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THE FLAT-EARTH APPROXIMATION TO THE
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THE FLAT-EARTH APPROXIMATION TO THE SOLUTION
OF ELECTROMAGNETIC PROPAGATION IN A STRATIFIED
TERRESTRIAL ATMOSPHERE

by

Robert Elgin Post

A Dissertation Submitted to the
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Approved:

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Dean of Graduate College

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Ames, Iowa

1962

October 4, 1962

I hereby request that the changes listed below which I desire to make in the manuscript copy of the thesis submitted for the degree Doctor of Philosophy be approved.

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The term $\exp(-j a_s \rho / 2K^2)$ be changed to $\exp(-j a_s k_o \rho / 2K^2)$ in Equation 143.

The term $\exp(-j a_s k_o \rho / 2K^2)$ be added to Equation 168 inside the summation sign.

The term $\exp[-(1.115 \times 10^{-5})(1 + a \frac{dn}{dr})^{2/3} \rho]$, be changed to

$\exp[-(9.65 \times 10^{-5})(1 + a \frac{dn}{dr})^{2/3} \rho]$ in Equation 194.

The term $\exp[-(6.7 \times 10^{-6})(1 + a \frac{dn}{dr})^{2/3} \rho]$, be changed to

$\exp[-(5.8 \times 10^{-5})(1 + a \frac{dn}{dr})^{2/3} \rho]$ in Equation 266.

These changes will affect the calculated curves plotted on Figures 2 and 4 by increasing the slope of these curves.

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I. INTRODUCTION

The problem of the propagation of electromagnetic energy around a spherical body where the diameter of the sphere is large with respect to the wavelength of the energy was first solved by Lord Rayleigh in terms of an infinite series of spherical harmonics. The solution was practically useless from an engineering point-of-view because of the large number of terms required to approximate the final answer. Subsequently, G. N. Watson developed a transformation which transformed the infinite series into a contour integral which is then evaluated by the method of residues. The residues of the integral involve asymptotic expansions of the Hankel functions of order one-third or the Airy integrals. A second approach to the problem, and the approach which serves as the basis of this paper is that developed by Pryce (16) and Pekar (15) in which the earth is assumed to be flat and the atmosphere homogeneous. The solution now reduces to an infinite integral of zero-ordered Bessel functions. Schelleng and Burrows (17) proposed a model wherein the earth was assumed to have a modified radius of about $4/3$ the actual radius and the earth's atmosphere was assumed to be homogeneous. This model was used to account for the gradient of the refractive index of the earth's atmosphere. All of these models refer to the so-called "normal mode" of propagation as distinguished from the "turbulent scatterer" theory of propagation of electromagnetic energy which will be discussed later. Very complete discussions of normal mode theory are included in: "Terrestrial Radio Waves" by Bremmer (3) and Volume 13 of the M.I.T. Radiation Laboratory Series: "Propagation of Short Radio Waves" by Kerr (12). In 1958, Carroll and Ring (4) published

the results of a very extensive investigation into the propagation of radio waves by the normal modes of the atmosphere. These results indicated that the classical "airless earth" modes were not valid for calculating the effects of the internal reflections of a stratified atmosphere. In 1959 Tukizi (18) published a theory which, while using a different approach to the problem, achieved results which had excellent comparison with experiment and, which are corroborated by this analysis. Tukizi's results indicate the utility of the normal mode theory in predicting the strength of the field in the diffraction region and especially in predicting the radial attenuation of the field in the diffraction region.

The term "diffraction region" as referred to by Tukizi (18) and a number of other authors stems from the theory that the presence of the field over the horizon from the source is due to a diffraction process caused by the curvature of the earth. Other authors call this same phenomenon by different names; for example, Carroll and Ring (4) use the term "Twilight region". In this paper, the terminology which is favored by the proponents of the "turbulent scatterer" theories will be used; "trans-horizon field". This will mean that field which is over the horizon from the source.

The other theory of propagation of short radio waves over the horizon was first introduced by Booker and Gordon (2) in 1950. Basically, the Booker-Gordon theory is that spherical or ellipsoidal anomalies in the refractive index act as scatterers of electromagnetic energy. These anomalies, often called "blobs", are supposedly located in the common volume of the transmitting and receiving antenna beams and serve to

scatter some of the transmitted energy down to the receiver. These "blobs" are being formed continuously, reducing in size, and finally dissipating, and are a direct manifestation of the turbulence of the atmosphere. The turbulence can be described in a statistical fashion and there are several theories concerning turbulence which serve as the basis for these arguments (9, 19, 20). The advantage of the turbulent scatterer theory is that the statistics of the turbulence lead directly to a statistical character for the field in the diffraction region. In normal mode theory, the atmosphere is assumed to be static, so that there is no statistical character to the transhorizon field. The turbulent scatterer theory is very attractive from a number of points-of-view, most of which involve the statistical character of the field in the transhorizon region. The most significant shortcoming in the turbulent scatterer theories is that the intensity of the turbulence, or the variations in the index of refraction in the "scattering blobs", is not sufficiently large to account for the fields observed since in most cases, the common volume of the antennas is very high in the troposphere.

At the present time the Booker-Gordon theory is most widely accepted with the majority of the work being done in correlating the meteorological phenomena with the field in the transhorizon region. A number of good resumes of the Booker-Gordon theory and the various turbulence theories are available (2, 9, 19, 20).

The present analysis uses the flat-earth approximation applied to the normal mode theory of transhorizon propagation. The results reported by Pryce (16) are achieved by a different technique and the analysis is extended

to include the effects of an inhomogeneous atmosphere with a constant gradient of refractive index. Finally, a technique for approximating an arbitrary profile of refractive index is developed and two model atmospheres are considered. Field strengths calculated on the basis of this analysis are compared with experimental data reported by Dinger, Garner, Hamilton and Teachman (5) with good results.

II. PROPAGATION IN A STRATIFIED ATMOSPHERE

A. Definition of the Problem

The problem to be considered is that of the propagation of electromagnetic energy around a spherical earth from a source located at some point on or above the surface of the earth. The atmosphere of the earth will be assumed to be spherically stratified; that is, the index of refraction is a function of radius. The coordinate system for the spherical earth is shown in Figure 1. The source is located at the point $r = a + d$, where a is the radius of the earth. Thus, d is the height of the source above the surface of the earth.

The analysis is concerned with the transhorizon field only, since the intent of the analysis is to develop an analytical technique which will allow a piecewise-linear approximation to any profile of refractive index. The transhorizon field will be solved for both magnetic and electric dipole sources located in an atmosphere having a linearly varying profile of refractive index. This solution will be compared with the result obtained by Pryce (16) for the case of a homogeneous atmosphere. This will be the case of a profile of refractive index with zero slope. The article by Pryce (16) is the basis of the present analysis since the "earth-flattening" transformation and the Airy Integral solution with its rapidly convergent asymptotic expansion is a very attractive technique. It should be pointed out that Pryce (16) solved for the cases of vertical and horizontal dipoles rather than the cases of electric and magnetic dipoles considered in this analysis. The essential difference lies in the fact that there are no

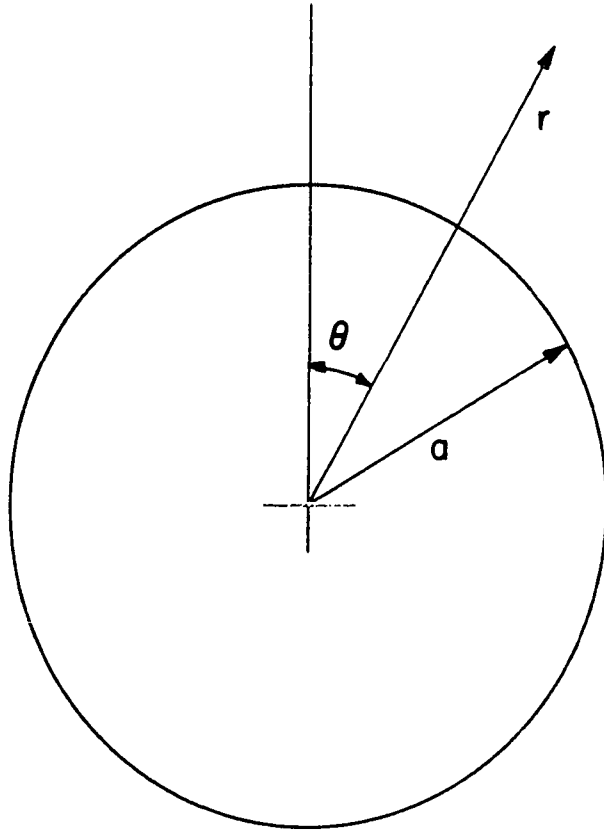


Figure 1. The spherical polar coordinates of the problem

azimuthal variations to consider when using the elementary dipoles as opposed to the more practical antennas of Pryce's analysis. The fact that the elementary antennas of this analysis are not practical antennas configurations does not detract from the significance of the results because it is the ratio of the energy in the transhorizon region to energy which has traveled a corresponding distance in free-space which has direct application in engineering problems. This ratio would be the same for any antenna configuration of a given polarization. Thus, it is a simple matter to estimate the strength of the field in the transhorizon region from the free-space field strength.

The analysis will begin with Maxwell's equations from which Hertzian vectors for both the magnetic and electric dipoles will be developed. The components of the electric and magnetic fields for both types of polarization will be expressed in terms of the Hertz vectors. The inhomogeneous wave equation for the Hertz vector will be reduced to a pair of single-dimensional differential equations by the separation-of-variables technique. At this point the "flat-earth" approximation will be made. The "flat-earth" approximation is a transformation from spherical-polar coordinates to cylindrical coordinates. After the transformation has been made, the two one-dimensional differential equations will be solved. The Hertz vector will be found by weighting each term of a complete orthonormal set of solutions in one variable by an appropriate Green's function of the other variable. The resulting expression will be the Hertz vector for a monochromatic elementary dipole of unit strength. The Green's function technique is well suited to making a piecewise or sectionally-linear

approximation to an arbitrary continuous curve. The resulting indefinite integrals are evaluated by means of the residue summation of the theory of complex variables. The asymptotic expansion of the Airy integrals allows the final expression, which is an infinite series, to be approximated by a few terms.

B. Maxwell's Equations for the Inhomogeneous Atmosphere

Maxwell's equations in spherical-polar form for application to an inhomogeneous atmosphere are:

$$\nabla \times \vec{E}(r, \theta, \phi, t) = -\mu \frac{\partial \vec{H}(r, \theta, \phi, t)}{\partial t}$$

$$\nabla \times \vec{H}(r, \theta, \phi, t) = \sigma \vec{E}(r, \theta, \phi, t) + \epsilon(r) \frac{\partial \vec{E}(r, \theta, \phi, t)}{\partial t}$$

1

$$\nabla \cdot \epsilon(r) \vec{E}(r, \theta, \phi, t) = \rho_c$$

$$\nabla \cdot \vec{H}(r, \theta, \phi, t) = 0$$

Where $\vec{E}(r, \theta, \phi, t)$ is the electric-field-intensity vector, $\vec{H}(r, \theta, \phi, t)$ is the magnetic-field-intensity vector, σ is the conductivity of the medium, ρ_c is the conduction charge in the medium, $\epsilon(r)$ is the dielectric constant of the medium; in this case, the permittivity of the medium is a function of the radial distance, and μ is the permeability of the medium. The radiation will be assumed to be monochromatic of frequency ω ; that is, the time-dependence can be written in the form; $e^{j\omega t}$. Under this condition, Maxwell's equations can be written in time-independent spherical-polar form as;

$$\nabla \times \vec{E}(r, \theta, \phi) = -j\omega\mu\vec{H}(r, \theta, \phi)$$

$$\nabla \times \vec{H}(r, \theta, \phi) = \sigma\vec{E}(r, \theta, \phi) + j\omega(r)\vec{E}(r, \theta, \phi)$$

2

$$\nabla \cdot \epsilon(r)\vec{E}(r, \theta, \phi) = \rho_c$$

$$\nabla \cdot \vec{H}(r, \theta, \phi) = 0$$

C. The Magnetic Dipole

A magnetic dipole of strength M is defined to be a loop of current of magnitude I and radius r such that $M = \lim_{\substack{r \rightarrow 0 \\ I \rightarrow \infty}} rI$, and is characterized by

the following field relationships:

$$E_r = E_\phi = H_\theta = 0.$$

The formulation of the Hertz vector for a magnetic dipole in an inhomogeneous atmosphere shown here is due to Friedman (7). Maxwell's equations for field components with no azimuthal variations in a charge-free inhomogeneous non-conducting medium are:

$$\nabla \times \vec{E}(r, \theta) = -j\omega\mu\vec{H}(r, \theta) \quad a$$

$$\nabla \times \vec{H}(r, \theta) = j\omega\epsilon(r)\vec{E}(r, \theta) \quad b$$

3

$$\nabla \cdot \epsilon(r)\vec{E}(r, \theta) = 0 \quad c$$

$$\nabla \cdot \vec{H}(r, \theta) = 0 \quad d$$

By Equation 3(c):

$$\nabla \cdot \epsilon(r)\vec{E}(r, \theta) = 0,$$

$\epsilon(r) \vec{E}(r, \theta)$ can be written as the curl of a vector $\epsilon(r) \vec{\Pi}_m(r, \theta)$. That is;

$$\epsilon(r) \vec{E}(r, \theta) = -j\omega\mu \nabla \times \epsilon(r) \vec{\Pi}_m(r, \theta) \quad . \quad 4$$

The seemingly arbitrary form of Equation 4 is justified by the fact that it is desired to develop a Hertz vector which is applicable to a magnetic dipole. For this reason, the operation above and other somewhat arbitrary definitions in the equations to follow are justified. Using Equation 4 in Maxwell's equation, 3(b) leads to:

$$\nabla \times \vec{H}(r, \theta) - \omega^2\mu \nabla \times \epsilon(r) \vec{\Pi}_m(r, \theta) = 0 \quad 5a$$

or

$$\nabla \times [\vec{H}(r, \theta) - \omega^2\mu \epsilon(r) \vec{\Pi}_m(r, \theta)] = 0 \quad 5b$$

The expression in parentheses can be written as the gradient of a scalar:

$$\vec{H}(r, \theta) - \omega^2\mu \epsilon(r) \vec{\Pi}_m(r, \theta) = \nabla \Psi(r, \theta) \quad . \quad 6$$

Solving Equation 6 for the magnetic field intensity $\vec{H}(r, \theta)$, results in:

$$\vec{H}(r, \theta) = \omega^2\mu \epsilon(r) \vec{\Pi}_m(r, \theta) + \nabla \Psi(r, \theta) \quad . \quad 7$$

Equation 3(c) can be rewritten in terms of the Hertz vector as:

$$\vec{E}(r, \theta) = \frac{-j\omega\mu}{\epsilon(r)} \nabla \times \epsilon(r) \vec{\Pi}_m(r, \theta) \quad . \quad 8$$

Substituting Equations 7 and 8 into Equation 3(a) results in:

$$\nabla \times \frac{1}{\epsilon(r)} \nabla \times \epsilon(r) \vec{\Pi}_m(r, \theta) - \omega^2\mu \epsilon(r) \vec{\Pi}_m(r, \theta) - \nabla \Psi(r, \theta) = 0. \quad 9$$

Substituting Equation 7 into Equation 3(d) results in:

$$\omega^2 \mu \nabla \cdot \epsilon(r) \vec{\Pi}_m(r, \theta) + \nabla^2 \Psi(r, \theta) = 0 . \quad 10$$

Friedman (7) points out that the divergence of Equation 9 will result in Equation 10, therefore it will be sufficient to consider only Equation 9.

Assume that the Hertz vector is radially directed and given by the equation:

$$\vec{\Pi}_m(r, \theta) = \vec{a}_r r U_m(r, \theta) .$$

Again, this somewhat arbitrary assumption is justified by the fact that it leads to the proper field components. Actually, this Hertz vector must reduce to the classical free space Hertz vector if the medium is assumed to be homogeneous and it is known that the free space Hertz vector has the same direction as the source dipole. In both cases, the source dipole is oriented radially.

$$\nabla \times \epsilon(r) \vec{\Pi}_m(r, \theta) = \vec{a}_\theta \epsilon(r) \frac{\partial U_m(r, \theta)}{\partial \theta} , \quad 11$$

and

$$\begin{aligned} \nabla \times \frac{1}{\epsilon(r)} \nabla \times \epsilon(r) \vec{\Pi}_m(r, \theta) &= \vec{a}_r \left\{ \frac{1}{r} \frac{\partial^2 U_m}{\partial \theta^2} + \frac{1}{r} \cot \theta \frac{\partial U_m}{\partial \theta} \right\} \\ &+ \vec{a}_\theta \left\{ \frac{1}{r} \frac{\partial U_m}{\partial \theta} + \frac{\partial^2 U_m}{\partial r \partial \theta} \right\} . \end{aligned} \quad 12$$

By Equation 9, Equation 12 must be equal to:

$$\omega^2 \mu \epsilon(r) \vec{a}_r r U_m(r, \theta) + \vec{a}_r \frac{\partial \Psi(r, \theta)}{\partial r} + \vec{a}_\theta \frac{1}{r} \frac{\partial \Psi(r, \theta)}{\partial \theta} . \quad 13$$

Equating the coefficients of the unit vectors \vec{a}_r and \vec{a}_θ results in:

$$\frac{1}{r} \frac{\partial^2 U_m}{\partial \theta^2} + \frac{1}{r} \cot \theta \frac{\partial U_m}{\partial \theta} + \omega^2 \mu \epsilon(r) r U_m + \frac{\partial \psi}{\partial r} = 0 \quad 14$$

and

$$\frac{1}{r} \frac{\partial U_m}{\partial \theta} + \frac{\partial^2 U_m}{\partial r \partial \theta} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad 15$$

Equation 15 can be integrated once with respect to θ to give

$$\psi(r, \theta) = \frac{\partial [r U_m(r, \theta)]}{\partial r} \quad 16$$

Substituting Equation 16 into Equation 14 results in:

$$\frac{1}{r} \frac{\partial^2 U_m}{\partial \theta^2} + \frac{1}{r} \cot \theta \frac{\partial U_m}{\partial \theta} + \omega^2 \mu \epsilon(r) r U_m + \frac{\partial^2 r U_m}{\partial r^2} = 0 \quad 17$$

Equation 17 is recognized as Helmholtz's equation in spherical polar coordinates applied to a function with no azimuthal dependence. Thus, it is shown that for the magnetic dipole case, the equation to be solved is

$$\nabla^2 U_m(r, \theta) + \omega^2 \mu \epsilon(r) U_m(r, \theta) = 0 \quad 18$$

regardless of the stratification of the medium.

Once the scalar function $U_m(r, \theta)$ has been determined the Hertz vector $\vec{\Pi}_m(r, \theta)$ can be written and the components of the electromagnetic field can be formulated. The field components are:

$$E_r = 0 \quad (a) \quad H_r = \frac{\partial^2 r U_m}{\partial r^2} + \omega^2 \mu \epsilon(r) U_m \quad (d)$$

$$E_\theta = 0 \quad (b) \quad H_\theta = \frac{1}{r} \frac{\partial^2 r U_m}{\partial r \partial \theta} \quad (e) \quad 19$$

$$E_\phi = -j\omega\mu \frac{\partial U_m}{\partial \theta} \quad (c) \quad H_\phi = 0 \quad (f)$$

The equation which must be solved to determine the components of an electromagnetic field propagating from a magnetic dipole located in an inhomogeneous atmosphere is:

$$\frac{1}{r} \frac{\partial^2 U_m}{\partial \theta^2} + \frac{1}{r} \cot \theta \frac{\partial U_m}{\partial \theta} + \omega^2 \mu \epsilon(r) r U_m + \frac{\partial^2 r U_m}{\partial r^2} = 0 . \quad 17$$

Assume a product-type solution of the form

$$U_m(r, \theta) = R(r) \theta(\theta) . \quad 20$$

Substituting Equation 20 into Equation 17 results in:

$$\frac{r^2 R''}{R} + \frac{2rR'}{R} + r^2 \omega^2 \mu \epsilon(r) + \frac{\theta''}{\theta} + \cot \theta \frac{\theta'}{\theta} = 0 . \quad 21$$

The separation constant λ_m^2 is defined such that:

$$\theta'' + \cot \theta \theta' + \lambda_m^2 \theta = 0 \quad 22$$

and

$$R'' + \frac{2R'}{r} + \left(\omega^2 \mu \epsilon(r) - \frac{\lambda_m^2}{r^2} \right) R = 0 \quad 23$$

and

$$\lambda_m^2 > 0 .$$

At this point it will be appropriate to develop a corresponding set of equations applicable to the electric dipole.

D. The Electric Dipole

An electric dipole of strength M is defined to be two charges of magnitude Q , having opposite polarity and located a distance d apart, such

that $M = \lim_{\substack{Q \rightarrow \infty \\ d \rightarrow 0}} dQ$.

As in the case of the magnetic dipole, the time-dependence of the radiation will be assumed to be of the form, $e^{j\omega t}$. Actually, the electric dipole is the radiating element, short compared with wavelength, which is used as an initial example in elementary texts on antenna theory (1). The formulation of the Hertz vector for the electric dipole again follows the work of Friedman (7). Starting from Equation 3(d);

$$\nabla \cdot \vec{H}(r, \theta) = 0$$

$\vec{H}(r, \theta)$ can be written as the curl of a Hertz vector,

$$\vec{H}(r, \theta) = \nabla \times \vec{\Pi}_e(r, \theta) \quad . \quad 24$$

Substituting Equation 24 into Equation 3(a) results in:

$$\nabla \times \vec{E}(r, \theta) = -j\omega\mu \nabla \times \vec{\Pi}_e(r, \theta) \quad 25a$$

or

$$\nabla \times \vec{E}(r, \theta) + j\omega\mu \vec{\Pi}_e(r, \theta) = 0 \quad . \quad 25b$$

The expression in parentheses can be written as the gradient of a scalar $\phi(r, \theta)$.

$$\vec{E}(r, \theta) + j\omega\mu \vec{\Pi}_e(r, \theta) = \nabla \phi(r, \theta) \quad 26$$

Solving Equation 26 for the electric field intensity vector results in:

$$\vec{E}(r, \theta) = -j\omega\mu \vec{\Pi}_e(r, \theta) + \nabla \phi(r, \theta) \quad 27$$

Substituting Equations 24 and 27 into Maxwell's equations 3(b) and 3(c) results in:

$$\nabla \times \nabla \times \vec{\Pi}_e(r, \theta) = \omega^2 \mu \epsilon(r) \vec{\Pi}_e(r, \theta) + j\omega \epsilon(r) \nabla \phi(r, \theta) \quad 28$$

and

$$-j\omega \mu \nabla \cdot \epsilon(r) \vec{\Pi}_e(r, \theta) + \nabla \cdot \epsilon(r) \nabla \phi(r, \theta) = 0. \quad 29$$

Again, it is noted that the divergence of Equation 28 is Equation 29; therefore, it will be sufficient to consider only Equation 28. Assume that the Hertz vector to be radially directed and given by the equation:

$$\vec{\Pi}_e(r, \theta) = \vec{a}_r r U_e(r, \theta) \quad . \quad 30$$

Substituting into Equation 28 and equating the coefficients of the unit vectors \vec{a}_r and \vec{a}_θ , results in the two equations:

$$\frac{1}{r^2} \cot \theta \frac{\partial U_e}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 U_e}{\partial \theta^2} + \omega^2 \mu \epsilon(r) U_e + \frac{j\omega \epsilon(r)}{r} \frac{\partial \phi}{\partial r} = 0 \quad 31$$

and

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial U_e}{\partial \theta} = \frac{j\omega \epsilon(r)}{r} \frac{\partial \phi}{\partial \theta} \quad 32$$

Integrating Equation 32 once with respect to θ results in:

$$\phi(r, \theta) = \frac{1}{j\omega \epsilon(r)} \frac{\partial r U_e(r, \theta)}{\partial r} \quad . \quad 33$$

Substituting Equation 33 into Equation 31 results in:

$$\frac{1}{r^2} \cot \theta \frac{\partial U_e}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 U_e}{\partial \theta^2} + \omega^2 \mu \epsilon(r) U_e + \frac{j\omega \epsilon(r)}{r} \frac{\partial}{\partial r} \frac{r U_e}{j\omega \epsilon(r)} = 0 \quad 34$$

or

$$\frac{\partial^2 U_e}{\partial r^2} + \frac{2}{r} \frac{\partial U_e}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U_e}{\partial \theta^2} + \frac{1}{r^2} \cot \theta \frac{\partial U_e}{\partial \theta} + \omega^2 \mu \epsilon(r) U_e - \frac{\epsilon'(r)}{\epsilon(r)} \frac{\partial U_e}{\partial r} - \frac{1}{r} \frac{\epsilon'(r)}{\epsilon(r)} U_e = 0. \quad 34$$

Equation 34 looks just like the corresponding equation which was developed for the magnetic dipole except for the terms which are due to the gradient in the permittivity of the medium. This difference is to be expected since the Hertz vector is a measure of the electric field; and, since the electric flux density must be continuous, the electric field intensity must vary on account of the inhomogeneities in the permittivity of the medium. Just as in the case of the magnetic dipole, once the scalar $U_e(r, \theta)$ has been determined, the Hertz vector is known and all the field components can be formulated. The field components for the electric dipole are:

$$\begin{aligned} E_r &= \frac{-j}{\omega \epsilon(r)} \left[\epsilon(r) \frac{\partial}{\partial r} \frac{1}{\epsilon(r)} r U_e + \omega^2 \mu \epsilon(r) r U_e \right] & a \\ E_\theta &= \frac{-j}{\omega \epsilon(r)} \frac{1}{r} \frac{\partial^2}{\partial \theta \partial r} r U_e & b \\ E_\phi &= 0 & c \\ H_r &= 0 & d \\ H_\theta &= 0 & e \\ H_\phi &= - \frac{\partial U_e}{\partial \theta} & f \end{aligned} \quad 35$$

In the electric dipole case, the equation which must be solved is:

$$\frac{\partial^2 U_e}{\partial r^2} + \frac{2}{r} \frac{\partial U_e}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U_e}{\partial \theta^2} + \frac{1}{r^2} \cot \theta \frac{\partial U_e}{\partial \theta} + \omega^2 \mu \epsilon(r) U_e - \frac{\epsilon'(r)}{\epsilon(r)} \frac{\partial U_e}{\partial r} - \frac{1}{r} \frac{\epsilon'(r)}{\epsilon(r)} U_e = 0$$

Assuming a product-type solution, that is:

$$U_e(r, \theta) = R(r) \theta(\theta) \quad 36$$

Substituting into the differential equation and separating the variables results in the two equations:

$$\theta'' + \cot \theta \theta' + \lambda_e^2 \theta = 0 \quad 37$$

$$R'' + \frac{1}{r} \left(2 - r \frac{\epsilon'(r)}{\epsilon(r)} \right) R' + \left(\omega^2 \mu \epsilon(r) - \frac{\lambda_e^2}{r^2} - \frac{\epsilon'(r)}{r \epsilon(r)} \right) R = 0 \quad 38$$

Where the separation constant λ_e^2 is defined such that $\lambda_e^2 > 0$. The Equations 37 and 38 are quite similar to the corresponding equations for the magnetic dipole, however, Equation 38 is more unwieldy than either of the other equations. This analysis is confined to an atmosphere with a linearly-varying profile of refractive index. This means that the gradient terms of Equation 38 will be a number rather than some function of radius. Thus, certain simplifying assumptions can be made during the analysis. In particular, when considering the portion of the atmosphere in which the gradient of the refractive index is very small, Equation 38 reduces to the equation developed for the magnetic dipole.

Having developed expressions suitable for determining the field components of the electromagnetic wave propagating from either of the two

source types, the next step is to convert from the spherical geometry of the problem to the cylindrical geometry of the "flat-earth" approximation.

E. The Flat-Earth Approximation

The earth-flattening approximation is nothing more than a transformation from a spherical-polar geometry to a cylindrical geometry. The transformation equations are:

$$\eta = a \ln r/a$$

$$\rho = a \theta$$

39

Where ρ represents radial distance or range and η represents height. These are the same transformation equations used by Pryce (16) but reduced to the case of no azimuthal variations. The transformation is such that a cone with its apex at the earth's center becomes a cylinder and the surface of the earth becomes a plane oriented such that the axis of the cylinder is normal to the surface of the plane. The introduction of the "earth-flattening" approximation is credited to Schelleng, Burrows and Ferrell (17) who, in an effort to simplify the analysis of the transhorizon field due to the curvature of the "rays" of energy from the source, decided to transform to a coordinate system where the rays became straight lines. This leads directly to the equivalent radius of the earth concept, or the so-called $4/3$ earth radius. Subsequent work considered the earth to be flat which meant that the rays were bent upward. This is no problem if one is interested in a solution of the differential equation governing the propagation of energy as opposed to the ray-tracing technique. Pryce (16) credits the final form of the transformation equations to C. L. Pekar

for the range transformation and to Prof. E. T. Copson for the height transformation. Pekarís has shown (15) that the error involved in making the small-angle approximation is less than 2% for ranges up to one-half the radius of the earth. Copson pointed out that the height transformation $\eta = a \ln r/a$ is preferable to the somewhat more intuitive $\eta = r - a$ because the geodesics correspond to straight lines in the first case and only approximate straight lines in the second case.

The differential equation with η , the height variable, as the independent variable will be referred to as the height-gain equation. The differential equation having ρ , the radial variable, as the independent variable, will be called the range equation. Pekarís analyzed the error involved in approximating the height-gain equation by Stoke's equation and concluded that the error becomes quite large at moderate heights. Koo and Katzin (13) have shown that the height-gain equation can be made exact by making a change-of-variable so that the height-gain equation becomes Stoke's equation. It will be seen that the range equation transforms into Bessel's equation of order zero with a parameter λ/a . The solutions to the transformed range equation will be zero-order Bessel functions and the solutions of the transformed height-gain equation will be Airy integrals.

F. The Airy Integral

The Airy integrals, signified by $Ai(z)$ and $Bi(z)$ are solutions of the differential equation known as Stoke's equation:

$$\frac{d^2 U(z)}{dz^2} - z U(z) = 0 \quad 40$$

The differential equation is of second order; therefore, there are two linearly independent solutions, $Ai(z)$ and $Bi(z)$, defined by the integrals:

$$Ai(z) = \frac{1}{\pi} \int_0^{\infty} \cos \left(sz + \frac{s^3}{3} \right) ds \quad 41$$

and

$$Bi(z) = \frac{1}{\pi} \int_0^{\infty} \left\{ \exp \left(tz - \frac{t^3}{3} \right) + \sin \left(tz + \frac{t^3}{3} \right) \right\} dt \quad 42$$

A complete discussion of the Airy integral is given in Jeffreys and Jeffreys (11). The relationship between Airy integrals and the general solution of Stoke's equation, Bessel functions of order one-third, is given in The Annals of the Computation Laboratory of Harvard University, Vol. II, entitled "Modified Hankel Functions of Order One-Third" (10). The Airy integral solutions are shown to be superpositions of the Hankel function solutions. The particular advantage of the Airy integral formulation is rapid convergence property of the asymptotic series expansion of the Airy integral solutions. The asymptotic expansions and their associated intervals of convergence are listed below.

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp \left(-\frac{2}{3} z^{3/2} \right) \left\{ 1 - \frac{1 \cdot 5}{1! 48} z^{-3/2} + \frac{1 \cdot 7 \cdot 5 \cdot 11}{2! 48^2} z^{-3} \right. \\ \left. - \frac{1 \cdot 7 \cdot 13 \cdot 5 \cdot 11 \cdot 17}{3! 48^3} z^{-9/2} + \dots \right. \quad 43$$

for $-\pi < \arg z < +\pi$.

$$\text{Bi}(z) \sim \frac{1}{\sqrt{\pi}} z^{-1/4} \exp\left(\frac{2}{3} z^{3/2}\right) \left\{ 1 + \frac{1 \cdot 5}{1! 48} z^{-3/2} + \frac{1 \cdot 7 \cdot 5 \cdot 11}{2! 48^2} z^{-3} \right. \\ \left. + \frac{1 \cdot 7 \cdot 13 \cdot 5 \cdot 11 \cdot 17}{3! 48^3} z^{-9/2} + \dots \right\} \quad 44$$

for $-1/3 \pi < \arg z < +1/3 \pi$.

When $\arg z = \pi$; set $z = \wp \exp(j\pi)$, which results in:

$$\text{Ai}(z) = \frac{1}{\sqrt{\pi}} \wp^{-1/4} \left\{ P(\wp) \sin\left(\frac{2}{3} \wp^{3/2} + \pi/4\right) - Q(\wp) \cos\left(\frac{2}{3} \wp^{3/2} \right. \right. \\ \left. \left. + \pi/4\right) \right\} \quad 45$$

$$\text{Bi}(z) = \frac{1}{\sqrt{\pi}} \wp^{1/4} \left\{ P(\wp) \cos\left(\frac{2}{3} \wp^{3/2} + \pi/4\right) + Q(\wp) \sin\left(\frac{2}{3} \wp^{3/2} \right. \right. \\ \left. \left. + \pi/4\right) \right\} \quad 46$$

where

$$P(\wp) \sim 1 - \frac{1 \cdot 7 \cdot 5 \cdot 11}{2! 48^2} \wp^{-3} + \frac{1 \cdot 7 \cdot 13 \cdot 19 \cdot 5 \cdot 11 \cdot 13 \cdot 19}{4! 48^4} + \dots \quad 47$$

and

$$Q(\wp) \sim \frac{1 \cdot 5}{1! 48} \wp^{-3/2} - \frac{1 \cdot 7 \cdot 13 \cdot 5 \cdot 7 \cdot 11}{3! 48^3} \wp^{-9/2} + \dots \quad 48$$

It should be noted that the second term of each expansion is about one-tenth of the first term for $|z| = 1$. The functions $\text{Ai}(z)$ and $\text{Bi}(z)$ are chosen as the fundamental pair in such a way that one of the solutions, $\text{Ai}(z)$, would decrease exponentially along the positive real axis and that both of the solutions would be of equal amplitude and oscillatory, but differing in phase by $\pi/2$ radians, along the negative real axis. These

solutions are most suitable for application to propagation problems because the first condition provides a solution applicable to wave propagation in a lossy medium such as the earth while the second condition provides a solution which represents an outgoing wave at large heights which is one of the boundary conditions of this problem. In the particular problem at hand, the dissipative medium, the earth, is the region of negative η and the medium suitable for lossless propagation is the region of positive η . It will be seen that on making a change of variable in the height-gain equation to transform it into Stoke's equation, an inversion of the η axis will be made thus making the Airy integral solutions applicable.

Two identities involving Airy integrals which will be of considerable utility in the analysis to follow are;

$$\text{Ai}[z \exp(j 2k\pi/3)] = \exp(j k\pi/3) \left[\begin{array}{l} \cos k\pi/3 \text{ Ai}(z) - \frac{j}{\sqrt{3}} \sin k\pi/3 \\ \text{Bi}(z) \end{array} \right] \quad 49$$

and

$$\text{Bi}[z \exp(j 2k\pi/3)] = \exp(j k\pi/3) \left[\begin{array}{l} -j \sqrt{3} \sin k\pi/3 \text{ Ai}(z) + \cos k\pi/3 \\ \text{Bi}(z) \end{array} \right] \quad 50$$

Complete tables of Airy integrals and their derivatives and a tabulation of the zeros of the Airy integrals and their derivatives are given in the Mathematical Tables Part-Vol. B (14). These tables allow ready evaluation of the expressions which will be developed for the field components.

The Airy integral solutions will be used to formulate a system of equations whose solution will be used to weigh each term of a complete

orthonormal set of solutions of the differential equation in the other variable. The resulting integral will be the complete field solution.

G. The Green's Function

The solutions of the differential equations governing the propagation of electromagnetic energy around the earth must be modified to account for the singularity at the source point. The Green's function is a function of η which satisfies the boundary conditions at the earth's surface, represents an exponentially decaying wave inside the earth, represents outgoing radiation at great heights above the surface of the earth, and whose derivative has the proper discontinuity at the source point.

Assume that a dipole of unit strength is located at a height d above the surface of the earth. It will be convenient to locate the $\theta = 0$ line through the dipole, such that the coordinates of the dipole are $(a + d, 0)$. This point will transform into the point $(\eta_d, 0)$ in the cylindrical geometry of the flat-earth approximation. The point discontinuity is represented by Dirac-delta functions in the spherical-polar coordinate system. The differential equation which must be solved can be written as;

$$[D^2(r, \theta) + k^2]U = \delta[r - (a + d)]\delta(\theta) \quad 51$$

where

$$D^2(r, \theta) \equiv D^2(r) + D^2(\theta) \quad 52$$

$D^2(r, \theta)$ represents a second order differential operation which in the case of a homogeneous atmosphere or a magnetic dipole in a spherically stratified atmosphere is the Laplacian operator. In the case of an

electric dipole in an inhomogeneous spherically stratified atmosphere the $D^2(r, \theta)$ operator is not the Laplacian operator because of the $\text{grad } \epsilon(r)$ terms. The point discontinuity will retain its essential characteristics through the transformation to the cylindrical coordinate system, thus Equation 51 transforms into:

$$[D'^2(\eta, \rho) + k'^2]U = \delta(\eta - \eta_d) \delta(\rho) \quad 53$$

Where the primed superscripts merely indicate that a transformation has been made. Since the dipole is monochromatic, the time-dependence can be removed from the source by assuming an exponential time-dependence of the form, $e^{j\omega t}$. All of the field components have the same time-dependence so the $e^{j\omega t}$ can be factored out of each term, thus reducing the problem to one of static fields.

The problem can be solved by finding a complete orthonormal set of solutions of the equation:

$$[D'^2(\rho) + \lambda'^2] \theta = 0 \quad 54$$

Each term of the complete orthonormal set of solutions of Equation 54 will be weighted by a Green's function in the η variable. To see that this procedure does indeed produce a solution of Equation 53, consider the separated differential equations after transformation. These transformed differential equations are:

$$[D'^2(\eta) + (k'^2 - \lambda'^2)]R = \delta(\eta - \eta_d) \quad 55$$

and

$$[D'^2(\rho) + \lambda'^2] \theta = \delta(\rho) \quad 56$$

A complete orthonormal set of solutions of the equation:

$$[D'^2(\rho) + \lambda'^2] \theta = 0 ,$$

is a series expansion of a Dirac-delta function, that is

$$[D'^2(\rho) + \lambda'^2] \sum_{i=0}^{\infty} \theta_{\lambda_i}(\rho) \theta_{\lambda_i}^*(0) = \delta(\rho) , \quad 57$$

where the superscript * means complex conjugate. It is necessary to find a solution to the equation

$$[D'^2(\eta) + k'^2 - \lambda'^2]G = \delta(\eta - \eta_d) \quad 58$$

Where $G(\eta, \eta_d)$ is the Green's function. The complete solution can be written as:

$$U = \sum_{i=0}^{\infty} G(\eta, \eta_d) \theta_{\lambda_i}(\rho) \theta_{\lambda_i}^*(0) . \quad 59$$

If Equation 59 is a solution of the differential equation, Equation 53, then substitution of Equation 59 into Equation 53 should result in the two-dimensional Dirac-delta function $\delta(\eta - \eta_d) \delta(\rho)$.

$$[D'^2(\eta, \rho) + k'^2] \sum_{i=0}^{\infty} G(\eta, \eta_d) \theta_{\lambda_i}(\rho) \theta_{\lambda_i}^*(0) = \delta(\eta - \eta_d) \delta(\rho) \quad 60$$

Separate $D'^2(\eta, \rho)$ into $D'^2(\eta) + D'^2(\rho)$ and move the operation inside the summation sign.

$$[D'^2(\eta, \rho) + k'^2] \sum_{i=0}^{\infty} G(\eta, \eta_d) \frac{\theta_{\lambda_i}(\rho)}{\lambda_i} \frac{\theta_{\lambda_i}^*(0)}{\lambda_i} = \sum_{i=0}^{\infty} [D'^2(\eta)$$

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$$+ D'^2(\rho) + k'^2] G(\eta, \eta_d) \frac{\theta_{\lambda_i}(\rho)}{\lambda_i} \frac{\theta_{\lambda_i}^*(0)}{\lambda_i}$$

Add and subtract λ'^2 inside the parenthesis which contains the differential operation and perform the indicated operation on the Green's function

$$\sum_{i=0}^{\infty} [D'^2(\eta) + k'^2 - \lambda'^2 + D'^2(\rho) + \lambda'^2] G(\eta, \eta_d) \frac{\theta_{\lambda_i}(\rho)}{\lambda_i} \frac{\theta_{\lambda_i}^*(0)}{\lambda_i} \quad 62$$

This results in

$$\sum_{i=0}^{\infty} \delta(\eta - \eta_d) [D'^2(\rho) + \lambda'^2] \frac{\theta_{\lambda_i}(\rho)}{\lambda_i} \frac{\theta_{\lambda_i}^*(0)}{\lambda_i} \quad 63$$

or

$$\delta(\eta - \eta_d) [D'^2(\rho) + \lambda'^2] \sum_{i=0}^{\infty} \frac{\theta_{\lambda_i}(\rho)}{\lambda_i} \frac{\theta_{\lambda_i}^*(0)}{\lambda_i}$$

which, by Equation 57, is

$$\delta(\eta - \eta_d) \delta(\rho).$$

Thus proving the validity of the technique.

The next step in the analysis will be to solve the separated equations for the two source types. The first source type to be considered will be the magnetic dipole.

III. THE COMPLETE SOLUTION FOR THE MAGNETIC DIPOLE

The separated differential equations and the field equations governing the propagation of electromagnetic energy from a magnetic dipole in an inhomogeneous medium are:

$$\theta'' + \cot \theta \theta' + \lambda_m^2 \theta = 0 \quad 64$$

$$R'' + \frac{2R'}{r} + (\omega^2 \mu \epsilon(r) - \frac{\lambda_m^2}{r^2}) R = 0 \quad 65$$

$$E_r = 0 \quad 66$$

$$E_\theta = 0 \quad 67$$

$$E_\phi = -j\omega\mu R\theta' \quad 68$$

$$H_r = \theta(R'' + \frac{2}{r} R' + \omega^2 \mu \epsilon(r) R) \quad 69$$

$$H_\theta = (R' + R/r) \theta' \quad 70$$

$$H_\phi = 0 \quad 71$$

The boundary conditions which must be satisfied at the earth-atmosphere interface or at any other spherically stratified discontinuity in the refractive index require matching of the tangential electric and magnetic fields. Matching the boundary conditions at the earth's surface requires that:

$$(E_\phi)_I = (E_\phi)_{II} \quad 72$$

and

$$(\mathbb{H}_\theta)_I = (\mathbb{H}_\theta)_{II} \quad 73$$

where the subscripts I and II refer to the earth and the atmosphere respectively. Substituting into the expressions for these tangential components of electric and magnetic field results in:

$$R_I \theta'_I \Big|_{r=a} = R_{II} \theta'_{II} \Big|_{r=a} \quad 74$$

and

$$(R'_I + R_I/r) \theta'_I \Big|_{r=a} = (R'_{II} + R_{II}/r) \theta'_{II} \Big|_{r=a} \quad 75$$

When considering the field inside the earth, it can be shown that the same differential equations which were developed to describe the fields in the earth's atmosphere will be applicable provided $\epsilon(r)$ is replaced by

$$\epsilon_0 \left(1 - \frac{j\sigma}{\omega\epsilon_0}\right)$$

where σ is the conductivity of the earth, ω is the angular frequency of the source, and ϵ_0 is the permittivity of vacuum. Transforming Equations 64 and 65 into the cylindrical coordinate system of the flat-earth approximation results in:

$$\frac{d^2\theta}{d\rho^2} + \frac{1}{a} \cot \rho/a \frac{d\theta}{d\rho} + \frac{\lambda^2}{a^2} \theta = 0 \quad 76$$

and

$$\frac{d^2R}{d\eta^2} + \frac{1}{a} \frac{dR}{d\eta} + [\omega^2\mu\epsilon(r) \exp(2\eta/a) + \lambda^2/a^2] R = 0 \quad 77$$

The range equation, Equation 76, will be simplified by using a series expansion for $\cot \rho/a$, that is:

$$\cot \rho/a \sim 1/(\rho/a) - (\rho/a)/3 + (\rho^3/a^3)/45 + \dots \quad 78$$

Equation 76 can be rewritten as:

$$\frac{d^2\theta}{d\rho^2} + \frac{1}{\rho} \frac{d\theta}{d\rho} + \frac{\lambda^2}{a^2} \theta = - \frac{1}{\rho} \frac{d\theta}{d\rho} \left[\frac{1}{3} \rho^2/a^2 - \frac{1}{45} \rho^4/a^4 + \dots \right] \quad 79$$

As mentioned in Section II-E, Pekaris (15) has shown that the error involved in ignoring the right-hand side of Equation 79 is less than two-percent at ranges up to one-half the earth's radius. The approximate range equation can be written as:

$$\frac{d^2\theta}{a\rho^2} + \frac{1}{\rho} \frac{d\theta}{d\rho} + \lambda^2/a^2 \theta \approx 0 \quad 80$$

This equation is seen to be Bessel's equation of zero order with a parameter λ/a . The solutions are $J_0(\lambda\rho/a)$ and $N_0(\lambda\rho/a)$, where $J_0(\lambda\rho/a)$ is the zero-ordered Bessel function of the first kind and $N_0(\lambda\rho/a)$ is the zero-ordered Bessel function of the second kind. Because of the unbounded nature of $N_0(\lambda\rho/a)$ as $\rho \rightarrow 0$, it will not be an allowed solution; therefore, the solution of the range equation is:

$$\theta(\rho) = C J_0(\lambda\rho/a) \quad 81$$

where C is an arbitrary constant.

The second equation to be solved is the height-gain equation:

$$\frac{d^2R}{d\eta^2} + \frac{1}{a} \frac{dR}{d\eta} + (\omega^2 \mu \epsilon(r) \exp(2\eta/a) - \lambda^2/a^2) R = 0 \quad 82$$

At this point, it would be appropriate to change from a formulation in

terms of the permittivity of the medium, $\epsilon(r)$, to a formulation in terms of the index of refraction of the medium, $n(r)$. The relationship between the permittivity and the index of refraction is:

$$\mu\epsilon = \frac{n^2}{C^2} \quad 83$$

where C is the velocity of light. Equation 83 can be written

$$\mu\epsilon(r) = n^2(r)/C^2$$

which can be substituted into the height-gain equation to give:

$$\frac{d^2R}{d\gamma^2} + \frac{1}{a} \frac{dR}{d\gamma} + \left(\frac{\omega^2}{C^2} n^2(r) \exp(2\gamma/a) - \gamma^2/a^2 \right) R = 0 \quad 84$$

The atmospheric model which will be used in this analysis will be assumed to have an index of refraction with a uniform slope, that is:

$$n(r) = n_0 + \frac{dn}{dr} (r - a) \quad r > a, \quad 85$$

from which $n^2(r)$ can be approximated by

$$n^2(r) \approx n_0^2 + 2n_0 \frac{dn}{dr} (r - a) \quad r > a \quad 86$$

provided $\frac{dn}{dr} \ll 1$. Transforming Equation 86 into cylindrical coordinates results in:

$$n^2(r) = n_0^2 + 2n_0 \left(\frac{dn}{dr} \right) a (\exp[\gamma/a] - 1) \quad 87$$

The term $\left(\frac{dn}{dr} \right)$ is not transformed because it is a number which would be known in any atmospheric model. Inserting Equation 87 into the height-gain equation results in:

$$\frac{d^2 R}{d\eta^2} + \frac{1}{a} \frac{dR}{d\eta} + \left\{ \omega_{n_0}^2 / c^2 \left[1 + \frac{2a}{n_0} \frac{dn}{dr} [\exp(\eta/a) - 1] \right] \exp(2\eta/a) - \lambda^2 / a^2 \right\} R = 0 \quad 88$$

Koo and Katzin (13) suggest that changing to a new function

$$V = (r/a)^{1/2a} R \quad \text{or} \quad R = V \exp(-\eta/2a).$$

When this substitution is made, the height-gain equation becomes:

$$\frac{d^2 V}{d\eta^2} + \left\{ k_0^2 \left[1 + \frac{2a}{n_0} \frac{dn}{dr} [\exp(\eta/a) - 1] \right] \exp(2\eta/a) - \frac{(\lambda^2 + 1/4)}{a^2} \right\} V = 0 \quad 89$$

where $k_0^2 = \omega_{n_0}^2 / c^2$. The factor $\exp(\eta/a) - 1$ will be approximated by η/a and the term $\exp(2\eta/a)$ will be approximated by $1 + 2\eta/a$. These approximations reduce Equation 89 to:

$$\frac{d^2 V}{d\eta^2} + \left[k_0^2 - \frac{(\lambda^2 + 1/4)}{a^2} + \frac{2k_0^2}{a} \left(1 + \frac{a}{n_0} \frac{dn}{dr} \right) \eta \right] V = 0 \quad 90$$

Making the substitution

$$-\xi = \beta^{-2/3} (\alpha - \beta\eta) \quad 91$$

where

$$\alpha = k_0^2 - \frac{(\lambda^2 + 1/4)}{a^2} \quad 92$$

and

$$\beta = \frac{2k_0^2}{a} \left(1 + \frac{a}{n_0} \frac{dn}{dr} \right) \quad 93$$

results in Stoke's equation:

$$\frac{d^2 V}{d\xi^2} - \xi V = 0 \quad 94$$

It should be noted that in the process of making the change in variable, the η axis undergoes translation and inversion; processes which lead to the proper Airy integral solutions as was discussed in Section II-F.

To get the function U , it is necessary to find a complete orthonormal set of solutions of the range equation. The solution must be finite at $\rho = 0$ and go to zero as $\rho \rightarrow \infty$. It has been established that the proper solutions are $J_0(\lambda\rho/a)$, where the λ 's take on a number of values. A complete orthonormal set of the solutions, applicable to a pipe of radius b and subject to the boundary condition that θ be finite at $\rho = 0$ and θ be zero at $\rho = b$, is given by the equation;

$$\delta(\rho-\rho') = \frac{2}{b^2} \sum_{i=1}^{\infty} \frac{J_0\left(\frac{\lambda_i \rho}{a}\right) J_0\left(\frac{\lambda_i \rho'}{a}\right)}{[J_1(\lambda_i b/a)]^2} \quad 95$$

To expand this solution to the boundary conditions of the present problem, it is necessary to take the limit as $b \rightarrow \infty$.

$$\delta(\rho-\rho') \xrightarrow{b \rightarrow \infty} \frac{\pi}{b} \sum_{i=1}^{\infty} \frac{\lambda_i}{a} J_0\left(\frac{\lambda_i \rho}{a}\right) J_0\left(\frac{\lambda_i \rho'}{a}\right) \quad 96$$

In the limit, the summation becomes an integral:

$$\delta(\rho-\rho') = \int_0^{\infty} \frac{\lambda}{a^2} J_0\left(\frac{\lambda \rho}{a}\right) J_0\left(\frac{\lambda \rho'}{a}\right) d\lambda \quad 97$$

For this problem, $\rho' = 0$, therefore:

$$\delta(\rho) = \int_0^{\infty} \frac{\lambda}{a^2} J_0\left(\frac{\lambda \rho}{a}\right) d\lambda \quad 98$$

To see that this is a dirac-delta function, it will be necessary to evaluate the integral, Equation 98. This can be done quite readily by means of Laplace Transform theory.

$$\int_0^{\infty} \frac{\lambda}{a^2} J_0\left(\frac{\lambda \rho}{a}\right) d\lambda = \lim_{s \rightarrow 0} \frac{1}{a^2} \int_0^{\infty} \exp(-s\lambda) J_0\left(\frac{\lambda \rho}{a}\right) \lambda d\lambda \quad 99$$

$$= \lim_{s \rightarrow 0} \frac{1}{a^2} \mathcal{L}\left[\lambda J_0\left(\frac{\lambda \rho}{a}\right)\right] \quad 100$$

Taking the Laplace Transform of the quantity in the brackets results in:

$$\frac{1}{a^2} \mathcal{L}\left[\lambda J_0\left(\frac{\lambda \rho}{a}\right)\right] = \frac{1}{a^2} \frac{s}{(s^2 + \rho^2/a^2)^{3/2}} \quad 101$$

Taking the limit as $s \rightarrow 0$

$$\lim_{s \rightarrow 0} \frac{s}{a^2 (s^2 + \rho^2/a^2)^{3/2}} = \delta(\rho) \quad 102$$

it is seen that the integral is zero everywhere except at $\rho = 0$, at which point it is undefined. Thus, the integral is a Dirac-delta function. To see that the complete orthogonal set is normalized, it is necessary to integrate the dirac-delta function over ρ .

$$\int_0^{\infty} \delta(\rho) \rho d\rho = \lim_{s \rightarrow 0} \int_0^{\infty} \frac{\rho s d\rho}{a^2 (s^2 + \rho^2/a^2)^{3/2}} \quad 103$$

$$= \lim_{s \rightarrow 0} (-) \frac{s}{(s^2 + \rho^2/a^2)^{1/2}} \Bigg|_0^{\infty} = 1 \quad 104$$

thus proving that the set is normalized.

The next step is to find the Green's function. The differential

equation which must be satisfied is:

$$\frac{d^2 V}{d\eta^2} + \left[k_0^2 - \frac{(\lambda^2 + 1/4)}{a^2} + \frac{2k_0^2}{a} \left(1 + \frac{a}{n_0} \frac{dn}{dr} \right) \eta \right] V = 0 \quad 105$$

which is transformed into Stoke's equation upon making a change of variable:

$$-\xi = \beta^{-2/3} (\alpha + \beta \eta) \quad 106$$

where α and β are functions of the medium. In the earth;

$$\alpha = k_0^2 \left(1 - \frac{\sigma}{\omega \epsilon_0} \right) - \frac{(\lambda^2 + 1/4)}{a^2} \quad 107$$

$$\beta = \frac{2k_0^2}{a} \left(1 - j \frac{\sigma}{\omega \epsilon_0} \right) \quad 108$$

In the atmosphere:

$$\alpha = k_0^2 - \frac{(\lambda^2 + 1/4)}{a^2} \quad 109$$

$$\beta = \frac{2k_0^2}{a} \left(1 + \frac{a}{n_0} \frac{dn}{dr} \right) \quad 110$$

In either case, the equation to be solved is:

$$\left(\frac{d^2}{d\xi^2} - \xi \right) G = (\eta - \eta_d) \quad 111$$

The allowed solutions from which the Green's function must be formulated are the Airy integrals, $Ai(-\xi)$ and $Bi(-\xi)$. The boundary conditions on the Green's function are that it must represent an outgoing wave for large positive η (large negative ξ), an exponentially decreasing wave for large negative η (large positive ξ), the solutions must be continuous at the earth's surface and at the source point, and have the proper discontinuity

in derivative at the source point. By considering the asymptotic form of the Airy integrals, the appropriate solution for large positive η is $\mathfrak{A}[\text{Bi}(-\xi) - j \text{Ai}(-\xi)]$. This is the solution which will be used above the source point ($\eta > \eta_q$). Between the source point and the earth's surface, the solution will just be a superposition of Airy integrals $\mathfrak{B}\text{Ai}(-\xi) + \mathfrak{C}\text{Bi}(-\xi)$. Inside the earth, the appropriate solution would be $\mathfrak{Q}\text{Ai}(-\xi)$, since this solution decays exponentially for large positive ξ . The boundary conditions are found by returning to Equations 74 and 75. Since θ is not a function of the index of refraction, it is the same function for all η ; therefore, Equations 74 and 75 can be written as:

$$G_I = G_{II} \quad 112$$

and

$$\frac{dG_I}{d\eta} + G_I/a = \frac{dG_{II}}{d\eta} + G_{II}/a \quad 113$$

Thus, the boundary conditions at the earth's surface require continuity of the function and its first derivative. This leads to two independent equations. The continuity of the Green's functions at the source point results in another equation and the last equation which is required in order to determine the coefficients of the Green's functions is obtained from the equation:

$$\left[\frac{d^2}{d\eta^2} + (\alpha + \beta\eta) \right] G = \delta(\eta - \eta_q) . \quad 114$$

Integrating this equation once with respect to η over the interval from $\eta_q - \epsilon$ to $\eta_q + \epsilon$ results in:

$$\frac{d}{d\eta} G \begin{pmatrix} \eta_d + \epsilon \\ \eta_d - \epsilon \end{pmatrix} = 1 \quad 115$$

Now the system of equations can be written:

$$\begin{aligned} B \text{Ai}(-\xi_d) + C \text{Bi}(-\xi_d) - \mathcal{D}[\text{Bi}(-\xi_d) - j \text{Ai}(-\xi_d)] &= 0 \\ B \text{Ai}'(-\xi_d) \beta_{II}^{1/3} + C \text{Bi}'(-\xi_d) \beta_{II}^{1/3} - \mathcal{D}[\text{Bi}'(-\xi_d) - j \text{Ai}'(-\xi_d)] \beta_{II}^{1/3} &= 1 \\ C \text{Ai}(-\xi_a) - B \text{Ai}(-\xi_a) - C \text{Bi}(-\xi_a) &= 0 \\ C \text{Ai}'(-\xi_a) \beta_I^{1/3} - B \text{Ai}'(-\xi_a) \beta_{II}^{1/3} - C \text{Bi}'(-\xi_a) \beta_{II}^{1/3} &= 0 \end{aligned} \quad 116$$

To simplify the analysis, the following notation will be employed.

$$\begin{aligned} A_{2d} &= \text{Ai}(-\xi_{II d}) & A'_{2d} &= \text{Ai}'(-\xi_{II d}) \\ B_{2d} &= \text{Bi}(-\xi_{II d}) & B'_{2d} &= \text{Bi}'(-\xi_{II d}) \\ A_{2a} &= \text{Ai}(-\xi_{II a}) & A'_{2a} &= \text{Ai}'(-\xi_{II a}) \\ B_{2a} &= \text{Bi}(-\xi_{II a}) & B'_{2a} &= \text{Bi}'(-\xi_{II a}) \\ A_{1a} &= \text{Ai}(-\xi_{I a}) & A'_{1a} &= \text{Ai}'(-\xi_{I a}) \end{aligned} \quad 117$$

It will also be very useful to denote the function $\text{Bi}(-\xi) - j \text{Ai}(-\xi)$ by $f(\xi)$ which is in agreement with the notation used by Pryce (16). The system of equations can be written in matrix notation as:

$$\begin{bmatrix} 0 & A_{2d} & B_{2d} & -f_{2d} \\ 0 & A'_{2d} \beta_2^{1/3} & B'_{2d} \beta_2^{1/3} & -f'_{2d} \beta_2^{1/3} \\ A_{1a} & -A_{2a} & -B_{2a} & 0 \\ A'_{1a} \beta_1^{1/3} & -A'_{2a} \beta_2^{1/3} & -B'_{2a} \beta_2^{1/3} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A} \\ \mathcal{B} \\ \mathcal{C} \\ \mathcal{D} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad 118$$

The Green's function appropriate to the region between the earth's surface and the source point is found by determining the coefficients \mathcal{B} and \mathcal{C} .

$$G(\eta < \eta_d) = \frac{f_{2d}}{(A_{2d} B'_{2d} - B_{2d} A'_{2d}) \beta_2^{1/3}} \left[\text{Ai}(-\xi) - \frac{A_{1a} \beta_2^{1/3} A'_{2a} - A'_{1a} \beta_1^{1/3} A_{2a}}{A_{1a} \beta_2^{1/3} f'_{2a} - A'_{1a} \beta_1^{1/3} f_{2a}} f(-\xi) \right] \quad 119$$

In the region above the source point, the Green's function is found by determining the coefficient \mathcal{D} .

$$G(\eta > \eta_d) = \frac{f(-\xi)}{(A_{2d} B'_{2d} - B_{2d} A'_{2d}) \beta_2^{1/3}} \left[A_{2d} - \frac{A_{1a} \beta_2^{1/3} A'_{2a} - A'_{1a} \beta_1^{1/3} A_{2a}}{A_{1a} \beta_2^{1/3} f'_{2a} - A'_{1a} \beta_1^{1/3} f_{2a}} f_{2d} \right] \quad 120$$

It can be seen that a general Green's function can be formulated.

$$G(\eta_<, \eta_>) = \frac{f(-\xi_<)}{(A_{2d} B'_{2d} - B_{2d} A'_{2d}) \beta_2^{1/3}} \left[\text{Ai}(\xi_>) - \frac{A_{1a} \beta_2^{1/3} A'_{2a} - A'_{1a} \beta_1^{1/3} A_{2a}}{A_{1a} \beta_2^{1/3} f'_{2a} - A'_{1a} \beta_1^{1/3} f_{2a}} f(-\xi_>) \right] \quad 121$$

Once the Green's function has been determined, U can be written as:

$$U = \int_0^{\infty} \exp(-\eta/2a) \lambda/a^2 J_0(\lambda\rho/a) G(\eta_<, \eta_>) d\lambda \quad 122$$

The expression $A_{2d} B'_{2d} - B_{2d} A'_{2d}$ is the Wronskian of Stoke's equation and is equal to $1/\pi$ for all arguments. To see that this is true, consider the two solutions of Stoke's equation, $Ai(-\xi)$ and $Bi(-\xi)$. Being solutions means that:

$$\frac{d^2 Ai(-\xi)}{d\xi^2} - \xi Ai(-\xi) = 0 \quad 123$$

$$\frac{d^2 Bi(-\xi)}{d\xi^2} - \xi Bi(-\xi) = 0 \quad 124$$

Multiply the first equation by $Bi(-\xi)$ and the second equation by $Ai(-\xi)$ and subtract the two resulting equations to get:

$$Bi(-\xi) \frac{d^2 Ai(-\xi)}{d\xi^2} - Ai(-\xi) \frac{d^2 Bi(-\xi)}{d\xi^2} = 0 \quad 125$$

This is exactly

$$\frac{d}{d\xi} \left[Bi(-\xi) \frac{dAi(-\xi)}{d\xi} - Ai(-\xi) \frac{dBi(-\xi)}{d\xi} \right] = 0 \quad 126$$

or

$$Bi(-\xi) \frac{dAi(-\xi)}{d\xi} - Ai(-\xi) \frac{dBi(-\xi)}{d\xi} = \text{const.} \quad 127$$

Thus, the Wronskian is independent of ξ .

The field component which must be known in order to determine the energy density in the atmosphere is the radial component of the magnetic field, H_r . Once H_r is known, it is a simple matter to compute the Poynting vector since the ratio of E to H must be the intrinsic impedance of the medium. The equation used to determine H_r is:

$$H_r = \left[\frac{d}{dr^2} + \frac{2}{r} \frac{d}{dr} + \omega^2 \mu \epsilon(r) \right] U \quad 128$$

In the cylindrical coordinates,

$$H_r = \exp(-2\eta/a) \left[\frac{d^2}{d\eta^2} + \frac{1}{a} \frac{d}{d\eta} + \omega^2 \mu \epsilon(r) \exp(2\eta/a) \right] U \quad 129$$

or

$$H_r = \int_0^\infty \exp(-2\eta/a) \exp(-\eta/2a) \left[\frac{d^2}{d\eta^2} + \frac{1}{a} \frac{d}{d\eta} + \omega^2 \mu \epsilon(r) \exp(2\eta/a) \right] \frac{\lambda}{a^2} J_0(\lambda \rho/a) G(\eta_<, \eta_>) d\lambda \quad 130$$

The height-gain equation in cylindrical coordinates is:

$$\left[\frac{d^2}{d\eta^2} + \frac{1}{a} \frac{d}{d\eta} + \omega^2 \mu \epsilon(r) \exp(2\eta/a) - \lambda^2/a^2 \right] U = 0 \quad 131$$

from which it can be seen that the radial component of the magnetic field is:

$$H_r = \exp(-5\eta/2a) \int_0^\infty \frac{\lambda^3}{a^4} J_0(\lambda \rho/a) G(\eta_>, \eta_<) d\lambda \quad 132$$

This integral must be evaluated in order to determine the field in the diffraction region. To facilitate evaluation of the integral, it will be appropriate to change the variable of integration to \mathcal{L} , defined by $\mathcal{L} = \beta_2^{1/3} \alpha_2$. When this substitution is made the integral becomes:

$$\exp(-5\eta/2a) \int_{-\infty}^{K^2} \frac{8K^{10}}{a^4} (1 - \mathcal{L}^2/K^2) J_0 [k_0 \rho (1 - \mathcal{L}^2/K^2)^{1/2}] G(\eta_<, \eta_>) d\mathcal{L} \quad 133$$

Where $K = \left(\frac{k_0 a}{2}\right)^{1/3}$ and $k_0 = \omega/c$.

The Green's function can be simplified somewhat by making an approximation

in the function defining propagation into the earth. That is:

$$\left. \text{Ai}'(-\mathcal{E}) \right|_{\eta=0} = \left. \text{Ai}'(\mathcal{E} + \beta_1^{1/3} \eta) \right|_{\eta=0} \sim -\mathcal{E}^{1/2} \text{Ai}(\mathcal{E}_1) \quad 134$$

This is the large argument approximation to the Airy integral and, since \mathcal{E}_1 contains a large imaginary part due to the complex dielectric constant of the earth, the asymptotic expansion is valid. The $(\mathcal{E}_1 + \beta_1^{1/3} \eta)$ term is approximated by \mathcal{E} . This is allowed because the wave is assumed to attenuate rapidly and only has appreciable amplitude in region of small η .

With this approximation, the Green's function becomes

$$G(\mathcal{E}, \eta_{<}, \eta_{>}) = \frac{\pi}{\beta_2^{1/3}} f(\mathcal{E} + \beta_2^{1/3} \eta_{>}) \left[g(\mathcal{E} + \beta_2^{1/3} \eta_{<}) - \frac{g'(\mathcal{E})\beta_2^{1/3} + \frac{1}{3}\beta_1^{1/3}g(\mathcal{E})}{f'(\mathcal{E})\beta_2^{1/3} + \frac{1}{3}\beta_1^{1/3}f(\mathcal{E})} f(\mathcal{E} + \beta_2^{1/3} \eta_{<}) \right] \quad 135$$

Where $g(\mathcal{E}) = \text{Ai}(\mathcal{E})$. This notation is used to show agreement with the results of Pryce since this is the notation that he used. If a further simplification is made by denoting $\beta_2^{1/3} \frac{-1/2}{1} \beta_1^{-1/3}$ by τ_1 , the Green's function can be written as:

$$G(\mathcal{E}, \eta_{<}, \eta_{>}) = \frac{\pi}{\beta_2^{1/3}} f(\mathcal{E} + \beta_2^{1/3} \eta_{>}) \left[g(\mathcal{E} + \beta_2^{1/3} \eta_{<}) - \frac{g'(\mathcal{E})\tau_1 + g(\mathcal{E})}{f'(\mathcal{E})\tau_1 + f(\mathcal{E})} f(\mathcal{E} + \beta_2^{1/3} \eta_{<}) \right] \quad 136$$

Substituting this into the integral defining H_r results in:

$$H_r = \exp(-5\eta/2a) \int_{-\infty}^{K^2} \frac{8K^{10}}{a^4 \beta_2^{1/3}} (1 - \mathcal{E}/K^2) J_0 [k_o \rho (1 - \mathcal{E}/K^2)^{1/2}] f(\mathcal{E} + \beta_2^{1/3} \eta_{>}) \left[g(\mathcal{E} + \beta_2^{1/3} \eta_{<}) - \frac{g'(\mathcal{E})\tau_1 + g(\mathcal{E})}{f'(\mathcal{E})\tau_1 + f(\mathcal{E})} f(\mathcal{E} + \beta_2^{1/3} \eta_{<}) \right] d\mathcal{E} \quad 137$$

Since the zero-ordered Bessel function decays very rapidly for imaginary argument, the range of integration can be extended to the entire \mathcal{L} axis without adding significantly to the integral. If the large argument form of the Bessel function is used, that is:

$$J_0 [k_0 \rho (1 - \mathcal{L}/K^2)^{1/2}] \sim \sqrt{\frac{2}{\pi k_0 \rho}} \cos (k_0 \rho - \mathcal{L} \rho / K^2 - \pi/4) \quad 138$$

the integral reduces to:

$$H_r = \exp(-5\pi/2a) \int_{-\infty}^{+\infty} k_0^2 \rho^2 \frac{1}{2} \sqrt{\frac{\pi}{2k_0 \rho}} \cos(k_0 \rho - \mathcal{L} \rho / K^2 - \pi/4) G(\mathcal{L}; \eta_1, \eta_2) d\mathcal{L} \quad 139$$

This simplification is possible because $k_0 \rho$ is a very large number; about 10^6 at a range of 50 miles and a frequency of 100 mcs. Equation 139 corresponds exactly to Equation 12.7 of Pryce (16) except that a magnetic field is considered here. The integral can be evaluated by the method of residues because the denominator of the Green's function has an infinite number of zeros in the complex \mathcal{L} plane. Using the relationship

$$\text{Ai} [\gamma \exp(j\pi/3)] = \frac{\exp(j\pi/6)}{2} f(\nu) \quad 140$$

the denominator of the Green's function and consequently of the integral, becomes:

$$f'(\mathcal{L}) \tau_1 + f(\mathcal{L}) = 2 \exp(-j\pi/6) \text{Ai}[\mathcal{L} \exp(j\pi/3)] \tau_1 + \text{Ai}[\mathcal{L} \exp(j\pi/3)] \quad 141$$

Since $\text{Ai}(z)$ is oscillatory along the line, $\arg z = \pi/3$, there are an infinite number of poles of the integrand, whenever

$$A_1 [a_s \exp(j\pi/5)] + A_2 [a_s \exp(j\pi/5)] = 0. \quad 142$$

Thus the integral can be evaluated by complex residue theory, and Pryce (16) has shown that the integral can be expressed as:

$$H_r = \exp(-5\eta/2a) k_0^2 \beta_2^{1/3} (2\pi/k_0 \rho)^{1/2} \exp[j(3\pi/4 - k_0 \rho)] \sum_s^{\infty} \frac{f(a_s + \beta_2^{1/3} n_>) f(a_s + \beta_2^{1/3} n_<)}{(1 + \tau_1^2 a_s) f'^2(a_s)} \exp(-ja_s \rho / 2K^2) \quad 143$$

Where $\beta_2 = 2k_0^2/a$, k_0^2/a , $k_0^2 = \frac{\omega_n^2}{c^2}$, and $\tau_1 = \beta_2^{1/3} \beta_1^{1/3}$.

IV. THE COMPLETE SOLUTION FOR THE ELECTRIC DIPOLE

The case of propagation from an electric dipole located in an inhomogeneous medium is somewhat more difficult to analyze than the case of the magnetic dipole because the differential equations are more complicated. However, it will be possible to make simplifying approximations at appropriate points to facilitate the analysis. The separated differential equations and the field equations governing the propagation of electromagnetic energy from an electric dipole in a spherically stratified inhomogeneous medium are:

$$\theta'' + \cot \theta \theta' + \lambda_e^2 \theta = 0 \quad 144$$

$$R'' + \frac{1}{r} \left(2 - \frac{r\epsilon'(r)}{\epsilon(r)} \right) R' + \left(\omega^2 \mu \epsilon(r) - \frac{\lambda_e^2}{r^3} - \frac{\epsilon'(r)}{r\epsilon(r)} \right) R = 0 \quad 145$$

$$E_r = \frac{-jr}{\omega \epsilon(r)} \left[R'' + \frac{1}{r} \left(2 - \frac{r\epsilon'(r)}{\epsilon(r)} \right) R' + \left(\omega^2 \mu \epsilon(r) - \frac{\epsilon'(r)}{r\epsilon(r)} \right) R \right] \theta \quad 146$$

$$E_\theta = \frac{-j}{\epsilon(r)} (R/r + R') \theta' \quad 147$$

$$E_\phi = 0 \quad 148$$

$$H_r = 0 \quad 149$$

$$H_\theta = 0 \quad 150$$

$$H_\phi = -R\theta' \quad 151$$

Equation 145 can be simplified somewhat by considering the term $\frac{\epsilon'(r)}{\epsilon(r)}$. This expression can be rewritten as $\frac{2n'(r)}{n(r)}$ by means of Equation 83. Typically, $n'(r)$ is a number on the order of 10^{-7} /meter near the surface of the earth; getting smaller at increasing heights, while $n(r)$ is very close

to unity. Therefore,

$$\frac{\epsilon'(r)}{\epsilon(r)} \sim 2 \times 10^{-7} / \text{meter} .$$

The function $\exp(\eta/a)$ can be approximated by $1 + \eta/a$, or $\exp(\eta/a) \sim 1$.

The earth's radius a is a large number, about 6.5×10^6 meters. The term

$\frac{\exp(\eta/a)}{a} \frac{\epsilon'(r)}{\epsilon(r)}$ can be neglected, but the term $a \frac{\exp(\eta/a)\epsilon'(r)}{\epsilon(r)}$ cannot be

neglected. Thus, Equation 145 can be approximated by:

$$\frac{d^2 R}{d\eta^2} + \frac{1}{a} \left(1 - \frac{2an'(r)}{n_0}\right) \frac{dR}{d\eta} + \left\{ \omega^2 \mu \epsilon(r) \exp(2\eta/a) - \lambda^2/a^2 \right\} R = 0 \quad 152$$

The change of variable suggested by Koo and Katzin (13) becomes

$$R = v \exp(-\eta/2a)^b \quad \text{where} \quad b = \left(1 - \frac{2an'(r)}{n_0}\right),$$

and the height-gain equation becomes:

$$\frac{d^2 v}{d\eta^2} + \left\{ k_0^2 \left[1 + \frac{2an'(r)}{n_0} [\exp(\eta/a) - 1]\right] \exp(2\eta/a) - \frac{\lambda^2 + b^2/4}{a} \right\} v = 0 \quad 153$$

Making the same approximations as in the magnetic dipole case and ignoring

the $b^2/4a^2$ term as being too small with respect to the other terms, re-

sults in

$$\frac{d^2 v}{d\eta^2} + \left[\left(k_0^2 - \frac{\lambda^2}{a^2}\right) + \frac{2k_0^2}{a} \left(1 + \frac{a}{n_0} \frac{dn}{dr}\right) \eta \right] v = 0 \quad 154$$

which is the same as Equation 90 which was developed for the magnetic dipole case.

The boundary conditions at the earth-atmosphere interface requires continuity of the tangential electric and magnetic fields, which means

that:

$$R_I \theta_I' \Big|_{r=a} = R_{II} \theta_{II}' \Big|_{r=a} \quad 155$$

and

$$\frac{1}{\epsilon_I} \left(\frac{R_I}{r} + R_I' \right) \theta_I' \Big|_{r=a} = \frac{1}{\epsilon_{II}} \left(\frac{R_{II}}{r} + R_{II}' \right) \theta_{II}' \Big|_{r=a} . \quad 156$$

When considering the field inside the earth, it can be shown that the same differential equations which were developed to describe the field in the earth's atmosphere will apply provided $\epsilon(r)$ is replaced by $\epsilon_0 \left(1 - \frac{j\sigma}{\omega\epsilon_0} \right)$.

The range equation for the electric dipole case is the same as the range equation for the magnetic dipole case, and is subject to the same boundary conditions. Therefore, the solution developed in the preceding section will be applicable in this case also. Thus, the problem is reduced to finding the appropriate Green's function. The height-gain equation is somewhat different than the height-gain equation for the magnetic dipole case, so the electric dipole analysis will begin with consideration of the height-gain equation. In the cylindrical coordinate system, the height-gain equation is:

$$\frac{d^2 R}{d\eta^2} + \frac{1}{a} \left(1 - \frac{a \exp(\eta/a) \epsilon'(r)}{\epsilon(r)} \right) \frac{dR}{d\eta} + \left\{ \omega^2 \mu \epsilon(r) \exp(2\eta/a) - \lambda^2/a^2 - \frac{\exp(\eta/a)}{a} \frac{\epsilon'(r)}{\epsilon(r)} \right\} R = 0 \quad 157$$

The Airy integral solutions will also apply in the electric dipole case, and the main difference will be due to the boundary conditions at the earth's surface. The boundary conditions require that

$$G_I = G_{II}$$

$$\frac{1}{\epsilon_0 \left(1 - \frac{j\sigma}{\omega\epsilon_0}\right)} \left(\frac{G_I}{a} + G'_I\right) = \frac{1}{\epsilon_0} \left(\frac{G_{II}}{a} + G'_{II}\right)$$

These two boundary conditions lead to two independent equations. The other two independent equations are the same as for the magnetic dipole case.

Thus the system of equations can be written as:

(a)

$$B \text{Ai}(-\xi_{II d}) + C \text{Bi}(-\xi_{II d}) - \mathcal{D} [\text{Bi}(-\xi_{II d}) - j \text{Ai}(-\xi_{II d})] = 0$$

(b)

$$B \text{Ai}'(-\xi_{II d}) \beta_{II}^{1/3} + C \text{Bi}'(-\xi_{II d}) \beta_{II}^{1/3} - \mathcal{D} [\text{Bi}(-\xi_{II d}) - j \text{Ai}(-\xi_{II d})] \beta_{II}^{1/3} = +1$$

(c)

$$A \text{Ai}(-\xi_{II a}) - B \text{Ai}(-\xi_{II a}) - C \text{Bi}(-\xi_{II a}) = 0$$

(d)

$$\frac{a}{\epsilon'} \left[\frac{\text{Ai}(-\xi_{II a})}{a} + \text{Ai}'(-\xi_{II a}) \right] B \left[\frac{\text{Ai}(-\xi_{II a})}{a} + \text{Ai}'(-\xi_{II a}) \right] - C \left[\frac{\text{Bi}(-\xi_{II a})}{a} + \text{Bi}'(-\xi_{II a}) \right] = 0$$

160

In matrix notation, and using the symbolism of the preceding section, the system of equation becomes

$$\begin{bmatrix} 0 & A_{2d} & B_{2d} & -f_{2d} \\ 0 & A'_{2d} \beta_2^{1/3} & B'_{2d} \beta_2^{1/3} & -f'_{2d} \beta_2^{1/3} \\ A_{1a} & -A_{2a} & -B_{2a} & 0 \\ \frac{1}{\epsilon'} \left(\frac{A_{1a}}{a} + A'_{1a} \beta_1^{1/3} \right) - \left(\frac{A_{2a}}{a} + A'_{2a} \beta_2^{1/3} \right) & - \left(\frac{B_{2a}}{a} + B'_{2a} \beta_2^{1/3} \right) & 0 & 0 \end{bmatrix} \begin{bmatrix} Q \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ +1 \\ 0 \\ 0 \end{bmatrix} \quad 161$$

Evaluation of the matrix is simplified if one considers the relative sizes of the terms in the fourth row. A good pictorial representation of the Airy integrals of real arguments and their derivatives is shown on page B-16 of The British Association For the Advancement of Science Mathematical Tables (14). Here it is seen that $Ai(x)$ is a very well behaved function, as is $Ai'(x)$ and both are of the same order of magnitude; $Bi(x)$ and $Bi'(x)$ are well behaved for negative arguments but increase exponentially for positive arguments. In either case it is reasonable to ignore the term which is divided by a , the earth's radius. Under this assumption, the matrix equation becomes:

$$\begin{bmatrix} 0 & A_{2d} & B_{2d} & -f_{2d} \\ 0 & A'_{2d} \beta_2^{1/3} & B'_{2d} \beta_2^{1/3} & -f'_{2d} \beta_2^{1/3} \\ A_{1a} & -A_{2a} & -B_{2a} & 0 \\ \frac{1}{\epsilon'} A'_{1a} \beta_1^{1/3} & -A'_{2a} \beta_2^{1/3} & -B'_{2a} \beta_2^{1/3} & 0 \end{bmatrix} \begin{bmatrix} Q \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ +1 \\ 0 \\ 0 \end{bmatrix} \quad 162$$

This matrix is the same as the matrix which was developed for the magnetic dipole case except for the term $\frac{1}{\epsilon'}$ which is multiplying the derivative of

the solution in the earth. The solution of the preceding section can be applied if $\beta_1^{1/3}/\epsilon'$ is substituted for $\beta_1^{1/3}$. The general Green's function for the electric dipole case can be written directly as:

$$G(\mathcal{L}; \eta_<, \eta_>) = \frac{\pi}{\beta_2^{1/3}} f(\mathcal{L} + \beta_2^{1/3} \eta_>) \left[g(\mathcal{L} + \beta_2^{1/3} \eta_<) + \frac{g'(\mathcal{L})\tau_2 + g(\mathcal{L})}{f'(\mathcal{L})\tau_2 + f(\mathcal{L})} f(\mathcal{L} + \beta_2^{1/3} \eta_<) \right] \quad 163$$

where

$$\tau_2 = (\beta_2^{1/3} \mathcal{L}^{-1/2} \beta_1^{-1/3} \epsilon_1^{-1}).$$

To determine the energy-density in the transhorizon region, the field component which must be known to the radial component of the electric field, E_r . The Equation which is used to determine E_r is:

$$E_r = -\frac{j'r}{\omega} \left[R'' + \frac{1}{r} \left(2 - \frac{r\epsilon'(r)}{\epsilon(r)} \right) R' + \left\{ \omega^2 \mu \epsilon(r) - \frac{\epsilon'(r)}{r\epsilon(r)} \right\} R \right] \theta = 0 \quad 164$$

Consideration of the height-gain equation for the electric dipole case

$$R'' + \frac{1}{r} \left(2 - \frac{r\epsilon'(r)}{\epsilon(r)} \right) R' + \left[\omega^2 \mu \epsilon(r) - \frac{\lambda^2}{r} - \frac{\epsilon'(r)}{r\epsilon(r)} \right] R = 0 \quad 165$$

indicates that the radial component of electric field is:

$$E_r = \frac{-j}{\omega} \int_0^\infty \exp(-\eta b/2a) \exp(-\eta/a) \frac{\lambda^3}{a} J_0(\lambda \rho/a) G(\mathcal{L}; \eta_<, \eta_>) d\lambda \quad 166$$

or

$$E_r = \frac{-ja \exp(-2+b)\eta/2a}{\omega \epsilon(r)} \int_0^\infty \frac{\lambda^3}{a^4} J_0(\lambda \rho/a) G(\mathcal{L}; \eta_<, \eta_>) d\lambda \quad 167$$

The integrand of Equation 167 is exactly the same as the integrand of Equation 132 except that the Green's function in this case has τ_2

substituted for τ_1 . The integral can be evaluated by means of complex residue theory just as in the magnetic dipole case, with the result

$$E_r = \frac{-ja \exp[-(2+b)\mathcal{N}/2a]}{\omega \epsilon(r)} k_o^2 \beta_2^{1/3} (2\pi/k_o \rho)^{1/2} \exp [j(3\pi/4 - k_o p)] \sum_{s=1}^{\infty} \frac{f(a_s + \beta_2^{1/3} \mathcal{N}_>) f(a_s + \beta_2^{1/3} \mathcal{N}_<)}{(1 + \tau_2^2 a_s^2) f(a_s^2)}$$

V. COMPARISON WITH EXPERIMENTAL DATA

There has been a great deal of published and unpublished experimental work concerning the strength of the electromagnetic signal beyond the radio horizon. In recent years, much of this experimental data has been taken between two fixed locations with emphasis on the statistics of the field in the transhorizon region. The statistics of the received signal would ideally be correlated with the statistics of the turbulence of the atmosphere. In fact, herein lies the key to the preference of the Booker-Gordon (2) "scattering blob" theory over the traditional normal mode theory which is the subject of this paper. The Green's function approach to the solution of propagation in a stratified atmosphere could permit the inclusion of a time-varying atmosphere by using a time-dependent Green's function. It would be assumed that the time-variations of the atmosphere occur much slower than the frequency of the source. This subject will be considered in somewhat greater detail as the Green's function solution is extended to include elevated stratifications in the next section. At this point it will be appropriate to compare a typical set of experimentally measured field strengths with the field strength calculated by using the results of Sections III and IV. The type of data which is most suitable for comparison with the theory of this paper is not that which is taken between two fixed points, but data which is taken from a radial path of varying length. This leads to a plot of field strength versus distance. A good set of data for this is given by Dinger, Garner, Hamilton and Teachman (5).

To show the validity of the flat-earth approximation, a comparison

of the theoretical results of Section III with the data of Dinger, Garner, Hamilton and Teachman (5) will be made. This data was taken from a transmission path extending from the south shore of Massachusetts, near the city of New Bedford, to a ship traveling along great circle courses to a maximum distance of 630 nautical miles from the transmitter. A 10-kw, 412.85 mc transmitter feeding a 28-foot paraboliod antenna and a 40-kw, 412.85 mc transmitter feeding a 60-foot paraboliod antenna were used, with the more powerful transmitter being used at the longer distances. The receiving antenna aboard the ship was a 17-foot paraboliod. Both the transmitting and the receiving antennas were horizontally polarized. Both the transmitting antenna and the receiving antenna were located at a height of about 95 feet above sea level. A plot of the results of this investigation is shown in Figure 2. The theoretical model which is used for comparison will be magnetic dipole. Actually, the comparison will be made between the Hertz vector for free-space propagation and the Hertz vector for propagation in a spherically-stratified atmosphere, which reduces to the free space Hertz vector in an isotropic medium. The free space Hertz vector for a unit dipole is:

$$\pi_{fs} = \frac{\exp(-jkr)}{4\pi\epsilon_0} \quad 169$$

The corresponding Hertz vector for a stratified medium is; by Equation 11,

$$\pi_m = r U \quad 170$$

For fields at the earth's surface,

$$\pi_m = a U . \quad 171$$

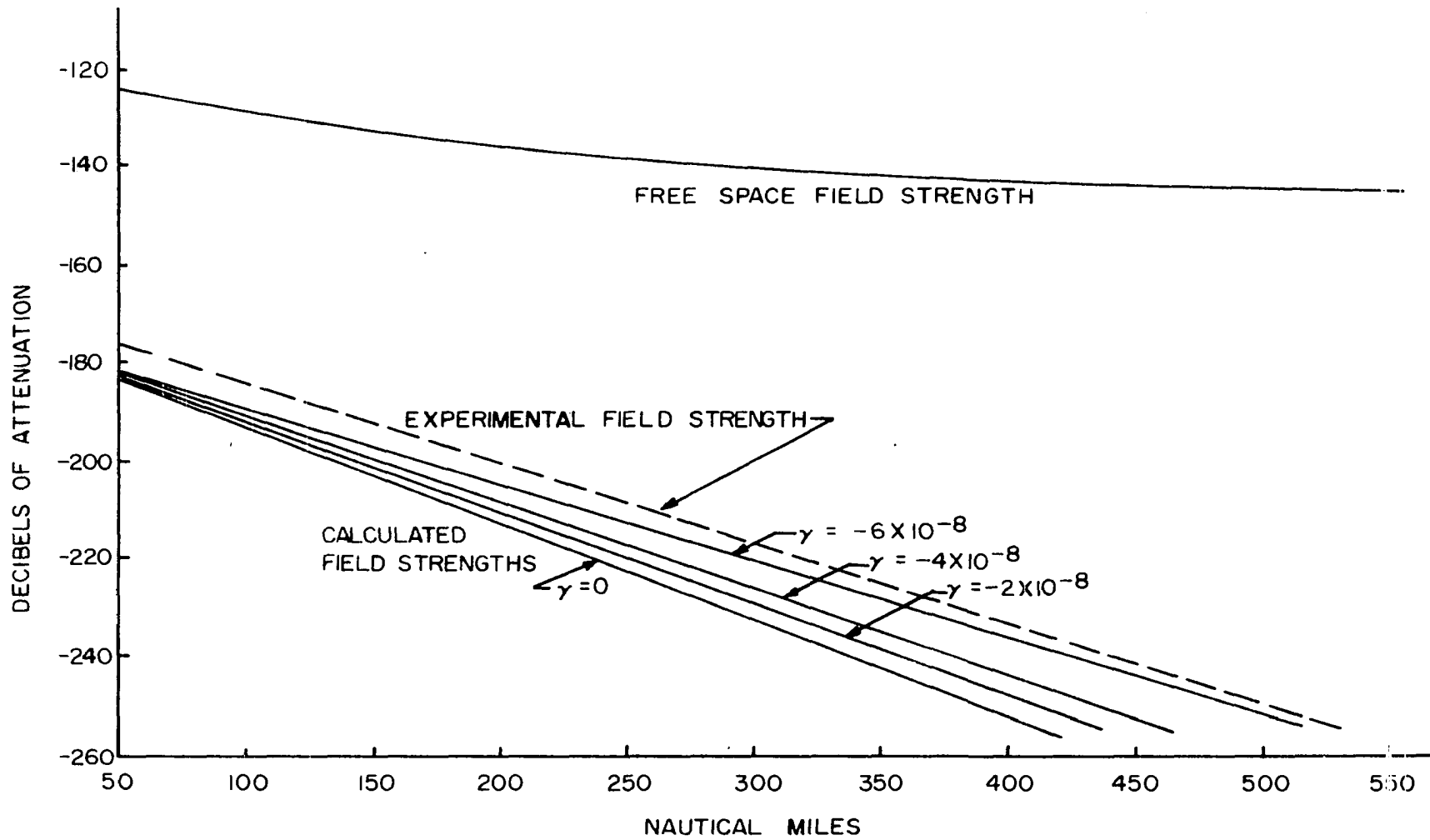


Figure 2. A comparison of experimental field strength with calculated field strengths for linearly varying profiles of refractive index

The function U is given by equation

$$U = \int_0^{\infty} \frac{\lambda}{a^2} J_0(\lambda \rho/a) G(\eta_<, \eta_>) d\lambda \quad 172$$

Or, upon making the change-of-variable

$$\xi = \beta^{-2/3} (k_0^2 - \lambda^2/a^2) , \quad 173$$

where $k_0 = \frac{\omega}{c}$ and $\beta = 2k_0^2/a(1 + \frac{a}{n_0} \frac{dn}{dr})$,

$$U = \beta_2^{1/3} \sqrt{\frac{\pi}{2k_0 \rho}} \int_{-\infty}^{+\infty} \cos(k_0 \rho + \xi \rho / 2K^2 - \pi/4) [g(\xi + \beta_2^{1/3} \eta_<) - \frac{g'(\xi) \tau_1 + g(\xi)}{f'(\xi) \tau_1 + f(\xi)} f(\xi + \beta_2^{1/3} \eta_<)] d\xi \quad 174$$

This is the same integral which was evaluated in Section III so the result can be written directly as:

$$U = \beta_2^{1/3} \sqrt{\frac{\pi}{2k_0 \rho}} \exp(j 3\pi/4 - k_0 \rho) \sum_s^{\infty} \frac{f(a_s + \beta_2^{1/3} \eta_>) f(a_s + \beta_2^{1/3} \eta_<)}{(1 + \tau_1^2 a_s) f'^2(a_s)} \exp(-j a_s \rho / 2K^2) \quad 175$$

The experiment was carried out over sea water at a frequency of 412.85 mc. Using this information to evaluate τ_1 ,

$$\tau_1 = \beta_2^{1/3} \beta_1^{-1/2} \beta_1^{1/3} = \frac{\left(\frac{2k_o^2}{a}\right)^{1/3} \left(1 + \frac{a}{n_o} \frac{dn}{dr}\right)^{1/3}}{\left(\frac{2k_o^2}{a}\right)^{1/3} \left(1 - \frac{j\sigma}{\omega\epsilon}\right) \left[k_o^2 \left(1 - \frac{j\sigma}{\omega\epsilon}\right) - \lambda^2/a^2\right]^{1/2}}$$

176

$$= \frac{1}{\left(\frac{2k_o^2}{a}\right)^{1/3} \left(1 - \frac{j\sigma}{\omega\epsilon}\right) \left[\beta_2 - \frac{j\sigma}{\omega\epsilon} \frac{K^2}{\left(1 + \frac{a}{n_o} \frac{dn}{dr}\right)}\right]^{2/3}}$$

For sea water $\sigma = 4$ mhos per meter and $\epsilon' = 81$. Therefore,

$$1 - \frac{j\sigma}{\omega\epsilon} = 1 - j 2.16 \quad 177$$

$$k_o = 8.65 \quad 178$$

$$K = 3.05 \times 10^2 \quad 179$$

and

$$\left(\frac{2k_o^2}{a}\right)^{1/3} = 2.87 \times 10^{-2} \quad 180$$

Thus,

$$\tau = \frac{1}{(2.87 \times 10^{-2})(1 - j 2.16) (a_s - j 20 \times 10^4)^{1/2}} \quad 181$$

The root of the Airy integral; a_s , is a small positive number. If consideration is limited to the first root, $a_s \approx 2.34$. Under these conditions Equation 181 can be approximated as:

$$\tau_1 \approx \frac{1}{(2.87 \times 10^{-2})(1 - j2.16) (4.45 \times 10^2 / -45^\circ)} \quad 182$$

$$\tau_1 = \frac{1}{30 \angle -110^\circ} \quad 183$$

$$|\tau_1| = \frac{1}{30} \therefore |\tau_1|^2 = \frac{1}{900} \quad 184$$

Thus, the integral expression for U can be simplified to:

$$U = \beta_2^{1/3} \sqrt{\frac{\pi}{2k_o \rho}} \exp(j 3\pi/4 - k_o \rho) \sum_s \frac{f(a_s + \beta_2^{1/3} \eta_{>}) f(a_s + \beta_2^{1/3} \eta_{<})}{f'^2(a_s)} \exp(-ia_s \rho / 2K^2) . \quad 185$$

Pryce (16) has shown that with this simplification,

$$U = \beta_2^{1/3} \sqrt{\frac{\pi}{2k_o \rho}} \exp(j 3\pi/4 - k_o \rho) \sum_s \Gamma_s(\beta_2^{1/3} \eta_{<}) \Gamma_s(\beta_2^{1/3} \eta_{>}) \exp[-(\sqrt{3}+j) \alpha_s \rho / 2K^2] \quad 186$$

where

$$\Gamma_s(x) = \exp(j \pi/3) \text{Ai}[-\alpha_s + \exp(j \pi/3) x] / \text{Ai}'(-\alpha_s)$$

and

$$\alpha_s = \exp(-2j \pi/3) a_s .$$

At great distances, the first term of the series is a good approximation,

that is:

$$|U| \approx \beta_2^{1/3} \sqrt{\frac{\pi}{2k_o \rho}} \Gamma_1(\beta_2^{1/3} \eta_{<}) \Gamma_1(\beta_2^{1/3} \eta_{>}) \exp(-2.0249/2K^2) . \quad 187$$

In this case

$$\rho_2^{1/3} = \left(\frac{2k_0^2}{a}\right)^{1/3} \left(1 + \frac{a}{n_0} \frac{dn}{dr}\right)^{1/3} = 2.67 \times 10^{-2} \left(1 + \frac{a}{n_0} \frac{dn}{dr}\right)^{1/3} \quad 188$$

$$\eta_{<} = \eta_{>} = 28.9 \text{ meters.}$$

Assume $n_0 = 1.000$.

Typically,

$$\frac{dn}{dr} = \gamma = -3 \text{ to } -4 \times 10^{-8} / \text{meter.} \quad 189$$

$$\beta_2^{1/3} = (3.1 \times 10^{-2}) \quad 190$$

$$\beta_2^{1/3} \eta_{<} = \beta_2^{1/3} \eta_{>} = 0.93$$

$$\frac{1}{K^2} = \frac{1}{k_0^2} \left(\frac{2k_0^2}{a}\right)^{2/3}$$

where

$$k_0'^2 = k_0^2 \left(1 + \frac{a}{n_0} \frac{dn}{dr}\right)$$

Therefore:

$$\frac{1}{K^2} = \left(\frac{2}{k_0 a}\right)^{2/3} \left(1 + \frac{a}{n_0} \frac{dn}{dr}\right)^{2/3} = \frac{\left(1 + \frac{a}{n_0} \frac{dn}{dr}\right)^{2/3}}{9.1 \times 10^4} \quad 191$$

The expression for magnitude U reduces to:

$$|U| = (3 \times 10^{-2}) \sqrt{\frac{\pi}{2_1 k_0 \rho}} \sqrt{1}^{.93} \exp \left[- \frac{(1.0125) \left(1 + \frac{a}{n_0} \frac{dn}{dr}\right)^{2/3} \rho}{9.1 \times 10^4} \right] \quad 192$$

Once the magnitude of U is known, the magnitude of the Hertz vector is easily obtained by multiplying by the earth's radius a

$$\pi_m = a |U| = 6.35 \times 10^6 (3 \times 10^{-2}) \sqrt{\frac{\pi}{2k_0 \rho}} \int_1^2 (.93) \exp \left[- \frac{-1.0125 \left(1 + \frac{a}{n_0} \frac{dn}{dr}\right)^{2/3} \rho}{9.1 \times 10^4} \right] \quad 193$$

$\int_1^2(x)$ has been tabulated by Pryce (16) so that $\int_1^2(.93)$ can be readily estimated, and is determined to be 1.04 or ≈ 1 . Thus π_m can be evaluated as a function of ρ only.

$$\pi = \frac{7.86 \times 10^4}{\sqrt{\rho}} \exp \left[-(1.115 \times 10^{-5}) \left(1 + a \frac{dn}{dr}\right)^{2/3} \rho \right] \quad 194$$

The desired quantity to be plotted will be the ratio of the transhorizon field to the value of the field in free space where there is no earth to influence the field. This ratio, when expressed in decibels, ends up as the difference between the amplitude of the transhorizon field in db and the free space field in db. This difference is plotted on Figure 2,, for four values of the gradient. It is seen that the attenuation rate would be well predicted gradient refractive index $\frac{dn}{dr}$, of about -5×10^{-8} /meter. This is not an unreasonable gradient above the ocean as is pointed out by Tukizi (18) in his paper. The significant difference lies in the fact that the experimental data is roughly 4 to 6 db stronger than predicted by the model.

There are a number of factors which could account for the 4 to 6 db offset. It is doubtful that the approximations made by considering only the first term of the asymptotic expansion of the Airy integrals would contribute much more than a 10 to 15 percent error. One possible source

of the error lies in the fact that the plot of experimental data shown in Figure 2 is really nothing more than a smooth approximation to a great number of data points. Consideration of the plotted data points as shown in Figure 7 of the paper by Dinger, et al. (5) indicates that the data points cover a range of 10 to 20 decibels at any given distance. Thus, the 4 to 6 decibel discrepancy between the calculated curve and the "smoothed" experimental curve is not as serious as it looks since the calculated curve is still well within the range of the actual data points.

VI. EXTENDING THE GREEN'S FUNCTION TO INCLUDE THE EFFECTS OF ELEVATED STRATIFICATIONS

The linearly-sloping profile of refractive index provides a very simple analytical result when considering the propagation of electromagnetic energy around a spherical earth with a spherically stratified inhomogeneous atmosphere. This atmospheric model is not very acceptable because the profile of the refractive index becomes infinitely negative at large heights. This fact, plus the fact that the atmosphere is seldom such that the profile of the refractive index can be described by a simple mathematical model, means that some technique must be developed whereby an arbitrary profile of the index of refraction can be approximated to a reasonably good degree of accuracy.

It will be shown that a Green's function can be formulated such that any profile of refractive index can be broken down into a series of straight-line segments each of which has a slope approximately equal to the slope of the continuous function which represents the profile of the refractive index evaluated at the appropriate point. In this way any analytical function which describes the profile of the refractive index can be approximated by representing the earth's atmosphere as a series of concentric shells each of which has a linearly-varying profile of refractive index. The general solution can be formulated by matching the tangential components of the fields at the boundaries of the shells. Since the permittivity of space is assumed to be continuous, the boundary conditions will be satisfied if the Green's function and its derivative are continuous at the boundary. Each "shell" adds two more unknown coefficients which

must be determined if it is desired to know the complete Green's function applicable at any height. The fact that there are two additional coefficients in the complete Green's function means that a complete solution requires two additional independent equations. These additional independent equations are given by matching the boundary conditions.

In general, it is not necessary to determine the complete Green's function applicable at any height, rather it is usually desired to know the field close to the surface of the earth due to a transmitter located over the horizon and close to the surface of the earth. This means that the Green's function applicable in the first strata above the earth's surface is sufficient and it is only necessary to solve for the coefficients in this region. The rest of the atmosphere above the source and receiver can be represented by an arbitrarily large number of stratifications. In fact, it will be possible to express the result of the stratifications as a summation of effects and, since there can be an arbitrary number of stratifications, the summation can be extended to an integral. This technique will allow rapid evaluation of a profile which can be expressed as an explicit mathematical function. Thus, the effect of an exponentially-tapering profile of refractive index can be considered. It will be seen that the simple linearly-varying model is a good approximation to the exponentially-varying atmosphere which is a more satisfying model because the refractive index goes to unity at large heights. This technique is also very useful to determine the effect of layered perturbations in the refractive index.

If the atmosphere is divided into L concentric shells, each of which

has a different slope of refractive index, the Green's function will be determined by solving a system of 2L equations for the unknown coefficients. The boundary conditions are such that the solution in the earth represents an exponentially decreasing wave, the solution above the L^{th} shell represents outgoing radiation, and the solutions and their derivatives are continuous at the boundaries in the atmosphere. The boundary conditions at the earth-atmosphere interface are the same as those considered in the simple model having a linearly-tapered profile of refractive index. The complete system of equations is written in Matrix notation as shown in Equation 196. To define the Green's function in the first stratum of the atmosphere it would be necessary to determine C_1 and C_2 since

$$G(\eta \leq \eta_s; \xi) = C_1(\xi) \text{Ai}(\xi + \beta_1^{1/3} \eta) + C_2(\xi) \text{Bi}(\xi + \beta_1^{1/3} \eta). \quad 195$$

It will be assumed that the receiver is not higher than the transmitter. This assumption is merely for convenience in analysis and other situations can be considered in the same fashion.

The coefficients C_1 and C_2 will be determined by applying Cramer's Rule to the 2L x 2L matrix of Equation 196. In order to simplify the analysis the matrix will be written as shown in Equation 197.

This $n \times n$ matrix, $n = 2L$, can be reduced to an $n - 1 \times n - 1$ matrix by means of the relationship (6):

$$\text{Det} \begin{pmatrix} A_1 & \delta \\ \gamma & d \end{pmatrix} = d^{2-n} \text{Det} (dA_1 - \delta\gamma) \quad 198$$

Where $d = a_{nn}$, A_1 is the $n - 1 \times n - 1$ matrix formed by deleting the n^{th} row and the n^{th} column of the $n \times n$ matrix of Equation 197. δ is the $n - 1 \times 1$ column matrix formed by deleting a_{nn} from column n of the original matrix and γ is the $1 \times n - 1$ row matrix formed by deleting a_{nn} from row n of the original matrix. Thus, to evaluate the determinant of the denominator matrix, the $n \times n$ matrix can be reduced to an $n - 1 \times n - 1$ matrix, and the result is a constant multiplying the determinant of an $n - 1 \times n - 1$ matrix.

$$\text{Det} (\text{Den } C_1, C_2) = a_{nn}^{2-n} \text{Det} (a_{nn} A_1 - \delta\gamma) \quad 199$$

where

$$a_{nn} A_1 = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & & \\ & & 0 \\ & & 0 \\ & a_{n-3,n-3} & a_{n-3,n-2} & a_{n-3,n-1} \\ & a_{n-2,n-3} & a_{n-2,n-2} & a_{n-2,n-1} \\ & 0 & a_{nn} a_{n-1,n-2} & a_{nn} a_{n-1,n-1} \end{array} \right] \quad 200$$

$$\delta\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \\ & & \\ & & \\ & & \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{n-1,n} a_{n,n-2} & a_{n-1,n} a_{n,n-1} \end{bmatrix} \quad 201$$

Thus

$$a_{nn} A_1 - \delta\gamma = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ 0 & 0 & a'_{n-1,n-2} & a'_{n-1,n-1} \\ & & a_{n-2,n-2} & a_{n-2,n-1} \\ & & a_{n-3,n-2} & a_{n-3,n-1} \\ & & & 0 \end{bmatrix} \quad 202$$

Where

$$a'_{n-1,n-2} = a_{nn} a_{n-1,n-2} - a_{n-1,n} a_{n,n-2}$$

$$a'_{n-1,n-1} = a_{nn} a_{n-1,n-1} - a_{n-1,n} a_{n,n-1} \quad 204$$

Then

$$\text{Det} (\text{Den } C_1, C_2) = a_{nn}^{2-n} \text{Det } A_1' \quad 205$$

Where A_1' is the $n-1 \times n-1$ matrix defined by Equation 202. If the process is repeated on the $n-1 \times n-1$ matrix A_1' , the result is:

$$\text{Det} (\text{Den } C_1, C_2) = a_{nn}^{2-n} a'_{n-1,n-1}{}^{3-n} \text{Det } A_2^2 \quad 206$$

Where A_2^2 is given by the matrix

$$\begin{array}{ccccccc}
 a_{11} & a_{12} & a_{13} & 0 & 0 & & \\
 a_{21} & a_{22} & a_{23} & 0 & 0 & & \\
 0 & a_{32} & & & & & \\
 & & & & & & 0 \\
 & & & & & & 0 \\
 & & & & & & a_{n-4,n-4} & a_{n-4,n-3} & 0 \\
 0 & & & & & & a_{n-3,n-4} & a_{n-3,n-3} & a_{n-3,n-2}^2 \\
 0 & & & & & & a_{n-2,n-4} & a_{n-2,n-3} & a_{n-2,n-2}^2
 \end{array} \quad 207$$

and

$$a_{n-3,n-2}^2 = a'_{n-1,n-1} a_{n-3,n-2} - a'_{n-1,n-2} a_{n-3,n-1} \quad 208$$

$$a_{n-2,n-2}^2 = a'_{n-1,n-1} a_{n-2,n-2} - a'_{n-1,n-2} a_{n-2,n-1} \quad 209$$

$a'_{n-1, n-1}$ and $a'_{n-1, n-2}$ are as defined earlier. At this stage, it is seen that the matrix looks exactly like the original $n \times n$ matrix, that is non-zero-entries in the last three positions of the bottom two rows, except that this is a $n - 2 \times n - 2$ matrix. By continuing this process, it would be possible to reduce the $n \times n$ matrix to a 4×4 matrix with some multiplicative constant in front.

$$\text{Det (Den } C_1, C_2) = \{ \text{Const} \} \text{Det} \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34}^{n-4} \\ 0 & a_{42} & a_{43} & a_{44}^{n-4} \end{bmatrix} \quad 210$$

The a_{34}^{n-4} and a_{44}^{n-4} terms are modified from the original terms in the $n \times n$ matrix because of the successive reductions in the size of the matrix.

The other terms of the 4×4 matrix are the same as the corresponding terms in the $n \times n$ matrix and incidently are the same terms as in the 4×4 matrix for the linearly-sloping profile.

This matrix reduction technique can also be used in evaluating the determinants of the numerator matrices to determine the coefficients C_1 and C_2 . The matrices, whose determinants are desired, are:

Num $C_1 = \text{Det}$

a_{11}	0	a_{13}	0	0
a_{21}	0	a_{23}	0	0
0	1	a_{33}	a_{34}	a_{35}
0	0	a_{43}	a_{44}	a_{45}
0	0	0	a_{54}	a_{55}

0
211

0	$a_{n-1,n-2}$	$a_{n-1,n-1}$	$a_{n-1,n}$
0	$a_{n,n-2}$	$a_{n,n-1}$	a_{nn}

and

Num $C_2 = \text{Det}$

a_{11}	a_{12}	0	0	0
a_{21}	a_{22}	0	0	0
0	a_{32}	1	a_{34}	a_{35}
0	a_{42}	0	a_{44}	a_{45}
0	0	0	a_{54}	a_{55}
0	0	0	a_{64}	a_{65}

0
212

0	$a_{n-2,n-1}$	0
0	$a_{n-1,n-2}$	$a_{n-1,n-1}$
0	$a_{n,n-2}$	$a_{n,n-1}$

It is noted that in either case, except for the 4×4 submatrix in the upper left corner, the rest of the $n \times n$ matrix is the same as the corresponding part of the denominator matrix. If these matrices are reduced in the same way that the denominator matrix was reduced, the result is

$$\text{Det (Num } C_1) = \{\text{Const}\} \text{Det} \begin{bmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 1 & a_{33} & a_{34}^{(n-4)} \\ 0 & 0 & a_{43} & a_{44}^{(n-4)} \end{bmatrix} \quad . \quad 213$$

and

$$\text{Det (Num } C_2) = \{\text{Const}\} \text{Det} \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & 1 & a_{34}^{(n-4)} \\ 0 & a_{42} & 0 & a_{44}^{(n-4)} \end{bmatrix} \quad 214$$

Where $a_{34}^{(n-4)}$ and $a_{44}^{(n-4)}$ and the multiplicative constant are the same as defined for denominator. Thus, the Green's function applicable to the first strata above the earth is given by the equation:

$$G = \begin{array}{c} \left| \begin{array}{cccc} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 1 & a_{33} & a_{34}^{(n-4)} \\ 0 & 0 & a_{43} & a_{44}^{(n-4)} \end{array} \right| \text{Ai} \left(\frac{\xi}{\beta} \right) + \left| \begin{array}{cccc} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & 1 & a_{34}^{(n-4)} \\ 0 & a_{42} & 0 & a_{44}^{(n-4)} \end{array} \right| \text{Bi} \left(\frac{\xi}{\beta} \right) \\ \hline \left| \begin{array}{cccc} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34}^{(n-4)} \\ 0 & a_{42} & a_{43} & a_{44}^{(n-4)} \end{array} \right| \end{array} \quad 215$$

At this point, it is appropriate to consider the terms $a_{34}^{(n-4)}$ and $a_{44}^{(n-4)}$. Once these terms are known, the Green's function can be determined since all the other terms are known.

These terms will be evaluated by going back to the original $n \times n$ matrix and examining the modified entries as the matrix is reduced. In the first reduction, the modified terms are $a'_{n-1,n-1}$ and $a'_{n-1,n-2}$. These terms are given by the equations

$$a'_{n-1,n-1} = a_{nn} a_{n-1,n-1} - a_{n-1,n} a_{n,n-1} \quad 216$$

$$a'_{n-1,n-2} = a_{n,n} a_{n-1,n-2} - a_{n-1,n} a_{n,n-2} \quad 217$$

where

$$a_{n-1,n-1} = +B'_L \beta^{1/3} \quad a_{n,n-1} = B_L \quad a_{n,n-2} = A_L \quad 218$$

$$a_{n,n} = -f_L \quad a_{n-1,n} = -f'_L \beta^{1/3} \quad a_{n-1,n-2} = A'_L \beta^{1/3}$$

It will be necessary to use the approximation

$$\beta_L^{1/3} = \beta_{L-1}^{1/3} \left(1 + \frac{1}{3} \frac{\Delta\beta_{L-1}}{\beta_{L-1}} \right) \quad 219$$

to simplify the ensuing analysis. Inserting the values in the expressions for the modified terms results in:

$$a'_{n-1,n-1} = j\beta_{L-1}^{1/3} \left(W - j \frac{f'_L B_L \Delta\beta_{L-1}}{3\beta_{L-1}} \right) \quad 220$$

and

$$a'_{n-1,n-2} = \beta_{L-1}^{1/3} \left(W + \frac{f'_L A_{L-1} \Delta\beta_{L-1}}{3\beta_{L-1}} \right) \quad 221$$

where W is the Wronskian of the differential equation from which the solutions were taken.

In the second matrix reduction, the modified terms are $a_{n-2,n-2}^2$ and $a_{n-3,n-2}^2$, given by the equations

$$a_{n-2,n-2}^2 = a'_{n-1,n-1} a_{n-2,n-2} - a'_{n-1,n-2} a_{n-2,n-1} \quad 222$$

$$a_{n-2,n-2}^2 = a'_{n-1,n-1} a_{n-2,n-3} - a'_{n-1,n-2} a_{n-3,n-1} \quad 223$$

$a'_{n-1,n-1}$ and $a'_{n-1,n-2}$ are carried over from the first matrix reduction and

$$\begin{aligned} a_{n-2,n-2} &= -A_{L-1} & a_{n-3,n-2} &= A'_{L-1} \beta_{L-1}^{1/3} \\ a_{n-2,n-1} &= -B_{L-1} & a_{n-3,n-1} &= B'_{L-1} \beta_{L-1}^{1/3} \end{aligned} \quad 224$$

Substituting into the equations for the modified terms results in:

$$a_{n-2,n-2}^2 = \beta_{L-1}^{1/3} \left[W f_{L-1} + \frac{f'_L \Delta \beta_{L-1}}{3 \beta_{L-1}} (A_L B_{L-1} - B_L A_{L-1}) \right] \quad 225$$

$$a_{n-3,n-2}^2 = \beta_{L-1}^{2/3} \left[W f'_{L-1} + \frac{f'_L \Delta \beta_{L-1}}{3 \beta_{L-1}} (A_L B'_{L-1} - B_L A'_{L-1}) \right] \quad 226$$

If the thickness of a shell is reduced to infinitesimal dimensions as will be done when the final summation process is reduced to an integral, the expressions for the modified terms of the $n-2 \times n-2$ matrix can be further reduced because $A_L B_{L-1} - B_L A_{L-1} \rightarrow 0$ and $A_L B'_{L-1} - B_L A'_{L-1} \rightarrow W$. Under these conditions:

$$a_{n-3,n-2}^2 \approx \beta_{L-1}^{1/3} f'_{L-1} W \beta_{L-1}^{1/3} \left(1 + \frac{\Delta \beta_{L-1}}{3 \beta_{L-1}} \right) \quad 227$$

$$a_{n-2,n-2}^2 \approx \beta_{L-1}^{1/3} W f_{L-1} \quad 228$$

In the third reduction, the result is an $n-3 \times n-3$ matrix, with the modified terms $a_{n-3,n-3}^3$ and $a_{n-3,n-4}^3$; given by the equations

$$a_{n-3,n-3}^3 = a_{n-2,n-2}^2 a_{n-3,n-3}^2 - a_{n-3,n-2}^2 a_{n-2,n-3}^2 \quad 229$$

$$a_{n-3,n-4}^3 = a_{n-2,n-2}^2 a_{n-3,n-4}^2 - a_{n-3,n-2}^2 a_{n-2,n-4}^2 \quad , \quad 230$$

where:

$$a_{n-3,n-3}^2 = B'_{L-2} \beta_{L-2}^{1/3} a_{n-3,n-4}^2 = A'_{L-2} \beta_{L-2}^{1/3} \quad 231$$

$$a_{n-2,n-3}^2 = B_{L-2} \quad a_{n-2,n-4}^2 = A_{L-2}$$

and

$$\beta_{L-1}^{1/3} = \beta_{L-2}^{1/3} \left(1 + \frac{\Delta\beta_{L-2}}{3\beta_{L-2}} \right) \quad 232$$

Substituting into the defining equations for the modified terms of the $n-3 \times n-3$ matrix and ignoring product terms in $\Delta\beta$, results in:

$$a_{n-3,n-3}^3 = -j \beta_{L-2}^{1/3} \left[W - j \left(\frac{f'_{L-1} \Delta\beta_{L-2}}{3\beta_{L-2}} + \frac{f'_L \Delta\beta_{L-1}}{3\beta_{L-1}} \right) B_{L-1} \right] \quad 233$$

and

$$a_{n-3,n-4}^3 = -W \beta_{L-2}^{1/3} \beta_{L-1}^{1/3} \left[W + \left(\frac{f'_L \Delta\beta_{L-1}}{3\beta_{L-1}} + \frac{f'_{L-1} \Delta\beta_{L-2}}{3\beta_{L-2}} \right) A_{L-1} \right] \quad 234$$

The next matrix reduction will reduce the matrix to an $n-4 \times n-4$ matrix with the modified terms $a_{n-4,n-4}^4$ and $a_{n-5,n-4}^4$. These modified terms reduce to:

$$a_{n-4,n-4}^4 = -W \beta_{L-1}^{1/3} \beta_{L-2}^{1/3} f_{L-2} \quad 235$$

$$a_{n-5,n-4}^4 = -W \beta_{L-1}^{1/3} \beta_{L-2}^{1/3} \beta_{L-2}^{1/3} \left[f'_{L-2} + \left(\frac{f'_L \Delta\beta_{L-1}}{3\beta_{L-1}} + \frac{f'_{L-1} \Delta\beta_{L-2}}{3\beta_{L-2}} \right) \right] \quad 236$$

It is evident that a pattern has been established and it would be possible to write general expressions for $a_{44}^{(n-4)}$ and $a_{34}^{(n-4)}$. These general expressions are:

$$a_{44}^{(n-4)} = \text{const } f_2 \quad 237$$

$$a_{34}^{(n-4)} = \text{const } \beta_2^{1/3} \left[f'_2 + \frac{1}{3} \left(\frac{f'_3 \Delta\beta_2}{\beta_2} + \frac{f'_4 \Delta\beta_3}{\beta_3} + \dots + \frac{f'_L \Delta\beta_{L-1}}{\beta_{L-1}} \right) \right] \quad 238$$

Since these two terms are the only non-zero entries in the fourth column of the reduced matrices which define the Green's function, the const can

be factored out of each matrix and consequently cancel out of Green's function completely. The Green's function for the stratified atmosphere is similar to the Green's function developed for the linearly-varying atmosphere with the difference being a result of the modified term $a_{34}^{(n-4)}$. If the Green's function given by Equation 215 is evaluated in terms of the solutions of the last section, the result is:

$$G = \frac{f(\zeta + \beta_2^{1/3} \eta)}{W \beta_2^{1/3}} \left[g(\zeta + \beta_2^{1/3} \eta) - \frac{[g(\zeta) + \tau_1 g'(\zeta)] f(\zeta + \beta_2^{1/3} \eta) - \frac{S}{W} \left[\begin{array}{l} \{g(\zeta) f(\zeta + \beta_2^{1/3} \eta_d) - f(\zeta) g(\zeta + \beta_2^{1/3} \eta_d) \\ + \tau_1 [g'(\zeta) f(\zeta + \beta_2^{1/3} \eta_d) - \\ g(\zeta + \beta_2^{1/3} \eta_d) f'(\zeta)] \} g(\zeta + \beta_2^{1/3} \eta) \end{array} \right]}{f(\zeta) + \tau_1 f'(\zeta) - \frac{S}{W} \left[\begin{array}{l} \{g(\zeta) f(\zeta + \beta_2^{1/3} \eta_d) - f(\zeta) g(\zeta + \beta_2^{1/3} \eta_d) \\ + \tau_1 [g'(\zeta) f(\zeta + \beta_2^{1/3} \eta_d) - f'(\zeta) g(\zeta + \beta_2^{1/3} \eta_d)] \} \end{array} \right]} \right] \quad 239$$

Where S is the stratification function for the atmosphere above the transmitter

$$S = \frac{\beta_2^{1/3}}{3} \left[\frac{f'_3 \Delta\beta_2}{\beta_2} + \frac{f'_4 \Delta\beta_3}{\beta_3} + \frac{f'_5 \Delta\beta_5}{\beta_5} + \dots + \frac{f'_L \Delta\beta_{L-1}}{\beta_{L-1}} \right] \quad 240$$

Below the transmitter, the atmosphere is assumed to have a linearly-varying profile of refractive index. It should be noted that if $S = 0$, the Green's function reduced to the Green's function developed in the linearly-varying profile analysis. This result is reasonable since for this case the stratification function is identically zero because the $\Delta\beta$'s are all

zero. This Green's function developed for this case is somewhat unwieldy, but in an actual example some simplifying assumptions can be made. At this point it is of interest to consider two examples. First, the effect of an exponentially varying profile of refractive index will be considered. The second example will consider the effect of a layered perturbation in the profile of refractive index.

The first model which will be considered is the exponentially varying profile of refractive index. This profile will be defined by the equation:

$$n(r) = 1.000000 + 4 \times 10^{-4} \left\{ 1 - \exp [-(r-a)/10^4] \right\} \quad r > a \quad 241$$

It is desired to know the stratification function:

$$S = \frac{\beta_2^{1/3}}{3} \left[\frac{f'(\zeta_3 + \beta_3^{1/3} \eta_3) \Delta\beta_2}{\beta_2} + \frac{f'(\zeta_4 + \beta_4^{1/3} \eta_4) \Delta\beta_3}{\beta_3} + \dots \right. \\ \left. + \frac{f'(\zeta_L + \beta_L^{1/3} \eta_L) \Delta\beta_{L-1}}{\beta_{L-1}} \right]. \quad 242$$

The various components of the stratification function are defined as follows.

$$\beta_L = \frac{2k^2}{a} \left(1 + \frac{a}{n_0} \frac{dn}{dr} \Big|_{r=r_L} \right) \quad 243$$

$$\zeta_L = \beta_L^{2/3} (k_0^2 - \lambda^2/a^2) \quad 244$$

In the Airy Integral solutions, whose derivatives are part of the stratification function, it will considerably simplify the analysis if the argument $\zeta_i + \beta_i^{1/3} \eta_i$ is replaced by $\zeta + \beta^{1/3} \eta_i$. This simplification introduces an error of less than 20%. The increased ease of analysis is a

small price to pay for the error. The derivative of the Airy integral solution $f'(\zeta + \beta\eta)$ can be written as:

$$f'(\zeta + \beta\eta_i) = \exp(j\pi/6) \text{Ai}'[(\zeta + \beta\eta_i) \exp(j\pi/3)] \quad 245$$

Equation 243 can be used to formulate:

$$\frac{\Delta\beta_L}{\beta_L} = \frac{\frac{2k^2}{a} \left(1 + \frac{a}{n_0} \frac{dn}{dr} \Big|_{r=L}\right) - \frac{2k^2}{a} \left(1 + \frac{a}{n_0} \frac{dn}{dr} \Big|_{r=r_{L+1}}\right)}{\frac{2k^2}{a}} \quad 246$$

This equation is simplified by using $\frac{2k^2}{a}$ for all the β 's. Equation 246 can be simplified to

$$\frac{\Delta\beta_i}{\beta_i} = 4 \times 10^{-8} \frac{a}{n_0} \exp[-(r-a)/10^4] [1 - \exp(-\Delta r/10^4)] \quad 247$$

This can be written in approximate form as:

$$\frac{\Delta\beta_i}{\beta_i} = \frac{4a \times 10^{-12}}{n_0} \exp[-(r_i - a/10^4)] \Delta r_i \quad 248$$

Or, setting $r_i - a = h_i$ and $a = 6.35 \times 10^6$ meters, Equation 248 can be written as:

$$\frac{\Delta\beta_i}{\beta_i} = 2.54 \times 10^{-5} \exp(-10^{-4} h_i) \Delta h_i \quad 249$$

The stratification function can now be written as

$$S = \frac{\beta^{1/3} \exp(j\pi/6)}{3W} (2.54 \times 10^{-5}) \sum_{i=3}^L \text{Ai}'[\zeta \exp(j\pi/3) + \beta \eta_i^{1/3} \exp(j\pi/3)] \exp(-10^{-4} h_i \Delta h_i) \quad 250$$

If the stratifications are taken to be very close together, the summation of Equation 250 can be extended into the integral

$$S = \beta^{1/3} \frac{2.54 \times 10^{-5}}{3 W} \int_{h_3}^{\infty} \text{Ai}'[-\alpha + \beta^{1/3} \exp(j\pi/3)] \exp(-10^{-4}h) dh, \quad 251$$

where $\mathcal{L} \exp(j\pi/3) = -\alpha$ which is the root of the denominator of the Green's function. The upper limit of the integral has been extended to infinity. The number $-\alpha$ has not been determined as yet for this case, but it should not be much different than the root a_s of the linearly-varying atmosphere. The derivative of the Airy Integral $\text{Ai}'(-\alpha + \beta^{1/3} \exp[j\pi/3])$ can be approximated by the first term of the asymptotic expansion

$$\text{Ai}'(z) \approx -\frac{1}{2\sqrt{\pi}} z^{1/4} \exp(-2/3 z^{3/2}) \quad 252$$

Substituting this expression into the integral defining S results in:

$$S = -\frac{25.4 \times 10^{-6}}{6\sqrt{\pi} W} \exp(j\pi/6) \beta^{1/3} \int_{h_0}^{\infty} [-\alpha + x \exp(j\pi/3)]^{1/4} \exp[-2/3(-\alpha + x \exp(j\pi/3))^{3/2} - 10^{-4}h] dh \quad 253$$

where $x = \beta^{1/3}$. To see the effect of the exponential atmosphere on the transhorizon field it is necessary to estimate the magnitude of the stratification function S. If S is not infinitely large, the linearly-tapering solution will be a reasonable approximation to the solution in an exponentially tapering atmosphere. To get an idea of the size of the integral, consider the behavior of the integrand at reasonably large values of x.

The integrand can be approximated as

$$\exp(j\pi/12) [\beta^{1/3} a \ln(\frac{a+h}{a})]^{1/4} \exp \left\{ -j \frac{2}{3} [\beta^{1/3} a \ln(\frac{a+h}{a})]^{3/2} \right\}$$

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$$\exp \left\{ [\beta^{1/3} a \ln(\frac{a+h}{a})]^{1/2} \frac{\sqrt{3} \alpha}{2} - 10^{-4} h \right\}$$

The integrand oscillates rapidly with increasing h and it rises exponentially to some value then levels off and drops to zero as the $\exp(-10^{-4}h)$ term takes control. There will be some contribution to the integral for some values of x but even though the integrand rises quite rapidly for intermediate values of x , the oscillations of the integrand occur with shorter and shorter periods so that the net contribution is probably quite small. This is a heuristic argument which is borne out by the comparison with experiment. An exact analytical solution would be desirable only in that it would determine a different root of the denominator of the Green's function, and once a different root of the denominator of the Green's functions were established, different gradients of the refractive-index would be used to establish a correlation with experiment. This is all true as long as the integral is of a reasonable value. If the integral is infinitely large, then there would be only one root α and it would be zero. This would in turn mean that the field would not exhibit an exponential radial dependence which contradicts experimental evidence.

A second stratification example which will be considered is that of an isolated layer in the upper atmosphere. Consider the profile of the refractive index to be as shown in Figure 3. If the effect of the exponential atmosphere is ignored, it is necessary to consider only the

portion of the atmosphere from h_0 to $h_0 + L$. The stratification function can be written as:

$$S = \exp(j\pi/6) \int_{h_0}^{h_0+L} \text{Ai}' [-\alpha + x \exp(j\pi/3)] \frac{\pi\Delta\beta}{L} \sin \frac{2\pi h}{L} dh \quad 255$$

Using an asymptotic expansion of the Airy integral for large x , the stratification function reduces to

$$S = \frac{\exp(j\pi/4)\pi\Delta\beta}{W L} \int_{h_0}^{h_0+L} x^{1/4} \sin \frac{2\pi h}{L} \exp\left(\frac{2}{3} x^{3/2}\right) \exp\left(-\frac{1}{2} x^{1/2}\right) dh \quad 256$$

Again, in order to evaluate this integral, a number of approximations will be made, namely:

$$\begin{aligned} x^{1/4} &\approx (\beta^{1/3} h)^{1/4} \\ x^{1/2} &\approx (\beta^{1/3} h)^{1/2} \\ x^{3/2} &\approx (\beta^{1/3} h)^{3/2} \end{aligned} \quad 257$$

Using the approximation; $h_0 \gg L$, the stratification integral can be written as:

$$\begin{aligned} \frac{S}{W} &= \frac{\pi\Delta\beta}{WL} (\beta h_0)^{1/4} \exp\left\{\frac{3}{2} \alpha (\beta h_0)^{1/2}\right\} \exp\left\{-j\left[\frac{2}{3} (\beta h_0)^{3/2} - \pi/4\right]\right\} \\ &\int_0^L \exp[-j(\beta^{1/3} h_0)^{1/2} h] \sin \frac{2\pi h}{L} dh \end{aligned}$$

$$= \frac{\frac{\Delta\beta}{2W} (\beta^{1/3} h_o)^{1/4} \exp[3\alpha(\beta h_o)^{1/2}/2] \exp\{-j[2/3(\beta^{1/3} h_o)^{3/2} - \pi/4]\}}{1 - \left(\frac{\beta^{1/3} h_o L}{2\pi}\right)^2 \left[1 - \exp(-j\beta^{1/3} h_o L)\right]^{-1}} \quad 258$$

The term $\beta^{1/3} h_o L/2\pi$ is, in general, much greater than one, therefore Equation 258 can be rewritten as:

$$\frac{S}{W} \approx \frac{\Delta\beta}{2W} \frac{(\beta^{1/3} h_o)^{1/4} \exp\left[\frac{3\alpha(\beta h_o)^{1/2}}{2}\right] \exp\{-j[2/3(\beta^{1/3} h_o)^{3/2} - \pi/4]\}}{\left(-\frac{\beta^{1/3} h_o L}{2\pi}\right) \left[1 - \exp(-j\beta^{1/3} h_o L)\right]^{-1}} \quad 259$$

$$\approx \frac{\frac{\beta}{2} (\beta_o h_o)^{1/4} \exp\left[\frac{3(\beta h_o)^{1/2}\alpha}{2}\right] \exp\left[-j\left(\frac{2}{3}(\beta h_o)^{3/2} - \frac{\pi}{4}\right)\right] \left[\exp(-j\beta^{1/3} h_o L) - 1\right]}{\left(\frac{\beta^{1/3} h_o L}{2\pi}\right)^2} \quad 260$$

There are several significant features of this result which are worthy of note. First, the integral becomes larger as the height of the perturbation becomes larger. This is reasonable because the perturbation acts as a source of partially reflected wavelets. Thus, the higher the source is above the earth, the stronger is the resulting field. Secondly, the reflected wave is a function only of the end points of the perturbation, that is, a function of the width or thickness of the perturbation. This fact was pointed out by Friis, Crawford and Hogg (8). The reflected wave is directly proportional to the intensity of the perturbation in the

refractive index and becomes larger with decreasing L . These characteristics are to be expected since increased $\Delta\beta$ or decreased L mean that the gradients are greater.

Consider an example using the preceding theory. The Green's function defined by Equation 248 can be simplified by setting $\tau_1 = 0$, in

$$G = \frac{f(\xi + \beta^{1/3} \frac{1}{n_d})}{W \beta^{1/3}} \left[g(\xi + \beta^{1/3} \frac{1}{n}) \right. \\ \left. - \frac{\left\{ g(\xi) f(\xi + \beta^{1/3} \frac{1}{n}) - \frac{S}{W} \left[\begin{array}{l} g(\xi) f(\xi + \beta^{1/3} \frac{1}{n_d}) \\ - f(\xi) g(\xi + \beta^{1/3} \frac{1}{n_d}) \end{array} \right] g(\xi + \beta^{1/3} \frac{1}{n}) \right\}}{f(\xi) - \frac{S}{W} \left[g(\xi) f(\xi + \beta^{1/3} \frac{1}{n}) - f(\xi) g(\xi + \beta^{1/3} \frac{1}{n}) \right]} \right]$$

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The stratification function will be evaluated for the following parameters:

$$\beta^{1/3} = 3 \times 10^{-2}$$

$$L = 100 \text{ meters}$$

$$h_o = 10^4 \text{ meters}$$

$$\Delta\beta = 10^{-3}$$

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Inserting these values, the stratification function S becomes:

$$\frac{S}{W} \approx 3 \times 10^{-10} \exp(15\alpha) [\exp(-j1732) - 1]$$

$$\approx 3 \times 10^{-10} \exp(15\alpha) (-1.94 + j 342)$$

$$\approx 6 \times 10^{-10} \exp(15\alpha) \exp(j 170^\circ)$$

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At this point one can draw some general conclusions which lead to an approximate form of the final answer. The complete integral is evaluated

by determining the sum of the residues of the poles of the integrand. The poles of the integrand are determined by finding the zeros of the denominator of the Green's function. That is, the equation

$$f(\zeta) - \frac{S}{W} [g(\zeta)f(\zeta + \beta^{1/3} \eta_d) - f(\zeta)g(\zeta + \beta^{1/3} \eta_d)] = 0 \quad 264$$

The functions $f(\zeta)$ and $g(\zeta)f(\zeta + \beta^{1/3} \eta_d) - f(\zeta)g(\zeta + \beta^{1/3} \eta_d)$ are of comparable magnitude, so the magnitude of $\frac{S}{W}$ has to be about unity. This would enable an estimate of the size of α .

$$6 \times 10^{-10} \exp(15\alpha) \approx 1$$

$$\alpha \approx 1.4 \quad 265$$

$$\text{or } a_s^* = -1.4$$

Now, this value of a_s can be used in the distance dependence of the Equation 194, developed earlier for the case of a dipole radiating into an atmosphere with a linearly-varying index-of-refraction. If the atmospheric perturbation of this example is combined with the model used to compare with the experimental results of Section IV, the resulting Hertz vector is:

$$\pi = \frac{7.86 \times 10^4}{\sqrt{\rho}} \exp[-(6.7 \times 10^{-6})(1 + a \frac{dn}{dr})^{2/3} \rho] \quad 266$$

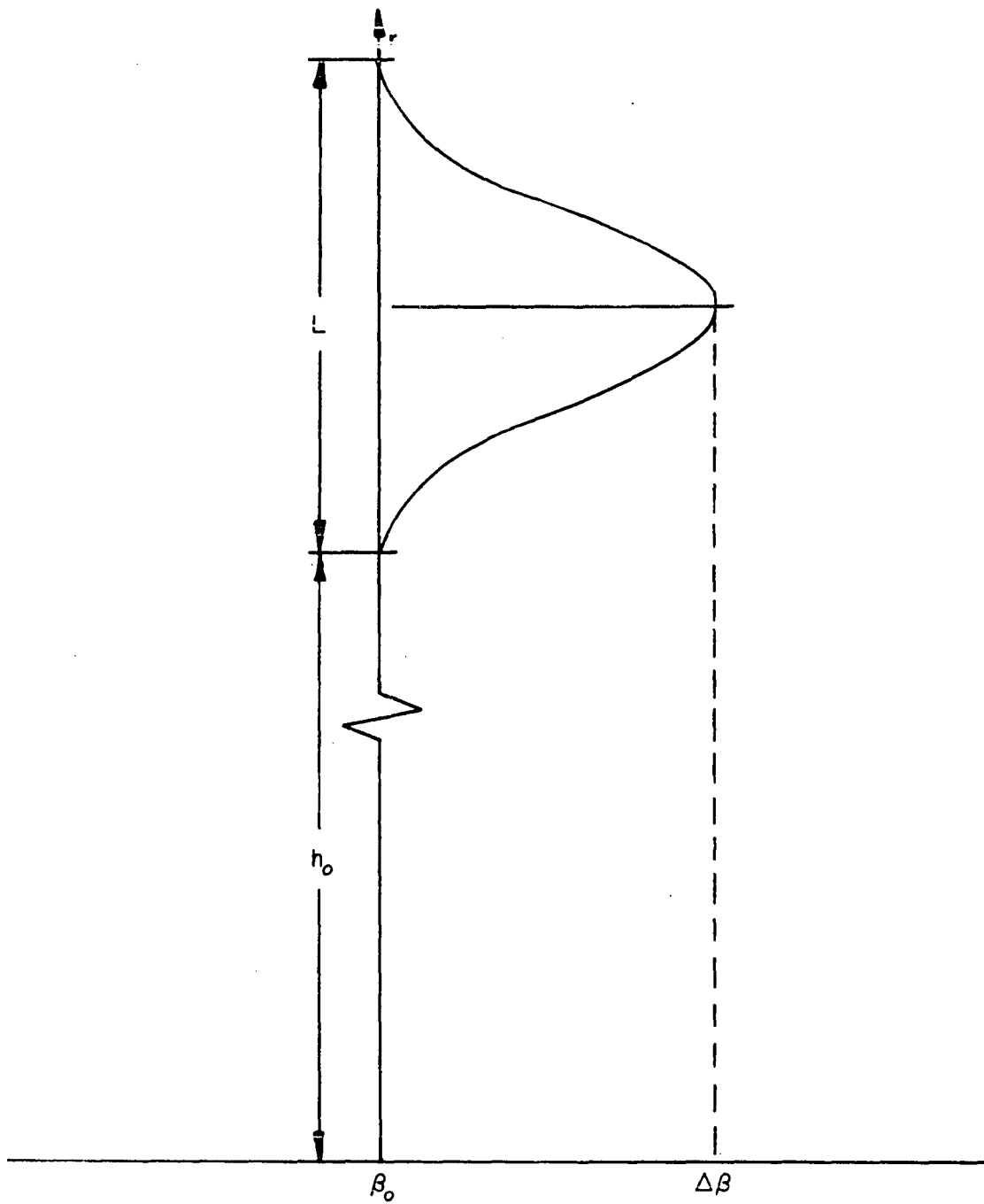
A comparison of the results of Section IV with a plot of Equation 266 is shown in Figure 4 to indicate the effect of the stratified perturbation.

The model of the preceding example is considerably simplified in an effort to get an idea of the effect of the stratification. A more detailed

look into the effect of the stratification of the atmosphere would reveal additional information which would be useful but it is questionable whether this information is worth the additional analytical effort. For example, a root of the denominator of the Green's function has been determined which is a negative real number. The actual root of the denominator would not, in general, be a real number; rather, it would be complex because the stratification is, in general, complex. The effect of a complex root would be to put an additional sinusoidal ρ dependence into the solution. This is not unreasonable since it would be expected that the reflections from the stratified anomaly would set up an interference pattern in the radial direction. The complex root would not be expected to have a phase angle much different than π radians, because the term which was suppressed in the phase of the stratification integral was

$$\exp [j(\beta^{1/3} h_0)^{1/2} \alpha / 2]$$

where α is a negative number. This means that the phase will change quite rapidly with α , since $(\beta^{1/3} h_0)^{1/2}$ is a reasonably large number. Thus, one would expect that the phase of the root of the denominator of the Green's function would be altered only slightly.



$$\beta = \beta_0 + \frac{\Delta\beta}{2} \left(1 - \cos \frac{2\pi r}{L}\right)$$

$$a + h_0 \leq r \leq a + h_0 + L$$

Figure 3. The profile of a stratified anomaly in the refractive index

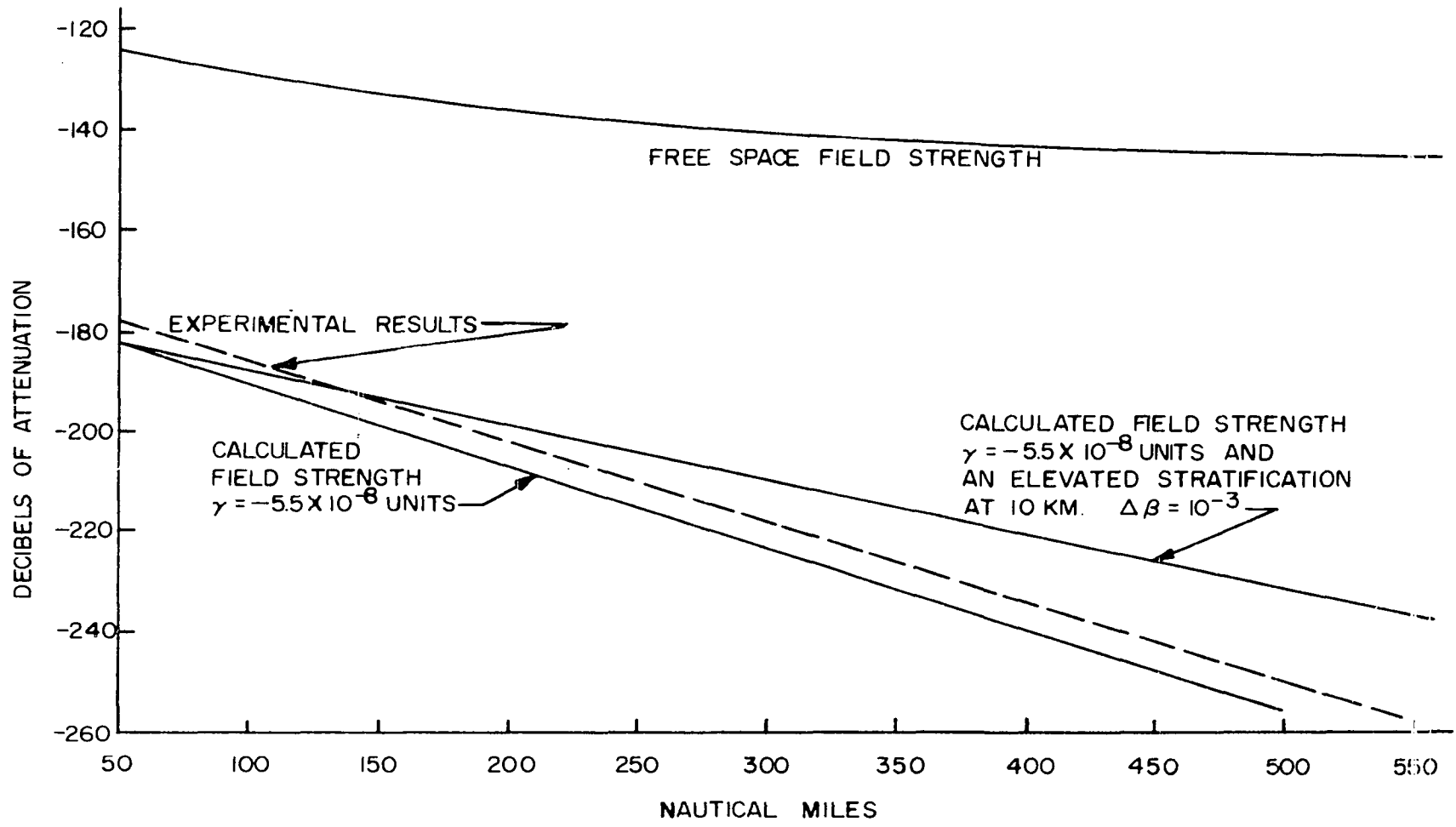


Figure 4. A comparison of the transhorizon field for two atmospheric models

VII. CONCLUSIONS

The earth-flattening approximation coupled with the Airy Integral solutions of the resulting differential equations provides a simple, readily understood technique for evaluating the strength of the fields over the horizon from a radio transmitter. The Green's function approach to the formulation of the complete solution allows considerable utility in extending the results obtained for the simple models of the earth's atmosphere. In this analysis, the Green's function is modified to include the effects of stratified perturbations in the refractive index of the atmosphere. Further, the Green's function can be formulated with an explicit time dependence which will permit the effect of slow temporal variations in the structure of the atmosphere to be considered. This modification would then bridge the gap between the "turbulent scatterer" theory and the normal mode theory.

The results of the analysis are compared with data taken from a trans-horizon propagation experiment conducted over the ocean. There is good agreement between the modest amount of reliable experimental data and the results of this analysis; thus substantiating in large measure the validity of this technique. The possibility of using a time-dependent Green's function to develop a model for the time-variant atmosphere was not pursued further because there is no experimental data with which to make a comparison.

It is impossible to go through analysis on a subject of this nature without slighting many ramifications of the problem. Such is the case in

this analysis. There are several areas of the investigation which should be carried through to a more detailed conclusion. These include the detailed evaluation of the effects of perturbations in the structure of the atmosphere and the use of a time-dependent Green's function to include the statistics of a time-varying atmosphere into normal-mode theory. The reason these topics were merely mentioned or given superficial treatment is that the subsequent analysis are considered worthy of individual reports. It is intended that this analysis will serve as the foundation on which to treat these topics in more detail.

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