THE LINEAR STABILITY OF THE FLOW IN THE NARROW GAP BETWEEN TWO CONCENTRIC ROTATING SPHERES

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SUMMARY

The flow of a viscous fluid contained in the narrow gap between two concentric spheres rotating with different angular velocities about a common rotation axis is considered. The onset of instability of this flow is investigated analytically, and two special cases are given particular attention. In the first case the outer sphere is at rest, while in the second case the fluid is in almost rigid rotation with the inner sphere rotating slightly faster than the outer. Instability first sets in near the equator, but the critical Taylor number is greater than that for the corresponding cylinder problem. The WKBJ method is used. Difficulties which arose in previous treatments are resolved by identifying the turning points. They are located not on the real latitude axis but in its extension to the complex plane. The implementation of the procedure leads to an ordinary-differential-equation eigenvalue problem which can be solved by standard numerical techniques.

1. Introduction

THE flow of an incompressible viscous fluid confined between concentric spheres rotating differentially about a common axis has been investigated recently by a number of authors. Laboratory experiments have been performed by Munson and Menguturk (**1**) and Wimmer (**2, 3**); numerical studies have been made by Bratukhin (**4**) and Munson and Menguturk (**1**), and analytical work has been done by Walton (**5**) and Hocking (**6**).

The system can be characterised by three independent dimensionless parameters. The gap ratio is

$$\epsilon = (R_2 - R_1)/R_1, \tag{1.1}$$

where R_1 and R_2 are the inner and outer radii, the angular velocity and related equatorial angular momentum ratios are

$$\tilde{\mu} = \Omega_2/\Omega_1, \qquad \mu = R_2^2\Omega_2/R_1^2\Omega_1, \tag{1.2a,b}$$

where Ω_1 and Ω_2 are the angular velocities of the inner and outer spheres, and the modified Taylor number is

$$T = \{2(R_2 - R_1)/(R_2 + R_1)\}^3\{(R_1^2\Omega_1)^2 - (R_2^2\Omega_2)^2\}/\nu^2, \tag{1.3a}$$

$$= \epsilon^{-1}(1 + \tfrac{1}{2}\epsilon)^{-3}\,\delta^{-1}R_M^2, \tag{1.3b}$$

where the Reynolds number R_M is

$$R_M = \epsilon^2(1-\mu)R_1^2\Omega_1/\nu, \quad \text{and} \quad \delta = (1-\mu)/(1+\mu). \tag{1.4a,b}$$

The advantages of the unusual representation (1.3a) are peculiar to the spherical geometry and occur in the narrow-gap limit

$$\epsilon \ll 1, \tag{1.5}$$

discussed in this paper. The stability of the system is particularly interesting when the equatorial angular momentum ratio μ is close to unity. It must be emphasised, however, that the corresponding limit $\delta \downarrow 0$ does not coincide with the case of rigid-body rotation, which is achieved when $\tilde{\mu} = 1$. Nevertheless, since μ and $\tilde{\mu}$ are almost identical in the small-ϵ limit, specifically $\mu/\tilde{\mu} = (1+\epsilon)^2$, it is legitimate to link a small value of δ with a state of almost rigid rotation.

Unlike the classical infinite cylinder geometry, for which the flow velocity at low Taylor number is purely azimuthal and of order $R_1\Omega_1$, the sphere problem has a slow axisymmetric meridional circulation whose velocity is of order $R_M(R_1\Omega_1)$. This circulation is symmetric about the equator and radial on it. Wimmer's experiments (**2, 3**) show that for narrow gaps there is a sharp transition at a Taylor number close to that predicted by the corresponding cylinder problem. Instability first sets in localised near the equator and the disturbance is characterised by Taylor vortices of almost square cross-section, whose intensity decreases towards the poles, where the underlying zonal shear is insufficient to drive the instability.

The experimental results, which are in accord with intuition, suggested to Walton (**5**) that an asymptotic representation of the solution for the case $\mu = 0$, when the outer sphere is at rest, should be possible based on the small parameter ϵ. Thus, a WKBJ solution was sought for which the disturbance in the vicinity of the latitude $-\lambda$ is characterised by a latitudinal wave number $\epsilon^{-1}k$ and a growth rate σ. At lowest order latitudinal variations of the basic state are ignored and the local analysis leads to an eigenvalue problem (see (2.15) below) which has a solution provided that k, λ, σ are related by a dispersion relation of the type

$$T = T(k, \lambda, \sigma) + O(\epsilon). \tag{1.6}$$

As expected, the smallest value of T for which σ has a non-negative real part is

$$T_{cyl} = 1694{\cdot}95, \quad \text{with} \quad \sigma = 0, \tag{1.7a}$$

which occurs when

$$k = k_{cyl} = 3{\cdot}1266, \qquad \lambda = 0. \tag{1.7b}$$

The values T_{cyl} and k_{cyl} are simply the critical values for the corresponding cylinder problem. Following this preliminary calculation Walton (**5**) anticipated that for some value of T, slightly in excess of T_{cyl}, two-scale methods would demonstrate the existence of a modulated amplitude solution similar to that found in the experiments.

The experimental and intuitive picture just described failed to emerge from the calculations in (**5**). To appreciate the difficulties, it is helpful to consider carefully the principles which lead to amplitude-modulation equations. As explained above, the analysis begins with the construction of the dispersion relation (1.6) and the subsequent minimisation of T. Regarding T as an analytic function of the three complex variables, k, λ, σ, the increments of T with respect to variations of k, λ, σ may be derived from the differential

$$dT = T_k\, dk + T_\lambda\, d\lambda - iT_\sigma\, d\omega, \tag{1.8}$$

where $\omega = i\sigma$ is the frequency and the subscripts are used to denote partial derivatives. On the other hand, since T is real, (1.6) also provides a constraint which links the admissible real values of k, λ, ω. Hence at the minimum, given any arbitrary real values of dk, $d\lambda$, there must exist some real value of $d\omega$ for which dT is zero. This result can be achieved only if

$$\operatorname{Re}(T_k/T_\sigma) = 0, \qquad \operatorname{Re}(T_\lambda/T_\sigma) = 0. \tag{1.9a,b}$$

In many problems, conditions (1.9) are met simply by

$$T_k = 0, \qquad T_\lambda = 0. \tag{1.10a,b}$$

In this case, solutions are given typically by the form (2.25) below, where the leading-order terms are proportional to

$$a \exp i\{\epsilon^{-1}k_0(\lambda - \lambda_0) - \omega_0 t\}, \tag{1.11}$$

where t is the dimensionless time. From the dispersion relation (1.6) it follows (see, for example, Kawahara (**7**)) that the modulation of the amplitude is governed by the equation

$$\tfrac{1}{2}\epsilon^2 T_{kk0} a_{\lambda\lambda} + i\epsilon T_{k\lambda 0}(\lambda - \lambda_0) a_\lambda + \\ + \{-\epsilon\beta + (T - T_0) - \tfrac{1}{2}T_{\lambda\lambda 0}(\lambda - \lambda_0)^2\} a = T_{\sigma 0} a_t, \tag{1.12}$$

where the suffix zero is used to denote values at the minimum. The additional constant β appears in (1.12) because the variable λ and the operator $\partial/\partial\lambda$ fail to commute. The equation can be solved by separating the variables to give the normal modes

$$\exp\{(\sigma^{(n)} + i\omega_0)t - \tfrac{1}{2}\epsilon^{-1}l^{-2}(\lambda - \lambda_0)^2\} H_n(\epsilon^{-\frac{1}{2}}L^{-1}(\lambda - \lambda_0)), \tag{1.13a}$$

where n is a non-negative integer, H_n are Hermite polynomials,

$$l^2 = T_{kk0}/S_0 (= ik_{\lambda 0}^{-1}), \qquad L^2 = T_{kk0}/\Delta_0, \tag{1.13b,c}$$

$$S = iT_{k\lambda} + \Delta, \qquad \Delta = (T_{kk}T_{\lambda\lambda} - T_{k\lambda}^2)^{\frac{1}{2}} \tag{1.13d,e}$$

and

$$T_{\sigma 0}(\sigma^{(n)}+i\omega_0)=T-T_0-\epsilon(\beta+\tfrac{1}{2}S_0+n\Delta_0). \tag{1.13f}$$

The case $n=0$ is of special interest because it provides the mode with the maximum growth rate and can be used to determine the critical Taylor number and frequency correct to order ϵ. On the other hand, any arbitrary initial disturbance can be resolved into the sum of normal modes (1.13a) at $t=0$, provided that it decays sufficiently rapidly as $\epsilon^{-\frac{1}{2}}\lambda \to \pm\infty$ in a sense defined precisely by Szego (**8**, p. 107). This decomposition is, of course, necessary to resolve the ultimate fate of such disturbances as shown from a slightly different point of view in the Appendix to (**6**). The asymptotic analysis of many physical systems leads to equation (1.12) and its solution (1.13). Nevertheless, one problem which reveals most of their features is the stability of free convection in a horizontal cylindrical annulus investigated by Walton (**9**) (see also (**10**)). In (**9**) an equation of the form (1.12) is given together with its solution (1.13).

Unfortunately not all problems are characterised by (1.10). In our spherical annulus problem, the minimum value $T=T_{cyl}$ is attained by axisymmetric modes at $k=k_{cyl}$ on $\lambda=0$ (see (1.7b)), but at this minimum, whereas T_k vanishes as before, T_λ is non-zero. The fact that $T_\lambda \neq 0$ makes a crucial difference, since the amplitude equation (1.12) is replaced by

$$\tfrac{1}{2}\epsilon^2 T_{kk0}a_{\lambda\lambda}+\{(T-T_0)-T_{\lambda 0}(\lambda-\lambda_0)\}a=T_{\sigma 0}a_t, \tag{1.14}$$

in which T_{kk0}, $iT_{\lambda 0}$ and $T_{\sigma 0}$ are all real. Here, since the λ-length scale is relatively long, $O(\epsilon^{\frac{2}{3}})$, the terms in (1.12) previously linked with $T_{k\lambda 0}$ and $T_{\lambda\lambda 0}$ are small and have been neglected. Unlike (1.12) there is no steady or oscillatory solution which decays to zero as $\epsilon^{-\frac{2}{3}}\lambda$ tends to both plus and minus infinity. Properties of the Airy-type equation (1.14) are discussed further in section 5 under the less restrictive conditions (5.1).

The conclusion we draw, as did Hocking (**6**), is that a minimum of T linked with $T_\lambda \neq 0$ does not lead to acceptable neutrally stable solutions, so the basic idea that the minimum critical Taylor number for the spherical annulus problem is simply the minimum value of the dispersion relation (1.6), T_{cyl}, is wrong. The correct answer is found by isolating those values $k=k_0$, $\lambda=\lambda_0$ at which conditions (1.10) are met even if this means accepting complex values of k_0 and λ_0. The requirement that the corresponding values of $T=T_0$, $\omega=\omega_0$ are real together with the real and imaginary parts of (1.10) provides six equations for the six unknowns $\mathrm{Re}(k_0)$, $\mathrm{Im}(k_0)$, $\mathrm{Re}(\lambda_0)$, $\mathrm{Im}(\lambda_0)$, T_0, ω_0. Since these six equations differ from (5.1) it is not surprising to find from our numerical calculations that the smallest T_0 (the solution of our six equations is not necessarily unique) exceeds T_{cyl}. The idea behind our proposal is simply that the realised solution of the physical problem, which decays exponentially towards the poles, when extended into the complex λ-plane, satisfies the amplitude equation (1.12) in the vicinity of $\lambda = \lambda_0$. The pertinence of this remark is clarified by the discussion of another special case in the paragraph below while the detailed application of the idea to Walton's (**5**) problem is described in section 3(a). For the moment it is sufficient to note that for axisymmetric modes

$$\mathrm{Re}\,(\lambda_0) = \mathrm{Im}\,(k_0) = \omega_0 = 0, \tag{1.15}$$

while for asymmetric modes of given azimuthal wave number m, all six of our unknowns are non-zero (see section 4 below).

Although normally the dispersion relation has to be computed numerically for complex λ, in the case of almost rigid-body rotation $\delta = (1-\mu)/(1+\mu) = O(\epsilon^{\frac{1}{2}})$ the axisymmetric mode solutions are simply perturbations of those for the corresponding cylinder problem. For that case the asymptotic analysis undertaken in section 3(b) is accomplished by considering only real values of λ. Conditions (1.9) are met at $\lambda = 0$ with $T_\lambda = O(\delta)$. Since this implies that the length scale appropriate to (1.14) is now $O(\epsilon^{\frac{1}{2}})$, the terms linked with $T_{k\lambda}$ and $T_{\lambda\lambda}$ in (1.12) must be reinstated. The appropriate amplitude equation (3.23) written in an alternative form is

$$\epsilon^2 a_{\lambda\lambda} + \kappa^2(\lambda) a = 0, \tag{1.16a}$$

where κ is defined by

$$\tfrac{1}{2} T_{kk0}\kappa^2(\lambda) = T - T_0(\delta) - T_{\delta\lambda 0}\delta\lambda - \tfrac{1}{2}T_{\lambda\lambda 0}\lambda^2 \tag{1.16b}$$

and

$$T_0(\delta) = T_0 + \tfrac{1}{2}T_{\delta\delta 0}\delta^2 + O(\delta^3). \tag{1.16c}$$

Here, $T_0(\delta)$ is the local minimum Taylor number for small but finite δ, whereas T_{kk0}, $T_{\delta\delta 0}$, $iT_{\delta\lambda 0}$, $T_{\lambda\lambda 0}$ are all evaluated at zero δ and are consequently all real. Upon introduction of the imaginary number

$$\lambda_0 = i\zeta_0, \quad \text{where} \quad \zeta_0/\delta = iT_{\delta\lambda 0}/T_{\lambda\lambda 0}\ (\text{real}), \tag{1.17a,b}$$

equation (1.16) can be recast in the form (1.12) with

$$\epsilon\beta = \tfrac{1}{2}T_{\lambda\lambda 0}\zeta_0^2. \tag{1.18}$$

As before, the eigenvalue problem has normal-mode solutions (1.13). The critical value of T is

$$T_c = T_0(\delta) + \tfrac{1}{2}T_{\lambda\lambda 0}\zeta_0^2 + \tfrac{1}{2}\epsilon S_0, \qquad \text{where} \qquad S^2 = T_{kk}T_{\lambda\lambda}, \tag{1.19}$$

which, because of the term ζ_0^2 of order δ^2, does not tend to $T_0(\delta)$ in the limit $\epsilon \to 0$.

In addition to the exact solutions of (1.16) given in the form (1.13) for discrete values of the Taylor number, there always exist two independent asymptotic solutions of the WKBJ type

$$a_\pm = A_\pm(\lambda) \exp\left\{\pm i\epsilon^{-1}\int_0^\lambda \kappa(\lambda')\, d\lambda'\right\}, \tag{1.20}$$

provided that λ is not close to a turning point at which $\kappa(\lambda)$ vanishes. For the particular form of $\kappa(\lambda)$ given by (1.16b), the solutions $a_\pm$ can be normalised so that, when λ is of order $\epsilon^{\frac{1}{2}}$, they behave like

$$a_\pm = \exp\{\pm i\epsilon^{-1}\kappa_0\lambda \pm \tfrac{1}{2}i\epsilon^{-1}\kappa_{\lambda 0}\lambda^2 + O(\epsilon^{-1}\lambda^3)\}, \tag{1.21}$$

where κ_0 and $i\kappa_{\lambda 0}(<0)$ are both real. Consequently, when $T > T_0$, there are no turning points on the real λ-axis and it might be supposed that a_+ always provides the asymptotic solution of (1.16) for all real values of λ. This view is too simplistic because it ignores the well-known Stokes phenomenon (see, for example, Heading (**11**)). The resolution of this difficulty is linked with the two turning points associated with (1.16) and located close together at

$$\lambda = \lambda_\pm = i\zeta_0 \pm \epsilon^{\frac{1}{2}}\lambda_1, \tag{1.22a}$$

where ζ_0 is given by (1.17) and

$$\epsilon\lambda_1^2 + \zeta_0^2 = 2\{T - T_0(\delta)\}/T_{\lambda\lambda 0}. \tag{1.22b}$$

As illustrated in Fig. 1, a Stokes line can be drawn from $P(\lambda = i\zeta_0)$ along the imaginary axis to meet the real axis at $S(\lambda = 0)$, while anti-Stokes lines can be drawn from P to meet the real axis at $A_\pm(\lambda = \pm\zeta_0)$. If a solution of (1.16) is chosen such that it is given asymptotically for $\lambda < 0$ by a_+, then on crossing the Stokes line at S it acquires the additional contribution

$$\tau a_-, \tag{1.23}$$

where τ is the Stokes constant whose magnitude depends on T. The term τa_- remains subdominant until the anti-Stokes line is crossed at A_+, after which it becomes dominant. Maximum dominance (i.e. the maximum value of the ratio $|a_+/a_-|$) is attained where the Stokes line crosses the real axis at

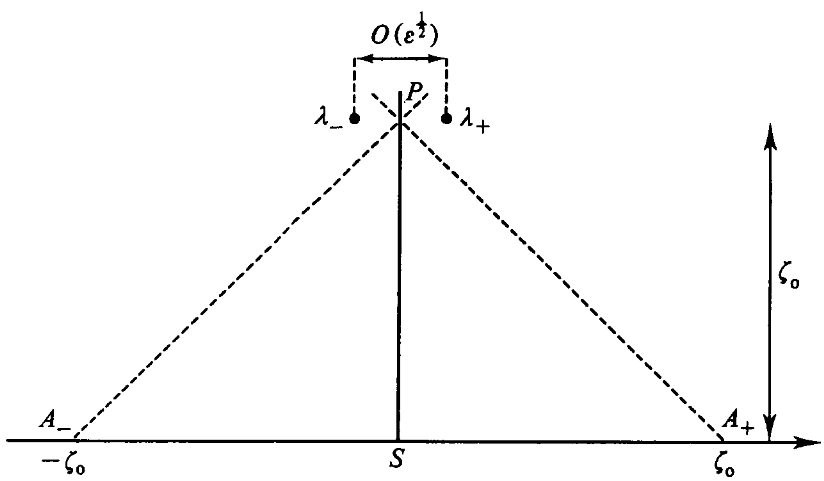

FIG. 1. The turning points $\lambda = \lambda_+, \lambda_-$ located near P at $\lambda = i\zeta_0$ are shown on the complex λ-plane. The Stokes line PS and the anti-Stokes lines PA_+, PA_-, which intersect the real axis are also indicated.

S. For this problem and for all other axisymmetric cases (e.g. see section 3(a)) characterised by (1.15) the maximum amplitude of a_+ also occurs at S. This is not the case, however, for asymmetric modes (see section 4) whose maximum amplitude occurs elsewhere at M (see Fig. 2). Either side of the maximum the solution decays on the order-$\epsilon^{\frac{1}{3}}$ length scale. The eigensolutions which we approximated for all real λ by a_+ with

$$\kappa_0 = (T_{\lambda\lambda 0}/T_{kk0})^{\frac{1}{2}}\zeta_0, \qquad i\kappa_{\lambda 0} = -\kappa_0/\zeta_0, \tag{1.24}$$

(see (1.16), (1.19)) are obtained when

$$\tau(T) = 0.$$

The point we wish to emphasise is that the resolution of the eigenvalue problem by asymptotic methods is accomplished by considering the nature of the solution not at $S(\lambda = 0)$ but off the real λ-axis in the vicinity of the two almost coincident turning points where (1.12) holds and the associated eigensolutions (1.13) are readily identified. The fact that the analysis can be accomplished by considering only real values of λ for the case $\delta = O(\epsilon^{\frac{1}{3}})$ simply reflects the fact that the domain of validity of (1.12), which is $|\lambda - \lambda_0| = O(\epsilon^{\frac{1}{3}})$, and the domain of validity of (1.16), which is $|\lambda| = O(\epsilon^{\frac{1}{3}})$, are identical because $\lambda_0 = O(\epsilon^{\frac{1}{3}})$ (see (1.17)).

2. The basic equations

In this section the mathematical formulation of the linear stability of steady axisymmetric flow between concentric spheres rotating differentially about a common axis is given. The problem is characterized by the three dimensionless parameters ϵ, δ, T defined by (1.1) to (1.4) and the governing equations are made dimensionless by adopting R_1 for the unit of length, $\epsilon^2 R_1^2/\nu$ for the unit of time and $R_1\Omega_1$ for the unit of velocity. The system is referred to spherical polar coordinates (r, θ, ϕ), where the radial coordinate r and the colatitude θ are expressed alternatively as

$$r = 1 + \epsilon x, \qquad \theta = \tfrac{1}{2}\pi + \lambda. \tag{2.1a,b}$$

The inner and outer spherical surfaces are located at $x = 0$ and 1 respectively, while the system is symmetrical about the equatorial plane $\lambda = 0$.

Following Walton (5) the basic meridional and azimuthal flow is described in terms of the stream function ψ and the angular momentum (circulation) h. Consequently, in component form the velocity is

$$(R_M w, R_M u, v), \tag{2.2a}$$

where

$$(w, u, v) = (r\cos\lambda)^{-1}(\epsilon r^{-1}\,\partial\psi/\partial\theta, -\partial\psi/\partial x, h). \tag{2.2b}$$

Though the exact solution of the governing equations and boundary conditions is not known, a power series representation for ψ and h is possible based on the small size of ϵ. The series expansion, which emphasises the state of rigid rotation at $\tilde{\mu}=1$, is

$$\psi=(1-\tilde{\mu})/(1-\mu)\sin 2\lambda\cos\lambda\{\tilde{a}_0+\epsilon\tilde{a}_1+O(\epsilon^2)\}, \tag{2.3a}$$

$$h=\cos^2\lambda[(1+\epsilon x)^2+(1-\tilde{\mu})\{\tilde{b}_0+\epsilon\tilde{b}_1+O(\epsilon^2)\}], \tag{2.3b}$$

where the order-ϵ terms are separated into two parts:

$$\tilde{a}_1=\tilde{a}_{01}(\tilde{\mu})+(1-\mu^2)^{-1}T\tilde{a}_{10}(\tilde{\mu}), \qquad \tilde{b}_1=\tilde{b}_{01}(\tilde{\mu})+(1-\mu^2)^{-1}T\tilde{b}_{10}(\tilde{\mu}). \tag{2.3c,d}$$

The functions $\tilde{a}_0$, $\tilde{b}_0$, $\tilde{a}_{01}$ and $\tilde{b}_{01}$ are given by the relatively simple expressions

$$\tilde{a}_0(\tilde{\mu})=x^2(1-x)^2\{3+2\tilde{\mu}-(1-\tilde{\mu})x\}/5!, \qquad \tilde{b}_0=-x, \tag{2.4a,b}$$

$$\tilde{a}_{01}(\tilde{\mu})=4x^2(1-x)^2\{3(1+2\tilde{\mu})-(5-2\tilde{\mu})x+2(1-\tilde{\mu})x^2\}/6!, \qquad \tilde{b}_{01}=-2x. \tag{2.4c,d}$$

The remaining terms $\tilde{a}_{10}$ and $\tilde{b}_{10}$ are complicated but their values are only required to calculate the order-ϵ terms for two special cases listed in (2.7) to (2.9) below.

Though the basic flow takes its simplest form of rigid rotation at $\tilde{\mu}=1$, it transpires that the stability of almost rigid rotation is discussed most readily in terms of the parameter δ based on μ. For this reason it is convenient to rewrite the expansion (2.3) in the alternative form

$$\psi=(1+\mu)\sin 2\lambda\cos\lambda\{a_0+\epsilon a_1+O(\epsilon^2)\}, \tag{2.5a}$$

$$h=(1+\mu)\cos^2\lambda\{b_0+\epsilon b_1+O(\epsilon^2)\}, \tag{2.5b}$$

where

$$a_0=\tfrac{1}{2}x^2(1-x)^2\{5-2\delta(x-\tfrac{1}{2})\}/5!, \qquad b_0=\tfrac{1}{2}-\delta(x-\tfrac{1}{2}). \tag{2.6a,b}$$

When the outer sphere is at rest ($\mu=0$, $\delta=1$) the order-ϵ terms take the values

$$a_1=a_{01}+T(a_{101}+a_{102}\cos^2\lambda), \qquad b_1=T(b_{101}+b_{102}\cos^2\lambda), \tag{2.7a,b}$$

where

$$a_{01}=-4x^2(1-x)^3(2x-3)/6!, \tag{2.8a}$$

$$a_{101}=48x^2(1-x)^2(10x^7-90x^6+345x^5-705x^4+603x^3+ \\ +273x^2-705x-63)/5!10!, \tag{2.8b}$$

$$a_{102}=-96x^2(1-x)^2(70x^7-630x^6+2520x^5-5220x^4+4860x^3+ \\ +1080x^2-4779x-243)/5!11!, \tag{2.8c}$$

$$b_{101}=\tfrac{4}{5}x(x-1)(x-3)(2x-3)(10x^3-15x^2+6x+3)/7!, \tag{2.8d}$$

$$b_{102}=-\tfrac{1}{5}x(x-1)(10x^5-60x^4+123x^3-102x^2+18x+18)/6!. \tag{2.8e}$$

Except for errors in a_{101} and a_{102}, (2.7), (2.8) were listed previously by Walton (**5**, 2.10, 2.11). In making comparisons, however, it should be emphasised that though similar notation has been employed, the definitions of a_{01}, a_{101} and a_{102} are different. When the system is in almost rigid rotation $(\delta \downarrow 0)$ and λ is small, Walton's equations (**5**, p. 676) can again be integrated subject to the appropriate boundary conditions and lead to

$$a_1 = \delta^{-1}a_0 + O(T, T\lambda^2/\delta, 1), \qquad b_1 = \delta T b_{10} + O(T\lambda^2), \tag{2.9a,b}$$

where

$$b''_{10} = -2a_0 \tag{2.9c}$$

and the prime is used to denote partial differentiation with respect to x.

Perturbations of the basic flow are represented by the addition of the velocity

$$(R_M\bar{w}, R_M\bar{u}, \delta\bar{v}) \tag{2.10}$$

to (2.2a). Since it satisfies linear equations with constant coefficients independent of ϕ and t, separable solutions are sought of the form

$$(\bar{w}, \bar{u}, \bar{v}) = (\bar{W}, \bar{U}, \bar{V}) \exp(i\delta R_M^{-1} m\phi + \sigma t), \tag{2.11}$$

where $\delta R_M^{-1} m$ is a large integer, subject to the no-slip conditions which in terms of $\bar{W}$ and $\bar{V}$ alone are

$$\bar{W} = \partial\bar{W}/\partial x = \bar{V} = 0 \tag{2.12}$$

on $x = 0$ and 1. Asymptotic solutions of the governing equations are sought of the WKBJ type

$$\mathbf{Y} \equiv (\bar{W}, \bar{V})^T = (W, V)^T \exp\left\{i\epsilon^{-1}\int_{\lambda_M}^{\lambda} k(\lambda')\, d\lambda'\right\}, \tag{2.13}$$

where λ_M is a constant to be chosen at our convenience. At lowest order the λ-derivatives of (W, V) are negligible and so the ensuing ordinary differential equations admit solutions of the form

$$(k^{-1}W, V)^T = b(\lambda)\mathbf{X}(\lambda, x). \tag{2.14}$$

The two components X_1, X_2 of the column vector $\mathbf{X}$ satisfy the eigenvalue problem

$$\mathscr{L}\mathbf{X} = (\mathscr{D}_0 + \mathscr{K}_0)\mathbf{X} = \mathbf{0}, \tag{2.15a}$$

with

$$X_1 = DX_1 = X_2 = 0 \tag{2.15b}$$

on $x = 0$ and 1, where the differential operators $\mathscr{D}_0$ and $\mathscr{K}_0$ are

$$\mathscr{D}_0 = (D^2 - k^2 - \sigma - imb_0)\begin{pmatrix} D^2 - k^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.16a}$$

$$\mathscr{K}_N = \begin{pmatrix} ikT\sin 2\lambda\{a'_N(D^2-k^2) - a'''_N\} & 2i\sin\lambda(b_N D + b'_N) - 2kb_N\cos\lambda \\ -k\delta^{-1}Tb'_N\cos\lambda & ikTa'_N\sin 2\lambda \end{pmatrix} \tag{2.16b}$$

with

$$N = 0 \quad \text{and} \quad D = \partial/\partial x. \tag{2.16c}$$

These equations provide the extension of Walton's equations (**5**, 4.1, 4.2) to non-zero values of m and μ. The WKBJ solution is completed by solving a first-order differential equation for $b(\lambda)$ obtained at the order-ϵ level as explained in (**5**).

The eigenvalue problem posed by (2.15), (2.16) has a solution only when the parameters are related by some equation of the form

$$\mathcal{F}(\mu, m, k, \lambda, \sigma, T) = 0. \tag{2.17}$$

The case $\mu = 0$, for which the outer sphere is at rest, has been discussed previously by two authors. Walton (**5**) investigated axisymmetric modes ($m = 0$), while Hocking (**6**) considered both zero and non-zero values of m using an alternative formulation of the problem. As pointed out in (**5**), when the values of μ, m, σ, T are prescribed (2.17) gives an equation with six roots. In view of the equatorial symmetries, they can be represented in terms of only three independent functions as

$$k = k^{(i)}(\lambda), -k^{(i)}(-\lambda); \qquad i = 1, 2, 3. \tag{2.18}$$

Associated with each of the $k^{(i)}$'s there exists an independent WKBJ solution $\mathbf{Y}^{(i)}$ defined by (2.13). The remaining solutions linked with $-k^{(i)}(-\lambda)$ are

$$\mathbf{Y}^{(i)}(-\lambda, x), \qquad i = 1, 2, 3. \tag{2.19}$$

To the lowest order the critical values of the Taylor number T_0 and the frequency ω_0 for fixed m are characterised by the conditions (1.10) irrespective of whether they are achieved for real or complex values of λ $(= \lambda_0)$ and k $(= k_0)$. The first condition $T_k = 0$ implies that (2.17) has a repeated root, which may be chosen to be

$$k^{(1)}(\lambda_0) = k^{(2)}(\lambda_0) = k_0. \tag{2.20}$$

Consequently, the point P $(\lambda = \lambda_0)$ in the complex λ-plane defines a turning point in the neighbourhood of which the WKBJ solutions $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ are indistinguishable. The second condition, $T_\lambda = 0$, on the other hand, ensures that in the limit $\epsilon \to 0$, two turning points are coincident. This property implies that there are distinct pairs of Stokes and anti-Stokes lines passing through P, which are defined by

$$\operatorname{Re}\left\{\int_{\lambda_0}^{\lambda} (k^{(2)} - k^{(1)})\, d\lambda\right\} = 0, \qquad \operatorname{Im}\left\{\int_{\lambda_0}^{\lambda} (k^{(2)} - k^{(1)})\, d\lambda\right\} = 0, \tag{2.21a,b}$$

respectively. The anti-Stokes lines divide the complex λ-plane up into four domains, three of which $\mathcal{D}_-$, $\mathcal{D}_S$ and $\mathcal{D}_+$ are illustrated in Fig. 2. As indicated in the figure it is anticipated that the anti-Stokes lines intersect the real

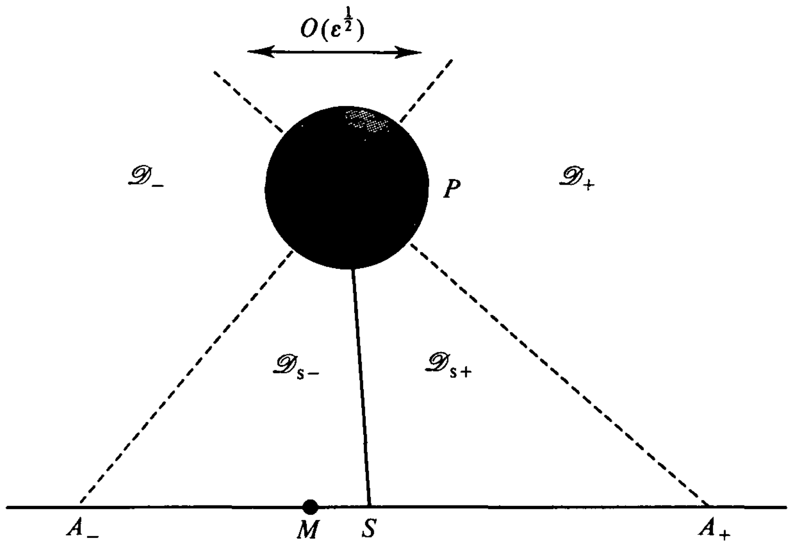

FIG. 2. A schematic diagram, which indicates the various domains in the complex λ-plane. The "inner" solution (2.25) is valid in the shaded region, while the "outer" eigensolution $\mathbf{Y}^{(1)}$ is valid elsewhere in $\mathscr{D}_-$, $\mathscr{D}_S(=\mathscr{D}_{S_-}+\mathscr{D}_{S_+})$, $\mathscr{D}_+$.

λ-axis at $A_-(\lambda=\lambda_{A_-})$ and $A_+(\lambda=\lambda_{A_+})$, while a Stokes line divides $\mathscr{D}_S$ into two separate subdomains, $\mathscr{D}_{S_-}$, $\mathscr{D}_{S_+}$, and intersects the real λ-axis at $S(\lambda=\lambda_S)$. If $\mathbf{Y}^{(1)}$ is the WKBJ solution dominant in $\mathscr{D}_S$ and subdominant in $\mathscr{D}_-$ and $\mathscr{D}_+$, then, of course, $\mathbf{Y}^{(2)}$ is subdominant in $\mathscr{D}_S$ and dominant in $\mathscr{D}_-$ and $\mathscr{D}_+$. In view of the Stokes phenomenon it is, in general, possible to find a particular integral of the governing equations, which is approximated uniformly along the real axis by

$$\mathbf{Y}=\begin{cases}\mathbf{Y}^{(1)}, & \lambda\leq\lambda_S, \quad (2.22a)\\ \mathbf{Y}^{(1)}+\tau\mathbf{Y}^{(2)}, & \lambda>\lambda_S. \quad (2.22b)\end{cases}$$

None of the remaining four distinct WKBJ solutions are acquired in crossing the Stokes line because, of course, they are not linked with the turning point $\lambda=\lambda_0$. Furthermore, Walton (**5**) has shown that $ik^{(3)}$ is real at all latitudes, indicating that the third independent solution $\mathbf{Y}^{(3)}$ is of no interest. As explained in section 1, the required solution is obtained when the Stokes constant τ vanishes. In that case, there exists a point $M(\lambda=\lambda_M\in\mathscr{D}_S)$ on the real axis at which the wave number k_M and its derivative $k_{\lambda M}$ have the properties

$$\operatorname{Im} k_M=0, \qquad \operatorname{Im} k_{\lambda M}>0. \tag{2.23}$$

Thus, the realised disturbance takes its maximum value at M and is approximated locally by

$$a(\lambda_M)\exp\{i\epsilon^{-1}k_M(\lambda-\lambda_M)+\tfrac{1}{2}i\epsilon^{-1}k_{\lambda M}(\lambda-\lambda_M)^2\}, \tag{2.24}$$

for $\lambda-\lambda_M=O(\epsilon^{\frac{1}{2}})$. It should be emphasised, however, that though M and S are coincident for the axisymmetric modes discussed in section 3 below they are distinct for the asymmetric modes discussed in section 4.

The asymptotic solutions (2.22a,b) are also valid for complex values of λ lying inside the domains $\mathcal{D}_-+\mathcal{D}_{S_-}$ and $\mathcal{D}_{S_+}+\mathcal{D}_+$ respectively, provided that $|\lambda-\lambda_0|\gg\epsilon^{\frac{1}{2}}$. The value of the Stokes constant τ is determined by obtaining an 'inner' solution valid when $|\lambda-\lambda_0|=O(\epsilon^{\frac{1}{2}})$ and matching with the outer solution (2.22). The 'inner' solution is obtained by forming the series expansion

$$\mathbf{Y}=[a\mathbf{X}+\{-i\epsilon a_\lambda\mathbf{X}^{(k)}+(\lambda-\lambda_0)a\mathbf{X}^{(\lambda)}\}+O(\epsilon)]\exp\{i\epsilon^{-1}k_0(\lambda-\lambda_0)\}, \tag{2.25}$$

in place of (2.13). Here $\mathbf{X}$ is defined as before to be the solution of the zeroth-order problem, while the amplitude function $a(\lambda)$ $(\neq b(\lambda))$ varies on the length scale $\epsilon^{\frac{1}{2}}$. At order $\epsilon^{\frac{1}{2}}$, the terms $\mathbf{X}^{(k)}$ and $\mathbf{X}^{(\lambda)}$ are simply the solutions of

$$\mathcal{L}\mathbf{X}^{(\tau)}=-\mathcal{L}_{\tau 0}\mathbf{X}, \qquad \tau=k,\lambda, \tag{2.26}$$

which satisfy the boundary conditions (2.15b). The partial derivatives $\mathcal{L}_k$, $\mathcal{L}_\lambda$ are defined in the usual way and solutions (not unique) of (2.26) are possible because the derivatives T_{k0}, $T_{\lambda 0}$ vanish (see also the Appendix). At order ϵ, care must be taken to allow for a variation in the frequency ω. Thus consistency conditions on the order-ϵ problem lead to equation (1.12) with a_t replaced by $(\sigma+i\omega_0)a$. The solutions, which match (2.22) with $\tau=0$ in the overlap region $\epsilon^{\frac{1}{2}}\ll|\lambda-\lambda_0|\ll 1$, are given by (1.13) for discrete values of

$$T_1^{(n)}=\epsilon^{-1}(T-T_0) \quad \text{and} \quad \omega_1^{(n)}=\epsilon^{-1}(i\sigma-\omega_0). \tag{2.27}$$

The smallest eigenvalue ($n=0$) determines the critical Taylor number and frequency as

$$T_c=T_0+\epsilon T_1+O(\epsilon^2), \tag{2.28a}$$

$$\omega_c=\omega_0+\epsilon\omega_1+O(\epsilon^2). \tag{2.28b}$$

Finally, it is noted that there exists in addition to $\mathbf{Y}^{(1)}$ a second solution $\mathbf{Y}^{(1)}(-\lambda,x)$ (see (2.19)), which defines a disturbance localised in the vicinity of $\lambda=-\lambda_M$.

3. Axisymmetric disturbances

For the special case of axisymmetric disturbances, it is convenient to represent the perturbation flow in terms of a stream function $\bar{\psi}$ and an angular momentum $\delta\bar{h}$ just as in (2.2b) above (cf. (2.10)). Instability for these $m=0$ modes is first manifest as steady convection, for which the frequency vanishes ($\omega_0=0$). Thus, instead of the more general form (2.11), it is possible, following Walton (**5**), to seek WKBJ solutions

$$\mathbf{Y}=(i\bar{\psi},\bar{h})^T=b(\lambda)\mathbf{X}(\lambda,x)\exp\left\{i\epsilon^{-1}\int_0^\lambda k(\lambda')\,d\lambda'\right\},\tag{3.1}$$

similar to (2.13), (2.14), where at zeroth order $\mathbf{X}$ satisfies the eigenvalue problem (2.15), (2.16) as before. The two cases $\delta=1$ and $\delta=O(\epsilon^{\frac{1}{2}})$ are discussed in detail in the two subsections which follow.

(*a*) *Outer sphere at rest*, $\delta=1$

The zeroth-order problem is resolved by locating the values of λ_0 and k_0, at which T_k and T_λ vanish. They are easy to find because the equatorial symmetry implies that λ_0/i $(=\zeta_0)$ and k_0 are both real. Indeed, with λ in (2.16b) replaced by $i\zeta$, all the coefficients of the operator $\mathscr{L}$ are real and as a result the eigenvalue problem (2.15) leads to the special form

$$T=T(k,i\zeta,0)\tag{3.2}$$

of (1.6), in which T is a real function of the real variables k and ζ. Thus, the numerical search is initiated with $\zeta=0$, for which T takes the minimum value T_{cyl} at $k=k_{cyl}$ (see (1.7)). The value of ζ is increased and the minimisation of T over real values of k is repeated. Eventually the value of the minimum reaches a local maximum at

$$\zeta_0=0{\cdot}1384,\quad\text{where}\quad T_0=1767{\cdot}90\quad\text{and}\quad k_0=3{\cdot}1769.\tag{3.3a,b,c}$$

There T has a saddlepoint characterised by

$$T_{k0}=T_{\zeta 0}=0,\qquad \Delta_0^2>0,\tag{3.4}$$

where the values of the partial derivatives are calculated from the formula in the Appendix, and the parameters S_0, Δ_0 (see (1.13d,e)) are

$$\left.\begin{aligned}T_{kk0}&=515{\cdot}5,\\ T_{k\zeta 0}(=iT_{k\lambda 0})&=-233{\cdot}4,\qquad S_0=1816{\cdot}7,\\ T_{\zeta\zeta 0}(=-T_{\lambda\lambda 0})&=-8047{\cdot}9,\qquad \Delta_0=2050{\cdot}1.\end{aligned}\right\}\tag{3.5}$$

As in Fig. 1, the imaginary λ-axis provides a Stokes line, on which $k^{(1)}$ and $k^{(2)}$ are both real. This means that the points M and S shown on Fig. 2 are coincident, being located at

$$\lambda_M=\lambda_S=0.\tag{3.6}$$

On the other hand, the anti-Stokes lines are not straight as indicated in Fig. 1 but bend slightly. Direct integration of (2.21b), when cast in the particular form

$$\operatorname{Im}\int_0^{\lambda_A}(k^{(1)}-k^{(2)})\,d\lambda=\int_0^{\zeta_0}(k^{(1)}-k^{(2)})d\zeta>0, \tag{3.7a}$$

shows that the points A_+ and A_- are given approximately by

$$-\lambda_{A_-}=\lambda_{A_+}=\lambda_A=0{\cdot}11. \tag{3.7b}$$

The realised disturbance is, of course, localised near the equator, where it is given asymptotically by (2.24) in which the coefficients k_M and $ik_{\lambda M}$ take the values

$$k_M=3{\cdot}6979, \qquad ik_{\lambda M}=-4{\cdot}050. \tag{3.8}$$

In the neighbourhood of the double turning point $\lambda=i\zeta_0$, the eigensolutions are given by (1.13) with $\sigma^{(n)}+i\omega_0=0$, where l^2 and L^2 (both real and positive) can be calculated from (3.5). Of greater practical interest is the order-ϵ correction T_1 to the critical Taylor number (2.28a), which is given by $\epsilon^{-1}(T-T_0)$ in (1.13f). In that expression the parameter β may be calculated from the formula

$$\beta=-i\langle\mathbf{X}^A,\mathscr{L}_{k0}\mathbf{X}^{(\lambda)}\rangle+\langle\mathbf{X}^A,(\tfrac{3}{2}T_0\mathscr{L}_{T0}+2x\mathscr{M}+\mathscr{N}+\mathscr{H}_1)\mathbf{X}\rangle, \tag{3.9}$$

(see the Appendix), where the coefficients of the matrix operators $\mathscr{M}$ and $\mathscr{N}$ are

$$\mathscr{M}_{11}=2k^2(D^2+x^{-1}D-k^2)-ikT_0\sin 2\lambda\{a_0'(D^2-2k^2)-a_0'''\}, \tag{3.10a}$$

$$\mathscr{M}_{12}=-2i\sin\lambda(b_0D+b_0')+3kb_0\cos\lambda, \tag{3.10b}$$

$$\mathscr{M}_{21}=k\delta^{-1}T_0b_0'\cos\lambda, \tag{3.10c}$$

$$\mathscr{M}_{22}=k^2-ikT_0a_0'\sin 2\lambda, \tag{3.10d}$$

$$\begin{aligned}\mathscr{N}_{11}=2ik\tan\lambda(D^2-k^2)-2T_0\{&(1-3\sin^2\lambda)a_0(D^2-k^2)D\\&-\sin^2\lambda a_0'(2D^2-3k^2)-\cos^2\lambda a_0''D-ika_0''\sin 2\lambda\},\end{aligned} \tag{3.11a}$$

$$\mathscr{N}_{12}=-4ib_0\sin\lambda, \tag{3.11b}$$

$$\mathscr{N}_{21}=2i\delta^{-1}T_0\sin\lambda b_0D, \tag{3.11c}$$

$$\mathscr{N}_{22}=ik\tan\lambda-2T_0(1-3\sin^2\lambda)a_0D, \tag{3.11d}$$

and $\mathscr{H}_1$ is given by (2.16b) with $N=1$; all are evaluated at $\delta=1$. In section 1 it was argued that the term β arises because the variable λ and the operator $\partial/\partial\lambda$ do not commute. Only the first term $-i\,\mathscr{L}_{k0}\mathbf{X}^{(\lambda)}$ in (3.9) is due to this property. The remaining terms provide convenient groupings of the additional terms neglected in the zeroth-order eigenvalue problem (2.15). The various terms which contribute to $(2x\mathscr{M}+\mathscr{N}+\mathscr{H}_1)\mathbf{X}$ were listed previously except for a few small errors in (**5**, p. 680). The additional term $\tfrac{3}{2}T_0\mathscr{L}_{T0}\mathbf{X}$ appears here because Walton's (**5**) Taylor number exceeds ours (see (1.3)) by a factor $(1+\frac{1}{2}\epsilon)^3$. The value of β was found numerically to be

$$\beta = -1323\cdot 0, \tag{3.12}$$

and so, with the help of (3.5), the smallest eigenvalue $T_1^{(0)}$ (see (1.13f), (2.27)) is

$$T_1 = -414\cdot 7. \tag{3.13}$$

(*b*) *Almost rigid rotation*, $\delta = O(\epsilon^{\frac{1}{2}})$

For values of μ other than zero, it is necessary to extend (3.2) and consider the Taylor number in the form

$$T = T(\mu, k, \lambda, 0). \tag{3.14}$$

For these values only the zeroth-order contribution $T_0(\delta)$ to the critical Taylor number was calculated. A plot of $T_0(\delta)$ versus μ is shown in Fig. 3, together with the corresponding cylinder value for comparison. As $\mu \uparrow 1$, the two values coincide and so permit an asymptotic expansion with

$$\delta \ll 1, \tag{3.15}$$

based on the solution for rotating cylinders, which is characterised by

$$T_{k0} = 0, \qquad T_{\lambda 0} = 0, \tag{3.16}$$

at

$$\delta = 0, \qquad k = k_0 = 3\cdot 116, \qquad \lambda = \lambda_0 = 0, \tag{3.17}$$

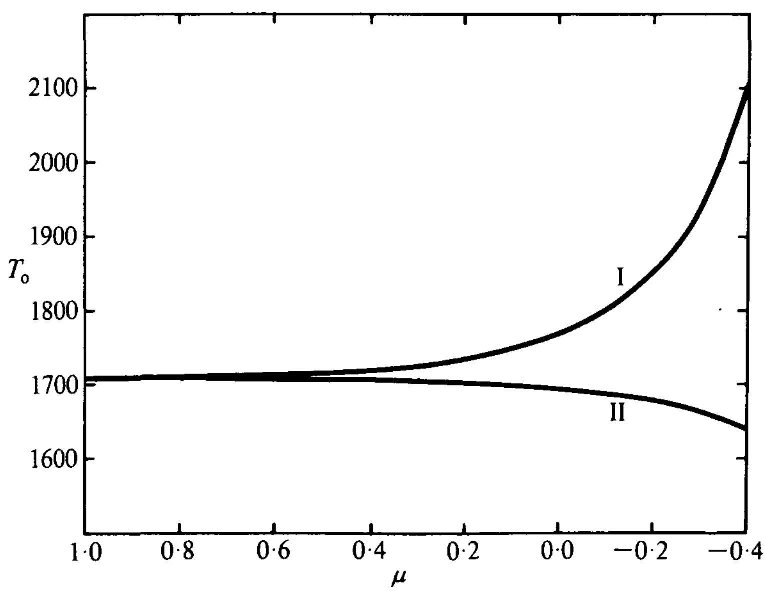

FIG. 3. The zeroth-order value $T_0(\delta)$ of the critical Taylor number is plotted versus μ on curve I. The second curve II is the corresponding plot for infinite coaxial cylinders.

where

$$T_0 = 1707\cdot76. \tag{3.18}$$

This has the advantage, previously taken in (**12**) (but see section 5(b) below), that the asymptotic analysis may be restricted to real values of λ.

The solution vector $\mathbf{Y}$ for non-zero values of δ can be found in the neighbourhood of $\lambda = 0$ from an extended version of (2.25), namely

$$\mathbf{Y} = [a\mathbf{X} + \{\delta a\mathbf{X}^{(\delta)} - i\epsilon a_\lambda \mathbf{X}^{(k)} + \lambda a\mathbf{X}^{(\lambda)}\} + O(\epsilon)] \exp(i\epsilon^{-1}k_0\lambda), \tag{3.19}$$

where $\delta, \epsilon\, \partial/\partial\lambda$ and λ are all assumed to be of order $\epsilon^{\frac{1}{2}}$. The order-$\epsilon^{\frac{1}{2}}$ contributions are determined from (2.26) and can be used to determine the various partial derivatives of T. In this respect, considerable advantage can be taken of the symmetries about $x = \frac{1}{2}$ as noted by Chandrasekhar (**13**, pp. 309–313) for the simpler cylinder problem. Specifically at $(\delta, k, \lambda) = (0, k, 0)$ the differential operator $\mathscr{L}$ and its partial derivatives $\mathscr{L}_{k0}, \mathscr{L}_{\delta\delta 0}(=0), \mathscr{L}_{kk0}$, $\mathscr{L}_{\lambda\lambda 0}$ and $\mathscr{L}_{\lambda\delta 0}$ are all symmetric in the sense that

$$\mathscr{L}_0(x - \tfrac{1}{2}, D) = \mathscr{L}_0(\tfrac{1}{2} - x, -D). \tag{3.20a}$$

The symmetry of $\mathscr{L}_0$, in particular, means that both $\mathbf{X}$ and its adjoint $\mathbf{X}^{\mathrm{A}}$ are symmetric about $x = \frac{1}{2}$. On the other hand, the partial derivatives $\mathscr{L}_{\delta 0}, \mathscr{L}_{\lambda 0}$, $\mathscr{L}_{k\delta 0}$ and $\mathscr{L}_{k\lambda 0}$ are all antisymmetric and so, for example,

$$\mathscr{L}_{\delta 0}(x - \tfrac{1}{2}, D) = -\mathscr{L}_{\delta 0}(\tfrac{1}{2} - x, -D). \tag{3.20b}$$

Consequently, the formulas of the Appendix imply that in addition to (3.16) the following partial derivatives also vanish:

$$T_{\delta 0} = T_{k\delta 0} = T_{k\lambda 0} = 0. \tag{3.21}$$

The remaining second-order partial derivatives of T were evaluated directly and together with the related quanties ζ_0, S_0 (see (1.17b), (1.19)) are

$$\left.\begin{aligned} T_{kk0} &= 506, & T_{\delta\delta 0} &= -26\cdot0, \\ T_{\lambda\lambda 0} &= 7395, & \zeta_0/\delta &= 0\cdot1384, \\ iT_{\delta\lambda 0} &= 1023, & S_0 &= 1934. \end{aligned}\right\} \tag{3.22}$$

It is shown below that the parameter β defined by (3.9) vanishes and as a result a satisfies the amplitude equation

$$\tfrac{1}{2}\epsilon^2 T_{kk0} a_{\lambda\lambda} + (T - T_0 - \tfrac{1}{2}T_{\delta\delta 0}\delta^2 - T_{\delta\lambda 0}\delta\lambda - \tfrac{1}{2}T_{\lambda\lambda 0}\lambda^2)a = 0. \tag{3.23}$$

The calculation of β requires some care. At $\lambda = 0$ the symmetries about $x = \frac{1}{2}$ clearly show that the contributions from $i\mathscr{L}_{k0}$, $\mathscr{N}$ and $\mathscr{K}_1$ vanish. On the other hand, there is an apparent contribution from $\frac{3}{2}T_0\mathscr{L}_{T0} + 2x\mathscr{M}$, namely

$$\beta = \langle \mathbf{X}^{\mathrm{A}}, \mathscr{B}\mathbf{X}\rangle_{\lambda=0}, \tag{3.24a}$$

where

$$\mathscr{B} = \begin{pmatrix} 2k^2(D^2-k^2) & \frac{3}{2}k \\ \frac{1}{2}kT_0 & k^2 \end{pmatrix} = -\tfrac{1}{2}k\mathscr{L}_{k0} + \begin{pmatrix} 0 & k \\ kT_0 & 0 \end{pmatrix}. \tag{3.24b}$$

Nevertheless, since T_{k0} is zero, the contribution from $-\frac{1}{2}k\mathscr{L}_{k0}$ vanishes, whereas the remaining term vanishes upon integration by parts. In addition to its value at $\lambda = 0$, β must also be evaluated when λ is of order δ (equivalently $\epsilon^{\frac{1}{2}}$). For example, a contribution might be expected from the terms $\mathcal{N}_{21}$, $(\mathcal{K}_1)_{11}$, $(\mathcal{K}_1)_{22}$, which by these estimates are each of order unity. The corresponding integrals do in fact vanish because the integrands are anti-symmetric about $x = \frac{1}{2}$.

With the help of (3.22), the formula (1.19) for the critical Taylor number and the expressions (1.21) and (1.24), which characterize the disturbance, yield

$$T_c = T_0(\delta) + 70{\cdot}8\delta^2 + 967\epsilon, \qquad k_M - k_0 = 0{\cdot}529\delta, \qquad ik_{\lambda M} = -3{\cdot}822, \tag{3.25a,b,c}$$

where $T_0(\delta)$ defined by (1.16c) is

$$T_0(\delta) = 1707{\cdot}76 - 13{\cdot}0\delta^2. \tag{3.25d}$$

The latter expression for $T_0(\delta)$ was derived by Davey (**14**, 8.4) from a slightly different point of view. These formulas are remarkable in as much as at $\delta = 1$ the zeroth-order values of T_c and $T_0(1)$ computed from them are

$$T_c = 1765{\cdot}6 + O(\epsilon), \qquad T_0(1) = 1694{\cdot}8, \tag{3.26}$$

which agree with the 'exact' results (3.3b) and (1.7a) to almost four significant figures! Furthermore, the values of ζ_0, k_0, k_M and $ik_{\lambda M}$ given by (3.22), (3.17) and (3.25b,c) at $\delta = 1$ compare favourably with (3.3a,c) and (3.8). In this respect it is also interesting to compare the values of the second partial derivatives of T and the quantities S and β, which contribute towards the order-ϵ term T_1.

The analysis of this section provided the motivation for our choice of Taylor number. The factor $(1+\frac{1}{2}\epsilon)^{-3}$ in (1.3b) was chosen specifically so that the contribution (3.24) to β would vanish. On the other hand the factor $(1-\mu^2)$ in (1.3a) was guided by the structure of the zonal shear defined by (2.6b) and (2.9b). The extremely simple forms for b_0 and b_1 are peculiar to the spherical geometry and depend ultimately on the different character of the Laplacian operator ∇^2 in two and three dimensions. In other words, the advantages gained by the use of our Taylor number do not extend to the cylindrical geometry because the corresponding parameter β does not vanish.

There is a separate issue of some concern, which relates to the size of δ; specifically the limit $\delta \downarrow 0$ with ϵ fixed is not straightforward. The difficulty stems from the fact that the state of "constant equatorial angular momentum" ($b_0' = 0$, $\mu = 1$) differs from the state of rigid rotation $\tilde{\mu} = 1$. In view of the scaling for the velocity in (2.2a) and the factor δ^{-1} in (2.9a), the meridional velocities become infinite in the limit $\mu \to 1$ with ϵ and T finite.

This factor δ^{-1} does not provide any contribution to T_c at the order-ϵ level. Nevertheless, it is reasonable to anticipate corrections of order ϵ^2/δ with the implication that the leading-order terms in the expansion of T_c are only given by (3.25) when

$$\epsilon \ll \delta. \tag{3.27}$$

4. Asymmetric modes

When $m \neq 0$ it is no longer the case that T_k and T_λ vanish on the imaginary λ-axis, so k_0 and λ_0 become complex numbers with non-zero real and imaginary parts. Also, instability occurs in the form of a travelling wave with frequency ω_0. In this case, the Taylor number has the form

$$T = T(\mu, m, k, \lambda, -i\omega). \tag{4.1}$$

To determine the location of the double turning point, we must solve (4.1) in conjunction with (1.10) for k, λ, T and ω with μ and m fixed. As explained in section 1, this leads to six equations for six unknowns. The method of solving them starts by calculating the complex eigenvalue T from equations (2.15) and (2.16) using an initial estimate of k, λ and ω. This is done by a shooting method using fourth-order Runge-Kutta integration with 51 mesh points; orthonormalizations (see, for example, Conte (**15**)) were needed to treat the cases where μ is negative. By giving the initial estimate small increments, the values of T_k and T_λ can be computed. This then provides a numerical construction of five of the required functions of five of the unknowns. Finally, an iteration procedure, based on Newton-Raphson iteration, is used to find those values of k, λ and ω which reduce T_k, T_λ and Im T to zero. Then (4.1) determines T, the remaining unknown.

The starting values used for an iteration procedure were obtained by gradually increasing m from zero (the axisymmetric solution already obtained) and using the solution from the previous m-value as the start for the next one. In this way we obtained the results given in Table 1, which refer to the case $\mu = 0$. The most important conclusion from Table 1 is that it is the axisymmetric mode which has lowest critical T and is therefore preferred. This result also holds for $-0{\cdot}4 \leq \mu < 1$, but for some values of μ below this

TABLE 1. *The critical values for the asymmetric solutions are listed for the special case* $\mu = 0$, *when* m *takes the values* 10, 20, 30, 40 *and* 50 *respectively.*

$T_0^{-\frac{1}{2}}m$	k_M	λ_M	ω_0	T_0
0·2376	3·7008	0·0327	5·0179	1771·81
0·4736	3·7097	0·0651	10·0382	1783·51
0·7065	3·7243	0·0967	15·0630	1802·88
0·9351	3·7445	0·1184	20·0943	1829·69
1·1582	3·7686	0·1567	25·1339	1863·69

range a non-axisymmetric mode becomes preferred; this behaviour is similar to that in an infinite cylinder (Krueger, Gross and DiPrima (**16**)).

5. Historical survey

The relation between the analysis presented here and the earlier attempts to solve the problem by asymptotic methods deserves elucidation. The two main avenues of attack are discussed separately in the subsections (a) and (b) below.

(a) The Airy-type equation

Equation (1.14) presented in section 1 arises whenever the conditions

$$T_k = 0, \qquad iT_\lambda/T_\sigma = \Gamma, \text{ say, real and non-zero,} \tag{5.1a}$$

are met. If, in addition,

$$\mathrm{Re}\,(T_{kk}/T_\sigma) > 0, \tag{5.1b}$$

as it generally is for stability problems, there exist no acceptable normal mode solutions

$$a = \tilde{a}(\lambda)e^{(\sigma + i\omega_0)t} \tag{5.2}$$

of (1.14). For whatever the value of σ, (1.14) leads after a shift of origin to Airy's equation with complex coefficients. This equation has no solutions which decay to zero as $\epsilon^{-\frac{2}{3}}\lambda \to \pm\infty$.

In the case of the spherical annulus problem, for which T_{kk0}, $iT_{\lambda 0}$ and $T_{\sigma 0}$ are all real, Walton (**5**) sought steady solutions of (1.14) with $T - T_0$ zero. Though this meant that a solution had to be accepted which grows exponentially as $\epsilon^{-\frac{2}{3}}\lambda$ tends to infinity, a solution of the full problem was attempted which embedded this local solution near the equator. The methods employed, however, were not rational in the sense that $T - T_0$ was not asymptotically small and the solution had the unsatisfactory feature that the amplitude increased towards the poles.

As a preliminary calculation to help focus our attention on the rôle of the new term $T_{\lambda 0}(\lambda - \lambda_0)a$ in (1.14), we consider long length-scale disturbances for the special case $T = T_0$. They satisfy

$$i\Gamma(\lambda - \lambda_0)a = a_t, \tag{5.3}$$

which has two solutions

$$a = \begin{cases} \hat{a}\exp\{i(\omega_0 - \omega)t\}, & \omega - \omega_0 = -\Gamma(\lambda - \lambda_0), \quad (5.4a) \\ \hat{a}\exp\{i\epsilon^{-1}(k - k_0)(\lambda - \lambda_0)\}, & k - k_0 = \epsilon\Gamma t, \quad (5.4b) \end{cases}$$

where $\hat{a}$ is a constant. Though identical, the solutions (5.4a,b) admit two physically distinct interpretations. The former says that for given wave number the frequency alters its value linearly with position, while the latter says that for given frequency the wave number evolves linearly with time ($k_t = \epsilon\Gamma$).

The second interpretation is particularly appropriate when (1.14) is considered in full. For then (5.4b) continues to be the exact solution provided that $\hat{a}$ evolves according to the equation

$$T_{\sigma 0}\hat{a}_t = \{T - T_0(k(t))\}\hat{a}, \tag{5.5a}$$

where $k(t)$ is given by (5.4b) and

$$T(k) = T(k, \lambda_0, -i\omega_0) = T_0 + \tfrac{1}{2}T_{kk0}(k-k_0)^2 + \dots. \tag{5.5b}$$

The most striking feature of this solution is that only one wave number $k(t)$ is present at any instant. In view of (5.1b), growth of this mode is only possible during the limited interval when

$$k_0^{(\alpha)} < k(t) < k_0^{(\beta)}, \tag{5.6}$$

where $k_0^{(\alpha)}$, $k_0^{(\beta)}$ are the two real solutions of $\mathrm{Re}\,[\{T - T_0(k)\}/T_{\sigma 0}] = 0$. If for definiteness it is supposed that Γ is positive, maximum amplification occurs when $k(t) = k_0^{(\beta)}$, while simple calculations reveal that order-one amplification is possible when

$$T - T_0 = O(\epsilon^{\frac{2}{3}}). \tag{5.7}$$

Since arbitrary initial disturbances can be represented as Fourier integrals of (5.4b), the discussion indicates that all disturbances ultimately decay very fast as $t \to \infty$.

The final decay predicted by (5.4b), (5.5a) can only be overcome by the addition of terms which transfer energy from the short length scale "sink" back into the wave-number band (5.6), where amplification is possible. With T close to T_0, the only terms which can be introduced in a rational way are those stemming from nonlinear effects. In a study of the stability of a rapidly rotating sphere containing a uniform distribution of heat sources, for which (1.14) is appropriate, Soward (**17**) obtained steady solutions of the nonlinear equations which relied on these finite amplitude effects. Hocking (**6**), on the other hand, has emphasised that no such solutions exist for the spherical annulus problem.

(*b*) *Hocking's problem*

The results outlined in the above subsection suggested to Soward (**17**, p. 43) that the critical value of the parameter T predicted by the conditions (5.1a) differed from the actual critical value by an amount of order unity. This idea was established mathematically by Hocking and Skiepko (**12**) who considered a modified version of the spherical annulus problem. In it the spheres are replaced by coaxial prolate spheroids; the inner spheroid rotates about the axis of symmetry while the outer spheroid is at rest. In the limit for which the minor/major axis ratio ϵ is small, an equation analogous to

(1.16) was derived. Their result similar to (1.19) showed that the difference $T_c - T_{cyl}$ remains finite in the limit $\epsilon \to 0$.

With regard to the spherical annulus problem of section 3(a), Hocking (**6**) assumed that the critical Taylor number exceeded T_{cyl} and that the asymptotic solution is a_+ defined by (1.21). On this much we are in agreement. On the other hand, he did not consider the Stokes phenomenon, so instead of demanding that the Stokes constant should vanish, he focussed attention on the asymptotic form of a_+ near its maximum at $\lambda = 0$. There the value of the wave number $k(\lambda)$ in the exponent of the WKBJ solution (3.1) is given by a dispersion relation of the form (3.2), namely

$$T(k(\lambda), \lambda) = T_H, \quad \text{a constant,} \tag{5.8}$$

where the suffix H is used to denote Hocking's predicted values of critical quantities. Since T_H exceeds T_{cyl}, (5.8) has two positive real roots at $\lambda = 0$ and one of them $k = k_H$ satisfies the conditions (2.23). As in (2.24) the Taylor series expansion of $k(\lambda)$ in the neighbourhood of $\lambda = 0$ leads to the expression

$$i\epsilon^{-1} \int_0^\lambda k(\lambda')\, d\lambda' = i\epsilon^{-1}\left(k_H\lambda + \tfrac{1}{2}k_{\lambda H}\lambda^2 + \frac{1}{3!}k_{\lambda\lambda H}\lambda^3 + \cdots\right) \tag{5.9}$$

for the exponent in (3.1). Direct differentiation of (5.8) shows that the coefficients $k_{\lambda H}$ and $k_{\lambda\lambda H}$ in (5.9) are given by

$$k_{\lambda H} = -T_{\lambda H}/T_{kH}, \tag{5.10a}$$

$$k_{\lambda\lambda H} = -(k_{\lambda H}^2 T_{kkH} + 2k_{\lambda H}T_{k\lambda H} + T_{\lambda\lambda H})/T_{kH}. \tag{5.10b}$$

Here the partial derivatives are related to Hocking's (**6**) numbers d_i, $i = 1, \ldots, 5$, by the identities

$$\frac{T_{kH}}{i\, d_1} = \frac{T_{\lambda H}}{d_2}; \qquad -\frac{T_{kkH}}{d_3} = \frac{2T_{k\lambda H}}{i\, d_4} = \frac{T_{\lambda\lambda H}}{d_5}. \tag{5.11}$$

In essence, Hocking (**6**) does not permit a general expression of the type (5.9). Instead, he assumes that the solution is given by a Fourier mode with real wave number $\epsilon^{-1}k_H$, whose amplitude is modulated by the factor

$$\exp\{-\tfrac{1}{2}(k_{\lambda H}/i)(\lambda^2/\epsilon)\}, \tag{5.12}$$

where $k_{\lambda H}/i$ is real and positive (**6**, 3.1). This means that higher-order terms in the expansion (5.9) are prohibited and so to the order of accuracy attempted the condition implies that

$$k_{\lambda\lambda H} = 0, \tag{5.13}$$

(cf. (5.10), (5.11) and (5.13) with (**6**, 3.2)).

Evidently, the criterion used in (**6**) to fix k_H and the corresponding critical Taylor number T_H is different from that adopted in this paper. The relation between T_H and our value T_0 is clarified if it is supposed that

$$(k_0 - k_H)/k_H = O(\zeta_0), \tag{5.14a}$$

where ζ_0 given by (3.3a) is taken to be a small quantity:

$$\zeta_0 \ll 1. \tag{5.14b}$$

With these assumptions the Taylor series expansion

$$T_0 - T_H = T_{kH}(k - k_H) + T_{\lambda H}\lambda + \\ + \tfrac{1}{2}\{T_{kkH}(k-k_H)^2 + 2T_{k\lambda H}(k-k_H)\lambda + T_{\lambda\lambda H}\lambda^2\} + O(\zeta_0^3 T_H), \tag{5.15}$$

may be used to evaluate T_0. Thus, application of the conditions $T_{k0} = T_{\lambda 0} = 0$ to the expansion (5.15) yields the identity

$$\begin{pmatrix} T_{kkH} & T_{k\lambda H} \\ T_{\lambda kH} & T_{\lambda\lambda H} \end{pmatrix} \begin{pmatrix} k_0 - k_H \\ \lambda_0 \end{pmatrix} = -\begin{pmatrix} T_{kH} \\ T_{\lambda H} \end{pmatrix} + O(\zeta_0^2 T_H), \tag{5.16}$$

and substitution of its solution $k_0 - k_H$, λ_0 into (5.15) leads to the result

$$T_0 - T_H = \tfrac{1}{2} k_{\lambda\lambda H} T_{kH}^3 / \Delta_H^2 + O(\zeta_0^3 T_H). \tag{5.17}$$

The resulting estimate (use (5.13))

$$(T_0 - T_H)/T_H = O(\zeta_0^3), \tag{5.18}$$

which is consistent with the numerical values given by (3.3a,b) and (6.1a), explains why Hocking's value T_H differs only slightly from our value T_0.

6. Concluding remarks

Three different methods have appeared in the literature and have been used to calculate the zeroth-order value of the critical Taylor number for axisymmetric modes, when the outer sphere is at rest ($\mu = 0$). Walton (**5**) on the basis of local analysis assumed that it is given by T_{cyl} (see (1.7a)), while Hocking† (**6**) obtained

$$T_H = 1757, \tag{6.1a}$$

which is the value of T on the neutral stability curve ($\sigma = 0$) for the classical infinite cylinder problem at

$$k = k_H = 3{\cdot}653, \quad \text{with} \quad \lambda = 0. \tag{6.1b}$$

The new value T_0 (see (3.3b)) obtained in this paper, however, exceeds both of them. Since the numerical values of T_{cyl}, T_H and T_0 are very close, it is difficult to distinguish between them on the basis of Wimmer's (**2, 3**)

† Hocking (**6**) also obtained an order-ϵ correction term T_{H1} which corresponds to our number $T_1 + \frac{3}{2}T_0$. Like T_H and T_0, the computed values $T_{H1} = 2149$ and $T_1 + \frac{3}{2}T_0 = 2237$ only differ by a small amount.

experiments. On the other hand, from a mathematical point of view we believe that our results provide the first correct asymptotic solution of the linear stability problem. Hopefully, more refined experiments will, in the future, confirm our predictions.

A separate issue of some concern may make it difficult to realise the features of the linear problem experimentally. The difficulty is that once the Taylor number exceeds T_{cyl} an initially imposed disturbance can grow significantly before it eventually decays. This means that given some background noise a small part of its spectrum may be subject to amplification. Experimentally rather than the sharp bifurcation predicted by linear theory, a slower and less marked transition would be observed. The realised picture is complicated further by nonlinear effects. Hocking (**6**) has shown that for T close to T_{cyl} nonlinearity acts to suppress disturbances. In our paper, however, no attempt has been made to assess the effect of nonlinear processes when T is close to T_0. Nevertheless, there is no reason to suppose that their role is significantly different from that at $T = T_{cyl}$ or that, in consequence, subcritical instability is possible.

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APPENDIX

Associated with the eigenvalue problem (2.15) is the adjoint equation

$$\mathscr{L}^{A}\mathbf{X}^{A} \equiv (\mathscr{D}_0 + \mathscr{K}_0^{A})\mathbf{X}^{A}, \qquad \mathbf{X}^{AT} = (X_1^{A}, X_2^{A}), \tag{A1a}$$

where $\mathbf{X}^{A}$ satisfies the boundary conditions (2.15b) and $\mathscr{K}_0^{A}$ is given by

$$\mathscr{K}_0^{A} = \begin{pmatrix} ikT\sin 2\lambda\{a_0'(D^2-k^2)+2a_0''D\} & -k\delta^{-1}Tb_0'\cos\lambda \\ -2i\sin\lambda b_0 D - 2kb_0\cos\lambda & ikTa_0'\sin 2\lambda \end{pmatrix}. \tag{A1b}$$

The inner product

$$\langle \mathbf{U}, \mathbf{V}\rangle = \int_0^1 (U_1V_1 + U_2V_2)\,dx \tag{A2}$$

is chosen so that if $\mathbf{X}^{A}$ is the solution of the adjoint problem (A1a), (2.15b) and $\mathbf{V}$ satisfies the boundary conditions (2.15b) also, then the identity

$$\langle \mathbf{X}^{A}, \mathscr{L}\mathbf{V}\rangle = \langle \mathscr{L}^{A}\mathbf{X}^{A}, \mathbf{V}\rangle = 0 \tag{A3}$$

holds true.

The eigenvalue problem (2.15) leads to the relation

$$T = T(m, k, \lambda, \sigma) \tag{A4}$$

for the existence of an eigensolution $\mathbf{X}$, where in general, T is a complex function. With $\mathbf{X}$ and $\mathbf{X}^{A}$ normalised such that

$$\langle \mathbf{X}^{A}, \mathscr{L}_T\mathbf{X}\rangle = -1, \tag{A5a}$$

the first-order partial derivatives of T are given simply by

$$T_\tau = \langle \mathbf{X}^{A}, \mathscr{L}_\tau\mathbf{X}\rangle, \qquad \tau = m,\ k,\ \lambda \text{ or } \sigma, \tag{A5b}$$

where the subscript τ is used to denote a partial derivative. When a derivative T_τ vanishes, the inhomogeneous equation

$$\mathscr{L}\mathbf{X}^{(\tau)} = -\mathscr{L}_\tau\mathbf{X} \tag{A6}$$

has a particular integral satisfying the boundary conditions (2.15b). Furthermore, if two such derivatives T_{τ_1} and T_{τ_2} (say) both vanish, then the second-order partial derivative $T_{\tau_1\tau_2}$ is given by

$$T_{\tau_1\tau_2} = \langle \mathbf{X}^{A}, \mathscr{L}_{\tau_1\tau_2}\mathbf{X}\rangle + \langle \mathbf{X}^{A}, \mathscr{L}_{\tau_1}\mathbf{X}^{(\tau_2)}\rangle + \langle \mathbf{X}^{A}, \mathscr{L}_{\tau_2}\mathbf{X}^{(\tau_1)}\rangle, \tag{A7}$$

where the special case $\tau_1 = \tau_2$ is also permitted.