$$

If we put in here the expression (2) p. 360 for $K_1 - K$, then we have for the adequate approximation of (7):

$$\varrho'' \frac{e^2 \sin 2\beta_1}{36} \frac{F^* (b \cos \mathfrak{A}_b + c \cos \mathfrak{A}_c)\, bc \cos(\mathfrak{A}_b - \mathfrak{A}_c)}{a_0^5}.$$

The maximum of this expression can only exist for $b = c$, $\mathfrak{A}_c = 360^0 - \mathfrak{A}_b$. But if we introduce $h_a = \pm b \cos \mathfrak{A}_b = \sqrt{\frac{3}{4}(2m^2 - a^2)}$, then it becomes

$$\pm \varrho'' \frac{e^2 \sin 2\beta_1}{144} \frac{a h_a^2 (4h_a^2 - a^2)}{a_0^5}. \tag{8}$$

With constant m this becomes largest for $a^2 = \frac{21 - \sqrt{201}}{20} m^2$ and with $m = 0{,}1 a_0$ equal to

$$\pm 0{,}00032'' \sin 2\beta_1. \tag{9}$$

This amount is quite insignificant. The formulae (1) of Par. 6 p. 362 are thus sufficient in fact just as far as (6) of Par. 4 p 359; it being presumed that a^2 is computed sufficiently accurately. If we calculate this, as assumed in Par. 6 p. 362 (1), then according to p. 98 (4) it is affected approximately by an error:

$$-\frac{b^2c^2\sin^2 A}{3a_0^4} \text{ i. e. } -\frac{F^{*2}}{12a_0^4}. \tag{10}$$

We easily see that its influence on ε and $A - A^*$ etc., becomes a maximum for the equilateral triangle and of course is approximately

$$\left.\begin{aligned} &\text{in } \varepsilon \text{ equal to } -\varrho''\frac{F^{*2}}{96a_0^4} \\ &\text{in } A - A^* \text{ equal to } +\varrho''\frac{F^{*2}}{1080a_0^4} \end{aligned}\right\} \text{ with } F^* = \frac{1}{4}m^2\sqrt{3}. \tag{11}$$

These values are fully insignificant for $m = 0{,}1a_0$ for they amount only to:

$$-0{,}00017'', \quad +0{,}00002'', \text{ respectively.} \tag{12}$$

Par. 15. Numerical Example III. The following geodetic triangle formed from shortest lines corresponds to the chord-triangle with horizontal angles dealt with as an example on p. 200:

		Azimuth α.	Angle	Meters
Berlin	K	239° 33′ 0,6889″	95° 19′ 5,1218″	$\log KW = 5{,}8907956.989$
	W	334 52 5,8107		
Königsberg	W	23 11 41,5017	42 4 27,8633	$\log WB = 5{,}7182160.562$
	B	65 16 9,3650		
Vienna	B	157 10 7,2216	42 48 6,1935	$\log BK = 5{,}7242591.353$
	K	199 58 13,4151		
		$180^0 + \varepsilon = 180^0\ 11'\ 39{,}1786''$		

Par. 15. Numerical Example III.

	Geographic Latitude	Reduced Latitude
Berlin	$B_1 = 52^0\ 30'\ 16{,}7''$	$\beta_1 = 52^0\ 24'\ 43{,}01137''$
Königsberg	$B_2 = 54\ 42\ 50{,}6$	$\beta_2 = 54\ 37\ 24{,}75639$
Vienna	$B_3 = 48\ 12\ 35{,}5$	$\beta_3 = 48\ 6\ 52{,}27604$

The azimuths and distances are derived from the data of p. 200 in the same manner, as is done for the line BK alone on pp. 339, 182 and 341. For the process a control was yielded through this by direct treatment of the line BK p. 256 ff.

If the points B, K, W are marked with the subscripts 1, 2, 3, then for the reduction of the azimuths according to formula (5) p. 337:

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$$a_{1.3} - \alpha_{1.3} = [13{,}4990]\frac{\cos^2 B_1 \sin 2\alpha_{1.3}}{\varrho_{\alpha_1}\varrho_{m_3}} + [18{,}616]\frac{\sin 2B_1 \sin \alpha_{1.3}}{a_0 \varrho_\alpha \varrho_{m_3}},$$

$$a_{3.1} - \alpha_{3.1} = [13{,}4990]\frac{\cos^2 B_3 \sin 2\alpha_{3.1}}{\varrho_{\alpha_3}\varrho_{m_3}} + [18{,}616]\frac{\sin 2B_3 \sin \alpha_{3.1}}{a_0 \varrho_{\alpha_3}\varrho_{m_3}},$$

$$\log \varrho_{\alpha_1} = 6{,}8047 \quad \log \varrho_{\alpha_3} = 6{,}8044 \quad \log \varrho_{m_1} = 6{,}8056 \quad \log \varrho_{m_3}' = 6{,}8055$$

whence

$$a_{1.3} - \alpha_{1.3} = -\,0{,}2205 - 0{,}0065 = -\,0{,}2270''$$

$$a_{3.1} - \alpha_{3.1} = -\,0{,}2460 + 0{,}0061 = -\,0{,}2399\,.$$

Further, we have:

$$a_{2.3} - \alpha_{2.3} = [13{,}8442]\frac{\cos^2 B_2 \sin 2\alpha_{2.3}}{\varrho_{\alpha_2}\varrho_{m_3}} + [19{,}133]\frac{\sin 2B_2 \sin \alpha_{2.3}}{a_0 \varrho_{\alpha_2}\varrho_{m_3}},$$

$$a_{3.2} - \alpha_{3.2} = [13{,}8442]\frac{\cos^2 B_3 \sin 2\alpha_{3.2}}{\varrho_{\alpha_3}\varrho_{m_3}} + [19{,}133]\frac{\sin 2B_3 \sin \alpha_{3.2}}{a_0 \varrho_{\alpha_3}\varrho_{m_3}},$$

$$\log \varrho_{\alpha_2} = 6{,}8048 \quad \log \varrho_{\alpha_3} = 6{,}8043 \quad \log \varrho_{m_2} = 6{,}8056 \quad \log \varrho_{m_3} = 6{,}8055$$

whence:

$$\alpha_{2.3} - \alpha_{2.3} = + 0{,}4142 + 0{,}0194 = + 0{,}4336''$$
$$\alpha_{3.2} - \alpha_{3.2} = + 0{,}4891 - 0{,}0177 = + 0{,}4714\,.$$

As a trial, the following equations were computed:

$$\cos\beta_1 \sin\alpha_{1.3} = -\cos\beta_3 \sin\alpha_{3.1}$$
$$\cos\beta_2 \sin\alpha_{2.3} = -\cos\beta_3 \sin\alpha_{3.2}.$$

Resulting in:

$\log\cos\beta_1 = 9{,}7853155.518 - 10$	$\log\cos\beta_3 = 9{,}8245448.818 - 10$
$\log\sin\alpha_{1.3} = 9{,}6280829.617_n - 10$	$\log\sin\alpha_{3.1} = 9{,}5888536.313 - 10$
Sum $= 9{,}4133985.135_n - 10$	Sum $= 9{,}4133985.131 - 10$
$\log\cos\beta_2 = 9{,}7626381.918 - 10$	$\log\cos\beta_3 = 9{,}8245448.818 - 10$
$\log\sin\alpha_{2.3} = 9{,}5953413.007 - 10$	$\log\sin\alpha_{3.2} = 9{,}5334346.092_n - 10$
Sum $= 9{,}3579794.925 - 10$	Sum $= 9{,}3579794.910_n - 10.$

The differences indicate inaccuracies of the azimuths of some few units of the fourth decimal.

For the reduction of the chord BW, formula (11) p. 181 was used. $\log\varrho = 6{,}8045141.$ belongs to the mean azimuth 156°1'6" in the mean latitude 50°21'26". Thus for the shortest distance BW we have:

$$\log s = 5{,}7180944.366 + 1215.440 + .748$$
$$= 5{,}7182160.554.$$

If, on the other hand, we compute with the mean of $\log\varrho$ for 52°30'17" latitude, and 154°52'6" azimuth on the one hand, and 48°12'36" latitude and 157°10'7" azimuth on the other, we have after rigorous computation:

$$\left.\begin{matrix}6{,}8046724\\6{,}8043503\end{matrix}\right\}\text{Mean} = 6{,}8045113,$$

respectively. Then according to the formulae (10) and (11), p.181, we obtain for BW:

$$\begin{aligned} \log s = \quad & 5{,}7180944.366 \\ + \quad & 1215.456 \\ + \quad & 0.748 \quad \text{according to (11)} \\ - \quad & 0.008 \quad \text{according to (10)} \\ \log s = \quad & 5{,}7182160.562. \end{aligned}$$

For KW, we find for the mean geographic latitude 51°27'44" and to the mean azimuth 21°34'57" in rigorous calculation

$$\log \varrho = 6{,}8055536$$

and from this, according to formula (11) p. 181:

$$\begin{aligned} \log s &= 5{,}8905264.742 + 2688.558 + 3.662 \\ &= 5{,}8907956.962 \end{aligned}$$

More accurate is the following computation of $\log s$, in which the sixth-order terms are fully considered.

We have $\log \varrho$ for 48°12'36" latitude and 19°58'13" azimuth, as well as 54°42'51" latitude and 23°11'42" azimuth after rigorous computation respectively:

$$\left.\begin{matrix} 6{,}8043073 \\ 6{,}8047889 \end{matrix}\right\} \text{Mean} = 6{,}8045481.$$

With this there follows according to p. 181 (10) and (11), with addition of an easily-inferred term from the first formula (4) p. 30:

$$\left\{\begin{aligned} \log k &= 5{,}8905264.742 \\ + \tfrac{1}{6} M \left(\tfrac{k}{2\varrho}\right)^2 &= + \quad 2688.623 \\ + \tfrac{11}{180} M \left(\tfrac{k}{2\varrho}\right)^4 &= + \quad 3.662 \\ + \tfrac{191}{5670} M \left(\tfrac{k}{2\varrho}\right)^6 &= + \quad 0.007 \\ + M \tfrac{e^2}{40} \tfrac{k^4}{\varrho^4} (\cos^2 B \cos^2 a - \sin^2 B) &= - \quad 0.045 \end{aligned}\right.$$

For KW: $\log s = 5{,}8907956.989.$

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For the purpose of application of formulae (1) p. 362, we now have for the angles of the triangle:

$$\left.\begin{aligned} K_1 &= 0{,}9982777 \\ K_2 &= 0{,}9977849 \\ K_3 &= 0{,}9992626 \end{aligned}\right\} \quad \begin{aligned} K &= 0{,}9984417 \\ \log K &= 9{,}9993227-10 \end{aligned}$$

$$\log \frac{\varrho'' K}{2a_0^2} = 1{,}4034309-10 \qquad \log \frac{MK}{24a_0^2} = 4{,}6476-20$$

$\log a = 5{,}8907957$	$A = 95^0\,19'\ 5{,}12''$	$\log \sin A = 9{,}9981266-10$
$\log b = 5{,}7182161$	$B = 42\ \ 4\ 27{,}86$	$\log \sin B = 9{,}8261364-10$
$\log c = 5{,}7242591$	$C = 42\ 48\ \ 6{,}19$	$\log \sin C = 9{,}8321660-10$

$a^2 = 10^{10}.\ 60{,}477$	$4a^2 - 3m^2 = 126{,}026 \,.\, 10^{10} = [12{,}1005\ ]$
$b^2 = 10^{10}.\ 27{,}317$	$4b^2 - 3m^2 = -\,6{,}614 \,.\, 10^{10} = [10{,}8205_n]$
$c^2 = 10^{10}.\ 28{,}088$	$4c^2 - 3m^2 = -\,3{,}530 \,.\, 10^{10} = [10{,}5478_n]$
$3m^2 = 10^{10}.\ 115{,}882$	Trial; Sum $= 115{,}882 \,.\, 10^{10}$.

Thus we obtain from

A, b and c:	$\log \varepsilon = 2{,}8440327$	$+\ 5598$	$= 2{,}8445925$
B, a and c:	$2{,}8446221$	$-\ 294$	$= \quad 5927$
C, a and b:	$2{,}8446087$	$-\ 157$	$= \quad 5930$

$$\left.\begin{aligned} \varepsilon &= 699{,}1856'' \\ &\quad\ 1860 \\ &\quad\ 1864 \end{aligned}\right\} \quad \text{in the mean } 11'\ 39{,}1860''.$$

This value is somewhat too large by 0",0074.

Further, for $A - A^*$, etc., according to the formulae (1) p. 362:

(See Illustrative Material on the Next Page)

$$\log (K_1 - K) = 6{,}2151_n - 10 \quad \log (m^2 - a^2) = 11{,}3395_n$$
$$\log (K_2 - K) = 6{,}8174_n - 10 \quad \log (m^2 - b^2) = 11{,}0535$$
$$\log (K_3 - K) = 6{,}9143 - 10 \quad \log (m^2 - c^2) = 11{,}0229$$
$$\log \frac{\varepsilon}{12K} = 1{,}7661 - 10 \qquad \log \frac{\varepsilon K}{60 a_n^2} = 7{,}4565 - 20$$

$$A - A^* = \frac{\varepsilon}{3} - 0{,}0096'' - 0{,}0625'' = \frac{\varepsilon}{3} - 0{,}0721''$$

$$B - B^* = \frac{\varepsilon}{3} - 0{,}0383 + 0{,}0324 = \frac{\varepsilon}{3} - 0{,}0059$$

$$C - C^* = \frac{\varepsilon}{3} + 0{,}0479 + 0{,}0301 = \frac{\varepsilon}{3} + 0{,}0780.$$

With $\varepsilon = 699{,}1786$ we have the values:

$$\begin{cases} A^* = 95^0\ 15'\ 12{,}1343'' \\ B^* = 42\ \ 0\ 34{,}8097 \\ C^* = 42\ 44\ 13{,}0560 \end{cases}$$
$$\text{Sum} = 180\ 00\ 00{,}0000.$$

The sines of these angles must behave as the sides a, b, c. In fact, we have:

$\log a = 5{,}8907956.989$	$\log b = 5{,}7182160.562$	$\log c = 5{,}7242591.353$
$\log \sin A^* = 9{,}9981719.226$	$\log \sin B^* = 9{,}8255922.810$	$\log \sin C^* = 9{,}8316353.602$
$5{,}8926237.763$	$5{,}8926237.752$	$5{,}8926237.751$

The conformity is sufficient.

We use previous results finally in a rigorous computation of the excess according to Weingarten's formula (1) p. 371.

We have first of all by means of the angle of the plane triangle $\log F^* = 11{,}4406471.1$ and from this by means of formula (16) p. 367:

$$\log F = 11{,}4406471.1 + 5147.7 + 3.5 = 11{,}4411622.$$

Now we have:

$$\varrho'' \frac{F}{a_0^2} K = [2{,}8445931] = 699{,}1866''.$$

Further:

$$\sin B_1 = 0{,}7934 \quad \sin B_2 = 0{,}8163 \quad \sin B_3 = 0{,}7456$$

and thus:

$$\varrho'' \frac{F}{a_0^2} \frac{e^2}{6} \left\{ \begin{matrix} (\sin B_1 - \sin B_2)^2 + (\sin B_2 - \sin B_3)^2 \\ + (\sin B_3 - \sin B_1)^2 \end{matrix} \right\} = +0{,}0061''$$

$$- \varrho'' \frac{F}{a_0^2} \frac{e^2}{12} \left[\frac{b^2+c^2}{a_0^2} \sin^2 B_1 + \frac{a^2+c^2}{a_0^2} \sin^2 B_2 + \frac{a^2+b^2}{a_0^2} \sin^2 B_3 \right] = -0{,}0136.$$

This gives together -0,0075". More exactly we therefore have:

$$\varepsilon = 699{,}1791'',$$

which value agrees to within 0,0005" of that which has been derived from the horizontal angles.

The higher terms, by which formulae pp. 371 ff.

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differ from those of p. 362, have an influence on $A - A^*$ etc., of only 0,0001" in average.

Par. 16. Bessel's Formulae for the Reduction of a Geodetic Triangle. Bessel in the year 1822 has already stated in the Astronomical News volume 1 No. 6 p. 85 (Abhandlungen (Treatises) volume 3, p. 3) a formula for the reduction of the angle of a geodetic triangle to a spherical triangle with the same sides, without, however, showing its development. The formula contains all terms that depend upon e^2 in full, but disregards the higher powers of e^2. It can be obtained by abbreviating the formulae of Par. 9 pp. 232 et seq. to terms with e^2, and then applying them to the three sides of a geodetic triangle.

Par. 16. Bessel's Formulae for Reduction of a Geodetic Triangle.

(The development is similar to that of Par. 4 pp. 329 et seq.; in the final results similar bracketed expressions also appear as in (10) p. 332.)

Hansen has carried through the development in his Geodätische Untersuchungen pp. 116-136 (cf. also p. 104) and finally gives an interesting abbreviation of the expressions for moderately large sides of a triangle. Hence we have for the reduction of the angles A, B, C in the geodetic triangle to the angles A'', B'', C'' of the spherical triangle with the same sides, the formulae:

$$\left.\begin{aligned} A - A'' &= \frac{1}{12} e^2 \frac{F}{a_0^2} (2\cos 2\beta_1 + \cos 2\beta_2 + \cos 2\beta_3) + Gl_6 \\ B - B'' &= \frac{1}{12} e^2 \frac{F}{a_0^2} (\cos 2\beta_1 + 2\cos 2\beta_2 + \cos 2\beta_3) + Gl_6 \\ C - C'' &= \frac{1}{12} e^2 \frac{F}{a_0^2} (\cos 2\beta_1 + \cos 2\beta_2 + 2\cos 2\beta_3) + Gl_6. \end{aligned}\right\} \quad (1)$$

Where the equatorial radius a_0 is presumed as the radius of the sphere. If we consider that, as well as the formulae:

$$\left.\begin{aligned} K_1 &= 1 + e^2 \cos 2\beta_1 + Gl_4. \\ K_2 &= 1 + e^2 \cos 2\beta_2 + Gl_4 \\ K_3 &= 1 + e^2 \cos 2\beta_3 + Gl_4, \end{aligned}\right\} \quad (2)$$

then we can easily arrive at the formulae (3) p. 358 from formulae (1), as follows.

Finally, $a_0 : \sqrt{K}$ is assumed as the spherical radius, and K as the arithmetic mean of K_1, K_2 and K_3. But if we first proceed from the spherical triangle on to the sphere with radius a_0, to the plane triangle and then from there to the sphere of radius $a_0 : \sqrt{K}$, then considering the formulae p. 92 we have:

$$A'' - A' = \frac{F^e}{3a_0^2}(1 - K) + Gl_6$$
$$= -\frac{1}{9}e^2\frac{F}{a_0^2}(\cos 2\beta_1 + \cos 2\beta_2 + \cos 2\beta_3) + Gl_6 .$$

If we add this equation to the first equation (1), then there follows an expression for $A - A'$, from which by analogy we will also have $B - B'$ and $C - C'$. The result is therefore:

$$\left.\begin{aligned} A - A' &= \frac{1}{12}e^2\frac{F}{a_0^2}\left(\cos 2\beta_1 - \frac{\cos 2\beta_1 + \cos 2\beta_2 + \cos 2\beta_3}{3}\right) + Gl_6 \\ B - B' &= \frac{1}{12}e^2\frac{F}{a_0^2}\left(\cos 2\beta_2 - \frac{\cos 2\beta_1 + \cos 2\beta_2 + \cos 2\beta_3}{3}\right) + Gl_6 \\ C - C' &= \frac{1}{12}e^2\frac{F}{a_0^2}\left(\cos 2\beta_3 - \frac{\cos 2\beta_1 + \cos 2\beta_2 + \cos 2\beta_3}{3}\right) + Gl_6 , \end{aligned}\right\} \qquad (3)$$

which formulae system now apparently agrees with (3) on p. 358.

It cannot be ignored that the development indicated previously has the advantage over that given on p. 348 ff. in that it requires fewer preparations. It will become much more simplified (in comparison with Hansen, ibid) when from the beginning only moderately large values of triangle sides are assumed. Conversely, it seems to be very difficult to also include in this development higher powers of e^2, which is by no means the case in the other.

Hansen on pp. 210-218 of his Geodätische Untersuchungen uses the general Bessel formulae for approximating the solution of many problems that relate to a specific geodetic line by three parts which can thereby be of considerable length. In particular, he considers the case of given latitudes of the end points and given linear length of the line. This rarely appearing problem can naturally also be solved by successive approximations by means of the formula given here in a similar way as the problem of defining a geodetic line from the

geographic position of its end-points. As remarked, Hansen's formulae are also rigid and possibly require likewise a further approximation computation.

Par. 17. Andrae's Developments. Triangles from Vertical Sections etc.

On p. 94 the manner how Andrae proceeds in the first volume of the Danische Gradmessung in the development of Legendre's theorem was already stated. For the ellipsoid he next connects, pp. 187-200, his observations to the vertical sections (to be made of equal length with the geodetic line) and to the horizontal angle lying between them, and shows in the way already indicated (with Zachariae, ibid, pp. 129-132)

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what reductions must be applied to the horizontal angles to be able to consider them as angles of a plane triangle, the sides of which correspond to the lengths of the vertical sections (and thus of the geodetic lines) between the endpoints.

In order to obtain those formulae of particular interest we here reverse the procedure, for we already have the reduction formulae for the geodetic triangle. We need only to introduce the horizontal angle (i.e., the angle between the vertical sections) instead of the angles between the geodetic lines, and we then have the formulae for the figure formed from the vertical sections, because there is no difference in the linear lengths of the vertical sections and of the geodetic lines as long as we confine ourselves to moderately large distances.

If we designate the horizontal angles by A_0, B_0 and C_0, but the angles between the geodetic lines by A, B and C, and confine ourselves to the largest terms in the differences $A - A_0$, etc., then with regard to formula (11) p. 332, when $A = a_{1.3} - a_{1.2}$, $A_0 = a_{1.3} - a_{1.2}$:

$$\left.\begin{aligned} A - A_0 &= \underset{\text{in seconds}}{\varrho''} \frac{e^2 \cos^2\beta_1}{12 a_0^2} (c^2 \sin 2a_{1.2} - b^2 \sin 2a_{1.3}) + Gl_5 \\ B - B_0 &= \underset{\text{in seconds}}{\varrho''} \frac{e^2 \cos^2\beta_2}{12 a_0^2} (a^2 \sin 2a_{2.3} - c^2 \sin 2a_{2.1}) + Gl_5 \\ C - C_0 &= \underset{\text{in seconds}}{\varrho''} \frac{e^2 \cos^2\beta_3}{12 a_0^2} (b^2 \sin 2a_{3.1} - a^2 \sin 2a_{3.2}) + Gl_5. \end{aligned}\right\} \quad (1)$$

If we combine this with formulae (6) p. 359, continuing only to fifth-order terms, we have:

$$\left.\begin{aligned} A_0 - A^* &= \underset{\text{in seconds}}{\frac{\varepsilon}{3}} + \underset{\text{in seconds}}{\varrho''} \frac{F^* K^2 (m^2 - a^2)}{60 a_0^4} \\ &\quad + \varrho'' \frac{e^2 \cos^2\beta_1}{12 a_0^2} (b^2 \sin 2a_{1.3} - c^2 \sin 2a_{1.2}) + Gl_5 \\ B_0 - B^* &= \underset{\text{in seconds}}{\frac{\varepsilon}{3}} + \underset{\text{in seconds}}{\varrho''} \frac{F^* K^2 (m^2 - b^2)}{60 a_0^4} \\ &\quad + \varrho'' \frac{e^2 \cos^2\beta_2}{12 a_0^2} (c^2 \sin 2a_{2.1} - a^2 \sin 2a_{2.3}) + Gl_5 \\ C_0 - C^* &= \underset{\text{in seconds}}{\frac{\varepsilon}{3}} + \underset{\text{in seconds}}{\varrho''} \frac{F^* K^2 (m^2 - c^2)}{60 a_0^4} \\ &\quad + \varrho'' \frac{e^2 \cos^2\beta_3}{12 a_0^2} (a^2 \sin 2a_{3.2} - b^2 \sin 2a_{3.1}) + Gl_5. \end{aligned}\right\} \quad (2)$$

From this formula we immediately see that in all cases where any such distances come into consideration for which

geodetic and astronomical azimuths are noticeably different from each other, it is always simpler to compute with the geodetic line than with the vertical section.

Par. 17. Andrae's Developments.

According to Andrae's argument, the advantage of the introduction of the geodetic line is that the calculation of the triangle becomes the simplest possible. In fact, comparison of formulae (2) with (6) p. 359 shows that the geodetic line gives the most simple reduction formula of all connecting lines that can possibly be considered as of equal length (and falling together on the sphere). The computation is just as easy, if we consider only terms up to the fourth-order inclusive, for the individual geodetic triangle on the ellipsoid as well as on the sphere. It is only somewhat complicated through the inclusion of fifth-order terms, but still in only a limited amount.

In so far as measurable triangles are now concerned, a preference for the geodetic connecting line is, to be sure, not necessarily present. Since the observed azimuths are astronomical azimuths, and the horizontal angles are differences of the azimuths, it is evident that the latter at the application of the geodetic line must first be reduced to geodetic azimuths. Here the terms present in the formulae (1) again become valid. The same are, however, merely insignificant quantities that are often omitted as quite insignificant against the observation errors. But if these corrections are considered, then the above argument shows that they are not to be avoided by the choice of any other connecting line. Further correction terms must enter, even if these lines do not just enclose again the horizontal angles A_0, B_0, C_0 with one another, into the right sides of the formulae (2).

The advantage of the geodetic line for the computation of the triangle becomes more evident on consideration of the computation of polar coordinates. Here, once the observed directions are corrected for all derived computations, we have only to deal with geodetic azimuths. Thus the (relatively) simple formulae (6), p. 359 enter

in very favorable contrast to the above formulae (2) or still more complicated formulae.

The computation with the geodetic line proves itself much superior to the one with the chord (cf. p. 197 and 211); and this applies also to measurable triangles because the chord computation has fourth-order terms that are dependent upon e^2 three times as large as the computation with the geodetic line, (cf. formulae (2) with those (1) p. 197).

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From different auxiliary lines taken into consideration, the line of flux particularly has been examined by Andrae, ibid, volume 1, p. 179 and by Bremiker, ibid, p. 59 ff. The line of flux is characterized by the fact that the horizontal angle in each point amounts to 180° from the end points, i. e., that the normal of the area is in a plane through this point and both the end points. It therefore joins in P_1 the vertical section after P_2 and in P_2 it joins P_1 tangentially. If Bremiker p. 64 finds that that plane is the general tangential plane of the curve, then that is an error and depends only on an apparent proof, as the exact computation and the circumstance show that then the curve would be a plane curve and that its plane would have to coincide with the vertical planes P_1P_2 and P_2P_1. In both our articles Zeitschrift für Math. und Phys. (Journal of mathematic and physics) by Schlömilch 1870 Litt. Ztg. p. 29 and Astronom. Vierteljahrsschrift (Astronomical Quarterly) volume 13 p. 73, normal plane is simply to be read in place of tangential normal

plane (vertical plane). (The misprint is merely an oversight, as we had become immediately attentive to the apparent proof of Bremiker in an exhaustive investigation of the line of flux immediately after the appearance of the first named article. This investigation was not publicized, as the line of flux offers so little interest from the geodetic standpoint that other works soon made it forgotten.)

One may also compare the main work of the English Survey Ordnance Survey, Principal Triangulation p. 237.

Par. 18. Calculation of a Triangle Chain. Polar Coordinates. The observed azimuths and their differences, and the horizontal angles, are, because of the difference of the astronomical and geodetic azimuths according to formula (11) reduced to their first term p. 332:

$$\alpha - a \underset{\text{in seconds}}{=} -\frac{1}{12}\rho''\, e^2\, \frac{s^2}{a_0^2} \cos^2 B \sin 2a + \cdots \qquad (1)$$

to be corrected, where s is the distance and B signifies the geographical latitude of the station point, α the geodetic and a the astronomical azimuth. Moreover, formulae (6) p. 359 are to be applied.

Often not only the reduction of the azimuths and horizontal angles are omitted, but also the computation of the triangle sides and polar coordinates carried out purely spherically, taking a mean curvature of the whole trigonometrical net as a basis. Bessel too took the spherical radius in the computation of the East Prussian arc measurement equal to the equatorial radius of the ellipsoid and finally considered the results of the spherical computation as valid for the geodetic lines between the grid points.

The question of errors that are made here is a threefold one.

It concerns: 1. the influence of insufficient reduction of the measured azimuths and angles, 2. the influence of the curvature on the triangle sides computed according to the Sine Theorem and the adjusted angles, and 3. the influence of the spherical computation by averaging out the polar coordinates.

The reduction of the azimuths according to (1) amounts for $s = 0{,}01 a_0$ i. e., 64 km in the maximum, to about $\pm 0{,}01''$. Since s as a rule, only somewhat exceeds that amount in directly observed triangle series in exceptional cases, one can well disregard the reduction as being well within the observation errors.

Favorable to this process is the fact that, except for fifth-order terms, the surplus of the sum of the three horizontal angles over 180° gives the excess of the geodetic triangle, as the addition of the equations (1) p. 398 shows, whereupon

$$A_0 + B_0 + C_0 = A + B + C + Gl_0. \qquad (2)$$

The same system of improvements is therefore obtained by the adjustment of a single triangle series without reduction of the azimuths and horizontal angles as when the azimuths and horizontal angles are corrected.

But the disregarded reductions $\alpha - a$ remain, adhering to the individual directions purely as errors, and their influence on the computation of sides and transfer of azimuths as well as for a straight extended series of the form of fig. 13 p. 198 is easily seen.

Since $\alpha - a$ depends upon sin $2a$, angles with respectively opposite directions and equally long sides become equally erroneous, and therewith the influence of the omissions on the computations of the sides a', a'', a''' etc., (which are inserted in fig. 13 in place of

the chords a', a'', a''' etc.) vanishes the more, the more exactly triangles group in pairs into parallelograms (cf. the observation on p. 198, lower part). In return all sides of c and d are erroneously obtained by almost equal amounts and thus the total length is also somewhat erroneous. However, the error grows only in simple proportion to the length because as noticed no aggregate of errors enters into the sides a', a'' etc.

If we now consider that the extended angles on the long sides of the series of $a-a$ are not considerably influenced, then only a limited rotation of the whole series as a result of the error in the azimuth of the sides c and d, remains as a total influence, as well as a limited alteration of the total length of the series.

The former will not substantially exceed 0",01, and the latter,

is only noticeable in the eighth decimal-place of the logarithm with well-formed triangles which we easily find by formation of $\log \frac{c}{a}$ by means of the by some 0,01" erroneously assumed opposite angle.

We can still notice that those equations of the sides that correspond to a double connection of two of the sides a, a', a'' in pairs are barely influenced by disregarding the azimuth corrections. If, for example, a and a', fig. 13 p. 198 are connected by a rectangle with both diagonals and if we consider the ratio $\frac{a}{a'}$ formed with the help of both diagonals and the equation of the side so formed, then this and both values of the ratio $\frac{a}{a}$ are equally affected by disregarding the azimuth corrections. It is clear that any other way of formulating

the side equation in that rectangle will also be free of the disregarded corrections if the above is, because we can prepare any other form of the side equation from this and the triangle angle equation, which are also free from the disregarded corrections.

If according to the foregoing, the omission of the reductions of the measured astronomical azimuths and azimuthal differences under geodetically favorable circumstances gives only limited error, then it is not rightly seen, in view of the great ease in applying these reductions everywhere, why they are omitted--at times in the case where individual long diagonals appear, the influence of the omissions can without doubt substantially exceed the above amount.

One would perhaps give more attention to this reduction if it were noticeable in the excesses or if in the computation of geographic latitudes and longitudes by use of the results of the triangulation, discrepancies were to arise between the computed results, which are obtained in different ways through the trigonometric net.

This latter is but as little the case as the former.

While we take the astronomical azimuthal differences for geodetic, errors are made which can become noticeable similar to observation errors only in the formulation of the conditional equations (nevertheless they often do not become noticeable here, as shown above), but which nevertheless lead no longer to contradictions <u>after</u> resultant adjustment.

This behavior is materially different from the behavior with the computation with chords. If we disregard here the corresponding fourth-order terms, then we have an error of the mathematical

condition, i. e., it does not become correctly adjusted, and it must therefore show an appropriate small difference in the results for common points in the computation of geographic positions along different sides.

Ch. H. Kummell in 1877 in the Astronom. Nachr. volume 89, No. 2116, p. 49, subjected the usual computation of the triangle net according to the simple Legendre theorem to a criticism. He used for it the hypothesis that one should be able to compute any chord triangle of the ellipsoid like a chord triangle of a sphere, the radius of which corresponds to the mean curvature of the three angles. Our investigations show, however, the impermissibility of this hypothesis.

Par. 19. Continuation. Spherical Calculation of a Chain. If a common value $K_m = K + \delta K$ is applied in the whole net instead of the special curvature K for each individual triangle, then ε increases by $\delta\varepsilon$ according to the equation:

$$\delta\varepsilon = \varepsilon \frac{\delta K}{K}, \qquad (1)$$

as differentiation of the equation for ε p. 359 shows. If now K corresponds to the geographic latitude B and $K_m = K + \delta K$ to the mean latitude of the net $B_m = B + \delta B$, then because generally $K = (1 - e^2 \sin^2 B)^2 : (1 - e^2)$:

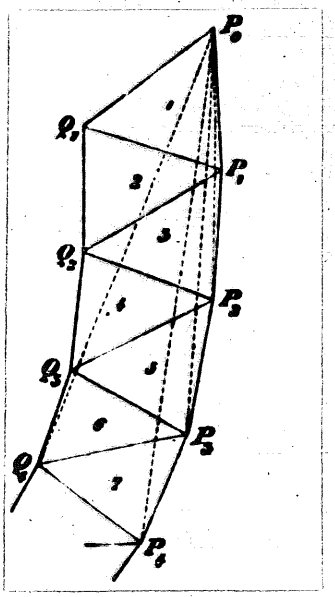


Figure 36

$$\frac{\delta K}{K_m} = -\frac{2e^2 \sin 2B_m}{1 - e^2 \sin^2 B_m} \delta B. \qquad (2)$$

Here δB is considered as small, as will be the case if K_m is a mean curvature of the net. We can now clearly give a sufficient approximation:

$$\delta\varepsilon = -\varepsilon \cdot 2e^2 \sin 2B_m \delta B + \cdots. \qquad (3)$$

For $\delta B = 0{,}05$ (taken as arc, as required by formula (3)) and for a mean latitude $B_m = 45^0$ as well as for $\varepsilon = 10''$, $\delta\varepsilon$ becomes equal to 0",007. That amount of ε belongs in the maximum to a mean side length $m = 0{,}01 a_0$ approximately, but $\delta B = 0{,}05$ appears first at the most extreme ends of a 637 km long chain extended along a meridian. We recognize from this that generally it is certainly permissible to compute with a mean curvature of the net.

If, however, $K_m = 1$, as with Bessel's computation of the East Prussian arc measurement, then $\delta K = K_m - K = 1 - K$ and we have in sufficient approximation

$$\delta\varepsilon = -\varepsilon \cdot e^2 \cos 2B + \cdots. \qquad (4)$$

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For B equal zero we have with $\varepsilon = 10''$ for $\delta\varepsilon$ the considerable value 0,07". Only by chance is $\delta\varepsilon$ far smaller for the East Prussian arc measurement, since here B is in the mean 55^0, thus cos $2B$ is equal to -0,34. $\delta\varepsilon$ is everywhere equal to $+\varepsilon : 450$, thus for $\varepsilon = 10''$, $+0{,}022''$.

The $\delta\varepsilon$ are now distributed in the equation as observation errors. The manner of the distribution is thus different according to the grouping of the triangles and the weights of the different observations. The matter is very clear with a simple triangle chain without diagonals.

the angles of which are observed with equal precision. Here $\delta\varepsilon$ is distributed equally on the three angles in each triangle through the routine course of computation and each angle increased by $\frac{\delta\varepsilon}{3}$. This has, however, no influence on the computation of the side, for A^*, B^*, C^* are obviously the same as for the right amount of the excess, because $\frac{1}{3}\delta\varepsilon$ is again subtracted at its formation, while we deduct $\frac{1}{3}$ of the excess from the adjusted angles.

The sides and, especially, the complete length P_0P_n of a directly extended triangle chain, fig. 36, are thus correctly obtained.

Not, however, the azimuths. The angles with P_1, P_2, P_3 etc., are too large by $\frac{1}{3}$ of the sum of the ε of the self adjoining triangles. For fig. 36 especially, the angles at P_1 are too large by $\frac{1}{3}(\delta\varepsilon_1 + \delta\varepsilon_2 + \delta\varepsilon_3)$ i. e., approximated as $\delta\varepsilon_2$. Similarly we have too much for P_2 approximated as $\delta\varepsilon_4$, etc. At the interchange of the azimuths from P_0P_1 to P_1P_2, P_2P_3 etc., there arise errors in the amount of $\delta\varepsilon_2$, $\delta\varepsilon_2 + \delta\varepsilon_4$, $\delta\varepsilon_2 + \delta\varepsilon_4 + \delta\varepsilon_6$, etc.

These amounts can become considerable; as soon as the $\delta\varepsilon$ are of one sign as with the East Prussian arc measurement, because the ε have been calculated with the equatorial radius a_0.

If in return a mean curvature of the net is chosen, then obviously not only the amount of the individual $\delta\varepsilon$ becomes smaller, but there appear the $\delta\varepsilon$ with different signs in the parts of the net situated north and south of the middle, whereby a too large aggregate is thus precluded.

If we put $P_0P_n = 0{,}1a_0$ and $n = 10$, then in case one (Bessel's method) the azimuth error of the side P_9P_{10} becomes equal to $9e^2\cos 2B_m.\varepsilon$ and since under the hypothesis of equilateral triangles $\varepsilon = 10$ seconds

it therefore amounts to about $0{,}6'' \cos 2B_m$, a value that is in general too large to be disregarded.

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In the second case (application of a mean curvature) error occurs only if the series does not have the east/west direction. If we take a meridian direction, then the azimuth error of P_9P_{10} also becomes equal to zero. That of the other sides is not zero, that is the azimuth error of the side P_4P_5 becomes a maximum. However, under the same hypothesis as before, it amounts to only $0{,}013'' \sin 2B_m$.

The errors for any directly extended triangle series of equal length should essentially be just as insignificant for the application of a mean curvature of the network, even if the distribution of the $\delta\varepsilon$ among the angles with dissimilar exactness of the same will be dissimilar, as well as for existence of side equations, and the influence of $\delta\varepsilon$ on the sides is not quite vanishing.

In return, the application of the value $K = 1$ (which corresponds to the latitude $B =$ approx. 45^0) must be designated as impermissible.

The question that may here be asnwered, is how does ε vary for an error in e^2 and an error in a_0. We easily find:

$$\delta\varepsilon = +\varepsilon \cdot \cos 2B\, \delta e^2 + \cdots \qquad (5)$$

$$\delta\varepsilon = -2\varepsilon \frac{\delta a_0}{a_0} \qquad (6)$$

and see from this that even such considerable values as $\delta e^2 = 0{,}0003$ and $\frac{\delta a_0}{a_0} = 0{,}0001$ influence only the thousandths of a second in ε.

Par. 20. Continuation. Spherical Calculation of Polar Coordinates. If in conjunction with fig. 36 polar coordinates with reference to P_0 as center are to be computed for P_1, P_2, P_3 etc., then triangles with

long sides come more and more into consideration and one will consider applying the strict formulae (1) p. 362. If, however, the chain is as extended as in fig. 36, so that only very small excesses arise for those triangles, then the simple Legendre theorem can again be applied in connection with the sphere corresponding to the mean curvature of the net. Only with triangle nets with large extensions in two dimensions are the rigorous formulae necessary, for here because of greater ε a substantial dissimilarity of the reductions $A - A^*$, $B - B^*$, $C - C^*$ arises in the triangles required for the calculation of the polar coordinates.

In the case of fig. 36, only the azimuths of the sides, and not their lengths are influenced by the spherical computation, as already proven.

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Due to this only the azimuths, but not the lengths of P_0P_1, P_0P_2, P_0P_3... become erroneous.

We obtain as the error of the azimuths at the point P_0, as observation of the figure shows, if for the extended angles in P_1, P_2 etc., the errors which are respectively equal to $\delta\varepsilon_2$, $\delta\varepsilon_4$ etc. are placed.

For P_0P_2: $\frac{1}{2}\delta\varepsilon_2$

„ P_0P_3: $\frac{2}{3}\delta\varepsilon_2 + \frac{1}{3}\delta\varepsilon_4$

„ P_0P_4: $\frac{3}{4}\delta\varepsilon_2 + \frac{2}{4}\delta\varepsilon_4 + \frac{1}{4}\delta\varepsilon_6$

. . . . . .

„ P_0P_{10}: $\frac{9}{10}\delta\varepsilon_2 + \frac{8}{10}\delta\varepsilon_4 + \cdots \frac{2}{10}\delta\varepsilon_{16} + \frac{1}{10}\delta\varepsilon_{18}$.

If we calculate now with $K = 1$, so that almost all the $\delta\varepsilon$ reach the same value (4) p. 403, then this amounts to $5\delta\varepsilon$ i. e., $5e^2 \cos 2B_m \cdot \varepsilon$ or about $0{,}3'' \cos 2B$. This azimuth error can become noticeable in the second decimal-place of the seconds of the geographic latitude and longitude transferred from P_0 to P_{10}.

But if we calculate with K_m and assume (as the most unfavorable case) a meridian extent of the chain, then in a manner of approximation we have:

$$\begin{array}{l|l} \delta\varepsilon_2 = -\delta\varepsilon_{18} = -0{,}005'' \sin 2B_m & \delta\varepsilon_4 = -\delta\varepsilon_{16} = -0{,}004'' \sin 2B_m \\ \delta\varepsilon_6 = -\delta\varepsilon_{14} = -0{,}003 \sin 2B_m & \delta\varepsilon_8 = -\delta\varepsilon_{12} = -0{,}001 \sin 2B_m \end{array}$$

and therefore the azimuth error for P_0P_{10} is only equal to $-0{,}008'' \sin 2B_m$, which will be regarded generally as trivial.

The error for P_0Q_{10}, becomes in general somewhat larger however, because in the calculation of the polar coordinates for Q_{10}, from P_{10} the still thin but not small triangle $P_0P_{10}Q_{10}$ shrinks. With consideration to what has been given above, we easily find by use of the formulae (1) p. 362, that here the computation according to Legendre's theorem, by means of K_m can add to the azimuth error computed above for P_0P_{10} for the transfer to P_0Q_{10} still 0,01 to 0,02", so that this spherical calculation is permissible just for thin, directly linear chains only.

Chapter 9.

Rectangular Geodetic Coordinates and Their Conversion from Spherical Coordinates.

Par. 1. Fundamental Theorem. Let us assume a rectangular coordinate system of geodetic lines in which the abscissae are counted on a geodesic and its ordinates will be normal to it, as indicated in Fig. 37.

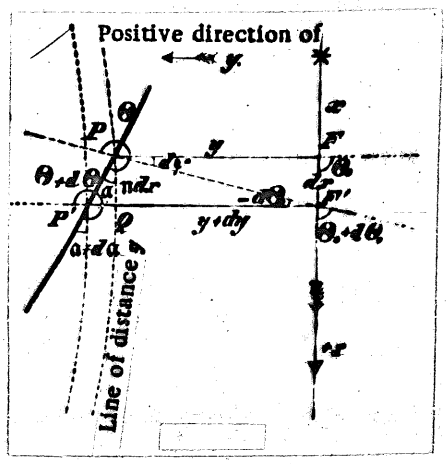


Figure 37.

When we move an ordinate y towards increasing values of x, its terminal point P will form a line of constant distance from the x-axis. If the initial point F is moved by only an infinitesimally small amount, which equals dx, P will be moved by the amount of $\mathfrak{n}dx$. In it, $\mathfrak{n}$ will be a function of the ordinate's length and location, and we have the equation

$$PQ = \mathfrak{n}dx. \tag{1}$$

To determine $\mathfrak{n}$, we shall first turn PF about P and get PF', which in turn is being moved about F' to its new location QF'. Since FF' is normal to PF, PF' must coincide with y and hence with QF', except for an infinitesimally small distance of the second order. From this, we may regard PQ to be normal to the ordinate QF'. (The deviation from the normal position is the same as in the case of the geodetic circle, cf. Par. 3, p. 271.)

Applying formula (4), p. 347 to Figures 28 and 37, and trans-

ferring the designations, we find

$$d\Theta_0 = -\frac{\sin\Theta_0}{m_y}\left(\frac{\partial m_y}{\partial y}\right)_{PF} dx_0$$

in which the index PF at $\frac{\partial m_y}{\partial y}$ means that y will be produced in the direction of PF. Since $\Theta_0 = 90^0$, it follows that:

$$-\frac{m_y d\Theta_0}{dx} = \left(\frac{\partial m_y}{\partial y}\right)_{PF}. \qquad (2)$$

The left-hand product $-m_y d\Theta_0$ equals PQ, which may be seen from Fig. 37 and from the fact

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that PQ does not deviate essentially from the geodetic circle which is described by the line QF' when revolving about F'. Now, since $-m_y d\Theta_0 = PQ$ and, by (1), is equal to $n dx$, we get from (2):

$$n = \left(\frac{\partial m_y}{\partial y}\right)_{PF}. \qquad (3)$$

This formula forms the foundation of the rectangular geodetic coordinates, together with the above-obtained theorem that all parallels to the x-axis, i.e. the lines of constant y, are normal to the ordinates. The differentiation required in (3) will be inconvenient for computing, however, and we shall try to re-write (3). First, instead of (3), we say:

$$n = \frac{dm_y}{dF}. \qquad (3^*)$$

in which the partial differentiation signs are replaced by ordinary differentiation signs since there will be no confusion about a change in the azimuth of y. In the denominator, we write dF instead of dy to indicate at the same time the index PF.

Par. 2. Differential Formula for the Direction Angle.

By (6), p. 275, we get, when using (3*) above:

$$\frac{d^2 m_y}{dP^2} + m_y \frac{K}{a_0^2} = 0. \qquad (4)$$

in which $K:a_0^2$ denotes the amount of curvature at point P. When we differentiate this equation with respect to y, which is implicitly lengthened beyond F, it follows that:

$$\frac{d^2 m_y}{dP\,dF} + \frac{dm_y}{dF} \frac{K}{a_0^2} = 0 \qquad (5)$$

Substituting $\mathfrak{n}$ from (3*) we have:

$$\frac{d^2 \mathfrak{n}}{dP^2} + \mathfrak{n} \frac{K}{a_0^2} = 0.$$

Now we shall again replace $\frac{d\mathfrak{n}}{dP}$ by $\frac{d\mathfrak{n}}{dy}$ in which y, in the sense of increasing ordinates, is implicitly lengthened beyond P, and we get:

$$\frac{d^2 \mathfrak{n}}{dy^2} + \mathfrak{n} \frac{K}{a_0^2} = 0. \qquad (6)$$

This formula corresponds to (6), p. 275, for m.

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Gauss proved, loc. cit. article 19, the validity of both differential equations in the form of another equation which may be applied even more generally. Formulae (3) and (6) are valid even if the abscissa-axis is not a geodesic but any line.

Par. 2. Differential Formula for the Direction Angle.

Though we shall not make use of this formula we are going to evolve it for the sake of completeness.

In Fig. 37, let us assume a geodesic PP' through P which together with the parallels to the x-axis, forms the direction angle

α in P. When this line is produced beyond the infinitesimally close point P', its direction angle in P' will be $\alpha + d\alpha$.

First of all, we refer this geodesic PP' to lines $F'P$ and FP' which we consider to be radii vectors. Formula (4), p. 347, indicates the amount by which the angle Θ will increase when we move from P to P'. Θ will be counted from the increasing radius vector to the increasing line PP'. When PP' is denoted by ds, we have:

$$d\Theta = -\frac{\sin\Theta}{m_y}\left(\frac{\partial m_y}{\partial y}\right)_{FP} ds.$$

Fig. 37 shows on one hand that PQ equals $-\sin\Theta ds$; on the other hand, it is equal to $-m_y d\Theta_0$. Therefore, we may also write

$$d\Theta = -\left(\frac{\partial m_y}{\partial y}\right)_{FP} d\Theta_0.$$

By (2), p. 407, we have further:

$$d\Theta_0 = -\left(\frac{\partial m_y}{\partial y}\right)_{FP} \frac{dx}{m_y}.$$

Substituting this value for $d\Theta_0$ in the preceding equation and considering further, from Fig. 37, the relation $dx = m_y d\psi$, we obtain:

$$d\Theta = \left(\frac{\partial m_y}{\partial y}\right)_{FP}\left(\frac{\partial m_y}{\partial y}\right)_{PF} d\psi. \qquad (1)$$

Again, the figure shows that:

$$\Theta + d\psi = 270^0 + \alpha \text{ and } \Theta + d\Theta = 270^0 + \alpha + d\alpha,$$

hence $d\Theta - d\psi = d\alpha$. Introducing (1) into this we have now:

$$d\alpha = \left(\left(\frac{\partial m_y}{\partial y}\right)_{FP}\left(\frac{\partial m_y}{\partial y}\right)_{PF} - 1\right) d\psi.$$

The relation $dx = m_y d\psi$ which was already mentioned gives an expression for $d\psi$. Substituting this in the preceding equation it follows that:

(See Illustrative Material on Next Page.)

Par. 2. Differential Formula for the Direction Angle.

$$d\alpha = \frac{\left(\frac{dm_y}{dy}\right)_{PF}\left(\frac{dm_y}{dy}\right)_{FP} - 1}{m_y}\, dx. \qquad (2)$$

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In this formula, the differential quotient for m_y has the ordinary differential sign because the opposition to changes of m_y and the azimuth does not take place.

To simplify (2) further, we remember equations (4) and (5) on page 408:

$$\frac{d^2 m_y}{dP^2} + m_y \frac{K}{a_0^2} = 0 \quad \text{and} \quad \frac{d^2 m_y}{dP^2 dF} + \frac{dm_y}{dF}\frac{K}{a_0^2} = 0.$$

Eliminating K from this and adding to both sides the term $\frac{dm_y}{dP}\frac{d^2 m_y}{dPdF}$ we obtain:

$$m_y \frac{d^3 m_y}{dP^2 dF} + \frac{dm_y}{dP}\frac{d^2 m_y}{dPdF} = \frac{dm_y}{dF}\frac{d^2 m_y}{dP^2} + \frac{dm_y}{dP}\frac{d^2 m_y}{dPdF}.$$

Now we integrate both members of the equation and get:

$$m_y \frac{d^2 m_y}{dPdF} = \frac{dm_y}{dF}\frac{dm_y}{dP} + c.$$

To determine the constant c, we observe that for $y = 0$, $m_y = 0$, and also $\frac{dm_y}{dF} = 1 = \frac{dm_y}{dP}$, which is obvious when considering the relations on the plane. The constant, then, will evidently be -1. Moreover, by (3), p. 408, we get $\frac{dm_y}{dF} = n$ and hence:

$$m_y \frac{dn}{dP} = \frac{dm_y}{dF}\frac{dm_y}{dP} - 1$$

or

$$\left(\frac{dm_y}{dy}\right)_{PF}\left(\frac{dm_y}{dy}\right)_{FP} - 1 = m_y \left(\frac{dn}{dy}\right)_P.$$

Introducing this into equation (2) and omitting the now superfluous index P at the right, we have:

$$d\mathfrak{x} = \frac{d\mathfrak{x}}{dy}\,dx. \qquad (3)$$

The above formula was presented by Gauss loc. cit. article 19. It is also valid if the abscissa-axis is not a geodesic.

We may also obtain the differential formulae for $\mathfrak{x}$ and $\mathfrak{y}$ by producing the ordinates y in F and F' to their intersection A and then considering AQF' as a position of APF created by revolution; PQ and FF' will then be elements of geodetic circles. The required formulae will be found by using the relations stated by Christoffel loc. cit. pp. 148, 149, for three points of a geodesic (Relation No. 3 will be especially useful).

Par. 3. Determination of $\mathfrak{x}$. For determining $\mathfrak{x}$, we are taking K for point P from (5), p. 276, in which K_1, $\alpha_{1.2}$, and β_1 refer to the initial point of the ordinate y which replaces s; we denote them better K_F, α_F, and β_F. When we let the x-axis coincide with the meridian we have:

$$\alpha_F = 90^0, \quad \cos\alpha_F = 0 \qquad (1)$$

by which several terms of K vanish. Expression (5) will now read:

$$K = K_F\left(1 + 2e^2\sin^2\beta_F\sin^2\frac{y}{a_0} + \cdots\right).$$

Supposing e and $y:a_0$ to be terms of the first order we get:

$$K = K_F + Gl_4. \qquad (2)$$

Substituting this expression for K in equation (6), p. 408, it

follows that:

$$\frac{d^2u}{dy^2} + u\frac{K_F}{a_0^2}(1 + Gl_3) = 0. \qquad (3)$$

Apart from the remainder Gl_4, this formula is the differential equation for u with respect to a sphere of the curvature $K_F : a_0^2$. We may easily verify that (3) is satisfied by:

$$u = \cos\left(\frac{y}{a_0}\sqrt{K_F}\right) + Gl_6. \qquad (4)$$

To introduce the curvature of a fixed point on the x-axis, we assume this point to be the origin of the coordinates, its K and β having the index zero. In formula (5), p. 276, we have now $\alpha_{1,2} = 0$, $\cos\alpha_{1,2} = 1$, $\beta_1 = \beta_0$, $K_1 = K_0$, $s = x$, from which we obtain:

$$K_F = K_0 + 2K_0e^2\sin 2\beta_0\frac{x}{a_0} + Gl_4. \qquad (5)$$

When we further expand $\cos\left(\frac{y}{a_0}\sqrt{K_F}\right)$ we get:

$$\cos\left(\frac{y}{a_0}\sqrt{K_F}\right) = \cos\left(\frac{y}{a_0}\sqrt{K_0}\right) - K_0e^2\sin 2\beta_0\frac{xy^2}{a_0^2} + Gl_6. \qquad (6)$$

This will also be the expression for u. To simplify further, we write:

$$\frac{1}{6}e^2\sin 2\beta_0 = E_0$$
$$\frac{x}{a_0}\sqrt{K_0} = \xi \qquad \frac{y}{a_0}\sqrt{K_0} = \eta; \qquad (7)$$

after adding the factor $\sqrt{K_0}$ to the second term of (6) which will result only in an error of a higher order, equation (4) will become by (6):

$$u = \cos\eta - 6E_0\xi\eta^2 + Gl_6. \qquad (8)$$

This formula is quite suitable for further application. According to (8) it may also be written as:

$$\frac{1}{\mathfrak{n}} = \sec\eta + 6E_0\xi\eta^2 + Gl_6. \qquad (9)$$

Par. 4. Course of Further Development. First of all, we formulate a differential equation for the shortest line between two points given by their coordinates, as illustrated in Fig. 38.

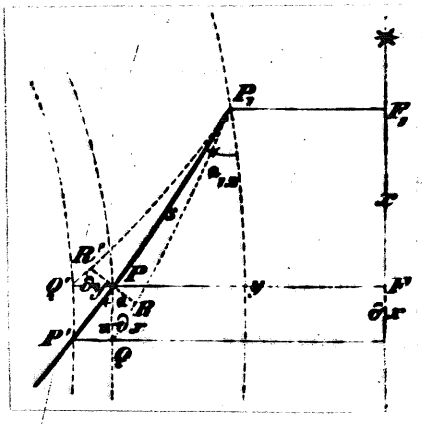


Figure 38.

This was first presented by Gauss in Disqu. c. sup. c., Article 19 (Gauss's works, Vol. 4, p. 251), in connection with a development of the theory of geodetic triangles.

Let us assume a geodesic commencing in P_1 at the direction angle $\mathfrak{a}_{1,2}$ and let its length be s to a point P. If the abscissa x, with y being constant, increases by ∂x, then line P_1P moves to the new position P_1Q. Drawing PR normal to P_1Q, we have:

$$QR = PQ\cos\mathfrak{a}.$$

However, QR is the partial increase of s, with y being constant and x variable. Hence $\partial s = \mathfrak{n}\partial x.\cos\mathfrak{a}$, and

$$\frac{\partial s}{\mathfrak{n}\partial x} = \cos\mathfrak{a}. \qquad (1)$$

If x remains constant and y increases by ∂y, we get an increase of $\partial s = QR$; since $QR = \partial y\sin\mathfrak{a}$, it follows that:

$$\frac{\partial s}{\partial y} = \sin\mathfrak{a}. \qquad (2)$$

Eliminating $\mathfrak{a}$ from (1) and (2) by the equation:

$$\cos^2\mathfrak{a} + \sin^2\mathfrak{a} = 1$$

results in the equation:

$$\left(\frac{\partial s}{\mathfrak{n}\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2=1.$$

We now multiply both sides by $4s^2$ and, in order to apply the convenient expression (8) for $\mathfrak{n}$, we replace x and y by ξ and η from relations (7) of the preceding pages. We state further:

$$\sigma=\frac{s}{a_0}\sqrt{K_0}, \tag{3}$$

and obtain finally:

$$\left(\frac{d(\sigma^2)}{\mathfrak{n}d\xi}\right)^2+\left(\frac{d(\sigma^2)}{d\eta}\right)^2=4\sigma^2. \tag{4}$$

This equation has to be integrated. When this is done by the method of undetermined coefficients, we shall obtain σ^2 as a power series. However if σ^2 is known as a function of ξ and η, equations (1) and (2) give, for determining $\mathfrak{n}$, the set:

$$\left.\begin{aligned}\sigma\cos\mathfrak{n}&=\frac{1}{2\mathfrak{n}}\frac{d(\sigma^2)}{d\xi}\\ \sigma\sin\mathfrak{n}&=\frac{1}{2}\frac{d(\sigma^2)}{d\eta}\end{aligned}\right\} \tag{5}$$

When exchanging the points in (5), we get formulae for the direction angle $\mathfrak{n}_{1,2}$ and from this we have the means to determine $\mathfrak{n}_{1,2}-\mathfrak{n}$.

We are now going to find σ^2 and $\mathfrak{n}$, and also $\mathfrak{n}_{1,2}-\mathfrak{n}$ by the above method. This will be quite convenient since all the expressions in question have already been evolved for the sphere and need only minor corrections.

Par. 5. Continuation. Determination of σ^2. In equation (4) of the preceding paragraph, let us write $\sec\eta$ for $1:\mathfrak{n}$. Then the

relation of σ^2 to the coordinates of points P_1 and P will be the one which is valid for a sphere having the radius $a_0 : \sqrt{K_0}$ which may be obtained from page 122. A more accurate value of 1:s requires a small addition to σ^2, which we call σ_0^2. If the correction is denoted V, we have:

$$\sigma^2 = \sigma_0^2 + V. \tag{1}$$

By this, the above-mentioned equation (4) changes to:

$$\left(\frac{\partial(\sigma_0^2)}{\partial\xi} + \frac{\partial V}{\partial\xi}\right)^2 (\sec\eta + 6E_0\xi\eta^2 + Gl_0)^2 + \left(\frac{\partial(\sigma_0^2)}{\partial\eta} + \frac{\partial V}{\partial\eta}\right)^2 = 4\sigma_0^2 + 4V. \tag{2}$$

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This will be simplified by the equality:

$$\left(\frac{\partial(\sigma_0^2)}{\partial\xi}\right)^2 \sec^2\eta + \left(\frac{\partial(\sigma_0^2)}{\partial\eta}\right)^2 = 4\sigma_0^2$$

and further by disregarding the second powers of the differential quotients of V in favor of its first powers. V itself comes only into existence by the fifth-order term $6E_0\eta^2\xi$, hence by (2), it will have the order of this term's product in $\left(\frac{\partial(\sigma_0^2)}{\partial\xi}\right)^2$, i.e. the seventh order. Then the differential quotients of V will be of the sixth order and their squares of the twelfth order. Since the differential quotients of σ_0^2 are of the first order, their products with the differential quotients of V will be terms of the seventh order.

If we omit all terms which exceed the seventh order, we get by (2):

$$\frac{\partial V}{\partial\xi}\frac{\partial(\sigma_0^2)}{\partial\xi} + \frac{\partial V}{\partial\eta}\frac{\partial(\sigma_0^2)}{\partial\eta} + 6E_0\xi\eta^2\left(\frac{\partial(\sigma_0^2)}{\partial\xi}\right)^2 + Gl_0 = 2V. \tag{3}$$

Par. 5. Continuation. Determination of σ^2.

From p. 122, we have $\sigma_0^2 = \Delta\xi^2 + \Delta\eta^2 + Gl_4$ when formulating:

$$\Delta\xi = \xi - \xi_1 \qquad \Delta\eta = \eta - \eta_1. \tag{4}$$

From this, we obtain for the differential quotients of σ_0^2:

$$\frac{\partial(\sigma_0^2)}{\partial\xi} = 2\Delta\xi + Gl_3 \qquad \frac{\partial(\sigma_0^2)}{\partial\eta} = 2\Delta\eta + Gl_3. \tag{5}$$

As mentioned above, we have to introduce a function of the seventh order for V. Since e^2 appears as a factor, it will be a function of the fifth order with respect to the coordinates of points P_1 and P. However, it is obvious that σ^2, and from this also V, are a function which is built symmetrically in regard to the coordinates of P and P_1.

We may first think of arranging this function for powers of $\Delta\xi$ because $\Delta\xi$ must be a factor in it; since when $\Delta\xi = 0$, $\sigma^2 = \Delta\eta^2$, and V will vanish. However, we need only consider $\Delta\xi^2$ and $\Delta\xi^4$ since the odd powers of $\Delta\xi$ do not exist according to the above mentioned symmetry. We shall be satisfied to formulate while omitting $\Delta\xi^4$:

$$V = E_0\Delta\xi^2\left(\xi(a\eta^2 + b\eta_1^2 + c\eta\eta_1) + \xi_1(b\eta^2 + a\eta_1^2 + c\eta\eta_1)\right) + Gl_6, \tag{6}$$

in which a, b, and c denote coefficients that have to be determined. This presumption will be verified.

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To find a, b, and c we substitute expression (6) in (3) disregarding the common factor $E_0\Delta\xi^2$ and obtain the following equation which is not yet reduced:

(See Illustrative Material on Next Page.)

$$2\{\xi(a\eta^2+b\eta_1^2+c\eta\eta_1)+\xi_1(b\eta^2+a\eta_1^2+c\eta\eta_1)\}$$
$$+(\xi-\xi_1)(a\eta^2+b\eta_1^2+c\eta\eta_1)+(\eta-\eta_1)\{\xi(2a\eta+c\eta_1)+\xi_1(2b\eta+c\eta_1)\}$$
$$+12\eta^2\xi=\xi(a\eta^2+b\eta_1^2+c\eta\eta_1)+\xi_1(b\eta^2+a\eta_1^2+c\eta\eta_1)+Gl_6;$$

after proper rearrangement, we get:

$$\left.\begin{array}{c}\xi\{4\eta^2(a+3)+\eta\eta_1(3c-2a)+\eta_1^2(2b-c)\}\\+\xi_1\{\eta^2(3b-a)+\eta\eta_1(c-2b)+\eta_1^2(a-b-c)\}+Gl_6\end{array}\right\}=0.$$

This equation must vanish simultaneously which will only happen if a, b, and c have the following values:

$$a=-3 \qquad b=-1 \qquad c=-2.$$

Substituting this into (6) and adding V to σ_0^2 it follows that:

$$\sigma^2=\sigma_0^2-E_0\Delta\xi^2\left\{\begin{array}{c}\xi(3\eta^2+\eta_1^2+2\eta\eta_1)\\+\xi_1(\eta^2+3\eta_1^2+2\eta\eta_1)\end{array}\right\}+Gl_8. \tag{7}$$

In connection with formulae (5) of the previous paragraph we shall now form the differential quotients of σ^2 and obtain:

$$\sigma\cos\alpha=(\sec\eta+6E_0\xi\eta^2+Gl_6)\left(\frac{1}{2}\frac{\partial(\sigma_0^2)}{\partial\xi}\right.$$
$$\left.-E_0\Delta\xi\left[\xi\left(\frac{9}{2}\eta^2+\frac{3}{2}\eta_1^2+3\eta\eta_1\right)+\xi_1\left(-\frac{1}{2}\eta^2+\frac{5}{2}\eta_1^2+\eta\eta_1\right)\right]+Gl_7.\right)$$

When we denote $\sec\eta\cdot\frac{1}{2}\frac{\partial(\sigma_0^2)}{\partial\xi}$ by $(\sigma\cos\alpha)_0$ which is the valid expression for the sphere, and recognize further the above-mentioned approximation value of $\frac{1}{2}\frac{\partial(\sigma_0^2)}{\partial\xi}$ we get:

$$\sigma\cos\alpha=(\sigma\cos\alpha)_0+E_0\Delta\xi\left\{\begin{array}{c}\xi\left(\frac{3}{2}\eta^2-\frac{3}{2}\eta_1^2-3\eta\eta_1\right)\\+\xi_1\left(\frac{1}{2}\eta^2-\frac{5}{2}\eta_1^2-\eta\eta_1\right)\end{array}\right\}+Gl_7. \tag{8}$$

The differentiation of σ^2 with respect to η gives further:

$$\sigma\sin\alpha=(\sigma\sin\alpha)_0-E_0\Delta\xi^2\{\xi(3\eta+\eta_1)+\xi_1(\eta+\eta_1)\}+Gl_4. \tag{9}$$

Par. 6. Determination of Distance and Direction Angles by the Coordinates. We are now going to apply formulae (7), (8) and (9) of the previous paragraph to two points P_1 and P_2. The index 2 will have to be added to those symbols ξ and η which are without indices; we shall further replace α by $\alpha_{2.1} + 180^0$.

We are introducing the linear coordinates again while observing formulae (3), (4), and (5), p. 122, which concern the sphere. After some reductions, we have:

$$s^2 = \left\{ \begin{array}{c} \Delta x^2 + \Delta y^2 - \left(\frac{y^2}{\varrho^2} + \frac{\Delta y^2}{12\varrho^2}\right)\Delta x^2 \\ + \left(\frac{y^4}{3\varrho^4} - \frac{y^2\Delta y^2}{6\varrho^4} - \frac{\Delta y^4}{720\varrho^4}\right)\Delta x^2 - \left(\frac{y^2}{12\varrho^2} + \frac{\Delta y^2}{720\varrho^2}\right)\frac{\Delta x^4}{\varrho^2} \\ - \left\{\left(\frac{2y^3}{\varrho^3} + \frac{\Delta y^3}{6\varrho^3}\right)\frac{x}{\varrho} + \frac{y\Delta x\Delta y}{3\varrho^3}\right\}\Delta x^2 e^2 \sin 2B_0 + \varrho^2 Gl_3. \end{array} \right\} \quad (1)$$

In this we put the following relations:

$$\left. \begin{array}{cc} x = \frac{x_2 + x_1}{2} & y = \frac{y_2 + y_1}{2} \\ \Delta x = x_2 - x_1 & \Delta y = y_2 - y_1 \\ \varrho^2 = \frac{a_0^2}{K_0} = \frac{b_0^2}{W_0^4} \text{ with } W = \sqrt{1 - e^2 \sin^2 B_0}, & \end{array} \right\} \quad (2)$$

The last expression is derived from (3), Par. 15, p. 59, and B_0 indicates the geographic latitude of the coordinates' origin.

When formulating the expressions $s \cos \alpha_{2.1}$ and $s \sin \alpha_{2.1}$ by (3) and (4), p. 122, we have to exchange indices 1 and 2, and we find:

$$s \cos \alpha_{2.1} = -\Delta x \left\{ \begin{array}{c} 1 - \left(\frac{y_1^2}{2\varrho^2} - \frac{\Delta y^2}{6\varrho^2}\right) \\ + \left(\frac{y_1^4}{24\varrho^4} - \frac{y_1^2\Delta y^2}{12\varrho^4} + \frac{7\Delta y^4}{360\varrho^4}\right) - \left(\frac{y_1 y_2}{2\varrho^2} + \frac{2\Delta y^2}{15\varrho^2}\right)\frac{\Delta x^2}{3\varrho^2} \\ - \left[\left(\frac{y_1^2}{2\varrho^2} - \frac{\Delta y^2}{12\varrho^2}\right)\frac{x_1}{\varrho} + \left(\frac{y_1^2}{2\varrho^2} - \frac{\Delta y^2}{4\varrho^2}\right)\frac{x_2}{\varrho}\right] e^2 \sin 2B_0 \end{array} \right\} + \varrho Gl_1; \quad (3)$$

$$s \sin \alpha_{2.1} = \left\{ \begin{array}{c} -\Delta y + \left(y_2 - \frac{1}{3}\Delta y\right)\frac{\Delta x^2}{2\varrho^2} \\ -\left\{\left(y_1 - 2\Delta y\right)\frac{y_1^2}{\varrho^2} + \left(y_1 - \frac{2}{15}\Delta y\right)\frac{\Delta y^2}{\varrho^2}\right\}\frac{\Delta x^2}{3\varrho^2} + \left(y_1 - \frac{7}{15}\Delta y\right)\frac{\Delta x^4}{24\varrho^4} \\ +\left[y\frac{x_1}{3\varrho} + (4y + \Delta y)\frac{x_2}{6\varrho}\right]\frac{\Delta x^3}{\varrho^3} e^2 \sin 2B_0 + \varrho\, Gl_1. \end{array} \right\} \quad (4)$$

Exchanging the indices 1 and 2 gives:

$$s \cos \alpha_{1.2} = \Delta x \left\{ \begin{array}{c} 1 - \left(\frac{y_2^2}{2\varrho^2} - \frac{\Delta y^2}{6\varrho^2}\right) \\ +\left(\frac{y_2^4}{24\varrho^4} - \frac{y_2^2\Delta y^2}{12\varrho^4} + \frac{7\Delta y^4}{360\varrho^4}\right) - \left(\frac{y_1 y_2}{2\varrho^2} + \frac{2\Delta y^2}{15\varrho^2}\right)\frac{\Delta x^2}{3\varrho^2} \\ -\left[\left(\frac{y_2^2}{2\varrho^2} - \frac{\Delta y^2}{4\varrho^2}\right)\frac{x_1}{\varrho} + \left(\frac{y_2^2}{2\varrho^2} - \frac{\Delta y^2}{12\varrho^2}\right)\frac{x_2}{\varrho}\right] e^2 \sin 2B_0 \end{array} \right\} + \varrho\, Gl_1; \quad (5)$$

$$s \sin \alpha_{1.2} = \left\{ \begin{array}{c} \Delta y + \left(y_1 + \frac{1}{3}\Delta y\right)\frac{\Delta x^2}{2\varrho^2} \\ -\left\{\left(y_1 + 2\Delta y\right)\frac{y_1^2}{\varrho^2} + \left(y_1 + \frac{2}{15}\Delta y\right)\frac{\Delta y^2}{\varrho^2}\right\}\frac{\Delta x^2}{3\varrho^2} + \left(y_1 + \frac{7}{15}\Delta y\right)\frac{\Delta x^4}{24\varrho^4} \\ +\left[(4y - \Delta y)\frac{x_1}{6\varrho} + y\frac{x_2}{3\varrho}\right]\frac{\Delta x^3}{\varrho^3} e^2 \sin 2B_0 + \varrho\, Gl_1. \end{array} \right\} \quad (6)$$

To check these formulae we make use of the following relations:

$$s^2 = s^2 \cos^2 \alpha_{2.1} + s^2 \sin^2 \alpha_{2.1} = s^2 \cos^2 \alpha_{1.2} + s^2 \sin^2 \alpha_{1.2}.$$

A further control will be obtained by computing s twice from sets (3), (4), and (5), (6).

To get still another verification, we use a formula which gives directly the difference:

$$\alpha_{2.1} - \alpha_{1.2} - 180^0 = \Delta\alpha \quad (7)$$

We know this difference for the sphere and have only to find the correction term.

Adding equations (4) and (6), we obtain with regard to (7):

(See Illustrative Material on Next Page.)

$$s\left(\sin \mathfrak{A}_{1.2} - \sin (\mathfrak{A}_{1.2} + \Delta\mathfrak{A})\right) = \left\{s\left(\sin \mathfrak{A}_{1.2} - \sin (\mathfrak{A}_{1.2} + \Delta\mathfrak{A})\right)\right\}_0$$
$$+ \left\{\left(y - \tfrac{1}{6}\Delta y\right)\tfrac{x_1}{\varrho} + \left(y + \tfrac{1}{6}\Delta y\right)\tfrac{x_2}{\varrho}\right\}\tfrac{\Delta x^2}{\varrho^2} e^2 \sin 2B_0 + \varrho Gl_7, \qquad (8)$$

in which the first term of the right member refers again to the sphere. When reducing $\sin(\mathfrak{A}_{1.2} + \Delta\mathfrak{A})$ and rearranging the term multiplied by e^2 we get by (8):

$$-s\cos\mathfrak{A}_{1.2}\cdot\Delta\mathfrak{A} + \cdots = -(s\cos\mathfrak{A}_{1.2}\cdot\Delta\mathfrak{A})_0 + \cdots$$
$$+\left\{2y\tfrac{x}{\varrho} + \Delta y\tfrac{\Delta x}{6\varrho}\right\}\tfrac{\Delta x^2}{\varrho^2} e^2 \sin 2B_0 + \varrho Gl_7.$$

Formula (5) shows that Δx differs from $s\cos\mathfrak{A}_{1.2}$ in general only by a small second-order fraction of its value; with this in mind, we divide by Δx and have (reserving our validity test for later):

$$\Delta\mathfrak{A} = (\Delta\mathfrak{A})_0 - \left(2\tfrac{xy}{\varrho^2} + \tfrac{\Delta x\,\Delta y}{6\varrho^2}\right)\tfrac{\Delta x}{\varrho} e^2 \sin 2B_0 + Gl_6.$$

The value of $(\Delta\mathfrak{A})_0$ results from (4), p. 119, in which we may substitute Δx and Δy for u and v in the terms of the fourth order.

From this, we obtain finally:

$$\underset{\text{in sec.}}{\Delta\mathfrak{A}} = -\varrho''\tfrac{\Delta x}{\varrho}\left\{\begin{array}{l}\tfrac{y}{\varrho}\left(1 - \tfrac{y^2}{6\varrho^2} + \tfrac{\Delta x^2}{12\varrho^2} + \tfrac{\Delta y^2}{8\varrho^2}\right) \\ + \left(2\tfrac{xy}{\varrho^2} + \tfrac{\Delta x\,\Delta y}{6\varrho^2}\right) e^2 \sin 2B_0 + Gl_5.\end{array}\right\} \qquad (9)$$

To ensure the validity of this development we have to rewrite (8) as follows:

$$s\sin(\mathfrak{A}_{1.2} + \Delta\mathfrak{A}) = s\sin\mathfrak{A}_{1.2} + h. \qquad (8^*)$$

The expansion of $\Delta\mathfrak{A}$ for powers of h is valid, as indicated

on page 31, if $h : s \cos^2 \alpha_{1.2}$ will be small enough that its powers may be disregarded from a certain point on. If only the terms concerning the sphere are considered, formula (9) will also be good for larger values of the coordinates. But it will remain valid for the ellipsoid since the terms in h which were created by e^2 all have the factor Δx^2; this factor is of necessity part of all higher terms of s^2 in (1), p. 416, as indicated by the development on page 414. This factor remains valid for $s \sin \alpha_{1.2}$ and we may divide h by $s^2 \cos^2 \alpha_{2.1}$; the latter differs from Δx^2 only by a factor which does not significantly depart from 1.

We conclude that $h : s \cos^2 \alpha_{2.1}$ will be a small term of the second order for such coordinate values which are terms of the first order in relation to a_0; its powers will decrease more and more.

The previous formulae will be more accurate numerically if we move the zero point of the abscissae, for which K_0 is valid, in such a way that the abscissae will be rather small.

It will be most convenient if K_0 is assumed in the center between the two initial points of the ordinates y_1 and y_2. Then $x = 0$ and $x_1 = -x_2$, and the values of those terms dependent on e^2 will be greatly reduced.

When the values of the distances s and of the coordinates y are smaller than $0{,}1 a_0$ the influence of the terms dependent on e^2 on $\log s$

will be no more than several units of the eighth decimal-place, at the most. At the same time, their influence upon $\Delta \alpha$ will be as many units of the second decimal place of seconds.

Par. 7. Determining the Difference of the Coordinates.

To see how accurate the formulae of this paragraph really are, let us evolve them by using the equations for the geodetic triangle on page 359. For this purpose, we divide the quadrangle $P_1P_2F_2F_1$ (Fig. 38, p. 412) into two triangles. With proper selection of K_0 in $\Delta\alpha$, we find by table (27), p. 388 and formula (3), p. 403, that the maximum error will be several hundredths of a second when the dimensions in question reach the value of $0{,}1a_0$. We notice further, and also by Par. 14, p. 388 ff. that in this case the eighth decimal place of $\log s$ will become doubtful.

Par. 7. Determining the Difference of the Coordinates and the Difference of the Direction Angles by the Distance P_1P_2, the Direction Angle at P_1, and the Coordinates of P_1. It will only be necessary to resolve formulae (5) and (6) of the previous paragraph for Δx and Δy. For those terms which are valid for the sphere the formulae are given on page 118. It will also suffice to substitute in the terms multiplied by e^2:

$$x_2 = x_1 + u + Gl_5 \qquad y_2 = y_1 + v + Gl_3,$$

and we get:

$$\left.\begin{aligned} v &= s\sin\alpha_{1,2} \qquad u = s\cos\alpha_{1,2} \\ \varrho^2 &= \frac{a_0^2}{K_0} = \frac{b_0^2}{W_0^4} \end{aligned}\right\} \tag{1}$$

$$y_2 - y_1 = \left\{\begin{aligned} &v - \left(y_1 + \frac{1}{3}v\right)\frac{u^2}{2\varrho^2} + \left(y_1 + \frac{1}{5}v\right)\left(\frac{u^4}{24\varrho^4} - \frac{u^2v^2}{3\varrho^4}\right) - (y_1 + 2v)\frac{u^2y_1^2}{6\varrho^4} \\ &- \left[\left(y_1 + \frac{1}{3}v\right)\frac{x_1}{\varrho} + \left(\frac{1}{3}y_1 + \frac{1}{6}v\right)\frac{u}{\varrho}\right]\frac{u^2}{\varrho^2}e^2\sin 2B_0 + \varrho\, Gl_7 \end{aligned}\right\} \tag{2}$$

$$x_2 - x_1 = u\left\{\begin{aligned} &1 + \left(\frac{y_2^2}{2\varrho^2} - \frac{v^2}{6\varrho^2}\right) + \left(\frac{y_2^2}{2\varrho^2} - \frac{v^2}{5\varrho^2}\right)\frac{u^2}{3\varrho^2} + \frac{5}{6}\left(\frac{y_2^2}{2\varrho^2} - \frac{v^2}{10\varrho^2}\right)^2 \\ &+ \left[\left(\frac{y_2^2}{\varrho^2} - \frac{v^2}{3\varrho^2}\right)\frac{x_1}{\varrho} + \left(\frac{y_2^2}{2\varrho^2} - \frac{v^2}{12\varrho^2}\right)\frac{u}{\varrho}\right]e^2\sin 2B_0 \end{aligned}\right\} + \varrho Gl_7. \tag{3}$$

To compute $\Delta\alpha$ we may either apply formula (9) of the previous paragraph or we may write with regard to (5), p. 119:

$$\left.\begin{aligned} \Delta\alpha_{\text{in sec.}} &= -\varrho'' \frac{u}{\varrho}\left\{\left(\frac{y_1}{\varrho}+\frac{v}{2\varrho}\right)+\left(\frac{y_1}{\varrho}+\frac{v}{4\varrho}\right)\frac{5v^2-u^2}{6\varrho^2}+\left(\frac{y_1}{3\varrho}+\frac{v}{\varrho}\right)\frac{y_1^2}{\varrho^2}\right\} \\ &\quad +\left(\frac{2xy}{\varrho^2}+\frac{uv}{6\varrho^2}\right)e^2\sin 2B_1+Gl_3 \\ x &= \tfrac{1}{2}(x_1+x_2) \qquad y=\tfrac{1}{2}(y_1+y_2). \\ \alpha_{2,1} &= \alpha_{1,2}+180^0+\Delta\alpha. \end{aligned}\right\} \tag{4}$$

As to the accuracy of these formulae, we refer to our remarks in the previous paragraph. In particular, we must observe that it will be convenient to have the coordinate origin to which K_0 refers, in the center between the ordinate's initial points; for this purpose, we write $x_1 = -\frac{1}{2}u$.

However, the smaller the dimensions concerned are the more we may disregard the above recommendations. It will be quite inconvenient to have another K_0 for every new side of a triangle when computing successively the coordinates of main triangulation points; especially so if K_0 and W_0 respectively have to be interpolated from the tables. It is always sufficient however, to take the nearest table value if the values are only 10' apart (even an interval of 20' will suffice).

Thus, correcting ϱ successively, we have the advantage that the terms multiplied by e^2 will become very small; in most cases they will be so small that we are allowed to omit them altogether.

We may then also be able to reduce formulae (2) and (4) to the terms of the two lowest orders which are emphasized in the print.

When computing an extensive network of triangles by these formulae we must use several abscissa-axes; their east and west

distances should not exceed 150 km. Points of a central location must be referred to both neighboring axes. (cf. page 120.)

To secure the exact north and south direction of these axes we are obliged to compute by more precise formulae; starting from the first axis which is fixed by astronomical measurements, we calculate by precise formulae until we reach those network points through which the two neighboring axes are to be laid. From these, we are able to determine the following neighboring axes, etc.

In Prussia, a different method is being used: in the main triangulation network, the spherical coordinates are computed directly for every point (they are needed anyway for cartographic purposes), and these will then be converted to local rectangular coordinates, if necessary.

This method has its advantages as the abscissa-axes can easily be run at a distance of exactly one degree of longitude, and a system of local rectangular coordinates may be established anywhere without difficulties.

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Par. 8. The Conversion of Spherical Coordinates. When this problem was discussed on page 123 for small distances on the sphere, we found it convenient first to compute rectangular coordinates for the terminal point P_2 of the line s with reference to the meridian of the starting point P_1. In Fig. 39, let us assume coordinates x and y for P_2 were computed first, then we moved by means of the meridian arc x from the geographic latitude B_1 of P_1 to latitude F of point F which is the initial point of the ordinate. That leaves

the problem of computing, from F and y, the geographic longitudinal difference and the geographic latitude of P_2.

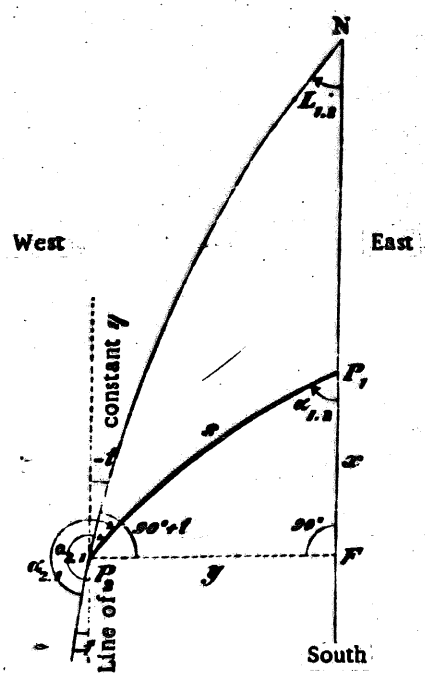


Figure 39.

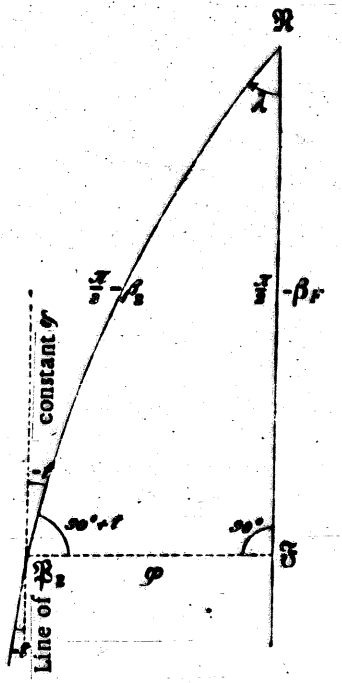


Figure 40.

When the distances are very small it will be sufficient to compute figure NFP_2 on the ellipsoid, as a spherical triangle on a sphere of the transverse radius of curvature ϱ_n, i.e. the normal in point F. When small distances are concerned the ellipsoid normals for points P_2 and F intersect each other very close to a point K_F' of the rotation axis; this point has the distance ϱ_n from F (Fig. 1, p. 40). The spherical triangle in question will be located on a sphere described by ϱ_n about K_F', between the points where the lines $K_F'F$, $K_F'P_2$, and $K_F'N$ intersect the sphere's surface. The sides and angles of this triangle depart very little from the angle between the normal and the earth's axis, and from the angles $90°$, $L_{1,2}$, and $90°+t$, of

the triangle NFP_2 on the ellipsoid. The difference will be no greater than the difference between $K'_F P_2$ and the accurate normal of the ellipsoid.

To correct the spherical calculation we write:

$$\frac{y}{\varrho_m} = \eta, \quad \varrho_m \text{ for the geographic latitude } F. \tag{1}$$

In connection with the formulae of Chapter 5, p. 218 ff., we introduce with Fig. 39 also Fig. 40. This spherical figure gives immediately the azimuths and the complements of the reduced latitudes.

By (9), p. 221, we get:

$$dy = a_0 \sqrt{\frac{1-e^2}{1-k^2}} \sqrt{1-k^2 \sin^2\varphi}\, d\varphi$$

$$k = e \sin F.$$

Since $\varrho_m = a_0 : \sqrt{1-e^2 \sin^2 F}$ we have also $\varrho_m = a_0 : \sqrt{1-k^2}$, and hence:

$$d\eta = \sqrt{1-e^2}\sqrt{1-k^2\sin^2\varphi}\, d\varphi. \tag{2}$$

Supposing η to be a term of the first order we obtain by series expansion:

$$\sqrt{1-k^2\sin^2\varphi} = 1 - \tfrac{1}{2}k^2\left(\varphi^2 - \tfrac{1}{3}\varphi^4\right) + Gl_8.$$

Substituting this equality in expression (2) and then integrating from $\varphi = 0$ to φ, it follows that:

$$\eta = \varphi\sqrt{1-e^2}\left(1 - \tfrac{1}{6}k^2\varphi^2 + \tfrac{1}{30}k^2\varphi^4 + Gl_8\right). \tag{3}$$

The inversion of this formula, which can be verified, will be:

$$\varphi = \frac{\eta}{\sqrt{1-e^2}}\left(1 + \tfrac{1}{6}k^2(1+e^2)\eta^2 - \tfrac{1}{30}k^2\eta^4 + Gl_8\right). \tag{4}$$

We get further by (14), p. 231, when writing $k_1 = \frac{1}{4}k^2 + \frac{1}{8}k^4 + \cdots$ and $n = \frac{1}{4}e^2 + \frac{1}{8}e^4 + \cdots$, and remembering that $\cos\beta_F = \cos F : \sqrt{1-k^2}$:

$$L_{1,2} = \lambda - \frac{e^2 \cos F}{2\sqrt{1-k^2}}\left\{\begin{array}{c}\left(1+\frac{1}{4}e^2-\frac{1}{8}k^2+\frac{1}{8}e^4-\frac{5}{64}k^4\right)\varphi-\left(\frac{1}{16}k^2+\frac{1}{32}k^4\right)\sin 2\varphi \\ +\frac{1}{256}k^4\sin 4\varphi + Gl_7\end{array}\right\}.$$

From now on, we shall omit the index of $L_{1,2}$ for the sake of brevity. Substituting the series $\sin 2\varphi = 2\varphi - \frac{4}{3}\varphi^3 + \cdots$, $\sin 4\varphi = 4\varphi + \cdots$ and $1 : \sqrt{1-k^2} = 1 + \frac{1}{2}k^2 + \frac{3}{8}k^4 + \cdots$, we get:

$$L = \lambda - \frac{1}{2}e^2\cos F\left\{\left(1+\frac{1}{4}k^2+\frac{1}{4}e^2+\frac{1}{8}k^4+\frac{1}{8}e^2k^2+\frac{1}{8}e^4\right)\varphi + \frac{1}{12}k^2\varphi^3 + Gl_7\right\}.$$

Finally, we replace φ by η from (4) and have:

$$L = \lambda - \frac{1}{2}e^2\cos F\left\{\begin{array}{c}\left(1+\frac{1}{4}k^2+\frac{3}{4}e^2+\frac{1}{8}k^4+\frac{5}{8}e^4+\frac{1}{4}e^2k^2\right)\eta \\ +\frac{1}{4}k^2\eta^3 + Gl_7.\end{array}\right\} \qquad (5)$$

By spherical trigonometry we obtain further:

$$\tan\lambda = \tan\varphi\sec\beta_F = \sqrt{1-k^2}\tan\varphi\sec F.$$

In this, we take the value of λ from (5) and the value of φ from (4). From this it follows, if the development below will be applied, by Taylor's Theorem:

$$\begin{array}{c}\tan(u+\delta) = \tan u + \delta\sec^2 u + \delta^2\sec^2 u\tan u \\ +\delta^3\sec^2 u\left(\tan^2 u + \frac{1}{3}\right) + \cdots,\end{array} \qquad (6)$$

in the case of $u = L$ with $\delta = \lambda - L$ and $u = \eta$ with $\delta = \varphi - \eta$:

$$\begin{array}{c}\tan L + (\lambda - L)\sec^2 L + (\lambda - L)^2\sec^2 L\tan L + Gl_9 \\ = \sqrt{1-k^2}\sec F\left\{\tan\eta + (\varphi-\eta)\sec^2\eta + (\varphi-\eta)^2\sec^2\eta\tan\eta + Gl_9\right\}.\end{array} \qquad (7)$$

The permissibility of this application will be examined later.

Expanding $1:\sqrt{1-e^2}$ in (4) results in:

$$\varphi-\eta=\eta\left(\tfrac{1}{2}e^2+\tfrac{3}{8}e^4+\tfrac{5}{16}e^6+\tfrac{1}{6}k^2\left(1+\tfrac{3}{2}e^2\right)\eta^2-\tfrac{1}{30}k^2\eta^4+Gl_5\right). \tag{8}$$

Substituting this expression in (7) and also the expression for $\lambda-L$ given by (5), and utilizing the relation $\sec^2=1+\tan^2$ we have:

$$\tan L+\tfrac{1}{2}e^2\eta\cos F\left\{\begin{array}{l}1+\tfrac{3}{4}e^2+\tfrac{1}{4}k^2+\tfrac{5}{8}e^4+\tfrac{1}{8}k^4+\tfrac{1}{4}e^2k^2\\ \quad+\tfrac{1}{4}k^2\eta^2+\tan^2L\left(1+\tfrac{1}{4}k^2+\tfrac{3}{4}e^2\right)\end{array}\right\}=$$
$$+\tfrac{1}{4}e^4\eta^2\cos^2F\tan L+Gl_9 \tag{9}$$

$$=\sqrt{1-k^2}\sec F\left\{\begin{array}{c}\tan\eta+\eta\left(\tfrac{1}{2}e^2+\tfrac{3}{8}e^4+\tfrac{5}{16}e^6\right)\\ +\tfrac{1}{6}\left(1+\tfrac{3}{2}e^2\right)k^2\eta^3-\tfrac{1}{30}k^2\eta^5\\ +\left(\tfrac{1}{2}e^2+\tfrac{3}{8}e^4+\tfrac{1}{6}k^2\eta^2\right)\eta\tan^2\eta+\tfrac{1}{4}e^4\eta^2\tan\eta+Gl_9\end{array}\right\}.$$

Expanding $\sqrt{1-k^2}$ in the series $1-\tfrac{1}{2}k^2-\tfrac{1}{8}k^4-\tfrac{1}{16}k^6-\ldots$ and by partial substitution of $\tan\eta=\eta+\tfrac{1}{3}\eta^3+\tfrac{2}{15}\eta^5+\ldots$, the <u>right</u> member of equation (9) transforms into the following expression:

$$\sec F\left\{\begin{array}{c}\tan\eta+\left\{\begin{array}{l}\tfrac{1}{2}e^2-\tfrac{1}{2}k^2+\tfrac{3}{8}e^4-\tfrac{1}{8}k^4-\tfrac{1}{4}e^2k^2\\ \quad+\tfrac{5}{16}e^6-\tfrac{1}{16}k^6-\tfrac{3}{16}e^4k^2-\tfrac{1}{16}e^2k^4\end{array}\right\}\eta\\ +\left(\tfrac{1}{2}e^2+\tfrac{5}{8}e^4-\tfrac{1}{8}k^4\right)\eta^3\\ +\left(\tfrac{1}{3}e^2+\tfrac{1}{15}k^2\right)\eta^5+Gl_9.\end{array}\right\} \tag{10}$$

Comparing with the left member of (9), we notice that:

$$\tan L=\sec F\tan\eta+\eta\sec F\left(\tfrac{1}{2}e^2-\tfrac{1}{2}k^2-\tfrac{1}{2}e^2\cos^2F\right)+Gl_5.$$

The parenthesis will vanish becuause $k^2=e^2\sin^2F$, and then we get:

$$\tan L = \sec F \tan \eta + Gl_3. \tag{11}$$

By this relation, we eliminate $\tan^2 L$ and $\tan L$ from the smaller terms of (9). The left member of (9) will then read:

$$\begin{aligned}\tan L + \eta \cos F\left(\tfrac{1}{2}e^2 + \tfrac{3}{8}e^4 + \tfrac{1}{8}e^2k^2 + \tfrac{1}{16}e^2k^4 + \tfrac{1}{8}e^4k^2 + \tfrac{5}{16}e^6\right)\\ + \eta^3 \cos F\left(\tfrac{1}{4}e^4 + \tfrac{1}{8}e^2k^2\right) + \eta^3 \sec F\left(\tfrac{1}{2}e^2 + \tfrac{3}{8}e^4 + \tfrac{1}{8}e^2k^2\right)\\ + \eta^5 \sec F \cdot \tfrac{1}{3}e^2 + Gl_9.\end{aligned}$$

Replacing $e^2 \cos F$ in the second and third term by $e^2 \cos^2 F \cdot \sec F$, i.e. $(e^2 - k^2)\sec F$, we obtain the following expression for the left member of (9):

$$\tan L + \sec F \left\{ \begin{aligned} &\left\{ \begin{aligned} &\tfrac{1}{2}e^2 - \tfrac{1}{2}k^2 + \tfrac{3}{8}e^4 - \tfrac{1}{8}k^4 - \tfrac{1}{4}e^2k^2 \\ &+\tfrac{5}{16}e^6 - \tfrac{1}{16}k^6 - \tfrac{3}{16}e^4k^2 - \tfrac{1}{16}e^2k^4 \end{aligned} \right\}\eta \\ &+\left(\tfrac{1}{2}e^2 + \tfrac{5}{8}e^4 - \tfrac{1}{8}k^4\right)\eta^3 + \tfrac{1}{3}e^2\eta^5 + Gl_9. \end{aligned} \right\} \tag{12}$$

Comparing this with expression (10) for the right member of (9), we finally obtain the simple formula:

$$\tan L_{1,2} = \tan \eta \sec F \left(1 + \tfrac{1}{15}k^2\eta^4 + Gl_8\right). \tag{13}$$

If we only go up to the seventh order, it will suffice to write $\tan L = \tan \eta \sec F$. Even if $\eta = 0{,}1$, the influence of $k^2\eta^4$ in the right-hand parenthesis on $\log \tan L$ will not exceed two units of the eighth decimal-place.

Regarding the convergence of the series in the right-hand parenthesis of (13), we must examine whether the use of series (6) was permissible. The only issue in doubt, however, will be its

application with respect to $u = L$ because L may become very great near the Pole although η was assumed to be a term of the first order. Multiplying (13) by $\cos F$ and using this factor, the above application will lead us to:

$$\tan\lambda\cos F = \tan L\cos F + (\lambda - L)\cos F\sec^2 L + (\lambda - L)^2\cos F\sec^2 L\tan L + (\lambda - L)^3\cos F\sec^2 L\left(\tan^2 L + \tfrac{1}{3}\right) + \cdots \qquad (13^*)$$

When we stop at $(\lambda - L)^2$ as we did, the remainder will be:

$$(\lambda - L)^3\cos F\sec^2 L'\left(\tan^2 L' + \tfrac{1}{3}\right), \qquad (14)$$

in which L' has an undetermined value between L and λ (p. 25). $(\lambda - L)$ may be replaced by $\frac{1}{2}e^2\eta\cos F$ which is a sufficient approximation. Further, we let $\sec^2 L'$ have the value of $1 + \tan^2 L'$; whether the remainder may be neglected near the Pole will then depend on the expression:

$$\tfrac{1}{8}e^6\eta^3(\tan L'\cos F)^4. \qquad (14^*)$$

In this, we set $L' = \lambda - \frac{1}{2}\varkappa e^2\eta\cos F$, with $\varkappa$ not greatly exceeding the value of 1. Hence:

$$\tan L'\cos F = \frac{\tan\lambda\cos F - \tan\left(\frac{1}{2}\varkappa e^2\eta\cos F\right)\cos F}{1 + \tan\lambda\tan\left(\frac{1}{2}\varkappa e^2\eta\cos F\right)}.$$

The above developments indicate, however, that $\tan\lambda\cos F = \sqrt{1-k^2}\tan\varphi$, i.e. it will be close to $\tan\eta$. Consequently, in the preceding right-hand expression, the denominator will always be approximately 1 for small values of η;

and we may easily see that, in general, $\tan L'\cos F$ very nearly

equals $\tan\eta$, hence it will be a magnitude of the first order.

We conclude that the partial remainder (14*) and consequently all of remainder (14) continue to be a small magnitude within the omitted order, even at the Pole.

Hence, formula (13) will be suitable near the Pole for small values of η.

Par. 9. Continuation: Meridian Convergence. The auxiliary spherical triangle yields the following formula for computing the meridian convergence t:

$$\tan t = -\sin\varphi \tan\beta_F. \tag{1}$$

In this, we set

$$\tan\beta_F = \sqrt{1-e^2}\tan F \tag{2}$$

and

$$\sin\varphi = \sin\left(\eta+(\varphi-\eta)\right) = \sin\eta+(\varphi-\eta)\cos\eta-\frac{1}{2}(\varphi-\eta)^2\sin\eta+Gl_3.$$

Remembering the value of $(\varphi-\eta)$ from (8), p. 423, we derive from this gradually:

$$\sqrt{1-e^2}\sin\varphi = \left\{\begin{array}{c}\sin\eta\left(1-\frac{1}{2}e^2-\frac{1}{8}e^4-\frac{1}{16}e^6\right)\\ +(\varphi-\eta)\left(1-\frac{1}{2}\eta^2+\frac{1}{24}\eta^4\right)\left(1-\frac{1}{2}e^2-\frac{1}{8}e^4\right)\\ -\frac{1}{2}\eta(\varphi-\eta)^2+Gl_5\end{array}\right\}$$

or

$$\sqrt{1-e^2}\sin\varphi = \sin\eta+\frac{1}{6}\eta^3(k^2-e^2)(1+e^2)+\frac{1}{60}\eta^5(e^2-7k^2)+Gl_7. \tag{3}$$

By (1) we have now, in connection with formula $k^2=e^2\sin^2F$:

$$\tan t = -\tan F\left(\sin\eta-\frac{1}{6}\eta^3e^2(1+e^2)\cos^2F+\frac{1}{60}\eta^5e^2(1-7\sin^2F)+Gl_7\right).$$

Factoring $\sin\eta$ in this and remembering $\eta:\sin\eta = 1+\frac{1}{6}\eta^2+Gl_4$, it follows that:

$$\tan t = -\sin\eta\tan F\left\{1-\frac{1}{6}\eta^2\frac{e^2}{1-e^2}\cos^2F+\frac{1}{90}\eta^4e^2(8\cos^2F-9)+Gl_6\right\}. \tag{4}$$

For small values of η, this formula is valid beyond any doubt.

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Par. 10. Continuation: Difference of Latitude. From the system of formulae (9), p. 126, we get the following equation which applies to the auxiliary spherical triangle:

$$\sin(\beta_F - \beta_2) = -\tan t \cos\beta_F \tan\frac{\lambda}{2}. \tag{1}$$

To obtain the latitude difference in the left member, we use the formulae:

$$\left.\begin{aligned} \sin\beta_F &= \frac{\sqrt{1-e^2}}{\sqrt{1-k^2}}\sin F \qquad \cos\beta_F = \frac{\cos F}{\sqrt{1-k^2}} \\ \sin\beta_2 &= \frac{\sqrt{1-e^2}\sin B_2}{\sqrt{1-e^2\sin^2 B_2}} \qquad \cos\beta_2 = \frac{\cos B_2}{\sqrt{1-e^2\sin^2 B_2}}. \end{aligned}\right\} \tag{2}$$

Forming $\sin\beta_F\cos\beta_2 - \cos\beta_F\sin\beta_2$, we have immediately:

$$\sin(\beta_F - \beta_2) = \sin(F - B_2)\frac{\sqrt{1-e^2}}{\sqrt{1-k^2}\sqrt{1-e^2\sin^2 B_2}}.$$

Hence:

$$\sin(F - B_2) = -\tan t \cos F \tan\frac{\lambda}{2}\frac{\sqrt{1-e^2\sin^2 B_2}}{\sqrt{1-e^2}}, \tag{3}$$

in which we now replace λ by L.

Because of the indentity $\frac{1}{2}\lambda = \frac{1}{2}L + \frac{1}{2}(\lambda - L)$ and expansion (6), p. 423, it follows that:

$$\cos F\tan\frac{\lambda}{2} = \cos F\left\{\tan\frac{L}{2} + \frac{\lambda-L}{2}\sec^2\frac{L}{2} + \left(\frac{\lambda-L}{2}\right)^2\sec^2\frac{L}{2}\tan\frac{L}{2} + \cdots\right\}, \tag{4}$$

in which we write from (5), p. 423:

$$\frac{\lambda - L}{2} = \frac{1}{4}e^2\eta\cos F\left(1 + \frac{1}{4}k^2 + \frac{3}{4}e^2\right) + Gl_4.$$

We now must observe that by (13), p. 425, the absolute value of L cannot exceed 90°; the maximum value of $\tan\frac{L}{2}$ will then equal 1.

(These limits will occur in the case of point F coinciding with the Pole.) We conclude from this that expansion (4) will also converge at the Pole since all its terms are smaller than the corresponding terms of expansion (13*), p. 425. In particular, $\cos F \tan \frac{L}{2}$ will be a small first-order term up to the Pole because the same applies to the greater value of $\cos F \tan L$.

The substitution of the value for $\frac{1}{2}(\lambda - L)$ from (4) gives:

$$\cos F \tan \frac{1}{2} = \cos F \left\{ \begin{array}{c} \tan \frac{L}{2} + \left(\frac{1}{4} e^2 + \frac{1}{16} e^2 k^2 + \frac{3}{16} e^4\right) \eta \cos F \\ + \frac{1}{4} e^2 \eta \cos F \tan^2 \frac{L}{2} + G l_7 \end{array} \right\}.$$

From this, expression (3) for $\sin(F - B_2)$ will be transformed into:

$$\sin(F - B_2) = -\tan t \cos F \left\{ \begin{array}{l} \tan \frac{L}{2} + \left(\frac{1}{4} e^2 + \frac{1}{16} e^2 k^2 + \frac{3}{16} e^4\right) \eta \cos F \\ + \frac{1}{4} e^2 \eta \cos F \tan^2 \frac{L}{2} + G l_7 \end{array} \right\} \frac{\sqrt{1 - e^2 \sin^2 B_2}}{\sqrt{1 - e^2}}. \qquad (5)$$

Expanding the last factor in the right-hand member will result in:

$$\frac{\sqrt{1 - e^2 \sin^2 B_2}}{\sqrt{1 - e^2}} = 1 + \frac{1}{2} e^2 \cos^2 B_2 + e^4 \left(\frac{3}{8} - \frac{1}{4} \sin^2 B_2 - \frac{1}{8} \sin^4 B_2\right) + G l_6, \qquad (6)$$

in which we shall first replace B_2 by the latitude F. We see from the preceding equation that we may write, with regard to (4), p. 426:

$$B_2 - F = -\eta \sin F \tan \frac{L}{2} (1 + G l_2).$$

We now substitute this in the following series which is always valid in this connection:

$$\cos^2 B_2 = \cos^2 F - (B_2 - F) \sin 2F - (B_2 - F)^2 \cos 2F + \frac{2}{3} (B_2 - F)^3 \sin 2F + \cdots.$$

Then, using formula $\cos 2F = 2\cos^2 F - 1$, we obtain the set:

$$\left.\begin{aligned} \cos^2 B_2 &= \cos^2 F + \eta \sin F \sin 2F \tan\frac{L}{2} + \eta^2 \sin^2 F \tan^2\frac{L}{2} + Gl_4 \\ \sin^2 B_2 &= \sin^2 F + Gl_2. \end{aligned}\right\} \qquad (7)$$

From this, we are able to write (6) in the following form:

$$\frac{\sqrt{1-e^2\sin^2 B_2}}{\sqrt{1-e^2}} = 1 + \frac{1}{2}e^2\cos^2 F + e^4\left(\frac{3}{8} - \frac{1}{4}\sin^2 F - \frac{1}{8}\sin^4 F\right) + e^2\eta \sin^2 F \cos F \tan\frac{L}{2} + \frac{1}{2}e^2\eta^2 \sin^2 F \tan^2\frac{L}{2} + Gl_6.$$

This will be even more simplified when making use of relation:

$$\tan\frac{L}{2} = \frac{1}{2}\tan L\left(1 - \tan^2\frac{L}{2}\right)$$

---

and state, with regard to (13), p. 425:

$$\cos F \tan\frac{L}{2} = \frac{1}{2}\eta\left(1 - \tan^2\frac{L}{2}\right) + \frac{1}{6}\eta^3\left(1 - \tan^2\frac{L}{2}\right) + Gl_5. \qquad (8)$$

We now obtain without difficulty:

$$\frac{\sqrt{1-e^2\sin^2 B_2}}{\sqrt{1-e^2}} = \left\{\begin{aligned} &1 + \frac{1}{2}e^2\cos^2 F + e^4\left(\frac{3}{8} - \frac{1}{4}\sin^2 F - \frac{1}{8}\sin^4 F\right) \\ &+ \frac{1}{2}e^2\eta^2\sin^2 F + Gl_6. \end{aligned}\right\} \qquad (9)$$

Substituting this in expression (5), we shall also write in the second term of the factor between braces in (5):

$$\eta = \eta\left(1 - \tan^2\frac{L}{2}\right) + \eta\tan^2\frac{L}{2} = 2\cos F \tan\frac{L}{2}\left(1 - \frac{1}{3}\eta^2\right) + \eta\tan^2\frac{L}{2} + Gl_5$$

The factor between braces in (5) reads now:

$$\tan\frac{L}{2}\left\{1 + \left(\frac{1}{2}e^2 + \frac{1}{8}e^2k^2 + \frac{3}{8}e^4 - \frac{1}{6}e^2\eta^2\right)\cos^2 F + \frac{1}{2}e^2\eta\cos F\tan\frac{L}{2} + Gl_6\right\}.$$

In the last term, we replace $\cos F \tan\frac{L}{2}$ by its equal value $\frac{1}{2}\eta\left(1 - \tan^2\frac{L}{2}\right) + Gl_3$ and after some reductions with regard to relation

$k = e \sin F$, we obtain:

$$\sin(F-B_2) = -\tan t \cos F \tan\frac{L}{2}\left\{\begin{array}{c} 1+(e^2+e^4)\cos^2 F + Gl_6 \\ +e^2\eta^2\left(\frac{3}{4}\sin^2 F + \frac{1}{12}\cos^2 F - \frac{1}{4}\tan^2\frac{L}{2}\right)\end{array}\right\}. \qquad (10)$$

In this, the value of $\tan t$ by (4), p. 426, will be substituted to yield:

$$\sin(F-B_2) = \sin\eta \sin F \tan\frac{L}{2}\left\{\begin{array}{c} 1+(e^2+e^4)\cos^2 F + Gl_6 \\ +e^2\eta^2\left(\frac{3}{4}\sin^2 F - \frac{1}{12}\cos^2 F - \frac{1}{4}\tan^2\frac{L}{2}\right)\end{array}\right\}. \qquad (11)$$

We may simplify these formulae further by introducing the relation $\varrho_n : \varrho_m$ for some as yet undetermined latitude between B_2 and F. Assuming the latitude to be $F + \varkappa(B_2 - F)$, then:

$$\frac{\varrho_n}{\varrho_m} = \frac{1-e^2\sin^2(F+\varkappa(B_2-F))}{1-e^2} = 1+(e^2+e^4)\cos^2\left(F+\varkappa(B_2-F)\right)+Gl_6.$$

Remembering the similar expansion of $\cos^2 B_2$ on page 428, we have:

$$\begin{aligned}\cos^2\left(F+\varkappa(B_2-F)\right) &= \cos^2 F - \varkappa(B_2-F)\sin 2F - \varkappa^2(B_2-F)^2\cos 2F + Gl_4 \\ &= \cos^2 F + 2\varkappa\eta\sin^2 F\cos F\tan\frac{L}{2} \\ &\quad + \varkappa^2\eta^2\sin^2 F\tan^2\frac{L}{2} + Gl_4 \\ &= \cos^2 F + \varkappa\eta^2\sin^2 F\left(1-(1-\varkappa)\tan^2\frac{L}{2}\right) + Gl_4.\end{aligned}$$

Hence, we get for the latitude $F + \varkappa(B_2 - F)$:

$$\frac{\varrho_n}{\varrho_m} = 1+(e^2+e^4)\cos^2 F + \varkappa e^2\eta^2\sin^2 F\left(1-(1-\varkappa)\tan^2\frac{L}{2}\right)+Gl_6. \qquad (12)$$

Comparison with (10) and (11) shows that it will be more convenient to set $\varkappa = \frac{3}{4}$. Then we obtain:

(See Illustrative Material on Next Page.)

Par. 11. Continuation: Summary.

$$\sin(F-B_2)=\begin{cases}=-\tan t\cos F\tan\frac{L_{1,2}}{2}\cdot\frac{\varrho_n}{\varrho_m}\\ \times\left\{1+e^2\eta^2\left(+\frac{1}{12}\cos^2F+\left[\frac{3}{16}\sin^2F-\frac{1}{4}\right]\tan^2\frac{L_{1,2}}{2}\right)+Gl_6\right\}\\ =+\sin\eta\sin F\tan\frac{L_{1,2}}{2}\cdot\frac{\varrho_n}{\varrho_m}\\ \times\left\{1+e^2\eta^2\left(-\frac{1}{12}\cos^2F+\left[\frac{3}{16}\sin^2F-\frac{1}{4}\right]\tan^2\frac{L_{1,2}}{2}\right)+Gl_6\right\}\\ \frac{\varrho_n}{\varrho_m}=\frac{W^2}{1-e^2}\text{ for the geographic latitude }F+\frac{3}{4}(B_2-F).\end{cases}\qquad(13)$$

To apply these formulae we have to use indirect calculation; if we prefer to compute directly, we have to take $x=$ zero and find:

$$\sin(F-B_2)=\begin{cases}=-\tan t\cos F\tan\frac{L_{1,2}}{2}\cdot\frac{\varrho_n}{\varrho_m}\\ \times\left(1+e^2\eta^2\left[\frac{3}{4}-\frac{2}{3}\cos^2F-\frac{1}{4}\tan^2\frac{L_{1,2}}{2}\right]+Gl_6\right)\\ =+\sin\eta\sin F\tan\frac{L_{1,2}}{2}\cdot\frac{\varrho_n}{\varrho_m}\\ \times\left(1+e^2\eta^2\left[-\frac{1}{12}+\frac{5}{6}\sin^2F-\frac{1}{4}\tan^2\frac{L_{1,2}}{2}\right]+Gl_6\right)\\ \frac{\varrho_n}{\varrho_m}=\frac{W^2}{1-e^2}\text{ for the geographic latitude }F.\end{cases}\qquad(14)$$

When we are not close to the pole, we may disregard $e^2\eta^2\tan^2\frac{L}{2}$ in (13) and (14) since, in that case, the term will only be of the sixth order.

Par. 11. Continuation: Summary. In the following summary, we rearranged those equations involving indirect calculation for a more convenient logarithmic form. The formula for B_1-F is the same as (6), p. 51.

Given: the rectangular coordinates x and y of P_2 referred to the meridian of P_1, and the latitude B_1 of P_1.

Wanted: B_2, $L_{1.2}$ and the meridian convergence t.

$$\log (B_1 - F) \underset{\text{in sec.}}{=} \log\left(\frac{\varrho'' x}{\varrho_m}\right) - \beta_2\left(\frac{\varrho'' x}{\varrho_m}\right)^2 - \beta_4\left(\frac{\varrho'' x}{\varrho_m}\right)^4 + Gl_3$$

$$\frac{\varrho''}{\varrho_m} = \frac{\varrho'' W_m^2}{a_0(1-e^2)} = [8{,}5126900.290 - 10]\, W_m^3, \quad W_m \text{ for argument } \tfrac{1}{2}(B_1 + F)$$

$$\beta_2 = \frac{M_0}{2\varrho''^2}\left(\cos(B_1 + F) - 7n\cos^2(B_1 + F) + 5n\right) \qquad \beta_4 = -\frac{1}{20\varrho''^2}\beta_2$$

$$\beta_2 = [2{,}93171 - 10]\left(\cos(B_1 + F) - [8{,}069 - 10]\cos^2(B_1 + F) + 0{,}00803\right) \quad \text{for units of the 7th decimal-place}$$

$$\beta_4 = -[8{,}070 - 20]\,\beta_2 \quad \text{for units of the 7th decimal-place}$$

(1)

$$\eta \underset{\text{in sec.}}{=} \varrho'' \frac{y}{\varrho_m} = y W_F \frac{\varrho''}{a_0} = y W_F\, [8{,}5097816{,}695 - 10]$$

W_F for argument F. (Note the change for 1".)

(2)

$$\log \tan L_{1.2} = \log(\tan \eta \sec F) + \frac{Me^2}{15\varrho''^4}\eta^4 \sin^2 F + Gl_4$$

$$\log \frac{Me^2}{15\varrho''^4} = 2{,}0284 - 20 \text{ for units of the seventh decimal-place.}$$

(3)

$$\log \tan t = \log(-\sin\eta \tan F) - \frac{Me^2}{6(1-e^2)\varrho''^2}\eta^2 \cos^2 F + \frac{Me^2}{90\varrho''^4}\eta^4(8\cos^2 F - 9) + Gl_4$$

$$\log \frac{Me^2}{6(1-e^2)\varrho''^2} = 3{,}05810 - 10 \qquad \log \frac{Me^2}{90\varrho''^4} = 1{,}250 - 20; \text{ both for units of the seventh decimal-place.}$$

(4)

$$\log \sin(F - B_2) = \begin{cases} \log\left(\sin\eta \sin F \tan\frac{L_{1.2}}{2} \cdot \frac{W^2}{1-e^2}\right) \\ -\frac{1}{2}\cdot\frac{Me^2}{6(1-e^2)\varrho''^2}\eta^2\cos^2 F - \frac{Me^2}{16\varrho''^2}\eta^2(1 + 3\cos^2 F)\tan^2\frac{L}{2} + Gl_6 \end{cases}$$

W for argument $F - \frac{3}{4}(F - B_2)_0$ to be derived from W_F.

$$-\log(1 - e^2) = 0{,}0029083.595$$

$$\log \frac{Me^2}{16\varrho''^2} = 2{,}629 - 10 \text{ for units of the seventh decimal-place.}$$

(5)

In the smaller terms, η is considered to be expressed in seconds.

The preceding formulae are still so accurate for distances of $0{,}1\,a_0$ that we would need nine-place logarithms to make full use of them. And the highest terms were only written to examine the

accuracy of abbreviated formulae.

Par. 12. Continuation: Computing of x, y, and $\alpha_{2.1}$ from s and $\alpha_{1.2}$. In the formulae on page 419, y_1 was considered to be zero, and K_0 was referred to a point which was located at two-thirds the distance from P_1 to F (cf. Fig. 39, p. 421), and had therefore the latitude $B_0 = B_1 - \frac{2}{3}(B_1 - F)$. Since P_1F is approximately u, we have to substitute:

$$y = \frac{1}{2}v \qquad x_1 = -\frac{2}{3}u \qquad x = -\frac{1}{6}u.$$

When we observe that $\mathfrak{a}_{1.2}$ and $\alpha_{1.2}$ now become identical, and from Fig. 39, p. 421, $\alpha_{2.1} = \mathfrak{a}_{2.1} + t$, we get from (1) to (4), p. 419:

$$u = s\cos\alpha_{1.2} \qquad v = s\sin\alpha_{1.2}$$

$$x = u\left\{1 + \frac{1}{3}\frac{v^2}{\varrho^2} - \frac{1}{15}\frac{u^2v^2}{\varrho^4} + \frac{2}{15}\frac{v^4}{\varrho^4} - \frac{1}{36}e^2\frac{uv^2}{\varrho^3}\sin 2B_0 + Gl_6\right\}$$

$$y = v\left\{1 - \frac{1}{6}\frac{u^2}{\varrho^2} - \frac{1}{15}\frac{u^2v^2}{\varrho^4} + \frac{1}{120}\frac{u^4}{\varrho^4} + \frac{1}{18}e^2\frac{u^2}{\varrho^2}\sin 2B_0 + Gl_6\right\}$$

$$\underset{\text{in sec.}}{\Delta\mathfrak{a}} = -\varrho''\frac{uv}{2\varrho^2}\left(1 + \frac{5}{12}\frac{v^2}{\varrho^2} - \frac{1}{12}\frac{u^2}{\varrho^2} + Gl_4\right)$$

x, y and $\Delta\mathfrak{a}$ vanish with u, v, and uv, respectively; therefore, we may factor the latter terms in the corresponding expressions. Hence it becomes possible to write these expressions in logarithmic forms as follows:

$$u = s\cos\alpha_{1.2} \qquad v = s\sin\alpha_{1.2} \tag{1}$$

$$\varrho = \frac{a_0}{\sqrt{K_0}} = \frac{b_0}{W_0^2} = \frac{[6{,}8031893]}{W_0^2} \tag{2}$$

W_0 for arg. $B_0 = B_1 + \frac{2}{3}(F - B_1)$, where $(F - B_1)$ in min. $= [6{,}7358 - 10]$ and W_0^2;

$$\log x = \log u + \frac{1}{3}M\frac{v^2}{\varrho^2} - \frac{1}{15}M\frac{u^2v^2}{\varrho^4} + \frac{7}{90}M\frac{v^4}{\varrho^4} - \frac{1}{36}M^2\frac{uv^2}{\varrho^3}\sin 2B_0 + Gl_6 \tag{3}$$

$$\log y = \log v - \frac{1}{6}M\frac{u^2}{\varrho^2} - \frac{1}{15}M\frac{u^2v^2}{\varrho^4} - \frac{1}{180}M\frac{u^4}{\varrho^4} + \frac{1}{18}M^2\frac{u^2}{\varrho^2}\sin 2B_0 + Gl_6 \tag{4}$$

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$$\log \underset{\text{in sec.}}{\Delta\alpha} = \log\left(-\frac{1}{2}\varrho''\frac{uv}{\varrho^2}\right) + \frac{5}{4}\cdot\frac{1}{3}M\frac{u^3}{\varrho^3} - \frac{1}{2}\cdot\frac{1}{6}M\frac{u^3}{\varrho^3} + Gl_4 \qquad (5)$$

$$\alpha_{2.1} = \alpha_{1.2} + 180^0 + \Delta\alpha + \varepsilon.$$

$\log\left(\frac{1}{3}M\right) = 6{,}1606630$	$\log\left(\frac{1}{15}M\right) = 5{,}4617$	$\log\left(\frac{1}{30}M\right) = 5{,}1286$	for units of the 7th decimal place
$\log\left(\frac{1}{6}M\right) = 5{,}8596330$	$\log\left(\frac{1}{180}M\right) = 4{,}383$	$\log\left(\frac{1}{18}Mc^2\right) = 3{,}207$	
$\log\left(\frac{1}{2}\varrho''\right) = 5{,}0133951$	$\log\left(\frac{1}{36}Mc^2\right) = 2{,}908$		

These equations are somewhat less accurate than those in the previous paragraph. To obtain a corresponding accuracy we would have to expand the second orders further.

Now we may solve the right triangle P_1FP_2 by means of the expanded theorem of Legendre. However, we would only obtain a very indirect solution. Of the formulae in question, set (1), p. 362, corresponds to the equations above, and we are able to establish them from the mentioned set by expansion (which we did for a check). When we apply the more accurate formulae of Par. 10, p. 370 ff., then (11), p. 374, will be most suited for computing the excess when b and c are taken as x and y, respectively.

We may then notice from Fig. 39, p. 421, that ε will be the same as $-\Delta\alpha$ when regarding the algebraic signs properly. The sign of ε has to be the same as that of xy.

For $(A - A^*)$, etc., it is still sufficient to use the simple relations (1), p. 362, as indicated in Par. 14, p. 388 ff.

Par. 13. Numerical Example I. We shall now apply the formulae of the two previous paragraphs to the example on page 244 ff. making use of seven-place logarithms with the eighth decimal place

interpolated.

Given:

$$B_1 = 52^\circ\,30'\,16{,}7'' \qquad \alpha_{1.2} = 239^\circ\,33'\,0{,}68921''$$
$$s = 529979{,}5784 \text{ meters.}$$

The formulae of Par. 12, p. 432 will result in the following numbers:

$$\left.\begin{array}{c} \log s = 5{,}7242591.6 \\ \log\cos\alpha_{1.2} = 9{,}7048223.3_n - 10 \\ \log\sin\alpha_{1.2} = 9{,}9355442.5_n - 10 \end{array}\right| \begin{array}{l} \log u = 5{,}4290814.9_n \\ \log v = 5{,}6598034.1_n \end{array}$$

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$$B_0 = 52^\circ\,30'{,}3 + \tfrac{2}{3}\,[6{,}735_n - 10 + 5{,}429_n + 9{,}997 - 10]$$
$$= 52^\circ\,30'{,}3 + \tfrac{2}{3}\cdot 145' = 54^\circ\,7'{,}0$$
$$\log W_0 = 9{,}9990465 - 10 \qquad \log \varrho = 6{,}8050963$$
$$\log \tfrac{u}{\varrho} = 8{,}6239852_n - 10 \qquad \log \tfrac{v}{\varrho} = 8{,}8547071_n - 10$$
$$\log\sin 2B_0 = 9{,}978 - 10.$$

Formula (3):	Formula (4):	Formula (5):
$5{,}4290814.9_n$	$5{,}6598034.1_n$	$2{,}4920874_n$
$+\ 7414.4$	$-\ 1281.2$	$+\ 7414$
$-\ 2.6$	$-\ 2.6$	$+\ 1854$
$+\ 8.9$	$-\ 0.1$	$-\ 641$
$+\ 0.2$	$-\ 0.1$	
$\log x = 5{,}4298235.8_n$	$\log y = 5{,}6596750.1_n$	$\log \underset{\text{in sec.}}{\Delta\alpha} = 2{,}4929501_n$
		$\Delta\alpha = -5'11{,}136''.$

The formulae of Par. 11, p. 431, will now give us the following values:

(See Illustrative Material on Next Page.)

Argument for W_m:

First approximation: $52^\circ 30',8 + \frac{1}{2} \cdot 145' = 53^\circ 42',8$

$\frac{\varphi'' x}{\varrho_m} = -8763,2'' = -2^\circ 26',053$

Second approximation: $52^\circ 30,278' + 1^\circ 12,527'$
$= 53^\circ 42,805'$

$B_1 + F = 107^\circ 25,6'$

$\log \cos (B_1 + F) = 9,476_n - 10$

$\log \beta_4 = 2,408_n - 10$

$\log x = 5,4298235.8_n$

$8,5126900.3 - 10$

Sum $= 3,9425136.1_n$

$3 \log W_m = 9,9971689 - 10$

$\log \left(\frac{\varphi'' x}{\varrho_m}\right) = 3,9396825_n$

$3 \log W_m = 9,9971688.1 - 10$

$\log \left(\frac{\varphi'' x}{\varrho_m}\right) = 3,9396824.2_n$

$-\beta_4 \left(\frac{\varphi'' x}{\varrho_m}\right)^3 = + 1.9$

$\log (B_1 - F) = 3,9396826.1_n$ in sec.

$B_1 - F = -2^\circ 25' 3,2722''$ $\quad F = 54^\circ 55' 19,9722''.$

Increment of $\log W_F$ for 1° equals $- 3.98$

$\eta = -4^\circ 5' 39,537''$

$\log y = 5,6596750.1_n$

$8,5097816.7 - 10$

$\log W_F = 9,9990271.7 - 10$

$\log \eta = 4,1684838.5_n$ in sec.

$\log \tan \eta = 8,8547988.0_n - 10$

$\log \sec F = 0,2405679.2$

$\log \tan L_{1.2} = 9,0953667.2_n - 10$

$L_{1.2} = -7^\circ 6' 0,001''$

$\alpha_{2.1} = 65^\circ 16' 9,368''$

$\log \sin \eta = 8,8536889.9_n - 10$

$\log \tan F = 0,1535189.7$

$-[3,058 - 10] \eta^2 \cos^2 F = -8.2$

$\log \tan t = 9,0072071.4 - 10$

$t = +5^\circ 48' 19,815''.$

$\log \sin \eta = 8,8536890_n - 10$

$\log \sin F = 9,9129511 - 10$

$\log \tan \frac{L_{1.2}}{2} = 8,7926620_n - 10$

$2 \log W_F = 9,9980543 - 10$

$-\log (1 - e^2) = 0,0029084$

$-\frac{1}{2} 8.2 \text{ (s. o.)} = -4$

$\log \sin (F - B_2) = 7,5602644 - 10$

$S = 4,6855739 - 10$

$\log (F - B_2) = 2,8746905$ in sec.

The value of $\log (F - B_2)$ as computed at left results in $F - B_2 = 12' 29,36''$.

From this, we get as argument of W: $F - 9,37'$

and $2 \log W = 2 \log W_F + 75$ units of the seventh decimal-place.

Adding the 75 to the value of $\log (F - B_2)$, as computed at left, we get the more accurate value:

$\log (F - B_2) = 2,8746980$ in sec.

$F - B_2 = 12' 29,3729''$

$B_2 = 54^\circ 42' 50,599''.$

Results:		From p. 247:
B_2	$= 54^\circ\ 42'\ 50{,}599''$	50,600''
$L_{1.2}$	$= 7\ \ 6\ \ 0{,}001$ Easterly	0,000
$\alpha_{2.1}$	$= 65\ 16\ \ 9{,}368$	9,368.

Par. 14. Continuation. For triangle P_1FP_2 we shall first apply the formulae of Par. 10, p. 370 ff. to obtain a more accurate value for x, y, and $\Delta\alpha$. After denoting the points P_1FP_2 temporarily by CAB, we have from the statements on page 244 ff. and the calculations of the preceding paragraph:

$C = 59^\circ\ 33'\ \ 0{,}68921''$	$\log c = 5{,}6596750.1$	$B_3 = 52^\circ\ 30'\ 16{,}7''$
$A = 90^\circ$	$\log a = 5{,}7242591.353$	$B_1 = 54\ \ 55\ \ 20{,}0$
$B = 30\ \ 32\ \ 10{,}447$	$\log b = 5{,}4298235.8$	$B_2 = 54\ \ 42\ \ 50{,}6$
$\varepsilon = \quad 5'\ 11{,}136''$		

From this, we get first of all:

$K_3 = 0{,}9982777$	Mean:
$K_1 = 0{,}9977392$	$K = 0{,}99793393$
$K_2 = 0{,}9977849$	$\log K = 9{,}9991017.9 - 10.$

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The area F^* of the plane triangle will be obtained sufficiently approximate by the formula $F^* = \frac{1}{2} bc \sin\left(90^\circ - \frac{1}{3}\varepsilon\right)$. Since $\log \sin\left(90^\circ - \frac{1}{3}\varepsilon\right) = 9{,}9999999.4 - 10$, we find

$$\log F^* = 10{,}7884685.3.$$

For the purpose of converting this to $\log F$ we have further:

$\log \frac{c}{a_0} = 8{,}85504$	$\frac{c^2}{a_0^2} = 0{,}0051296$	$\frac{c^4}{a_0^4} = 0{,}0000263$
$\log \frac{a}{a_0} = 8{,}91962$	$\frac{a^2}{a_0^2} = 0{,}0069062$	$\frac{a^4}{a_0^4} = 0{,}0000476$
$\log \frac{b}{a_0} = 8{,}62518$	$\frac{b^2}{a_0^2} = 0{,}0017797$	$\frac{b^4}{a_0^4} = 0{,}0000032$
	$\frac{m^2}{a_0^2} = 0{,}0046052$	$\frac{m^4}{a_0^4} = 0{,}0000257$
	$= [7{,}66325 - 10]$	$= [5{,}410 - 10].$

Hence, formula (16), p. 367, yields:

$$\begin{cases} \log F^* = 10{,}7884685.3 \\ \frac{1}{8} M \frac{m^2 K}{a_0^2} = + \quad 2494.8 \\ M \frac{8n^4+9m^4}{1920 a_0^4} = + \quad 1.0 \\ \hline \log F = 10{,}7887181.1 \end{cases} \qquad \log \tfrac{1}{8} M = 5{,}73469 \text{ for units of the 7th decimal-place.}$$

$$\begin{cases} \log F = 10{,}7887181.1 \\ \log \frac{\varrho''}{a_0^2} = 1{,}70513820-10 \\ \hline \log \left(\varrho'' \frac{F}{a_0^2}\right) = 2{,}4938563.1 \end{cases}$$

Applying formula (1), p. 371, we obtain now:

$$\sin B_3 = 0{,}79340$$
$$\sin B_1 = 0{,}81837$$
$$\sin B_2 = 0{,}81628$$

$$(\sin B_1 - \sin B_2)^2 + (\sin B_2 - \sin B_3)^2 + (\sin B_3 - \sin B_1)^2 = 0{,}0011512$$

$$\frac{b^2+c^2}{a_0^2} \sin^2 B_1 + \frac{a^2+c^2}{a_0^2} \sin^2 B_2 + \frac{a^2+b^2}{a_0^2} \sin^2 B_3 \qquad = 0{,}01808$$

$$\begin{cases} \varrho'' \frac{F}{a_0^2} K = [2{,}4929581.0]\ \text{Sek.} = 5' 11{,}14161'' \\ \varrho'' \frac{F}{a_0^2} \left(0{,}0011512 \frac{e^2}{6} - 0{,}01808 \frac{e^2}{12}\right) = [7{,}4369_n - 10] = - \ 0{,}00273'' \\ \hline \varepsilon = 5' 11{,}13888''. \end{cases}$$

If we use formula (11), p. 374, instead, we find:

$$\varepsilon = \varrho'' \frac{F}{a_0^2} K + \varrho'' \frac{F}{a_0^2} \left\{ \begin{array}{l} - \frac{m^4}{2a_0^2} e^2 \sin^2 B_1 + \frac{b^2}{3a_0^2} e^2 \cos^2 B_1 \\ + \frac{a^2+3b^2-c^2}{12a_0^2} bc^2 \sin 2B_1 \end{array} \right\} + \cdots$$

and we get for the three smaller terms: $-0{,}00321'' + 0{,}00041'' + 0{,}00002''$ which add up to $-0{,}00278''$. This is essentially the same as we obtained above, and we continue to use the first value which is presumably more accurate (p. 382).

To compute $A - A^*$, etc., we shall be satisfied with set (1),

p. 362, because the higher terms will effect only the fifth decimal place of the seconds, as shown by (2), p. 389. We then have:

$$C - C^* = 103{,}71296'' \left(1 - \tfrac{1}{20}\, 0{,}000524\, K + \tfrac{1}{4K}\, 0{,}0003437\right)$$

$$A - A^* = 103{,}71296 \left(1 - \tfrac{1}{20}\, 0{,}002301\, K - \tfrac{1}{4K}\, 0{,}0001947\right)$$

$$B - B^* = 103{,}71296 \left(1 + \tfrac{1}{20}\, 0{,}002825\, K - \tfrac{1}{4K}\, 0{,}0001490\right)$$

or:

$$\left.\begin{aligned} C - C^* &= 103{,}71296'' + 0{,}00621'' = 1'\ 43{,}71917'' \\ A - A^* &= 103{,}71296 \ - 0{,}01697 \ = 1\ 43{,}69599 \\ B - B^* &= 103{,}71296 \ + 0{,}01076 \ = 1\ 43{,}72372 \end{aligned}\right\}$$

$$\varepsilon = 5'\ 11{,}13888''$$

and in the plane triangle:

$C^* =$ 59° 31′ 16,97004″	log sin = 9,9354158.096 − 10	log c = 5,6596749.999
$A^* =$ 89 58 16,30401	9,9999999.450 − 10	log a = 5,7242591.353
$B^* =$ 30 30 26,72595	9,7055643.912 − 10	log b = 5,4298235.815
Sum = 180 0 0,00000		

Hence,

$$\log x = 5{,}4298235.815_n \qquad \log y = 5{,}6596749.999_n .$$

The above value of ε which would not differ noticeably by applying the just found values of x and y, will give us further:

$$\Delta\alpha = -\ 5'\ 11{,}13888''.$$

When computing x and y by the series of Par. 12, p. 432, we obtain:

(See Illustrative Material on Next Page.)

5,4290814.956$_n$	5,6598033.984$_n$
+ 7414.427	− 1281.161
− 2.625	− 2.625
+ 8.859	− 0.076
+ 0.165	− 0.114
$\log x = 5{,}4298235.782_n$	$\log y = 5{,}6596750.008_n$

Using the formulae of Par. 11, p. 431, we get the following figures:

$$B_1 + F = 107^0\,25'\,36{,}672''$$

Argument for $W_m = 53^0\,42{,}8056'$

$$\cos(B_1 + F) = -0{,}29949$$
$$-[8{,}069-10]\cos^2(B_1+F) = -\ 105 + 803$$
Sum: $-0{,}29251 = [9{,}46614_n - 10]$

$$\log \beta_4 = 2{,}39785_n - 10 \qquad \log \beta_6 = 0{,}47 - 20$$

$$\log x = 5{,}4298235.815_n$$
$$8{,}5126900.290 - 10$$
$$3\log W_m = 9{,}9971687.926 - 10$$

$$\log\left(\frac{\varrho'' x}{\varrho_m}\right) = 3{,}9396824.031_n$$
$$-\beta_4\left(\frac{\varrho'' x}{\varrho_m}\right)^2 = +\ 1.893$$
$$-\beta_6\left(\frac{\varrho'' x}{\varrho_m}\right)^4 = 0$$

$$\log (B_1 - F)_{\text{in sec.}} = 3{,}9396825.924_n$$

$$B_1 - F = -\ 2^0\,25'\,3{,}27273'' \qquad F = 54^0\,55'\,19{,}97273''.$$

Argument for $W_F = 54^0\,55{,}33288'$

$$\log y = 5{,}6596749.999_n$$
$$8{,}5097816.695 - 10$$
$$W_F = 9{,}9990271.646 - 10$$

$$\eta = -\ 4^0\,5'\,39{,}536740'' \qquad \log \eta_{\text{in sec.}} = 4{,}1684838.340_n$$

$$\log \tan \eta = 8{,}8547988.144_n - 10$$
$$\log \sec F = 0{,}2405678.736$$
$$[2{,}028-20]\,\eta^4 \sin^2 F = +\ 0.034$$

$$\log \tan L_{1.2} = 9{,}0953666.914_n - 10$$
$$L_{1.2} = -\ 7^0\,6'\,0{,}00006''$$

$$\left\{\begin{array}{rl} & \log \sin \eta = 8{,}8536890.221_n - 10 \\ & \log \tan F = 0{,}1535189.519 \\ -\,[3{,}05810-10]\,\eta^2 \cos^2 F = & -\ 8.202 \\ +\,[1{,}250-20]\,\eta^4 (8\cos^2 F - 9) = & -\ 0.037 \end{array}\right.$$

$$\log \tan t = 9{,}0072071.501 - 10$$

$$\alpha_{2.1} = 65^\circ\ 16'\ 9{,}36537'' \left\{\begin{array}{rl} t = & 5^\circ\ 48'\ 19{,}81504'' \\ \Delta\alpha = & -\ 5\ 11{,}13888 \\ \alpha_{1.2} - 180^\circ = & 59\ 33\ 0{,}68921 \end{array}\right.$$

$$\left\{\begin{array}{rl} \text{Argument for } w & \log \sin \eta = 8{,}8536890.22_n - 10 \\ = 54^\circ\ 45{,}9657' & \log \sin F = 9{,}9129510.78 - 10 \\ & \log \tan \frac{L_{1.2}}{2} = 8{,}7926619.64_n - 10 \\ & 2 \log W = 9{,}9980617.99 - 10 \\ & -\log (1 - c^2) = 0{,}0029083.60 \\ & -\frac{1}{2}\ 8.202\ (\text{s. o.}) = -\ 4.10 \\ -\,[2{,}629-10]\,\eta^2 (1 + 3\cos^2 F) \tan^2 \frac{L_{1.2}}{2} = & -\ 0.07 \end{array}\right.$$

$$\log \sin (F - B_2) = 7{,}5602718.06 - 10$$

$$\log \underset{\text{in sec.}}{(F - B_2)} = \log \left(\varrho'' \sin (F - B_2)\right) + \frac{1}{6} M \sin^2 (F - B_2) + \cdots$$

$$\log \underset{\text{in sec.}}{(F - B_2)} = \left\{\begin{array}{r} 7{,}5602718.06 - 10 \\ 5{,}3144251.33 \\ +\ 9.55 \end{array}\right\} = 2{,}8746978.94$$

$$F - B_2 = +\ 12'\ 29{,}37275'' \qquad B_2 = 54^\circ\ 42'\ 50{,}59998''.$$

The above calculations give us the following:

		from p. 244 ff.	from p. 256 ff.
Results:	$B_2 = 54^\circ\ 42'\ 50{,}59998''$	60000	59998
	$L_{1.2} = 7\ \ 6\ \ 0{,}00006$ Easterly	00002	00002
	$\alpha_{2.1} = 65\ 16\ \ 9{,}36537$	36534	36539

From this, we discern the great accuracy of the formulae in Par. 11, p. 431, which was the primary intention of the preceding

Using the formulae in such cases as shown in the above paragraph will be quite inconvenient since we must solve the right spherical triangle.

Par. 15. Rectangular Coordinates, Distance, and Azimuths by Geographic Positions. It will be best to proceed from the formulae of Par. 11, p. 431, which have to be rearranged with regard to the spherical formulae on page 129. In Par. 11, we proved the validity of the formulae up to the Pole for $s:a_0$ as a magnitude of the first order. For the sake of simplicity, we assume now that, besides $s:a_0$, $L_{1.2}$ will also be a term of the first order. At the Pole, $\eta \sec F$ will then be of the first order and the ordinate η must be a small term of a higher order.

From (13), p. 425, we have:

$$\tan \eta = \tan L_{1.2} \cos F \left(1 - \tfrac{1}{15} k^2 \eta^4 + Gl_8\right), \qquad (1)$$

From this we derive, since on the sphere $\sin \eta = \sin L \cos B_2$:

$$\sin \eta = \sin L \cos F . \cos \eta \sec L \left(1 - \tfrac{1}{15} k^2 \eta^4 + Gl_8\right).$$

Replacing $\cos \eta$ and $\sec L$ by their expansions we get:

$$\sin \eta = \sin L \cos F \left(1 + \tfrac{1}{2} L^2 + \tfrac{5}{24} L^4 + \tfrac{61}{720} L^6 + Gl_8\right) \times \left(1 - \tfrac{1}{2} \eta^2 + \tfrac{1}{24} \eta^4 - \tfrac{1}{720} \eta^6 - \tfrac{1}{15} k^2 \eta^4 + Gl_8\right). \qquad (2)$$

By (11), p. 429, we find further:

$$\sin (F - B_2) = \sin \eta \tan \tfrac{L}{2} \sin F \left(1 + e^2 \cos^2 F + e^4 \cos^2 F - \tfrac{1}{12} e^2 \eta^2 + \tfrac{5}{6} e^2 \eta^2 \sin^2 F + Gl_6\right).$$

With regard to the formula $\sin(F-B_2) = \sin^2\frac{L}{2}\sin 2B_2 \sec\eta$ which is valid for the sphere, and by substituting expression (2), we convert the above equality to the following:

$$\sin(F-B_2) = \sin^2\frac{L}{2}\sin 2F\left\{\begin{array}{l} 1+\frac{L^2-\eta^2}{2}+e^2\cos^2F \\ \quad +e^2\cos^2F+\frac{\eta^4+5L^4}{24} \\ -\frac{1}{4}\eta^2L^2-\frac{1}{12}e^2\eta^2+\frac{5}{6}e^2\eta^2\sin^2F \\ \quad +\frac{L^2-\eta^2}{2}e^2\cos^2F+Gl_6. \end{array}\right\} \qquad (3)$$

When we further expand η with respect to powers of L from (1), we get:

$$\eta = L\cos F\left(1+\frac{1}{3}L^2\sin^2F+\frac{1}{15}L^4\sin^2F[3\sin^2F-1]+Gl_6\right). \qquad (4)$$

Hence:

$$\eta^2 = L^2\cos^2F\left(1+\frac{2}{3}L^2\sin^2F+\frac{1}{45}L^4\sin^2F[23\sin^2F-6]+Gl_6\right). \qquad (5)$$

From this, (2) and (3) will now read respectively:

$$\sin\eta = \sin L\cos F\left\{\begin{array}{l} 1+\frac{1}{2}L^2\sin^2F+\frac{1}{24}L^4(5-9\cos^2F)\sin^2F \\ \quad +\frac{1}{720}L^6\sin^2F(61-270\cos^2F+225\cos^4F) \\ \quad -\frac{1}{15}e^2L^4\sin^2F\cos^4F+Gl_6 \end{array}\right\} \qquad (6)$$

$$\sin(F-B_2) = \sin^2\frac{L}{2}\sin 2F\left\{\begin{array}{l} 1+\frac{1}{2}L^2\sin^2F+e^2\cos^2F+e^2\cos^2F \\ \quad -\frac{1}{12}e^2L^2\cos^2F+\frac{4}{3}e^2L^2\sin^2F\cos^2F \\ \quad +\frac{1}{24}L^4\sin^2F(5-9\cos^2F)+Gl_6 \end{array}\right\} \qquad (7)$$

To eliminate F from the right members of these formulae we formulate first of all:

$$\cos B_2 = \cos\left(F-(F-B_2)\right) = \cos F\cos(F-B_2)+\sin F\sin(F-B_2)$$
$$= \cos F\left(1+\sin(F-B_2)\tan F-\tfrac{1}{2}\sin^2(F-B_2)+Gl_6\right),$$

in which we must observe that $\sin(F-B_2)$ is a magnitude of the second order. We find now:

$$\cos F = \cos B_2\left\{\begin{array}{l}1-\sin(F-B_2)\tan F+\sin^2(F-B_2)\left(\frac{1}{2}+\tan^2 F\right)\\ \quad -\sin^3(F-B_2)\tan F\sec^2 F+Gl_8.\end{array}\right. \tag{8}$$

From this, we get by (7) when at the same time we replace $\sin^2\frac{L}{2}$ by the series $\frac{1}{4}L^2\left(1-\frac{1}{12}L^2+\frac{1}{360}L^4+Gl_6\right)$:

$$\sin(F-B_2)=\tfrac{1}{4}L^2\sin 2F\left\{\begin{array}{l}1-\frac{1}{12}L^2+\frac{1}{2}L^2\sin^2F+e^2\cos^2F+e^4\cos^2F\\ \quad -\frac{1}{6}e^2L^2\cos^2F+\frac{4}{3}e^2L^2\cos^2F\sin^2F\\ \quad +\frac{1}{360}L^4(61-195\cos^2F+135\cos^4F)+Gl_6.\end{array}\right. \tag{9}$$

Substituting this in (8), we find:

$$\cos F = \cos B_2\left\{\begin{array}{l}1-\frac{1}{2}L^2\sin^2F+\frac{1}{24}L^4\sin^2F(4-3\sin^2F)\\ \quad -\frac{1}{2}e^2L^2\sin^2F\cos^2F-\frac{1}{2}e^4L^2\sin^2F\cos^2F\\ \quad +\frac{1}{12}e^2L^4\sin^2F\cos^2F(4-5\sin^2F)\\ \quad +\frac{1}{720}L^6\sin^2F(-1+30\cos^2F-45\cos^4F)+Gl_8\end{array}\right. \tag{10}$$

Hence we get by (6):

$$\sin\eta = \sin L\cos B_2\left\{\begin{array}{c}1-\frac{1}{2}e^2L^2\sin^2F\cos^2F\\ -\frac{1}{15}e^2L^4\sin^2F\cos^2F(5-9\cos^2F)\\ -\frac{1}{2}e^4L^2\sin^2F\cos^2F+Gl_8.\end{array}\right. \tag{11}$$

To determine the left-hand term we must know F first. For the introduction of B_2 we develop as follows:

$$\sin\eta = \sin\left(\frac{y}{a_0}\sqrt{1-e^2\sin^2 F}\right) = \sin\left(\frac{y}{a_0}\sqrt{1-e^2\sin^2 B_2}\cdot\frac{\sqrt{1-e^2\sin^2 F}}{\sqrt{1-e^2\sin^2 B_2}}\right). \tag{12}$$

Now we obtain by means of Taylor's Theorem:

$$1-e^2\sin^2 B_2 = 1-e^2\sin^2 F - e^2\left(\sin 2F(B_2-F)+\cos 2F(B_2-F)^2+Gl_6\right)$$

and from this it follows by using (9):

$$1-e^2\sin^2 B_2 = 1-e^2\sin^2 F + e^2L^2\sin^2 F\cos^2 F\left(1-\tfrac{1}{3}L^2+L^2\sin^2 F\right)+Gl_6.$$

Hence:

$$\frac{\sqrt{1-e^2\sin^2 F}}{\sqrt{1-e^2\sin^2 B_2}} = 1-\frac{1}{2}e^2L^2\sin^2 F\cos^2 F - \frac{1}{2}e^4L^2\sin^2 F\cos^2 F + \frac{1}{2}e^2L^4\sin^2 F\cos^2 F\left(\frac{1}{3}-\sin^2 F\right)+Gl_6. \tag{13}$$

We are now denoting:

$\frac{y}{a_0}\sqrt{1-e^2\sin^2 B_2} = \frac{y}{a_0}W_2 = \frac{y}{\varrho_m}$ by η'; W_2 and ϱ_m respectively for the arg. B_2, (14)

For the time being, we let the radical quotient (13) equal $1+\zeta$ in which ζ will be a term of the fourth order. Then we get by (12):

$$\sin\eta = \sin(\eta'+\zeta\eta') = \sin\eta'\cos\zeta\eta' + \cos\eta'\sin\zeta\eta'$$
$$= \sin\eta'\left(1+\zeta\left[1-\tfrac{1}{3}\eta'^2\right]+Gl_6\right).$$

Substituting the value of ζ yields now:

$$\sin\eta = \sin\eta'\left\{1-\frac{1}{2}e^2L^2\sin^2 F\cos^2 F - \frac{1}{2}e^4L^2\sin^2 F\cos^2 F + e^2L^4\sin^2 F\cos^2 F\left(\frac{2}{3}\cos^2 F-\frac{1}{3}\right)+Gl_8.\right\} \tag{15}$$

Introducing this into the left-hand member of (11), then transposing for $\sin\eta'$ and finally replacing F in the highest-order terms by B_2, we obtain:

$$\sin\eta' = \sin L_{1.2}\cos B_2\left(1-\frac{1}{15}e^2L^4\sin^2 B_2\cos^4 B_2+Gl_8\right). \tag{16}$$

The above development will doubtless converge if both $s:a_0$ and $L_{1,2}$ are small first-order terms as we did assume. The formula seems to satisfy even at the Pole without the above condition since L is multiplied by $\cos B_2$. But this is not so. Let us suppose P_1 coincides with the Pole, then $B_1 = 90^0$; y will equal the meridian arc from B_2 to 90^0, and $L = 90^0$. In this case, $\sin \eta'$ will not equal $\cos B_2 (1 - Gl_6)$ but $\cos B_2 (1 - Gl_4)$ which may be verified from the formulae for the meridian arc (3), p. 50.

To introduce B_2 also into the right-hand member of (7) we write by Taylor's Theorem:

$$\sin 2B_2 = \sin 2F \left(1 + 2 (B_2 - F) \cot 2F - 2 (B_2 - F)^2 + Gl_6\right).$$

Introducing (9) and transposing for $\sin 2F$ results in:

$$\sin 2F = \sin 2B_2 \left(1 + \tfrac{1}{2} L^2 \cos 2F + \tfrac{1}{2} e^2 L^2 \cos^2 F \cos 2F - \tfrac{1}{4} L^4 \left(\tfrac{1}{3} \sin^2 F - \tfrac{5}{12}\right) + Gl_6\right). \qquad (17)$$

Formulae (7) will now read:

$$\sin (F - B_2) = \sin^2 \frac{L}{2} \sin 2B_2 \left\{ \begin{array}{c} 1 + \frac{1}{2} L^2 \cos^2 F + \frac{1}{24} L^4 \cos^2 F (8 - 3\cos^2 F) \\ + e^2 \cos^2 F + e^4 \cos^2 F \\ + e^2 L^2 \cos^2 F \left(\frac{1}{4} + \frac{2}{3} \cos^2 F\right) + Gl_6 \end{array} \right\}. \qquad (18)$$

Since the corresponding equation for the sphere has $\sec \eta'$ as a factor in the right-hand member it will also be introduced above. As $\sin \eta$ and $\sin \eta'$ are identical except for the terms of the fifth order we have

$$\sec \eta' = \sqrt{1 + \tan^2 \eta} + Gl_6 = \sqrt{1 + \tan^2 L \cos^2 F} + Gl_6$$

hence:

$$\sec \eta' = 1 + \tfrac{1}{2} L^2 \cos^2 F + \tfrac{1}{24} L^4 \cos^2 F (8 - 3 \cos^2 F) + Gl_6. \qquad (19)$$

Making use of this, we may transform (18) into:

$$\sin(F-B_2)=\sin^2\frac{L}{2}\sin 2B_2\sec\eta'\left\{\begin{array}{c}1+(e^2+e^4)\cos^2F\\+e^2L^2\cos^2F\left(\frac{1}{4}+\frac{1}{6}\cos^2F\right)+Gl_6\end{array}\right\}. \qquad (20)$$

In the right-hand factor, we better introduce $\frac{\varrho_n}{\varrho_m}$ for the argument B_2. From (12), p. 430, we have when taking $x=1$ in it:

$$\frac{\varrho_n}{\varrho_m}=1+(e^2+e^4)\cos^2F+e^2L^2\sin^2F\cos^2F+Gl_6;\ \text{arg. } B_2.$$

Substituting this in (20) we find:

$$\sin(F-B_2)=\left\{\begin{array}{c}\sin^2\frac{L}{2}\sin 2B_2\sec\eta'\cdot\frac{\varrho_n}{\varrho_m}\\\times\left(1+e^2L^2\cos^2B_2\left[-\frac{3}{4}+\frac{7}{6}\cos^2B_2\right]+Gl_6\right)\\\frac{\varrho_n}{\varrho_m}=\frac{W_2^3}{1-e^2};\ \text{arg. } B_2.\end{array}\right\} \qquad (21)$$

Par. 16. Continuation: Convergence of the Meridian, etc.

In connection with the development on pp. 422 ff., let us state the following equality in the auxiliary spherical triangle:

$$\tan t=-\tan\lambda\sin\beta_2.$$

Now, from page 423 we have $\tan\lambda$ equal to the left member of (9), p. 423, hence:

$$\tan\lambda=\tan L\left\{\begin{array}{c}1+\frac{1}{2}e^2\frac{\eta}{\tan L}\cos F\left[1+\frac{3}{4}e^2+\frac{1}{4}k^2+\frac{5}{8}e^4+\frac{1}{8}k^4\right.\\\left.+\frac{1}{4}e^2k^2+\frac{1}{4}k^2\eta^2+L^2\left(1+\frac{2}{3}L^2+\frac{1}{4}k^2+\frac{3}{4}e^2\right)\right]\\+\frac{1}{4}e^4\eta^2\cos^2F+Gl_8.\end{array}\right\} \qquad (1)$$

By (1) and (5), pp. 440 and 441, we get further:

$$\frac{\eta}{\tan L}=\cos F\left(1-\frac{1}{3}\eta^2-\frac{1}{45}\eta^4+Gl_6\right)$$

and

$$\eta^2=L^2\cos^2F\left(1+\frac{2}{3}L^2\sin^2F+Gl_4\right).$$

Substituting this in (1) to eliminate η we find:

$$\tan\lambda = \tan L\left\{1+\tfrac{1}{2}e^2\cos^2F\left[\begin{array}{l}1+\tfrac{1}{4}e^2(3+\sin^2F)+L^2\left(1-\tfrac{1}{3}\cos^2F\right)\\ +e^2L^2\left(1+\tfrac{1}{6}\sin^2F\cos^2F\right)\\ +\tfrac{1}{45}L^4(30-10\sin^2F\cos^2F-\cos^4F)\\ +\tfrac{1}{8}e^4(5+2\sin^2F+\sin^4F)\end{array}\right]+Gl_8\right\}. \qquad (2)$$

We have further:

$$\sin\beta_2 = \sin B_2\frac{\sqrt{1-e^2}}{\sqrt{1-e^2\sin^2B_2}} = \sin B_2\left\{\begin{array}{l}1-\tfrac{1}{2}e^2\cos^2B_2\left[1+\tfrac{1}{4}e^2(1+3\sin^2B_2)\right.\\ \left.+\tfrac{1}{8}e^4(1+2\sin^2B_2+5\sin^4B_2)\right]+Gl_8\end{array}\right\}. \qquad (3)$$

In the parenthesis at the right hand, we now eliminate B_2 by means of F. For this purpose, we get the following relation from (7), p. 428:

$$\cos^2B_2 = \cos^2F + L\eta\sin^2F\cos F\left(1+\tfrac{1}{12}L^2\right)+\tfrac{1}{4}\eta^2L^2\sin^2F+Gl_6,$$

which by means of (4), p. 441, namely:

$$\eta = L\cos F\left(1+\tfrac{1}{3}L^2\sin^2F\right)+Gl_5,$$

will be transformed into:

$$\cos^2B_2 = \cos^2F\left\{1+L^2\sin^2F+\tfrac{1}{3}L^4\sin^2F+\tfrac{1}{3}L^4\sin^4F+Gl_6\right\}. \qquad (4)$$

From this, we obtain further:

$$\sin^2B_2 = \sin^2F\left(1-L^2\cos^2F\right)+Gl_4. \qquad (5)$$

Substituting the values of $\cos^2B_2$ and $\sin^2B_2$ in (3), we find:

(See Illustrative Material on Next Page.)

$$\sin\beta_2 = \sin B_2\left\{1-\tfrac{1}{2}e^2\cos^2F\left[\begin{array}{l}1+\frac{1}{4}e^2(1+3\sin^2F)+L^2\sin^2F\\ +\frac{1}{2}e^2L^2\sin^2F(-1+3\sin^2F)\\ +\frac{1}{3}L^4\sin^2F+\frac{1}{5}L^4\sin^4F\\ +\frac{1}{8}e^4(1+2\sin^2F+5\sin^4F)\end{array}\right]+Gl_6\right\}. \qquad (6)$$

When we multiply each member of this equation by its respective member of (2), we get at the left hand $\tan\lambda\sin\beta_2$ which equals $-\tan t$ according to the equality at the beginning of this paragraph. Hence we obtain:

$$\tan t = -\tan L\sin B_2\left\{1+\tfrac{1}{3}e^2L^2\cos^4F\left[\begin{array}{l}1+\frac{1}{2}e^2(5-3\cos^2F)\\ +L^2\left(\frac{7}{6}-\frac{1}{5}\cos^2F\right)\end{array}\right]+Gl_6\right\}.$$

In the right-hand member, we substitute the value of B_2 from (4) and find:

$$\tan t = -\tan L_{1.2}\sin B_2\left\{1+\tfrac{1}{3}e^2L^2\cos^4B_2\left[\begin{array}{l}1+\frac{1}{2}e^2(5-3\cos^2B_2)\\ -L^2\left(\frac{5}{6}-\frac{9}{5}\cos^2B_2\right)\end{array}\right]+Gl_6\right\}. \qquad (7)$$

Having determined η', F and t by (16) and (21) of the previous paragraph and by (7) above, we then evaluate y from η' by (14), p. 442 and x from $F-B_1$ by (3), p. 50. x and y will then give the distance and azimuths by means of the equations in Par. 6, p. 416.

In these, we substitute as follows: $a_{1.2}=a_{1.2}$, $y_1=0$, hence $\Delta y = y_2 = y$ and $y=\frac{1}{2}y$, further $\Delta x = x$; and in the terms dependent on e^2: $x_1 = \frac{2}{3}x$, $x_2 = +\frac{1}{3}x$ and $x = -\frac{1}{6}x$. Hereby K_0 will be referred to the latitude $F+\frac{1}{2}(B_1-F)$, and we obtain:

$$\left.\begin{aligned} s\cos\alpha_{1.2} &= x\left\{1-\frac{1}{3}\frac{y^2}{\varrho^2}-\frac{1}{45}\frac{y^4}{\varrho^4}-\frac{2}{45}\frac{x^2y^2}{\varrho^4}+\frac{1}{36}\frac{xy^2}{\varrho^3}e^2\sin 2B_0+Gl_6\right\}\\ s\sin\alpha_{1.2} &= y\left\{1+\frac{1}{6}\frac{x^2}{\varrho^2}+\frac{7}{360}\frac{x^4}{\varrho^4}-\frac{2}{45}\frac{x^2y^2}{\varrho^4}-\frac{1}{18}\frac{x^3}{\varrho^3}e^2\sin 2B_0+Gl_6\right\}. \end{aligned}\right| \qquad (8)$$

This set of formulae is easily brought into logarithmic form since x and y respectively are factors of every term. However, it will be more convenient at first to turn to $s\cos\left(\alpha_{1.2}+\frac{1}{2}\Delta\alpha\right)$ and $s\sin\left(\alpha_{1.2}+\frac{1}{2}\Delta\alpha\right)$ by means of the following expansions:

$$\begin{aligned} s\cos\left(\alpha_{1.2}+\tfrac{1}{2}\Delta\alpha\right) &= s\cos\alpha_{1.2}\left(1-\frac{1}{8}\Delta\alpha^2+Gl_5\right)\\ &\quad -\frac{1}{2}\Delta\alpha\, s\sin\alpha_{1.2}\,(1+Gl_6)\\ s\sin\left(\alpha_{1.2}+\tfrac{1}{2}\Delta\alpha\right) &= s\sin\alpha_{1.2}\left(1-\frac{1}{8}\Delta\alpha^2+Gl_5\right)\\ &\quad +\frac{1}{2}\Delta\alpha\, s\cos\alpha_{1.2}\,(1+Gl_6). \end{aligned}$$

Now we have:

$$\Delta\alpha = -\frac{xy}{2\varrho^2}\left(1+\frac{x^2+y^2}{12\varrho^2}+Gl_4\right) \qquad (9)$$

and hence:

$$\left.\begin{aligned} s\cos\left(\alpha_{1.2}+\tfrac{1}{2}\Delta\alpha\right) &= x\left\{1-\frac{1}{12}\frac{y^2}{\varrho^2}-\frac{1}{720}\frac{y^4}{\varrho^4}-\frac{19}{1440}\frac{x^2y^2}{\varrho^4}\right.\\ &\qquad \left.+\frac{1}{36}\frac{xy^2}{\varrho^3}e^2\sin 2B_0+Gl_6\right\}\\ s\sin\left(\alpha_{1.2}+\tfrac{1}{2}\Delta\alpha\right) &= y\left\{1-\frac{1}{12}\frac{x^2}{\varrho^2}-\frac{1}{720}\frac{x^4}{\varrho^4}-\frac{19}{1440}\frac{x^2y^2}{\varrho^4}\right.\\ &\qquad \left.-\frac{1}{18}\frac{x^3}{\varrho^3}e^2\sin 2B_0+Gl_6\right\} \end{aligned}\right\}. \qquad (10)$$

Par. 17. Summary. Given: B_1, B_2 and $L_{1.2}$.

Wanted: x, y of P_2 with respect to the meridian of P_1; further s, $\alpha_{1.2}$ and $\alpha_{2.1}$.

The formulae developed in the two previous paragraphs will at

once be written in their logarithmic forms. In the smaller terms, L and $B_1 - F$ are always expressed in seconds.

$$\log \sin \eta' = \log (\sin L_{1,2} \cos B_2) - \frac{M e^2}{15 \varrho''^4} L^2 \sin^2 B_2 \cos^4 B_2 + Gl_3 \qquad (1)$$

$\log \frac{M e^2}{15 \varrho''^4} = 2{,}0284 - 20$ for units of the seventh decimal-place.

$$y = \frac{\eta' \text{ in sec.}}{W_2} \cdot \frac{a_0}{\varrho''} = \frac{\eta'' \text{ in sec.}}{W_2} \quad [1{,}4902183.305] \qquad (2)$$

W_2 for argument B_2.

$$\log \sin (F - B_2) = \log \left(\sin^2 \frac{L_{1,2}}{2} \sin B_2 \cos B_2 \sec \eta' \cdot \frac{2 W_2^2}{1 - e^2} \right) + \frac{M e^2}{\varrho''^2} L^2 \cos^2 B_2 \left(-\frac{3}{4} + \frac{7}{6} \cos^2 B_2 \right) + Gl_4 \qquad (3)$$

$\log \frac{2}{1 - e^2} = 0{,}3039383.552$; $\log \frac{M e^2}{\varrho''^2} = 3{,}2334 - 10$ for units of the 7th decimal-place.

$$\log x = \log \left(\frac{\varrho_m (B_1 - F) \text{in sec.}}{\varrho''} \right) + \beta_2 (B_1 - F)^2 + \beta_4 (B_1 - F)^4 + Gl_5 \qquad (4)$$

$\frac{\varrho_m}{\varrho''} = \frac{a_0 (1 - e^2)}{\varrho'' W_m^3} = [1{,}4873099.710]$, W_m for arg. $\frac{1}{2}(B_1 + F)$

$\beta_2 = \frac{M e^2}{2 \varrho''^2} \left(\cos (B_1 + F) - 7 n \cos^2 (B_1 + F) + 5 n \right)$ $\beta_4 = -\frac{1}{20 \varrho''^2} \beta_2$

$\beta_2 = [2{,}93171 - 10] \left(\cos (B_1 + F) - [8{,}069 - 10] \cos^2 (B_1 + F) + 0{,}00803 \right)$ for units of the seventh decimal-place.

$\beta_4 = -[8{,}070 - 20] \beta_2$

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$$\log \tan t = \log (-\tan L_{1,2} \sin B_2) + \frac{M e^2}{3 \varrho''^2} L^2 \cos^2 B_2 + \frac{M e^2}{6 \varrho''^2} L^2 \cos^2 B_2 (5 - 8 \cos^2 B_2) - \frac{M e^2}{3 \varrho''^4} L^4 \cos^4 B_2 \left(\frac{5}{6} - \frac{9}{5} \cos^2 B_2 \right) + Gl_6 \qquad (5)$$

$\log \frac{M e^2}{3 \varrho''^2} = 3{,}35622 - 10$

$\log \frac{M e^2}{6 \varrho''^2} = 0{,}880 - 10$ $\log \frac{M e^2}{3 \varrho''^4} = 2{,}727 - 20$ for units of the seventh decimal-place

$$\varrho = \frac{a_0}{\sqrt{K_0}} = \frac{b_0}{W_0^2} = \frac{[6{,}8031893]}{W_0^3} \qquad (6)$$

W_0 for arg. $B_0 = F + \frac{1}{2}(B_2 - F)$

$$\log \left\{ s \cos \left(\alpha_{1,2} + \frac{1}{2} \Delta \alpha \right) \right\} = \log x - \frac{1}{12} M \frac{y^2}{\varrho^2} - \frac{7}{1440} M \frac{y^4}{\varrho^4} - \frac{19}{1440} M \frac{x^2 y^2}{\varrho^4} + \frac{1}{36} M^2 \frac{x y^2}{\varrho^2} \sin 2 B_0 + Gl_6 \qquad (7)$$

$$\log\left\{s\sin\left(\alpha_{1.2}+\frac{1}{2}\Delta\alpha\right)\right\}=\log y-\frac{1}{12}M\frac{x^2}{\varrho^2}-\frac{7}{1440}M\frac{x^4}{\varrho^4}-\frac{19}{1440}M\frac{x^2y^2}{\varrho^4}\Bigg\}$$
$$-\frac{1}{18}Me^2\frac{x^2}{\varrho^2}\sin 2B_0+Gl_3 \tag{8}$$

$$\log\underset{\text{in sec.}}{\Delta\alpha}=\log\left\{-\frac{1}{2}\varrho''\frac{xy}{\varrho^2}\right\}+\frac{1}{12}M\frac{x^2+y^2}{\varrho^2}+Gl_4 \tag{9}$$

$$\alpha_{2.1}=\alpha_{1.2}+180^0+\Delta\alpha+t; \tag{10}$$

$$\log\left(\frac{1}{12}M\right)=5{,}5586030\left\{\begin{array}{ll}\log\frac{7M}{1440}=4{,}325 & \log\frac{19M}{1440}=4{,}758\\ \log\left(\frac{1}{36}Me^2\right)=2{,}906 & \log\left(\frac{1}{18}Me^2\right)=3{,}207\end{array}\right\}\text{for units of the seventh decimal-place}$$

$$\log\left(\frac{1}{2}\varrho''\right)=5{,}0133951.$$

Par. 18. Numerical Example I. $B_1=52^0\,30'\,16{,}7''$ $B_2=54^0\,42'\,50{,}6''$

$$L_{1.2}=7^0\,6'\,0''\text{Easterly}=-25560''.$$

The formulae of Par. 17 give in turn:

$$\left\{\begin{array}{rl}\log\sin L_{1.2}= & 9{,}0920236.596_n-10\\ \log\cos B_2= & 9{,}7616703.379-10\\ -[2{,}028-20]\,L^4\sin^2B_2\cos^4B_2= & -\quad 0.034\\ \hline \log\sin\eta'= & 8{,}8536939.941_n-10\end{array}\right.$$

$$\eta'=-4^0\,5'\,39{,}705771''=-14739{,}705771''.$$

$$\left\{\begin{array}{rl}\log\underset{\text{in sec.}}{\eta'}= & 4{,}1684888.149_n\\ & 1{,}4902183.305\\ -\log W_2= & 0{,}0009678.539\\ \hline \log y= & 5{,}6596749.993_n.\end{array}\right.$$

(See Illustrative Material on Next Page.)

$$\left\{\begin{array}{l|l} & 2\log\sin\frac{L_{1,2}}{2} = 7{,}5836556.26 - 10 \\ \text{Argument for } W_2 & \log\cos B_2 = 9{,}7616703.38 - 10 \\ = 54^\circ\,42{,}8433' & \log\sin B_2 = 9{,}9118387.96 - 10 \\ & \log\sec\eta' = 0{,}0011098.18 \\ & 2\log W_2 = 9{,}9980642.92 - 10 \\ & \qquad\qquad 0{,}3039383.55 \\ [3{,}83334 - 10]\, L^2\cos^2 B_2\left(-\frac{3}{4}+\frac{7}{6}\cos^2 B_2\right) = & \qquad\quad -\ 53.57 \end{array}\right.$$

$$\log\sin(F - B_2) = 7{,}5602718.68 - 10$$

$$F = 54^\circ\,55'\,19{,}97285'' \qquad F - B_2 = +\ 12'\,29{,}37285''$$

$$B_1 - F = -\ 2^\circ\,25'\,3{,}27285'' \qquad B_1 - F = -\ 8703{,}27285''$$

$$\left\{\begin{array}{ll} \log(B_1 - F) = & 3{,}9396825.990_n \\ \text{in sec.} & 1{,}4873099.710 \\ -3\log W_m = & 0{,}0028312.074 \\ \text{For this, cf. p. 438.} - & \qquad 1.893 \end{array}\right.$$

$$\log x = 5{,}4298235.881_n.$$

$$\left\{\begin{array}{rl} \log\tan L_{1,2} = & 9{,}0953666.904_n - 10 \\ \log\sin B_2 = & 9{,}9118387.956 - 10 \\ +\,[3{,}35622 - 10]\, L^2\cos^4 B_2 = & +\ 16.520 \\ +\,[0{,}880 - 10]\, L^2\cos^4 B_2\,(5 - 3\cos^2 B_2) = & +\ 0.221 \\ -\,[2{,}727 - 20]\, L^4\cos^4 B_2\left(\frac{5}{6}-\frac{9}{5}\cos^2 B_2\right) = & -\ 0.059 \end{array}\right.$$

$$\log\tan t = 9{,}0072071.542 - 10$$

$$t = +\,5^\circ\,48'\,19{,}8152''.$$

The above value of $\log x$ differs from the one determined on page 437 by a considerable amount. The reason for this is the neglect of the higher terms for $F - B_2$ in (3).

This formula in particular is not as precise as the corresponding formula (5), p. 431; this is indicated by the relative magnitude of the fourth-order terms in the two equations. Formula (3) will

be precise enough, however, if the following formulae of the preceding paragraph are applied in further calculation.

We find by:

Formula (7):	Formula (8):	Formula (9):
5,4298235.88	5,6596749.99	2,4927011
− 1852.51	− 642.77	+ 1853
− 0.55	− 0.07	+ 643
− 0.52	− 0.52	2,4929507
− 0.17	+ 0.11	$= \log \Delta\alpha$ in sec.
5,4296382.13	5,6596106.74	
$= \log\left\{s \cos\left(\alpha_{1.2} + \frac{1}{2}\Delta\alpha\right)\right\}$	$= \log\left\{s \sin\left(\alpha_{1.2} + \frac{1}{2}\Delta\alpha\right)\right\}$	$\Delta\alpha = -\ 5'11{,}1363''$
		$t = 5^0 48' 19{,}8152''$
		$\alpha_{1.2} = 239^0 33'\ 0{,}6868''$
		$\alpha_{2.1} = 65^0 16'\ 9{,}3657''$

$$\alpha_{1.2} + \frac{1}{2}\Delta\alpha = 239^0\ 30'\ 25{,}1186''$$

$$\log s = 5{,}7242591.37$$

From p. 261:

Results: $\log s = 5{,}7242591.37$ .358

$\alpha_{1.2} = 239^0\ 33'\ 0{,}6868''$ 0,6889''

$\alpha_{2.1} = 65\ 16\ 9{,}3657$ 9,3650

For a more precise computation, we may apply (10), p. 429, in which the value of $\frac{\varrho_n}{\varrho_m}$ for $x = 1$ has to be substituted from (12), p. 430. After a simple reduction we get:

$$\log \sin(F - B_2) = \left\{ \begin{array}{c} \log\left(-\tan t \cos F \tan \frac{L_{1.2}}{2} \cdot \frac{W_2^2}{1 - e^2}\right) \\ + \frac{M e^2}{\varrho''^2} L^2 \cos^2 B_2 \left(-\frac{1}{4} + \frac{1}{3}\cos^2 B_2 - \frac{1}{4}\tan^2 \frac{L}{2}\right) + G l_0 \end{array} \right\}$$

Introducing the above-calculated values of t and F into this, we find:

$$\log \sin(F - B_2) = 7{,}5602718.20 - 10 \qquad F - B_2 = 12'\ 29{,}37277''.$$

With this value, $\log x$ will decrease by 40 units of the tenth decimal-place and we obtain a much better agreement than before, provided that $s = -\Delta s$ will be computed from the sides x and y and the included angle 90^0 by one of the precise

Par. 19. Conversion of Geographic Coordinates by the Sides of Triangles. When we assume that $s \leq 0{,}02 a_0$, i.e. 127km, the formulae will become simplified for even the most precise calculation. First of all, we abbreviate the formulae for x, y and $\Delta\alpha$ in Par. 12, p. 432, as follows:

$$\log x = \log u + \frac{1}{3}\frac{M}{b_0^2} v^2 W_0^2 + \cdots \qquad (1)$$

$$\log y = \log v - \frac{1}{6}\frac{M}{b_0^2} u^2 W_0^2 + \cdots \qquad (2)$$

$$\underset{\text{in sec.}}{\Delta\alpha} = -\frac{1}{2}\frac{\varrho''}{b_0^2} xy W_0^2 + \cdots . \qquad (3)$$

The greatest omissions of the first formula will influence x by:

$$\left.\begin{aligned} &+\tfrac{1}{90}\varrho''\left(\tfrac{s}{a_0}\right)^5 \sin^2\alpha_{1.2}\cos\alpha_{1.2}\,(7\sin^2\alpha_{1.2} - 6\cos^2\alpha_{1.2} \\ &-\tfrac{1}{36}\varrho'' e^2\left(\tfrac{s}{a_0}\right)^4 \sin^2\alpha_{1.2}\cos^2\alpha_{1.2}\sin 2B_0, \end{aligned}\right\} \text{ to (1)}$$

as expressed in equator seconds; or as the central angle for a sphere having the radius a_0 in which the error of x will be considered as an arc of the great circle. The maximum error will then be:

$$\left.\begin{aligned} &\pm 3900''\left(\tfrac{s}{a_0}\right)^5 \text{ for } \tan\alpha_{1.2} = \pm 2{,}5 \quad \alpha_{1.2} \text{ in quad. I} = 68\tfrac{1}{3}^\circ \quad B_0 \text{ any value} \\ &\pm\ 9{,}5''\left(\tfrac{s}{a_0}\right)^4 \text{ for } \tan\alpha_{1.2} = \pm 1 \quad \alpha_{1.2} \text{ in quad. I} = 45^\circ \quad B_0 = \pm 45^\circ. \end{aligned}\right\} \text{ to (1)}$$

For y, the principal error-causing terms, when reduced to equator seconds, will give:

$$\left.\begin{aligned} &-\tfrac{1}{15}\varrho''\left(\tfrac{s}{a_0}\right)^4 \sin^3\alpha_{1.2}\cos^2\alpha_{1.2} \\ &+\tfrac{1}{18}\varrho'' e^2\left(\tfrac{s}{a_0}\right)^4 \sin\alpha_{1.2}\cos^3\alpha_{1.2}\sin 2B_0. \end{aligned}\right\} \text{ to (2)}$$

They will reach a maximum of about:

$$\left.\begin{array}{l|l} \pm 2600''\left(\frac{s}{a_0}\right)^6 \text{ for } \tan\alpha_{1.2} = \pm\sqrt{\frac{3}{2}} & \alpha_{1.2} \text{ in quad. I.} = 50\frac{2}{3}^\circ \quad B_0 \text{ any value} \\ \pm\ 25''\left(\frac{s}{a_0}\right)^6 \text{ for } \tan\alpha_{1.2} = \pm\sqrt{\frac{1}{3}} & \alpha_{1.2} \text{ in quad. I.} = 30^\circ \quad B_0 = \pm 45^\circ. \end{array}\right\} \quad \text{to (2)}$$

Comparing (3) with (9), p. 448, we find the error of $\Delta\alpha$ computed by (3) to be:

$$\left.\begin{array}{l} -\frac{1}{24}\rho''\left(\frac{s}{a_0}\right)^6 \sin\alpha_{1.2}\cos\alpha_{1.2} \\ \text{and the maximum will be approximately:} \\ \pm 4300''\left(\frac{s}{a_0}\right)^6 \text{ for } \tan\alpha_{1.2} = \pm 1;\ \alpha_{1.2} \text{ in quad. I.} = 45^\circ. \end{array}\right\} \quad \text{to (3)}$$

In the case of $s = 0{,}02a_0$, we get x and y as accurately as 0,000012" and 0,000008" respectively, and $\Delta\alpha$ will only differ by 0,0007".

In the case of $s = 0{,}05a_0$, the respective errors will be 0,0012", 0,0008", and 0,027", at the maximum. They also stay within the limits in which they may be disregarded for certain purposes.

Further, for computing the argument B_0 for W_0 it will suffice to use the formulae:

$$B_0 = B_1 + \frac{2}{3}(F - B_1), \quad \underset{\text{in min.}}{F - B_1} = [6{,}732_n - 10]\,u. \qquad (4)$$

In the corresponding equations (2), p. 432, we introduced W_0 as a mean value, i.e. 9,9993 - 10; therefore, $F - B_1$ will only differ by fractions of a minute from the accurate value, which may be verified from the tables of $\log W$ as being insignificant.

Par. 20. Continuation. We replace the formulae of Par. 11, p. 431, by the simpler ones:

$$\log (B_1 - F)_{\text{in sec.}} = \log\left(\frac{\varrho'' x}{\varrho_m}\right) - \frac{M e^2}{8 \varrho''^2}\left(\frac{\varrho'' x}{\varrho_m}\right)^2 \cos (B_1 + F) + \cdots \quad (1)$$

$$\tan L_{1,2} = + \tan \eta \sec F + \cdots \quad (2)$$

$$\tan t \quad = - \sin \eta \tan F + \cdots \quad (3)$$

$$\sin (F - B_2) = \sin \eta \sin F \tan \frac{L_{1,2}}{2} \cdot \frac{W^2}{1 - e^2} + \cdots . \quad (4)$$

When applying these formulae, the resulting error will depend on F and $L_{1,2}$ and on x or y. We substitute in s the two last terms and obtain the following expressions which exert the maximum influence on errors:

$$\text{in } B_1 - F: + \frac{1}{8} \varrho'' e^4 \left(\frac{s}{a_0}\right)^3 \left(6 \sin^2 \frac{B_1 + F}{2} - 7 \sin^4 \frac{B_1 + F}{2}\right)$$

$$\text{in } L_{1,2}: + \frac{1}{15} \varrho'' e^2 \left(\frac{s}{a_0}\right)^5 \tan F \sin F \cos^2 L$$

$$\text{in } t: + \frac{1}{6} \varrho'' e^2 \left(\frac{s}{a_0}\right)^3 \sin F \cos F \cos^2 t$$

$$\text{in } F - B_2: \begin{cases} - \frac{1}{12} \varrho'' e^2 \left(\frac{s}{a_0}\right)^3 \sin F \cos^2 F \tan \frac{L}{2} \\ - \frac{1}{16} \varrho'' e^2 \left(\frac{s}{a_0}\right)^3 (1 + 3 \cos^2 F) \sin F \tan^3 \frac{L}{2} \end{cases}.$$

For the errors in $L_{1,2}$ and t, it was considered that an error δ in $\tan u$ would correspond to an error $\delta \cos^2 u$ in u itself.

In addition to the above errors, we have strictly speaking those caused by incorrect computing of x and y from Par. 19. We shall assume, however, that the simplifications introduced there were permissible; hence we shall disregard errors of x and y.

The error in $B_1 - F$ will become a maximum for $\frac{1}{2}(B_1 + F)$ which is about $\pm 41^0$; i.e., a mean latitude of about $\pm 41^0$; the maximum error in $B_1 - F$ will be

$$+ 1{,}5'' \left(\frac{s}{a_0}\right)^3,$$

to (1)

which will give 0,000012" in the case of $s = 0{,}02a_0$, and 0,0002" in the case of $s = 0{,}05a_0$.

To determine the maximum of the factor $\tan F \sin F \cos^2 L$ in the error expression for $L_{1,2}$, we observe that by (13), p. 425, $\eta \sec F$ is an approximate value of $\tan L$.

From this, we get the following approximation which is sufficient to estimate the amount of error:

$$\tan F \sin F \cos^2 L = \frac{\tan F \sin F}{1 + \eta^2 \sec^2 F} = \frac{\sin^2 F \cos F}{\cos^2 F + \eta^2}.$$

When we differentiate this, we obtain this condition as the maximum of the last expression:

$$\cos^4 F + (1 + 3\eta^2)\cos^2 F = \eta^2$$

which results in $\cos^2 F = \eta^2$, disregarding the higher terms. The maximum value of the above factor is then a satisfactory approximation to $1:2\eta$. Now we have to replace η by $s:a_0$ and obtain the maximum error in $L_{1,2}$ as:

$$\pm 46'' \left(\frac{s}{a_0}\right)^4, \qquad \text{to (2)}$$

which will be about 0,000006" in the case of $s = 0{,}02a_0$, and about 0,0003" for $s = 0{,}05a_0$. If we exclude the proximity of the Pole, the error will be much smaller.

To determine the maximum of the factor $\sin F \cos F \cos^2 t$ we find that from page 426 $-\eta \tan F$ will be a suitable approximation for $\tan t$.

We then have with sufficient accuracy:

$$\sin F \cos F \cos^2 t = \frac{\sin F \cos F}{1 + \eta^2 \tan^2 F}.$$

Because of the factor $\cos F$, the denominator will not exert any considerable influence upon the maximum; it will be reached when F equals about $\pm 45^0$ which will also be found by differentiation. From this, we get the maximum error in t to be:

$$\pm 115'' \left(\frac{s}{a_0}\right)^6, \qquad \text{to (3)}$$

which yields 0,0009" for $s = 0{,}02a_0$, and may therefore be omitted. In the case of $s = 0{,}05a_0$, the maximum error will be 0,014" which is sufficiently small for certain purposes.

In any case, the error will be relative to the accuracy in computing Δu.

In the first term of the expression for $F - B_2$ we substitute:

$$\cos F \tan \frac{L}{2} = \frac{1}{2}\eta.$$

Since by (8), p. 429, $\cos F \tan \frac{L}{2} = \frac{1}{2}\eta\left(1 - \tan^2 \frac{L}{2}\right)$ and $\tan^2 \frac{L}{2}$ will be equal to 1 at the most, this supposition corresponds to the maximum. Substituting $\frac{s}{a_0}$ for η and remembering that the maximum of $\frac{1}{2}$ for $\sin F \cos F$ will occur when $F = 45^0$, we arrive at the maximum error in $F - B_2$, except for the vicinity of the Pole:

$$\pm 57'' \left(\frac{s}{a_0}\right)^4, \qquad \text{to (4)}$$

which gives 0,000009" for $s = 0{,}02a_0$, and for $s = 0{,}05a_0$ not quite 0,0004".

In (4) we assumed the argument for W to be $F + \frac{3}{4}(B_2 - F)$; if we consider F to be the argument, then for $s = 0{,}02a_0$ the error will exert a considerable influence upon the fourth decimal place of seconds.

The second term of the expression for $F - B_2$ will be significant only very close to the Pole. While the factor $(1 + 3\cos^2 F)\sin F$ decreases slowly near the Pole with increasing F, $\tan^3 \frac{L}{2}$ will

increase very rapidly. Because when $F = 90^0 - \eta$, $L = 45^0$ approximately, and

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when $F = 90^0$ then $L = 90^0$. This term will therefore reach its maximum in the case of $F = 90^0$ and its value will be:

$$\pm 86'' \left(\frac{s}{a_0}\right)^3, \qquad \text{to (4)}$$

which yields 0,0007" for $s = 0{,}02 a_0$. This error is of no practical importance since it will decrease to 0,00005" for $F = 90^0 - \eta$. Even in the case of $s = 0{,}05 a_0$ it will only be significant very close to the Pole.

Par. 21. Expansions for $L_{1.2}$, t and $F - B_2$. When we assume that not only η but also $L_{1.2}$ or its equivalent, $\eta \sec F$, are small magnitudes, then we may, under certain circumstances, prefer to use the series which were already presented in Par. 8, p. 126 ff., instead of applying the formulae of the previous paragraph.

We get at once:

$$\log \underset{\text{in sec.}}{L_{1.2}} = \left\{ \begin{array}{l} \log(\varrho'' \eta \sec F) - \frac{1}{3} M \eta^2 \tan^2 F \\ + \frac{1}{90} M \eta^4 \tan^2 F [13 \tan^2 F + 6] + Gl_6 \end{array} \right\} \qquad (1)$$

$$\log \underset{\text{in sec.}}{t} = \left\{ \begin{array}{l} \log(-\varrho'' \eta \tan F) - \frac{1}{6} M \eta^2 [2 \tan^2 F + 1] \\ + \frac{1}{180} M \eta^4 [26 \tan^4 F + 20 \tan^2 F - 1] + Gl_6 \end{array} \right\} \qquad (2)$$

$$\log \underset{\text{in sec.}}{(F - B_2)} = \left\{ \begin{array}{l} \log\left(\frac{1}{2} \varrho'' \eta^2 \tan F \cdot \frac{W^2}{1 - e^2}\right) - \frac{1}{12} M \eta^2 [3 \tan^2 F + 1] \\ + \frac{1}{1440} M \eta^4 [135 \tan^4 F + 90 \tan^2 F - 1] + Gl_6. \end{array} \right\} \qquad (3)$$

We reduce $\tan^2 F + 1$ to $\sec^2 F$ in the second right-hand term of

(2) and (3) and substitute in the same term the following relations which may easily be derived from (1) and (2):

$$\eta^2 \sec^2 F = L^2 + \frac{2}{3}\eta^4 \tan^2 F(\tan^2 F + 1) + Gl_6$$
$$\eta^2 \tan^2 F = t^2 + \frac{1}{3}\eta^4 \tan^2 F(2\tan^2 F + 1) + Gl_6.$$

Then formulae (2) and (3) will read more conveniently:

$$\log \underset{\text{in sec.}}{t} = \begin{Bmatrix} \log(-\varrho''\eta \tan F) - \frac{1}{6} M[\eta^2 \tan^2 F + L^2] \\ + \frac{1}{180} M\eta^4 [6\tan^4 F - 1] + Gl_6 \end{Bmatrix} \qquad (4)$$

$$\log \underset{\text{in sec.}}{(F - B_3)} = \begin{Bmatrix} \log\left(\frac{1}{2}\varrho''\eta^2 \tan F \cdot \frac{W^3}{1-e^2}\right) - \frac{1}{12} M[2t^2 + \eta^2 \sec^2 F] \\ - \frac{1}{1440} M\eta^4 [25\tan^4 F - 10\tan^2 F + 1] + Gl_6. \end{Bmatrix} \qquad (5)$$

In the case of $\eta = 0{,}02$ and $F = 65^0$ with $\tan = 2{,}14$, and $\tan^2 = 4{,}6$, and $\sec = 2{,}37$, the terms multiplied by η^4 in (1), (4), and (5) exert the following influence:

upon	$L_{1.2}$	:	0,0054"
"	t	:	0,0010"
"	$F - B_3$	:	0,000004".

From this we infer that for $s \leq 0{,}02 a_0$ and $F < 65^0$ we may omit the fourth-order terms of (4) and (5), but should retain them for (1). Only for $\eta \sec F < 0{,}012$ will the influence of the fourth-order term in this formula become smaller than 0,00001" (replacing this term by the greater value of $\frac{1}{90} M\eta^4 \sec^2 F . 13 \sec^2 F$, etc. will prove this easily). On the other hand, it may be seen that for the assumed limits of η and F the sixth-order terms omitted in (1) will not influence the

first five decimal-places of seconds in $L_{1.2}$.

Par. 22. Summary. Given: B_1, s, and $\alpha_{1.2}$; or alternatively, x and y for P_2 referred to the meridian of P_1.

Wanted: B_2, $L_{1.2}$ and $\alpha_{2.1}$.

$$\left.\begin{aligned} u &= s\cos\alpha_{1.2} \\ v &= s\sin\alpha_{1.2} \end{aligned}\right| \left.\begin{aligned} \log x &= \log u + \frac{1}{3}\frac{M}{b_0^2}v^2W_0^4 + \cdots \\ \log y &= \log v - \frac{1}{6}\frac{M}{b_0^2}u^2W_0^4 + \cdots \\ \underset{\text{in sec.}}{\Delta\alpha} &= -\frac{1}{2}\frac{\varrho''}{b_0^2}xyW_0^4 + \cdots \end{aligned}\right\} \quad (1)$$

$$\log\left(\frac{1}{3}\frac{M}{b_0^2}\right) = 2{,}55428-10 \qquad \log\left(\frac{1}{6}\frac{M}{b_0^2}\right) = 2{,}25325-10,$$

both for units of the seventh decimal-place;

$$\log\left(-\frac{1}{2}\frac{\varrho''}{b_0^2}\right) = 1{,}407017_n-10$$

W_0 for arg. $B_0 = B_1 + \frac{2}{3}(F-B_1)$, where $(F-B_1)$ in min. $= [6{,}732_n-10]_n$.

$$\underset{\text{in sec.}}{\log(B_1-F)} = \log\left(\frac{\varrho''x}{\varrho_m}\right) - \frac{Me^2}{8\varrho''^2}\left(\frac{\varrho''x}{\varrho_m}\right)^2\cos(B_1+F) + \cdots \quad (2)$$

$$\frac{\varrho''}{\varrho_m} = \frac{\varrho''W_m^3}{a_0(1-e^2)} = [8{,}5126900.29-10]\,W_m^3$$

W_m for arg. $\frac{1}{2}(B_1+F)$, to be derived from the above value of $F-B_1$ by the 1st approximation.

(Note the change for 1 minute.)

$\log\frac{Me^2}{8\varrho''^2} = 2{,}930-10$ for units of the seventh decimal-place.

$$\underset{\text{in sec.}}{\eta} = \varrho''\frac{y}{\varrho_m} = yW_F\frac{\varrho''}{a_0} = yW_F\,[8{,}5097816.70-10] \quad (3)$$

W_F for arg. F. (Note the change for 1 minute.)

$$\left.\begin{aligned} \tan L_{1.2} &= +\tan\eta\sec F + \cdots \\ \tan t &= -\sin\eta\tan F + \cdots \\ \sin(F-B_2) &= \sin\eta\sin F\tan\frac{L_{1.2}}{2}\cdot\frac{W^2}{1-e^2} + \cdots \end{aligned}\right\} \quad (4)$$

W for arg. $F-\frac{2}{4}(F-B_2)$; to be derived from W_F

$$-\log(1-e^2) = 0{,}0029083.6.$$

Should the absolute value of the latitude be smaller than 65°, we use the following formulae in place of (4):

$$\left.\begin{aligned} L_0 &= \sec F\,(\eta \text{ in sec.}) \qquad t_0 = -\tan F\,(\eta \text{ in sec.}) \\ \log \underset{\text{in sec.}}{L_{1.2}} &= \log L_0 - \frac{M}{3\varrho''^2}\,t_0^2 + \left|\frac{7M}{90\varrho''^4}\,t_0^4 + \frac{M}{15\varrho''^4}\,t_0^2 L_0^2 + \cdots\right| \\ &\qquad \text{to be omitted for } \log L_0 < 8{,}40 \\ \log \underset{\text{in Sek.}}{t} &= \log t_0 - \frac{1}{2}\cdot\frac{M}{3\varrho''^2}\,t_0^2 - \frac{M}{6\varrho''^2}\,L_{1.2}^2 + \cdots \\ \log \underset{\text{in sec.}}{(B_2 - F)} &= \log \frac{\eta t_0 W^3}{2(1-e^2)\varrho''} - \frac{1}{2}\cdot\frac{M}{3\varrho''^2}\,t^2 - \frac{1}{2}\cdot\frac{M}{6\varrho''^2}\,L_0^2 + \cdots \end{aligned}\right\} \quad (4^*)$$

W for argument $F + \frac{3}{4}(B_2 - F)$

$$\log \frac{M}{3\varrho''^2} = 5{,}531813 - 10 \qquad \log \frac{7M}{90\varrho''^4} = 4{,}271 - 20$$
$$\log \frac{M}{6\varrho''^2} = 5{,}23078 - 10 \qquad \log \frac{M}{15\varrho''^4} = 4{,}204 - 20$$

} for units of the 7th decimal-place

$$\log \frac{1}{2(1-e^2)\varrho''} = 4{,}3874532.3 - 10\,.$$

And finally we get:

$$\alpha_{2.1} = \alpha_{1.2} + 180^0 + \Delta\alpha + t. \qquad (5)$$

In the case of $s = 0{,}02\,a_0$, the above formulae give B_2 and $L_{1.2}$ accurate for one or two units more or less of the fifth decimal-place, and $\alpha_{2.1}$ for as many units more or less of the third decimal-place of seconds.

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More details about the accuracy may be obtained from the last two paragraphs.

When using our formulae it will be necessary twice to calculate indirectly; which is not inconvenient, however, when applying logarithmic differences. We may avoid this by some simple series expansions especially when remembering the series on page 298. In these, the substitution of

$\alpha_{1,2} = 90°$ gives formulae for converting the latitude, longitude, and the azimuth by means of an ordinate.

The Bavarian State Survey was using that kind of formulae which were developed by Soldner and Orff.

After the author had deduced his formulae, O. Schreiber recently published formulae for the Prussian State Survey Office which only differ from (1), (2), (3), and (4*) by avoiding indirect calculation. Cf. the Rechnungsvorschriften für die trigonometrische Abteilung der Landesaufnahme. Formeln und Tafeln zur Berechnung der geographischen Koordinaten aus den Richtungen und Längen der Dreiecksseiten. Berlin 1878. Zu beziehen durch die Königliche Hofbuchhandlung von E. S. Mittler and Sohn (Computation Instructions for the Trigonometric Section of the Cadastral Service. Formulae and Tables for the Computation of Geographic Coordinates from Azimuths and Longitudes of Triangle Sides. Berlin 1878. Royal Court Book Shop of E.S. Mittler and Son). (They are special formulae for triangles of the first, second and third order.)

Our formulae should not be more inconvenient than those of Schreiber; on the contrary, they may be advantageous insofar as we would have to compute fewer corrections when the latitudes are known approximately.

Andrae indicated recently the possibility of applying formulae (4) for distances of some equator degrees; cf. Dänische Gradmessung (Danish Arc Measurement), Vol. 3 (recast of formulae from Astronomische Nachrichten (Journal of Astronomy), Vol. 50, No. 1187, p. 161 and Vol. 53, No. 1272, p. 369,

1859/60). This very interesting and in part geometrical method of development may be clearly conceived from No. 1272 of the Astronomische Nachrichten and Zachariae loc. cit. p. 198 (for sides of triangles up to 65° latitude only). We attained our formulae by a quite different development and in Par. 11, p. 431, obtained formulae which were even more precise. Moreover, for the sides of a triangle we avoided the indirect computing of x and y by Legendre's theorem in contrast to Andrae and by making use of higher terms we obtained formulae (4*) more precise than those of Zachariae; therefore, our formulae permit a wider application.

Referring to the Astronomische Nachrichten, Vol. 71, 1868, No. 1690, p. 147, we find that the trigonometric section of the Prussian general staff until recently used formulae similar to those of Zachariae since von Müffling († 1851) became its chief in 1820.

Börsch, like Andrae, computed x and y by means of Legendre's theorem in his Anleitung zur Berechnung rechtwinkliger sphärischer Koordinaten (Instructions for the Computation of Rectangular Spherical Coordinates), (Kassel 1868); otherwise, he applied formulae (4) and used argument F in its third equation. Confer his Tafeln für geodätische Berechnungen zwischen den geographischen Breiten von 35° to 71°, nach Bessel's Elementen (Tables for Geodetic Computations Between Geographic Latitudes from 35° to 71°, According to Bessel's Elements). (Appendix to the program of the higher technical schools in Kassel, 1869.)

In Volume 2 of his Handbuch, p. 280 ff., Jordan, according to Bohnenberger (1826), used expansions only with respect to x and y, like Zachariae.

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These series correspond to our set (I), but Jordan did not bring his formulae into logarithmic form.

Hansen's formulae differ from ours insofar as they, in addition, contain the reduced latitude, but they also use reduction by means of rectangular coordinates. Cf. Geodätische Untersuchungen (Geodetic Investigations), p. 38, Par. 31.

The method of converting geographic coordinates by reducing them to rectangular coordinates originates probably with Legendre. After explaining in the Memoirs of the French Academy of Sciences of the year 1787 his idea of determining a degree of latitude, he gave some addenda in Delambre, Méthodes analytiques pour la Détermination d'un Arc du Méridien (Analytical Methods for the Determination of the Arc of the Meridian), 1799, pp. 1 ff. On pp. 14 ff., he determined the latitude, longitude, and azimuth of the terminal point for a perpendicular to the meridian when the latitude of its initial point was known. His formulae and series including s^3 were developed on a sphere with K_1' as its center (see Fig. 8, p. 134). Legendre did not examine the influence exerted upon the ellipsoid by its difference from the sphere and by shortest lines being assumed on the former. This was already mentioned

by Soldner (Baeyerische Landesvermessung (Cadastral Surveying), p. 533). Soldner is mistaken, however, when he also names Delambre in this connection because the latter calculated by means of chords which do never represent a shortest line on the surface (cf. p. 211). With regard to Legendre, we may refer to his treatise of 1806, cf. p. 300.

Formulae of a different kind are given by Gauss, 1847, in the second part of Geodätische Untersuchungen, pp. 26 ff. (cf. p. 312). He did not use the reduction to rectangular coordinates but applied indirect calculation by introducing the arithmetic means of the latitudes and azimuths.

This seems to favor the convergence of the series; however, the series for L and t contain, apart from the terms corresponding to the ones in (4*), other terms which cancel this advantage in part. (Cf. also example VII.) The coefficients of the terms, moreover, are not as simple as above, and one needs more than one table in contrast to ours for $\log\sqrt{1-e^2\sin^2 B}$.

Gauss' formulae are given by Jordan in the second volume of his handbook, p. 421 (not by means of Gauss' direct derivation, however, but by means of another indirect one which also comes from Gauss). Jordan supplements these tables by auxiliary tables for variable eccentricities.

Par. 23. Rectangular Coordinates, Distance, and Azimuths from Spherical Coordinates for Distances of the Order of the Sides. When we replace, in connection with Par. 17, p. 447, formulae (1), (3), and (5) by the following ones:

$$\left.\begin{aligned} \sin\eta' &= \sin L_{1.2}\cos B_2 + \cdots \\ \sin(F-B_2) &= \sin^2\frac{L_{1.2}}{2}\sin B_2\cos B_2\cdot\frac{2W_2^2}{1-e^2}+\cdots \\ \tan t &= -\tan L_{1.2}\sin B_2+\cdots, \end{aligned}\right\} \qquad (1)$$

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then the committed errors will be given by the following expressions if L is considered to be expressed in radius:

$$\text{for } \eta' : -\frac{1}{15}\varrho''\,e^2\,L^5\sin^2 B_2\cos^5 B_2$$

$$\text{for } F-B_2: +\frac{1}{2}\varrho''\,e^2\,L^4\sin B_2\cos^3 B_2\left(-\frac{3}{4}+\frac{7}{6}\cos^2 B_2\right)$$

$$\text{for } t : -\frac{1}{3}\varrho''\,e^2\,L^3\sin B_2\cos^4 B_2.$$

We assume as on page 447 before that L will be a small magnitude, hence t will also be limited to small values. With respect to B_2, we obtain as maximum values:

for η' : $-\ 11''\,L^5$ at $\sin^2 B_2 = \frac{2}{7}$	$B_2 = \pm\ 32\frac{1}{2}^{\circ}$
for $F-B_2$: $\pm\ 55''\,L^4$ at $\sin^2 B_2 = 0{,}093$	$B_2 = \pm\ 17\frac{3}{4}$
for t : $\pm\ 130''\,L^3$ at $\sin^2 B_2 = \frac{1}{5}$	$B_2 = \pm\ 26\frac{1}{2}$.

For $L = 0{,}02$, we get the following error: in $F-B_2$, 0,000009", in t, 0,0010", and in η' it will be so small that it is insignificant. In the case of $L = 0{,}05$, we get: in η', 0,000003", in $F-B_2$, 0,00034", and in t, 0,016".

Now we write from (4), p. 447:

$$\log x = \log\left(\frac{a_0(1-e^2)(B_1-F)\text{in sec}}{\varrho''\,W_m^3}\right)+\frac{Me^2}{8\varrho''^2}\left(\div\right)^2\cos(B_1+F)+\cdots. \qquad (2)$$

The error committed in x will be obtained as on page 453; its maximum will be in equator seconds:

$$\text{for } x: -1{,}5''\,(B_1-F)^3, \text{where } \frac{1}{2}(B_1+F) = \pm\ 41^{\circ}$$

and $B_1 - F$ expressed in radians. For $B_1 - F = 0{,}02$, this will give -0,000012". For $B_1 - F = 0{,}05$, we get -0,0002".

For set (1) we now find the following expansions from (4), p. 130, in which L is considered to be in radians:

$$\left.\begin{aligned}
\log_{\text{in sec.}} \eta' &= \left\{\begin{aligned} &\log(\varrho'' L \cos B_2) - \tfrac{1}{6} ML^2 \sin^2 B_2 \\ &- \tfrac{1}{120} ML^4 \sin^2 B_2 [12 - 11 \sin^2 B_2] + Gl_6 \end{aligned}\right. \\
\log_{\text{in sec.}} (F - B_2) &= \left\{\begin{aligned} &\log\left(\tfrac{\varrho''}{2(1-e^2)} L^2 \sin B_2 \cos B_2 \cdot W_2^2\right) + \tfrac{1}{12} ML^2 [5 - 6\sin^2 B_2] \\ &+ \tfrac{1}{1440} ML^4 [119 - 420 \sin^2 B_2 + 300 \sin^4 B_2] + Gl_6 \end{aligned}\right. \\
\log_{\text{in sec.}} t &= \left\{\begin{aligned} &\log(-\varrho'' L \sin B_2) + \tfrac{1}{3} ML^2 \cos^2 B_2 \\ &+ \tfrac{1}{90} ML^4 \cos^2 B_2 [7 - 13 \sin^2 B_2] + Gl_6 . \end{aligned}\right.
\end{aligned}\right\} \qquad (3)$$

When we disregard the terms multiplied by L^4, we obtain the following maximum errors for η', $F - B_2$, and t:

in η' :	$-4000'' L^5$ for $\sin^2 B_2 = 0{,}42$	$B_2 = \pm 41^\circ$
in $F - B_2$:	$\pm 1700'' L^6$ for $\sin^2 B_2 = 0{,}10$	$B_2 = \pm 18^\circ$
in t :	$\pm 3000'' L^5$ for $\sin^2 B_2 = 0{,}14$	$B_2 = \pm 21^\circ$.

In the case of $L = 0{,}02$, only η' will become noticeably incorrect at the maximum, namely by -0,000013". For $L = 0{,}05$, the maximum error in η', $F - B_2$, and t will be $-0{,}0013''$, $\pm 0{,}000025''$, and $\pm 0{,}0009''$, respectively.

From this we deduce that we may substitute formulae (3) for formulae (1) and omit the terms multiplied by L^4 if L does not exceed about 0,02, i.e. approximately 1,2°. For greater values of L, the omitted terms in (3) become noticeable and also those omitted in (1). Therefore, we have to use the formulae of Par. 17,

p. 447, if the absolute value of L exceeds 0,02.

Par. 24. Continuation. To obtain convenient formulae for computing s and $\alpha_{1.2}$ we do not refer to (10), p. 447, again; these formulae will become unsuitable when $\Delta\alpha$ is calculated by the simplified formula $-\varrho'' xy : 2\varrho^2$.

Now, we introduce (8) and (9), pp. 446 and 447, into the derivations:

$$s\cos\left(\alpha_{1.2}+\frac{1}{3}\Delta\alpha\right)=s\cos\alpha_{1.2}\left(1-\frac{1}{18}\Delta\alpha^2+Gl_6\right)-\frac{1}{3}\Delta\alpha\, s\sin\alpha_{1.2}\,(1+Gl_6)$$

$$s\sin\left(\alpha_{1.2}+\frac{1}{3}\Delta\alpha\right)=s\sin\alpha_{1.2}\left(1-\frac{1}{18}\Delta\alpha^2+Gl_6\right)+\frac{1}{3}\Delta\alpha\, s\cos\alpha_{1.2}\,(1+Gl_6)$$

hence we obtain the following formulae:

$$s\cos\left(\alpha_{1.2}+\frac{1}{3}\Delta\alpha\right)=x\left\{1-\frac{1}{6}\frac{y^2}{\varrho^2}-\frac{1}{120}\frac{y^4}{\varrho^4}-\frac{1}{60}\frac{x^2y^2}{\varrho^4}+\frac{1}{36}\frac{xy^2}{\varrho^2}e^2\sin 2B_0+Gl_6\right\}$$

$$s\sin\left(\alpha_{1.2}+\frac{1}{3}\Delta\alpha\right)=y\left\{1+\frac{1}{180}\frac{x^4}{\varrho^4}-\frac{1}{60}\frac{x^2y^2}{\varrho^4}-\frac{1}{18}\frac{x^3}{\varrho^3}e^2\sin 2B_0+Gl_6\right\}$$

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or in logarithmic form:

$$\left.\begin{aligned}\log\left(s\cos\left(\alpha_{1.2}+\frac{1}{3}\Delta\alpha\right)\right)&=\left\{\log x-\frac{1}{6}M\frac{y^2}{\varrho^2}-\frac{1}{45}M\frac{y^4}{\varrho^4}-\frac{1}{60}M\frac{x^2y^2}{\varrho^4}+\frac{1}{36}Me^2\frac{xy^2}{\varrho^2}\sin 2B_0+Gl_6\right\}\\ \log\left(s\sin\left(\alpha_{1.2}+\frac{1}{3}\Delta\alpha\right)\right)&=\left\{\log y+\frac{1}{180}M\frac{x^4}{\varrho^4}-\frac{1}{60}M\frac{x^2y^2}{\varrho^4}-\frac{1}{18}Me^2\frac{x^3}{\varrho^3}\sin 2B_0+Gl_6.\right\}\end{aligned}\right\}\qquad(1)$$

In these, we have:

$$\varrho^2 = b_0^2 : W_0^4 \qquad (2)$$

$$\log \underset{\text{in sec.}}{\Delta\alpha} = \log\left(-\frac{1}{2}\varrho''\frac{xy}{\varrho^2}\right) + \frac{1}{12}M\frac{e^2}{c^2} + Gl_4. \qquad (3)$$

Omitting in the above formulae the terms in small print, the committed error for $s\cos\left(\alpha_{1.2} + \frac{1}{3}\Delta\alpha\right)$ and for $s\sin\left(\alpha_{1.2} + \frac{1}{3}\Delta\alpha\right)$ in equator seconds (cf. p. 451), and for $\Delta\alpha$ in seconds respectively, will be:

$$\left.\begin{aligned} &-\frac{1}{180}\varrho''\left(\frac{s}{a_0}\right)^5 \sin^2\alpha_{1.2}\cos\alpha_{1.2}\,(4\sin^2\alpha_{1.2} + 3\cos^2\alpha_{1.2}) \\ &\quad + \frac{1}{36}\varrho'' e^2\left(\frac{s}{a_0}\right)^4 \sin^2\alpha_{1.2}\cos^2\alpha_{1.2}\sin 2B_0; \\ &+\frac{1}{180}\varrho''\left(\frac{s}{a_0}\right)^5 \sin\alpha_{1.2}\cos^2\alpha_{1.2}\,(\cos^2\alpha_{1.2} - 3\sin^2\alpha_{1.2}) \\ &\quad - \frac{1}{18}\varrho'' e^2\left(\frac{s}{a_0}\right)^4 \sin\alpha_{1.2}\cos^3\alpha_{1.2}\sin 2B_0; \end{aligned}\right\} \text{ to (1)}$$

$$-\frac{1}{24}\varrho''\left(\frac{s}{a_0}\right)^4 \sin\alpha_{1.2}\cos\alpha_{1.2}. \qquad \text{to (3)}$$

The maximum values for these expressions are partially indicated on page 451. We get the following maxima for the terms independent of e^2:

$$\left.\begin{aligned} &\pm 1600''\left(\frac{s}{a_0}\right)^5 \text{ for } \sin^2\alpha_{1.2} = 0{,}7 \quad \alpha_{1.2} \text{ in quad. I.} = 57^\circ \\ &\pm\ 520''\left(\frac{s}{a_0}\right)^5 \text{ „ } \sin^2\alpha_{1.2} = 0{,}7 \quad \text{„} \quad 57^\circ \end{aligned}\right\} \text{ to (1)}$$

$$\pm 4300''\left(\frac{s}{a_0}\right)^4 \text{ „ } \tan\alpha_{1.2} = \pm 1 \quad \text{„} \quad 45^\circ, \qquad \text{to (3)}$$

for the terms dependent on e^2 we have, on the other hand:

$$\left.\begin{aligned} &\pm 9{,}5''\left(\frac{s}{a_0}\right)^4 \text{ for } \tan\alpha_{1.2} = \pm 1 \quad \alpha_{1.2} \text{ in quad. I.} = 45^\circ \quad B_0 \text{ any value} \\ &\pm\ 25\ \left(\frac{s}{a_0}\right)^4 \text{ „ } \tan\alpha_{1.2} = \pm\sqrt{\frac{1}{3}} \quad \text{„} \quad 30 \quad B_0 = \pm 45^\circ. \end{aligned}\right\} \text{ to (1)}$$

From this, we obtain as maximum error for $s : a_0 = 0{,}02$ in $s \cos\left(\alpha_{1.2} + \frac{1}{3}\Delta\alpha\right)$, $s \sin\left(\alpha_{1.2} + \frac{1}{3}\Delta\alpha\right)$ and $\Delta\alpha$ respectively:

$$\pm 0{,}000006'', \pm 0{,}000005'' \text{ and } \pm 0{,}0007'';$$

for $s : a_0 = 0{,}05$, the respective maximum error will be $\pm 0{,}0005''$, $\pm 0{,}0002''$, and $\pm 0{,}027''$.

Par. 25. Summary. Given: B_1, B_2 and $L_{1.2}$.

Wanted: s, $\alpha_{1.2}$, and $\alpha_{2.1}$, or alternatively x and y of P_2 referred to the meridian of P_1.

$$\eta'_0 = +\cos B_2 \,.\, \underset{\text{in sec.}}{L_{1.2}} \qquad \log \underset{\text{in sec.}}{\eta'} = \log \eta'_0 - \frac{M}{6\varrho''^2} t_0^2 + \cdots$$

$$t_0 = -\sin B_2 \,.\, \underset{\text{in sec.}}{L_{1.2}} \qquad \log \underset{\text{in sec.}}{t} = \log t_0 + \frac{M}{3\varrho''^2} \eta'^2_0 + \cdots$$

$$\log \frac{M}{6\varrho''^2} = 5{,}230783 - 10 \qquad \log \frac{M}{3\varrho''^2} = 5{,}531813 - 10,$$

both for units of the seventh decimal-place; (1)

$$\log \underset{\text{in sec.}}{(F - B_2)} = \log \frac{-\eta'_0 t_0 W_2^2}{2(1-e^2)\varrho''} - \frac{1}{2} \cdot \frac{M}{6\varrho''^2} t_0^2 + \frac{5}{4} \cdot \frac{M}{3\varrho''^2} \eta'^2_0 + \cdots$$

W_2 for argument B_2

$$\log \frac{-1}{2(1-e^2)\varrho''} = 4{,}3874532.3_8 - 10$$

$$\log x = \log \left(\frac{\varrho_m (B_1 - F)\text{in sec.}}{\varrho''}\right) + \frac{Me^2}{8\varrho''^2}\left(\frac{\cdot}{\cdot}\right)^4 \cos(B_1 + F) + \cdots$$

$$\frac{\varrho_m}{\varrho''} = \frac{a_0 (1 - e^2)}{\varrho'' W_m^3} = [1{,}4873099.71]\, W_m^3; \quad W_m \text{ Arg. } \frac{1}{2}(B_1 + F) \qquad (2)$$

$\log \frac{Me^2}{8\varrho''^2} = 2{,}930 - 10$ for units of the seventh decimal-place.

$$y = \varrho_m \frac{\eta' \text{in sec.}}{\varrho''} = \frac{a_0}{\varrho''} \cdot \frac{\eta' \text{in sec.}}{W_2} = \frac{\eta' \text{in sec.}}{W_2} [1{,}4902183.30] \qquad (3)$$

(See Illustrative Material on Next Page.)

Par. 25. Summary.

$$\left.\begin{aligned}
&\log\left\{s\cos\left(\alpha_{1.2}+\tfrac{1}{3}\Delta\alpha\right)\right\} = \log x - \tfrac{1}{6}\,\frac{M}{b_0^2}\,y^2 W_0^{-4} + \cdots \\
&\log\left\{s\sin\left(\alpha_{1.2}+\tfrac{1}{3}\Delta\alpha\right)\right\} = \log y + \cdots \\
&\underset{\text{in sec.}}{\Delta\alpha} = -\tfrac{1}{2}\,\frac{\varphi''}{b_0^2}\,xy\,W_0^{-4} + \cdots \\
&\log\left(-\tfrac{1}{2}\,\frac{\varphi''}{b_0^2}\right) = 1{,}407017_n - 10 \qquad \text{W. for argument } F + \tfrac{1}{3}(B_1 - F) \\
&\log\left(\tfrac{1}{6}\,\frac{M}{b_0^2}\right) = 2{,}25325 \quad -10 \text{ for units of the seventh decimal-place.}
\end{aligned}\right\} \qquad (4)$$

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$$\alpha_{2.1} = \alpha_{1.2} + 180^\circ + \Delta\alpha + t. \qquad (5)$$

These formulae will be satisfactory for even the most precise calculation if $s : a_0 \leq 0{,}02$ and, at the same time, $L_{1.2} \leq 1{,}2^\circ$. For a detailed account of their accuracy refer to the two previous paragraphs.

Scope of application for above formulae. The formulae of this paragraph, or similar ones, will be used to a great extent, for example, when we have to establish a rectangular coordinate system for a small part of a country, as in the case of the Prussian State Survey. One point, say P_1 having the latitude B_1, will be considered as the origin of the coordinates, and the geographic locations of other points will then be related to P_1 by means of rectangular coordinates. (cf. page 420.)

The problem of this paragraph and that of Par. 17, p. 447, was solved in a different way, especially by Hansen who used the vertical section (cf. the remark on p. 265). The respective formulae will be found in Par. 53 and partially in

Par. 52, pp. 70 ff. of his Geodätische Untersuchungen. When computing the numerical example given by Hansen in Par. 70, p. 91, we come to the conclusion that the formulae developed above ought to be preferred to Hansen's. For the sake of comparison, we are going to compute Hansen's example in the following paragraph.

Oudemans, in Astronomische Nachrichten, Vol. 81, No. 1940, pp. 305 ff., reviewed several older and inadequate formulae for determining distance and azimuths from spherical coordinates. In the end, he transformed Hansen's formulae but avoided their expansion which he should have done by computing the sine and tangent of small angles by means of arc and secant as was mentioned by us on page 32. These formulae seem to be very convenient indeed but their degree of accuracy was not completely determined by the author; moreover, one of them (that for the distance) fails in the case of small azimuths, because of its denominator, sine azimuth. For that reason, the set of formulae in question requires corrections before it is applied in practice.

In the case of distances and azimuths only, however, we prefer the formulae given on page 313 to all others.

For modifications of above formulae to suit even smaller distances, cf. Zachariae loc. cit. p. 218, and Jordan loc. cit. Vol. 2, pp. 291 ff.

Par. 26. Numerical Example VI. Compare Hansen, Geodätische Untersuchungen, p. 47 and p. 91 (Par. 37, 70).

Par. 26. Numerical Example VI.

Given: $B_1 = 20^g \quad s = \frac{2^\circ}{\varrho^\circ} a_0 \quad \alpha_{1,2} = 30^g.$

We apply the formulae on page 456 and use seven-place logarithms with the eighth decimal-place interpolated from proportional parts.

The formulae's accuracy will therefore be sufficient although s is relatively great. (Even eight-place logarithms may be used in this respect.)

To obtain s in meters, also u, and v, formulae (1), p. 456, give:

$$\begin{cases} \log 2a_0 = 7{,}1056734.6 \\ \log \varrho^\circ = 1{,}7581226.3 \\ \hline \log s = 5{,}3475508.3 \end{cases} \qquad \begin{aligned} \log \sin \alpha_{1,2} &= 9{,}6989700 - 10 \\ \log \cos \alpha_{1,2} &= 9{,}9375306 - 10 \end{aligned}$$

$$\log u = 5{,}2850814.3 \qquad \log v = 5{,}0465208.3.$$

From this, we get in first approximation:

$$F - B_1 = [6{,}732_n + 5{,}285 - 10] = -104{,}0'',$$

whence $B_0 = 20^g - 1^g\,9' = 18^g\,51'$, with $\log W_0 = 9{,}99849 - 10$, and $\log W_0^4 = 9{,}99940 - 10.$ We now find:

$$\begin{cases} \log u = 5{,}2850814.3 \\ + [2{,}55428 - 10]\, v^2 W_0^4 = + \quad 443.3 \\ \hline \log x = 5{,}2851257.6 \end{cases}$$

$$\begin{cases} \log v = 5{,}0465208.3 \\ - [2{,}25325 - 10]\, u^2 W_0^4 = - \quad 665.0 \\ \hline \log y = 5{,}0464543.3 \end{cases}$$

$$1{,}40702_n - 10$$

$$\begin{cases} \log (xy) = 10{,}33158 \\ \log W_0^4 = 9{,}99940 - 10 \\ \hline \log \Delta a = 1{,}73800_n; \quad \Delta a = -54{,}702'' \end{cases}$$

(in sec.)

Formula (2), p. 456, gives then:

In the first approximation, the argument for W'_m will be $20° - 52' = 19°8'$, from which we form $3 \log W'_m$; its increase for 1° equals -7.86.

$\frac{1}{2}(B_1 - F)$ will become more accurately $\frac{1}{2}[3,79735]$ seconds, or 52.26", hence the increase to $3 \log W'_m$ will be $+7.86 \times 0,26$.

$B_1 + F = 38° 15'$.

$$\begin{cases} \log x = 5,2851257.6 \\ 8,5126900.3 - 10 \\ 3 \log W'_m = 9,9995327.3 - 10 \end{cases}$$

$$\begin{cases} \log \left(\frac{e''}{e_m}\right) = 3,7973485.2 \\ -[2930 - 10]\left(\frac{e''}{e_m}\right)^2 \cos(B_1 + F) = \quad -2.6 \\ \text{Increase to } 3 \log W'_m = \quad +2.0 \end{cases}$$

$$\log (B_1 - F) = 3,7973484.6$$

in sec.

$$B_1 - F = 1° 44' 31,168'' \qquad F = 18° 15' 28,832''.$$

Formulae (3) and (4*), p. 457, yield:

$$\begin{cases} 8,5097816.7 - 10 \\ \log y = 5,0464543.3 \\ \log W_F = 9,9998577.0 - 10 \end{cases}$$

Increment of $\log W_F$ for 1' equals -2.51.

$$\begin{cases} \log \eta = 3,5560937.0 \\ \text{in sec.} \\ \log \cos F = 9,9775660.2 - 10 \\ \log \tan F = 9,5183896.5 - 10 \end{cases}$$

$$\log L_0 = 3,5785276.8$$

$$\log t_0 = 3,0744833.5_n$$

$$\begin{cases} \log L_0 = 3,5785276.8 \\ -[5,53181 - 10]\, t_0^2 = \quad -48.0 \\ +[4,271 - 20]\, t_0^4 = \quad 0.0 \\ +[4,204 - 20]\, t_0^2 L_0^2 = \quad 0.0 \end{cases}$$

$$\log L_{1.2} = 3,5785228.8$$

in sec.

$$L_{1.2} = 1° 3' 8,985''$$

$$\begin{cases} \log t_0 = 3,0744833.5_n \\ -\frac{1}{2} \times 48,0 = \quad -24.0 \\ -[5,23078 - 10]\, L_{1.2}^2 = \quad -244.3 \end{cases}$$

$$\log t = 3,0744565.2_n$$

in sec.

$$\begin{cases} t = - \ 19'\,47{,}016'' \\ \Delta a = - \ \ 54{,}702 \\ \alpha_{2.1} = \ 30 \ \ 0 \ \ 0 \end{cases}$$

$$\alpha_{2.1} = 209^0\,39'\,18{,}282''.$$

Instead of W_F we ought to apply the argument $F + \frac{3}{4}(B_2 - F)$. However, the increment of $2 \log W$ for $- 0{,}13'$ will be insignificant.

$$\begin{cases} 4{,}38745 - 10 \\ \log \eta = 3{,}55609 \\ \log t_0 = 3{,}07448. \\ 2 \log W_F = 9{,}99972 - 10 \\ -\frac{1}{2} \times 48.0 = \ \ 0 \\ -\frac{1}{2} \times 244.3 = - 1 \end{cases}$$

$$\log (B_2 - F) \underset{\text{in sec.}}{=} 1{,}01773.$$

$$B_2 - F = - 10{,}417'' \qquad B_2 = 18^0\,15'\,18{,}415''.$$

Results:

		Hansen loc. cit. p. 48:
$B_2 =$	$18^0\,15'\,18{,}415''$	18,417"
$L_{1.2} =$	$1 \ \ 3 \ \ 8{,}985$	8,983
$\alpha_{2.1} =$	$209 \ \ 39 \ \ 18{,}282$	18,279.

Par. 27. Numerical Example VI. Reversal of last paragraph's problem.

Given: $B_1 = 20^0 \quad B_2 = 18^0\,15'\,18{,}415'' \quad L_{1.2} = 1^0\,13'\,8{,}985''$.(*)

With $L_{1.2} = 3788{,}985''$, the formulae of Par. 25, p. 463, give the following values:

(See Illustrative Material on Next Page.)

(*)Editor's note: number should read as $1^0 3' 8{,}985''$. -L.D.

$$\left\{\begin{array}{rl} \log L_{1.2} \text{ in sec.} & = 3{,}5785228.8 \\ \log \cos B_2 & = 9{,}9775732.9 - 10 \end{array}\right.$$

$$\left\{\begin{array}{rl} \log \eta'_0 & = 3{,}5560961.7 \\ -\,[5{,}23078 - 10]\, t_0^2 & = -\quad 24.0 \end{array}\right.$$

Argument for $\log W_2 = 18^\circ\,15{,}307'$

$$\left\{\begin{array}{rl} \log \eta' \text{ in sec.} & = 3{,}5560937.7 \\ -\log W_2 & = 0{,}0001422.6 \\ & 1{,}4902183.3 \end{array}\right.$$

$$\log y = 5{,}0464543.6$$

$$\left\{\begin{array}{rl} \log L_{1.2} \text{ in sec.} & 3{,}5785228.8 \\ \log \sin B_2 & = 9{,}4958890.8 - 10 \end{array}\right.$$

$$\left\{\begin{array}{rl} \log t_0 & = 3{,}0744119.6. \\ +\,[5{,}53181 - 10]\, \eta'^2_0 & = +\quad 440.6 \end{array}\right.$$

$$\log t \text{ in sec.} = 3{,}0744560.2.$$

$$t = -\,19'\,47{,}015''.$$

$$\left\{\begin{array}{rl} & 4{,}38745. - 10 \\ \log \eta'_0 & = 3{,}55610 \\ \log t_0 & = 3{,}07441 \\ 2 \log W_2 & = 9{,}99972 - 10 \\ -\frac{1}{2} \times 24.0 & = \quad 0 \\ +\frac{5}{4} \times 440.6 & = +\quad 6 \end{array}\right.$$

$$\log (F - B_2) \text{ in sec.} = 1{,}01774.$$

$$F - B_2 = +\ 10{,}417''$$

$$F = 18^\circ\ 15'\ 28{,}832''$$

$$B_1 - F = 1^\circ\ 44'\ 31{,}168''$$

$$= 6271{,}168''$$

Formula (2) p. 463 gives:

$$\left\{\begin{array}{rl} & 1{,}4873099.7 \\ \log (B_1 - F) \text{ in sec.} & = 3{,}7973484.6 \\ -\,3 \log W_m & = 0{,}0004670.7 \\ +\,[2{,}930 - 10](B_1 - F)^2 \cos (B_1 + F) & = +\quad 2.6 \end{array}\right.$$

$$\log x = 5{,}2851257.6$$

Argument $\frac{1}{2}(B_1 + F)$ for $\log W_m = 19^\circ\ 7{,}740'$. $B_1 + F = 38^\circ\ 15'$.

With $\log W_0 = 9{,}999849 - 10$ for the argument $B_0 = 18^\circ\,51'$, formulae (4), p. 463 and (5), p. 465 result in:

Par. 28. Numerical Example VII.

$$\log x = 5{,}2851257.6$$
$$-[2{,}25325 - 10]y^2 W_0^4 = - \quad 221.6$$
$$\log\left\{s\cos\left(\alpha_{1.2} + \tfrac{1}{3}\Delta\alpha\right)\right\} = 5{,}2851036.0$$
$$\log\left\{s\sin\left(\alpha_{1.2} + \tfrac{1}{3}\Delta\alpha\right)\right\} = 5{,}0464543.6$$
$$\alpha_{1.2} + \tfrac{1}{3}\Delta\alpha = 29^0\,59'\,41{,}780''$$
$$\log s = 5{,}3475508.3$$

$$1{,}40702.-10$$
$$\log(xy) = 10{,}33158$$
$$4\log W_0 = 9{,}99940 - 10$$
$$\log \Delta\alpha = 1{,}73800$$
in sec.
$$\Delta\alpha = - \quad 54{,}702''$$
$$t = - \quad 19'\,47{,}015''$$
$$\alpha_{1.2} = 30^0\ 0'\ 0{,}014''$$
$$\alpha_{2.1} = 209^0\,39'\,18{,}297''.$$

Results:		According to Previous Paragraph:	According to Hansen loc. cit. p. 92:
$\log x =$	5,2851257.6	7.6	$\frac{s}{a_0}$ is greater by 0,004° than
$\log y =$	5,0464543.6	3.3	2°, increasing the logarithm
$\log s =$	5,3475508.3	8.3	by 2.5.
$\alpha_{1.2} =$	30° 0′ 0,014″	0,000″	− 0,020″
$\alpha_{2.1} =$	209 39 18,297″	18,282	18,256.

Par. 28. Numerical Example VII. $B_1 = 51^0\,48'\ 1{,}9294''$

$\log s = 5{,}0252128.6 \quad \alpha_{1.2} = 5^0\,42'\,21{,}7699''$

for the triangle side Brocken - Inselsberg of the Hanoverian State Survey from Gauss, Untersuchungen über Gegenstände der höheren Geodäsie, 2. Abtheilung (Investigations of the Subject of Higher Geodesy, Part 2), p. 33.

Using seven-place logarithms, we have for formulae (1) and (2), p. 456:

$$\log \sin \alpha_{1.2} = 8{,}9974945.7 - 10 \qquad \log \cos \alpha_{1.2} = 9{,}9978427.5 - 10$$
$$\log v = 4{,}0227074.3 \qquad \log u = 5{,}0230556.1$$
$$F - B_1 = [6{,}732. - 10 + 5{,}023] = - 56{,}8'$$

Then $\log W_0$ with the argument $B_0 = 51^0\,48' - \frac{2}{3}.57'$ or $51^0\,10'$ becomes $9{,}999119 - 10$, and $\log W_0^4 = 9{,}99648 - 10$.

From this follows:

(See Illustrative Material on Next Page.)

$$\begin{cases} \log u = 5{,}0230556.1 \\ +\,[2{,}5543 - 10]\, v^2 W_0^4 = +\quad 4.0 \end{cases}$$
$$\log x = 5{,}0230560.1$$
$$\begin{cases} \log v = 4{,}0227074.3 \\ -\,[2{,}2533]\, u^2 W_0^4 = -\quad 197.6 \end{cases}$$
$$\log y = 4{,}0226876.7$$

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$$\begin{cases} 1{,}40702_n - 10 \\ \log (xy) = 9{,}04574 \\ 4 \log W_0 = 9{,}99648 - 10 \end{cases}$$
$$\log \Delta u = 0{,}44924_n; \quad \Delta u = -2{,}8135''.$$
in sec.

Now, formula (2), p. 456, gives:

In the first approximation, the argument for W_m equals $51^\circ 48{,}03' - 28' = 51^\circ 20'$ from which we form $3 \log W_m$; its increase for $1' = -12.39$.

$$\begin{cases} \log x = 5{,}0230560.1 \\ 8{,}5126900.3 - 10 \\ 3 \log W_m = 9{,}9973438.9 - 10 \end{cases}$$

$\frac{1}{2}(B_1 - F)$ will be more accurately $\frac{1}{2}$ [3,53309] seconds, or 28, 44', hence the argument will be $51^\circ 19{,}59'$. $B_1 + F = 102^\circ 39'$.

$$\begin{cases} \log \left(\frac{\varrho'' x}{\varrho_m}\right) = 3{,}5330899.3 \\ -\,[2{,}930 - 10]\left(\frac{\varrho'' x}{\varrho_m}\right)^2 \cos (B_1 + F) = +\quad 0.2 \\ \text{Increase to } 3 \log W_m \text{ for } -0{,}41' = +\quad 5.1 \end{cases}$$
$$\log (B_1 - F) = 3{,}5330904.6$$
in sec.

$$B_1 - F = 56'\ 52{,}6398'' \qquad F = 50^\circ\ 51'\ 9{,}2896''.$$

Formulae (3) and (4*), p. 457, yield:

$$\begin{cases} 8{,}5097816.7 - 10 \\ \log y = 4{,}0226876.7 \\ \log W_F = 9{,}9991265.7 - 10 \end{cases}$$

Increment of $\log W_F$ for 1' equals -4.0

$$\begin{cases} \log \eta = 2{,}5315959.1 \\ \text{in sec.} \\ \log \cos F = 9{,}8002480.4 - 10 \\ \log \tan F = 0{,}0893472.5 - 10 \end{cases}$$

$\log L_0 = 2{,}7313478.7$

$\log t_0 = 2{,}6209431.6_n$

Par. 29. Numerical Example II.

$$\begin{cases} \log L_0 = 2{,}7313478.7 \\ -[5{,}532 - 10]\, t_0^2 = - \quad 5.9 \\ \text{Terms of higher order} = \quad 0.0 \end{cases}$$

$$\log L_{1.2} = 2{,}7313472.8 \quad \text{in sec.}$$

$$L_{1.2} = 8'\ 58{,}7003''$$

$$\begin{cases} \log t_0 = 2{,}6209431.6_n \\ -\frac{1}{2} \times 5.9 = - \quad 3.0 \\ -[5{,}231 - 10]\, L_{1.2}^2 = - \quad 4.9 \end{cases}$$

$$\log t = 2{,}6209423.7_n \quad \text{in sec.}$$

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$$\left.\begin{aligned} t &= -\ 6'\ 57{,}7749'' \\ \Delta\alpha &= -\ \ 2{,}8135 \\ \alpha_{1.2} &= 5^0\ 42'\ 21{,}7699 \end{aligned}\right\} \alpha_{2.1} = 185^0\ 35'\ 21{,}1815''$$

Since $B_2 - F$ amounts to fractions of a second only, W_F itself will be sufficiently accurate for computing.

$$\begin{cases} 4{,}38745 - 10 \\ \log \eta = 2{,}53160 \\ \log t_0 = 2{,}62094_n \\ 2 \log W_F = 9{,}99825 - 10 \\ \text{Terms of higher order} = \quad 0 \end{cases}$$

$$\log(B_2 - F) = 9{,}53824_n - 10 \quad \text{in sec.}$$

$$B_2 - F = -\ 0{,}3453'' \qquad B_2 = 50^0\ 51'\ 8{,}9443''.$$

Results:

		According to Gauss:
$B_2 =$	$50^0\ 51'\ 8{,}9443''$	8,9444"
$L_{1.2} =$	$0\ \ 8\ 58{,}7003$	58,7002
$\alpha_{2.1} =$	$185\ 35\ 21{,}1815$	21,1815.

Par. 29. Numerical Example II. Example 7 in Formeln und Tafeln etc. (für Dreiecksseiten 1. Ordnung), (Formulae and Tables etc. (For Triangle Sides of First Order)) by O. Schreiber.

Given: $B_1 = 57^0 \quad s = 120^{km} \quad \alpha_{1.2} = 315^0.$

In this case, we shall use logarithms which are accurate to the

ninth decimal place.

Set (1), p. 456, yields:

$$\begin{cases} \log s = 5{,}0791812.46 \\ \log \sin \alpha_{1.2} = 9{,}8494850.02_n - 10 \\ \log \cos \alpha_{1.2} = 9{,}8494850.02 - 10 \end{cases} \qquad \begin{matrix} \log v = 4{,}9286662.48_n \\ \log u = 4{,}9286662.48 \end{matrix}$$

$$F - B_1 = [6{,}732_n - 10 + 4{,}929] = -45{,}8',$$

hence

$$B_0 = 57^0 - 30{,}6' = 56^0\,29{,}4',$$

$$\log W_0 = 9{,}998990 - 10 \text{ and } 4 \log W_0 = 9{,}995960 - 10.$$

$$\begin{cases} \log u = 4{,}9286662.48 \\ +[2{,}55428 - 10]\, v^2 W_0^4 = + \quad 255.61 \end{cases}$$

$$\log x = 4{,}9286918.09$$

$$\begin{cases} \log v = 4{,}9286662.48_n \\ -[2{,}25325 - 10]\, u^2 W_0^4 = - \quad 127.80 \end{cases}$$

$$\log y = 4{,}9286534.68_n$$

$$\begin{cases} 1{,}407017_n \\ \log (xy) = 9{,}857345_n \\ 4 \log W_0 = 9{,}995960 - 10 \end{cases}$$

$$\log \underset{\text{in sec.}}{\Delta a} = 1{,}260322;\ \Delta a = 18{,}2105''.$$

Now we get by (2), p. 456:

In first approximation, the argument for W_m equals $57^0 - 23' = 56^0 37'$, from which we form $3 \log W_m$. Its increase for 1° will be -11.67.

$\frac{1}{2}(B_1 - F)$ becomes more accurately: $\frac{1}{2}$ [3,43834] seconds, or 22,865'. Hence the increase for $3 \log W_m$ equals $-0{,}135 \times 11{,}67$.

$B_1 + F = 113^0 14'$.

$$\begin{cases} \log x = 4{,}9286918.09 \\ 8{,}5126900.29 - 10 \\ 3 \log W_m = 9{,}9969613.57 - 10 \end{cases}$$

$$\begin{cases} \log \left(\frac{\varrho'' x}{\varrho_m}\right) = 3{,}4383431.95 \\ -[2{,}930 - 10]\left(\frac{\varrho'' x}{\varrho_m}\right)^2 \cos(B_1 + F) = + \quad 0.25 \end{cases}$$

Increase for $3 \log W_m$ $-0{,}135 \times 11{,}67 = - \quad 1.57$

$$\log \underset{\text{in sec.}}{(F - B_1)} = 3{,}4383430.63$$

$$F - B_1 = 45'\ 43{,}74068'' \qquad F = 56^0\ 14'\ 16{,}25932''$$

Par. 29. Numerical Example II.

Formulae (3) and (4*), p. 457, give:

$$\begin{cases} 8{,}5097816.70 - 10 \\ \log y = 4{,}9286534.68_n \\ \log W_F = 9{,}9989959.90 - 10 \end{cases}$$

Increment of $\log W_F$ for 1' equals $-3{,}92$.

$$\begin{cases} \log \eta = 3{,}4374311.28_n \text{ in sec.} \\ \log \cos F = 9{,}7448767.69 - 10 \\ \log \tan F = 0{,}1749080.61 \end{cases}$$

$$\log L_0 = 3{,}6925543.59_n$$
$$\log t_0 = 3{,}6123391.89$$

$$\begin{cases} \log L_0 = 3{,}6925543.59_n \\ -[5{,}531813 - 10]\, t_0^2 = - \quad 570.80 \\ +[4{,}271 - 20]\, t_0^4 = + \quad 0.05 \\ +[4{,}204 - 20]\, t_0^2 L_0^2 = + \quad 0.07 \end{cases}$$

$$\log L_{1.2} = 3{,}6924972.91_n \text{ in sec.}$$

$$L_{1.2} = -1^\circ\, 22'\, 6{,}03270''$$

$$\begin{cases} \log t_0 = 3{,}6123391.89 \\ -\frac{1}{2} \times 570.80 = - \quad 285.40 \\ -[5{,}23078 - 10]\, L_{1.2}^2 = - \quad 412.84 \end{cases}$$

$$\log t = 3{,}6122693.65 \text{ in sec.}$$

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$$\begin{cases} t = 1^\circ\, 8'\, 15{,}1458'' \\ \Delta s = 18{,}2105 \\ \alpha_{1.2} = 315\ 0\ 0 \end{cases}$$

$$\alpha_{2.1} = 136\ 8\ 33{,}3563''$$

(See Illustrative Material on Next Page.)

$$\begin{cases} 4{,}3874532.3 - 10 \\ \log \eta = 3{,}4374311.3_n \\ \log t_0 = 3{,}6123391.9 \\ 2 \log W_F = 9{,}9979919.8 - 10 \end{cases}$$

$$\begin{cases} \log (B_2 - F)\text{ in approx. } 1 = 1{,}4352155.3_n \\ \text{Increase for } 2 \log W = + \quad 2.7 \\ -[5{,}23078 - 10]\, t^2 = - \quad 285.4 \\ -\tfrac{1}{2}[5{,}23078 - 10]\, L_0^2 = - \quad 206.5 \end{cases}$$

$$\log (B_2 - F)_{\text{in sec.}} = 1{,}4351666.1_n$$

$B_2 - F = -27{,}24 = -0{,}454$

hence $\frac{3}{4}(B_2 - F) = -0{,}34$,

from which we obtain the following increase of $2 \log W_F$:

$3 \times 0{,}34 \times 3{,}92$

$B_2 - F = -27{,}23746''$

$B_2 = 56^0\ 13'\ 49{,}02186''$.

Results:

		According to p. 304:	According to Schreiber loc. cit. p. 12:
$B_2 =$	$56^0\ 13'\ 49{,}02186''$	49,02182"	49,0218"
$L_{1.2} =$	$1\ 22\ 6{,}03270$ easterly	6,03263	6,0327
$\alpha_{2.1} =$	$136\ 8\ 33{,}3563$	33,3566	33,356.

The results on page 304 are doubtful in the fifth decimal place.

Par. 30. Continuation of Numerical Example II. Reversal of the previous problem.

Given:

$B_1 = 57^0 \qquad B_2 = 56^0\ 13'\ 49{,}02186'' \qquad L_{1.2} = 1^0\ 22'\ 6{,}03270''$ easterly

With $L_{1.2} = -4926{,}03270''$, formulae (1) and (3), p. 463, yield:

$$\begin{cases} \log L_{1.2} = 3{,}6924972.91_n \\ \log \cos B_2 = 9{,}7449625.47 - 10 \end{cases}$$

$$\begin{cases} \log \eta_0' = 3{,}4374598.38_n \\ -[5{,}230783 - 10]\, t_0^2 = - \quad 285.28 \end{cases}$$

Argument for W_2 $= 56^0\ 13{,}8170'$:

$$\begin{cases} \log \eta'' = 3{,}4374313.10_n \\ \quad \text{in sec.} \\ -\log W_2 = 0{,}0010038.30 \\ \qquad\qquad 1{,}4902183.30 \end{cases}$$

$$\log y = 4{,}9286534.70_n$$

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$$\left\{\begin{array}{r} \log L_{1.2} = 3{,}692497291_n \\ \log \sin B_2 = 9{,}919746488 - 10 \end{array}\right.$$

$$\left\{\begin{array}{r} \log t_0 = 3{,}612243779 \\ +[5{,}531813 - 10]\,\eta_0'^2 = + \quad 255.51 \end{array}\right.$$

$$\log t = 3{,}612269330$$

in sec.

$$t = 1^0\,8'\,15{,}1454''$$

$$\left\{\begin{array}{r} \log \eta_0' = 3{,}437459841_n \\ \log t_0 = 3{,}612243778 \\ 2\log W_2 = 9{,}997992334 - 10 \\ 4{,}387453233_n - 10 \\ -\frac{1}{2} \times 285.28 = - \quad 142.6 \\ +\frac{5}{4} \times 255.51 = + \quad 319.4 \end{array}\right.$$

$$\log (F - B_2) = 1{,}435166877$$

in sec.

$$F - B_2 = + \quad 27{,}23748''$$

$$F = 56^0\,14'\,16{,}25934''$$

$$B_1 - F = + \; 45'\,43{,}74066''$$

$$= + \quad 2743{,}74066''.$$

Formula (2), p. 463, will give us further:

$$\left\{\begin{array}{r} 1{,}487309971 \\ \log (B_1 - F) = 3{,}438343059 \\ -3\log W_m = 0{,}003038800 \\ +[2{,}930 - 10](B_1 - F)^3 \cos(B_1 + F) = - \quad 0.25 \end{array}\right.$$

$$\log x = 4{,}928691805$$

Argument $\frac{1}{2}(B_1 + F)$ for $\log W_m$ $= 56^0\,37.1355'$.

$B_1 + F = 113^0\,14'$.

With $\log W_0 = 9{,}998990 - 10$, according to the argument $B_0 = 56^0\,29{,}5'$, we obtain finally from formulae (4) and (5), pp. 463, 464:

$$\left\{\begin{array}{r} \log x = 4{,}928691805 \\ -[2{,}25325 - 10]\,y^2 W_0^4 = - \quad 127.80 \end{array}\right.$$

$$\left\{\begin{array}{r} \log\left\{s\cos\left(\alpha_{1.2} + \frac{1}{3}\Delta\alpha\right)\right\} = 4{,}928679025 \\ \log\left\{s\sin\left(\alpha_{1.2} + \frac{1}{3}\Delta\alpha\right)\right\} = 4{,}928653470_n \end{array}\right.$$

$$\alpha_{1.2} + \tfrac{1}{3}\Delta\alpha = 315^0\,0'\,6{,}0686''$$

$$\log s = 5{,}079181246$$

$$\left\{\begin{array}{r} 1{,}407017_n - 10 \\ \log(xy) = 9{,}857345_n \\ 4\log W_0 = 9{,}995960 - 10 \end{array}\right.$$

$$\log \Delta\alpha = 1{,}260322$$

in sec.

$$\left\{\begin{array}{r} \Delta\alpha = \quad 18{,}2105'' \\ t = \quad 1^0\,8'\,15{,}1454'' \\ \alpha_{1.2} = 314\;\;59\;\;59{,}9984 \end{array}\right.$$

$$\alpha_{2.1} = 136^0\;\;8'\;33{,}3543''$$

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Results:		According to the Previous Paragraph:
	$\log x = 4{,}9286918{,}05$	.89
	$\log y = 4{,}9286534{,}70_n$	.89n
	$\log s = 5{,}0791812{,}46$	.46
	$\alpha_{1\ 2} = 314°\ 59'\ 59{,}9984''$	59,9986"
	$\alpha_{2\ 1} = 136\ \ 8\ 33{,}3543$	33,3545

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H37mE
pt.1 v.4

3 1818 00008683 3

Chapter 10

Calculation of Small Figures on the Ellipsoid of Rotation by Means of Projection on a Plane

Par. 1. Various Transitions of the Formulae for Rectangular Spherical Coordinates. On p. 420 it was already indicated that in the survey of an extensive country as e.g., Prussia, it is necessary to turn to a large number of special abscissae-axes in order to calculate with convenient formulae. Insofar as it is a question of coordinates from distances and angles of direction the formula system:

$$\left.\begin{array}{l|l} v = s \sin \alpha_{1,2} & y_2 - y_1 = v - \left(y_1 + \frac{v}{3}\right)\frac{u^2}{2\varrho^2} + \varrho\, Gl_5 \\ u = s \cos \alpha_{1,2} & x_2 - x_1 = u\left(1 + \frac{y_2^2}{2\varrho^2} - \frac{v^2}{6\varrho^2} + Gl_4\right) \end{array}\right\} \qquad (1)$$

$$\varrho^2 = \frac{a_0^2}{K_0} = \frac{b_0^2}{W_0^4}$$

is fully adequate as long as the distances and ordinates remain smaller than 100 km and that point of the abscissae-axis for which ϱ is taken corresponds within 10 minutes in geographical width or 20 km to the argument $x_1 + \frac{1}{2}u$. For then, the influence of the disregarded parts is not 1 mm.

In the second formula (1), u is written as a factor in order to facilitate the following application. Since $x_2 - x_1$ disappears with u, this method is admissible. (The same is shown from p. 414 onward, in the development of the general formulae.)

A modification of the formula (1), which Zachariae and others use is already mentioned on p. 121. It considers the third-order terms

in part through small reductions in $\alpha_{1,2}$ and s, which are however different in both formulae:

$$\left.\begin{aligned} y_2 - y_1 &= s \sin\left(\alpha_{1,2} - (E+\varepsilon)\right) + \varrho\, Gl_5 \\ x_2 - x_1 &= s \cos\left(\alpha_{1,2} - 2(E+\varepsilon)\right).\left(1 + \frac{y_1^2}{2\varrho^2}\right) + \varrho\, Gl_5 \end{aligned}\right\} \qquad (2)$$

$$E = \frac{1}{2}\varrho'' \frac{u y_1}{\varrho^2} \qquad \varepsilon = \frac{1}{6}\varrho'' \frac{u v}{\varrho^2}.$$

The origin of this modification is probably due more to the manner of the derivation than to the purpose of reducing the inconveniences of the formulae (1).

This purpose, according to a first idea, may perhaps be arrived at sooner by placing:

$$\left.\begin{aligned} y_2 - y_1 &= (s + \delta_s) \sin(\alpha_{1,2} + \delta_\alpha) + \cdots \\ x_2 - x_1 &= (s + \delta_s) \cos(\alpha_{1,2} + \delta_\alpha) + \cdots, \end{aligned}\right\} \qquad (3)$$

consequently s and $\alpha_{1,2}$ change in similar amounts in both formulae, $s + \delta_s$ is nothing else but the distance between two points in a plane with the rectangular coordinates x_1, y_1 and x_2, y_2; $\alpha_{1,2} + \delta_\alpha$ is the pertinent angle of direction. There is no doubt that δ_s and δ_α are functions of the coordinates of the points.

When these functions are sufficiently simple, then the advantage of calculating the exact values of the coordinate differences will result not only according to the formula (3), that is by means of plane polygonometry but on the whole also all figures between the reference-points especially in the introduction of reference-points into the main net, may be treated simply according to the formulae of plane geometry, after the horizontal angles are adequately corrected. Since the correction δ_α has to be applied to every direction angle α, then the correction of a horizontal

angle is the difference of the δ_a for directions of both its sides: δ_a to the right minus δ_a to the left. δ_1 only comes into question when plane distance $s + \delta_1$ is to be compared with the true distance; but not in the calculation of the coordinates, since this will be supported on the plane triangle calculation which, by means of the reduced angle, gives $s + \delta_1$ directly.

Utilization of the formulae (3) in itself, is however not expedient, because δ_1 and δ_a retain inconvenient expressions. But it is possible to obtain all described advantages and to simplify these expressions, when we multiply one of the coordinates x or y with a suitable factor before we consider them as plane coordinates. It is easy to see that it

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is an advantage to leave x, lying in the meridian, unchanged. The proper factor of y will now presumably deviate from 1 only by a magnitude of the second order. We shall place y tentatively equal to

$$1 + \frac{cy^2}{\varrho^2}, \qquad (4)$$

where c signifies an unknown constant for the time being. In addition when we designate the plane distance $s + \delta_a$ with s';
the plane direction angle $\alpha_{1.2} + \delta_a$ with $\alpha'_{1.2}$;
and the plane coordinates x' and y'; (5)

then:

$$\left.\begin{aligned} y'_2 - y'_1 &= y_2\left(1 + \frac{cy_2^2}{\varrho^2}\right) - y_1\left(1 + \frac{cy_1^2}{\varrho^2}\right) = s' \sin \alpha'_{1.2} \\ x'_2 - x'_1 &= x_2 - x_1 \quad . \quad . \quad . \quad . \quad = s' \cos \alpha'_{1.2}. \end{aligned}\right\} \qquad (6)$$

In consideration of the formulae (1) the following results are

obtained from this without difficulty:

$$s' \sin \alpha'_{1.2} = v\left(1 + c\frac{y_1^2 + y_1 y_2 + y_2^2}{\varrho^2}\right) - \left(y_1 + \frac{v}{3}\right)\frac{u^2}{2\varrho^2} + \varrho\, Gl_5$$

$$s' \cos \alpha'_{1.2} = u\left(1 + \frac{y_2^2}{2\varrho^2} - \frac{v^2}{6\varrho^2} + Gl_4\right).$$

The division of the left and right sides yields, when we introduce on the right the factor

$$1 - \frac{y_2^2}{2\varrho^2} + \frac{v^2}{6\varrho^2} + Gl_4$$

for the factor appearing in the denominator of u, into the numerator and simultaneously also place $u^2 = s^2 - v^2 + Gl_4$ and $v^2 = (y_2 - y_1)^2 + Gl_4$:

$$\tan \alpha'_{1.2} = \frac{1}{u}\left\{v\left[1 + \left(c - \frac{1}{6}\right)\frac{y_1^2 + y_1 y_2 + y_2^2}{\varrho^2}\right] - \frac{1}{2}\left(y_1 + \frac{1}{3}v\right)\frac{s^2}{\varrho^2} + \varrho\, Gl_5\right\}.$$

This becomes especially simple for the assumption

$$c = \frac{1}{6}. \qquad (7)$$

Under this supposition, in consideration of the relation:

$$v : u = \tan \alpha_{1.2}$$

then:

$$\tan \alpha'_{1.2} = \tan \alpha_{1.2} - \frac{1}{u}\left\{\frac{1}{2}\left(y_1 + \frac{1}{3}v\right)\frac{s^2}{\varrho^2} + \varrho\, Gl_5\right\}.$$

When we imagine that, in place of the tangent, the cotangent of the complementary angle is written, we can immediately apply Taylor's series, as explained on pages 30 and 31. It yields:

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$$\alpha'_{1.2} = \alpha_{1.2} - \left\{\frac{1}{2}\left(y_1 + \frac{1}{3}v\right)\frac{s^2}{\varrho^2} + \varrho\, Gl_5\right\}\frac{\cos^2 \alpha_{1.2}}{u} \qquad (8)$$

and this development is indeed useful as long as the second power of $\{\} \cos \alpha_{1.2} : u$ can be disregarded. Since $u = s \cos \alpha_{1.2}$, then this expression is a small second-order quantity, the square of which has exactly the same order as the others which were disregarded in (8).

The formula (8) is after some reduction, especially substitution of $u = s \cos \alpha_{1.2}$, generally valid within the limits stated in the

beginning:

$$\underset{\text{in seconds}}{\alpha'_{1,2} - \alpha_{1,2}} = -\frac{1}{2}\varrho'' \frac{(x_2 - x_1)\left(y_1 + \frac{1}{3}(y_2 - y_1)\right)}{\varrho^2} + Gl_4, \tag{9}$$

wherein the plane coordinates may be substituted in the right-hand side. When we further consider (4), (7) and (6), we find:

$$x' = x \qquad y' = y\left(1 + \frac{y^2}{6\varrho^2}\right) \tag{10}$$

and

$$s'^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2\left(1 + \frac{y_1^2 + y_1 y_2 + y_2^2}{3\varrho^2} + Gl_4\right). \tag{11}$$

On the other hand, according to the formulae (1) corresponding to (1) page 416:

$$s^2 = (x_2 - x_1)^2\left(1 - \frac{y_1^2 + y_1 y_2 + y_2^2}{3\varrho^2} + Gl_4\right) + (y_2 - y_1)^2. \tag{12}$$

From this, the following results after subtraction and collection of terms:

$$s'^2 = s^2\left(1 + \frac{y_1^2 + y_1 y_2 + y_2^2}{3\varrho^2} + Gl_4\right),$$

whereby the disregarded terms could be taken into the parentheses, while their factors $(y_2 - y_1)^2$ and $(x_2 - x_1)^2$ in (8) and (11) are real fractions of s^2. Now, we can immediately set:

$$s' = s\left(1 + \frac{y_1^2 + y_1 y_2 + y_2^2}{6\varrho^2} + Gl_4\right). \tag{13}$$

Herewith, the relations are developed which exist between the parts of a shortest geodesic line, the ordinates of its terminal points and of the trapezoids defined by the abscissae-axis on the ellipsoid and of those of its plane transformation.

Par. 2. Plane Projections. If we imagine a point as the end-point of different lines, then, according to (10) the plane ordinates y' each taken strictly according to a specific line, obtain different values because ϱ^2 should always be approximately related to a point of the abscissae-axis lying within the foot-points of the extreme ordinates. Meanwhile the error does not become

important, if we determine y' according to the formula:

to the formula:

$$y' = y\left(1 + \frac{y^2}{6\varrho_F^2}\right) \qquad (1)$$

referred to y, whereby ϱ_F is related to the foot of the ordinates y itself. The error is equal to

$$\frac{1}{6} y^3 \left(\frac{1}{\varrho^2} - \frac{1}{\varrho_F^2}\right) \qquad (2)$$

and since $\varrho^2 = b_0^2 : W^4$ we have to substitute herein:

$$\frac{1}{\varrho^2} - \frac{1}{\varrho_F^2} = \frac{1}{b_0^2}\left((1 - e^2 \sin^2 B_0)^2 - (1 - e^2 \sin^2 B_F)^2\right),$$

when B_F is the geographical latitude of the foot-point of the y-ordinate and B_0 denotes that of the geographical latitude, which formula (10) p. 477 assumes as an argument. When $B_F - B_0$ is regarded as a differential, we obtain the following as a satisfactory approximate value for the last expression:

$$\frac{1}{\varrho^2} - \frac{1}{\varrho_F^2} = \frac{2}{b_0^2} e^2 \sin 2B_0 . (B_F - B_0),$$

by means of which the error (2) of the ordinates y' assumes the following form:

$$\frac{1}{3} e^2 \frac{y^3}{b_0^2} \sin 2B_0 . (B_F - B_0). \qquad (3)$$

When we place sides and ordinates in the maximum equal to 100 km, then $B_F - B_0$ cannot exceed the $50 : b_0^m$; and the maximum value of the ordinate-error amounts to only about 1/2 millimeter.

This insignificant linear error also has a very slight influence on the azimuth. As a result of (3), the error in $y_2' - y_1'$ is equal to:

$$\frac{1}{3} e^2 \sin 2B_0 \left(\frac{y_2^3}{b_0^2}(B_2 - B_0) - \frac{y_1^3}{b_0^2}(B_1 - B_0)\right),$$

in which B_2 and B_1 designate the geographical latitudes of the foot-points of the ordinates y_2 and y_1. But since we have to

place $B_0 = \frac{1}{2}(B_1 + B_2)$ the above expression is reduced to:

$$\frac{1}{3} e^2 \sin 2B_0 \frac{y_1^2 + y_2^2}{2b_0^2} (B_2 - B_1).$$

The error in $\tan \alpha'_{1,2}$ results from this when the almost similar expression $(x_2 - x_1) : b_0$ is introduced for $B_2 - B_1$, and is equal to:

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and this at the same time is the maximum of the error

$$\frac{1}{3} e^2 \sin 2B_0 \frac{y_1^2 + y_2^2}{2b_0^2} \qquad (4)$$

and this at the same time is the maximum of the error in $\alpha_{1,2}$ itself, in arc units. Multiplied by ϱ'' and setting $B_0 = 45^0$, and letting $y_1 = y_2 = 100^{km}$ then the result is not quite $0{,}002^y$ for the extreme amount of error in seconds.

Let us now refer the plane coordinates to the geodetic, by means of the relations:

$$x' = x \qquad y' = y\left(1 + \frac{y^2}{6\varrho_F^2}\right) \qquad (5)$$

we obtain a plane representation (or projection) of part of the ellipsoid of rotation.

For we can imagine that on both sides of a meridian all points up to a distance of 100 km are transferred without considering the special surfaces to which they are referred.

This plane projection within the parts utilized is identical with that of Gauss for calculating the Hannover land-survey (more correct: arc measurement, cf. O. Schreiber, Theorie der Projektionsmethode der hannoverischen Landesvermessung (Theory of the Projection Method of the Hannover Land-Survey) 1866, p. 91) by application of the conformal projection method. Similar to this method, it only provides advantages when the survey to be used extends up to negligible distances from a central meridian (of the abscissae-

axis) while along the same, it may be expanded quite arbitrarily.

But a diagram is called conformal when it resembles the original in the most minute parts. Infinitesimally minute rectangles must therefore not only be represented as rectangles in the diagram, but must also be rectangles similar to the original. When we first imagine the ordinate y in the foregoing paragraph to be simply applied as perpendicular to the abscissae-axis in the plane, then this diagram gives infinitesimally minute rectangles as rectangles adjacent to y but not as similar rectangles. According to p. 407 the infinitesimally minute distance dy traces, when the foot-point of the ordinate y is displaced by dx, a rectangle with the sides $\mathrm{II}\, dx$ and dy. But in that plane representation, which simply transforms y, the representation of the rectangle is a rectangle with sides dx and dy.

On the other hand, in the above detailed description, which assumes $y\left(1+\frac{y^2}{6\varrho_f^2}\right)$ as the ordinate, the diagram very nearly has the form of a rectangle with the sides dx and $dy\left(1+\frac{y^2}{2\varrho_f^2}\right)$ that is, with the sides in the ratio of $\mathrm{II}\, dx : dy$.

The divergence from the rectangular figure may be recognized from the fact that a slightly variable y' corresponds to a constant y in a variable x, since ϱ^2 is a function of x. In seconds, it amounts to:

$$\frac{1}{3}\varrho'' e^2 \sin 2B \frac{y^2}{b_0^2}, \qquad (6)$$

which shows a calculation quite identical with that indicated on p. 478 (see especially formula (4)). B, herein, refers to the

foot-points of the ordinates. The maximum of the expression (6) does not quite amount to 0,002"; the divergence is therefore insignificant.

Similarly insignificant is the divergence of the side-relationship in the plane from that of the ellipsoid. As a ratio of the sides we have the following:

$$\left.\begin{array}{l} \text{in the plane: } dx : dy\left(1 + \frac{y^2}{2\varrho_F^2}\right) \\ \text{in the ellipsoid: } u\,dx : dy \text{ or } dx : \frac{1}{u}\,dy \end{array}\right\} \quad (7)$$

According to p. 411 (4) however $u = \cos\frac{y}{\varrho_F} + Gl_6$ and therefore:

$$\frac{1}{u} = 1 + \frac{y^2}{2\varrho_F^2} + \frac{5y^4}{24\varrho_F^4} + \cdots \quad (8)$$

The side ratios (7) thus are identical with exception of the fraction $5y^4 : 24\varrho_F^4$. The difference also appears to be even less in its effect on y'. For when the differential $\frac{1}{u}\,dy$ is integrated from zero to y, we obtain that length of the ordinate which corresponds to that of the ellipsoidic side ratio. It is greater by $y^5 : 24\varrho_F^4$ than formula (5) gives it. This, in the maximum, amounts to $\frac{1}{4}$ millimeter.

In addition, we will also mention that Gauss prescribes as argument for the formula (9) p. 477 for φ^2 not $x_1 + \frac{1}{2}u$ but $x_1 + \frac{1}{3}u$. But the difference of the two arguments always falls within the admissible variation of the argument for sides less than 100 km.

Par. 3. Summary of the Formulae for the Plane Projection. Hypotheses: Sides and ordinates, not over 100 km in length; abscissae, any length.

According to (5) of the foregoing paragraph the reduction of the ellipsoid of rotation to the plane is:

$$\left.\begin{aligned} x' &= x \quad y' = y\left(1 + \frac{y^2}{6\varrho^2}\right) \\ \log y' &= \log y + \frac{M}{6}\frac{y^2}{\varrho^2} + Gl_4 \\ \varrho^2 &= \frac{a_0^2}{K} = \frac{b_0^2}{W^4}, \end{aligned}\right\} \quad (1)$$

Argument for K & W: the geographical latitude of the foot-points of the ordinates

The logarithmic formula does introduce small omissions, the linear not quite 0,1 mm. This in the logarithm is only at the most four units of the 10 decimal-place.

For the reduction of the linear lengths to the plane, according to (13) p. 477 we have:

$$\left.\begin{aligned} s' &= s\left(1 + \frac{y_1^2 + y_1 y_2 + y_2^2}{6\varrho^2} + Gl_4\right) \\ \log s' &= \log s + \frac{M}{2}\left(\frac{y_2 + y_1}{2\varrho}\right)^2 + \frac{M}{24}\left(\frac{y_2 - y_1}{\varrho}\right)^2 + Gl_4, \end{aligned}\right\} \quad (2)$$

whereby for greater convenience the calculation $\frac{3}{4}(y_2 + y_1)^2 + \frac{1}{4}(y_2 - y_1)^2$ was substituted in the second formula instead of $y_1^2 + y_1 y_2 + y_2^2$. The errors resulting by the introduction of the logarithmic form do not amount to 1 mm.

For ϱ in (2), an argument is to be employed which along with the arithmetical mean of the geographic latitudes of the foot-points of the ordinates amounts to about 10'.

The reduction of the direction-angles to the plane, according to (9) p. 477, when ϱ is formed as above, is:

$$\underset{\text{in seconds}}{\mathfrak{a}'_{1.2} - \mathfrak{a}_{1.2}} = -\frac{1}{2}\varrho'' \frac{(x_2 - x_1)\left(\frac{y_2 + y_1}{2} - \frac{y_2 - y_1}{6}\right)}{\varrho^2} + Gl_4. \quad (3)$$

For individual horizontal angles the difference of the reductions for both sides, appears as a reduction. Insofar as it here concerns objects whose coordinates are still unknown, approximate values of the same must first be ascertained. Then, it is generally

Par. 3. Summary of the Formulae for the Plane Projection.

expedient to insert the plane ordinates into the right side of formula (3), which does not result in an error worth mentioning.

For the reduction from the plane to the ellipsoid, as we may easily deduce from the previous formula:

$$\left.\begin{aligned} \log y &= \log y' - \frac{M}{6}\frac{y'^2}{\varrho^2} + Gl_4 \\ \log s &= \log s' - \frac{M}{2}\left(\frac{y'_2+y'_1}{2\varrho}\right)^2 - \frac{M}{24}\left(\frac{y'_2-y'_1}{\varrho}\right)^2 + Gl_4 \\ \alpha_{1.2} - \alpha'_{1.2} &= +\frac{1}{2}\varrho''\frac{(x_2-x_1)\left(\frac{y'_2+y'_1}{2} - \frac{y'_2-y'_1}{6}\right)}{\varrho^2} + Gl_4, \end{aligned}\right\} \quad (4)$$

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whereby ϱ, has to correspond in the first of these formulae with the foot-point of the ordinates; in both of the others, however, with the mean geographic latitude of both foot-points, to the nearest 10'.

Accuracy: The accuracy of the above formulae fully corresponds with that of the formula system (1) p. 474. In part, this is already proven but not as yet in regard to the formulae (9) and (13) p. 477. But is is possible to conduct a summary proof, if one shows that with the given y_1, s and $\alpha_{1.2}$ the former as well as the latter formulae produce the same results up to magnitudes of $y_2 - y_1$ and $x_2 - x_1$ which do not come into consideration. For abbreviation we place:

$$\begin{aligned} \log s' &= \log s + M\delta_1, \\ \alpha'_{1.2} - \alpha_{1.2} &= \varrho''\delta_2, \end{aligned}$$

in seconds

Then first of all, $s' = s\left(1 + \delta_1 + \frac{1}{2}\delta_1^2 + \cdots\right)$ and furthermore, by the formation of the expressions for $s' \cos \alpha'_{1.2}$ and $s' \sin \alpha'_{1.2}$ and when $s \cos \alpha_{1.2}$ is designated with u and $s \sin \alpha_{1.2}$ with v:

$$x_2 - x_1 = u + u\delta_1 - v\delta_2 + \frac{1}{2}u(\delta_1^2 - \delta_2^2) - v\delta_1\delta_2 + \cdots$$
$$y_2' - y_1' = v + v\delta_1 + u\delta_2 + \frac{1}{2}v(\delta_1^2 - \delta_2^2) + u\delta_1\delta_2 + \cdots \quad (5)$$

Noting further that, according to pp. 476 and 477, we can set:

$$y_2' - y_1' = (y_2 - y_1)\left(1 + \delta_1 + \frac{1}{2}\delta_1^2 + \cdots\right)$$

or

$$y_2 - y_1 = (y_2' - y_1')\left(1 - \delta_1 + \frac{1}{2}\delta_1^2 + \cdots\right),$$

then, through the above expression for $y_2' - y_1'$ there will follow:

$$y_2 - y_1 = v + u\delta_2 - \frac{1}{2}v\delta_2^2 + \cdots. \quad (6)$$

When we disregard the highest terms in the formulae (5) and (6) and simply put

$$x_2 - x_1 = u + u\delta_1 - v\delta_2 + \cdots \quad (5^*)$$
$$y_2 - y_1 = v + u\delta_2, \quad (6^*)$$

then we commit, in the maximum, errors of not quite 1 mm, as the examination of the maxima of the neglected terms shows, in which we may take

$$\delta_1 = \frac{y_2(y_2 - v) + \frac{1}{3}v^2}{2\varrho^2}$$

and

$$\delta_2 = -\frac{u\left(y_2 - \frac{2}{3}v\right)}{2\varrho^2}.$$

Now, in order to compare (5*) and (6*) with the formulae (1) p. 474, we have to utilize the exact values for (5*) and (6*)

$$\delta_1 = \frac{y_1^2 + y_1 y_2 + y_2^2}{6\varrho^2}$$
$$\delta_2 = -\frac{(x_2 - x_1)\left(\frac{2}{3}y_1 + \frac{1}{3}y_2\right)}{2\varrho^2}$$

and insert on the right-hand side, instead of y_2, y_1 and $x_2 - x_1$ the quantities y_2, y_1, v and $\varkappa$, according to formula (1). This may easily be done by using formulae (5*) and (6*) themselves. The differences with (1) p. 474 will again appear as small terms, the maximal amount does not significantly exceed 1 mm. Now, since these formulae (1) themselves are exact only up to 1 mm, then we can indeed, indicate formulae (1) to (3) of the plane transfer as exactly equal to the former.

Finally, considering formulae (4), the circumstance that y and y' only diverge from each other by $\frac{1}{10000}$ of their amount and considering the maximum values of the second-order terms, show immediately that they differ from formulae (1) to (3), only by very insignificant amounts.

Par. 4. Calculation of the Plane Coordinates from Geographical Positions and Vice Versa.

When, in addition to the geographical position of the coordinate origin P_1, that of the point P_2 is given, whose plane coordinates x and y' are to be determined--a task which arises in the transfer to the systems of detail coordinates--then without more ado, we can utilize formulae (1) to (3) of Par. 25, p. 463, with the change, the formula

$$\log y' = \log y + \frac{M}{6 b_0^2} y^2 W_r^2 \qquad (1)$$

is added to formula (3) in order to obtain the plane ordinate y'. The numerical value of the coefficient $M : 6b_0^2$ is indicated on p. 463; for W_r however, we may simply insert W_2, which is already necessary otherwise. This corresponds to the other

omissions in the formula system on p.463 and in y' gives only fractions of millimeters.

For the reverse task also (which appears when a triangulation net is calculated in the plane projection) we will retain the earlier formulae for geodetic coordinates i.e., the formulae of Par. 22, p. 456 and first of all only think of the plane ordinates reduced on the ellipsoid by means of the formula:

$$\log y = \log y' - \frac{M}{6b_0^2} y'^2 W_P' + \cdots. \qquad (2)$$

As the given point P_1, the beginning of the coordinates is assumed, and the additional calculation is contained in the formulae (2), (3) and (4*) (loc.cit.). The calculation of t is omitted as well as the calculation s, $\alpha_{1.2}$ and $\alpha_{2.1}$, the values of which do not immediately come into question. However, if we wish to ascertain the direction of the meridian at P_2, t must also be determined. Then, since, according to pp. 479 and 480, the parallels of the abscissae-axis on the ellipsoid are very closely portrayed as such in the plane, and the diagram is essentially conformal, then the meridian in this also forms the angle t with the parallels to the abscissae-axis (Fig. 39, p. 421).

Par. 5. General Remarks about the Method of the Plane Projection. A detailed theory of Gauss' projection method has been given elsewhere by O. Schreiber (op. cit.).

For the use of the method in narrow meridional districts, for the purpose of detailed measurement which we alone considered as advantageous, the above modification could suffice. Writers have already developed this in less rigorous representation in

the Zeitschr. f. Verm.-Wesen (Journal of Surveying) 1876, p. 238, in which a logarithmic diagram, which may easily be set up, is used for the approximate reduction of the direction-angle to the plane (which is often adequate). The practical advantages of the coordinates on the ellipsoid are also made known. (In this connection, also compare an opposite view in Jordan, Handbuch (Handbook) Vol. 2, pp. 293 ff.).

The chief advantage of the plane projection it seems to us, lies in the fact, that following the resulting reduction of the directions on the plane all further calculations only take place according to the rules of plane geometry, i.e., the consideration of the ellipsoid shape of the earth is brought into a very convenient and clear form. Consideration of the small terms in the calculations of the detail triangulation on the ellipsoid itself is very troublesome, even though these points are very small.

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Since for the scale relationship of the dimensions, in the restriction to absolute values of the ordinates $y \leq 0{,}01\,a_0$, i.e., 64 km:

$$1 + \frac{y^2}{2\varrho^2} \qquad (1)$$

only exceeds the unit value by $\frac{1}{20000}$, at the boundary of the survey region. Then in this case, for purposes of the detailed survey, the plane-dimensions may even be regarded, as true dimensions and may be connected with measured-lengths in the adjustment of traverse lines.

Gauss himself did not publish anything about the conformal plane projection of the Hannoverian land survey; what we know is derived from the exchange of letters with Schumacher. However, in his Geodätischen Untersuchungen (Geodetic Investigations), he developed another projection-method which conforms excellently, i.e., the one on the sphere. This is useful for very extensive surveys and only requires very small reductions of the measured directions. Regarding this method, we refer to the original work, as well as the elaboration given by Jordan in his Handbuch der Vermessungskunde (Handbook for the Science of Measurement). We only note here, that in this method, in contrast to the direct calculation on the ellipsoid, the advantage simply consists in the spherical calculation of the geographic positions. It also uses the following convenient transfer of the geographical latitudes, by means of an auxiliary-table and the geographical differences of length through the multiplication by a constant on the ellipsoid. The direct calculation on the ellipsoid on the other hand has the advantage, that it is easier to conduct the calculation with different elements of the ellipsoid--at least for short distances. The correlation between the geodetic results and the geographic coordinates is more clear.

Finally, we might mention as a disadvantage to this method, that in the transfer of the measured angles on the sphere, seemingly unimportant corrections may be omitted which may still accumulate to appreciable amounts for extensive territories.

Chapter 11

The Calculations for a Land Survey*)

Par. 1. The Significance of Geographic Coordinates. For a land survey it is always adequate to make the assumption that the projection is made from an ellipsoid of revolution with straight plumb lines (see p. 134).

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Although this is not rigorously true, the influence on the opposed positions of the points is insignificant, as later examination will show. The elements of this ellipsoid also come into consideration to only a minor extent, insofar as the distances (and surface areas) are concerned (cf. p. 405). However, these factors have a fundamental influence on geographic positions. The calculated values of these can in addition show considerable differences from the true values because of the deviation of the plumb lines. But, first of all, in a large land survey, the geographic coordinates are used only as a means of describing the opposed location of the points. In this connection they are much more convenient to use than rectangular coordinates; even though the latter depend much less on the elements of the ellipsoid, their practical usefulness is confined to small areas (p. 474). The reason for this convenience is that

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*)We do not consider here the additive method or the method of chords. The pertinent information for these has already been considered in the appropriate places.

in the successive transfer of geographic coordinates from point to point only the simple formulae of Par. 22, p. 456 are required, while in the application of rectangular coordinates to the projection of an area the by-no-means simple formulae of p. 419 and possibly, still more complicated formulae are required immediately because of the dependence of the formulae on the distance of the points from the abscissae-axis.

Because of the significance of the calculated geographic position it is entirely correct to state them to one unit of the 3, 4 or even 5 decimal-place of the seconds of arc. This is proper since these figures reflect the accuracy of geodetic and not of astronomical measurements, and since one is often tempted to push the precision of calculations one or two places beyond the precision of measurement*).

Par. 2. Preliminary Calculations. In order to make the exact calculation possible, it is necessary first to make a preliminary calculation of sides from the length of the base (see Par. 3) reduced to sea level. In doing this one makes use of the Law of Sines of plane trigonometry to adjust the sum of the angles to 180°, whereupon it is immediately possible to apply Legendre's Theorem.

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*)It is nevertheless desirable that the calculated positions correspond to the true ones as nearly as possible. If possible, one therefore selects as the astronomical point of origin a locality which is free of mass inequalities which would deflect the plumb line. It would be still better to observe the deviation of the plumb line at the designated point of origin in order to find local plumb-line disturbances which arise as a result of invisible mass inequalities. This would be done by comparison with approximated triangulation points before making final measurements.
This procedure is also to be recommended for use in finding the error in making projections of areas being surveyed.

Subsequently an approximate determination of geographic latitudes is made from the starting point, whose latitude has been precisely determined astronomically. This is done with the help of the astronomically-determined azimuth of a side of a triangle which originates here (the so-called primary azimuth), in case an older map is not available.

This determination only has the purpose of providing a sufficiently accurate estimate of the curvature of the ellipsoid to permit the calculation of the triangle excesses. Since the excesses which come into consideration mostly amount to only a few seconds of arc, in general an error of as much as $\frac{1}{2}$ degree in latitude would not give as much as 0,001" error in the excess. This is shown by the differential formula (3), p. 403, and by the appended calculations. Even with an excess of 10" a precision of 10' in latitude always suffices.

Following this (and perhaps more thoroughly after the above-mentioned formula) one can see that an older map or a superficial calculation of the geographic latitudes suffices for the stated purpose, as set forth in the formulae on p. 456 and p. 457, assuming a constant value for W and neglecting the higher terms.

Par. 3. Reduction of the Base to a Normal Horizon. If the reduction of the geographic location of the base, as described in Par. 2, was not sufficiently well determined, then a renewed reduction to sea-level must be carried out after the preliminary calculation has made the geographic location by latitude and azimuth sufficiently well known. (An approximate direct observation of both

elements will also suffice.) The reduction follows from the shortness of the distance, without doubt, and quite sufficiently, from the assumption that the directly-measured horizontal distance is taken as one arc of a circle at mean sea-level of the base. Likewise, the projection may be viewed as the arc of a circle. The radius of curvature for the latter is the specific radius of curvature of the ellipsoid at the mean geographic latitude of the base and at its mean azimuth; the radius of curvature for the former is greater by the distance to mean sea-level.

The calculation can be passed over as it has already been stated in principle on p. 6; especially since its treatment involves no difficulties.

The reduction of the base to a so-called mean land horizon is only to be recommended for small countries, where the transition from the primary triangulation to rectangular coordinates may be made directly. For the calculation of geographic positions, however, this procedure leads to complications (as will be shown in subsequent paragraph), while on the other hand, for

detail work and for the practical purposes of surveying it is entirely a matter of indifference if the distances deviate from those of the mean level by fractions of the order of $\frac{h}{\rho}$, since these are for the most part less than $\frac{1}{10000}$. Eventually, a corresponding reduction will also be easy to carry out.

Therefore we assume the reduction to a sea-level in the following section.

Chapter 11. The Calculations for a Land Survey.

Par. 4. Reduction of the Measured Angles and Directions. The angles and directions observed at reference points (stations) can first be adjusted by stations. The result of the station adjustment will be a list of numbers whose differences represent the angles, but which are properly called direction values and which differ from the azimuths by a constant. These values are to be corrected before the adjustment of the net in the following manner:

1. According to (6) p. 190 the reduction for a south-west astronomic azimuth, taken for an object at a sea level H amounts to in seconds:

$$+\frac{1}{2}\rho''e^2\frac{H}{a_0}\cos^2 B\sin 2a+\cdots. \tag{1}$$

This reduction is necessary to obtain the astronomic azimuth of the projection of this object on the ellipsoid of rotation; and then, using Bessel's dimensions of the ellipsoid:

$$+0{,}108''\cos^2 B\sin 2a\,(H\text{ in kilometers})+\cdots. \tag{1*}$$

2. Further, the reduction from the astronomic to the geodetic azimuth, according to (11), p. 332, amounts, in seconds, to:

$$-\frac{1}{12}\rho''e^2\frac{s^2}{a_0^2}\cos^2 B\sin 2a+\cdots, \tag{2}$$

or

$$-0{,}028''\cos^2 B\sin 2a\left(\frac{s\text{ in kilometers}}{100}\right)^2+\cdots \tag{2*}$$

Here a signifies the measured azimuth, B the latitude of the point of observation and s the horizontal distance of the object.

According to these formulae every observed azimuth and every observed direction of the net, corrected at the station is, strictly speaking, to be corrected. For the angles the correction is obtained as the difference of the directional corrections of both sides.

Par. 4. Reduction of the Measured Angles and Directions.

If the distances are very great, then the higher terms of the equations on p. 190 and p. 332 can become significant; ordinarily, however, the above expressions will suffice.

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We have already discussed the errors which arise as a result of the omission of the reductions (2) on p. 400 ff.

The above corrections are not sufficient to reveal the true nature of the earth; nevertheless they do represent the most important factors and those to be considered first. We will now name the other pertinent corrections. They are:

3. The correction for the deviation of the plumb which arises because the plumb lines are not exact normals of an ellipsoid.

4. The correction due to the deviation of the geoid surface from the surface of an ellipsoid of revolution.

5. The correction for the lateral refraction of the light rays, i.e., because of the deviation from a vertical plane, from the observation point to the object, due to the ellipsoid form of the air layers.

In Grunerts Archiv, (Grunerts Archive), 1869, part 51, p. 20 ff. Sonderhof treats all the reductions of the horizontal angle under the hypothesis of an ellipsoidal figure for the mathematical surface of the earth. Villarceau discusses the reduction due to elevation above sea level, in Comptes rendus (Transactions), 1866, volume 63, p. 850; Andrae cited the formulae with partial development in Dänische Gradmessung, (Danish Arc Measurement), 1867, volume 1 also cf. the review in Vierteljahrsschrift der Astronom. Ges., (Quarterly of Astronomy and Geodesy), 1878, volume 13, p. 72 ff.

That Gauss and Bessel already knew these reductions is not to be doubted, although the former did not develop them, and Bessel only developed the reduction from the astronomic to the geodetic azimuth. In this connection cf. a communication by Peters in Astronom. Nachr.(Journal of Astronomy), 1856, volume 43, No. 1022, p. 209.

Par. 5. Computation of the Spherical Excesses of the Triangles. For the purpose of the customary adjustment of the net from the specified observations, it is first necessary to calculate the spherical excesses of those triangles of which all angles have been determined. For this one needs the enclosed areas, which are most conveniently calculated from two sides and the angle between them. The logarithms of the sides yield the preliminary value.

The exactness of these data for the sides can always be considered sufficient for the calculation of excesses, for if the exactness amounted to only $\frac{1}{10000}$ then the excesses of a 10" quantity would still only be in error by 0,001". But even the points which are farther removed from the base have the precision required for these calculations since one can get values for the angles within at least ±1" mean error, from the station correction.

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On the other hand, it is superfluous to retain more than three decimal places in the adjustment of the triangle angles.

Regarding the calculation of spherical excess, the admissibility of a purely spherical calculation for a moderately extended chain

has already been discussed on p. 403 ff. It is easy, using the table on p. 213 in Albrecht's collection of tables, to consider the special curvature of single triangles or of small complexes of triangles. Formula (1), p. 362, for log ε is ordinarily considered next. As has been stated (p. 99), the first term suffices in most cases, and we may therefore write:

$$\varepsilon = \rho'' \frac{bc \sin A}{2a_0^2} K + \cdots$$

A side length of 120 km is required before the influence is felt in the third decimal place, as has been stated in detail on p. 99. In this case formula (3), p. 100, should perhaps be used, as the higher terms may be more conveniently considered here.

Par. 6. Adjustment. In order to accomplish the adjustment of observational errors, a preferably wide-meshed net of the largest triangles (the first-order triangles) must be built up, all of which are adjusted among themselves. These will then serve as the fixed foundation for smaller triangles, which are interspersed in systems of second-, third-, and fourth-order. Correspondingly, the first-order triangles receive the maximum precision in measurement of angles. (For the best methods of doing this cf. O. Schreiber, Anordnung der Horizontalwinkelmessungen, etc., Zeitschr. f. Vermessungswesen, (Classification of Horizontal Angle Measurements, etc., Journal of Surveying), 1878, volume 7, p. 209 ff; also, by the same author, Richtungsbeobachtungen und Winkelbeobachtungen (Azimuth and Angle Observations), ibid, volume 8, p. 97 ff.)

First-order nets attain such extension in large countries that, for this reason, the detailed work is commenced without waiting for completion and adjustment of the entire primary net. It then becomes necessary to carry out a step-by-step preliminary adjustment at the same time.

Then the results of previous adjustments must always be retained and this limits the parts of the net which follow. This, however, is harmless as long as the mean value of the corrections of the angles, contained in the equations of condition, of the part to be joined is greater than the mean value of the suppressed corrections of the parts of the net which is to be retained unaltered.

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Or, in other words, so long as the figure is less well-determined between the new part and those points which are common to both parts of the net, than it is between these points and the part which has already been finished.

Otherwise, a considerable constraint can appear, as for instance, in the insertion of a narrow, strip-like net composed of a few triangles into a completely corrected group of triangles.

In England, where the main triangulation has been completed for three decades, the first-order net covers the entire country uniformly and is adjusted in uniformly-ranked parts.

In Prussia, however, where first-order triangulation is still in progress, a wide-meshed net of triangle chains, principally, is being built up. After adjustment of this system of chains, which must be carried out piecemeal, the open meshes are filled. Here, however, a considerable constraint remains. Even though this is negligible for land survey purposes, it still makes the utilization of the calculated, geodetically-extended azimuths, seem inconvenient for arc measurement, since these azimuths are doubtlessly often

erroneous by several seconds (cf. especially the Moravian-Silesian net, p. 253 ff. and the Posnanian net p. 88 ff. in volume 3 of Hauptdreiecke--First-Order Triangles).

The English East Indian Survey is likewise built of triangle chains and these are arranged in grid-iron form. (cf. the review in Vierteljahrsschrift der Astronom. Ges., 1873, volume 8, pp. 14 ff.)

For the simultaneous utilization of the triangulation net for land surveying and arc measurement it would be advantageous to give up the customary adjustment procedure employing conditioned observations and to apply instead the procedure using adjusted observations, as has long been done with secondary nets. Although the former procedure is shorter, at least in the case of simpler chains (cf. the Correction Calculation, p. 329), when it finally comes to the overall adjustment of large net complexes it loses clarity, while the adjustment using adjusted observations may always be organized lucidly. We will treat this subject in the last paragraph of this chapter, however, in what immediately follows we will describe the customary method.

In this connection we want only to mention that, in setting up the equation for the sides by means of the Law of Sines, the simple Legendre's Theorem always suffices, as has been proven in Par. 5, p. 361.

The correction of triangulation nets using conditioned observations is given by Andrae in Dänische Gradmessung, volume 1, and also, it is given in detail with an interesting modification, including a device applied by Hansen, in the Hauptdreiecke der preuss. Landestriangulation (First-Order Triangles of the Prussian Land Triangulation), volume 2, pp. 303 ff.

Andrae's treatment is a completion of Bessel's Ausgleichungsverfahrens (Adjustment Procedure). The procedure of the author, which is illustrated by an example on p. 240 of his Ausgleichungsrechnung, coincides with Andrae's in that he utilizes for the adjustment of the net unknown angle corrections which are to be determined, while the previously-mentioned modification is distinguished from both of the other methods by the introduction of direction adjustment as the unknown to be determined.

The author has clearly developed and listed Andrae's formulae in Vierteljahrsschrift der Astronom. Ges., 1877, volume 12, p. 192 ff.

In order to simplify the adjustment it is desirable to arrange the observations so that one can regard the result of the adjustment, either exactly or with small error, as being given by a single set of very accurate direction measurements (Ausgleichungsrechnung, p. 332). Then the adjustment procedure in the net becomes very much simplified, and it assumes the form stated by Gauss, which he used in the arc measurement in Hanover (cf. also the example given by him on p. 185 of Ausgleichungsrechnung). By this means O. Schreiber has made great savings in calculation in the newer chains of the Prussian triangulation (cf. his above-mentioned publication).

Par. 7. Calculation of Geographic Coordinates. After adjustment has been completed there follows the final calculation of sides by means of the Law of Sines and the application of Legendre's Theorem (Par. 5, p. 361) and then the calculation of coordinates. To what

has been remarked in Par. 1 there remains only to be added that, for the calculation of short distances the formulae on p. 456 and p. 457, or the formulae of O. Schreiber which were mentioned on p. 458 are best adapted. For the latter, as has already been mentioned, there are tables available for latitudes from 47° to 57°. Schreiber's formulae are not essentially different from those on pp. 456 and 457.

Controls for the calculation may be had by double calculation of the points of intersection of the sides along which the translation of the position is accomplished.

Par. 8. Insertion of Points into the First Order Net. Here a considerable complication may easily arise in the adjustment procedure, since a great number of condition equations appear, even in the case where one always successively inserts only one point at a time. For this reason it has long been recognized that it is advantageous to adjust using mean observations. Then for every point which is to be inserted there appear only the two coordinates as unknowns.

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As long as the distances between adjacent points amount to several miles (as in the second-order net), the choice of unknowns will be similar to that in the first-order net. Par. 10 can serve to illustrate the adjustment; for the procedure is altered only to the extent that the coordinates of the points of the already adjusted net must be strictly retained.

For still shorter distances, but also under circumstances for these first insertions, the application of rectangular, plane

coordinates recommends itself. The formulae for this purpose were developed in the previous chapter, on pp. 476 ff. Even when one introduces rectangular geodetic coordinates for mapping details it is most convenient to take into account the ellipsoidal figure of the projection surface by making the transition to the plane rectangular coordinates for the adjustment and to return to geodetic rectangular coordinates once the adjustment is completed.

In relation to the details of the adjustment we refer the reader to pp. 158 ff. of the Ausgleichungsrechnung. One will easily see from this how substitution may be carried out, not only for isolated points, but for entire nets, if the observations at the stations can be incorporated simply into complete equations. Andrae has developed the formulae for independent observations in volume 2 of the Dänische Gradmessung (cf. also Vierteljahrsschrift der Astronom. Ges., 1878, volume 13, pp. 65 ff); except for the special coefficients of the error equation one can directly apply here all that was given in Par. 10 in an essentially identical process.

See also in this connection the procedure of substitution, and adjustment procedures, in the paper by Mauck about the survey in Mecklenburg-Schwerin, published in Zeitschrift für Vermessungswesen, 1879, volume 8.

As regards the situation of the meridian which serves as the abscissa, one can either, as in the Prussian cadastral survey, choose it according to local circumstances, or systematically according to round values of geographic longitude, as from degree to degree. In the latter case it is a matter of little consequence whether any point of the main net coincides with this meridian or not.

Par. 9. Land Horizon.

(cf. also the article about the organization of a land survey in Zeitschrift für Vermessungswesen, 1877, volume 6, p. 607 ff.)

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Par. 9. Land Horizon. If the base is not reduced to sea level but instead to some convenient level surface which is closer to the mean level of the land, the mean land horizon, then one must take this into account in the calculation of the spherical excesses and the geographic positions.

The spherical excesses are very little influenced by the dimensions of the ellipsoid; if one suspects influence from this source, however, one can either calculate the reference ellipsoid by means of them, or one can simply increase a_0 by the land horizon's elevation above sea level. When it comes to the geographic position, it is most convenient in practice to ascertain the quantity by which the logarithm of the base line is changed through the reduction to sea level, and then to apply this change to the logarithms of all the sides, thence to calculate geographic positions with those dimensions of the earth ellipsoid for which tables exist. This procedure is in accord with the assumption that the land horizon and the geoid can be treated as similar ellipsoids. Naturally, this is not rigorous, for the supposition of an ellipsoid of revolution as the geoid, and the straight (or, even curved, as in fact is true) plumb lines, require reference surfaces which are no longer exactly in the form of ellipsoids of revolution.

It would be more rigorous, while adopting the ellipsoidal form, to increase both of the semi-axes, a_0 and b_0, by the elevation

above sea level of the land horizon (thus not retaining e^2). Or, still better: to determine the corresponding values of φ_m and φ_0 for the mean latitude of the country from the values of a_0 and e^2 which have been corrected to sea level for the land horizon ellipsoid.

However, here one must dispense with the convenience of available tables and strive only for the estimation of minor influences which one otherwise mostly neglects. For, if one neglects the reduction for the height of objects above sea level (or above the land horizon) in the measured angles, then one is also justified in considering the projection of the net on the land horizon and on the sea horizon as similar figures. This gives rise to a uniform omission, and thus the practical, convenient procedure is justified.

One gets around all objections regarding pertinent mathematical treatment by immediately projecting the base lines to the sea horizon, as is customary in extended surveys.

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One can after all go to the spherical hypothesis, which is surely sufficient for this purpose, for detail surveying on higher horizons, if desired.

Par. 10. Adjustment Using Mean Observations. In large countries it is best to introduce geographic latitude and longitude differences as unknowns (referred to the arbitrary prime meridian). In smaller countries the unknowns can be direct geodetic or planar rectangular coordinates. We will only consider the former case.

In doing this it is necessary, based on the preliminary calculation of sides, to carry out as accurately as possible a preliminary calculation of latitudes and longitudes from the astronomical starting point. This is done in order that the adjustment can yield only very small changes, the squares of which are negligible, so that simple differential formulae suffice for the calculation of the changes in azimuth due to changes in geographic position.

These differential formulae have already been derived. The second or third formula of (4), p. 282 will immediately yield the change $d\alpha_{ik}$, of the azimuth at point P_i relative to point P_k, which is associated to the changes dB_i and dB_k of the geographic latitudes and to the changes dL_i and dL_k of the geographic longitudes of points P_i and P_k. We have:

$$d\alpha_{ik} = -\frac{\rho}{m_{ik}}\left\{\begin{array}{c} -\frac{1-e^2}{W_i^3}\left(\frac{dm}{ds}\right)_{ki}\sin\alpha_{ik}\,dB_i - \frac{1-e^2}{W_k^3}\sin\alpha_{ki}\,dB_k \\ +(dL_i - dL_k)\frac{\cos B_k}{W_k}\cos\alpha_{ki} \end{array}\right\}. \qquad (1)$$

The calculation of the magnitudes of the m and their derivatives has already been discussed on pp. 283 ff.

If we consider $\overline{B}$ and $\overline{L}$ as the geographic positions of the azimuths of those geodetic lines which correspond to the sides of the triangulation net to be calculated, and if $\bar{\alpha}_{ik}$ is one of these calculated azimuths, α_{ik} the observed (and properly reduced according to Par. 4, p. 488) value, and λ_{ik} is the corresponding probable correction for one of its observational errors, then one will choose dB_i, dB_k, dL_i and dL_k of such nature as to yield the value of $d\alpha_{ik}$ which is, according to the above equation,

$$\alpha_{ik} + \lambda_{ik} - \bar{\alpha}_{ik} \qquad (2)$$

equal to*). The observation of the azimuth α_{ik} would then give the error equation:

$$\lambda_{ik} = \bar{\alpha}_{ik} - \alpha_{ik} + \frac{\sigma_0}{m_{ik}} \left\{ \begin{array}{c} -\frac{1-e^2}{W_i^2}\left(\frac{dm}{ds}\right)_{ki} \sin\alpha_{ik}\, dB_i - \frac{1-e^2}{W_k^2}\sin\alpha_{ki}\, dB_k \\ +(dL_i - dL_k)\frac{\cos B}{W_k}\cos\alpha_{ki} \end{array} \right\}. \qquad (3)$$

Here, of course, it suffices to calculate the coefficients with the approximate values of α and B.

However, in general, it is not the azimuth which is measured but a set of values of two or more directions. By addition of a suitable value one can, however (and we assume it), derive the azimuth by successive approximation of the given set; but an unknown correction $-u$ will still remain (u denotes the direction errors of the set), which we will have to think of as added to all given values of α (Ausgleichungsrechnung, p. 160).

In the right member of the above equation, then, only the unknown $+u$ still appears. This has the same value for all equations of one set but changes for every new set. The error equations for one set, which contains all directions which occur, take the form:

$$\left.\begin{array}{l} \lambda_1 = -l_1 + u + a_1 x + b_1 y + c_1 z + \cdots \text{ weight } g_1 \\ \lambda_2 = -l_2 + u + a_2 x + b_2 y + c_2 z + \cdots \quad " \quad g_2 \\ \lambda_3 = -l_3 + u + a_3 x + b_3 y + c_3 z + \cdots \quad " \quad g_3 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}, \qquad (4)$$

*)The formulae of Par. 23, p. 313 can serve to calculate $\bar{\alpha}$. Also, the formulae of Par. 11, p. 157 should be adaptable, and naturally, the reduction from the astronomical to the geodetic azimuth vanishes here.

in which, in accordance with equation (3), l is the difference of observed azimuth - calculated azimuth, and $x, y, z \cdots$ are the unknown (differential) corrections to the geographic coordinates of all points; $a, b, c \cdots$ represent their coefficients in abbreviated form. We set the weights, g equal to 1 as long as the directions are really observed, as against zero if they are not.

A second set from the same observation point gives a system of the same form, such as:

$$\left.\begin{aligned} \lambda'_1 &= -l'_1 + u' + a_1 x + b_1 y + c_1 z + \cdots \quad \text{weight } g'_1 \\ \lambda'_2 &= -l'_2 + u' + a_2 x + b_2 y + c_2 z + \cdots \quad \text{"} \quad g'_2 \\ \lambda'_3 &= -l'_3 + u' + a_3 x + b_3 y + c_3 z + \cdots \quad \text{"} \quad g'_3 \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\}, \qquad (5)$$

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every other set from the same observation point follows the same pattern, and likewise, those from other observation points.

There is only <u>this</u> distinction for different observation points; that new combinations of four unknowns always appear, while in the system of equations for <u>one</u> station the combinations $ax+by+cz+\cdots$ in the equations with the same subscript (i. e. which have the same direction) are identical.

Aside from this, a modification appears for the first station, the astronomical station. For this station dB and dL are equal to zero. Further, here <u>one</u> azimuth is truly given.

For the time being, we disregard the fact that one azimuth is measured, and we will show in the discussion of the base line how

this is to be introduced. One is inevitably led to the conclusion that the calculation is the same if the latitude and azimuth are not measured at the same station.

In order now to minimize the sum of the squares of all $\lambda\lambda g$, one has only the usual procedure of forming normal equations to consider (Ausgleichungsrechnung, p. 116). In connection with the unknowns a distinction arises because $u, u' \ldots$ always appear in only one system of error equations, but $x, y, z \ldots$ appear in various different systems. One will first eliminate the former, local unknowns, before the entire normal equations for $x, y, z \ldots$ are formed. The unknowns, however, cannot be introduced one at a time but only as combinations, since the station $x, y, z \ldots$ is not determined singly but only in combination.

If we then call these unknown combinations at the station, for which the above equations have been written $A, B, C \ldots$, then the error equations are:

$$\left.\begin{array}{l|l|l} \lambda_1 = -l_1 + u + A \text{ weight } g_1 & \lambda'_1 = -l'_1 + u' + A \text{ weight } g'_1 & \cdot\cdot \\ \lambda_2 = -l_2 + u + B \quad \text{„} \quad g_2 & \lambda'_2 = -l'_2 + u' + B \quad \text{„} \quad g'_2 & \cdot\cdot \\ \lambda_3 = -l_3 + u + C \quad \text{„} \quad g_3 & \lambda'_3 = -l'_3 + u' + C \quad \text{„} \quad g'_3 & \cdot\cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot\cdot \end{array}\right\} \qquad (6)$$

From which we get the normal equations:

$$\left.\begin{array}{l} 0 = -[gl] + [g]\,u + Ag_1 + Bg_2 + Cg_3 + \cdots \text{ for } u \\ 0 = -[g'l'] + [g']\,u' + Ag'_1 + Bg'_2 + Cg'_3 + \cdots \text{ „ } u' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \qquad (7)$$

and:

$$\left.\begin{array}{l} 0 = -[g_1 l_1] + ug_1 + u'g'_1 + \cdots + A[g_1] \text{ for } A \\ 0 = -[g_2 l_2] + ug_2 + u'g'_2 + \cdots + B[g_2] \text{ „ } B \\ 0 = -[g_3 l_3] + ug_3 + u'g'_3 + \cdots + C[g_3] \text{ „ } C \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}. \qquad (8)$$

Eliminating the unknowns $u, u' \ldots$ from the latter equation by means of the first, the equations assume the following form:

$$\left.\begin{array}{lllllll} 0 = -(al) + A(aa) + B(ab) + C(ac) + \cdots & \text{for } A \\ 0 = -(bl) + A(ab) + B(bb) + C(bc) + \cdots & \text{" } B \\ 0 = -(cl) + A(ac) + B(bc) + C(cc) + \cdots & \text{" } C \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} . \qquad (9)$$

In order to recognize how further to treat these data for the individual stations, we assume that we have formed the normal equation for x. To this the station for which the error equations (4) and (5) have been specially written makes a contribution to the right-hand member, which one can write as follows, using the abbreviations $A, B, C \ldots$:

$$\left.\begin{array}{l} a_1\{-[g_1 l_1] + ug_1 + u'g_1' + \cdots + A[g_1]\} \\ + a_2\{-[g_2 l_2] + ug_2 + u'g_2' + \cdots + B[g_2]\} \\ + a_3\{-[g_3 l_3] + ug_3 + u'g_3' + \cdots + C[g_3]\} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \qquad (10)$$

or also:

$$\left.\begin{array}{l} a_1\{\text{right side of the normal equation (8) for } A\} \\ + a_2\{\text{" " " " " " } B\} \\ + a_3\{\text{" " " " " " } C\} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \qquad (11)$$

Since the normal equations (7) remain the same for $u, u' \ldots$, one may regard $A, B, C..$ as unknowns, or operate directly on $x, y, z..$ (in which case $A, B, C..$ only signify the temporary abbreviations), so that in expression (11) normal equation (8) for $A, B, C..$ may be taken in form (9) free from $u, u' \ldots$

Thus we also obtain the addend of a station to the normal equation for x, if one multiplies and adds the terms of the right side of the station normal equation (9) in order by the coefficients of x, in

$A, B, C..$ in the proportions determined by (11)--and if one finally substitutes the expression in $x, y, z..$ for $A, B, C..$

This should be sufficient explanation of the procedure of adjustment. It is seen that one treats each station according to the usual procedure up to the formation of the normal equations, but that a station-by-station solution of the latter is not necessary.

Par. 11. Equations for the Measurement of Azimuth and Base. Each such measurement yields an equation relating the four corrections to the coordinates of the end points of the respective lines.

Equation (3), p. 496 is the condition equation corresponding to the azimuth measurement when λ_{ik} is set equal to zero.

The equation $s_{ik}=\bar{s}_{ik}+ds_{ik}$, where s_{ik} is the measured length, $\bar{s}_{ik}$ the precisely calculated base length derived from the approximate coordinate values, and ds_{ik} is the increment of $\bar{s}_{ik}$ due to changes in the geographic coordinates of the end points, is the condition equation for base measurement.

Following the first equation (4), p. 282 this condition equation is therefore the following:

$$0 = \bar{s}_{ik} - s_{ik} + a_0 \left\{ \begin{array}{c} \frac{1-e^2}{W_i^3} \cos \alpha_{ik} dB_i + \frac{1-e^2}{W_k^3} \cos \alpha_{ki} dB_k \\ + (dL_i - dL_k) \frac{\cos B_k}{W_k} \sin \alpha_{ki} \end{array} \right\}. \qquad (1)$$

If a number of ground lines is measured, then one sets up such a condition equation for each one, thus strictly satisfying each base measurement. Correction appears superfluous for the reason

that the relationship of two adjoining base lengths results much more precisely from their direct measurement than from the angle measurements of the intervening part of the net. Another, more important, reason which makes the treatment of (1) as an error equation seem in error, we shall recognize in Par. 18 of the following chapter (cf. also the footnote below same).

In order to take the condition equations for the azimuth and the ground line into consideration, one must always (Ausgleichungsrechnung, p. 195, procedure I), by means of this equation, eliminate one unknown before forming the general normal equation. This elimination follows, as one readily recognizes, simply in the expressions for $A, B, C..$, so that (as is also clear from the outset) the formation of the station normal equation, (9), p. 498 is not at all affected by it.

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If one has already ascertained the general normal equations from the contributions of (11), p. 498, neglecting the mentioned restrictions, then the calculations subsequently take the course, by means of the coefficients of the aforementioned normal equations, which is illustrated in the following example.

If x and y are two unknowns which have been chosen for elimination, one can put the condition equations in the form:

$$\left.\begin{aligned} x &= i_0 + i_3 z + i_4 t + \cdots \\ y &= k_0 + k_3 z + k_4 t + \cdots \end{aligned}\right\} \qquad (2)$$

Chapter 11. The Calculation for a Land Survey.

Any error equation which is originally in the form:

$$\lambda = -l + ax + by + cz + dt + \cdots \text{ weight } g \tag{3}$$

could have been given the following new form, by elimination of x and y:

$$\lambda = -(l - ai_0 - bk_0) + (c + ai_3 + bk_3)z + (d + ai_4 + bk_4)t + \cdots \text{ weight } g. \tag{4}$$

In undeveloped form (Ausgleichungsrechnung, p. 80, (6) the normal equations, in reference to (3) are:

$$\underset{\text{for } x}{[a\lambda g] = 0} \qquad \underset{\text{for } y}{[b\lambda g] = 0} \qquad \underset{\text{for } z}{[c\lambda g] = 0} \quad \ldots; \tag{5}$$

In contrast, in undeveloped form, the normal equation for z, in reference to (4) is:

$$[(c + ai_3 + bk_3)\lambda g] = 0. \tag{6}$$

From this one recognizes that the normal equation for z, is put together by means of normal equation (5), which is itself based on error equation (3), as follows:

$$(Z) + (X)i_3 + (Y)k_3 = 0, \tag{7}$$

where (X), (Y) and (Z) successively designate the latter normal equations in developed form. In these equations one must naturally consider x and y to have been determined by means of (2), and through $z, t \ldots$.

In the place of (5) there appear, in general, equations (9), p. 498.

If one applies procedure II of p. 196 of Ausgleichungsrechnung to the above example then by successive elimination of unknowns from the normal equations using Gauss' Algorithm, a number of quadratic coefficients of the unknowns become zero, because not all of them can be determined without azimuth and

base measurement. Nevertheless, the calculation may be continued entirely without difficulty, if as soon as no more unknowns with quadratic coefficients occur, the following normal equations for the correlatives k are used in the elimination (cf. also Ausgleichungsrechnung, p. 212 D).

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Par. 12. About Details. It may be mentioned that it is superfluous to carry out the reduction described in Par. 4, p. 488 before the station adjustment, as has already been mentioned at that place. Since these same reductions always appear, where ever A, B or C... appear, one can consider them to be taken care of for these values for the time being, and in the general normal equations we set (Par. 10, pp. 496 and 497):

$$\left.\begin{aligned} A &= a_1x + b_1y + c_1z + \cdots - \text{Reduction for azimuth of first direction} \\ B &= a_2x + b_2y + c_2z + \cdots - \quad " \quad " \quad " \quad " \text{ second } " \\ C &= a_3x + b_3y + c_3z + \cdots - \quad " \quad " \quad " \quad " \text{ third } " \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\}, \qquad (1)$$

in which the reduction has the sign of the observed azimuth α in the error equations.

In entirely analogous fashion one can next substitute any other preliminary value $\bar{\alpha}'$ for $\bar{\alpha}$ in the formation of the error equations. Then one has only, finally, in the substitution of the expressions for A, B, C... in the right-hand members of the above, to add, respectively:

$$\left.\begin{array}{l} -\bar{\alpha}_1' + \bar{\alpha}_1 \text{ for first direction} \\ -\bar{\alpha}_2' + \bar{\alpha}_2 \text{ " second " } \\ -\bar{\alpha}_3' + \bar{\alpha}_3 \text{ " third " } \\ \cdots\cdots\cdots \end{array}\right\}. \qquad (2)$$

If local directions occur, i. e. those going to points which are not included in the net adjustment, then the preceding substitutions are not made, since the net adjustment serves only in further improving the same directions. Then, if the third direction, for instance, is such a one, one simply carries the symbol C in the general adjustment.

One can easily see, however, that it is both permissible and profitable to eliminate this C from the station normal equations before forming the general normal equation. The normal equation for C, in which the substitutions for A, B, D,.. are to be made supplementarily, serves after the completion of the general adjustment only to find C, and in addition, to make a contribution in the summarizing formation of $[\lambda\lambda g]$ (Ausgleichungsrechnung, p. 107).

If one wishes to know the results of the station adjustments in separated form, one must set one of the magnitudes of A, B, C.. equal to zero (following Bessel's procedure), or otherwise assume zero values for them, and then omit their normal equation.

The reason for this is that the individual station does not obviously completely determine these magnitudes, but only their differences. One recognizes this, moreover, from the fact that the sum of the station normal equations (9), p. 498 always is identically zero.

Par. 12. About Details.

The results of the station adjustments are generally in the form:

$$\bar{\alpha}_1 + A_0,\ \bar{\alpha}_2 + B_0,\ \bar{\alpha}_3 + C_0, \ldots, \tag{3}$$

where A_0, B_0, C_0... signify the respective assumed values for these symbols.

If we assume in advance that the station adjustments are at hand, completed according to Bessel's, or a related procedure, then the normal equations are to be converted in such a fashion that they take the form which we have here assumed in advance, (9), p. 498. But if one assumes A and the first equation to be non-existent, in this system of normal equations corresponding to Bessel's procedure, then it is a very simple matter to accomplish the completion since the sum of the coefficients in this system, (9), p. 498, is always zero. Therefore, one need only substitute $B - A, C - A, \ldots$ for the unknowns $B', C' \ldots$ in Bessel's normal equations, combine the terms in A in every equation and then form the negative sum equation, in which one has the normal equation for A. The total system of equations is the system (9), p. 498. In this transformation (a1), (b1), (c1)...drop out, since one accepts the station adjustment. One must, however, give such values to the $\bar{\alpha}'$ in the members $\bar{\alpha}' - \bar{\alpha}$ which are added to the expressions for $A, B, C \ldots$ (see system (2) above) that their differences correspond to the result for the station.

In carrying out the Prussian land triangulation, in a certain case, there were introduced for all directions of a station the unknowns $A, B, C \ldots$, right from the start. The resulting normal equations were immediately to be retained and, specifically, in their original form. For these, as is customary there, were now in a modified form useless for the general adjustment because of the

addition of a condition equation relating $A, B, C \ldots$. These equations serve in this case only to obtain quickly the station results. For the purpose of illustration we examine the case where all angles (each a combination of two directions) are often measured immediately (Schreiber's observation procedure, see above). This is the only case in which the above-mentioned procedure is used in land triangulation; otherwise Bessel's is employed.

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Then, therefore, the station normal equations assume the form:

$$\left.\begin{aligned} 0 &= -(al) + (aa)A + (ab)B + (ab)C + \cdots \\ 0 &= -(bl) + (ab)A + (aa)B + (ab)C + \cdots \\ 0 &= -(cl) + (ab)A + (ab)B + (aa)C + \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \qquad (4)$$

i. e. all quadratic coefficients are the same, as also are all non-quadratic ones; here we have for n directions, because of the identical vanishing of the sum of all equations:

$$(aa) = -(n-1)(ab). \qquad (5)$$

The land triangulation now adds the condition equation:

$$0 = A + B + C + \cdots \qquad (6)$$

If this is multiplied by (ab) and subtracted from (4) then one gets the modified system:

$$\left.\begin{aligned} 0 &= -(al) + \frac{n}{n-1}(aa)A \\ 0 &= -(bl) + \frac{n}{n-1}(aa)B \\ 0 &= -(cl) + \frac{n}{n-1}(aa)C \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\}. \qquad (7)$$

An entirely direct trial easily leads to the conviction that

in the general adjustment using adjusted observations this system may not be used for the formation of the normal equations.

One can also recognize this as follows, wherewith a clear representation of this procedure is obtained (which can likewise be applied in more complicated cases).

We derive from the basic system (4) the expression:

$$(ab)(A+B+C+\cdots)$$

in $(ab)nu$ and write, then, considering (5):

$$\left.\begin{aligned} 0 &= -(al) - u \cdot \tfrac{n}{n-1}(aa) + \tfrac{n}{n-1}(aa)A \\ 0 &= -(bl) - u \cdot \tfrac{n}{n-1}(aa) + \tfrac{n}{n-1}(aa)B \\ 0 &= -(cl) - u \cdot \tfrac{n}{n-1}(aa) + \tfrac{n}{n-1}(aa)C \\ &\cdots\cdots\cdots\cdots \\ u &= \tfrac{1}{n}(A+B+C+\cdots). \end{aligned}\right\} \qquad (8)$$

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However, if one imagines that only one complete set of all directions has been observed with the weight $\frac{n}{n-1}(aa)$, then one directly obtains the same form of the equation that the customary sequence of calculations shows (pp. 497 and 498 (6) to (9)). One can, therefore, conceive the complex of angle observations as a complete set with the weight $\frac{n}{n-1}(aa)$: equivalent set (cf. Ausgleichungsrechnung, p. 164).

Since the station normal equations now assume the form (8) and in the procedure of Par. 10 the assumption (6) is inadmissible, we must retain (8), i. e., after restitution of the expression for u, we must , for the adjustment of net, retain system (4) and not (7).

The modified form (7) nevertheless gives the angles in the station adjustment (and it is only these that we are concerned about in this case) entirely correctly, since $A, B, C \ldots$ according to (7) and (8) differ only by a constant, namely u.

One can now see also, incidentally, why it is permissible, in adjusting the net, to calculate equations (7) using condition observations without specifying the condition equation $0 = A + B + C + \cdots$, farther than it is customary in doing land triangulation. For, if one supposes that in the above case $A - u, B - u, C - u \cdots$ corresponding to (8) were written instead of $A, B, C \ldots$ then u would always drop out of the condition equations since magnitudes of $A - u, B - u, C - u, \ldots$ always appear in pairs rather than singly.

Par. 13. Summary Calculation of $[\lambda\lambda g]$. The general normal equations for $x, y, z \ldots$ assume the form:

$$\left.\begin{aligned} 0 &= -(al) + (aa)x + (ab)y + (ac)z + \cdots \\ 0 &= -(bl) + (ab)x + (bb)y + (bc)z + \cdots \\ 0 &= -(cl) + (ac)x + (bc)y + (cc)z + \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\}, \qquad (1)$$

whereby it must be entirely inconsequential if one arrives at them by way of the station normal equations, or directly. In any case, the $u, u' \ldots$ are to be considered eliminated.

One has now, following the general formulae in Gaussian notation (Ausgleichungsrechnung, p. 107, (9) or p. 121 (8) :

$$[\lambda\lambda g] = \mathrm{I} - \mathrm{II} \qquad (2)$$

$$\left.\begin{aligned} \mathrm{I} &= \left\{[llg] - \frac{[gl]^2}{[g]} - \frac{[g'l']^2}{[g']} - \cdots\right\} \qquad (3) \\ \mathrm{II} &= \left\{\frac{(al)^2}{(aa)} + \frac{(bl.1)^2}{(bb.1)} + \frac{(cl.2)^2}{(cc.2)} - \cdots\right\} \end{aligned}\right\},$$

where in $[\lambda\lambda g]$ and $[llg]$ the summation is extended over upper and lower indices and over all stations. In the case of the other brackets the summation is only over the lower indices, i. e. over every single set.

Because part II of the above expression for $[\lambda\lambda g]$ may be calculated by means of the general normal equations (1), and since it can also be put into the form:

$$II = [(al)x + (bl)y + (cl)z + \cdots] \qquad (4)$$

it is necessary to go further back for the first part. However, in the case of the station adjustments, the sums for part I have already been determined, for the calculation of the contribution of every station (cf. (7), p. 497), provided that one has already applied the final l, which had been derived from the calculated azimuths $\bar{\alpha}$, which in turn came from the preliminary values of the geographic coordinates. However, as has already been stated previously, other values $\bar{\alpha}'$ are related thereto and these are also omitted at the beginning of the reduction. Therefore, a new formula is now necessary which, using the station calculations, permits as simple a formation of part I as is possible. We now turn to the development of the same.

The station adjustments give, instead of λ, other corrections $e;\ u, u' \ldots$, as well as $A, B, C \ldots$ do not have the same values as those which they assume in the general adjustment. Rather, they have other values which are distinguished by the subscript 0. Since the l are also not necessarily those of the general adjustment, another symbol $\mathfrak{l}$ must be substituted here. For the direction sets of a

station one has, then, first of all, with regard to station adjustment only, the following error equations:

$$\left.\begin{array}{l|l|l} e_1 = -l_1 + u_0 + A_0 \text{ weight } g_1 & e'_1 = -l'_1 + u'_0 + A_0 \text{ weight } g'_1 & \dots \\ e_2 = -l_2 + u_0 + B_0 \quad " \quad g_2 & e'_2 = -l'_2 + u'_0 + B_0 \quad " \quad g'_2 & \dots \\ e_3 = -l_3 + u_0 + C_0 \quad " \quad g_3 & e'_3 = -l'_3 + u'_0 + C_0 \quad " \quad g'_3 & \dots \\ \dots\dots\dots & \dots\dots\dots & \dots \end{array}\right\} \qquad (5)$$

The relation of the l' to the l results from the fact that for any direction (corresponding to the procedure advanced in Par. 12, p. 501) we set:

$$\left.\begin{array}{l} l = \alpha - \bar{\alpha}' \quad l = \alpha - \bar{\alpha} + \text{Reduction for this direction} \\ \text{One has then generally} \\ l = l + \bar{\alpha}' - \bar{\alpha} + \text{Reduction for this direction} \end{array}\right\} \qquad (6)$$

The station error equations now give in combination with the latter relationship the relations:

$$\left.\begin{array}{l|l|l|l|l} l_1 = -e_1 + u_0 + (1) & g_1 & l'_1 = -e'_1 + u'_0 + (1) & g'_1 & \dots \\ l_2 = -e_2 + u_0 + (2) & g_2 & l'_2 = -e'_2 + u'_0 + (2) & g'_2 & \dots \\ l_3 = -e_3 + u_0 + (3) & g_3 & l'_3 = -e'_3 + u'_0 + (3) & g'_3 & \dots \\ \dots\dots\dots & \dots & \dots\dots\dots & \dots & \dots \end{array}\right\}, \qquad (7)$$

if one uses the following abbreviations:

$$\left.\begin{array}{l} (1) = (\bar{\alpha}'_1 + A_0) + Red_1 - \bar{\alpha}_1 \text{ for 1st direction} \\ (2) = (\bar{\alpha}'_2 + B_0) + Red_2 - \bar{\alpha}_2 \text{ " 2nd " } \\ (3) = (\bar{\alpha}'_3 + C_0) + Red_3 - \bar{\alpha}_3 \text{ " 3rd " } \\ \dots\dots\dots\dots\dots\dots \end{array}\right\}, \qquad (8)$$

in which $\bar{\alpha}'_1 + A_0$, $\bar{\alpha}'_2 + B_0$, $\bar{\alpha}'_3 + C_0$, ... signify the contributions for the directions (Par. 12, p. 502).

Par. 13. Summary Calculation of $[llg]$.

If one next forms the expressions for one station from the relations (7) by squaring and adding the $[llg]$, there follows, if llg and eeg are summed over the upper and lower indices:

$$\left.\begin{aligned}[llg]=[eeg]&+\{u_0+(1)\}^2g_1+\{u_0+(2)\}^2g_2+\{u_0+(3)\}^2g_3+\cdots\\&+\{u'_0+(1)\}^2g'_1+\{u'_0+(2)\}^2g'_2+\{u'_0+(3)\}^2g'_3+\cdots\\&\cdots\cdots\cdots\cdots\cdots\end{aligned}\right\} \qquad (9)$$

All double products drop out in this procedure, in consequence of the station normal equations whose factors:

$$[e_1g_1]=[e_2g_2]=[e_3g_3]=\cdots$$

as well as:

$$[eg]=[e'g']=[e''g'']=\cdots$$

are equal to zero.

Otherwise, the following relations hold for the individual series (summation over the lower index only, then):

$$\left.\begin{aligned}[lg]&=u_0[g]+(1)g_1+(2)g_2+(3)g_3+\cdots\\[l'g']&=u_0[g']+(1)g'_1+(2)g'_2+(3)g'_3+\cdots\\&\cdots\cdots\cdots\cdots\end{aligned}\right\}. \qquad (10)$$

One easily obtains the values of $[lg]^2:[g]$, $[l'g']^2:[g']$, ... from there and with this result we have for the individual stations, the equation:

$$[llg]-\frac{[lg]^2}{[g]}-\frac{[l'g']^2}{[g']}-\cdots=\left\{\begin{aligned}&[eeg]+(1)^2[g_1]+(2)^2[g_2]+(3)^2[g_3]+\cdots\\&-\frac{\{(1)g_1+(2)g_2+(3)g_3+\cdots\}^2}{[g]}\\&-\frac{\{(1)g'_1+(2)g'_2+(3)g'_3+\cdots\}^2}{[g']}-\cdots\end{aligned}\right\} \qquad (11)$$

This equation holds for every station and one, therefore, has to carry on the calculation for part I as follows, in order to avoid having to set up the l of the individual observations. Thus one can write:

$$I = \left\{ \begin{array}{l} \text{Sum of all } [eeg] \text{ for all stations in accordance with all station adjustments} \\ + \left\{ \begin{array}{l} \text{Sum of all } (i)^2[g_i] \text{ for all directions observed in the net, where } [g_i] \\ \text{signifies the sum of weights of all inserted values of i direction.} \end{array} \right\} \\ - \left\{ \begin{array}{l} \text{Sum of all } \dfrac{\{(1)g_1 + (2)g_2 + (3)g_3 + \cdots\}^2}{[g]} \text{ for all sets, where } [g] \\ \text{signifies the sum of the weights of all directions of one set.} \end{array} \right\} \end{array} \right\} \qquad (12)$$

If parts I and II and therewith $[\lambda\lambda g]$ are ascertained according to the above formulae, then one thus obtains $[\lambda\lambda g]$ by a summary calculation; we therefore assume the $[eeg]$ for the station derived by means of the well-known summation formulae.

As a check, one can henceforth also ascertain $[\lambda\lambda g]$ from the individually calculated λ, just as one can calculate the $[eeg]$ from the numerical values of the individually calculated e -- or, one can avail oneself of the following control formulae in order not to have to return once more to the individual observations, as is the case with the e.

If one subtracts the corresponding values for e from each error equation for λ, (cf. (6), p. 497 and (7), p. 506), and notes the relation of l to l, then one obtains the direction sets of any station:

$$\left. \begin{array}{l|l||l|l||l} \lambda_1 = e_1 + \delta - (1') & g_1 & \lambda'_1 = e'_1 + \delta' - (1') & g'_1 & \cdots \\ \lambda_2 = e_2 + \delta - (2') & g_2 & \lambda'_2 = e'_2 + \delta' - (2') & g'_2 & \cdots \\ \lambda_3 = e_3 + \delta - (3') & g_3 & \lambda'_3 = e'_3 + \delta' - (3') & g'_3 & \cdots \\ \cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots \end{array} \right\}, \qquad (13)$$

wherein one sets:

$$\delta = u - u_0; \quad \delta' = u' - u'_0; \cdots \qquad (14)$$

$$\left. \begin{array}{l} (1') = (1) - (a_1 x + b_1 y + c_1 z + \cdots) \\ (2') = (2) - (a_2 x + b_2 y + c_2 z + \cdots) \\ (3') = (3) - (a_3 x + b_3 y + c_3 z + \cdots) \\ \cdots\cdots\cdots\cdots\cdots \end{array} \right\}. \qquad (15)$$

Par. 13. Summary Calculation of $[\lambda\lambda g]$

One now obtains the following expression by means of the preceding equations by the formation of all $\lambda\lambda g$ for one station; so that, then, the sums $[\lambda\lambda g]$ and $[eeg]$ refer to upper and lower index within a station:

$$[\lambda\lambda g] = \left\{ \begin{array}{c} [eeg] + (1')^2[g_1] + (2')^2[g_2] + (3')^2[g_3] + \cdots \\ + \delta^2[g] + \delta'^2[g'] + \cdots \\ - 2(1')[\delta g_1] - 2(2')[\delta g_2] - 2(3')[\delta g_3] - \cdots \end{array} \right\} \qquad (16)$$

But now the normal equation for u_0 and u may immediately be derived (cf. (7), p. 497):

$$u_0 = \frac{[gl] - A_0 g_1 - B_0 g_2 - C_0 g_3 - \cdots}{[g]}$$

$$u = \frac{[gl] - (a_1 x + b_1 y + c_1 z + \cdots) g_1 - (a_2 x + b_2 y + c_2 z + \cdots) g_2 - (a_3 x + b_3 y + c_3 z + \cdots) g_3 - \cdots}{[g]}$$

and from this, considering (6) and (8):

as also:

$$\left. \begin{array}{l} \delta = \dfrac{(1')g_1 + (2')g_2 + (3')g_3 + \cdots}{[g]} \\ \delta' = \dfrac{(1')g'_1 + (2')g'_2 + (3')g'_3 + \cdots}{[g']} \end{array} \right\} \qquad (17)$$

and so on for all other sets. By means of these relations there follows, through multiplication of the equations for δ by $[g]\delta$, and the multiplications of the equation for δ' by $[g']\delta'$, etc., and then adding:

$$\delta^2[g] + \delta'^2[g'] + \cdots = (1')[\delta g_1] + (2')[\delta g_2] + (3')[\delta g_3] + \cdots. \qquad (18)$$

One next obtains for the individual station:

$$[\lambda\lambda g] = \left\{ \begin{array}{c} [eeg] + (1')^2[g_1] + (2')^2[g_2] + (3'^2)[g_3] + \cdots \\ - \dfrac{\{(1')g_1 + (2')g_2 + (3')g_3 + \cdots\}^2}{[g]} - \dfrac{\{(1')g'_1 + (2')g'_2 + (3')g'_3 + \cdots\}^2}{[g']} - \cdots \end{array} \right\} \qquad (19)$$

Through formation of the sum $[\lambda\lambda g]$ for the entire net there is finally obtained the control formula:

$$[\lambda\lambda g] = \left\{ \begin{array}{l} \{\text{Sum of all } [ecg] \text{ for all stations; calculated from individual } e \\ + \{\text{Sum of all } (i'')^2[g_i] \text{ for all directions of the net, where } [g_i] \\ \quad \text{is the sum of weights of all settings of the } i\text{-direction} \} \\ - \{\text{Sum of all } \frac{\{(1')g_1 + (2')g_2 + (3')g_3 + \cdots\}^2}{[g]} \text{ for all sets,} \\ \quad \text{where } [g] \text{ is the sum of weights of all directions of one set.} \} \end{array} \right\} \qquad (20)$$

Par. 14. Continuation. The function of (1), (2), (3)...in the expression for I after formula (12), p. 507, and the entirely analogously-formed function of (1'), (2'), (3'),...in the expression for $[\lambda\lambda g]$ after (20) above, can be formed in many cases by use of the coefficients of the station normal equation.

If one sets, in the following combinations:

$$\left. \begin{array}{l} ug_1 + u'g_1' + \ldots + A[g_1] \\ ug_2 + u'g_2' + \ldots + B[g_2] \\ ug_3 + u'g_3' + \ldots + C[g_3] \\ \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \right\}, \qquad (1)$$

the weights $g, g' \ldots$ taken corresponding to the direction sets of a station, and if $u, u' \ldots$ as also $A, B, C \ldots$ are preliminarily understood to have entirely arbitrary values; and substitute for $u, u' \ldots$ the values:

$$\left. \begin{array}{l} u = -\dfrac{Ag_1 + Bg_2 + Cg_3 + \ldots}{[g]} \\ u' = -\dfrac{Ag_1' + Bg_2' + Cg_3' + \ldots}{[g']} \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{array} \right\}, \qquad (2)$$

then they are converted, as comparison with (7) to (9), p. 497 and p. 498 shows, into the following combinations:

$$\left.\begin{array}{l}(aa)A+(ab)B+(ac)C+\ldots\\(ab)A+(bb)B+(bc)C+\ldots\\(ac)A+(bc)B+(cc)C+\ldots\\\ldots\ldots\ldots\ldots\end{array}\right\}, \tag{3}$$

the coefficients of which agree completely with the station normal equations. If one now multiplies the combinations of both forms in order by $A, B, C\ldots$, adds them and eliminates $u, u'\ldots$ from the first form by means of (2), there is obtained:

$$\left.\begin{array}{l}\left\{\begin{array}{l}A^2[g_1]+B^2[g_2]+C^2[g_3]+\ldots\\-\frac{\{Ag_1+Bg_2+Cg_3+\ldots\}^2}{[g]}\\-\frac{\{Ag'_1+Bg'_2+Cg'_3+\ldots\}^2}{[g']}\\-\ldots\ldots\ldots\ldots\end{array}\right\}-\\=\left\{\begin{array}{l}A\{(aa)A+(ab)B+(ac)C+\ldots\}\\+B\{(ab)A+(bb)B+(bc)C+\ldots\}\\+C\{(ac)A+(bc)B+(cc)C+\ldots\}\\+\ldots\ldots\ldots\ldots\end{array}\right\}\end{array}\right\}. \tag{4}$$

Herein one can set the still arbitrary values $A, B, C\ldots$ equal to:

$$A=(1)\quad B=(2)\quad C=(3)\ldots \tag{5}$$

or

$$A=(1')\quad B=(2')\quad C=(3')\ldots \tag{6}$$

and obtain therewith the formulae for the station-by-station calculation of the respective members of the formulae (12), p. 507 and (20), p. 508. The station normal equations are assumed in the form (9), p. 498, as found necessary according to Par. 10, for the formation of the general normal equations belonging to the general net adjustment.

One is however not bound to use this form, rather one can also apply Bessel's form. In order to demonstrate this, one subtracts from the right-hand member of (4) the expression:

$$\left\{\begin{array}{l} A\{(aa)A+(ab)A+(ac)A+\ldots\} \\ +B\{(ab)A+(bb)A+(bc)A+\ldots\} \\ +C\{(ac)A+(bc)A+(cc)A+\ldots\} \\ +\ldots\ldots\ldots\ldots \end{array}\right\},$$

i. e., sets it equal to zero, and then sets $B-A=B'$, $C-A=C'\ldots$, then there follows the equivalent expression:

$$\left\{\begin{array}{l} B'\{(bb)B'+(bc)C'+\ldots\} \\ +C'\{(bc)B'+(cc)C'+\ldots\} \\ \ldots\ldots\ldots\ldots \end{array}\right\}, \qquad (7)$$

as one finds without difficulty, with regard to the relationships $(aa)+(ab)+(ac)+\ldots=0$, $(ab)+(bb)+(bc)+\ldots=0$, and so forth.

This shows that one can also calculate with the coefficients of the Besselian station normal equations, if one introduces the equivalent expression (7) into the right member of (4) and therein sets:

$$\left.\begin{array}{l} B'=(2)-(1)\quad C'=(3)-(1)\ldots \text{for the application to (12) p. 507} \\ B'=(2')-(1')\quad C'=(3')-(1')\ldots \text{" " " " (20) p. 508} \end{array}\right\} \qquad (8)$$

Now a third form of the right-hand side of (4) finally arises by means of the coefficients of the reduced normal equations (Ausgleichungsrechnung, p. 101). One can write in place of (7):

$$\left\{\begin{array}{l} (bb)\left\{B'+\frac{(bc)}{(bb)}C'+\frac{(bb)}{(bb)}D'+\cdots\right\}^2 \\ +(cc.1)\left\{C'+\frac{(bb.1)}{(cc.1)}D'+\cdots\right\}^2 \\ +(dd.2)\{D'+\cdots\}^2 \\ \ldots \end{array}\right\}, \qquad (7^*)$$

for which, through successive elimination from (7), we can in the familiar fashion of conversion, complete the proof.

Par. 15. Mean Error of the Unit Weights; General Remarks.

With the last expression we have arrived at the individual station measurement equivalent system of angle functions. A comparison with Ausgleichungsrechnung, (5), p. 166, will show that the curved brackets of the individual lines of expression (7*) represent systems of functions for the angles between the first and other directions, equivalent to the complex of station measurements, if one replaces $B', C', D' \ldots$ with the numerical values according to a station adjustment. If one, however, employs the first line of the substitutions (8), then one obtains the negative changes of these functions through introduction of the approximate value δ and the reduction during transition to the net adjustment.

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If, on the contrary, one inserts the second substitution of (8), then one obtains the negative corrections of the same function following the net adjustment. The expression (7*), then, represents the sum of the squares of these corrections multiplied by their weights.

Par. 15. Mean Error of the Unit Weights; General Remarks. The given adjustment procedure is not only applicable when $g = 1$ or zero, but rather, it applies generally; it is presupposed, however, that no considerable sources of error will appear in the net adjustment, that therefore, the observational errors do not in part arise as a result of constant or systematic causes. We do not consider here the following questions which are noteworthy in this connection: how

would the adjustment be modifiable in the latter case, if and when unlike weights are to be introduced for individual directions or measurements; and what is the significance of the mean error which follows as a consequence of the adjustment.*) We make mention only of the purely formal relation, that the mean error of the unit weight for the individual stations yields, following the familiar formula for the station adjustment and by means of the associated $[eeg]$:

$$\pm\sqrt{\frac{[eeg]}{R-S-m+1}}, \qquad (1)$$

where R is the number of direction observations, S is the number of sets and m is the number of directions.

The discrepancies of the net adjustments alone give as the mean error of a unit weight:

$$\pm\sqrt{\frac{[llg]-[eeg]}{M-3N+F+G+3}}, \qquad (2)$$

where M is the number of directions from the $N-F$ stations to the N net points, in general, and G signifies the number of ground lines. From (1) and (2) together there follows, finally, through addition of the numerators and denominators:

$$\pm\sqrt{\frac{[llg]}{[R]-[S]-2N+M-[m]+G+3}}. \qquad (3)$$

The denominators correspond in every case to the redundant observations (Ausgleichungsrechnung, p. 208 and p. 325**).

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*)In reference to these questions cf. the above-mentioned Abhandlungen by Schreiber and the often mentioned Dänische Gradmessung, as well as the survey itself.

**)Relative to our formula for the number of angle equations on p. 326 it should here be noted incidentally as a reference, since, due to its derivation, it holds only for the case where every station is linked with at least two other through reciprocal directions. This, to be sure, is always the case in good triangulation nets.

In its formation it is naturally assumed that such points which were not or could not be included in the net adjustment are not counted as net points although they appear in the station adjustment. If such points are entirely absent then $M - [m] = 0$.

Although the adjustment method given in the preceding paragraph is more complicated when applied to simple chains than the customary one using conditioned observations, it permits the easy joining of different nets, which meet at common points. This is done not merely in such a manner as to permit the results of the individual adjustments of a number of nets to be retained entirely or in part, but also so that the entire complex undergoes a simultaneous adjustment.

For the latter purpose a master system must be built from the normal equations of the various net parts through addition of those equations which share the same unknowns. For the sake of clarity one should only undertake this addition after the unknowns which do not appear in several net parts have been eliminated: a procedure which is completely rigorous.

How the adjustment takes shape when the results of the adjustments of the individual nets are to be partially retained, requires just as little discussion as the already-mentioned insertion of points into a completed net, since here the general procedure undergoes no further changes beyond the assumption of individual given values.

Chapter 12.

Measurements Performed on the Physical Surface of the Earth and Approximate Determinations of Single Parts of the Geoid.

Par. 1. Reference Ellipsoid and Deflection of the Vertical. In all considerations, from p. 134 on, it was accepted that the plumb lines of the geodetic operation-point on the physical surface of the earth are straight plumb lines of a mathematical surface of the earth which has the shape of a slightly flattened ellipsoid of revolution. But we explained already the reason for this error in the introduction to Chapters 1 and 2. According to that, neither the mathematical surface of the earth nor, as it were, the level surfaces near the physical surface of the earth have that rigorous shape but only approximately the shape of the flattened ellipsoid of revolution.

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Besides, these level surfaces are not exactly parallel to one another and, therefore, the verticals are slightly curved. In mathematical considerations we have to distinguish the verticals from the plumb lines.

It is, by no means, exact to imagine such an ellipsoid of revolution for the reduction of geodetic measurements, even if this has been done up to now. But ignorance about the more exact shape of the equipotential(*) surfaces forces us to make that as-

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(*)Editor's note: heretofore, the German word niveaufläche has been translated in its literal form "level surface." Hereafter, it is believed that understanding will be facilitated by use of the more sophisticated mathematical and mechanical term "equipotential surface."

sumption also in practice in the beginning; it is a first approximation.

Let us suppose, in the meantime, that the shape of the equipotential surfaces and, correspondingly, also the directions of the verticals were known. How would the geodetic measurements on the physical surface of the earth be reduced then? Would it especially be reduced to one of the equipotential surfaces?

Here the answer is: no. As already mentioned in the introduction, the geoid as well as all equipotential surfaces which are close to the physical surface of the earth show, beside the continental irregularities, also local ones with a rapid change. The latter manifest themselves especially as considerable variations in the magnitude of the radii of curvature, where discontinuities or continuous variations which are practically to be considered as the same, are not excluded. Considering the speed, the local variability is the same as in a hilly region, and valleys (reverse curvature) are not formed because the curvature radius remains always directed toward the inside.

Such irregular surfaces are not suitable for a convenient mathematical treatment. Especially expansions in a series would be impracticable even for short geodetic lines of the length of first-order triangle sides or, rather, almost impossible, because already numerous curvature variations are present on such distances as is already shown on p. 22.

The formulae of geodesy were partially developed, as it were, for surfaces taken at random; among others are those for geodetic triangles and those for the deviation of the vertical section from

the geodetic line. But solely for true equipotential surfaces, these derivations are completely useless as they have not enough terms and, therefore, would often not converge. (The conclusions which are drawn from those derivations are, of course, useless for the general validity of the usual geodetic methods).

The mathematical treatment requires a simple surface and, as formerly mentioned, the slightly flattened ellipsoid of revolution does not only suffice but is very suitable because of its convenient formulae.

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The objection which can be raised against its application is that a non-homogeneous liquid body does not have the equipotential surfaces of an ellipsoid of revolution, and even in this ideal case, the introduction of the ellipsoid theoretically produces deflections of the vertical. The above-mentioned objection is irrelevant, because the true distribution of the masses creates much larger deflections of the vertical.

Now, the ellipsoid which replaces the mathematical surface of the earth as a projection surface is generally called Reference Ellipsoid. It is not to be confused with the most favorable ellipsoid of revolution which, in general, is the nearest to the form of the geoid, of which the axis is the earth's axis and of which the center is the gravity center of the earth: that is the earth ellipsoid. It is, rather, and as we will see later, one whose axis is parallel to the earth's axis, but at an undetermined distance from the earth's axis, and which is, furthermore, very close

to the geoid, and intersects it, in a menner to be discussed further, in the area of the considered geodetic operations.

The plumb line of a geodetic point deviates from the vertical of the reference ellipsoid which intersects this geodetic point by as much as the deflection of the vertical. These deflections of the vertical are, of course, different according to the chosen dimensions of the ellipsoid as well as its position, and they are not to be confused with the deflections of the vertical against the vertical of the above-mentioned best fitting ellipsoid (the earth ellipsoid). Those latter (deflections) are called absolute deflections of the vertical or they may also be called deviations of the vertical whenever it is necessary to distinguish them from those deflections which are only relative ones.

In general, we consider the term deflection of the vertical as the most correct because it excludes the thought that we are dealing only with local irregularities in mass distribution.

Par. 2. Reduction of Horizontal Angles. Assuming now that the deflections of the vertical at the geodetic operation points on the physical surface of the earth against the reference ellipsoid are known, the amounts can be computed by which the measured horizontal angles have to be corrected in order to refer them to those vertical sections which connect their projections on the ellipsoid by means of the ellipsoidal verticals.

Around the observation point P_1 on the physical surface of the earth, we put an auxiliary sphere with an arbitrary radius which serves as unity for the length, and we move all linears to

be taken into consideration parallel through P_1. Fig. 41 shows the sections of these parallels on the surface of the auxiliary sphere. The great-circle arcs between them

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indicate the angles between the lines, and the angles between the great-circles indicate the surface angles. Especially Z represents the intersection of the ellipsoid's vertical which crosses P_1, and Z' is the intersection of the true plumb line in P_1. The former is the ellipsoidal zenith, and the latter is the real zenith of P_1 on the auxiliary sphere (celestial sphere).

Furthermore, let N be the North pole, as intersection of the Northern end of a parallel to the axis of the reference ellipsoid with the auxiliary sphere at P_1. This intersection line is assumed to be parallel to the earth's axis. P_2 denotes the intersection of the apparent direction from P_1 to P_2 (that means the last tangent of the light-rays) with the auxiliary sphere.

Moreover, the notations of the lines of Fig. 41 will be comprehensible.

$ZZ' = \Theta$ means the deflection of the vertical: it has the ellipsoidal azimuth A and the true azimuth $180^0 + A'$.

$A + W_{1,2}$ is the ellipsoidal azimuth of the apparent direction toward the object P_2 and $A' + W_{1,2}$ is the corresponding true azimuth.

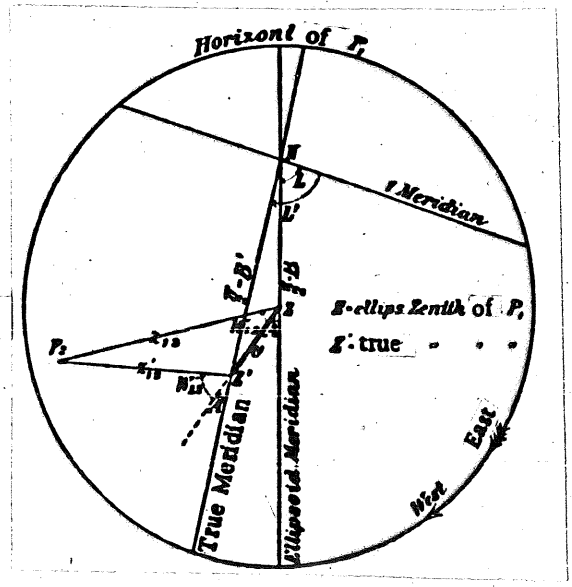


Figure 41.

$z_{1.2}$ is the ellipsoidal zenith distance, $z'_{1.2}$ is the true zenith distance of the apparent direction toward P_2. Furthermore, imagnining the apparent direction to another object P_3 as given in Fig. 41, then $A + W_{1.3}$ becomes the ellipsoidal azimuth, and $A' + W_{1.3}$ the true azimuth of the apparent direction toward the object P_3. Therefore the reduction of the measured horizontal angle $\binom{P_2 P_3}{P_1}$ to the ellipsoidal vertical is equal to

$$(W_{1.3} - W_{1.2}) - (W'_{1.3} - W'_{1.2}) \text{ or } (W_{1.3} - W'_{1.3}) - (W_{1.2} - W'_{1.2}). \tag{1a}$$

The spherical triangle $ZZ'P_2$ is now computed easily; according to Napier's first Analogy on p. 78 (6), the following equation for the determination of $W_{1.2} - W'_{1.2}$:

$$\tan \frac{z_{1.2} + z'_{1.2}}{2} = \frac{\sin \frac{W'_{1.2} + W_{1.2}}{2}}{\sin \frac{W'_{1.2} - W_{1.2}}{2}} \tan \frac{\Theta}{2}. \tag{1}$$

From this it follows that:

$$\sin \frac{W'_{1.2} - W_{1.2}}{2} = \tan \frac{\Theta}{2} \sin \frac{W'_{1.2} + W_{1.2}}{2} \cot \frac{z'_{1.2} + z_{1.2}}{2} \tag{2}$$

or, to a sufficient approximation:

$$W_{1.2} - W'_{1.2} = -\Theta \cot z'_{1.2} \sin W'_{1.2} + \cdots. \tag{3}$$

If the true azimuth of P_2 in P_1 is denoted as $a'_{1.2}$ then it can be written also:

$$W_{1.2} - W'_{1.2} = -\Theta \cot z'_{1.2} \sin (a'_{1.2} - A') + \cdots. \tag{4}$$

According to this, for the reduction of the horizontal angle

$\binom{P_2 P_3}{P_1}$ to the ellipsoidal vertical, the term:

$$\Theta\left(\cot z'_{1.2}\sin(a'_{1.2}-A')-\cot z'_{1.3}\sin(a'_{1.3}-A')\right)+\cdots, \tag{5}$$

is obtained, whereby Θ has to be taken in seconds in order to get the reductions in seconds.

The solution of the sine in (5) renders for the reduction, instead of (5) the term:

$$\left\{\begin{array}{l}\xi\left(\cot z'_{1.2}\sin a'_{1.2}-\cot z'_{1.3}\sin a'_{1.3}\right)\\ -\eta\left(\cot z'_{1.2}\cos a'_{1.2}-\cot z'_{1.3}\cos a'_{1.3}\right)+\cdots\end{array}\right\}, \tag{6}$$

whereby:

$$\Theta\cos A'=\xi \quad \Theta\sin A'=\eta. \tag{7}$$

After having corrected the horizontal angle through the above-mentioned reduction (6), the angle refers to the assumption that the vertical axis of the theodolite in P_1 follows the ellipsoidal vertical. Strictly speaking, the angle must furthermore be corrected because of the lateral refraction (p. 489).

But, because of the elevation of the objects P_2 and P_3 above the ellipsoid, the angle has to be corrected more in order to have it corresponding to the surface angles between the vertical sections which lead from the projection of point P_1 effected by the ellipsoidal vertical toward the corresponding projections of the points P_2 and P_3. As is already mentioned above in reference to the ellipsoid, the reduction can not be carried out by means of the object's elevation above sea level. But the elevation of the station point P_1 is not important.

Furthermore, as already mentioned above, the transfer is to be effected from the vertical sections to the geodetic lines.

This is, according to the theory, a series of reductions. But the neglects occuring in practice are considerable. More about this will be mentioned further below.*)

Par. 3. Reduction of the Azimuths, Geographical Latitudes, and Longitudes. The spherical triangle NZZ', for the computation of the difference $A - A'$ by means of Napier's first Analogy p. 78 (6), first gives:

$$\cot \frac{B' + B}{2} = \frac{\sin \frac{A + A'}{2}}{\sin \frac{A - A'}{2}} \tan \frac{\Theta}{2} \qquad (1)$$

or:

$$\sin \frac{A - A'}{2} = \tan \frac{\Theta}{2} \tan \frac{B' + B}{2} \sin \frac{A' + A}{2}. \qquad (2)$$

From this always follows with sufficient accuracy, inasfar as approaches near the Poles are impossible at the present time, and considering the small amount of Θ, that $\sin(A - A')$ will always remain small:

$$A - A' = \Theta \tan B' \sin A' + \cdots. \qquad (3)$$

But the true azimuth of P_2 in P_1, that means $a'_{1.2}$ is equal $A' + W'_{1.2}$ and the corresponding ellipsoid azimuth $a_{1.2}$ is equal $A + W_{1.2}$. Reducing the true azimuth to the ellipsoidal azimuth, we have therefore the value:

*)Here we may state firstly that properly we should begin with the correction referring to lateral refraction. But, this correction can be carried out only after the assumption of an ellipsoidal shape of the geoid surface is made and, therefore, the reduction referring to the deflection of the vertical preceds quite conveniently.

Par. 3. Reduction of the Azimuths, Geographical Latitudes, and Longitudes.

$$a_{1.2} - a'_{1.2}, \text{i.e.} (A - A') + (W_{1.2} - W'_{1.2})$$

or:

$$\Theta (\tan B' \sin A' - \cot z' \sin (a'_{1.2} - A')) + \cdots \quad (4)$$

In reference to (7), we may also write instead of (4):

$$-\xi \cot z' \sin a'_{1.2} + \eta (\tan B' + \cot z' \cos a'_{1.2}) + \cdots \quad (5)$$

Herewith may also be mentioned that the azimuth as well as the horizontal angles are, strictly speaking, to be corrected because of the lateral refraction. Furthermore, inasfar as we deal with terrestrial objects, we have to reduce the angle values because of the altitude of the objects above the reference ellipsoid and, eventually, because of the transfer from the vertical section to the geodetic line.

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In reference to the geographical latitude the triangle NZZ' in Fig. 41 renders, applying Napier's second Analogy (6) on p. 78, the equation:

$$\tan \frac{B' - B}{2} = - \frac{\cos \frac{A + A'}{2}}{\cos \frac{A - A'}{2}} \tan \frac{\Theta}{2}. \quad (6)$$

From this it follows that, not considering the area close to the Poles, (in a completely sufficient approximation) the reduction of the true geographical latitude to the ellipsoidal latitude is:

$$B - B' = + \Theta \cos A' + \cdots = + \xi + \cdots \quad (7)$$

Counting from an optional prime meridian, the difference of longitude becomes in the triangle NZZ':

$$\sin(L' - L) = \sin\Theta \frac{\sin A}{\cos B'}. \qquad (8)$$

Excluding again the areas in close proximity to the Poles, we obtain the reduction of the real geographical longitude to the ellipsoidal longitude in a sufficient approximation:

$$L - L' = -\Theta \sin A \sec B' + \cdots = -\eta \sec B' + \cdots. \qquad (9)$$

It is noticeable that the reductions of the azimuth and of the geographical longitude depend essentially, respectively, exclusively on η, the western component of Θ, that is the deflection of the vertical, but they (the reductions) depend only to a small extent or not at all on ξ, the southern component. By this, a valuable connection of both reductions is given, and this will prove advantageous from now on.

Par. 4. Reduction of the Zenith Distances; Trigonometric Measurement of Altitudes.

The spherical triangle $ZZ'P_2$ in Fig. 41 gives:

$$\cos z_{1.2} = \cos z'_{1.2} \cos\Theta - \sin z'_{1.2} \sin\Theta \cos W'_{1.2}. \qquad (1)$$

From this it follows that:

$$\cos z_{1.2} - \cos z'_{1.2} = -\sin\Theta \sin z'_{1.2} \cos W'_{1.2} - 2\sin^2\frac{\Theta}{2}\cos z'_{1.2};$$

as long as the value z' refers to terrestrial objects, i.e., is by no means insignificant, it is possible to set, accordingly and with sufficient approximation for the reduction of the true zenith

distance to the ellipsoidal zenith distance:

$$z_{1.2} - z'_{1.2} = +\Theta \cos W'_{1.2} + \cdots = \Theta \cos (a'_{1.2} - A') + \cdots,$$

or:

$$z_{1.2} - z'_{1.2} = \xi \cos a'_{1.2} + \eta \sin a'_{1.2} + \cdots \qquad (2)$$

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But the value $z'_{1.2}$ which denotes the observed zenith distance, in order to be applied in a trigonometric altitude-measurement, has to be corrected first because of the altitude refraction and secondly in accordance with formula (2). It is possible to express the first correction in the form $\frac{1}{2} k_1 \gamma'$ when k_1 signifies the refraction coefficient in P_1 and γ' the convergence angle between the plumb lines in P_1 and P_2. For the present we consider the coefficient k_1 as known.

Before we draw a diagram, we remember that the ellipsoidal verticals have no intersections among themselves, and that there is no common vertical plane between the points P_1 and P_2. Therefore we choose a plane as a projection surface which is parallel to and passes between the verticals and project them onto that plane.

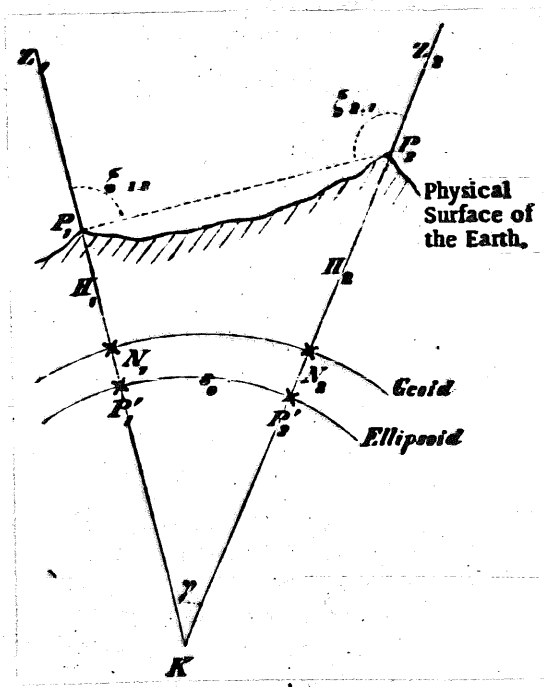


Figure 42.

The distances H and N (Fig. 42) measured off on those verticals are, of course, correctly projected, as well as γ, the angle between them. But it is

neither completely correct to consider the arc s_0 of Fig. 42 as the length of the geodetic line between P_1' and P_2', nor is the angle designated $\zeta_{1.2}$ completely equal to $\zeta_{1.2}$, which is the ellipsoidal zenith distance corrected for refraction. The distance between the points P_1 and P_2, in the present case, will always be a short one in comparison with a_0, and therefore also the shortest distance between the ellipsoidal verticals will be small in relation to $P_1'P_2'$.*) It follows, consequently, that we may call s_0 in fact, in Fig. 42,

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the geodetic distance $P_1'P_2'$ on the reference ellipsoid, and denote the angle $Z_1P_1P_2$ as $\zeta_{1.2}$ without any noticeable error, because the points are located relatively close to the projection plane.

The triangle P_1P_2K now gives according to the Law of Tangents of plane trigonometry:

$$(P_2K - P_1K) : (P_2K + P_1K) = \tan\tfrac{1}{2}(\zeta_{2.1} - \zeta_{1.2}) : \tan\left(180^0 - \tfrac{1}{2}(\zeta_{2.1} + \zeta_{1.2})\right), \qquad (3)$$

or, in view of the fact that $\zeta_{2.1} = \gamma + 180^0 - \zeta_{1.2}$, it follows that:

*)In reference to p. 137 (9) and Fig. 10 as well as according to p. 41 (9) not considering the sign:

$$\varepsilon \cos^2 B \sin 2\alpha \cdot s_0,$$

follows approximately as the shortest distance between the verticals, wherein B is the mean geographical latitude, and α the mean azimuth; and wherein ε denotes the flattening. According to this the distance of the vertical from the plane is less than $\frac{1}{300}$ of the distance between the verticals and the angle of inclination of the lines $P_1'P_2'$ and s_0 toward the projection plane and is in its maximum not even $\frac{1}{2}''$.

Par. 4. Reduction of the Zenith Distances; Trigonometric Measurement of Altitudes.

$$(H_2 - H_1) + (N_2 - N_1) = 2\left(\varrho_{1.2} + \frac{1}{2}(H_2 + H_1 + N_2 + N_1)\right)\tan\frac{1}{2}\gamma\tan\frac{1}{2}(\zeta_{2.1} - \zeta_{1.2}) - E_{2.1}. \quad (4)$$

Here the term $E_{2.1}$ denotes the difference $P_2'K - P_1'K$ which is set by authors generally to be equal to zero.

$\varrho_{1.2}$ is introduced for the arithmetic mean of $P_2'K$ and $P_1'K$; obviously it is possible to set, in a close approximation:

$$s_0 = \gamma\varrho_{1.2}.$$

Substituting $\tan\frac{1}{2}\gamma$ with the expansion in a series and eliminating also in the last factor $\zeta_{2.1}$, then follows a formula for so-called non-reciprocal zenith distances:

$$\left.\begin{aligned} &(H_2 - H_1) + (N_2 - N_1) = s_m\left(1 + \frac{1}{12}\frac{s_0^2}{\varrho_{1.2}^2}\right)\cot\left(\zeta_{1.2} - \frac{1}{2}\gamma\right) - E_{2.1} + \cdots \\ &\text{wherein} \\ &\qquad s_m = s_0\left(1 + \frac{H_1 + H_2 + N_1 + N_2}{2\varrho_{1.2}}\right) \\ &\qquad \gamma = \varrho'' \frac{s_0}{\varrho_{1.2}}, \quad \text{in seconds} \\ &\qquad \zeta_{1.2} = \varepsilon_{1.2}' + \frac{1}{2}k_1\gamma' + \xi_1\cos\alpha_{1.2}' + \eta_1\sin\alpha_{1.2}' + \cdots. \end{aligned}\right\} \quad (5)$$

The symbols ξ and η here are furnished with the indices 1 in order to indicate that they denote components of the deflection of the vertical at the station point P_1.

Furthermore the following formula for reciprocal zenith distances is obtained:

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned}(H_2-H_1)+(N_2-N_1)&=s_m\left(1+\tfrac{1}{12}\tfrac{s_m^2}{\varrho_{1.2}^2}\right)\tan\tfrac{1}{2}(\zeta_{2.1}-\zeta_{1.2})-E_{2.1}\\ \text{with:}\quad s_m&=s_0\left(1+\frac{H_1+H_2+N_1+N_2}{2\varrho_{1.2}}\right)\\ \zeta_{2.1}-\zeta_{1.2}&=\varepsilon'_{2.1}-\varepsilon'_{1.2}+\left\{\begin{matrix}\frac{\varkappa}{2}(k_2-k_1)\gamma'\\ +\xi_2\cos\alpha'_{2.1}+\eta_2\sin\alpha'_{2.1}\\ -\xi_1\cos\alpha'_{1.2}-\eta_1\sin\alpha'_{1.2}+\cdots\end{matrix}\right\}.\end{aligned}\right\}\qquad(6)$$

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In order to obtain γ', we take into consideration the equation $\zeta_{1.2}+\zeta_{2.1}-180^0=\gamma$ which corresponds to the equation $\zeta'_{1.2}+\zeta'_{2.1}-180^0=\gamma'$. According to this:

$$\gamma'=\gamma-\left\{\begin{matrix}\xi_1\cos\alpha'_{1.2}+\eta_1\sin\alpha'_{1.2}\\ +\xi_2\cos\alpha'_{2.1}+\eta_2\sin\alpha'_{2.1}\end{matrix}\right\}+\cdots.\qquad(7)$$

In the equation (6) and (7) we denote the components of the deflection of the vertical in P_2 as ξ_2 and η_2. But s_m which is inserted into the formulae (5) and (6) as an approximation corresponds evidently very closely to the horizontal distance between P_1 and P_2, measured in an equipotential surface which is located at a mean altitude between the two points.

The formulae (5) and (6) demonstrate that not only the deflections of the vertical but also the distances of the reference ellipsoid from the geoid have bearing on a trigonometric altitude measurement. If we consider the H as known besides the deflection of the vertical which will not differ noticeably from the altitudes above sea level, (p. 8) then, according to (5) and (6), the zenith dis-

tance measurements could serve the purpose of determining the distances N of the reference ellipsoid from the geoid (except for a constant), if we would have a sufficient knowledge of the refraction coefficient; but exactly that is missing. We do not intend to amplify this further here, we rather save this amplification for Volume II of this book, because especially the theory of refraction is essentially of a physical nature.

Then also, the small completion term $E_{2,1}$ will be dealt with.

Par. 5. The Reduction of the Base Line of a Triangulation Net. As a rule, the base line has only a short length in relation to a_0, generally less than 10^{km}. We may assume, by all means, that the measurement be carried out in one of the two ellipsoidal vertical planes passing through the points $P_1 P_2$ which, moreover, may be considered as coinciding in the exposed case because, for practical reasons, the physical surface of the earth should be plane in the direction of the base line, at least in rough approximation.

Imagining the measuring as starting at P_1 or P_2, then the measuring is, strictly speaking carried out in one of the two true vertical planes on the physical surface of the earth. A turn of the vertical plane, corresponding to the angle of the deflection of the vertical, around the straight line $P_1 P_2$ does not provide a noticeably different section line on the physical surface of the earth as it does not differ sufficiently from the mathematical straight line because of the assumed approximate planeness.

This assumption is permitted, if the alignment is carried out from a central point C and

if C is in a position that the horizontal angle toward P_1 and P_2 is exactly 180°. In this case the measurement is also carried out in a plane passing through P_1P_2, which is the vertical plane for C. This plane forms small angles with the other mentioned vertical planes because of the assumed shortness of the distance. These angles depend essentially only on the deflection of the vertical.*) The maximum separation of the lines which are measured under different circumstances on the surface is thus approximately equal to an arc of a circle, the radius of which is the maximum vertical distance of the section line from the mathematical straight line P_1P_2 and the central angle which reaches a maximum of only the amount of the deflection of the vertical; that means only a very small magnitude in practical cases.

In Fig. 43 as well as in Fig. 42, a projection plane parallel to the ellipsoidal verticals P_1P_1' and P_2P_2' is used. It is immediately noticeable that, in this figure, the proportion of the length of the section line and the shortest line between P_1' and P_2' (which is essentially identical with the two vertical sections) is, except for insignificant magnitudes, the same as in

*) According to p. 189, the distances of the ellipsoidal vertical sections running from P_1 to P_2 and vice versa are, for the assumed distance, smaller than 0,01 mm. But, of the two, the intersection of the plane which is perpendicular in C with the surface has, by all means, the shorter distance in comparison with the two (above-mentioned). This becomes evident if we imagine the point C as moving and taking all positions from P_1 as far as P_2. As far as the geographical latitude herewith generally increases or decreases, the vertical plane turns continuously only toward the same side.

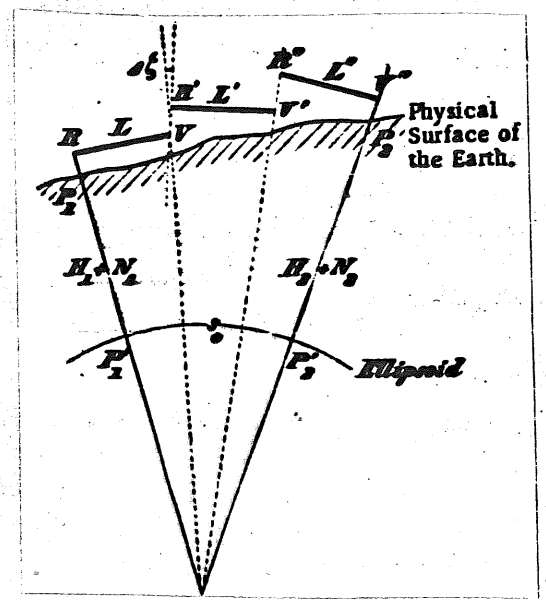


Figure 43.

reality. That is because the two lines have evidently almost the same inclination toward the projection plane, and, furthermore, the dimensions change only by a small amount.

We insert, therefore, in Fig. 43 directly the notations of the true length.

Now we have to conclude from the result of the measurement in the section to the shortest distance s_0 from P_1' and P_2'.

But the measurement is essentially a graduated one.

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Fig. 43 shows, as an illustration, three horizontal lines L, L' and L'' (instead of the numerous positions of the measuring rods which are present in practice). If we would use now the verticals of the ellipsoid, then, where each line may be considered as an arc which is concentric to its projection on the ellipsoid, we could set directly:

$$s_0 = L\frac{\varrho}{\varrho + H + N} + L'\frac{\varrho}{\varrho + H' + N'} + \cdots, \qquad (1)$$

whereby $H + N$, $H' + N'$, etc. means the altitudes of the lines above the ellipsoid, and where it is permissible to substitute ϱ completely sufficiently in all terms by the radius of curvature of the ellipsoid in the mean azimuth of the base line . The always-sufficient approximation formula:

$$s_0 = L\left(1 - \frac{H+N}{e}\right) + L'\left(1 - \frac{H'+N'}{e}\right) + \cdots, \qquad (2)$$

is obtained from the above-mentioned by expansion in a series. This formula can be simplified more by introduction of a mean value for the factors L, L' etc., which coincide with each other except for small details, in order to compute the small terms $L\frac{H+N}{e}$ etc.:

$$s_0 = (L + L' + \cdots)\left(1 - \frac{H_m + N_m}{e} + \cdots\right). \qquad (3)$$

Herein H_m and N_m denote the sums H, $H' \cdots$ respectively N, $N' \cdots$ for all lines divided by their number. It is possible to consider H_m as the mean altitude of the measured section above the geoid. N_m, however, is the mean altitude of the geoid above the reference ellipsoid below the base line.

The above-mentioned formula assumes that the rods are perpendicular to the verticals against the vertical to be measured, and that these verticals be used as plumb lines. As this is not the case, we consider, for instance, the measurement of the line L' in Fig. 43, and assume conveniently that L', be horizontal and the terminal point R' be perpendicular above the terminal point V of the line L. Then this horizontal and perpendicular direction differs from the corresponding ellipsoidal directions in that component of the deflection of the vertical which is in the direction of the base line. If this is denoted by $\Delta\xi$, then, strictly speaking:

$$L' \cos \Delta\xi + (H' - H) \sin \Delta\xi. \qquad (4)$$

has to be introduced into (3) instead of L'. It is easily recognizable that this is an insignificant change from L' which, as a rule, even by its summation along the whole base line does not amount to much, even with considerable values of $\Delta\xi$.

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Thus we may consider formula (3) as sufficiently accurate; as a rule, even when the measurement operation differs a little from the above assumed procedure.

In Grunert's Archives, p. 28, part 51, 1869, Sonderhof compares the geodetic line on an equipotential surface parallel to the surface of the ellipsoid of revolution with the one on the ellipsoid of revolution if the two lines are located between the same verticals. The result corresponds to the reduction procedure adhered to above.

Par. 6. First Approximation in the Computation of a Triangulation Net. If there is nothing known about the relation between geoid and a reference ellipsoid, then, in a first approximation, the geoid has to be considered as identical to some convenient reference ellipsoid (the dimensions accepted by Bessel are quite suitable), so that the deflections of the vertical and the altitudes N have not to be taken into consideration. At the reduction of the base line measurement according to formula (3) on the previous page, N_m has not to be taken into consideration.

But this alone does not produce any error.

Assuming, as always done so far, that the axis of the reference

ellipsoid should be parallel to the earth's axis, but not coinciding with it, then the assumtion N_m = zero means only that the reference ellipsoid and the geoid intersect one another below the center of the base line.

The situation definitely becomes different when there are several base measurements taken into consideration, because the assumption N_m = zero is doubtlessly only acceptable for one base line as will be shown in the following Par. 10. If we assume now for all base lines N_m = zero, then this consequently produces errors in reducing their length. The magnitude of these errors generally depends on the mutual separation of the base lines. It is already easily noticeable that N_m values of only a few meters are sufficient to produce an uncertainty which is far larger than that of the error of pure measuring, even if we assume them as to be $\frac{1}{500000}$ of the length. Yet this uncertainty is not very serious because the linear length of the geodetic lines obtained from the base lines by means of triangle chains are, by far, less accurate than the error of angle observation.*)

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We will not follow this discussion further, but beside the error in reducing the base lines, we have to mention the error in

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*)According to present experiences, the mean error of the triangle sides in the basic net raises from $\frac{1}{150000}$ to $\frac{1}{250000}$ and increases easily further to $\frac{1}{100000}$ and more according to the formula which will be mentioned below in Par. 8. See the report about the Dänische Gradmessung (Danish Arc Measurement) p. 63, Vol. 13 of the Vierteljahrsschrift der Astron. Ges. (Quarterly of the Astronomical Society).

reducing the angles. Strictly speaking, every horizontal angle has to be reduced according to (6) p. 516, in order to become valid for the reference ellipsoid. This reduction has to be left undone because the value of the components ξ and η of the deflection of the vertical are not known at this first approximation. The errors which are herewith produced amount however, as a rule, to no more than a few tenths of seconds. Only in high mountain chains where the deflections of the vertical as well as the $\cot z'$ reach generally the highest values, the errors may increase to 1" and more. But the danger that this will happen in first-order triangles is reduced by the fact that extremely inclined lines of sight, that means large values of $\cot z'$ are rare and mostly can be avoided. Let us assume that these influences are, in general, a little smaller than those of the observation errors. In the estimation of their influence on the final results regarding the triangle nets, it has to be taken into consideration whether they are subject to a partial compensation as random errors as well as observational errors, or whether their constant systematic character can effect an increase of their influence.

It is easy to understand that one as well as the other will occur. The deflections of the vertical have, on the one hand, a systematic character which depends on the choice of the reference ellipsoid and the continental irregularities of the geoid. Partially on the other hand, they have a rapidly changing character which is governed by the local irregularities of the geoid. The variability of the reductions of the horizontal angles depends

not only on the local irregularities but also on the variations of cot z' values which are however, as a rule, small negative values in first-order triangles.*)

Now we will consider the results of the neglect of the reductions of partially systematic errors and errors which partially change accidentally.

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Par. 7. Influence of the Deflections of the Vertical on the Results of a Triangulation Net. Let us assume that $\cot z'$ has the same value for the three sides of an equilateral triangle which has its three points at the same elevation, and assuming furthermore that the deflection of the vertical is the same at these points, then the angular sum of this triangle is not changed by neglect of the reductions. Disregarding the factor $\Theta \cot z'$ the reduction for the three angles is, according to (5) p. 516:

$$\begin{array}{l} \sin(a'_{1.2} - A') - \sin(a'_{1.3} - A') \\ \sin(a'_{2.3} - A') - \sin(a'_{2.1} - A') \\ \sin(a'_{3.1} - A') - \sin(a'_{3.2} - A') \end{array}$$

if $180^0 + A'$ is the true azimuth of the ellipsoidal zenith which

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*)It follows from the known approximate formula for trigonometric altitude measurements:

$$H_2 - H_1 = s \cot z'_{1.2} + \frac{s^2}{2\varrho}(1-k),$$

it can also be derived from (5) p. 520, that:

$$\cot z'_{1.2} = \frac{H_2 - H_1}{s} - \frac{s}{2\varrho}(1-k).$$

is common to all three points. However, in reality, this sum becomes zero because all azimuths differ by 120° from one another in consequence of the equilaterality of the triangle. Thus, each vertical column is eliminated.*) Now, of course, the correction does not change the bearing which the neglected reductions have on the single angle.

A simple triangulation chain, consisting of equilateral triangles, thus will be deformed in a way which is easily understood. We consider firstly two neighboring triangles 1.2.3. and 2.4.3, and then, taking the side 1.2 as a base, we find the errors of the sides 2.4 and 3.4.

But there is

$$(2.4) = (1.2)\frac{\sin\binom{2.3}{1}\sin\binom{2.4}{3}}{\sin\binom{1.2}{3}\sin\binom{3.2}{4}}$$

$$(3.4) = (1.2)\frac{\sin\binom{2.3}{1}\sin\binom{4.3}{2}}{\sin\binom{1.2}{3}\sin\binom{3.2}{4}}.$$

Disregarding the factor $\Theta \cot z'$, the neglected reductions are

$$\text{in } \binom{1.2}{3}: \quad \sin(a'_{3.1} - A') - \sin(a'_{3.2} - A')$$

*)The same results are obtained in the case of scalene triangles with equal elevations of their points and letting $\cot z'$ be $\frac{s}{2\varrho}$ (Bessel, Abhandl. (Transactions) p. 32, Vol. 3). Under certain circumstances which are often approximately true, the central systems in their approximations occurring through the adjustment, are not affected by the constant deflection of the vertical.

$$\text{in } \binom{3.2}{4}: \quad \sin(a'_{4.3} - A') - \sin(a'_{4.2} - A')$$

$$\text{in } \binom{2.3}{1}: \quad \sin(a'_{1.2} - A') - \sin(a'_{1.3} - A')$$

$$\text{in } \binom{2.4}{3}: \quad \sin(a'_{3.2} - A') - \sin(a'_{3.4} - A')$$

$$\text{in } \binom{4.3}{2}: \quad \sin(a'_{2.4} - A') - \sin(a'_{2.3} - A').$$

In reference to the parallelism of the sides 1.2 and 3.4 as well as 1.3 and 2.4, these reductions may, in an easily seen shortened form, also be written

$$\text{for } \binom{1.2}{3}: \quad +r_2 + r_1$$

$$\text{» } \binom{3.2}{4}: \quad -r_3 - r_2$$

$$\text{» } \binom{2.3}{1}: \quad +r_3 + r_2$$

$$\text{» } \binom{2.4}{3}: \quad -r_1 - r_3$$

$$\text{» } \binom{4.3}{2}: \quad -r_2 - r_1.$$

Herewith, if all angles are taken to be 60°, the influence of these values on $\log \frac{2.4}{1.2}$ is obtained equal to

$$\frac{i}{\rho} M \cot 60° . (r_2 + r_3 - 2r_1) \qquad (1)$$

and on $\log \frac{3.4}{1.2}$ as equal to

$$\frac{1}{\rho} M \cot 60° . (2r_3 - 2r_1), \qquad (2)$$

wherein the r values are expressed in seconds. If there is another pair of adjoining triangles, having the side 3.4 as a common side and others follow it up to $2i$ triangles forming a straight

chain in the azimuth $a'_{1.3}$ (see Fig. 13, p. 198), and denoting the principal side 1.2 by a and the parallel, neighboring sides 3.4 etc. of the triangles with $a', a'' \ldots a^{(i)}$, and, however, denoting the parallel sides following one another in the direction 2.4 with $d, d' \ldots d^{(i-1)}$, then the constant component of the deflection of the vertical will have the following influence on $\log \frac{a^{(i)}}{a}$:

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$$\frac{1}{\varrho''} M \cot 60^0 . (2ir_3 - 2ir_1). \qquad (3)$$

On the other hand, the influence on $\log \frac{d^{(i-1)}}{a}$ is equal to

$$\frac{1}{\varrho''} (i-1) M \cot 60^0 . (2r_3 - 2r_1) + \frac{1}{\varrho''} M \cot 60^0 . (r_2 + r_3 - 2r_1). \qquad (4)$$

Computing the influence on $\frac{d^{(i-1)}}{a}$ from this, and then by summing up all its values for $i = 1$ to i, the influence on $\frac{[d]}{a}$, where $[d]$ is equal to the long side of the net in the direction 2.4, then it follows finally as the influence on $\log \frac{[d]}{a}$:

$$\frac{1}{\varrho''} M \cos 60^0 . \{ir_3 + r_2 - (i+1) r_1\}. \qquad (5)$$

Beside the influences on the ratios of sides we also consider the influence on the horizontal angle between the long side $[d]$ of the net in the direction 2.4 and the base side 1.2. The error in the angle $\binom{1.4}{2}$ is, however, equal to $-r_2 + r_3$ and the errors of the straight angles between the lines $d, d', \ldots d^{(i-1)}$ are in succession all equal to $-2r_2$. Through this, a curvature of the long side is formed, and, as can be found through a projection onto a direction transverse to 2.4, the total error in the men-

tioned angle is equal to:

$$(-r_2 + r_3) - 2r_2 \frac{1+2+3+\ldots+(i-1)}{i}$$

i.e.,

$$-ir_2 + r_3. \qquad (6)$$

On the other hand the error of the difference of the azimuths of the sides $a^{(i)}$ and a is equal to:

$$-2ir_2. \qquad (7)$$

According to these formulae it is possible to estimate the influence of that portion of the deflection of the vertical which is common to all stations, not only in the case of the ideal triangulation net, but often also in real and simple triangulation chains whilst, another changing portion is connected with the errors of measuring. For purposes of comparison we consider now the accumulation of the errors of measuring as far as they are to be considered as accidental errors.

Par. 8. The Influence of Accidental Errors. Next we begin again with the side ratios $\frac{2.4}{1.2}$ and $\frac{3.4}{1.2}$ in the first pair of triangles, assuming that the triangles are equilateral ones. If now μ is the mean accidental error

of an angle of the triangle, then it is easy to see that the mean error in $\log\frac{2.4}{1.2}$ as well as in $\log\frac{3.4}{1.2}$ becomes equal to:

$$\pm\frac{1}{\rho''} M \cot 60^\circ . \mu\sqrt{4}, \qquad (1^*) \text{ and } (2^*)$$

since, in these side ratios, four angular errors independent of

one another occur. At this, the existence of the condition equation between the three triangles does not change anything, because this equation effects equally large corrections on the angles measured with equal accuracy, and therefore the sine ratio of angles of equal size is not affected.*)

Now we obtain the influence on $\log \frac{a^{(i)}}{a}$ as well as on $\log \frac{d^{(i-1)}}{a}$, because these side ratios are composed of i side ratios, independent of one another, which include the same mean error (1*):

$$\pm \frac{1}{\varrho''} M \cot 60^\circ \cdot \mu \sqrt{4i} \qquad (3^*) \text{ and } (4^*)$$

The computation for the mean error of $\log \frac{[d]}{a}$ is more complicated. If we take into account that the sides, $d, d' \ldots$ are all equal to a, then, following Fig. 13, p. 198, the expression for $\frac{[d]}{a}$ becomes:

$$\frac{\sin\binom{2\,3}{1}}{\sin\binom{1.2}{3}\sin\binom{3.2}{4}}\left\{\sin\binom{2.4}{3}+\frac{\sin\binom{4.3}{2}\sin\binom{4.5}{3}}{\sin\binom{3.4}{5}\sin\binom{5.4}{6}}\left\{\sin\binom{4.6}{5}+\frac{\sin\binom{6.5}{4}\sin\binom{6.7}{5}}{\sin\binom{5.6}{7}\sin\binom{7.6}{8}}\left\{\sin\binom{6.8}{7}+\cdots\right\}\right\}\right\}.$$

The terms set up here refer to three pairs of triangles. In order to obtain the mean square error now, the differential quotients corresponding to all appearing angles have to be formed;

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*)In Article 49, Par. 4 of the Studien über rationelle Vermessungen (Studies Regarding Rational Surveys), Zeitschrift f. Math. u. Phys. (Journal of Mathematics and Physics) 1868, we already called attention to the small insignificant effect of the angular equations of the triangles on the side ratios. Only the side equations increase the accuracy of the side ratios essentially. (See the Report concerning the Dänische Gradmessung (Danish Arc Measurements), p. 63, Vol. 13 of the Vierteljahrsschrift der Astronomischen Ges. (Quarterly of the Astronomical Society). Of course, it is not permissible to conclude that it might not be necessary to measure all three triangle angles. If one wanted to measure only two of them, then one of the side ratios of the triangle would become considerably more inaccurate.

wherein all angles have to be equal to 60°, and the squared sum of the mentioned quotients has to be multiplied by μ^2.

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Not considering the factor $\frac{1}{\rho''^2}\mu^2\cot^2 60^0$, the components of the mean square error become

i^2 for each of the three angles $\binom{2.3}{1}, \binom{1.2}{3}, \binom{3.2}{4}$,

$(i-1)^2$ for each of the four angles $\binom{4.3}{2}, \binom{4.5}{3}, \binom{3.4}{5}, \binom{5.4}{6}$,

$(i-2)^2$ " " " " four " $\binom{6.5}{4}, \binom{6.7}{5}, \binom{5.6}{7}, \binom{7.6}{8}$,

. . . . . .

1 " " " " i " $\binom{2.4}{3}, \binom{4.6}{5}, \binom{6.8}{7}, \cdots$

or, stated altogether:

$$-i^2 + 4(i^2 + (i-1)^2 + \cdots + 1) + i;$$

That gives the value

$$\frac{1}{\rho''}\mu\cot 60^0 . \sqrt{\frac{1}{6} i (8i^2 + 6i + 10)}.$$

for the mean error of $\frac{[d]}{a}$, because the sum of the squares from 1 to i^2 is equal to $\frac{1}{6} i (i+1)(2i+1)$.

Thus, finally, the mean error is equal to:

$$\pm \frac{1}{\rho''} M \cot 60^0 . \mu \sqrt{\frac{4}{3} i + \frac{5}{3} \frac{1}{i} + 1}, \qquad (5^*)$$

in $\log \frac{[d]}{a}$ because $\frac{[d]}{a}$ is equal to i.

The expressions for the condition of complete (or equivalent) observations of sets are somewhat different; however, the fact

which is important to us, i.e. the increase with the factor $\sqrt{i}$ and not with i*), remains a characteristic of the expression (5*).

In order to obtain the mean error of the horizontal angle between the base side 1.2 and the long side $[d]$ in the direction 2.4, we denote the true error of the angles $\binom{4.1}{2}, \binom{6.2}{4}, \binom{8.4}{6}, \cdots$ as $\varepsilon_2, \varepsilon_4, \varepsilon_6 \cdots$.

Then, one finds immediately by projection onto a direction perpendicular to the side 2.4 that the error in the mentioned horizontal angle is equal to:

$$\varepsilon_2 + \frac{1}{i}\left((i-1)\,\varepsilon_4 + (i-2)\,\varepsilon_6 + \cdots + 2\varepsilon_{2(i-1)} + \varepsilon_{2i}\right).$$

If we assume that all these angles with the mean error μ are obtained by direct observation, then the mean square error follows immediately:

$$\mu^2\left(1 + \frac{(i-1)^2 + (i-2)^2 + \cdots + 1}{i^2}\right).$$

The mean error in the mentioned horizontal angle is therefore equal to:

$$\pm\mu\sqrt{\frac{1}{3}i + \frac{1}{2} + \frac{1}{6i}}.$$

Herewith, the remaining measurements of the net are not yet taken into consideration. But these, furthermore, render a second

*)It is easy to see, the alteration of the expressions cannot be significant because only elements which are essentially independent from one another are taken into account for the successive computation of the values $a', a'', a''' \ldots$

determination of an equal accuracy by means of the long side [c]. Therefore we set the mean error as being equal to:

$$\pm\mu\sqrt{\frac{1}{6}i+\frac{1}{4}+\frac{1}{12i}}. \qquad (6^*)$$

The difference of the azimuths of the sides $a^{(i)}$ and a can be computed by [d] and [c] i.e., the two length sides of the net as well as by the sequence of the sides a and b which are located between [d] and [c]. These determinations are essentially independent of one another and the adjustment is connecting them according to their weights which are in the proportion $1:1:\frac{1}{i}$. Since the mean error of the difference of the azimuths of the sides $a^{(i)}$ and a from a long side will be equal to:

$$\pm\mu\sqrt{i+1}$$

thus, by taking the adjustment into account, the mean error for that azimuth difference becomes approximately equal to:

$$\pm\mu\sqrt{\frac{i+1}{2,5}}. \qquad (7^*)$$

Par. 9. Comparison, Moment of Total Adjustment, etc. The expression derived in the preceding paragraphs will be altered a little if one takes the adjustment conditions of complete observations of directions rigorously into account. But the obtained material in its present form is already sufficient to show (which was the reason for the developments) that the influence of the accidental errors with somewhat large values of i can be surpassed by corresponding effects which result from the reduction errors according to certain conditions mentioned in Par. 7. This occurs

even if the constant values of r_1, r_2 and r_3 only amount to as many hundredths of a second as μ amounts to tenths of a second.

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Larger triangulation chains which are separately adjusted and referred to special base lines may consequently show discrepancies in their adjoining sides as well as in the conditions of their junction. Considering the angles, these discrepancies do not depend on the accidental errors of angle measurements and the variable component of the influence of the deflection of the vertical, but also depend noticeably on the constant component of the influence of the deflection of the vertical upon the angles. The latter may show up more the better the quality of the measurements are. The reduction errors of the base lines which follow because of the unknown elevation of the geoid in comparison with the reference ellipsoid are, according to p. 523, generally not insignificant. Therefore, a complete adjustment of the complex of all triangle chains which are extended across a part of the earth's surface, can not be carried out, before one of the following has been determined. Evidence by preliminary computations that the considered effects of the deflections of the vertical and the elevation of the geoid do not significantly exceed the influences of the accidental errors of measurement should be found; or a reduction of the directions in the triangulation net and of the base lines has to be carried out, which is based on a preliminary determination of at least the systematic part of the deflections of the vertical, and the elevation of the geoid. This reduction of

the direction would, strictly speaking, require even a new adjustment of the single chains, (with the exception of the station adjustments which would not have to be repeated). But, as a rule, in the computation of the triangulation nets, one can remain without the reduction of the directions in consequence of relative deflections of the vertical, if only a small extent of the geoid is studied, or if a land survey is the purpose of the measurements.

In the first instance one has primarily to determine a reference ellipsoid which corresponds to the measurements as much as possible, be it by dimensions given or those yet to be computed. Relative deflections of the vertical against this ellipsoid will, usually, show a character which varies considerably from point to point. Therefore, no essentially systematic influence of the deflection of the vertical is to be feared. The influence of the elevation of the geoid can not be significant either. An additional, favorable circumstance is the very slight effect which the choice of the dimensions of the ellipsoid has on the computations of the triangles, as can be seen in the analysis on p. 405. A new computation does not have to be made if the dimensions of the ellipsoid are changed in the end.

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In the second instance, the land survey, it would always be sufficient not to take into consideration those reductions resulting from the not completely correct assumption of an ellipsoid. Here, for the purpose of a detailed survey, it is usually sufficient

to neglect the often significant difference between the distances measured directly on the physical surface of the earth, and the corresponding distances derived from the main triangulation without reference to the elevations. Of course, those angle reductions can, in intercalated measurements, become larger than the observational errors, because of steep lines of sight. However, on the other hand, the accumulation of errors is prevented through the intercalation in an established net.*)

Par. 10. The Computation of the Deflections of the Vertical with reference to assumed dimensions of the reference ellipsoid requires either the setting of the deflection of the vertical for one point equal to zero, or the introduction of the deflection as an unknown magnitude. Here, of course, we assume that all triangle nets are connected with one another--or, better said in reverse: nets which are connected with the initial point chosen for the computation, drop out completely from the computations which are to be discussed in the following.

In the first instance, i.e., setting one deflection of the vertical equal to zero, it is evidently assumed that the real direction of the vertical of the considered point is normal to the reference ellipsoid. If, in order to apply Fig. 41 and the formulae of this chapter, one now assumes parallelism of the ellipsoid's axis to the earth's axis, then its position is completely determined except for a displacement magnitude parallel

*)It should be mentioned that the propriety of spherical computation of the figures within a small part of the ellipsoid of rotation which replaces the geoid can easily be demonstrated by the method of deflections of the vertical. Then, the angles between the normals to the ellipsoid and the normals to the sphere are the deflections of the vertical.

to that direction of the vertical which depends on the neglect of the N_m at the reduction of the base lines. If there is only one base line, then the ellipsoid intersects the geoid below the center of the base line. If only one base line is present, then the ellipsoid intersects the geoid below the center of this base line. If there are several base lines measured, then, as long as there is no reduction applied with approximate values of N_m, the ellipsoid gets a mean position (which can not be described exactly without further ado), in such a manner that some N_m values are positive and others negative. (The distance between the earth's axis and the axis of the reference ellipsoid can not be given, with the exception of the position of the above-mentioned initial point in relation to the earth's axis being known.

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This knowledge is impossible to obtain by the ordinary astro-geodetic methods).

It is not convenient to set the deflection of the vertical for a point equal to zero. The adjustment of the reference ellipsoid can be achieved much better by giving the deflection of the vertical an indefinite value at the initial point.

This being the more general case, we will base on it the problems in the following, i.e., to demonstrate the successive computation of the deflections of the vertical; to improve eventually the dimensions of the ellipsoid, and to derive the shape of the geoid from the deflections of the vertical as far as possible.

Par. 11. The Influence of the Deflection of the Vertical on Geographical Coordinates.

We will not enter into the problem of finding deflections of the vertical for a reference ellipsoid the axis of which is not parallel to the earth's axis. It may be mentioned that the respective formulae should easily be derived from Fig. 41 by distinguishing therein two North Poles N and N'. Following the dimensions of the ellipsoid and the components of the deflections of the vertical at an initial point, that means four unknowns, there would furthermore appear two unknowns which determine the position of N against N'. But this procedure will be disregarded because the ellipsoidal geographical latitudes can differ considerably from the real latitudes, the more so when the relative deflections of the vertical are usually caused by local and quick changes of the shape of the equipotential surfaces. In addition to that, the computation method becomes considerably complicated if the difference between N and N' is not assumed to be especially small, as can be seen in the repeatedly mentioned treatise by Fergola, respectively in Vol. 11 of the Vierteljahrsschrift der Astronom. Ges. (Quarterly of the Astronomical Society). The reason is that the higher powers of the difference between N and N' can not be neglected without further ado (which is partially done by Fergola).

Par. 11. The Influence of the Deflection of the Vertical on Geographical Coordinates which are Geodetically Determined: Formulae for the Computation of the Deflection of the Vertical.

If we imagine the computation of the latitude, longitude and azimuth at P_2 from the real values of the geographical latitude B_1, the longitude L_1, and the azimuth $\alpha_{1,2}$ at P_1 by means of the

linear length of the shortest connection between P_1 and P_2, and assuming definite dimensions of a reference ellipsoid, then these values will generally deviate from the true values of these magnitudes.

We denote the computed values as B_2', L_2'', and $\alpha_{2.1}''$.

If one had carried out the computation with the ellipsoidal values B_1, L_1,

and $\alpha_{1.2}$, then the ellipsoidal values B_2, L_2, and $\alpha_{2.1}$ would have resulted also. In order to compare these ellipsoidal values with those mentioned above, we apply the differential formulae.

According to pp. 517 and 518, Par. 3, we have as a satisfactory approximation:

$$\left.\begin{aligned} B_1 - B_1' &= +\xi_1 + \cdots \\ L_1 - L_1' &= -\eta_1 \sec B_1' + \cdots \\ \alpha_{1.2} - \alpha_{1.2}' &= +\eta_1 \tan B_1' + \cdots, \end{aligned}\right\} \qquad (1)$$

where, in the last formula as well as in the computation of the triangle net, the small terms multiplying the cotangents of the zenith distances as well as the influence of the distance between geoid and ellipsoid on the reduction of the directions because of the altitude of the objects are not considered--or, where it is assumed that all directions, including $\alpha_{1.2}'$, are already corrected for these small terms by means of preliminary values of the deflections of the vertical.

Introducing the values $+\xi_1$, $-\eta_1 \sec B_1'$ and $+\eta_1 \tan B_1'$, in the form of δB_1, δL_1, and $\delta\alpha_{1.2}$ into the formulae (7) to (9), p. 282,

we obtain:

$$\left.\begin{aligned} B_2 - B_2' = &+ \frac{W_2^2}{W_1^2}\left\{\cos\Delta\alpha + \left[1 - \left(\frac{dm}{ds}\right)_{2.1}\right]\sin\alpha_{1.2}\sin\alpha_{2.1}\right\}\xi_1 \\ &- \frac{m}{a_0}\frac{W_2^2}{1-e^2}\tan B_1 \sin\alpha_{2.1}\,.\,\eta_1 + \cdots \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} L_2 - L_2' = &- \frac{(1-e^2)W_2}{W_1^2}\left\{\sin\Delta\alpha - \left[1 - \left(\frac{dm}{ds}\right)_{2.1}\right]\sin\alpha_{1.2}\cos\alpha_{2.1}\right\}\sec B_2\,.\,\xi_1 \\ &- \left\{\sec B_1 + \frac{m}{a_0} W_2 \sec B_2 \tan B_1 \cos\alpha_{2.1}\right\}\eta_1 + \cdots \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} \alpha_{2.1} - \alpha_{2.1}' = &\frac{a_0}{m}\frac{1-e^2}{W_1^2}\left\{\begin{aligned}&\left[1 - \left(\frac{dm}{ds}\right)_{1.2} + 1 - \left(\frac{dm}{ds}\right)_{2.1}\right]\cos\Delta\alpha\sin\alpha_{2.1} - \frac{\sin^2\Delta\alpha}{\sin\alpha_{1.2}} \\ &+ \left[1 - \left(\frac{dm}{ds}\right)_{1.2}\right]\left[1 - \left(\frac{dm}{ds}\right)_{2.1}\right]\sin^2\alpha_{2.1}\sin\alpha_{1.2}\end{aligned}\right\}\xi_1 \\ &+ \left\{\left(\frac{dm}{ds}\right)_{1.2} + \frac{m}{a_0} W_2 \tan B_2 \cos\alpha_{2.1}\right\}\tan B_1\,.\,\eta_1 + \cdots . \end{aligned}\right\} \quad (4)$$

If, corresponding to formulae (1), one applies the substitutes:

$$\left.\begin{aligned} B_2 &= B_2' + \xi_2 + \cdots \\ L_2 &= L_2' - \eta_2 \sec B_2' + \cdots \\ \alpha_{2.1} &= \alpha_{2.1}' + \eta_2 \tan B_2' + \cdots , \end{aligned}\right\} \quad (5)$$

here instead of B_2, L_2, and $\alpha_{2.1}$, then one obtains the equation establishing a relation between the components ξ and η of the

deflection of the vertical of the two points P_1 and P_2 with reference to the reference ellipsoid. They are:

$$\left.\begin{aligned} B_2' - B_2 - \xi_2 + \frac{W_2^2}{W_1^2}\left\{\cos\Delta\alpha + \left[1 - \left(\frac{dm}{ds}\right)_{2.1}\right]\sin\alpha_{1.2}\sin\alpha_{2.1}\right\}\xi_1 \\ - \frac{m}{a_0}\frac{W_2^2}{1-e^2}\tan B_1 \sin\alpha_{2.1}\,.\,\eta_1 + \cdots = 0 \end{aligned}\right\} \quad (6)$$

$$L_2'' - L_2' + \sec B_2 . \eta_2 - \frac{(1-e^2)W_2}{W_1^3}\left\{\sin\Delta\alpha - \left[1-\left(\frac{dm}{ds}\right)_{2.1}\right]\sin\alpha_{1.2}\cos\alpha_{2.1}\right\}\sec B_2 . \xi_1 \Bigg\} - \left\{1+\frac{m}{a_0}W_2\sin B_1 \sec B_2 \cos\alpha_{2.1}\right\}\sec B_1 . \eta_1 + \cdots = 0 \tag{7}$$

$$\left.\begin{aligned} &\alpha_{2.1}'' - \alpha_{2.1}' - \tan B_2 . \eta_2 \\ &- \frac{a_0}{m}\frac{1-e^2}{W_1^3}\left\{\left[1-\left(\frac{dm}{ds}\right)_{1.2}+1-\left(\frac{dm}{ds}\right)_{2.1}\right]\cos\Delta\alpha\sin\alpha_{2.1} - \frac{\sin^2\Delta\alpha}{\sin\alpha_{1.2}} \right. \right. \\ &\left. \left. + \left[1-\left(\frac{dm}{ds}\right)_{1.2}\right]\left[1-\left(\frac{dm}{ds}\right)_{2.1}\right]\sin^2\alpha_{2.1}\sin\alpha_{1.2}\right\}\xi_1 \right. \\ &\left. + \left\{\left(\frac{dm}{ds}\right)_{1.2} + \frac{m}{a_0}W_2\tan B_2\cos\alpha_{2.1}\right\}\tan B_1 . \eta_1 + \cdots = 0. \right. \end{aligned}\right\} \tag{8}$$

Regarding the exact and approximate computation of the coefficients appearing herein, see p. 283, Par. 11.

In any case, one can apply the real values of the geographical coordinates and of the azimuths in the coefficients instead of the ellipsoidal values which are introduced because of simplicity.

If we consider furthermore that, on the one hand, the observation errors in B', L' and α' appear already in the first decimal place of the seconds, and that, on the other hand, the components of the deflections of the vertical, generally amount only to a few seconds, then it appears completely sufficient to state the coefficients of ξ_1, and η_1, to three decimal-places. This is the same as the neglect of $e^2 s : a_0$, that is the application of formulae (10) on p. 285, letting s be $< 0{,}1 a_0$, and (as usual) assuming no high latitudes.

According to this, and under the assumptions mentioned shortly before, formulae:

$$\left.\begin{aligned} &B_2'' - B_2' - \xi_2 + \xi_1 \cos L_{1.2} + \eta_1 \sin L_{1.2} \sin B_1 + \cdots = 0 \\ &L_2'' - L_2' + \eta_2 \sec B_2 + \xi_1 \sin L_{1.2} \tan B_2 \\ &\qquad - \eta_1 (\sec B_1 + \sin L_{1.2} \tan B_1 \csc \alpha_{1.2} \cos \alpha_{2.1}) + \cdots = 0 \\ &\alpha_{2.1}'' - \alpha_{2.1}' - \eta_2 \tan B_2 - \xi_1 \sin L_{1.2} \sec B_2 \\ &\qquad + \eta_1 \cos L_{1.2} \sin B_1 \sec B_2 + \cdots = 0, \end{aligned}\right\} \qquad (9)$$

follow correctly, including terms of the order $e^2 s : a_0$ in the coefficients of ξ_1 and η_1. Here, B_2', L_2', $\alpha_{2.1}'$ denote the true values of the geographical latitude and longitude as well as the azimuth of the geodetic line at P_2. On the other hand, B_2'', L_2'', $\alpha_{2.1}''$ denote the values computed from P_1 by means of B_1', L_1', $\alpha_{1.2}'$, and s. Furthermore, it is to be taken that $L_{1.2} = L_2 - L_1$ in the coefficients.

In the coefficients of η_1 in the second equation (9), one can, instead of:

$$\sin L_{1.2} \tan B_1 \csc \alpha_{1.2} \cos \alpha_{2.1}$$

also write:

$$\sin \frac{s}{a_0} \tan B_1 \sec B_2 \cos \alpha_{2.1}.$$

If one equates ξ_1 and η_1 to zero, then:

$$\left.\begin{aligned} \xi_2 &= B_2'' - B_2' + \cdots \\ \eta_2 &= -(L_2'' - L_2') \cos B_2 + \cdots \\ \eta_2 &= (\alpha_{2.1}'' - \alpha_{2.1}') \cot B_2 + \cdots. \end{aligned}\right\} \qquad (9^*)$$

follow from equations (9).

Therefore, if the geographical longitudes as well as the azimuths are existent, one obtains the equation:

$$(\alpha''_{2.1} - \alpha'_{2.1}) + (L''_2 - L'_2)\sin B_2 + \cdots = 0, \qquad (10)$$

from the double determination of η_2. This condition equation (10) provides a control of the accuracy of the geodetic and astronomic operations. If one does not set ξ_1 and η_1 equal to zero, then the term on the left side of the equation is equal to the sum of the two fractions of η_1 and ξ_1 which, with reference to $s:a_0$, are of the first-order.

The third equation (9*) loses its usefulness near the equator because, here, the errors of the azimuth measurement enter into the value η_2 very enlarged. But equation (10) is here also practical and useful.

This equation is already mentioned by Laplace (see the end of this chapter), and it is also called: Laplace's Theorem.

Assuming that ξ_1 and η_1 are equal to zero for Munich, Carl von Orff, on pp. 55 and following, of the treatise: Bestimmung der geographischen Breite der Königl. Sternwarte bei München, (Determination of the Geographical Latitude of the Royal Observatory near Munich), published in the supplement of the 21st volume of the annals of the same observatory, computes deflections of the vertical of several points north and south of the Alps and, p. 57, obtains six control equations from the double determination of the value η:

	$L''-L'=$	$\alpha''-\alpha'=$	$(L''-L')\sin B=$
Mannheim	— 4,71"	+ 2,94"	— 3,58"
Pfänder	—19,98	+14,84	—14,78
Zürich	+ 0,71	— 0,1	+ 0,5
Rigi	— 2,16	— 2,31	— 1,6
Bern	+ 6,14	+ 0,4	+ 4,5
Mailand	+ 6,21	+ 0,2	+ 4,4

	$\alpha''-\alpha'+(L''-L')\sin B=$	Length of the Chains
Mannheim	— 0,64"	60 Miles
Pfänder	+ 0,11	22 „
Zürich	+ 0,4	38 „
Rigi	— 3,9	33 „
Bern	+ 4,9	48 „
Mailand	+ 4,6	52 „

Here, first, the polar coordinates were computed, and then, the formulae on p. 298 were used advantageously for the geodetic transfer of the latitude and longitude as well as for the transfer of the azimuth.

On p. 269 of Vol. 14 of the Astronom. Nachr. (Astronomical Reports), in the article: Über den Einfluss der Unregelmässigkeiten der Figur der Erde auf geodätische Arbeiten (Regarding the bearing of the irregularities of the figure of the earth on geodetic work) etc. (cf. Abhandl. Vol. 3 on pp. 19 et seq., especially p. 36) 1837, Bessel has developed the terms for the coefficients ξ_1 and η_1, without the application of the reduced length m, which is only possible by means of expansion in a series. However, his data are in the same way distorted as the others mentioned on p. 295; but, the same as those data on p. 295, without this distortion having

any significant influence on the results of the numerical computation.

Par. 12. The Accuracy of the Computed Relative Deflections of the Vertical. Let the values δs, $\delta\alpha'$, $\delta B'$, $\delta L'$ be improvements of the observed real values s, α, B and L; equations (10) with reference to (10*) on p. 285 immediately yield the following terms:

$$\cos L_{1.2}\,\delta B'_1 - \delta B'_2 + \varrho'' \cos\alpha_{2.1}\frac{\delta s}{a_0} + \sin L_{1.2}\cos B_1\,\delta\alpha'_{1.2} \qquad (1)$$

$$\left.\begin{aligned}&\delta L'_1 - \delta L'_2 + \sin L_{1.2}\tan B_2\,\delta B'_1\\ &-\varrho''\sec B_2\sin\alpha_{2.1}\frac{\delta s}{a_0} - \sin\frac{s}{a_0}\sec B_2\cos\alpha_{2.1}\,\delta\alpha'_{1.2}\end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned}&-\sin L_{1.2}\sec B_2\,\delta B'_1\\ &+\varrho''\tan B_2\sin\alpha_{2.1}\frac{\delta s}{a_0} + \cos L_{1.2}\cos B_1\sec B_2\,\delta\alpha'_{1.2} - \delta\alpha'_{2.1}.\end{aligned}\right\} \qquad (3)$$

which are to be introduced into the equations (6), (7), and (8) on p. 536, or into the left side of the three equations (9) in p. 536 respectively. Imagining that the values ξ_2 and η_2 are computed by the equations (9)

on p. 536, then the formula (1)states now, at the same time, the influence of the observation errors on ξ_2 in the sense of an improvement. The formulae $-(2)\cos B_2$ and $+(3)\cot B_2$ are, on the other hand, the corresponding influences on η_2. They are equal to:

$$\left.\begin{aligned}&(\delta L'_2 - \delta L'_1)\cos B_2 - \sin L_{1.2}\sin B_2\,\delta B'_1\\ &+\varrho''\sin\alpha_{2.1}\frac{\delta s}{a_0} + \sin\frac{s}{a_0}\cos\alpha_{2.1}\,\delta\alpha'_{1.2}\end{aligned}\right\} \qquad (2^*)$$

and:

Par. 12. The Accuracy of the Computed Relative Deflections of the Vertical.

$$\left.\begin{array}{l} -\sin L_{1.2}\csc B_2\,\delta B'_1 \\ +\varphi''\sin\alpha_{2.1}\dfrac{\delta s}{a_0}+\cos L_{1.2}\cos B_1\csc B_2\,\delta\alpha'_{1.2}-\cot B_2\,\delta\alpha'_{2.1}. \end{array}\right\} \quad (3^*)$$

respectively.

Furthermore, the influence on the left side of the control equation (10) on p. 537 is equal to:

$$\left.\begin{array}{l} (\delta L'_1-\delta L'_2)\sin B_2-\sin L_{1.2}\cos B_2\delta B'_1 \\ +\left(\cos L_{1.2}\cos B_1\sec B_2-\sin\dfrac{s}{a_0}\tan B_2\cos\alpha_{2.1}\right)\delta\alpha'_{1.2}-\delta\alpha'_{2.1}. \end{array}\right\} \quad (4)$$

This demonstrates that the control equation is not dependent on the error δs, and that, in addition, this control equation, near the equator, becomes free from $\delta L'$, and that, at least, this equation provides a good control of the azimuth transfer by means of the triangle net.

Regarding the determination of η_2, the expressions (2*) and (3*), of course, show, in the first place, again the uselessness of the azimuth measurements at the equator; but then these terms generally show a superiority of the geographical longitude determination over the azimuth measurements. For $\delta\alpha'_{1.2}$ in formula (2*) has, even at the maximum, only the factor $\sin\frac{s}{a_0}$, whereas the factor in formula (3*) can have any value. In formula (2*) however, the term of error $(\delta L'_2-\delta L'_1)\cos B_2$ still appears which has no analogy in formula (3*), but this fact does not considerably decrease the degree of accuracy of the determination of η_2, if the telegraphic determinations of the longitude according to the best methods are taken for granted. The mean error of a telegraphic determination

is in fact, not larger than that of an azimuth measurement.*)

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The value $\delta\alpha'_{1,2}$ is composed of the mean error of the azimuth measurements and the transfer errors from the triangle net, whereas the value $(\delta L'_2 - \delta L'_1)$ corresponds immediately to the mean error of the longitude difference. This part increases inasmuch as it is of an accidental character and according to formula (6*) on p. 531, in proportion to the square root of the distance, but insofar as it depends on constant causes and, according to formula (6) on p. 528, it increases directly with the distance.

The terms (1), (2*), and (3*) permit computation of the mean error in the determination of ξ_2 and η_2, if the mean errors in B', L', s and α' are given.

If we, for example, assume a chain of 20 triangles, connected with one another at 45° mean latitude, being of a length of 637 km at the azimuth of 45°, then, firstly, we obtain the mean values

$$\delta s = \pm 3{,}5''' \quad \text{and} \quad \delta w = \pm 0{,}7'',$$

according to formula (5*) on p. 530 and (6*) on p. 531 with

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*)According to the preliminary adjustment of some longitude determinations by Albrecht, on pp. 305 following, No. 2132, Vol. 89 of Astronom. Nachr. (Astronomical Reports), we find the value ± 0,75" as the mean error of the best determinations of longitude. However, after five double determinations of azimuth, the mean error of a double determination of azimuth by the Geodetic Institute (with two different instruments following two different methods) would be equal to ± 0,58". But these are extreme values which will be considerably reduced by a definite revision of the final results. We suppose these values will be reduced to ± 1/3" in both cases. However, after seven double determinations of the geographical latitude by the Geodetic Institute, the mean error of such a double determination is ± 0,29".

$\mu = \pm 0{,}5''$ and $i = 10$, if ω denotes the angle between the initial side and the long side of the net. If we furthermore assume as mean errors of the determinations of the latitude, longitude, and azimuth: ± 0,3", ± 0,5", and ± 0,5" respectively, then we obtain as mean values of:

$$\delta B_1' \text{ and } \delta B_2' = \pm 0{,}3'', \quad (\delta L_2' - \delta L_1') = \pm 0{,}5''$$
$$\delta \alpha_{1.2}' = \pm 0{,}5'' \pm 0{,}7'' = \pm 0{,}9''$$
$$\delta \alpha_{2.1}' - \delta \alpha_{1.2}' = \pm 0{,}5'' \pm 0{,}5'' \pm 1{,}0'' = \pm 1{,}2'',$$

whereby the third term of the last equation was computed according to formula (7*) on p. 531. The terms (1), (2*), and (3*) yield now:

$$\pm 0{,}3 \pm 0{,}3 \pm 0{,}7.0{,}11 \pm 0{,}07.0{,}9 = \pm 0{,}4'' \text{ for } \xi,$$
$$\pm 0{,}7.0{,}5 \pm 0{,}07.0{,}3 \pm 0{,}7.0{,}11 \pm 0{,}07.0{,}9 = \pm 0{,}4'' \text{ „ } \eta,$$
$$\pm 0{,}14.0{,}3 \pm 0{,}7.0{,}11 \pm 1{,}2 = \pm 1{,}2'' \text{ „ } \eta_T$$

The great superiority of the geographical longitude determinations is here outstandingly clear.

Yet the azimuth measurements deserve some attention because they are easier to carry out than measurements of geographical longitudes. In most cases where one deals with the computation of a larger number of astronomical stations, one will only be able to determine the latitude and the azimuth. But in order to prevent an accumulation of errors in the η values, also geographical longitudes are to be determined for

some larger distances. The azimuth measurements have then to serve as a means to interpolate the η values between the terminal points of larger distances.

The aspect of the above-mentioned error terms gives reason to remark that the determination of ξ and η from the geographical latitudes and longitudes can be very inaccurate because of the errors in s and α' at very considerable extensions. Besides, the error in ξ and η becomes considerably smaller than the above-mentioned terms state it, if several base lines are always measured at a large extension of the triangulation area, and if, at the adjustment of the net, this is taken into consideration as well as the control equation (10) on p. 537 which are to be created in a sufficient number.

In Vol. 90, No. 2144 and Vol. 91, No. 2170 of the Astronom. Nachr. (Astronomical Reports), Sadebeck states the influence of the deflection of the vertical on the measurement of the horizontal angles at the Brocken, (highest peak of the Hartz Mountains). For the computation of the deflection of the vertical at the Brocken, taking Seeberg near Gotha as a base, with $\xi_1 = 0$ and $\eta_1 = 0$, the following values are given:

astronomic	$B'_2 = 51°48'10{,}59''$	$L'_{1.2} = 26{,}757^s$	$\alpha'_{2.1} = 355°18'29{,}665''$
geodetic	$B''_2 = 1{,}41$	$L''_{1.2} = 26{,}986$	$\alpha''_{2.1} = 21{,}430$
It is, therefore, according to our method of computation	$\xi_2 = -9{,}18''$	$\eta_2 \sec B_2 = -3{,}435''$ $\eta_2 = -2{,}12''$	$\eta_2 \tan B_2 = -8{,}235''$ $\eta_2 = -6{,}48''$.

The error of the control equation amounts to 5,5", a

value which points to a gross error. Sadebeck applies an adjustment which is erroneous because it is based on two condition equations although the three determinations for two elements of the deflection of the vertical can only be subject to one condition. This is, of course, the control equation. Those two equations are brought about by the introduction of the direction of the deflection of the vertical given as a fixed value, in a hidden way, after a first computation.

An adjustment appears inconvenient because of the suspected gross error. But if one wants to form a mean value, then the simple arithmetical mean $\eta_2 = -4{,}3''$ suffices completely. Because after formulae (2*) and (3*) on p. 539, the most essential parts of the errors in the two values of η_2 are equal to:

$(\delta L'_2 - \delta L'_1)\cos B_2$ respectively $(\delta \alpha'_{1.2} - \delta \alpha'_{2.1})\cot B_2$

with

$$\cos B_2 = 0{,}62 \qquad \cot B_2 = 0{,}79 .$$

Under the assumption of equal mean values of $\delta L'_2 - \delta L'_1$ and $\delta \alpha'_{1.2} - \delta \alpha'_{2.1}$ the proportion of the weights is, therefore, equal to $0{,}79^2 : 0{,}62^2$. But, considering the complete uncertainty about the permissibility of that assumption, the inequality of the weights is, however, without importance.

Besides, the strong contradiction appears probably only because the value $\alpha''_{2.1}$ is not based on an azimuth measurement at Seeberg, but on an azimuth at Inselsberg (compare also p. 568). The second of the above-mentioned

values of η_s, therefore, would be very much influenced by the deflection of the vertical at Inselsberg.

Par. 13. The Method of Computation will have different forms in practice depending on the frequency of the astronomical stations and the completeness of the observations at the stations. Let us consider three cases a little more detailed.

In the first case, we assume that no equations for the determination of the deflection of the vertical result in excess, because at each astronomical station (except one of them) there is measured only: either the azimuth of a triangle side, or the longitude difference with reference to another astronomical station.

In the second case, we assume that the latitude and the azimuth, and partly also longitude differences are measured at all astronomical stations.

In the third case, we assume the same, but, along with it, we assume that all geodetic stations are astronomic stations, too.

In all three cases we assume that the determination of the longitude differences form a continuous net, and that the conditional equations which result from determinations of longitude differences in excess, are satisfied already by an adjustment and, therefore, can be considered as final ones.*) This procedure is completely permissible as long as errors of the azimuth transfer dominate in the control equations of the net, formula (10) on

*)Regarding this, among others, compare Albrecht's publication, mentioned on p. 539.

p. 537, so that the correction of the geographical longitude values would be a very dubious advantage. But, at present, this is the case in all arc measurement nets because the number of the stations with measurements of longitude and azimuth will always remain relatively small because of the difficulty of longitude determinations.

The latitude determinations can also be subject to a separate adjustment, perhaps advantageously, as it happened several times. Beside the geographical latitudes, the declination of the stars (and at α Ursae Minoris as well as the polar stars in general, also the right ascensions) are here considered as unknown, and all is computed by means of a common adjustment.**)

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Then, firstly, one obtains only the latitude differences, but very accurate at that, and then one can also compute the geographical latitudes themselves, retaining rigorously the values of the differences. However, this proceeding will be advantageous only if common stars in a sufficient number are observed, according to the same method, at neighboring stations, so that the latitude

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**)W. Struve, in the arc measurement of the Russian-Baltic provinces, and James, in the English arc measurement, proceeded in this manner (cf. the main publication of the English land survey which is often mentioned). Very precise star locations with special reference to the purposes of arc measurement having been derived in recent years, this proceeding may perhaps be modified in the sense that one takes the star locations as given, but that one tries rigorously to eliminate the instrumental errors which can hardly be eliminated completely at the single station. This is accomplished by a nearest possible proceeding at the latitude determination at the different stations.

differences are freed from declinations and instrumental errors. But the very large deviations of the determinations of the same latitudes following different methods demonstrate that it is necessary to increase the accuracy of the latitude differences (which, evidently, interest the most).

Par. 14. First Case of the Practical Determination of the Deflection of the Vertical. Formulae (6) to (8) or (9) on p. 536 permit the computation of ξ and η for all astronomical stations if the formulae for a first station P_1 are left indeterminate. As such we consider as fixed a station which exists within the net of the longitude determinations, but for which also the geographical latitude and an azimuth are measured.

Lastly we also assume that the triangle net be on the whole adjusted and, therefore, without discrepancies; that, furthermore, polar coordinates be computed from P_1 to all astronomical stations, that means the radius vectors as well as the angles between them and the following triangle sides in their terminal points being computed. As the azimuth of a triangle side is directly observed at P_1, one can immediately tell the values $\alpha'_{1.i}$ for any station P_i. The value $\alpha'_{i.1}$ is obtained from the azimuth of a triangle side observed at P_i, and the computed angles.

Then one can turn to B''_i, L''_i, and $\alpha''_{i.1}$ from $s_{1.i}$ and $\alpha'_{1.i}$, according to formulae given before. One obtains an equation involving ξ_i and an equation involving η_i, for every point $P_2, P_3 \ldots P_i \ldots$, where the latitude and the longitude, or the azimuth are measured. These equations can be reduced to the following, symbolic expressions:

Par. 14. First Case of the Practical Determination of the Deflection of the Vertical.

$$\left.\begin{aligned} \xi_i &= -l_i + a_i\xi_1 + b_i\eta_1 \\ \eta_i &= -l'_i + a'_i\xi_1 + b'_i\eta_1 \end{aligned}\right\}. \qquad (1)$$

Herein, l_i and l'_i denote values of seconds of arc, and a_i and b_i are the coefficients. This determination of ξ_i and η_i is evidently only a just sufficient one without any control, so that all observation errors and the reduction

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error of the azimuths enter completely into the values ξ_i and η_i. A usefulness of the longitude determinations for the improvement of the η values from the azimuths is not evident because twice-determined η values can not be compared anywhere.

In conclusion, in order to obtain definite values of ξ and η, an assumption has to be made for ξ_1 and η_1, that means, the reference ellipsoid has to be brought into a certain position with reference to the direction of the vertical at P_1:

Choosing ξ_1 and η_1, it is best to proceed so that the sum of $\xi_i^2 + \eta_i^2$ reaches a minimum with reference to ξ_1 and η_1 as unknowns (variables). This choice is at least then a very convenient one when the astronomical stations are uniformly distributed across a part of the surface of geoid.

The adjustment is according to the arranged observations and without any difficulties. Denoting ξ_1 and η_1, inasmuch as they are unknowns, as x and y, then the error equations of the errors $\xi_1, \xi_2 \cdots \eta_1, \eta_2 \cdots$ read in symbolic form:

$$\left.\begin{array}{ll} \xi_1 = \ldots + x \ldots & \eta_1 = \ldots \ldots + y \\ \xi_2 = -l_2 + a_2 x + b_2 y & \eta_2 = -l'_2 + a'_2 x + b'_2 y \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots, \end{array}\right\} \qquad (2)$$

which are to be further dealt with, according to the adjustment computation on pp. 115 et seq.

Here we remark only that the two systems of error equations and all single equations have to be considered as having the same weight as long as the errors of observation and reduction can not, according to the probability calculations, increase so much that the values of ξ and η deviate from their true values in amounts which are of the order of these values, that means amounts of several seconds of arc. According to Par. 12, p. 540, this will occur especially in azimuth measurements.

Regarding the computation of B''_i, L''_i and $\alpha''_{i.1}$ under the above-mentioned assumption of the use of polar coordinates, it must still be pointed out that the computation can also be carried out without polar coordinates. It is sufficient and, under circumstances more convenient, to transfer the latitude and the longitude as well as the azimuth from point to point. This transfer may be carried out either by means of the triangle sides themselves, or by inserted auxiliary lines which perhaps connect the neighboring astronomical stations. Of course, this method renders the same results as the other above-mentioned computation if it is wholly based on the triangle net and on the astronomical observation values obtained for P_1, and if another observed azimuth for further transfer

is nowhere applied. But, regarding the form, there exists the difference that the value $\alpha''_{i.k}$ is not the azimuth of one point P_i but of the last line used for the transfer which here, temporarily, shall be denoted by $\alpha''_{i.k}$. The azimuth observation in P_i, utilizing the angles of the net, provides here directly a value $\alpha'_{i.k}$, which can immediately be compared with $\alpha''_{i.k}$.

In the equation which corresponds to the azimuth measurement, we must now, at this proceeding, use $\alpha''_{i.k} - \alpha'_{i.1}$ instead of $\alpha''_{i.1} - \alpha'_{i.1}$ Both values coincide with each other because the differences $\alpha''_{i.1} - \alpha''_{i.k}$ and $\alpha'_{i.1} - \alpha'_{i.k}$ designate the same angle value which is easily found with reference to the computation method for $\alpha'_{i.1}$ mentioned on p. 543. (A difference appears only if the net has a different orientation for the different computation, and this causes also a difference between the angles. But, as can be seen in Par. 19 on p. 405, this difference becomes insignificantly small, because the resulting differences corresponding to the two orientations, as a rule, are so insignificant at the geographical latitudes of the points that they remain without influence on the reduction terms which are computed at the application of Legendre's extended theorem).

If one applies the lately discussed, successive transfer instead of the polar coordinates, then the values $\alpha_{1.i}$ and $\alpha_{i.1}$, as well as eventually $s_{1.i}$ are to be computed for the differential formulae because these magnitudes can not be missed without consequences in those formulae. Of course, a rather superficial com-

putation with a few decimals, which will apply the formulae of Par. 23, on p. 313, in a reduced form, is here sufficient.

The survey of Great Britain and Ireland is an example of the above-mentioned problem, cf. the main publication regarding the survey: Ordnance Trigonometrical Survey, Principal Triangulation pp. 686 et seq. Here, use is made of the latitude of 35 stations, the longitude-difference between five stations and Greenwich, and the azimuths of 35 stations which are not identical with longitude stations, and which do not all coincide with the latitude stations. The computations are based on Airy's elements for the earth's ellipsoid (of 1830).

However, one did not stop at that but added terms to the equations for ξ and η, regarding an increase of a_0 and e^2, and then determined the minimum of $\xi_i^2 + \eta_i^2$ with reference to x, y and the two unknown increases of the elements of the meridian ellipse. Several adjustments following different points of view are carried out with reference to the weight assumption.

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This excellent publication which is most commendable for study is also interesting because the deflections of the vertical of a larger number of stations were computed from evident mass-irregularities in the vicinity of the stations. The comparison (to which we will refer in the second volume) shows that the considered deflection of the vertical against the reference ellipsoid computed as mentioned above, can indeed be explained to a large

extent by the local causes (p. 699; cf. p. 700 in order to know to which extent one can utilize local causes).*)

Some of the mentioned adjustment computations are carried out after a previous correction because of a local attraction. Finally, an ellipsoid is chosen (pp. 710-713), which has the flattening of 1 : 280,4 as the most plausible value which is taken as a base for the definite land surveying as well as for the geographical positions to be retained (p. 717).

It would be worth while to look up the details.

Par. 15. Second Case of a Practical Determination of the Deflection of the Vertical. This case differs from the former one in the determination of η for some points being done twice, computed from the longitude difference as well as from the azimuth measurement. If we now assume the azimuth as being very well determined, and the mutual distances between the points introduced into the net of the longitude determinations as not being particularly small, so that the azimuth differences become considerably more inaccurate than the longitude differences, then it appears suitable, and at the same time convenient, to retain the value η from the longitude determination.

In the following, we disregard an improvement of the geographical latitudes and longitudes and improve only the values of the azimuths and the elements of the geodetic net.

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*)With reference to this see also the publication: Die bayerische Landesvermessung (The Bavarian Land Survey).

In the beginning, the computation method can be followed the same as in the first case. But then, some difficulty arises at the utilization of the controls for the azimuth measurements provided by the double determination of η made at some points.

Strictly speaking, the controls are to be taken into consideration already in the net adjustment. If this, as far as we know, has not been done anywhere, then the main reason is probably the inaccuracy of the determination of the geographical longitudes up to the present time. In fact, because of this inaccuracy, the controls had hardly any significance for the angles but solely for those longitude determinations.

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Furthermore, the control equations under consideration take a somewhat unusual form, as can be seen if one does not set ξ_1 and η_1 equal to zero in the derivation of equation (10) on p. 537 but retains them as unknown magnitudes. Then, two new terms with ξ_1 and η_1 appear in the modified equation (10). The coefficients of these two magnitudes are however, as a rule, small, but it is undoubtedly necessary to take them into consideration which is to be done in such a way that all results of the adjustment computation are rendered in the form of linear functions of ξ_1 and η_1 .

The herewith produced complication of the computation in connection with the difficulty of correctly introducing the relative accuracies of the longitude determinations and the observations of azimuths and angles, may also be at present considered as

solid reasons not at first to mix the geodetic net adjustment with those controls.

Disregarding a direct and rigorous treatment by imagining the net adjustment made as usual, then the problem of how to utilize those controls afterwards remains to be solved.

A method of approximation is the following: As only the astronomical stations are of interest, we substitute for the primary triangle net an ideal one which connects neighboring astronomical stations, so that triangles and polygons are formed.

In this ideal net, we do not only take into consideration the directions, i.e. the azimuths, except a constant which is characteristic for every station, but also, as independent observed quantities, the sides, the weights of which are estimated according to the side length (Par. 8 on pp. 528 et seq., hints on this matter).

The choice of the ideal net is more or less arbitrary but not indifferent. In order not to sacrifice any rigorosity, one should connect the astronomical points N_a by an open tracing of $N_a - 1$ lines, and then one should introduce these lines as well as the $N_a - 2$ intermediate angles as results of the geodetic net adjustment into the general net adjustment.

Herewith, however, the $2N_a - 3$ values should not be taken as independent, but a system of $2N_a - 3$ functions of them should be formed which should be equivalent to the result of the geodetic net adjustment, and which should be introduced primarily as independent magnitudes into the general adjustment.

The formation of such equivalent functions is shown generally

in Par. 27 on pp. 222 et seq. of our adjustment computation. However, it is disadvantageous to want a rigorous adjustment in this way, because it is too troublesome. But, being solely an approximation,

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the discussion just presented shows advantageously that one has to strive to obtain a system of observations in the ideal net which is, as much as possible, equivalent to the result of the geodetic net adjustment.

Assuming more than $2N_e - 3$ values in the ideal net, it will be more feasible than with only $2N_e - 3$, to choose them in such a way as to consider them independent. Generally it is perceivable that it is advantageous if the lines of the ideal net cover the primary net as uniformly as possible, and that the lines are to avoid places where the primary net does not provide a direct connection. Introducing not only angles but also linear elements (lengths) into the ideal net as assumed observations, we have the advantage that we are not only dependent on well-formed triangles and on triangles in general, but that we can also utilize polygons.

For the practical utilization of the above-mentioned, it will be necessary to study the most convenient choice of ideal nets in a number of cases. Here, this would lead too far. But we hope to return to this study later on and, at present, it may only be remarked that the study may be carried out

with plane figures, using to the best advantage rectangular coordinates. In a similar way as is mentioned in Par. 10, on pp. 495 et seq., one has to establish the normal equations for the coordinates of the net points and then eliminate those of the non-astronomical points. Following, the same system is to be derived approximately by means of an ideal net.

Let us assume, now, any line P_iP_k of the ideal net and imagine the values B_k'', L_k'', and α_{ki}'' computed from P_i by means of $s_{ik}, \alpha_{ik}', B_i'$ and L_i'. If we now designate the improved values of s_{ik}, α_{ik}' and α_{ki}' with the symbols $\delta s_{ik}, \delta\alpha_{ik}'$ and $\delta\alpha_{ki}'$, then, with all terms reduced to seconds, we obtain the following three equations (cf. Pars. 11 and 12 on pp. 534 et seq.):

$$\left.\begin{aligned} B_k'' - B_k' - \xi_k &+ p_1\xi_i + p_2\eta_i + p_3\delta s_{ik} + p_4\delta\alpha_{ik}' + \cdots = 0 \\ L_k'' - L_k' + \eta_k \sec B_k &+ q_1\xi_i + q_2\eta_i + q_3\delta s_{ik} + q_4\delta\alpha_{ik}' + \cdots = 0 \\ \alpha_{ki}'' - \alpha_{ki}' - \eta_k \tan B_k &+ r_1\xi_i + r_2\eta_i + r_3\delta s_{ik} + r_4\delta\alpha_{ik}' - \delta\alpha_{ki}' + \cdots = 0. \end{aligned}\right\} \quad (1)$$

Here, the coefficients p, q, r have the following approximate values:

$$\left.\begin{aligned} p_1 &= \cos L_{ik} & q_1 &= \sin L_{ik} \tan B_k \\ p_2 &= \sin L_{ik} \sin B_i & q_2 &= -\left\{\begin{matrix}\sec B_i \\ + \sin\frac{s_{ik}}{a_0} \tan B_i \sec B_k \cos\alpha_{ki}\end{matrix}\right\} \\ p_3 &= \frac{\varrho''}{a_0}\cos\alpha_{ki} & q_3 &= -\frac{\varrho''}{a_0}\sec B_k \sin\alpha_{ki} \\ p_4 &= -\sin\frac{s_{ik}}{a_0}\sin\alpha_{ki}; & q_4 &= -\sin\frac{s_{ik}}{a_0}\sec B_k\cos\alpha_{ki} \\ &= \sin L_{ik}\cos B_i & & \end{aligned}\right\} \quad (2)$$

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned} r_1 &= -\sin L_{ik}\sec B_k \\ r_2 &= \cos L_{ik}\sin B_i \sec B_k \\ r_3 &= \frac{\varrho''}{a_0}\tan B_k \sin \alpha_{ki} \\ r_4 &= \cos\frac{s_{ik}}{a_0} + \sin\frac{s_{ik}}{a_0}\tan B_k \cos\alpha_{ki} \\ &= \cos L_{ik}\cos B_i \sec B_k . \end{aligned}\right\}$$ belonging to (2)

However, these coefficients read in a rigorous form according to formulae (6) and (8) on p. 536 as well as (6) on p. 280 as follows:

$$\left.\begin{aligned} p_1 &= \frac{W_k^3}{W_i^3}\left\{\cos\Delta\alpha + \left[1-\left(\frac{dm}{ds}\right)_{ki}\right]\sin\alpha_{ik}\sin\alpha_{ki}\right\} \\ q_1 &= -\frac{(1-e^2)W_k}{W_i^3}\left\{\sin\Delta\alpha - \left[1-\left(\frac{dm}{ds}\right)_{ki}\right]\sin\alpha_{ik}\cos\alpha_{ki}\right\}\sec B_k \\ r_1 &= -\frac{a_0}{m_{ik}}\frac{1-e^2}{W_i^3}\left\{\begin{aligned}&\left[1-\left(\frac{dm}{ds}\right)_{ik}+1-\left(\frac{dm}{ds}\right)_{ki}\right]\cos\Delta\alpha\sin\alpha_{ki} - \frac{\sin^2\Delta\alpha}{\sin\alpha_{ik}} \\ &+\left[1-\left(\frac{dm}{ds}\right)_{ik}\right]\left[1-\left(\frac{dm}{ds}\right)_{ki}\right]\sin^2\alpha_{ki}\sin\alpha_{ik}\end{aligned}\right\} \\ p_2 &= -\frac{m_{ik}}{a_0}\frac{W_k^3}{1-e^2}\tan B_i \sin\alpha_{ki} \\ q_2 &= -\left\{1+\frac{m_{ik}}{a_0}W_k \sin B_i \sec B_k \cos\alpha_{ki}\right\}\sec B_i \\ r_2 &= \left\{\left(\frac{dm}{ds}\right)_{ik}+\frac{m_{ik}}{a_0}W_k\tan B_k\cos\alpha_{ki}\right\}\tan B_i \\ p_3 &= \frac{\varrho''}{a_0}\frac{W_k^3}{1-e^2}\cos\alpha_{ki} \qquad p_4 = -\frac{m_{ik}}{a_0}\frac{W_k^3}{1-e^2}\sin\alpha_{ki} \\ q_3 &= -\frac{\varrho''}{a_0}W_k\sec B_k\sin\alpha_{ki} \qquad q_4 = -\frac{m_{ik}}{a_0}W_k\sec B_k\cos\alpha_{ki} \\ r_3 &= \frac{\varrho''}{a_0}W_k\tan B_k\sin\alpha_{ki} \qquad r_4 = \left(\frac{dm}{ds}\right)_{ik}+\frac{m_{ik}}{a_0}W_k\tan B_k\cos\alpha_{ki} . \end{aligned}\right\}$$ (3)

Here $\Delta\alpha = \alpha_{ki} - \alpha_{ik} - 180^0$ and $W_k = \sqrt{1-e^2\sin^2 B_k}$; cf. Par. 11 p. 283 for the rest.

If the ideal net contains Q astronomical points and among them, except point P_1, A points with a geographical longitude

determination, and if there are furthermore assumed R connection lines, each of them with three equations (1) established, then it must first be considered that $2(Q-1)$ components ξ and η, of the deflection of the vertical occur as unknown values, except for ξ_1 and η_1, which remain indeterminate, and which depend on the definition of the reference ellipsoid. Furthermore

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$Q-A-1$ geographical longitudes L', are unknown, which can be introduced only symbolically into the corresponding equations (1). (If, for instance, L'_i is unknown, then L''_k appears in the form $L''_k = L'_i +$ a numerical value).

If one eliminates these $3(Q-1)-A$ unknowns then:

$$3(R-Q+1)+A \qquad (4)$$

conditional equations between the values δs and $\delta\alpha$ remain, which now permit derivation of the values δs and $\delta\alpha$ as functions of ξ_1 and η_1.

If there would not be any geographical longitude determination (or more generally speaking: if there were not any astronomical point determined by its azimuth together with its geographical longitude), then one would have only $3(R-Q+1)$ equations which should be satisfied without any contradiction because they combine only terms of the already-adjusted net. But the polygon equations are joined by A equations which correspond to the control equation (10) on p. 537.

Based on these condition equations, the adjustment is now

carried out in a way that the sum:

$$g_s \delta s^2 + g_v v^2 + g_u u^2 \qquad (5)$$

which is to be formed for all lines of the ideal net, becomes a minimum, whereat the g symbols designate weights, and whereat the u symbol corresponds to the error of the astronomical azimuth determination at the single stations, whereas v denotes solely the error of the geodetic direction value.*)

Correspondingly, we take $\delta\alpha' = v - u$ in the equations (1); u remains constant for all directions from a station.

Instead of (5) one better sets for the numerical computation:

$$g\left(\frac{\varrho''}{a_0}\delta s\right)^2 + g'\left(\frac{m}{a_0}v\right)^2 + g''\left(\frac{m}{a_0}u\right)^2, \qquad (6)$$

where now g, g' and g'' respectively are taken inversely proportional to the mean values of the three squares

$$\left(\frac{\varrho''}{a_0}\delta s\right)^2 \qquad \left(\frac{m}{a_0}v\right)^2 \qquad \left(\frac{m}{a_0}u\right)^2$$

(m can be replaced by s without any damage to these values).

Now, one has the advantage of dealing with error magnitudes which

*)If n longer direction lines of which the mutual positions were computed, start from a station, then the result consists of n numbers the differences of which correspond to the mutual position of the directions. These geodetic direction values become astronomical values, i.e., azimuths, by addition of the orientation error on the base of an azimuth determination. But as this determination does not refer directly to any of the directions, so it seems the most adequate to split the value $\delta\alpha'$ for all directions into $v - u$, and not simply to omit v even in one direction, an omission which is, above all, by no means permissible.

are reduced to the same scale unit.

If the values of δs and $\delta\alpha'$ are computed, then the values of ξ and η result from equations (1). Further treatment can be continued according to Par. 14, pp. 543 et seq.

Par. 16. Case 2, Continuation: Approximate Adjustment by Means of Adjusted Observations. Because of the smaller number of equations, as well as for achievement of greater clarity, the approximate adjustment by means of adjusted observations is highly recommended.

We determine first of all, somehow through exact calculations, a system of correlated values of the B, L, s and α, derived for all astronomical stations; and we denote these values for instance for the line P_iP_k as

$$\bar{B}_i\ \bar{B}_k\ \bar{L}_i\ \bar{L}_k\ \bar{s}_{ik}\ \bar{\alpha}_{ik}\ \bar{\alpha}_{ki}\ .$$

By use of (1) p. 535 we obtain for the correction of these values (thus bringing them to their most favorable ellipsoid value):

$$\left.\begin{aligned}
dB_i &= B'_i + \xi_i - \bar{B}_i + \cdots \\
dL_i &= L'_i - \eta_i \sec B_i - \bar{L}_i + \cdots \\
d\alpha_{ik} &= \alpha'_{ik} + \delta\alpha'_{ik} + \eta_i \tan B_i - \bar{\alpha}_{ik} + \cdots \\
ds_{ik} &= s_{ik} + \delta s_{ik} - \bar{s}_{ik} + \cdots \\
dB_k &= B'_k + \xi_k - \bar{B}_k + \cdots \\
dL_k &= L'_k - \eta_k \sec B_k - \bar{L}_k + \cdots \\
d\alpha_{ki} &= \alpha'_{ki} + \delta\alpha'_{ki} + \eta_k \tan B_k - \bar{\alpha}_{ki} + \cdots .
\end{aligned}\right\} \qquad (1)$$

Now, either we may introduce these $dB, dL, d\alpha$ and ds instead of $\delta B, \delta L, \delta\alpha$ and δs in equations (4) p. 282, wherein only the indices i and k respectively, ought to be applied for one and two or, which is preferable, we retain the dB and dL as unknown quantities (instead of ξ and η in the other case).

Whilst we follow this path, we have to eliminate the quantity η from the expressions for $d\alpha$. With reference to (1) we therefore state

$$\left.\begin{aligned} d\alpha_{ik} &= \alpha'_{ik} - \bar{\alpha}_{ik} + \delta\alpha'_{ik} + (L'_i - \bar{L}_i - dL_i)\sin B_i + \cdots \\ d\alpha_{ki} &= \alpha'_{ki} - \bar{\alpha}_{ki} + \delta\alpha'_{ki} + (L'_k - \bar{L}_k - dL_k)\sin B_k + \cdots \end{aligned}\right\} \quad (2)$$

Further on we observe, that, with regard to equation (5) p. 275 it may be stated thus:

$$\left.\begin{aligned} \frac{m_{ik}}{a_0}\sin B_k + \frac{\cos B_i}{W_i}\cos\alpha_{ik} &= -\frac{\cos B_k}{W_k}\left(\frac{dm}{ds}\right)_{ik}\cos\alpha_{ki} \\ \frac{m_{ik}}{a_0}\sin B_i + \frac{\cos B_k}{W_k}\cos\alpha_{ki} &= -\frac{\cos B_i}{W_i}\left(\frac{dm}{ds}\right)_{ki}\cos\alpha_{ik}, \end{aligned}\right\} \quad (3)$$

by which we may essentially simplify the dL coefficients.

By taking dB, dL and $d\alpha'$ in seconds, (4) p. 282 changes, after a few more simple transformations, into:

$$\left.\begin{aligned} \frac{\varrho''}{a_0}\delta s_{ik} &= \left\{\begin{aligned} &\frac{\varrho''}{a_0}(\bar{s}_{ik} - s_{ik}) + \frac{1-e^2}{W_i^3}\cos\alpha_{ik}dB_i + \frac{1-e^2}{W_k^3}\cos\alpha_{ki}dB_k \\ &+ \frac{\cos B_k}{W_k}\sin\alpha_{ki}(dL_i - dL_k) + \cdots \end{aligned}\right\} \\ \frac{m_{ik}}{a_0}\delta\alpha'_{ik} &= \left\{\begin{aligned} &\frac{m_{ik}}{a_0}\left(\bar{\alpha}_{ik} - \alpha'_{ik} + (\bar{L}_i - L'_i)\sin B_i\right) \\ &- \frac{1-e^2}{W_i^2}\left(\frac{dm}{ds}\right)_{ki}\sin\alpha_{ik}dB_i - \frac{1-e^2}{W_k^3}\sin\alpha_{ki}dB_k \\ &- \frac{\cos B_i}{W_i}\left(\frac{dm}{ds}\right)_{ki}\cos\alpha_{ik}dL_i - \frac{\cos B_k}{W_k}\cos\alpha_{ki}dL_k + \cdots \end{aligned}\right\} \\ \frac{m_{ik}}{a_0}\delta\alpha'_{ki} &= \left\{\begin{aligned} &\frac{m_{ik}}{a_0}\left(\bar{\alpha}_{ki} - \alpha'_{ki} + (\bar{L}_k - L'_k)\sin B_k\right) \\ &- \frac{1-e^2}{W_i^3}\sin\alpha_{ik}dB_i - \frac{1-e^2}{W_k^3}\left(\frac{dm}{ds}\right)_{ik}\sin\alpha_{ki}dB_k \\ &- \frac{\cos B_i}{W_i}\cos\alpha_{ik}dL_i - \frac{\cos B_k}{W_k}\left(\frac{dm}{ds}\right)_{ik}\cos\alpha_{ki}dL_k + \cdots \end{aligned}\right\} \end{aligned}\right\} \quad (4)$$

These equations are to be established for each line; then $\delta\alpha'$ has to be split into $v - u$ and u to be brought over to the right. Along with the error equations for δs and v there is now, so that we can use the known adjustment formulae, for each station an identical equation:

$$u = u, \qquad (5)$$

u is naturally different for each station. On the right side u means an unknown quantity, on the left side a correction.

According to expression (6) p. 550, normal equations now have to be established for the unknown quantities dB and dL, except dB_1 and dL_1, which remain as unknown quantities in the calculations; further on for the u of all the Q stations and finally for the L' of the $Q - A - 1$ in geographic longitude undertermined stations; together $4Q - A - 3$ equations.

From these it is easy to eliminate those for u and L'. When dB and dL or L' respectively have been found, then

equations (1) and the observed B' and L' give ξ and η as functions of ξ_1 and η_1 (dL_1 respectively).

Par. 17. Modification of Case 2 for the European Triangulation Net. The European triangulation net consists of many separately-adjusted geodetic systems, some of which have an essentially linear form as triangulation chains, and others which have an aerial extent in the form of triangulation nets. While we, as in the previous case, substitute ideal nets for the latter, which must

have, in additon to astronomical stations, connecting points with neighboring units; we should regard the first-mentioned one simply as a broken line, which connects the astronomical stations and to which the connecting sides are joined directly or through an auxiliary line.

In general therefore, when adjusting by use of conditional observations, for each line the three-equation system (1) p. 548 has to be used and as on p. 550 $\delta\alpha'$ has to be split into $v - u$. For the non-astronomical stations the members multiplied by ξ and η vanish, since they are indeterminate; on the other hand instead of the real values B', L' and α' for these stations the unknown ellipsoidal values B, L and α are substituted (simply by omitting the upper index). The development in Par. 11, pp. 534 et seq. demonstrates without doubt that this is so.

Regarding the connecting sides, the above-mentioned calculations apply only to the nets, while in the chains they generally do not apply, while we presume that they connect a broken path of lines unilaterally. (The calculation of three equations is only required for those connecting sides, which join on both sides to lines of the same system).

The unilaterally attached connecting sides contribute however, two equations to the appropriate system, as follows:

$$\left.\begin{aligned} \text{Side } pq &= \text{Num. value} + \delta s_{pq} \\ \alpha_{pq} - \alpha_{po} &= \text{Num. value} + v_{pq} - v_{po}, \end{aligned}\right\} \qquad (1)$$

where the connecting side is indicated by pq and the side of the line path which is connected is indicated by po. δs_{pq} indicates

the correction to the linear length; furthermore v_{p_1} and v_{p_2} indicate corrections to the preliminary geodetic directions. Obviously δs_{p_1} and v_{p_1} receive distinguishing indices in adjacent systems.

When all these equations are established, then as in the previous

case (p. 550) the unknown quantities are eliminated, so that condition equations will be obtained between δs and $\delta\alpha' = v - u$.

ξ and η appear as unknowns of the Q astronomical stations of the two values ξ_1 and η_1 (of an arbitrarily chosen station), which remain as unknown quantity; also the $Q-A-1$ geographic longitudes of those astronomical stations, which are determined only in latitude and azimuth, and the B, L and α on the non-astronomical stations; finally yet the connecting sides.

For those chains (nets), which have no base line, an additional unknown quantity appears for each in the correction δs of the linear line length; thus because of the provisional calculations of these dimensions, members in the form of results of an unknown quantity have to be attached to the ordinary symbol δs in the linear length.

Now before we are able to proceed to the adjustment according to conditions (5) or (6) p. 550 it should be considered, whether parts of a simple plotted line, which are replacing the triangulation chains, should be regarded as independent or not. In the first case the control points (1) do not achieve, any influence on the chains, through the connecting sides, since there is no connec-

tion between the δs of these sides and the corrections of the other parts.

However it is evident, that a change in the length of a connecting side necessitates also changes in length and azimuth of the other lines; therefore when for the complex of two attached chains, the opposite signs at the corrections of connecting sides do not cause always an essential change in length in both chains, the change still remains in particular. Azimuth changes will also occur.

An approximate consideration of the connecting side influences would appear itself, if we would also presume the construction of a closed ideal net for the chains. Meanwhile we regard it as simpler to stop here at the simple line plotting as a rule; we imagine, however, an (at least approximate) equivalent system of the correction functions introduced, for the parts as it has been already mentioned on p. 547.

Instead of δs and v (the correction of geodetic direction data, p. 550) linear functions of the same values appear, which we indicate by δl. The derivation of the functional connection between δl and δs and v

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results also in a determination of the mean square belonging to δl and with this of its weight g.

In the condition equations the δs and v are to be eliminated with the δl and finally the adjustment should be made in such a

way that the total of the δl squares multiplied by their weight and the remaining corrections (considered as independent) cf. p. 550 (5) respectively (6) become a minimum.

We hope to be able to deal with the detailed explanations to the above more fully, especially the construction of equivalent systems, somewhere else. Here we refer only to the remarks shown on p. 548 and we remind only that a satisfactory solution for chains will be achieved relatively easily, if the calculation of the mean error of a few main results is attached to the special adjustment. Then simpler chains from equilateral triangles with nearly the same accuracy of the relative results may be substituted for the given chains for the purpose of equivalent system construction.

In the treatment of the present case according to adjusted observations, which is to be recommended because of its clarity one has to proceed for each ideal grid (each path of a line) as in Par. 16, p. 551 ff. At the non-astronomical stations the members with ξ and η disappear and instead of B', L' and α' the ellipsoid values of B, L and α are to be placed in the equations.

These α are known, except for one value u common to all directions of a station, so that $v - u$ is substituted for $\delta\alpha$. But since no observations exist for these u, so for the non-astronomical stations the error equations

$$u = u,$$

are not applicable cf. p. 552. At the non-astronomical stations the error equations (4) p. 552 for $\delta\alpha_{ik}$ take after some reductions

at the right hand side the simpler form (3) p. 496.

The following unknown quantities appear (cf. p. 552 end of Par. 16) ξ and η of the $Q-1$ astronomical stations, the $Q-A-1$ values L' of stations not determined in geographic longitude, the B and L of the non-astronomical stations, and finally the u of all stations and the unknown quantities which result from lack of base measurements.

Before we proceed to the formation of normal equations we have to insert the expressions for the δs and v in the equivalent functions, which are indicated with δl at the end of the previous page; as far as these δs and v have not been already considered as independent values.

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The great clarity of the procedure of adjustment by adjusted observations is based upon the fact, that for each net (each chain) the normal equations can be set up separately and those unknown quantities which are appearing only in the relative section may be eliminated and therefore are determinable.

Essentially only the unknowns appearing in at least two nets (chains) transform into the main system of the normal equations. They are formed for each unknown quantity by addition of the modified relative normal equations of the particular parts. Thus, the smallest number of unknown quantities are used for mutual adjustment.

Par. 18. Strict Compensation of the European Triangulation Grid. This may take place, if practically feasible, on the basis of adjusted observations, since the number of condition equations, even for single grids and chains is often only bordering on clearness. This is improved very little, however, when, neglecting Andrae's strictness and ignoring all diagonals*), we substitute into chains imaginary ones, which are as simple as possible.

We imagine therefore the adjustment as carried out for each partial net (for each chain) according to p. 495 et seq. If not originally, then subsequently, after the already made conditional adjustment, in which case the absolute terms disappear, provided we begin with approximate values which are satisfactory for the results of these adjustments. (Especially in the latter case we have to be satisfied with coefficients of few figures for the unknown values).

At the construction of the general normal equations of each partial net the improvements of the geographic coordinates will be suppressed for no point, (as those of the astronomical starting points), but normal equations will be constructed for all points. We presume also that the base measurements and azimuth determination (as all astronomical determinations), will stay disregarded at first.

From these normal equations one will eliminate first of all the coordinate adjustments for these points, which do not contain

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*) Compare p. 244, about Verhandlungen der 5. allgemeinen Konferenz der Europäischen Gradmessung (Negotiations of the 5. General Conference of European Degree Measurement, Berlin 1878.
As soon as chains combine into polygons the compensation can not be completed as simply as indicated here. Also the checks on geographic longitudes and azimuths, which we mentioned several times, have not been taken into account (not even mentioned).

astronomical determinations, exclusive of those of the connecting points

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of adjacent nets, as well as end points of bases and target points for astronomical azimuth determinations.

If such a target point is a local marker, then the adjustment in the direction towards it (according to *C* p. 501) should be preserved in the normal equation system.

The absolute members of these normal equations disappear as soon as we consider further adjustments in the unknowns, which are to be applied to the results of a separate adjustment of this partial net. (At this separate dissolution four of the unknowns would have to equal zero or be assumed arbitrarily, when as assumed we do not introduce the base and azimuth measurements).

For the combination of the above modified normal equations with the different partial nets, the unknowns appearing in them must refer, as far as they are of the same value, to corrections of the same approximate value. This is easily achieved:

If in a system Z indicates an approximate value of an unknown quantity, z its correction appearing as an unknown quantity in the adjustment, and in the other system the corresponding values Z_1 and z_1 are indicated respectively, then

$$Z + z = Z_1 + z_1 \qquad (1)$$

and we have to substitute in the second system for z_1 simply $Z - Z_1 + z$ in order to introduce Z also in this system as an

approximate value.

In this way we may obtain again absolute members in some normal equations. Now we add up all normal equations, whose squared coefficients belong to the same unknown and obtain a main system of normal equations, which is obviously equivalent to the total angular observations of the net as a whole. (Its sum is identical with zero).

Joining them are first of all the equations for base measurements. Presuming several, we regard them here chiefly as error equations. The measurement of the line P_iP_k gives us, when δs_{ik} the correction of the linear length, while dB and dL is the correction of approximate values of the elliptical geographic latitudes and longitudes according to the above main system of normal equations of angular measurements (compare p. 499 (1) or p. 552 (4)):

$$\frac{\varrho''}{a_0}\delta s_{ik} = \frac{\varrho''}{a_0}(\bar{s}_{ik} - s_{ik}) + \left\{ \begin{array}{c} \frac{1-e^2}{W_i^3}\cos\alpha_{ik}\,dB_i + \frac{1-e^2}{W_k^3}\cos\alpha_{ki}\,dB_k \\ + (dL_i - dL_k)\frac{\cos B_k}{W_k}\sin\alpha_{ik} \end{array} \right\}, \qquad (2)$$

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To keep these equations as error equations in the ordinary sense, would not be proper. The main part of the δs arises from the lack of reduction due to the height of the geoid above the ellipsoid. As soon as we presume an extension of the net as the one of the European net, we are not able to disregard this fact, in a strict treatment of the extension. As the reduction is unknown for the time being, we leave the δs for the base lines in

the calculation as unknown quantities.*) No special consideration is needed for the errors of measurement, they diminish completely as was already mentioned on p. 499.

We now imagine that with the help of base equations in each partial net, as has been indicated on pp. 499 and 500, an unknown quantity may be eliminated, so that these equations (2) do not enter in the general adjustment.

However, in opposition to the previous, a difference appears, each base leaves an unknown quantity δs in the absolute members of the normal equations.

The measured azimuths give us according to p. 282 (4) and p. 551 (1) error equations in the following form, which are especially related to α'_{ik} in P_i for P_iP_k:

$$\delta\alpha'_{ik} = \bar{\alpha}_{ik} - \alpha'_{ik} - \eta_i \tan B_i - \frac{a_0}{m_{ik}} \left\{ \begin{array}{l} \frac{1-e^2}{W_i^2}\left(\frac{dm}{ds}\right)_{ki} \sin\alpha_{ik}\, dB_i + \frac{1-e^2}{W_k^2}\sin\alpha_{ki}\, dB_k \\ + \frac{\cos B_k}{W_k}\cos\alpha_{ki}\,(dL_k - dL_i) \end{array} \right\} + \cdots \qquad (3)$$

Then there are also the relations for each point:

$$\left. \begin{array}{l} \bar{L} + dL = L' - \eta \sec B + \cdots \\ \bar{B} + dB = B' + \xi + \cdots, \end{array} \right\} \qquad (4)$$

where dL and dB again indicate the corrections of the approximate values $\bar{L}$ and $\bar{B}$, which are appearing already at the above-mentioned main system of normal equations of the angular measurements.

*)However for land surveying, the supposition δs = zero is very expedient, because through this we obtain dimensions at sea level in all parts. As a result of the somehow erroneous mathematical connection only the correspondence of the results is somewhat more unfavorable, than that of the very accurate treatment.

With (4) we may either introduce instead of dB and dL, ξ and η, or which is easier by far eliminate η from (3)

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that is by means of the first equation (4) to be expressed by dL. With regard to the second equation (3) p. 552, (3) now goes over into:

$$\delta\alpha'_{ik} = \bar{\alpha}_{ik} - \alpha'_{ik} + (\bar{L}_i - L'_i)\sin B_i - \frac{a_0}{m_{ik}}\left\{\begin{array}{l} \frac{1-e^2}{W_i^3}\left(\frac{dm}{ds}\right)_{ki}\sin\alpha_{ik}\,dB_i + \frac{1-e^2}{W_k^3}\sin\alpha_{ki}\,dB_k \\ + \frac{\cos B_i}{W_i}\left(\frac{dm}{ds}\right)_{ki}\cos\alpha_{ik}\,dL_i + \frac{\cos B_k}{W_k}\cos\alpha_{ki}\,dL_k \end{array}\right\} + \cdots \qquad (5)$$

This equation has to be regarded as an error equation only then when L' can be found. Since each L' appears only in <u>one</u> equation, therefore $\delta\alpha'$ must be assumed equal to zero where L' can not be found.

We therefore establish the equations (5) only for points determined by azimuth <u>and</u> geographic longitudes and form the additional terms of these equations for normal equations of dL and dB with regard to the weights.

When a local marker is the target point of the azimuth measurement then on the right hand side in (5) the polynomial in the large parenthesis, is replaced by the <u>one</u> unknown $+\sin B_i dL_i$ which is indicated with C before on pp. 501 and 557.

<u>The determination of weights</u> of the azimuth observations in relation to the directional observations of the survey must follow the principle, that the weights are inversely proportional to the mean error squares. We presume that the net parts are completely

equalized separately, or at least the triangular closures are set; from this the weight unit of the directional or angular observations respectively gives us the mean error square. The determining value of the mean error square is always the one which has been given by the discrepancy of the net alone. The surest way to find the mean error square of the azimuths is by comparing the double and multiple determinations of the azimuths at different times and with different instruments with each other.*)

Finally from the addition of the azimuth measurement contributions to the relative equations of the normal equation systems we get the general normal equation system. This should be left undeterminate for the point P_1, dL_1, and dB_1, so that the normal equations disappear and all the others dL and dB finally appear as functions of dL_1, and dB_1.

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In order to present an explicit solution of the normal equations, it is advisable to deal with each partial grid in the above-mentioned way. Thus, all the unknowns which appear in only one of the partial normal equation systems may then be eliminated. Only then the addition of the equations belonging to the same unknown quantities is performed resulting in the construction of a system which will be dealt with.

In passing over, with the help of relations (4), from dL and dB

*)See note on p. 539.

to ξ and η, we obtain all the ξ and η as functions of values ξ_1 and η_1 and the δs of the base lines.

In the first approximation set $\delta s =$ zero and then arrange ξ_1 and η_1 as in Par. 14, p. 543 ff. Afterwards we should try to establish the height of the geoid in relation to the chosen reference ellipsoid with the help of ξ and η in the way which will be stated later. This will give us approximate values for the δs and enable us to carry through a new calculation for ξ and η. Finally it is possible also to establish the mean error of the weight unit from the discrepancies, respectively the adjustments which are required for the general adjustment. For this the instructions given on pp. 504-511 are sufficient with some modifications.

Par. 19. Third Case of the Determinations from Deflections of the Vertical: Each Station of the Triangulation Net is Also an Astronomical Station. In this case it is possible to consider the small influences of the deflection of the vertical depending on $\cot z'$ without having previously established their approximate values by approximation calculations (p. 535 respectively); one is in this case nearly completely independent from the hypothesis of the shape of the earth. A nearly ellipsoid shape remains only as a presumption with neglect of the very small errors in the reduction of directions because of the height of objects above the reference ellipsoid (for which, not quite correctly, the sea level has been taken).

The adjustment begins exactly as on p. 495 ff. with the adjustment of the stations. When proceeding to the net-adjustment

the meaning of the polynomials $ax + by + cz + \cdots$ will be different.

In considering the expression (5) p. 517 we obtain now an error equation corresponding to one of the azimuth measurements, compare pp. 552 and 558:

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$$\delta\alpha'_{ik} = \bar{\alpha}_{ik} - \alpha'_{ik} - \eta_i \tan B_i - \eta_i \cot z_{ik} \cos \alpha_{ik} + \xi_i \cot z_{ik} \sin \alpha_{ik}$$
$$- \frac{a_0}{m_{ik}} \left\{ \begin{array}{c} \frac{1-e^2}{W_i^3} \left(\frac{dm}{ds}\right)_{ki} \sin \alpha_{ik} dB_i + \frac{1-e^2}{W_k^3} \sin \alpha_{ki} dB_k \\ + \frac{\cos B_k}{W_k} \cos \alpha_{ki} (dL_k - dL_i) \end{array} \right\} + \cdots.$$

From which follows with regard to the relations:

$$\left. \begin{array}{l} \bar{L} + dL = L' - \eta \sec B + \cdots \\ \bar{B} + dB = B' + \xi + \cdots \end{array} \right\} \qquad (1)$$

and considering the two equation (3) p. 552:

$$\left. \begin{array}{l} \delta\alpha'_{ik} = \bar{\alpha}_{ik} - \alpha'_{ik} + (\bar{B}_i - B'_i) \cot z_{ik} \sin \alpha_{ik} \\ \qquad + (\bar{L}_i - L'_i)(\sin B_i + \cot z_{ik} \cos B_i \cos \alpha_{ik}) \\ - \frac{a_0}{m_{ik}} \left\{ \begin{array}{l} \left[\frac{1-e^2}{W_i^3}\left(\frac{dm}{ds}\right)_{ki} - \frac{m_{ik}}{a_0} \cot z_{ik}\right] \sin \alpha_{ik} dB_i + \frac{1-e^2}{W_k^3} \sin \alpha_{ki} dB_k \\ + \left[\frac{1}{W_i}\left(\frac{dm}{ds}\right)_{ki} - \frac{m_{ik}}{a_0} \cot z_{ik}\right] \cos B_i \cos \alpha_{ik} dL_i + \frac{\cos B_k}{W_k} \cos \alpha_{ki} dL_k \end{array} \right\} + \cdots. \end{array} \right\} \qquad (2)$$

In the error equations corresponding to the geodetic direction observations, we substitute for $\delta\alpha'_{ik}$ as on p. 496 (3) a correction λ_{ik} and we add on the right hand side for each set an additional unknown quantity u. The comparison of (2) with (3) and (4) p. 496 shows now the somehow changed importance of the

polynomials $ax + by + cs + \cdots$; on the one hand it consists in an alteration of the coefficients dB and dL, on the other hand where L_i' is not observed, this magnitude (or more easily $\bar{L}_i - L_i'$) appears as a new unknown quantity.

The further procedure for the formation of net normal equations first of all is exactly the same as on p. 497 ff. We will mention that in all error equations λ_{ik} the part $(\bar{L}_i - L_i') \sin B_i$ may simply be left out. For we may think of w the orientation error of each set which is independent from the special object as added to the equation. Through this the coefficient of $(\bar{L} - L')$ becomes substantially smaller and so the calculation especially since L_i' is unknown, becomes somehow easier.

The normal equations obtained up till now correspond only to the geodetic direction observations. For each station we obtain however one error equation of the form (2) due to the azimuth measurements. $\delta a_{ik}'$ indicates here immediately the improvement of the latter.

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The additions of all these equations to the normal equations of the different unknown quantities with regard to the weights have to be formed and to be added to the corresponding previous normal equations.

At least the strict procedure is like this. The determination of the unknown L', where the geographic longitude determination is missing, is essentially based only on the azimuth measurement,

since here only L' has a large coefficient. Therefore in those equations (2) which correspond with azimuth measurements at stations, where L' is unknown, the best thing to do is to set $\delta\alpha' =$ zero and to use the equation for the determination of L' as a function of the other unknown quantities.

Hence the procedure becomes modified since, from the polynomials $ax+by+cz+\cdots$ the unknown L' has to be eliminated with the help of those azimuth equations (2) before the formation of the normal equations.

Only those equations (2) remain as azimuth error adjustments for which L' may be directly observed.

We take these L' (as well as B') without corrections, because of the previously mentioned reason (Par. 15, p. 546); therefore introduction of this would result only in much effort without essential advantages, as long as stations with geographic longitude determination are relatively rare, which case is practically the only one for consideration.

(For the B' the reason for this is, that it is sufficient to ignore corrections due to the small amount of the coefficients of these values in the error equations).

For consideration of base lines, as well as regarding other details, compare the previous paragraph. When the general system is solved, then (1) leads to ξ and η etc., as in that paragraph.

Par. 20. Reference Ellipsoid of the most Favorable Dimensions. Up till now, the meridian ellipse elements a_0 and e^2 have been presumed as given. However already in Par. 14, p. 545 it has been

mentioned for a special case, how a favorable choice of ξ_1, and η_1 also allows for a favorable choice of the reference ellipsoid.

The same method may be applied also in other cases. After all the ξ and η are retained in the general equation as functions of ξ_1 and η_1, we may as in the first mentioned case, add terms to these expressions which depend on the changes da_0 and de^2 of the values of a_0 and e^2 used until then. In each case (compare p. 543) equations of the following form result:

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$$\left.\begin{aligned} \xi_i &= -l_i + a_i\xi_1 + b_i\eta_1 + c_i\frac{da_0}{a_0} + d_i\cdot\frac{1}{2}\frac{de^2}{1-e^2} \\ \eta_i &= -l'_i + a'_i\xi_1 + b'_i\eta_1 + c'_i\frac{da_0}{a_0} + d'_i\cdot\frac{1}{2}\frac{de^2}{1-e^2}. \end{aligned}\right\} \qquad (1)$$

To obtain the expressions for the coefficients c and d, we observe the equations (6), (7) and (8) p. 536. The same indicate that the changes $\delta B''_i, \delta L''_i$ and $\delta\alpha''_{i.1}$ in the calculated relative values of B''_i, L''_i and $\alpha''_{i.1}$ for ξ_i, η_i and η_i cause the changes $\delta B''_i, -\delta L''_i \cos B_i, \delta\alpha''_{i.1}\cot B_i$ respectively. However how these changes $\delta B''_i, \delta L''_i$ and $\delta\alpha''_{i.1}$ are produced by da_0 and de^2 is given in the formulae in Par. 16, p. 294. It follows:

$$c_i = -\varrho''\frac{s_{i.1}}{a_0}\frac{W_i^3}{1-e^2}\cos\alpha_{i.1} \qquad d_i = \varrho''\frac{W_i^3}{1-e^2}\left(\frac{\mathfrak{S}}{a_0}\cos\alpha_{i.1} + \frac{\mathfrak{E}}{a_0}\sin\alpha_{i.1}\right), \qquad (2)$$

further on

$$c'_i = -\varrho''\frac{s_{i.1}}{a_0}W_i\sin\alpha_{i.1} \qquad d'_i = \varrho''W_i\left(\frac{\mathfrak{S}}{a_0}\sin\alpha_{i.1} - \frac{\mathfrak{E}}{a_0}\cos\alpha_{i.1}\right), \qquad (3)$$

when η_i is obtained with the help of geographic longitudes; on the other hand

$$c_i' = -\varrho'' \frac{s_{i.1}}{a_0} W_i \sin \alpha_{i.1} \quad d_i' = \varrho'' W_i \left(\frac{\mathfrak{S}}{a_0} \sin \alpha_{i.1} - \frac{\mathfrak{T}}{a_0} \cos \alpha_{i.1} + \frac{1}{W_i} \mathfrak{M} \right), \qquad (4)$$

when η is obtained with help of the azimuth.

The importance of $\mathfrak{S}$, $\mathfrak{T}$ and $\mathfrak{M}$ is evident from p. 294, where index 2 has always to be replaced by index i.

As can be seen, the two forms of d_i' do not agree completely; they differ by the certainly small term with $\mathfrak{M}$. In the case when ξ and η are determined by the methods of Pars. 15 to 19 p. 546 ff., the expression (3) still has to be applied for d_i' since all η are adapted as far as possible to the determination of the geographic longitude.

This method however is not completely accurate, just as little as generally the whole procedure; because in the calculation of members depending from da_0 and de^2, we imagine the geodetic lines as unchanged between P_1 and the different points P_i not only in regard to their linear length, but also in regard to their azimuths; while the grid becomes somehow deformed by a change in a_0 and e^2 and the linear length and also azimuth also suffer changes, which could be very considerable for a net which is extended as far as the European net is.

A completely correct procedure may be achieved, when during the adjustment by <u>adjusted</u> observations (pp. 551 and 555 ff.) we add to the expressions for $\delta \alpha_{ik}'$ and δs_{ik} according to Par. 14 p. 291 the terms depending on da_0 and de^2.

When for instance as in the previous paragraph we deal only with triangulation sides of the first order, that is short distances, then we obtain on the right hand side of the equations increases according to formulae (6) p. 291

for

$$\left.\begin{aligned} \delta\alpha'_{ik} &: \quad \frac{1}{2}\frac{de^2}{1-e^2}\frac{\varrho''}{W^2}(2\cos^2 B\sin\alpha\cos\alpha+\cdots) \\ \delta s_{ik} &: \frac{da_0}{a_0}s_{ik}-\frac{1}{2}\frac{de^2}{1-e^2}\frac{s_{ik}}{W^2}(2\cos^2 B\cos^2\alpha-[1-e^2]\sin^2 B+\cdots), \end{aligned}\right\} \qquad (5)$$

when

$$B=\frac{B_i+B_k}{2} \qquad \alpha=\frac{\alpha_{ik}+\alpha_{ki}-180^\circ}{2}$$

with argument B

should be taken.

The first expression has to be added not only to all error equations for measured azimuths, but also to all error equations for geodetic directional observations at the stations; it also appears as an increase of the expressions $ax+by+cz+\cdots$.

For conditional adjustment in connection with formulae (1) p. 548 ff. we have to add to B''_i, L''_i and α''_{ki} values of $\delta B_i, \delta L_i$ and $\delta\alpha_{ki}$ given by formulae Par. 16, p. 294, when the indices 1 and 2 must be replaced by i and k. The unknowns da_0 and de^2 have to be also eliminated, as the others, if one does not prefer to deal with the equation according to Par. 26, p. 215 of our adjustment calculation.

The determination of da_0 and de^2 obtains a proper value only if the astronomical points are distributed over the area at least approximately uniformly.

The best principle for the determination of da_0 and de^2 is to make the total of the N^2 extending over the whole area to

a minimum. But there are great practical difficulties in the way of its realization.

Par. 21. First Approximation for the Determination of the Geoid. When the plumb line differences ξ and η are determined against a real snug ellipsoid, with a number of points, located that closely that these do not show an irregular run from point to point, but show a more regular character, then the determination of the geoid surface is possible; naturally only under the condition, that the plumb lines within their expansion from the stations on the physical earth surface, to the geoid, may be regarded as straight lines. Because of the actual curvature

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of the plumb lines an inaccuracy remains, the value of which, as we shall later see, is slight, as far as it does not originate in local causes; which however, if this is the case, can at present not at all be submitted to the calculation for practical reasons.

Moreover the error disappears practically when all the stations are at sea level, or are situated nearly at the same height. In the latter case we may presume anyway, that the construction refers to a level surface instead of the geoid surface proper in accuracy.

To differentiate from the geoid, we wish to call the surface adjusted for the deflections of the vertical spheroid.

The construction may be tackled in the empiric-graphical manner as follows, when both components ξ and η are always given.

Par. 21. First Approximation for the Determination of the Geoid.

We unite in a general map the stations to nearly straight lines, and first of all calculate the γ components of the deflections of the vertical at each point, which are directed along the line. If the azimuth is α' then according to Fig. 41, p. 515, the components $\gamma = \Theta \cos(A' - \alpha')$, therefore because of (7) p. 516:

$$\gamma = \xi \cos \alpha' + \eta \sin \alpha'. \tag{1}$$

The reason for this is that in the azimuth α' the spheroid's tangent, has the depression angle γ relative to the tangent of the reference ellipsoid.

All γ values should be plotted as ordinates in a large scale on millimeter square paper at right angles to the abscissae, the differences of which correspond to the horizontal distances. Through the obtained points we plot a freehand line, where for rounding off of the curve, deviations from the points are allowable; but they have to stay within the value of the feasible observation error influences in ξ and η on γ; as a rule they have to be taken alternatively positive and negative.

With completely sufficient approximation we may state for the increase of the elevation $\Delta N = N_k - N_i$, which contains the spheroid in relation to the ellipsoid from P_i to P_k, that:

$$\Delta N = -\int \frac{\gamma''}{\varrho''} ds, \tag{2}$$

whereby the integration is to be extended over the whole length of the shortest line P_i to P_k.

If in the drawing c_1 seconds of the ordinates and c_2 meter of the abscissae amount to 1^{mm}, then $c_1 \gamma^{mm} = \gamma''$ and $c_2 s^{mm} = s^m$, therefore

or

$$\left.\begin{aligned} \Delta N &= -\frac{c_1 c_2}{\varrho''} \int \gamma^{mm} ds^{mm} \\ \Delta N &= -\frac{c_1 c_2}{\varrho''} F_{ik}^{mm}. \end{aligned}\right\} \qquad (3)$$

F_{ik} designates the surface between ordinates γ_i and γ_k in square millimeters. This may be calculated with the polar planimeter or by mechanical squaring.

In the latter case we measure all ordinates in sufficiently small intervals Δs^{mm} of the abscissae, so that between two adjacent ordinates the interpolated curve may be considered straight. It will be sufficient due to the inaccuracy of the whole procedure, apart from more strict formulae, to state:

$$F_{ik} = \left(\sum_i^k \gamma - \frac{1}{2}(\gamma_i + \gamma_k)\right) \Delta s, \qquad (4)$$

where $\Sigma\gamma$ means the sum of ordinates in millimeters from γ_i to γ_k inclusive.

The expression becomes still easier when for each part Δs the <u>average</u> ordinate will be measured. Whereby it is simply

$$F_{ik} = \Sigma(\text{of the average ordinates})\,\Delta s. \qquad (5)$$

On the basis of these formulae we calculate the values ΔN for a sufficient number of points relative to one of them. We may finally make a list of them or even plot (naturally distorted for greater clarity) a profile of the spheroid or ellipsoid respectively.

As a rule we plot a large part of the line points in meridional direction not only because it is easy, but also because the number

of the determined ξ will prevail. Then of course we will have to, sufficient points are rarely found on one meridian, interpolate ξ for ideal meridian points from those to be found on both sides. The geographic latitudes will also serve best directly as abscissae.

Meridional traverses are in themselves not sufficient because they give only the shape of a number of cuts without connection. To be able to join them together, we have to know also so many points in η that one can establish either along the parallel circles with the η alone, or with some azimuths (which are crossing the meridian cuts) joining profiles.

When several connections can be established, then this gives us controls for the accuracy of the construction. We may utilize them

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by means of an adjustment. It will be conducted in the same way as leveling and it is best on the basis of adjusted observations, where only the heights of the profile junctions above an arbitrary zero point are regarded as unknown; the $\Delta N'$ of each profile run, between two junction points appear as observation. (Compare further details in the adjustment calculation p. 310 below).*)

The weights of these observations have to be taken in general inversely proportional to the distances, otherwise under similar conditions the mean error squares in the $\Delta N'$ obviously are increas-

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*)The heights should be introduced above an unknown zero point, so that the sum of the normal equations will be identically zero, which gives a greater convergence in the solution by approximation expedient in this case. (loc. cit. p. 133).
(Gauss's dodge by v. Freeden, The Praxis of M. d. kl. Qu., p. 29.)

ing with the number n of the ordinates, which have been used for their construction. The graphic interpolations of the deflections of the vertical can be produced only very unevenly for the different profiles, which has to be taken in account also at the weight determination. Therefore we take the weights inversely proportional to the distances multiplied by the mean error squares of the ordinates; for the latter the uncertainty of the drawing gives basis for their determination (see example). When there are only a few intermediate points in a path even the weight may become inversely proportional to the square of distances multiplied by the mean error square of the ordinates, because then the ordinate errors will all be towards one side.

When the height difference of the junction points is obtained, then the adjustments for the intermediate points are determined (<u>adjustment calculation</u> p. 311), and now only the height of one of the points of the spheroid above the ellipsoid is necessary for the complete determination of the spheroid. When only <u>one</u> base line is measured, then both surfaces intersect at the location of this base line; when there are several base lines there remains however an obscurity (compare pp. 524 and 533), and we have to imagine the spheroid placed in such a way that it will have a central position to the ellipsoid with regard to the location of the base lines. The uncertainty is moreover of slight importance, because the position of the reference ellipsoid's axis to the earth axis is still not yet known (p. 533); so as mentioned above the spheroid is more of a level surface at the medium height of the physical earth surface of the respective area, than the geoid

itself.

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Finally the spheroid may be characterized by curves of equal $\Delta N'$ in the plan drawing.

Par. 22. Numerical Example: Deflections of the Vertical in the Harz. The Royal Prussian Geodetic Institute has established a set of deflections of the vertical ξ, which show a definitely regular course. We confine ourselves here to a set of points, which are situated near the Brocken meridian, Fig. 44. According to p. 150 in the publications of the astronomical works in 1875 (published 1876), for which we have the following figures:

Station	Sea level	Easterly position from Ferro	Polar altitude		ξ
			geodetic B'	astronomic B'	
Heldburg	395m	28° 24′	50° 17′ 23,57″	19,25″	+ 4,32″
Dollmar	740	28 9	37 32,45	27,35	+ 5,10
Inselsberg	916	28 8	51 8,66	11,47	— 2,81
Craula	456	28 9	51 3 30,44	28,38	+ 2,06
Mühlhausen	227	28 9	12 10,44	6,18	+ 4,26
Löwenburg	422	28 13	26 34,26	33,93	+ 0,33
Tettenborn	323	28 13	34 22,39	17,29	+ 5,10
Hohegeis	640	28 20	39 58,38	57,02	+ 1,36
Osterode	260	27 54	43 23,20	23,18	+ 0,02
Brocken	1143	28 17	48 1,41	10,59	— 9,18
Ilsenburg	249	28 21	52 24,86	35,71	— 10,85
Harzburg	217	28 13	53 25,74	39,25	— 13,51
Fallstein	167	28 18	52 1 5,91	9,34	— 3,43
Asse	203	28 19	8 20,38	20,38	0,00

Chapter 12. Measurements Performed on the Physical Surface of the Earth.

For the calculation of the geodetic latitude the one of the observatory in Seeberg near Gotha, has been taken for a base. As azimuth served the one measured in 1869 on the Inselsberg (compare p. 148 loc. cit). The ξ refer also to a Bessel ellipsoid, which gives for Seeberg $\xi = 0$ and for Inselsberg $\eta = 0$.*)

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Regarding its height relative to the geoid, referring to the end of Par. 21, the remark is sufficient, that both will intersect somewhere in the vicinity of the studied area; that the spheroid conforms rather to a level surface at several hundred meters above sea level, than to a geoid itself.

For obtaining a nearly meridional profile Hohegeis and Osterode on one hand, and Harzburg and Ilsenburg on the other hand have been united to an ideal point:

	Longitude	Geodetic polar altitude	ξ
$\left(\frac{4}{5}$ Hohegeis $+ \frac{1}{5}$ Osterode$\right)$	28° 15′	51° 40′ 39″	+ 1,1″
$\left(\frac{1}{2}$ Ilsenburg $+ \frac{1}{2}$ Harzburg$\right)$	28 17	51 52 55	− 12,2

Since the azimuth α' of the path is slight, ξ may be substituted directly for γ in the formulae on p. 565, when proceeding from South to North, that is $\alpha' = 180^0$, $-\xi$.

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*)This is not completely correct, but with regard to the slight geographic longitudinal difference L of Inselsberg and Seeberg it is sufficient. Formula (9) p. 536 shows of course, that at the transfer of the ellipsoidal azimuth to Seeberg from Inselsberg, since ξ is not quite zero here, a small influence proportionate to sin L arises, which finally however affects the ξ calculated from Seeberg only in a negligible way.

Par. 22. Numerical Example: Deflections of the Vertical in the Harz.

In the graphic illustration two extreme interpolation curves have been formed, whereby

$$1^{mm}\ \text{Ordinate} = 0{,}1''$$

and

$$1^{mm}\ \text{abscissa} = 30''\ \text{in geographic latitude}$$

hence:

$$= 927^{m},$$

$$c_1 = 0{,}1$$

$$c_2 = 927^{m}$$

$$\frac{c_1 c_2}{\varrho''} = \frac{1}{2220}.$$

Fig. 45 shows this graphic illustration in half size.

The following numbers have been obtained:

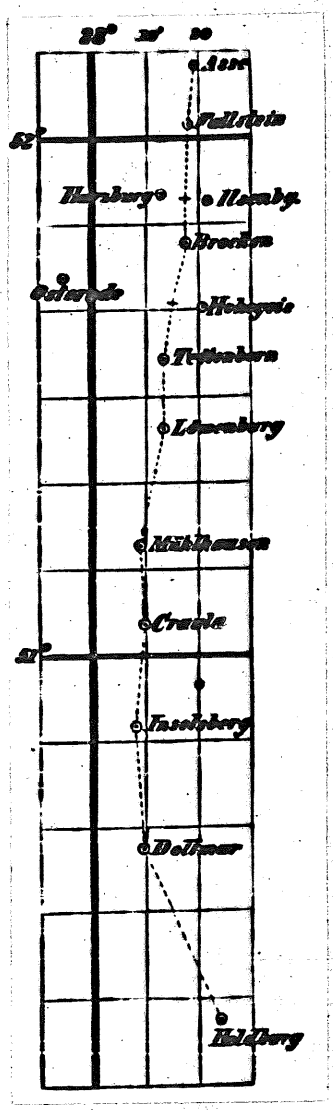


Figure 44.

(See Illustrative Material on Next Page.)

	I.		II.		Mean
	$-F$	$\Sigma \Delta N'$	$-F$	$\Sigma \Delta N'$	
		0,00m*)		0,00m	0,00m
From 50°20′ to 50°30′	+1001^{qmm}	+0,45	+1220^{qmm}	+0,55	0,50
„ 30 „ 40	+1034	+0,92	+1110	+1,05	0,98
„ 40 „ 50	+ 465	+1,12	— 45	+1,03	1,07
„ 50 „ 51 0	— 523	+0,89	— 318	+0,88	0,88
„ 51 0 „ 10	+ 505	+1,12	+ 505	+1,11	1,11
„ 10 „ 20	+ 650	+1,41	+ 850	+1,49	1,45
„ 20 „ 30	+ 65	+1,44	+ 285	+1,62	1,53
„ 30 „ 40	+ 800	+1,80	+ 760	+1,96	1,88
„ 40 „ 50	— 978	+1,36	—1070	+1,48	1,42
„ 50 „ 52 0	—1875	+0,52	—1925	+0,62	0,57
„ 52 0 „ 10	— 305	+0,38	— 295	+0,48	0,43

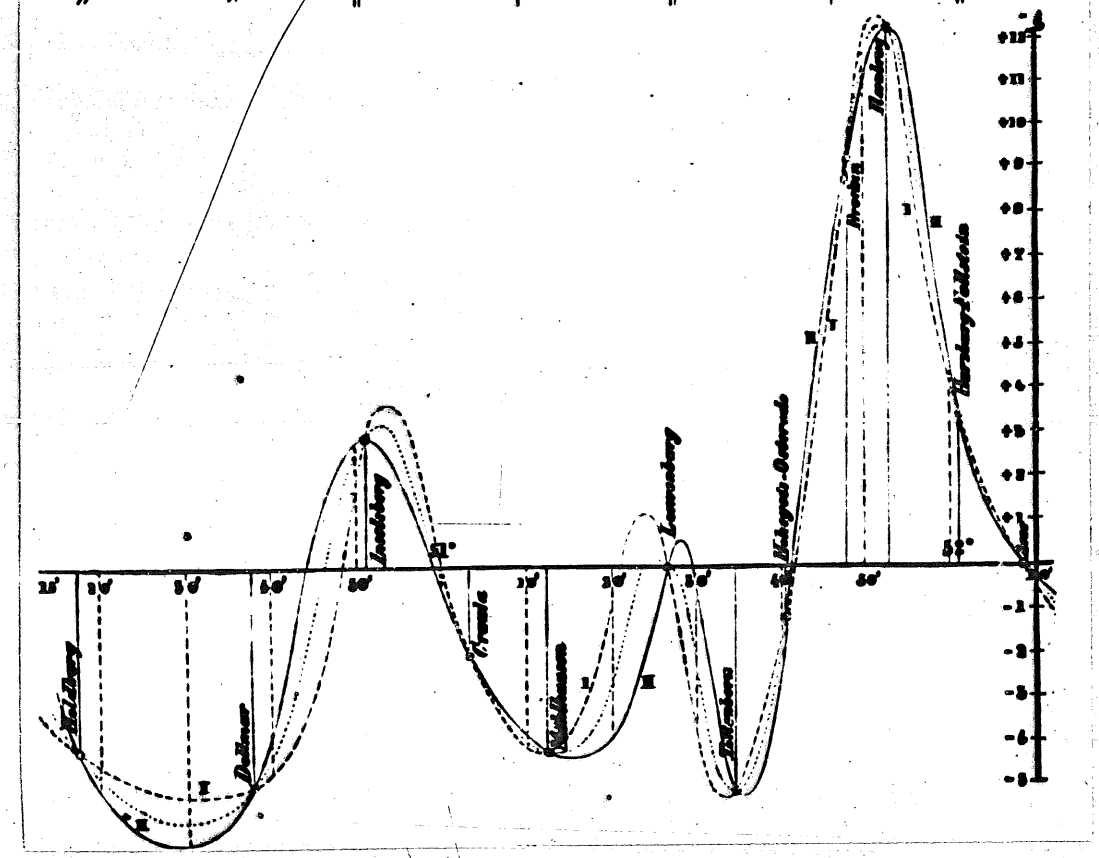


Figure 45.

*)Arbitrarily taken as zero point.

Par. 22. Numerical Example: Deflections of the Vertical in the Harz.

The two interpolation curves, which are carried both exactly through the

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points, are giving fairly well corresponding results. It is of course possible that this is only an accident.*) After looking at the drawing the differences F, may be regarded as accidental for both curves, from 10' to 10' geographic latitude. Constructing from the squares of the eleven differences of the F, the mean difference, in the sense of a mean error in the same line, then the mean error of a $\Delta N'$ for a distance of 10' in geographic latitude follows as:

$$\pm 0^m,10.$$

For the total height difference in I or II then according to this the mean error is equal to:

$$\pm 0{,}1\sqrt{11}, \text{ that is } \pm 0{,}33^m$$

and for the mean of both results

$$\pm 0{,}23^m.$$

The inaccuracy of ξ itself, which is not yet obtained in here, increases this mean error only slightly. On the other hand the same is estimated rather too high than to low, while in particular the two interpolation curves retained as far as possible, extreme,

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*)On p. 206 of the Zeitschrift für Vermessungswesen (Journal of Surveying) Vol. 4, 1875 Börsch gives similar results. which show differences up to $0{,}4^m$ with the results I and up to $0{,}2^m$, with the results II.

although feasible locations. Fig. 45 shows also the interpolation curve, which corresponds to the mean of I and II.

Par. 23. Deflection of the Vertical in the Alpine Regions. Carl von Orff has attached in the writing determination of the geographic latitude of the Royal Observatory near München, 1877 (supplement to the Vol. 21 of the observatory's records), a very interesting appendix, in which by means of South German, Austrian, Swiss and Italian triangulation nets, plumb line deviations are obtained North and South of the Alps. Here also a course is unmistakeable, but in spite of 31 points in latitude, eight in longitude and 14 in azimuth (except the starting point München with $\xi_1 = 0\ \eta_1 = 0$) with the large size of the distribution area, only a very uncertain determination of the spheroid is possible. For a chain of points close to the meridian of München there were the following values (on p. 60 loc. cit):

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(See Illustrative Material on Next Page.)

No.	Station	Geographic latitude approximated	ξ
9	Nürnberg	49°27,5′	+ 4,9″
8	Wülzburg	49 1,5	— 1,8
7	Ingolstadt	48 45,8	+ 4,8
6	München	48 8,3	0,0*)
5	Holzkirchen, Peifsenberg } Mean	47 50,5	— 3,5
4	Benediktbeuren, Wendelstein, Mittenwald, Lanserkopf } Mean	47 31,5	— 12,1
3	S. Salvatore	45 51,1	+ 13,2
2	Padua, Venice } Mean	45 24,9	+ 5,5
1	Bologna	44 29,8	— 6,6

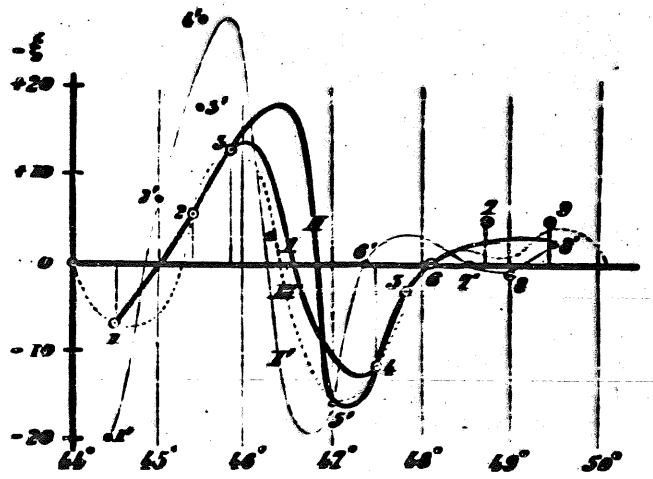


Figure 46.

The graphic illustration now gives for two coinciding, but in general still extreme adjustment curves, while adjusting the small fluctuations at the northerly points:

*)Presumed.

	I.		II.		Mean	Reduced Mean
	$-F$	$\Sigma N'$	$-F$	$\Sigma N'$		
44,5°		0,0m	as in I	0,0m	0,0m*)	
From 44,5° to 45	− 16,5mm	− 0,9		− 0,9	− 0,9	− 2,0m
„ 45 „ 45,5	+ 17	+ 0,0		+ 0,0	+ 0,0	− 1,1
„ 45,5 „ 46	+ 52	+ 2,8		+ 2,8	+ 2,8	+ 1,4
„ 46 „ 46,5	+ 44	+ 5,2	+ 85	+ 7,4	+ 6,3	+ 5,2
„ 46,5 „ 47	− 17,5	+ 4,3	+ 14	+ 8,2	+ 6,3	+ 5,2
„ 47 „ 47,5	− 56,5	+ 1,2	− 69	+ 4,5	+ 2,8	+ 1,7
„ 47,5 „ 48	− 33	− 0,6	as in I	+ 2,7	+ 1,1	+ 0,0*)
„ 48 „ 48,5	+ 2,5	− 0,4		+ 2,8	+ 1,2	+ 0,1
„ 48,5 „ 49	+ 12,5	+ 0,2		+ 3,5	+ 1,9	+ 0,8
„ 49 „ 49,5	+ 12,5	+ 0,9		+ 4,2	+ 2,6	+ 1,5

Whereupon

$$c_1 = 1 \text{ and } c_2 = 11150, \frac{c_1 c_2}{\varrho''} = 0{,}054.$$

The reduced means arise through subtraction of 1,1m. The uncertainty of the total difference in height from 45,5° to 49° geographic latitude should not exceed 2 to 3m.

Fig. 46 shows yet another interpolation curve I' for a second nearly meridional profile. Moreover the numbers 1', 2', 3', 4', 5', 6', 7', 8', refer in this order to the points Mondovi, Torino, Milano, Andrate, Rigi, Zürich, Strassburg-Tübingen, Mannheim.

The outcome of ξ is here very similar as for the first profile and we may expect also a similar process in the heights in accordance with the areas and the same (probably somehow higher)

*)Presumed.

extremes.

Par. 24. Application of the Calculation for the Determination of the Spheroid. To avoid the arbitrariness of the graphic procedure, we may follow the path of calculation. We will of course find very soon, that here the arbitrary choice of the interpolation formula is also doubtful. It is probably even more doubtful than the arbitrariness at the graphic procedure, because it is more difficult to verify the degree of the arbitrariness. Only at very close designations, which permit a real adjustment is calculation to be preferred.

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Supposing that the spheroid surface could be represented by one equation*) then we may apply for a moderate extension of the area, under investigation for N' a series of the form:

$$\left.\begin{aligned} N' = N_0' &+ \mathfrak{A}_1 \Delta B + \mathfrak{A}_2 \Delta L \\ &+ \tfrac{1}{2}\mathfrak{B}_1 \Delta B^2 + \mathfrak{B}_2 \Delta B \Delta L + \tfrac{1}{2}\mathfrak{B}_3 \Delta L^2 \\ &+ \tfrac{1}{6}\mathfrak{C}_1 \Delta B^3 + \tfrac{1}{2}\mathfrak{C}_2 \Delta B^2 \Delta L + \tfrac{1}{2}\mathfrak{C}_3 \Delta B \Delta L^2 + \tfrac{1}{6}\mathfrak{C}_4 \Delta L^3 + \cdots, \end{aligned}\right\} \quad (1)$$

where $\Delta B = B - B_0$, $\Delta L = L - L_0$ and N_0' indicates the value of N' at the starting point $(B_0 L_0)$. We may regard this sequence either as Taylor's sequence, or as an interpolation formula, which derives from the one of Lagrange for a meridian cut, and from a power series

*)This will be approximately the case in spite of the curvature discontinuations of the geoid (p. 22). However a great difficulty remains in any case because of it, since there are many terms for representation whose required coefficients are to be determined.

of ΔB, when the coefficients have been again developed as a power series of ΔL.

Now is dM a meridian element and dP an element of the parallel circle, then $\frac{\partial N'}{\partial M} dM$ is the increase of N' towards North for dM and $\frac{\partial N'}{\partial P} dP$ the increase of N' towards West for dP. As these values are also equal to ξdM and $-\eta dP$ respectively, therefore, we obtain:

$$\left.\begin{aligned} \xi &= \frac{\partial N'}{\partial M} \quad \text{i.e.} \quad \xi = \frac{\partial N'}{\varrho_m \partial \Delta B} \\ \eta &= -\frac{\partial N'}{\partial P} \quad \text{i.e.} \quad \eta = -\frac{\partial N'}{\varrho_n \cos B \partial \Delta L'} \end{aligned}\right\} \qquad (2)$$

where ϱ_m is the meridian radius of curvature, ϱ_n indicates the perpendicular radius for the geographic latitude B.

Carrying out the differentiation, we obtain the expressions for ξ and η, which show, how expression (1) can be determined from a series of given ξ and η for N' up to the constant N_0'.

For a ith degree function we must have $(2+3+\cdots i+1)$, i.e. $\frac{1}{2} i(i+3)$ components of the plumb line deviation. When these are not distributed evenly over ξ and η, then at least i of the given values have to be components ξ and η, according to whether η or ξ is the dominant number.

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When we consider, that an averaging of the function N' will be worthwhile only, if ξ and η are given for a larger number of points; therefore by strict representation of these values there are also many coefficients to be determined. A reduction of their

number demands an adjustment, which however is very troublesome, because of the unavoidably large numerical coefficients (linked with the nature of the exponential series), and as a rule it is even not feasible. A simple numerical example will further explain and prove the difficulties.

The calculation is naturally simpler when we have to deal only with a single profile. The mentioned example refers indeed only to one profile; even here the numerical work of the adjustment calculation becomes already significant.

It is still simplest to represent all the deflections of the vertical components for one profile exactly, whereby the Lagrange interpolation formula may be applied in its original form, without having to solve the equations first.*) We have for instance, for a meridional profile, when the values $\xi_1, \xi_2, \ldots \xi_i \ldots \xi_n$ for n points are given, which have, in relation to an arbitrary starting point, the distances $\Delta B_1, \Delta B_2, \ldots \Delta B_i \ldots \Delta B_n$, in geographic latitude, and for the point ΔB the value ξ determined by interpolation as:

$$\xi = \sum_1^n \xi_i \frac{\Delta B - \Delta B_1}{\Delta B_i - \Delta B_1} \frac{\Delta B - \Delta B_2}{\Delta B_i - \Delta B_2} \cdots \frac{\Delta B - \Delta B_{i-1}}{\Delta B_i - \Delta B_{i-1}} \frac{\Delta B - \Delta B_{i+1}}{\Delta B_i - \Delta B_{i+1}} \cdots \frac{\Delta B - \Delta B_n}{\Delta B_i - \Delta B_n} \qquad (3)$$

The development of n terms of the sum on the right hand

*)This interpolation formula is nothing else than the general solution of the problem. To achieve it the coefficients of the exponential series may be taken from the equation systems to be solved by determinants, which gives us the correctly represented values of the exponential series. Otherwise another especially the well-known principle, regarding the conversion of a product of all differences from given values into a determinant is to be applied.

side gives us ξ as exponential series of ΔB.**) However as a rule it would be easier to calculate ξ for a number of ΔB in the same interval and derive $\Delta N'$ by mechanical squaring.

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Another fairly easy way for the interpolatory representation of the deflections of the vertical of single profiles can be obtained, when we renounce to satisfy all given values by one equation.

The simplest method is always to interpolate with regard to two neighboring deflections of the vertical, i.e. to set for a meridional profile between ξ_i and ξ_k:

$$\xi = \xi_i \frac{\Delta B - \Delta B_k}{\Delta B_i - \Delta B_k} + \xi_k \frac{\Delta B - \Delta B_i}{\Delta B_k - \Delta B_i}. \qquad (4)$$

In the profile of ξ this results in straight lines between two points at a time, producing an interpolation with instabilities,

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**)It comes to mind to look for the general formula for ξ also for that case, when, with the use of adjustment calculations from n observations, an experimental series of less than n coefficients has to be deducted. This may indeed be done also by the well-known determinant principles. However in spite of the theoretic simplicity of the formula its application seems to be too troublesome.

Tchébychef has dealt in several treatises (compare especially the Mémoires de l'Acad. imp. des sciences de St. Pétersbourg from 1858 and 1859) with the problem of interpolation, between others also with the interpolation by the method of the least square (1859). The large figures are not avoided by this, as after all an essential simplification against the common procedure for this method has not been achieved and does not even seem possible.

which is however hidden in N profile, since the equality of the ξ in the places of instability of the former profiles produces converging tangents (see the first equation (2)) in the profile of the N.

It is preferable in many cases, always to consider three adjacent deflections of the vertical, thus to set for a meridional profile

$$\xi = \left\{ \begin{array}{c} \xi_h \dfrac{\Delta B - \Delta B_i}{\Delta B_h - \Delta B_i} \dfrac{\Delta B - \Delta B_k}{\Delta B_h - \Delta B_k} + \xi_i \dfrac{\Delta B - \Delta B_h}{\Delta B_i - \Delta B_h} \dfrac{\Delta B - \Delta B_k}{\Delta B_i - \Delta B_k} \\ + \xi_k \dfrac{\Delta B - \Delta B_h}{\Delta B_k - \Delta B_h} \dfrac{\Delta B - \Delta B_i}{\Delta B_k - \Delta B_i} \end{array} \right\} \qquad (5)$$

whereby ξ_h, ξ_i, ξ_k are three consecutive given values; as interpolation curve in the profile of ξ the second-grade parabola appears now. Naturally there are also instabilities here, but only in profile of ξ.

For the practical application, as the most functional procedure, a method should be used, where four adjacent deflections of the vertical are always taken in one interpolation formula (3), but each time only the central part of the interpolation curve should be used.*) In this way the smallest discontinuations may be obtained. The formula is here, if $\xi_h, \xi_i, \xi_k, \xi_l$ follow one another:

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(See Illustrative Material on Next Page.)

*)Of course we have to deviate somewhat from this procedure at both ends of the amplitude to be interpolated, which however is of no importance just at the ends.

$$\xi = \left\{ \begin{array}{l} \xi_h \dfrac{\Delta B-\Delta B_i}{\Delta B_h-\Delta B_i}\dfrac{\Delta B-\Delta B_k}{\Delta B_h-\Delta B_k}\dfrac{\Delta B-\Delta B_l}{\Delta B_h-\Delta B_l} + \xi_i \dfrac{\Delta B-\Delta B_h}{\Delta B_i-\Delta B_h}\dfrac{\Delta B-\Delta B_k}{\Delta B_i-\Delta B_k}\dfrac{\Delta B-\Delta B_l}{\Delta B_i-\Delta B_l} \\ + \xi_k \dfrac{\Delta B-\Delta B_h}{\Delta B_k-\Delta B_h}\dfrac{\Delta B-\Delta B_i}{\Delta B_k-\Delta B_i}\dfrac{\Delta B-\Delta B_l}{\Delta B_k-\Delta B_l} + \xi_l \dfrac{\Delta B-\Delta B_h}{\Delta B_l-\Delta B_h}\dfrac{\Delta B-\Delta B_i}{\Delta B_l-\Delta B_i}\dfrac{\Delta B-\Delta B_k}{\Delta B_l-\Delta B_k}. \end{array} \right\} \qquad (6)$$

Compare the American Journal of Mathematics pure and applied Vol. II No. 4, 1879 for two other interpolation formulae; these are less advantageous for the case at hand. -- There are also formulae for the interpolation of given values ξ for points of a surface; Lambert mentioned these already in the third part of his Beiträge zum Gebrauch der Mathematik (Appendix to the Application of Mathematics) (1772) p. 68 ff. In it the factor pairs $(\Delta B-\Delta B_i)(\Delta L-\Delta L_i)$ appear in the place of $\Delta B-\Delta B_i,\ldots$. If such a formula were applied to the Harz, then its integration with respect to B would yield the expression for N' except for an unknown fuction of L. A number of values of η are necessary for the determination of that function. Besides Lambert already draws one's attention to the advantage of periodic series.

Par. 25. Continuation. We may obtain for the accurate representation and for the equalization in general a much better suited formula, than the exponential series, when N' is developed for one meridian according to Fourier's series and the coefficients again are developed as functions of the geographic longitudinal difference according to Fourier. Thus, we state:

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned} N' &= b_0 + b_1\cos x + b_2\cos 2x + b_3\cos 3x + \cdots \\ &\quad + a_1\sin x + a_2\sin 2x + a_3\sin 3x + \cdots \\ \text{whereby}\quad b_i &= b_{i.0} + b_{i.1}\cos y + b_{i.2}\cos 2y + \cdots \\ &\quad + b'_{i.1}\sin y + b'_{i.2}\sin 2y + \cdots \\ \text{and}\quad a_i &= a_{i.0} + a_{i.1}\cos y + a_{i.2}\cos 2y + \cdots \\ &\quad + a'_{i.1}\sin y + a'_{i.2}\sin 2y + \cdots \end{aligned}\right\} \qquad (1)$$

x and y are such multiples of ΔB and ΔL, that the amplitudes of x and y becoming interpolated, are unadequately smaller than 2π. We have to avoid of course, that the outmost given values of N' should belong to values of x or y respectively, which are nearly different by 2π, since expression (1) renders the same N' value for variations of x and y about 2π. The ignoring of this circumstance makes the representation obviously difficult.

For further facilitation of the same, a modification of the expression (1) is also necessary. The differential quotients of N', which with the relations (2) of the previous paragraph lead to

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the expressions for ξ and η, which with the present form of the expression (1) contain no term free of cosine or sine. Therefore the sum of ξ or η for all values is zero over a period beginning from x or y.

If we add to the expressions (1) on the right hand side

(See Illustrative Material on Next Page.)

$$\left.\begin{array}{lll} \text{to } N': & \text{the term } a_0 x & \\ \text{„ } b_i: & \text{„ „ } & b'_{i.0}\, y \\ \text{„ } a_i: & \text{„ „ } & a'_{i.0}\, y, \end{array}\right\} \qquad (1^*)$$

then constant members will also appear in the differential quotients, so that the sum of ξ or η over a period will not necessarily be zero, which makes the junction much easier.

The differential quotients are now Fourier's series in complete generality; they remain so even when we regard a_0 fully constant and take all coefficients of y except the one coefficient $b'_{0.0}$ as equal to zero; only then the constant members of the series for ξ and η do not change from meridian to meridian, or from parallel to parallel respectively, whereby the junction is again made very difficult. We therefore retain the general expressions. (Only $a'_{0.0}$ could be omitted in any case).

These expressions are suited very well, in comparison with exponential series, for interpolation of given ξ and η (in accuracy or with adjustment), because the coefficients of the unknown quantities do vary not only due to size, but also due to sign.

Par. 26. Numerical Example. We apply the numerical values of the example from Par. 23, p. 572 to the exponential series. Fig. 46 shows, that ξ has to be assumed at least from the fourth degree on. While we go still somewhat further and put in the expression (1) p. 574 ΔL equal to zero, as well as partly changing the meaning of coefficients somewhat, we obtain:

$$\left.\begin{array}{l} N' = N'_0 + \mathfrak{A}_1 \Delta B + \frac{1}{2}\mathfrak{B}_1 \Delta B^2 + \frac{1}{3}\mathfrak{C}_1 \Delta B^3 + \frac{1}{4}\mathfrak{D}_1 \Delta B^4 \\ \qquad + \frac{1}{5}\mathfrak{E}_1 \Delta B^5 + \frac{1}{6}\mathfrak{F}_1 \Delta B^6. \end{array}\right\} \qquad (1)$$

Differentiation yields, when ΔB is understood in degrees and ξ for $\Delta B =$ zero (München) has been also taken as zero:

$$\underset{\text{in seconds}}{\xi =} \left\{ \mathfrak{B}_1 \Delta B + \mathfrak{C}_1 \Delta B^2 + \mathfrak{D}_1 \Delta B^3 + \mathfrak{E}_1 \Delta B^4 + \mathfrak{F}_1 \Delta B^5 \right\} \frac{e'' e''}{e_m} \qquad (2)$$

or in easily visible abbreviation:

$$\underset{\text{in seconds}}{\xi =} \mathfrak{B}_1' \Delta B + \mathfrak{C}_1' \Delta B^2 + \mathfrak{D}_1' \Delta B^3 + \mathfrak{E}_1' \Delta B^4 + \mathfrak{F}_1' \Delta B^5. \qquad (3)$$

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We obtain now the following error equations, while we imagine a correction λ to be added to each ξ and then transfer ξ to the right:

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	$-\xi$	$\mathfrak{A}_1$ 1.	$\mathfrak{C}_1$ 2.	$\mathfrak{D}_1$ 3.	$\mathfrak{E}_1$ 4.	$\mathfrak{F}_1$ 5.
$\lambda_1 =$	+ 6,6″	− 3,64	13,2496	− 48,2285	175,5519	− 639,0089
$\lambda_2 =$	− 5,5	− 2,72	7,3984	− 20,1236	54,7363	− 148,8828
$\lambda_3 =$	− 13,2	− 2,29	5,2441	− 12,0090	27,5006	− 62,9763
$\lambda_4 =$	+ 12,1	− 0,61	0,3721	− 0,2269	0,1384	− 0,0844
$\lambda_5 =$	+ 3,5	− 0,30	0,0900	− 0,0270	0,0081	− 0,0024
$\lambda_7 =$	− 4,8	+ 0,63	0,3969	+ 0,2500	0,1575	+ 0,0992
$\lambda_8 =$	+ 1,8	+ 0,89	0,7921	+ 0,7050	0,6274	+ 0,5584
$\lambda_9 =$	− 4,9	+ 1,32	1,7424	+ 2,3000	3,0360	+ 4,0075
Totals:			29,2856	− 77,3600	261,7562	− 846,2897

Higher potentials of ΔB for the formation of normal equations

6.	7.	8.	9.	10.
2325,9924	− 8466,6125	30818,470	− 112179,230	408332,396
404,9612	− 1101,4945	2996,065	− 8149,296	22166,087
144,2158	− 330,2542	756,282	− 1731,886	3966,019
0,0515	− 0,0314	0,019	− 0,012	0,007
0,0007	− 0,0000	0,000	0,000	0,000
0,0625	+ 0,0394	0,025	+ 0,016	0,010
0,4970	+ 0,4423	0,394	+ 0,350	0,312
5,2899	+ 6,9826	9,217	+ 12,166	16,060
2881,0710	− 9890,9283	34580,472	− 122047,892	434480,891

The normal equations appear then as follows:

	$\mathfrak{A}'_1$	$\mathfrak{C}_1$	$\mathfrak{D}_1$	$\mathfrak{E}_1$	$\mathfrak{F}_1$
1.	29,2856	− 77,3600	261,7562	− 846,2897	2881,0710
2.	− 77,3600	261,7562	− 846,2897	2881,0710	− 9890,9283
3.	261,7562	− 846,2897	2881,0710	− 9890,9283	34580,472
4.	− 846,2897	2881,0710	− 9890,9283	34580,472	− 122047,892
5.	2881,0710	− 9890,9283	34580,472	− 122047,892	434480,891

for 1	= −	4,843
for 2	= +	26,66565
for 3	= +	63,15049
for 4	= −	481,78488
for 5	= +	2587,45361.

An estimate has indicated, that the coefficient of $\mathfrak{F}_1$ will finally be reduced to a few units in the decimals, just as the right hand side absolute member, with successive elimination of $\mathfrak{B}'_1$, $\mathfrak{C}'_1$, $\mathfrak{D}'_1$, and $\mathfrak{E}'_1$ in the fifth equation. It is therefore even doubtful whether the fixed coefficients are calculated to enough decimals to obtain the unknown quantities with the needed accuracy. For this reason, since the procedure proved itself completely unpractical, the calculation has been discontinued.

By using the straight line interpolation or the interpolation by means of a parabola of second degree respectively (compare formulae (4) and (5) p. 576) von Orff obtained p. 61 loc. cit. the following values from N', whereby N' for München (Nr. 6) was taken as zero:

(See Illustrative Material on Next Page.)

	N'		Geogr. latitude
	1.	2.	
1	− 0,80m	+ 8,41m	44,50°
2	− 1,08	+ 7,91	45,42
3	+ 1,12	+ 10,09	45,85
4	+ 1,62	+ 1,66	47,53
5	+ 0,28	+ 0,27	47,84
6	0,00	0,00	48,14
7	+ 0,81	+ 0,91	48,76
8	+ 1,02	+ 1,03	49,03
9	+ 1,38	+ 0,96	49,46

Par. 27. Continuation of the Numerical Example. The series (1.1*) pp. 577 and 578 gives after differentiation with respect to x:

$$\frac{\partial N'}{\partial x} = \left\{ \begin{array}{c} -b_1 \sin x - 2b_2 \sin 2x - 3b_3 \sin 3x - \cdots \\ + a_0 + a_1 \cos x + 2a_2 \cos 2x + 3a_3 \cos 3x + \cdots \end{array} \right\} \qquad (1)$$

In this the coefficients a and b have constant values, when the series has been applied to a meridian profile as in the above case.

The graphical presentation shows us immediately, that it is convenient to put $60\,\Delta B = x$.*) Hence $dx = 60\,d\Delta B$ and therefore:

$$\underset{\text{in seconds}}{\xi} = \frac{60\,\varrho''}{\varrho_m} \left\{ \begin{array}{c} -b_1 \sin x - 2b_2 \sin 2x - 3b_3 \sin 3x - \cdots \\ + a_0 + a_1 \cos x + 2a_2 \cos 2x + 3a_3 \cos 3x + \cdots \end{array} \right\} \qquad (2)$$

or abbreviated:

*)A somewhat smaller factor would have been still more adequate.

Par. 27. Continuation of the Numerical Example.

$$\xi = \underset{\text{in seconds}}{\left\{ \begin{array}{c} S_1 \sin x + S_2 \sin 2x + S_3 \sin 3x + \cdots \\ + T_0 + T_1 \cos x + T_2 \cos 2x + T_3 \cos 3x + \cdots \end{array} \right\}} \qquad (3)$$

Since for München $\xi = 0$ has been taken and at the same time $x = 0$, the following condition, to retain ξ exactly equal to zero, has to be met:

$$0 = T_0 + T_1 + T_2 + T_3 + \cdots, \qquad (4)$$

therefore finally:

$$\xi = \underset{\text{in seconds}}{\left\{ \begin{array}{c} S_1 \sin x + S_2 \sin 2x + S_3 \sin 3x + \cdots \\ - T_1 (1 - \cos x) - T_2 (1 - \cos 2x) - T_3 (1 - \cos 3x) - \cdots \end{array} \right\}} \qquad (5)$$

Hereby with restriction to six coefficients, the following error equations are obtained:

$x =$				
$-218^0\,24'$	$\lambda_1 =$	$+ 6{,}6 + 0{,}621\,S_1$	$- 1{,}783\,T_1$	$- 0{,}974\,S_2$
$-163\;12$	$\lambda_2 =$	$- 5{,}5 - 0{,}289$	$- 1{,}957$	$+ 0{,}553$
$-137\;24$	$\lambda_3 =$	$- 13{,}2 - 0{,}677$	$- 1{,}736$	$+ 0{,}996$
$-\;36\;36$	$\lambda_4 =$	$+ 12{,}1 - 0{,}596$	$- 0{,}197$	$- 0{,}957$
$-\;18\;\;0$	$\lambda_5 =$	$+ 3{,}5 - 0{,}309$	$- 0{,}049$	$- 0{,}588$
$+\;37\;48$	$\lambda_7 =$	$- 4{,}8 + 0{,}613$	$- 0{,}210$	$+ 0{,}969$
$+\;53\;24$	$\lambda_8 =$	$+ 1{,}8 + 0{,}803$	$- 0{,}404$	$+ 0{,}957$
$+\;79\;12$	$\lambda_9 =$	$- 4{,}9 + 0{,}982$	$- 0{,}813$	$+ 0{,}368$*)

for λ_1	$- 0{,}772\,T_2$	$+ 0{,}905\,S_3$	$- 0{,}574\,T_3$
for λ_2	$- 0{,}167$	$- 0{,}771$	$- 1{,}637$
for λ_3	$- 0{,}916$	$- 0{,}790$	$- 0{,}387$
for λ_4	$- 0{,}711$	$- 0{,}941$	$- 1{,}339$
for λ_5	$- 0{,}191$	$- 0{,}809$	$- 0{,}412$
for λ_7	$- 0{,}751$	$+ 0{,}918$	$- 1{,}397$
for λ_8	$- 1{,}289$	$+ 0{,}339$	$- 1{,}941$
for λ_9	$- 1{,}928$	$- 0{,}844$	$- 1{,}536.$

**)Should more exactly be equal to 0,371.

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The coefficients of the normal equations have been formed to four decimal places; meanwhile two would be adequate, so that a slide-rule would be sufficient here. These equations are:

1.	3,3631 S_1	− 0,4853 T_1	+ 1,0369 S_2	− 2,7170 T_2	+ 2,1366 S_3	− 2,6194 T_3
2.	− 0,4853	+ 10,9321	− 1,7466	+ 5,6888	+ 1,8481	+ 7,5092
3.	+ 1,0369	− 1,7466	+ 5,4983	− 2,1308	+ 0,1849	− 2,9844
4.	− 2,7170	+ 5,6888	− 2,1308	+ 7,9476	+ 1,4782	+ 8,6142
5.	+ 2,1366	+ 1,8481	+ 0,1849	+ 1,4782	+ 5,2475	+ 1,9976
6.	− 2,6194	+ 7,5092	− 2,9844	+ 8,6142	+ 1,9976	+ 13,2001

for 1 = − 0,0226
for 2 = − 23,6202
for 3 = + 40,9866
for 4 = − 9,3747
for 5 = − 6,7633
for 6 = − 3,4178.

They are checked by direct formation of their summation equation, which has been carried over for continuous control even in the successive elimination (Adjustment calculation p. 104).

The reduced normal equations are:

(See Illustrative Material on Next Page.)

Par. 27. Continuation of the Numerical Example.

$3{,}3631\,S_1 -$	$0{,}4853\,T_1 +$	$1{,}0369\,S_2 -$	$2{,}7170\,T_2 +$	$2{,}1366\,S_3 -$	$2{,}6194\,T_3$	1.
$10{,}8621$	$-1{,}5970$	$+5{,}2967$	$+2{,}1564$	$+7{,}1312$		2.
	$4{,}9438$	$-0{,}5144$	$-0{,}1568$	$-1{,}1283$		3.
		$3{,}1163$	$+2{,}1365$	$+2{,}9032$		4.
			$1{,}9922$	$+0{,}2198$		5.
				$3{,}4917$		6.

for 1 $= -\ 0{,}0226$
for 2 $= -\ 23{,}6235$
for 3 $= +\ 37{,}5204$
for 4 $= +\ 6{,}0305$
for 5 $= -\ 5{,}0035$
for 6 $= +\ 15{,}5709,$

which yield the following values for the unknowns:

$S_1 =$	$2{,}1674$	$\log = 0{,}33594$
$T_1 =$	$-\ 3{,}1793$	$0{,}50233_n$

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$S_2 = +\ 8{,}4951$	$0{,}92917$
$T_2 = -\ 0{,}1601$	$9{,}2044_n - 10$
$S_3 = -\ 3{,}0035$	$0{,}47763_n$
$T_3 = +\ 4{,}4594$	$0{,}64927.$

The remaining errors, that is the corrections of the given ξ (except of ξ for München) will be:

$$\left.\begin{array}{ll} \lambda_1 = +\ 0{,}187'' & \lambda_5 = -\ 1{,}385'' \\ \lambda_2 = -\ 0{,}163 & \lambda_7 = -\ 3{,}439 \\ \lambda_3 = +\ 0{,}107 & \lambda_8 = +\ 3{,}487 \\ \lambda_4 = +\ 0{,}273 & \lambda_9 = -\ 1{,}067 \end{array}\right\} [\lambda\lambda] = 27{,}190.$$

On the other hand, according to the two well-known control

formulae*) in the mean, $[\lambda\lambda] = 27{,}194$. Of these, the one which uses the terms of the reduced normal equations is interesting; for it shows, how $[\lambda\lambda]$ decreases with the removal of one unknown at a time. We obtain:

$$[\lambda\lambda] = [\xi\xi] - \frac{0{,}0226^2}{3{,}3631} - \frac{23{,}6235^2}{10{,}8621} - \cdots$$
$$= \left.\begin{array}{c} 457{,}00 - 0{,}000 - 51{,}378 - 284{,}756 \\ - 11{,}670 - 12{,}566 - 69{,}437 \end{array}\right\} = 27{,}193.$$

The formula for ξ will now be, since according to (4) $T_0 = -1{,}1200$, the following:

$$\xi_{\text{in seconds}} = \left\{\begin{array}{c} +2{,}1674 \sin x + 8{,}4951 \sin 2x - 3{,}0035 \sin 3x \\ -1{,}1200 - 3{,}1793 \cos x - 0{,}1601 \cos 2x + 4{,}4594 \cos 3x \end{array}\right\}$$

or

$$\xi_{\text{in seconds}} = \left\{\begin{array}{c} -1{,}1200 + 3{,}8478 \sin (x - 55^0\,43') \\ +8{,}4967 \sin (2x - 1^0\,5') \\ +5{,}3765 \sin (3x + 123^0\,59'). \end{array}\right\}$$

Further:

$$N = \left\{\begin{array}{c} b_0 - 2{,}1674 \cos x - 4{,}2476 \cos 2x + 1{,}0012 \cos 3x \\ -1{,}1200 x - 3{,}1793 \sin x - 0{,}0801 \sin 2x + 1{,}4865 \sin 3x \end{array}\right\} \frac{e_m}{60\,e''}$$

or

$$N = \left\{\begin{array}{c} b_0 - 1{,}1200\,x - 3{,}8478 \cos (x - 55^0\,43') \\ -4{,}2483 \cos (2x - 1^0\,5') \\ -1{,}7922 \cos (3x + 123^0\,59') \end{array}\right\} \frac{e_m}{60\,e''},$$

whereby x is considered in the second member as an arc, or x is equal

*) Adjustment calculation pp. 106 and 107 (6) and (9).

in degrees: ϱ^0. Also $\varrho_m : \varrho''$ for 46,5° latitude equals 30,87; hence

$$\frac{\varrho_m}{60\,\varrho''} = 0{,}5145.$$

Computing also, for checking purposes, ξ for the eight observations by the second of the first found formulae for ξ, we obtain:

$\lambda_1 = +0{,}193''$	$\lambda_5 = -1{,}386''$
$\lambda_2 = -0{,}163$	$\lambda_7 = -3{,}442$
$\lambda_3 = +0{,}101$	$\lambda_8 = +3{,}493$
$\lambda_4 = +0{,}271$	$\lambda_9 = -1{,}036,$

which is in accord with the above obtained values, to a very satisfactory degree, when we realize that the coefficients of the error equations are established only to three decimals. (λ_9 is indeed deviating from the result of the other calculation, due to a gross error of 0,03" in one of the coefficients, however even this is of no importance yet).

We find finally, using the formulae for ξ and N', where $(5{,}42 - b_0) \cdot 0{,}5145$ has to be added to all N', to make N' for München equal to zero:

(See Illustrative Material on Next Page.)

B	ΔB	x	ξ	N'
44,5°	− 3,64°	− 218° 24′	− 6,41″	+ 5,3m
45	− 3,14	− 188 24	− 5,59	+ 3,3
45,5	− 2,64	− 158 24	+ 7,55	+ 3,5
46	− 2,14	− 128 24	+ 12,78	+ 6,7
46,5	− 1,64	− 98 24	− 1,00	+ 8,6
47	− 1,14	− 68 24	− 15,32	+ 6,1
47,5	− 0,64	− 38 24	− 12,43	+ 2,0
48	− 0,14	− 8 24	− 1,89	+ 0,1
48,5	+ 0,36	+ 21 36	+ 1,61	+ 0,2
49	+ 0,86	+ 51 36	+ 1,58	+ 0,6
49,5	+ 1,36	+ 81 36	+ 4,00	+ 1,4
50	+ 1,86	+ 111 36	+ 1,68	

The results for N' deviate by 3 to 5m between 45,5° and 49° from the ones obtained graphically on p. 573. However when we observe the course of ξ according to the above calculation, in the drawing of Fig. 46, where it has been represented by curve III.

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we have to admit that there is at least no more certainty in the present result, than in the one produced graphically.

A close correspondence to all nine values of ξ would not be of much use, because it considerably influences the curve only in the vicinity of the points 7, 8 and 9, where at present the correspondence is still bad.*)

*)We have to mention finally, that we have applied the adjustment procedure to the numerical values of Par. 23, p. 572 only for the sake of an example. Strictly speaking it is completely irrelevant in the above case. The observations ξ are also too incomplete for a somewhat accurate definition.

Par. 28. Historical Notes on the Development of the Theory of Deflections of the Vertical. The relation (10) p. 537 between deflection of the vertical in longitude and azimuth, i.e. the repeatedly mentioned control equation, has already been given by Laplace in 1799, in the Mecanique celeste t. II, l. III, p. 117 and only specifically for the end points of a meridian arc.

Gauss utilized them in 1830 (compare Gauss' works, Bd. 4, p. 376) to check the reliability of measurements, according to a notice in the Opérations géodésique et astronom. pour la mesure d'un arc du parallèle moyen, exécutées en Piémont et en Savoie, etc. en 1821-23 (Astronomic and Geodetic Operations for the measuring of the prime meridian executed in Piedmont and Savoy, etc. in 1821-23), in these surveys the geographic longitude differences were established by powder signals; the controls therefore were of a far lower geodetic value, than now since the introduction of the electromagnetic telegraph.

Bessel 1837 in his repeatedly mentioned treatise, "Über den Einfluss der Unregelmässigkeiten der Figue der Erde auf geodätische Arbeiten und ihre Vergleichung mit den astronomischen Bestimmungen (About the Influence of the Irregularities of the Shape of the Earth on Geodetic Work and their Comparison with the Astronomic Determinations), (Astronom. Nachr. (Astronomic News) Vol. 14, Nr. 329-331, p. 269 or Treatises Vol. 3, p. 19 ff) dealt with all essentials. Since then there was no doubt left, about the justification or even necessity to calculate using the hypothesis of the ellipsoid of rotation; and about calculation instructions in

determining deflections of the vertical, etc. from measurements.

The main work of English survey Ordnance Survey, Principal Triangulation 1858 deals also with the problem, p. 609 ff.

Also General von Schubert has developed, in 1860 in the Astronom. Nachr.(Astronomic News) Vol. 52, Nr. 1245-47, p. 321, the formulae for the influence of a deflection of the vertical on the azimuth, latitude and longitude.

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Zenith distance of earthly objects was taken as 90°, so that it has no influence here. However he states the formula for horizontal angles.

According to Puissant, Traité de Géodésie, (Treatise on Geodesy) part II, in the middle of the 18th century Clairaut has made an attempt already before Laplace, to consider deviations from the shape of the ellipsoid of rotation.

Later descriptions are from Villarceau, Sadebeck (Astronom. Nachr. (Journal of Astronomy) 1877 Vol. 20, p. 113 and Vol. 91 p. 145) and Bremiker (written by v. Morozowicz loc. cit. Vol. 90, p. 353). The two latter authors occupy themselves especially with its influence on the horizontal angle.

Villarceau, in his first theorem, deals first with Laplace's theorem and then deals with different methods for the determination of the geoids elevation above the ellipsoid; first with those using zenith distances in connection with geometric

leveling (compare p. 521), and secondly with those using interpolations of the deflection of the vertical. For the latter case he states different interpolation formulae. But if he in order, to find an expression for N, assumes that N is a (analytic) function of ΔB and ΔL, so that one may state without doubt:

$$\frac{\partial \xi}{\partial P} = -\frac{\partial \eta}{\partial M}$$

since both differential quotients represent only the same value $\frac{\partial^2 N}{\partial P \partial M}$, then this is not quite correct (as mentioned before) and allowed only for an interpolation formula (compare p. 574).

His second theorem states the influence of the deflections of the vertical on the zenith distance. His third theorem states the last mentioned differential equation. Compare Verhandlungen der permanenten Kommission der europäischen Gradmessung (Negotiations of the Permanent Commission for European Arc Measurements), of 1875 (published 1875) p. 111 ff. where an assembly of his works, except the first theorem, may be found; or Journal de Mathem. pures et appl. (Journal of Pure and Applied Mathematics) (2. Ser., Vol. 12), 1867); Comptes rendus (Reports, or Transactions) 1868 Vol. 67 p. 1275, as well as 1871, Vol. 73, p. 808 (periodic series of Bertrand) and 1873, Vol. 76, p. 851 (Exponential series).

Bessel states (compare Treatises Vol. 3, p. 39) in his above-mentioned treatise also formulae for the determination of the special curvature conditions at one point, with which

the determination of an osculating ellipsoid is closely combined. Similarly Laplace 1799 in the Méc. cél. (Celestial Mechanics) Vol. II, Book III, p. 124 deals with the task of obtaining, from latitude and azimuth measurements, an ellipsoid of rotation osculating at one spot. (Compare also Ordnance Survey, Principal Triangulation, p. 622 ff. and Gauss in the Exchange of Letters between Gauss and Schumacher Vol. 2, p. 197).

Since these questions have no practical value whatsoever, they have not been dealt with here.

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Chapter 13

Determination of the Earth Ellipsoid from Arc Measurements.

Par. 1. Preliminary Remarks. Although the general formulae for the determination of the ellipsoid of rotation, which corresponds to the many astro-geodetic measurements distributed over the earth's surface, have essentially already been given in Par. 20, p. 562, it is still necessary to treat the determination of this earth ellipsoid in a special way. This leaves the generally used methods not unmentioned and also illustrates the importance of the results.

We again imagine the measurements reduced according to the manner presented on p. 485 ff. for land surveying, where any temporary elements of the meridian ellipse are used and all base lines are to be referred to sea level. As was already emphasized earlier, the purely geodetic results of the deflection of the vertical and the chosen elements of the meridian ellipse, compared with the desired earth ellipsoid, are affected less than the astronomical results. We will therefore be satisfied to assume the purely geodetic results as correct in the first approximation, with the exception of a reduction factor of the linear lengths, which depends upon the elevation between the geoid and the earth ellipsoid.

Par. 2. Two Latitude Measurements. Already, on p. 12, it has been mentioned that the meridian arcs were measured directly as horizontal distances in the oldest measurements for the determination of the size and shape of the earth. We image two meridian arcs,

in (taken absolutely) as different geographic latitudes as possible, determined directly in this manner, or indirectly by triangulation, and assume that their amplitudes do not exceed 1° to 2° of geographic latitude.

But, according to p. 52, one can regard a meridian arc of slight extension in relation to its length with great accuracy as a spherical arc. Its radius would correspond to the radius of curvature of the average latitude in the meridian, and its central angle would correspond to the difference of geographic latitude of the end points.

If now for an arc measurement, the geographic latitudes of the end points are designated by B_1 and B_2, the radius of curvature by ϱ_m for the latitude $B = \frac{1}{2}(B_1 + B_2)$, in the meridian, then we get

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$$\varrho_m = \varrho'' \frac{M}{\Delta B} + \cdots, \qquad (1)$$

there M designates the length of the meridian arc and ΔB is equal to the absolute value of the difference $B_2 - B_1$ in seconds.

e^2 is determined from the two values ϱ_m and ϱ'_m for the average latitudes B and B' as follows. We have:

$$\varrho_m = \frac{a_0(1-e^2)}{\sqrt{1-e^2\sin^2 B^3}}, \qquad \varrho'_m = \frac{a_0(1-e^2)}{\sqrt{1-e^2\sin^2 B'^3}}. \qquad (2)$$

From this it follows by division and involution:

$$\frac{1-e^2\sin^2 B}{1-e^2\sin^2 B'} = \left(\frac{\varrho'_m}{\varrho_m}\right)^{\frac{2}{3}}.$$

Reducing this to e^2 or $1-e^2$, then one obtains:

$$\left.\begin{aligned} e^2 &= \frac{q-1}{q\sin^2 B' - \sin^2 B} \quad \text{and} \quad 1-e^2 = \left(\frac{b_0}{a_0}\right)^2 = \frac{\cos^2 B - q\cos^2 B'}{q\sin^2 B' - \sin^2 B} \\ q &= \sqrt[3]{\left(\frac{\varrho'_m}{\varrho_m}\right)^2}. \end{aligned}\right\} \qquad (3)$$

Knowing e^2 first, then a_0 follows from the equations (2) for ϱ_m and ϱ'_m without any difficulty.

If one considers that ϱ_m and ϱ'_m differ relatively little from one another, then one can derive approximation formulae for e^2 or the flatness α with the omission of the second power of $(\varrho'_m - \varrho_m)$. These formulae are hardly any simpler than those above, but they nevertheless have been used previously. If we let $\varrho'_m > \varrho_m$, then we can equate successively:

$$\left.\begin{aligned} q &= \left(1+\frac{\varrho'_m-\varrho_m}{\varrho_m}\right)^{\frac{2}{3}} = 1+\frac{2}{3}\,\frac{\varrho'_m-\varrho_m}{\varrho_m}+\cdots \\ e^2 &= \frac{2}{3}\,\frac{\varrho'_m-\varrho_m}{\sqrt[3]{\varrho'^2_m\varrho_m}\sin^2 B' - \varrho_m\sin^2 B}+\cdots \end{aligned}\right\}. \qquad (4)$$

In the second term of the numerator one can substitute $\sqrt[3]{\varrho'^2_m\varrho_m}$ for ϱ_m, since this produces only an error of the order $\left(\frac{\varrho'_m-\varrho_m}{\varrho_m}\right)^2$ in the result. This is more insignificant yet, since the factor $\sin^2 B$ also supervenes (especially since B, taken absolutely, is the smaller geographic latitude).

If one also multiplies the numerator and denominator by $3600:\varrho''$, so that ϱ'_m and ϱ_m change to the arc lengths G' and G, and

regarding the relation of e^2 to α, as well as the relation $\sin^2 B' - \sin^2 B = \sin(B'+B)\sin(B'-B)$, then one obtains:

$$\alpha = \frac{1}{3}\,\frac{(G'-G):\sqrt[3]{G'^2G}}{\sin(B'+B)\sin(B'-B)} + \cdots. \qquad (5)$$

Equating, corresponding to the Peruvian and the revised older French measurement of arc*),

$$G' = 57012' \text{ for } B' = 45^0$$
$$G = 56734 \;\; „ \;\; B = -1^0\,31',$$

it follows that:

$$\log\frac{G'}{G} = 0{,}0021229 \qquad \log q = 0{,}0014153$$
$$q = 1{,}0032642 \qquad q - 1 = 0{,}0032642$$
$$q\sin^2 B' = 0{,}5016321 \qquad \sin^2 B = 0{,}0007005.$$

Formulae (3) for e^2 yields therefore $e^2 = 0{,}0065163$; from this it follows accurately:

$$\alpha = 0{,}0032635 = 1:306{,}4.$$

On the other hand formula (5) yields for α approximately:

$$\alpha = 1:306{,}7.$$

According to Todhunter, Figure of the Earth Vol. 1, p. 70 the problem of finding the axes of the earth ellipsoid from two latitude measurements was solved by Maupertuis in a treatise entitled Sur la Figure de la Terre (On the Shape of the Earth) in the memoirs of the French Scientific Academy of 1733 (publ. 1735). Maupertuis equates $\alpha = \frac{1}{3}(G'-G):(G'\sin^2 B' - G\sin^2 B)$, while E. Schmidt 1829 derives formula (5) in his Lehrbuch der mathematischen Geographie (Textbook of Mathematical Geography) Vol. 1, p. 178, with the difference, that $\sqrt[3]{G'^2G}$ is replaced by

*)According to R. Wolff, Handbuch der Math., Physik, Geodäsie und Astronomie Vol. 2, p. 130 and E. Mayer, Gestalt und Grösse der Erde p. 37. If one takes $G = 56732$, the same as Zach (cf. Mayer p. 36 elsewhere), then $\alpha = 1:304$ follows corresponding to the usual data reproduced by us on p. 13.

G. According to Gehlers physikalischem Wörterbuch (Gehler's Physical Dictionary), article on the earth p. 861, Euler and Roumovsky gave the formula attributed to Maupertuis above.

Formula (5) shows, that the accuracy of the determination of α for the same values of B and B' is especially dependent on the errors in $G' - G$.

If we modify and simplify this formula, for the purpose of further investigation in that direction, to:

$$\alpha = \left(\frac{G'}{G} - 1\right) : 3 \sin (B' + B) \sin (B' - B) + \cdots, \qquad (6)$$

then we see, that the numerator on the right-hand side takes the value $\frac{1}{100}$, its maximum value, for $B' = 90^\circ$ and $B = 0^\circ$ when $\alpha = \frac{1}{300}$. But as a rule, this numerator is much smaller and it is now easy to realize how much the deviations of the geoid from the earth ellipsoid influence the calculations of α.

For the above example, $\frac{G'}{G} - 1 = \frac{1}{200}$. If now for G the geoid is 1000^m above the ellipsoid, then G has to be diminished by $\frac{1}{6370}$ of its magnitude. The numerator in (6) increases therefore by $\frac{1}{6370}$ or $\frac{1}{3}$ of its magnitude, i.e., $\alpha = \frac{1}{297}$.

Even more considerable is the influence of the errors in geographic latitude. If the value of G is only upset by $5''$ due to deflections of the vertical or observation errors, then this results in errors of $\frac{5}{3600}$ or $\frac{1}{720}$ in G and therefore errors of $\frac{1}{36}$ in α in fractions of its magnitude. Due to this α changes to

approximately $\frac{1}{112}$ or $\frac{1}{110}$ (according to the sign of the error).

This example proves sufficiently, that not even a tolerably useful value of s can be obtained from two latitude measurements which are of only slight extent, since influences on the plumb line due to local attractions can easily occur in the magnitude of several seconds.

Par. 3. Reduction to the Spacing of the Parallels. The end points of an arc for the determination of ϱ_m, measured by triangulation, are in general not exactly on the same meridian. In order to find the curvature in the meridian, the meridian arc has to be deduced first from the distance between these points, which corresponds to their geographic latitudes and is called the spacing of the parallels.

The reduction formula turns out different, if one assumes, besides the geodetic line between points P_1 and P_2, the azimuth in P_1 or P_2, or the geographic difference of longitudes as given. But the dimensions of the ellipsoid of rotation, as the magnitudes to be determined, have to have as little influence as possible in the reduction. With regard to that, it is advantageous to assume, as Bessel has done, that both azimuths are given, that is, they are deduced directly from astronomical observations.

The formula, which is to be used for this, is (16) p. 308, where s designates the given geodetic line, while M is the spacing of the parallels. The influence of e^2 amounts, in fractions of M, indeed only to

$$\frac{1}{12} e^2 \frac{s^3}{a_0^3} \sin^2\alpha \cos 2B,$$

therefore nearly disappearing for practical cases, and we will be able, in the following, at least to neglect the influence of an error in e^2 on the reduction. Also the behavior of the formula in relation to a small error in a_0 is favorable.

Much more unfavorable is a formula which assumes only one of the azimuths, say $\alpha_{1.2}$, as given.

For the first terms of such a formula are:

$$M = -s\left(\cos\alpha_{1.2} + \frac{1}{2}\frac{s}{a_0}\tan B_1 \sin^2\alpha_{1.2}\left(1 - \frac{1}{2}e^2\sin^2 B_1\right) + \cdots\right) \quad (1)$$

and one sees, that the influence of a_0 and e^2 is here greater particularly since s appears only as a factor of e^2 to the first power.

Furthermore the (completed) formula (1) is obviously much more inconvenient for the numerical calculation, with regard to the number of reduction terms, than formula (16) p. 308 mentioned above.

As far as the inaccuracy of the latter formula in relation to the omitted terms (of the sixth order in relation to e and $\frac{s}{a_0}$) in the brackets is concerned, one can see directly that for the utmost the 8th decimal place of $\log M$ reaches a slight uncertainty for the first time when $s : a_0 = 0{,}1$, an accuracy which is always sufficient.

The reduction to the meridian arc can also be made by dropping a perpendicular from one end point of the meridian to the other, and determining the geographic latitude of the base as well as the length of the meridian arc to it. This method was used by Delambre

in the repeatedly mentioned Méthodes analytiques pour la Détermination d'un Arc du Méridien (Analytical Methods for the Determination of the Arc of a Meridian) (1799).

Several other formulae are also given in Ordnance Survey, Principal Triangulation pp. 248 ff.

Par. 4. Several Latitude Measurements. The newer latitude measurements determine the geographic latitudes not only for the extreme points of meridian chains of triangulation, but also for points of a more central location. Therefore each operation breaks down into several latitude measurements. Since the total number of measurements now exceeds two, the calculation of the values

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for the elements of the meridian ellipse requires an adjustment of all these measurements.

We designate the astronomical points of a single chain of triangulation, taken from south to north, by $P_0 P_1 \dots P_m$. If we imagine, as Bessel did, since the azimuth is also presumed as measured at all these stations, that the distance M of the parallels has been calculated for two neighboring stations at a time with the formula (16) of p. 308, mentioned in the previous paragraph, then one can form the meridian arcs between the parallels of P_0 and $P_1, P_2, \dots P_m$ by successive addition. We will designate them by $M_{0.1}, M_{0.2}, \dots M_{0.m}$.

But from these M, the spacing of the geographic latitudes can be calculated according to the formulae of Par. 10, pp. 53 ff, if

the elements of the meridian ellipse are presumed known. The formula for the calculation of the difference in latitude between P_0 and P_i from $M_{0.i}$ is for example:

$$B_i - B_0 = \underset{\text{in seconds}}{\Delta\sigma_{0.i}} + 3\varrho'' n \cos 2\sigma_{0.i} \sin \Delta\sigma_{0.i} + \frac{21}{8}\varrho'' n^2 \cos 4\sigma_{0.i} \sin 2\Delta\sigma_{0.i} + \cdots,$$

where:

$$2\sigma_{0.i} = 2\sigma_0 + \Delta\sigma_{0.i}$$

$$2\sigma_0 = \underset{\text{in seconds}}{2B_0} - \underset{\text{in seconds}}{3\varrho'' n \sin 2B_0} + \cdots$$

as well as:

$$\underset{\text{in seconds}}{\Delta\sigma_{0.i}} = 3600 \frac{M_{0.i}}{G}$$

and G is connected with a_0 by the equation:

$$G = \frac{a_0}{\varrho^\circ} \frac{1 + \frac{1}{4} n^2 + \cdots}{1 + n}.$$

(1)

Terms with n^3, which even for $M_{0.i} = 0{,}1\,a_0$ amount to hardly 0,001", are hereby not taken into account, since observation errors and deflections of the vertical produce much greater discrepancies in the equations, that is, from experience, two to three seconds on the average.

Now if G'' is an approximate value of G (average length of a degree) and n'' an approximate value of n (the auxiliary quantity introduced for e^2 and characterized by its significance on p. 37), then one can equate finally:

$$\frac{1}{G} = \frac{1}{G''}(1 + x) \quad \text{and} \quad n = n''(1 + y). \qquad (2)$$

where the x and y are only small fractions.

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The substitution of these values into (1) leads to the following formula:

$$B_i - B_0 = \underset{\text{in seconds}}{(B_i' - B_0')} + \underset{\text{in seconds}}{\Delta\sigma''_{0.i}}(1 + 3n'' \cos 2\sigma_i'')\,x + \left(\begin{matrix} 3\varrho'' n'' \cos 2\sigma''_{0.i} \sin \Delta\sigma''_{0.i} + \frac{21}{4}\varrho'' n''^2 \cos 4\sigma''_{0.i} \sin 2\Delta\sigma''_{0.i} \\ + 9\varrho'' n''^2 \sin 2B_0 \sin 2\sigma''_{0.i} \sin \Delta\sigma''_{0.i} \end{matrix}\right) y + \cdots, \qquad (3)$$

where the first term on the right-hand side designates the difference in latitude calculated with the approximate values; that is:

$$\underset{\text{in seconds}}{B_i'} - \underset{\text{in seconds}}{B_0'} = \underset{\text{in seconds}}{\Delta\sigma''_{0.i}} + 3\varrho'' n'' \cos 2\sigma''_{0.i} \sin \Delta\sigma''_{0.i} + \frac{21}{8}\varrho'' n''^2 \cos 4\sigma''_{0.i} \sin 2\Delta\sigma''_{0.i} + \cdots$$

$$2\sigma''_{0.i} = 2\sigma''_0 + \Delta\sigma''_{0.i} \qquad 2\sigma''_i = 2\sigma''_0 + 2\Delta\sigma''_{0.i}$$

$$\underset{\text{in seconds}}{2\sigma''_0} = \underset{\text{in seconds}}{2B_0} - 3\varrho'' n'' \sin 2B_0 + \cdots$$

$$\underset{\text{in seconds}}{\Delta\sigma''_{0.i}} = 3600\,\frac{M_{0.i}}{G''}. \qquad (4)$$

The terms multiplied by xy and y^2, which appear in formula (3) with the substitution of the expressions (2), have been omitted. They amount to, at the most, only several thousandths of a second for $x = 0{,}0001$ and $y = \frac{1}{10}$. (Moreover, one has to omit the mentioned terms in the later treatment according to the method of least squares; but this is unobjectionable since it is quite sure that, for example, the values found by Bessel for a_0 and n do not need corrections which significantly exceed the mentioned magnitudes of x and y.)

The observation values for B_0 and B_i naturally have to be introduced into the formulae (3) and (4), as well as those for $M_{0.i}$. But if we distinguish the observation values by a superscript and if we designate the corrections made necessary by the adjustment of B_0' and B_i' with λ_0 and λ_i, but $\delta_{0.i}$ for $M'_{0.i}$, then the error equation, which corresponds to the measurement of the arc $M_{0.i}$, follows as:

$$\left.\begin{aligned}\lambda_i-\lambda_0-3600\,\frac{\delta_{0,i}}{G''}\,(1+3n''\cos 2\sigma_i'')+\cdots &= (B_i''-B_0'')-(B_i'-B_0')\\ &\text{in seconds}\qquad\qquad\text{in seconds}\\ +\underset{\text{in seconds}}{\Delta\sigma_{0,i}''}(1+3n''\cos 2\sigma_i'')x+\Big(3\varrho''n''\cos 2\sigma_{0,i}''\sin\Delta\sigma_{0,i}''+\tfrac{21}{4}\varrho''n''^2\cos 4\sigma_{0,i}''\sin 2\Delta\sigma_{0,i}''&\\ +9\varrho''n''^2\sin 2B_0\sin 2\sigma_{0,i}''\sin\Delta\sigma_{0,i}''\Big)y.&\end{aligned}\right\}\quad(5)$$

The relations (4) remain, only $2\sigma_0''$ will be calculated with B_0' instead of B_0. The error term produced by this, as well as several other quite insignificant ones, has been omitted.

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In all the adjustments up to now, only the λ, but not the δ have been considered. The justification for it would be very dubious, if the λ would only correspond to observation errors; only according to (1) p. 535 the correction of the observed latitude B_i' to the ellipsoidal B_i is equal to the observation error $+\xi_i$, of the meridian components of the deflection of the vertical. Since this is mainly dependent on local deflections of the vertical, one can expect average magnitudes of several seconds for the λ.

The influence of the δ is now indeed substantially smaller. Only a δ of 30^m would produce about 1" in the left-hand side of (5). But errors in M can be produced by observation errors in the angles of the base lines, as well as by reduction errors in the calculation due to the inadequate knowledge of the geoid and the inadequate knowledge of a_0 and e^2. The influence of all these errors, except for that of the deflection of the vertical, has already been discussed in the reduction of the geodetic lines to the spacing of the parallels. According to formula (16) p. 308, the influence of the

deflection of the vertical on α is of especial importance, and one obtains with regard to its magnitude (cf. p. 535) the influence on the meridian arc M_{ik} between two adjacent points essentially as:

$$-M_{ik}\tan\alpha\,\frac{\eta_0\tan B_0+\eta_i\tan B_i}{2\varrho''}. \qquad (6)$$

This is always unimportant for meridian triangulation chains.

The influence of all error sources remains generally below 1", as long as one does not take meridian arcs over 1000^{km} long into account. It seems to us, that the omission of the reduction due to the height N of the geoid above the ellipsoid is to be feared most. It produces, e.g., with the assumption of $N = 200^{m}$ for $M = 1000^{km}$ already 1".

In spite of it one can still assume on the average, that the deflections of the vertical exert a much greater influence through the λ, than all the observation and reduction errors combined in δ.

If one now neglects the terms with δ in the error equations (5), then this means that one regards the M as accurately measured meridian arcs and projects all the errors onto the latitudes. The λ have, from then on, therefore, one more cause as a basis, which obviously has a systematic nature in the single degree measurement operation. This is more insignificant yet, with the still small numerical value, since the λ, insofar as they originate from deflections of the vertical, have, aside from it, also systematically depreciating constituents, which far exceed these amounts from experience.

Par. 5. Continuation: Adjustment. The error equation (5) assumes the following form after the omission of the δ:

$$\lambda_i - \lambda_0 = \{(B_i'' - B_0'') - (B_i' - B_0')\} + a_i x + b_i y, \qquad (1)$$

where the significance of the coefficients a_i and b_i is immediately apparent by comparison with (5).

The quantity in the braces has the significance: Calculated difference in latitude—observed difference in latitude, in short calculation—observation.

The other arcs are to be utilized in the same manner as arc $M_{0,i}$ is used for the setting up of the error equation (1). There are m arcs which are independent of one another between $m+1$ parallels. The best way of calculating them will be, as assumed till now, to proceed systematically from one starting point, or also, to take them in succession. The result of the adjustment is naturally quite independent from this, as well as from the choice of the origin. (If something is dubious, then it is only the motivation in Par. 4 of the admissibility of the abbreviation of the error equations by the omission of δ.)

We retain the most southern point P_0 as the origin of all arcs M, and we obtain therefore a system of error equations, for the singular latitude measurement with m astronomical points, of the form:

$$\left.\begin{aligned} \lambda_1 - \lambda_0 &= -l_1 + a_1 x + b_1 y \\ \lambda_2 - \lambda_0 &= -l_2 + a_2 x + b_2 y \\ &\cdots\cdots\cdots\cdots \\ \lambda_m - \lambda_0 &= -l_m + a_m x + b_m y. \end{aligned}\right\} \qquad (2)$$

The l serves as abbreviation for the differences obs.--calc. Each degree measurement produces such a system with the same unknowns x and y, while the l obviously always have different values. According to our assumption, degree measurements, which have a connection, are taken here as a single one.

For the adjustment according to the method of least squares, the sum of the squares $[ll]$ for all the different systems (2) taken together has to be minimized without the introduction of different weights. This becomes immediately apparent with regard to the most fundamental cause of the l, that is, the deflections of the vertical.

The remainder of the calculation can be done according to the foot of p. 143 of our adjustment calculation. Accordingly, one can most conveniently substitute the following modified system of error equations in place of (2):

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$$\left.\begin{aligned} l_0 &= -l'_0 + u + a'_0 x + b'_0 y \\ l_1 &= -l'_1 + u + a'_1 x + b'_1 y \\ l_2 &= -l'_2 + u + a'_2 x + b'_2 y \\ &\cdots\cdots\cdots\cdots \\ l_m &= -l'_m + u + a'_m x + b'_m y, \end{aligned}\right\} \qquad (3)$$

where, if the square brackets denote summation, the following has been equated:

$$l'_0 = -\frac{[l]}{m+1} \qquad a'_0 = -\frac{[a]}{m+1} \qquad b'_0 = -\frac{[b]}{m+1} \qquad (4)$$

and the following relations exists for a subscript i:

$$\left.\begin{aligned} l_i' &= l_i + l_o' \\ a_i' &= a_i + a_o' \\ b_i' &= b_i + b_o' \end{aligned}\right\} \qquad (5)$$

The complete systems (3) are now used in the usual manner for the formation of normal equations. Each system gives one equation for its characteristic constant u, which reduces to zero, since the normal equation in question is:

$$(m+1)u + [a']x + [b']y = [l']$$

and $[a']$, $[b']$, $[l']$ are zero for each system as shown by (4) and (5).

Therefore the normal equations for x and y follow from the systems (3) according to the scheme:

$$\left.\begin{aligned} [a'a']x + [a'b']y &= [a'l'] \\ [a'b']x + [b'b']y &= [b'l'], \end{aligned}\right\} \qquad (6)$$

where the summation is to be extended to all systems (3). G and n are obtained with x and y by means of the relations (2) p. 592, and therefore a_0 from the last equation (1) p. 592, as well as ε^2 or s from n by means of the relations stated on p. 37.

The weight calculations for x and y can be based on the system (6) in the usual manner. The weights of the values derived from x and y are produced from system (6) according to the usual rules of the method of the least squares (adjustment calculation pp. 81 and 137).

With the help of these weights, the mean error can be deduced after the mean error of an equation of the reduced systems (3) has been calculated; whereby the formula:

$$\mu = \pm\sqrt{\frac{[ll]}{[m]-2}} \qquad (7)$$

is applied, where $[m]$ designates the sum of all m, i.e., the total number of all independent arcs, from which the unknowns x and y have to be determined. Only, since the λ are not caused exclusively by the actions of random error causes, the result of (7), as also that of the weight calculations, is in any case of a dubious value; and one has to be prepared to accept the fact, that the uncertainty of the results for x and y is always much greater than that shown by the calculations.

In closing it also has to be emphasized that one cannot identify the λ required by the adjustment with the deflections of the vertical ξ. In this respect one should not be deceived by the similarity of the preceding problem with that of Par. 20, p. 562. Of course for the meridian arcs the equations (1), in the place cited, change over into the form of the equations (2), since in it the $a_i = 1$ and $b_i =$ zero, so that λ_i appears to be identical with ξ_i. But there, only <u>one</u> system of equations has been presumed, corresponding to <u>one</u> coherent operation. While now it is only a case of a combination of isolated operations. The adjustment substitutes for each sum:

$$[\lambda] = 0 \qquad (8)$$

and furthermore for all systems together also $[a'\lambda] = 0 = [b'\lambda]$, and one sees immediately, that the deflections of the vertical will not at all satisfy equation (8) for each operation separately.

The λ are therefore not the meridional components of the deflection of the vertical ξ, entirely apart from the above-mentioned

influences, which make themselves noticed along with the deflections of the vertical in the λ (cf. p. 607).

One cannot blame the adjustment procedure for it, since no calculation procedure can replace the lack of coherence of the operations (and even with this coherence assumed, it is still a long way from the reference ellipsoid now definable to an ellipsoid which would correspond to the complete geoid).

The first attempt to derive favorable values for the dimensions of the earth from several arc measurements seems to have been made by Boscovich. Boscovich and Maire, two Jesuits, made measurements (according to Todhunter, Figure of the Earth Vol. 1, p. 332) since 1750 in the Papal state and described their efforts in the work: De litteraria expeditione per Pontificiam ditionem; the French translation of it of 1755: Voyage astronomique et géographique dans l'État de l'Église (Astronomic and Geographic Journey through the Papal State) renders also a report on the adjustment procedure of Boscovich in an appendix, but applies it to nine meridian arcs, while Boscovich treats only five.

He equates the sums of the positive and negative λ and makes them as small as possible. The geometric development

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is replaced by Laplace in a transaction of the memoirs of the French Academy of 1789 (publ. 1793) by an interesting analytic development. This method is the third of the three adjustment procedures which he applied in his Mécanique céleste (Celestial

Mechanics) to the problem, namely that method, to whose results Laplace gave the most trust (Tome II, book III, p. 135 etc.). The other two methods are indeed less free of arbitrary acts.

The first of the methods (ibid. p. 126 ff.) was already applied by Laplace to the latitude measurements (and pendulum observations) in one of the transactions in the mentioned memoirs of 1783 (publ. 1786), and later again, but simplified, in the memoirs of 1789. In this method the largest error is made as small as possible.

Laplace found in the Méc. cél. p. 139 the flatness from seven latitude measurements with the second and third methods as $\frac{1}{177}$ and $\frac{1}{215}$. This huge difference in the results is explained by the great variations of the individual measurements, consequently quite different dimensions result from the combination in pairs. Therefore also considerably different values have to arise from the somewhat different combinations of seven measurements.

In Gehlers physikalischem Wörterbuch, article on the earth p. 872, one can find the flatness, according to 18 different combinations in pairs, as $\frac{1}{307,7}$ on the average (1827).

The method of least squares was probably first applied by Walbeck 1819, namely to six degree measurements, whereby he considered only the polar elevation at the extreme points and retained also only the first power of the flatness in the formulae. Ed. Schmidt included also the average polar eleva-

tions in several calculations undertaken by him between 1827 and 1830 due to the suggestion by Gauss.

Schmidt explained his method in 1829 in the first volume of his Mathematische Geographie (Mathematical Geography) p. 190 ff. Essentially the same method was later applied by Bessel, Astronom. Nachr. (Journal of Astronomy), 1837, Vol. 14, No. 333, p. 333 (transactions vol. 3, p. 41 ff.). Our presentation is essentially the same as the model given by Schmidt. The differences consist only of the derivation of the error equations (2) from the meridian arcs, whereby we did not follow Bessel either, who starts from formula (4) p. 49. Schmidt took the simpler formula and used s instead of n.

Bessel criticized the observation material quite vigorously and discovered particularly an error in the French Arc Measurement. It has been found lately, that several of the measurements of the meridian arcs (where one would have expected it least) which were used by him are affected by considerable errors. The Danish arc Lauenburg-Lysabbel is $8\frac{3}{4}''$ too small, as Andrae's revision has shown (Danish Degree Measurement Vol. 2, p. 489, also Vierteljahrsschr. der Astronom. Ges. (Quarterly of Astronomy and Geodesy) Vol. 12, p. 187) and the Hannover arc, which is based on the Braacker Base has to be lengthened by about $4''$ (that is $\frac{1}{50000}$ of the length and not $\frac{1}{10000}$ of the length as was stated at the last mentioned place due to printing errors). According to Ph. Fischer, Gestalt der Erde (The Shape of the Earth), changes up to $33{,}6''$ have been applied in the third edition of

Puissant, Traité de Géodésie (Treatise on Geodesy), to some arcs of the French Arc Measurements (in comparison to Bessel's

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last assumption of 1841?) and considerable changes in the southern part of the East India Arc Measurements can also be expected (Vierteljahrsschr. der Astronom. Ges. Vol. 8, p. 18).

Par. 6. Longitude Degree-Measurements. We will imagine beside the geographic longitude differences also the geographic latitude and azimuth as determined for the astronomical stations of an east-west chain of triangles. Therefore formula (13) p. 311 can be used immediately for the establishment of an equation for the elements of a meridian ellipse for two adjacent stations P_i and P_k. But by virtue of formula (3) p. 313 one can designate the product of $s \sin \alpha$ and the quantities in braces in that formula by P_{ik}, the parallel arc in the geographic latitude $B = \frac{1}{2}(B_i + B_k)$, which pertains to the geographic longitude differences L_{ik}. It then takes the form:

$$\left.\begin{array}{l} L_{ik} = \underset{\text{in seconds}}{\frac{\varrho''}{a_0}} P_{ik} W \sec B \\ W = \sqrt{1 - e^2 \sin^2 B}. \end{array}\right\} \qquad (1)$$

P_{ik} is to be calculated with approximate values of a_0 and e^2 and (just as M according to the reduction formula (16) p. 308) is dependent only to a very slight extent on the errors of these values. In the following discussion, same as in the discussion of M, we will not consider these insignificant errors.

The influence of the deflections of the vertical on P_{ik} is essentially given by the expression:

$$P_{ik} \cot \alpha \frac{\eta_i \tan B_i + \eta_k \tan B_k}{2\varrho''} \qquad (2)$$

(cf. (6) p. 594). This is always unimportant for east-west chains.

As far as the accuracy of formula (13) p. 311 is concerned, reduced to P_{ik}, it might be mentioned, that the terms of the sixth order which were neglected in the braces can influence the eighth decimal of the logarithm slightly, only when $s \sec B = 0{,}1 a_0$.

The unknowns which appear in equation (1), which is a basis for the following, are $\frac{1}{a_0}$ and e^2; in order to have the same unknowns as in the latitude degree-measurements, which is desirable for easily seen reasons, we will introduce G and n as unknowns as we did there. But we have

$$\frac{1}{G} = \frac{\varrho^0}{a_0} \frac{1+n}{1+\frac{1}{4}n^2+\cdots} \quad \text{and} \quad e^2 = \frac{4n}{(1+n)^2}. \tag{3}$$

With this we obtain by a simple calculation:

$$\frac{\sqrt{1-e^2\sin^2 B}}{a_0} = \frac{1+\frac{1}{4}n^2+\cdots}{\varrho^0 G (1+n)^2}\sqrt{1+n^2+2n\cos 2B} - \frac{1}{\varrho^0 G}\left\{\begin{array}{c} 1-n(2-\cos 2B) \\ +n^2\left(\frac{15}{4}-2\cos 2B-\frac{1}{2}\cos^2 2B\right)+Gl_0 \end{array}\right\}.$$

If we substitute this into (1), we have to take into consideration that $\varrho'' = 3600\,\varrho^0$. We will then obtain:

$$\underset{\text{in seconds}}{L_{ik}} = p_{ik}\left\{\begin{array}{c} 1-n(2-\cos 2B) \\ +n^2\left(\frac{15}{4}-2\cos 2B-\frac{1}{2}\cos^2 2B\right)+Gl_0 \end{array}\right\} \tag{4}$$

$$p_{ik} = 3600\frac{P_{ik}}{G}\sec B.$$

We will now again apply the approximate values G'' and n'' by equating:

$$\frac{1}{G} = \frac{1}{G''}(1+x) \qquad n = n''(1+y) \tag{5}$$

Therefore, from (4), the following equation results:

$$L_{ik} = L''_{ik} + a_{ik}x + b_{ik}y, \tag{6}$$

where the abbreviating symbols on the right-hand side are to be calculated according to the following formulae:

$$\left.\begin{aligned} L''_{ik} &= p''_{ik}\left\{\begin{matrix} 1 - n''(2-\cos 2B) \\ + n''^2\left(\frac{15}{4} - 2\cos 2B - \frac{1}{2}\cos^2 2B\right) + Gl_6 \end{matrix}\right\} \\ p''_{ik} &= 3600\,\frac{P_{ik}}{G''}\sec B \qquad a_{ik} = L''_{ik} \\ b_{ik} &= p''_{ik}\left\{\begin{matrix} -n''(2-\cos 2B) \\ + 2n''^2\left(\frac{15}{4} - 2\cos 2B - \frac{1}{2}\cos^2 2B\right) + \cdots \end{matrix}\right\}. \end{aligned}\right\} \tag{7}$$

We will finally also introduce the astronomically-observed values L'_{ik} and $B' = \frac{1}{2}(B'_i + B'_k)$. The latter have to be corrected for deflections of the vertical and observation errors. The corrections are respectively $\lambda'_k - \lambda'_i$ and $\frac{1}{2}(\xi_i + \xi_k)$, the latter without special

emphasis on the observation errors for the sake of simplicity. The component of λ'_i originating from the deflection of the vertical is, according to p. 535 (1), equal to $-\eta_i \sec B'_i$ and the one for λ'_k is corresponding. Designating finally also the correction of P_{ik} by δ'_{ik} then (6) changes over into the error equation corresponding to the measurement of the geographic difference in longitude L_{ik}:

$$\left.\begin{aligned} \lambda'_k - \lambda'_i - \left(\frac{\xi_i + \xi_k}{2\varrho''} \tan B' + \frac{\sigma''_{ik}}{P_{ik}}\right) L''_{ik} + \cdots \\ = L''_{ik} - L'_{ik} + a_{ik}x + b_{ik}y, \end{aligned}\right\} \qquad (8)$$

whereby it is assumed, that the average value of the astronomically-observed latitudes stated above is substituted for B in (7). Furthermore, different insignificant error terms have been omitted.

If we now retain only the difference $\lambda'_k - \lambda'_i$ on the left-hand side, i.e., if we project all the errors onto the geographic longitudes, then this is a useful approximation insofar as, at least for moderately large L_{ik}, the term on the left-hand side containing the brackets will be smaller, on the average, than the difference (with regard to δ'_{ik}, compare also the now appropriate remarks p. 594).

If we now apply formula (8) to the subscripts 0 to m, namely always in conformity with two adjacent points in pairs, the following system of equations results:

$$\left.\begin{aligned} \lambda'_1 - \lambda'_0 &= L''_{0.1} - L'_{0.1} + a_{0.1} \cdot x + b_{0.1} \cdot y \\ \lambda'_2 - \lambda'_1 &= L''_{1.2} - L'_{1.2} + a_{1.2} \cdot x + b_{1.2} \cdot y \\ \lambda'_3 - \lambda'_2 &= L''_{2.3} - L'_{2.3} + a_{2.3} \cdot x + b_{2.3} \cdot y \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda'_m - \lambda'_{m-1} &= L''_{m-1.m} - L'_{m-1.m} + a_{m-1.m}x + b_{m-1.m}y. \end{aligned}\right\} \qquad (9)$$

By adding successively the first and second equation, then the third, then the fourth, etc., one obtains a more convenient system, which on the left-hand side is equal to system (2) p. 595 and which can symbolically also be written on the right-hand side as:

$$\left.\begin{aligned} \lambda'_1 - \lambda'_0 &= -l_1 + a_1 x + b_1 y \\ \lambda'_2 - \lambda'_0 &= -l_2 + a_2 x + b_2 y \\ &\cdots\cdots\cdots\cdots \\ \lambda'_m - \lambda'_0 &= -l_m + a_m x + b_m y. \end{aligned}\right\} \qquad (10)$$

The same transformation as on p. 596 agrees with this, and the equations (3) to (7) can be transferred here, however with the modification that the weights have to be taken into account.

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If we give the weight 1 to the error equations of the latitude measurements p. 596 (3), then the weight of an error equation (3) p. 596, if they refer to longitude measurements, will have to be equated to $\cos^2 B_i$ where B_i is the geographic latitude of that point, to which the subscript of the correction λ'_i on the left-hand side of the error equation refers. This also takes into account the dominating influence of the deflections of the vertical on the error of measurement (except for older longitude measurements) and the fact that the magnitude $\eta_i \sec B_i$ is contained in λ'_i on the part of the deflection of the vertical.

These modified error equations with their weights are therefore, with the designations of p. 596, equal to:

$$\left.\begin{array}{lll} \lambda'_0 = -l'_0 + u + a'_0 x + b'_0 y & \text{weight} & \cos^2 B_0 \\ \lambda'_1 = -l'_1 + u + a'_1 x + b'_1 y & \text{„} & \cos^2 B_1 \\ \lambda'_2 = -l'_2 + u + a'_2 x + b'_2 y & \text{„} & \cos^2 B_2 \\ \cdots\cdots\cdots\cdots & & \\ \lambda'_m = -l'_m + u + a'_m x + b'_m y & \text{„} & \cos^2 B_m, \end{array}\right\} \qquad (11)$$

and the quantities on the right-hand side are determined (adjustment calculation p. 145) according to the formulae:

(See Illustrative Material on Next Page.)

$$\left.\begin{aligned} l'_0 &= -\frac{[l\cos^2 B]}{[\cos^2 B]} & l'_i &= l_i + l'_0 \\ a'_0 &= -\frac{[a\cos^2 B]}{[\cos^2 B]} & a'_i &= a_i + a'_0 \\ b'_0 &= -\frac{[b\cos^2 B]}{[\cos^2 B]} & b'_i &= b_i + b'_0. \end{aligned}\right\} \qquad (12)$$

The formation of the normal equation for u shows now that this value becomes zero; on the other hand the following normal equations result for x and y:

$$\left.\begin{aligned} [a'a'\cos^2 B]\,x + [a'b'\cos^2 B]\,y &= [a'l'\cos^2 B] \\ [a'b'\cos^2 B]\,x + [b'b'\cos^2 B]\,y &= [b'l'\cos^2 B], \end{aligned}\right\} \qquad (13)$$

which are to be combined with the corresponding normal equations of other longitudinal arc measurement operations, and possibly with those of the latitude measurements, by addition.

In other respects the conclusion of Par. 5 pp. 596 and 597 should be compared.

Par. 7. Azimuth Measurements. If we designate the product $s \sin \alpha$ and the quantities in the braces in formula (5) p. 312 by $\mathfrak{p}_{ik}$, if it is a question of two points P_i and P_k, then the difference of the measured azimuths gives the following equation:

$$\left.\begin{aligned} \underset{\text{in seconds}}{\Delta\alpha} &= -\frac{\varrho''}{a_0}\,\mathfrak{p}_{ik}\,W\tan B \\ \Delta\alpha &= \alpha_{ki} - \alpha_{ik} - 180^\circ. \end{aligned}\right\} \qquad (1)$$

Equation (1) shows a great similarity to equation (1) p. 599, which corresponds to the observed geographic differences in longitudes; since, except for the small differences in the braces appear-

ing in $P_{i,k}$ and $\mathfrak{P}_{i,k}$, these expressions, and therefore the right sides of the mentioned equations, are identical except for the factors $-\tan B$ and $\sec B$ respectively. For this reason, the equation for x and y, which can be deduced from the difference of the azimuths, will not be essentially different from that which results from the geographic difference in longitude, since the coefficients of x and y are in the same ratio in both equations.

The further treatment of the formulae is identical to that of the longitude determinations, only, neglecting observation errors, the weight of the error equations (11) p. 602 is not $\cos^2 B$, but $\cot^2 B$, since the deflection of the vertical influences the individual azimuth by $\eta \tan B$.

One can see immediately, that this weight determination becomes unreliable in the proximity of the equator, since here $\eta \tan B$ vanishes completely with respect to the observation errors. Then equation (1) is only a measure for the observation errors, but not a means for determination of the elements of the meridian ellipse due to the small factor $\tan B$.

The observation errors, by the way, come into consideration much more in the azimuth equations than in the equations of geographic longitudes due to a different reason, if we assume here, as always up to now, that an electromagnetic telegraph was used. For the observation errors in the latter case for the difference in longitude $L_{0.m}$ become not particularly larger than for any $L_{i.k}$, as long as (which will always be the case) the most distant points P_0 and P_m are directly connected to one another.

This is different for the azimuth equations. Here it is obvious that the observation errors have much more significance in the last equations of the system (10) p. 601, than in the first. The mentioned adjustment procedure can therefore only then be regarded as reliable, when the extent of the complete operation is not so considerable, that in the average the influences of the observation errors have to be regarded as equal to those of the deflections of the vertical--or if a control is created by the observation of geographic longitudes.

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Actually, when the geographic difference in longitude, as well as the azimuth difference, is available, then a control with regard to the influence of the observation error results. Therefore according to the above remarks, the absolute terms of the error equation, which are multiplied by $\sin B$, should correspond to those from the azimuth measurements. We will not enter into the application of these controls any further, since we consider it advisable, in the presence of such, to apply the formulae of Par. 15, p. 546 ff. for each individual operation to the open trend of geodetic lines $P_0P_1, P_1P_2, \ldots P_{m-1,m}$. To this one would have to add the terms for δa_0 and δc^2 according to Par. 20, p. 562 and possibly transform the above-used unknowns x and y (G and n).

On the other hand, one will be able to neglect the equations following from the azimuth measurements completely, for longitude measurements with far distant adjacent astronomical stations, with respect to the equations following from the geographic difference in longitudes, due to the limited certainty.

It can also be mentioned that the deflections of the vertical ξ make themselves noticed much more unfavorably in the error equations which correspond to the azimuth measurements as in those of the difference in longitude; compare (8) p. 601. The term in question on the left-hand side equals:

$$+\frac{\xi_i+\xi_k}{\varrho''\sin 2B}\Delta\alpha''. \qquad (2)$$

If one reduces it to the same weight as that in (8) p. 601, then it shows itself increased by a factor $\csc^2 B'$. The influence of the term (2) is generally only insignificant for geographic latitudes which are not small; in the vicinity of the equator it can increase quite considerably. But here, as was mentioned above, azimuth equations are useless anyway for the determination of the elements of the meridian ellipse.

The significance of the azimuth measurements in not-too-small latitudes was already recognized by Euler and later Laplace (according to Soldner, Bayerische Landesvermessung (Bavarian Cadastral Survey) p. 533). Only Legendre denies it due to an omission of p. 15 of the repeatedly mentioned Delambre, Méthodes analytiques, etc.

Par. 8. Arc Measurements Inclined to the Meridian. A reasonably extended arc measurement close to the meridian or parallel gives essentially only one equation for the unknown elements of the meridian ellipse. In order to obtain at least two of them, two latitude and two longitude measurements in (according to the absolute value) essentially different geographic latitudes are required of operations of that kind.

With measurements in the same geographic latitudes one can only obtain a solution, if one of the operations is a latitude measurement while the other is a longitude measurement. This can only be obtained in one single triangulation if the values of the azimuth are between zero and 90°.

If we imagine the geographic latitude and azimuth being measured at both ends of a geodetic line P_1P_2, then one can first calculate the spacing M of the parallels, and the auxiliary quantity $\mathfrak{P}$ which comes very close to the parallel arc of the mean latitude $B = \frac{1}{2}(B_1 + B_2)$ [cf. p. 308 (16) and p. 312 (5) or p. 603 (1)]. It then follows [cf. p. 588 (1) and (2)]:

$$\left.\begin{aligned} \frac{a_0(1-e^2)}{\sqrt{1-e^2\sin^2 B^3}} &= \varrho_m = \varrho'' \frac{M}{\Delta B} + \cdots \\ \frac{a_0}{\sqrt{1-e^2\sin^2 B}} &= \varrho_n = -\varrho'' \frac{\mathfrak{P}}{\Delta\alpha}\tan B, \end{aligned}\right\} \qquad (1)$$

where ϱ_m and ϱ_n are the radii of curvature of the meridian and the prime vertical for the mean latitude B, while ΔB and $\Delta\alpha$ designate the difference of the geographic latitudes and azimuths of the points P_1 and P_2 in accordance with known formulae.

The (1) yield now easily, with the introduction of an abbreviation q':

$$\frac{\varrho_n}{\varrho_m} = \frac{1-e^2\sin^2 B}{1-e^2} = q' \qquad (2)$$

$$e^2 = \frac{q'-1}{q'-\sin^2 B}, \qquad (3)$$

and for the flatness:

$$\alpha = \frac{1}{2}(q'-1)\sec^2 B + \cdots. \qquad (4)$$

If e^2 is also known, then (1) yield a_0 also without any difficulty. But the determination becomes completely unreliable for the vicinity of the equator, since here the $\Delta\alpha$ approaches zero, and the observation errors obtain a more and more dominating influence on ϱ_m, as the differentiation of $\log \varrho_m$ will immediately show.

In this and also in other respects, the determination of ϱ_m from the difference in longitude is more advantageous. We obtain therefore [p. 599 (1)]:

$$\varrho_m = \varrho'' \frac{P}{L} \sec B, \qquad (5)$$

where now P designates the parallel arc of mean latitude B.

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Now since P and $\mathfrak{P}$ are nearly equal for the same geodetic line the comparison of (5) with the second equation (1) shows that L taken absolutely is $> \Delta\alpha$. Observation errors of equal magnitude are therefore always of slighter influence in the determination of ϱ_m from (5) as from the second equation (1), but the difference decreases with increasing latitude.

On the other hand, the influence of the deflection of the vertical is always constant for both determinations. It amounts quite closely to $(\eta_2 - \eta_1)\tan B$ in $\Delta\alpha$ and $(\eta_1 - \eta_2)\sec B$ in L, therefore for ϱ_m in fractions of its value it amounts in both cases quite closely to

$$\frac{\eta_2 - \eta_1}{\varrho''} \frac{a_0}{s \sin\alpha}, \qquad (6)$$

if $s \sin\alpha$ is substituted in short for P and $\mathfrak{P}$.

The influence of the deflection of the vertical on φ_m is therefore:

$$\frac{\xi_2 - \xi_1}{\rho''} \frac{a_0}{s \cos\alpha}. \qquad (7)$$

The influence of these magnitudes on a becomes least for the equator, since here $q' - 1$ is greatest, that is about $\frac{1}{150}$. If one gives the line s the same extensions as for two latitude measurements for the purpose of comparison with p. 589, then a is influenced only half as much by the deflection of the vertical than in that example, keeping all the other conditions the same.

On the other hand, for higher latitudes the determination of a according to the method of this paragraph is increasingly unfavorable due to the accelerated decrease of $q' - 1$. It will be found, that generally, due to the influence of (6) and (7), two latitude measurements whose mean latitude taken absolutely differs by about 45° will be more advantageous than an arc measurement at 45° azimuth. The latter would be just as extended as the former together, since for it the mean latitude taken absolutely becomes $> 45°$.

Bessel, at the time, took the East Prussian arc measurement inclined to the meridian direction, in order to obtain two equations by it alone, which would have been sufficient for the determination of the meridian ellipse, if the deflections of the vertical and observation errors did not exist. In order to decrease the latter as much as possible for $\Delta\alpha$, he tried as far as possible to obtain long sides for the azimuth transfer.

The significance of such arc measurements was realized, by the way, already in the previous century, as Bessel mentions

also Abhandlungen (Transactions) Vol. 3, p. 63 , by Tobias Mayer the older (1723-1762), whose posthumous, very interesting manuscript about this topic is presented in the Astronom. Nachr. No. 310, p. 354 (1836). Mayer imagines the latitude and azimuth being measured at the end points of a geodetic line and shows, that one can determine the ratio $a_0 : b_0$ without any knowledge of the linear length of the line.

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Par. 9. Calculation of the Earth Ellipsoid and Arc Measurements in General. The method applied in Paragraphs 4 to 7, which sets up error equations corresponding to the astro-geodetic measurements, is not accurate, since always only certain influences of the deflection of the vertical are assumed as essential. The omission is indeed the more unimportant, the more triangle chains follow the direction of the meridians or parallels. Yet already for the latter case the remark about the [stricter adjustment] was necessary (cf. p. 604).

Now as soon as geodetic lines appear in random azimuths, a more rigorous treatment, especially due to the influence of the deflection of the vertical on M and P or $\mathfrak{P}$, is necessary [p. 594 (6) and 599 (2)].

If one wanted to do a rigorous calculation with the method of those paragraphs, then one would have to retain all the individual terms which give the successive deflection of the vertical and also

specify error terms, which would depend on the observation errors in longitude, latitude, azimuth and s. In the meantime, the originating form of the equation is very inconvenient for an accurate calculation.

For an accurate calculation one will proceed much more comfortably by the regulations of the previous chapter (as a rule Par. 15, p. 546 ff. applies, in any case also Par. 20, p. 562). One has to look, for the normal equations for ξ_1, η_1, δa_0 and δe^2 for each coherent operation. The deflections of the vertical ξ_1 and η_1 of the arbitrary and for each operation isolated special origin are then eliminiated by means of their normal equations and finally all normal equations for δa_0 as well as all for δe^2, which are produced by the different systems, are added. The two equations for δa_0 and δe^2 thus formed are the final normal equations, which now have to be solved for the two mentioned unknowns.

The significance of this calculation is that the dimensions of the reference ellipsoids of the individual adjustment systems are taken identical to axes of rotation parallel to the earth axis. Thus it remains undecided in what measure the axes of rotation and the centers of the individual ellipses coincide with one another and with the earth axis and the earth's center of gravity.

Only in the ideal case, when all arc measurements would be combined into one coherent operation, all the reference ellipsoids would change over into one. But this one ellipsoid could still in no manner be brought into relation with the earth's center of gravity by the arc measurements alone.

But in this case the elevations of the geoid above the ellipsoid could be considered

with adequately close positions of the astronomical stations. One would, for example, have to add terms of the form $-s\frac{N'}{a_0}$, where N' designates the elevation for the respective base line, to the δs in the equations (1) with the application of the directions of Par. 15, p. 546.*) Equations, which represent the difference N' of adjacent base lines as linear functions of the deflections of the vertical of the astronomical intermediate points, would have to be added to the present equations. Such equations would be established in surplus (probably for the N' of all adjacent astronomical stations) and they would at the same time be used with the other equations (the systems (1) p. 548) in the adjustment.

Besides we would hardly obtain a more accurate result, in this manner, for the most advantageous reference ellipsoid, since the errors, which result because of the lacking reduction of the lines s due to N', would presumably not accumulate in the end result, but will essentially compensate one another.

Par. 10. Inadequacy of the Arc Measurements for the Exact Determination of the Earth Ellipsoid. It has been already hinted in the previous paragraph, that the result of the adjustment of the error equations of several isolated arc measuring systems refers to just as many reference ellipsoids. These ellipsoids do not in general coincide among each other, but individually, they coincide approximately with the geoid surface in the region of the respec-

*) N is provided with the superscript, since the geoid does not come into consideration in the accuracy, but the spheroid defined on p. 565.

tive arc measuring system. One cannot obtain any means from the arc measurements alone, by which one could deduce that ellipsoid, which corresponds best with the measurements and at the same time coincides in its small axis with the earth axis and with its center of gravity with the earth's center of gravity.

They do not even offer such an ellipsoid in the ideal case, when the whole earth could be encompassed by a triangle net, whose entire points were astronomical stations, since a geometric connection between the position of the center of gravity and the deflection of the vertical does not exist, but only a dynamic one. Without further approaching the matter from this point of view one can still realize that much, that one will get closer to the ellipsoid, that is in regard to its dimensions, by numerous arc measurements, than by few. The latter case is indeed at present the actual one, and it will always remain so to a certain degree,

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since only $\frac{1}{11}$ of the earth's surface is covered by mainland and therefore is accessible for arc measurements.

It is now of interest to get an idea of the accuracy of the results for the elements of the meridian ellipse, as the present state of arc measurement supplies it. We will consider it in the following paragraph; but in doing so we naturally again have to accept the estimations of the absolute deflections of the vertical and elevations N of the geoid above the earth ellipsoid, as they are given by considerations of mechanics.

Chapter 13. Determination of the Earth Ellipsoid from Arc Measurements.

Par. 11. Degree of Accuracy. The dimensions of the earth ellipsoid have been calculated repeatedly with an increase in the number and extension of the latitude measurements especially. For it, the method given by Ed. Schmidt and Bessel which was applied above, and the method of least squares, has been applied. Ph. Fischer thought to have noticed (compare his Untersuchungen über die Gestalt der Erde (Investigations of the Shape of the Earth)) a great deficiency in it and has generally attacked the formal treatment. His expositions are not valid in this respect.*) Criticism can only be raised insofar as one has calculated the weights and mean errors of the end results as if the remaining errors were only the result of random observation errors and accidental deflections of the vertical, but consequently regarded them as much more accurate than they are (p. 597 top). For since the absolute deflections of the vertical within an isolated operation have essentially a systematic nature, it can happen (Ausgleichungsrechnung (Adjustment Computation) p. 257), that the remaining errors are small in spite of the significant magnitude of the deflections of the vertical, while the systematical component of the latter still exerts a very considerable influence on the results.

This is indeed the case in the East India arc measurement,

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*)The well-founded part of the criticism by Ph. Fischer with regard to the conception of the problem should in general not concern the calculators, but those who simply accept their work without knowing it in its formation. As far as the calculators are concerned it can be mentioned among others, that Paucker, whose detailed calculations were discussed on p. 17, realized not only the nature of the remaining errors of the adjustment, especially their relation to the absolute deflections of the vertical, but also noticed correctly the unfavorable influence of systematic deflections of the vertical on the quality of the calculation results.

for which Pratt thought great deflections of the vertical possible due to the external terrain shape. Now since this measurement was for all calculations the only one of great extension in small latitudes and therefore remained without significant control, and since the European measurements,

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with which it combined itself, acted as a control mainly only among themselves, therefore a certain systematic part of the deflections of the vertical had to remain unknown. This is a part, which one realizes immediately, except for an absolute constant and harmless part, which increases with the meridian arc length and has the same effect as a constant error of the base lines or the mass unit respectively--or as an error in the reduction of the base lines due to the height N of the geoid above the ellipsoid.

Among the many interesting calculations, which are presented in the repeatedly mentioned main publication of the English survey Ordnance Survey, Principal Triangulation, an indication of the influences of the changes of the units of measurement on the semiaxes $a_0 b_0$ of the meridian ellipse and the auxiliary quantities $\frac{1}{2}\frac{a_0+b_0}{a_0-b_0}$, i.e., $\frac{1}{2n}$ is given on p. 775. We will use here only the indication in relation to the latter magnitude.

One can now deduce from the nine isolated latitude measurement operations:

$$\begin{aligned}\frac{1}{2n} = 293{,}76 &+ 0{,}1139\,\Theta'_0 + 0{,}0144\,\Theta'_1 - 0{,}3589\,\Theta'_2 + 0{,}2222\,\Theta'_3 \\ &+ 0{,}0018\,\Theta'_4 - 0{,}0004\,\Theta'_5 + 0{,}0076\,\Theta'_6 - 0{,}0004\,\Theta'_7 - 0{,}0003\,\Theta'_8.\end{aligned} \qquad (1)$$

Chapter 13. Determination of the Earth Ellipsoid from Arc Measurements.

Whereby the linear lengths, which correspond to the results of the arc measurements, are still multiplied by $1 + \frac{\Theta_i}{100000}$, if the subscript i belongs to the individual operations according to the following chart:

Subscript	0	English	Arc	Measurement	from	49,9°-60,8°	Northern Latitude
"	1	French	"	"	"	38,7 -51,0	"
"	2	Russian	"	"	"	45,3 -70,7	"
"	3	Indian	"	"	"	8,1 -29,5	"
"	4	Older Indian	"	"	"	11,7 -13,3	"
"	5	East Prussian	"	"	"	54,2 -55,7	"
"	6	Peruvian	"	"	"	0,0 - 3,1	Southern
"	7	Hannover	"	"	"	51,5 -53,5	Northern
"	8	Danish	"	"	"	53,4 -54,9	"

The formula now gives a very good means to estimate the degree of accuracy of the determination of $\frac{1}{2n} = 293{,}76$.

It shows, moreover, that $\Theta_3 = -30$ would already create seven units change in $\frac{1}{2n}$.

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Such a magnitude of Θ_3 can of course hardly still be thought of as an influence of the elevation N of the geoid above the ellipsoid, since it would require a value of N of the magnitude $a_0 \cdot \frac{3}{10000}$, i.e., about 1,9 km. But the very plausible difference of the deflections of the vertical of 24" for the Great Indian Arc comprising 21°.4 is sufficient to explain that magnitude, if these 24" distribute themselves evenly about the arc and signify a leveling off.*) The shortening of the arc

*)Compare Pratt, A Treatise on Attractions etc. and the Figure of the Earth, Cambridge, 1860, p. 56.

by $\frac{24}{21,4.60.60}$ of its amount, that is a $\Theta_s = -31$, corresponds to such a difference.

The founded reproaches of Ph. Fischer will be eliminated to a certain extent (as well as the conditions permit) when the arc measurements have attained a more even extension than at the present and also, when those arc measurements, as long as they are not numerous, are excluded from the determination of the earth ellipsoid, for which strong irregularities can be expected due to the external shape of the earth. In relation to the valuation of the Indian Arc Measurement it should never be forgotten, that it was undertaken primarily, to study the shape of the geoid in India (compare our review in the Vierteljahrsschr. der Astronom. Ges. Vol. 8, p. 14).

We also have the attempt in mind, which was made by General V. Schubert and others, to free the geographic latitudes of the arc measuring stations of the local irregularities of the vertical by calculating the mass attractions (Astronom. Nachr., Vol. 52, No. 1245-47; cf. also p. 546). But with regard to the subterraneous mass irregularities it is much better, according to experience, to study and remove local anomalies by means of closely adjacent astronomical stations.

Par. 12. Conclusiveness of the Arc Measurement for the Existence of the Approximately Rotational, Ellipsoidal Shape of the Geoid.

Up to now the following question has not yet been discussed. Does it necessarily follow from the fact, that the arc measurements correspond, within a small deviation, to a surface of rotation, especially to a flattened ellipsoid of rotation, that the geoid would be, except for small deviations, a surface of rotation or especially

flattened ellipsoid of rotation? For we have only asked for the ellipsoid of rotation which best corresponds either to one

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complex of coherent measurements (the reference ellipsoid), or to several such complexes, which are not connected among themselves. For the latter case it has been investigated in the last preceding paragraphs, to what extent the results correspond to the earth ellipsoid, whose existence was assumed established by observations and measurements of a non-geometric nature. Even if it is completely without doubt, that one cannot treat the question of the determination of a rotational ellipsoid, possibly corresponding to the geoid as a whole, from a purely geometric point of view in order to arrive at the most confirmed results, it is still of interest to consider the question, whether the geometric measurements alone are already capable of proving the approximately rotational ellipsoidal shape of the geoid. Since, if we were able to measure astronomically a little less accurately than it is the case, the arc measurements could be completely adapted to an ellipsoid of rotation within the error of observation.

Therefore if we assume that the arc measurements would suffice as far as mathematical accuracy is concerned, we would still have to show that the geoid has really this shape and no other.

The problem will take different shapes according to the assumption of the result of the arc measurement. It would be the most complete, if it would determine the curvature in all azimuths

for a sufficient number of locations. But if the curvature corresponds to that of a sphere or an ellipsoid over all, then the geoid surely has also that form, since one can construct only one surface from the given radii of curvature, due to the assumption, that the geoid is a continuous curved surface.*)

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Taken as a whole, the geoid surely has a spherical shape. This follows already from the journeys around the earth by water and land alone, which established that the curvature relationships do always correspond to the spherical shape in general. But we are lacking material for a closer investigation. The arc measurements in the majority of cases are only done as

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*)In the treatise: Über die Bestimmung der Gestalt einer Krummen Oberfläche durch lokale Messungen auf derselben (On the Determination of the Shape of a Curved Surface by Local Measurements), Journal für reine und angewandte Mathematik von Crelle Bd. 64 S. 193 (Journal for Pure and Applied Mathematics of Crelle Vol. 64, p. 193), Christoffel showed in 1864 how the measurement of the sum of the two main radii of curvature alone in a sufficient number of points on the surface can lead to a knowledge of it. This assumes a continuous curved form which is arched everywhere. He also showed how, due to the fact that that sum would correspond to the known expression for the ellipsoid of rotation, power of proof for the existence of that form would be inherent in it. But since the measurements are regulated differently by the surveyors and since the study of the local terrain, presumed by Christoffel, is very difficult, it seemed appropriate to us to supply the proof, that the usual procedure of meridional latitude measurements also places material at one's disposal, whereby the existence of the shape of rotation can be proven purely geometrically without real hypotheses.

latitude measurements since they can be achieved easiest in that form. Therefore the investigation of the curvature in all azimuths is lacking;

In order to explain here to some extent the Christoffel method, we have to image it being applied to the determination of an enclosed plane curve which is everywhere convex towards the exterior (possibly the meridian curve of a body of rotation). Now let B be the angle of the normal directed inwards at the point (x, y) with the x-axis, and ϱ be the radius of curvature, then

$$\left.\begin{aligned} dx &= -ds \cdot \sin B \\ dy &= +ds \cdot \cos B \end{aligned}\right\} \quad ds = \varrho\, dB$$

and from it

$$x = -\int \varrho \sin B\, dB, \qquad y = +\int \varrho \cos B\, dB.$$

For the purpose of integration, ϱ has to be determined as a function of B from the measurements, for which the interpolative presentation in the form of a series is used.

One can here remark that it is simpler and more accurate not to deduce ϱ, but to stop at the arcs s and the corresponding B of the end points in the immediate results of the measurements. If one then presents s interpolatively as a periodic function of B one obtains:

$$x = -\int \sin B\, ds, \qquad y = +\int \cos B\, ds.$$

We would prefer this method for the determination of curves, since it corresponds completely to the usual procedure for latitude measurements. For the determination of surfaces we would like to give preference to the corresponding method of the 12th chapter due to practical reasons. That is we would prefer a simple surface as a basis, with the introduction of deflections of the vertical.

We will mentioned in passing, that the equations on the bottom of p. 5 are not applicable in the use of the developments in the mentioned treatise for other purposes than those of the author (that is possibly for the determination of the shape from the curvature measurements in certain azimuths). This is not emphasized by the author on that page. They are only valid, when, as in the treatise, one deals with the main curvatures, that is the curvature of planes which completely contain the normals of infinitely adjacent points; but they are not valid for the curvatures in random planes. Indeed one cannot equate $d\varrho$ or $d\varphi$ to zero in those formulae, because $\frac{\partial s}{\partial \varphi}$ = zero follows every time, which contradicts the assumptions.

but in general there is still a great lack of curvature measurements, so that we could not even assert with some certainty, that a general property of the geoid is: that it has equal arc lengths on the geographic meridians in the different geographic longitudes. Very many latitude measurements would indeed be necessary for the proof of the reliability of such an assertion.

With regard to the azimuth measurements the matter is much more favorable. They are always made with the latitude measurements. They can be used, with certain assumptions, to establish the coinciding of the geographic meridians (that is of the lines of equal geographic longitude) with the astronomical meridian planes. From this, as we will see, results also the rotational nature. Since each pair of opposite azimuth measurements contributes a share to the investigation of those properties, therefore measurements of that kind in the azimuth obviously give much more convincing material than an equal number of latitude measurements.

In the following we will first discuss an ideal case, which corresponds quite closely to those measurements, which are made most frequently due to practical reasons; that is the meridional latitude measurements with several azimuth determinations. In a single case we will always imagine two points of the surface for which it is established directly by astronomic measurements, that they are mutually situated in the azimuth zero and therefore have a common meridian plane as well as the geographic difference in

longitude of zero. There should be a sufficient number of such point pairs, that is, as many as are necessary for the assumption of a continuous development of the surface. Therefore it would appear as a general property of the latter, that the meridian plane of any point is also the meridian plane of all adjacent points of the surface through which it passes.

Without taking consideration of the curvature measurements themselves for the time being, we want to investigate first, what contribution is made by the establishment of the just mentioned property of the surface to its nature.

But if one follows the points of intersection of the meridian plane of a point with the surface in order, then one realizes, that all meridian planes of these points of intersection will coincide. That is, the geographic meridians would have to be plane curves whose planes are identical with the astronomical meridian planes of their points.

Now already the conjecture, that the desired surface would have to be a surface of rotation suggests itself.

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This can indeed be shown as follows.

We refer the surface to three rectangular coordinate axes, taking the z-axis parallel to the earth's axis of rotation and therefore the xy-plane parallel to the equatorial plane. The zx-plane will be used as the origin for the counting of all geographic longitudes L. If now, at the point P, the geographic latitude is

equal to B and the longitude equal to L, and designating the inclination of the normals at P to the three axes by λ, μ, ν, then:

$$\cos\lambda = \cos B \cos L \quad \cos\mu = \cos B \sin L \quad \cos\nu = \sin B, \tag{1}$$

This can be found immediately by the inspection of the line proportions of a rectangular parallelepiped, whose one diagonal is parallel to the normal at P and whose edges are parallel to the three axes. On the other hand one can find in a similar manner, that the cosines of the inclination of an element of line ds of the area to the three axes are equal, to in order:

$$\frac{dx}{ds}, \quad \frac{dy}{ds}, \quad \frac{dz}{ds}.$$

But if this element originates at P, then it is perpendicular to the normal and one obtains therefore with a well known theorem:

$$\frac{dx}{ds}\cos\lambda + \frac{dy}{ds}\cos\mu + \frac{dz}{ds}\cos\nu = 0,$$

or with regard to (1) and with the omission of the common denominator ds:

$$dx\cos B\cos L + dy\cos B\sin L + dz\sin B = 0. \tag{2}$$

This equation is the equation of condition that the transition of P to an infinitely close point of the surface takes place. But if we move a distance ds in the meridian plane of P, then the meridian plane and L remain constant; that is, the projection of P on the xy-plane is displaced, where the meridian plane appears as a line at an inclination L to the x-axis, in that line itself, and one obtains therefore:

$$dx : dy = \cos L : \sin L.$$

In order to express the fact that L is constant for dx and dy, that is only B is variable, we will write, after a common division by dB and with the application of the sign for partial differentiation:

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$$\frac{\partial x}{\partial B} : \frac{\partial y}{\partial B} = \cos L : \sin L. \qquad (3)$$

This equation has to be combined with (2), whereby it seems fitting to take x, y and z as functions of B and L and to completely neglect an equation between x, y and z. In general one has to equate in equation (2):

$$dx = \frac{\partial x}{\partial B} dB + \frac{\partial x}{\partial L} dL,$$

etc.

it yields therefore, depending on whether one varies B or L only:

$$\left.\begin{aligned} \frac{\partial x}{\partial B} \cos B \cos L + \frac{\partial y}{\partial B} \cos B \sin L + \frac{\partial z}{\partial B} \sin B = 0 \\ \frac{\partial x}{\partial L} \cos B \cos L + \frac{\partial y}{\partial L} \cos B \sin L + \frac{\partial z}{\partial L} \sin B = 0. \end{aligned}\right\} \qquad (4)$$

But these equations are identities, insofar as one considers x, y and z as functions of B and L, since there exists no equation between B and L. If we now differentiate on the left-hand side with respect to B or L, then we again obtain diminishing expressions. We will use only the differentiation of the first equation with respect to L and the differentiation of the second equation with respect to B, since the same higher differential quotients appear only for this case. We will then obtain:

$$\left.\begin{aligned}&\frac{\partial^2 x}{\partial B\partial L}\cos B\cos L+\frac{\partial^2 y}{\partial B\partial L}\cos B\sin L+\frac{\partial^2 z}{\partial B\partial L}\sin B\\&-\frac{\partial x}{\partial B}\cos B\sin L+\frac{\partial y}{\partial B}\cos B\cos L\end{aligned}\right\}=0$$

$$\left.\begin{aligned}&\frac{\partial^2 x}{\partial L\partial B}\cos B\cos L+\frac{\partial^2 y}{\partial L\partial B}\cos B\sin L+\frac{\partial^2 z}{\partial L\partial B}\sin B\\&-\frac{\partial x}{\partial L}\sin B\cos L-\frac{\partial y}{\partial L}\sin B\sin L+\frac{\partial z}{\partial L}\cos B\end{aligned}\right\}=0.$$

If we now assume that we are dealing with a surface whose curvature changes constantly, then

$$\frac{\partial^2 x}{\partial B\partial L}=\frac{\partial^2 x}{\partial L\partial B},\qquad (5)$$

etc.

The subtraction of the two preceding equations yields therefore:

$$-\frac{\partial x}{\partial B}\cos B\sin L+\frac{\partial y}{\partial B}\cos B\cos L$$
$$=-\frac{\partial x}{\partial L}\sin B\cos L-\frac{\partial y}{\partial L}\sin B\sin L+\frac{\partial z}{\partial L}\cos B.$$

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If one adds the right-hand side of this equation to the left-hand side of the second equation (4) which was first multiplied by $\tan B$, then one obtains easily:

$$\frac{\partial z}{\partial L}=-\frac{\partial x}{\partial B}\cos^2 B\sin L+\frac{\partial y}{\partial B}\cos^2 B\cos L.\qquad (6)$$

The three equations (4) and (6) are valid generally; but with the introduction of the special condition (3) they take the following form:

$$\frac{\partial z}{\partial B}\sin B=-\frac{\partial x}{\partial B}\cos B\sec L,\qquad (7)$$

$$\frac{\partial s}{\partial L}\sin B = -\frac{\partial x}{\partial L}\cos B\cos L - \frac{\partial y}{\partial L}\cos B\sin L, \tag{8}$$

$$\frac{\partial s}{\partial L} = 0. \tag{9}$$

The last equation shows, that as a result of condition (3)

$$s \text{ is a function of } B \text{ only} \tag{10}$$

The meridian sections are therefore (plane) congruent curves; further, the parallels (i.e., the lines of same latitude B) are also plane curves, whose planes are normal to the axis of rotation. One also realizes further, that the line element of the parallel is at right angles to the meridian plane of every point, because it is in the surface and the plane of the parallel, and both are perpendicular to the meridian plane at P.

If we now call R the radius of curvature of the parallel at P, then one obtains for the element of arc ds of the same, with regard to the above-mentioned, equation:

$$ds = R\,dL,$$

and from it, since the latitude B is constant for ds:

$$R = \frac{\partial s}{\partial L}. \tag{11}$$

But now the inclination of the element ds to the x-axis is equal to $L + 90^0$, therefore $ds = dx\cos(L + 90^0) + dy\sin(L + 90^0)$ or better:

$$\frac{\partial s}{\partial L} = -\frac{\partial x}{\partial L}\sin L + \frac{\partial y}{\partial L}\cos L. \tag{12}$$

From (11) and (12) follows a relation of R to $\frac{\partial x}{\partial L}$ and $\frac{\partial y}{\partial L}$,

where the latter two are dependent upon one another in accordance with (8) and (9). One obtains therefore:

$$\left.\begin{aligned} R &= -\frac{\partial x}{\partial L}\sin L + \frac{\partial y}{\partial L}\cos L \\ 0 &= \frac{\partial x}{\partial L}\cos L + \frac{\partial y}{\partial L}\sin L . \end{aligned}\right\} \qquad (13)$$

If one eliminates $\frac{\partial y}{\partial L}$ from both, then one obtains:

$$R = -\frac{\partial x}{\partial L}\csc L \qquad (14)$$

and

$$\frac{\partial R}{\partial B} = -\frac{\partial^2 x}{\partial B \partial L}\csc L .$$

But the differentiation of equation (7), which is multiplied by $\cos L$, with respect to L yields, since, because of (9),

$$\frac{\partial^2 z}{\partial B \partial L} = 0 :$$

$$\frac{\partial^2 x}{\partial B \partial L}\csc L = \frac{\partial z}{\partial B}\tan B$$

and one obtains therefore:

$$\frac{\partial R}{\partial B} = -\frac{\partial z}{\partial B}\tan B . \qquad (15)$$

This equation shows, in conjunction with (10), that $\frac{\partial R}{\partial B}$ is a function of B alone; therefore the following has to be true:

$$R = f(B) + \varphi(L), \qquad (16)$$

where f and φ designate functions.

If this surface should be a surface of rotation, then $\varphi(L)$ has to disappear or be constant.

This necessitates L being indeterminate in at least one point of the surface; for if $\varphi(L)$ yields the same for indeterminate values of L, then it is no longer a function of L.

Our surface therefore has at least two such points, since the geoid is without any doubt spherically enclosed (see above). If we now imagine the plane of the parallel normal to the z-axis in motion and possibly approaching the surface from the exterior, then it can be easily seen, that the mentioned nature of the enclosure requires a diminishing of the parallel to zero at two points. It further requires that infinitely small parallels are situated around these points and that therefore an infinite number of meridians would have to pass through those points.

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The azimuth measurements at the occasion of the latitude measurements were therefore alone sufficient to decide the question, whether measurements on a surface of rotation can be made or not with no other assumption than that of a surface which would be constantly curved and arched, and also spherically closed. But according to the general theorems of the potential theory (neglecting local curvature inconstancies) a constant development in relation to flexure and curvature exists and the surface is closed without cutting itself. The spherical nature (the simple relationship) follows, as already mentioned, from the rough curvature measurements at all azimuths during surface observation trips (apart from obvious arguments of another kind, as for example the shape of the earth's shadow).*)

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*)If the earth could have a ring shape, then the azimuth measurements alone would not be sufficient to prove the rotational shape, since L does not become indeterminate at any point for rings because the parallels do nowhere decrease to zero.

Par. 12. Conclusiveness of the Arc Measurement for the Existence of the Approximately Rotational, Ellipsoidal Shape of the Geoid.

The curvature measurements in the meridian yield, after the rotational nature has been proven, the shape of the meridian curve quite explicitly, if the latitude at the poles is 90°, which is required by the constant curvature.

The practical conditions claim of course that deviation from our ideal case, that the azimuth measurements cannot be made exactly mutually opposite for points of the same geographic meridian. In order to compare the azimuths it is therefore necessary to introduce approximate values for the curvature relations. This alone is quite insignificant if the astronomical stations to be compared have always only a small difference in geographic longitude and if it can be assumed, that the station situated northerly of the two adjacent stations falls to the east as often as it does to the west. An error in those values will then produce deviations of an accidental nature, just as the local irregularities.

These local irregularities now postulate in relation with the existence of the inevitable observation errors an extension of the azimuth measurements to as many parts of the surface as possible, as well as that of the latitude measurements to many meridians in general. As far as the small systematic deviations which appear are concerned, they point to small deviations from the rotational shape, but one does not have to be afraid that they completely obliterate this complete nature of the surface.

Par. 13. Continuation: Latitude Measurements. We will now treat the ideal case already mentioned above, which was declared as being of little practical value. It has been found by numerous latitude measurements that a general property of the geoid is, that it has arcs of equal length between two parallels at a time on all geographic meridians. Presented mathematically this means:

$$\frac{\partial s}{\partial B} \text{ is a function of } B \text{ only.} \tag{1}$$

Now, however,

$$\left(\frac{\partial s}{\partial B}\right)^2 = \left(\frac{\partial x}{\partial B}\right)^2 + \left(\frac{\partial y}{\partial B}\right)^2 + \left(\frac{\partial z}{\partial B}\right)^2. \tag{2}$$

We will eliminate x and y from this equation by means of the first equation (4) and equation (6) of the previous paragraph, which yield:

$$\left.\begin{aligned} \frac{\partial x}{\partial B} &= -\frac{\partial z}{\partial B}\tan B\cos L - \frac{\partial z}{\partial L}\sec^2 B\sin L \\ \frac{\partial y}{\partial B} &= -\frac{\partial z}{\partial B}\tan B\sin L + \frac{\partial z}{\partial L}\sec^2 B\cos L. \end{aligned}\right\} \tag{3}$$

Therefore it follows from (2) and (1), if F designates an arbitrary function and if the polar distance $P = 90^\circ - B$ is substituted for B:

$$\left(\frac{\partial z}{\partial P}\right)^2 \sin^2 P + \left(\frac{\partial z}{\partial L}\right)^2 = F(P). \tag{4}$$

In order to integrate this equation with regard to the condition, that our surface has at least two points where P is equal to zero, we will limit ourselves, for the present, to the consideration of such a small region located around one of these points that

a convergent development of s according to powers of P is possible for it. At the same time we will displace the coordinate origin to that point and equate therefore:

$$s = P^2 L_2 + P^3 L_3 + P^4 L_4 + \cdots, \tag{5}$$

where $L_2, L_3, L_4 \cdots$ are functions of L alone. The term with the first power of P has been omitted, since $\frac{\partial s}{\partial P}$, for $P=0$, has to become zero according to our assumptions about the surface. These assumptions postulate also, that L_2 is either only positive or negative for all values of L. Because otherwise there would be no plane normal to the s-axis, which does not intersect the surface infinitely close to the chosen point, and the point would therefore not belong to

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those two necessarily existing points, of which one should have become the coordinate origin.

We will now form the following expressions:

$$\begin{aligned}
\frac{\partial s}{\partial P} &= 2PL_2 + 3P^2 L_3 + 4P^3 L_4 \cdots \\
\sin P &= P - \tfrac{1}{6} P^3 + \cdots \\
\frac{\partial s}{\partial P} \sin P &= 2P^2 L_2 + 3P^3 L_3 + \left(4L_4 - \tfrac{1}{3} L_2\right) P^4 + \cdots \\
\frac{\partial s}{\partial L} &= P^2 L_2' + P^3 L_3' + P^4 L_4' + \cdots.
\end{aligned}$$

In the last line, the upper dash on $L_2, L_3 \cdots$ designates the first derivative with respect to L. If we substitute the above formulae into (4), it follows:

$$\begin{aligned}
&(4L_2^2 + L_2'^2) P^4 + (12 L_2 L_3 + 2 L_2' L_3') P^5 \\
&+ \left(-\tfrac{4}{3} L_2^2 + 16 L_2 L_4 + 9 L_3^2 + 2 L_2' L_4' + L_3'^2\right) P^6 + \cdots = F(P).
\end{aligned} \tag{6}$$

This equation is only satisfied, if the coefficients of P^4, P^5, P^6... are individually constant. First we would have to have:

$$4L_2^2 + L_2'^2 = k^2, \qquad (7)$$

k constant. From this follows either $L_2 =$ constant or

$$dL = \frac{\pm dL_2}{\sqrt{k^2 - 4L_2^2}}, \quad L_2 = \pm \frac{k}{2} \sin 2L. \qquad (8)$$

Of these two solutions only the first one is usable, because the second one contradicts for $k \gtrless$ zero (except for the assumption $L_2 = \pm$ abs. val. $\frac{k}{2} \sin 2L$, according to which the surface would have edges) the condition, that L_2 can only be positive or negative. But for $k =$ zero the radius of curvature at the coordinate origin would become zero and therefore the direction of the normal would acquire an indeterminacy there.

Therefore only the possibility to substitute $L_2 =$ constant remains. But since the coefficient of P^6 in (6) has also to be constant, L_3 also becomes necessarily constant. And therefore each following coefficient shows the constancy of a following value in the series L_4, L_5... Therefore it results, that z can only be a function of P and therefore a function of the geographic latitude B alone at the coordinate origin.

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Our surface is therefore a surface of rotation in the vicinity of the two critical points; but it is a surface of rotation in general, since it should be consistent and since all properties are already revealed in each small, but finite part of such a surface.

That is, one can assume close to the boundary of the region, where (5) stops to converge, a new origin of a Taylor expansion which converges in part in the first region and in part outside of it. Here also s can only be a function of B alone. In this manner the whole surface can be enclosed little by little.

The assumptions in the foregoing solution of the problem were essentially the same as in the previous problem. Only the development (5) requires a constancy and finiteness not only for the first and second differential quotient (i.e., in flexure and curvature) but also for all higher differential quotients. But one can regard these conditions as fulfilled for the higher differential quotients for the geoid, except for local deviations, for the same reasons as for the first two alone.

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(See Illustrative Material on Next Page.)

TABLE

for

$$\log W = \log \sqrt{1 - e^2 \sin^2 B},$$

the geographic latitude B from 47° to 57°

$$\log e^2 = 7{,}8244104.15 - 10$$

Remark: The eleventh decimal place is only included, in order to obtain the tenth place only uncertain by one unit at the most for the threefold and fourfold magnitudes of log W. Itself it can probably be incorrect by a maximum of up to three units.

B	log W	1. Diff.	2. Diff.	B	log W	1. Diff.	2. Diff.
47° 0′	9,9992234.0243			49°30′	9,9991603.5613		
		21.1014	4			20.8959	9
5	2212.9229		4	35	1582.6654		10
		21.0972	5			20.8864	10
10	2191.8257		5	40	1561.7790	20.8765	10
		21.0927	5			20.8666	10
15	2170.7330		5	45	1540.9025	20.8565	10
		21.0881	5			20.8462	10
20	2149.6449		5	50	1520.0359	20.8357	11
		21.0833	5			20.8251	11
25	2128.5616		5	55	1499.1794	20.8142	11
		21.0784	5			20.8031	11
47°30′	9,9992107.4832			50° 0′	9,9991478.3333	20.7919	11
		21.0732	5			20.7805	12
35	2086.4100		5	5	1457.4976	20.7688	12
		21.0678	6			20.7572	12
40	2065.3422		6	10	1436.6725	20.7453	12
		21.0622	6			20.7334	12
45	2044.2800		6	15	1415.8583	20.7213	13
		21.0567	6			20.7087	13
50	2023.2233		6	20	1395.0552	20.6960	13
		21.0508	6			20.6832	13
55	2002.1725		6	25	1374.2633	20.6702	13
		21.0447	6			20.6571	13
48° 0′	9,9991981.1278			50°30′	9,9991353.4828	20.6437	13
		21.0384	6			20.6303	14
5	1960.0894		6	35	1332.7140	20.6167	14
		21.0320	7			20.6030	14
10	1939.0574		7	40	1311.9568	20.5890	14
		21.0255	7			20.5748	15
15	1918.0319		7	45	1291.2115	20.5603	15
		21.0188	7			20.5458	15
20	1897.0131		7	50	1270.4781		
		21.0118	7				
25	1876.0013		7	55	1249.7568		
		21.0047	7				
48°30′	9,9991854.9966			51° 0′	9,9991229.0481		
		20.9974	8				
35	1833.9992		8	5	1208.3521		
		20.9898	8				
40	1813.0094		8	10	1187.6689		
		20.9823	8				
45	1792.0271		8	15	1166.9987		
		20.9745	8				
50	1771.0526		8	20	1146.3416		
		20.9663	8				
55	1750.0863		8	25	1125.6979		
		20.9581	8				
49° 0′	9,9991729.1282			51°30′	9,9991105.0676		
		20.9497	8				
5	1708.1785		8	35	1084.4505		
		20.9413	9				
10	1687.2372		9	40	1063.8479		
		20.9326	9				
15	1666.3046		9	45	1043.2589		
		20.9236	9				
20	1645.3810		9	50	1022.6841		
		20.9144	9				
25	1624.4666		9	55	1002.1238		
		20.9053	9				
49°30′	9,9991603.5613			52° 0′	9,9990981.5780		

B	log W	1. Diff.	2. Diff.	B	log W	1. Diff.	2. Diff.
52° 0′	9,9990981.578₀		15	54°30′	9,9990372.818₆		20
		20.531₀				20.009₇	
5	0961.047₀		15	35	0352.808₉		20
		20.516₁				19.989₇	
10	0940.530₉		15	40	0332.819₂		20
		20.501₁				19.969₅	
15	0920.029₈		15	45	0312.849₇		20
		20.486₀				19.949₁	
20	0899.543₈		15	50	0292.900₆		21
		20.470₆				19.928₄	
25	0879.073₂		16	55	0272.972₂		21
		20.455₀				19.907₇	
52°30′	9,9990858.618₂		16	55° 0′	9,9990253.064₅		21
		20.439₄				19.887₀	
35	0838.178₈		16	5	0233.177₅		21
		20.423₅				19.866₀	
40	0817.755₃		16	10	0213.311₅		21
		20.407₁				19.844₇	
45	0797.348₂		16	15	0193.466₈		21
		20.390₉				19.823₃	
50	0776.957₃		16	20	0173.643₅		22
		20.374₅				19.801₈	
55	0756.582₈		17	25	0153.841₇		22
		20.358₀				19.780₀	
53° 0′	9,9990736.224₈		17	55°30′	9,9990134.061₇		22
		20.341₃				19.757₉	
5	0715.883₅		17	35	0114.303₈		22
		20.324₂				19.735₈	
10	0695.559₃		17	40	0094.568₀		22
		20.307₀				19.713₈	
15	0675.252₃		17	45	0074.854₂		22
		20.289₆				19.691₄	
20	0654.962₇		17	50	0055.162₈		23
		20.272₃				19.668₉	
25	0634.690₄		18	55	0035.493₉		23
		20.254₇				19.646₂	
53°30′	9,9990614.435₇		18	56° 0′	9,9990015.847₇		23
		20.236₉				19.623₁	
35	0594.198₈		18	5	9,9989996.224₆		23
		20.218₉				19.600₁	
40	0573.979₉		18	10	9976.624₅		23
		20.200₈				19.577₀	
45	0553.779₁		18	15	9957.047₅		24
		20.182₄				19.553₅	
50	0533.596₇		19	20	9937.494₀		24
		20.163₈				19.529₉	
55	0513.432₉		19	25	9917.964₁		24
		20.145₀				19.506₂	
54° 0′	9,9990493.287₉		19	56°30′	9,9989898.457₉		24
		20.126₃				19.482₃	
5	0473.161₆		19	35	9878.975₆		24
		20.107₅				19.458₂	
10	0453.054₁		19	40	9859.517₄		24
		20.088₂				19.434₀	
15	0432.965₉		20	45	9840.083₄		24
		20.068₇				19.409₆	
20	0412.897₂		20	50	9820.673₈		24
		20.049₁				19.385₂	
25	0392.848₁		20	55	9801.288₆		25
		20.029₅				19.360₃	
54°30′	9,9990372.818₆		20	57° 0′	9,9989781.928₃		25

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TABLE

for

$\log W = \log \sqrt{1 - e^2 \sin^2 B}$ and $\log w = \log \sqrt{1 - e^2 \cos^2 \beta}$,

Geographic Latitude B and the Reduced Latitude β from 0 to 90°

$\log e^2 = 7{,}8244104.15 - 10$

B	log W			B	log W		
0° 0′	0,0000000.0	0.1	90° 0′	5° 0′	9,9999889.9	7.4	85° 0′
10	9,9999999.9	0.4	50	10	9882.5	7.7	50
20	9999.5	0.6	40	20	9874.8	7.9	40
30	9998.9	0.9	30	30	9866.9	8.2	30
40	9998.0	1.1	20	40	9858.7	8.4	20
50	9996.9	1.3	10	50	9850.3	8.7	10
1° 0′	9,9999995.6	1.6	89° 0′	6° 0′	9,9999841.6	8.8	84° 0′
10	9994.0	1.9	50	10	9832.8	9.2	50
20	9992.1	2.1	40	20	9823.6	9.3	40
30	9990.0	2.3	30	30	9814.3	9.6	30
40	9987.7	2.5	20	40	9804.7	9.9	20
50	9985.2	2.9	10	50	9794.8	10.1	10
2° 0′	9,9999982.3	3.0	88° 0	7° 0′	9,9999784.7	10.3	83° 0′
10	9979.3	3.3	50	10	9774.4	10.5	50
20	9976.0	3.6	40	20	9763.9	10.8	40
30	9972.4	3.8	30	30	9753.1	11.1	30
40	9968.6	4.0	20	40	9742.0	11.2	20
50	9964.6	4.3	10	50	9730.8	11.5	10
3° 0′	9,9999960.3	4.5	87° 0′	8° 0′	9,9999719.3	11.8	82° 0′
10	9955.8	4.8	50	10	9707.5	12.0	50
20	9951.0	5.0	40	20	9695.5	12.2	40
30	9946.0	5.3	30	30	9683.3	12.4	30
40	9940.7	5.5	20	40	9670.9	12.7	20
50	9935.2	5.7	10	50	9658.2	12.9	10
4° 0′	9,9999929.5	6.0	86° 0′	9° 0′	9,9999645.3	13.2	81° 0′
10	9923.5	6.2	50	10	9632.1	13.3	50
20	9917.3	6.5	40	20	9618.8	13.6	40
30	9910.8	6.7	30	30	9605.2	13.9	30
40	9904.1	7.0	20	40	9591.3	14.1	20
50	9897.1	7.2	10	50	9577.2	14.3	10
5° 0′	9,9999889.9		85° 0′	10° 0′	9,9999562.9		80° 0′
	log w		β		log w		β

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B	log W			B	log W		
10° 0′	9,9999562.9		80° 0′	17° 0′	9,9998760.8		73° 0′
		14.5				23.7	
10	9548.4		50	10	8737.1		50
		14.8				23.9	
20	9533.6		40	20	8713.2		40
		15.0				24.1	
30	9518.6		30	30	8689.1		30
		15.2				24.3	
40	9503.4		20	40	8664.8		20
		15.5				24.5	
50	9487.9		10	50	8640.3		10
		15.6				24.7	
11° 0′	9,9999472.3		79° 0′	18° 0′	9,9998615.6		72° 0′
		15.9				24.9	
10	9456.4		50	10	8590.7		50
		16.2				25.1	
20	9440.2		40	20	8565.6		40
		16.3				25.3	
30	9423.9		30	30	8540.3		30
		16.6				25.5	
40	9407.3		20	40	8514.8		20
		16.8				25.7	
50	9390.5		10	50	8489.1		10
		17.1				25.8	
12° 0′	9,9999373.4		78° 0′	19° 0′	9,9998463.3		71° 0′
		17.3				26.1	
10	9356.1		50	10	8437.2		50
		17.4				26.3	
20	9338.7		40	20	8410.9		40
		17.8				26.4	
30	9320.9		30	30	8384.5		30
		17.9				26.7	
40	9303.0		20	40	8357.8		20
		18.1				26.8	
50	9284.9		10	50	8331.0		10
		18.4				27.0	
13° 0′	9,9999266.5		77° 0′	20° 0′	9,9998304.0		70° 0′
		18.6				27.3	
10	9247.9		50	10	8276.7		50
		18.8				27.4	
20	9229.1		40	20	8249.3		40
		19.1				27.6	
30	9210.0		30	30	8221.7		30
		19.2				27.7	
40	9190.8		20	40	8194.0		20
		19.5				28.0	
50	9171.3		10	50	8166.0		10
		19.7				28.1	
14° 0′	9,9999151.6		76° 0	21° 0′	9,9998137.9		69° 0′
		19.9				28.4	
10	9131.7		50	10	8109.5		50
		20.1				28.5	
20	9111.6		40	20	8081.0		40
		20.4				28.6	
30	9091.2		30	30	8052.4		30
		20.5				28.9	
40	9070.7		20	40	8023.5		20
		20.8				29.0	
50	9049.9		10	50	7994.5		10
		21.0				29.3	
15° 0′	9,9999028.9		75° 0′	22° 0′	9,9997965.2		68° 0′
		21.2				29.4	
10	9007.7		50	10	7935.8		50
		21.4				29.6	
20	8986.3		40	20	7906.2		40
		21.6				29.7	
30	8964.7		30	30	7876.5		30
		21.8				29.9	
40	8942.9		20	40	7846.6		20
		22.1				30.1	
50	8920.8		10	50	7816.5		10
		22.2				30.3	
16° 0′	9,9998898.6		74° 0′	23° 0′	9,9997786.2		67° 0′
		22.5				30.5	
10	8876.1		50	10	7755.7		50
		22.6				30.6	
20	8853.5		40	20	7725.1		40
		22.9				30.8	
30	8830.6		30	30	7694.3		30
		23.1				30.9	
40	8807.5		20	40	7663.4		20
		23.3				31.1	
50	8784.2		10	50	7632.3		10
		23.4				31.3	
17° 0′	9,9998760.8		73° 0′	24° 0′	9,9997601.0		66° 0′
	log w		β		log w		β

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B	log W			B	log W		
24° 0′	9,9997601.0	31.5	66° 0′	31° 0′	9,9996152.1	37.4	59° 0′
10	7569.5	31.6	50	10	6114.7	37.4	50
20	7537.9	31.7	40	20	6077.3	37.6	40
30	7506.2	32.0	30	30	6039.7	37.7	30
40	7474.2	32.1	20	40	6002.0	37.8	20
50	7442.1	32.2	10	50	5964.2	37.9	10
25° 0′	9,9997409.9	32.4	65° 0′	32° 0′	9,9995926.3	38.0	58° 0′
10	7377.5	32.6	50	10	5888.3	38.2	50
20	7344.9	32.7	40	20	5850.1	38.2	40
30	7312.2	32.9	30	30	5811.9	38.3	30
40	7279.3	33.0	20	40	5773.6	38.5	20
50	7246.3	33.2	10	50	5735.1	38.5	10
26° 0′	9,9997213.1	33.4	64° 0′	33° 0′	9,9995696.6	38.6	57° 0′
10	7179.7	33.5	50	10	5658.0	38.8	50
20	7146.2	33.6	40	20	5619.2	38.8	40
30	7112.6	33.8	30	30	5580.4	39.0	30
40	7078.8	33.9	20	40	5541.4	39.0	20
50	7044.9	34.1	10	50	5502.4	39.1	10
27° 0′	9,9997010.8	34.2	63° 0′	34° 0′	9,9995463.3	39.2	56° 0′
10	6976.6	34.4	50	10	5424.1	39.3	50
20	6942.2	34.5	40	20	5384.8	39.4	40
30	6907.7	34.7	30	30	5345.4	39.5	30
40	6873.0	34.8	20	40	5305.9	39.6	20
50	6838.2	34.9	10	50	5266.3	39.7	10
28° 0′	9,9996803.3	35.1	62° 0′	35° 0′	9,9995226.6	39.7	55° 0′
10	6768.2	35.2	50	10	5186.9	39.8	50
20	6733.0	35.3	40	20	5147.1	40.0	40
30	6697.7	35.5	30	30	5107.1	39.9	30
40	6662.2	35.6	20	40	5067.2	40.1	20
50	6626.6	35.8	10	50	5027.1	40.2	10
29° 0′	9,9996590.8	35.8	61° 0′	36° 0′	9,9994986.9	40.2	54° 0′
10	6555.0	36.0	50	10	4946.7	40.3	50
20	6519.0	36.2	40	20	4906.4	40.4	40
30	6482.8	36.2	30	30	4866.0	40.4	30
40	6446.6	36.4	20	40	4825.6	40.5	20
50	6410.2	36.5	10	50	4785.1	40.6	10
30° 0′	9,9996373.7	36.7	60° 0′	37° 0′	9,9994744.5	40.7	53° 0′
10	6337.0	36.7	50	10	4703.8	40.7	50
20	6300.3	36.9	40	20	4663.1	40.8	40
30	6263.4	37.0	30	30	4622.3	40.9	30
40	6226.4	37.1	20	40	4581.4	40.9	20
50	6189.3	37.2	10	50	4540.5	41.0	10
31° 0′	9,9996152.1		59° 0′	38° 0′	9,9994499.5		52° 0′
	log w		β		log w		β

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B	log W			B	log W		
38° 0′	9,9994499.5	41.0	52° 0′	45° 0′	9,9992741.3	42.3	45° 0′
10	4458.5	41.1	50	10	2699.0	42.3	50
20	4417.4	41.2	40	20	2656.7	42.3	40
30	4376.2	41.2	30	30	2614.4	42.3	30
40	4335.0	41.2	20	40	2572.1	42.3	20
50	4293.8	41.2	10	50	2529.8	42.3	10
39° 0′	9,9994252.4	41.4	51° 0′	46° 0′	9,9992487.5	42.3	44° 0′
10	4211.1	41.3	50	10	2445.2	42.3	50
20	4169.7	41.4	40	20	2403.0	42.2	40
30	4128.2	41.5	30	30	2360.7	42.3	30
40	4086.7	41.5	20	40	2318.5	42.2	20
50	4045.1	41.6	10	50	2276.2	42.3	10
40° 0′	9,9994003.5	41.6	50° 0′	47° 0′	9,9992234.0	42.2	43° 0′
10	3961.8	41.7	50	10	2191.8	42.2	50
20	3920.1	41.7	40	20	2149.6	42.2	40
30	3878.4	41.7	30	30	2107.5	42.1	30
40	3836.6	41.8	20	40	2065.3	42.2	20
50	3794.8	41.8	10	50	2023.2	42.1	10
41° 0′	9,9993752.9	41.9	49° 0′	48° 0′	9.9991981.1	42.1	42° 0′
10	3711.1	41.8	50	10	1939.1	42.0	50
20	3669.1	42.0	40	20	1897.0	42.1	40
30	3627.2	41.9	30	30	1855.0	42.0	30
40	3585.2	42.0	20	40	1813.0	42.0	20
50	3543.2	42.0	10	50	1771.1	41.9	10
42° 0′	9,9993501.2	42.0	48° 0′	49° 0′	9,9991729.1	42.0	41° 0′
10	3459.1	42.1	50	10	1687.2	41.9	50
20	3417.0	42.1	40	20	1645.4	41.8	40
30	3374.9	42.1	30	30	1603.6	41.8	30
40	3332.7	42.2	20	40	1561.8	41.8	20
50	3290.6	42.1	10	50	1520.0	41.8	10
43° 0′	9,9993248.4	42.2	47° 0′	50° 0′	9,9991478.3	41.7	40° 0′
10	3206.2	42.2	50	10	1436.7	41.6	50
20	3164.0	42.2	40	20	1395.1	41.6	40
30	3121.8	42.2	30	30	1353.5	41.6	30
40	3079.5	42.3	20	40	1312.0	41.5	20
50	3037.3	42.2	10	50	1270.5	41.5	10
44° 0′	9,9992995.0	42.3	46° 0′	51° 0′	9,9991229.0	41.5	39° 0′
10	2952.7	42.3	50	10	1187.7	41.3	50
20	2910.5	42.2	40	20	1146.3	41.4	40
30	2868.2	42.3	30	30	1105.1	41.2	30
40	2825.9	42.3	20	40	1063.8	41.3	20
50	2783.6	42.3	10	50	1022.7	41.1	10
45° 0′	9,9992741.3	42.3	45° 0′	52° 0′	9,9990981.6	41.1	38° 0′
	log w		β		log w		β

B	log W			B	log W		
52° 0′	9,9990981.6	41.1	38° 0′	59° 0′	9,9989325.1	37.3	31° 0′
10	0940.5	41.0	50	10	9287.8	37.3	50
20	0899.5	40.9	40	20	9250.5	37.1	40
30	0858.6	40.8	30	30	9213.4	37.0	30
40	0817.8	40.8	20	40	9176.4	36.9	20
50	0777.0	40.8	10	50	9139.5	36.7	10
53° 0′	9,9990736.2	40.6	37° 0′	60° 0′	9,9989102.8	36.6	30° 0′
10	0695.6	40.6	50	10	9066.2	36.6	50
20	0655.0	40.6	40	20	9029.6	36.3	40
30	0614.4	40.4	30	30	8993.3	36.3	30
40	0574.0	40.4	20	40	8957.0	36.1	20
50	0533.6	40.3	10	50	8920.9	36.0	10
54° 0′	9,9990493.3	40.2	36° 0′	61° 0′	9,9988884.9	35.9	29° 0′
10	0453.1	40.2	50	10	8849.0	35.7	50
20	0412.9	40.1	40	20	8813.3	35.6	40
30	0372.8	40.0	30	30	8777.7	35.5	30
40	0332.8	39.9	20	40	8742.2	35.4	20
50	0292.9	39.8	10	50	8706.8	35.2	10
55° 0′	9,9990253.1	39.8	35° 0′	62° 0′	9,9988671.6	35.0	28° 0′
10	0213.3	39.7	50	10	8636.6	34.9	50
20	0173.6	39.5	40	20	8601.7	34.8	40
30	0134.1	39.5	30	30	8566.9	34.7	30
40	0094.6	39.4	20	40	8532.2	34.5	20
50	0055.2	39.4	10	50	8497.7	34.3	10
56° 0′	9,9990015.8	39.2	34° 0′	63° 0′	9,9988463.4	34.3	27° 0′
10	9,9989976.6	39.1	50	10	8429.1	34.0	50
20	9937.5	39.0	40	20	8395.1	34.0	40
30	9898.5	39.0	30	30	8361.1	33.7	30
40	9859.5	38.8	20	40	8327.4	33.7	20
50	9820.7	38.8	10	50	8293.7	33.4	10
57° 0′	9,9989781.9	38.6	33° 0′	64° 0′	9,9988260.3	33.4	26° 0′
10	9743.3	38.6	50	10	8226.9	33.1	50
20	9704.7	38.4	40	20	8193.8	33.1	40
30	9666.3	38.3	30	30	8160.7	32.8	30
40	9628.0	38.3	20	40	8127.9	32.7	20
50	9589.7	38.1	10	50	8095.2	32.6	10
58° 0′	9,9989551.6	38.0	32° 0′	65° 0′	9,9988062.6	32.4	25° 0′
10	9513.6	37.9	50	10	8030.2	32.2	50
20	9475.7	37.8	40	20	7998.0	32.1	40
30	9437.9	37.7	30	30	7965.9	31.9	30
40	9400.2	37.6	20	40	7934.0	31.8	20
50	9362.6	37.5	10	50	7902.2	31.5	10
59° 0′	9,9989325.1		31° 0′	66° 0′	9,9987870.7		24° 0′
	log w		β		log w .		β

B	log W			B	log W		
66° 0′	9,9987870.7		24° 0′	73° 0′	9,9986705.1		17° 0′
10	7839.2	31.5	50	10	6681.4	23.7	50
20	7808.0	31.2	40	20	6658.0	23.4	40
30	7776.9	31.1	30	30	6634.8	23.2	30
40	7746.0	30.9	20	40	6611.8	23.0	20
50	7715.2	30.8	10	50	6589.0	22.8	10
67° 0′	9,9987684.6	30.6	23° 0′	74° 0′	9,9986566.5	22.5	16° 0′
10	7654.2	30.4	50	10	6544.1	22.4	50
20	7624.0	30.2	40	20	6521.9	22.2	40
30	7593.9	30.1	30	30	6500.0	21.9	30
40	7564.0	29.9	20	40	6478.2	21.8	20
50	7534.3	29.7	10	50	6456.7	21.5	10
68° 0′	9,9987504.7	29.6	22° 0′	75° 0′	9,9986435.4	21.3	15° 0′
10	7475.4	29.3	50	10	6414.2	21.2	50
20	7446.2	29.2	40	20	6393.3	20.9	40
30	7417.2	29.0	30	30	6372.7	20.6	30
40	7388.3	28.9	20	40	6352.2	20.5	20
50	7359.7	28.6	10	50	6332.0	20.2	10
69° 0′	9,9987331.2	28.5	21° 0′	76° 0′	9,9986312.0	20.0	14° 0′
10	7302.9	28.3	50	10	6292.2	19.8	50
20	7274.8	28.1	40	20	6272.6	19.6	40
30	7246.9	27.9	30	30	6253.2	19.4	30
40	7219.2	27.7	20	40	6234.0	19.2	20
50	7191.7	27.5	10	50	6215.1	18.9	10
70° 0′	9,9987164.3	27.4	20° 0′	77° 0′	9,9986196.4	18.7	13° 0′
10	7137.1	27.2	50	10	6177.9	18.5	50
20	7110.2	26.9	40	20	6159.6	18.3	40
30	7083.4	26.8	30	30	6141.6	18.0	30
40	7056.8	26.6	20	40	6123.8	17.8	20
50	7030.4	26.4	10	50	6106.2	17.6	10
71° 0′	9,9987004.2	26.2	19° 0′	78° 0′	9,9986088.8	17.4	12° 0′
10	6978.2	26.0	50	10	6071.7	17.1	50
20	6952.3	25.9	40	20	6054.8	16.9	40
30	6926.7	25.6	30	30	6038.1	16.7	30
40	6901.3	25.4	20	40	6021.6	16.5	20
50	6876.1	25.2	10	50	6005.4	16.2	10
72° 0′	9,9986851.0	25.1	18° 0′	79° 0′	9,9985989.4	16.0	11° 0′
10	6826.2	24.8	50	10	5973.6	15.8	50
20	6801.6	24.6	40	20	5958.0	15.6	40
30	6777.1	24.5	30	30	5942.7	15.3	30
40	6752.9	24.2	20	40	5927.6	15.1	20
50	6728.9	24.0	10	50	5912.7	14.9	10
73° 0′	9,9986705.1	23.8	17° 0′	80° 0′	9,9985898.1	14.6	10° 0′
	log w		β		log w		β

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B	log W			B	log W		
80° 0′	9,9985898.1	14.4	10° 0′	85° 0′	9,9985569.0	7.2	5° 0′
10	5883.7	14.1	50	10	5561.8	7.0	50
20	5869.6	14.0	40	20	5554.8	6.8	40
30	5855.6	13.7	30	30	5548.0	6.5	30
40	5841.9	13.4	20	40	5541.5	6.3	20
50	5828.5	13.3	10	50	5535.2	6.0	10
81° 0′	9,9985815.2	13.0	9° 0′	86° 0′	9,9985529.2	5.8	4° 0′
10	5802.2	12.7	50	10	5523.4	5.5	50
20	5789.5	12.6	40	20	5517.9	5.3	40
30	5776.9	12.2	30	30	5512.6	5.1	30
40	5764.7	12.1	20	40	5507.5	4.8	20
50	5752.6	11.8	10	50	5502.7	4.5	10
82° 0′	9,9985740.8	11.6	8° 0′	87° 0′	9,9985498.2	4.3	3° 0′
10	5729.2	11.3	50	10	5493.9	4.1	50
20	5717.9	11.1	40	20	5489.8	3.8	40
30	5706.8	10.9	30	30	5486.0	3.6	30
40	5695.9	10.6	20	40	5482.4	3.3	20
50	5685.3	10.4	10	50	5479.1	3.1	10
83° 0′	9,9985674.9	10.2	7° 0′	88° 0′	9,9985476.0	2.9	2° 0′
10	5664.7	9.9	50	10	5473.1	2.6	50
20	5654.8	9.6	40	20	5470.5	2.3	40
30	5645.2	9.5	30	30	5468.2	2.1	30
40	5635.7	9.1	20	40	5466.1	1.9	20
50	5626.6	9.0	10	50	5464.2	1.6	10
84° 0′	9,9985617.6	8.7	6° 0′	89° 0′	9,9985462.6	1.3	1° 0′
10	5608.9	8.4	50	10	5461.3	1.1	50
20	5600.5	8.3	40	20	5460.2	0.9	40
30	5592.2	7.9	30	30	5459.3	0.6	30
40	5584.3	7.8	20	40	5458.7	0.4	20
50	5576.5	7.5	10	50	5458.3	0.1	10
85° 0′	9,9985569.0		5° 0′	90° 0′	9,9985458.2		0° 0′
	log w		β		log w		β

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