Relativity in Rotating Frames

Fundamental Theories of Physics

An International Book Series on The Fundamental Theories of Physics: Their Clarification, Development and Application

Volume 135

Relativity in Rotating Frames

Relativistic Physics in Rotating Reference Frames

Edited by

Guido Rizzi

Politecnico di Torino and Istituto Nazionale di Fisica Nucleare,
Torino, Italy

and

Matteo Luca Ruggiero

Politecnico di Torino and Istituto Nazionale di Fisica Nucleare,
Torino, Italy

SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-90-481-6514-8 ISBN 978-94-017-0528-8 (eBook)
DOI 10.1007/978-94-017-0528-8

Printed on acid-free paper

Contents

Preface

For today's physicist, steeped in the Newtonian and post-Newtonian traditions, it takes a real effort of the imagination to realize that, for by far the longest stretch of time, the prevailing scientific efforts to understand the universe (to say nothing of pre- or non-scientific views, past or contemporary) were based on the concept that uniform circular motion is higher, nobler, and more natural than any other form of motion. Michel Blay has well described the situation:

> Since Antiquity and more precisely since the elaboration of the Aristotelian conceptual outlook, circular motion was conceived as both the prime motion and the natural motion. This motion, for example that of the stars, proceeded from an internal principle and, contrary to violent motion, such as that of a stone that is thrown, did not presuppose the action of some exterior motor in order to continue. In the Aristotelian Cosmos, divided into two worlds, the movement of the stars belonged to the celestial sphere of perfect motions and incorruptible bodies, the motion of a thrown stone belonged to the sublunar sphere (the earth and its surrounding space) of more or less chaotic motions and bodies subject to decay.

Of course, in the ancient world the atomists opposed this viewpoint, their atoms falling straight downward perpetually through the void, with an occasion clinamen, or random deviation from their downward fall, to explain the formation of worlds such as ours. But, for whatever scientific or sociological reasons, the Aristotelian world view triumphed and atomism languished for almost two millennia.

Even Galileo, a determined opponent of the peripatetics, as the followers of Aristotle were called, succumbed to the appeal of circular motion in his first attempts to formulate the concept of inertial motion. It was only with Descartes and then Newton that the law of inertia in its modern form triumphed, and uniform motion in a straight line came to be regarded as the natural standard, deviations from it being attributed to forces, notably gravitation as understood by Newton. Now it was uniform circular motion that required an external explanation. And, as Newton's famous bucket experiment demonstrated, there was a great dynamical difference between non-rotating and rotating frames of reference. Since we live on such a rotating frame of reference, this difference is of great practical as well as theoretical significance. But as long as the Newtonian kinematical framework stood intact, both rotating and non-rotating observers shared a common, absolute time.

With the advent of the special theory of relativity and its new kinematics, the absolute time concept shattered, falling apart into two distinct concepts.

One is the proper time, associated with every time-like world-line and directly measurable by a clock travelling along that world-line. The proper time is an invariant but one that is no longer independent of the world-line. The time elapsed between two events depends on the history of the clock travelling between the two. However, the concept of proper time survives the transition to general relativity more-or-less unscathed.

In special relativity, one can also define a family of global times, one associated with each inertial frame of reference, but each depending on a conventional stipulation (such as the constancy of the one-way velocity of light in that frame) and only measurable indirectly, in terms of the appropriate proper time and proper length measurements in the frame in question. Once one ventures beyond the class of inertial frames, the concept of a global time becomes even more problematic, and the problem only becomes more acute when one ventures into the general theory of relativity. Solutions exist, such as the Gödel universe, for which it is impossible to formulate such a concept.

What are the implications of the new kinematics for rotating frame of reference? This question early became the subject of an intense discussion among relativists (to say nothing of their opponents!), a discussion that continues down to our times. This book constitutes an important and fascinating contribution to this ongoing discussion. One can hardly agree with everything found in it, since many of the authors disagree among themselves; and it is not the function of a preface to pass judgement on merit of the individual contributions. That must be left to each reader in the short run, and to the scientific community in the long run. But what is certain is that each reader will spend many absorbing hours reading the papers in this book, and perhaps even more trying to form and justify his or her opinions on the questions raised in them.

JOHN STACHEL

This book is in memory of
Jeeva Anandan

Contributing Authors

Neil Ashby (1934) is Professor at the University of Colorado in Boulder, and Guest Researcher at National Institute of Standards and Technology, Boulder. His Ph.D. is from Harvard University (1961). He is interested in electrons in solids, solar system tests of relativity, relativistic effects on clocks in satellites.

Dennis Dieks (1949) is Professor of Philosophy and Foundations of Science at Utrecht University, The Netherlands. His Ph.D. is from Utrecht University (1981). He is interested in the foundations of physics: quantum theory and its interpretation, questions about space and time, and problems relating to causality and determinism.

Bahram Mashhoon (1947) is Professor of Physics at the University of Missouri - Columbia. His Ph.D. in Physics is from Princeton (1972). He is interested in theoretical physics, relativity and gravitation.

Franco Selleri (1946) is Professor of Physics at the University of Bari, Italy. He is interested in elementary particles, quantum physics, relativistic physics.

Guido Rizzi (1939) is Professor of Physics at Politecnico di Torino, Italy. He is interested in special and general relativity.

Alessio Serafini (1977) is Ph.D. student at the University of Salerno, Italy. His *laurea* is from the University of Pisa, Italy (2001). He is interested in relativity, quantum mechanics and quantum optics.

Robert D. Klauber (1943) is retired (from industry) and part time teacher. His Ph.D. is from Virginia Polytechnic Institute and State University (1982). He is interested in relativity, cosmology and quantum field theory.

Thomas A. Weber (1934) is Professor of Physics at Iowa State University. His Ph.D is from the University of Notre Dame (1961). He is interested in mathematical physics, relativity, formal scattering theory, inverse problems, isospectral and phase equivalent hamiltonians.

Francesco Sorge (1961) is member of INFN in Padua, Italy, and teaches Mathematics and Physics at high schools. His Ph.D. in Physics is from the University of Padua (2002). He is interested in relativity and theory of measure, relativistic astrophysics and gravitational waves.

José-Fernando Pascual-Sánchez (1954) is Associate Professor of Applied Mathematics at the University of Valladolid, Spain. His Ph.D. is from the University of Valladolid (1983). He is interested in gravitomagnetism, GNSS and relativity, inhomogeneous cosmology.

Angel San Miguel (1956) is Associate Professor of Applied Mathematics at the University of Valladolid, Spain. His Ph.D. is from the University of Valladolid (1985). He is interested in relativity, geometrical mechanics.

Francisco Vicente (1956) is Associate Professor of Applied Mathematics at the University of Valladolid, Spain. His Ph.D. is from the University of Valladolid (1985). He is interested in relativistic mechanics, general relativity.

Matteo Luca Ruggiero (1975) is Ph.D. student at Politecnico di Torino, Italy. His *laurea* is from the University of Pisa, Italy (2000). He is interested in relativity, Einstein-Cartan theory, foundations of relativity, theoretical physics.

Donato Bini (1963) is Researcher at Istituto per le Applicazioni del Calcolo "M. Picone" CNR, Rome, Italy. His Ph.D. in Physics is from the University of Rome "La Sapienza" (1995). He is interested in differential geometry with applications to relativity, gravitational waves, spacetime splitting techniques, black hole perturbations, spinning test particles.

Robert T. Janzten (1952) is Professor at Villanova University, USA. His Ph.D. in Physics is from the University of California at Berkeley (1978). He is interested in differential geometry and Lie group theory with applications to relativity, spatially homogeneous cosmology, higher dimensional spacetimes with symmetry, spacetime splitting techniques.

Lluís Bel (1933) is Visiting Professor at the University of the Basc Country, Spain. His Ph.D. in Mathematical Sciences his from the University of Paris (1960). He is interested in dynamical systems, frames of reference, cosmology.

Angelo Tartaglia (1943) is Professor of Physics at Politecnico di Torino, Italy. He is interested in special and general relativity, gravitomagnetism.

Hrvoje Nikolić (1970) is Post-Doc at the University of Zagreb, Croatia. His Ph.D. is from the University of Zagreb (2001). He is interested in the theory of relativity, gravitation, foundations of quantum mechanics, quantum field theory.

Øyvind Grøn (1944) is Professor of Physics at the University of Oslo, Norway. He is interested in the theory of relativity, cosmology, history of physics.

Giorgio Papini (1934) is Professor Emeritus at the University of Regina, Canada. His Ph.D. is from the University of Saskatechewan, Canada. He is interested in the interaction of inertia and gravitation with quantum systems, and in conformal invariant theories of Weyl-Dirac type.

Jeeva Anandan (1948) is Professor of Physics at the University of South Carolina. His Ph.D. in Physics is from the University of Pittsburgh (1978), his Ph.D. in Philosophy is from Oxford (1997). He is interested in the foundations of quantum mechanics, general relativity and gauge fields.

Jun Suzuki (1976) is Ph.D. student at the University of South Carolina. His B.A. is from the University of Tokio (2001). He is interested in the foundations of quantum theory.

Fernando de Felice (1943) is Professor of Relativity at the University of Padua, Italy. He is interested in relativity and theoretical astrophysics.

Introduction

When Alwyn Van der Merwe asked us to edit a book on relativistic physics in rotating reference frames, about a year ago, we had just published a paper on the space geometry of a rotating disk. Our interest in this field dates back to some years ago, when one of us came across the Sagnac effect, and thoroughly studied this intriguing issue. Since then, we have been studying this field with increasing interest and now we are well acquainted with the pertaining literature, which encompasses a lot of seminal discussions and debates that have stimulated the development of relativistic physics for a century.

We must admit that it was not easy to find our way through the great number of papers: a real *mare magnum* without any guiding light, where the same topics were treated using very different approaches, and often different notations, which made it difficult to compare them. Indeed, we think that many physicists believe that nothing new or interesting can be found in this field, and the only open questions are a matter of philosophy rather than of some interest in the current development of theoretical and experimental physics. On the contrary, carefully digging in the literature, we discovered problems still controversial, viewpoints in contention, and open issues pertaining to the very foundations of relativity.

Therefore a monograph on these subjects seemed to us really necessary, not only to tidy up a little the various papers published here and there over the years, but also to critically re-examine and discuss the most controversial issues. For these reasons we accepted Van der Merwe's kind invitation with enthusiasm.

We started our work by collecting contributions from scientists who, though having different and at times opposite viewpoints on these subjects, share a common physical-mathematical background. We asked them to write their papers with a clear and plain style and, also, with a somewhat pedagogical aim: according to our purposes, this book should be a guide and a reference for all those who, now and in the future, have an interest in this field. As you well know, the higher the ambitions, the greater the efforts; as to the results, only the reader can judge.

We deliberately decided to confine ourselves to special relativity, and not to consider rotating frames in curved space-times. This would have enormously increased the material at our disposal: on the other hand we found it expedient (and easier, of course) to understand the crucial problems and concepts in flat space-time, avoiding the formal complications deriving from gravitation.

The first historical contributions to the study of rotating frames in relativity date back to first decades of the last century, when the papers by Ehrenfest and Sagnac shattered the foundations of the brand-new theory of relativity: these papers, translated into English, are published in our book, as an ideal beginning to this long-standing debate.

A glance at the contributing papers shows that, even now, a century later, interest in these problems is not purely academic or philosophical. In his paper, Ashby explains the relevance of the relativistic study of rotation for a modern technical device, such as the Global Positioning System. After all, we must remember that we are living in a rotating frame, the Earth, and every experimental expectation, based on a relativistic approach, should take this into account as an obvious fact. Another fundamental issue, developed by Mashhoon, is the so-called hypothesis of locality and its limitations in accelerating systems. However, this hypothesis, together with other fundamental ones, are questioned by those who find, in the relativistic approach to rotation, arguments against the self-consistency of the theory. For instance, this is the case of Klauber and Selleri who, though adopting different approaches, claim that the special theory of relativity is not valid when it is applied to rotating frames, and to this purpose they raise several stimulating arguments. The standard formulation of the theory is defended against these ”attacks”, by Dieks, Grøn, Weber, Rizzi and Serafini, and an interesting debate develops about the fundamental problems of measurements of space, time, synchronization, that also involves Nikolić, Bel and Tartaglia. Other topics related to these fundamental issues, such as the isotropy of the velocity of light and the universality of the Sagnac effect for matter and light beams, are dealt with in the papers by Sorge, Pascual-Sánchez, San Miguel and Vicente, Rizzi and Ruggiero. The relativistic approach to inertial forces and the role of rotating observers are examined by Bini and Jantzen, while the mathematical properties of rotating space-times are studied by De Felice. Finally, halfway between the classical theory of relativity and the quantum world, the quantum-inertial effects are thoroughly described in the papers by Papini, Anandan and Suzuki.

In order to give the reader a deeper insight into the most controversial issues, we organized a sort of virtual round table. After the publication of the drafts of the contributing papers at our web site, we asked the authors to comment on the papers, and confront the various viewpoints. Then, we collected their comments, which resulted in a lively on-line discussion. The dialogues that you can read at the end of this book are based on this discussion. However the raw contributions of the authors to the on-line debate, which could not be published as they were, have been supplemented by fragments that we borrowed from their papers. During this editing job, we aimed at composing the material at our disposal to obtain something like a real discussion about the main topics of the book. We are well aware of the arbitrariness of this job; however we did it with the greatest care in order to accurately quote the ideas and the opinions of the authors. The final result should not be considered just as an appendix, but as an integral and fundamental part of the book: we have attempted to offer the reader a *vademecum* to find her/his way through the papers. Indeed, the round table can also be read independently of the papers, to get a bird's eye view of the main subjects, or it can be read after the papers, for a direct comparison of the viewpoints on the most controversial and interesting topics. The underlying aim, which is actually the aim of the whole book, is to offer the reader the possibility of understanding these opinions, and the subtleties on which their differences are based: to this end, we have tried to write as clearly as possible, having in mind this pedagogical purpose. We hope that, in this way, the reader can shape his own ideas, which is our ultimate goal.

We do not know whether we have succeeded or not: what we know for sure is that, at the end of this work we feel richer in knowledge and also in doubts, and ready and willing to go on in this stimulating field. If you share these feelings, there is a chance that we did not fail in our purposes; however, if we bored you, be assured, we did not do it on purpose.

Last, but definitely not least, a thought for Prof. Jeeva Anandan, who passed away during the making of this book, and whose last paper we are honoured to publish here: a great loss for all physicists, and one which we feel deeply, first of all because he was one of our collaborators, then because he wrote, during his long scientific activity, many important and fundamental papers about the main matter of this book, such as the Sagnac effect and relativistic rotation. This book is dedicated to him.

Guido Rizzi and Matteo Luca Ruggiero
Torino, September 15, 2003

Acknowledgments

We are grateful to Alwyn Van der Merwe, for giving us the opportunity of editing this book, and to all contributing authors, for their willingness, patience and clever collaboration during all phases of the project. During our work, friendly relationships were established among all collaborators: it is all thanks to them that we have succeeded in carrying out this project. In addition, we also thank those who could not participate in our book, because of previous commitments, but who strongly encouraged our work.

A special word of thanks to John Stachel, who wrote the Preface with excellent insight.

We would also like to thank the staff at Kluwer Academic Publisher, in particular publishing editor Sabine Freisem and her assistant Kirsten Theunissen, and Berendina Schermers-van Straalen who helped us for the copyright permissions.

We thank Alessio Serafini and Maria Blandin Savoia, for carefully reading the virtual round table, and Ugo Montrucchio for translating the historical papers.

G Rizzi and ML Ruggiero

Of my indebtedness to my mother, my father and my brother Marco, for their encouragement, and to Dara, for her infinite patience, I can make no adequate acknowledgments.

ML Ruggiero

I

HISTORICAL PAPERS

Chapter 1

UNIFORM ROTATION OF RIGID BODIES AND THE THEORY OF RELATIVITY

Paul Ehrenfest*
St. Petersburg, September 1909

In order to generalize the relativistic kinematics of rigid bodies in rectilinear uniform motion to whatever kind of motion, following Minkowski's ideas we obtain the following statement:

To say that a body remains relativistically rigid means: it deforms continuously by arbitrary motion so that each of its infinitesimal elements Lorentz contract (relative to its rest length) all the time in accordance with the instantaneous velocity of each of its elements, as observed by an observer at rest.

According to me, as I am going to explain, the consequences of this statement, applied to a very simply motion, result in a contradiction.

Recently, Born[2] has given a definition of relativistic rigidity, which applies to all motions. Born's definition - which is in agreement with the relativistic principles - is not based on the system of measurements performed by an observer at rest; rather, its definition is based on the (Minkowskian) measurements performed by a continuum of inertial observers: as measured by these observers, co-moving with each point of an arbitrarily moving body, each element of the body remains undeformed.

Both definitions of relativistic rigidity, as far as I can understand, are equivalent.

*Translated from *Physikalische Zeitschrift*, **10**, 918 (1909), Courtesy of S. Hirzel Verlag.

[2]M. Born, Die Theorie des starren Electrons in der Kinematik des Relativitäts Prinzipes, *Ann. d. Phys*, **30**, 1 (1909), *Phys. Zeits.*, **9**, 844 (1908).

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 3-4.

Let me show a very simple kind of motion, for which the previous definition bring about a contradiction: namely, the motion I am going to refer to is the uniform rotation around a fixed axis.

Consider a relativistically rigid cylinder with radius R and height H. It is given a rotating motion around its axis, which finally becomes constant. As measured by an observer at rest, the radius of the rotating cylinder is R'. Then R' has to fulfill the following two contradictory requirements:

1 The circumference of the cylinder must obtain a contraction

$$2\pi R' < 2\pi R$$

relative to its rest length, since each of its elements moves with an instantaneous velocity $R'\omega$.

2 If one considers each element along a radius, then the instantaneous velocity of each element is directed perpendicular to the radius. Hence, the elements of a radius cannot show any contraction relative to their rest length. This means that:

$$R' = R$$

Chapter 2

THE EXISTENCE OF THE LUMINIFEROUS ETHER DEMONSTRATED BY MEANS OF THE EFFECT OF A RELATIVE ETHER WIND IN AN UNIFORMLY ROTATING INTERFEROMETER

M. Georges Sagnac*
introduced by M.E. Bouty

1. Principles of the Method

I let a horizontal platform rotate uniformly, at one or two revolutions per second, around a vertical axis; on the platform I have firmly fixed the pieces of an interferometer equal to that I used in my previous experiments, described in 1910 (*Comptes rendus*, **150**, 1676). The two interfering beams, which are reflected by four mirrors placed on the rim of the rotating platform, are superimposed after the propagation in opposite directions along the same circular horizontal circuit which encloses an area S. The rotating system contains also the light source, a little electric lamp, the receiver, a fine grain photographic plate which records the interference fringe. On the photographs d and s, obtained respectively by a *dextrorsum* and a *sinistrorsum* rotation with the same frequency, the centre of the central fringe has different positions. I measure this displacement with respect to the centre of interference.

*Translated from *Comptes rendus de l'Académie des sciences*, **157**, 708 (1913), Courtesy of Elsevier-France.

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 5-7.

1.1 First Method

On d and then on s, I look for the position of the central fringe with respect to the images of the micrometrical vertical traits in the focal plane of the collimator.

1.2 Second Method

I measure directly the distance between the vertical central fringe of a d photograph and the central fringe of a s photograph exactly contiguous to the first, below a sharp horizontal separation line. I get these photographs in a direct way without touching the photographic chassis, giving, before each of the images d and s, the contiguous positions that correspond to the illuminating slit with cutting horizontal edges, in the focal plane of the collimator.

2. Optical vortex effect

The fringe shift z of the interference centre that I measured by the method just outlined, turns out to be a particular case of the optical vortex effect defined in my previous works (*Congrés de Bruxelles de septembre*, 1910 **I**, 217; *Comptes Rendus*, **152** 1911, 310; *Le Radium*, **VIII**, 1911, 1) and which, according to the current ideas, should be thought of as a direct manifestation of the luminiferous ether.

In a system, moving as a whole with respect to the ether, the time of propagation between any couple of points should be modified as though the system were at rest under the action of an ether wind, whose relative velocity turns out to be equal and opposite to the one of those points, and dragging the light just like the atmosphere wind drags the sound waves.

The observation of the optical effect of the wind relative to the ether will constitute a proof of the ether, in the same way as the observation of the influence of the wind relative to atmosphere - in a moving system - on the speed of sound allows to deduce the existence of the atmosphere around the moving system, if no other effects are present.

The necessity of getting from the same point-like source the light waves that are recombined in another point to obtain interference, cancels the first order interference effect due to the motion of the whole optical system, unless the matter that drags the ether provokes a circulation C of ether in the light path, spanning an area S, that is an ether vortex bS (*Comptes rendus*, **141**, 1905, 1220; 1910 and 1911, *loc.cit.*). I showed by interferometrical techniques (1910 and 1911 *loc.cit.*), using a circuit having a vertical projection of 20 m^2 that the ether dragging near the ground does not produce a vortex density greater than $1/1000$ radians per second.

In a horizontal optical circuit, at a given latitude α, the daily terrestrial rotation, if the ether is at rest, should produce a ether-relative vortex whose density is $4\pi \sin \alpha / T$, where T is the period of the sidereal day, or $4\pi \sin \alpha / 86164$, which is noticeably smaller than the upper limit $1/1000$ that I established using a vertical circuit. I hope to be able to find out whether the corresponding small optical vortex effect exists or not.

At first, it was easier to find out the proof of the existence of the ether using a small rotating optical circuit. A frequency N of two revolutions per second produces a vortex density $b = 4\pi N$, that is 25 radians per second. A *dextrorsum* uniform rotation of the interferometer produces a *sinistrorsum* ether wind; the rotation causes a backward shift x of the phase of the beam **T**, whose propagation around the area S is *dextrorsum*, and an equal forward shift of the phase of the counter-propagating beam **R**, so that the relative displacement of the fringes turns out to be $2x$. The displacement z that I notice between a photograph s and a photograph d should be twice the previous one. Hence, using the value of x that I gave before (*loc.cit.* 1910 and 1911), I obtained:

$$z = 4x = 4\frac{bS}{\lambda V_0} = \frac{16\pi NS}{\lambda V_0}$$

where V_0 is the velocity of light in vacuum and λ is the wavelength of light. Using a frequency $N = 2Hz$ and an area $S = 860\ cm^2$, the displacement z (for the indigo light) turns out to be 0.07, and it can be easily seen; the interfringe is from $0,5\ mm$ to $1\ mm$.

The interference displacement z, which is a constant fraction of the interfringe for a given frequency of rotation N, is not visible on the photographs when the fringes are too close; this shows that the observed effect depends on a phase difference due to the rotational motion of the system, and the displacement of the interference centre, observed by a comparison between a photograph d and a photograph s, does not depend on the random or elastic displacement of the optical pieces during rotation (thanks to the counter-screws that block the regulation screws of the optical system).

An air vortex, produced above the interferometer by a ventilator with vertical axis (blowing down the air towards the interferometer) does not displace the interference centre, thanks to the careful regulated superimposition of the counter-propagating beams. The air vortex, similar but less strong, produced by the rotation of the interferometer, does not cause any noticeable effect.

Hence, the observed interference effect turns out to be the optical vortex effect due to the motion of the system with respect to the ether and it is a direct proof of the existence of the ether, which is a necessary support for the light waves of Huygens and Fresnel.

II

PAPERS

Chapter 1

THE SAGNAC EFFECT IN THE GLOBAL POSITIONING SYSTEM

Neil Ashby
Department of Physics, UCB 390.
University of Colorado.
Boulder, CO 80309-0390 USA.
n_ashby@mobek.colorado.edu

Abstract In the Global Positioning System (GPS) the reference frame used for navigation is an earth-centered, earth-fixed rotating frame, the WGS-84 frame. The time reference is defined in an underlying earth-centered locally inertial frame, freely falling with the earth but non-rotating, with a time unit determined by atomic clocks at rest on earth's rotating geoid. Therefore GPS receivers must apply significant Sagnac or Sagnac-like corrections, depending on how information is processed by the receiver. These corrections can be described either from the point of view of the local inertial frame, in which light travels with uniform speed c in all directions, or from the point of view of an earth-centered rotating frame, in which the Sagnac effect is described by terms in the fundamental scalar invariant that couple space and time. Such corrections are very important for comparing time standards world-wide.

1. Introduction

The purpose of the Global Positioning System (GPS) is accurate navigation on or near earth's surface. GPS also provides an accurate world-wide clock synchronization and timing system. Most GPS users are interested in knowing their position on earth; the developers of GPS have therefore adopted an Earth-Centered, Earth-Fixed (ECEF) rotating reference frame as the basis for navigation. Specifically, in the WGS-84(873) frame, the model earth rotates about a fixed axis with a defined rotation rate, $\omega_E = 7.2921151247 \times 10^{-5}$ rad s^{-1}.[1],[2]

In an inertial frame, a network of self-consistently synchronized clocks can be established either by transmission of electromagnetic signals that propagate

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 11-28.

with the universal constant speed c (this is called *Einstein synchronization*), or by slow transport of portable atomic clocks. On the other hand it is well-known[3] that in a rotating reference frame, the Sagnac effect prevents a network of self-consistently synchronized clocks from being established by such processes. This is a significant issue in using timing signals to determine position in the GPS. The Sagnac effect can amount to hundreds of nanoseconds; a timing error of one nanosecond can lead to a navigational error of 30 cm.

To account for the Sagnac effect, a hypothetical non-rotating reference frame is introduced. Time in this so-called Earth-Centered Inertial (ECI) Frame is adopted as the basis for GPS time; this is discussed in Section 2. Of course the earth's mass encompasses the origin of the ECI frame and has significant gravitational effects. To an extremely good approximation in the GPS, however, gravitational effects can be simply added to other effects arising from special relativity. In this article gravitational effects will not be considered. Even time dilation, which is an effect of second order in the small parameter v/c, where v is the velocity of some clock, will be neglected. I shall confine this discussion to effects which are of first order (linear) in velocities. The Sagnac effect is such an effect.

A description of the GPS system, of the signal structure, and the navigation message, needed to understand how navigation calculations are performed, is given in Section 3. In comparing synchronization processes in the ECI frame with those in the ECEF frame, taking into account relativity principles, it becomes evident that the Sagnac effect is a manifestation of the relativity of simultaneity. Observers in the rotating ECEF frame using Einstein synchronization will not agree that clocks in the ECI frame are synchronized, due to the relative motion. In fact observers in the rotating frame cannot even globally synchronize their own clocks, due to the rotation. This is discussed in Section 4. Section 5 discusses Sagnac corrections that are necessary when comparing remote clocks on earth by observations of GPS satellites in common-view. Section 6 introduces the GPS navigation equations and discusses synchronization processes from the point of view of the rotating ECEF frame. Section 7 develops implications of the fact that GPS navigation messages provide satellite ephemerides in the ECEF frame.

2. Local Inertial Frames

Einstein's Principle of Equivalence allows one to discuss frames of reference which are freely falling in the gravitational fields of external bodies. Sufficiently near the origin of such a freely falling frame, the laws of physics are the same as they are in an inertial frame; in particular electromagnetic waves propagate with uniform speed c in all directions when measured with standard rods and atomic clocks. Such freely falling frames are called *locally inertial* frames. For the GPS, it is very useful to introduce such a frame that is non-rotating, with its origin fixed at earth's center, and which falls freely along with the earth in the gravitational fields of the other solar system bodies. This is called an Earth-Centered Inertial (ECI) frame.

Clocks in the GPS are synchronized in the ECI frame, in which self - consistency can be achieved. Thus imagine the underlying ECI frame, unattached to the spin of the earth, but with its origin at the center of the earth. In this non-rotating frame a fictitious set of standard clocks is introduced, available anywhere, all of them synchronized by the Einstein synchronization procedure and running at agreed rates such that synchronization is maintained. These clocks read the coordinate time t. Next one introduces the rotating earth with a set of standard clocks distributed around upon it, possibly roving around. One applies to each of the standard clocks a set of corrections based on the known positions and motions of the clocks. This generates a "coordinate clock time" in the earth-fixed, rotating system. This time is such that at each instant the coordinate clock agrees with a fictitious atomic clock at rest in the local inertial frame, whose position coincides with the earth-based standard clock at that instant. Thus coordinate time is equivalent to time which would be measured by standard clocks at rest in the local inertial frame. [4]

In the ECEF frame used in the GPS, the unit of time is the SI second as realized by the clock ensemble of the U. S. Naval Observatory, and the unit of length is the SI meter. In summary, the reference frame for navigation is the rotating WGS-84 frame, but clocks are synchronized in the underlying hypothetical ECI frame with a unit of time defined by clocks (essentially on the geoid) and a unit of length determined by the defined value of the speed of light, $c = 299792458$ m/s.

3. The GPS

The Global Positioning System can be described in terms of three principal "segments:" a Space Segment, a Control Segment, and a User Segment. The Space Segment consists essentially of 24 satellites carrying atomic clocks. (Spare satellites and spare clocks in satellites exist.) There are four satellites in each of six orbital planes inclined at 55° with respect to earth's equatorial plane, distributed so that from any point on the earth, four or more satellites are almost always above the local horizon. Tied to the clocks are navigation and timing signals that will be discussed below.

The Control Segment is comprised of a number of ground-based monitoring stations which continually gather information from the satellites. These data are sent to a Master Control Station in Colorado Springs, CO, which analyzes the constellation and projects the satellite ephemerides and clock behavior forward for the next few hours. This information is then uploaded into the satellites for retransmission to users.

The User Segment consists of all users who, by receiving signals transmitted from the satellites, are able to determine their position, velocity, and the time on their local clocks.

The timing signals transmitted from each satellite are right circularly polarized. A carrier signal of frequency 1.542 MHz is modulated with a series of phase reversals; these phase reversals carry information bits from the transmitter to the receiver. Such phase reversals are conceptually important because the phase of an electromagnetic wave is a relativistic scalar. The phase reversals correspond to physical points in spacetime at which - for all observers - the electric and magnetic fields vanish.

The navigation message contained in these bit streams include values of parameters from which the receiver can compute the satellite's position *in the rotating ECEF frame*, as a function of time of transmission. Also the GPS time on the satellite clock is indicated by a particular phase reversal in the sequence. A receiver distinguishes the signal from a particular satellite by comparing the bit streams, that are unique to each satellite, with bit streams generated by electronic circuitry within the receiver.

Additional information contained in the messages includes an almanac for the entire satellite constellation, information about satellite vehicle health, and information from which Universal Coordinated Time as maintained by the U. S. Naval Observatory–UTC(USNO)–can be determined.

The GPS is a navigation and timing system that is operated by the United States Department of Defense (DoD), and therefore has a number of aspects to it which are classified. Several organizations monitor GPS signals independently and provide services from which satellite ephemerides and clock behavior can be obtained. Accuracies in the neighborhood of 5-10 cm are not unusual. Carrier phase measurements of the transmitted signals are commonly done to better than a millimeter.

For purposes of the remainder of this article, I shall think of a signal from a GPS satellite as containing within itself information about the position and time of a transmission "event". The position is specified in the rotating ECEF frame. GPS time is time in an underlying local inertial frame. The signal propagates with speed c in a straight line in the ECI frame to the receiver, where it is decoded and its arrival time t_R is compared to the time of transmission t_T. The receiver can then form the so-called pseudoranges

$$\rho = c(t_R - t_T). \tag{1.1}$$

A receiver continually forms such pseudoranges for each satellite being observed. A signal can be imagined abstractly as propagating with speed c from transmitter to receiver in a straight line in the ECI frame, with position and

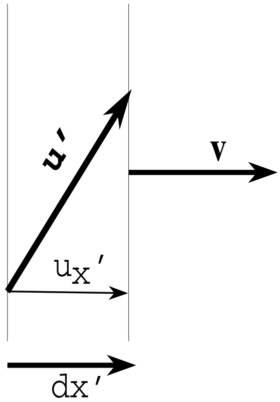

Figure 1.1. Synchronization by transmission of a signal

time of the transmission event “known" by the receiver. Possible clock biases in the receiver prevent the GPS time of the reception event from being known a priori.

4. Relativity of Simultaneity

To establish the connection between the Sagnac effect and the relativity of simultaneity, consider an observer moving with velocity $\mathbf{v}$ in the x direction relative to an inertial frame such as the ECI frame. To be specific, one can imagine measurements of unprimed quantities such as $\mathbf{v}$ and signal velocity $\mathbf{u}$ to be performed in the ECI frame, while primed quantities such as $\mathbf{u}'$ are measured in the rest frame of the moving observer. Referring to Figure 1.1, let a signal be travelling with speed components (u'_x, u'_y) (measured in the moving observer's frame). The vertical lines represent planes at x' and $x'+dx'$. The signal travels a distance dx' in the x direction and the moving observer desires to use this signal to transfer time from clocks in the plane at x' to clocks in the plane at $x' + dx'$. Here I am neglecting higher-order terms in the velocity so $dx = dx'$, there being no appreciable Lorentz contraction. Let the components of signal speed in the ECI frame be (u_x, u_y). The well-known Lorentz transformations for speed include the expression

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \,. \tag{1.2}$$

The terms in the denominator of this expression arise from the time-component of the ordinary Lorentz transformation. In particular the second term in the denominator arises from the relativity of simultaneity, a consequence of the constancy of the speed of light. We wish to compare the propagation time of this signal, measured by the moving observer, with the propagation time measured in the ECI frame. The analysis is performed in the ECI frame.

If the moving observer moves a distance vdt in time dt, then the total distance travelled by the signal in the x-direction is $u_x dt$, which is comprised of two contributions: the distance dx, plus the distance vdt required to catch up to the plane at $x' + dx'$. Thus

$$u_x dt = dx + vdt, \tag{1.3}$$

and therefore the time required is

$$dt = \frac{dx}{u_x - v}\,. \tag{1.4}$$

But from the expression for the Lorentz transformation of speed, keeping only terms of linear order in v,

$$u_x - v \approx \frac{u'_x}{1 + \frac{u'_x v}{c^2}}\,. \tag{1.5}$$

and therefore

$$dt = \frac{dx}{u'_x} + \frac{vdx}{c^2}\,. \tag{1.6}$$

The first term in this result is just the time required, in the moving frame, for the signal to travel from the x' plane to the $x' + dx'$ plane. If the moving observer ignores the motion relative to the ECI frame, this would be the time used to synchronize clocks in the $x' + dx'$ plane to clocks in the x' plane. The second term is the additional time required to *synchronize the clocks in the ECI frame*. Note that in this second term, the value of u'_x has cancelled out, so that the value of the signal speed is irrelevant. The signal could be a light signal travelling in a fiber of index of refraction n, or it could even be an acoustic signal. The signal speed could even be variable, the last term would not be affected.

Consider for example an optical fiber loop of length L and index of refraction n which by means of a system of pulleys is made to move with speed v around in a closed circuit, relative to an inertial frame. The circuit itself could be of any shape, such as a figure 8 or an oval. In such a case it is not useful to speak of rotation, although Eq. 1.6 applies to the rotational case as well. Eq. 1.6 applies to each infinitesimal segment of the moving loop, since one can imagine a sequence of moving reference frames each of which is instantaneously at rest with respect to the moving fiber loop and in which Eq. 1.6 is valid. If a signal travels around the loop in a direction parallel to the velocity, then from Eq. 1.6, the total time required for the signal to make one circuit is

$$\Delta t_+ = \oint dt = \oint \frac{dx}{u'_x} + \frac{vL}{c^2}\,, \tag{1.7}$$

and the time required for the signal to make one circuit in the direction opposite to the velocity is

$$\Delta t_- = \oint \frac{dx}{u'_x} - \frac{vL}{c^2}\,, \tag{1.8}$$

The difference is

$$\Delta t = \Delta t_+ - \Delta t_- = \frac{2vL}{c^2}\,, \tag{1.9}$$

and for two counterpropagating monochromatic beams this can be converted into an observable interference fringe shift. If the beams are recombined in the ECI frame where they have angular frequency ω, then the phase difference will be

$$\Delta\phi = \omega\Delta t\,. \tag{1.10}$$

The Sagnac effect in a moving fiber loop is independent of the fiber's index of refraction or of the shape of the loop. This has been confirmed in recent experiments.[5]

For example for electrons of energy $E = \hbar\omega$, the phase difference will be

$$\Delta\phi = \frac{2EvL}{\hbar c^2}\,. \tag{1.11}$$

Interference experiments with electrons have been reported in reference [6], which also has a comprehensive discussion of the many different points of view of the Sagnac effect that can be taken.

In the GPS, a decision was made to synchronize GPS clocks in the ECI reference frame. The above discussion demonstrates that observers on earth, in the ECEF frame, must apply a “Sagnac" correction (the second term in Eq. 1.6) to their synchronization processes in order to synchronize their clocks to GPS time.

The correction can be generalized slightly by noting that the distance dx is in the same direction as the relative velocity $\mathbf{v}$. If $d\mathbf{r}$ is the vector increment of path in the direction of signal propagation, then the Sagnac correction term can be written

$$dt_{Sagnac} = \frac{\mathbf{v}\cdot d\mathbf{r}}{c^2}\,. \tag{1.12}$$

For applications in the GPS, it is useful to describe this correction term another way, in terms of accounting for motion of the receiver during propagation of signals from transmitters to receivers. Henceforth only signals propagating with speed c will be considered. This assumption also applies to measurements made locally by the moving observer in the ECEF frame, since at each instant the measurements of distance and time intervals are the same as they would be in an inertial frame which instantaneously coincides with the observer in the ECEF frame and which moves with the instantaneous velocity $\mathbf{v}$ of the ECEF observer. In Eq. 1.3, the velocity v is present to account for the fact that the signal must catch up to the position at $x' + dx'$ which is moving with velocity $\mathbf{v}$, and to first order in the small quantity v/c leads directly to the Sagnac correction term in Eqs. 1.3 and 1.12. The Sagnac correction can thus be interpreted as an effect which arises in the ECEF frame when one accounts for motion of the receiver during propagation of the electromagnetic signal with speed c.

5. Time Transfer with the GPS

In the GPS navigation is accomplished by means of signals from four or more satellites, whose arrival times are measured at the location of the receiver. I now consider one such signal in space, transmitted from satellite position $\mathbf{r}_T$ at GPS time t_T. Let the receiver position at GPS time t_T be $\mathbf{r}_R$, and let the receiver have velocity $\mathbf{v}$ in the ECI frame. Let the signal (considered abstractly as a pulse) arrive at the receiver at time t_R. During the time interval $\Delta t = t_R - t_T$, the displacement of the receiver is $\mathbf{v}\Delta t$. Since the signal travels with speed c, the constancy of the speed of light c implies that

$$c^2(\Delta t)^2 = (\mathbf{r}_R + \mathbf{v}\Delta t - \mathbf{r}_T)^2 \,. \tag{1.13}$$

To simplify the equation, I define

$$\mathbf{R} = \mathbf{r}_R - \mathbf{r}_T \,. \tag{1.14}$$

Then to leading order in v,

$$c^2(\Delta t)^2 = (\mathbf{R} + \mathbf{v}\Delta t)^2 \approx R^2 + 2\mathbf{v}\cdot\mathbf{R}\Delta t \,. \tag{1.15}$$

Taking the square root of both sides of Eq. (1.15) and again expanding to leading order in v gives

$$c\Delta t = R + \frac{\mathbf{v}\cdot\mathbf{R}\Delta t}{R} \,. \tag{1.16}$$

This equation can be solved approximately for Δt to give

$$\Delta t = \frac{R}{c} + \frac{\mathbf{v}\cdot\mathbf{R}}{c^2} \,. \tag{1.17}$$

The second term in Eq. 1.17 is the Sagnac correction term, which arises when one accounts for motion of the receiver while the signal propagates from transmitter to receiver. This is illustrated in Figure 1.2.

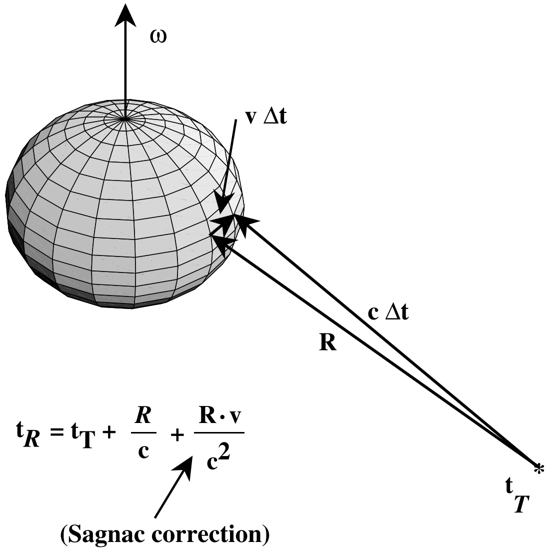

Figure 1.2. Sagnac correction arising from motion of the ECEF observer.

Suppose that the receiver is fixed to the surface of the earth, at a well-surveyed location so that the receiver position $\mathbf{r}_R$ is well known at all times. The velocity of the receiver will be just that due to rotation of the earth with angular velocity $\omega_{\mathbf{E}}$, so

$$\mathbf{v} = \omega_{\mathbf{E}} \times \mathbf{r}_R\,, \tag{1.18}$$

We take $\mathbf{r}_R$ to be the vector from earth's center to the receiver position. Then the Sagnac correction term can be rewritten as

$$\Delta t_{Sagnac} = \frac{\omega_{\mathbf{E}} \times \mathbf{r}_R \cdot \mathbf{R}}{c^2} = \frac{2\omega_{\mathbf{E}}}{c^2} \cdot \left(\frac{1}{2}\mathbf{r}_R \times \mathbf{R}\right)\,. \tag{1.19}$$

The quantity $2\omega_E/c^2$ has the value

$$\frac{2\omega_E}{c^2} = 1.6227 \times 10^{-21}\ \mathrm{s/m^2} = 1.6227 \times 10^{-6}\ \mathrm{ns/km^2}\,. \tag{1.20}$$

The last factor in Eq. 1.19 can be interpreted as a vector area $\mathbf{A}$:

$$\mathbf{A} = \frac{1}{2}\mathbf{r}_R \times \mathbf{R}\,. \tag{1.21}$$

The only component of $\mathbf{A}$ which contributes to the Sagnac correction is along earth's angular velocity vector $\omega_{\mathbf{E}}$, because of the dot product that appears in the expression. This component is the projection of the area onto a plane normal to earth's angular velocity vector. This leads to a simple description of the Sagnac correction: Δt_{Sagnac} is $2\omega_E/c^2$ time the area swept out by the electromagnetic pulse as it travels from the GPS transmitter to the receiver, projected onto earth's equatorial plane. This is depicted in Figure 1.3, in which the receiver is on earth's surface at the tip of the path vector $\mathbf{R}$.

In the early 1980s clocks in remotely situated timing laboratories were being compared by using GPS satellites in "common view", that is when one GPS satellite is observed at the same time by more than one timing laboratory. In one experiment[7] signals from GPS satellites were utilized in simultaneous common view between three pairs of earth timing centers to accomplish a circumnavigation of the globe. The centers were the National Bureau of Standards (now the National Institute of Standards and Technology) in Boulder, Colorado; Physikalisch-Technische Bundesanstalt in Braunschweig, West Germany; and Tokyo Astronomical Observatory. A typical geometrical configuration of ground stations and satellites, with the corresponding projected areas, is illustrated in Figure 1.4. The size of the Sagnac effect calculated varies from about 240 ns to 350 ns depending on the location of the satellites at a particular moment. Sufficient data were collected to perform 90 independent circumnavigations. As Figure 1.4 shows, when a satellite is eastward of one timing center and westward of another, one of the Sagnac corrections is positive and the other is negative, so when computing the difference of times between the two terrestrial clocks, the Sagnac corrections actually add up in a positive sense.

The mean value of the residuals over 90 days of observation was 5 ns, less than 2 percent of the magnitude of the calculated total Sagnac correction. A significant part of these residuals can be attributed to random noise processes in the clocks.

Sagnac corrections of the form of Eq. 1.19 are routinely used in comparisons between distant time standards laboratories on earth.

6. GPS Navigation Equations and the ECEF Frame

The navigation problem in GPS is to determine the position of the receiver in the ECEF reference frame. A by-product of this process is the accurate determination of GPS time at the receiver. In general neither the position nor the time is known, so the assumptions used in previous sections regarding the Sagnac effect are of little use. The principles of position determination and time transfer in the GPS can be very simply stated. Let there be four synchronized atomic clocks which transmit sharply defined pulses from the positions $\mathbf{r}_j$ at times t_j, with $j = 1, 2, 3, 4$ an index labelling the different transmission events. Suppose that these four signals are received at position $\mathbf{r}$ at one and the same instant t. This is called "time-tagging at the receiver", meaning that observations of the various signals are made simultaneously at the receiver at

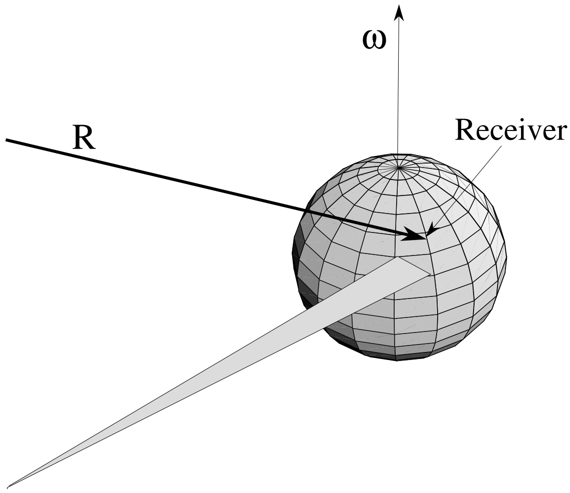

Figure 1.3. Sagnac correction arising from motion of the ECEF observer.

time t. Then from the principle of the constancy of the speed of light,

$$c^2(t - t_j)^2 = |\mathbf{r} - \mathbf{r}_j|^2, \quad j = 1, 2, 3, 4. \tag{1.22}$$

These four equations can be solved for the unknown space-time coordinates of the reception event, $(t, \mathbf{r})$. The solution will provide the position of the receiver at the time of the simultaneous reception events, t. No knowledge of the receiver velocity is needed. The Sagnac effect becomes irrelevant. At most one can say that because the solution gives the final position and time of the reception event, the Sagnac effect has been automatically accounted for.

However there are complications from the fact that the navigation equations, Eqs. 1.22, are valid in the ECI frame, whereas users almost always want to know their position in the ECEF frame. For discussions of relativity, the particular choice of ECEF frame is immaterial. Also, the fact the the earth truly rotates about an axis slightly different from the WGS-84 axis, with a variable rotation rate, has little consequence for relativity and I shall not go into this here.

It should be emphasized strongly that the transmitted navigation messages provide the user only with a function from which the satellite position can be calculated *in the ECEF* as a function of the transmission time. Usually the satellite transmission times t_j are unequal, so the coordinate system in which the satellite positions are specified changes orientation from one measurement

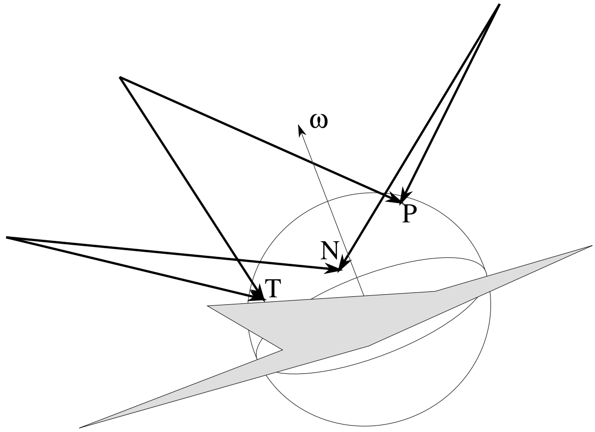

Figure 1.4. Common-view signals from three satellites provide an Around-the-World Sagnac experiment.

to the next. Therefore to implement Eqs. (1.22), the receiver must generally perform a different rotation for each measurement made, into some common inertial frame, so that Eqs. (1.22) apply. After solving the propagation delay equations, a final rotation must then be performed into the ECEF to determine the receiver's position. This can become exceedingly complicated and confusing. I shall discuss this in a later section.

The purpose of the present discussion is to examine first-order relativistic effects from the point of view of the ECEF frame. Consider the simplest instance of a transformation from an inertial frame, in which the space-time is Minkowskian, to a rotating frame of reference. Thus ignoring gravitational potentials, the metric in an inertial frame in cylindrical coordinates is

$$-ds^2 = -(c\,dt)^2 + dr^2 + r^2 d\phi^2 + dz^2\,, \qquad (1.23)$$

and the transformation to a coordinate system $\{t', r', \phi', z'\}$ rotating at the uniform angular rate ω_E is

$$t = t', \quad r = r', \quad \phi = \phi' + \omega_E t', \quad z = z'\,. \qquad (1.24)$$

This results in the following well-known metric (Langevin metric) in the rotating frame:

$$-ds^2 = -\left(1 - \frac{\omega_E^2 r'^2}{c^2}\right)(cdt')^2 + 2\omega_E r'^2 d\phi' dt' + (d\sigma')^2\,, \qquad (1.25)$$

where the abbreviated expression $(d\sigma')^2 = (dr')^2 + (r'd\phi')^2 + (dz')^2$ for the square of the coordinate distance has been used.

The time transformation $t = t'$ in Eqs. (1.24) is a result of the convention to determine time t' in the rotating frame in terms of time in the underlying ECI frame.

Now consider a process in which observers in the rotating frame attempt to use Einstein synchronization (that is, the principle of the constancy of the speed of light) to establish a network of synchronized clocks. Light travels along a null worldline so I may set $ds^2 = 0$ in Eq. (1.25). Also, it is sufficient for this discussion to keep only terms of first order in the small parameter $\omega_E r'/c$. Then

$$(cdt')^2 - \frac{2\omega_E r'^2 d\phi'(cdt')}{c} - (d\sigma')^2 = 0\,, \tag{1.26}$$

and solving for (cdt'),

$$cdt' = d\sigma' + \frac{\omega_E r'^2 d\phi'}{c}\,. \tag{1.27}$$

The quantity $r'^2 d\phi'/2$ is just the infinitesimal area dA'_z in the rotating coordinate system swept out by a vector from the rotation axis to the light pulse, and projected onto a plane parallel to the equatorial plane. Thus the total time required for light to traverse some path is

$$\int_{\rm path} dt' = \int_{\rm path} \frac{d\sigma'}{c} + \frac{2\omega_E}{c^2}\int_{\rm path} dA'_z. \qquad [\,\text{light}\,] \tag{1.28}$$

Observers fixed on the earth, who were unaware of earth rotation, would use just $\int d\sigma'/c$ for synchronizing their clock network. Observers at rest in the underlying inertial frame would say that this leads to significant path-dependent inconsistencies, which are proportional to the projected area encompassed by the path. Consider for example a synchronization process which follows earth's equator in the eastwards direction. For earth, $2\omega_E/c^2 = 1.6227 \times 10^{-21}$ s/m^2 and the equatorial radius is $a_1 = 6,378,137$ m, so the area is $\pi a_1^2 = 1.27802 \times 10^{14}$ m^2 . Thus the last term in Eq. (1.28) is

$$\frac{2\omega_E}{c^2}\int_{\rm path} dA'_z = 207.4 \text{ ns}. \tag{1.29}$$

From the underlying inertial frame, this can be regarded as the additional travel time required by light to catch up to the moving reference point. Simple-minded use of Einstein synchronization in the rotating frame gives only $\int d\sigma'/c$, and thus leads to a significant error. Traversing the equator once eastward, the last clock in the synchronization path would lag the first clock by 207.4 ns.

Traversing the equator once westward, the last clock in the synchronization path would lead the first clock by 207.4 ns.

In an inertial frame a portable clock can be used to disseminate time. The clock must be moved so slowly that changes in the moving clock's rate due to time dilation, relative to a reference clock at rest on earth's surface, are extremely small. On the other hand, observers in a rotating frame who attempt this find that the proper time elapsed on the portable clock is affected by earth's rotation rate. Factoring $(dt')^2$ out of the right side of Eq. (1.25), the proper time increment $d\tau$ on the moving clock is given by

$$(d\tau)^2 = (ds/c)^2 = dt'^2 \left[1 - \left(\frac{\omega_E r'}{c}\right)^2 - \frac{2\omega_E r'^2 d\phi'}{c^2 dt'} - \left(\frac{d\sigma'}{cdt'}\right)^2\right]. \tag{1.30}$$

For a slowly moving clock $(d\sigma'/cdt')^2 << 1$ so the last term in brackets in Eq. (1.30) can be neglected. Also, keeping only first order terms in the small quantity $\omega_E r'/c$,

$$d\tau = dt' - \frac{\omega_E r'^2 d\phi'}{c^2} \tag{1.31}$$

which leads to

$$\int_{\text{path}} dt' = \int_{\text{path}} d\tau + \frac{2\omega_e}{c^2}\int_{\text{path}} dA'_z. \qquad [\text{portable clock}] \tag{1.32}$$

This should be compared with Eq. (1.28). Path-dependent discrepancies in the rotating frame are thus inescapable whether one uses light or portable clocks to disseminate time, while synchronization in the underlying inertial frame using either process is self-consistent.

Eqs. 1.28 and 1.32 can be reinterpreted as a means of realizing coordinate time $t' = t$ in the rotating frame, if after performing a synchronization process appropriate corrections of the form $+2\omega_E \int_{path} dA'_z/c^2$ are applied. It is remarkable how many different ways this can be viewed. The different ways discussed so far in this article include the fact that from the inertial frame it appears that the reference clock from which the synchronization process starts is moving, requiring light to traverse a different path than it appears to traverse in the rotating frame. The Sagnac effect can also be regarded as arising from the relativity of simultaneity in a Lorentz transformation to a sequence of local inertial frames co-moving with points on the rotating earth, or as the difference between proper times of a slowly moving portable clock and a Master reference clock fixed on earth's surface.

This was recognized in the early 1980s by the Consultative Committee for the Definition of the Second and the International Radio Consultative Committee who formally adopted procedures incorporating such corrections for the comparison of time standards located far apart on earth's surface. For the GPS it means that synchronization of the entire system of ground-based and orbiting atomic clocks is performed in the local inertial frame, or ECI coordinate system.

7. Sagnac-like effects due to rotation of the ECEF frame

By design, the ephemerides (positions) of the GPS satellites are broadcast in such a way that the receiver can compute their positions at the instant of transmission *in the rotating WGS-84 reference frame*. For time-tagging at the receiver, the propagation delays from different satellites can vary from about 67 ms to 86 ms. During this approximately 19 ms transmission time variation, the ECEF reference frame can rotate more than a microradian and the positions of the satellites due to this rotation alone can vary by over 30 meters while the satellites move in inertial space by as much as 60 meters. If this is not carefully accounted for, unacceptable navigation errors can occur.

It would lead to serious error to assert Eqs. 1.22 were valid in the ECEF frame. What the receiver must do is rotate the positions of each of the satellites, that have been computed in the rotating frame, into some chosen ECI frame. Then Eqs. 1.22 are valid and can be solved in the ECI frame. The resulting position found in the ECI frame is finally rotated into the WGS-84 frame and used for navigation.

To illustrate that these rotations give rise to Sagnac-like effects, suppose the chosen ECI frame instantaneously coincides with the WGS-84 frame at the instant of arrival of the earliest of the four signals. I denote the GPS time of arrival of this particular signal by t_1, and the position of this particular satellite at this time as $\mathbf{r}_1$. Let the time intervals between the arrival of this signal and the other three signals be denoted by

$$\Delta t_i = t_i - t_1 \,, \quad i = 1, 2, 3, 4 \tag{1.33}$$

where for simplicity I have taken $\Delta t_1 = 0$. During the time interval Δt_i the ECEF frame has rotated the amount $\omega_E \Delta t_i$. An active rotation of the satellite position $\mathbf{r}_i(ECEF)$ by the amount $+\omega_E \Delta t_i$ is necessary in order to express the position of satellite i in the inertial frame in which the position $\mathbf{r}_1$ is expressed. This rotation operation can be expressed as

$$\mathbf{r}_i(ECI) = \mathbf{r}_i(ECEF) + \boldsymbol{\omega}_E \times \mathbf{r}_i(ECEF)\Delta t_i \,. \tag{1.34}$$

The navigation equations then become

$$c^2(t - t_i)^2 = |\mathbf{r} - \mathbf{r}_i(ECEF) - \boldsymbol{\omega}_E \times \mathbf{r}_i(ECEF)\Delta t_i|^2 \tag{1.35}$$

and if I put $\Delta t = t - t_1$ (no subscript on t) and $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i(ECEF)$ I obtain

$$c^2(\Delta t - \Delta t_i)^2 = |\mathbf{R}_i - \boldsymbol{\omega}_E \times \mathbf{r}_i(ECEF)\Delta t_i|^2 \tag{1.36}$$

Eqs. 1.36 have within them the four unknowns $(\Delta t, \mathbf{r})$. The position solution for $\mathbf{r}$ will be in the ECI frame chosen for computation. After finding this position, the result must then be rotated into the ECEF frame for navigation. Since the ECEF frame rotates an amount $\omega_E \Delta t$ during the time interval Δt, the final solution for the position in the ECEF frame will be

$$\mathbf{r}(ECEF) = \mathbf{r} - \omega_E \times \mathbf{r}\Delta t\,. \tag{1.37}$$

The size of the correction term in this last equation can easily be estimated, since $\Delta t \approx .015$ s and $r \approx 6.4 \times 10^6$ m. A typical value will be about 9 meters. Eq. 1.36 can be solved approximately for Δt by expanding the square on the right side, keeping only linear terms in ω_E, and then taking a square root, similar to the approximations made in deriving Eq. 1.17. The result is

$$\Delta t = \Delta t_i + \frac{R_i}{c} + \frac{\omega_E \times \mathbf{r}_i(ECEF) \cdot \mathbf{R}_i}{cR_i} \Delta t_i\,. \tag{1.38}$$

The last term in the above equation is a Sagnac-like correction. I can estimate its magnitude by substituting in an approximate expression for Δt_i:

$$\Delta t_i \approx \frac{R_i}{c} - \frac{R_1}{c} \tag{1.39}$$

So the correction term becomes, after interchanging dot and cross products,

$$\frac{\omega_E \cdot \mathbf{r}_i(ECEF) \times \mathbf{R}_i}{c^2} \left(1 - R_1/R_i\right)\,. \tag{1.40}$$

Is this really a Sagnac correction? It is linear in the rotational velocity, the coefficient can be interpreted in terms of an area, and it is relativistic (there is a factor $1/c^2$).

In the case of time-tagging at the transmitters, signals are chosen for processing which leave the transmitters at some chosen time t_T. Then the broadcast ephemerides will all be calculated by the receiver in one and the same ECEF frame. It would then be natural to choose for application of the navigation equations (Eqs. 1.22) an inertial frame which coincides with this ECEF frame at the instant t_T of GPS time. But then the signals do not arrive simultaneously at the receiver, and the receiver motion during the interval between arrival of the first and last signals must be accounted for.

To illustrate the size of the Sagnac-like effects that occur in this situation, let $\mathbf{r}$ denote the receiver position at transmission time t_T, and let $\mathbf{r}_i$ denote the transmitter position at time t_T. Imagine these positions to be expressed in an inertial frame which coincides instantaneously with the ECEF frame at time t_T. Let t_i denote the arrival time at the receiver, of the signal from the ith satellite. The receiver position at time t_i will be modified by earth rotation and will be

$$\mathbf{r} + \omega_E \times \mathbf{r}(t_i - t_T)\,. \tag{1.41}$$

The navigation equations in this inertial reference frame will be

$$c^2(t_i - t_T)^2 = |\mathbf{r} + \omega_E \times \mathbf{r}(t_i - t_T) - \mathbf{r}_i|^2 \tag{1.42}$$

Because of the similarity of this equation to Eq. 1.13 it is clear that Sagnac-like corrections will enter solution of the equations. The times t_i are however known only to within an added constant, because of a possible error or systematic bias in the receiver's clock. If the arrival times actually measured in the receiver are t'_i, then

$$t_i = t'_i + b\,. \tag{1.43}$$

where b is the receiver clock bias then the navigation equations become

$$c^2(t'_i + b - t_T)^2 = |\mathbf{r} + \omega_E \times \mathbf{r}(t'_i + b - t_T) - \mathbf{r}_i|^2 \tag{1.44}$$

and the unknowns are $(b, \mathbf{r})$. Obviously there are many other ways of formulating the problem of accounting for receiver motion. A technical note[8] discusses these issues in more detail, with numerical examples.

8. Summary

In the GPS, the Sagnac effect arises because the primary reference frame of interest for navigation is the rotating Earth-Centered, Earth-Fixed frame, whereas the speed of light is constant in a locally inertial frame, the Earth-Centered Inertial frame. Additional Sagnac-like effects arise because the satellite ephemerides are broadcast in a form allowing the receiver to compute satellite positions in the ECEF frame. In the case of time-tagging of observations at the receiver, it is necessary to rotate the satellite positions into a common ECI reference frame in order apply the principle of the constancy of c. In the rotating frame of reference the effect appears to arise from a Coriolis-like term in the fundamental scalar invariant. Whether synchronization procedures are performed by using electromagnetic signals or slowly moving portable clocks, to leading order the same Sagnac effect arises. The effect is of significant magnitude and must be taken into account for accurate navigation. It is also necessary to apply Sagnac corrections when comparing remote clocks on earth's surface.

References

[1] *World Geodetic System 1984*, (National Imagery and Mapping Agency, 1994) Report No. 8350.2, Third Edition, Amendment 1. NIMA Stock No. DMATR83502WGS84, MSN 7643-01-402-0347.

[2] Malys, S., and Slater, "Maintenance and Enhancement of the World Geodetic System 1984", in *Proceedings of the 7th International Technical Meeting of the Satellite Division of the Institute of Navigation, Sept.*

20-23, Salt Lake City, UT (ION GPS-94), 17-24. The Institute of Navigation (1994).

[3] Landau, L., and E. Lifshitz, *The Classical Theory of Fields*, 4th ed., Pergamon, New York p. 254 (1997).

[4] Ashby, N., and Allan, D.W., "Practical Implications of Relativity for a Global Coordinate Time Scale", *Radio Science* **14**, 649-669 (1979).

[5] Wang, R., Y. Zheng, A. Yao, and D. Langley, "Modified Sagnac experiment for measuring travel-time difference between counter-propagating light beams in a uniformly moving fiber", *Phys. Letts*. **A 312**, pp 7-10, (2003).

[6] Hasselbach, F., and M. Nicklaus, *Phys. Rev.* **A48**, pp 143-151, (1993).

[7] Allan, D. W., M. Weiss, and N. Ashby, "Around-the-World Relativistic Sagnac Experiment", *Science* **228**, pp 69-70 (5 April 1985).

[8] Ashby, N., and M. Weiss, "Global Positioning Receivers and Relativity", NIST Technical Note 1385, U. S. Government Printing Office, Washington, D.C., March (1999).

Chapter 2

SPACE, TIME AND COORDINATES IN A ROTATING WORLD

Dennis Dieks
Institute for the History and Foundations of Science
Utrecht University, P.O.Box 80.000
3508 TA Utrecht, The Netherlands
d.g.b.j.dieks@phys.uu.nl

Abstract The peculiarities of rotating frames of reference played an important role in the genesis of general relativity. Considering them, Einstein became convinced that coordinates have a different status in the general theory of relativity than in the special theory. This line of thinking was confused, however. To clarify the situation we investigate the relation between coordinates and the results of space-time measurements in rotating reference frames. We argue that the difference between rotating systems (or accelerating systems in general) and inertial systems does not lie in a different status of the coordinates (which are conventional in all cases), but rather in different global chronogeometric properties of the various reference frames. In the course of our discussion we comment on a number of related issues, such as the question of whether a consideration of the behavior of rods and clocks is indispensable for the foundation of kinematics, the influence of acceleration on the behavior of measuring devices, the conventionality of simultaneity, and the Ehrenfest paradox.

1. Introduction

In his Autobiographical Notes [1], Einstein relates how important Machian empiricist ideas were for his discovery of a theory that could reconcile the idea that all inertial frames are equivalent with the principle that the velocity of light has a fixed value that is independent of the velocity of the emitting source. It was essential, he states, to realize what the meaning of *coordinates* in physics is: they are nothing but the outcomes of length and time measurements by means of rods, clocks and light signals. This idea led Einstein to his famous critique of the classical notion of simultaneity, one of the cornerstones of the special theory of relativity.

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 29-42.

It soon turned out, however, that the special theory of relativity was not able to accommodate gravitation, and the principle of equivalence, in a natural way. Einstein fully recognized this problem in 1908, but it took him another seven years before he succeeded in constructing the general theory. As he explains in his Autobiographical Notes, the main reason for the slowness of his progress in this period was the difficulty of *abandoning* again, in the context of the general theory, the idea that coordinates should possess immediate metrical meaning.

From a systematical (as opposed to a historical or psychological) point of view this emphasis on the different meaning of coordinates, in the context of the two theories, is very odd. For the practice of physics before, during and after Einstein's days, even if governed by the severest empiricist norms, does not at all indicate that coordinates should possess a metrical significance, relating to the indications of rods and clocks. Think, for example, of the way coordinates are used in observational astronomy: the essential thing is that the coordinates are assigned to celestial objects in an objective and reproducible way; how the coordinates relate to distances is a matter to be found out subsequently. Coordinates are even routinely attributed to regions of the universe in which rods and clocks could not possibly exist. This is obviously unobjectionable from an empiricist point of view, as long as the method by which the coordinates are assigned is operationally specified. So, even within the framework of special relativity general coordinate systems that do not reflect the indications of rods and clocks are entirely permissible.

What finally led Einstein to abandon his special relativistic analysis of the meaning of coordinates, he tells us, was the lack of metrical significance of coordinates in accelerating frames of reference; the consideration of coordinates on a rotating disc played an important role in reaching this conclusion [2]. But, as we will see, there is confusion here: the metrical significance of coordinates in accelerating frames can be determined completely through application of the principles of *special* relativity, so there can be no need to revise the meaning of the notion of coordinates, or to invoke a new epistemological analysis.

As it turns out, the difference between inertial and non-inertial frames of reference, and between special and general relativity, is not in the epistemological status of the coordinates. Rather, the difference is that chronogeometric characteristics become globally different. This is a physical rather than a philosophical difference, and has nothing to do with the meaning or permissibility of coordinate systems.

The rotating frame of reference nicely illustrates these points. There is no problem in defining operationally meaningful coordinates in a rotating (and therefore accelerating) frame. Furthermore, relating these coordinates to distances and time intervals, and the behavior of moving objects, can be done by the means provided by special relativity. However, the spatial geometry becomes non-Euclidean, and local Einstein synchrony does not lead to a global notion of time. These latter features constitute the essential differences from the situation in an inertial frame.

In the course of our discussion we will have occasion to comment on a number of related issues, such as the status of rods and clocks, the behavior of accelerating measuring devices, the conventionality of simultaneity, and the Ehrenfest paradox.

2. The rotating frame of reference

Let us start from Minkowski space-time, coordinatized by inertial coordinates r, φ, z and t: r and φ are polar coordinates in a plane, z is a Cartesian coordinate orthogonal to this plane, and t is the standard time coordinate. It so happens that r, z, and t can be thought of as representing the indications of rods and clocks, but that is not important for their role as coordinates, which is just to pinpoint events unequivocally. The choice of coordinates is conventional and pragmatic. In this case we choose polar coordinates because we are going to describe a system that possesses axial symmetry: polar coordinates simplify the description.

Once we have laid down coordinates, the metrical aspects should be introduced via further stipulations. This is ordinarily done through the introduction of the 'line element' $ds^2 = c^2dt^2 - dr^2 - r^2d\varphi^2 - dz^2$, plus a specification of what this mathematical expression represents physically. The traditional approach is to invoke standard rods and clocks: ds/c is the time measured by a standard clock whose r, φ and z coordinates are constant. Furthermore, $\sqrt{-ds^2}$ is the length of a rod with a stationary position in the coordinates and with constant coordinates and differences dr, $d\varphi$, dz between its endpoints, taken at one instant according to standard simultaneity ($dt = 0$). However, it would be a mistake to think that rods and clocks are indispensable to relate the coordinates to metrical concepts. In section 4 below we will discuss an approach that does not make use of rods and material clocks.

We now introduce alternative coordinates for the events in this Minkowski world: $t' = t$, $r' = r$, $\varphi' = \varphi - \omega t$ and $z' = z$, with ω a constant. Since rest in the new coordinates obviously means uniform rotation with respect to the old frame, we call the frame of reference defined by these new coordinates the *rotating frame of reference*.

It is clear that if operational methods are at hand to fix the old coordinates, the same methods can be used to assign values to the new coordinates (we assume ω to be known). So from an empiricist or operational point of view the new coordinates are impeccable. However, from the special theory of relativity we know that material bodies at rest in the new coordinates may not exist (ωr may be greater than c, the velocity of light). It is true, therefore, that the new coordinates will not always have a direct interpretation in terms of co-moving bodies–but this is something to be distinguished sharply from the more general question of whether they have adequate empirical significance at all.

Substitution of the rotating coordinates into the expression for the line element yields $ds^2 = (c^2 - r'^2\omega^2)dt'^2 - dr'^2 - r'^2 d\varphi'^2 - dz'^2 - 2\omega r'^2 d\varphi' dt'$. As we already mentioned, it is a basic principle of the special theory of relativity that the line element supplies all information about the physics of the situation, as described in the given coordinates. It was also mentioned above that the traditional link between ds and physical concepts makes use of clocks and measuring rods. However, there is another and more fundamental physical interpretation available that only makes use of the basic laws of motion: as long as no disturbing forces act, point particles follow time-like geodesics and light follows null-geodesics in the metric defined by ds^2. The relation between these dynamical aspects (how particles and light move) and the metrical aspects (rods and clocks) will be the subject of comments in section 4.

3. Rods and clocks

Let us for the moment stay with the physical interpretation of ds in terms of measurements performed with rods and clocks. Concerning time, the coordinating principle is that ds/c represents proper time, measured by a clock whose world line connects the events between which ds is calculated. This principle entails that a clock at rest in the rotating frame will indicate the proper time

$$ds/c = \sqrt{(1 - r'^2\omega^2/c^2)}dt'. \tag{2.1}$$

Because $t' = t$ and t has the physical meaning of the time indicated by a clock at rest in the old frame, this implies that clocks at rest in the rotating frame are slow compared to clocks in the original ("laboratory") frame.

With regard to spatial distances, the interpretative principle is that $\sqrt{-ds^2}$ gives the length of an infinitesimal rod whose endpoints are simultaneous according to standard simultaneity in the rod's rest frame ([3], p.187). (A rod is a three-dimensional object, so we need a stipulation about the instants at which its endpoints should be considered in order to get a four-dimensional interval for which ds can be calculated.) When we apply this rule to rods that are at rest in the rotating frame of reference, we encounter the complication that $dt' = 0$ does not automatically correspond to standard simultaneity in the rotating frame. The definition of standard synchrony of two (infinitesimally near) clocks A and B is that a light signal sent from A to B and immediately reflected to A, reaches B when B indicates a time that is halfway between the instants of emission and reception, respectively, as measured by A. Suppose that A and B, both at rest in the rotating frame, have positions with coordinate differences dr, $d\varphi$ and dz–from now on we drop the primes of the rotating coordinates. A light signal between A and B follows a null-geodesic:

$$ds^2 = (c^2 - r^2\omega^2)dt^2 - dr^2 - r^2 d\varphi^2 - dz^2 - 2\omega r^2 d\varphi dt = 0. \quad (2.2)$$

This equation gives the following solutions for dt when it is applied to the signals from A to B and back, respectively:

$$dt_{1,2} = \frac{\pm\omega r^2 d\varphi + \sqrt{(c^2 - \omega^2 r^2)(dz^2 + dr^2) + c^2 r^2 d\varphi^2}}{c^2 - \omega^2 r^2}. \quad (2.3)$$

If t_0 is the time coordinate of the emission event at A, the event at A with time coordinate $t_0 + 1/2(dt_1 + dt_2)$ is standard-simultaneous with the event at B with time coordinate $t_0 + dt_1$. It follows that standard synchrony between infinitesimally close events corresponds to the following difference in t-coordinate:

$$dt = (t_0 + dt_1) - (t_0 + 1/2dt_1 + 1/2dt_2) = (\omega r^2 d\varphi)/(c^2 - \omega^2 r^2). \quad (2.4)$$

As was to be expected, it is only for events that differ in their φ-coordinates that $dt = 0$ is not equivalent to standard simultaneity; indeed, the instantaneous velocity of the rotating frame is tangentially directed, and the relativistic dilation and contraction effects only take place in the direction of the velocity.

The spatial distance between two infinitesimally near points, as measured by a rod resting in the rotating frame, is found by substituting the just-derived value of dt, (2.4), in the expression for ds^2. The result is the following expression for the 3-dimensional spatial line element:

$$dl^2 = dr^2 + \frac{r^2 d\varphi^2}{1 - \omega^2 r^2/c^2} + dz^2. \quad (2.5)$$

We could have found (2.1) and (2.5) in a simpler way by making use of the standard expressions for the time dilation and Lorentz contraction undergone by clocks and rods, respectively, that possess the instantaneous velocity ωr. However, the use of the line element as the central theoretical quantity provides us with a unifying framework that makes it easier to discuss the relation between metrical and dynamical concepts.

4. Space and time without rods and clocks

In his Autobiographical Notes, Einstein already points out that from a fundamental point of view it is unsatisfactory to interpret ds via measuring procedures with complicated macroscopic instruments. Indeed, this could create the false impression that rods and clocks are basic entities without which the theory would have no physical content. However, it is clear that rods and clocks themselves consist of more fundamental entities, like atoms and molecules. In principle it would therefore be better to base the interpretation of the theory directly on what it says about the fundamental constituents of matter. It is only because no complete theory of matter was available, Einstein explains, that it was expedient to introduce the theory through measurements by rods and clocks. In principle they should be eliminated at a later stage.

This desideratum, to do without rods and clocks, becomes even more urgent when accelerated frames of reference are considered, as in the case of our rotating world. Obviously the motions of clocks and rods that are stationary in the rotating frame are not inertial. Centrifugal and Coriolis forces will therefore arise, which will distort the rotating instruments. It is not a priori clear that such deformed instruments will keep on functioning as indicators of ds. Indeed, one could easily think of rods or clocks that would be torn apart by centrifugal forces and would therefore certainly not indicate any length or time intervals.

Fortunately, it *is* possible to found the space-time description of our rotating world on a more fundamental level than that of macroscopic measuring devices. In fact, in general space-times one can use the basic principles that time-like geodesics are physically realized by inertially moving point-particles and that null-geodesics represent light rays, to define space-time distances between neighboring events ([4], section 16.4). In our case, Minkowski space-time, we can start by constructing a set of elementary 'light clocks' by letting light signals bounce back and forth between neighboring parallel particle geodesics. If we confine our attention to the plane $z = 0$, we can take the geodesics defined in the laboratory frame (the inertial system we started with) by constant r, φ and $r + dr$, φ, respectively. The thus constructed clock has a constant period (the dt between two 'ticks') of $2dr/c$. In other words, we have here an elementary process that provides a physical realization of t; and we have come to this conclusion on the basis of the dynamical postulates alone (the only ingredient is that light follows null-geodesics). Length can be determined in a similar way: let a light signal depart from A, with fixed r and φ and go to a neighboring position B with $r + dr$ and $\varphi + d\varphi$ from which it returns immediately to A. Let the round trip time measured at A be dt. We can now define the spatial distance dl between A and B as $cdt/2$. From the postulate that light follows null-geodesics it follows that $dl^2 = dr^2 + r^2 d\varphi^2$. In this way the laboratory coordinates obtain metrical significance, without reliance on macroscopic clocks and rigid rods. When such (complicated) systems are introduced at a later stage, we can study their workings on the basis of the fundamental laws of physics governing their constituents and see, on that basis, whether they are indeed suitable to measure the just-defined intervals.

We now turn our attention to measurements performed within the rotating system, i.e. with instruments resting in the rotating coordinates. From Eq.

(2.3) we see that the round trip time dt needed by a light signal between two neighboring points that are stationary in the rotating frame of reference is given by

$$dt = dt_1 + dt_2 = 2\frac{\sqrt{(c^2 - \omega^2 r^2)dr^2 + c^2 r^2 d\varphi^2}}{c^2 - \omega^2 r^2}.$$

If the laboratory coordinate t is used as the measure of time, and if the definition $dl = cdt/2$ is used to fix spatial distances, we arrive at the metric

$$dl^2 = \frac{(1 - \omega^2 r^2/c^2)dr^2 + r^2 d\varphi^2}{(1 - \omega^2 r^2/c^2)^2}.$$

However, it is more natural to link the measure of time intervals in the rotating system to the indications furnished by light clocks that are co-moving, i.e. stationary in the rotating coordinates instead of stationary in the laboratory frame. So let a light ray bounce back and forth between two points that only differ in their r-coordinate, by the amount dr, in the rotating frame. It follows from the expression (2.2) that the period of the thus defined clock is $2dr/\sqrt{c^2 - \omega^2 r^2}$, whereas the period of the similar and instantaneously coinciding clock in the laboratory frame is $2dr/c$. The period of the rotating light clock is therefore longer, by a factor $1/\sqrt{1 - \omega^2 r^2/c^2}$, than the period of the laboratory clock. When we now define distances as $cd\tau/2$, with τ measured in the new 'co-moving' time units, we have to multiply the distances we found a moment ago by $\sqrt{1 - \omega^2 r^2/c^2}$. The final result is

$$dl^2 = dr^2 + r^2 d\varphi^2/(1 - \omega^2 r^2/c^2).$$

This is the same result as we found in Eq. (2.5).

5. Accelerating measuring devices

The above sketch shows how we can achieve a physical implementation of the two systems of coordinates, and give them metrical meaning, by the sole use of point-particles and light. The thus defined space-time distances can be used to calibrate macroscopic measuring rods and clocks. Indeed, it is clear that in general such instruments will be deformed by the rotational motion, and that this will introduce inaccuracies in their readings.

The general effect of accelerations can be illustrated by the consideration of a light-clock of the kind mentioned above: a light signal bouncing back and forth between two particle world-lines. Light travelling to and fro between two mirrors resting in an inertial system, with mutual distance L, defines a clock with half period $T = L/c$. When the two mirrors move uniformly with the same velocity $\vec{v}$, in a direction parallel to their planes, a simple application of the Pythagorean theorem shows that the half period of the moving clock becomes $L/(c\sqrt{1 - v^2/c^2}) = T/\sqrt{1 - v^2/c^2}$. This demonstrates the presence of time dilation in the case of a moving light-clock (by means of the relativity principle this result can be extended to other time-keeping devices). Consider now what happens if the velocity is not uniform but the system starts accelerating when the light leaves the first mirror, with a small acceleration $\vec{a}$ in the direction of $\vec{v}$. As judged from the inertial frame, the light now needs a time T' to reach the second mirror; during this time the accelerating mirror system has covered a distance $s \approx vT' + 1/2aT'^2$. Application of Pythagoras now yields $c^2T'^2 = L^2 + s^2$. It follows that

$$c^2T'^2 = L^2 + v^2T'^2 + avT'^3 + 1/4a^2T'^4. \tag{2.6}$$

The half period T' that follows from this equation obviously depends on a. However, it is also obvious that the extent of the change in the period caused by a depends on the magnitude of T' itself. If we make T' in Eq. (2.6) very small, by reducing L, we find in the limiting situation $T' = T/\sqrt{1 - v^2/c^2}$, just as in the case of the uniformly moving clock. In other words, the acceleration has an effect, but the magnitude of this effect depends on the peculiarities of the specific clock we are considering (in this case on L). This acceleration-dependent effect can be made as small as we wish, by using suitably constructed clocks (in the example: by reducing L). What remains in all cases is the universal effect caused by the velocity.

This shows in what sense velocities have a universal effect on length and time determinations, but accelerations not. There is no independent postulate involved here; everything can be derived from the dynamical principles of special relativity theory, by considering the inner workings of the measuring devices. It turns out that acceleration-dependent effects are there, but can be varied, and corrected for, by varying the characteristics of the devices. This is the real content of the textbook statement that acceleration has no metrical effects. It should be stressed again that this does not constitute a new hypothesis that has to be *added* to the dynamical principles of the theory of relativity. Quite to the contrary, the effects of accelerations on any given clock or measuring rod can be computed from the dynamical principles applied to these devices.

Of course, that the magnitudes of distortions will depend on the specific constitutions of the rods or clocks in question is only to be expected. Robust rods and clocks will be less affected accelerations than fragile ones. One way of correcting for the deformations is to gauge the accelerating instruments against the light measurements results described in section (4). The expressions (2.1) and (2.5) should be understood as applying to the results of space-time measurements performed with thus corrected measuring devices.

6. Space and time in the rotating frame

The spatial geometry defined by the line element (2.5) is non-Euclidean, with a negative r-dependent curvature (see [5], pp. 330-337). One of the notorious characteristics of this geometry is that the circumference of a circle with radius r (in the plane $z = 0$) is $2\pi r/(1 - \omega^2 r^2/c^2)^{1/2}$, which is greater than $2\pi r$. The recognition that the geometry in accelerated frames of reference will in general be non-Euclidean, which through the equivalence principle suggests that the presence of gravitation will also cause deviations from Euclidean geometry, played an important role in Einstein's route to General Relativity. We will restrict ourselves to the special theory, however.

The properties of time in the rotating frame are perhaps even more interesting than the spatial characteristics. Expression (2.4) demonstrates that standard simultaneity between neighboring events in the rotating frame corresponds to a non-zero difference dt. It follows that if we go along a circle with radius r, in the positive ϕ-direction, while establishing standard simultaneity along the way, we create a 'time gap' $\triangle t = 2\pi\omega r^2/(c^2 - \omega^2 r^2)$ upon completion of the circle. Doing the same thing in the opposite direction results in a time gap of the same absolute value but with opposite sign. So the total time difference generated by synchronizing over a complete circle in one direction, and comparing the result with doing the same thing in the other direction is $\triangle t = 4\pi\omega r^2/(c^2 - \omega^2 r^2)$.

Now suppose that two light signals are emitted from a source fixed in the rotating frame and start travelling, in opposite directions, along the same circle of constant r. We follow the two signals while locally using standard synchrony; this has the advantage that locally the standard constant velocity c can be attributed to the signals. We therefore conclude that the two signals use the same amount of time in order to complete their circles and return to their source, as calculated by integrating the elapsed time intervals measured in the successive local comoving inertial frames (the signals cover the same distances, with the same velocity c, as judged from these frames). However, because of the just-mentioned time gaps the two signals do not complete their circles simultaneously, in one event. There is a time difference $\triangle t = 4\pi\omega r^2/(c^2 - \omega^2 r^2)$ between their arrival times, as measured in the coordinate t. This is the celebrated Sagnac effect (see [6], p. 652 for a related derivation).

The Sagnac effect directly reflects the space-time geometry of the rotating frame; it does not depend on the specific nature of the signals that propagate in the two directions. Indeed, as long as the two signals have the same velocities in the locally defined inertial frames with standard synchrony, the difference in arrival times is given by the above time gap. So the same Sagnac time difference is there not only for light, but for any two identical signals running into two directions. The Sagnac experiment directly probes the space-time relations in the rotating frame.

Because of the difference in arrival times of the two light signals, the velocity of light obviously cannot be everywhere the same in the rotating coordinates. This is a consequence of the fact that in the rotating frame events with equal time coordinate t are not standard simultaneous. So t may appear as an unnatural time coordinate for the rotating frame: it would be desirable to have a time coordinate that *would* reflect standard simultaneity everywhere. The question can therefore be asked whether we could define a coordinate $\tilde{t}$ in such a way that $d\tilde{t} = 0$ would imply standard synchrony in the local inertial frame. Suppose that $\tilde{t} = \tilde{t}(t, r, \varphi)$, then we should have that $d\tilde{t} = 0$ if Eq. (2.4) holds. This implies that $\omega^2 r^2/(c^2 - \omega^2 r^2)\partial\tilde{t}/\partial t + \partial\tilde{t}/\partial\varphi = 0$ and $\partial\tilde{t}/\partial r = 0$. In view of the axial symmetry in our frame we may assume that $\partial\tilde{t}/\partial\varphi = 0$. The only solution of our partial differential equations is therefore that $\tilde{t}$ is independent of r, φ and t, which clearly is unacceptable. Therefore, it turns out to be a basic characteristic of the rotating frame that the locally defined Lorentz frames do not mesh: they cannot be combined into one frame with a globally defined standard simultaneity. Evidently it *is* possible to define global time coordinates, like t; but the description of physical processes in terms of these coordinates must necessarily differ from the standard description in inertial systems. The non-constancy of the velocity of light in the rotating system furnishes an example. It should be noted that this peculiarity of the description of physical processes in the rotating system is not a consequence of the presence of centrifugal and Coriolis forces: indeed, in our space-time determinations we have compensated for the effects of such forces. It is the space-time geometry itself that is at issue.

7. Simultaneity, slow clock transport and conventionality

As we saw in the previous section, the Sagnac effect is independent of the nature of the signals that propagate into the two directions on the rotating disc. So, if we transport two clocks along a circle with radius r around the center of the disk, one clockwise and one counter-clockwise, while keeping their velocities the same in the locally co-moving inertial frames, there will be a difference $\triangle t = 4\pi\omega r^2/(c^2 - \omega^2 r^2)$ between their return times (measured in the laboratory time t). It is well known that the indications of the clocks will conform to standard simultaneity in the limiting situation of vanishing velocities. That is, if the clocks are transported very slowly with respect to the rotating disc, they will remain synchronized according to standard simultaneity in the local inertial frames. It follows that slow clock transport cannot be used to define an unambiguous global time coordinate on the rotating disc: in the just-mentioned case the result will depend on whether a clockwise or counter-clockwise path is chosen. In general, the result of synchronization by slow clock transport will be path dependent.

With regard to time in inertial frames, there has been a long-standing and notorious debate about whether standard simultaneity ($\varepsilon = 1/2$ according to Reichenbach's formulation) is conventional or not. One of the arguments often put forward against the conventionality thesis is that the natural procedure of slow clock transport leads to $\varepsilon = 1/2$, thus showing its privileged status. In the case of the rotating world, this argument can only be applied locally. Neither the Einstein light signal procedure, nor the slow transport of clock can be used to establish a global notion of simultaneity on the rotating disc.

More generally, it cannot be denied that in inertial frames standard simultaneity has a special status: it allows a simple formulation of the laws, conforms to slow clock transport and other physically plausible synchronization procedures, and agrees with Minkowski-orthogonality with respect to world lines representing the state of rest [7]. So time coordinates t that correspond to this notion of simultaneity (in the sense that $dt = 0$ expresses simultaneity) may be said to be privileged. In non-inertial frames this still is so, though now the argument applies only locally. The rotating system illustrates the situation very well: in each point on the disc standard simultaneity can be defined just as in an inertial system, but this does not result in a global time coordinate. This supports the general conclusion of this paper, namely that the difference between the status of coordinates in inertial and non-inertial frames of reference, or special and general relativity, is not so much a matter of epistemology—or philosophical analysis of the meaning of coordinates—but rather a matter of physical facts. In global inertial systems privileged coordinates can be chosen that have a global metrical interpretation. In reference frames that are not globally inertial such privileged coordinates do not exist in general. This is not a matter of a different philosophical status of coordinates, but rather a reflection of different global space-time symmetry properties—a factual physical difference rather than a philosophical distinction.

The purpose of *coordinates* is to label events unambiguously, which can be done in infinitely many different ways. The choice between these different possibilities is a matter of pragmatics; though there may be very good reasons to prefer one choice over another. Thus, in inertial frames of reference time coordinates that reflect standard simultaneity lead for many purposes to an especially simple description. In this case there exists a physically significant global temporal relation between events, and coordinates that are adapted to this relation inherit its special status. But in the general case no physically significant simultaneity relation exists. Global "simultaneity" can then only refer to some global time coordinate, which is chosen conventionally. This is true in non-inertial frames of reference, like the rotating disc, and in generally relativistic space-times in which there are no global temporal symmetries.

These non-inertial frames of reference, and general relativistic space-times, seem an arena where the thesis that (global) simultaneity is conventional can be defended without controversy.

8. The rotating Ehrenfest cylinder

Not only in its temporal aspects, but also in its spatial physical properties the rotating frame differs globally from an inertial frame. Until now we spoke about a rotating frame of reference as defined by a set of rotating *coordinates*, without discussing a possible material realization of this frame. It is clear from the outset that the special theory of relativity sets limits to such a realization: objects at rest in the rotating frame should not move faster than light as judged from the inertial laboratory frame. This implies that $\omega r < c$ should hold for such an object. In other words, there is an upper bound to the value of r that can be realized materially.

However, even if this condition is satisfied there remain interesting questions, as made clear by Ehrenfest in his famous note on the subject [8]. Suppose that a solid cylinder of radius R is gradually put into rotation about its axis; finally it reaches a state of uniform rotation with angular velocity ω. It would seem that in the final state the cylinder has to satisfy contradictory requirements: on the one hand the Lorentz contraction should make the circumference shorter, on the other hand the radial elements should not contract because their motion is normal to their lengths. From symmetry it is clear that the form of a cross section of the moving cylinder remains a circle, as judged from the laboratory frame; but this would apparently mean that the circumference of the circle has become smaller while the radius has stayed the same. This is inconsistent (remember that Euclidean geometry holds in the laboratory frame).

The solution of this paradox is that the various parts of the cylinder, being fastened to each other, cannot move freely and therefore cannot Lorentz contract as freely moving infinitesimal measuring rods would do. What will happen to the cylinder during its acceleration depends on the elastic properties of the material: tensions will develop because the tangential elements want to shrink, whereas the radial elements do not. A possible scenario is that the tangential elements will be stretched as compared to their natural (i.e. Lorentz contracted) lengths. Another possibility, if the material is sufficiently strong, is that the radius will contract, allowing the circumference to contract too. However, if ω becomes big enough one would have to expect that the tensions and strains grow to such an extent that they cause the cylinder to explode. This makes it clear that the Lorentz contraction can be responsible for clearly dynamical effects—the contractions are not just a matter of "perspective" (see [9] and [10]). (Of course, this whole discussion is rather academical because centrifugal forces will tear the cylinder apart before the relativistic effects become noticeable.)

As long as the cylinder survives, and keeps its cylindrical shape (as judged from the laboratory frame), not all its elements will be free from deformations, tensions or strains. However, the length determinations by measuring rods at rest in the rotating frame, as discussed in section 3, were supposed to be carried out with freely movable rods that are not hampered in their Lorentz contractions. So measuring rods laid out along the circumference of the circle will have undergone a Lorentz contraction, whereas rods laid out along a radius will have retained their rest length (as judged from the laboratory system). The measurement would reveal that the circumference is longer than 2π times the radius, in conformity with equation (2.5).

The spatial geometry of the disc is therefore non-Euclidean. That means that distance relations must be represented by a metrical tensor that cannot be put into the Euclidean diagonal form everywhere. It remains possible, of course, to choose coordinates locally in such a way that the Euclidean form results at the point in question. The difference from the inertial system concerns global aspects, not local ones. The impossibility to define a global coordinate system in which the metrical tensor reduces to its Euclidean standard form implies that there cannot be coordinates whose differences correspond to distances everywhere. The situation is analogous to the one we discussed in the context of time coordinates: nothing changes in the status and meaning of coordinates when we go from inertial to non-inertial systems. The things that do change are the global characteristics of the physical geometry, which are coordinate-independent.

Conclusion

The transition from inertial to non-inertial frames of reference, and the transition from special to general relativity, does not imply a change in the status and meaning of coordinate systems. It is therefore a misunderstanding to think that general relativity allows a wider class of coordinate systems than classical physics or special relativity. In classical physics and in relativity theory, both in inertial systems and non-inertial systems, coordinates just serve to label events. The choice for a particular coordinate system from the infinity of possible ones is dictated by pragmatic considerations.

What *does* change in the transition from inertial to non-inertial systems, and from special to general relativity, are the global aspects of the physical spatial and temporal relations. Pragmatic arguments for choosing one coordinate system over another may therefore lead to different choices in the different situations: if geometrical relations have become different, coordinate systems with different characteristics, adapted to the new geometry, may lead to a simpler description. But this does not change the conventional nature of the coordinates.

References

[1] *Albert Einstein: Philosopher-Scientist*, P.A. Schilp (ed.), Open Court, La Salle, 1949.

[2] J. Stachel, "The Rigidly Rotating Disc as the 'Missing Link' in the History of General Relativity", pp. 48-62 in *Einstein and the History of General Relativity*, D.Howard and J. Stachel, (eds.), Birkhäuser, Basel, 1989.

[3] H. Reichenbach, *The Philosophy of Space and Time*, Dover, New York, 1957.

[4] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, Freeman, San Francisco, 1973.

[5] M.-A. Tonnelat, *The Principles of Electromagnetic Theory and of Relativity*, Reidel, Dordrecht, 1966.

[6] D. Dieks and G. Nienhuis, "Relativistic Aspects of Nonrelativistic Quantum Mechanics", *American Journal of Physics* **58** (1990) 650-655.

[7] D. Malament, "Causal Theories of Time and the Conventionality of Simultaneity", *Noûs* **11** (1977) 293-300.

[8] P. Ehrenfest, "Gleichformige Rotation starrer Körper und Relativitätstheorie", *Physikalische Zeitschrift* **10** (1909) 918.

[9] D. Dieks, "Time in Special Relativity and its Philosophical Significance", *European Journal of Physics* **12** (1991) 253-259.

[10] D. Dieks, "The 'Reality' of the Lorentz Contraction", *Zeitschrift für allgemeine Wissenschaftstheorie* **15** (1984) 330-342.

Chapter 3

THE HYPOTHESIS OF LOCALITY AND ITS LIMITATIONS

Bahram Mashhoon
Department of Physics and Astronomy
University of Missouri-Columbia
Columbia, Missouri 65211, USA

Abstract The hypothesis of locality, its origin and consequences are discussed. This supposition is necessary for establishing the local spacetime frame of accelerated observers; in this connection, the measurement of length in a rotating system is considered in detail. Various limitations of the hypothesis of locality are examined.

1. Introduction

The basic laws of microphysics have been formulated with respect to ideal inertial observers. However, all actual observers are accelerated. To interpret the results of experiments, it is therefore necessary to establish a connection between actual and inertial observers. This is achieved in the standard theory of relativity by means of the hypothesis of locality, namely, the assumption that an accelerated observer at each instant along its worldline is physically equivalent to an otherwise identical momentarily comoving inertial observer. In this way a noninertial observer passes through a continuous infinity of hypothetical momentarily comoving inertial observers [1].

The hypothesis of locality stems from Newtonian mechanics, where the state of a particle is given at each instant of time by its position and velocity. Thus the accelerated observer and the hypothetical inertial observer share the same state and are therefore equivalent. Hence, the treatment of accelerated systems in Newtonian mechanics requires no new assumption. More generally, if all physical phenomena could be reduced to pointlike coincidences of classical point particles and electromagnetic *rays*, then the hypothesis of locality would be exactly valid. However, an electromagnetic *wave* has intrinsic scales of

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 43-55.

length and time characterized by its wavelength λ and period λ/c. For instance, the measurement of the frequency of the wave necessitates observation of a few oscillations before a reasonable determination can become possible. If the state of the observer does not change appreciably over this period of time, then the hypothesis of locality would be essentially valid. This criterion may be expressed as $\lambda/\mathcal{L} << 1$, where $\mathcal{L}$ is the relevant acceleration length of the observer. That is, the observer has intrinsic scales of length $\mathcal{L}$ and time $\mathcal{L}/c$ that characterize the degree of variation of its state. For instance, $\mathcal{L} = c^2/a$ for an observer with translational acceleration a, while $\mathcal{L} = c/\Omega$ for an observer rotating with frequency Ω [1, 2].

The consistency of these ideas can be seen in the case of an accelerating charged particle. Imagine a particle of mass m and charge q moving under the influence of an external force $\mathbf{F}_{ext}$. The particle radiates electromagnetic waves that have a characteristic wavelength $\lambda \sim \mathcal{L}$, where $\mathcal{L}$ is the acceleration length of the particle. Thus the interaction of the particle with the electromagnetic field violates the hypothesis of locality since $\lambda/\mathcal{L} \sim 1$. The radiating charged particle is therefore not momentarily equivalent to an otherwise identical comoving inertial particle. This agrees with the fact that in the nonrelativistic approximation the Abraham-Lorentz equation of motion of the particle is

$$m\frac{d\mathbf{v}}{dt} - \frac{2}{3}\frac{q^2}{c^3}\frac{d^2\mathbf{v}}{dt^2} + ... = \mathbf{F}_{ext} \quad , \tag{3.1}$$

so that the state of a radiating particle is not determined by its position and velocity alone.

Imagine an accelerated measuring device in Minkowski spacetime. The internal dynamics of the device is then subject to inertial effects that consist of the inertial forces of classical mechanics together with their generalizations to electromagnetic and quantum domains. If the net influence of these inertial effects integrates — over the relevant length and time scales of a measurement — to perturbations that do not appreciably disturb the result of the measurement and can therefore be neglected, then the hypothesis of locality is valid and the device can be considered standard (or ideal). Consider, for instance, the measurement of time dilation in terms of muon lifetime by observing the decay of muons in a storage ring. It follows from the hypothesis of locality that $\tau_\mu = \gamma\tau_\mu^0$, where γ is the Lorentz factor and τ_μ^0 is the lifetime of the muon at rest in the background inertial frame. On the other hand, the lifetime of such a muon has been calculated on the basis of quantum theory by assuming that the muon occupies a high-energy Landau level in a constant magnetic field [3]. One can show that the result of [3] can be expressed as [4]

$$\tau_\mu \simeq \gamma \tau_\mu^0 \left[1 + \frac{2}{3} \left(\frac{\lambda}{\mathcal{L}} \right)^2 \right] \quad . \tag{3.2}$$

Here $\lambda = \hbar/(mc)$ is the Compton wavelength of the muon, m is the muon mass and $\mathcal{L} = c^2/a$, where $a = \gamma^2 v^2/r$ is the effective centripetal acceleration of the muons in the storage ring. The hypothesis of locality is completely adequate for such experiments since $\lambda/\mathcal{L}$ is extremely small. In fact, the hypothesis of locality is clearly valid in many Earth-bound experimental situations since $c^2/g_\oplus \simeq 1$ lyr and $c/\Omega_\oplus \simeq 28$ AU.

The hypothesis of locality plays a crucial role in Einstein's theory of gravitation: Einstein's principle of equivalence together with the hypothesis of locality implies that an observer in a gravitational field is locally inertial. Indeed, the equivalence between an observer in a gravitational field and an accelerated observer in Minkowski spacetime is useless operationally unless one specifies what an accelerated observer measures.

The hypothesis of locality was formally introduced in [1] and its limitations were pointed out. To clarify the origin of this conception, some background information is provided in section 2. The implications of this assumption for length determination in rotating systems are pointed out in section 3. Section 4 contains a discussion.

2. Background

Maxwell's considerations regarding optical phenomena in moving systems implicitly contained the hypothesis of locality [5]. The fundamental form of Maxwell's theory of electromagnetism, derived from Maxwell's original electrodynamics of media, is essentially due to Lorentz's development of the theory of electrons.

Lorentz conceived of an electron as an extremely small charged particle with a certain smooth volume charge density. A free electron at rest was regarded as a spherical material system with certain internal forces that ensured the constancy of its size and form. An electron in translational motion would then be a flattened ellipsoid according to Lorentz, since it would be deformed from its original spherical shape by the Lorentz-FitzGerald contraction in the direction of its motion. The internal dynamics of electrons therefore became a subject of scientific inquiry and in 1906 Poincaré postulated the existence of a particular type of internal stress that could balance the electrostatic repulsion even in a moving (and hence flattened) electron. These issues are discussed in detail in the fifth chapter (on optical phenomena in moving bodies) of Lorentz's book [6] on the theory of electrons.

In extending the Lorentz transformations in a pointwise manner to accelerating electrons, Lorentz encountered a problem regarding the dynamical equilibrium of the internal state of the electron. To avoid this problem, Lorentz introduced a basic assumption that is discussed in section 183 of his book [6]:

> "... it has been presupposed that in a curvilinear motion the electron constantly has its short axis along the tangent to the path, and that, while the velocity changes, the ratio between the axes of the ellipsoid is changing at the same time."

To elucidate this assumption, Lorentz explained its approximate validity as follows (§183 of [6]):

> "... If the form and the orientation of the electron are determined by forces, we cannot be certain that there exists at every instant a state of equilibrium. Even while the translation is constant, there may be small oscillations of the corpuscle, both in shape and in orientation, and under variable circumstances, i.e. when the velocity of translation is changing either in direction or in magnitude, the lagging behind of which we have just spoken cannot be entirely avoided. The case is similar to that of a pendulum bob acted on by a variable force, whose changes, as is well known, it does not instantaneously follow. The pendulum may, however, approximately be said to do so when the variations of the force are very slow in comparison with its own free vibrations. Similarly, the electron may be regarded as being, at every instant, in the state of equilibrium corresponding to its velocity, provided that the time in which the velocity changes perceptibly be very much longer than the period of the oscillations that can be performed under the influence of the regulating forces."

It is therefore clear that the hypothesis of locality and its limitation were discussed by Lorentz for the case of the motion of electrons.

Einstein, in conformity with his general approach of formulating symmetry-like principles that would be independent of the specific nature of matter, simply adopted the same general assumption for rods and clocks. In fact, in discussing the rotating disk problem, Einstein stated in a footnote on page 60 of [7] that:

> "These considerations assume that the behavior of rods and clocks depends only upon velocities, and not upon accelerations, or, at least, that the influence of acceleration does not counteract that of velocity."

The modern experimental foundation of Einstein's theory of gravitation necessitates that this assumption be extended to all (standard) measuring devices; therefore, the hypothesis of locality supersedes the clock hypothesis, etc.

Though the hypothesis of locality originates from Newtonian mechanics, one should point out that the state of a relativistic point particle differs from that in Newtonian mechanics: the magnitude of velocity is always less than c. Moreover, the hypothesis of locality rests on the possibility of defining instantaneous inertial rest frames along the worldline of an arbitrary point particle. In fact, Minkowski raised this possibility and hence the corresponding hypothesis of locality to the level of a fundamental axiom [8].

Another aspect of Lorentz's presupposition must be mentioned here that involves the extension of the notion of rigid motion to the relativistic domain: the electron moves rigidly as it is always undeformed in its momentary rest frame. The notion of rigid motion in the special and general theories of relativity has been discussed by a number of authors [9, 10, 11, 12, 13]. It is important to note that the concept of an infinitesimal rigid rod is indispensable in the theory of relativity (cf. section 3).

In some expositions of relativity theory, such as [10] and [14], the hypothesis of locality is completely implicit. For instance, in Robertson's paper on "Postulate *versus* Observation in the Special Theory of Relativity" [14], attention is simply confined to "the kinematics *im kleinen* of physical spacetime" [14]. However, when interpreting the observational foundations of special relativity, one must recognize that actual observers are all accelerated and that the difference between accelerated and inertial observers must be investigated; in fact, this problem is ignored in [14] by simply asserting that physics is essentially local.

3. Length measurement

To illustrate the nature of the hypothesis of locality, it is interesting to consider spatial measurements of rotating observers. Imagine observers A and B moving on a circle of radius r about the origin in the (x, y)-plane of a background global inertial frame with coordinates (t, x, y, z). Expressed in terms of the azimuthal angle φ, the location of A and B at $t = 0$ can be chosen such that $\varphi_A = 0$ and $\varphi_B = \Delta$ with no loss in generality. The motion of A and B is then assumed to be such that for $t > 0$ they rotate in *exactly the same way* along the circle with angular frequency $\hat{\Omega}_0(t) > 0$. Thus for $t > 0$ observers A and B can be characterized by the azimuthal angles

$$\varphi_A(t) = \int_0^t \hat{\Omega}_0(t')dt' \quad , \quad \varphi_B(t) = \Delta + \int_0^t \hat{\Omega}_0(t')dt' \quad . \tag{3.3}$$

According to the static inertial observers in the background global frame, the angular separation of A and B is constant at any time $t > 0$ and is given by $\varphi_B(t) - \varphi_A(t) = \Delta$; moreover, the spatial separation of the two observers along the circular arc at time $t > 0$ is $\ell(t) = r\Delta$.

Consider now a class of observers O populating the whole arc from A to B and moving exactly the same way as A and B. At any time $t > 0$, it appears to inertial observers at rest in the background frame that these rotating observers are all at rest in the (x', y', z') system that is obtained from (x, y, z) by a simple rotation about the z-axis with frequency $\hat{\Omega}_0(t)$. What is the length of the arc according to these rotating observers? It follows from an application of the hypothesis of locality that for $t > 0$ the spatial separation between A and B as measured by the rotating observers is $\ell' = \hat{\gamma}\ell(t)$, where $\hat{\gamma}$ is the

Lorentz factor corresponding to $\hat{v} = r\hat{\Omega}_0(t)$. Units are chosen here such that $c = 1$ throughout this section. Indeed at any time $t > 0$ in the inertial frame, each observer O is momentarily equivalent to a comoving inertial observer and the corresponding infinitesimal element of the arc $\delta\ell$ has a rest length $\delta\ell'$ in the momentarily comoving inertial frame such that from the Lorentz transformation between this local inertial frame and the global background inertial frame one obtains

$$\sqrt{1-\hat{v}^2}\ \ \delta\ell' = \delta\ell \tag{3.4}$$

in accordance with the Lorentz-FitzGerald contraction. Defining

$$\ell' = \Sigma\ \delta\ell'\ \ , \tag{3.5}$$

where each $\delta\ell'$ is the infinitesimal length at rest in a different local inertial frame, one arrives at $\ell' = \hat{\gamma}\ell$, since $\hat{v}(t)$ is the same for the class of observers O at time t. The same result is obtained if length is measured using light travel time over infinitesimal distances between observers O, since in each local inertial frame the two methods give the same answer. As is well known, the light signals could also be used for the synchronization of standard clocks carried by observers O.

It is important to remark here that equation (3.5) is far from a proper geometric definition of length and one must question whether it is even physically reasonable, since each $\delta\ell'$ in equation (3.5) refers to a different local Lorentz frame. In any case, in this approach the length of the arc as measured by the accelerated observers is

$$\ell' = \hat{\gamma}(t) r\Delta\ \ . \tag{3.6}$$

The sum in equation (3.5) involves infinitesimal rest segments each from a separate local inertial frame. Perhaps the situation could be improved by combining these infinite disjoint local inertial rest frames into one continuous *accelerated* frame of reference. The most natural way to accomplish this would involve choosing one of the noninertial observers on the arc and establishing a geodesic coordinate system along its worldline. In such a system, the measure of separation along the worldline (proper time) and away from it (proper length) would also be determined by the hypothesis of locality. That is, at any instant of proper time the rules of Euclidean geometry are applicable as the accelerated observer is instantaneously inertial. It turns out that the length of the arc determined in this way would in general be different from ℓ' and would depend on which reference observer $O : A \to B$ is chosen for this purpose [15]. To illustrate this state of affairs and for the sake of concreteness, in the rest of this section the length of the arc will be determined in a geodesic coordinate system along the worldline of observer A and the result will be compared with equation (3.6).

In the background inertial frame, the coordinates of observer A are

$$x^\mu_A = (t, r\cos\varphi_A, r\sin\varphi_A, 0) \ , \tag{3.7}$$

and the proper time along the worldline of A is given by

$$\tau = \int_0^t \sqrt{1 - \hat{v}^2(t')} \ dt' \ , \tag{3.8}$$

where $\tau = 0$ at $t = 0$ by assumption. It is further assumed that $\tau = \tau(t)$ has an inverse and the inverse function is denoted by $t = F(\tau)$. Thus $dt/d\tau = dF/d\tau = \gamma(\tau) = (1 - v^2)^{-1/2}$ is the Lorentz factor along the worldline of A, so that $v(\tau) := \hat{v}(t)$ and $\gamma(\tau) := \hat{\gamma}(t)$. Moreover, it is useful to define $\phi(\tau) := \varphi_A(t)$ and $d\phi/d\tau = \gamma\Omega_0(\tau)$, where $\Omega_0(\tau) := \hat{\Omega}_0(t)$. With these definitions, the natural orthonormal tetrad frame along the worldline of A for $\tau > 0$ is given by

$$\lambda^\mu{}_{(0)} = \gamma(1, -v\sin\phi, v\cos\phi, 0) \ , \tag{3.9}$$

$$\lambda^\mu{}_{(1)} = (0, \cos\phi, \sin\phi, 0) \ , \tag{3.10}$$

$$\lambda^\mu{}_{(2)} = \gamma(v, -\sin\phi, \cos\phi, 0) \ , \tag{3.11}$$

$$\lambda^\mu{}_{(3)} = (0 \ , \ 0 \ , \ 0 \ , \ 1) \ , \tag{3.12}$$

where $\lambda^\mu{}_{(0)} = dx^\mu_A/d\tau$ is the temporal axis and the spatial triad corresponds to the natural spatial frame of the rotating observer. To obtain this tetrad in a simple fashion, first note that by setting $r = 0$ and hence $v = 0$ and $\gamma = 1$ in equations (3.9) - (3.12) one has the natural tetrad of the fixed noninertial observer at the spatial origin — as well as the class of noninertial observers at rest in the background inertial frame — that refers its observations to the axes of the (x', y', z') coordinate system alluded to before; then, boosting this tetrad with speed v along the second spatial axis tangent to the circle of radius r results in equations (3.9) - (3.12).

It follows from the orthonormality of the tetrad system (3.9) - (3.12) that the acceleration tensor $\mathcal{A}_{\alpha\beta}$ defined by

$$\frac{d\lambda^\mu{}_{(\alpha)}}{d\tau} = \mathcal{A}_\alpha{}^\beta \lambda^\mu{}_{(\beta)} \tag{3.13}$$

is antisymmetric. The translational acceleration of observer A, which is the "electric" part of the acceleration tensor ($a_i = \mathcal{A}_{0i}$), is given by

$$\mathbf{a} = (-\gamma^2 v\Omega_0 \ , \ \gamma^2 \frac{dv}{d\tau} \ , \ 0) \tag{3.14}$$

with respect to the tetrad frame and similarly the rotational frequency of A, which is the "magnetic" part of the acceleration tensor ($\Omega_i = \frac{1}{2}\epsilon_{ijk}\mathcal{A}^{jk}$), is given by

$$\mathbf{\Omega} = (0, 0, \gamma^2\Omega_0) \ . \tag{3.15}$$

Moreover, in close analogy with electrodynamics, one can define the invariants of the acceleration tensor as

$$I = -a^2 + \Omega^2 = \gamma^2\Omega_0^2 - \gamma^4 \left(\frac{dv}{d\tau}\right)^2 \tag{3.16}$$

and $I^* = -\mathbf{a} \cdot \mathbf{\Omega} = 0$. The analogue of a null electromagnetic field is in this case a null acceleration tensor; that is, an acceleration tensor is null if both I and I^* vanish. A rotating observer with a null acceleration tensor is discussed in the appendix.

The translational acceleration $\mathbf{a}$ consists of the well-known centripetal acceleration $\gamma^2 v^2/r$ and the tangential acceleration $\gamma^2 dv/d\tau$. The latter formula is consistent with the corresponding result in the case of linear acceleration along a fixed direction. To interpret equation (3.15) as the frequency of rotation of the spatial frame with respect to a local nonrotating frame, it is necessary to construct a nonrotating, i.e. Fermi-Walker transported, orthonormal tetrad frame $\tilde{\lambda}^\mu{}_{(\alpha)}$ along the worldline of observer A. Let $\tilde{\lambda}^\mu{}_{(0)} = \lambda^\mu{}_{(0)}$, $\tilde{\lambda}^\mu{}_{(3)} = \lambda^\mu{}_{(3)}$ and

$$\tilde{\lambda}^\mu{}_{(1)} = \cos\Phi \ \lambda^\mu{}_{(1)} - \sin\Phi \ \lambda^\mu{}_{(2)} \ , \tag{3.17}$$

$$\tilde{\lambda}^\mu{}_{(2)} = \sin\Phi \ \lambda^\mu{}_{(1)} + \cos\Phi \ \lambda^\mu{}_{(2)} \ , \tag{3.18}$$

where the angle Φ is defined by

$$\Phi = \int_0^\tau \ \Omega(\tau')d\tau' \ , \tag{3.19}$$

so that $d\Phi/d\tau = \gamma^2\Omega_0$. It remains to show that $\tilde{\lambda}^\mu{}_{(i)}$, $i = 1, 2, 3$, correspond to local ideal gyroscope directions. This can be demonstrated explicitly using equations (3.17) - (3.19) and one finds that

$$\frac{d\tilde{\lambda}^\mu{}_{(i)}}{d\tau} = \tilde{a}_i \ \tilde{\lambda}^\mu{}_{(0)} \ , \tag{3.20}$$

where $\tilde{\mathbf{a}}$ is the translational acceleration with respect to the nonrotating frame, as expected. It is straightforward to study the average motion of the spatial frame $\tilde{\lambda}^{\mu}{}_{(i)}$ with respect to the background inertial axes and illustrate Thomas precession with frequency $(1-\hat{\gamma})\hat{\Omega}_0$ per unit time t. That is, the frame of the accelerated observer rotates with frequency $\hat{\Omega}_0(t)$ about the background inertial axes, while the Fermi-Walker transported frame rotates with frequency $-\hat{\gamma}\hat{\Omega}_0$ per unit time t with respect to the frame of the accelerated observer according to equations (3.17) - (3.19); therefore, the unit gyroscope directions precess with respect to the background inertial frame with frequency $(1-\hat{\gamma})\hat{\Omega}_0$ as measured by the static background inertial observers.

Along the worldline of observer A, the geodesic coordinates can be introduced as follows: At a proper time τ, consider the straight spacelike geodesics that span the hyperplane orthogonal to the worldline. An event $x^{\mu} = (t, x, y, z)$ on this hyperplane is assigned geodesic coordinates $X^{\mu} = (T, \mathbf{X})$ such that

$$x^{\mu} = x^{\mu}_{A}(\tau) + X^{i}\lambda^{\mu}{}_{(i)}(\tau) \ , \quad \tau = T \ . \tag{3.21}$$

Let $\mathbf{X} = (X, Y, Z)$ and recall that along the worldline of $A, t = F(\tau)$ and $\varphi_A(t) = \phi(\tau)$; then, the transformation to the new coordinates is given by

$$\begin{aligned} t &= F(T) + \gamma(T)v(T)Y \ , &(3.22)\\ x &= (X + r)\cos\phi(T) - \gamma(T)Y\sin\phi(T) \ , &(3.23)\\ y &= (X + r)\sin\phi(T) + \gamma(T)Y\cos\phi(T) \ , &(3.24)\\ z &= Z \ . &(3.25) \end{aligned}$$

For $r = 0$, the geodesic coordinate system reduces to (t', x', y', z'), where $t' = t$; that is, the standard rotating coordinate system is simply the geodesic coordinate system constructed along the worldline of the noninertial observer at rest at the origin of spatial coordinates.

The form of the metric tensor in the geodesic coordinate system has been discussed in [1, 15, 16]. It turns out that in the case under consideration here the geodesic coordinates are admissible within a cylindrical region [16]. The boundary of this region is a real elliptic cylinder for $I > 0$, a parabolic cylinder for $I = 0$ or a hyperbolic cylinder for $I < 0$, where the acceleration invariant I is given by equation (3.16).

The class of observers $O : A \to B$ lies on an arc of the circle $x^2+y^2 = r^2$ in the background coordinate system; therefore, it follows from equations (3.23) and (3.24) that in the geodesic coordinate system the corresponding figure is an ellipse

$$\frac{(X + r)^2}{r^2} + \frac{Y^2}{(r\gamma^{-1})^2} = 1 \tag{3.26}$$

with semimajor axis r, semiminor axis $r\sqrt{1-v^2}$ and eccentricity v. The latter quantities are in general dependent upon time T, hence at a given time t each observer lies on a different ellipse. It is natural to think of the ellipse (3.26) as a circle of radius r that has suffered Lorentz-FitzGerald contraction along the direction of motion [1, 15].

The measurement of the length from A to B in the new system involves the integration of $dL, dL^2 = dX^2 + dY^2$, along the curve from $A : (T_A, 0, 0, 0)$ to $B : (T_B, X_B, Y_B, 0)$ corresponding to $A : (t, r\cos\varphi_A,\ r\sin\varphi_A, 0)$ and $B : (t, r\cos\varphi_B, r\sin\varphi_B, 0)$ in the background inertial frame. To clarify the situation, it is useful to introduce — in analogy with the elliptic motion in the Kepler problem — the eccentric anomaly θ by

$$X + r = r\cos\theta \ \ , \ \ Y = r\sqrt{1-v^2}\sin\theta \ \ . \tag{3.27}$$

Then, for a typical rotating observer $O : (t, r\cos\varphi, r\sin\varphi, 0)$ on the arc from $A \to B$ with

$$\varphi = \delta + \int_0^t \hat{\Omega}_0(t')dt' \tag{3.28}$$

one has in geodesic coordinates $O : (T, X, Y, 0)$, where X and Y are given by equations (3.27), and equations (3.22) - (3.24) imply that

$$t = F(T) + rv(T)\sin\theta \ \ , \tag{3.29}$$

$$\varphi = \theta + \phi(T) \ \ . \tag{3.30}$$

As O ranges from A to B, $\delta : 0 \to \Delta$ in equation (3.28) and hence $\theta : 0 \to \Theta$. For a fixed t, $t = F(T_A)$, equation (3.29) can be solved to give T as a function of θ; then, a detailed calculation involving equations (3.27) - (3.30) shows that

$$L = r\int_0^\Theta \sqrt{1 - v^2 W\cos^2\theta}\ d\theta \ \ . \tag{3.31}$$

Here W is defined by

$$W = \gamma^2\,\frac{1 - r^2\dot{v}^2\sin^2\theta}{(\gamma + r\dot{v}\sin\theta)^2} \ \ , \tag{3.32}$$

$\dot{v} = dv/dT$ and Θ can be found in terms of Δ by solving equations (3.29) and (3.30) at B:

$$t = F(T_B) + rv(T_B)\sin\Theta \ \ , \tag{3.33}$$

$$\Delta + \int_0^t \hat{\Omega}_0(t')dt' = \Theta + \phi(T_B) \ . \tag{3.34}$$

In practice, the explicit calculation of L can be rather complicated; therefore, for the sake of simplicity only the case of constant v (i.e. uniform rotation) will be considered further here [1, 15]. Then, $W = 1$ and equation (3.31) simply refers to the arc of a constant ellipse for which a Kepler-like equation

$$\theta - v^2 \sin\theta = \delta \tag{3.35}$$

follows from equations (3.29) and (3.30). Furthermore, the proper acceleration length of the uniformly rotating observer A is given by $\mathcal{L} = I^{-1/2} = (\gamma\Omega_0)^{-1}$. The case of uniform rotation, where L and ℓ' are independent of time and $L \neq \ell'$ in general, has been treated in detail in [1, 15] and it is clear that irrespective of the magnitude of $\Delta, L/\ell' \to 1$ as $r/\mathcal{L} = v\gamma \to 0$; on the other hand for $\Delta \to 0, L/\ell' \to 1$ irrespective of $v < 1$. That is, consistency can be achieved only if the length under consideration is negligibly small compared to the acceleration length of the observer.

4. Discussion

It is important to recognize that the hypothesis of locality is an essential element of the theories of special and general relativity. In particular, it is indispensable for the measurement of spatial and temporal intervals by accelerated observers. Therefore, relativistic measurement theory must take this basic assumption and its limitations into account. This has been done for the measurement of time in [17]. In connection with the measurement of distance, it has been shown that there is a lack of uniqueness; however, this problem can be resolved if the distance under consideration is much smaller than the relevant acceleration length of the observer [1, 15]. This means that from a basic standpoint the significance of noninertial reference frames is rather limited [16]. In practice, however, the difference between L and ℓ' (discussed in section 3) is usually rather small; for instance, in the case of the equatorial circumference of the Earth this difference amounts to about 10^{-2} cm [15].

The application of these concepts to standard accelerated measuring devices that are by definition consistent with the hypothesis of locality results in a certain maximal acceleration [18, 19] that is imposed by the quantum theory. For a classical device of mass M, the dimensions of the device must be much larger than $\hbar/(Mc)$ according to the quantum theory of measurement [20, 21]. On the other hand, the dimensions of the device must be much smaller than its acceleration length $\mathcal{L}$. It follows that $\mathcal{L} >> \hbar/(Mc)$ for any standard classical measuring device [2, 4]. Thus for $\mathcal{L} = c^2/a$, the translational acceleration a must be much smaller than $Mc^3/\hbar$, while for $\mathcal{L} = c/\Omega$, the rotational frequency Ω must be much smaller than $Mc^2/\hbar$. Further discussion of the notion of maximal acceleration is contained in [22].

The hypothesis of locality is compatible with wave phenomena only when the latter are considered in the ray limit $(\lambda/\mathcal{L} \to 0)$. To go beyond the basic limitations inherent in the hypothesis of locality regarding the treatment of wave phenomena, a nonlocal theory of accelerated observers has been developed [23, 24, 25]. In this theory, the amplitude of a radiation field as measured by an accelerated observer depends on its history, namely, its past worldline in Minkowski spacetime. This acceleration-induced nonlocality constitutes the first step in the program of developing a nonlocal theory of gravitation.

Appendix: Null acceleration

The relativistic theory of an observer in arbitrary circular motion is treated in section 3. In this case, the proper acceleration length of the observer is defined to be $|I|^{-1/2}$, where I is given by equation (3.16). It is interesting to study the circular motion of an observer with a constant prescribed magnitude of I. In fact, equation (3.16) can be written as

$$\left(\frac{d\hat{v}}{dt}\right)^2 = \frac{1}{r^2}\,\hat{v}^2(1-\hat{v}^2)^2 - I(1-\hat{v}^2)^3\ ,$$

which for constant I can be simply integrated. For the null acceleration case $I = 0$, the solution is

$$\hat{v}^{-2} = 1 + \eta\, e^{\mp 2\frac{t}{r}}$$

for $\eta > 0$. The upper sign refers to motion that asymptotically $(t \to \infty)$ approaches the speed of light, while the lower sign corresponds to an asymptotic state of rest.

References

[1] B. Mashhoon, Phys. Lett. A 145 (1990) 147.
[2] B. Mashhoon, Phys. Lett. A 143 (1990) 176.
[3] A. M. Eisele, Helv. Phys. Acta 60 (1987) 1024.
[4] B. Mashhoon, in: Black Holes: Theory and Observation, Lecture Notes in Physics 514, edited by F. W. Hehl, C. Kiefer and R. J. K. Metzler (Springer, Heidelberg, 1998) pp. 269-284.
[5] J. C. Maxwell, Nature 21 (1880) 314.
[6] H. A. Lorentz, The Theory of Electrons (Dover, New York, 1952).
[7] A. Einstein, The Meaning of Relativity (Princeton University Press, Princeton, 1950).
[8] H. Minkowski, The Principle of Relativity, by H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl (Dover, New York, 1952) p. 80.
[9] M. Born, Ann. Phys. (Leipzig) 30 (1909) 1.

[10] W. Pauli, Theory of Relativity (Dover, New York, 1981).
[11] J. L. Synge, Relativity: The Special Theory, 2nd edition (North-Holland, Amsterdam, 1965).
[12] W. Rindler, Introduction to Special Relativity, 2nd edition (Clarendon Press, Oxford, 1991).
[13] G. Salzman and A. H. Taub, Phys. Rev. 95 (1954) 1659.
[14] H. P. Robertson, Rev. Mod. Phys. 21 (1949) 378.
[15] B. Mashhoon and U. Muench, Ann. Phys. (Leipzig) 11 (2002) 532.
[16] B. Mashhoon, in: Advances in General Relativity and Cosmology, edited by G. Ferrarese (Pitagora, Bologna, 2003).
[17] S. R. Mainwaring and G. E. Stedman, Phys. Rev. A 47 (1993) 3611.
[18] E. R. Caianiello, Lett. Nuovo Cimento 41 (1984) 370.
[19] E. R. Caianiello, Riv. Nuovo Cimento 15 (1992) no. 4.
[20] E. Schrödinger, Preuss. Akad. Wiss. Berlin Ber. 12 (1931) 238.
[21] H. Salecker and E. P. Wigner, Phys. Rev. 109 (1958) 571.
[22] G. Papini, Phys. Lett. A 305 (2002) 359.
[23] B. Mashhoon, Phys. Rev. A 47 (1993) 4498.
[24] C. Chicone and B. Mashhoon, Ann. Phys. (Leipzig) 11 (2002) 309.
[25] C. Chicone and B. Mashhoon, Phys. Lett. A 298 (2002) 229.

Chapter 4

SAGNAC EFFECT: END OF THE MYSTERY

Franco Selleri
Università di Bari - Dipartimento di Fisica
INFN - Sezione di Bari
I 70126 Bari, Italy
selleri@ba.infn.it

Abstract Transformations of space and time depending on a synchronization parameter , e_1, indicate the existence of a privileged inertial system S_0. The Lorentz transformations are obtained for a particular $e_1 \neq 0$. No classical experiment on inertial frames depends on the choice of e_1, but if accelerations are considered only $e_1 = 0$ remains possible. The choice $e_1 = 0$ provides a rational resolution of the long standing mystery connected with the relativistic interpretation of the Sagnac effect.

1. History: 1913 - 2003

In the Sagnac 1913 experiment a circular platform was made to rotate uniformly around a vertical axis at a rate of 1-2 full rotations per second. In an interferometer mounted on the platform, two interfering light beams, reflected by four mirrors, propagated in opposite directions along a closed horizontal circuit defining a certain area A. The rotating system included also the luminous source and a detector (a photographic plate recording the interference fringes). On the pictures obtained during a clockwise and a counterclockwise rotation with the same frequency, the interference fringes were observed to be in different positions. Sagnac measured the relative displacement Δz by overlapping the two figures.

This displacement Δz is strictly tied to the time delay with which a light beam reaches the detector with respect to the other one and turned out to depend on the disk angular velocity. Sagnac observed a shift of the interference fringes every time the rotation was modified. Considering his experiment

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 57-77.

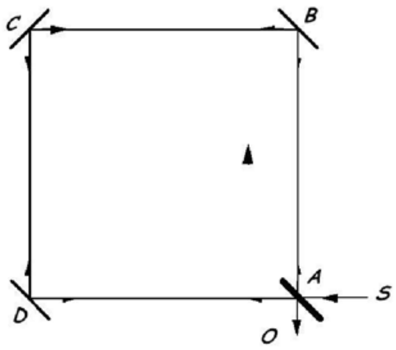

Figure 4.1. Simplified configuration of the Sagnac apparatus. Light from a source S is divided in two parts by the semitransparent mirror A. The first part follows the path $ABCDAO$ concordant with the platform rotation, the second part follows $ADCBAO$ discordant from rotation. The interference fringes are observed in O.

conceptually similar to the Michelson-Morley one, he informed the scientific community with two papers (in French) bearing the titles *"The existence of the luminiferous ether demonstrated by means of the effect of a relative ether wind in an uniformly rotating interferometer"*[1] and *"On the proof of reality of the luminiferous ether with the experiment of the rotating interferometer"*[2].

The experiment was repeated many times in different ways, with the full confirmation of the results obtained by Sagnac. Famous is the 1925 repetition by Michelson and Gale[3] for the very large dimensions of the optical interference system (a rectangle about 650m x 360m); in this case the disk was the Earth itself at the latitude concerned. The light propagation times were not the same, as evidenced by the resulting fringe shift. Full consistency was found with the Sagnac formula [Eq. (4.3) below] if the angular velocity of the Earth rotation was used.

An important question is the following: can light propagate with the usual velocity c relatively to the rotating platform? The question was directly faced in the 1942 experiment by Dufour e Prunier[4], in which the mirrors defining the paths of the interfering light beams were partly fixed in the laboratory (directly above the disk) and partly in the spinning disk. The fringe shifts were the same as in a repetition of the test with all mirrors fixed on the disk, confirming that the light does not adapt to the movement of the disk, and that it is physically connected with some other reference system, in all probability inertial.

Surprisingly theoreticians were little interested in the Sagnac effect, as if it did not pose a conceptual challenge. As far as I know Einstein's publications never mentioned it, for example. A first discussion by Langevin came only 7-

8 years later[5] and was as much formally self-assured as substantially weak. One of the opening statements is this: "I will show how the theory of general relativity explains the results of Sagnac's experiment in a quantitative way". Langevin argues that Sagnac's is a first order experiment, on which all theories (relativistic or prerelativistic) must agree qualitatively and quantitatively, given that the experimental precision does not allow one to detect second order effects: therefore it cannot produce evidence for or against any theory. Then he goes on to show that an application of Galilean kinematics explains the empirical observations! In fact his approach is only slightly veiled in relativistic form by some words and symbols, but is really 100% Galilean.

The impression that Langevin, beyond words, could not be satisfied with his explanation is reinforced by his second article of 1937[6] in which two(!) relativistic treatments are presented. The first one is still that of 1921, this time deduced from the strange idea that the time to be adopted everywhere on the platform is that of the rotational centre (which is motionless in the laboratory). The second one is to define "time" in such a way as to enforce a velocity of light constant and equal to c by starting from a non total differential, falling so flatly in the problem of the discontinuity for a tour around the disk that we will discuss later.

In 1963 was published the very influential review paper by Post[7], who seems to agree with the idea that two relativistic proofs of the Sagnac effect are better than one. The first proof (in the main text) uses arbitrarily the laboratory to platform transformation of time $t' = t\,R$ where R is the usual square root factor of relativity, here written with the rotational velocity. The second proof (in an appendix) starts from the Lorentz transformation $t' = \left(t + \vec{v} \cdot \vec{r}/c^2\right)/R$, but it hastens to make the second term disappear with the (arbitrary) choice of $\vec{r}$ perpendicular to $\vec{v}$.

The tendency to cancel the spatial variables in the transformation of time is thus common to Langevin and Post and shows once more the great difficulty in explaining the physics of the rotating platform with the TSR. The final result can only be a great confusion, to the point that Hasselbach and Nicklaus, describing their own experiment[8], list about twenty different explanations of the Sagnac effect and comment: "This great variety (if not disparity) in the derivation of the Sagnac phase shift constitutes one of the several controversies ... that have been surrounding the Sagnac effect since the earliest days of studying interferences in rotating frames of reference".

In the present paper we will show that the problems concerning the Sagnac effect are overcome by adopting on the rotating platform the one way velocity of light given by

$$c_1(\theta) = \frac{c}{1 + \beta\cos\theta} \tag{4.1}$$

with $\beta = \omega\, r/c$, where r is the distance from the platform rotation centre, ω the angular velocity and θ the angle between the light propagation direction and the rotational velocity in the point where light is moving. In general $c_1(\theta)$ varies from point to point of the light path, but it equals constantly either $c_1(0)$ or $c_1(\pi)$ if light is moving on a circle centered in the platform rotation centre.

Equation (4.1) is valid in inertial systems, where the "inertial transformations" give the best description of the empirical evidence[9]: see Appendix 4.B for a short review. It applies also to the rotating platform, a small segment of which for a short time can be considered as practically belonging to a comoving inertial system.

2. The Sagnac Correction on the Earth Surface

As recounted by Kelly[10], in 1980 the CCDS (*Comitè Consultatif pour la Dèfinition de la Seconde*) and in 1990 the CCIR (*International Radio Consultative Committee*) suggested rules - later universally adopted - for synchronizing clocks in different points of the globe. Two are the methods used to accomplish this task. The first one is to transport a clock from one site to another and to regulate clocks at rest in the second site with the time reading of the transported clock. The second method is to send an electromagnetic signal informing the second site of the time reading in the first site. The rules of the committee establish that three corrections should be applied before comparing clock readings:

(**a**) the first correction keeps into account the velocity effect of the theory of special relativity (TSR). It is proportional to $v^2/2c^2$, where v is the velocity of the airplane, and corresponds to a slower timing of the transported clock;

(**b**) the second correction keeps into account the gravitational effect of the theory of general relativity (TGR). It is proportional to $g(\phi)h/c^2$ where g is the total acceleration (gravitational and centrifugal) at sea level at the latitude ϕ and h is the height over sea level. It corresponds to a faster timing of the transported clock;

(**c**) the "Sagnac correction" is assumed proportional to $2A_E\omega/c^2$, where A_E is the equatorial projection of the area enclosed by the path of travel of the clock (or of the electromagnetic signal) and the lines connecting the two clock sites to the centre of the Earth, and ω is the angular velocity of the Earth.

There are no doubts about nature and need of the first two corrections, but the justification of the third one is unconvincing. I agree completely with Kelly [11] when he says that the only possible reason to include (**c**) is that the eastward velocity of light relative to the Earth is different from the westward.

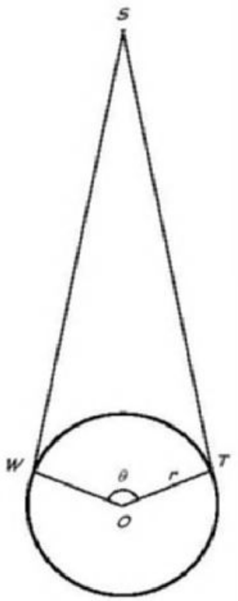

Figure 4.2. An electromagnetic signal travels between two points W and T on the Earth via a geostationary satellite S (seen from the North pole).

In fact we will next deduce, for a real experiment, the "Sagnac correction" from Eq. (4.1) applied to a geostationary satellite, for which the satellite itself and the Earth surface can be thought to be at rest on the same rotating platform.

Saburi *et al.* carried out their experiment in 1976, before the CCDS and CCIR deliberations, and made clear that "corrections" were indeed necessary already in the title of their paper[12] ("*High-Precision Time Comparison via Satellite and Observed Discrepancy of Synchronization*"). They had two atomic clocks, not quite synchronous, one in a first station W (near Washington, USA) the other one in a second station T (near Tokyo, Japan) practically on the same parallel of the two cities. The time difference between the two clocks at 02h 34m UTC on August 27, 1976 was measured with two different methods:

(i) by sending an airplane carrying a third clock (initially synchronous with the one in W) from W to T, via Hawaii (westward);

(ii) by sending an electromagnetic signal, via a geostationary satellite, from W to T, again westward.

The uncorrected airplane clock found the T clock $9.42\ \mu s$ fast with respect to the W clock. The velocity correction and the gravitational correction together were estimated to be about $0.080\ \mu s$ (to be subtracted to the time shown by the transported clock). By applying such a correction the $T - W$ time difference increased to $9.50\ \mu s$.

The electromagnetic signal carried with itself the time shown by the clock of the transmitting station. Assuming that the signal velocity was c, it was found that the T clock was $9.11\ \mu s$ fast with respect to the W clock.

Thus, the discrepancy between the two measurements was about $0.39\ \mu s$.

Let L_{WS} and L_{ST} the Washington-satellite and Tokyo-satellite distances, respectively (see figure 4.2). As most physicists in similar experiments, Saburi and collaborators synchronized clocks by imposing that the velocity of light is c, that is in such a way that

$$t_T - t_W = \frac{L_{WS} + L_{ST}}{c} \tag{4.2}$$

t_W and t_T being the times of signal departure from W and arrival in T as marked by the respective clocks. In order to ensure that Eq. (4.2) applied to their clocks they had to apply the so called "Sagnac correction" to the clock of the receiving station. Such a correction is given by

$$\Delta t_T = \frac{2\,\omega\,A_E}{c^2} \tag{4.3}$$

where A_E is the area of the quadrangle $OWSTO$ of figure 4.2.

By adopting (4.2) Saburi and collaborators made an error because, as we now know (see Appendix 4.B), the correct velocity of light is that given by the inertial transformations, which in the appropriate directions is

$$c_{WS} = \frac{c}{1 + \beta \cos \alpha_{WS}}\,;\, c_{ST} = \frac{c}{1 + \beta \cos \alpha_{ST}} \tag{4.4}$$

where $\beta = \omega\, r/c$ (r is the radius of the $W - T$ parallel and ω is the Earth angular velocity), α_{WS} is the angle between the line WS and the local velocity (normal to the radius OW), α_{ST} is the angle between the line ST and the normal to the radius OT in figure 4.2. Therefore

$$\alpha_{WS} = \theta_W - \frac{\pi}{2}\,;\, \alpha_{ST} = \theta_T - \frac{\pi}{2} \tag{4.5}$$

where θ_W and θ_T are the angles $O\hat{W}S$ and $O\hat{T}S$ of figure 4.2, respectively.

But (4.4) is not the velocity adopted in this experiment. Having imposed the impossible condition (4.2) the quoted authors had now to apply the mysterious "Sagnac correction" Δt_T on the time of arrival in T. Such a correction, from our point of view, can be calculated by replacing c with c_{WS} and c_{ST} as follows

$$\Delta t_T = \frac{L_{WS}}{c_{WS}} - \frac{L_{WS}}{c} + \frac{L_{ST}}{c_{ST}} - \frac{L_{ST}}{c} \tag{4.6}$$

which is positive because $c > c_{WS},\ c_{ST}$. One can also write

$$\Delta t_T = \frac{\beta \left(L_{WS} \cos \alpha_{WS} + L_{ST} \cos \alpha_{ST}\right)}{c} \tag{4.7}$$

whence, using (4.5)

$$\Delta t_T = \frac{\omega\, r\ \left(L_{WS}\ sin\theta_W + L_{ST}\ sin\theta_T\right)}{c^2} \tag{4.8}$$

But $r L_{WS} \sin\theta_W + r L_{ST} \sin\theta_T = 2A_E$, where A_E is the area of the quadrangle $OWSTO$ of figure 4.2. We have thus provided a full physical justification of (4.3). We see that the mystery of the "Sagnac correction" of Earth physics is fully eliminated by adopting the inertial transformations. The procedure which we can suggest to experimentalists is to avoid using a wrong velocity of light and correcting the result with an *ad hoc* term, but rather to use from the beginning the velocity of light (4.1) of the inertial transformations.

Our present results confirm the following qualitative observation of Hayden [13]: electromagnetic signals need more time for a full tour around our planet toward east than toward west and this can only mean that relatively to the Earth the velocity of light in the two senses is not the same.

3. Rotating Platforms

In this section we review earlier results and show that the comparison between the relativistic descriptions of rotating platforms and inertial reference systems points out to the existence of a fundamental difficulty. Furthermore in the next section we show that this difficulty can be overcome only by substituting the Lorentz transformations between inertial systems with the "inertial" ones[14]. The problem is tightly bound to the Sagnac effect.

It is well known that no perfectly inertial frame exists in practice because of Earth rotation, of orbital motion around the Sun, of Galactic rotation. All knowledge about inertial systems has therefore been obtained in frames having small but non zero acceleration a. For this reason the mathematical limit $a \to 0$ taken in the theoretical schemes should be smooth and no discontinuities should arise between systems with small acceleration and inertial systems. From such a point of view the existing relativistic theory will be shown to be inconsistent.

Consider an inertial reference system S_0 and assume that it is isotropic so that the one-way velocity of light relative to S_0 has the usual value c in all directions. In relativity the latter assumption is true in all inertial frames, while in other theories only one such frame exists.

In this system there is a circular platform having radius r and centre constantly at rest in S_0 which rotates around its axis with constant angular velocity ω and peripheral velocity $v = \omega\, r$. On its rim, consider a single clock C_Σ

(marking the time t) and assume it to be set as follows: When a clock of the laboratory momentarily very near C_Σ shows time $t_0 = 0$ then also C_Σ is set at time $t = 0$. When the platform is not rotating, C_Σ constantly shows the same time as the laboratory clocks. When it rotates, however, motion modifies the pace of C_Σ and the relationship between the times $t\ and\ t_0$ is taken to have the general form

$$t_0 \; = \; t\, F\,(v,\; ...) \tag{4.9}$$

where F is a function of velocity v and eventually acceleration and higher derivatives of position (not shown). Eq. (4.9) is a consequence of the isotropy of S_0. Its validity can be shown in three steps:

1. In the inertial system S_0 all directions are physically equivalent. If a clock is moving on a straight line ℓ with a certain speed v relative to S_0, the modification of the rate of advancement of its hands cannot depend on the orientation of ℓ.

2. A similar case is the clock C_Σ at rest on the rim of a platform, whose centre is at rest in S_0, rotating with constant angular velocity. If space is isotropical the speed of its hands cannot depend on the angle between the clock instantaneous velocity vector and any given direction in S_0 but only on speed v and eventually acceleration.

3. This conclusion, clearly correct by symmetry reasons, was confirmed experimentally by the 1977 CERN measurements of the anomalous magnetic moment of the muon[15]. The decay of muons was followed very closely in different parts of the storage ring and the results showed a decay rate constant in the different points of the trajectory.

Thus we have every reason to believe (4.9) to be correct. We are of course far from ignorant about the function F. There are strong experimental indications [15] that the dependence on the acceleration is totally absent and that:

$$F\,(v,\; ...) \; = \; \frac{1}{\sqrt{1 - v^2/c^2}} \tag{4.10}$$

Important as it is, Eq. (4.10) is however irrelevant for our present needs, because the results obtained below hold for all possible factors F.

On the rim of the platform besides clock C_Σ there is a light source Σ placed in a fixed position very near C_Σ. Two light flashes leave Σ at the time t_1 of C_Σ and are forced to move on a circumference, by "sliding" on the internal surface of a cylindrical mirror placed at rest on the platform, all around it and very near its border. Mirror apart, the light flashes propagate in the vacuum. The motion of the mirror cannot modify the velocity of light, because the mirror behaves like a source ("virtual") and a source motion never changes the velocity of the emitted light signals. Thus, relative to the laboratory, the light flashes propagate with the usual velocity c.

The description of light propagation given by the laboratory observers is the following: two light flashes leave Σ at time t_{01}. The first one propagates on a circumference, in the sense discordant from the platform rotation, and comes back to S at time t_{02} after a full circle around the platform. The second flash propagates on the same circumference, in the sense concordant with the platform rotation, and comes back to Σ at time t_{03} after a full circle around the platform. These laboratory times, all relative to events taking place in a fixed point of the platform very near C_Σ, are related to the corresponding platform times via (4.9):

$$t_{0i} = t_i\, F\,(v,\ ...)\ \ (i = 1{,}2{,}3) \tag{4.11}$$

The circumference length is assumed to be L_0 and L, measured in the laboratory S_0 and on the platform, respectively. Light propagating in the direction opposite to the disk rotation, must cover a distance smaller than L_0 by $x = \omega\, r\ (t_{02} - t_{01})$, the shift of Σ during the time $t_{02} - t_{01}$ taken by light to reach Σ. Therefore

$$L_0 - x = c\,(t_{02} - t_{01})\,;\, x = \omega\, r\,(t_{02} - t_{01}) \tag{4.12}$$

From these equations it follows:

$$t_{02} - t_{01} = \frac{L_0}{c\,(1 + \beta)} \tag{4.13}$$

with $\beta = \omega\, r\,/c$. Light propagating in the rotational direction of the disk, must instead cover a distance larger than the disk circumference length L_0 by a quantity $y = \omega\, r\,(t_{03}-t_{01})$ equalling the shift of Σ during the time $t_{03}-t_{01}$ taken by light to reach Σ. Therefore

$$L_0 + y = c\,(t_{03} - t_{01})\,;\, y = \omega\, r\,(t_{03} - t_{01}) \tag{4.14}$$

One now gets

$$t_{03} - t_{01} = \frac{L_0}{c\,(1 - \beta)} \tag{4.15}$$

The difference between (4.15) and (4.13) gives the delay between the arrivals of the two light flashes back in Σ which is

$$t_{03} - t_{02} = \frac{2\, L_0\, \beta}{c\ (1 - \beta^2)} \tag{4.16}$$

This is the well known delay time for the Sagnac effect calculated in the laboratory. We show next that these relations fix to some extent the velocity of light relative to the disk. In fact Eq. (4.11) applied to (4.13) and (4.15) gives

$$(t_2 - t_1)\ F = \frac{L_0}{c\,(1 + \beta)}\,;\ (t_3 - t_1)\ F = \frac{L_0}{c\,(1 - \beta)} \tag{4.17}$$

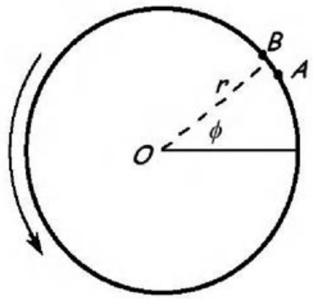

Figure 4.3. By symmetry reasons, the velocity of light relative to the rotating disk between two nearby points A and B does not depend on the angle ϕ fixing the position of the segment AB on the rim of the disk.

If $\tilde{c}\,(0)$ and $\tilde{c}\,(\pi)$ are the light velocities, relative to the disk, for the flash propagating in the direction of the disk rotation and in the opposite direction, respectively, we have from the very definition of velocity

$$\frac{1}{\tilde{c}\,(\pi)} = \frac{t_2 - t_1}{L} = \frac{L_0/L}{F\,c\,(1+\beta)}\,;\,\frac{1}{\tilde{c}\,(0)} = \frac{t_3 - t_1}{L} = \frac{L_0/L}{F\,c\,(1-\beta)} \tag{4.18}$$

From (4.18) it follows :

$$\frac{\tilde{c}(\pi)}{\tilde{c}(0)} = \frac{1\,+\,\beta}{1\,-\,\beta} \tag{4.19}$$

Notice that the function F has disappeared in the ratio (4.19).

Next comes an important remark. Clearly, Eq. (4.19) gives us not only the ratio of the two global light velocities for full trips around the platform, but *the ratio of the instantaneous velocities as well.* In fact the isotropy of the inertial system S_0 ensures, by symmetry, that the instantaneous velocities of light are the same in all points of the rim of the rotating circular disk whose centre is at rest in S_0. There is no reason why the light instantaneous velocities relative to the disk in the different points of the rim should not be equal to one another. With reference to figure 4.3 we can therefore write the equations

$$\tilde{c}_{\phi_1}(0) \;=\; \tilde{c}_{\phi_2}(0)\,;\,\tilde{c}_{\phi_1}(\pi) \;=\; \tilde{c}_{\phi_2}(\pi)$$

where ϕ_1 and ϕ_2 are arbitrary values of the angle ϕ.

Therefore the light instantaneous velocities relative to the disk will also coincide with the average velocities $\tilde{c}(0)$ and $\tilde{c}(\pi)$, and Eq. (4.19) will apply also to the ratio of the instantaneous velocities [thus we do not need a different symbol for the instantaneous velocities].

The consequences of (4.19) applied to instantaneous velocities will be discussed in the next section.

4. Absolute simultaneity in inertial systems

The result (4.19) holds with the same numerical value for platforms having different radius, but the same peripheral velocity v. Let a set of circular platforms be given with centres at rest in S_0. Let their radii be $r_1,\ r_2,\ ...\ r_i,\ ...$, with $r_1 < r_2 <\ ... < r_i <\ ...$, and suppose they are made to spin with angular velocities $\omega_1,\ \omega_2,\ ...\ \omega_i,\ ...$ such that

$$\omega_1 r_1 \ = \omega_2 r_2 = \ ... \ = \omega_i r_i = \ ... \ = \ v \tag{4.20}$$

where v is constant. Obviously, then, (4.19) applies to all such platforms with the same β $(\beta = v/c)$. The centripetal accelerations decrease regularly with increasing r_i. Therefore, a small part AB of the rim of a platform, having peripheral velocity v and large radius, for a short time is completely equivalent to a small part of a "comoving" inertial reference frame (endowed with the same velocity). For all practical purposes the segment AB will belong to that inertial reference frame. But the velocities of light in the two directions AB and BA has to satisfy (4.19). It follows that the one way velocity of light relative to the comoving inertial frame cannot be c and must instead satisfy

$$\frac{c_1(\pi)}{c_1(0)} = \frac{1 + \beta}{1 - \beta} \tag{4.21}$$

As shown in Appendix 4.A the equivalent transformations (of which the Lorentz transformations are a particular case) predict the inverse one way velocity of light relative to the comoving system S:

$$\frac{1}{c_1(\theta)} = \frac{1}{c} + \left[\frac{\beta}{c} + e_1\, R\right] \cos\theta \tag{4.22}$$

where θ is the angle between the light propagation direction and the absolute velocity $\vec{v}$ of S. Eq. (4.22) applied to the cases $\theta = 0$ and $\theta = \pi$ gives

$$\frac{1}{c_1(0)} = \frac{1}{c} + \left[\frac{\beta}{c} + e_1 R\right] ; \frac{1}{c_1(\pi)} = \frac{1}{c} - \left[\frac{\beta}{c} + e_1 R\right] \tag{4.23}$$

whence

$$\frac{c_1(\pi)}{c_1(0)} = \frac{1 + \beta + c\, e_1 R}{1 - \beta - c\, e_1 R} \tag{4.24}$$

Clearly Eq. (4.24) is compatible with (4.21) only if

$$e_1 \ = \ 0 \tag{4.25}$$

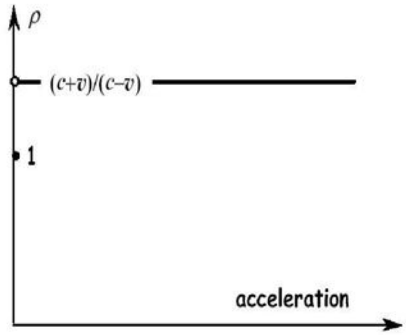

Figure 4.4. The ratio $\rho = \tilde{c}(\pi)/\tilde{c}(0)$ plotted as a function of acceleration for rotating platforms of constant peripheral velocity and decreasing radius (increasing acceleration). The prediction of the TSR is 1 (black dot on the ρ axis) and is not continuous with the ρ value of the rotating platforms.

We thus see that our fundamental result (4.21) is consistent with the physics of the inertial systems only if absolute simultaneity is adopted.

For a better understanding of the reasons why the TSR does not work consider again the ratio

$$\rho \equiv \frac{\tilde{c}(\pi)}{\tilde{c}(0)} \tag{4.26}$$

which, owing to (4.19), is larger than unity. Therefore the light velocities parallel and antiparallel to the disk peripheral velocity are different. For the TSR this conclusion is unacceptable, because a set of platforms, all endowed with the same peripheral velocity locally approximates an inertial system better and better with increasing radius. The logical situation is shown in figure 4.4.

Thus the TSR predicts for ρ a discontinuity at zero acceleration.

While all the experiments are performed in the real physical world [where, of course, $a \neq 0, \rho = (1+\beta)/(1-\beta)$], the theory has gone out of the world $(a = 0, \rho = 1)$!

This discontinuity is the origin of the problems met with clock synchronization on or near the Earth surface we discussed in section 2. This is not a surprise: after all also the Earth is some kind of rotating platform!

Notice that the velocity of light given by Eq. (4.22) with $e_1 = 0$ is required for all inertial systems but one, the isotropic system S_0. In fact, for every small region AB of every such system it is possible to imagine a large rotating platform with center at rest in S_0 and rim locally comoving with AB and the result

(4.25) can be applied. Therefore the velocity of light depends on direction in all inertial systems with the sole exception of the privileged one S_0.

5. The impossible defense of orthodoxy

It was pointed out by T. Van Flandern[16] that an "orthodox" way of dealing with the rotating platform problem should assume a position dependent desynchronization, with respect to the laboratory clocks, as an objective phenomenon, concretely applicable to the clocks placed in different fixed points of the rim of the platform. The Lorentz transformation of time

$$t\,R\;-\;t_0\;=\;-\,x_0\,v/c^2 \tag{4.27}$$

can be read as follows: the difference between the time t of the "moving" frame S (corrected by a R factor in order to cancel the time dilation effect) and the time t_0 of the "stationary" frame S_0 has a linear dependence on the coordinate x_0 of S_0 as given by (4.27). This difference is called "desynchronization".

Applied to the rim of the circular platform of radius r Eq. (4.27) would become

$$t\,R\;-\;t_0\;=\;-\,r\,\theta_0\,v/c^2 \tag{4.28}$$

where θ_0 is the angle between the radius on which the given clock is placed and any fixed direction.

Why the orthodox idea cannot work? Well, for at least three reasons. First of all because the whole argument leading to (4.19) and then to (4.21) was based on a single clock in a fixed position, for which it does not make sense to assume a position dependent desynchronization. Secondly because experimental evidence shows that many small clocks (muons) injected in different points of the CERN muon storage ring behave in the same way, independently of their position: in such a case it is certainly not possible to conceive a human intervention desynchronizing the muons! Thirdly because, anyway, such an approach would end in a mess.

Let us see how. Assume that the disk rotation is counterclockwise in figure 4.5. Two flashes of light are emitted at laboratory time $t_0 = 0$ by the source Σ in opposite directions along the platform border. Assuming Eq. (4.28) to be applicable, when the right-moving (left-moving) flash reaches point A (point B) at lab. time t_{0A} (t_{0B}) after covering a distance x_{0A} (x_{0B}) (measured in the lab. along the platform border), it will find a local clock desynchronized by $\Delta t_A > 0$ ($\Delta t_B < 0$) with respect to the laboratory clocks given by

$$\Delta t_A\;=\;x_{0A}v/c^2\;+\;\alpha\,;\quad \Delta t_B\;=\;-\;x_{0B}v/c^2\;+\;\alpha \tag{4.29}$$

where α represents whatever desynchronization the clock placed in Σ might have had at time $t_0\;=\;0$ with respect to the laboratory clocks. We will now show that the desynchronization (4.29), far from eliminating the discontinuity

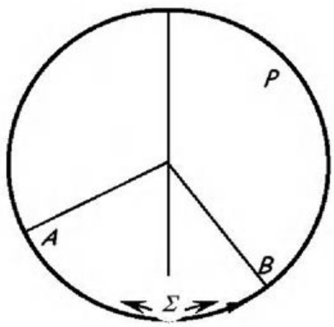

Figure 4.5. Two light flashes emitted by Σ in opposite directions along the platform border at the laboratory time $t_0 = 0$ meet in the point P in which the local platform clock should mark two different times.

of figure 4.4, gives rise to a further discontinuity in the time shown by clocks placed on the platform rim.

From the point of view of the laboratory observers the space between Σ and the right-moving (left-moving) flash opens at a rate $c - v$ $(c + v)$. Therefore:

$$(c-v)\ t_{0A}\ =\ x_{0A}\ ;\ (c+v)\ t_{0B}\ =\ x_{0B}$$

From Eq. (4.29) we then get

$$\Delta t_A =\ (c-v)\ t_{0A}\, v/c^2 +\ \alpha\, ; \Delta t_B =\ -(c+v)\ t_{0B}\, v/c^2 +\ \alpha \quad (4.30)$$

There will be a time t_{0P} in which the two flashes meet in point P after describing different paths. When this happens one should have

$$t_{0A}\ =\ t_{0B}\ =\ t_{0P} \quad (4.31)$$

The problem is that Eqs. (4.30) should both apply to the same clock at the common time t_{0P}, but they are instead incompatible if (4.31) is satisfied, as $\Delta t_A =\ \Delta t_B$ would then give $c = -c$.

We can add that Eq. (4.29) is in sharp contradiction with the rotational invariance assumed above to prove the existence of the discontinuity of figure 4.4: if the inertial system S_0 (in which the centre of the circular platform is at rest) is isotropic and if the platform is set in rotation in a regular way, no difference between clocks on its border can ever arise. It is impossible to understand why the clock in point A should be desynchronized differently from the clock in B, unless this is achieved artificially by some observer.

In conclusion this "orthodox" way of dealing with the rotating platform problem is only a useless complication. The only rational possibility remains the adoption of our inertial transformations.

6. New proofs of absolute simultaneity

We will now present a more general proof of the absolute simultaneity condition $e_1 = 0$, by deducing it in the broader context of the "general transformations" [Eqs. (4.32) below]. In this way absolute simultaneity will be shown to be necessary in all theories avoiding the discontinuity between inertial and accelerated systems.

Given the inertial frames S_0 and S one can set up Cartesian coordinates and make the following usual assumptions:

(**i**) Space is homogeneous and isotropic and time homogeneous, at least if judged by observers at rest in S_0, so that relatively to S_0 the velocity of light is the same ("c") in all directions, clocks can be synchronized in S_0 with Einstein's method, and the one way velocities can be measured in S_0 ;

(**ii**) The origin of S, observed from S_0, is seen to move with velocity $v < c$ parallel to the $+x_0$ axis, that is according to the equation $x_0 = v\, t_0$;

(**iii**) The axes of S and S_0 coincide at $t = t_0 = 0$;

The general transformations from S_0 to S are then necessarily

$$\left\{ \begin{array}{rcl} x &=& f_1(x_0 - vt_0) \\ y = g_2\, y_0\, ; & z & = g_2\, z_0 \\ t &=& e_1\, x_0 + e_4 t_0 \end{array} \right. \tag{4.32}$$

where $f_1,\ g_2,\ e_4$ and e_1 are v dependent parameters. The transformations inverse of (4.32) can easily be shown to be

$$\left\{ \begin{array}{rcl} x_0 &=& \frac{(e_4/f_1)x + v\, t}{e_4 + e_1\, v} \\ y_0 = \frac{1}{g_2}\, y\, ; & z_0 & = \frac{1}{g_2}\, z \\ t_0 &=& \frac{t - (e_1/f_1)\, x}{e_4 + e_1\, v} \end{array} \right. \tag{4.33}$$

The one way velocity of light relative to the moving system S, $c_1(\theta)$, can be found by applying (4.33) to the equation

$$x_0^2 + y_0^2 + z_0^2 = c^2 t_0^2 \tag{4.34}$$

describing a spherical wave front born at time $t_0 = 0$ in the origin of the axes of the isotropic system S_0. A calculation lengthy but devoid of conceptual difficulties leads to the following result:

$$\frac{1}{c_1(\theta)} = \frac{1}{f_1\ (1-\beta^2)} \left\{ \left(e_1 + \frac{e_4}{c}\, \beta \right)\, cos\theta + \left(\frac{e_4}{c} + e_1\, \beta \right)\, \left[cos^2\theta + \gamma^2 sin^2\theta \right]^{1/2} \right\} \tag{4.35}$$

where $\beta = v/c$, θ is the angle between the light propagation direction and the absolute velocity $\vec{v}\ of\ S$, and

$$\gamma^2 = \frac{f_1^2\left(1-\beta^2\right)}{g_2^2} \tag{4.36}$$

From Eq. (4.35) one obviously gets

$$\begin{cases} \frac{1}{c_1(0)} = \frac{1}{f_1\ (1-\beta^2)} \left(\frac{e_4}{c} + e_1\right)(1 + \beta) \\ \frac{1}{c_1(\pi)} = \frac{1}{f_1\ (1-\beta^2)} \left(\frac{e_4}{c} - e_1\right)(1 - \beta) \end{cases} \tag{4.37}$$

whence

$$\frac{c_1(\pi)}{c_1(0)} = \frac{\left(\frac{e_4}{c} + e_1\right)(1 + \beta)}{\left(\frac{e_4}{c} - e_1\right)(1 - \beta)} \tag{4.38}$$

As we know, the assumed continuity between rotating platforms and inertial systems leads to Eq. (4.21) which we repeat here:

$$\frac{c_1(\pi)}{c_1(0)} = \frac{1 + \beta}{1 - \beta} \tag{4.39}$$

By comparing (4.38) and (4.39) it follows

$$e_1 = 0 \tag{4.40}$$

Therefore the most general transformations of space and time between inertial systems allowed by continuity are

$$\begin{cases} x = f_1(x_0 - vt_0) \\ y = g_2\ y_0\ ;\ z = g_2\ z_0 \\ t = e_4 t_0 \end{cases} \tag{4.41}$$

and imply the necessary existence of absolute simultaneity. In fact, two point-like events with coordinates x_{10} and x_{20} ($x_{01} \neq x_{02}$) taking place at the same time t_0, according to the fourth of (4.41) are judged simultaneous also in S. Once more the absolute simultaneity is seen to be unavoidable. With $e_1 = 0$ the velocity of light becomes

$$\frac{1}{c_1(\theta)} = \frac{e_4}{c\ f_1\ (1-\beta^2)} \left\{\left[cos^2\theta + \gamma^2 sin^2\theta\right]^{1/2} + \beta\ cos\theta\right\} \tag{4.42}$$

showing that the β cosθ term in the one way velocity of light is a fixed ingredient in all theories of inertial systems satisfying the continuity condition with the accelerated ones.

The Galilean transformations are of the type (4.32) with

$$f_1^G = g_2^G = e_4^G = 1\ ; e_1^G = 0 \tag{4.43}$$

Using (4.43) the one way velocity of light of the Galilean theory can be obtained as a particular case from (4.42) and turns out to be given by

$$\frac{1}{c_1^G(\theta)} = \frac{1}{c\ (1\ -\ \beta^2)} \left\{ \left[1 - \beta^2 sin^2\theta \right]^{1/2}\ +\ \beta\, cos\theta \right\} \tag{4.44}$$

Naturally Eq. (4.44) contains the characteristic term $\beta\cos\theta$ of all theories treating inertial systems in a way continuous with the accelerated ones. The absence of this term in the TSR, in which $c_1 = c$ is isotropic, gives rise to the discontinuity of figure 4.4 which, as we saw, can be eliminated only by adopting $e_1 = 0$.

We can conclude that the famous synchronization problem is solved by nature itself: it is not true that the synchronization procedure can be chosen freely as the usually adopted convention leads to an unacceptable discontinuity in the physical theory.

Appendix: A - The Equivalent Transformations

According to Poincarè[17], Reichenbach[18], Jammer[19] and Mansouri and Sexl[20] the clock synchronization in inertial systems is conventional, but the choice based on the invariance of the one way velocity of light made in the TSR was legitimate on reasons of simplicity. In [21] I showed that a suitable parameter e_1 can be introduced to allow for different synchronizations in the transformations of the space and time variables. The TSR is obtained for a particular nonzero value of e_1.

These developments are briefly reviewed in the present section. I also found, however, that the choice $e_1 = 0$ is the only one allowing for a treatment of accelerations rationally connected with the physics of inertial systems. This result is deduced once more in the main text as far as centripetal accelerations are concerned. Another proof of $e_1 = 0$ is reviewed in Appendix 4.B.

Given the inertial frames S_0 and S one can set up Cartesian coordinates and make the following assumptions:

(**i**) Space is homogeneous and isotropic and time homogeneous, at least if judged by observers at rest in S_0;

(**ii**) In the isotropic system S_0 the velocity of light is "c" in all directions, so that clocks can be synchronized in S_0 and one way velocities relative to S_0 can be measured;

(**iii**) The origin of S, observed from S_0, is seento move with velocity $v < c$ parallel to the $+x_0$ axis, that is according to the equation $x_0 = v\ t_0$;

(**iv**) The axes of S and S_0 coincide for $t = t_0 = 0$;

The system S_0 turns out to have a privileged status in all theories satisfying the assumptions (**i**) and (**ii**), with only one exception, the TSR. Two further assumptions based on direct experimental evidence can be added:

(**v**) The two way velocity of light is the same in all directions and in all inertial systems[22];

(**vi**) Clock retardation takes place with the usual velocity dependent factor when clocks move with respect to the isotropic reference frame S_0 ([23]–[26]).

These conditions were shown[21] to imply for the transformations of the space and time variables from S_0 to S;

$$\begin{cases} x = \frac{x_0 - v\, t_0}{R} \\ y = y_0 \; ; \; z = z_0 \\ \\ t = R\, t_0 + e_1\, (x_0 - v\, t_0) \end{cases} \tag{4.A.1}$$

where

$$R = \sqrt{1 - v^2/c^2} \tag{4.A.2}$$

Eqs. (4.A.1) and the assumption (ii) imply that relative to the moving system S the one way velocity of light propagating at an angle θ from the velocity $\vec{v}$ of S relative to S_0 ("absolute velocity") is[21]:

$$c_1(\theta) = \frac{c}{1 + \Gamma \cos\theta} \tag{4.A.3}$$

with

$$\Gamma = \frac{v}{c^2} + e_1\, R \tag{4.A.4}$$

while, of course, the two way velocity of light is c in all directions.

The inverse transformations of (4.A.1) are

$$\begin{cases} x_0 = (R - e_1 v)\, x + \frac{v\, t}{R} \\ y_0 = y \; ; \; z_0 = z \\ t_0 = \frac{t - R\, e_1\, x}{R} \end{cases} \tag{4.A.5}$$

All theories with different values of e_1 imply the existence of a privileged inertial system, S_0, in which the velocity of light is isotropic, as it is clear also from (4.A.3)-(4.A.4) since $c_1(\theta) \rightarrow c$ if $v \rightarrow 0$, given that in this limit the transformations (4.A.1) must become identities and therefore $e_1 \rightarrow 0$ and $\Gamma \rightarrow 0$. This is very important. If our theory describes correctly the physical reality a particular inertial system has to exist in which simultaneity and time are not conventional but truly physical. This should be the system in which the Lorentz ether is at rest, of course.

The TSR is a particular case of the previous theory, obtained for

$$e_1 = -\frac{v}{c^2 R} \tag{4.A.6}$$

giving $\Gamma = 0$ and $c_1(\theta) = c$ and reducing (4.A.1) to the Lorentz form.

Appendix: B - The Inertial Transformations

The inertial transformations are obtained from the equivalent transformations, by setting $e_1 = 0$:

$$\begin{cases} x = \frac{x_0 - v\, t_0}{R} \\ y = y_0 \; ; \; z = z_0 \\ \\ t = R\, t_0 \end{cases} \tag{4.B.1}$$

where R is the usual square root given by (4.A.2).

Eqs. (4.B.1) imply that relative to the moving system S the one way velocity of light propagating at an angle θ from the velocity $\vec{v}$ of S relative to S_0 ("absolute velocity") is[21]:

$$c_1(\theta) = \frac{c}{1 + \beta\cos\theta} \tag{4.B.2}$$

with

$$\beta = v/c \tag{4.B.3}$$

while, of course, the two way velocity of light is c in all directions.

The inverse transformations of (4.B.1) are

$$\begin{cases} x_0 = R\left[x + \frac{v\,t}{R^2}\right] \\ y_0 = y\,;\; z_0 = z \\ t_0 = \frac{t}{R} \end{cases} \tag{4.B.4}$$

The theory of the inertial transformations implies the existence of a privileged inertial system, S_0, in which the velocity of light is isotropic, as it is clear from (4.B.2) if $\beta = 0$.

The usually assumed indifference of the physical reality about clock synchronization exists only insofar as one neglects accelerations. When these come into play every inertial system exists, so to say, only for a vanishingly small time interval and it is physically impossible in the accelerated frame to adopt any time-consuming procedure for the synchronization of distant clocks (such as Einstein's procedure based on light signals). Yet physical events take place and synchronization must somehow be fixed by nature itself. In the text we saw how this happens for rotating platforms. Here we will review the argument for linear accelerations.

The accelerating spaceships In the isotropic system S_0 clocks have been synchronized with the Einstein method, by using light signals. Two identical spaceships **A** and **B** initially at rest on the x_0 axis of S_0 have internal clocks synchronized with those of S_0. At time $t_0 = 0$ the spaceships start accelerating in the direction +x_0, and they do so in exactly the same way, so that they have the same velocity $v(t_0)$ at every time t_0 of S_0. At time $\bar{t}_0$ they reach a preestablished velocity $v = v(\bar{t}_0)$ and their acceleration ends. For $t_0 \geq \bar{t}_0$ the spaceships are at rest in a different inertial system S (which they concretely determine) in motion with velocity v with respect to S_0. The relationship between the coordinates of S_0 and S is given by the transformations (1) with $e_1 = 0$ (not by the Lorentz transformations), because the delay between the times marked by clocks on board of **A** and **B** and those in S_0 does not depend on position: since **A** and **B** had at every time exactly the same velocity, their clocks accumulated exactly the same delay with respect to S_0. Therefore two events simultaneous in S_0, taking place in points of space near which **A** and **B** are passing, must be simultaneous also for the travellers in **A** and **B**, and thus also in the rest system of the spaceships, S. This is clearly a situation of absolute simultaneity which cannot be accounted for if the Lorentz transformations are applied, but is obtained from (4.B.1) and (4.B.4).

Not only the absolute simultaneity arises spontaneously in S, but it provides the only reasonable description of the physical reality. To see this, suppose that in **A** and **B** there are two passengers who are homozygous twins. Naturally nothing can stop them from re-synchronizing their clocks, once the acceleration has ceased. However, if they do so, they discover to have different biological ages at the same time of S, as they cannot re-synchronize their bodies! Everything is regular, instead, if they do not modify the times of their clocks.

References

[1] M.G. Sagnac, *Compt. Rend.* **157**, 708, 1410 (1913).

[2] M.G. Sagnac, *J. de Phys.* **4**, 177 (1914).

[3] A.A. Michelson and H.G. Gale, *Astroph. J.* **61**, 137 (1925).

[4] A. Dufour and F. Prunier, *Compt. Rend.* **204**, 1925 (1937); A. Dufour and F. Prunier, *J. de Phys.* **3**, 153 (1942)

[5] P. Langevin, *Compt. Rend.* **173**, 831 (1921);

[6] P. Langevin, *Compt. Rend.* **205**, 304 (1937).

[7] E.J. Post, *Rev. Mod. Phys.* **39**, 475 (1967).

[8] F. Hasselbach and M. Nicklaus, *Phys. Rev.* A **48**, 143 (1993).

[9] F. Selleri, *Found. Phys.* **26**, 641 (1996); F. Selleri, *Chin. J. Syst. Eng. Electr.* **6**, 25 (1995).

[10] A.G. Kelly in: *Open Questions in Relativistic Physics,* F. Selleri, ed. (Apeiron Press, Montreal, 1998), p. 25.

[11] A.G. Kelly, *Electronics World*, September 2000, p. 722.

[12] Y. Saburi *et al.*, *IEEE Trans.* IM**25**, 473 (1976).

[13] H.C. Hayden, *Physics Essays,* **8**, 366 (1995).

[14] F. Selleri, in *New Developments on Fundamental Problems in Quantum Physics*, M. Ferrero *et al.*, eds. (Kluwer, Dordrecht, 1997), p. 381; F. Goy and F. Selleri, *Found. Phys. Lett.* **10**, 17 (1997); F. Selleri, *Found. Phys. Lett.* **10**, 73 (1997).

[15] J. Bailey *et al.*, *Nature* **268**, 301 (1977) .

[16] Tom Van Flandern, private communication.

[17] H. Poincarè, *Rev. Metaphys. Morale* **6**, 1 (1898).

[18] H. Reichenbach, *The Philosophy of Space and Time*, (Dover, New York, 1958).

[19] M. Jammer, *Some fundamental problems in the special theory of relativity*, in: G. Toraldo di Francia, ed., *Problems in The Foundations of Physics*, pp. 202-236 (Società Italiana di Fisica, Bologna and North Holland, Amsterdam, 1979).

[20] R. Mansouri and R. Sexl, *General Relat. Gravit.* **8**, 497, 515, 809 (1977).

[21] F. Selleri, *Found. Phys.* **26**, 641 (1996); *Found. Phys. Lett.* **9**, 43 (1996).

[22] Presently the two way velocity of light in vacuum is known with an error of 20 cm/s. See: P.T. Woods, K.C. Sholton, W.R.C. Rowley, *Appl. Opt.* **17**, 1048 (1978); D.A. Jennings, R.E. Drullinger, K.M. Evenson, C.R. Pollock, J.S Wells, *J. Res. Natl. Bur. Stand.* **92**, 11 (1987).

[23] B. Rossi and D.B. Hall, *Phys. Rev.* **59**, 223 (1941).

[24] J. Bailey, K. Borer, F. Combley, H. Drumm, F. Krienen, F. Lange, E. Picasso, W. von Ruden, F.J.M. Farley, J.H. Field, W. Flegel and P.M. Hettersley, *Nature* **268**, 301 (1977).

[25] J.C. Hafele and R.E. Keating, *Science* **177**, 166 (1972).

[26] T. Van Flandern, *What the Global Positioning System Tells Us about Relativity* in *Open Questions in Relativistic Physics*, F. Selleri, ed., pp. 81-90, (Apeiron, Montreal, 1998).

Chapter 5

SYNCHRONIZATION AND DESYNCHRONIZATION ON ROTATING PLATFORMS

Guido Rizzi
Dipartimento di Fisica, Politecnico di Torino,
Corso Duca Degli Abruzzi 24, I-10129 Torino, Italy.
INFN, Sezione di Torino,
Via Pietro Giuria 1, I-10125 Torino, Italy
guido.rizzi@polito.it

Alessio Serafini
Dipartimento di Fisica "E. R. Caianiello", Università di Salerno
Via S. Allende, I-84081 Baronissi (SA), Italia
serale@sa.infn.it

Abstract The Sagnac effect allows a plain geometrical interpretation in Minkowskian spacetime, involving a geometrical time-like gap known in the literature as the "time-lag". However the time-lag seems to be a mere 'theoretical artefact', whose crucial feature is the use of Einstein synchronization, extrapolated from local to global along the rim of the rotating disk. This paper straightforwardly shows the deep physical, non-conventional, nature of the time-lag. In particular, the physical root of the Sagnac effect is revealed as the desynchronization of slowly travelling clocks after a complete round-trip in opposite directions. Moreover, the paper shows that slow transport synchronization agrees with Einstein synchronization also in rotating frames. These results are compared with the experimental results found by Hafele and Keating about thirty years ago, by means of atomic clocks flying around the world in eastward and westward directions, respectively. Our approach makes use of a general synchronization gauge, which allows for the description of different synchrony choices in any (local or global) inertial frame. In such a context, we point out that the "unacceptable discontinuity" found by Selleri in Special Relativity is not a physical discontinuity, but a mere artefact: indeed it expresses nothing but the adoption of different synchrony choices in different (local or global) inertial frames. In fact,

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 79-102.

this discontinuity disappears if the same synchronization procedure is adopted everywhere.

1. Introduction

One of the most outstanding features revealing the non time-orthogonality of rotating frames is the appearance of a global desynchronization effect, experimentally revealed by Sagnac interferometry. Consider a rigid circular platform of radius R, rotating with constant angular velocity Ω with respect to a central inertial frame S_0, and two counter-propagating coherent monochromatic light or matter beams travelling along the rim of the platform[1]. As well known, the Sagnac effect is a shift of the interference fringes occurring in a suitable interferometer at rest on the platform, and is due to the proper time difference between the arrivals on the detector, after a complete round-trip, of the co-rotating and counter-rotating beam.

In the case of light beams in vacuum, which is the most investigated in the literature, the customary explanation [1, 52, 3, 4] ascribes such a time difference to the anisotropy of light propagation along the rim of the platform, due to rotation.

On the one hand, since a rotating frame is not a (global) inertial frame, such an anisotropy can be easily understood in the background of relativistic physics.[2]

On the other hand, it is often taken for granted that global anisotropy should imply local anisotropy of light.

This seems to be more difficult to understand, because it apparently challenges some of the most sacred principles of relativistic physics. In fact, according to the *hypothesis of locality*[5], an infinitesimal region of a rotating disk should be indistinguishable, by means of any physical experiment, from the local comoving inertial frame (LCIF), where Special Relativity Theory (SRT) must hold; so that light signals, according to the principle of invariance of the velocity of light, should propagate isotropically.

We have already showed that this is not really a problem [6]. In fact the principle of locality implies that the light velocity must be the same, both for the accelerated frame and the LCIF, if and only if both frames share the same synchronization procedure (see also [7]). But if the global (round-trip) and the

[1] With the same (absolute) velocity with respect to any local comoving inertial frame, provided that such a frame is Einstein synchronized (see [13] for details).

[2] According to Peres [2], on the rim of a rotating disk "there is no reason to demand that the speed of light be the same eastward and westward."

local synchronizations are defined in different operational ways, as often - but rarely explicitly - authors do, the global synchronization holds no implications at all about the local synchronization (*i.e.* about the local velocity of light).[3]

However, the synchronization usually shared (at least locally) in the relativistic community is Einstein synchronization, that implies local isotropic propagation of light. Notice that an explanation of the time difference responsible of the Sagnac effect, which is in tune with this widespread assumption, acually exists. First proposed by Cantoni [9] in 1968 and, more extensively, by Anandan [10] in 1981, it has been recently developed by Rizzi-Tartaglia [11], and independently achieved by Bergia-Guidone [12]. Moreover, in the present book it has been extended to a more general case (light beams in a comoving refracting medium, matter beams etc.), with a deeper insight about the crucial role of synchronization [13]. This explanation is consistent with both global anisotropy (as measured by a single clock) and local isotropy (as measured by a couple of near Einstein-synchronized clocks) of light propagation. It ascribes the proper time difference responsible of the Sagnac effect to the "non-uniformity of time" on the rotating platform, and in particular to the synchronization gap

$$\tau_{lag} = \frac{2\pi\Omega R^2}{c^2}\gamma(\Omega) \ , \quad \gamma(\Omega) \doteq \sqrt{1 - \frac{\Omega^2 R^2}{c^2}} \, . \tag{5.1}$$

The time interval τ_{lag} has been called "time-lag" by Anandan himself [10]. It arises in synchronizing clocks along the rim according to the standard Einstein procedure (in spite of the lack of self-consistency of Einstein procedure *at large* in non time-orthogonal reference frames).

This explanation has the agreeable feature of allowing a geometrical interpretation of the time-lag (5.1), but encounters a serious hitch. The time-lag is defined [10, 11] as a measure of the geometrical gap between the circumference of the disk (as viewed in the central inertial frame) and a suitable open curve in spacetime, defined as the locus of the events simultaneous, *according to Einstein synchronization*, along the rim of the disk (see Sec. 4). As a consequence, the geometrical interpretation of the time-lag seems to be a mere 'theoretical artefact', due to the use of Einstein synchronization arbitrarily extrapolated from local to global (we recall, once again, that the extension of

[3] Usually the global (round-trip) velocity of light is measured by a single clock; therefore this velocity is a synchrony-independent concept. On the other hand, the local velocity of light is measured by a couple of near clocks, synchronized according to some procedure [6, 8]: therefore this velocity is a synchrony-dependent concept. As a consequence, the global and local velocity of light turn out to be operationally different concepts.

Einstein synchronization from local to global is not allowed in a non time-orthogonal frame).[4]

In this paper we will see why and how the time-lag, although actually introduced as a theoretical artefact, hides a strict physical meaning.

First of all, we will point out that the synchronization is not "given by God", as often both relativistic and anti-relativistic Authors assume, but it can be arbitrarily chosen within the Cattaneo gauge (which is the set of all the possible parametrizations of a given physical reference frame). In particular, we will show that the Cattaneo gauge encapsulates the "Anderson-Vetharaniam-Stedman" gauge [14], which in turn encapsulates the "Selleri gauge" [3]; the latter includes, as a particular case, the Einstein synchrony choice.

That having been said, we will see that the relativistic explanation of the Sagnac effect is independent on the choice of any of such conventions about synchronization (in particular, it is independent on the Einstein synchronization procedure).

In Sec. 5, a straightforward calculation will bring to light the dark physical ("non conventional") root of the Sagnac effect. The underlying physical root could be expressed in the following way: "a clock, slowly transported along the rim of the platform, turns to be out of synchronization, after a complete round trip, with respect to a clock enduringly at rest on the platform".

To better clarify this point, let us now tell a short tale. Two twins live in a (point-like) town Σ on the rim of a (large) platform. They are not aware of the rotation: they just experience an outwards gravitational field. The twins share a standard secondary education; in particular, they share a firm faith in classical physics and traditional Euclidean geometry. One day, the twins (which are first-class experimental physicists and explorers) leave their home country and move along the rim in opposite directions, as slowly as possible. Both explorers carry standard rods and standard clocks,[5] carefully synchronized in the starting-point Σ. At each step the explorers lay down one rod, tail to head of the previous one; and carefully keep count of them. After many years, the two explorers meet again in their town Σ, and confront their measurements. Of course they agree about the circumference lenght, as measured by the total number of rods laid down along the rim, but they find something weird in their counting: the length is not $2\pi R$, as expected, but $2\pi R\gamma(\Omega)$ (see *f.i.* [15] for demonstration).

This is astonishing enough, but it is not the actual matter ot this paper. The most astonishing thing - which we have to deal with - has yet to come: when

[4]We would like to thank F. Selleri and R. D. Klauber, who brought this remark (the time-lag as a 'theoretical artefact') to our attention.

[5]The rods could be the wavelengths of a monochromatic light source at rest on the platform, while the clocks could be ensembles of slow unstable particles, whose decay time defines a natural unity of time.

they compare the readings of their two clocks, they find a desynchronization of the amount $\Delta\tau = 2\tau_{lag}$. The twins, open-mouthed, realize that their ages are different!

In particular, the co-rotating twin is younger: his clock is late of an amount given by the time-lag (5.1) with respect to the clock C_Σ at rest in Σ. The counter-rotating twin is older: his clock is ahead of the same amount.

The faith in classical physics, in the Euclidean geometry, and more generally in the rationality of the physical world is lost. The consequent nervous breakdown disappears much later, when the younger twin finds the solution of the mistery in the local library. Of course, the reader knows the title of the timely book: “Relativity on Rotating Frames”!

In the real world, a very similar experiment has been actually realized by Hafele and Keating [16] about thirty years ago, by means of commercial flights (carrying caesium beam atomic clocks) around the world in eastward and westward directions. The result seems to confirm the first order theoretical predictions. However, the approximations used by these Authors seem to us a bit foolhardy; in fact, any trustworthy theoretical prediction at first order approximation should be found out as a first order Taylor expansion of an exact theoretical prediction. Strangely enough, an exact theoretical prediction in full theory seems to be lacking (or maybe lost somewhere in the literature); therefore a straightforward computation is performed in this paper.

Let us make a further remark on the time-lag (5.1). Being a difference between proper times, such a desynchronization is obviously a synchrony-independent quantity. Therefore the computation performed in this paper straightforwardly reveals the deep physical, non conventional, nature of the time-lag, characterizing the non time-orthogonality of the rotating frame and responsible of the fringes' shifts in Sagnac interferometers.

Moreover, this computation also reveals that *slow transport synchronization agrees with Einstein synchronization also in rotating frames*, although in these frames Einstein synchronization is not allowed at large. As far as we know, this (absolutely non trivial) fact is usually maintained[6] but nowhere demonstrated - with the significant exception of Ashby[18], though his demonstration is confined to first order approximation.

We have found instructive to carry out the computation of the desynchronization effect both in a coordinate system adapted to the central inertial frame and in a coordinate system adapted to the rotating platform.

Summing up, though acknowledging the well known ‘privilege’ of the inertial time of the central inertial frame (which is the only one providing a consistent global synchronization of the rotating frame), our analysis shows once

[6] See for instance Dieks[17].

more that, contrary to a reiterate claim of Selleri [3, 19, 20, 21], the desynchronization effects arising on a rotating platform cannot force the choice of a particular synchronization procedure. This complies with the fact that the observable physical quantities do not depend on the local synchronization choices (which are arbitrary within the Cattaneo gauge) in the LCIFs along the rim.

2. The many choices of synchronization in a physical reference frame

Although in this paper we intend not to go beyond the special relativistic context, we shall adopt the most general description in order to take into account arbitrary (in general, 'non-Einsteinian') synchronization procedures.

The formalism and concepts introduced in this section refer to inertial frames in the Minkowskian spacetime of Special Relativity. Of course, they also apply to LCIF's in a non-inertial frame: in this way the 'synchronization gauges' to which this section will be devoted can be *locally* extended to non-inertial frames - in particular to rotating frames.

2.1 A physical reference frame and its parametrization

The Minkowskian spacetime of Special Relativity is an affine pseudo-Euclidean manifold $\mathcal{M}^4$, with signature $(1,-1,-1,-1)$. A physical reference frame is a time-like congruence Γ in $\mathcal{M}^4$made up by the set of world lines of the test-particles constituting the "reference fluid".[7] The congruence Γ is identified by the field of unit vectors tangent to its world lines. Briefly speaking, the congruence is the (history of the) physical reference frame.

Let $\{x^\mu\} = \left\{x^0, x^1, x^2, x^3\right\}$ be a system of coordinates in a suitable neighborhood U_P of a point $p \in \mathcal{M}^4$; these coordinates are said to be admissible (with respect to the congruence Γ) when[8]

$$g_{00} > 0 \quad g_{ij}dx^i dx^j < 0\,. \tag{5.2}$$

Thus the coordinate lines $x^0 = var$ can be seen as describing the world lines of the ∞^3 particles of the reference fluid, while the label coordinates $\{x^i\} = \left\{x^1, x^2, x^3\right\}$ can be seen as the name of any particle of the reference fluid.

When a reference frame has been chosen, together with a set of admissible coordinates, the most general coordinate transformation which does not change the physical frame (i.e. the congruence Γ) has the form [22, 23, 24, 25]

$$\left\{ \begin{array}{l} x'^0 = x'^0(x^0, x^1, x^2, x^3) \\ x'^i = x'^i(x^1, x^2, x^3) \end{array} \right. \quad , \tag{5.3}$$

[7]The concept of 'congruence' refers to a set of word lines filling the manifold, or some part of it, smoothly, continuously and without intersecting.

[8]Greek indices run from 0 to 3, Latin indices run from 1 to 3.

with the additional condition $\partial x'^0/\partial x^0 > 0$, which ensures that the change of time parameterization does not change the arrow of time. The coordinate transformation (5.3) is said 'internal' to the physical frame Γ or, more simply, an *internal gauge transformation*. In particular, Eq. $(5.3)_2$ just changes the names of the reference fluid's particles, while Eq. $(5.3)_1$ changes the "coordinate clocks" of the particles. In the following we will refer to this gauge as to the *Cattaneo gauge*.

Remark. An "observable" physical quantity is, in general, frame-dependent, but its physical meaning requires that it cannot depend on the particular parameterization of the physical frame. This means that, once a physical frame is given, any observable must be gauge-invariant.

2.2 Re-synchronization as a particular case of re-parametrization of the physical frame

Inside Cattaneo gauge, which is the set of all the possible parametrizations of a given physical reference frame, the transformation

$$\begin{cases} x'^0 = x'^0(x^0, x^1, x^2, x^3) \\ x'^i = x^i \end{cases} \tag{5.4}$$

defines a sub-gauge (the *synchronization gauge*) that describes the set of all the possible synchronizations of the physical reference frame; [9] it will play a key role in the following. it will play a key role in the following.

Within the synchronization gauge (5.4), Einstein synchronization is the only one which does not discriminate points and directions, i.e. which is homogeneous and isotropic [26]. Starting from Einstein synchronization, any inertial frame can be re-synchronized according to the following transformation [14]

$$\begin{cases} \widetilde{t} = t - \tilde{\mathbf{k}} \cdot \mathbf{x} \\ \widetilde{x}^i = x^i \end{cases} , \tag{5.5}$$

where t is the Einstein coordinate time of the physical inertial frame, and $\tilde{\mathbf{k}} = \tilde{\mathbf{k}}(\mathbf{x})$ is an arbitrary smooth vector field.[10] The set of all the possible synchronizations described by Eqns. (5.5) defines a gauge: the Anderson-Vetharaniam-Stedman gauge (briefly "AVS-gauge").[11]

---

[9] Borrowing a picturesque phrase from [8], the synchronization gauge (5.4) describes the set of all the possible ways to "spread time over space".

[10] A non-null vector field breaks isotropy, and the possible dependence of this vector field on the vector position **x** breaks homogeneity.

[11] Such a gauge freedom, actually in the instance $\tilde{\mathbf{k}}(\mathbf{x}) = const$, had already been brought to attention by Mansouri and Sexl [29].

The re-synchronization (5.5) redefines the simultaneity hypersurfaces, that are now described by $\widetilde{t} = const$. Therefore, the set of these hypersurfaces defines a foliation of spacetime which depends not only on the physical frame, but also on the field $\tilde{\mathbf{k}}(\mathbf{x})$.

A trivial particular case is $\tilde{\mathbf{k}}(\mathbf{x}) \cdot \mathbf{x} = const(\mathbf{x})$: in this instance, Eq. (5.5) simply means a change in the origin of Einstein coordinate time.

Now, let us stress a more interesting particular case. Let S_0 be an inertial reference frame (IF) in which an Einstein synchronization procedure is adopted (by stipulation) and let S be an IF travelling along the x-axis (of unit vector $\mathbf{e}_1$) with constant adimensional velocity $\beta \equiv v/c$. If both S_0 ans S are Einstein synchronized, the standard Lorentz transformation follows

$$\begin{cases} t = \gamma(t_0 - \frac{\beta}{c}x_0) \\ x = \gamma(x_0 - \beta c t_0) \\ y = y_0 \\ z = z_0 \end{cases} , \tag{5.6}$$

where $\gamma \doteq \left(1 - \beta^2\right)^{-1/2}$ is the Lorentz factor. Now, let us re-synchronize S according to transformation (5.5), in which the vector field $\tilde{\mathbf{k}}$ is chosen as follows:

$$\tilde{\mathbf{k}} \doteq -\frac{\Gamma(\beta)}{c}\mathbf{e}_1 , \tag{5.7}$$

$\Gamma(\beta)$ being an arbitrary function of β. We get:

$$\begin{cases} \widetilde{t} = t + \frac{\Gamma(\beta)}{c}x = \gamma(t_0 - \frac{\beta}{c}x_0) + \frac{\Gamma(\beta)}{c}\gamma(x_0 - \beta c t_0) \\ \widetilde{x} = x = \gamma(x_0 - \beta c t_0) \\ \tilde{y} = y = y_0 \\ \tilde{z} = z = z_0 \end{cases} . \tag{5.8}$$

If the function $\Gamma(\beta)$ is written as follows:

$$\Gamma(\beta) \equiv \beta + e_1(\beta)c\gamma^{-1} , \tag{5.9}$$

where $e_1(\beta)$ is an arbitrary function of β, then Eqns. (5.8) take the form

$$\begin{cases} \widetilde{t} = \gamma^{-1}t_0 + e_1(\beta)(x_0 - \beta c t_0) \\ \widetilde{x} = x = \gamma(x_0 - \beta c t_0) \\ \tilde{y} = y = y_0 \\ \tilde{z} = z = z_0 \end{cases} . \tag{5.10}$$

This is the 'Selleri gauge' [19]. It is interesting to note that, as Selleri exhaustively shows, this is the most general gauge (inside the AVS-gauge) compatible with the standard Einstein expression of time dilation with respect to S_0. The arbitrary function $e_1(\beta)$ is Selleri's "synchronization parameter" . In particular, the synchrony choice $e_1(\beta) \doteq -\beta\gamma/c$ (i.e. $\Gamma(\beta) \doteq 0$) gives the standard Einstein synchronization, whereas the synchrony choice $e_1(\beta) \doteq 0$ gives Selleri synchronization[12]. In Selleri synchrony choice, Eqns. (5.10) read

$$\begin{cases} \widetilde{t} = \gamma^{-1} t_0 \\ \widetilde{x} = x = \gamma(x_0 - \beta c t_0) \\ \tilde{y} = y = y_0 \\ \tilde{z} = z = z_0 \end{cases} . \tag{5.11}$$

Remarks and discussion about Selleri's viewpoint: absolute synchronisation in SRT. As one can see from the first equation in (5.11), the resynchronization defined by the choice $e_1(\beta) \doteq 0$ cancels out the term $-\gamma\frac{\beta}{c}x_0$ appearing in Lorentz time transformation and responsible for the relativity of simultaneity [12, 6]. Thus the notion of simultaneity between "spatially separated" events turns out to be, in such a gauge choice, independent from the considered inertial frame. Because of this fact the synchronization resulting from $e_1(\beta) \doteq 0$ has been called "absolute" by Selleri himself [3, 19, 20].[13] In other words, the relativity of time, expressed by the first equation in (5.11), does not clash with the absolute character of simultaneity: $\Delta t = 0 \Leftrightarrow \Delta t_0 = 0$. As already remarked by Mansouri and Sexl [29], an 'absolute simultaneity' can be reintroduced in Special Relativity without affecting neither the logical structure, nor the predictions of the theory. Actually, it's just a simple parametrization effect.

In particular the choice $e_1 = -\beta\gamma/c$, corresponding to Einstein synchronization, involves a spacetime foliation depending on the chosen reference frame. Such a foliation is realized by the hypersurfaces which are Minkowski-orthogonal (briefly 'M-orthogonal') with respect to the world lines of the test-particles by which the reference frame is made up.

On the other hand, the choice $e_1 = 0$ involves a frame-invariant foliation (which is Einstein foliation of the IF S_0, that we assumed to be optically isotropic *by stipulation*).

The main difference between our point of view and Selleri's lies in a completely different paradigmatic background. Which, in our opinion, does not properly belong to physics, but rather to the interpretation of physics. Sell-

---

[12]The synchrony choice $e_1(\beta) \doteq 0$ is equivalent to the choice

$$\tilde{\mathbf{k}} \doteq -\frac{v}{c^2}\mathbf{e}_1$$

for the AVS vector field.

[13]Operatively, the absolute synchronization in an IF S can be realized by setting to $\gamma^{-1}t_0$ the reading of each clock of S just when its spatial position matches the one of a clock of the optically isotropic IF S_0 displaying the value t_0.

eri's work develops on the paradigmatic background of the ether hypothesis, which we consider as an unnecessary (and misleading) superstructure - a sort of 'ideological fossil'. Nevertheless, the actual possibility of an 'absolute synchronization' in the orthodox background of SRT is not affected by such considerations.

A major point of disagreement is constituted by Selleri's position about the relativity principle. According to him "transformations (5.11) violate the relativity principle for any value of e_1, except for the relativistic value".

This belief rests on a misunderstanding. The transformation described by Eqns. (5.11), although formally correct, could be misleading if a pertinent physical interpretation is lacking. Indeed, from the mathematical point of view, Selleri's belief arises from the formal asimmetry of Eqns. (5.11) between the optically isotropic frame S_0 and the optically anisotropic frame S. However, it should be clear that one cannot require the transformations' simmetry without imposing the fundamental condition: "provided that the synchronization is the same in every IF". Since this condition is manifestly violated in Selleri gauge's synchrony choice, there is no reason at all to claim that the non-symmetrical character of Eqns. (5.11) does violate the relativity principle. Actually, the asymmetry of kinematical transformations (5.11) does not expresse any kind of *physical* privilege of frame S_0 (it is merely the frame in which one has performed an Einstein synchronization procedure by a "conventional agreement"): rather, such an asymmetry simply reflects the asymmetry of the synchronization procedures in the IF's S_0 and S!

A further subject of dispute with Selleri constitutes a crucial issue of this paper. We agree with him that synchronization is a matter of convention as far as inertial frames, translating with respect to one another, are concerned. However, Selleri claims that, as soon as rotating frames are considered, synchronization cannot be conventional any longer: when rotation is taken into account, the synchronization parameter e_1 is forced to take the value zero and the 'absolute synchronization' is the only legitimate one [3, 19, 20, 21].

Instead we believe that no empirical evidence is able to force the choice of e_1: neither the Sagnac interference fringes, nor the ages of the slow travelling twins depend on the choice of e_1. More generally, no observable effect, neither in translational instances nor in rotational ones, can distinguish between Selleri's "theory" and SRT. Actually, Selleri's "theory" is not properly a theory, but just a syncrony choice inside the synchronization gauge - the only physical theory being the SRT, for any synchrony choice. This will be better clarified in section 6.

3. Sagnac effect and its universality

We can now move back to rotating platforms. Let K be a rigid circular platform of radius R, rotating with constant angular velocity Ω with respect to the central inertial frame S_0. Let Σ be a point of the rim where a clock, a source of "entities" and an absorber are located. The "entities" are supposed to be couples of equal physical objects of whatever nature (e.g., classical particles, De Broglie matter waves, electromagnetic waves in vacuum or in a refracting comoving medium, acoustic waves, Cooper pairs, ...), travelling in opposite directions along the rim. Besides, the two objects' velocities are supposed to be the same (in absolute value) with respect to any LCIF, provided that the LCIF's are Einstein-synchronized [13].

The world line of Σ is a timelike helix, say γ_Σ, that wraps around the cylinder representing the disk in $2+1$ dimensions. The world line of the counter-propagating "entities" are two (timelike for massive, null for non massive "entities") helixes $\gamma_{E_\pm}$. The two "entities" meet the source again at two different events $\Sigma_{\tau_\pm}$, which correspond to the intersections between the helixes $\gamma_{E_\pm}$ and the timelike helix γ_Σ. The "entities" take different proper times $\tau_\pm$, as measured by the standard clock C_Σ located in Σ, for a complete round-trip. Moreover, such times $\tau_\pm$ do obviously depend on the velocity of the "entities" with respect to the LCIF's. However, it can be proved [13] that the difference between these times is always given by the following equation

$$\delta\tau \equiv \tau_+ - \tau_- = \frac{4\pi\Omega R^2}{c^2}\gamma(\Omega) \tag{5.12}$$

This proper time difference[14] is always the same, both for matter and electromagnetic waves, regardless of the physical nature of the interfering beams and of the possible presence of a comoving refracting medium. The astounding "universality" of this effect (experimentally well proved at first order approximation) commandingly invokes a geometrical interpretation in minkowskian spacetime. Next section will be devoted to this aim.

4. The time-lag as a "theoretical artefact"

Let Σ_o be the event "emission of the two counter-propagating beams from the source located in Σ". Then the locus of events *simultaneous to* Σ_o *ac-*

[14] The proper time difference (5.12) can be experimentally measured (in first order approximation) by an interferometric device, located in Σ, as a fringe shift $\delta z = c\delta\tau/\lambda T$, provided that the two "entities" are monochromatic coherent beams. This means, in particular, that the massive "entities" are not single particles, but ensembles of particles prepared as monochromatic coherent wave beams (briefly, "De Broglie matter waves").

cording to Einstein synchronization can be defined *only locally* in a consistent way. It is well known that this local simultaneity criterium cannot be extended *globally* on the whole platform. In fact if we transport, step by step, this local synchronization on a complete round-trip along the rim, a synchronization gap, with respect to the proper time of clock C_Σ in Σ, arises [10, 11, 12]. Let us now show how one can evaluate this gap.

To this end, we start from the event Σ_o and move along the rim of the platform, transporting Einstein simultaneity criterium as defined in the LCIF attached to Σ_o. Although the expression "Einstein simultaneity" cannot be properly used at large, the outlined procedure is well defined on the operational ground, and establishes an equivalence relation between events in spacetime. In order to avoid any confusion, we will refer to this equivalence relation as to "Extended Einstein local simultaneity" (briefly "EE-simultaneity"). As a result, the set of events taking place on the rim and EE-simultaneous to Σ_o is mapped[15] into a spacelike helical curve γ_S, belonging to the cylindrical surface σ_R generated by the wordlines of the circumference $r = R$ at rest in the central inertial frame. The spacelike helix γ_S starts from Σ_o and is everywhere M-orthogonal to the timelike helixes associated to the points of the rim, whose tangent vectors form a constant angle with respect to the time axis of S. The M-orthogonality is shown in the plot by the E-symmetry, with respect to the local light cone,[16] between the tangent vector to γ_S and the tangent vectors to the timelike helixes of the rim's points (see Fig. 5.1).

Let Σ_τ be the first intersection of the helix γ_Σ with the helix γ_S (Fig. 5.1).[17] The timelike distance between Σ_o and Σ_τ along γ_Σ, as measured by the clock C_Σ at rest in Σ, is given by [11]

$$\tau_{lag} = \frac{1}{c}\int_{\Sigma_o}^{\Sigma_\tau} \mathrm{d}s = \frac{2\pi\Omega R^2}{c^2}\gamma(\Omega)\,, \tag{5.13}$$

which is exactly half the proper time interval (5.12) between the arrivals of the co-rotating and counter-rotating beams on the absorber in Σ. Expression (5.13) gives the synchronization gap which, because of rotation, arises in synchronizing clocks around the rim *according to Einstein prescription.*

[15] In the 3-dimensional plot of the $(2+1)$ Minkowskian spacetime (see Fig. 5.1).

[16] As it is well known, a spacetime diagram is a topological map from a Minkowskian space to an Euclidean space, which changes the metrical relations (angles and lengths) in a well defined way; in particular, the M-orthogonality between two directions is depicted in the diagram as an E-symmetry with respect to the light cone.

[17] Of course the event Σ_τ, though "EE-simultaneous" to Σ_o according to the previous definition, belongs to the future of Σ_o! This is a well known argument which displays the impossibility of a self-consistent definition of Einstein simultaneity *at large* on the disk.

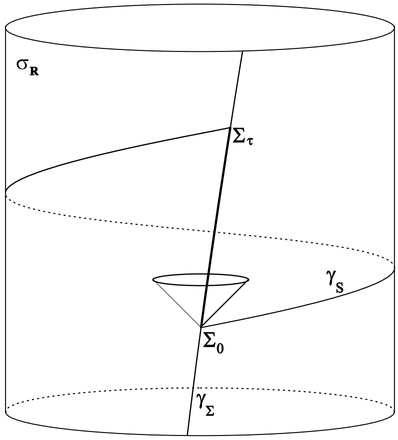

Figure 5.1. The cylinder lateral surface σ_R represents the spacetime belonging to the rim of the rotating disk, the vertical axis of the cylinder coinciding with the t-axis of the central intertial frame frame S_o. The curve γ_Σ is the world-line of the point Σ, while the helix γ_S is the locus of space-time events "EE-symultaneous" to Σ_o. The proper length of the path laying between Σ_o and Σ_τ is what we have called the "time-lag". The E-symmetry between the tangent vectors to γ_Σ and γ_S, with respect to the light-cone in Σ_o, is a pictorial manifestation of their M-orthogonality.

This gap (the so-called "time-lag" [10, 11]) exactly coincides with the synchronization gap found by Bergia and Guidone [12] with a rather similar approach.[18]

Some Authors [27] take issue with this approach, pointing out that the time-lag seems to be a mere theoretical artefact, due to the use of Einstein synchronization incorrectly extrapolated from local to global. We partially share this remark but we stress that the time-lag, although introduced by Cantoni [9], Anandan [10], Rizzi-Tartaglia [11] and ourselves in a rather formal way, hides a deep physical meaning: it is nothing but the geometrical interpretation of the Sagnac effect in spacetime. As a matter of fact, the time-lag (5.13) turns out to be exactly (half) the proper time interval (5.12) between the arrivals of the co-rotating and of the counter-rotating beams on the absorber in Σ. So, the

[18]Gluing togheter all the Lorentz transformations from the central inertial frame to any LCIF along the rim, Bergia and Guidone find the same synchronization gap, without a geometrical interpretation in spacetime. Of course the use of the Lorentz transformation shows that also this procedure is strictly dependent on the Einstein synchronization extrapolated from local to global.

physical Sagnac effect can be fully explained as an observable manifestation of the geometrical time-lag (5.13).

However, we feel that the physical meaning of the time-lag should be better clarified in an even more direct way, in which Einstein synchronization plays no role at all. We will address this issue in next section.

5. The time-lag as an observable desynchronization

Two clocks A and B, initially at rest in the point Σ of the rim and here synchronized, are separated: clock A stays in point Σ, while clock B travels along the rim, in co-rotating (cr) or counter-rotating (crr) sense, with a constant angular velocity $\pm\omega$ with respect to the rotating disk (time of the central inertial frame, see Sec. 5.1 below). After a complete round-trip, the clocks meet again, and compare their times. They find different times: the two clocks, initially synchronized, are now desynchronized because of their different kinematical histories. We now propose to compute such a desynchronization.

Since this result is simply due to the difference in Lorentz factors with respect to the central inertial frame S, such a desynchronization is just a variance of the well known twin paradox. Of course, in the case $\Omega = 0$ (traditional formulation of the twin paradox), the limit $\omega \to 0$ makes the desynchronization disappear. But what about the case $\Omega \neq 0$? If one performs this calculation and takes the limit $\omega \to 0$, one encounters an amazing surprise: the difference between the times showed by the two clocks goes to a finite non null value, depending on Ω. This is known in the first order approximation [16, 28]. But we want to stress a very interesting feature of this time difference: its expression, as calculated in full theory, exactly coincides with the time-lag (5.13). This reveals, in a straightforward way, the concrete physical meaning hidden in the time-lag.

Let us now directly prove the previous statements. Since this paper also retains a didactic purpose, we find instructive to carry out the calculation of the desynchronization effect both in the central inertial frame and in the rotating frame.

5.1 The time-lag seen from the central inertial frame

Let t be the (inertial) time of the central inertial frame S_0, where light propagates isotropically and the standard SRT holds. Let (r, ϑ, t) be a cylindrical chart adapted to the rotating frame K. In particular, let φ_+ (φ_-) and ϑ_+ (ϑ_-) be the azimuthal coordinate of the cr (crr) clock with respect to S_0 and K, respectively. In the following, a $+$ ($-$) label will refer to the cr (crr) case. The passage from the rotating frame K to the central inertial one S_0 is described by the external transformation

$$\varphi_\pm = \Omega t + \vartheta_\pm \,. \tag{5.14}$$

If the relative angular velocity ω is the same in the cr and crr instance, derivation of Eq. (5.14) with respect to t gives

$$\omega_{\pm} = \Omega \pm \omega \,, \tag{5.15}$$

where the superior (inferior) sign stands for the cr (crr) travelling clock. The time interval Δt after which the two clocks A and B meet again in Σ is defined by

$$(\Omega \pm \omega)\Delta t = \Omega\Delta t \pm 2\pi \Longrightarrow \Delta t = \frac{2\pi}{\omega} \,.$$

We will call $\Delta\tau_A$ and $\Delta\tau_B$ the proper time intervals measured, respectively, by clock A and B between the "separation" event and the "meeting again" event. The usual time-dilation formula, with respect to the clocks of the inertial frame S_0, gives

$$\begin{aligned} \Delta\tau_A &= \frac{\Delta t}{\gamma(\Omega)} = \frac{2\pi}{\omega\gamma(\Omega)} \,, \\ \Delta\tau_B &= \frac{\Delta t}{\gamma(\Omega \pm \omega)} = \frac{2\pi}{\omega\gamma(\Omega \pm \omega)} \,, \end{aligned}$$

where $\gamma(x) = \left(1 - \frac{x^2R^2}{c^2}\right)^{-1/2}$ is Lorentz time-dilation factor.

Performing the Taylor expansion of $1/\gamma(\Omega \pm \omega)$ with respect to ω, one can write[19]

$$\begin{aligned} \Delta\tau_B &= \frac{2\pi}{\omega}\frac{1}{\gamma(\Omega \pm \omega)} \\ &= \frac{2\pi}{\omega\gamma(\Omega)} + \frac{2\pi}{\omega}\left(\mp\gamma(\Omega)\frac{R^2}{c^2}\Omega\omega - \frac{1}{2}\gamma(\Omega)\frac{R^2}{c^2}\omega^2 + O(\frac{\omega^3R^3}{c^3})\right) \\ &= \frac{2\pi}{\omega\gamma(\Omega)} \mp 2\pi\gamma(\Omega)\frac{R^2}{c^2}\Omega - \pi\gamma(\Omega)\frac{R^2}{c^2}\omega + O(\frac{\omega^2R^2}{c^2}) \,, \end{aligned}$$

where $O(\frac{\omega^2R^2}{c^2})$ stands for a term of the second order in $\frac{\omega R}{c}$.

---

[19]Taylor development of $1/\gamma(x)$ centered in Ω, reads:

$$\begin{aligned} \frac{1}{\gamma(\Omega \pm \omega)} &\doteq \left(1 - \frac{(\Omega \pm \omega)^2 R^2}{c^2}\right)^{1/2} \\ &= \frac{1}{\gamma(\Omega)} \mp \gamma(\Omega)\frac{R^2}{c^2}\Omega\omega - \frac{1}{2}\gamma(\Omega)\frac{R^2}{c^2}\omega^2 + O(\frac{\omega^3R^3}{c^3}) \,, \end{aligned}$$

where we call $O(x^3)$ a term of the third order in x.

Even if the results of this section could be achieved just by a two-terms Taylor expansion, the second order term will be required in order to discuss the Hafele-Keating experiment, in which the limit $\omega \to 0$ is not performed (see sect. 5.2 below).

Therefore, the desynchronization $\Delta\tau_B^+ - \Delta\tau_A$ of the cr-clock, at the end of its eastward trip[20], is given by

$$\Delta\tau_B^+ - \Delta\tau_A = -2\pi\gamma(\Omega)\frac{R^2}{c^2}\Omega - \pi\gamma(\Omega)\frac{R^2}{c^2}\omega + O(\frac{\omega^2 R^2}{c^2}) . \qquad (5.16)$$

Likewise, the desynchronization $\Delta\tau_B^- - \Delta\tau_A$ of the crr-clock, at the end of its westward trip, is given by

$$\Delta\tau_B^- - \Delta\tau_A = 2\pi\gamma(\Omega)\frac{R^2}{c^2}\Omega - \pi\gamma(\Omega)\frac{R^2}{c^2}\omega + O(\frac{\omega^2 R^2}{c^2}) . \qquad (5.17)$$

From Eqns. (5.16), (5.17) the following important result follows

$$\lim_{\omega\to 0}\left(\Delta\tau_B^- - \Delta\tau_A\right) = \lim_{\omega\to 0}\left(\Delta\tau_A - \Delta\tau_B^+\right) = \frac{2\pi\Omega R^2}{c^2}\gamma(\Omega) \equiv \tau_{lag} . \quad (5.18)$$

Conclusion 1. The physical root of the Sagnac effect: desynchronization of slowly travelling clocks.

Eq. (5.18) shows that:
(i) the limit $\omega \to 0$ does not cancel the desynchronization;
(ii) such a limit is just the time-lag (5.13) responsible of the Sagnac effect.

Conclusion 2. Compatibility between Einstein synchronization and slow transport synchronization.

We have directly shown that the slow transport of a clock on a complete round-trip along the rim results in the same synchronization gap as the one obtained through the equivalence relation introduced in Sec. 4, with the label "EE-synchronization", between events belonging to the spacetime surface σ_R.[21]

Such a conclusion shows that slow transport synchronization agrees with Einstein synchronization not only in inertial frames [29, 14], but also in rotating ones. As far as we know, this (absolutely non trivial) fact is often maintained, but until now never demonstrated in full theory.[22]

Remark. The appearance of ω (that is $|d\vartheta_\pm/dt|$, where t is the time of the central IF) in Eq. (5.18) could arouse the suspicion that this equation depends

[20] The terms "eastward" and "westward" refer to a trip around the Earth, along a parallel. In particular, these terms hint at the Hafele-Keating experiment.
[21] The label "EE-synchronization" (sorry for this disagreeable acronym) recalls that this equivalence relation comes from the formal extension of the local Einstein synchronization along the whole rim.
[22] The only demonstration we know is given by Ashby[18], and is carried out at first order approximation.

on the local synchronization, namely the synchronization of the central IF. Of course this is a nonsense, because Eq. (5.18) is an observable quantity, and any observable quantity must be independent on any convention about local synchronization. In Appendix we show that, actually, Eq. (5.18) does not depend on the local synchronization.

5.2 Remark on the Hafele-Keating experiment

Eq. (5.18) is jocularly commented in the twins' introductory tale. However, we can rely on something more tangible than a tale. In fact, a famous experiment has been actually realized in October 1971 by Hafele and Keating [16], by means of two jet aircrafts flying along a parallel in opposite directions. Therefore, further comments are in order.

For the sake of simplicity, just consider two equatorial around-the-world trips, in eastward and westward direction respectively. The velocity of the Earth at the equator, with respect to the IF of its axis, is 1670 km/h; the velocity of a (commercial) jet aircraft with respect to the Earth is about 1000 km/h. This means that $\Omega R/c \cong 1.548 \times 10^{-6}$ and $\omega R/c \cong 9.259 \times 10^{-7}$. Therefore, both $\Omega R/c$ and $\omega R/c$ are same order small with respect to the unity.

As a consequence, Eqns. (5.16) and (5.17) reduce, at first order approximation with respect to $\omega R/c$ and $\Omega R/c$, to

$$\Delta\tau_B^{east} - \Delta\tau_A = -2\pi\frac{R^2}{c^2}\Omega - \pi\frac{R^2}{c^2}\omega = -\pi R\frac{2\Omega R + \omega R}{c^2}, \tag{5.19}$$

$$\Delta\tau_B^{west} - \Delta\tau_A = 2\pi\frac{R^2}{c^2}\Omega - \pi\frac{R^2}{c^2}\omega = \pi R\frac{2\Omega R - \omega R}{c^2}, \tag{5.20}$$

respectively. The numeric computation gives $\Delta\tau_B^{east} - \Delta\tau_A \cong -268$ ns and $\Delta\tau_B^{west} - \Delta\tau_A \cong 145$ ns, so that $\Delta\tau_B^{west} - \Delta\tau_B^{east} \cong 413$ ns. Taking into account the latitude (and the variations of the direction of fly with respect to the parallel), Hafele and Keating found $\Delta\tau_B^{east} - \Delta\tau_A \cong -184$ ns; $\Delta\tau_B^{west} - \Delta\tau_A \cong 96$ ns. Of course a gravitational red-shift term, coming from General Relativity, must be added, due to the different gravitational potential experienced by the flying clock with respect to the one on the ground. Such a term is about the same for both flies. The net observable desynchronization effect is just the algebraic sum of the kinematical and gravitational effect.

The experiment seems to confirm quite well the first order predictions of both special and general theory of relativity. In the words of the Authors, "the effects of travel on the time recording behavior of macroscopic clocks (...) can be observed in a straightforward and unambiguous manner with relatively inexpensive commercial jet flights and commercially available caesium beams clocks".

5.3 The time-lag seen from the rotating platform

We now perform the calculation of the time-lag τ_{lag} in a coordinate system adapted to the rotating frame K. Such a choice results in the adoption of the 'generalized Born metric', which can be derived from the usual pseudo-euclidean one by means of two simple tranformations:

- $\varphi = \vartheta + \Omega t$: an external transformation characterizing the passage from the central inertial frame S_0 to the rotating one K, φ and ϑ being, respectively, the azimuthal coordinates of S_0 and K;
- $\tilde{t} = \sqrt{1 - \left(\frac{\Omega r}{c}\right)^2}\, t \equiv \gamma^{-1}(\Omega)\, t$: a temporal rescaling which introduces the coordinate $\tilde{t}$, measured by actual clocks on the rim of the disk.

Let us notice that the adoption of time $\tilde{t}$ reflects the adoption of Selleri "absolute" synchronization in every LCIF, which can be achieved by setting each clock on the rim to 0 when it coincides with a clock of S_0 displaying the time $t = 0$. This complies with the fact (previously stated) that Selleri choice provides a consistent parametrization of the rotating frame at large.

The generalized Born metric reads [15, 30]

$$c^2 \mathrm{d}\tau^2 = c^2 \mathrm{d}\tilde{t}^2 - R^2 \mathrm{d}\vartheta^2 - 2\gamma(\Omega)\Omega R^2 \mathrm{d}\vartheta \mathrm{d}\tilde{t}\,, \tag{5.21}$$

R being the radius of the disk.

The world-lines of clocks A and B, respectively standing on the rim and performing a round-trip, can be parametrized by the proper times τ_A and τ_B and are described by the following equations

$$\begin{aligned} \mathrm{d}\vartheta_A &= 0\,, \\ \mathrm{d}\vartheta_B &= \tilde{\omega}\mathrm{d}\tilde{t} \end{aligned} \tag{5.22}$$

($\tilde{\omega}$ stands for the angular velocity of B with respect to the temporal coordinate $\tilde{t}$).

In terms of $\tilde{t}$, the time interval between the departure and the arrival of clock B is given by

$$2\pi = \tilde{\omega}\Delta\tilde{t} \Longrightarrow \Delta\tilde{t} = \frac{2\pi}{\tilde{\omega}}\,.$$

Exploiting (5.21) and (5.22) one has, for the proper time intervals $\Delta\tau_A$ and $\Delta\tau_B$ measured by the two clocks while B is moving along the rim:

$$\begin{aligned} \Delta\tau_A &= \Delta\tilde{t} = \frac{2\pi}{\tilde{\omega}}\,, \\ \Delta\tau_B &= \oint \sqrt{\mathrm{d}\tilde{t}^2 - \frac{R^2}{c^2}\mathrm{d}\vartheta_B^2 - \frac{2\gamma(\Omega)\Omega R^2}{c^2}\mathrm{d}\vartheta_B \mathrm{d}\tilde{t}} \end{aligned}$$

$$= \oint \mathrm{d}\tilde{t}\sqrt{1 - \frac{R^2}{c^2}\tilde{\omega}^2 - \frac{2\gamma(\Omega)\Omega R^2}{c^2}\tilde{\omega}}$$
$$= \frac{2\pi}{\tilde{\omega}}\sqrt{1 - \frac{R^2}{c^2}\tilde{\omega}^2 - \frac{2\gamma(\Omega)\Omega R^2}{c^2}\tilde{\omega}}\ .$$

The development in $\tilde{\omega} = 0$ of the square root factor yields $1 - \frac{\gamma(\Omega)\Omega R^2}{c^2}\tilde{\omega} + O(\frac{\tilde{\omega}^2 R^2}{c^2})$, so that

$$\Delta\tau_B = \frac{2\pi}{\tilde{\omega}}\sqrt{1 - \frac{R^2}{c^2}\tilde{\omega}^2 - \frac{2\gamma(\Omega)\Omega R^2}{c^2}\tilde{\omega}} = \frac{2\pi}{\tilde{\omega}} - \frac{2\pi\gamma(\Omega)\Omega R^2}{c^2} + O(\frac{\tilde{\omega}R}{c})\ .$$

Eventually

$$\Delta\tau_A - \Delta\tau_B = \frac{2\pi\gamma(\Omega)\Omega R^2}{c^2} + O(\frac{\tilde{\omega}R}{c})\ ,$$

from which

$$\lim_{\tilde{\omega}\to 0} \Delta\tau_A - \Delta\tau_B = \frac{2\pi\gamma(\Omega)\Omega R^2}{c^2} \equiv \tau_{lag}\ .$$

A result that, as one should expect, matches the preceding computation, which had been performed using standard coordinates of the central inertial frame S_0.[23]

6. Exploiting Selleri gauge freedom

We know from Sec. 2 that the synchronization is not "given by God", as often both relativistic and anti-relativistic Authors assume, but it can be arbitrarly chosen within the Cattaneo gauge (5.3). In particular, it can be chosen within the narrower Selleri gauge. Amidst Selleri gauge, Einstein synchrony choice (corresponding, on the rotating frame, to $e_1 = -\Omega R\gamma/c^2$ in every LCIF) has the agreeable feature of being the only one which complies with slow transport synchronization; its remarkable drawback consists in being restricted to the infinitesimal domain of any LCIF. On the contrary, Selleri synchrony choice (corresponding to $e_1 = 0$ in every LCIF), although being incompatible with slow transport synchronization, is the only procedure allowing a global synchronization of clocks on the platform.

In fact, Selleri's absolute time is nothing but the inertial time t read on the clock of S_0 by which an arbitrary clock on the platform K passes at a given instant, rescaled by a factor γ^{-1} [see Eq. (5.11)]. Because of this, the synchrony

[23] Actually, one could object that the limit process has been taken here for $\tilde{\omega} \to 0$, while, in section 5.1, it had been taken for $\omega \to 0$. However, it is easy to see that the two limits imply each other.

criterium in K is nothing but the synchrony criterium in S_0: this is Selleri's synchrony choice.[24]

An important application is the GPS. In fact, the synchronization of the clocks carried by the GPS satellites requires a Selleri-like synchrony choice: no other synchronization can be carried out in the rotating frame of the Earth [31],[18]. More explicitly, because of the elliptical orbit of the satellites, the synchronization must be performed in the central inertial frame S_0. This is the so-called "absolute" (in Selleri's term) synchronization, which agrees with Selleri's synchronization on the rim[25].

We agree with Selleri that synchronization is a matter of convention as far as inertial frames, translating with respect to one another, are concerned. However, as we have anticipated in Sec. 2.2, a strong objection can be raised against Selleri's belief according to which synchronization is a matter of convention only in case of translation, but not in case of rotation. In his words, "the famous synchronization problem is solved by nature itself: it is not true that the synchronization procedure can be chosen freely, because all conventions but the absolute one lead to an unacceptable discontinuity in the physical theory" [3], [21].

The discontinuity to which Selleri refers descends from the comparison of the global velocity of light on a round-trip along the rim, as measured by a single clock, and the local velocity of light, as measured by two clocks in a LCIF on the rim. Such a discontinuity persists even when the limit $\Omega \to 0$ is considered.[26] This is the reason why Selleri properly regards this discontinuity as an "unacceptable" one.

However, we are in a position to give a very simple solution of this 'paradox'. The discontinuity is uniquely originated by the fact that Selleri compares two velocities resulting from different synchronization conventions; in particular, an isotropic synchronization is chosen in S_0 and an anisotropic synchronization is chosen, according to Eqns. (5.11), in any LCIF on the rim [26]. In other terms, the discontinuity found by Selleri is not a physical discontinuity, but merely expresses different synchrony choices. This discontinuity disap-

[24]The Selleri's synchrony choice results in the choice of a coordinate time $\tilde{t}$ on the platform that coincides with the proper time of the clocks at rest on the platform ($\tilde{t} = \gamma^{-1}t = \gamma^{-1}(\gamma\tau) = \tau$). However, the global simultaneity criterium is given by the spacetime foliation $t = const$, that implies $\tilde{t} = const$ on the rim ($t = const$, $r = R = const \Rightarrow \tilde{t} = \gamma^{-1}t \equiv \gamma^{-1}(\Omega R)t = const$), but not on the whole platform. As a consequence, one can see that the time coordinate allowing a global synchronization on the whole platform (*i.e.* for $0 \le r \le R$) is actually the time t of the central inertial frame S_o.

[25]On the rim the Lorentz factor is constant; so it plays the irrelevant role of a factor scale.

[26]More explicitly, the limit is: $\Omega \to 0$, $R \to \infty$ in such a way that ΩR turns to be unchanged (see [3] for details).

pears if the same synchronization procedure is adopted in any (local or global) inertial frame: this is a necessary condition (altough often not explicitly stated) for the validity of a proper formulation of the relativity principle.

Let us stress that it doesn't matter whether the "absolute" or the Einstein synchronization is used: according to a meaningful formulation of the relativity principle, the only necessary condition is that in any (local or global) inertial frame the same synchronization procedure must be adopted. This sort of obviousness is, unfortunately, quite clouded by the standard formulation of SRT. As a matter of fact, the optical isotropy of every IF is usually considered as a physical property of the IF itself, rather than a consequence of a "suitable" synchrony gauge choice.[27]

7. Conclusions

The time-lag (characterizing the non time-orthogonality of the rotating frame and responsible of the fringes' shifts in Sagnac interferometers) has been introduced by Cantoni and Anandan in Minkowski spacetime as a 'theoretical artefact', whose crucial (and odd) feature is the use of Einstein synchronization extrapolated from local to global along the rim. However, the computation performed in this paper straightforwardly reveals the deep physical, non-conventional, nature of the time-lag. In particular, it reveals the dark physical root of the Sagnac effect: desynchronization of slowly travelling clocks.

Moreover, the same computation also reveals that slow transport synchronization agrees with Einstein synchronization also in rotating frames, although in these frames Einstein synchronization is not allowed at large.

To conclude, we claim that, in the light of the results shown and of the considerations presented here, no empirical evidence related to rotation can force the choice of a privileged synchronization procedure, in the context of Cattaneo internal gauge freedom.

The physical nature of the desynchronization effect, also in connection with the Hafele-Keating-like experiments, has been thoroughly investigated. Quite remarkably, this effect lets only one synchronization procedure to be consistent *at large* on the rim of a rotating turntable.

Despite of this fact, the desynchronization time-lag, although introduced as a 'theoretical artefact' strictly related to a well particular synchrony choice, turns to be inherently 'gauge-invariant': it can be explained by adopting any synchronization convention on the LCIF's of the rim of the turntable.

[27]The previous statement obviously concerns the *one-way* velocity of light. Any IF is isotropic on "two-way" paths, regardless of the synchronization choice. According to SRT, this is a crucial physical property of any IF.

Appendix

According to Eq. (5.15), let $v_\pm = \omega_\pm R = (\Omega \pm \omega) R$ be the velocity of the cr/crr-particle relative to the central IF.[28] Now, let $v'_\pm = \omega'_\pm R$ be the velocity of the cr/crr-particle *relative to an Einstein synchronized LCIF*.[29] Then the velocities $v_\pm$ and $v'_\pm$ are related through the standard Lorentz law

$$v_\pm = \frac{v'_\pm + \Omega R}{1 + \frac{\Omega R v'_\pm}{c^2}} \, .$$

As a consequence, the angular velocities are related through the following law:

$$\Omega \pm \omega = \frac{\omega'_\pm + \Omega}{1 + \frac{\Omega R \cdot \omega'_\pm R}{c^2}} \, ,$$

which reduces, at first order approximation with respect to $\omega'_\pm R/c$, to

$$\Omega \pm \omega = \Omega \pm \gamma^{-2}(\Omega)\omega'_\pm + O(\frac{\omega^2_\pm R^2}{c^2}) \, .$$

Therefore, at first order approximation the following relationship holds:

$$\omega = \pm\gamma^{-2}(\Omega)\omega'_\pm \, .$$

Therefore, Eqns. (5.16), (5.17) can take the form

$$\begin{aligned} \Delta\tau_B^+ - \Delta\tau_A &= -2\pi\gamma(\Omega)\frac{R^2}{c^2}\Omega - \pi\gamma^{-1}(\Omega)\frac{R^2}{c^2}\omega'_+ + O(\frac{\omega'^2_+ R^2}{c^2}) \, , \\ \Delta\tau_B^- - \Delta\tau_A &= 2\pi\gamma(\Omega)\frac{R^2}{c^2}\Omega + \pi\gamma^{-1}(\Omega)\frac{R^2}{c^2}\omega'_- + O(\frac{\omega'^2_- R^2}{c^2}) \, , \end{aligned}$$

from which

$$\lim_{\omega'_+ \to 0} \left(\Delta\tau_A - \Delta\tau_B^+\right) = \lim_{\omega'_- \to 0} \left(\Delta\tau_B^- - \Delta\tau_A\right) = \frac{2\pi\Omega R^2}{c^2}\gamma(\Omega) \equiv \tau_{lag} \, ,$$

that is nothing but Eq. (5.18), with ω'_+ and ω'_- instead of ω.

This shows that Eq. (5.18), which gives the (observable) desynchronization between the clocks A and B at the end of the slow round-trip performed by B, turns out to be independent on any convention about local synchronization: just as one should expect for an observable quantity.

References

[1] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, (Butterworth-Heinemann, Woburn, 1997).
[2] A. Peres, *Phys. Rev. D* **18**, 2173 (1978).
[3] F. Selleri, *Found. Phys. Lett.* **10**, 73 (1997).
[4] R. D. Klauber, *Found. Phys. Lett.* **11**, 405 (1998).

[28]"Absolute velocity" in Selleri's terms

[29]Notice that all the angular velocities can be positive or negative, except for ω which is positive defined

[5] B. Mashhoon, 3 **145**, 147 (1990).
[6] G. Rizzi and A. Serafini, *G. Fis.* **44**, 3, www.sif.it/pub-reviews-it.shtml (2003).
[7] T. Matolesi, *Found. Phys.* **28,** 1685 (1998).
[8] A. Macdonald, *Am. J. Phys.* **51**, 759 (1983).
[9] V. Cantoni, *Il nuovo Cimento* **57**, 220 (1968).
[10] J. Anandan, *Phys. Rev. D* **24**, 338 (1981).
[11] G. Rizzi and A. Tartaglia, *Found. Phys.* **28,** 1663 (1998).
[12] S. Bergia and M. Guidone, *Found. Phys. Lett.* **11**, 549 (1998).
[13] G. Rizzi, and M. L. Ruggiero, in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)
[14] R. Anderson, I. Vetharaniam and G. E. Stedman, *Phys. Rep.* **295**, 93 (1998).
[15] G. Rizzi and M. L. Ruggiero, *Found. Phys.* **32**, 1525 (2002).
[16] J. C. Hafele and R. E. Keating, *Science* **177**, 166 (1972).
[17] D. Dieks, in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)
[18] N. Ashby, in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)
[19] F. Selleri, *Teorie alternative alla relatività e natura del tempo*, in "La scienza e i vortici del dubbio", L. Conti and M. Mamone Eds. (Capria, Ed. Scientifiche It., Napoli, 1998).
[20] F. Selleri, *Found. Phys.* **26**, 641 (1996).
[21] F. Selleri, in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)
[22] C. Cattaneo, *Il Nuovo Cimento* **10,** 318 (1958).
[23] C. Cattaneo, *Introduzione alla teoria einsteiniana della gravitazione*, (Veschi, Roma, 1961).
[24] C. Møller, *The Theory of Relativity*, (Oxford University Press, New Delhi, 1972).
[25] Ø. Grøn, *Found. Phys.* **9**, 353 (1979).
[26] A. Serafini, *Problemi di sincronizzazione in Relatività Ristretta e Generale: aspetti operativi e convenzionali*, (Degree Thesis, Università di Pisa, 2001).
[27] R. Klauber, private communication.
[28] R. Schlegel, *Am. J. Phys.* **42**, 183 (1974).
[29] R. Mansouri and R.U. Sexl, *Gen. Rel. Grav.* **8,** 515 (1977).
[30] E.J. Post, *Rev. of Modern Phys.* **39**, 475 (1967).

[31] N. Ashby, *Relativity in the Future of Engineering*, IEEE Transactions on Instrumentation and Measurement **43**, 505 (1994).

Chapter 6

TOWARD A CONSISTENT THEORY OF RELATIVISTIC ROTATION

Robert D. Klauber
1100 University Manor Dr., 38B, Fairfield, IA 52556, USA
klauber@kdsi.net, rklauber@netscape.net

Abstract *Part 1: Traditional Analysis Conundrums.* Although most physicists presume the theoretical basis of relativistically rotating systems is well established, there may be grounds to call the traditional analysis of such systems into question. That analysis is argued to be inconsistent with regard to its prediction for circumferential Lorentz contraction, and via the hypothesis of locality, the postulates of special relativity. It is also contended that the traditional analysis is in violation of the continuous and single valued nature of physical time. It is further submitted to be in disagreement with the empirical finding of Brillet and Hall, the global positioning system satellite data, and a light pulse arrival time analysis of the Sagnac experiment.

Part 2: Resolution of the Conundrums: Differential Geometry and Non-time-orthogonality. It is postulated that physical constraints on time (its continuous and single-valued nature) limit the set of possible synchronization/simultaneity schemes in rotation to one, the "flash from center" scheme. A differential geometry analysis based on this simultaneity postulate is presented in which the rotating frame metric is constrained to be locally non-time-orthogonal (NTO) and due to which, all inconsistencies and disagreements with experiment are resolved. The hypothesis of locality is shown to be invalid for rotation specifically, and generally valid only for non-inertial frames in which the metric can have all null off diagonal space-time components (i.e., time is orthogonal to space.) The analysis approach presented does not contravene traditional relativistic theory for translating systems and makes many (but not all) of the same predictions for rotating systems as does the traditional (time orthogonal) analysis.

Part 3: Experiment and Non-time-orthogonal Analysis. Experiments performed from the 1880s to the present to test special relativity are summarized, and their relevance to NTO analysis is presented. One test, that of Brillet and Hall, appears capable of discerning between the NTO and traditional approaches to relativistic rotation. It yielded a signal predicted by NTO analysis, but not by the traditional approach. Other evidence in favor of the NTO approach may be inherent in the global positioning system data, and the Sagnac experiment.

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 103-137.

1. Traditional Analysis Conundrums

1.1 Introduction

Part 1 outlines the traditional approach to relativistic rotation and discusses various apparent inconsistencies associated therewith. Following an analysis of synchronization/simultaneity in rotating frames and seeming traditional approach problems therein, a new postulate is introduced, which will be used in Part 2 to pose an alternative approach to resolving the inconsistencies.

1.2 Relevant Relativity Principles

Special relativity theory (SRT) is restricted to inertial systems and is derived from two symmetry postulates:

1 The speed of light is the same for all inertial observers (it is invariant) and equals c.

2 There is no preferred inertial reference frame. (Velocity is relative, and the laws of nature are covariant, i.e., the same for all inertial observers.)

The first postulate, applied to the one-way speed of light, is equivalent to demanding that Einstein synchronization of clocks holds. In Einstein synchronization, one starts from a first clock at time t_A on that clock and sends a light pulse to a second clock fixed in the same frame as the first. The light pulse is reflected back at the second clock and returns to the first clock at time t_B on the first clock. The time on the second clock is then set such that its reading when the light was reflected would have been $(t_A + t_B)/2$, the time on the first clock half way between the emission and reception times. This ensures the one-way speed of light, measured as the distance travelled between clocks divided by the time difference of the two clocks, is always c.

In recent years, many relativists have come to consider Einstein synchronization merely a convention, or gauge, that affects no measurable quantities [1],[2]. For example, in all such gauge theories of synchronization, the round trip speed of light is c (though the one-way speed of light need not be.) Nevertheless, underlying SRT is the assumption that Einstein synchronization is always one of the possible conventions that makes valid predictions about inertial frames in the physical world.

General relativity is applicable to non-inertial systems and is based on additional postulates, including the equivalence principle and the hypothesis of locality (or sometimes, the "surrogate frames postulate"). The hypothesis of locality stands as a linchpin in the traditional approach to relativistic rotation, and thus, I number it among the postulates of importance to this article.

3 Hypothesis of locality: Locally (i.e., over infinitesimal regions of space and time), neither gravity nor acceleration changes the length of a standard rod or the rate of a standard clock relative to a nearby freely falling (i.e., inertial) standard rod or standard clock instantaneously co-moving with it. See Møller[3], Einstein[4], and Mashoon[5].

Stated another way, a local inertial observer is equivalent to a local co-moving non-inertial observer in all matters having to do with measurements of distance and time. It follows immediately that Einstein synchronization can be carried out locally, and that for such synchronization, the local one-way speed of light measured in a *non-inertial* frame is c. Hence, a Lorentz frame can be used as a local surrogate for the non-inertial frame. This has a basis in differential geometry, in which a curved space is locally flat and can be represented locally by Cartesian coordinates.

Minguzzi[6] and Møller[7], among others, note that the hypothesis of locality is only an assumption. It is, however, an assumption that, historically, has worked very well in a large number of applications. See, for example, the treatment of acceleration by Misner, Thorne, and Wheeler[8] using instantaneous local Lorentz frames.

1.3 The Traditional Approach

The traditional approach to relativistic rotation assumes the hypothesis of locality is a fundamental and universal truth. As done successfully in other, non-rotating, cases, values in local co-moving Lorentz frames are integrated to determine global values for quantities such as distance and time, which would, in principle, be measured with standard meter sticks and clocks by an observer in the rotating frame.

The oft - cited example, first delineated by Einstein[9], is the purported Lorentz contraction of the rim of a rotating disk. (Or alternatively, the circumferential stresses induced in the disk when the rim tries to contract but is restricted from doing so via elastic forces in the disk material.) A local Lorentz frame instantaneously co-moving with a point on the rim, it is argued, exhibits Lorentz contraction of its meter sticks in the direction of the rim tangent, via its velocity, $v = \omega r$. This infinitesimal length contraction is subsequently integrated over all of the local Lorentz frames instantaneously at rest with respect to each successive point along the rim. The result is a number of meter sticks that is greater than $2\pi r$, and thus, the disk surface is concluded to be non-Euclidean, or Riemann curved[10],[11].

1.4 Inconsistency of Circumferential Lorentz Contraction

According to SRT, an observer does not see his own lengths contracting. Only a second observer moving relative to him sees the first observer's length dimension contracted. Hence, from the point of view of the disk observer, her own meter sticks are not contracted[12], and there can be no curvature of the rotating disk surface. The traditional analysis is thus, inconsistent[13].

Consider further the disk observer looking out at the meter sticks at rest in the lab close to the disk's rim. Via the hypothesis of locality (in which she is equivalent to a local co-moving Lorentz observer), she sees the lab meter sticks as having a velocity with respect to her. Hence, by the traditional logic, she sees them as contracted in the circumferential direction. She must therefore conclude that the lab surface is curved. But those of us living in the lab know this is simply not true, and again the analysis is inconsistent.

Although these arguments seem to be rarely considered by traditionalists, when they are brought to their attention, the usual defense is that "the rotating frame is not an inertial frame and thus is different". Yet, the hypothesis of locality, the starting point for the analysis, assumes that they are not different in this regard.

Furthermore, if the non-inertial argument has any validity, then it must imply that the length contraction of the rim is absolute, i.e., both the lab and disk observer agree that the disk meter sticks are contracted. Yet, consider the limit case of low ω, high r, such that $a = \omega^2 r \approx 0$, while $v = \omega r$ is close to the speed of light (the "limit case"). Advocates of the traditional approach contend that, since the limit case observer fixed to the rotating disk rim feels no inertial "force", she becomes, effectively, a Lorentz observer. In this case, each of the lab and disk observers must see the other's meter sticks as contracted and their own as normal. Yet, the non-inertial argument started with the assumption that the disk observer's meter sticks contracted in an absolute way, agreed to by all observers[14].

Conclusion: Length contraction applied via the traditional analysis to rotating systems appears self-contradictory.

1.5 Second SRT Postulate Not Valid in Rotation

Without looking outside, an observer on the rim of a rotating disk can determine her angular velocity ω, using, for example, a Foucault pendulum. She can also use a spring mass system to measure $kx/m = \omega^2 r$, and hence determine r, the distance to the center of rotation. (The Newtonian limit is used to simplify the example. The conclusion is also true for relativistic calculations.)

That is, contrary to the dictate of the second postulate, there are experiments an observer can perform locally from entirely within the rotating frame to determine her speed in an absolute sense. (To be precise, her speed with respect to the inertial frame in which her center of rotation is fixed.) Her velocity is not relative. Both the lab and the rim fixed observers determine the same value for

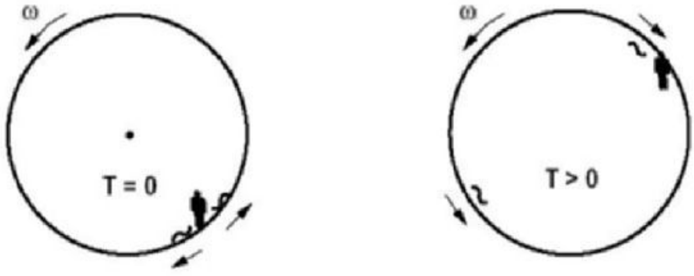

Figure 6.1. Rotating Disk Observer Measuring Light Speed

it. With respect to circumferential speed, there is a preferred frame, and both observers agree it is the one where such speed is zero, i.e., the non-rotating lab frame.

Conclusion: The second relativity postulate does not appear to hold for rotating systems

1.6 First SRT Postulate: Thought and Sagnac Experiments

1.6.1 Thought Experiment. Consider the following thought experiment (see Selleri[15]) involving an observer fixed to the rotating disk of Figure 6.1 who measures the speed of light.

The observer shown has already laid meter sticks along the rim circumference and determined the distance around that circumference. As part of her experiment, she has also set up a cylindrical mirror, reflecting side facing inward, all around the circumference. She takes a clock with her and anchors herself to one spot on the disk rim. When her clock reads time $T = 0$ (left side of Figure 6.1) she shines two short pulses of light tangent to the rim in opposite directions. The mirror will cause these light pulses to travel circular paths around the rim, one clockwise (cw) and one counterclockwise (ccw).

From the lab, we see the cw and ccw light pulses having the same speed c. However, as the pulses travel around the rim, the rim and the observer fixed to it move as well. Hence, a short time later, as illustrated in the right side of Figure 6.1, the cw light pulse has returned to the observer, whereas the ccw pulse has yet to do so. A little later (not shown) the ccw pulse will have caught up to the observer.

For the observer, from her perspective on the disk, both light rays travel the same number of meters around the circumference. But her experience and her clock readings tell her that the cw pulse took less time to travel the same distance around the circumference than the ccw pulse.

She can only conclude that, from her point of view, the cw pulse travelled faster than the ccw pulse. Hence, the speed of light as measured on the rotating disk is anisotropic and different from that measured in the lab. Thus, one could conclude that the first relativity postulate, in the context of the hypothesis of locality, is violated.

Arguments against this conclusion are rooted in two interrelated concepts: i) purported differences between the global (as measured in the above thought experiment) and local, physical speeds of light[16],[17],[18], and ii) the synchronization/simultaneity employed[19],[20]. The author has extensive remarks on this in a subsequent section, but for now, submits that the appropriate synchronization scheme comprises the following.

Consider the ccw light pulse and the time difference on the clock held by the observer in Figure 6.1 between the emission (assume initial clock time is t_A = 0) and reception (clock time t_B) events. Employ the synchronization method of Section 1.2, only instead of using a back and forth round trip for light (Einstein synchronization), use a circular round trip. That is, the time on the clock half way along the round trip ccw path (at 180° of the disk here), at the instant the ccw light pulse was there, should be $(t_A + t_B)/2 = t_B/2$. With this time, the ccw speed of light will be the same as that computed for the round trip, i.e., it will be less than c. Now synchronize the clock at 90° the same way. Assume its setting at the instant the light pulse passed was half that on the clock at 180° when the light passed that clock (or equivalently, $^1/_4$ of t_B.) Doing this, one again finds the ccw speed of light to be the same value, which is $< c$. Repeat over smaller intervals until, in the limit, one finds the local speed of light to be the same, and therefore not equal to c.

The entire procedure can be repeated for the cw light pulse, and one will find the clocks at each location done via the cw and ccw methods are synchronized, i.e., they are the same clocks. One will also find that the local speeds of light are anisotropic and equal to the same values as the global ones determined using a single clock at the emission/reception point.

Conclusion: While it may appear that the local speed of light, being anisotropic, violates the first relativity postulate, there is still the possibility that Einstein synchronization may be valid locally (as one of the possible local synchronization schemes). This would mean that for such synchronization, the local speed of light would be c, and one could get correct results using the hypothesis of locality and local Lorentz frame analyses.

Challenge to traditionalists: The author does not believe this and challenges advocates of the traditional approach to begin with the assumption of local isotropic light speed, and without looking outside of the rotating frame, kinematically derive the result that the cw light pulse arrives back at the emission point before the ccw one.

1.6.2 The Sagnac Experiment. In 1913, G. Sagnac[21] carried out a now famous experiment, similar in many ways to the thought experiment of Figure 6.1. On a rotating platform, he split light from a single source into cw and ccw rays that travelled identical paths in opposite directions around the platform. He combined the returning rays to form a visible interference pattern, and found that the fringes shifted as the speed of rotation changed. A number of others[22],[23] subsequently performed the same test with the same results.

If the speed of light were locally invariant and always equal to c, then speeding up or slowing of the rotation rate of the platform should not change the location of the fringes. However, the fringes do change with speed and once again we have a test (Sagnac) whereby we can determine a preferred frame, in seeming violation of the second relativity postulate and the hypothesis of locality.

Putative explanations for this in the context of the traditional approach hinge, once again, on synchronization/simultaneity and global vs. local arguments. These are addressed in the following section.

I do contend that the thought experiment of Figure 6.1 makes it clear that any explanation for the Sagnac experiment, from the point of view of the disk reference frame, must account for different *arrival times* for the cw and ccw light pulses. Analyses based on Doppler shifts[24] or DeBroglie momentum/wave length[25] changes are simply not sufficient to explain this.

The calculation of this arrival time difference, derived from the lab frame, is well known and is repeated for reference in the Appendix.

1.7 Synchronization/Simultaneity in the Traditional Approach

1.7.1 The Traditional Approach "Time Gap". Consider the non-rotating (lab) frame as K; the rotating (disk) frame as k. Figure 6.2 depicts inertial measuring rods in inertial frames K_1 to K_8 with speeds ωr instantaneously at rest with respect to eight points on the rotating disk rim as shown. For practical reasons, only eight finite length rods are shown, and one can consider them as a symbolic representation of an infinite number of such rods of infinitesimal length. A and B are *events* located in space at the endpoints of the K_1 rod which are simultaneous as seen from K_1; B and C are events located in space at the endpoints of the K_2 rod which are simultaneous in K_2; and so on for the other events C to J. A,B, ...J can be envisioned as flashes of light emitted by bulbs situated equidistantly around the disk rim.

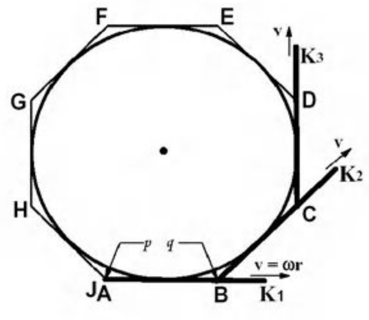

Figure 6.2. Inertial Co-Moving Frame

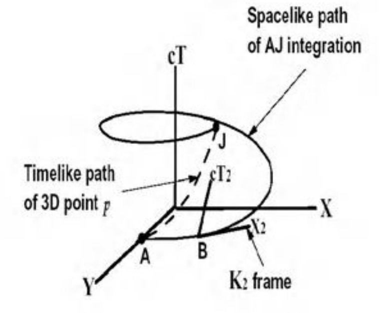

Figure 6.3. Co-Moving Frames Integration Path

p is a spatial (three-dimensional) point fixed to the disk at which both A and J occur. q is the spatial point on the disk at which B occurs. In principle, A, B, ... J, as well as p and q are located on the disk rim though they may not look so in Figure 6.2, since the co-moving rods shown are not infinitesimal in length.

In the traditional analysis, the hypothesis of locality is invoked to claim that times and distances measured by standard measuring rods and clocks in the local co-moving inertial frames are identical to those riding with the disk.

Note that although events A and B are simultaneous as seen from K_1, they are not simultaneous as seen in K (via SRT for two inertial frames in relative motion). As seen from K, A occurs before B. Similarly, B occurs before C, and so on around the rim. If the events are light flashes, a ground based observer looking down on the disk would see the A flash, then B, then C, etc. Hence we conclude that as seen from K, A occurs before J even though A and J are both located at the same 3D point p fixed to the rim. As seen from K, during the time interval between events A and J, the disk rotates, and hence the point p moves. (As an aside, Figure 6.2 can now be seen to be merely symbolic since events A to J would not in actuality be seen from K to occur at the locations shown in Figure 6.2. That is, by the time the K observer sees the B flash, the disk has rotated a little. It rotates a little more before he sees the C flash, etc.)

According to the traditional treatment of the rotating disk, one then uses the K_i rods and integrates (adds the rod lengths) along the path AB ...J, moving sequentially from co-moving inertial frame to co-moving inertial frame. This path is represented by the solid line in Figure 6.3, and one can visualize small Minkowski coordinate frames situated at every point along the curve AJ (see K_2 in Figure 6.3) with integration taking place along a series of spatial axes (such as X_2 in Figure 6.3). By doing this one arrives at a length for AJ, the presumed circumference of a disk of radius r, of

$$\mathrm{AJ} = \frac{2\pi r}{\sqrt{1-\omega^2 r^2/c^2}} > \text{ Circumference in lab,} \tag{6.1}$$

and thus, the disk surface is concluded to be non-Euclidean (Riemann curved.)

But consider that since point p moves along a timelike path as seen from K (see dotted line in Fig. 6.3), a time difference between events A at 0^o and J at 360^o must therefore exist as measured by a clock attached to point p. To continue from J to A requires a jump in time, and thus, the traditional analysis approach leads to a discontinuity in time (or alternatively, multi-valued time), a seemingly impossible physical situation. Further, as noted by Weber[26], this means that if light rays are sent 360^o around the rim to synchronize the clock at J with that at A, then the two clocks (which are really one and the same clock) are not in synchronization. That is, each clock on the disk is out of synchronization with itself.

Still further, according to the traditional analysis, time all along the path AJ is fixed. Thus, by that analysis, which depends on the locality hypothesis and integration of values (time in this case) from local frame to local frame, A and J must be simultaneous. But they are not.

1.7.2 Traditional Approach to Resolve the "Time Gap". In recent years, this problem has been treated as if this "time gap" were a mathematical artifact, and approaches labelled "desynchronization"[27],[28],[29] and "discontinuity in synchronization"[30],[31] have been proposed that entail multiple clocks at a given event. These approaches seem motivated by the gauge theory of synchronization philosophy that time settings on clocks are inherently arbitrary.

Furthermore, the time gap is often said to be identical in nature to travelling at constant radius in a polar coordinate system. The ϕ value is discontinuous at 360^o. Similar logic applies for time, with the International Date Line for the time zone settings on the earth. If one starts at that line and proceeds 360^o around the earth, one returns to find one must jump a day in order to re-establish one's clock/calendar correctly.

1.7.3 Arguments for Physical Interpretation of the "Time Gap". The gauge synchronization philosophy champions innumerable, equally valid, synchronization schemes. Yet, within any one of those schemes, time is single valued and continuous, and clocks are all in synchronization with themselves. For a given synchronization method, each event within a given frame has a single time associated with it.

In the desynchronization approaches, a given event in a given rotating frame, for a given synchronization method, can have any number of possible times on it. For example, the clock at point p in Figure 6.2 has one time on it at event A. If one Einstein synchronizes the clock at 360^o (i.e., the same clock at p) with ccw light rays, one gets another time setting. Thus, one has a choice of which of two times one prefers for any given event at point p. If, on the other hand, one synchronizes the clock at 360^o via cw light rays, one gets yet another setting, and a third possible time to choose for any given event. Consider yet another path in which the light ray goes radially inward 1 meter, then 360^o around, then radially outward 1 meter. One then gets yet another setting for the clock at p. Since there are an infinite number of possible paths by which one could synchronize the clock at p, there are an infinite number of possible times for each event at p. (This does not happen in translation. Any possible path for the light rays results in the same unique setting, for a given synchronization scheme, on each clock in the frame.)

This plethora of possible settings for the same clock results from insisting on "desynchronization" of clocks in order to keep the speed of light locally c everywhere. And thus, one is in the position of choosing whichever value for time one needs in a given experiment in order to get the answer one insists one must have (i.e., invariant, isotropic local light speed.) One can only then ask if this is really physics or not. Can an infinite number of possible readings on a single clock at a single event for a single method of synchronization be anything other than meaningless?

The polar coordinate analogy, I believe, confuses physical discontinuity with coordinate discontinuity. In 2D, place a green X at 0^o, travel 360^o at constant radius, and then place a red X. The red and green marks coincide in space. There is no discontinuity in space between them, although there is a discontinuity in the coordinate ϕ.

Flash a green light on the equator at the International Dateline, then trace a path once around the equator along which no time passes. If you flash a red light at the end of that path, the red and green flashes are coincident in space and time. There is no physical discontinuity, although your time zone clocks show a coordinate discontinuity.

In Figure 6.3, flash a green light at event A. Travel 360^o on the disk along the space-time path AJ (along which no time passes according to the traditional analysis), then flash a red light at event J. The two lights are not coincident. There is real world space-time gap between them, and they exist at different points in 4D. The discontinuity is physical, not merely coordinate.

Peres was aware of this time discontinuity, calling it a *"heavy price which we are paying to make the [circumferential] velocity of light ... equal to c"* [32]. Dieks[33] noted that though arbitrary in certain senses, time in relativity must be *"directly linked to undoubtedly real physical processes"*. This author agrees.

1.7.4 The Only Physically Possible Synchronization/Simultaneity.

There are potential choices for synchronization/simultaneity in the rotating frame other than Einstein's. The traditional one with local Einstein synchronization around the disk rim is based on the Lorentz transformation from the lab to the local co-moving inertial frame, i.e.

$$cdT_i = \frac{1}{\sqrt{1 - v^2/c^2}}(cdT - \frac{v}{c}dX) = \gamma(cdT - \frac{v}{c}dX) \tag{6.2}$$

where $v = \omega r$, *dT* is the time interval in the lab, *dX* is the space interval in the lab along the disk rim, dT_i is the time interval in the local co-moving inertial frame, which we presume, by the locality hypothesis, equals the time interval on the disk. We could just as well have chosen[34]

$$cdT_i = \gamma(cdT - \kappa dX) \tag{6.3}$$

where κ could have any value other than v/c.

However, for any $\kappa \neq 0$, we would again have a time discontinuity, and all the issues with multiple event times and clocks being out of synchronization with themselves of the prior section.

I suggest that, prior to all else, any theory of rotation must be compliant with the physical world constraint that time be continuous and single valued (within a given frame, and for a given synchronization scheme.) That is only possible for a synchronization/simultaneity scheme where $\kappa = 0$. For this scheme, events in the lab that are simultaneous (i.e., have *dT* = 0 between them) are also simultaneous in the rotating frame (have *dt* = 0).

Postulate: Any synchronization/simultaneity scheme for the rotating frame for which that frame and the lab do not share common simultaneity results in a physical time discontinuity and is thus unacceptable on physical grounds.

The traditional approach to rotation is at odds with this postulate.

1.8 Experiment and the Traditional Approach

In Part 3, virtually all of the experiments that have been carried out to verify SRT are reviewed. One of these, the Michelson-Morley type experiment performed by Brillet and Hall[35], found a persistent, anomalous, non-null signal at the 10^{-13} level, which is not predicted by SRT. The approach to relativistic rotation of Part 2, which is based on the above postulate, predicts this signal, and otherwise, makes the same predictions as the traditional approach for the remaining tests.

Furthermore, as a result of studies on the Global Positioning System (GPS) data for the rotating earth, recognized world leading GPS expert Neil Ashby states

"Now consider a process in which observers in the rotating frame attempt to use Einstein synchronization..... Simple minded use of Einstein synchronization in the rotating frame ... thus leads to a significant error".[36]

He also recently noted in *Physics Today*,

" .. the principle of the constancy of c [the speed of light] cannot be applied in a rotating reference frame .." [37].

1.9 Summary of Part 1

Thought experiments, actual experiments, and the physical nature of the space-time continuum appear discordant with the traditional approach to relativistic rotation.

2. Resolution of the Conundrums: Differential Geometry and Non-time-orthogonality

".. a good part of science is distinguishing between useful crazy ideas and those that are just plain nutty." Princeton University Press advertisement for the book "Nine Crazy Ideas in Science"

2.1 Introduction

Part 2 poses an alternative approach to relativistic rotation that resolves the inconsistencies, and as will be seen in Part 3, appears to have better agreement with experiment than the traditional approach. There are two fundamental steps to the alternative approach.

1 Postulate that, in accord with physical world logic as presented in Part 1, simultaneity/synchronization in the rotating frame can only be such that time in that frame is continuous and single valued.

2 Apply differential geometry and note resulting predictions.

Before beginning the analysis, relevant background material from differential geometry is presented in Section 2.2.

2.2 Physical vs. Coordinate Components

If one has coordinate components, found from generalized coordinate tensor analysis, for some quantity, such as stress or velocity, one needs to be able to translate those into the values measured in experiment. For some inexplicable reason, the method for doing this is not typically taught in general relativity (GR) texts/classes, so it is reviewed here. (Note that often in GR, one seeks invariants like $d\tau$, ds, etc., which are the same in any coordinate system, and in such cases, this issue does not arise. The issue does arise with vectors/tensors, whose coordinate components vary from coordinate system to coordinate system.)

The measured value for a given vector component, unlike the coordinate component, is unique within a given reference frame. In differential geometry (tensor analysis), that measured value is called the "physical component".

Many tensor analysis texts show how to find physical components from coordinate components[38]. A number of continuum mechanics texts do as well [39]. The only GR text known to the present author that mentions physical components is Misner, Thorne, and Wheeler[40]. Those authors use the procedure to be described below, but do not derive it[41]. The present author has written an introductory article on this, oriented for students, that may be found at the Los Alamos web site[42]. The following is excerpted in part from that article.

The displacement vector $d\mathbf{x}$ between two points in a 2D Cartesian coordinate system is

$$d\mathbf{x} = dX^1\hat{\mathbf{e}}_1 + dX^2\hat{\mathbf{e}}_2 \tag{6.4}$$

where the $\hat{\mathbf{e}}_i$ are unit basis vectors and dX^i are physical components (i.e., the values one would measure with meter sticks). For the same vector $d\mathbf{x}$ expressed in a different, generalized, coordinate system we have different coordinate components $dx^i \neq dX^i$ (dx^i do not represent values measured with meter sticks), but a similar expression

$$d\mathbf{x} = dx^1\mathbf{e}_1 + dx^2\mathbf{e}_2, \tag{6.5}$$

where the generalized basis vectors $\mathbf{e}_i$ point in the same directions as the corresponding unit basis vectors $\hat{\mathbf{e}}_i$, but are not equal to them. Hence, for $\hat{\mathbf{e}}_i$, we have

$$\hat{\mathbf{e}}_i = \frac{\mathbf{e}_i}{|\mathbf{e}_i|} = \frac{\mathbf{e}_i}{\sqrt{\mathbf{e}_{\underline{i}} \cdot \mathbf{e}_{\underline{i}}}} = \frac{\mathbf{e}_i}{\sqrt{g_{\underline{ii}}}} \tag{6.6}$$

where underlining implies no summation.

Substituting (6.6) into (6.4) and equating with (6.5), one obtains

$$dX^1 = \sqrt{g_{11}}dx^1 \qquad\qquad dX^2 = \sqrt{g_{22}}dx^2, \tag{6.7}$$

which is the relationship between displacement physical (measured with instruments) and coordinate (mathematical value only) components.

Consider a more general case of an arbitrary vector $\mathbf{v}$

$$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 = v^{\hat{1}}\hat{\mathbf{e}}_1 + v^{\hat{2}}\hat{\mathbf{e}}_2 \tag{6.8}$$

where, $\mathbf{e}_1$ and $\mathbf{e}_2$ here do not, in general, have to be orthogonal, $\mathbf{e}_i$ and $\hat{\mathbf{e}}_i$ point in the same direction for each index i, and carets over component indices indicate physical components. Substituting (6.6) into (6.8), one readily obtains

$$v^{\hat{i}} = \sqrt{g_{\underline{ii}}}v^i, \tag{6.9}$$

which we have shown here to be *valid in both orthogonal and non-orthogonal systems.*

As a further aid to those readers familiar with anholonomic coordinates (which are associated with non-coordinate basis vectors superimposed on a generalized coordinate grid), physical components are special case anholonomic components for which the non-coordinate basis vectors have unit length.

It is important to recognize that anholonomic components do not transform as true vector components. So one can not simply use physical components in tensor analysis as if they were. Typically, one starts with physical components as input to a problem. These are converted to coordinate components, and the appropriate tensor analysis carried out to get an answer in terms of coordinate components. One then converts these coordinate components into physical components as a last step, in order to compare with values measured with instruments in the real world.

As a basis vector is derived from infinitesimals (derivative at a point), one sees (6.9) is valid locally in curved, as well as flat, spaces, and can be extrapolated to 4D general relativistic applications. So, very generally, for a 4D vector v^{μ} and a metric signature (-,+,+,+)

$$v^{\hat{i}} = \sqrt{g_{\underline{ii}}}v^{i} \qquad v^{\hat{0}} = \sqrt{-g_{00}}v^{0}, \tag{6.10}$$

where Roman sub and superscripts refer solely to spatial components (i.e. i = 1,2,3.)

2.3 Alternative Analysis Approach

We begin with the simultaneity postulate of Section 1.7.4, repeated below for convenience.

Postulate: Any synchronization/simultaneity scheme for the rotating frame, for which that frame and the lab do not share common simultaneity, results in a physical time discontinuity and is thus unacceptable on physical grounds.

2.3.1 Disk Transformation and Metric. As will be discussed, the global transformation from the lab to the rotating frame apparently first used by Langevin[43] to find a suitable metric for the rotating frame incorporates the above postulate. This transformation is used in the following analysis, which parallels that of Klauber[44]. The correctness of the transformation can be judged by whether the predictions made by using it match experiment.

This transformation, between the non-rotating (lab, upper case symbols) frame to a rotating (lower case) frame, is

$$\begin{aligned} cT &= ct \\ R &= r \\ \Phi &= \phi + \omega t \\ Z &= z\,. \end{aligned} \tag{6.11}$$

ω is the angular velocity of the rotating frame as seen from the lab, and cylindrical spatial coordinates are used. The coordinate time t for the rotating system equals the proper time of a standard clock located in the lab. Substituting the differential form of (6.11) into the line element in the lab frame

$$ds^2 = -c^2 dT^2 + dR^2 + R^2 d\Phi^2 + dZ^2 \tag{6.12}$$

results in the line element for the rotating frame

$$ds^2 = -c^2(1 - \tfrac{r^2\omega^2}{c^2})dt^2 + dr^2 + r^2 d\phi^2 + 2r^2\omega d\phi dt + dz^2 = g_{\alpha\beta}dx^\alpha dx^\beta. \tag{6.13}$$

Note that the metric in (6.13) is not diagonal, since $g_{\phi t} \neq 0$, and this implies that time is not orthogonal to space (i.e., a non-time-orthogonal, or NTO, frame.)

2.3.2 Time on the Disk. Time on a standard clock at a fixed 3D location on the rotating disk, found by taking $ds^2 = -c^2 d\tau$ and $dr = d\phi = dz = 0$ in (6.13), is

$$d\tau = \sqrt{1 - r^2\omega^2/c^2}dt = \sqrt{-g_{tt}}dt, \tag{6.14}$$

This varies with radial position r. At the axis of rotation (where $r = 0$), the standard clock agrees with the clock in the lab. At other locations, standard clock time is diluted by the Lorentz factor, as in traditional SRT. The coordinate time everywhere on the disk is t, and that equals the time T in the lab.

The time difference between two events at two locations (each having its own clock) on the disk, in coordinate components, is *dt*. The corresponding physical time difference is

$$dt_{phys} = d\hat{t} = \sqrt{-g_{tt}}dt = \sqrt{1 - r^2\omega^2/c^2}dt. \tag{6.15}$$

If the two locations happen to be one and the same location, one obviously gets (6.14).

Note that two events seen as simultaneous in the lab have *dT* = 0 between them. From the first line of (6.11) and (6.15), the same two events must also have $dt_{phys} = 0$, and thus they are also seen as simultaneous on the disk. This statement is true for all standard (physical) clocks on the disk. Though the standard clocks at different radii thereon run at different rates, and thus can not be synchronized, they *can* share common simultaneity. The lab shares this common simultaneity with all of the disk clocks, and thus our postulate above holds for transformation (6.11), and the resulting (NTO) metric of (6.13).

Note further, that the simultaneity chosen here is equivalent in the physical world to what is sometimes called "flash from center" simultaneity (or synchronization if one is confined to clocks at fixed radius). In that scheme, one imagines a flash of light on the axis of rotation whose wave front propagates outwardly in all radial directions. Events when the wave front impacts individual points along a given circumference are considered simultaneous.

It is significant that the "flash from center" synchronization is the same as that proposed (via other logic) near the end of Section 1.6.1.

2.3.3 Local Speed of Light on the Disk. For light $ds^2 = 0$. Inserting this into (6.13), taking $dr=dz=0$, and using the quadratic equation formula, one obtains a local coordinate velocity (generalized coordinate spatial grid units per coordinate time unit) in the circumferential direction

$$v_{light,coord,circum} = \frac{d\phi}{dt} = -\omega \pm \frac{c}{r}, \qquad (6.16)$$

where the sign before the last term depends on the circumferential direction (cw or ccw) of travel of the light ray. The local physical velocity (the value one would measure in experiment using standard meter sticks and clocks in units of meters per second) is found from this to be

$$v_{light,phys,circum} = \frac{\sqrt{g_{\phi\phi}}d\phi}{\sqrt{-g_{tt}}dt} = \frac{-\,r\omega\ \pm\ c}{\sqrt{1-\frac{\omega^2 r^2}{c^2}}} = \frac{-\,v\ \pm\ c}{\sqrt{1-\frac{v^2}{c^2}}}. \qquad (6.17)$$

Note that for this approach, the local physical speed of light in rotating frames is not invariant or isotropic, and that this lack of invariance/isotropy depends on ω, the angular velocity seen from the lab. Note particularly that this result is a direct consequence of the NTO nature of the metric in (6.13). If ω=0, local physical (measured) light speed is isotropic and invariant, the metric is diagonal, and time is orthogonal to space.

I thus call this alternative analysis method, the NTO approach to relativistic rotation.

2.4 Implications of NTO Approach

2.4.1 Hypothesis of Locality. Local physical light speed in the rotating frame, according to the NTO approach, is not equal to c. Yet, in a local, co-moving, Lorentz frame, which via the hypothesis of locality is equivalent locally to the non-inertial frame, the physical speed of light is always c. Thus, a local co-moving Lorentz frame is *not* equivalent locally to the rotating frame, and the hypothesis of locality is not valid for such frames. One does *not* measure the same values for velocity, and hence by implication for time and space, in the two frames.

This is a direct result of the simultaneity postulate, required to keep time in the rotating frame continuous and single-valued. That requirement results in a rotating frame that can only be NTO, i.e., it can only have a metric with off diagonal terms in the metric.

I submit that the hypothesis of locality remains true only for those non-inertial frames in which it is possible for the metric to have all null off-diagonal space-time components. This set of frames comprises the vast majority of problems encountered in GR. Rotation is a critical exception.

2.4.2 Absolute Nature of Simultaneity in Rotation. In NTO analysis, simultaneity/synchronization on the rotating disk, unlike that in the gauge theory of synchronization, is unique (absolute.) The gauge theory validity, it is submitted, is restricted to translating frames and does not apply to rotation. This is not unlike other differences between rotation and translation. Velocity in rotation, for example, has an absolute quality, whereas in translation it does not. There is a preferred frame in rotation, upon which everyone agrees (the frame with no Coriolis effect, for example); in translation, there is no such frame.

2.4.3 Lorentz Contraction Revisited. To determine Lorentz contraction of meter sticks, we merely need to compare physical length in the circumferential direction in both the lab and rotating frames, i.e., look at the physical component for $d\Phi$ and $d\phi$. This is equivalent to finding the proper length when $dT = 0$ in the first frame (lab here), and $dt = 0$ in the second frame (disk here), which is what one does in SRT.

The distance between two points along the circumference in the lab in meter sticks is

$$d\Phi_{phys} = d\hat{\Phi} = \sqrt{g_{\Phi\Phi}}d\Phi = Rd\Phi = R(\Phi_2 - \Phi_1), \tag{6.18}$$

which is not surprising, and which (for $dR = dT = dZ = 0$) equals ds. (6.18) represents the number of meter sticks between points 1 and 2 in the lab. Now, consider two 3D points on the disk located instantaneously at the same place as points 1 and 2 in (6.18). We ask, how many meter sticks span that distance as measured on the disk? That distance between points 1 and 2 in meter sticks is

$$d\phi_{phys} = d\hat{\phi} = \sqrt{g_{\phi\phi}}d\phi = rd\phi = r(\phi_2 - \phi_1). \tag{6.19}$$

According to (6.11), $\phi_1 = \Phi_1 - \omega t_1$ and $\phi_2 = \Phi_2 - \omega t_2$. Since, $r = R$ and $dt = t_2 - t_1 = 0$, (6.18) and (6.19) are equal. The disk observer sees the same number of meter sticks between two points as the lab observer does between those points, and hence, there is no Lorentz contraction.

Note that we would need a metric component $g_{\phi\phi} \neq r^2$ in the rotating frame to have Lorentz contraction. The postulate of simultaneity/time continuity leads to the metric of (6.13), which is NTO, and which has $g_{\phi\phi} = r^2$.

The Lorentz contraction issue is treated more extensively, and with graphical illustrations, in Klauber[44]. The limit case for NTO analysis is also discussed therein though it is treated at great length in Klauber[45], and found to be free of inconsistencies.

2.4.4 Sagnac and Thought Experiments. A complete and general derivation of the Sagnac result from the rotating frame using NTO analysis can be found in Klauber[46]. Shown below is the simpler derivation for a circumferential light path whose center is the axis of rotation. Note that different speeds for light in the cw and ccw directions is inherent in the NTO approach, and thus that approach is completely consonant with the thought experiment of Part 1, Section 1.6.1.

The difference in time measured on a ccw rotating disk between two pulses of light traveling opposite directions along a circumferential arc of length *dl* is

$$dt_{phys} = \frac{dl}{v_-} - \frac{dl}{v_+}, \tag{6.20}$$

where v_+ is the speed for the cw light ray and v_- is the speed for the ccw ray. Using (6.17) this becomes

$$dt_{phys} = \frac{dl\sqrt{1 - v^2/c^2}}{c - v} - \frac{dl\sqrt{1 - v^2/c^2}}{c + v} = \frac{v}{c^2}\frac{2dl}{\sqrt{1 - v^2/c^2}}. \tag{6.21}$$

By integrating the RHS of (6.21) from 0 to $2\pi r$ (recall there is no Lorentz contraction), the LHS becomes the time difference on the clock fixed at the emission/reception point,

$$\Delta t_{phys} = \frac{\omega r}{c^2}\frac{2(2\pi r)}{\sqrt{1 - \omega^2 r^2/c^2}} = \frac{4\omega A}{c^2\sqrt{1 - \omega^2 r^2/c^2}}, \tag{6.22}$$

which agrees with the derivation from the lab frame of the Appendix.

2.4.5 Brillet and Hall. The Brillet and Hall[35] experiment is described in Part 3. It remains to this day the only test of sufficient accuracy to detect any non-null Michelson-Morley (MM) effect due to the surface speed of the earth rotating about its axis. Brillet and Hall found null signals for the solar and galactic orbit speeds. However, they noted a persistent non-null signal at 2×10^{-13}, which had fixed phase in the lab frame.

This signal is not predicted by traditional SRT, which insists on local Lorentz invariance for light speed, and was thus simply deemed "spurious" without further explanation. However, this signal is predicted by NTO analysis due to the earth surface speed (see Klauber[47]).

2.4.6 Gravitational Orbit vs. True Rotation. One could ask why any test should get a null signal for the solar and galactic orbital velocities, but a non-null signal for the earth surface speed from its own rotation.

The answer is that a body in gravitational orbit is in free fall, and is therefore Lorentzian. No centrifugal "force" is felt, and no Foucault pendulum moves, as a result of revolution in orbit. There is no experimental means by which one could determine (without looking outside at the stars) one's rate of revolution in orbit. Hence, you can not determine any absolute circumferential speed, and the second postulate of relativity holds. Related logic[48] leads to the conclusion that the speed of light on such a body is invariant and equal to c as well.

Thus, the usual form of relativity should hold for gravitational orbits and one should expect a null Michelson-Morley result for orbital speeds, which is just what is measured. However, one *can* use instruments to determine the speed of the earth's surface about its axis, and therefore we should expect that relativity theory will not hold in precisely the same form for that case. It is submitted that the Brillet and Hall result justifies that expectation. This subject is treated in depth in Klauber[48].

2.4.7 NTO vs. Selleri Transformations. In treating rotation, Selleri [49] uses the same simultaneity as the lab (though he advocates an "absolute" simultaneity that pervades translation as well.) He finds anisotropic one-way light speed on a rotating disk as

$$v_{light,phys,circum,Selleri} = \frac{-\omega r \pm c}{1 - \omega^2 r^2 / c^2} \tag{6.23}$$

for the cw and ccw speeds of light along the circumference.

Comparing this with the NTO relation of (6.17), one finds the two differ by a factor of $(1 - \omega^2 r^2 / c^2)^{-1/2}$.

Selleri shows that his relation (6.23) results in a circular round trip speed for light (one way around the rim) that agrees with (the first order) Sagnac experimental results. However, for a back and forth round trip for light along the same path, he shows his relation results in a round trip speed of precisely c. Thus, he predicts a null result for any Michelson-Morley type experiment.

NTO analysis on the other hand, due to the Lorentz factor difference from (6.23), predicts a back and forth round trip speed for light as not equal to c. Therefore, it predicts a non-null result for MM experiments (which are sensitive enough to detect effects from the earth surface speed due to its own rotation.)

This difference can be attributed to the lack, in the NTO approach, of circumferential Lorentz contraction, as opposed to the inclusion of such contraction in the Selleri approach. Given Lorentz contraction, light rays will travel a greater number of meter sticks, and thus speed will be increased by the magnitude of that contraction. This is the difference between (6.23) and (6.17).

2.4.8 Co-moving vs. Disk-fixed Observers. It should be clearly noted that in the NTO approach, the rotating disk fixed observer and the local co-moving Lorentz observer are not equivalent. They do not, for example, see the lab meter sticks as having the same length. This is in accord with earlier statements regarding the invalidity of the hypothesis of localityfor rotating frames.

From another perspective, it could be claimed that the two observers are not truly co-moving, as the disk observer at r is rotating (at ω), whereas the local Lorentz observer is not.

2.5 Summary of Part 2

By adopting

1) the postulate that time in a rotating frame must be continuous and single valued (each clock must be in synchronization with itself), and

2) the specific transformation of form (6.11) that incorporates that postulate, one can develop an NTO theory for rotation that resolves all conundrums of Part 1, and in which the physical speed of light is constrained to be locally anisotropic. One finds agreement with experiment, including the Brillet and Hall test result, which is not predicted by the traditional approach to relativistic rotation.

One also finds the hypothesis of locality can only be true for non-inertial frames in which the metric can be expressed in diagonal form and still maintain continuity in time. In rotation, this is not true, and the local co-moving observer does not see events (in particular, meter stick lengths) in the same way as the disk-fixed observer.

3. Experiment and Non-time-orthogonal Analysis

3.1 Introduction

Part 3 reviews the experiments that have been performed to test special relativity, and implicitly therein, the hypothesis of locality and the traditional approach to relativistic rotation. Results of these experiments are examined in order to compare the predictive capacity of the NTO and traditional analysis approaches.

3.2 The Experiments

Table 6.1 is an extensive list of experiments performed since 1887 capable of evaluating at least one aspect of SRT. Particular experiments are referred to herein via the symbol in the first column. A terse description of each is given in column two, with the year and author citations in column three. Note the acronym SRT implies both special relativity theory and the traditional approach to rotation. Column four briefly summarizes how the NTO effect in a given experiment compares with the traditional approach effect. The last two columns compare the predictions of NTO and the traditional approach (Trad) for the given experiment. For a summary of JPL, Mössbauer, TPA, and GPA, see Will[50]. For a summary of Hughes-Drever, BH, NBS, UWash, and Mössbauer see Haugan and Will[51].

Three experiments known to the author are not included in the table because their results were contrary to both SRT and NTO theories. What these results mean is subject to debate, though most physicists who are aware of them believe they must be in error. The earliest of these was by Miller[52], a highly respected colleague of Michelson. He repeated the Michelson-Morley test four times over many years, with various equipment in various places, and much of the work was done jointly with Morley. The other experiments reporting results contrary to SRT were by Silvertooth[53] and Marinov[54]. In any event, these experiments do not discern between the Trad and NTO approaches, and are referenced here for completeness.

3.3 The Comparisons

Both the traditional and NTO approaches predict time dilation, and experiments measuring this, such as PartAcc (see Table 6.1), would, for the most part, provide no capability of differentiating between approaches. Also, Doppler shift effects tend to be the same in NTO and Trad, though, for certitude, each experiment comprising Doppler measurement needs to be evaluated on its own.

Tests of the speed of light itself, such as MM, should be more directly indicative. These must, however, have i) sufficient accuracy to detect any effect from the relatively low earth surface speed about its axis, and ii) apparatus that turns with respect to the earth surface. The MM, Post MM, and Joos tests, for example, lacked the first of these. The JPL and CORE experiments lacked the second. The LFV test did not meet either criterion.

For some tests, there is uncertainty. For example, in the ODM experiment, rotation of the apparatus yielded a persistent $\sim$1.5 km/sec variation, which was attributed to the earth's magnetic field. This would, however, mask any NTO effect (at $\sim$0.35 km/sec), and yield uncertainty as to whether Trad predicts the result or not. In the Hughes-Drever test, Doppler shifts are measured and NTO usually predicts the same shift as Trad. Extensive analysis would be required, however, to be certain.

The most interesting of the tests is BH (Brillet and Hall), as this is the only one for which NTO and Trad differ with certainty with regard to results. BH used a Fabry-Perot interferometer that rotated with respect to the lab. A fraction of the light ray incident on the interferometer emerged directly from the far end. Another portion of the ray was reflected at the far end and forced to travel round trip, rear to front to rear, before emerging. The different portions interfered to form a fringe pattern. If the round trip speed of light were anisotropic, the time for it to travel back and forth inside the interferometer would vary with orientation of the apparatus. This, in turn, would cause the fringe pattern, and thus, the signal BH monitored, to vary. In Newtonian theory, this variation, peak-to-peak and to second order, is $\frac{1}{2}v^2/c^2$, where v is the maximum change from c of the speed of light. As shown by Klauber[47], the NTO effect on light transit time is quantitatively the same (though subtle calculational differences exist from the Newtonian analysis.)

The speed of the earth surface about its axis at the location of the BH test is .355 km/sec. For this, the amplitude of the variation via NTO theory should be

$$\frac{1}{4}\frac{v^2}{c^2} = \frac{1}{4}\left(\frac{.355}{3 \times 10^5}\right)^2 = 3.5 \times 10^{-13} \tag{6.24}$$

at twice the apparatus rotation rate. The BH test found a "persistent" $\sim 1.9 \times 10^{-13}$ signal at that rate and with fixed phase relative to the earth surface. They deemed this signal "spurious", because it seemed inexplicable. The character of the BH signal and its proximity in value to (6.24) should, of themselves, be intriguing. However, there is a secondary effect of light speed anisotropy on the BH signal.

The path of travel is altered slightly when the light ray direction is transverse to the principle direction of anisotropy. In a heuristic sense, the ray seems to be pushed "sideways". In the BH experiment, this would result in a shifting of the fringe pattern, and a concomitant change in the measured signal. Klauber [47] calculated this effect and found it dependent on certain dimensions of the apparatus, which are not known. However, by using values for these dimensions estimated from the figure of the apparatus shown in the BH report, he found an expected net signal from all effects of $\sim 2 \times 10^{-13}$.

3.4 Comparison to Selleri

As noted in Section 2.4.7, the Selleri theory, like the NTO approach, is based on what this author considers a physically defensible simultaneity scheme. The Selleri theory, on the other hand, predicts a null signal for the BH experiment.

It would be interesting to compare predictions of the Selleri theory to results of other tests such as Mössbauer.

3.5 GPS and Sagnac

I do not profess expertise in the GPS system, though I have noted earlier the remarks by Ashby, who does have extensive expertise. Those remarks appear consonant with NTO analysis and its requisite non-Einstein synchronization and local light speed anisotropy.

Furthermore, in the context of the thought experiment of Section 1.6.1, the traditional approach seems incapable of deriving the Sagnac effect from within the rotating frame. That is, considering the local physical speed of light to be isotropic does not seem sufficient to derive different arrival times for the cw and ccw light rays. This is not the case for the NTO approach, and in this context, the Sagnac experiment may be considered empirical support for it.

3.6 Future Experiments

Tobar[55][56] (WSMR in Table 6.1) expects to complete a modified version of the Michelson-Morley experiment, accurate to several orders of magnitude beyond that of BH, by the end of 2004. He will use a whispering spherical mode resonator and rotate it with respect to the lab. Preliminary analysis by the present author suggests that the WSMR experiment may be capable of detecting an NTO effect on light speed, if it exists, due to the surface speed of the earth.

3.7 Summary of Part 3

Only one non GPS/Sagnac experiment appears capable of distinguishing between the traditional and NTO approaches to relativistic rotation, that of Brillet and Hall. That test, sensitive to 10^{-15}, found a signal at $\sim 1.9 \times 10^{-13}$, which is strikingly close to the signal predicted by the NTO approach from the earth surface speed about its axis of rotation, and which is not predicted by the traditional approach.

Table 6.1: History of SRT Experiments

Symbol	**Description of the test**	**Authors (Year)**	**Effect NTO**	**Trad**	**NTO**
MM	Original Michelson-Morley experiment	Michelson & Morley [57] (1887)	Accuracy too low ~7-10 km/sec	Y	Y

WW	Electric field effect of rotating magnetic insulator in magnetic field	Wilson and Wilson[58] (1913); Hertzberg et al[59] (2001)	NTO prediction = Trad [60]	Y	Y
Post MM	Repeats of MM interferometer tests	Kennedy (1926); Piccard & Stahel (1926-8); Michelson et al ((1929)	Null results: 1 km/sec to 7 km/sec accuracy	Y	Y
Joos	Version of MM	Joos[61] (1930)	Accuracy too low ~1.5 km/sec	Y	Y
KT	Original experiment on time dilation	Kennedy and Thorndike [62] (1932)	Not rotated. Low accuracy ~10 km/sec	Y	Y
Ives-Stilwell	Doppler frequency time dilation in H canal rays	Ives and Stilwell [63](1938, 1941)	Accuracy 100X too low for NTO effect[64]	Y	Y
PartAcc	Particle accelerator time dilation on half lives	Mid 1900s to present	NTO prediction = Trad	Y	Y
ODM	Two opposite direction NH_3 maser beams. Ether wind Doppler variation as rotate	Cedarholm et al [65] (1958)	Rotated 180^o, ~1.5 km/sec variation. Attributed to earth mag field	?	Y

Hughes-Drever	Isotropy of nuclear energy levels. Doppler shift of photons emitted by 2 different atoms	Hughes et al [66] and Drever (1960)	Significant analysis needed for NTO prediction	Y	?
PDM	Perpendicular direction He-Ne masers. Rotated	Jaseva, Townes et al. [67] (1964)	Accuracy too low. Systematic signal as rotated	?	Y
Mössbauer	Mössbauer rotor. Classical frequency shift different from SRT	Champeney et al [68] (1963); Turner and Hill [69] (1964)	NTO predicts same frequency change as Trad	Y	Y
HK	Time dilation on atomic clock flown around world	Hafele and Keating[70] (1972)	NTO prediction = Trad	Y	Y
BH	Fringe shift in interferometer as rotate	Brillet and Hall [35] (1978)	2^{nd} order effect at 10^{-13}. NTO predicts[47]	N	Y
GPA	Gravity probe A. Maser on rocket and maser on ground. Classical Doppler varies from SRT	Vessot and Levine [71] (1979), Vessot et al [72] (1980)	NTO shift = Trad. Analysis done in non-rotating earth centered frame	Y	Y

Refract	Light split in air and glass. 1st order fringe effect in Galilean & Fresnel ether drag theories	Byl et al. [73] (1985)	1^{st} order effect NTO = Trad $\neq$ Galilean or Fresnel drag [74]	Y	Y
NBS	Isotropy of nuclear energy levels. Doppler shift. Rotation of earth changed orientation	Prestage [75] et al (1985) at National Bureau Standards	Apparatus not rotated. NTO effect = Trad	Y	Y
TPA	2 photon absorption in atomic beam. Doppler shift opposite directions affected by ether wind	Kaivola et al. [76] (1985); Riis et al. [77] (1988)	Apparatus not rotated. Beam aligned N-S [78]. NTO effect = Trad	Y	Y
UWash	Isotropy of nuclear energy levels. Doppler shift. Rotation of earth changed orientation	Lamoreaux et al. [79] at Univ Washington (1986)	Apparatus not rotated. NTO effect = Trad	Y	Y
JPL	Jet Propulsion Lab. 2 earth fixed masers. Fiberoptic comparison	Krisher et al. [80] (1990)	Apparatus not rotated. NTO effect = Trad	Y	Y

LFV	Laser frequency variation as earth rotates: stabilized laser compared to stable cavity locked laser	Hils and Hall [81] (1990)	Apparatus not rotated, plus accuracy too low for NTO effect	Y	Y
Sat	GPS satellite test of SRT	Wolf and Petit [82] (1997)	Analysis done in non-rotating earth centered frame. NTO effect = Trad.	Y	Y
CORE	Cryogenic Optical Resonators measure anisotropy of light speed as earth rotates	Braxmaier et al. [83] (2002)	Apparatus not rotated. NTO effect = Trad.	Y	Y
WSMR	Whispering spherical mode resonator Michelson-Morley experiment	Tobar [55], [56] (2004)	Appears capable of discerning between NTO and Trad	TBD	TBD

Appendix: Deriving Sagnac Result from the Lab Frame

Consider Figure 6.1 of section 1.6.1 with time ($T > 0$) in the right side of the figure when the cw light pulse reaches the disk observer designated as T_1. Consider the time when the ccw pulse reaches the disk observer (not shown) as T_2. Then lengths travelled as seen from the lab by the ccw light pulse and the observer at T_1 must sum to equal the circumference, i.e.

$$cT_1 + \omega R T_1 = 2\pi R \qquad \rightarrow \qquad T_1 = \frac{2\pi R}{c + \omega R}. \tag{6.A.1}$$

Similarly, at time T_2

$$cT_2 = \omega R T_2 + 2\pi R \qquad \rightarrow \qquad T_2 = \frac{2\pi R}{c - \omega R}. \tag{6.A.2}$$

Hence, the arrival time difference in the lab is

$$\Delta T = T_2 - T_1 = \frac{2\omega R}{c^2}\frac{2\pi R}{(1-\omega^2R^2/c^2)} = \frac{4\omega A}{c^2(1-\omega^2R^2/c^2)}. \tag{6.A.3}$$

As is well known, the standard (physical) clocks on the disk rim run more slowly than the lab clocks by $\sqrt{1-\omega^2R^2/c^2}$, so the observer on the disk must measure an arrival time difference of

$$\Delta t_{phys} = \frac{4\omega A}{c^2\sqrt{1-\omega^2R^2/c^2}}. \tag{6.A.4}$$

References

[1] Anderson, R., Vetharaniam, I., Stedman, G.E., "Conventionality of Synchronisation, Gauge Dependence and Test Theories of Relativity", *Phys. Rep.* **295**(3&4), 93-180 (March 1998). Many articles have appeared on this subject. These authors cite virtually all of them prior to 1998.

[2] Minguzzi, E., "On the Conventionality of Simultaneity", *Found. Phys. Lett.* **15**, 153 (2002).

[3] Møller, C., *The Theory of Relativity,* Oxford at the Clarendon Press (1969), pp. 223.

[4] Stachel, J., "Einstein and the Rigidly Rotating Disk", Chapter 1 in Held, General Relativity and Gravitation (Plenum Press, New York, 1980), Stachel, p. 9; A. Einstein, *The Meaning of Relativity* (Princeton University Press, 1950), footnote on pg 60.

[5] Mashoon, B., "Gravitation and Nonlocality", gr-qc/0112058; "The Hypothesis of Locality and Its Limitations", in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)

[6] Minguzzi, E., "Simultaneity and generalized connections in general relativity", *Class. Quant. Grav.* **20**, 2443-2456 (2003), gr-qc/0204063. See Section III.

[7] See ref. [3], p. 223.

[8] Misner, C.W., Thorne, K.S., and Wheeler, J.A., *Gravitation* (Freeman, New York, 1973). See Chapter 6.

[9] Ref. [4], Stachel, p. 7.

[10] Transforming from the 4D Riemann flat space of the lab to the 4D space of the rotating frame implies the 4D space of the rotating frame is also Riemann flat. The traditional analysis does not contradict this, but claims the 2D surface of the rotating disk is curved within the flat 4D rotating frame space.

[11] Grøn, Ø., "Relativistic Description of a Rotating Disk", *Amer. J. Phys.* **43**, 869-876 (1975); "Rotating Frames in Special Relativity Analyzed in

Light of a Recent Article by M. Strauss", *Int. J. Theor. Phys.* **16**, 603-614 (1977); "Space Geometry in Rotating Frames: A Historical Perspective", in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003). Grøn's treatment has become a classic for traditional approach advocates.

[12] Some may contend that other meter sticks in the rotating frame (such as a meter stick on the far side of the rim) appear to have velocity with respect to a given rim observer. However, an observer fixed on the rim of the disk does *not* see the far side of the disk moving relative to her, since she, unlike the local Lorentz frame with the same instantaneous linear velocity, is rotating. There are two different local frames of interest here. Both have the same instantaneous rectilinear velocity as a point on the rim. But one rotates relative to the distant stars (at the same rate as the disk itself), and one does not. The latter is a local Lorentz frame. The former is not, and represents the true state of an observer anchored to the disk. The difference in kinematics between these two frames is significant, and is dealt with in Klauber, R., "Non-time-orthogonality, gravitational orbits and Thomas precession", gr-qc/0007018.

[13] Tartaglia, A., "Lengths on Rotating Platforms", *Founds. Phys. Lett.,* **12**, 17-28 (1999). Tartaglia and I have made the point in this paragraph independently.

[14] Nikolic', H., "Relativistic contraction and related effects in noninertial frames", *Phys. Rev. A.* **61**, 032109 (2000), gr-qc/9904078; "Fermi coordinates and rotation", in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003). Nikolic' presents a resolution of this based on the postulate that "although there is no relative motion among different points on a rotating disc, each point belongs to a different non-inertial frame". Though Nikolic' focuses on Fermi frames, for which this postulate may hold, the present author disagrees with it for the physical frame of the rotating disk. The present author considers the definition of the term "frame" to mean precisely that "there is no relative motion among different points therein."

[15] Selleri, F., "Noninvariant One-Way Speed of Light and Locally Equivalent Reference Frames", *Found. Phys. Lett.* **10**, 73-83 (1997). Selleri presents this argument quantitatively and makes a cogent argument for violation of the first postulate of SRT.

[16] For example, see Budden, T., "Geometric Simultaneity and the Continuity of Special Relativity", *Found. Phys. Lett.* ***11***, 343-357 (Aug 1998).

[17] Also, Sorge, F., "Local and Global Anisotropy in the Speed of Light", in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the

series “Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003).

[18] Also, Weber, T. “Some Elementary Relativistic Considerations of the Time and Geometry of Rotating Reference Frames”, in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series “Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003).

[19] See, for example, Bergia, S., and Guidone, M., “Time on a Rotating Platform and the One-Way Speed of Light”, *Found. Phys. Lett.,* **11**, 549-560 (Dec 1998).

[20] Also, refs. [11] and [18]

[21] Sagnac, M.G., *Académie des sciences - Comptes rendus des séances,* **157**, 708-718 (1913); “Effet tourbillonnaire optique. La circulation de l’éther lumineux dans un interférographe tournant”, *Journal de Physique Théorique et Appliquée*, Paris, Sociète française de physique, Series 5, Vol 4 (1914), 177-195.

[22] Dufour, A. et Prunier, F., “Sur l’observation du phénomène de Sagnac avec une source éclairante non entraînée”, *Académie des sciences - Comptes rendus des séances*, **204**, 1322-1324 (3 May 1937); “Sur un Déplacement de Franges Enregistre sur une Plate-forme en Rotation Uniforme”, *Le Journal de Physique et Le Radium*, serie VIII, T. III, No 9, 153-161 (Sept 1942).

[23] See other citations in E.J. Post, "Sagnac effect," *Mod. Phys*. **39**, 475-493 (1967).

[24] Dresden, M., and Yang, C.N., “Phase shift in a rotating neutron or optical interferometer”, *Phys. Rev. D.* **20**, 1846-1848 (15 Oct 1979).

[25] Mashoon, B., Neutze, R., Hannam, M., Stedman, G.E., “Observable frequency shifts via spin-rotation coupling”, *Phys. Lett. A,* **249**, 161-166 (1998). The authors consider the Sagnac effect to be a “manifestation of the coupling of orbital angular momentum of a particle .. to rotation”. For a wave this perturbation in the Hamiltonian induces a “frequency perturbation ... [which] ... recovers the Sagnac phase shift”. Again, correct arrival and departure times are not predicted by this approach.

[26] Weber, T.A., “Measurements on a rotating frame in relativity, and the Wilson and Wilson experiment,” *Am. J. Phys.* **65**, 946-953 (Oct 1997).

[27] Ref. [19].

[28] Rizzi, G., and Tartaglia, A., “Speed of Light on Rotating Platforms”, *Found. Phys.* **28**(11), 1663 (Nov 1998); “On Local and Global Measurements of the Speed of Light on Rotating Platforms”, *Found. Phys. Lett.* **12**, 179-186 (April 1999).

[29] Rizzi, G., and Ruggiero, M.L., “Space Geometry of Rotating Platforms: An Operational Approach”, *Found. Phys.* (in press), gr-qc/0207104.

[30] Cranor, M.B., Heider, E.M., Price, R.H., "A circular twin paradox", *Am. J. Phys.* **68**, 1016-1020 (Nov 2000).

[31] Anandan, J., "Sagnac effect in relativistic and nonrelativistic physics", *Phys. Rev. D* **24**, 338-346 (15 July 1981).

[32] Peres, A., "Synchronization of clocks in a rotating frame", *Phys Rev D*, **18**, 2173-2174 (15 Sept 78).

[33] Dieks, D., "Time in special relativity and its philosophical significance", *Eur. J. Phys.* **12**, 253-259 (1991).

[34] See ref. [1] for details.

[35] Brillet, A., and Hall, J.L., "Improved laser test of the isotropy of space," *Phys. Rev. Lett.*, **42**, 549-552 (1979).

[36] Ashby, N., "Relativistic Effects in the Global Positioning System", *15th Intl.Conf.Gen.Rel. and Gravitation*, Pune, India (Dec 15-21, 1997), available at www.colorado.edu/engineering/GPS/Papers/RelativityinGPS.ps. See pp. 5-7.

[37] Ashby, N., "Relativity and the Global Positioning System", *Phys. Today*, May 2002, 41-47. See pg 44.

[38] The texts and article listed below are among those that discuss physical vector and tensor components (the values one measures in experiment) and the relationship between them and coordinate components (the mathematical values that depend on the generalized coordinate system being used.) D. Savickas, "Relations between Newtonian Mechanics, general relativity, and quantum mechanics", *Am. J. Phys.*, 70, 798-806; I.S. Sokolnikoff, *Tensor Analysis*, Wiley & Sons (1951) pp. 8, 122-127, 205; G.E. Hay, *Vector and Tensor Analysis*, Dover (1953) pp 184-186; A. J. McConnell, *Application of Tensor Analysis*, Dover (1947) pp. 303-311; Carl E. Pearson, *Handbook of Applied Mathematics*, Van Nostrand Reinhold (1983 2nd ed.), pp. 214-216; Murry R. Spiegel, *Schaum's Outline of Vector Analysis*, Schaum, pg. 172; Robert C. Wrede, *Introduction to Vector and Tensor Analysis,* Dover (1972), pp. 234-235.

[39] Malvern, L.E., *Introduction to the Mechanics of a Continuous Medium* Prentice-Hall, Englewood Cliffs, New Jersey (1969). See Appendix I, Sec. 5, pp. 606-613. Fung, Y.C., *Foundations of Solid Mechanics* Prentice-Hall, Inc., Englewood Cliffs, NJ (1965). See pp. 52-53 and 111-115. Eringen, A.C., *Nonlinear Theory of Continuous Media,* McGraw-Hill, NY (1962). pp. 437-439. Chung, T.J., *Continuum Mechanics,* Prentice Hall, Inc., Englewood Cliffs, NJ (1988. pp. 40-53, 246-251.

[40] See ref. [8].

[41] Physical components are introduced in ref. [8] on pg. 37, and used in many places throughout the text, though surprisingly, the relation between physical and coordinate components is not derived. It is used, however.

See, for example, physical velocity components found in equation (31.5) on pg. 821.

[42] Klauber, R.D., "Physical components, coordinate components, and the speed of light", gr-qc/0105071;

[43] Langevin, P., "Théorie de l'expérience de Sagnac", *Académie des sciences - Comptes rendus des séances,* **173**, 831-834 (1921); "Relativité – Sur l'expérience de Sagnac", *Académie des sciences - Comptes rendus des séances,* **205**, 304-306 (2 Aug 1937.)

[44] Klauber, R.D., "New perspectives on the relatively rotating disk and non-time-orthogonal reference frames", *Found. Phys. Lett.* **11** 405-443 (1998). qr-qc/0103076.

[45] Klauber, R.D., "Relativistic Rotation in the Large Radius, Small Angular Velocity Limit", gr-qc/0209025.

[46] Klauber, R.D., "Derivation of the General Case Sagnac Result using Non-time-orthogonal Analysis", (in press) *Found. Phys. Lett.* (Oct 2003), gr-qc/0206033.

[47] Klauber, R.D., "Analysis of the Anomalous Brillet and Hall Experimental Result", gr-qc/0210106.

[48] Klauber, R.D., "Non-time-orthogonality, gravitational orbits and Thomas precession", gr-qc/0007018

[49] Selleri, F., "Noninvariant One-Way Velocity of Light", *Found. Phys.* **26**, 641-664 (May 1996); "Noninvariant One-Way Speed of Light and Locally Equivalent Reference Frames", *Found. Phys. Lett.* **10**, 73-83 (1997); "Space, Time, and Their Transformations", *Chinese J. Sys. Eng. & Elect.* **6**, 25-44 (1995); "The Relativity Principle and the Nature of Time", *Found. Phys.* **27**, 1527-1548 (1997).

[50] Will, C. M., "Clock synchronization and isotropy of the one-way speed of light", *Phys Rev. D*, **45**, 403-411 (15 Jan 1992).

[51] Haugan, M.P., and Will, C.M., "Modern Tests of Special Relativity", *Phys. Today*, May 1987, 69-76.

[52] Miller, D.C., "The Ether-Drift Experiment and the Determination of the Absolute Motion of the Earth", *Rev. Mod. Phys.* **3**, 203-242 (July 1933).

[53] Silvertooth, E.W., "Motion through the ether", *Elect. & Wireless World*, May 1989, 437-438; "Special Relativity", *Nature* **322**, 590 (Aug 1985); "Experimental detection of the ether", *Spec. Sci. & Tech.* **10**, 3-7. Silvertooth, E.W., and Whitney, C.K., "A New Michelson-Morley Experiment", *Phys. Essays*, **5**, 82-89 (1992).

[54] Marinov, S., "The Experimental Measurement of the One-Way Light Velocity and its Possibilities for Absolute Velocity Measurement", *Spec. Sci. & Tech.* **3**, 57-82 (1980); See Maddox, J., "Stefan Marinov wins friends", *Nature* **316**, 209 (18 July 1985).

[55] Tobar, M.E., Hartnett, J.G., Anstie, J.D., "Proposal for a new Michelson-Morley experiment using a single whispering mode resonator", *Phys. Lett. A.*, **300**, 33-39 (22 July 2002).

[56] Wolf, P., Bize, S., Clairon, A., Luiten, A.N. Gantarelli, G., Tobar, M.E., "Test of relativity using a microwave resonator", gr-qc/0210049 (Oct 2002).

[57] Michelson, A.A., and Morley, E.W., *Am. J. Sci.* **34**, 333 (1887).

[58] Wilson, M., and Wilson, H.A., "On the electric effect of rotating a magnetic insulator in a magnetic field", *Proc.R. Soc. London, Ser A* **89**, 99-106 (1913).

[59] Hertzberg, J.B., Bickman, S.R., Hummon, M.T., Krause, D., Jr., Peck, S.K. and Hunter, L.R., "Measurement of the relativistic potential difference across a rotating magnetic dielectric cylinder", *Am J. Phys.* **69**, 648-654 (June 2001).

[60] Burrows, M.L., "Comment on 'Maxwell's equations in a rotating medium: Is there a problem?,' by G. N. Pellegrini and A. R. Swift", *Am. J. Phys.* **65**, 929-931 (1997); Klauber, R., "Generalized Tensor Analysis Method and the Wilson and Wilson Experiment", gr-qc/0107035.

[61] Joos, V.G., *Ann. Phys.* **7**(4), 385-407 (1930)

[62] Kennedy, R.J., and Thorndike, E.M., "Experimental Establishment of the Relativity of Time", *Phys Rev.* **42**, 400-418 (1 Nov 1932).

[63] Ives, H.E. and Stilwell, G.R., "An Experimental Study of the Rate of a Moving Atomic Clock", *Jour. Opt. Soc. Am.*, **28**, 215-226 (July 1938); "An Experimental Study of the Rate of a Moving Atomic Clock II", *Jour. Opt. Soc. Am.*, **31**, 369-374 (May 1941).

[64] There may not be a difference between the NTO and SRT predictions for this experiment, and even with sufficient accuracy, the test might not be capable of discerning between the two approaches. Significant analysis would be required to answer this question.

[65] Cedarholm, J.P., Bland, G.F., Havens, B.L., and Townes, C.H., "New Experimental Test of Special Relativity", *Phys Rev. Lett.*, **1**, 342-343 (1 Nov 1958).

[66] Hughes, V.W., Robinson, H.G., Beltran-Lopez, V., *Phys Rev Lett.* **4**, 342 (1960); Dreve, R.W.P., *Philos. Mag.* **6**, 683 (1961).

[67] Jaseva, T.S., Javan, A., Murray, J., and Townes, C.H., "Test of Special Relativity or of the Isotropy of Space by Use of Infrared Masers", *Phys. Rev.* **133**(5A), A1221-A1225 (2 March 1964).

[68] Champeney, D.C., Issak, G.R., and Khan, A.M., "An 'Aether Drift' Experiment Based on the Mössbauer Effect", *Phys. Lett.*, **7**, 241-243 (1 Dec 1963)

[69] Turner, K.C., and Hill, H.A., "New Experimental Limit on Velocity-Dependent Interactions of Clocks and Distant Matter", *Phys. Rev.*, **134**(1B), B252-B256 (13 April 1964).

[70] Hafele, J.C., and Keating, R.E., "Around-the-World Atomic Clocks: Predicted Relativistic Time Gains", *Science* **177**, 166-167 (14 July 1972); "Around-the-World Atomic Clocks: Observed Relativistic Time Gains", *Science* **177**, 168-170 (14 July 1972);

[71] Vessot, R.F.C., and Levine, M.W., "A Test of the Equivalence Principle Using a Space-Borne Clock", *Gen. Rel. and Grav.* **10**, 181-204 (1979).

[72] Vessot, R.F.C., Levine, M.W., Mattison, E.M., Blomberg, E.L., Hoffman, T.E., Nystrom, G.U., Farrell, B.F., Decher, R., Eby, P.B., Baugher, C.R., Watts, J.W., Teuber, D.L., and Wills, F.D., "Test of Relativistic Gravitation with a Space-Born Hydrogen Maser", *Phys Rev Lett.* **45**, 2081-2084 (29 Dec 1980).

[73] Byl, J., Sanderse, M., van der Kamp, W., "Simple first-order test of special relativity", *Am. J. Phys.* **53**, 43-45 (Jan 1985).

[74] Klauber, R.D., "Non-time-orthogonality and Tests of Special Relativity", gr-qc/0006023.

[75] Prestage, J.D., Bollinger J.J., Itano, W.M., Wineland, D.J., "Limits for Spatial Anisotropy by Use of Nuclear-Spin-Polarized $^9Be^+$ Ions", *Phys. Rev. Lett.* **54**, 2387 (1985).

[76] Kaivola, M., Poulsen, O., Riis, E., "Measurement of the Relativistic Doppler Shift in Neon", *Phys. Rev. Lett.* **54**, 255-258 (28 Jan 1985).

[77] Riis, E., Andersen, L.A., Bjerre, N., Poulsen, O., Lee, S.A., , Hall, J.L., "Test of the Isotropy of the Speed of Light Using Fast-Beam Laser Spectroscopy", *Phys. Rev. Lett.* **60**, 81-84 (11 Jan 1988); Bay, Z., and White, J.A., "Comment on 'Test of the Isotropy of the Speed of Light Using Fast-Beam Laser Spectroscopy', *Phys. Rev. Lett.* **62**, 841 (13 Feb 1989); Riis et al, "Reply", *Phys. Rev. Lett.* **62**, 841 (13 Feb 1989).

[78] With the beam aligned north-south, as it was in the experiment, there is definitely no difference between the NTO and SRT predictions. For an east-west alignment, there may, or may not be a difference. Significant analysis would be required to answer this question.

[79] Lamoreaux, S.K., Jacobs, J.P., Heckel, B.R., Raab, F.J., Fortson, E.N., "New Limits on Spatial Anisotropy from Optically Pumped ^{201}Hg and ^{199}Hg", Phys. *Rev. Lett.*, **57**, 3125 (1986).

[80] Krisher, T.P., Maleki, L., Lutes, G.F., Primas, L.E., Logan, R.T., Anderson, J.D., and Will, C.M., "Test of the Isotropy of the One-way Speed of Light Using Hydrogen-maser Frequency Standards", *Phys Rev D* **42**, 731-734 (15 July 1990).

[81] Hils, D., and Hall, J.L., "Improved Kennedy-Thorndike Experiment to Test Special Relativity", *Phys. Rev. Lett.*, **64**, 1697-1700 (9 April 1990).

[82] Wolf, P., and Petit, G., "Satellite Test of Special Relativity Using the Global Positioning System", *Phys. Rev. A.* **56**, 4405-4409 (Dec 1997).

[83] Braxmaier, C., Müller, H., Pradl, O., Mlynek, J., Peters, A., "Test of Relativity Using a Cryogenic Optical Resonator", *Phys. Rev. Lett.* **88**, 010401-1 to 010401-4 (7 Jan 2002).

Chapter 7

ELEMENTARY CONSIDERATIONS OF THE TIME AND GEOMETRY OF ROTATING REFERENCE FRAMES

Thomas A. Weber
Department of Physics and Astronomy
Iowa State University, Ames IA, USA 50011
taweber@iastate.edu

Abstract Elementary methods of synchronization on rotating systems are discussed. It is argued that the continuous time synchronization preferred by Klauber and others is not the time synchronization for making distance measurements but rather leads to a velocity of light which depends on direction. A paradox discovered by Selleri will also be discussed. This paradox involves a limiting case of the rotating disk in which the edge of the disk approximates an inertial frame where the velocity of light depends on direction. In addition, a paradox on the conservation of charge will be resolved by referring to the geometry of the rotating disk. Finally, the isotropy of the velocity of light on rotating frames is discussed along with the experimental evidence of Brillet and Hall.

1. Introduction

Since special relativity was first introduced, there have been numerous disagreements on the interpretation of the time and geometry for a rotating coordinate system. More recently, Pellegrini and Swift [1] question the use of local co-moving inertial frames to describe a rotating system and thus cast doubt on the interpretation of the famous Wilson and Wilson experiment[2] which employed a rotating cylindrical shell of a magnetizable insulator in a constant magnetic field. The experiment purported to show that a moving magnetic dipole generates the field of an electric dipole as predicted by special relativity. Pellegrini and Swift argue that using local co-moving frames for the rotating system does not fully account for the absolute nature of the acceleration of

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 139-153.

rotation, and furthermore, introduces a discontinuity in the synchronization of time. Weber [3] analyzed the experiment along traditional lines and resolved some of the questions raised by Pellegrini and Swift by evaluating physical quantities observed in the laboratory in terms of the values as would be measured in the rest (rotating) frame.

Klauber [4] also contends that the traditional approach to the description of a rotating disk is unsatisfactory and proposes a time synchronization that is continuous. He develops a new perspective for the rotating disk that replaces the special relativistic postulate of the invariance of the speed of light with the postulate that the speed depends on direction as indicated by the Sagnac experiment [5].

In the next section, it is shown that the traditional method of accounting for time on the rotating earth leads to Universal Time, a synchronized time with no discontinuity. Similarly, the time synchronization of clocks on a rotating disk to a master clock at the center of the disk also provides a synchronized time with no discontinuity. In a subsequent section, time and space coordinates are introduced for the rotating frame along with their interpretation. Next, a paradox on the conservation of charge is resolved along with a paradox discovered by Selleri[6]. Selleri's paradox involves the apparent lack of a smooth approach of the velocity of light to c in a limit for which the edge of a rotating disk approximates an inertial frame. In the penultimate section, it is argued that the velocity of light is isotropic in a rotating frame. The experimental evidence of Brillet and Hall supporting isotropy is examined.

2. Time synchronization on rotating systems

Time synchronization by the standard method of sending a light signal back and forth between the clocks can not be extended globally over the rotating system. As will be shown in Sec.3, a discontinuity in time is introduced in any attempt to synchronize clocks in the standard way around a closed path. There are two other apparently equivalent methods, however, that lead to a continuous time synchronization. One is Universal Time or Mean Greenwich Time on the surface of the earth and the other uses light signals from a master clock placed at the center of a rotating disk. Klauber uses the latter in his development of special relativity on the rotating disk.

As will be seen, it appears that the two methods of synchronization, Universal Time on the rotating earth and signals from a master clock at the origin of the rotating disk, are equivalent. It will be noted that in either case, the synchronized time can be used as a time coordinate of an inertial frame from which the rotating system is observed. Furthermore, the rotating earth (without gravity) could be used as the master clock at the center of the rotating disk.

2.1 Clocks synchronized on the rotating earth

The determination of time requires some periodic phenomena as a clock. The earliest clock was simply the rotating earth. Noon, for an observer at a given location on the earth, is defined as the time the sun crosses the meridian [7]. The meridian is the great circle that passes through the celestial poles (extension of the rotation axis of the earth) and the zenith (overhead) of the observer. The hour angle of the sun is the angle between the meridian and the great circle that passes through the sun and the celestial poles measured positively toward the west. Apparent solar time is simply the hour angle of the sun plus 12 hours. This means, of course, that each longitude on the surface of the earth would have its own local time. Now there are corrections necessary to account for the seasonal variation in the length of the day and so one speaks of the "mean" sun. This gives us local mean time (LMT). Universal Time (UT), usually stated in the 24-hr system, is the local mean time at 0° longitude, by definition, the position of a line engraved in a brass plate in the floor of the Old Royal Observatory at Greenwich, England. The relation between local mean time, UT, and longitude is given by

$$UT = LMT + \frac{\Phi}{180^\circ} 12hrs, \tag{7.1}$$

where Φ is the longitude arbitrarily taken positive toward the west and where $24hrs$ should be added or subtracted as needed. Using this equation, all clocks on the surface of the earth can be synchronized to read UT. Note that UT can be used as the time in an inertial frame from which the rotating earth is observed.

There are some humorous stories related to longitude and time. In the 17th and 18th centuries much effort was made developing methods of determining longitude since great wealth was lost at sea due to ships getting lost. The above equation would solve the problem if seamen had an accurate clock giving them Mean Greenwich Time or UT. Sobel [8], in her delightful book on longitude, recounts the efforts made with the Powder of Sympathy or the wounded dog theory put forth in 1687. The Powder of Sympathy could supposedly heal at a distance by simply applying the powder to a bandage from the wound. The idea was to send aboard a wounded dog as the ship set sail while leaving ashore a trusted individual to dip the dog's bandage in a solution of the powder every day at noon. The dog's reaction would give the captain a time cue of when it was noon in London. Umberto Eco [9] gives us an esoteric recounting of this and other methods in his fictional story "The Island of the Day Before".

There are other corrections to UT due to the varying rotation rate of the earth and the definition of the second by atomic clocks instead of a fraction of the mean solar day. This gives us Coordinated Universal Time which is closely related to the time used in the Global Positioning System.

2.2 Clocks synchronized on the rotating disk

On a rotating disk clocks may be synchronized by sending a light signal from a master clock at the origin (axis of rotation) to a clock at some fixed radial distance r and then back to the master clock at the origin. The clock at radius r is synchronized by setting the time of arrival of the signal to the time midway between the sending and the receiving of the signal at the origin. This is the time synchronization used by Klauber. Because of symmetry, all the clocks at a fixed radius r will read the same time. But these clocks at radius r are not synchronized with each other in the standard way of sending a light signal back and forth between them. Note that the time on the master clock at the center can be used as the coordinate time of an inertial system from which the rotating system is observed.

3. Time and space coordinates on a rotating disk

The invariant line interval has the general form

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \tag{7.2}$$

where repeated Greek indices are summed over $0, 1, 2$, and 3. Start with the invariant line interval of the inertial frame,

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2, \tag{7.3}$$

where the spatial part is described by cylindrical coordinates with the z coordinate suppressed. Transformation to arbitrary coordinates leaves this interval unchanged; only the space - time description of events will be different.

In the transformation to a rotating frame of reference, the following notation will be useful:

$$v = \omega r, \tag{7.4}$$

$$\beta \equiv v/c, \tag{7.5}$$

and

$$\gamma \equiv (1 - \beta^2)^{-1/2}, \tag{7.6}$$

where ω is the angular velocity of the rotating frame as observed from the inertial or laboratory system. The transformation to the rotating frame, with coordinates denoted by primes, has the Galilean form,

$$t = t', \ r = r', \ \phi = \phi' + \omega t', \tag{7.7}$$

where the coordinates ϕ and ϕ' both have the range 0 to 2π. In terms of these new coordinates the invariant line interval given by Eq.(7.3) is

$$ds^2 = \gamma^{-2} c^2 dt'^2 - 2c\beta r' d\phi' dt' - dr'^2 - r'^2 d\phi'^2. \tag{7.8}$$

Note that coordinate clocks on the rotating frame, although fixed in that frame, read the same time as the clocks in the inertial system. These clocks have the continuous time synchronization of Sec.2.2. In the following, a rotating disk is used as a concrete realization of a rotating reference frame. The problem of accelerating a disk from rest is not discussed. The coordinates of Eq.(7.8) are just one set of an infinite number that could be used to describe the rotating frame. For example, one can go about the disk changing the time on the clocks to a new time $\bar{t}$ given by $\bar{t} = \bar{t}(r', \phi', t')$ without leaving the frame of the rotating disk. Furthermore, a fixed $r' = r$ and ϕ' give a point on the disk that has velocity $v = \omega r$ with respect to the inertial frame. These coordinate markers (r', ϕ') can be changed to a new set of markers so that, for given values of these new markers, one again has a point fixed on the disk. The problem comes in the interpretation of a given set of markers. Einstein [10], basing his arguments on the principle of equivalence, concluded that all coordinates are equally valid. But the laws of nature are usually given in reference to inertial frames where lengths are measured in standard rods and time is measured by standard clocks. Einstein states "We can always regard an infinitesimally small region of the space - time continuum as Galilean. For such an infinitely small region there will be an inertial system relative to which we are to regard the laws of the special theory of relativity as valid." In general, even with a metric that describes a gravitational field, a transformation can always be made to a local inertial frame such that material particles behave as if "free" of gravitational or inertial forces.

The invariant line interval, Eq.(7.8), shows that the proper time interval at a fixed position in the new coordinates is given by $\triangle\tau = \triangle t'/\gamma = \triangle t/\gamma$. This demonstrates the time dilation of a standard clock at rest in the rotating frame as compared to standard clocks in the inertial frame.

There is an obvious difficulty with the invariant interval for the rotating frame. With the cross-term, $dt'd\phi'$, clocks in the rotating system are not synchronized in the standard way (Einstein synchronization) of sending a light signal back and forth between the clocks. To synchronize two clocks, say clock B with clock A, the time on clock B must be adjusted by the amount [12]

$$c\triangle t' = \int_A^B (g'_{0i}/g'_{00})dx'^i = -\int_A^B \beta\gamma^2 r' d\phi', \tag{7.9}$$

where the integration is from the location of clock A to the location of B over some chosen path. Different paths may require different time adjustments. The Latin index i is summed over the spatial indices and the index 0 refers to the time coordinate ct'. This expression can be used to synchronize successive clocks along an open curve but integrating around a closed path, say the circumference of the disk, shows a time discontinuity which is directly related to the Sagnac effect; the clock at the start of the path, which is identical to the clock at the end of the path, must have a time adjustment with itself. Around the equator of the earth, the discontinuity is about $207ns$, the same as $1/2$ the difference in time for light signals to travel around the earth in opposite directions. Such a time difference is an important consideration in the Global Positioning System since it corresponds to a distance of $62m$.

It appears that a transformation eliminating the cross term from Eq.(7.8) will automatically synchronize the clocks. This transformation should be such that the clocks are reset by varying amounts depending on their positions with no changes in their spacial coordinates. The infinitesimal transformation [11] that accomplishes this is

$$cdt' = cdt^* + \beta\gamma^2 r^* d\phi^*,\ r' = r^*,\ \phi' = \phi^*. \tag{7.10}$$

It should come as no surprise that this transformation is closely related to the Lorentz transformation of time as discussed in Sec. 5. In these new coordinates the line interval becomes

$$ds^2 = \gamma^{-2}c^2 dt^{*2} - dr^{*2} - \gamma^2 r^{*2} d\phi^{*2}. \tag{7.11}$$

The expression for dt' in Eq.(7.10) is not integrable and so clocks cannot be synchronized throughout the space of the disk as already indicated by Eq.(7.9). But the coordinates given in Eq.(7.10) are the sort that one would expect an observer on the disk to set up in his local neighborhood. Therefore, from Eq.(7.11), one takes the spatial metric to be,

$$dl^2 = dr^{*2} + \gamma^2 r^{*2} d\phi^{*2}. \tag{7.12}$$

Einstein has said physics is only simple when analyzed locally so one divides all problems into a network of local questions. In this case, the observer measures a length on the disk locally and then moves to the adjacent position and repeats the procedure. Summing these measured lengths, the observer finds that the circumference of a circle of radius r^* is

$$C^* = \int_0^{2\pi} \gamma r^* d\phi^* = 2\pi\gamma r^*. \tag{7.13}$$

This shows that the space of the disk is not Euclidean. It may be objected that measurements around the periphery of the disk are not done simultaneously in the standard way. But an observer on the disk need not measure any lengths simultaneously; he merely lays a standard rod of length much smaller than r^* on the surface and marks off the ends, say, with a piece of chalk. The chalk marks on either end of the rod can be made at any time since the rod is at rest with respect to the surface. Repeating the procedure, the circumference of the disk is measured.

One may object with the above picture for measuring the circumference of the disk since one must first determine the length of the standard rod. The length will surely be different for a rod at rest in the rotating frame compared to a rod at rest in an inertial system. One could calculate such changes knowing the stresses induced by the rotating system. But such a procedure would require knowledge of the rotating frame's geometry which we are attempting to measure. So it is impossible to proceed in that way. The simplest way to determine the length of a rod oriented in any direction and at rest on the rotating frame is to compare the length directly to a standard rod at rest in the co-moving inertial frame. Or equivalently, one can use light signals according to the definition of the SI meter, that is, the distance traveled by a light signal in vacuum with speed c during a specified time interval as measured by an atomic clock. Such a procedure will lead to a result consistent with Eq.(7.13).

The spatial metric given in Eq.(7.12) follows from the analysis of Landau and Lifshitz [12] and many others. They measure the distance by taking the proper time for a light signal to go back and forth between two nearby points, multiplying by c and dividing by 2. They obtain

$$dl^2 = \left(-g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}}\right) dx^i dx^j. \tag{7.14}$$

Substituting the metric from Eq.(7.8) and taking $dx^1 = dr'$ and $dx^2 = d\phi'$ (remember $r' = r^*$ and $\phi' = \phi^*$) one obtains Eq.(7.12).

Klauber [4] does not agree that the space of the disk is non Euclidean. One of his arguments is based on the continuous time synchronization of Sec. 2.2. in which the time on the disk is the same as the time in the inertial frame. Under such circumstances, Klauber claims there is no length contraction of the edge of the disk.

It cannot be overemphasized that the coordinate markers describing a spacetime are arbitrary except for a few needed properties. Much has been made of the fact that clocks synchronized in the standard way (Einstein synchronization) around the periphery of a rotating disk results in a discontinuity in the time. Can such synchronized clocks be used for coordinate time? Ordinarily such discontinuities do not render a system of coordinate markers unusable. Even so, continuous time coordinates are usually favored. But claims that such time coordinates have physical meaning does not derive from their continuity but must be carefully demonstrated in relation to the physical phenomenon being analyzed. Sec. 5 contains more on the subject of synchronized time coordinates.

4. Paradoxes

There are many paradoxes associated with the rotating disk; two of these will be discussed here. One is on the conservation of charge in a rotating system and the other is the Selleriparadox [6].

4.1 Conservation of charge

Pellegrini and Swift [1] have proposed that there will be creation of charge with rotation if one insists on analyzing a rotating current loop by the traditional method of a series of co-moving inertial frames. One simple example, taken from Ref.[3], is discussed here; more detailed examples are given in that reference.

There appears to be an apparent creation of charge with the initiation of a current in a neutral circular wire. This paradox can be resolved by reference to the geometry of the rotating frame. Suppose a copper wire is bent into a circle of radius r. The copper, with no current, has a charge density ρ_0 of electrons and an equal but opposite charge density of positive ions at fixed positions within the wire. Let a current be established in the wire. The four current density [13] of the moving electrons as observed from the laboratory is

$$J = \gamma_d \rho_0' \left(c, v_d\right), \tag{7.15}$$

where ρ_0' is the charge density in a frame at rest with respect to the moving electrons. For simplicity, the random motion of the electrons is neglected and it is assumed that all electrons travel at the same speed, the drift velocity v_d. The relativistic factor $\gamma_d = (1 - v_d^2/c^2)^{-1/2}$ compensates for the contraction of lengths in the rest system of the electrons as measured by an observer in the laboratory. It may seem reasonable to take the rest density of the moving electrons to be the same as the original rest density as measured in the laboratory, that is, $\rho_0' = \rho_0$. But then there is the creation of charge in the amount

$$\triangle Q = \rho_0(\gamma_d - 1) 2\pi r A, \tag{7.16}$$

where A is the cross sectional area of the wire. The resolution of this paradox is found in the realization that the circumference of the circle in the rotating frame in which the electrons are at rest has increased by the factor γ_d. With the same charge distributed over this increased length, the charge density in the rest frame of the moving electrons is

$$\rho_0' = \rho_0/\gamma_d, \tag{7.17}$$

so that the density of the moving charges as observed from the laboratory remains unchanged at ρ_0.

One can see this more directly. Consider two adjacent electrons with angular separation $\triangle\phi$ before the application of the electric field. In this idealized example, let them both start at rest and have the same angular acceleration as observed from the laboratory. Then elementary kinematics tells us that the angular separation as observed from the laboratory does not change. The proper length of each element of arclength between charges, when they have reached the drift velocity, must be increased by the factor γ_d to compensate for the Lorentz contraction observed from the laboratory, that is, the arclength between the charges in the rotating frame in which they are at rest is $\gamma_d r' \triangle\phi'$ rather than $r' \triangle\phi'$ so that the length as observed in the lab frame remains fixed at $r \triangle\phi$. This is an alternate way to see that the circumference on the rotating frame has increased by the amount γ_d. See Ref.[14] for similar arguments leading to the resolution of the Ehrenfest paradox [15].

The problem of the two electrons with the same angular acceleration parallels a problem posed by Dewan and Beran [16]. Consider two identical rockets at rest in an inertial frame S. Let them face the same direction and be situated one behind the other. A thin thread that is just long enough links the two rockets center to center. The rockets are then fired simultaneously and have identical acceleration programs measured at their centers. As observed from S, the rockets remain displaced from each other by a fixed distance center to center. What happens to the string? J. S. Bell [17] relates the humorous story about the discussion that once took place at the CERN canteen. "A distinguished experimental physicist refused to accept that the string would break, and regarded my assertion, that indeed it would, as a personal misrepresentation of special relativity. We decided to appeal to the CERN Theory Division for arbitration and make a canvas of opinion... There emerged a clear consensus that the thread would not break!"

4.2 Selleri's paradox

Selleri[6] calculates the speed of light on a rotating disk by first calculating the times needed to go completely around in opposite directions as observed from the laboratory. He obtains the time delay of the well known Sagnac effect calculated in the laboratory. Selleri then takes the following formulas to relate time and lengths in the laboratory S_0 to those on the disk.

$$L_0 = LF_1(v, a), \tag{7.18}$$

where the circumference length is taken to be L_0 and L as measured in the laboratory S_0 and on the disk, respectively. Here F_1 is a function of velocity, acceleration $a = v^2/r$, and perhaps higher derivatives of position. Similarly for time

$$t_0 = tF_2(v, a). \tag{7.19}$$

Using these relations he shows that the ratio of the light velocities in opposite directions around the rim of the disk as observed from the disk is

$$\chi = \frac{1+\beta}{1-\beta}, \tag{7.20}$$

where the unspecified functions F just cancel out. This is an unusual result in that it does not depend on acceleration and is independent of the size of the disk but only on the speed of the edge of the disk. One can imagine taking the radius larger and larger while keeping β constant, that is, taking $r \longrightarrow \infty$ and $\omega \longrightarrow 0$ keeping $\beta = \omega r/c$ fixed. Then $a = v^2/r$ tends to zero. In this limit the edge of the disk is equivalent to an inertial frame wherein the velocity of light depends on direction! Herein lies the paradox.

Let us approach this problem from a different point of view. Set the invariant line interval in Eq.(7.8) equal to zero to obtain for light traveling in the $\mp\phi$ directions at fixed r',

$$r' \frac{d\phi'}{dt'} = \mp c(1 \pm \beta). \tag{7.21}$$

If the proper time and distance is used for the velocity on the left, the right hand side of the equation should be multiplied by γ^2. The magnitude of the ratio of these velocities is exactly the same as that obtained by Selleri, Eq.(7.20) above. One gets different velocities depending on direction because the coordinate clocks on the rotating disk reading laboratory time are not synchronized in the standard way. Note that in Selleri's Eq.(7.19), time intervals on the disk are proportional to the intervals in the laboratory. This is the same as using laboratory time scaled by a fixed factor F_2. Therefore, the clocks on the disk have a continuous synchronization equivalent to the time synchronization of Klauber even in the limit $r \rightarrow \infty$. Such a time synchronization guarantees a directional dependence of the speed of light.

One can see more clearly that no paradox exists since the same result obtains [18] [19] if one transforms the Minkowski metric,

$$ds^2 = c^2 dt^2 - dx^2, \tag{7.22}$$

using the Galilean transformation,

$$dx = dx' + v dt', \; dt = dt', \tag{7.23}$$

to get

$$ds^2 = c^2 \gamma^{-2} dt'^2 - dx'^2 - 2\beta c dx' dt'. \tag{7.24}$$

This is a perfectly good description of the moving inertial reference frame. One can make a transformation to eliminate the cross term while remaining on the moving frame and then a rescaling gives the Minkowski line element. The overall transformation is the Lorentz transformation as expected. Choose not to do this but rather take the interval as given in Eq.(7.24) where the time is that of the original inertial frame. To get the velocity of light with respect to the moving frame, set the line interval in Eq.(7.24) equal to zero to get

$$\frac{dx'}{dt'} = \mp c(1 \pm \beta), \tag{7.25}$$

that is, the velocity is $(c - v)$ to the right and $-(c + v)$ to the left. This does not contradict special relativity since, in the traditional view, the clocks reading time t' are not synchronized, that is, clocks with larger x' have later times than if they were synchronized in the standard way. Taking the magnitude of the ratio of the above velocities on the moving frame leads to Selleri's result.

One should be a bit uneasy about the development so far since it appears that acceleration plays no role. It is well known, however, that the gravitational potential Φ enters the temporal part of the metric so that the proper time is given by

$$d\tau = (1 + 2\Phi/c^2)^{1/2} dt. \tag{7.26}$$

An observer at rest on the rotating frame will experience the centrifugal force

$$ma = mv^2/r = m\omega^2 r \tag{7.27}$$

with a corresponding potential

$$\Phi = -\omega^2 r^2/2. \tag{7.28}$$

The zero of potential is at the origin where the proper time is the same as the time of the inertial system. With this potential, the proper time at rest with respect to the rotating frame is

$$d\tau = (1 - \omega^2 r^2/c^2)^{1/2} dt = (1 - v^2/c^2)^{1/2} dt, \tag{7.29}$$

as was obtained before. The observer at rest with respect to the rotating frame feels no motion but instead experiences a centrifugal force. That is, the same result comes about in different ways depending on the point of view of the observer. It should be noted that Einstein [20], using the principle of equivalence, proceeded in the opposite direction and deduced the dependence on the potential, Eq.(7.26), from the metric for the rotating system.

5. Synchronization and the Brillet and Hall experiment

In this section, some of the earlier comments about coordinate systems are summarized and extended. It appears that much of the problem describing rotating reference frames is related to a cavalier handling of coordinate systems. It cannot be overemphasized that the coordinate markers describing a space-time are arbitrary except for a few needed properties. The mapping between the coordinates and the space-time points should be one to one and a certain smoothness to the coordinate markers is required. Let us concentrate on the synchronization of coordinate clocks which, at fixed positions, keep the temporal order of events.

There appears to be no need for the coordinate clocks to be synchronized in any manner whatsoever. Much has been made of the fact that clocks synchronized around the periphery of a rotating disk in the standard way (Einstein synchronization) results in a discontinuity. But such a discontinuity causes little difficulty. Recall that angular measurements result in a discontinuity such that 0 and 2π are identified as the same angle yet we find such angular coordinates very useful even though they fail, in the large, one of our requirements. Perhaps a better comparison is to the branch lines of multivalued functions. At the discontinuity one can continue the synchronization onto a new or second "sheet" of the time function. In any case, using Einstein synchronized clocks as the coordinate clocks around the periphery of the disk seems perfectly valid. If we synchronize adjacent clocks (infinitesimally separated) around the periphery we will encounter the discontinuity of which we must be mindful. The coordinate clocks are then synchronized by light signals constrained to travel along the periphery of the disk. With this synchronization and the accompanying proper distances, the velocity of light on the periphery will be c in either direction, as guaranteed by Einstein synchronization. This is in contradistinction to claims made by several other papers in this book. Integrating Eq.(7.10) for fixed r^* yields

$$ct^* = ct' - \beta\gamma^2 r^* \phi^*, \tag{7.30}$$

which can be used to reset the coordinate clocks on the periphery of the disk to the Einstein synchronized time t^*. This equation not only shows the discontinuity in t^* at $\varphi^* = 2\pi$ compared to $\varphi^* = 0$, but allows the synchronization to be continued onto a new sheet where the time on the clocks differs from the time of the clocks on the original sheet by the amount $2\pi\beta\gamma^2 r^*$.

Note that Eq.(7.30) is just the special Lorentz transformation for time. To see this write the time on the rotating disk in terms of the proper time instead of t^* and use Eq.(7.12) for the distance, $x^* = \gamma r^* \phi^*$ and note that $t' = t$, the time in the laboratory or inertial frame, to get

$$c\Delta t = \gamma(c\Delta\tau^* + \beta\Delta x^*). \tag{7.31}$$

The continuous time variable t' gives $c - v$ for the velocity of light traveling in the direction of increasing ϕ^*. But it follows from Eq.(7.30) that for increasing ϕ^*, the time interval in t^* will be less than the time interval in t' so that the use of t^* results in the larger velocity c (calculate $\gamma r^* \Delta\phi^* / \Delta\tau^*$where $\Delta\tau^*$ is $\gamma^{-1}\Delta t^*$ and use $r^* \Delta\phi^* / \Delta t' = c(1 - \beta)$).

The experiment of Brillet and Hall [21] is a test of the isotropy of space. They measure the apparent length of a Fabry-Perot cavity mounted horizontally on a table rotated at a rate f (about once every 10s). The condition of standing waves within the cavity will change if the propagation of light varies due to a preferred direction in space. Such an anisotropy would show up as a signal at frequency $2f$. They obtain a null result after averaging out a spurious signal at frequency $2f$. The source of this spurious signal at $2f$ in not specifically identified. Klauber and others suggest that this spurious signal is due to the rotating frame of the earth. But the previous analysis shows that there should be no anisotropy to the velocity of light due to rotation.

Compelling evidence for or against isotropy due to the rotating earth would be given by the orientation of the interferometer at the maximum amplitude of the spurious signal $2f$. There is a graph in the paper giving the direction with respect to the lab frame but the axes are not identified.

If the instruments were especially sensitive there is the possibility that the experiment measured nonlocal effects even though the experiment was of short duration and spatial extent. A simple calculation shows that any nonlocal effects of the metric were negligible.

Brillet and Hall state that a major factor in limiting the sensitivity of the experiment is the actual change in length of the cavity due to the variable gravitational stretching of the interferometer. This variation comes about because the axis of rotation of the interferometer is not perfectly vertical. This stretching of the interferometer produces a strong signal at the table rotation frequency f which can be eliminated since the signal of interest is at $2f$. Brillet and Hall refer to the strong signal at frequency f as *nearly* sinusoidal. It is reasonable to conclude that the spurious signal at frequency $2f$ was due to a second harmonic of this variable gravitational stretching of the interferometer and not due to an anisotropy in the velocity of light. Even so, a repeat of this experiment at greater sensitivity would be welcome.

6. Conclusion

It has been shown that the continuous time synchronization which is favored by Klauber leads to the directional dependence on the velocity of light, a result consistent with the Sagnac effect. But this synchronization is not appropriate for measuring distances although it appears that one could make such measurements using the modified velocities of Eq.(7.21). But then one must interpret the spacial coordinates. Such interpretation is not necessary in the distance formulas presented in Sec.3 since distance is measured in terms of c, the velocity of light in vacuum, with time measured, for example, by an atomic clock. In fact, the SI meter is defined in this manner.

Rizzi and Ruggiero[22], in a recent paper, use an operational approach suitable for the analysis of non-time-orthogonal frames thereby giving a sound mathematical basis for the study of the space geometry of the rotating disk. It is satisfying that their results support some of the results of this paper since it is hard to see how the simple methods suggested by Einstein, if used properly, could lead to erroneous results.

Acknowledgments

I thank my colleaque, Robert A. Leacock, who helped me in presenting some of the material in my contribution to this book. He was always generous with his time in helping me and others with our diverse problems. We will miss him now that he has passed away.

References

[1] G. N. Pellegrini and A. R. Swift, "Maxwell's equations in a rotating medium: Is there a problem?", Am. J. Physics. **63** 694-705 (1995).

[2] M. Wilson and H. A. Wilson, "On the electric effect of rotating a magnetic insulator in a magnetic field," Proc. R. Soc. London, Ser. A **89,** 99-106 (1913).

[3] T. A. Weber, "Measurements on a rotating frame in relativity and the Wilson and Wilson experiment", Am. J. Phys. **65,** 946-953 (1997).

[4] R. D. Klauber,"New perspectives on the relativistically rotating disk and non-time- orthogonal reference frames", Found. Phys. Lett. **11** (5) 405-443 (1998).

[5] E. J. Post, "Sagnac effect," Rev. Mod. Phys. **39**, 475-493 (1967).

[6] F. Selleri, "Noninvariant one way speed of light and locally equivalent reference frames", Found. of Phys. Lett. **10,** 73-83 (1997).

[7] Alan M. MacRobert, "Time and the amateur astronomer", Sky and Telescope **94** (4) 48-53 (1997).

[8] Dava Sobel, *Longitude (*Penguin Books, New York, 1995).

[9] Umberto Eco, *The island of the day before (*Harcourt Brace, New York, *1995).*

[10] Albert Einstein, *The Meaning of Relativity (*Princeton U.P., Princeton, 1955), pp.55-63.

[11] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity (*McGraw-Hill, New York, 1975), pp.120-131.

[12] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975) pp.234-238.

[13] D. J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Englewood Cliffs, NJ, 1989) p. 503.

[14] T. A. Weber, “A note on rotating coordinates in relativity”, Am. J. Phys. **65** (6), 486-487 (1997).

[15] P. Ehrenfest, “Gleichförmige Rotation starrer Körper und Relativitätstheorie,” Phys. Z. **10**, 918 (1909).

[16] E. Dewan and M. Beran, “ Note on stress effects due to relativistic contraction,” Am. J. Phys. **31** 517-518 (1959).

[17] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge U.P., Cambridge, 1987), pp.67-68.

[18] Robert D. Klauber “Comments regarding recent articles on relativistically rotating frames” Am. J. Phys. **67** (2), 158-159 (1999).

[19] T. A. Weber, “Response to [Am J. Phys. **67** (2), 158 (1999)]” Am. J. Phys. **67** (2), 159-160 (1999).

[20] Albert Einstein, *Relativity,* (Henry Holt and Company, New York, 1921), Appendix III, pp.155-159.

[21] A. Brillet and J. L. Hall, “Improved laser test of the isotropy of space” Phys. Rev. Lett. **42** 549-552 (1979).

[22] Guido Rizzi and Matteo Luca Ruggiero “Space geometry of rotating platforms: an operational approach” Found. of Phys. **32** (10), 1525-1556 (2002).

Chapter 8

LOCAL AND GLOBAL ANISOTROPY IN THE SPEED OF LIGHT

Francesco Sorge
Dipartimento di Fisica "G. Galilei", Universita' di Padova, I-35131 Padova, Italy
I.N.F.N. Istituto Nazionale di Fisica Nucleare, Universita' di Padova, I-35131 Padova, Italy
francesco.sorge@pd.infn.it

Abstract We briefly review Lorentz invariance and the locality principle, which are the grounds of the theory of Relativity. Subsequently we discuss some recent claims about local anisotropy in the speed of light, as observed in a non inertial frame, and especially in a rotating frame. We show that a standard analysis of a typical physical measurement of the speed of light performed in a rotating frame leads to global anisotropic effects which vanish as the size of the experimental apparatus becomes negligible with respect to the typical lengthscales involved in the non inertial motion of the observer. The expected effects seem too small to be detected with the actual sensitivities at our disposal.

1. Introduction

The Special Theory of Relativity has its grounds on two main assumptions: a) the principle of relativity (or Lorentz invariance) and b) the hypothesis of locality. The former tells us that all the laws of physics have the same form in any reference frame; in particular it guarantees that the speed of light is the same for any physical observer: it is a universal constant, c. This is strictly true for an inertial (i.e. not accelerated) observer. The locality hypothesis allows Lorentz invariance to be extended to non inertial (i.e. accelerated) observers, too. This can be done supposing that, at any given time, an accelerated observer is equivalent to an inertial one, having at that time (and locally) the same position and velocity; such an observer is usually referred to as a locally comoving inertial observer. From an operational point of view, the locality principle has to be regarded with some care. The idea of locality is generally incompatible with a physical measurement, which is intrinsically non-local: an accelerated observer who is performing a measurement is really equivalent to

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 155-165.

an infinite series of locally comoving inertial observers; so to say, the observer changes during the measurement. This obviously causes some difficulties in defining what really the observer measures. Things become more and more interesting when the speed of light, c, is involved, since the experimental check of its *constant* value is a fundamental test for Special Relativity. During the last two decades, a lot of experiments with increasing sensitivities have been performed, in order to test the invariance in the speed of light. In a very accurate reply of the experiment of Michelson and Morley [1], Brillet and Hall [2] have detected a non null signal which they interpreted as a spurious signal (see also Haugan and Will [3]). Aspden [4] pointed out that such spurious signal could be related to the motion of the experimental apparatus due to the terrestrial rotation. Indeed, there are several well-known experimental evidences about the anisotropic behaviour of the speed of light; presumably the most famous is the Sagnac effect [5]. It is important to stress, at this point, that such anisotropy is a global (i.e. non-local) effect, as it is detected after the light rays have performed a complete trip along the rim of a rotating disk. What is then measured is a sort of mean velocity of propagation. It is not a local measurement of the speed of light, c. In recent years, however, some authors, like Selleri[6] and Klauber [7, 8], claimed the local feature of speed anisotropy in light propagation. Selleri's ideas have been critically discussed by Rizzi and Tartaglia [9], who pointed out that local and global measurements of the speed of light on a rotating platform are quite different, so that global anisotropy is not in contrast with local isotropy of the speed of light. We recall, in this respect, the papers of Mashhoon [10, 11, 12] and Nikolić [13] about the concepts of Lorentz invariance and locality. Here anisotropy appears as a global effect, vanishing when the typical size of the experimental apparatus becomes small with respect to the characteristic spatial and temporal scales of the non-inertial frame. In this paper we will follow Mashhoon's approach, specializing it to the case of a specific experiment performed by Byl, Sanderse and van der Kamp [14]. Such experiment was devoted to reveal the influence (or uninfluence) of the terrestrial motion on the speed of light.

The plan of the paper is as follows. In Section 2 we review some aspects of locality and Lorentz invariance in the framework of the Theory of Relativity. In Section 3 we briefly recall the experiment performed by Byl *et al.* Such experiment was considered by Klauber [8] in his analysis of Non-Time Orthogonal (NTO) frames, as the rotating ones. In Section 4 we analyze the above experiment in further detail, taking into account the terrestrial rotation, in order to explore possible effects on the measured speed of light. In Section 5 we discuss the results, showing that (according to Rizzi *et al* [9]) the anisotropy in the *measured* speed of light is a *global* effect; indeed, such effect vanishes as the typical size of the experimental apparatus becomes negligible with respect to the parameters describing the non-inertial character of the reference frame (in the present case, the radius of the terrestrial parallel where the experiment is performed and the rotation period of the Earth). Finally, Section 6 is devoted to some concluding remarks.

2. Light Speed, Locality and Lorentz-invariance

The speed of light, c, appears in Maxwell's equations as a *constant* quantity. The invariance of Maxwell's equations under Lorentz transformations is obviously related to such constant value. As a consequence, the experimental check of such invariance represents a fundamental test for the theory. We emphasize that what is *measured* in any experiment *is not* the value of the universal constant c. All that we may infer from an experimental measurement is merely the speed of propagation of an electromagnetic field (or photons) in the observer's reference frame. Any experimental apparatus has a typical size; moreover, any experiment has a typical time duration. As a consequence, any physical measurement is *non-local*. So, possible anisotropies in the observed speed of light (photons) may stem from a (non-local) measurement performed in a non-inertial reference frame. It is important to stress that such anisotropies simply reflect the inadequacy of the observer, who - so to say - "was changing" during the measurement. We stress that the result of a non-local measurement is not necessarily the constant c; c would be the result of an hypothetical local measurement. Since - strictly speaking - local measurements do not exist, we need to find the conditions which have to be met in order a non-local measurement to give the same result as a local one. Such conditions ([10, 11]) are the requirement of inertiality of the observer who is performing the measurement. Lorentz invariance will then guarantee that the *measured* velocity is exactly c.

For a non-inertial observer, the result of a measurement is *observer - dependent*, since it also encodes information about the observer's acceleration. Consequently, the *measured* value c_{meas} will usually differ from the universal constant c. Such difference will depend on a parameter, say Δ, comparing the typical size of the experimental apparatus, L, with the length-scale Λ describing the non-inertial features of the employed reference frame.

The main conclusion is that any observer (inertial or not), performing an experimental measurement of the speed of light must obtain a value which, in the limit of vanishing intervals of space and time involved in the measurement, coincides with c. This means that the spatial extension of the experimental apparatus, as well as the time duration of the measurement needs to be small when compared with the length and time scales characterizing the observer's non-inertiality.

In the limit of an inertial observer these scales become infinite [10, 11], thus allowing a non local measurement to give the same result of an hypothetical local one. Taking into account the above considerations, we may write:

$$c = \lim_{\Delta \to 0} c_{\text{meas}}(\Delta) = \lim_{\Delta \to 0} [c + \delta c(\Delta)], \tag{8.1}$$

where δc represents a (usually small) correction, namely the *anisotropy*, vanishing in the limit $\Delta \to 0$. This may be the case of an *inertial* observer (in such case the lengthscale Λ becomes infinite), or the case of an *ideal* local measurement ($L = 0$); so we look for somewhat as $\Delta \propto L/\Lambda$. Since, as we pointed out above, any physical measurement is unavoidably *non-local*, non-inertial effects are expected to appear through small corrections affecting the universal constant value c.

Summarizing, the experimental detection of any anisotropy in the speed of light is not in conflict with the grounds of the Theory of Relativity: according to the locality principle, it appears as a consequence of the unavoidable non-locality of the physical measurement. In this respect, any speculation intended to introduce local anisotropies in the value of c in the Theory of the Relativity is intrinsically inconsistent.

3. The Byl *et al.* Experiment

In this Section we briefly recall, for the sake of clarity, the experiment performed by Byl *et al* [14]. A laser beam is split in two rays which propagate along the same direction, one in air, the other in water. At the end of their path, the rays recombine and give rise to an interference pattern. The latter gives information about the time delay in the propagation of the two rays. Performing a 180^o rotation of the whole apparatus, a change in the interference pattern would confirm the presence of anisotropy in the speed of light, due to the terrestrial rotation. The authors notice that the anisotropy should appear if the light had to obey a Galilean composition law. In such a case, one would observe a displacement of the interference pattern, through a number M of interference fringes given by:

$$M = 2\frac{L}{\lambda}(n-1)\frac{v}{c} + 0[(v/c)^3] \tag{8.2}$$

where L is the length of the path followed by the two rays, n is the refraction index of water (we suppose $n = 1$ in air), λ is the laser wavelength and v is the velocity of the Earth with respect to the "ether". Eq. (8.2) explains why the experiment is considered to have a sensitivity of the first order in (v/c). The null experimental result found by Byl and collaborators simply means that the speed of light does not obey a Galilean addition law.

Obviously, this does not prove that light obeys a Lorentzian addition law as well; in this case one would have a null result at any order in (v/c), supposing an ideal local measurement.

Although it was not designed for such a specific scope, the experiment proposed and performed by Byl *et al* actually represents an interesting starting point for a detailed analysis of anisotropies due to the non-inertial motion of the physical observer; the latter is in fact a *rotating* observer. In his analysis of the Byl *et al* experiment, Klauber [8] has proposed to give the following coordinate transformation, from the non rotating (unprimed) to the rotating (primed) frame, a *physical* meaning:

$$\begin{aligned} t &= t' \\ r &= r' \\ \phi &= \phi' + \omega t \\ z &= z' \end{aligned} \tag{8.3}$$

The primed frame is the NTO frame (Non-Time Orthogonal). Following the calculations of Klauber [8], we actually get a result of the second order in (v/c), hence not detectable in the Byl's experiment; however, according to Klauber, an experiment having a sensitivity of $(v/c)^2$ could detect a non-null effect *also* in the limit of a (ideal or quasi-) local measurement; in other words, such effect should appear no matter what the extension of the experimental apparatus is, so violating the Locality postulate. We wish to notice that in Relativity, coordinates do not have a physical meaning (except in some particular case, as the minkowskian one). One is allowed to perform any coordinate transformation [15, 16, 17], without affect the underlying physics. So, it does not appear clear why (8.3) has to be preferred to any other admissible coordinate transformation. In the following we will rediscuss the experiment of Byl *et al*, taking into account Earth's rotation. We will show that the light speed anisotropy does appear, while remaining in the framework of the *orthodox* theory of Relativity. Such effect is however of the *third* order in the small quantity (v/c); furthermore, it depends on the spatial extension of the experimental apparatus, vanishing when the above extension becomes negligible with respect to the length scale related to the non-inertial motion of the terrestrial rotation.

4. The Byl *et al.* Experiment Revisited

Let's consider the coordinate transformation from an inertial frame O to a non inertial frame of an observer O', rotating together with the experimental apparatus. In Fig. 8.1 it is shown the terrestrial parallel on which we suppose the experimental apparatus is located. R is the radius of the parallel, ω is the terrestrial angular velocity. L' is the length of the path followed by the two light rays (one in air, the other in water), before they interfere. We have chosen the reference frames of the two observers $\{O, x, y\}$, $\{O', x', y'\}$ so that the reference axes coincide at $t = t' = 0$, when O' sends the two light rays along L'. O' rotates carrying along with him his reference axes, so that the y' axis is

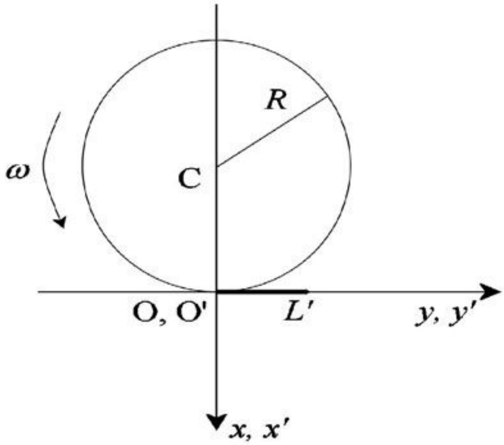

Figure 8.1. A schematic drawing of the employed reference frames.

always tangent to the terrestrial parallel. O remains fixed in his initial position. O' detects the photons which have travelled a distance L' at a time T' in a point of coordinates $x' = 0, y' = L'$ where the detector (screen, photodetector,...) is located. In order to find the relationship between the reference frames of the observers O and O', we will employ the generalized Lorentz transformation for an accelerated, rotating frame, derived by Nelson [18, 19]. Such transformation can be obtained in two steps. First, a coordinate transformation acting upon the Minkowski metric gives a set of non inertial metric coefficients, containing the Thomas precession, as well as the acceleration of the moving frame. Second, a rotation of axes allows to absorb the Thomas precession and to add an ordinary spatial rotation. Using the generalized Lorentz transformation in the useful and concise form presented by Nikolić [13], we get the coordinates of the event "photon arrival to the detector" according to the inertial observer O:

$$\begin{aligned} x &= -\gamma L' \sin \gamma\omega T' + R(\cos \gamma\omega T' - 1) \\ y &= \gamma L' \cos \gamma\omega T' + R \sin \gamma\omega T' \\ T &= \gamma\left(T' + \frac{\omega R L'}{c^2}\right), \end{aligned} \tag{8.4}$$

where $\gamma \equiv 1/\sqrt{1-(\omega R/c)^2}$ is the usual Lorentz factor. T is the photon fly-time according to the clock of O. Being L the spatial separation between the events "photon departure from O" and "photon arrival to the detector", one has exactly $L/T = c$, since O is inertial and the results of a local measurement agree with those of a non-local one (just as the measurement we are considering). From (8.4) we have:

$$L^2 = x^2 + y^2 = \gamma^2 L'^2 + 4R^2 \sin^2\left(\frac{\gamma\omega T'}{2}\right) + 2\gamma L' R \sin\gamma\omega T' \qquad (8.5)$$

Reasonably, $T' \simeq L'/c$ (for the observer O'). So we can expand (8.5) in the limit $L \to 0$, thus obtaining:

$$L = \gamma(L' + \omega R T') \qquad (8.6)$$

Recalling that $L/T = c$, and using (8.4), (8.6), we immediately get the value of the speed of light measured by O':

$$c' = L'/T' = c, \qquad (8.7)$$

i.e. no anisotropy, as expected in the limit of an almost local measurement ($L' \to 0$).

However, no local measurements exist. So, to what extent does the value of L' influence the result of the measurement of c' in the frame of O'? Let's go back to (8.5): comparison with $(8.4)_3$ yields, with $L/T = c$:

$$c^2\gamma^2\left(T' + \frac{\omega R L'}{c^2}\right)^2 = \gamma^2 L'^2 + 4R^2 \sin^2\left(\frac{\gamma\omega T'}{2}\right) + 2\gamma L' R \sin\gamma\omega T' \quad (8.8)$$

If along the path L' is located a medium (water) with refractive index n, then the speed of propagation of photons (as measured by an inertial observer) will differ from c. For an inertial observer O'', instantaneously comoving with O', such velocity will be:

$$u = c/n. \qquad (8.9)$$

Using the standard Lorentz transformation for velocities we have, for the fixed inertial observer O:

$$w = \frac{c}{n} + v - \frac{v}{n^2} + O(1/c), \qquad (8.10)$$

where $v \equiv \omega R$. According to O, $w = L/T$. When a refractive medium is present, (8.8) has to be rewritten as follows:

$$w^2\gamma^2\left(T' + \frac{\omega R L'}{c^2}\right) = \gamma^2 L'^2 + 4R^2 \sin^2\left(\frac{\gamma\omega T'}{2}\right) + 2\gamma L' R \sin\gamma\omega T' \quad (8.11)$$

Let's introduce the parameter:

$$\tau \equiv \gamma L'/c' \qquad (8.12)$$

On lack of anisotropy the measurements of O' and O'' must coincide: $c' = u = c/n$. Due to the acceleration of O' and the non-locality of the measurement, we expect an anisotropic correction to the above value:

$$c' = c/n + \delta c \tag{8.13}$$

Then, assuming a small correction δc:

$$\tau = \frac{\gamma L'}{c'} = \frac{\gamma L'}{\frac{c}{n} + \delta c} \simeq \frac{\gamma L'}{c} n\left(1 - n\frac{\delta c}{c}\right) \equiv \tau_0 n\left(1 - n\frac{\delta c}{c}\right), \tag{8.14}$$

where we have put $\tau_0 \equiv \gamma L'/c$. Using (8.11) and (8.12) and putting $T' = L'/c'$, we obtain:

$$w^2\gamma^2\left(T' + \frac{\omega R L'}{c^2}\right) = \gamma^2 L'^2 + 4R^2 \sin^2\left(\frac{\omega\tau}{2}\right) + 2\gamma L' R \sin\omega\tau. \tag{8.15}$$

Since $\omega\tau \ll 1$ [cf. (8.14)], we expand (8.15) up to the third order in $\omega\tau$ and linearize it in δc, using (8.14). After some algebra, we get:

$$\frac{\delta c}{c}\left[\frac{2R\omega n^2}{c} - \frac{2n^2w^2}{c^2} + \frac{2n^3R^2\omega^2}{c^2} - \frac{2\omega R n^2 w^2}{c^3} - \frac{\gamma^2 L'^2 R\omega^3 n^4}{c^3}\right] =$$
$$\left[1 + \frac{2R\omega n}{c} - \frac{w^2n^2}{c^2} + \frac{R^2\omega^2 n^2}{c^2} - \frac{2\omega R n w^2}{c^3} - \frac{\gamma^2 L'^2 R\omega^3 n^3}{3c^3} - \frac{w^2\omega^2 R^2}{c^4}\right] \tag{8.16}$$

5. Discussion

Let's now discuss in some detail the consequences of eq.(8.16), focusing our attention on a few particular cases, which will turn out to be useful later.

Consider the case of a local measurement in presence of a refractive medium ($n > 1$). From (8.16) we get:

$$\frac{\delta c}{c}\left[-\frac{2n^2w^2}{c^2}\right] \simeq \left[1 + \frac{2R\omega n}{c} - \frac{w^2n^2}{c^2} - \frac{2\omega n R w^2}{c^3} + O(1/c^2)\right] \tag{8.17}$$

(notice that $w/c = O(1)$). Since in the limit of a local measure Lorentz invariance has to be obeyed, we expect that $c' = u = c/n$ (the measurements of O' and O'' must coincide in such a limit, by virtue of the locality postulate). Using (8.13) we have $\delta c = 0$. This implies that the r.h.s. of eq.(8.17) is zero. Solving with respect to w we obtain:

$$w \simeq \frac{c}{n}\left[1 - \frac{R\omega}{nc}\left(1 - n^2 + \frac{2R\omega n}{c}\right)\right] = \frac{c}{n} + R\omega - \frac{R\omega}{n^2} + O(1/c), \tag{8.18}$$

in agreement with (8.10) (remember that $v = \omega R$). If Lorentz invariance has to be obeyed in the limit of a local measurement, then eq.(8.16), rewritten as follows:

$$\frac{\delta c}{c}\left[\frac{2R\omega n^2}{c}-\frac{2n^2w^2}{c^2}+\frac{2n^3R^2\omega^2}{c^2}-\frac{2\omega Rn^2w^2}{c^3}\right]-\frac{\delta c}{c}\left[\frac{\gamma^2L'^2R\omega^3n^4}{c^3}\right]=$$
$$\left[1+\frac{2Rn\omega}{c}-\frac{n^2w^2}{c^2}+\frac{R^2\omega^2n^2}{c^2}+\frac{2nRw^2\omega}{c^3}-\frac{\omega^2Rw^2}{c^4}\right]-\frac{\gamma^2L'^2R\omega^3n^3}{3c^3}, \quad (8.19)$$

is identically satisfied (in the limit $L' \to 0$) with $\delta c = 0$. This means that the quantity in square brackets in the r.h.s. of (8.19) is zero. But this quantity *does not depend* on L', so it must be zero also when $L' \neq 0$. As a consequence, with $L' \neq 0$, we have $\delta c \neq 0$, and to the lowest order of approximation:

$$\frac{\delta c}{c}\left[-\frac{2n^2w^2}{c^2}\right]\simeq-\frac{\gamma^2L'^2R\omega^3n^3}{3c^3}. \quad (8.20)$$

Using (8.7) we get:

$$\frac{\delta c}{c}\simeq\frac{\gamma^2L'^2R\omega^3n^2}{6c^3}, \quad (8.21)$$

and, finally, from (8.13), the value of c' as measured by the observer O':

$$c'\simeq\frac{c}{n}+\frac{\gamma^2L'^2R\omega^3n^2}{6c^2} \quad (8.22)$$

We stress that the anisotropy in the value of c' is present also in a vacuum ($n = 1$):

$$c'\simeq c+\frac{\gamma^2L'^2R\omega^3}{6c^2}. \quad (8.23)$$

Eq.(8.22), as well as (8.23), can be recast in a more convenient form:

$$c'\simeq\frac{c}{n}\left[1+\frac{1}{6}n^2\gamma^2\left(\frac{L'}{R}\right)^2\left(\frac{v}{c}\right)^3\right], \quad (8.24)$$

[compare this result with (8.1)]. Recall that $v = \omega R$ is the velocity of the experimental apparatus, due to the Earth's rotation (see Section 4). Notice that the anisotropy vanishes in the limit of an ideal measurement ($L' \to 0$). Actually the above anisotropy is controlled by the ratio L'/R (recall the parameter $\Delta \sim L/\Lambda$ introduced in Section 2), which compares, so to say, the extension L' of the experimental setup with the radius R ($\sim \Lambda$) of the circular motion of the non-inertial observer O'. We point out that such anisotropy is sensitive to the direction of the rotation.

We are now in a position to apply the results (8.22) and (8.23) to the analysis of the experiment of Byl *et al*. For convenience let's put:

$$\frac{\delta c}{c}=\frac{1}{6}\gamma^2\left(\frac{L'}{R}\right)^2\left(\frac{v}{c}\right)^3$$
$$\frac{\delta c_n}{c}=\frac{1}{6}n^2\gamma^2\left(\frac{L'}{R}\right)^2\left(\frac{v}{c}\right)^3=n^2\frac{\delta c}{c}. \quad (8.25)$$

The fly-time of photons in air ($n = 1$) is, according to O' [cf. (8.23)]:

$$T'_0 = L'/(c + \delta c) \tag{8.26}$$

In the refractive medium we have:

$$T'_n = L'/(c/n + n^2 \delta c) \tag{8.27}$$

The relative delay is then:

$$T'(0^o) = T'_n - T'_0 \simeq \frac{L'}{c}\left[(n-1) - \frac{\delta c}{c}(n^4 - 1)\right]. \tag{8.28}$$

After a 180^o rotation (as in the experiment of Byl and collaborators), $v \to -v$ and $\delta c \to -\delta c$; the relative delay is now:

$$T'(180^o) = T'_n - T'_0 \simeq \frac{L'}{c}\left[(n-1) + \frac{\delta c}{c}(n^4 - 1)\right]. \tag{8.29}$$

The difference is:

$$\Delta T' = \frac{2L'}{c}\frac{\delta c}{c}(n^4 - 1), \tag{8.30}$$

corresponding to a displacement in the interference pattern of M fringes, with M given by:

$$M \simeq \frac{1}{3}\frac{L'}{\lambda}(n^4 - 1)\left(\frac{L'}{R}\right)^2\left(\frac{v}{c}\right)^3, \tag{8.31}$$

where use has been made of (8.25), and $\gamma \simeq 1$, as it is reasonable in usual experimental conditions.

We notice that the anisotropy effect appears at the third order (not at the first, nor at the second one, as suggested by Klauber). Moreover, it is suppressed by the typically small factor $(L'/R)^2$. This is in agreement with the locality principle: when the typical size (L') of the experimental apparatus becomes negligible with respect to the length scale which defines the non-inertiality of the reference frame (R), the experimental results cannot differ from those which would be observed in an inertial reference frame.

6. Concluding Remarks

The intrinsic non-local character of any physical measurement causes the observed physical quantities to be sensitive to the possible non-inertial motion of the observer. In this respect, any measured physical quantity carries an imprint of the non-inertiality of the reference frame which was employed during the measurement. All this can also affect the measurement of the speed of light, giving rise to (expected small) anisotropies in the *observed* value of c. In this paper we have critically discussed the recent claims of Klauber (and other authors) about the *local* character of such anisotropies in rotating reference frames. Also, we have proved that a standard analysis, which rests on the postulates of Lorentz invariance and locality, leads to anisotropic effects which are, however, *non-local*; actually these effects disappear in the limit of a (quasi-) local measurements, since in that limit the non-inertial character of the employed reference frame becomes negligible.

References

[1] Michelson A. A. and Morley E. W., *Am. J. Sci.*, **34**, (1887)

[2] Brillet A. and Hall J. L., *Phys. Rev. Lett.*, **42**, 549, (1979)

[3] Haugan M. P. and Will C. M., *Phys. Today*, , 69, (1987)

[4] Aspden H., *Phys. Lett A*, **85**, 411, (1981)

[5] Sagnac G., *C.R. Acad. Sci. Paris*, **157**, 708, (1913)

[6] Selleri F., *Found. Phys. Lett.*, **10**, 73, (1997)

[7] Klauber R. D., *Found. Phys. Lett.*, **11**, 405, (1998)

[8] Klauber R. D., *Non-Time Orthogonality and Tests of Special Relativity*, arXiv: gr-qc/0006023 v2, 19 Jun 2000

[9] Rizzi G. and Tartaglia A., *Found. Phys. Lett.*, **12**, 179, (1999)

[10] Mashhoon B., *Phys. Lett. A*, **143**, 176, (1990)

[11] Mashhoon B., *Phys. Lett. A*, **145**, 147, (1990)

[12] Mashhoon B., *Measurement Theory and General Relativity*, Talk given at WE-Heraeus Seminar on Black Holes Theory and Observations, Germany, 18-20 Aug 1997)

[13] Nikolic H., *Phys. Rev. A*, **61**, 032109, (2000)

[14] Byl J., Sanderse M. and van der Kamp W., *Am. J. Phys.*, **53**, 43, (1985)

[15] Trocheris M., *Philos. Mag.*, **40**, 1143, (1949)

[16] Takeno H. *Progr. Theor. Phys.*, **7**, 367, (1952)

[17] Herrera L., *Nuovo Cimento B*, **115**, 307, (2000)

[18] Nelson R. A., *J. Math. Phys.*, **28**, 2379, (1987)

[19] Nelson R. A., *J. Math. Phys.*, (*Erratum*), **35**, 6224, (1994)

Chapter 9

ISOTROPY OF THE VELOCITY OF LIGHT AND THE SAGNAC EFFECT

José-Fernando Pascual-Sánchez
Angel San Miguel
Francisco Vicente
Dept. Matemática Aplicada Fundamental
Sección Facultad de Ciencias
Universidad de Valladolid
47005, Valladolid, Spain
jfpascua@maf.uva.es
asmiguel@maf.uva.es
fvicente@maf.uva.es

Abstract In this paper, it is shown, using a geometrical approach, the isotropy of the velocity of light measured in a rotating frame in Minkowski space-time, and it is verified that this result is compatible with the Sagnac effect. Furthermore, we find that this problem can be reduced to the solution of geodesic triangles in a Minkowskian cylinder. A relationship between the problems established on the cylinder and on the Minkowskian plane is obtained through a local isometry.

1. Introduction

One of the most celebrated results of the Theory of Relativity is the one known as the Sagnac effect [1], which appears when two photons describe, in opposite directions, a closed path on a rotating disk returning to the starting point. Physically, the Sagnac effect is essentially a phase shift between two coherent beams of light travelling along paths in opposite senses in an interferometer placed on a rigidly rotating platform [2]. This phase shift can be explained as a consequence of a time delay, so the Sagnac effect can also be measured with atomic clocks timing light rays sent, e.g., around the rotating Earth via the satellites of the Global Positioning System (GPS) [3]. From a geometrical approach, such phase shift has also been related [4] to the fact that

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 167-178.

the time component of the anholonomity object, corresponding to the choice of an orthonormal frame on the space-time, is different from zero.

The Sagnac effect outlines the problem of the isotropy of the velocity of light with respect to a non-inertial observer fixed on the rotating disk. This problem has been treated from different points of view. In [5], it is pointed out that the Sagnac time delay, measured by one single clock, is due to an anisotropy in the global speed of light for the non-inertial observer, in contradiction with the local Einstein synchronization convention. Another approach is found in [6]. There, the speed of light in opposite directions is the same, both locally and globally. The proof is performed using three clocks located at the initial and final positions of the two photons, and by extrapolating point to point, the local Einstein synchronization procedure to the whole periphery of the disk. The disagreements between both approaches are connected with the problem of the global time synchronization of points on the periphery of a rotating disk. Only if this global synchronization were possible there would exist a well defined spatial length between different points on the boundary of the rotating disk.

In this paper we consider an ideal rotating disk with negligible gravitational effects, thus the effects due to gravitational fields —that require the application of general relativistic techniques as those in [7] or [8], where exact and post-Minkowskian solutions are used— are not considered here. We will also show the isotropy of the velocity of light measured in a rotating frame in the Minkowski space-time. We verify that this isotropy is compatible with the Sagnac effect. For this we take into account that every kinematical problem in special relativity can always be translated into a geometrical problem on space-time.

Note on this respect that some authors have need to introduce some dynamical explanations for explaining the rotating disk problem [9].

An outline of the paper is as follows. In Sec. 2 we give a brief account of the technique used by Rizzi and Tartaglia [6] and describe how the use of the *hypothesis of locality*, (see [10]) offers an explanation of the Sagnac effect in the framework of special relativity, without using the anisotropy of a global speed of light. In Sec. 3, we solve this problem in terms of the world-function associated to the geodesic determined by the world-lines of the observer and the photon and the simultaneity space corresponding to the observer. In Sec. 4, a formulation of the problem using the solution of geodesic triangles is obtained. Finally, in Sec. 5, a relationship between the problem stated on a Minkowskian cylinder and on a Minkowskian plane is obtained.

2. The rotating disk and the Sagnac effect

Let $\mathcal{D} \subset \mathbb{R}^3$ be a disk of radius ρ, and let us denote by $\partial\mathcal{D}$ the circle bounding $\mathcal{D}$. We consider an inertial reference frame $F : (O', \{\boldsymbol{e}_i\}_{i=1}^3)$, where O' is the center of $\mathcal{D}$ and $\{\boldsymbol{e}_i\}_{i=1}^3$ is an orthonormal basis for the Euclidean space $\mathbb{R}^3$. In the coordinate system (x, y, z) associated to F, the points in $\mathcal{D}$ have coordinate z equal to zero. It will also be useful to consider polar coordinates (r, θ) on $\mathcal{D}$. Now we assume that the disk $\mathcal{D}$ is uniformly rotating about the O' axis, with angular velocity ω. In the space-time $(\mathcal{M}, \boldsymbol{\eta})$ of Special Relativity in Minkowski coordinates, with $\boldsymbol{\eta} = \mathrm{diag}\,(-1, -1, -1, c^2)$, the motion of the points $P \in \partial\mathcal{D}$ with polar coordinates (θ, t), is given by world-lines $\gamma_{_P} : t \mapsto \gamma_{_\theta}(t)$, that in coordinates (x, y, z, t) can be expressed as

$$\gamma_{_\theta}(t) : \;(\rho\cos(\omega t + \theta), \;\; \rho\sin(\omega t + \theta), \;\; 0, \;\; t). \tag{9.1}$$

This congruence of time-like curves determines a cylinder $\mathcal{C} \subset \mathcal{M}$. On the cylinder $\mathcal{C}$ both a metric $\boldsymbol{g}$ is induced by the metric $\boldsymbol{\eta}$, which in comoving coordinates (θ, t), reads

$$\boldsymbol{g} = -\rho^2 d\theta^2 - 2\,\omega\rho^2 dt d\theta + \alpha^2(\omega) c^2\, dt^2, \tag{9.2}$$

where

$$\alpha^2(\omega) := 1 - (c^{-1}\omega\rho)^2, \tag{9.3}$$

and a Killing vector field $\boldsymbol{\Gamma}$ given by a combination of a rotation and a time translation, that, at each point $P = (\theta, t)$, is $\boldsymbol{\Gamma}(P) = \dot{\gamma}_{_P}(t)$, are defined. The associated Killing congruence has non null vorticity within the cylinder but is zero outside it. So, the vorticity and the 4-velocity play an analogous role to the magnetic field and the 4-electromagnetic potential, respectively, of the Aharonov-Bohm effect in electrodynamics, [11]. The metric (9.2) is globally stationary and locally static; therefore a *local* splitting of $\mathcal{C}$ can be obtained using local hypersurfaces locally orthogonal to the trajectory of the rotating observer, as in [12]. Even a *global* operational quotient space by the Killing congruence can be build, by using the radar distance as a spatial distance [13].

In general, for every two points A, B, joined by a geodesic $\gamma(u)$, being u a special parameter with $\gamma(u_1) = A, \gamma(u_2) = B$, there is a function $\Omega(A\,B)$, which gives half the square of the space-time measure of the geodesic arc between A and B —the world function in Synge's terminology, [14]— defined by

$$\Omega(A\,B) := \tfrac{1}{2}\int_{u_1}^{u_2} \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v})\, ds \tag{9.4}$$

where $\boldsymbol{v} = \dot{\gamma}(u)$ denotes the tangent vector to the geodesic $\gamma(u)$. Let us now consider, at the time $t = t_1$, a point $O \in \partial\mathcal{D}$ and the world-line $\gamma_{_O}(t)$ corresponding to a curve in the congruence (9.1), with $\theta = 0$. On $\gamma_{_O}(t)$ one may build a field of non-inertial reference frames $F'(t)$. The proper time interval between two events P_0, P_1 with coordinate times t_1 and t_2 measured by the observer F' is given in terms of the world-function (9.4) as

$$\tau_2 - \tau_1 := c^{-1}\sqrt{2\,\Omega(P_0\,P_1)} = \alpha(\omega)(t_2 - t_1). \tag{9.5}$$

Suppose that the rotating observer fixed on the circle $\partial\mathcal{D}$ carries a device which emits, at the time $t = 0$, two photons in opposite directions along the periphery of the disk. The world-lines of both photons are null helices. Their equations in the inertial reference frame F read

$$\gamma_{L\pm}(t) : \ (x_{_L} = \rho\cos(\pm\varpi t),\ y_{_L} = \rho\sin(\pm\varpi t),\ t = t)\,, \tag{9.6}$$

where $\pm\varpi$ denotes the angular speeds of the photons given by $\varpi\rho = c$, being the plus (resp. minus) sign associated to the photon moving in the same (resp. opposite) sense as the rotating disk.

At the initial time $t = 0$ it is assumed that $\gamma_{_O}(0) = \gamma_{_L}(0)$. The world-line corresponding to each photon cuts the curve $\gamma_{_O}$ at times $t^*_\pm$, for which it is satisfied the condition $\gamma_{_L}(t_\pm) = \gamma_{_O}(t_\pm)$. Therefore one obtains

$$t^*_\pm = \frac{2\pi}{\varpi \mp \omega}. \tag{9.7}$$

The relationship between proper time on $\gamma_{_O}$ and the inertial coordinate time given in (9.5) establishes that the proper time in F' runs slow with respect to an inertial one. Hence, using (9.7) one obtains

$$\tau_\pm = 2\pi\frac{\alpha(\omega)}{\varpi \mp \omega}. \tag{9.8}$$

The proper time increment measured by the observer F' among the arrival times of the two photons $P_1 = \gamma_{_O}(t_1)$ and $P_2 = \gamma_{_O}(t_2)$ is (see, e.g. [11]):

$$\tau_+ - \tau_- = \frac{4\,\omega S}{c^2}\alpha^{-1}(\omega) \tag{9.9}$$

that, in the limit of small rotational speeds, takes the classical form given in [1]:

$$\tau_+ - \tau_- = \frac{4\,\omega S}{c^2} + O(c^{-4}) \tag{9.10}$$

where $S = \pi r_0^2$ is the disk area.

The Sagnac time delay is the desynchronization of a pair of clocks after a complete round trip, which has been initially synchronized and sent by the rotating observer to travel in opposite directions, [6]. In this case, the time differences along a complete round trip on the periphery $\partial\mathcal{D}$ of the disk, are not uniquely defined and the measurement of each one must be corrected by half the Sagnac time delay when compared with an identical clock remaining fixed at the initial position. After this correction is made, the global light speed is the same for the photon moving on $\partial\mathcal{D}$ in opposite sense. This is in fact what is done in the Global Positioning System, [3]. Note, on the other hand, that if the readings of both clocks are not corrected by half the Sagnac time (9.9), one obtains an anisotropic velocity of light, as in [5].

3. Measurement of relative speeds in Minkowski space-time

Let us assume that, at the time $t = 0$, in the inertial reference frame F, a non-inertial observer F' at a fixed point $P_0 \in \partial\,\mathcal{C}$ emits a pulse of light in the same direction of the movement of the disk. The event P_0 corresponds in the cylinder $\mathcal{C}$ to the point with cylindrical coordinates $(\theta, t) = (0, 0)$. We now determine the relative speed of the ray of light with respect to the non-inertial frame F'.

The world-lines for the observer and the photon can be expressed in cylindrical coordinates, in the form

$$\gamma_{_O}(t):\ (\theta = 0,\ t = t), \qquad \gamma_{_L}(t):\ (\theta = (\varpi - \omega)t,\ t = t) \tag{9.11}$$

respectively. The curve $\gamma_{_O}(t)$ is the time-like helix corresponding to a non-inertial observer fixed at the point O on the disk. The curve $\gamma_{_L}(t)$ describes the null helix of the photon co-rotating with the disk.

On the world-line $\gamma_{_O}$ one can determine a vector field $\mathbf{\Lambda}$ such that the orthogonality condition, $\boldsymbol{g}(\mathbf{\Lambda}, \dot{\gamma_{_O}}(t)) = 0$, is satisfied. For each point P on $\gamma_{_O}$, one builds a space-like geodesic $\gamma_{_S}$ on the cylinder $(\mathcal{C}, \boldsymbol{g})$, corresponding to the initial data $(P, \mathbf{\Lambda}(P))$

$$\gamma_{_S}:\ \big(\theta = (\varpi^2 - \omega^2)\,\omega^{-1}\,(t - t_0),\ t = t\big)\,. \tag{9.12}$$

The geodesic $\gamma_{_S}$ can be interpreted as the locus of locally simultaneous events on an arc of the circle $\partial\,\mathcal{C}$ as seen by the rotating observer. For the construction of the simultaneity space $\gamma_{_S}$, the hypothesis of locality given in [10] is used, which establishes the local equivalence of an accelerated frame and a local inertial frame with the same local speed. In this way, a slicing of the cylinder $\mathcal{C}$ through a family of sections $\gamma_{_S}$ orthogonal to the congruence of curves $\gamma_{_P}$ is obtained.

When the rotating frame reaches the point $P_1 = (0, t_1)$ in the curve $\gamma_{_O}$, the world-function $\Omega(P_0\,P_1)$ is the square of the proper time (up to a constant factor) between the events P_0 and P_1 of space-time, measured by the non-

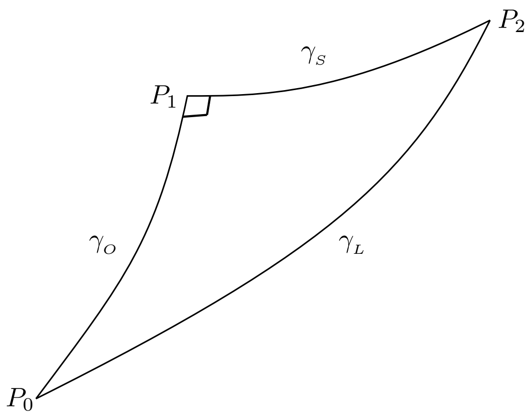

Figure 9.1. Geodesic triangle on the cylinder $(\mathcal{C}, \boldsymbol{g})$. γ_O, γ_L denote, respectively, the world-line of the observer and the photon. γ_S represents the simultaneity space at the point P_1.

inertial observer. At the time t_1 the photon lies on the point P_2 of the local simultaneity space relative to the non-inertial rotating observer. Since both the non-inertial observer and the photon move on $\partial\,\mathcal{C}$, the corresponding points in space-time remain on the Minkowskian cylinder $\mathcal{C}$. Consider the point P_2 given by the intersection of the curves γ_S and γ_L, (see Fig. 9.1):

$$P_2 = \left((\varpi^2 - \omega^2)\,\varpi^{-1}\,t_0,\ (\varpi + \omega)\,\varpi^{-1}\,t_0\right). \tag{9.13}$$

Using the metric (9.2), the world-function corresponding to the pairs of points (P_0, P_1) and (P_1, P_2), calculated along the curves γ_O and γ_S, are

$$\Omega(P_0\,P_1) = -\Omega(P_1\,P_2) = ct_0\,\alpha(\omega). \tag{9.14}$$

respectively. Therefore, taking into account (9.5), the relative speed of the photon with respect to the non-inertial frame defined by

$$v^2_{L,O} := -c^2\,\frac{\Omega(P_1\,P_2)}{\Omega(P_0\,P_1)}, \tag{9.15}$$

coincides with c^2.

4. Equivalent formulation of the problem

Result (9.15) can be compared with that obtained by using the solution of geodesic triangles on the semi-Riemannian manifold $(\mathcal{S}, \boldsymbol{g})$ For this we consider a geodesic triangle $P_0P_1P_2$ on a 2-manifold $(\mathcal{S}, \boldsymbol{g})$ as shown in Fig. 9.2.

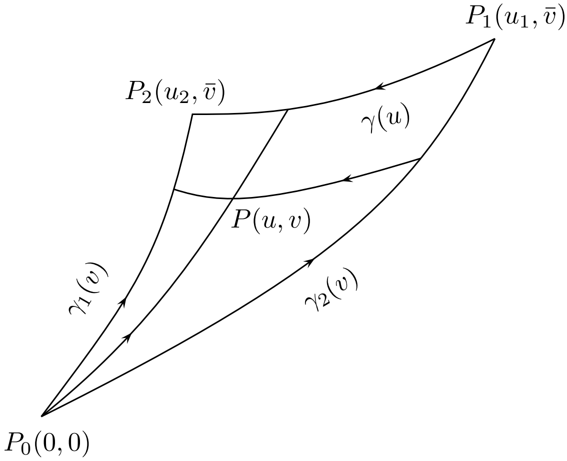

Figure 9.2. Geodesic triangle on a surface with small curvature. The family of curves $\gamma(v)$ emanating from the point P_0, are geodesics parameterized by $u \in [0, \bar{v}]$. Transversal curves are geodesics parameterized by $u \in [u_1, u_2]$.

For arbitrary points A and B, let $\Omega_a(A\,B)$ denote the covariant derivative of (9.4) with respect to the coordinates of A, and denote by $\Omega^a(A\,B)$ the vector associated to $\Omega_a(A\,B)$ by means of the metric $\boldsymbol{g}$.

Let us assume that the Riemannian curvature of a surface $\mathcal{S}$ is small and we will use the same notation as in [14], Chapter II. If $\{\lambda_0(P_0), \lambda_1(P_0)\}$ is an orthonormal basis on $T_{P_0}\mathcal{S}$, one can build a field of reference frames on $\mathcal{C}$ by parallel transport of this frame along all geodesics $x^i(v)$ through P_0. On the field $\{\lambda_0(P), \lambda_1(P)\}$, the vector field $V^i := \partial x^i/\partial v$, tangent to one of these geodesics on an arbitrary point, has constant components $V^{(a)} = V^i \lambda_i^{(a)}$. On the other hand, the components of the symmetrized Riemann tensor

$$S_{ijkl} := -\tfrac{1}{3}(R_{ijkm} + R_{imjk}), \tag{9.16}$$

will be denoted by $S_{(abcd)}$.

For the geodesic triangle determined by the curves $P_0P_1 : \gamma_1(v)$, $P_0P_2 : \gamma_2(v)$ and $P_1P_2 : \gamma(u)$ (with $u \in [u_1, u_2], v \in [0, \bar{v}]$, see Fig. 9.2) a relationship between the world-functions of the sides of this triangle is obtained in [14]:

$$\Omega(P_1\,P_2) = \Omega(P_0\,P_1) + \Omega(P_0\,P_2) - \Omega_a(P_0\,P_1)\,\Omega^a(P_0\,P_2) + \phi, \tag{9.17}$$

where

$$\phi := \tfrac{1}{6}\int_0^{\bar{v}} (\bar{v} - v)^3 D_v^4 \Omega\, dv \tag{9.18}$$

and $D_v^4\Omega$ denotes the covariant derivative of fourth order of the world-function $\Omega(\gamma_1(v), \gamma_2(v))$ for an arbitrary $v \in [0, \bar{v}]$. An explicit approximate expression for ϕ appears in [14] p. 73, written in terms of the Riemann tensor and its covariant derivatives. An application of this solution to build Fermi coordinates in general space-times of small curvature is given in [15]. In general it is satisfied that

$$\phi_0 = \frac{3}{(u_2 - u_1)^3} \int_0^{\bar{v}} \int_{u_1}^{u_2} q(u,v)\,[1122]\,du\,dv, \tag{9.19}$$

where $q(u,v)$ is the polynomial

$$q(u,v) := (\bar{v} - v)^3((u_2 - u)^2 + (u - u_1)^2), \tag{9.20}$$

and symbol [1122], defined as

$$[1122] := -\tfrac{1}{3} S_{(a_1 b_1 c_2 d_2)} V^{(a_1)} V^{(b_1)} V^{(c_2)} V^{(d_2)}, \tag{9.21}$$

is constant on $\mathcal{S}$, so that ϕ_0 vanishes. In (9.21) $V^{(i_1)}, V^{(i_2)}$ are the components of V at points P_1, P_2 respectively. In the problem considered in this work, the metric (9.2) is uniform on the cylinder $\mathcal{C}$, and the Riemannian curvature is zero, therefore expression (9.18) vanish.

Therefore, one obtains for the solution of the same triangle in the point P_1

$$\Omega(P_0\,P_2) = \Omega(P_1\,P_0) + \Omega(P_1\,P_2) - \Omega_a(P_1\,P_0)\,\Omega^a(P_1\,P_2), \tag{9.22}$$

where the covariant derivatives are calculated now at the point P_1. Now, since the geodesic $P_0\,P_2$ is null and the geodesics $P_1\,P_0$ and $P_1\,P_2$ are orthogonal in P_1; then, from (9.22) one obtains

$$\Omega(P_1\,P_2) = -\Omega(P_0\,P_1) \tag{9.23}$$

Consequently, the ratio

$$v^2_{L,O} := -c^2\,\frac{\Omega(P_1\,P_2)}{\Omega(P_0\,P_1)}, \tag{9.24}$$

coincides with (9.15).

5. Reduction to the Minkowskian plane

In this section, we will see that the rotating observer on the disk has a specific characteristic which other different non-inertial observers do not have in general. In the first place, it is observed that expression (9.5), which relates the proper time τ of a non-inertial observer fixed on the rotating disk (moving with constant angular speed ω, such that $\omega\rho = v$) to the coordinate time t,

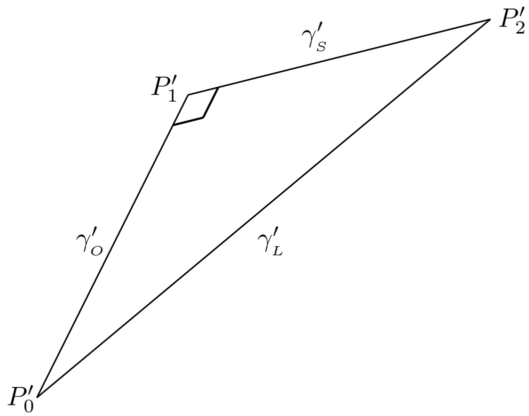

Figure 9.3. Geodesic triangle on the Minkowskian plane $(\mathcal{P}, \boldsymbol{\eta})$. γ_O', γ_L' denote, respectively, the world-line of the observer and the photon. γ_S' represents the simultaneity space at the point P_1'.

coincides with the expression relating the inertial observer's time to the time of another inertial reference frame boosted with rectilinear speed v. Then one concludes that only by measuring proper time, a rotating observer will not be able to determine the local inertial or non-inertial character of the frame rotating uniformly on the disk. The only magnitude that he will be able to measure in that case is the speed modulus v.

Now, let us consider a boosted rectilinear inertial frame K. To measure the speed of a photon moving in the same direction as K with respect to this frame we consider the configuration shown in Fig. 9.3.
Here γ_O' represents the straight line described by the observer (we are assuming that the speed is v) in a Minkowskian plane $(\mathcal{P}, \boldsymbol{\eta})$. On the other hand, the null straight line γ_L' represents the trajectory that one photon describes, and, finally, the line γ_S' is the space-like straight line of simultaneous events to the emission event of the photon. This line is everywhere η–orthogonal to the observer line at the event $P_1' : (vt_0, t_0)$. Explicitly, taking $P_0' = (0, 0)$, these curves are given by

$$\gamma_L : (x = ct, t), \qquad \gamma_S : (x = v^{-1}c^2(t - t_0\alpha^2(v)), t) \tag{9.25}$$

where now $\alpha^2(v) := 1 - v^2/c^2$. This can be verified directly from Figure 9.3.

Moreover, point P_2' at which γ_S' cuts to γ_L' has the coordinates

$$P_2' : \left(t_0(c + v),\, c^{-1}t_0(c + v) \right). \tag{9.26}$$

Therefore, keeping in mind again that $P_1' = (vt_0, t_0)$, one obtains that the distances between P_1' and P_2' along γ_L' and between P_0' and P_1' along γ_O' are

$$-\tilde{\Omega}(P_1'\,P_2') = \tilde{\Omega}(P_0'\,P_1') = \alpha(v)ct_0 \tag{9.27}$$

where $\tilde{\Omega}(AB)$ denotes the world-function associated to points A, B and the metric $\boldsymbol{\eta}$. The relative speed between the light ray and the boosted rectilinear inertial observer, defined through the ratio

$$v^2_{L,O'} = -c^2\,\frac{\tilde{\Omega}(P_1'\,P_2')}{\tilde{\Omega}(P_0'\,P_1')}, \tag{9.28}$$

coincides with c^2.

The identity between these expressions and those obtained before in Sec. 3 is clear. Indeed, if v is substituted for $\omega\rho$ those expressions are coincident. The fact that the values of $\tilde{\Omega}(P_0'\,P_1')$ and $\tilde{\Omega}(P_1'\,P_2')$ coincide with the values $\Omega(P_0\,P_1)$ and $\Omega(P_1\,P_2)$ obtained in the problem solved on the cylinder is due to a local isometry between the Minkowskian plane $(\mathcal{P}, \boldsymbol{\eta})$, which contains the line of universe of the boosted rectilinear inertial observer, and the Minkowskian cylinder $(\mathcal{C}, \boldsymbol{g})$, which contains the world-line of the non-inertial rotating frame. As pointed out at the beginning of this section, the non-inertial rotating observer on the disk has a specific characteristic which other different non-inertial observers do not have, in general. In this case, the expression (9.5), relating the proper time τ and the coordinate time t, is the same as in inertial frames. This allows to build an isometry between cylinder $\mathcal{C}$ and the plane $\mathcal{P}$ as follows.

Let $\phi : \mathcal{U} \subset \mathcal{C} \to \mathcal{P}$ be a smooth map between a neighborhood of $\mathcal{U}$, which contains the geodesic triangle considered above, and the plane $\mathcal{P}$. Denote by $T_P\mathcal{C}$ and $T_{\phi(P)}\mathcal{P}$ the tangent spaces to $\mathcal{C}$ and $\mathcal{P}$ at the points $P, \phi(P)$ respectively. The map ϕ is such that its differential, $d\phi\, : T_P\mathcal{C} \to T_{\phi(P)}\mathcal{P}$, is a linear isometry for every point $P \in \mathcal{U}$:

$$\boldsymbol{\eta}(d\phi\,(\boldsymbol{v}), d\phi(\boldsymbol{w})) = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{w}), \tag{9.29}$$

for every $\boldsymbol{v}, \boldsymbol{w} \in T_P\mathcal{C}$. Let us consider a map ϕ such that $\phi(t, \theta) = (t, x(t, \theta))$. We determine a function $x(t, \theta)$ satisfying condition (9.29). This function is determined through the partial differential system

$$\frac{\partial x}{\partial t} = \omega\rho, \qquad \frac{\partial x}{\partial \theta} = \rho, \tag{9.30}$$

whose solution is the function $x(t, \theta) = \rho(\omega t + \theta)$. Therefore an isometry as

$$\phi\, : (t, \theta) \longmapsto (t, \rho(\omega t + \theta))\,, \tag{9.31}$$

maps $(\mathcal{C}; \boldsymbol{g})$ into $(\mathcal{P}; \boldsymbol{\eta})$, retaining the same coordinate time in both manifolds.

The geodesic triangle of vertices P_0, P_1, P_2 in $\mathcal{C}$ is mapped into the straight triangle P_0', P_1', P_2' in $\mathcal{P}$.

Therefore, it is possible to translate the problem of measuring the speed of light with respect to a non-inertial reference frame, which describes a circumference rotating uniformly, to the problem of measuring the speed of light by an inertial reference frame, being the velocity equal to c in both cases. By means of this local isometry, for the point P_2 on the cylinder there exists a corresponding P_2' in the plane, which has the same coordinates as the event P_2, obtained in Sec. 3 by means of the hypothesis of locality with the slicing of $\mathcal{C}$.

Returning to the initial problem of two photons describing the periphery of a rotating disk in opposite senses, it is observed that one obtains the same result for both photons, as it may be verified solving the corresponding problem on the Minkowskian plane, where the speed of light is independent of the direction followed by the photons.

6. Concluding remarks

In [6], using the locus of locally simultaneous events to the non-inertial rotating observer (given by space-like helices in a Minkowski space-time), it is shown that the speed of light measured by a non-inertial observer fixed on the disk rim always turns out to be c both locally and globally. The local isometry (9.31), shows how this coincidence is obtained. In fact, this local isometry allows to calculate relative speeds (9.24) and (9.28) in the problem of the rotating disk, mapping the problem from the multiply connected Minkowskian cylinder to another one established in the simply connected Minkowskian plane.

From the above reasoning, one observes that although the observer is non-inertial this is not reflected on the measurements of relative speeds. This is because the module of the centripetal acceleration of the observer $\omega^2 \rho\, \alpha^{-2}(\omega)$, coincides with the module of the normal curvature of the world-line of the observer on the cylinder $(\mathcal{C}, \boldsymbol{g})$. A rotating observer corresponds to a Killing trajectory, so its world-line is a geodesic on this cylinder. Moreover, the Gaussian curvature of the cylinder is zero. So, the non-inertiality of the rotating observer is not reflected in the measurement procedure, because this is only based on the first fundamental form of $\mathcal{C}$.

Finally, we remark that the frame of reference considered in the problem of a rotating disk is very special, so the problem can be established on a circular cylinder. A more general case would be, for example, that of a deformable closed loop filament moving and preserving a non-circular shape.

Acknowledgments

The authors wish to thank A. Tartaglia for a discussion on the subject of this work. This work was completed with the support of the Junta de Castilla y León (Spain), project VA014/02.

References

[1] G. Sagnac, *C.R. Acad. Sci. Paris*, **157**, 708 (1913).

[2] E.J. Post, *Rev. Mod. Phys.*, **39**, 475, (1967); G.E. Stedman, *Rep. Prog. Phys.*, **60**, 615 (1997).

[3] D. Allan, N. Ashby and M. Weiss, *Science*, **228**, 69 (1985).

[4] J.F. Corum, *J. Math. Phys.*, **18**, 770 (1977); J.-F. Pascual-Sánchez, Summaries of workshop D.1 at 14th International Conference on General Relativity and Gravitation, Florence, 1995, (unpublished).

[5] F. Selleri, *Found. Phys. Lett.*, **10**, 73 (1997); A. Peres, *Phys. Rev. D*, **18**, 2173 (1978).

[6] G. Rizzi and A. Tartaglia, *Found. Phys.*, **28**, 1663 (1998).

[7] A. Ashtekar, *J. Math. Phys.*, **16**, 341 (1975); A. Tartaglia, *Phys. Rev. D*, **58**, 064009 (1998).

[8] S. Kopeikin and G. Schäfer, *Phys. Rev. D*, **60**, 124002 (1999).

[9] G. Cavalleri, *Nuovo Cimento*, **53 B**, 415 (1968); Ø. Grøn, *Am. J. Phys.*, **43**, 869 (1975), *Found. Phys.*, **10**, 499 (1980); T. E. Phipps, *Found. Phys.*, **10**, 289 (1980), **10**, 811 (1980); D. Dieks, *Eur. J. Phys.*, **12** , 253 (1991); T.A. Weber, *Am. J. Phys.*, **65**, 946 (1997).

[10] B. Mashhoon, *Phys. Lett. A*, **145**, 147 (1990); A. Tartaglia, in *Reference Frames and Gravitomagnetism*, eds. J.-F. Pascual-Sánchez, L. Floría, A. San Miguel, F. Vicente, (World Scientific, Singapore, 2001).

[11] J. Anandan, *Phys. Rev. D* **24**, 338 (1981).

[12] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, (Pergamon Press, Oxford, 1971).

[13] Ll. Bel, J. Martín, A. Molina, *J. Phys. Soc. Japan*, **63**, 4350 (1994); G. Rizzi, M. L. Ruggiero, *Found. Phys.*, **32**, 1525 (2002).

[14] J.L. Synge, *Relativity: The General Theory*, (North–Holland, Amsterdan, 1960).

[15] J.M. Gambi, P. Romero, A. San Miguel and F. Vicente, *Int. J. Theor. Phys.*, **30**, 1097 (1991).

Chapter 10

THE RELATIVISTIC SAGNAC EFFECT: TWO DERIVATIONS

Guido Rizzi
Matteo Luca Ruggiero
Dipartimento di Fisica, Politecnico di Torino,
Corso Duca degli Abruzzi 24, I-10129 Torino, Italy.
INFN, Sezione di Torino,
Via Pietro Giuria 1, I-10125 Torino, Italy
guido.rizzi@polito.it
matteo.ruggiero@polito.it

Abstract The phase shift due to the Sagnac Effect, for relativistic matter and electromagnetic beams, counter-propagating in a rotating interferometer, is deduced using two different approaches. From one hand, we show that the relativistic law of velocity addition leads to the well known Sagnac time difference, which is the same independently of the physical nature of the interfering beams, evidencing in this way the universality of the effect. Another derivation is based on a formal analogy with the phase shift induced by the magnetic potential for charged particles travelling in a region where a constant vector potential is present: this is the so called Aharonov-Bohm effect. Both derivations are carried out in a fully relativistic context, using a suitable 1+3 splitting that allows us to recognize and define the space where electromagnetic and matter waves propagate: this is an extended 3-space, which we call *relative space*. It is recognized as the only space having an actual physical meaning from an operational point of view, and it is identified as the 'physical space of the rotating platform': the geometry of this space turns out to be non Euclidean, according to Einstein's early intuition.

1. Introduction

The effects of rotation on space-time have always been sources of stimulating and fascinating physical issues for the last centuries. Indeed, even before the introduction of the concept of space-time continuum, the peculiarity of the rotation of the reference frame was recognized and understood. A beautiful ex-

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 179-220.

ample is the Foucault's pendulum, which shows, in the context of Newtonian physics, the absolute character of rotation.

At the dawn of the modern scientific era, the notions of absolute space and time, which are fundamental in the formulation of classical laws of physics, were criticized by Leibniz[1] and Berkeley[2]; consequently, the concepts of absolute motion, and hence, of absolute rotation, were questioned too. Mach's [3] analysis of the relativity of motions revived the debate at the dawn of Theory of Relativity. As it is well known, Mach's ideas played an important role and influenced Einstein's approach. However, the peculiarity of rotation, which is inherited by Newtonian physics, leads to bewildering problems even in the relativistic context. Actually, after the publication of Einstein's theory, those who were prejudicely contrary to Relativity found, in the relativistic approach to rotation, important arguments against the self-consistency of the theory. In 1909 an apparent logical contradiction in the Special Theory of Relativity (SRT), applied to the case of a rotating disk, was pointed out by Ehrenfest [4]. Subsequently, in 1913, Sagnac[5] evidenced an apparent contradiction in the SRT with respect to experiments performed with rotating interferometers: according to him

> L'effet interférentiel observé [...] manifeste directment l'existence de l'éther, support nécessaire des ondes lumineuses de Huygens et de Fresnel.

Since those years, both the so-called 'Ehrenfest's paradox' and the theoretical interpretation of the Sagnac effect have become topical arguments of a discussion on the foundations of the SRT, which is not closed yet, as the number of recent contributions confirms.

We studied elsewhere[6] the Ehrenfest's paradox, and we showed that it can be solved on the basis of purely kinematical arguments in the SRT. In this paper we are concerned with the Sagnac effect: we are going to show that it can be completely explained in the SRT. To this end, we are going to give two derivations of the effect.

On the one hand, using relativistic kinematics and, namely, the law of velocity addition, we are going to provide a "direct" derivation of the effect. In particular, the universality of the effect - i.e. its independence from the physical nature and the velocities (relative to the turntable) of the interfering beams - will be explained.

On the other hand, we are going to give a "derivation by analogy" which generalizes a previous work written by Sakurai[7]. Indeed, Sakurai outlined a beautiful and far-reaching analogy between the Sagnac effect and the Aharonov-Bohm effect[8], and obtained a first order approximation of the Sagnac effect. By generalizing Sakurai's result, we shall obtain the Sagnac effect in full theory, without any approximation, evidencing that the analogy holds also in a fully relativistic context. To this end, we shall use Cattaneo's 1+3 splitting [9], [10], [11], [12], [13], that will enable us to describe the geometrodynamics of the rotating frame in a simple and powerful way: in particular, some Newtonian elements used by Sakurai will be generalized to a relativistic context.

The present paper is organized as follows: in Section 2 a historical review of the Sagnac effect is made; in Section 3 the direct derivation is given, while the derivation by analogy is outlined in Section 4. Finally, a thorough exposition of the foundations of Cattaneo's splitting is given in Appendix.

2. A little historical review of the Sagnac effect

2.1 The early years

The history of the interferometric detection of the effects of rotation dates back to the end of the XIX century when, still in the context of the ether theory, Sir Oliver Lodge[14] proposed to use a large interferometer to detect the rotation of the Earth. Subsequently[15] he proposed to use an interferometer rotating on a turntable in order to reveal the rotation effects with respect to the laboratory frame. A detailed description of these early works can be found in the paper by Anderson *et al.*[16], where the study of rotating interferometers is analyzed in a historical perspective. In 1913 Sagnac[5] verified his early predictions[17], using a rapidly rotating light-optical interferometer. In fact, on the ground of classical physics, he predicted the following fringe shift (with respect to the interference pattern when the device is at rest), for monochromatic light waves in vacuum, counter-propagating along a closed path in a rotating interferometer:

$$\Delta z = \frac{4\mathbf{\Omega} \cdot \mathbf{S}}{\lambda c} \tag{10.1}$$

where $\mathbf{\Omega}$ is the (constant) angular velocity vector of the turntable, $\mathbf{S}$ is the vector associated to the area enclosed by the light path, and λ is the wavelength of light in vacuum (as seen in the local co-moving inertial frame of the light source). The time difference associated to the fringe shift (10.1) turns out to be

$$\Delta t = \frac{\lambda}{c}\Delta z = \frac{4\mathbf{\Omega} \cdot \mathbf{S}}{c^2} \tag{10.2}$$

Even if his interpretation of these results was entirely in the framework of the classical (not Lorentz's!) ether theory, Sagnac was the first scientist who reported an experimental observation of the effect of rotation in space-time, which, after him, was named "Sagnac effect". It is noteworthy to notice that the Sagnac effect has been interpreted as a disproval of the SRT since the early years of relativity (in particular by Sagnac himself) up to now (in particular by Selleri[18],[19], Croca-Selleri[20], Goy-Selleri[21], Vigier[22], Anastasowski *et al.*[23], Klauber[24],[25]). However, this claim is incorrect. As a matter of fact, the Sagnac effect can be completely explained in the framework of the SRT: see for instance Weber[26], Dieks[27], Anandan[28], Rizzi-Tartaglia [29], Bergia-Guidone [30], Rodrigues-Sharif[31], Pascual-Sánchez *et al.*[32]. According to the SRT, eq. (10.2) turns out to be just a first order approximation of the relativistic proper time difference between counter-propagating light beams. Moreover, in what follows, it will be apparent that the relativistic interpretation of the Sagnac effect allows a deeper insight into the very foundations of the SRT.

Few years before Sagnac, Franz Harres[33], graduate student in Jena, observed (for the first time but unknowingly) the Sagnac effect during his experiments on the Fresnel-Fizeau drag of light. However, only in 1914, Harzer [34] recognized that the unexpected and inexplicable bias found by Harres was nothing else than the manifestation of the Sagnac effect. Moreover, Harres's observations also demonstrated that the Sagnac fringe shift is unaffected by refraction: in other words, it is always given by eq. (10.1), provided that λ is interpreted as the light wavelength in a co-moving refractive medium. So, the Sagnac phase shift depends on the light wavelength, and not on the velocity of light in the (co-moving) medium.

If Harres anticipated the Sagnac effect on the experimental ground, Michelson[35] anticipated the effect on the theoretical side. Subsequently, in 1925, Michelson himself and Gale[36] succeeded in measuring a phase shift, analogous to the Sagnac's one, caused by the rotation of the Earth, using a large optical interferometer.

The field of light-optical Sagnac interferometry had a revived interest after the development of laser (see for instance the beautiful review paper by Post [37], where the previous experiments are carefully described and their theoretical implications analyzed). After that, there was an increasing precision in measurements and a growth of technological applications, such as inertial navigation[38], where the "fiber-optical gyro"[39] and the "ring laser"[40] are used.

2.2 Universality of the Sagnac Effect

The experimental data show that: (i) the Sagnac fringe shift (10.1) does not depend either on the presence of a co-moving optical medium or on the group velocity of the counter-propagating beams; (ii) the Sagnac time difference (10.2) does not depend either on the light wavelength or on the presence of a co-moving optical medium. This is a first important clue of the universality of the Sagnac effect. However, the most compelling claim for the universal character of the Sagnac effect comes from the validity of eq. (10.1) not for light beams only, but also for any kind of "entities" (such as electromagnetic and acoustic waves, classical particles and electrons Cooper pairs, neutron beams and De Broglie waves and so on...) travelling in opposite directions along a closed path in a rotating interferometer, with the same (in absolute value) velocity with respect to the turntable. This fact is well proved by experimental texts (see Subsection 2.3).

Of course the entities take different times for a complete round-trip, depending on their velocity relative to the turntable; *but the difference between these times is always given by eq. (10.2).* So, the amount of the time difference is always the same, both for matter and light waves, independently of the physical nature of the interfering beams.

This astounding but experimentally well proved "universality" of the Sagnac effect is quite inexplicable on the basis of the classical physics, and invokes a geometrical explanation in the Minkowskian space-time of the SRT.

2.3 Experimental tests and derivation of the Sagnac Effect

The Sagnac effect with matter waves has been verified experimentally using Cooper pairs[41] in 1965, using neutrons[42] in 1984, using ${}^{40}Ca$ atoms beams[43] in 1991 and using electrons, by Hasselbach-Nicklaus[44], in 1993. The effect of the terrestrial rotation on neutron phase was demonstrated in 1979 by Werner et al.[45] in a series of famous experiments.

The Sagnac phase shift has been derived, in the full framework of the SRT, for electromagnetic waves in vacuum (Weber[26], Dieks[27], Anandan[28], Rizzi-Tartaglia[29], Bergia-Guidone [30], Rodrigues-Sharif[31]). However, a clear and universally shared derivation for matter waves is not available as far as we know, or it is at least difficult to find it in the literature. Indeed, the Sagnac phase shift for matter waves has been derived, *in the first order approximation* with respect to the velocity of rotation of the interferometer, by many authors (see Ashby's paper in this book[46] and the paper by Hasselbach-Nicklaus for discussions and further references). These derivations are often based on an heterogeneous mixture of classical kinematics and relativistic dynamics, or non relativistic quantum mechanics and some relativistic elements.

An example of such derivations is given in a well known paper by Sakurai [7], on the basis of a formal analogy between the classical Coriolis force

$$\mathbf{F}_{Cor} = 2m_o\mathbf{v} \times \mathbf{\Omega} \ , \tag{10.3}$$

acting on a particle of mass m_o moving in a uniformly rotating frame, and the Lorentz force

$$\mathbf{F}_{Lor} = \frac{e}{c}\mathbf{v} \times \mathbf{B} \tag{10.4}$$

acting on a particle of charge e moving in a constant magnetic field $\mathbf{B}$.

Sakurai considers a beam of charged particles split into two different paths and then recombined. If S is the surface domain enclosed by the two paths, the resulting phase difference in the interference region turns out to be

$$\Delta\Phi = \frac{e}{c\hbar}\int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} \tag{10.5}$$

Therefore, $\Delta\Phi$ is different from zero when a magnetic field exists inside the domain enclosed by the two paths, even if the magnetic field felt by the particles along their paths is zero. This is the well known Aharonov-Bohm[8] effect. By formally substituting

$$\frac{e}{c}\mathbf{B} \rightarrow 2m_o\mathbf{\Omega} \tag{10.6}$$

Sakurai shows that the phase shift (10.5) reduces to

$$\Delta\Phi = \frac{2m_o}{\hbar}\int_S \mathbf{\Omega}\cdot d\mathbf{S} \tag{10.7}$$

If $\mathbf{\Omega}$ is interpreted as the angular velocity vector of the uniformly rotating turntable and $\mathbf{S}$ as the vector associated to the area enclosed by the closed path along which the two counter-propagating material beams travel, then eq. (10.7) can be interpreted as the Sagnac phase shift for the considered counter-propagating beams:[1]

$$\Delta\Phi = \frac{2m_o}{\hbar}\mathbf{\Omega}\cdot\mathbf{S} \tag{10.8}$$

This result has been obtained using non relativistic quantum mechanics: the relations between the Aharonov-Bohm effect and the wave equations are discussed in Subsection 4.1.

The time difference corresponding to the phase difference (10.8), turns out to be:

$$\Delta t = \frac{\Delta\Phi}{\omega} = \frac{\hbar}{E}\Delta\Phi = \frac{\hbar}{mc^2}\Delta\Phi = \frac{2m_o}{mc^2}\mathbf{\Omega}\cdot\mathbf{S} \tag{10.9}$$

Let us point out that eq. (10.9) contains, inconsistently but unavoidably, some relativistic elements ($\hbar\omega = E = mc^2$). Of course in the first order approximation, i.e. when the relativistic mass m coincides with the rest mass m_o, eq. (10.9) reduces to eq. (10.2); that is, as we stressed before, a first order approximation of the relativistic time difference associated to the Sagnac effect.[2]

This simple and beautiful procedure will be generalized and extended to a fully relativistic context in Sec. 4.

---

[1] In the case of the Aharonov-Bohm effect, the magnetic field $\mathbf{B}$ is zero along the trajectories of the particles, while in Sakurai's derivation, which we are going to generalize, the angular velocity $\mathbf{\Omega}$, which is the analogue of the magnetic field for particles in a rotating frames, is not null: therefore the analogy with the Aharonov-Bohm effect seems to be questionable. However, the formal analogy can be easily recovered when *the flux* of the magnetic field, rather than the magnetic field itself, is considered: this is just what we are going to do (see Section 4, below).

[2] Formulas (10.2) and (10.9) differ by a factor 2: this depends on the fact that in eq. (10.2) we considered the complete round-trip of the beams, while in this section we refer to a situation in which the emission point and the interference point are diametrically opposed.

3. Direct derivation: Sagnac effect for material and light particles

3.1 Direct derivation

In this section we are going to give a description of light or matter beams counter-propagating in a rotating interferometer, based on relativistic kinematics; here and henceforth, we shall refer to both light and matter beams by calling them simply "beams". Indeed, it is our aim to show that, under suitable conditions, the Sagnac time difference does not depend on the very physical nature of the interfering beams.

The beams are constrained to travel a circular path along the rim of the rotating disk, with constant angular velocity, in opposite directions. Let us suppose that a beam source and an interferometric detector are lodged on a point Σ of the rim of the disk. Let K be the central inertial frame, parameterized by an adapted (see Appendix A.6) set of cylindrical coordinates $\{x^\mu\} = (t, r, \vartheta, z)$, with line element given by[3]

$$\mathrm{d}s^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + \mathrm{d}z^2 \tag{10.10}$$

In particular, if we confine ourselves to a the disk ($z = const$), the metric turns out to be

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\vartheta^2 \tag{10.11}$$

With respect to K, the disk (whose radius is R) rotates with angular velocity Ω, and the world-line γ_Σ of Σ is

$$\gamma_\Sigma \equiv \begin{cases} x^0 &= ct \\ x^1 &= r = R \\ x^2 &= \vartheta = \Omega t \end{cases} \tag{10.12}$$

or, eliminating t

$$\gamma_\Sigma \equiv \begin{cases} x^0 &= \frac{c}{\Omega}\vartheta \\ x^1 &= R \\ x^2 &= \vartheta \end{cases} \tag{10.13}$$

The proper time read by a clock at rest in Σ is given by

$$\tau = \frac{1}{c}\int_{\gamma_\Sigma} |ds| = \frac{1}{c}\int_{\gamma_\Sigma} \sqrt{c^2dt^2 - R^2d\vartheta^2} = \frac{1}{\Omega}\sqrt{1-\beta^2}\int_{\gamma_\Sigma} d\vartheta \tag{10.14}$$

[3]The signature is (-1,1,1,1), Greek indices run from 0 to 3, while Latin indices run from 1 to 3.

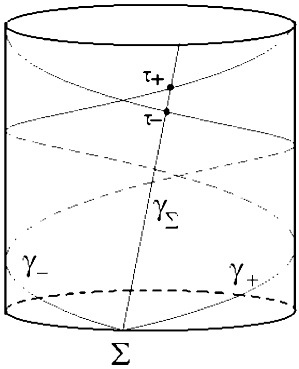

Figure 10.1. The world-line of Σ (a point on the disk where a beam source and interferometric detector are lodged) is γ_Σ; γ_+ and γ_- are the world-lines of the co-propagating (+) and counter-propagating (-) beams. The first intersection of γ_+ (γ_-) with γ_Σ takes place at the time τ_+ (τ_-), as measured by a clock at rest in Σ.

The world-lines of the co-propagating (+) and counter-propagating (-) beams emitted by the source at time $t = 0$ (when $\vartheta = 0$) are, respectively:

$$\gamma_+ \equiv \begin{cases} x^0 & = & \frac{c}{\omega_+}\vartheta \\ x^1 & = & R \\ x^2 & = & \vartheta \end{cases} \tag{10.15}$$

$$\gamma_- \equiv \begin{cases} x^0 & = & \frac{c}{\omega_-}\vartheta \\ x^1 & = & R \\ x^2 & = & \vartheta \end{cases} \tag{10.16}$$

where ω_+, ω_- are their angular velocities, as seen in the central inertial frame.[4] The first intersection of γ_+ (γ_-) with γ_Σ is the event "absorption of the co-propagating (counter-propagating) beam after a complete round trip" (see figure 10.1).

[4]Notice that ω_- is positive if $|\omega'_-| < \Omega$, null if $|\omega'_-| = \Omega$, and negative if $|\omega'_-| > \Omega$; see eq.(10.22) below.

This event takes place when

$$\frac{1}{\Omega}\vartheta_{\pm} = \frac{1}{\omega_{\pm}}(\vartheta_{\pm} \pm 2\pi) \tag{10.17}$$

where the $+$ $(-)$ sign holds for the co-propagating (counter-propagating) beam. The solution of eq. (10.17) is:

$$\vartheta_{\pm} = \pm\frac{2\pi\Omega}{\omega_{\pm} - \Omega} \tag{10.18}$$

If we introduce the dimensionless velocities $\beta = \Omega R/c$, $\beta_{\pm} = \omega_{\pm}R/c$, the ϑ-coordinate of the absorption event can be written as follows:

$$\vartheta_{\pm} = \pm\frac{2\pi\beta}{\beta_{\pm} - \beta} \tag{10.19}$$

Taking into account eq. (10.14), the proper time elapsed between the emission and the absorption of the co-propagating (counter-propagating) beam, read by a clock at rest in Σ, is given by

$$\tau_{\pm} = \pm\frac{2\pi\beta}{\Omega}\frac{\sqrt{1-\beta^2}}{\beta_{\pm} - \beta} \tag{10.20}$$

and the proper time difference $\Delta\tau \equiv \tau_{+} - \tau_{-}$ turns out to be

$$\Delta\tau = \frac{2\pi\beta}{\Omega}\sqrt{1-\beta^2}\frac{\beta_{-} - 2\beta + \beta_{+}}{(\beta_{+} - \beta)(\beta_{-} - \beta)} \tag{10.21}$$

Without specifying any conditions, the proper time difference (10.21) appears to depend upon $\beta, \beta_{+}, \beta_{-}$: this means that it does depend, in general, both on the velocity of rotation of the disk and on the velocities of the beams. Let $\beta'_{\pm}$ be the velocities of the beams as measured in any Minkowski inertial frame, locally co-moving with the rim of the disk, or briefly speaking in any locally co-moving inertial frame (LCIF). Provided that each LCIF is Einstein-synchronized (see Subsection 3.5 below), the Lorentz law of velocity addition gives the following relations between $\beta'_{\pm}$ and $\beta_{\pm}$:

$$\beta_{\pm} = \frac{\beta'_{\pm} + \beta}{1 + \beta'_{\pm}\beta} \tag{10.22}$$

By substituting (10.22) in (10.21) we easily obtain

$$\Delta\tau = \frac{4\pi\beta^2}{\Omega}\frac{1}{\sqrt{1-\beta^2}} + \frac{2\pi\beta}{\Omega}\frac{1}{\sqrt{1-\beta^2}}\left(\frac{1}{\beta'_{+}} + \frac{1}{\beta'_{-}}\right) \tag{10.23}$$

Now, let us impose the condition "equal relative velocity in opposite directions":

$$\beta'_+ = -\beta'_- \tag{10.24}$$

This condition means that the beams are required to have the same velocity (in absolute value) in every LCIF,[5] provided that every LICF is Einstein-synchronized. If condition (10.24) is imposed, the proper time difference (10.23) reduces to

$$\Delta\tau = \frac{4\pi\beta^2}{\Omega}\frac{1}{\sqrt{1-\beta^2}} = \frac{4\pi R^2\Omega}{c^2}\left(1-\frac{\Omega^2R^2}{c^2}\right)^{-1/2} \tag{10.25}$$

which is the relativistic Sagnac time difference.

A very relevant conclusion follows. According to eq. (10.20), the beams take different times - as measured by the clock at rest on the starting-ending point Σ on the platform - for a complete round trip, depending on their velocities $\beta'_\pm$ relative to the turnable. However, when condition (10.24) is imposed, the difference $\Delta\tau$ between these times does depend *only* on the angular velocity Ω of the disk, and it does not depend on the velocities of propagation of the beams with respect the turnable.

This is a very general result, which has been obtained on the ground of a purely kinematical approach. The Sagnac time difference (10.25) applies to any couple of (physical or even mathematical) entities, as long as a velocity with respect the turnable can be consistently defined. In particular, this result applies as well to photons (for which $|\beta'_\pm| = 1$) and to any kind of classical or quantum particles under the given conditions (or electromagnetic/acustic waves in presence of an homogeneous co-moving medium). [6] This fact highlights, in a clear and straightforward way, the universality of the Sagnac effect.

More in particular, the Sagnac time difference (10.25) also applies to the Fourier components of the wave packets associated to a couple of matter beams counter-propagating (with the same relative velocity) along the rim.[7] This remark is important to studying the interferometric detectability of the Sagnac effect (see Subsection 3.3 below).

[5]Or, differently speaking, with respect to any observer at rest in the "relative space" (see below) along the rim of the platform.

[6]Provided that a group velocity can be defined.

[7]Of course only matter beams are physical entities, while Fourier components are just mathematical entities, to which no energy transport is associated.

3.2 The Sagnac effect as an empirical evidence of the SRT

As we said in the Introduction, the Sagnac effect for electromagnetic waves in vacuum was first interpreted by Sagnac himself as an experimental evidence of the physical existence of the classical (non relativistic) ether, and an experimental disproval of the SRT. Sagnac's interpretation can be easily understood, on the basis of the relativistic eq. (10.21), as a casual consequence of a well known kinematical feature of light propagation through the ether. Indeed, the light velocity with respect to the ether (at rest in the central IF) must be c in both directions; as a consequence, if we set $\beta_{\pm} = \pm 1$ in eq. (10.21), the proper time difference $\Delta\tau$ reduces to eq. (10.25); the latter, in turn, reduces, at first order approximation, to the time difference given in eq. (10.2), which was actually predicted and experimentally tested by Sagnac.

However, any non relativistic explanation completely fails for subluminally travelling entities (such as matter waves, sound waves, electromagnetic waves in an homogeneous co-moving medium, and so on). In fact, for subluminally travelling entities the vital condition is: "equal relative velocity in opposite directions". If this condition is expressed by eq. (10.24) (which explicitly requires the local Einstein synchronization) the Sagnac proper time difference (10.25) arises, as we carefully showed before.

On the contrary, if the condition "equal relative velocity in opposite directions" is expressed by an analogous relation, in which the local Einstein synchronization is replaced by a synchronization borrowed from the global synchronization of the central IF (see Subsection 3.5), no time difference arises. Let us prove this claim.

Let $\varphi_{\pm}$ and $\vartheta^r_{\pm}$ be the azimuthal coordinates of the co-rotating/counter-rotating entity with respect to the central IF and the LCIF, respectively. These coordinates are related by the transformation

$$\varphi_{\pm} = \Omega t + \vartheta^r_{\pm} \tag{10.26}$$

Derivation of eq. (10.26) with respect to the central inertial time t gives

$$\omega_{\pm} = \Omega + \omega^r_{\pm} \tag{10.27}$$

where $\omega_{\pm} \equiv d\varphi_{\pm}/dt$ is the angular velocity relative to the central IF, and $\omega^r_{\pm} \equiv d\vartheta^r_{\pm}/dt$ is the angular velocity relative to the LCIF - provided that it is synchronized by means of the central inertial time t.

Eq. (10.27), multiplied by R/c, takes the dimensionless form

$$\beta_{\pm} = \beta + \beta^r_{\pm} \tag{10.28}$$

which replaces eq. (10.22). Let us stress that both eqs. (10.22) and (10.28) are correct: the former refers to the local Einstein synchronization, the latter to the local synchronization according to the simultaneity criterium of the central IF.

Introducing eq. (10.28) into eq. (10.21), the proper time difference $\Delta\tau$ reduces to

$$\Delta\tau = \frac{2\pi\beta}{\Omega}\sqrt{1-\beta^2}\frac{(\beta+\beta_-^r)-2\beta+(\beta+\beta_+^r)}{\beta_-^r\beta_+^r} = \frac{2\pi\beta}{\Omega}\sqrt{1-\beta^2}\frac{\beta_-^r+\beta_+^r}{\beta_-^r\beta_+^r} \tag{10.29}$$

If the vital condition ”equal relative velocity in opposite directions” is expressed by eq.

$$\beta_+^r = -\beta_-^r \tag{10.30}$$

instead of eq. (10.24), it is plain from eq. (10.29) that no time difference arises: $\Delta\tau = 0$ (Q.E.D.)

This calculation shows that the choice of the local Einstein synchronization is crucial even in non-relativistic motion. Indeed, the choice of the classical eq. (10.28), instead of the relativistic eq. (10.22), could be naively presumed as a reasonable approximation in non-relativistic motion: however, this choice simply cancels the effect!

This shows that, according to a very appropriate remark by Dieks and Nienhuis [47], the observed Sagnac effect is an experimental evidence of the SRT at first order approximation with respect to $\beta = \Omega R/c$, and an experimental disproval of the classical (non relativistic) ether.

Remark It could be always possible to substitute eq. (10.24) with an alternative suitable condition, so that eq. (10.29) turns out to be equal to eq. (10.25). Such a condition is:

$$\beta_-^r = -\beta_+^r\frac{1-\beta^2}{1-2\beta\beta_+^r-\beta^2} \tag{10.31}$$

But, of course, this is an extremely ”ad hoc” condition, which translates the simple and expressive condition (10.24): it is clear that the physical interpretation of (10.31) is not as evident as that of (10.24).

3.3 Interferometric detectability of the Sagnac effect

With regard to the interferometric detection of the Sagnac effect, the crucial point is the following one. Consider the Fourier components of the wave packets associated to a couple of matter beams counter-propagating, with the same group velocity, along the rim. Despite the lack of a direct physical meaning and energy transfer, the phase velocity of these Fourier components complies with the Lorentz law of velocity addition (10.22), and is the same for both the co-rotating and counter-rotating Fourier components. As a consequence, the Sagnac time difference (10.25) also applies to any couple of Fourier components with the same phase velocity.

Moreover, the interferometric detection of the Sagnac effect requires that the wave packet associated to the matter beam should be sharp enough in the frequency space, to allow the appearance, in the interferometric region, of an observable fringe shift.[8] It may be worth recalling that:

(i) the observable fringe shift Δz depends on the phase velocity of the Fourier components of the wave packet;

(ii) with respect to an Einstein-synchronized LCIF, the velocity of every Fourier component of the wave packet associated to the matter beam, moving with the velocity (in absolute value) $v \equiv c|\beta'_{\pm}|$, is given by the De Broglie expression $v_f = c^2/v$.

The consequent Sagnac phase shift, due to the relativistic time difference (10.25), is

$$\Delta\Phi = 2\pi\Delta z = 2\pi\left(\frac{v_f}{\lambda}\Delta\tau\right) = \frac{8\pi^2 R^2\Omega}{\lambda v}\left(1 - \frac{\Omega^2 R^2}{c^2}\right)^{-1/2} \tag{10.32}$$

3.4 Comparing to some results found in the literature

As we mentioned, the Sagnac time delay for matter beams has been derived by many authors in many different ways, but nearly always in the first order approximation. However, digging into the literature, we eventually found, between the preliminary and the final version of this paper, a couple of derivations of the Sagnac effect for matter beams in full SRT. In this section we are going to compare these approaches to ours.

3.4.1 Dieks-Nienhuis's approach. Dieks and Nienhuis[47] move from the standard Lorentz transformation from the LCIF to the central IF. In order to be as clear and self-consistent as possible, we shall translate everything into our notations.

Consider two near events, happening along the rim, belonging to the world-line of the co-rotating/counter-rotating beam; let dx, $d\tau$ be the space and time separation between these events, as measured in an Einstein-synchronized LCIF. Then the corresponding time separation dt, as measured in the central

[8] That is, the Fourier components of the wave packet should have slightly different wavelengths.

IF, is given by the usual Lorentz transformation

$$dt = \left(d\tau \pm \frac{\Omega R dx}{c^2} \right) \left(1 - \frac{\Omega^2 R^2}{c^2} \right)^{-1/2} \tag{10.33}$$

where the $+$ and $-$ hold for the co-rotating and counter-rotating beams, respectively. Gluing together all LCIFs, at the end of the round trip we have

$$\left(1 - \frac{\Omega^2 R^2}{c^2} \right)^{1/2} t_{\pm} = \tau(\gamma_{\pm}) \pm \frac{\Omega R}{c^2} \int_D dx \tag{10.34}$$

where D is the length of the rim of the platform, as seen on the platform itself.

The difference between the equations for the co-rotating and counter-rotating beams is:

$$\left(1 - \frac{\Omega^2 R^2}{c^2} \right)^{1/2} (t_+ - t_-) = \tau(\gamma_+) - \tau(\gamma_-) + \frac{2\Omega R}{c^2} \int_D dx \tag{10.35}$$

In the derivation given by Dieks and Nienhuis, two hypotheses play a vital role in order to get the correct conclusion: (i) the integration domain D is $2\pi R \left(1 - \frac{\Omega^2 R^2}{c^2}\right)^{-1/2}$; (ii) $\tau(\gamma_+) = \tau(\gamma_-) = \left(\frac{2\pi R}{v'}\right) \left(1 - \frac{\Omega^2 R^2}{c^2}\right)^{-1/2}$, where v' is the relative velocity (in absolute value) of both the co-rotating and the counter-rotating beam (as a consequence, $\tau(\gamma_+) - \tau(\gamma_-) = 0$).

Of course, the two hypotheses are correct, but both deserve further remarks. First of all, we could say that hypothesis (i) is unnecessary: indeed, our derivation of the Sagnac effect does not depend on the hyperbolic features of the space geometry of the disk (see Subsection 4.2 and Appendix A.14). On the other hand, this hypothesis establishes a very interesting link between the (observable) Sagnac time delay and the (unobservable) lengthening of the rim of the rotating platform, which, in turn, depends on the peculiar space geometry of the disk. This is a challenge to those authors (see for instance [25],[48]) who try to conciliate the Sagnac time delay with the Euclidean space geometry of the disk.

3.4.2 Malykin's approach. The long review paper by Malykin[51] (almost 300 references!) starts with the remark that the Sagnac effect for matter beams "is explained in several totally different ways", but - strangely enough - most of them give "correct results despite their obvious incorretness". After a wide overview of these "incorrect explanations", the author provides his attempt of solution on the basis of the relativistic law of velocity addition. This approach is the only one that turns out to be similar to ours; also the problem of the interferometric detection of the Sagnac effect is taken into account. Of course, this leads to the issue of the kinematical behavior of both the phase velocity and the group velocity. Here some misinterpretations and confusions arise. In particular, the author tries to show that ”both the group velocity and the phase velocity have identical translational properties during the transition from the frame of reference K [the central IF] to the frame of reference K’ [the LCIF]”.

Actually, the group velocity behaves contrary to the phase velocity with respect to the (local) Lorentz transformation group. However, this has no consequences on the interferometric detection of the Sagnac effect, because the only relevant requirement is that, with respect to any Einstein-synchronized LCIF: (i) the group velocities should be the same; (ii) the phase velocities of the Fourier components should be the same. This is exactly what happens: the transformation properties have no role. Anyway, we agree with Malykin when he says that ”the Sagnac effect constitutes a kinematical effect of SRT; (...) all explanations of the Sagnac effect are incorrect except the relativistic one”. We agree also with his final remark: the existence of a large number of incorrect explanations which give correct results (at least in first approximation) depends on the fact that the Sagnac effect is a first-order effect in v/c.

3.4.3 Anderson, Stedman and Bilger’s approach. It is interesting to compare our results to this approach, even though Anderson, Stedman and Bilger’s results[40],[16] are confined to a first order approximation. Indeed, they find, at first order approximation, the following time difference:

$$\Delta t = \frac{4\mathbf{\Omega} \cdot \mathbf{S}}{v^2} \tag{10.36}$$

and the following phase shift:

$$\Delta\Phi = \frac{8\pi\mathbf{\Omega} \cdot \mathbf{S}}{\lambda v} \tag{10.37}$$

where v is the ”undragged” velocity of the beams. Of course, the time difference (10.36) is not in agreement[9] with the first order approximation (with respect to $\beta = \Omega R/c$) of eq. (10.25). However, it is consistent with the first order approximation of eq. (10.21) provided that $\beta_+ = -\beta_- \equiv v/c$: this represents a completely different physical situation,[10] in which the two beams are injected into the rotating platform (tangentially to the rim) in opposite directions *with the same velocity with respect to the central inertial frame.*

In addition, the phase shift (10.37), which is the only observable quantity through an interferometric device, is not in agreement with the physical situation considered by these authors. However, it is worthwhile to notice that,

[9] Except for light beams in vacuum.
[10] Except for light beams in vacuum.

oddly enough, it perfectly agrees with the first order approximation of eq. (10.32).[11]

3.4.4 Ashby's approach. The best approach we know, at first order approximation, is the one suggested in this book by Ashby [46]. The peculiarity of this approach is that it is independent of the shape of the loop: this is a very important feature if one has to deal with the Global Positioning System (GPS). In particular, Ashby shows that the Sagnac time delay depends only on the area swept out by the electromagnetic pulse, as it travels from the GPS transmitter to the receiver, projected onto the terrestrial equatorial plane. A great care is devoted to synchronization problems: this issue is in complete agreement with our fully relativistic approach (see Subsection 3.5 below).

3.5 Synchronization in a LCIF: a free choice

As pointed out by Rizzi-Serafini[49], in a local or global inertial frame (IF) the synchronization is not "given by God", as often both relativistic and anti-relativistic authors assume, but it can be arbitrarily chosen within the *synchronization gauge*

$$\begin{cases} t' = t'\,(\,t,\, x^1, x^2, x^3) \\ x'_i = x_i \end{cases} \tag{10.38}$$

The synchronization gauge (10.38) is a subset of the *Cattaneo gauge* (10.A.2) (see Appendix A.4), which is the set of all possible parameterizations of the given physical inertial frame (IF). In eq. (10.38) the coordinates (t, x_i) are Einstein coordinates, and (t', x'_i) are re-synchronized coordinates of the IF under consideration. Of course, the IF turns out to be optically isotropic if and only if it is parameterized by Einstein coordinates (t, x_i). Then the following question arises: if the parameterization (in particular the synchronization) of a LCIF is a matter of choice, which is the most profitable choice in order to describe the Sagnac effect?

Since the synchronization gauge (10.38) is too general for a clear and useful discussion, it is advantageous to introduce a suitable subset of the synchronization gauge. Such a sub-gauge actually exists; it has been introduced by Selleri [18],[19]. Let us briefly summarize the so-called "Selleri gauge".

Let K be a "formally privileged" IF, in which an isotropic synchronization (that is Einstein synchronization) is assumed *by stipulation;* and let S be a (generally anisotropic) IF moving along the $x'_1 = x_1$ axis with dimensionless

[11] If condition (10.24) is imposed, eq. (10.36) is wrong and eq. (10.37) is right. However, both of them are right for light beams in vacuum.

velocity β with respect to K. The *Selleri gauge* is defined by

$$\begin{cases} t' = t + \frac{\Gamma(\beta)}{c} x \\ x'_i = x_i \end{cases} \tag{10.39}$$

where the unprimed coordinates refer to the Einstein-synchronized IF K, the primed coordinates refer to the arbitrarily-synchronized IF S and $\Gamma(\beta)$ is an arbitrary function of β. It is convenient to write this function as follows:

$$\Gamma(\beta) \equiv \beta + e_1(\beta) c \gamma^{-1} \tag{10.40}$$

The function $e_1(\beta)$ is Selleri's "synchronization parameter", which, in principle, can be arbitrarily chosen. Any choice of the function $e_1(\beta)$ is a choice of the synchronization in the IF under consideration; in principle, the synchronization can be freely chosen inside the Selleri gauge (10.39). In particular, the synchrony choice $e_1(\beta) \doteq -\beta\gamma/c$ (i.e. $\Gamma(\beta) \doteq 0$) gives the standard Einstein synchronization, which is "relative" (that is, frame-dependent); whereas the synchrony choice $e_1(\beta) \doteq 0$ gives the Selleri synchronization, which is "absolute" (that is, frame-independent).

The term "absolute" sounds rather eccentric in a relativistic framework, but it simply means that the Selleri simultaneity hypersurfaces $t' = const$ (contrary to the Einstein simultaneity hypersurfaces $t = const$) define a frame-invariant foliation of space-time - which is nothing but the Einstein foliation of the particular IF K assumed (by stipulation, once and for all) as optically isotropic for any choice of the synchronization parameter .

According to Selleri, the synchronization is a matter of convention in the case of translation, but not in the case of rotation, where the synchronization parameter e_1 is forced to take the value zero. On the contrary, as it is shown in [49], the choice of e_1 is not compelled by any empiric evidence: that is, also when rotation is taken into account, no physical effect can discriminate Selleri's synchrony choice $e_1(\beta) \doteq 0$ from the Einstein's synchrony choice $e_1(\beta) \doteq -\beta\gamma/c$.

Therefore, we have the opportunity of taking a very pragmatic view: both Selleri "absolute" synchronization and Einstein relative synchronization can be used, depending on the aims and circumstances. In particular, (i) if we look for a global synchronization on the rotating platform, Selleri "absolute" synchronization is required; (ii) if we look for a plain kinematical relationship between local velocities, Einstein synchronization is required in any LCIF.

Let us outline the advantages of the local Einstein synchronization on a rotating platform. First, let us recall[49] that the local isotropy or anisotropy of the velocity of light in a LCIF is not a fact, with a well defined ontological meaning, but a convention which depends on the synchronization chosen in the LCIF. Of course the velocity of light has the invariant value c in every LCIF, both in co-rotating and counter-rotating direction, if and only if the

LCIFs are Einstein-synchronized. We are aware that this statement sounds irrelevant or arbitrary to some authors[24], [18], [19], [52], so we try to suggest a more significant one. As we showed in Subsection 3.1, the Sagnac time difference (10.25) holds for two beams travelling along the rim in opposite directions with the same velocity with respect the turnable. This is a plain and meaningful condition: but it must be stressed that *this condition requires that every LCIF should be Einstein-synchronized*. Of course this condition could be translated also into Selleri absolute synchronization, but it would result in a very artificial and convoluted requirement, namely eq. (10.31). Only Einstein synchronization allows the clear and meaningful requirement: "equal relative velocity in opposite directions".[12]

4. The Sagnac effect from an analogy with the Aharonov-Bohm effect

In this section we shall give another derivation of the Sagnac effect for relativistic material beams counter-propagating on a rotating disk[53]. This derivation is based on a (formal) analogy with the Aharonov-Bohm effect which has been outlined by Sakurai[7]. However, Sakurai's approach, which rests upon the use of relativistic and Newtonian elements, gives only the first order approximation of Sagnac time difference (10.25). We want to show that, using Cattaneo's splitting techniques, it is possible to state the analogy between the Aharonov-Bohm effect and the Sagnac effect in a fully relativistic context, getting rid of the Newtonian elements and recovering the relativistic self-consistency of the derivation. On equal footing, our approach allows us to obtain the Sagnac time difference in full theory and not in its first order approximation, to which Sakurai and other authors confined themselves, exploiting the analogy with the Aharonov-Bohm effect.

According to us, our derivation highlights the common geometric nature of the two effects, which is the basis of their far-reaching analogy.

4.1 The Aharonov-Bohm effect

Let us start by briefly describing the Aharonov-Bohm effect. Consider the two slits experiment (figure 10.2) and imagine that a single coherent charged beam is split into two parts, which travel in a region where only a magnetic field is present, described by the 3-vector potential $\mathbf{A}$; then the beams are recombined to observe the interference pattern. The phase of the two wave functions, at each point of the pattern, is modified, with respect to the case of

[12] Formally expressed by condition (10.24).

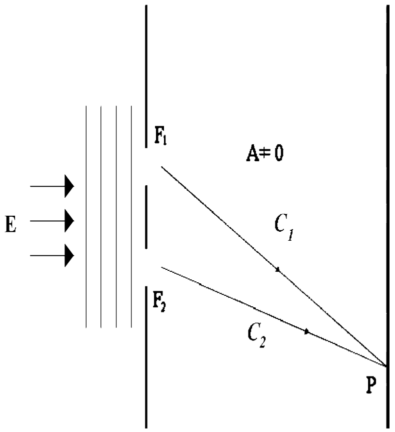

Figure 10.2. A single coherent charged beam, originating in E, is split into two parts (passing through the two slits F_1 and F_2) that propagate, respectively, along the paths C_1 and C_2 (in the figure these paths are represented, respectively, by EF_1P and EF_2P). The beams travel in a region where a vector potential $\mathbf{A}$ is present. In P, the beams interfere and an additional phase shift is provoked by the magnetic field.

free propagation ($\mathbf{A} = 0$), by the magnetic potential. The magnetic potential-induced phase shift has the form[8]

$$\Delta\Phi = \frac{e}{c\hbar}\oint_C \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \frac{e}{c\hbar}\int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} \tag{10.41}$$

where C is the oriented closed curve, obtained as the sum of the oriented paths C_1 and C_2 relative to each component of the beam (in the physical space, see figure 10.2). Eq. (10.41) expresses (by means of Stoke's Theorem) the phase difference in terms of the flux of the magnetic field across the surface S enclosed by the curve C. Aharonov and Bohm[8] applied this result to the situation in which the two split beams pass one on each side of a solenoid inserted between the paths (see figure 10.3). Thus, even if the magnetic field $\mathbf{B}$ is totally contained within the solenoid and the beams pass through a $\mathbf{B} = 0$ region, a resulting phase shift appears, since a non null magnetic flux is associated to every closed path which encloses the solenoid.

Tourrenc[54] showed that no explicit wave equation is demanded to describe the Aharonov-Bohm effect, since its interpretation is a pure geometric one: in fact eq. (10.41) is independent of the nature of the interfering charged beams, which can be spinorial, vectorial or tensorial. So, if we deal with relativistic charged beams, their propagation is described by a relativistic wave equation, such as the Dirac equation or the Klein-Gordon equation, depending on the nature of the beams themselves . From a physical point of view, spin has no influence on the Aharonov-Bohm effect because there is no coupling with the magnetic field which is confined inside the solenoid. Moreover, if the magnetic field is null, the Dirac equation is equivalent to the Klein-Gordon equation, and this is the case of a constant potential. As far as we are concerned, since in what follows we neglect spin, we shall just use eq. (10.41) and we shall not refer explicitly to any relativistic wave equation.

Indeed, things are different when a particle with spin moving in a rotating frame is considered. In this case a coupling between the spin and the angular velocity of the frame appears (this effect is evaluated by Hehl-Ni[55], Mashhoon[56] and Papini[57]).

Hence, the formal analogy that we are going to outline between matter waves moving in a uniformly rotating frame, and charged beams moving in a region where a constant magnetic potential is present, holds only when the spin-rotation coupling is neglected.

4.2 The relative space of a rotating disk

Before going on with our "demonstration by analogy", we want to recall the definition of the "relative space" of a rotating disk, that we introduced elsewhere[6]. Since our analogy with the Aharonov-Bohm effect is based on the measurements performed by the observers on the disk, the concept of relative space is necessary to define, in a proper mathematical way, the physical context in which measurements are made. Even though a global *isotropic* 1+3 splitting of the space-time is not possible when we deal with rotating observers (see Appendix A.8 and A.14), the introduction of the relative space allows well defined procedures for the space and time measurements that can be performed (at large) by the observers in rotating frames, and which reduce to the standard space and time measurements locally (that is, in every Einstein-synchronized LCIF). Let us outline the main points that lead to the definition of the relative space.

The world-lines of each point of the rotating disk are time-like helixes (whose pitch, depending on Ω, is constant), wrapping around the cylindrical surface $r = const$, with $r \in [0, R]$. These helixes fill, without intersecting, the whole space-time region defined by $r \leq R < c/\Omega$; they constitute a time-like con-

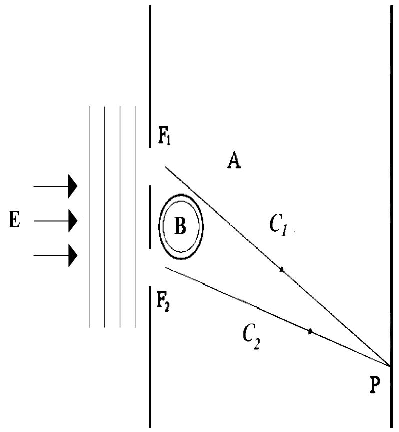

Figure 10.3. A single coherent charged beam, originating in E, is split into two parts (passing through the two slits F_1 and F_2) that propagate, respectively, along the paths C_1 and C_2 (in the figure these paths are represented, respectively, by EF_1P and EF_2P). Between the paths a solenoid is present; the magnetic field $\mathbf{B}$ is entirely contained inside the solenoid, while outside there is a constant vector potential $\mathbf{A}$. In P, the beams interfere and an additional phase shift, provoked by the magnetic field confined inside the solenoid, is observed.

gruence Γ which defines the rotating frame K_{rot}, at rest with respect to the disk.[13] Let us introduce the coordinate transformation

$$\left\{ \begin{array}{lcl} x'^0 & = & ct' = ct \\ x'^1 & = & r' = r \\ x'^2 & = & \vartheta' = \vartheta - \Omega t \\ x'^3 & = & z' = z \end{array} \right. . \tag{10.42}$$

The coordinate transformation $\{x^\mu\} \to \{x'^\mu\}$ defined by (10.42) has a kinematical meaning, namely it defines the passage from a chart adapted to the inertial frame K to a chart adapted to the rotating frame K_{rot}. In the chart $\{x'^\mu\}$

[13]The constraint $R < c/\Omega$ simply means that the velocity of the points of the disk cannot reach the speed of light.

the metric tensor is written in the form:[14]

$$g'_{\mu\nu} = \begin{pmatrix} -1+\frac{\Omega^2 r^2}{c^2} & 0 & \frac{\Omega r^2}{c} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\Omega r^2}{c} & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.43}$$

This is the so called Born metric, and in the classic textbooks (see, for instance Landau-Lifshits[58] and Møller [59]) it is commonly referred to as the space-time metric in the rotating frame of the disk.

Moreover, we can calculate the space metric tensor γ'_{ij} of the congruence which defines K_{rot} (see Appendix A.6 and A.14):

$$\gamma'_{ij} = g'_{ij} - \gamma'_i \gamma'_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{r^2}{1-\frac{\Omega^2 r^2}{c^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{10.44}$$

As it is shown explicitly in Appendix A.14, the congruence Γ of time-like helixes, wrapping around the cylindrical hypersurfaces σ_r $(r = cost \in]0, R])$, defines a Killing field not in the Minkowski space-time $\mathcal{M}^4$, but in the sub-manifolds $\sigma_r \subset \mathcal{M}^4$.

Consequently, we can point out the following interesting property. Let $T_p = \Theta_p \oplus \Sigma_p$ be the tangent space to $\mathcal{M}^4$ in p, where Θ_p, and Σ_p are the local time direction and the local space platform (see Appendix A.6). Then *the splitting $T_p = \Theta_p \oplus \Sigma_p$ and the space metric tensor $\gamma'_{ij}(p)$* are invariant along the lines of Γ. Then it is possible to define a one-parameter group of diffeomorphisms with respect to which both the splitting $T_p = \Theta_p \oplus \Sigma_p$ and the space metric tensor $\gamma'_{ij}(p)$ are invariant. The lines of Γ constitute the trajectories of this "space $\oplus$ time" isometry. This important property suggests a procedure to define an *extended* 3-space, which we call *'relative space'*: according to[6], it is recognized as the only space having an actual physical meaning from an operational point of view, and it is identified as the 'physical space of a rotating platform'. Let us briefly recall the definition of the relative space. First of all, we introduce the following equivalence relation between points and local space platforms:

RE: *" Two points (two local space platforms) are equivalent if they belong to the same line of the congruence Γ".*

[14] For the sake of simplicity, we substitute $r' = r$, from (10.42) $_{II}$.

The *relative space* is the "quotient space" of the world tube of the disk, with respect to the equivalence relation RE, among points and local space platforms belonging to the lines of the congruence Γ. In other words, each element of the relative space is an equivalence class of *points and local space platforms* which verify the equivalence relation RE.

This definition simply means that the relative space is the manifold whose ”points” *are* the lines of the congruence. We point out that the presence of the local space platforms in the equivalence relation RE is intended to emphasize the vital role of the local Einstein synchronization associated to each point of the relative space. Moreover we stress that it is not possible to describe the relative space in terms of space-time foliation, i.e. in the form $x^0 = const$, where x^0 is an appropriate coordinate time, because the space of the disk, as we show in Appendix A.14, is not time-orthogonal. Hence, thinking of the space of the disk as a sub-manifold or a subspace embedded in space-time is misleading and meaningless. The best we can do, if we long for some kind of visualization, is to think of the relative space as the union of the infinitesimal space platforms, each of which is associated, by means of the request of M-orthogonality, to one and only one line of the congruence.

In the relative space, an observer can perform measurements of space and time. His reference frame, defined by the relative space, coincides *everywhere* with the local rest frame of the rotating disk. As a consequence, space measurements are performed on the basis of the spatial metric (10.44), without caring of time, since γ'_{ij} does not depend on time.[15] Moreover, the observer can measure time intervals using his own standard clock, on which he reads the proper time.

4.3 The Sagnac effect in the relative space

Now, we are going to describe the interference process of material beams counter-propagating in a rotating ring interferometer, from the viewpoint of the rotating frame. As we showed before, the physical space of the rotating frame *is* the relative space. Then, a formal analogy between matter beams, counter-propagating in the rotating frame, and charged beams, propagating in a region where a magnetic potential is present, can be outlined on the basis of Cattaneo's formulation of the "standard relative dynamics". In particular, the equation of motion of a particle relative to the rotating frame K_{rot} can

[15] This is a consequence of the stationarity of the rotating frame. However, in order to get rid of any misunderstandings (see for instance[24],[25],[60]), we stress again that "without caring of time" does not mean "without caring of synchronization". As a matter of fact, if synchronization is not taken into account, rotation itself is not taken into account.

be given in terms of the Gravitoelectromagnetic (GEM) fields (see Appendix A.13). The introduction of the GEM fields leads to an analogy between the Aharonov-Bohm effect and the Sagnac effect in a fully relativistic context.

In eq. (10.A.60), the general form of the standard relative equation of motion of a particle is given in terms of the gravito-electric field $\widetilde{\mathbf{E}}_G$, the gravito-magnetic field $\widetilde{\mathbf{B}}_G$ and the external fields.[16] In particular, in eq. (10.A.60) a gravito-magnetic Lorentz force appears

$$\widetilde{\mathcal{F}}_i = m\gamma_0 \left(\frac{\widetilde{\mathbf{v}}}{c} \times \widetilde{\mathbf{B}}_G \right)_i \tag{10.45}$$

On the basis of this description, we want to apply the formal analogy between the gravito-magnetic and magnetic field to the phase shift induced by rotation on a beam of massive particles which, after being split, propagate in two opposite directions along the rim of a rotating disk. When they are recombined, the resulting phase shift is the manifestation of the Sagnac effect.

To this end, let us consider the analogue of the phase shift (10.41) for the gravito-magnetic field

$$\Delta\Phi = \frac{2m\gamma_0}{c\hbar} \oint_C \widetilde{\mathbf{A}}^G \cdot \mathrm{d}\mathbf{r} = \frac{2m\gamma_0}{c\hbar} \int_S \widetilde{\mathbf{B}}^G \cdot \mathrm{d}\mathbf{S} \tag{10.46}$$

which is obtained on the basis of the formal analogy between eq. (10.45) and the magnetic force (10.4):

$$\frac{e}{c}\mathbf{B} \rightarrow \frac{m\gamma_0}{c}\widetilde{\mathbf{B}}^G \tag{10.47}$$

To calculate the phase shift (10.46) we must explicitly express the gravito-magnetic potential and field corresponding to the congruence Γ relative to the rotating frame K_{rot}. In particular (see Appendix A.14) the non null components of the vector field $\boldsymbol{\gamma}(x)$, evaluated on the trajectory $R = const$ along which both beams propagate, are:

$$\left\{ \begin{array}{c} \gamma^0 \doteq \frac{1}{\sqrt{-g_{00}}} = \gamma \\ \gamma_0 \doteq \sqrt{-g_{00}} = \gamma^{-1} \\ \gamma_2 = \gamma_\vartheta \doteq g_{\vartheta 0}\gamma^0 = \frac{\gamma \Omega R^2}{c} \end{array} \right. \tag{10.48}$$

where $\gamma = \left(1 - \frac{\Omega^2 R^2}{c^2}\right)^{-\frac{1}{2}}$.

---

[16]For instance, the constraints that force the particle to move along the rim of the disk are "external fields".

As to the gravito-magnetic potential, we then obtain

$$\widetilde{A}_2^G = \widetilde{A}_\vartheta^G \doteq c^2 \frac{\gamma_\vartheta}{\gamma_0} = \gamma^2 \Omega R^2 c \tag{10.49}$$

As a consequence, the phase shift (10.46) becomes

$$\Delta\Phi = \frac{2m}{c\hbar\gamma} \int_0^{2\pi} \widetilde{A}_\vartheta^G d\vartheta = \frac{2m}{c\hbar\gamma} \int_0^{2\pi} \left(\gamma^2 \Omega R^2 c\right) d\vartheta = 4\pi \frac{m}{\hbar} \Omega R^2 \gamma \tag{10.50}$$

According to Cattaneo's terminology (Appendix A.10), the proper time is the "standard relative time" for an observer on the rotating platform; the proper time difference corresponding to (10.50) is obtained according to

$$\Delta\tau = \frac{\Delta\Phi}{\omega} = \frac{\hbar}{E}\Delta\Phi = \frac{\hbar}{mc^2}\Delta\Phi \tag{10.51}$$

and it turns out to be

$$\Delta\tau = 4\pi \frac{\Omega R^2 \gamma}{c^2} \equiv \frac{4\pi R^2 \Omega}{c^2} \left(1 - \frac{\Omega^2 R^2}{c^2}\right)^{-1/2} \tag{10.52}$$

Eq. (10.52) agrees with the proper time difference (10.25) due to the Sagnac effect, which, as we pointed out in Subsection 2.2, corresponds to the time difference for any kind of matter entities counter-propagating in a uniformly rotating disk. As we stressed before, this time difference does not depend on the standard relative velocity of the particles and it is exactly twice the *time lag* due to the synchronization gap arising in a rotating frame.

Remark In order to generalize Sakurai's procedure, which refers to neutron beams, in this section we always referred to material beams. However, the procedure that leads to the time difference (10.52) can be carried out also for light beams. Actually, in Appendix A.12, we show that a standard relative mass of a photon $m \doteq \frac{h\nu}{c^2}$ can be consistently defined. Consequently, the relative formulation of the equation of motion of a photon is described in a way analogous to that of a material particle, and the procedure that we have just outlined can be applied in a straightforward way to massless particles too.

The phase shift (10.50) can be expressed also as a function of the area S of the surface enclosed by the trajectories:

$$\Delta\Phi = 2\beta^2 S\Omega \frac{m}{\hbar} \frac{\gamma^2}{\gamma - 1} = 2\frac{m}{\hbar} S\Omega \left(\gamma + 1\right) \tag{10.53}$$

where $\beta = \frac{\Omega R}{c}$ and

$$S = \int_0^R \int_0^{2\pi} \frac{r dr d\vartheta}{\sqrt{1-\frac{\Omega^2 r^2}{c^2}}} = 2\pi \frac{c^2}{\Omega^2}\left(1-\sqrt{1-\frac{\Omega^2 R^2}{c^2}}\right) = 2\pi \frac{c^2}{\Omega^2}\left(\frac{\gamma-1}{\gamma}\right) \tag{10.54}$$

We notice that (10.53) reduces to (10.8)[17] only in the first order approximation with respect to $\frac{\Omega R}{c}$ (when $\gamma \to 1$): the formal difference between (10.53) and (10.8) is due to the non Euclidean features of the relative space (Appendix A.14).

5. Conclusions

The relativistic Sagnac effect has been deduced by means of two derivations.

In the first part of this paper a direct derivation has been outlined on the basis of the relativistic kinematics. In particular, only the law of velocity addition has been used to obtain the Sagnac time difference, and to show, in a straightforward way, its independence from the physical nature and the velocities (relative to the turntable) of the interfering beams.

In the second part of this paper, an alternative derivation has been presented. In particular, the formal analogy outlined by Sakurai, which explains the effect of rotation using a "ill-assorted" mixture of non-relativistic quantum mechanics and Newtonian mechanics (which are Galilei-covariant), and intrinsically relativistic elements[18] (which are Lorentz-covariant), has been extended to a fully relativistic treatment, using the 1+3 Cattaneo's splitting technique. The space in which the waves propagate has been recognized as the relative space of a rotating frame, which turns out to be non-Euclidean. In this way, we have obtained a derivation of the relativistic Sagnac time difference (whose first order approximation coincides with Sakurai's result) in a self-consistent way.

Both derivations are carried out in a fully relativistic context, which turns out to be the natural arena where the Sagnac effect can be explained. Indeed, its universality can be clearly understood as a purely geometrical effect in the Minkowski space-time of the SRT, while it is hard to grasp in the context of classical physics.

---

[17]Apart a factor 2, whose origin has been explained in footnote 2 in Subsection 2.3.

[18]Indeed, the lack of self-consistency, due to the use of this "odd mixture", is present not only in Sakurai's derivation, but also in all known approaches based on the formal analogy with the Aharonov-Bohm effect.

Appendix: Space-Time Splitting and Cattaneo's Approach

The tools for splitting space-time have had a great (even though heterogenous) development in the years, and they have been used in various application in General Relativity(GR). Indeed, the common aim of the different approaches to splitting techniques is the description of what is measured by a test family of observers, moving along certain curves in the four-dimensional continuum.

In this way, locally, along the world-lines of these observers, space+time measurements can be recovered, and the description of the physical phenomena borrowed from the SRT can be transferred into GR.

There are various approach to splitting of space-time, and a great work has be done, recently, to describe everything in a common framework, by stressing the connections among the different techniques[61],[62],[63].

Probably, the most well known and used splitting is the so called "ADM splitting"[64] (see also *Gravitation*[65]), which is based on the use of a family of space-like hypersurfaces ("slicing" point of view); on the other hand, the approach based on a congruence of time-like observers ("threading" point of view) was developed independently by various authors, such as Cattaneo[9], [10], [11], [12], [13], Møller[59] and Zel'manonv[66] during the 1950's, but it has remained greatly unknown for a long time, also because some of the original works were not published in English, but in Italian, French or Russian. Because of the pedagogical aim that we have in writing this paper, we decided to present here a very introductory primer to the original Cattaneo's works on splitting of space-time.

After the publication of his works during the 1950's and 1960's, a lot of work has been done, in order to improve his techniques (see [61],[62],[63], and references therein). However, we believe that the foundations of Cattaneo's approach can be understood, in the easiest and most enlightening way, by referring to his original works.

Moreover, since our aim is not historical but pedagogical, we shall translate his "relative formulation of dynamics" in terms of the Gravitoelectromagnetic analogy: indeed, we exploited this analogy in our derivation of the Sagnac effect starting from the Aharonov-Bohm effect.

A.1 It is important to define correctly the properties of the physical frames with respect to which we describe the measurement processes. We shall adopt the most general description, which takes into account non-inertial frames (for instance rotating frames) in the SRT, and arbitrary frames in GR.

The physical space-time is a (pseudo)riemannian manifold $\mathcal{M}^4$, that is a pair $(\mathcal{M}, \mathbf{g})$, where $\mathcal{M}$ is a connected 4-dimensional Haussdorf manifold and $\mathbf{g}$ is the metric tensor.[19] Let the signature of the manifold be $(-1, 1, 1, 1)$. Suitable differentiability conditions, on both $\mathcal{M}$ and $\mathbf{g}$, are assumed.

[19]The riemannian structure implies that $\mathcal{M}$ is endowed with an affine connection compatible with the metric, i.e. the standard Levi-Civita connection.

A.2 A physical reference frame is a time-like congruence Γ: the set of the world lines of the test-particles constituting the "reference fluid".[20] The congruence Γ is identified by the field of unit vectors tangent to its world lines. Briefly speaking, the congruence is the (history of the) physical frame or the reference fluid (they are synonymous).

A.3 Let $\{x^\mu\} = (x^0, x^1, x^2, x^3)$ be a system of coordinates in the neighborhood of a point $p \in \mathcal{M}$; these coordinates are said to be *admissible* (with respect to the congruence Γ) when[21]

$$g_{00} < 0 \quad g_{ij}dx^i dx^j > 0 \tag{10.A.1}$$

Thus the coordinates $x^0 = var$ can be seen as describing the world lines of the ∞^3 particles of the reference fluid.

A.4 When a reference frame has been chosen, together with a set of admissible coordinates, the most general coordinates transformation which does not change the physical frame, i.e. the congruence Γ, has the form (see [59],[67],[9]):

$$\left\{ \begin{array}{c} x'^0 = x'^0(x^0, x^1, x^2, x^3) \\ x'^i = x'^i(x^1, x^2, x^3) \end{array} \right. \tag{10.A.2}$$

with the additional condition $\partial x'^0/\partial x^0 > 0$, which ensures that the change of time parameterization does not change the arrow of time. The coordinates transformation (10.A.2) is said to be *internal to the physical frame* Γ, or *internal gauge transformation*, or more simply *Cattaneo's gauge transformation.*

A.5 An "observable" physical quantity is in general frame-dependent, but its physical meaning requires that it cannot depend on the particular parameterization of the physical frame: in brief, it cannot be gauge-dependent. Then a problem arises. In the mathematical model of GR, physical quantities are expressed by absolute entities,[22] such as world-tensors, and the physical laws, according to the covariance principle, are just relations among these entities. So, given a reference frame, how do we relate these absolute quantities to the relative, i.e. reference-dependent, ones? And how do we relate world equations to reference-dependent ones? In other words: how do we relate, by a suitable 1+3 splitting, the mathematical model of space-time to the observable quantities which are relative to a reference frame?

A.6 In order to do that, we are going to introduce the *projection technique* developed by Cattaneo. Let $\gamma(x)$ be the field of unit vectors tangent to the world lines of the congruence Γ. Given a time-like congruence Γ it is always possible to choose a system of admissible coordinate so that the lines $x^0 = var$ coincide with the lines of Γ; in this case, such coordinates are said to be *'adapted to the physical frame'* defined by the congruence Γ.

[20]The concept of 'congruence' refers to a set of word lines filling the manifold, or some part of it, smoothly, continuously and without intersecting. The concept of 'reference fluid' is an obvious generalization of the 'reference solid' which can be used in flat space-time, where the test particles constitute a global inertial frame. In this case, their relative distance remains constant and they evolve as a rigid frame. However: (i) in GR test particles can be subject to a gravitational field (curvature of space-time); (ii) in the SRT test particles can be subject to an acceleration field. In both cases, global inertiality is lost and tidal effects arise, causing a variation of the distance between them. So we must speak of "reference fluid", dropping the compelling request of classical rigidity.

[21]Greek indices run from 0 to 3, Latin indices run from 1 to 3.

[22]'Absolute' means 'independent of any reference frame'.

Being $g_{\mu\nu}\gamma^\mu\gamma^\nu = -1$, we get

$$\begin{cases} \gamma^0 = \frac{1}{\sqrt{-g_{00}}} \\ \gamma^i = 0 \end{cases} \qquad \begin{cases} \gamma_0 = \sqrt{-g_{00}} \\ \gamma_i = g_{i0}\gamma^0 \end{cases} \tag{10.A.3}$$

In each point $p \in \mathcal{M}$, the tangent space T_p can be split into the direct sum of two subspaces: Θ_p, spanned by γ^α, which we shall call *local time direction* of the given frame, and Σ_p, the 3-dimensional subspace which is supplementary (orthogonal) with respect to Θ_p; Σ_p is called *local space platform* of the given frame. So the tangent space can be written as the direct sum

$$T_p = \Theta_p \oplus \Sigma_p \tag{10.A.4}$$

A vector $\mathbf{v} \in T_p$ can be projected onto Θ_p and Σ_p using the *time projector* $\gamma_\mu\gamma_\nu$ and the *space projector* $\gamma_{\mu\nu} \doteq g_{\mu\nu} - \gamma_\mu\gamma_\nu$:

$$\begin{cases} \bar{v}_\mu = P_\Theta(\mathbf{v}) \doteq \gamma_\mu\gamma_\nu v^\nu \\ \widetilde{v}_\mu = P_\Sigma(\mathbf{v}) \doteq v_\mu - v^\nu\gamma_\nu\gamma_\mu = (g_{\mu\nu} - \gamma_\mu\gamma_\nu)\, v^\nu = \gamma_{\mu\nu}v^\nu \end{cases} \tag{10.A.5}$$

Notation The superscripts $^-$,$^\sim$ denote respectively a *time vector* and a *space vector*, or more generally, a *time tensor* and *a space tensor* (see below).

Equation (10.A.5) defines the *natural splitting* of a vector. The tensors $\gamma_\mu\gamma_\nu$ and $\gamma_{\mu\nu}$ are called *time metric tensor* and *space metric tensor*, respectively. In particular, for each vector $\mathbf{v}$ it is possible to define a 'time norm' $\| \mathbf{v} \|_\Theta$ and a 'space norm' $\| \mathbf{v} \|_\Sigma$ as follows:

$$\| \mathbf{v} \|_\Theta \doteq \bar{v}_\rho\bar{v}^\rho = \gamma_\rho\gamma_\nu v^\nu\, (\gamma^\rho\gamma_\mu v^\mu) = \gamma_\mu\gamma_\nu v^\mu v^\nu = (\gamma_\mu v^\mu)^2 \leq 0 \tag{10.A.6}$$

$$\| \mathbf{v} \|_\Sigma \doteq \widetilde{v}_\nu\widetilde{v}^\nu = \gamma_{\mu\nu}v^\mu(v^\nu - \gamma_\eta\gamma^\nu v^\eta) = \gamma_{\mu\nu}v^\mu v^\nu \geq 0 \tag{10.A.7}$$

For a tensor field $T \in T_p$, every index can be projected onto Θ_p and Σ_p by means of the projectors defined before:

$$P_\Theta(T_{\dots\mu\dots}) \doteq \gamma_\mu\gamma_\nu T^{\dots\nu\dots} \quad P_\Sigma(T_{\dots\mu\dots}) \doteq \gamma_{\mu\nu}T^{\dots\nu\dots} \tag{10.A.8}$$

A tensor field of order two can be split into the sum of four tensors

$$\begin{array}{lll} P_{\Sigma\Sigma}\,(t_{\mu\nu}) \doteq \gamma_{\mu\rho}\gamma_{\nu\eta}t^{\rho\eta} & & P_{\Sigma\Theta}\,(t_{\mu\nu}) \doteq \gamma_{\mu\rho}\gamma_\nu\gamma_\eta t^{\rho\eta} \\ P_{\Theta\Sigma}\,(t_{\mu\nu}) \doteq \gamma_\mu\gamma_\rho\gamma_{\nu\eta}t^{\rho\eta} & & P_{\Theta\Theta}\,(t_{\mu\nu}) \doteq \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\eta t^{\rho\eta} \end{array} \tag{10.A.9}$$

belonging to four orthogonal subspaces

$$T_p \otimes T_p = (\Sigma_p \otimes \Sigma_p) \oplus (\Sigma_p \otimes \Theta_p) \oplus (\Theta_p \otimes \Sigma_p) \oplus (\Theta_p \otimes \Theta_p) \tag{10.A.10}$$

In particular, every tensor belonging entirely to $\Sigma_p \otimes \Sigma_p$ is called a *space tensor* and every tensor belonging to $\Theta_p \otimes \Theta_p$ is called a *time tensor*. Of course, these entities have a tensorial behavior only with respect to the group of the coordinates transformation (10.A.2). It is straightforward to extend these procedures and definitions to tensors of generic order n (see below) .

Remark 1 The natural splitting of a tensor is gauge-independent: it depends only on the physical frame chosen. The projection technique gives gauge-invariant quantities that have an operative meaning in our physical frame; namely, they represent the objects of our measurements.

Remark 2 In Γ-adapted coordinates, a *time vector* $\bar{\mathbf{v}} \in \Theta_p$ is characterized by the vanishing of its controvariant space components ($\bar{v}^i = 0$); a *space vector* $\tilde{\mathbf{v}} \in \Sigma_p$ by the vanishing of its covariant time component ($\tilde{v}_0 = 0$). As a generalization: (i) a given index of a tensor $\mathbf{T}$ is called a *time-index* if all the tensor components of the type $T^{\cdot i \cdot}_{\cdots}$ ($i \in [1, 2, 3]$) vanish; (ii) a given index of a tensor $\mathbf{T}$ is called a *space-index* if all the tensor components of the type $T^{\cdot\cdot\cdot}_{\cdot 0 \cdot}$ vanish. For a *time tensor*, i.e. for a tensor belonging to $\Theta_p \otimes ... \otimes \Theta_p$, property (i) holds for all its indices; for a *space tensor*, i.e. a tensor belonging to $\Sigma_p \otimes ... \otimes \Sigma_p$, property (ii) holds for all its indices.

A.7 To formulate the physical equations relative to the frame Γ, we need the following differential operator

$$\tilde{\partial}_\mu \doteq \partial_\mu - \gamma_\mu \gamma^0 \partial_0 \tag{10.A.11}$$

which is called *transverse partial derivative*. It is a "space vector" and (its definition) is gauge-invariant.

It is easy to show that, for a generic scalar field $\varphi(x)$ we obtain:

$$P_\Sigma(\partial_\mu \varphi) = \tilde{\partial}_\mu \varphi \tag{10.A.12}$$

So $\tilde{\partial}_\mu$ defines the *transverse gradient,* i.e. the space projection of the local gradient.

The projection technique that we have just outlined allows to calculate the projections of the Christoffel symbols. It is remarkable that the total space projections turn out to be

$$\begin{aligned} P_{\Sigma\Sigma\Sigma}(\mu\nu, \lambda) &= \frac{1}{2}\left(\tilde{\partial}_\mu \gamma_{\nu\lambda} + \tilde{\partial}_\nu \gamma_{\lambda\mu} - \tilde{\partial}_\lambda \gamma_{\mu\nu}\right) \doteq \widetilde{\Gamma}^*_{\mu\nu\lambda} \\ P_{\Sigma\Sigma\Sigma}\left\{ {\lambda \atop \mu\nu} \right\} &= \widetilde{\Gamma}^*_{\mu\nu\sigma}\gamma^{\sigma\lambda} \doteq \widetilde{\Gamma}^{*\ \lambda}_{\mu\nu} \end{aligned} \tag{10.A.13}$$

where the space metric tensor $\gamma_{\mu\nu}$ substitutes the metric tensor $g_{\mu\nu}$ and the transverse derivative substitutes the "ordinary" partial derivative.

A.8 The differential features of the congruence Γ are described by the following tensors

$$\widetilde{C}_\mu = \gamma^\nu \nabla_\nu \gamma_\mu \tag{10.A.14}$$

$$\widetilde{\Omega}_{\mu\nu} = \gamma_0 \left[\tilde{\partial}_\mu \left(\frac{\gamma_\nu}{\gamma_0}\right) - \tilde{\partial}_\nu \left(\frac{\gamma_\mu}{\gamma_0}\right)\right] \tag{10.A.15}$$

$$\widetilde{K}_{\mu\nu} = \gamma^0 \partial_0 \gamma_{\mu\nu} \tag{10.A.16}$$

$\widetilde{C}_\mu$ is the *curvature vector*, $\widetilde{\Omega}_{\mu\nu}$ is the *space vortex tensor*, which gives the local angular velocity of the reference fluid, $\widetilde{K}_{\mu\nu}$ is the *Born space tensor*, which gives the deformation rate of the reference fluid; when this tensor is null, the frame is said to be rigid according to the definition of rigidity given by Born[68].

In a relativistic context the classical concept of rigidity, which is dynamical in its origin, since it is based on the presence of forces that are responsible for rigidity, becomes meaningless. Born's definition of rigidity is the natural generalization of the classical one. It depends on the motion of the test particles of the congruence: hence, it is a kinematical constraint. According to Born, a body moves rigidly if the space distance $\sqrt{\gamma_{ij}dx^i dx^j}$ between neighbouring points of the body, as measured in their successive (locally inertial) rest frames, is constant in time.[23] For Born's condition of rigidity see also Rosen[69], Boyer[70], Pauli[71].

Definitions The following definitions[24] are referred to the (geometry of the) physical frame Γ:

- *constant* - when there exists at least one adapted chart, in which the components of the metric tensor are not depending on the time coordinate: $\partial_0 g_{\mu\nu} = 0$
- *time-orthogonal* - when there exist at least one adapted chart in which $g_{0i} = 0$; in this system the lines $x^0 = var$ are orthogonal to the 3-manifold $x^0 = cost$
- *static* - when there exists at least one adapted chart in which $g_{0i} = 0$ and $\partial_0 g_{\mu\nu} = 0$.
- *stationary* when it is constant and non time-orthogonal

Remark 3 The condition of being time-orthogonal is a property of the physical frames, and not of the coordinate systems: for a reference frame to be time-orthogonal it is necessary and sufficient that the space vortex $\widetilde{\Omega}_{\mu\nu}$ tensor vanishes.

When the space vortex tensor is null, moreover, the fluid is said to be irrotational; if both the curvature vector and the space vortex tensor are zero, the fluid is said irrotational and geodesic. When the space vortex tensor is not null, a global synchronization of the standard clocks in the frame is not possible.

The irrotational, rigid and geodesic motion (of a frame) is characterized by the condition $\nabla_\mu \gamma_\nu = 0$: this is the generalization, in a curved space-time context, of the translational uniform motion in flat space-time.

A.9 The natural splitting also permits to calculate the Riemann curvature tensor of the 3-space of the reference frame. The complete space projection of the curvature tensor of space-time is [9]:

$$P_{\Sigma\Sigma\Sigma\Sigma}\left(R_{\mu\nu\sigma\rho}\right) = \widetilde{R}^*_{\mu\nu\sigma\rho} - \frac{1}{4}\left(\tilde{\Omega}_{\sigma\mu}\tilde{\Omega}_{\rho\nu} - \tilde{\Omega}_{\rho\mu}\tilde{\Omega}_{\sigma\nu}\right) - \frac{1}{2}\tilde{\Omega}_{\mu\nu}\tilde{\Omega}_{\sigma\rho} \tag{10.A.17}$$

where

$$\begin{aligned}\widetilde{R}^*_{\mu\nu\rho\sigma} \doteq\ & \frac{1}{4}\left(\widetilde{\partial}_{\rho\mu}\gamma_{\nu\sigma} - \widetilde{\partial}_{\sigma\mu}\gamma_{\nu\rho} + \widetilde{\partial}_{\sigma\nu}\gamma_{\mu\rho} - \widetilde{\partial}_{\rho\nu}\gamma_{\mu\sigma}\right) + \frac{1}{4}\left(\widetilde{\partial}_{\mu\rho}\gamma_{\nu\sigma} - \right. \\ & \left. \widetilde{\partial}_{\nu\rho}\gamma_{\mu\sigma} + \widetilde{\partial}_{\nu\sigma}\gamma_{\mu\rho} - \widetilde{\partial}_{\mu\sigma}\gamma_{\nu\rho}\right) + \gamma^{\alpha\beta}\left[\widetilde{\Gamma}^*_{\sigma\nu,\alpha}\widetilde{\Gamma}^*_{\rho\mu,\beta} - \widetilde{\Gamma}^*_{\rho\nu,\alpha}\widetilde{\Gamma}^*_{\sigma\mu,\beta}\right]\end{aligned} \tag{10.A.18}$$

The space Christoffel symbols are defined in eq. (10.A.13). Since it has all space indices (see Remark 2, Subsection A.6), the curvature tensor (10.A.18) is a space tensor. Then the curvature tensor which is adequate to describe the space geometry of the physical frame Γ is the space part $\widetilde{R}^*_{ijkl}$ of the tensor (10.A.18).

---

[23]In the simple case of translatory motion, a body moves rigidly if, at every moment, it has a Lorentz contraction corresponding to its observed instantaneous velocity, as measured by an inertial observer.

[24]It is worthwhile to notice that, in the literature, there is not common agreement about these definitions, see for instance Landau-Lifshits[58].

In particular, if we deal with flat space-time, since the curvature tensor $R_{\mu\nu\sigma\rho}$ is null, from (10.A.17) we get

$$P_{\Sigma\Sigma\Sigma\Sigma}\left(R_{\mu\nu\sigma\rho}\right) = \widetilde{R}^{*}_{\mu\nu\sigma\rho} - \frac{1}{4}\left(\tilde{\Omega}_{\sigma\mu}\tilde{\Omega}_{\rho\nu} - \tilde{\Omega}_{\rho\mu}\tilde{\Omega}_{\sigma\nu}\right) - \frac{1}{2}\tilde{\Omega}_{\mu\nu}\tilde{\Omega}_{\sigma\rho} = 0 \qquad (10.A.19)$$

Eq. (10.A.19) shows that, in this case, the space components $\widetilde{R}^{*}_{ijkl}$ are completely defined by the terms containing the space vortex tensor, which is related to rotation: hence, the non Euclidean nature of the space of a rotating frame depends only on rotation itself.

A.10 Let us consider two infinitesimally close events in space-time, whose coordinates are x^α and $x^\alpha + dx^\alpha$. We can introduce the following definitions:
"standard relative time"

$$dT = -\frac{1}{c}\gamma_\alpha dx^\alpha \qquad (10.A.20)$$

"standard relative space element"

$$d\sigma^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta \equiv \gamma_{ij} dx^i dx^j \qquad (10.A.21)$$

It is evident that these quantities are strictly dependent on the physical frame defined by the vector field $\boldsymbol{\gamma}$. They have a fundamental role in the standard relative formulation of the kinematics and dynamics of a particle in an inertial or gravitational field. To this end, it is worthwhile to notice that both dT and $d\sigma$ are invariant with respect to the internal gauge transformations (10.A.2). More generally speaking, all laws of relative kinematics and dynamics that we are going to illustrate, are invariant with respect to (10.A.2): in other words, their formulation will depend only on the choice of the congruence Γ, and it will be independent of the (adapted) coordinates chosen to parameterize the physical frame defined by Γ.

By using (10.A.20) and (10.A.21) it is easy to show that the space-time invariant ds^2 can be written in the form

$$ds^2 = d\sigma^2 - c^2 dT^2 \qquad (10.A.22)$$

Let us consider the motion of a point in $\mathcal{M}^4$. The world-line of a material particle is time-like ($ds^2 < 0$), while it is light-like ($ds^2 = 0$) for a photon (or for a generic massless particle) . The following definition applies to a particle P in the physical frame Γ: P is at rest if its world line coincides with one of the lines of the congruence. In other words, $dP \parallel \gamma$ and $dx^i \equiv 0$.

On the contrary, when the world-line of the point P does not coincide with any of the lines of Γ, the point is said to be in motion in the given physical frame. Since $dx^i \neq 0$, we can write the parametric equation of the world-line of P in terms of a parameter λ, $x^\alpha = x^\alpha(\lambda)$; dP is either time-like or light-like and in both cases $dT \neq 0$, so we can express the world-coordinates of the moving particle using the standard relative time as a parameter: $x^\alpha = x^\alpha(T)$.

Remark From the very definition (10.A.20), it is evident that dT represents the proper time measured by an observer at rest ($dx^i = 0$) in Γ.

A.11 Let $v^\alpha = \frac{dx^\alpha}{dT}$ be the relative 4-velocity. We shall call *"standard relative velocity"* its spatial projection

$$\widetilde{v}_\beta \doteq P_\Sigma(v_\beta) = \gamma_{\beta\alpha}\frac{dx^\alpha}{dT} = \gamma_{\beta i}\frac{dx^i}{dT} \qquad (10.A.23)$$

Since $\widetilde{v}_\beta \in \Sigma_p$, then $\widetilde{v}_0 = 0$. The controvariant components of the standard relative velocity are

$$\widetilde{v}^i = \frac{dx^i}{dT} \quad \widetilde{v}^0 = -\gamma_i \frac{\widetilde{v}^i}{\gamma_0} \qquad (10.A.24)$$

(because $\gamma_\alpha \widetilde{v}^\alpha = 0$).

As a consequence, eq. (10.A.23) can be written as

$$\widetilde{v}_i = \gamma_{ij}\widetilde{v}^j \tag{10.A.25}$$

The (space) norm of the standard relative velocity is (see eq. (10.A.7))

$$\| \mathbf{v} \|_\Sigma \doteq \widetilde{v}^2 = \gamma_{ij}\widetilde{v}^i\widetilde{v}^j = \frac{d\sigma^2}{dT^2} \tag{10.A.26}$$

In particular, for a photon, since $ds^2 = 0$, we get from eq. (10.A.22($\widetilde{v}^2 = c^2$, which is the same result that one would expect in the SRT. Dealing with material particles, we can introduce the proper time $d\tau^2 = -\frac{1}{c^2}d\sigma^2$, and, using (10.A.20), we can write

$$\frac{d\sigma^2}{dT^2} = -c^2\frac{d\tau^2}{dT^2} \tag{10.A.27}$$

Taking into account (10.A.26) we obtain

$$\frac{dT}{d\tau} = \frac{1}{\sqrt{1 - \frac{\widetilde{v}^2}{c^2}}} \tag{10.A.28}$$

This relation is formally identical to the one that is valid in the SRT. By using the definitions of standard relative time (10.A.20) and standard relative velocity (10.A.23), it is possible to obtain the following relation between dT and the coordinate time interval $dt = \frac{dx^0}{c}$:

$$dT\left(1 + \gamma_i\frac{\widetilde{v}^i}{c}\right) = -\gamma_0 dt \tag{10.A.29}$$

Summarizing, we have shown that Cattaneo's projection technique, endowed with the standard relative quantities defined before, allows to extend formally the laws of the SRT to any physical reference frame, in presence of gravitational or inertial fields.

Remark In general, the standard relative time that we have introduced is not an exact differential: this means that, in a generic frame Γ we cannot define a unique standard time, or, in other words, the global synchronization of the standards clocks is not possible. In order to have a globally well defined standard relative time, the γ_α must be identified as the partial derivatives of a scalar function f: $\gamma_\alpha = \partial_\alpha f$, and this is possible iff $\Omega_{\alpha\beta} \equiv \partial_\alpha\gamma_\beta - \partial_\beta\gamma_\alpha = 0$, that is when the physical frame is both irrotational (i.e. $\widetilde{\Omega}_{\alpha\beta} = 0$) and geodesic(i.e. $\widetilde{C}_\alpha = 0$).[25]

A.12 The equation of motion of a free mass point is a geodesic of the differential manifold $\mathcal{M}^4$, endowed with the Levi-Civita connection. The connection coefficients, in the coordinates $\{x^\mu\}$ adapted to the physical frame are $\Gamma^\alpha_{\beta\gamma}$. Explicitly, the geodesics equation is written as

$$\frac{DU^\alpha}{d\tau} = 0 \Leftrightarrow \frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}U^\beta U^\gamma = 0 \tag{10.A.30}$$

in terms of the 4-velocity U^α and the proper time τ. Let m_0 be the proper mass of the particle: then the energy-momentum 4-vector is $P^\alpha = m_0 U^\alpha$. We can write the geodesics equation also in the covariant and contravariant forms

$$\frac{DP_\alpha}{d\tau} = 0 \quad \frac{DP^\alpha}{d\tau} = 0 \tag{10.A.31}$$

[25] It is easy to verify that $\Omega_{\alpha\beta} = \widetilde{\Omega}_{\alpha\beta} + \widetilde{C}_\alpha\gamma_\beta - \gamma_\alpha\widetilde{C}_\beta$.

or, equivalently, using the standard relative time

$$\frac{DP_\alpha}{dT} = 0 \quad \frac{DP^\alpha}{dT} = 0 \tag{10.A.32}$$

Now we want to re-formulate the geodesics equation in its relative form, i.e. by means of the standard relative quantities that we have introduced so far. To this end, let us define the *standard relative momentum*

$$\widetilde{p}_\alpha \doteq P_\Sigma\left(P_\alpha\right) = \gamma_{\alpha\beta}P^\beta = m_0\gamma_{\alpha i}\frac{dx^i}{dT}\frac{dT}{d\tau} = m\widetilde{v}_\alpha \tag{10.A.33}$$

where the *standard relative mass*

$$m \doteq \frac{m_0}{\sqrt{1-\frac{\widetilde{v}^2}{c^2}}} \tag{10.A.34}$$

has been introduced, in formal analogy with the SRT. Since $\widetilde{p}_\alpha \in \Sigma_p$, then $\widetilde{p}_0 = 0$.

We can also define the *standard relative energy*

$$E \doteq -c\gamma_\alpha P^\alpha = -m_0 c\gamma_i\frac{dx^i}{d\tau} = m_0c^2\frac{dT}{d\tau} = mc^2 \tag{10.A.35}$$

recovering the well known relation which is used in the SRT. Notice also that

$$P_\Theta\left(P_\alpha\right) = \frac{E}{c}\gamma_\alpha \tag{10.A.36}$$

For a massless particle, like a photon, we can define the energy-momentum 4-vector

$$P^\alpha = \frac{h\nu}{c^2}\frac{dx^\alpha}{dT} \tag{10.A.37}$$

where h is the Planck constant and, in terms of relative quantities the relation that links the wavelength and the frequency of the photon to the velocity of light is $\lambda\nu = \frac{d\sigma}{dT} = c$.

So, for a photon , we can introduce the *standard relative energy*

$$E = -c\gamma^\alpha P_\alpha \doteq h\nu \tag{10.A.38}$$

the *standard relative mass*

$$m \equiv \frac{E}{c^2} = \frac{h\nu}{c^2} \tag{10.A.39}$$

and the *standard relative momentum*

$$\widetilde{p}_\alpha \equiv m\widetilde{v}_\alpha \tag{10.A.40}$$

The equation of motion of a free photon is a null geodesic

$$\frac{DP_\alpha}{dT} = 0 \tag{10.A.41}$$

where the standard relative time has been used to parameterize it.

The spatial projection of the geodesics equations for matter (10.A.32) and light-like particles (10.A.41) is written in the form

$$P_\Sigma\left(\frac{DP_\alpha}{dT}\right) \equiv \frac{\hat{D}\widetilde{p}_\alpha}{dT} - m\widetilde{G}_\alpha = 0 \Leftrightarrow \frac{\hat{D}\widetilde{p}_i}{dT} - m\widetilde{G}_i = 0 \tag{10.A.42}$$

where

$$\frac{\hat{D}\widetilde{p}_i}{dT} \doteq \frac{d\widetilde{p}_i}{dT} - \widetilde{(ij,k)^*}\widetilde{p}^k \frac{dx^j}{dT} \tag{10.A.43}$$

and

$$\widetilde{G}_i = -c^2\widetilde{C}_i + c\widetilde{\Omega}_{ij}\widetilde{v}^j \tag{10.A.44}$$

Hence, we can write the space projection of the geodesics equation in the simple form

$$\frac{\hat{D}\widetilde{p}_i}{dT} = m\widetilde{G}_i \tag{10.A.45}$$

where it is shown that the variation of the spatial momentum vector is determined by the field $\widetilde{G}_i$.

It is often useful to split the field $\widetilde{G}_i$ into the sum of two fields $\widetilde{G'_i}, \widetilde{G''_i}$, defined as follows:

$$\begin{aligned} \widetilde{G'}_i &\doteq -c^2\widetilde{C}_i = -c^2\left(\gamma_0\tilde{\partial}_i\gamma^0 - \partial_0\left(\frac{\gamma_i}{\gamma_0}\right)\right) \\ \widetilde{G''}_i &\doteq c\widetilde{\Omega}_{ij}\widetilde{v}^j \end{aligned} \tag{10.A.46}$$

The field $\widetilde{G'}_i$ can be interpreted as a dragging gravitational-inertial field (c^2C_α is the 4-acceleration a_α of the particle of the reference frame) and the field $\widetilde{G''}_i$ can be interpreted as a Coriolis-like gravitational-inertial field. Actually, starting from the space vortex tensor of the congruence

$$\widetilde{\Omega}_{hk} \doteq \gamma_0\left[\tilde{\partial}_h\left(\frac{\gamma_k}{\gamma_0}\right) - \tilde{\partial}_k\left(\frac{\gamma_h}{\gamma_0}\right)\right] \tag{10.A.47}$$

we can introduce $\widetilde{\boldsymbol{\omega}}\ (x) \in \Sigma_p$, which is the axial 3-vector associated to $\widetilde{\Omega}_{hk}$, by means of the relation

$$\widetilde{\omega}^i \doteq \frac{c}{4}\varepsilon^{ijk}\widetilde{\Omega}_{jk} = \frac{c}{2}\varepsilon^{ijk}\gamma_o\tilde{\partial}_j\left(\frac{\gamma_k}{\gamma_o}\right) \tag{10.A.48}$$

where $\epsilon^{ijk} \doteq \frac{1}{\sqrt{det(\gamma_{ij})}}\delta^{ijk}$ is the Ricci-Levi Civita tensor, defined in terms of the completely antisymmetric permutation symbol δ^{ijk} and of the spatial metric tensor γ_{ij}. As a consequence, we can write $\widetilde{G''_i}$ in the form

$$\widetilde{G''_i} = 2m(\widetilde{\mathbf{v}} \times \widetilde{\boldsymbol{\omega}})_i \tag{10.A.49}$$

which corresponds to a generalized Coriolis-like force. So, the equation of motion (10.A.45) can be written in the form

$$\frac{\hat{D}\widetilde{\mathbf{p}}}{dT} = -m\widetilde{\mathbf{a}} + 2m(\widetilde{\mathbf{v}} \times \widetilde{\boldsymbol{\omega}}) \tag{10.A.50}$$

where $\widetilde{\mathbf{a}}$ is the spatial projection of the 4-acceleration a_α.

From (10.A.50) we see that the relative formulation of the equation of motion of a free particle is identical to the expression of the classical equation of motion of a particle which is acted upon by inertial fields only. Moreover, if m is defined by eq. (10.A.39) the equation of motion (10.A.50) holds also for massless particles.

A.13 Now let us turn back to eq. (10.A.45). We can introduce the "gravito-electric potential" ϕ^G and the "gravito-magnetic potential" $\widetilde{A}_i^G$ defined by

$$\begin{cases} \phi^G \doteq -c^2\gamma^0 \\ \widetilde{A}_i^G \doteq c^2\frac{\gamma_i}{\gamma_0} \end{cases} \tag{10.A.51}$$

As we shall see in a while, these names are justified by the fact that, introducing the "Gravitoelectromagnetic" (GEM) potentials and fields, eq. (10.A.45) can be written as the equation of motion of a particle under the action of a generalized Lorentz force.

In terms of these potentials, the vortex 3-vector $\widetilde{\omega}^i$ is expressed in the form

$$\widetilde{\omega}^i = \frac{1}{2c}\varepsilon^{ijk}\gamma_0\left(\tilde{\partial}_j\widetilde{A}_k^G\right) \tag{10.A.52}$$

and, by introducing the "gravito-magnetic field"

$$\widetilde{B}_G^i \doteq \left(\widetilde{\nabla}\times\widetilde{\mathbf{A}}_G\right)^i \tag{10.A.53}$$

eq. (10.A.52) can be written as

$$\widetilde{\omega}^i = \frac{1}{2c}\gamma_0\widetilde{B}_G^i \tag{10.A.54}$$

As a consequence, the velocity-dependent force (10.A.49) becomes

$$\widetilde{G}_i'' = m\gamma_0\left(\frac{\widetilde{\mathbf{v}}}{c}\times\widetilde{\mathbf{B}}_G\right)_i \tag{10.A.55}$$

Moreover, the dragging term $\widetilde{G}_i'$ (see eq. 10.A.46)

$$\widetilde{G}_i' = -\left(-\widetilde{\partial}_i\phi_G - \partial_0\widetilde{A}_i^G\right) \tag{10.A.56}$$

can be interpreted as a "gravito-electric field":

$$\widetilde{E}_i^G \doteq -\left(-\widetilde{\partial}_i\phi_G - \partial_0\widetilde{A}_i^G\right) \tag{10.A.57}$$

Then, the equation of motion (10.A.45) can be written in the form

$$\frac{\hat{D}\widetilde{p}_i}{dT} = m\widetilde{E}_i^G + m\gamma_0\left(\frac{\widetilde{\mathbf{v}}}{c}\times\widetilde{\mathbf{B}}_G\right)_i \tag{10.A.58}$$

which looks like the equation of motion of a particle acted upon by a "generalized" Lorentz force.

If the particle is not free, its equation of motion is

$$\frac{DP^\alpha}{dT} = F^\alpha \tag{10.A.59}$$

where the external field is described by the 4-vector F^α. The space projection of (10.A.59) then becomes

$$\frac{\hat{D}\widetilde{p}_i}{dT} = m\widetilde{E}_i^G + m\gamma_0\left(\frac{\widetilde{\mathbf{v}}}{c}\times\widetilde{\mathbf{B}}_G\right)_i + \widetilde{F}_i \tag{10.A.60}$$

where the space projection $\widetilde{F}_i$ of the external field has been introduced.

Remark 1 As we have just outlined, the splitting in curved space-time leads to a non-linear analogy with electromagnetism in flat space-time, which is commonly referred to as "Gravitoelectromagnetism"[61]. Namely, the local fields, due to the "inertial forces" felt by the test observers, are associated to Maxwell-like fields: in particular, a gravito-electric field is associated to the local linear acceleration, while a gravito-magnetic field is associated to local angular acceleration (that is, to local rotation).[26]

Remark 2 We want to point out that while the field $\widetilde{G}_i$ is gauge invariant, its components $\widetilde{G}_i''$ and $\widetilde{G}_i''$ are not separately gauge invariant. In other words, the gravito-electric field $\widetilde{E}_i^G$ and gravito-magnetic field $\widetilde{B}_i^G$ are not invariant with respect to gauge transformations (10.A.2).[27]

A.14 In this subsection, a great number of calculations are explicitly carried out. The geometric objects which we deal with, always refer to the physical frame K_{rot}; the lines of the congruence which identifies this physical frame, are described in Subsection 4.2, and the passage from the inertial frame K to the rotating frame K_{rot} is defined by the coordinates transformations $\{x^\mu\} \rightarrow \{x'^\mu\}$

$$\left\{ \begin{array}{lcl} x'^0 & = & ct' = ct \\ x'^1 & = & r' = r \\ x'^2 & = & \vartheta' = \vartheta - \Omega t \\ x'^3 & = & z' = z \end{array} \right. . \tag{10.A.62}$$

However, here and henceforth, for the sake of simplicity, we shall not use primed letters. In particular, all space vectors belong to the (tangent bundle to the) relative space of the disk, which has been defined in Subsection 4.2.

The metric tensor expressed in coordinates adapted to the rotating frame is

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{\Omega^2 r^2}{c^2} & 0 & \frac{\Omega r^2}{c} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\Omega r^2}{c} & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.A.63}$$

and its contravariant components are:

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & \frac{\Omega}{c} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\Omega}{c} & 0 & \frac{1 - \frac{\Omega^2 r^2}{c^2}}{r^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.A.64}$$

The non zero Christoffel symbols turn out to be

$$\begin{array}{lcl} \Gamma_{00}^{1} & = & -\dfrac{\Omega^2}{c^2} r \\ \Gamma_{01}^{2} & = & \dfrac{\Omega}{cr} \\ \Gamma_{02}^{1} & = & -\dfrac{\Omega}{c} r \end{array} \tag{10.A.65}$$

---

[26]This analogy, built in fully non linear GR, in its linear approximation corresponds to the well known analogy between the theory of electromagnetism and the linearized theory of General Relativity[72],[73].

[27]It can be shown that they are invariant with respect to a smaller group of gauge transformations. For instance, they are invariant with respect to

$$x'^0 = ax^0 + b \qquad x'^i = x'^i(x^1, x^2, x^3) \tag{10.A.61}$$

where a, b are constants.

$$\begin{array}{rcl} \Gamma_{12}^{\ 2} & = & \frac{1}{r} \\ \Gamma_{22}^{\ 1} & = & -r \end{array}$$

The non null components of the vector field $\boldsymbol{\gamma}(x)$ are:

$$\left\{ \begin{array}{l} \gamma^0 \doteq \frac{1}{\sqrt{-g_{00}}} = \frac{1}{\sqrt{1-\frac{\Omega^2 r^2}{c^2}}} \\ \gamma_0 \doteq \sqrt{-g_{00}} = \sqrt{1-\frac{\Omega^2 r^2}{c^2}} \\ \gamma_2 \doteq g_{20}\gamma^0 = \frac{\Omega r^2}{c}\frac{1}{\sqrt{1-\frac{\Omega^2 r^2}{c^2}}} \end{array} \right. \tag{10.A.66}$$

and the components of the space metric tensor are turn out to be

$$\gamma_{ij} = g_{ij} - \gamma_i\gamma_j = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{r^2}{1-\frac{\Omega^2 r^2}{c^2}} & 0 \\ 0 & 0 & 1 \end{array} \right) \quad . \tag{10.A.67}$$

The non null components of the space vortex tensor are:

$$\widetilde{\Omega}_{12} = \gamma_0 \tilde{\partial}_1 \left(\frac{\gamma_2}{\gamma_0} \right) = \left(\sqrt{1-\frac{\Omega^2 r^2}{c^2}} \right)^{-3} \frac{2\Omega r}{c} \tag{10.A.68}$$

As a consequence, the rotating frame is not time orthogonal.

Moreover, the spatial Born tensor is null:

$$\widetilde{K}_{ij} \doteq \gamma^0 \partial_0 \gamma_{ij} = 0 \tag{10.A.69}$$

since the space metric (10.A.67) does not depend on the time coordinate. Hence the rotating frame K_{rot} is rigid, in the sense of Born's definition of rigidity (Section A.8).

The covariant components of the Killing tensor of the congruence Γ turn out to be

$$K_{\mu\nu} \equiv \gamma_{\mu;\nu} + \gamma_{\nu;\mu} = \frac{\partial \gamma_\mu}{\partial x^\nu} + \frac{\partial \gamma_\nu}{\partial x^\mu} - 2\Gamma_{\mu\nu}^{\ \alpha}\gamma_\alpha \tag{10.A.70}$$

Taking into account (10.A.65) and (10.A.66), we obtain that the only non null components in $\mathcal{M}^4$ are:

$$K_{01} \equiv \gamma_{0;1} + \gamma_{1;0} = \frac{\partial \gamma_0}{\partial r} - 2\Gamma_{01}^{\ 2}\gamma_2 = \frac{\partial \gamma_0}{\partial r} - g^{2\alpha}\frac{\partial g_{0\alpha}}{\partial r} \tag{10.A.71}$$

$$K_{21} \equiv \gamma_{2;1} + \gamma_{1;2} = \frac{\partial \gamma_2}{\partial r} - 2\Gamma_{21}^{\ 2}\gamma_2 = \frac{\partial \gamma_2}{\partial r} - g^{2\alpha}\frac{\partial g_{2\alpha}}{\partial r} \tag{10.A.72}$$

The components K_{01}, K_{21} depend solely on the partial derivatives with respect to r of some functions of r. If we evaluate these components in $\mathcal{M}^4$, we obtain a non zero result, while if we evaluate the same components on the cylindrical hypersurface $\sigma_r \equiv \{r = const \ (> 0)\}$, they result identically zero. Summing up, we get:

$$K_{01} \equiv \gamma_{0;1} + \gamma_{1;0} = \left\{ \begin{array}{l} -\frac{\Omega^2 r}{c^2} \left(\sqrt{1-\frac{\Omega^2 r^2}{c^2}} \right)^{-1} \neq 0 \ \ in \ \mathcal{M}^4 \\ 0 \ in \ the \ submanifold \ \sigma_r \ (r = const) \end{array} \right. \tag{10.A.73}$$

$$K_{21} \equiv \gamma_{2;1} + \gamma_{1;2} = \begin{cases} -\frac{\Omega^3 r^3}{c^3} \left(\sqrt{1 - \frac{\Omega^2 r^2}{c^2}}\right)^{-3} \neq 0 \quad in\ \mathcal{M}^4 \\ 0\ in\ the\ submanifold\ \sigma_r\ (r = const) \end{cases} \tag{10.A.74}$$

Equations (10.A.73) and (10.A.74) show that the time-like helixes congruence Γ defines a Killing field in the submanifold $\sigma_r \subset \mathcal{M}^4$ but this is not a Killing field in $\mathcal{M}^4$.

We get the same result if we express the Killing tensor using the Born tensor $\widetilde{K}_{\mu\nu}$ and the curvature vectors $\widetilde{C}_\nu$ of the lines of the congruence Γ:

$$K_{\mu\nu} \equiv \gamma_{\mu;\nu} + \gamma_{\nu;\mu} = \widetilde{K}_{\mu\nu} + \gamma_\mu \widetilde{C}_\nu + \widetilde{C}_\mu \gamma_\nu$$

For the rotating disk ($\widetilde{K}_{\mu\nu} = 0$) we simply obtain:

$$K_{\mu\nu} = \gamma_\mu \widetilde{C}_\nu + \widetilde{C}_\mu \gamma_\nu \tag{10.A.75}$$

Equation (10.A.75) is very interesting, because it shows the geometrical meaning of the fact that the Killing tensor is zero in the submanifold $\sigma_r \subset \mathcal{M}^4$, but it is not zero in $\mathcal{M}^4$. In fact, the congruence Γ of time-like helixes is geodesic on σ_r (where $\widetilde{C}_\mu = 0$), but of course not in $\mathcal{M}^4$,[28] where the curvature vector $\widetilde{C}_\mu = \gamma^\alpha \nabla_\alpha \gamma_\mu$ has the following non-null component:

$$\widetilde{C}_1 = \gamma^0 \frac{\frac{\Omega^2 r}{c^2}}{\sqrt{1 - \frac{\Omega^2 r^2}{c^2}}} = -\frac{\Omega^2 r}{c^2 - \Omega^2 r^2} \tag{10.A.76}$$

As a consequence, the Killing tensor has the following non-null components:

$$K_{01} = \gamma_0 \widetilde{C}_1 = -\frac{\Omega^2 r}{c^2} \left(\sqrt{1 - \frac{\Omega^2 r^2}{c^2}}\right)^{-1} \tag{10.A.77}$$

$$K_{21} = \gamma_2 \widetilde{C}_1 = -\frac{\Omega^3 r^3}{c^3} \left(\sqrt{1 - \frac{\Omega^2 r^2}{c^2}}\right)^{-3} \tag{10.A.78}$$

Equations (10.A.77) and (10.A.78) are in agreement, respectively, with equations (10.A.73) and (10.A.74).

The only non zero components of the curvature tensor of space of the disk are

$$\widetilde{R}^*_{1212} = -3 \frac{\left(\frac{\Omega r}{c}\right)^2}{\left(1 - \left(\frac{\Omega r}{c}\right)^2\right)^3} \tag{10.A.79}$$

and those which are obtained by the symmetries of the indices.

The non null components of the space projection of the Ricci tensor, are:

$$\begin{aligned} \widetilde{R}^*_{11} &= -3 \frac{\Omega^2}{c^2} \left(1 - \frac{\Omega^2 r^2}{c^2}\right)^{-2} \\ \widetilde{R}^*_{22} &= -3\, \frac{\Omega^2}{c^2} \left(1 - \frac{\Omega^2 r^2}{c^2}\right)^{-3} \end{aligned} \tag{10.A.80}$$

[28] Apart from the degenerate case $r = 0$, which corresponds to a straight line in $\mathcal{M}^4$.

Finally the curvature scalar is:

$$\widetilde{R}^* = -6\,\frac{\Omega^2}{c^2}\left(1-\frac{\Omega^2 r^2}{c^2}\right)^{-2} \tag{10.A.81}$$

The calculation of the curvature of the space of the rotating disk, which turns out to be non Euclidean (in particular hyperbolic), confirms Einstein's early intuition[74] about the relations between curvature and rotation.[29]

References

[1] Leibniz G., *Leibniz: Selections*, edited by WienerP.P. Charles Scribner's Sons, New York (1951)

[2] Berkeley G., *Berkeley's Philosophical Writings*, edited by Armstrong D.M., Collier-MacMillan, New York (1965)

[3] Mach E., *The Science of Mechanics*, Open Court, La Salle (1960)

[4] Ehrenfest P., *Phys. Zeits.*, **10**, 918 (1909)

[5] Sagnac M.G., *C.R. Acad. Sci. Paris*, **157**, 708,1410 (1913)

[6] Rizzi G. and Ruggiero M.L., *Found. Phys.*, **32** 1525 (2002), gr-qc/0207104

[7] Sakurai J.J.,*Phys. Rev. D*, **21**, 2993 (1980)

[8] Aharonov Y. and Bohm D., *Phys. Rev.*, **115**, 485 (1959)

[9] Cattaneo C., *Introduzione alla teoria einsteiniana della gravitazione*, Veschi, Roma (1961)

[10] Cattaneo C., *Il Nuovo Cimento,* **10**, 318 (1958)

[11] Cattaneo C., *Il Nuovo Cimento,* **11**, 733 (1959)

[12] Cattaneo C., *Il Nuovo Cimento,* **13**, 237 (1959)

[13] Cattaneo C., *Rend. Acc. Lincei*, **27**, 54 (1959)

[14] Lodge O.J., *Philos. Trans. R. Soc. London*, **184**, 727 (1893)

[15] Lodge O.J., *Philos. Trans. R. Soc. London*, **189**, 149 (1897)

[16] Anderson R., Bilger H.R. and Stedman G.E., *Am. J. Phys.*, **62**, 975 (1994)

[17] Sagnac M.G.,*C.R. Acad. Sci. Paris*, **141**, 1220 (1905)

[18] Selleri F., *Found. Phys.*, **26**, 641 (1996)

[19] Selleri F., *Found. Phys. Lett.*, **10**, 73 (1997)

[20] Croca J. and Selleri F., *Nuovo Cimento B*, **114**, 447 (1999)

[21] Goy F. and Selleri F., *Found. Phys. Lett.*, **10**, 17 (1997)

[22] Vigier J.P. , *Phys. Lett. A*, **234**, 75 (1997)

[23] Anastasowksi P.K. *et al.*, *Found. Phys. Lett*, **12**, 579 (1999)

[24] Klauber R.D., *Found. Phys. Lett.*, **11** (5), 405 (1998)

[29] See, for instance, Stachel[75].

[25] Klauber R.D., in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)

[26] Weber T.A., *Am. J. Phys*, **65**, 946 (1997)

[27] Dieks D., *Eur. J. Phys.*, **12**, 253 (1991)

[28] Anandan J., *Phys. Rev. D*, **24**, 338 (1981)

[29] Rizzi G. and Tartaglia A., *Found. Phys.*, **28** 1663 (1998)

[30] Bergia S. and Guidone M., *Found. Phys. Lett.*, **11**, 549 (1998)

[31] Rodrigues W.A. Jr. and Sharif M., *Found. Phys.*, **31**, 1767 (2001)

[32] Pascual-Sanchez F. *et al.*, in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)

[33] Harres F., Ph.D. Thesis, University of Jena, Germany (1912)

[34] Harzer P., *Astron. Nachr.*, **199**, 377 (1914)

[35] Michelson A.A., *Philos. Mag.*, **8**, 716 (1904)

[36] Michelson A.A., Gale H.G., *Astrophys. J.*, **61**, 137 (1925)

[37] Post E.J., *Rev. Mod. Phys.*, **39** (2), 475 (1967)

[38] Chow W.W. *et al.*, *Rev. Mod. Phys.*, **57**, 61 (1985)

[39] Vali V. and Shorthill R.W., *Appl. Opt.*, **15**, 1099 (1976)

[40] Stedman G.E., *Rep. Prog. Phys.*, **60**, 615 (1997)

[41] Zimmermann J.E. and Mercerau J.E., *Phys. Rev. Lett.*, **14**, 887 (1965)

[42] Atwood D.K. *et al.*, *Phys. Rev. Lett.*, **52**, 1673 (1984)

[43] Riehle F. *et al.*, *Phys. Rev. Lett.*, **67**, 177 (1991)

[44] Hasselbach F. and Nicklaus M., *Phys. Rev. A*, **48**, 143 (1993)

[45] Werner S.A. *et al.*, *Phys. Rev. Lett.*, **42**, 1103 (1979)

[46] Ashby N., in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)

[47] Diecks D., Nienhuis G., *Am. J. Phys.* **58**, 650 (1990)

[48] Tartaglia A. in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)

[49] Rizzi G, Serafini A., in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)

[50] Schlegel R., *Am. J. Phys.* **42**, 183 (1974)

[51] Malykin G.B., *Phys.-Uspekhi* **43**, 1229 (2000)

[52] Peres A., *Phys. Rev. D*, **18**, 2173 (1978)

[53] Rizzi G., Ruggiero M.L., *Gen. Rel. Grav.* **35**, 1743 (2003), gr-qc/0305046

[54] Tourrenc P., *Phys. Rev. D*, **16**, 3421 (1977)

[55] Hehl F.W. and Ni W.T., *Phys. Rev. D*, **42**, 2045 (1990)
[56] Mashhoon B., *Phys. Rev. Lett.*, **61**, 2639 (1988)
[57] Papini G., in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)
[58] Landau L.D. and Lifshitz E.M., *The Classical Theory of Fields*, Pergamon Press, New York (1971)
[59] Møller C., *The Theory of Relativity*, Oxford University Press, Oxford (1972)
[60] Tartaglia A., *Found. Phys. Lett.*, **12**, 17 (1999)
[61] Jantzen R.T., Carini P., Bini D., in *Proceedings of the Seventh Marcel Grossmann Meeting (Stanford 1994)*, eds. Jantzen R.T and Keiser G.M., World Scientific, Singapore (1996), gr-qc/0105096
[62] Jantzen R.T., Carini P., Bini D., *Annals of Physics*, **215**, 1 (1992) gr-qc/0106043
[63] Bini D.. Jantzen R.T., *Il Nuovo Cimento B*, **115**, 713 (2000)
[64] Arnowitt, R., Deser, S. and Misner, C.W., in *Gravitation: An Introduction to Current Research*, ed. Witten L., Wiley, New York (1962)
[65] Misner C.W., Thorne K.S., Wheeler J.A., *Gravitation*, Freeman, S. Francisco (1973)
[66] Zel'manov, A.L., *Dokl. Akad. Nauk. USSR*, **107**, 805 (1956), English translation in *Sov. Phys. Doklady*, **1**, 227 (1956)
[67] Grøn Ø., *Found. Phys.*, **9**, 353 (1979)
[68] Born M., *Ann. Phys. Lpz.*, **30**, 1 (1909)
[69] Rosen N., *Phys. Rev.*, **71**, 54 (1947)
[70] Boyer R.H., *Proc. Roy. Soc.*, **283**, 343 (1965)
[71] Pauli W., *Theory of Relativity*, Pergamon Press, New York (1958)
[72] Mashhoon B., Gronwald F., Lichtenegger H.I.M., *Lect. Notes Phys.*, **562**, (2001) 83, gr-qc/9912027
[73] Ruggiero M.L., Tartaglia A., *Il Nuovo Cimento B*, **117**, 743 (2002), gr-qc/0207065
[74] Einstein A., *The Meaning of Relativity*, Princeton University Press, Princeton N.J. (1950)
[75] Stachel J., in *General Relativity and Gravitation*, ed. Held A., Plenum Press, New York and London (1980)

Chapter 11

INERTIAL FORCES: THE SPECIAL RELATIVISTIC ASSESSMENT

Donato Bini
Istituto per le Applicazioni del Calcolo "M. Picone", C.N.R., I–00161 Roma, Italy
International Center for Relativistic Astrophysics (ICRA), University of Rome, I–00185 Rome, Italy
binid@icra.it

Robert T. Jantzen
Department of Mathematical Sciences, Villanova University, Villanova, PA 19085, USA
International Center for Relativistic Astrophysics (ICRA), University of Rome, I–00185 Rome, Italy
robert.jantzen@villanova.edu

Abstract Rotating observers and circular test particle orbits in Minkowski spacetime are used to illustrate the transport laws and derivative operators needed to define the various "inertial forces" one can introduce using the natural relative observer approach to describing spacetime. Various centripetal accelerations (often called centrifugal forces when multiplied by the mass) are evaluated and compared with the familiar value ν^2/r of nonrelativistic physics.

1. Introduction

Since the rotational aspects of general relativity are where a clear departure from Newtonian gravitation takes place, they have been a continued source of fascination for relativists since the birth of Einstein's theory of gravity in 1916. While Gödel's surprising cosmological solution [1] of 1949 stimulated thinking about rotation and gravitational fields a half century ago, followed by the Kerr black hole solution [2] about 15 years later in 1963, even flat Minkowski spacetime has been a source of controversy since before general relativity ever arrived on the scene regarding rotating frames, starting with Ehrenfest's para-

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 221-239.

dox of 1909 [3]. A recent summary of this history can be found in Rizzi and Ruggiero [4] and Grøn has provided a detailed historical account in this volume [5]. Apart from some outright errors, most of the controversy over the years is due to ambiguity of the questions posed or the quantities being measured, or a misunderstanding of the local Lorentz structure of Lorentzian manifolds.

An early edition of Landau and Lifshitz in the 1940s [6] was the first easily identified introduction of tools for examining curved spacetimes in terms of quantities that can be identified with local rest space and temporal measurements that correspond with our usual space plus time interpretation of kinematics in pre-relativistic and special relativistic mechanics. This (1+3) formalism, based on a family of timelike observers, was developed in the 1950s by, among others, Cattaneo, Zel'manov and Möller but was later eclipsed by the newer (3+1) version in the 1960s based on spacelike hypersurfaces that quickly found its way into the relativity community through the ADM approach to gravitation, then nicely promoted by the classic Misner, Thorne and Wheeler text *Gravitation* [7] in the 1970s and beyond.

In the second half of the 1980s, the old timelike observer based formalism was reawakened by a rediscovery of some known but not really exploited properties of static spacetimes by Abramowicz and collaborators (in an article whose citations link to one earlier article by de Felice, which in turn refers back to Landau and Lifshitz and Cattaneo), who proceeded to generate a large number of papers developing the seeds from that idea (optical geometry). Thus the industry of examining inertial forces in general relativity was born, centered primarily on circular orbits, the natural starting point for theoretical experimentation. This involved a further refinement of the 4 = 1+3 or 4 = 3+1 orthogonal splitting of the spacetime tangent space by 3 = 1+2 (parallel and perpendicular to the direction of relative motion in the local rest space). De Felice, after some scattered preliminary papers, re-entered this industry in the late 1980s, when Jantzen and Carini, later joined by Bini, stumbled into this topic, soon after joined by Rindler and Perlick, Iyer and Vishweshwara, and Semerak among others in the 1990s. This history is summarized in a review article [8] where relevant references can be found.

All of this discussion is interpretational and any value it may have lies only in helping us understand how properties of given spacetimes are either like or unlike pre-relativistic or special relativistic ideas about kinematics and gravitational field behavior that we may have. In fact, some of these constructions do help us see the ways in which the spacetime geometry affects particle motion and gyro transport along particle trajectories, through its spatial geometry and gravitoelectric and gravitomagnetic fields that the splitting formalisms naturally define. Gravitoelectromagnetism summarizes these splitting methods which bring a natural analogy with electromagnetism into the mix [9, 10, 11]. While not wanting to overstate their importance, studying these questions has

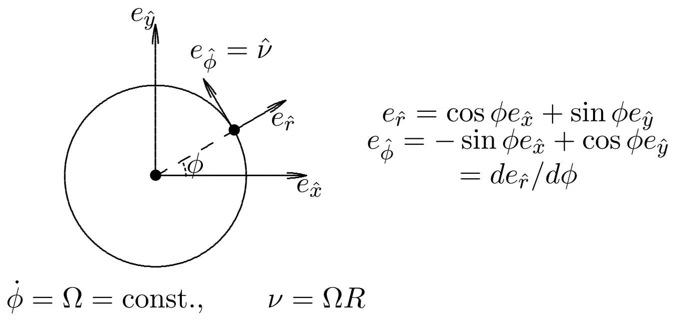

Figure 11.1. Circular motion in classical mechanics.

given us a handle on trying to make some sense of how global and local rotation is both similar and different but certainly much richer in general relativity compared to its predecessor theories.

The problem of introducing a relativistically correct definition of inertial forces in relativity has been approached from various directions. Abramowicz and coworkers introduced “inductive" definitions, starting from the special situation of a circular orbit around a black hole, while others starting with Landau and Lifshitz have given “deductive" definitions, starting from spacetime splitting techniques employed in at least stationary spacetimes. The former, which generalize the classical notion of inertial forces in certain ways based on the special circumstances, are somewhat unsatisfactory in that they fail to generalize further to more general situations. The optical geometry, while powerful for static spacetimes, is not very useful even for stationary spacetimes because of the disconnect between the optical geometry and null geodesics. On the other hand the most general setting in which the latter definitions arise require a great deal of effort introducing the appropriate formalism which in turn may overshadow the physical content of general formulas, i.e. “sufficient generality" requires a high price too. We outline the problem here and consider applications to circular orbits in Minkowski spacetime.

2. Inertial forces in classical mechanics

Inertial forces first enter our educational lives in the context of circular orbits in a central force field, so this is the natural starting point for re-examining them in a more general context. Consider the classical example of a particle in uniform circular motion in the x-y plane with angular velocity Ω along the positive z-axis (counter-clockwise motion, for example, as indicated in Figure 11.1). The orbit of the particle can be parametrized by the spatial arclength ℓ

$$\ell = R\phi = R\Omega t\,, \tag{11.1}$$

so that its Cartesian coordinate representation is

$$(x, y) = (R\cos(\ell/R), R\sin(\ell/R)) \ . \tag{11.2}$$

The intrinsic (Frenet-Serret) frame consists of the vectors (unit tangent, normal, binormal)

$$\mathbf{t} = e_{\hat\phi} = \hat\nu, \qquad \mathbf{n} = -e_{\hat r}, \qquad \mathbf{b} = e_{\hat\phi} \times (-e_{\hat r}) = e_{\hat z} \tag{11.3}$$

satisfying

$$\frac{d}{d\ell}[\mathbf{t}, \mathbf{n}, \mathbf{b}] = [\kappa\mathbf{n}, -\kappa\mathbf{t} + \tau\mathbf{b}, -\tau\mathbf{n}] = [\boldsymbol{\omega} \times \mathbf{t}, \boldsymbol{\omega} \times \mathbf{n}, \boldsymbol{\omega} \times \mathbf{b}] \tag{11.4}$$

where

$$\boldsymbol{\omega} = \kappa\mathbf{b} + \tau\mathbf{t} = \kappa e_{\hat z} \ , \tag{11.5}$$

and the curvature $\kappa = 1/R$ and torsion $\tau = 0$ values easily follow from the relations

$$\frac{de_{\hat\phi}}{d\ell} = \frac{1}{R}\frac{de_{\hat\phi}}{d\phi} = -\frac{1}{R}e_{\hat r} = \kappa\mathbf{n}, \qquad \frac{de_{\hat r}}{d\ell} = \frac{1}{R}\frac{de_{\hat r}}{d\phi} = \frac{1}{R}e_{\hat\phi} = \kappa\mathbf{t} \ . \tag{11.6}$$

The torsion describes the rotation of the frame vectors in the plane orthogonal to the direction of motion, while the curvature describes the rotation of the direction of motion itself. Its reciprocal, the radius of curvature, in this case is just the radius of the circle.

For a particle of mass m in this circular orbit, the classical notion of centripetal force ($=-$ centrifugal force) is

$$F^{(C)} = -\frac{m\nu^2}{R}e_{\hat r} = -m\kappa\nu^2 e_{\hat r} = m\nu^2\frac{de_{\hat\phi}}{d\ell} = m\nu^2\frac{d\hat\nu}{d\ell} \ , \tag{11.7}$$

which must be the value of the force which is responsible for keeping the particle in this orbit. Through the force law it must equal the mass times the constant inward radial (centripetal) acceleration which characterizes the circular motion. Alternatively, from the point of view of the rest frame of the particle, one can conveniently think of a “balancing" of this inward force by an equal but outward centrifugal force obtained by just reversing the sign of the centripetal force and putting it on the other side of the force equation.

This single circular orbit discussion is only the first step in introducing inertial forces. There are two distinct directions in which one can proceed initially. If one considers a rigidly rotating frame with the same angular velocity, then the points fixed in the rotating grid all undergo such circular motion. If one then considers the path of a particle in arbitrary motion, and describes its motion with respect to the rotating frame, then not only does the fictitious centrifugal force of the rotating frame prove useful, but the equivalent Coriolis force arising from its motion relative to the frame itself now plays a role. Alternatively, one can generalize the single circular orbit to an arbitrary trajectory and use the Frenet-Serret machinery to decompose its acceleration into a linear component along the direction of motion and a centripetal component along the normal direction, all in a nonrotating frame. The radius of curvature $1/\kappa$ then takes the place of the circular radius R in the above discussion. In this case one only has a centripetal force responsible for the transverse acceleration (although traditionally the term centripetal is restricted to the case of purely transverse acceleration).

Finally one can combine the two discussions by performing a Frenet-Serret analysis of the trajectory in the rotating frame. One then has the combined effects of the centrifugal and Coriolis forces associated with the noninertial observer motion and the centripetal force associated with the curvature of the trajectory as seen by those observers. No one ever considers such a description in Minkowski spacetime, but if one wants to make some serious generalization to curved spacetime, it pays to think about it. One must also see how to transfer the differential properties of the description to the spacetime tangent space with no underlying flat spacetime that glues them together in the same way that the space-fixed axes in the Euclidean vector space structure may be identified with a covariant constant orthonormal frame field on the corresponding flat Riemannian 3-manifold, then extended to a flat spacetime by taking the (orthogonal) cross-product with a time line. This preferred global inertial mathematical structure is not available in a general spacetime and is not geometrically relevant to the description with respect to a family of rotating observers.

Interpreting Euclidean space with a choice of origin as a vector space, let $\mathbf{e}_i$ be an orthonormal basis which is rigidly rotating with constant angular velocity $\mathbf{\Omega}$ and let $\dot{f} = df/dt$ be the time derivative, so that $\dot{\mathbf{e}}_i = \mathbf{\Omega} \times \mathbf{e}_i$. Then the time derivative of the position vector

$$(x^i \mathbf{e}_i)^{\cdot} = [\dot{x}^i + V^i]\mathbf{e}_i \tag{11.8}$$

picks up an extra term due to the velocity field $\mathbf{V} = \mathbf{\Omega} \times \mathbf{x}$ of the observers fixed in the rotating frame. The second time derivative picks up two terms

$$(x^i \mathbf{e}_i)^{\cdot\cdot} = [\ddot{x}^i + A^i + (2\mathbf{\Omega} \times \dot{x}^j \mathbf{e}_j)^i]\mathbf{e}_i \,, \tag{11.9}$$

first the acceleration

$$\mathbf{A} = \mathbf{\Omega} \times \mathbf{V} = \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) = \mathbf{\nabla}[-\mathbf{V} \cdot \mathbf{V}/2] \equiv -\mathbf{g} \tag{11.10}$$

of the rigidly rotating observers and second a term arising from the curl of their velocity field

$$\mathbf{H} = 2\mathbf{\Omega} = \mathbf{\nabla} \times \mathbf{V} \,. \tag{11.11}$$

The equations of motion of a particle under the influence of a force $\mathbf{F}$ then take the form

$$m[\ddot{x}^i + A^i + (\mathbf{H} \times \dot{x}^j \mathbf{e}_j)^i] = F^i \ . \qquad (11.12)$$

Simply transferring the extra terms from the left hand side to the right hand side of the equation

$$m\ddot{x}^i = m[g^i + (\dot{x}^j \mathbf{e}_j \times \mathbf{H})^i] + F^i \qquad (11.13)$$

leads to their interpretation as equivalent to new force terms, referred to as fictitious or inertial or pseudo- forces and which are due to the noninertial motion of the rotating observers: the centrifugal force $m\mathbf{g}$ and Coriolis force $m(\dot{x}^j \mathbf{e}_j \times \mathbf{H})$ entering through the acceleration and curl of the velocity field of these observers. The analogy with the Lorentz force law of electromagnetism in a nonrotating system of coordinates is obvious, suggesting the terminology of a gravitoelectric field $\mathbf{g}$ and a gravitomagnetic field $\mathbf{H}$ in this noncovariant flat spacetime discussion, each arising respectively from a scalar and vector potential associated with the observer velocity field.

However, to promote this to a discussion which makes sense in the context of the spacetime splitting of the tangent space to Minkowski spacetime associated with both the rotating and nonrotating observers, one must take into account the fact that their local rest spaces do not coincide and the time and space derivatives should be geometrized. This should be done so that one can apply the results to any family of observers moving arbitrarily (but smoothly) in space at less than the speed of light. (An observer horizon limits the validity of this description where the observer velocity field reaches the speed of light.) The choice of spatial geometry used to describe the flat spacetime situation is also important if one is to generalize to a nonflat scenario. In fact one would need to do this in order to unambiguously introduce a Frenet-Serret description of the left hand side of (11.13) relative to the rotating frame as well as interpret geometrically the operations defining the inertial forces on the right hand side. On the other hand, setting the angular velocity of the observers to zero, centrifugal effects would then be confined to the centripetal acceleration term of the left hand side.

Abramowicz refers to the centrifugal force associated with the noninertial observer in circular motion as Newton's definition and that associated with the Frenet-Serret decomposition as Huygens' definition [12], and chooses the latter (really a centripetal force) as the appropriate one to call centrifugal force in relativity. In reality both aspects are present, and one can smoothly interpolate between both descriptions of the situation in Minkowski spacetime by studying a circular orbit with angular velocity Ω about the z-axis from the point of view of a family of rotating coordinate systems with angular velocity varying from zero to the given value. The endpoint values were conveniently used by Rindler and Perlick [13] to calculate the precession of a spin vector under Fermi-Walker transport around an accelerated circular orbit in Minkowski spacetime as well as around geodesic circular orbits in the Schwarzschild, Kerr and Gödel spacetimes.

Although formally one can continue to use the Euclidean geometry in representing the differential equations of motion of a particle in the rotating frame, this geometry is not directly measurable by the rotating observers. This is where much confusion arises in deciding whether the circumference of a circle in the rotating frame is Lorentz contracted or not. The splitting of the tangent space into a local rest space and local time direction for the rotating observer family is directly connected with a local spatial measurement process. Spatial distances in the local rest space of a rotating observer may be interpreted as describing the separation of (infinitesimally) nearby observers in the family (identified with points in the tangent space) as determined by halving the light travel time between them and the given observer. This was carefully derived and presented by Landau and Lifshitz in their section discussing distances and time intervals in general relativity [6], and is the basis for considering such splitting schemes in the first place. There is no need to consider measuring the circumference of a circle at "the same time"—simultaneity in fact does not exist. Instead, nearby observers around the circle can determine their relative separations by exchanging light signals, from which the total circumference can then be calculated (using a limiting polygonal approximation construction, as in the Taylor-Wheeler discussion of Thomas precession [14]).

A simple figure helps explain the result one must find and is necessary to keep from identifying ruler lengths and nearby observer separations. In fact the paradoxes of special relativity are usually best explained with a spacetime diagram and the present case is no exception. Figure 11.2 shows 1) the world lines of the ends of a ruler of fixed length $r\Delta\phi$ separating two infinitesimally separated space-fixed observers separated by an angle $\Delta\phi$ on a circle of radius r, 2) the world lines of two rotating observers with the same angular separation, and 3) the world lines of the ends of a ruler of the same fixed length carried by the rotating observers (so that OC is boosted from OA). The rotating ruler appears Lorentz contracted as seen by the nonrotating observers: $||OE|| = ||OC||/\gamma$, but the proper separation between the two rotating observers in their own rest space is instead Lorentz expanded: $||OD|| = \gamma||OC||$, namely tangential lengths expand by exactly the gamma factor of the spatial metric expressed in the rotating coordinate system. The result of the calculation of the total circumference of the circle by the rotating observers is therefore Lorentz expanded, since the local relative distances are Lorentz expanded. The stationary situation saves us from having to think further about the question of time in this indirect measurement.

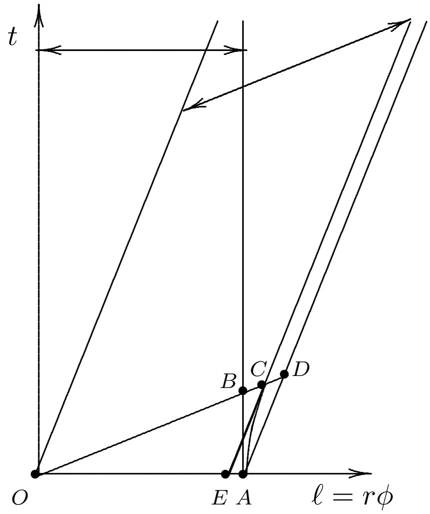

Figure 11.2. The boost between nonrotating and rotating rulers and the Lorentz contraction and expansion of nearby observer separations. The world sheets of the nonrotating ruler and rotating ruler of the same length $r\Delta\phi$ oriented along the tangential direction of the circle (hence related by the boost AC) are indicated by the double arrows. OE is the contracted rotating ruler as seen by the nonrotating observer, while OB is the contracted nonrotating ruler as seen by the rotating observer. The nearby rotating observers separated by the same angle $\Delta\phi$ marking the ends of the nonrotating ruler OA appear to be separated by the length of the nonrotating ruler as seen by the nonrotating observers, but by the Lorentz expanded length OD as seen by those rotating observers themselves. Thus while rulers contract, the relative distance between the corresponding observers expands.

To summarize, centripetal and centrifugal forces are closely associated notions in Newtonian physics, because of its universal time shared by the reference frame and the rotating particle, but arise in two different contexts: analyzing rotational motion as seen by a nonrotating observer family (centripetal force) or as seen by a rotating observer family (centrifugal and Coriolis forces). The connection between centripetal and centrifugal force is made only when the rotating observer family is corotating with a circularly orbiting particle, so in a more general context of relative motion of the particle and the rotating observers, this direct connection is lost.

In special and general relativity, the local rest space of the particle is distinct from that of the observer, each with its own local direction of time, so centripetal and centrifugal forces must differ even in the corotating case, and one must re-examine the time derivative itself. One must also keep in mind that the rotating frame description really involves an observer family and and a particle world line and their relative velocity along that world line. Implicit in the discussion is the family of observers fixed in space, simply following the time lines in the associated inertial coordinates on Minkowski spacetime.

These space-fixed observers, which are “nonrotating" in a number of senses (individually as world lines, and locally and globally as families), anchor the rotating frame to the inertial properties of that spacetime. Any generalization or geometrization of centrifugal forces will indirectly involve these inertial observers through the choice of local rest space frame along the given world line with respect to which the change of a spatial vector is measured.

To illustrate some of these problems without getting lost in formalism and semantics, it is useful to discuss two simple examples: 1) an accelerated particle in circular motion with angular velocity Ω as seen by the nonrotating observers and by the corotating observers, and 2) a particle fixed in space with respect to the nonrotating observers, seen as moving in a circular orbit by a family of rotating observers having angular velocity Ω. Although one can then formally discuss each of these two situations from the point of view of the particle in its own local rest space in terms of comoving relative Frenet-Serret quantities, for circularly orbiting observers and particles where the relative acceleration is transverse to the direction of relative motion, all interesting accelerations belong to the intersection of the two local rest spaces and nothing new is seen.

How does the family of rotating observers view the test particle trajectory? One must first parametrize the given world line by a measurable parameter. The natural proper time parametrization as measured by the test particle can be converted by the Lorentz dilation gamma factor to correspond to the local observer time of the observers. The observers along the world line measure the relative speed, which can then be used to convert to a spatial arc length parametrization as seen by the observers, which is needed to establish a correspondence with the nonrelativistic Frenet-Serret discussion. But then how does the observer family measure the change in the relative velocity vector in order to express the relative acceleration and force law? Now one needs a fiducial orthonormal spatial frame along the world line (equivalently a way of transporting such a frame) that defines what it means not to change. For the rotating observers, which follow Killing trajectories, there are two choices of transport along the world line which preserve the observer local rest space: spatially projected parallel transport and spatially projected Lie transport (with respect to the observers). These collapse to the same transport with respect to the nonrotating observers.

3. Inertial forces geometrized

To go into the details [9, 10, 11, 15, 16], one must first quantify the relative observer description by introducing definitions for the (relativistic) relative Frenet-Serret analysis. Let the test particle have 4-velocity U and consider a family of test observers, i.e. a congruence of timelike world lines, with 4-

velocity u. There arise two relative velocity unit vector fields belonging to the local rest space (LRS) of U and u: $\hat\nu(U,u)\in LRS_u$ and $\hat\nu(u,U)\in LRS_U$, related by a boost, and defined **only** along the particle world line U

$$\begin{aligned} U &= \gamma(U,u)[u+||\nu(U,u)||\hat\nu(U,u)]\ ,\\ u &= \gamma(u,U)[U+||\nu(u,U)||\hat\nu(u,U)]\ , \end{aligned} \tag{11.14}$$

where

$$\begin{aligned} &||\nu(U,u)|| = ||\nu(u,U)|| \equiv \nu\ ,\\ &\gamma(U,u) = [1-||\nu(U,u)||^2]^{-1/2} = \gamma(u,U)\equiv\gamma \end{aligned} \tag{11.15}$$

are the magnitude of the relative velocity and the Lorentz factor respectively. The test particle world line can be parameterized by its proper time τ_U, or by the relative observer proper time or proper length

$$d\tau_U = \gamma^{-1}d\tau_{(U,u)} = (\gamma\nu)^{-1}d\ell_{(U,u)}\ . \tag{11.16}$$

Studying the evolution along of the direction of relative motion $\hat\nu(U,u)$ along the particle world line as measured by its covariant derivative along U and projecting it onto the local rest space of the observer u using the spatial projection operator $P(u)$ (in components: $P(u)^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta$), one has the Fermi-Walker temporal derivative (spatial with respect to u in the sense that derivatives of spatial fields remain spatial with respect to u)

$$\frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}}\hat\nu(U,u) = (\gamma\nu)^{-1}P(u)\frac{D}{d\tau_U}\hat\nu(U,u) = k_{(\mathrm{fw},U,u)}\hat\eta_{(\mathrm{fw},U,u)} \tag{11.17}$$

which naturally defines the “relative centripetal force" in the context of the acceleration evaluation in analogy with (11.7), where the last equality here is just the decomposition of the preceding expression into its direction (unit vector) and magnitude (Fermi-Walker relative curvature). Similarly one can introduce a variation of this derivative in which the temporal part corresponds to the spatially projected Lie derivative along the observer congruence [9]

$$\begin{aligned} \frac{D_{(\mathrm{lie},U,u)}}{d\ell_{(U,u)}}\hat\nu(U,u) &= \frac{D_{(\mathrm{fw},U,u)}}{d\ell_{(U,u)}}\hat\nu(U,u) + \nu^{-1}\hat\nu(U,u)\times\omega(u)\\ &= k_{(\mathrm{lie},U,u)}\hat\eta_{(\mathrm{lie},U,u)}\ , \end{aligned} \tag{11.18}$$

where $\omega(u)$ is the vorticity vector of u, thus defining the Lie relative curvature.

Studying instead the evolution along the particle world line of $-\hat\nu(u,U)$ (the relative velocity of U with respect to u, as seen by U) and projecting its covariant derivative along U into the local rest space of the particle itself using the projection operator $P(U)$ (namely $P(U)^\alpha{}_\beta = \delta^\alpha{}_\beta + U^\alpha U_\beta$), one has the Fermi-Walker temporal derivative (spatial with respect to U)

$$\frac{D_{(\text{fw},U)}}{d\ell_{(U,u)}}[-\hat\nu(u,U)] = P(U)\frac{D}{d\ell_{(U,u)}}[-\hat\nu(U,u)] = \mathcal{K}_{(\text{fw},u,U)}\hat{\mathcal{N}}_{(\text{fw},u,U)} \tag{11.19}$$

which naturally defines the "relative centrifugal force" in the context of the acceleration evaluation.

While the practical meaning of these definitions may not be obvious, this seems to be the natural way of geometrizing inertial forces in relativity. In Eqs. (11.17) and (11.19), $k_{(\text{fw},U,u)}$ and $\mathcal{K}_{(\text{fw},u,U)}$ are respectively the Fermi-Walker relative curvatures of the relative motion as seen by u and U respectively. The vanishing of a Fermi-Walker relative curvature defines the notion of *Fermi-Walker relatively straight* curves, while the inverse of the Fermi-Walker relative curvature defines the Fermi-Walker relative curvature radius

$$\begin{aligned} R_{(\text{fw},U,u)} &= ||k_{(\text{fw},U,u)}||^{-1}, \\ \mathcal{R}_{(\text{fw},U,u)} &= ||\mathcal{K}_{(\text{fw},u,U)}||^{-1}. \end{aligned} \tag{11.20}$$

Each of the relative velocity unit vector fields $\hat\nu(U,u)$ and $-\hat\nu(u,U)$ can be used to introduce a "relative Frenet-Serret" frame and scalars (with the option of using either the Fermi-Walker or Lie relative derivative: let $\text{tem} = \text{fw}, \text{lie}$) to define spatial frames

$$(\hat\nu(U,u), \hat\eta_{(\text{fw},U,u)}, \hat\beta_{(\text{fw},U,u)})$$

in LRS_u and

$$(-\hat\nu(u,U), \hat{\mathcal{N}}_{(\text{fw},u,U)}, \hat{\mathcal{B}}_{(\text{fw},u,U)})$$

in LRS_U, satisfying the following relations

$$\begin{aligned} \frac{D_{(\text{tem},U,u)}}{d\tau_U}\hat\eta_{(\text{tem},U,u)} &= \gamma\nu[-k_{(\text{tem},U,u)}\hat\nu(U,u) + \tau_{(\text{tem},U,u)}\hat\beta_{(\text{tem},U,u)}]\,, \\ \frac{D_{(\text{tem},U,u)}}{d\tau_U}\hat\beta_{(\text{tem},U,u)} &= -\gamma\nu\tau_{(\text{tem},U,u)}\hat\eta_{(\text{tem},U,u)} \end{aligned} \tag{11.21}$$

and

$$\begin{aligned} \frac{D_{(\text{tem},U)}}{d\tau_U}\hat{\mathcal{N}}_{(\text{tem},u,U)} &= \gamma\nu[\mathcal{K}_{(\text{tem},u,U)}\hat\nu(u,U) + \mathcal{T}_{(\text{tem},u,U)}\hat{\mathcal{B}}_{(\text{tem},u,U)}]\,, \\ \frac{D_{(\text{tem},U)}}{d\tau_U}\hat{\mathcal{B}}_{(\text{tem},u,U)} &= -\gamma\nu\mathcal{T}_{(\text{tem},u,U)}\hat{\mathcal{N}}_{(\text{tem},u,U)} \end{aligned} \tag{11.22}$$

In (11.22) and (11.21) $\tau_{(\text{tem},U,u)}$ and $\mathcal{T}_{(\text{tem},u,U)}$ are the Fermi-Walker relative torsions. Their vanishing defines the Fermi-Walker or Lie relative flatness of a curve as seen by the observer or particle.

The relative observer decomposition of the force equation $DU/d\tau_U = f(U)$ for a test particle with unit mass $m = 1$ (for simplicity) is obtained by substituting the decomposition (11.14) of $U = \gamma u + \gamma\nu(U,u)$ and projecting into the local rest space of u. The derivative of the first term leads to gravitoelectric and gravitomagnetic force terms arising from the acceleration and vorticity of the observer 4-velocity analogous to our earlier nonrelativistic discussion of rotating observers. The derivative of the second term can be decomposed into components along the direction of motion and in the transverse directions most easily using the relative Frenet-Serret frame [10]. The transverse acceleration term lying along $\hat{\eta}_{(\mathrm{tem},u,U)}$ is the relative centripetal acceleration seen by the observer and depends on the choice of temporal derivative used in the measurement. There is no other natural way of introducing these quantities that is based only on the geometry of the successive $4 = 1 + 3$ (time plus space) and $3 = 1 + 2$ (longitudinal plus transverse) splittings of the spacetime tangent space associated with a pair consisting of an observer congruence and a test particle world line and which does not depend on any special spacetime symmetry. Admittedly in a nonstationary spacetime, these quantities may not be as useful as they seem to be in aiding our interpretation of the geometry of stationary spacetimes. Moreover, the decomposition in the particle local rest space of the relative velocity of the observers is one further step removed from directly measurable quantities and is given here for geometrical completeness.

The equations defining the centripetal or centrifugal forces may be interpreted as the balance of forces along the (relative) transverse normal direction in the respective local rest spaces. The relative centripetal/centrifugal forces for each choice of temporal derivative along the transverse directions $\hat{\eta}_{(\mathrm{tem},U,u)}$ and $\hat{\mathcal{N}}_{(\mathrm{tem},u,U)}$ in the local rest spaces of the observer and the particle are

$$\begin{aligned}
\mathcal{F}^{(C)}_{(\mathrm{tem},U,u)} &= \gamma\nu^2 k_{(\mathrm{tem},U,u)} &&= [F_{(U,u)} + F^{(G)}_{(\mathrm{tem},U,u)}]\cdot\hat{\eta}_{(\mathrm{tem},U,u)}\,,\\
\mathcal{F}^{(C)}_{(\mathrm{tem},u,U)} &= \gamma\nu^2 \mathcal{K}_{(\mathrm{tem},u,U)} &&= [f(U) + \mathcal{F}^{(G)}_{(\mathrm{tem},u,U)}]\cdot\hat{\mathcal{N}}_{(\mathrm{tem},u,U)}
\end{aligned} \tag{11.23}$$

where the spatial gravitational forces in each local rest space are

$$\begin{aligned}
F^{(G)}_{(\mathrm{tem},U,u)} &= -\frac{D_{(\mathrm{tem},U,u)}}{d\tau_U}u = \gamma[g(u) + \epsilon_{(\mathrm{tem})}\nu(U,u)\times_u H(u)]\,,\\
\mathcal{F}^{(G)}_{(\mathrm{fw},u,U)} &= \gamma^{-1}P(U,u)F^{(G)}_{(\mathrm{fw},U,u)} = -\gamma^{-1}\frac{D_{(\mathrm{fw},U)}}{d\tau_U}u\,.
\end{aligned} \tag{11.24}$$

In the observer local rest space gravitational force expression, the gravitoelectric and gravitomagnetic fields are just the sign-reversed acceleration $g(u) = -a(u)$ and twice the vorticity $H(u) = 2\omega(u)$ of the observer 4-velocity u, and represent the inertial forces due to the motion of the observers analogous to the centrifugal and Coriolis forces of the nonrelativistic discussion, while $\gamma(U,u)F(u,u)$ is the projection of the force $f(u) = ma(u)$ (unit mass for simplicity) into the observer local rest space. In the particle local rest space, only the projection of the spacetime Fermi-Walker derivative is available as a natural temporal derivative operator.

4. Application to rotating observers in Minkowski spacetime

To make this concrete, we return to the problem of an observer family (u) and test particle (U) both in circular motion in Minkowski spacetime

$$\begin{aligned} u &= \gamma_\omega[\partial_t + \omega\partial_\phi], \qquad \gamma_\omega = (1-\omega^2 r^2)^{-1/2}\,, \\ U &= \gamma_\Omega[\partial_t + \Omega\partial_\phi], \qquad \gamma_\Omega = (1-\Omega^2 r^2)^{-1/2} \end{aligned} \tag{11.25}$$

with constant angular velocities ω and Ω referred to nonrotating cylindrical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + dz^2\,. \tag{11.26}$$

The natural cylindrically symmetric spatial orthonormal frame adapted to the family of nonrotating observers ($e_0 = \partial_t$) is

$$e_1 = \partial_r\,,\ e_2 = (1/r)\partial_\phi\,, e_3 = \partial_z\,. \tag{11.27}$$

This frame can be boosted into the local rest space of the rotating observers ($E_0 = u$)

$$E_1 = e_1\,,\ E_2 = \gamma_\omega(e_2 + \omega r e_0)\,, E_3 = e_3\,, \tag{11.28}$$

which coincides with the relative Frenet-Serret frame up to signs and a permutation

$$\hat\nu(U,u) = \mathrm{sgn}(\Omega-\omega)E_2\,,\ \hat\eta_{(\mathrm{fw},U,u)} = -E_1\,,\ \hat\beta_{(\mathrm{fw},U,u)} = \mathrm{sgn}(\Omega-\omega)E_3\,. \tag{11.29}$$

Similarly from the spacetime point of view, this frame coincides with the Frenet-Serret frame of u (modulo signs and a permutation), along which the frame is Lie dragged (see the Appendix).

Solving the transport equations

$$\frac{D_{(\mathrm{tem},U,u)}}{d\ell_{(U,u)}} E_{(\mathrm{tem},U,u)} = 0\,, \quad \mathrm{tem=fw,lie} \tag{11.30}$$

determines two more geometrically meaningful observer adapted frames related to E_1, E_2, E_3 by a rotation of the first two vectors with angular velocities $\Omega_{(\mathrm{tem},U,u)}$ about E_3

$$R(\Omega_{(\mathrm{tem},U,u)}) \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}\,, \tag{11.31}$$

where $(c, s) = (\cos(\Omega_{(\mathrm{tem},U,u)} t), \sin(\Omega_{(\mathrm{tem},U,u)} t))$. A direct calculation leads to the result

$$\Omega_{(\mathrm{fw},U,u)} = -\gamma_\omega \Omega \; , \; \Omega_{(\mathrm{lie},U,u)} = -\gamma_\omega \Omega + \gamma_\omega^2 \gamma_\Omega^{-1} \omega \; . \tag{11.32}$$

The spacetime Fermi-Walker transported frame along u (see the Appendix) is of the same form as these with the angular velocity

$$\Omega_{(\mathrm{fw},u)} = -\gamma_\omega \omega \; . \tag{11.33}$$

The speed of the particle relative to the observers is

$$\nu = \frac{r|\Omega - \omega|}{1 - \Omega \omega r^2} \; . \tag{11.34}$$

Evaluating the Fermi-Walker and Lie relative curvatures (see Table 2 of [11]) leads to

$$\kappa_{(\mathrm{fw},U,u)} = \frac{|\Omega|}{r|\Omega - \omega|} \; , \quad \kappa_{(\mathrm{lie},U,u)} = \frac{\gamma_\omega^2}{r} \; . \tag{11.35}$$

Note that the Lie relative curvature is independent of Ω. Since the relative Lie derivative (11.18) used to define it consists of a Lie temporal derivative term [9] which vanishes for the Lie dragged ϕ direction of motion, and a spatial covariant derivative term, only the spatial covariant derivative contributes here. Thus the gamma-squared factor comes only from the spatial geometry of the rotating observer congruence, whose spatial metric is represented by the last three terms in the line element expressed in nonrotating cylindrical coordinates but adapted to the rotating observers

$$ds^2 = -\gamma_\omega^{-2}(dt - \gamma_\omega^2 r^2 \omega d\phi)^2 + dr^2 + \gamma_\omega^2 r^2 (d\phi + \omega dt)^2 + dz^2 \; . \tag{11.36}$$

No matter what the (nonzero) relative speed of the particle, it orbits around a circle of coordinate radius r as seen by the rotating observers. Although the circumferential radius of the circle as seen by the rotating observers is $\gamma_\omega r$, the radius of curvature comes from the rotation of the direction of motion as determined by parallel transport in the spatial geometry, and here one has the spatial geometry effect easily explained with a tangent cone in the embedding diagram of the r-ϕ plane [18].

For this circular motion of both the observers and test particle, all the spatial forces and accelerations are along the radial direction and the relative centripetal force balance equations for a unit mass particle are simply

$$\mathcal{F}^{(C)}_{(\mathrm{tem},U,u)^r} = [F_{(U,u)} + F^{(G)}_{(\mathrm{tem},U,u)}]_r \; , \tag{11.37}$$

where the spatial force

$$F_{(U,u)} = -\frac{\gamma_\Omega \Omega^2 r}{\gamma_\omega (1 - \Omega \omega r^2)} \partial_r \tag{11.38}$$

arises from the projection and rescaling of the particle 4-acceleration

$$a(U) = \gamma_\Omega^2 \Omega^2 r \partial_r \,. \tag{11.39}$$

The gravitoelectric and gravitomagnetic fields are

$$g(u) = \gamma_\omega^2 \omega^2 r \partial_r \,, \quad H(u) = 2\gamma_\omega^2 \omega \partial_z \,, \tag{11.40}$$

the spatial gravitational forces are

$$\begin{aligned} F^{(G)}_{(\mathrm{fw},U,u)} &= \gamma_\omega \gamma_\Omega \omega \Omega r \partial_r \,, \\ F^{(G)}_{(\mathrm{lie},U,u)} &= \gamma_\omega \gamma_\Omega \omega r [(\Omega - \omega)\gamma_\omega^2 + \Omega] \partial_r \,, \end{aligned} \tag{11.41}$$

and the centripetal accelerations are

$$\begin{aligned} F^{(C)}_{(\mathrm{fw},U,u)} &= -\gamma_\omega \gamma_\Omega \Omega r \frac{(\Omega - \omega)}{1 - \Omega \omega r^2} \partial_r \,, \\ F^{(C)}_{(\mathrm{lie},U,u)} &= -\gamma_\omega^3 \gamma_\Omega r \frac{(\Omega - \omega)^2}{1 - \Omega \omega r^2} \partial_r \,. \end{aligned} \tag{11.42}$$

There are three interesting cases to discuss:

1. $\omega = 0$: nonrotating observers,
2. $\Omega = 0$: nonrotating particle, and
3. $\Omega = \omega$: observers corotating with the particle.

$\omega = 0$ When the observers are nonrotating, the relative transport angular velocity $\Omega_{(\mathrm{lie},U,u)} = \Omega_{(\mathrm{fw},U,u)} = -\Omega$ reduces to the sign-reversal of the particle angular velocity in order to exactly compensate for the rotation of the cylindrical frame vectors relative to the Cartesian ones. The spatial gravitational forces vanish $F^{(G)}_{(\mathrm{fw},U,u)} = 0 = F^{(G)}_{(\mathrm{lie},U,u)}$ and the Fermi-Walker and Lie centripetal forces coincide $F^{(C)}_{(\mathrm{fw},U,u)} = F^{(C)}_{(\mathrm{lie},U,u)} = -\gamma_\Omega \Omega^2 r \partial_r$ and balance the spatial force responsible for maintaining the circular orbit.

$\Omega = 0$ When the particle is nonrotating, i.e., fixed in space relative to the nonrotating observers, it is a spacetime geodesic with zero acceleration. The relative Fermi-Walker angular velocity is zero $\Omega_{(\mathrm{fw},U,u)} = 0$ since the cylindrical axes are always located at the same point, so these axes boosted into the local rest space of the rotating observers do not rotate with respect to the Fermi-Walker transported space-fixed axes, while the Lie relative angular velocity $\Omega_{(\mathrm{lie},U,u)} = \gamma_\omega^2 \omega$ acquires an extra gamma factor relative to the similar result for the complementary situation of Fermi-Walker transported frame vectors along a rotating orbit (see the Appendix), apparently due to effects of the spatial geometry as seen by the rotating observers. The Fermi-Walker forces vanish $F^{(G)}_{(\text{fw},U,u)} = 0 = F^{(C)}_{(\text{fw},U,u)}$ and the Lie forces coincide $F^{(G)}_{(\text{lie},U,u)} = F^{(C)}_{(\text{lie},U,u)} = -\gamma_\omega^3\omega^2 r\partial_r$, their equality representing the force balance.

$\boldsymbol{\Omega = \omega}$ When the rotating observers are corotating with the orbiting particle, then $\Omega_{(\text{lie},U,u)} = 0$ (the frame is already Lie dragged and so is also relatively Lie dragged along u), while $\Omega_{(\text{fw},U,u)} = -\gamma_\Omega\Omega$. In this latter case one has an extra factor of gamma compared to the angular velocity needed to compensate for the rotation of the cylindrical frame similar to the case of the spacetime Fermi-Walker frame, responsible for the Thomas precession effect [18]. The relative centripetal forces vanish $F^{(C)}_{(\text{fw},U,u)} = 0 = F^{(C)}_{(\text{lie},U,u)}$ and the Fermi-Walker and Lie spatial gravitational forces coincide $F^{(G)}_{(\text{fw},U,u)} = F^{(G)}_{(\text{lie},U,u)} = \gamma_\Omega\Omega^2 r\partial_r$ and balance the spatial force responsible for maintaining the circular orbit.

Finally one could consider the relative centrifugal forces in the particle local rest space, but in this case of circular motion in which all the interesting acceleration fields are in the radial direction, the various force terms only get rescaled by gamma factors.

5. Conclusions

If this exercise proves one thing, it is that it makes little sense to speak of "the" centrifugal force or to speak of inertial forces in the context of special or general relativity as though a given situation has only one realization of these nonrelativistic concepts. Instead one has various possibilities depending on how one measures the changes in the relative velocity and which observer family is chosen. In other words, the centrifugal force of our nonrelativistic picture can contribute to either the spatial gravitational force or the relative centripetal force or both. To keep things as simple as possible, the nonrotating observer congruence eliminates the inertial forces in this flat spacetime example and puts all the action into the relative centripetal force. Indeed in a general stationary axisymmetric spacetime like a black hole spacetime, locally nonrotating observers like the zero angular momentum observers (ZAMOs) probably make the most sense to use in interpreting how the nontrivial gravitational field affects test particle motion, eliminating the Coriolis force and giving some sense to the relative centripetal force as the best representation of “centrifugal force." Indeed the black hole spacetimes have been the prime motivation for considering this whole approach. Apart from the conformal transformation of the spatial metric in the static case, this encompasses the Abramowicz formalism, whose standard presentation is hampered by the confusion about the geometry of differentiation along a world line in spacetime of quantities which are only defined along that world line [10].

Appendix: Adapted spacetime frames

There are two natural spacetime frames which are adapted to the family of rotating observers: the spacetime Frenet-Serret frame defined by the differential properties of each individual world line and the Fermi-Walker transported frame along each world line.

Spacetime Frenet-Serret frame along *U*

This frame is just the boost of the orthonormalized nonrotating cylindrical coordinate frame which maps ∂_t onto U, explicitly

$$\begin{aligned} E_{(\mathrm{FS},U)0} &= U = \gamma_\Omega[\partial_t + \Omega\partial_\phi] \,, \\ E_{(\mathrm{FS},U)1} &= \partial_r \,, \\ E_{(\mathrm{FS},U)2} &= \gamma_\Omega[\Omega r\partial_t + \frac{1}{r}\partial_\phi] \,, \\ E_{(\mathrm{FS},U)3} &= \partial_z \,, \end{aligned} \qquad (11.A.1)$$

and satisfies the relations

$$\begin{aligned} \frac{D}{d\tau_U}E_{(\mathrm{FS},U)0} &= \kappa E_{(\mathrm{FS},U)1}, \\ \frac{D}{d\tau_U}E_{(\mathrm{FS},U)1} &= \kappa E_{(\mathrm{FS},U)0} + \tau_1 E_{(\mathrm{FS},U)2} \,, \\ \frac{D}{d\tau_U}E_{(\mathrm{FS},U)2} &= -\tau_1 E_{(\mathrm{FS},U)2} + \tau_2 E_{(\mathrm{FS},U)3} \,, \\ \frac{D}{d\tau_U}E_{(\mathrm{FS},U)3} &= -\tau_2 E_{(\mathrm{FS},U)2} \,. \end{aligned} \qquad (11.A.2)$$

The spacetime Frenet-Serret curvature κ (magnitude of the 4-acceleration $DU/d\tau_U$, always nonzero unless $\Omega = 0$) and torsions τ_1 and τ_2 are

$$\kappa = -\gamma_\Omega^2\Omega^2 r \,, \quad \tau_1 = \gamma_\Omega^2\Omega \,, \quad \tau_2 = 0 \,, \qquad (11.A.3)$$

where $\nu = |\Omega r| = |\kappa|/|\tau_1| < 1$ for all the timelike orbits. According to the classification of Synge [17], who systematically studied timelike helices in flat spacetime, the orbit is a helix of type II which is degenerate (it lies in the plane $z = 0$) and of subtype IIc (torsion dominated: $|\kappa| < |\tau_1|$).

Fermi-Walker frame along *U*

The Fermi-Walker frame along U is obtained from the spacetime Frenet-Serret frame by an additional rotation by the angle $-\gamma_\Omega\Omega t$ of the vectors $E_{(\mathrm{FS},U)1}$ and $E_{(\mathrm{FS},U)2}$, and corresponds to axes in the local rest space of U aligned with three mutually orthogonal gyroscopes [19]

$$\begin{aligned} E_{(\mathrm{FW},U)0} &= U \,, \\ E_{(\mathrm{FW},U)1} &= \cos(\gamma_\Omega\Omega t)\, E_{(\mathrm{FS},U)1} - \sin(\gamma_\Omega\Omega t)\, E_{(\mathrm{FS},U)2} \,, \\ E_{(\mathrm{FW},U)2} &= \sin(\gamma_\Omega\Omega t)\, E_{(\mathrm{FS},U)1} + \cos(\gamma_\Omega\Omega t)\, E_{(\mathrm{FS},U)2} \,, \\ E_{(\mathrm{FW},U)3} &= E_{(\mathrm{FS},U)3} \,. \end{aligned} \qquad (11.A.4)$$

Integrating the Frenet-Serret angular velocity $\omega_{FS} = |\tau_1| = \gamma_\Omega^2 |\Omega|$ over an interval of proper time τ_U (the natural Frenet-Serret parameterization) and using the relation $d\tau_U = 1/\gamma_\Omega dt$ converts the proper time to coordinate time yielding the coordinate angular velocity $\gamma_\Omega \Omega$ in these expressions.

The boost of the inertial frame $(\partial_t, \partial_x, \partial_y, \partial_z)$ which maps ∂_t onto U is instead represented by a rotation with the gamma factor missing, leading to a relative rotation by the angle $(\gamma_\Omega - 1)\phi$, where $\phi = \Omega t$. This is the angular velocity of the Thomas precession [18].

References

[1] Gödel K., An example of a new type of cosmological solutions to Einstein's field equations of gravitation, *Rev. Mod. Phys.* **21**, 447 (1949); reprinted with commentary By G.F.R. Ellis in *Gen. Rel. Grav.* **32**, 1399 (2000)

[2] Kerr R.P., Gravitational field of a spinning mass as an example of algebraically special metrics, *Phys. Rev. Lett.* **11**, 237 (1963)

[3] Ehrenfest P., Gleichförmige Rotation starrer Körper und Relativitätstheorie, *Phys. Zeits.* **10**, 918 (1909)

[4] Rizzi G. and Ruggiero M.L., Space geometry of rotating platforms: an operational approach, *Found. Phys.* **32**, 1525 (2002); gr-qc/0207104

[5] Grøn G., in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series "Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)

[6] Landau L.D. and Lifshitz E.M., *The Classical Theory of Fields*, Second edition, Permagon Press, New York, 1975; first edition in English 1951; first Russian edition 1941

[7] Misner C.W., Thorne K.S. and Wheeler J.A., *Gravitation*, Freeman, San Francisco, 1973

[8] Bini D., Carini P. and Jantzen R.T., The inertial forces / test particle motion game, in *Proceedings of the Eighth Marcel Grossmann Meeting on General Relativity (1997)* ed. T. Piran, World Scientific, Singapore, 1999, 376; e-print gr-qc/9710051

[9] Jantzen R.T., Carini P. and Bini D., The many faces of gravitoelectromagnetism, *Ann. Phys. (N.Y.)* **215**, 1 (1992)

[10] Bini D., Carini P. and Jantzen R.T., The intrinsic derivative and centrifugal forces in general relativity: I. Theoretical foundations, *Int. J. Mod. Phys.* **D6**, 1 (1997); e-print gr-qc/0106013

[11] Bini D., Carini P. and Jantzen R.T., The intrinsic derivative and centrifugal forces in general relativity: II. applications to circular orbits in some familiar stationary axially symmetric spacetimes, *Int. J. Mod. Phys.* **D6**, 143 (1997); e-print gr-qc/0106014

[12] Abramowicz M.A., Centrifugal forces: a few surprises, *Mon. Not. R. Ast. Soc.* **245**, 733 (1990)

[13] Rindler W. and Perlick V., Rotating coordinates as tools for calculating circular geodesics and gyroscopic precession, *Gen. Rel. Grav.* **22**, 1067 (1990)

[14] Taylor E.F. and Wheeler J.A., *Spacetime Physics*, First Edition, W.H. Freeman, San Francisco, 1966: exercise 103 (dropped from Second Edition, 1992)

[15] Bini D., de Felice F. and Jantzen R.T., Relative Frenet-Serret frames and Fermi-Walker transport, *Class. Quantum Grav.* **16**, 2105 (1999)

[16] Bini D., Jantzen R.T. and Merloni A., Geometric interpretation of the Frenet-Serret frame description of circular orbits in stationary axisymmetric spacetimes, *Class. Quantum Grav.* **16**, 1333 (1999)

[17] Synge J.L., Timelike helices in flat spacetime, *Proc. Roy. Irish Acad.* **65**, 27 (1967)

[18] Jantzen R.T., Circular holonomy, clock effects and gravitoelectromagnetism: still going around in circles after all these years . . ., in *Proceedings of the Ninth ICRA Network Workshop on Fermi and Astrophysics*, eds. V. Gurzadyan and R. Ruffini, World Scientific, Singapore, 2003; e-print gr-qc/0202085

[19] Pirani, F.A.E., On the physical significance of the Riemann tensor, *Acta Phys. Polon.* **15**, 389 (1956)

Chapter 12

EPPUR, SI MUOVE !

Lluís Bel
Fisika Teorikoa, Euskal Herriko Unibertsitatea,
P.K. 644, 48080 Bilbo, Spain
wtpbedil@lg.ehu.es

Abstract Simple ideas that shed new light on the physics of rotation as it concerns two famous experiments: The Wilson and Wilson, and the Michelson and Morley experiments.

Introduction

Yes indeed! The Earth rotates, the Moon and the Planets rotate, and rotation is an ubiquitous state of motion in the Universe. But also in laboratory physics rotation is often an essential component in some experiments, among them the Michelson and Morley experiment, the Wilson and Wilson, and some others [1] that we shall not consider in this paper. Despite of this, many fundamental aspects of the physics of rotation remain not well understood if Special relativity is needed, and in particular when rotation and electromagnetism both play a role in an experiment.

The very definition of a rotating frame of reference in Special relativity has been questioned in the past and still is by some ([2] - [4]). It puzzles some authors the fact that the usual definition looks like non relativistic and leads to time-like congruences that do not fill the whole Minkowski's space. Many authors also are tempted to deal with rotation invoking the Principle of local Lorentz invariance but this amounts to overlooking some subtle points and the conditions that would justify this simplification need to be clarified ([5] - [6]). Also is not always clear whether it makes a difference to distinguish a rigid body that rotates from incoherent dust that rotates as if it were rigid. These and other thoughts recurrently come to the mind.

In Section 1 we review and develop a decomposition of the Lorentz transformations as a product of three transformations with different meanings and

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 241-260.

different importance. This is to clarify what is essential and what is just convenient in dealing with a particular problem involving only Galilean frames of reference.

In Section 2 we extend the point of view of the preceding section to deal with the coordinate transformations between adapted coordinates from a Galilean frame of reference to a rotating one and viceversa. The deep novelty is that to be able to have the analog of the third factor in the decomposition to select Cartesian-like coordinates of space, we feel that a departure of common ideas about the concept of space in a rotating frame of reference is necessary, as it was in some other problems ([7]-[9]). This has implications on the Principle of local Lorentz invariance.

Section 3 is our late contribution to a lively recent polemic about the outcome and the theory of the Wilson and Wilson experiment ([10]-[17]). Second order relativistic effects are completely negligible in the conditions of the experiment and therefore usually only the first order result that agrees with the experiment is calculated. We derive here an exact formula as a test case of the procedure that we propose that is complete and remains simple.

As it is well known the Michelson and Morley experiment ([18]) has been usually understood as a test of Special relativity in its most restrictive sense ([19]-[22]), but not always ([23]-[25]). The usual attitude excludes taking into account the rotation of the Earth except for changing the orientation of the interferometer with respect to an hypothetic absolute space whose existence would violate Special relativity. We have opposed this point of view in three preceding papers ([26]-[28]) predicting a positive result which by no means would contradict Special relativity, but only some restrictive ways of dealing with this theory. In these papers, besides the rotation of the Earth, also its gravity and its oblateness, or the influence of the Sun and the Moon, were part of the discussion and this somehow obscured a little the principal role of the first. In Section 4 we summarize a bare-bones presentation of our point of view emphasizing the role of the rotation of the Earth.

1. Galilean frames of reference

Let us consider Minkowski's space-time and a Cartesian system of coordinates T and X^i such that the line element is:

$$ds^2 = -dT^2 + \delta_{ij} dX^i dX^j, \quad c = 1, \quad i, j, k, .. = 1, 2, 3 \tag{12.1}$$

This simple formula hides the rich geometrical structure of a Galilean frame of reference, say S_0, which is a structure with two ingredients that it is useful to exhibit explicitly.

i) The first ingredient is the congruence $\mathcal{C}_0$ of time-like world-lines with parametric equations:

$$T = T, \quad X^i = const. \tag{12.2}$$

It is a Killing congruence, and therefore the motion that it defines is rigid:

$$\frac{d}{dT}\int_{L_g} ds = 0, \tag{12.3}$$

where L_g is any segment of geodesic whose end-points lie on any two world-lines of $\mathcal{C}_0$ and is orthogonal to them.

ii) The second ingredient is the foliation with equation:

$$T = const. \tag{12.4}$$

It is a family of hyperplanes orthogonal to the congruence $\mathcal{C}_0$. The associated synchronization is the scale that to an interval of T along any of the wold-lines, say L_0, corresponds the proper time interval:

$$T_1 - T_0 = \int_{L_0} \frac{ds}{dT}\, dT \tag{12.5}$$

The purpose of this paper is twofold. First of all to unravel the meaning and the role of the two ingredients in the relationship between two different Galilean frames of reference. And secondly to apply our findings to reach a better comprehension of uniformly rotating frames of reference which will be the subject of the remaining of this paper starting in Sect. 2.

Let us consider the following formulae:

$$T = T, \quad X^i = x^i + v^i T \tag{12.6}$$

and the inverse:

$$T = T, \quad x^i = X^i - v^i T \tag{12.7}$$

where v^i are constants. They make sense both in Classical mechanics and in Special relativity and in both cases they have the same dual meaning. Here we consider them in the framework of the Galilean frame of reference considered above, and therefore we assume that:

$$v^2 < 1, \quad v^2 = v_i v^i, \quad v_i = \delta_{ij} v^j \tag{12.8}$$

Formulae (12.6) can be interpreted as the parametric equations $X^\alpha(T; x^i)$ $(\alpha, \beta, \gamma .. = 0, 1, 2, 3)$ of a family of time-like world-lines with parameter T and initial conditions x^i. This family is, as it was the case with $\mathcal{C}_0$, a new Killing congruence $\mathcal{C}_1$ of time-like geodesics. They can therefore be considered as the first ingredient of a new Galilean frame of reference, say S_1.

But the same formulae can be interpreted as defining a coordinate transformation $X^\alpha(T, x^i)$ from coordinates T, x^i to coordinates T, X^i. From this point of view they define a coordinate transformation between two Galilean frames of reference whose first ingredient are two different congruences $\mathcal{C}_0$ and $\mathcal{C}_1$ but share the same foliation. This does not pose any problem but one has to keep in mind that the synchronization which was associated with proper-time along the geodesics of $\mathcal{C}_0$ does not correspond now to proper-time along $\mathcal{C}_1$. This asymmetry may be inconvenient in some occasions and convenient in some others. More on that later.

Using the coordinate transformation (12.6) the line-element (12.1) becomes:

$$ds^2 = -(1-v^2)dT^2 + 2v_i dx^i dT + \delta_{ij} dx^i dx^j \tag{12.9}$$

that can be decomposed as follows:

$$ds^2 = -{\theta^0}^2 + ds_1^2 \tag{12.10}$$

where:

$$\theta^0 = -\sqrt{1-v^2}dT + \frac{v_i dx^i}{\sqrt{1-v^2}} \tag{12.11}$$

$$ds_1^2 = (\delta_{ij} + \frac{v_i v_j}{1-v^2})dx^i dx^j \tag{12.12}$$

Let us now choose as a new foliation the family of hyperplanes orthogonal to $\mathcal{C}_1$ and as an associated synchronization that corresponding to choosing as scale of time the proper-time along any of the world-lines of this congruence, say for instance:

$$L_0 \ : \ X^i = x_0^i + v^i T \tag{12.13}$$

This amounts to choosing the new time coordinate t such that:

$$t - t_0 = \int_0^{T_0} \sqrt{1 - \delta_{ij}\frac{dX^i}{dT}\frac{dX^j}{dT}}\, dT = \sqrt{1-v^2}T_0 \tag{12.14}$$

t_0 being an arbitrary origin of t and $T_0(T, X^i; x_0^i)$ being the value of T at the intersection of L_0 with the hyperplane Π of the new foliation passing through the event with coordinates T, X^i.

From the equation of Π:

$$(X^i - X_0^i)v_i - (T - T_0) = 0 \tag{12.15}$$

and from:

$$X_0^i = x_0^i + v^i T_0 \tag{12.16}$$

we obtain:

$$t - t_0 = \frac{T - (X^i - x_0^i)v_i}{\sqrt{1 - v^2}} \tag{12.17}$$

and also, using (12.6):

$$T = \frac{t - t_0}{\sqrt{1 - v^2}} + \frac{(x^i - x_0^i)v_i}{1 - v^2} \tag{12.18}$$

Substituting T in (12.9) we get:

$$ds^2 = -dt^2 + ds_1^2 \tag{12.19}$$

with ds_1^2 being the unchanged metric given in (12.12). This metric is obviously flat because its coefficients are constants but the coordinates x^i are not Cartesian coordinates. Particular Cartesian ones are :

$$\bar{x}^i = (\delta_j^i + bv^i v_j)x^j, \quad b = \frac{1}{v^2}(-1 + \frac{1}{\sqrt{1 - v^2}}) \tag{12.20}$$

as can be seen substituting the inverse functions:

$$x^i = (\delta_j^i + av^i v_j)\bar{x}^j, \quad a = \frac{1}{v^2}(-1 + \sqrt{1 - v^2}) \tag{12.21}$$

into (12.12) that becomes:

$$ds_1^2 = \delta_{ij} d\bar{x}^i d\bar{x}^j \tag{12.22}$$

At the end we obtain thus:

$$ds^2 = -dt^2 + \delta_{ij} d\bar{x}^i d\bar{x}^j \tag{12.23}$$

We have therefore succeeded in decomposing a Lorentz transformation leading from (12.1) to (12.23) in three steps as a product of three particular transformations with different meanings and different importance.

i) Step 1.- The first transformation is given by (12.6) and it is the one that picks the congruence, i.e. the rigid motion of a second Galilean frame of reference. Any other choice would not define a new frame of reference with uniform constant velocity. We shall see that this is the only step of a Lorentz transformation that is necessary to transform tensor fields between two different Galilean frames of reference.

ii) Step 2.- The second transformation is the time transformation given by (12.18). This defines a convenient foliation and it is important because the corresponding synchronization implements a fundamental postulate of Special, as well as General, relativity identifying proper space-time intervals along time-like world-lines with physical time intervals measured by real clocks of reference. But this time-transformation is by no means necessary in many occasions.

iii) Step 3.- The third step is the three dimensional coordinate transformation given by (12.21). This is a passive, innocuous, change of names to refer to the world-lines of the congruence. Its physical meaning importance comes from being the last step to implement the Relativity principle by the invariance of the line-element (12.1) under the Lorentz transformations. But in practice its role is at most a mere simplifying convenience which is not necessary in many occasions.

Let us consider an example to clarify the above remarks. Let $F^{\alpha\beta}(T, X^i)$ be a second rank skew-symmetric tensor describing an electromagnetic field in the Galilean frame of reference S_0. The electric and magnetic fields are then:

$$E^i = F^i_{.0} \quad B^{ij} = F^{ij} \tag{12.24}$$

Let us now accept that for any other Galilean frame of reference with congruence $\mathcal{C}_1$ the electric E^a and magnetic B^{ab} fields be given by:

$$E^a = F^{\alpha}_{.\beta}\theta^a_\alpha U^\beta, \quad B^{ab} = F^{\alpha\beta}\theta^a_\alpha\theta^b_\beta \tag{12.25}$$

where U^α is the unit tangent to the world-lines of the congruence $\mathcal{C}_1$:

$$U^0 = \frac{1}{\sqrt{1-v^2}}, \quad U^i = \frac{v^i}{\sqrt{1-v^2}} \tag{12.26}$$

and where θ^a_α are three arbitrary covariant vectors fields orthogonal to U^α.

Using the space coordinates transformation (12.6) and keeping the time coordinate T unchanged leads to the following components:

$$u^0 = \frac{1}{\sqrt{1-v^2}}, \quad u^i = 0 \tag{12.27}$$

Three particular covariant vectors N^α orthogonal to U^α are:

$$N^a = dx^a : N^a_0 = -v^a, \quad N^a_i = \delta^a_i \quad a, b, c, .. = 1, 2, 3 \tag{12.28}$$

From their definition we know that their components in the Galilean system of reference S_1 are:

$$n_0^a = 0, \quad n_i^a = \delta_i^a \tag{12.29}$$

The same coordinate transformation leads to the following expressions for the electromagnetic field in the Galilean frame of reference S_1:

$$f^{i0} = F^{i0} \tag{12.30}$$

$$f^{ij} = F^{ij} + F^{i0}v^j - F^{j0}v^i \tag{12.31}$$

Each term in the r-h-s of these expressions is a function of T and X^i and the l-h-s are functions of T and x^i obtained using (12.6) in the r-h-s.

In particular if we calculate the preceding scalars (12.25) in the system of reference S_1 with $\theta_i^a = n_i^a$ we obtain:

$$e^a = f^\alpha_\beta n^a_\alpha u^\beta = \frac{1}{\sqrt{1-v^2}} f^a_{.0}, \quad b^{ab} = f^{\alpha\beta} n^a_\alpha n^b_\beta = f^{ab} \tag{12.32}$$

To obtain the result corresponding to a standard Lorentz transformation we have to choose:

$$\theta^a = d\bar{x}^a \tag{12.33}$$

where $\bar{x}^a$ have been defined in (12.20)

Notice that to transform vector or tensor components only the coordinate transformations (12.6) have been used. The time transformation (12.18) will be needed only if the original electromagnetic field depends on T and we want to use a time reference based on a clock measuring proper-time along geodesics of $\mathcal{C}_1$. This may not be convenient if the master clock on which a time distribution is based is a clock which is at rest with respect to S_0. Neither is always the most convenient choice to calculate the scalars (12.32) using (12.33).

2. Uniformly rotating frames of reference

Let us consider again Minkowski's space-time with line element (12.1). It it will be now and then convenient to use cylindrical space coordinates:

$$X = \rho\cos(\phi), \quad Y = \rho\sin(\phi), \quad Z = z \tag{12.34}$$

in which case the line-element is:

$$ds^2 = -dT^2 + d\rho^2 + \rho^2 d\Phi^2 + dz^2 \tag{12.35}$$

Let us consider the time-like congruence $\mathcal{C}_2$ defined by the following parametric equations:

$$T = T \tag{12.36}$$
$$X = x\cos(\omega T) - y\sin(\omega T) \tag{12.37}$$
$$Y = x\sin(\omega T) + y\cos(\omega T) \tag{12.38}$$
$$Z = z \tag{12.39}$$

where x, y and z are the initial conditions labelling each of the world-lines of the congruence and ω is a constant. Using cylindrical coordinates the same congruence has parametric equations:

$$T = T, \quad \rho = \rho, \quad \phi = \varphi + \omega T, \quad z = z \tag{12.40}$$

Considering the dual meaning of (12.36)-(12.39), or (12.40), as a change of coordinates we get:

$$ds^2 = -(1-\omega^2\rho^2)dT^2 + 2\omega(xdy - ydx)dT + dx^2 + dy^2 + dz^2 \tag{12.41}$$

which can be split as follows:

$$ds^2 = -(\theta_2^0)^2 + ds_2^2 \tag{12.42}$$

with:

$$\theta_2^0 = -\sqrt{1-\omega^2\rho^2}dT + \frac{\omega(xdy - ydx)}{\sqrt{1-\omega^2\rho^2}} \tag{12.43}$$
$$d\hat{s}_2^2 = dx^2 + dy^2 + dz^2 + \omega^2\frac{y^2dx^2 + x^2dy^2 - 2xydxdy}{1-\omega^2\rho^2} \tag{12.44}$$

or:

$$ds^2 = -(1-\omega^2\rho^2)dT^2 + 2\rho^2\omega d\varphi dT + d\rho^2 + \rho^2 d\varphi^2 + dz^2 \tag{12.45}$$

which can be split as follows:

$$\theta_2^0 = -\sqrt{1-\omega^2\rho^2}dT + \frac{\omega\rho^2 d\varphi}{\sqrt{1-\omega^2\rho^2}} \tag{12.46}$$
$$d\hat{s}_2^2 = d\rho^2 + \frac{\rho^2 d\varphi^2}{1-\omega^2\rho^2} + dz^2 \tag{12.47}$$

The fact that (12.41) does not depend on T proves that the congruence $\mathcal{C}_2$ is again a Killing congruence and therefore it has an intrinsic meaning in the geometrical framework of Minkowski's space-time. Each of its world-lines describes a circular motion with constant angular velocity ω and thus $\mathcal{C}_2$ is the first ingredient of a uniformly rotating frame of reference.

The second ingredient of a frame of reference must be a synchronization associated to a foliation. To this end we choose any of the world-lines of $\mathcal{C}_2$, say L_0, corresponding to initial conditions (x_0, y_0, z_0) and define a foliation $\mathcal{F}_0$ as being the family of hyperplanes orthogonal to L_0 and as associated synchronization the proper-time scale along L_0. More precisely we define a new coordinate t such that:

$$t - t_0 = \int_0^{T_0} \sqrt{1 - \rho_0^2 \frac{d\phi^2}{dT^2}}\, dT = \sqrt{1 - \omega^2 \rho_0^2} T_0 \tag{12.48}$$

where $\rho_0^2 = x_0^2 + y_0^2$, t_0 is an arbitrary origin of t and $T_0(T, X, Y, Z; x_0, y_0, z_0)$ is the value of T at the intersection E_0 of L_0 with the hyperplane Π of the foliation $\mathcal{F}_0$ passing through the event with coordinates T, X, Y, Z. Let X_0, Y_0 and Z_0 be the values of X, Y and Z at E_0, and let $\dot{X}_0$ $\dot{Y}_0$ and $\dot{Z}_0$ be the values of its derivatives with respect to T. Then the equation of the hyperplane Π is:

$$(X_0 - X)\dot{X}_0 + (Y_0 - Y)\dot{Y}_0 + (Z_0 - Z)\dot{Z}_0 - T_0 + T = 0 \tag{12.49}$$

Taking into account that:

$$\dot{X}_0 = -\omega Y_0, \quad \dot{Y}_0 = \omega X_0, \quad \dot{Z}_0 = 0 \tag{12.50}$$

Eq. (12.49) simplifies to:

$$\omega(XY_0 - YX_0) - T_0 + T = 0 \tag{12.51}$$

To find $T_0(T, X, Y, Z; x_0, y_0, z_0)$ this equation has to be solved keeping in mind that X_0 and Y_0 in:

$$X_0 = x_0 \cos(\omega T_0) - y_0 \sin(\omega T_0) \tag{12.52}$$

$$Y_0 = x_0 \sin(\omega T_0) + y_0 \cos(\omega T_0) \tag{12.53}$$

are functions of x_0, y_0 as well as the unknown T_0. And since this equation is transcendental it has to be solved at some approximation. The approximation that we consider below consists in using the Taylor expansions of X_0, Y_0 and T_0 with respect to the variables x_0 and y_0 neglecting all monomials of order 3 or greater with respect to ω.

If $x_0 = y_0 = 0$ then $X_0 = Y_0 = 0$ and we have:

$$T_0 = T \tag{12.54}$$

Derivating (12.51) with respect to x_0 we get:

$$\omega\left[X(\frac{\partial Y_0}{\partial x_0} + \frac{\partial Y_0}{\partial T_0}\frac{\partial T_0}{\partial x_0}) - Y(\frac{\partial X_0}{\partial x_0} + \frac{\partial X_0}{\partial T_0}\frac{\partial T_0}{\partial x_0})\right] - \frac{\partial T_0}{\partial x_0} = 0 \tag{12.55}$$

or:

$$\frac{\partial T_0}{\partial x_0} = \omega[X\sin(\omega T) - Y\cos(\omega T)] \tag{12.56}$$

Similarly we get:

$$\frac{\partial T_0}{\partial y_0} = \omega[X\cos(\omega T) + Y\sin(\omega T)] \tag{12.57}$$

Equivalently, since from (12.37) and (12.38) we have:

$$x = X\cos(\omega T) + Y\sin(\omega T) \tag{12.58}$$
$$y = -X\sin(\omega T) + Y\cos(\omega T) \tag{12.59}$$
$$z = Z \tag{12.60}$$

we can write:

$$\frac{\partial T_0}{\partial x_0} = -\omega y, \quad \frac{\partial T_0}{\partial y_0} = \omega x \tag{12.61}$$

This process could be continued but the following derivatives would be already of order ω^3 and we shall stop it here.

T_0 is then at our approximation:

$$T_0 = T + \omega x_0[X\sin(\omega T) - Y\cos(\omega T)] + \omega y_0[X\cos(\omega T) + Y\sin(\omega T)] \tag{12.62}$$

and therefore from (12.48) we get at the corresponding approximation:

$$t - t_0 = (1 - \frac{1}{2}\rho_0^2\omega^2)T + \omega x_0[X\sin(\omega T) - Y\cos(\omega T)] + \omega y_0[X\cos(\omega T) + Y\sin(\omega T)] \tag{12.63}$$

This equation together with (12.58)- (12.60) completes the coordinate transformation from the Galilean frame of reference S_0 to the uniformly rotating frame of reference S_2.

From (12.63), (12.58) and (12.59) we obtain:

$$T = (1 + \frac{1}{2}\rho_0^2\omega^2)(t - t_0) + \omega x_0 y - \omega y_0 x \tag{12.64}$$

which is the time-component transformation from the uniformly rotating frame of reference S_2 to the Galilean frame of reference S_0. Substituting T from (12.64) into (12.37) and (12.38) would yield the space part of the transformation.

Up to this point we have completed two steps which are similar to those of the preceding section:

i) Step 1 picked a rotating Killing congruence as the first ingredient of a new frame of reference.

ii) Step 2 defined some convenient synchronizations. But despite the similarities some relevant differences with the pure Galilean case deserve to be mentioned explicitly:

1) The congruences $\mathcal{C}_0$ and $\mathcal{C}_2$ share the bunch of world-lines corresponding to the points of the axis of rotation but have notorious well-known different intrinsic geometries. Moreover the domain of $\mathcal{C}_2$ must be restricted to the domain $\omega\rho < 1$ to keep it time-like.

2) The synchronizations of S_2 depend on the world-line L_0 that defines the scale of time, but those world-lines which are common to $\mathcal{C}_0$ and $\mathcal{C}_2$ are equally well adapted to both frames of reference.

There is another important qualitative difference between the case considered here and that of the preceding section: namely that now the quotient metric (12.47) is not Euclidean and therefore Step 3 there does not make sense here because there are not Cartesian coordinates for this metric. The far-reaching consequences of this fact which, as we are told ([29]), played an important historical role in the genesis of General relativity by A. Einstein, has been in our opinion under-estimated by the relativity community ever since. First of all this means that rigid bodies can not be compared in general if they are in different locations or have different orientations. Another way of saying this is to say that a rigid body can not be moved around. This shatters the very foundations of metrology and therefore of physics. Similarly with the concept of parallelism on which is based the idea that it makes sense for two astronomers in two different locations to point their telescopes in the same direction.

In our opinion this unsatisfactory situation stems from a misinterpretation of the line-element (12.47) as describing the geometry of space in a rotating frame of reference. The point of view that we develop below consists in defining the geometry of space by the principal transform of (12.47), a concept that we introduced in [7], and in re-interpreting (12.47) as defining an optical length, i.e. a length measured by a round trip transit time of light, instead of a physical length, i.e. measured for instance with an stretched ideally inextensible thread.

The principal transform of (12.47) is by definition a metric with line-element:

$$d\bar{s}^2 = e^{2\mu}\left(d\rho^2 + \frac{\rho^2}{1-\omega^2\rho^2}d\varphi^2\right) + e^{2\nu}dz^2 \tag{12.65}$$

with μ and ν such that:

$$\bar{R}_{ijkl} = 0 \tag{12.66}$$

and:

$$\hat{g}^{ij}\left(\hat{\Gamma}^k_{ij} - \bar{\Gamma}^k_{ij}\right) = 0 \tag{12.67}$$

The first condition (12.66) tells us that (12.65) is Euclidean and the second condition tells us that Cartesian coordinates of (12.65) are harmonic coordinates of (12.47). Both conditions are necessary to make the association intrinsic and non ambiguous.

Requiring the function μ to be regular on the axis, the solutions for μ and ν of Eqs. (12.66) and (12.67) are:

$$\mu = \int_0^\rho \frac{du}{u}\left(\sqrt{1-\omega^2u^2} - \frac{1}{1-\omega^2u^2}\right), \qquad \nu = 0 \tag{12.68}$$

Step 3.- Now it makes sense to proceed with Step 3 requiring, if convenient, the use of Cartesian coordinates of (12.65). They are the following:

$$\bar{x} = \frac{e^\mu x}{\sqrt{1-\omega^2\rho^2}}, \quad \bar{y} = \frac{e^\mu y}{\sqrt{1-\omega^2\rho^2}}, \bar{z} = z \tag{12.69}$$

so that (12.65) becomes:

$$d\bar{s}^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 = d\bar{\rho}^2 + \bar{\rho}^2 d\varphi^2 + d\bar{z}^2 \tag{12.70}$$

with $\bar{\rho}^2 = \bar{x}^2 + \bar{y}^2$. A system of orthonormal axes equally oriented all over the uniformly rotating frame of reference could now be obtained as 1-forms differentiating (12.69).

Neglecting from now on all terms of order higher than $(\omega\rho)^2$ we have:

$$\mu \approx -\frac{3}{4}\omega^2\rho^2 \tag{12.71}$$

$$\bar{\rho} \approx \rho\left(1 - \frac{1}{4}\omega^2\rho^2\right) \tag{12.72}$$

and:

$$d\hat{s}^2 \approx \left(1 + \frac{3}{2}\omega^2\rho^2\right)(d\bar{\rho}^2 + \bar{\rho}^2 d\varphi^2) + dz^2 \tag{12.73}$$

or equivalently:

$$d\hat{s}^2 \equiv \frac{1}{c_1^2}d\bar{\rho}^2 + \frac{1}{c_2^2}\bar{\rho}^2 d\varphi^2 + \frac{1}{c_3^2}dz^2 \tag{12.74}$$

where:

$$c_1 = c_2 \approx 1 - \frac{3}{4}\omega^2\rho^2, \qquad c_3 = 1 \tag{12.75}$$

At this approximation the formulas (12.69) and its inverse become:

$$\bar{x} = (1 - \frac{1}{4}\omega^2\rho^2)x, \quad \bar{y} = (1 - \frac{1}{4}\omega^2\rho^2)y, \quad \bar{z} = z \tag{12.76}$$

$$x = (1 + \frac{1}{4}\omega^2\bar{\rho}^2)\bar{x}, \quad y = (1 + \frac{1}{4}\omega^2\bar{\rho}^2)\bar{y}, \quad z = \bar{z} \tag{12.77}$$

We are going to check an intuitive belief that is often promoted to a fundamental principle called the Principle of local Lorentz invariance. According to it one assumes that if the relevant time interval and domain of space are small enough then, at any location of a rotating frame of reference, the usual Lorentz transformations can be used ignoring every thing else except the instantaneous velocity of the location with respect to a Galilean frame of reference at rest with respect to the axis of rotation.

Let us calculate dT using (12.64), (12.77) and:

$$x_0 = (1 + \frac{1}{4}\omega^2\bar{\rho}_0^2)\bar{x}_0, \quad y_0 = (1 + \frac{1}{4}\omega^2\bar{\rho}_0^2)\bar{y}_0, \quad z_0 = \bar{z}_0 \tag{12.78}$$

Then, keeping the approximation to order ω^2, let us calculate dX and dY using (12.37)-(12.39). And finally choose a world-line of $\mathcal{C}_2$ with initial conditions:

$$y_0 = 0, \quad t_0 = 0 \tag{12.79}$$

and evaluate the result when:

$$\bar{x} = \bar{x}_0, \quad \bar{y} = \bar{y}_0, \quad \bar{z} = \bar{z}_0 \tag{12.80}$$

The final result is the following:

$$dT = (1+\frac{1}{2}\omega^2\bar{x}_0^2)dt + \omega\bar{x}_0 d\bar{y} \quad (12.81)$$

$$dX = (1+\frac{3}{4}\omega^2\bar{x}_0^2)d\bar{x} \quad (12.82)$$

$$dY = (1+\frac{5}{4}\omega^2\bar{x}_0^2)d\bar{y} + \omega x_0 dt \quad (12.83)$$

$$dZ = d\bar{z} \quad (12.84)$$

These infinitesimal transformations have to be compared, at the appropriate approximation, with the Lorentz transformations corresponding to two Galilean frames of reference when one of them moves with respect to the other with constant velocity $v = \omega x_0$ in the Y direction:

$$dT = (1+\frac{1}{2}v^2)dt+vd\bar{y}, \quad dX = d\bar{x}, \quad dY = (1+\frac{1}{2}v^2)dy+vdt, \quad dZ = d\bar{z} \quad (12.85)$$

A glance to (12.82)and (12.83) shows that these transformations do not coincide, this meaning that the so-called Principle of Local Lorentz Invariance is not valid in the framework that we have described that includes the third step which led to (12.77), but it is acceptable if one is willing to renounce to (12.77) and accept instead as meaningful the local infinitesimal change of space coordinates:

$$d\tilde{x} = dx, \quad d\tilde{y} = (1+\frac{1}{2}\omega^2\rho^2)dy, \quad d\tilde{z} = dz \quad (12.86)$$

On the other hand one sees, neglecting terms of order ω^2 that the Principle of local Galilean invariance is always satisfied as it was obvious from the beginning.

3. The Wilson and Wilson experiment

Let $(F^{\alpha\beta},\ K^{\gamma\delta})$ be the Minkowski's description of an electromagnetic field in a medium with electric permittivity ϵ and magnetic permeability μ. The physical interpretation of this couple of 4-dimensional skew-symmetric tensors comes from the following identifications, where E^α is the electric field, B^α is the magnetic induction, D^α is the electric displacement and H^α is the magnetic field:

$$E^\alpha = F^\alpha_{\ \beta}u^\beta, \quad B^\alpha = -\tilde{F}^\alpha_{\ \beta}u^\beta \quad (12.87)$$

$$D^\alpha = K^\alpha_{\ \beta}u^\beta, \quad H^\alpha = -\tilde{K}^\alpha_{\ \beta}u^\beta \quad (12.88)$$

u^α being the unit vector tangent to the congruence defining the motion of the frame of reference and $\tilde{F}^{\alpha\beta}$ being the dual of $F^{\alpha\beta}$. In a Galilean frame of reference co-moving with the medium ($u^i = 0$) these formulas translate as follows:

$$E^i = F^i{}_0, \quad B_k = \frac{1}{2}\delta_{ijk}F^{ij} \tag{12.89}$$

$$D^i = K^i{}_0, \quad H_k = \frac{1}{2}\delta_{ijk}K^{ij} \tag{12.90}$$

The remaining components being zero.

The constitutive equations are:

$$D^i = \epsilon E^i, \quad H_k = \frac{1}{\mu}B_k \tag{12.91}$$

and we shall use units such that for vacuum $\epsilon_0 = \mu_0 = 1$. The matching conditions at the boundary of a neutral medium with vacuum are:

$$(D^i_+ - D^i_-)n_i = 0, \quad \delta_{ijk}(E^i_+ - E^i_-)n^j = 0 \tag{12.92}$$

$$(B^+_i - B^-_i)n^i = 0, \quad \delta^{ijk}(H^+_i - H^-_i)n^j = 0 \tag{12.93}$$

where n_i is the normal to the boundary and where a super or sub index $+$ will refer to vacuum and $-$ will refer to the dielectric medium.

If the medium is rigid, is uniformly rotating with respect to a Galilean frame of reference and adapted coordinates to the co-moving frame of reference are used, then the identifications (12.89) and (12.90) do not correspond anymore to. (12.89) and (12.90). Let us assume that cylindrical coordinates are used and therefore the line-element of Minkowski's metric in a Galilean frame of reference is (12.35), and (12.45) in the rotating one. The appropriate identification is then given by the following formulas, invariant under arbitrary synchronizations, derived from (12.87) and (12.88):

$$e^i = \xi^{-1}f^i{}_0 = g_{0\alpha}f^{i\alpha}, \quad b_k = \frac{1}{2}\sqrt{\hat{g}}\delta_{ijk}f^{ij} \tag{12.94}$$

$$d^i = \xi^{-1}k^i{}_0 = g_{0\alpha}g^{i\alpha}, \quad h_k = \frac{1}{2}\sqrt{\hat{g}}\delta_{ijk}k^{ij} \tag{12.95}$$

where $f^{\alpha\beta}$, $k^{\alpha\beta}$ are the images of $F^{\alpha\beta}$, $K^{\alpha\beta}$ by the congruence transformation (12.40); $\xi = \sqrt{-g_{00}}$; $\hat{g}$ is the determinant of the 3-dimensional metric (12.47) and $g_{0\alpha}$ are the corresponding coefficients of (12.45). Therefore:

$$\xi = \sqrt{1-\omega^2\rho^2}, \quad \hat{g} = \frac{\rho}{\sqrt{1-\omega^2\rho^2}} \tag{12.96}$$

$$g_{00} = -(1-\omega^2\rho^2), \quad g_{11} = 1, \quad g_{22} = \rho^2 \quad g_{33} = 1, \quad g_{02} = \omega\rho^2 \quad (12.97)$$

The constitutive equations (12.91) and matching conditions (12.92) and (12.93) remain unchanged in form but they hold now for the transformed fields:

$$d^i = \epsilon e^i, \quad h_k = \frac{1}{\mu} b_k \quad (12.98)$$

and:

$$(b^i_+ - b^i_-)n_i = 0, \quad \delta^{ijk}(e^i_+ - e^i_-)n_j = 0 \quad (12.99)$$

$$(d^i_+ - d^i_-)n_i = 0, \quad \delta^{ijk}(h^i_+ - h^i_-)n_j = 0 \quad (12.100)$$

We are going to use the preceding considerations to discuss the Wilson and Wilson experiment. In this experiment a hollow dielectric cylinder is rotated with constant angular velocity ω in a uniform and constant magnetic field B parallel to the axis of rotation. Two brushes fixed with respect to the laboratory rub the inner and outer cylindrical surfaces of radius, say ρ_1 and ρ_2, and the electric potential difference between them $\triangle V$ is measured. The results obtained in the experiments agree quite well with the approximate formula which one obtains neglecting terms of order $\omega^2\rho^2$ or smaller:

$$\triangle V = \frac{1}{2}\mu B\omega(1 - \frac{1}{\epsilon\mu})(\rho_2^2 - \rho_1^2) \quad (12.101)$$

For our purposes though it is interesting to consider the following fully relativistic formula:

$$\triangle V = -\frac{\mu B}{2\omega}(1 - \frac{1}{\epsilon\mu}) \ln \frac{1-\omega^2\rho_2^2}{1-\omega^2\rho_1^2} \quad (12.102)$$

This formula can be derived using a variety of methods ([10]-[16]). Our goal below is to show that (12.102) can be derived from (12.40) and the line-element(12.45), without using any non trivial time transformation nor any re-definition of the space coordinates, thus demonstrating that these two simple ingredients (12.40) and (12.45) are all that it takes in some cases to implement Special relativity physics. This explains why so many methods lead to the correct result.

The tensor $F^{\alpha\beta}$ in Wilson and Wilson's experiment has a single non zero component outside the cylinder, namely in cylindrical coordinates:

$$B_3^+ = \rho F_+^{12} = B \tag{12.103}$$

where B is the uniform magnetic field. With the coordinate transformation (12.40) this contravariant field remains unchanged:

$$f_+^{12} = F_+^{12} \tag{12.104}$$

so that:

$$b_3^+ = \frac{\rho}{\sqrt{1-\omega^2\rho^2}} f_+^{12} = \frac{B}{\sqrt{1-\omega^2\rho^2}} \tag{12.105}$$

but the line-element is now (12.45) and therefore from (12.94) and (12.97) we get:

$$e_+^1 = \frac{\omega\rho B}{\sqrt{1-\omega^2\rho^2}} \tag{12.106}$$

the remaining components being zero.

The non trivial matching conditions (12.99) (12.100) are:

$$d_+^1 = d_-^1, \quad h_+^3 = h_-^3 \tag{12.107}$$

wherefrom we get:

$$e_-^1 = \frac{\omega\rho B}{\epsilon\sqrt{1-\omega^2\rho^2}}, \quad b_3^- = \frac{\mu B}{\sqrt{1-\omega^2\rho^2}} \tag{12.108}$$

and therefore:

$$f_{-0}^1 = \frac{\omega\rho B}{\epsilon}, \quad f_-^{12} = \frac{\mu B}{\rho} \tag{12.109}$$

Finally from:

$$f_{-0}^1 = g_{00} f_-^{10} + g_{02} f_-^{12} \tag{12.110}$$

we have:

$$f_-^{10} = \frac{1}{g_{00}}(f_{-0}^1 - g_{02} f_-^{12}) \tag{12.111}$$

or:

$$f_-^{10} = \frac{\mu\omega\rho B}{1-\omega^2\rho^2}(1 - \frac{1}{\epsilon\mu}) \tag{12.112}$$

Transforming back to the Galilean frame of reference to take into account the fact that the brushes do not move we obtain:

$$E^1 = F^1_{-0} = -F^{10}_{-} = -f^{10}_{-} \tag{12.113}$$

Using (12.112) and

$$E^1 = -\partial_\rho V \tag{12.114}$$

we obtain by a simple integration the formula (12.102)

4. The Michelson-Morley experiment

At the end of Sect. 2 we proposed to deal with the unsatisfactory situation to which it leads the fact that the space metric (12.47) is not Euclidian by denying to it the role of describing the geometry of space and to attribute this role to its principal transform (12.65). This raises the following question: what is then the meaning of (12.47)? The answer that we favor is that this metric describes a crystal-like structure of vacuum that is responsible for an anisotropy of the round trip speed of light coming from a distinction between optical length, which is defined using (12.47), and geometrical (or mechanical) length, which is defined using (12.65). In other words we propose to predict that the round trip velocity of light v_γ propagating in a direction γ^i in a location with coordinates $\bar{x}^k(\bar{x}, \bar{y}, \bar{z})$ will be given by:

$$v_\gamma(\bar{x}^k) = \frac{\sqrt{\bar{g}_{ij}(\bar{x}^k)\gamma^i\gamma^j}}{\sqrt{\hat{g}_{ln}(\bar{x}^k)\gamma^l\gamma^n}} \tag{12.115}$$

With this interpretation it follows that the unit vectors defined covariantly by the 1-forms:

$$\bar{\theta}^1 = d\bar{\rho}, \quad \bar{\theta}^2 = \bar{\rho}d\varphi, \quad \bar{\theta}^3 = d\bar{z} \tag{12.116}$$

define the principal directions of the anisotropy, and that the scalars (12.75) are the corresponding speeds. A statement that can be summarized by the following formula:

$$\frac{1}{v_\gamma^2} = \frac{\gamma_1^2}{c_1^2} + \frac{\gamma_2^2}{c_2^2} + \frac{\gamma_3^2}{c_3^2} \tag{12.117}$$

γ^i being the cosines of the direction of propagation with respect to the principal directions (12.116).

This point of view has been used to predict a non null outcome for those experiments of the Michelson-Morley type that rotate on the horizontal plane whatever is used as an oriented rigid standard of length.

If the location is at a colatitude θ and the direction of propagation lies on a horizontal plane making an angle A with the East direction then:

$$\vec{\gamma} = -\cos\theta \sin A\, \vec{e}_1 + \cos A\, \vec{e}_2 + \sin\theta \sin A\, \vec{e}_3 \tag{12.118}$$

where $\vec{e}_i$ are the unit vectors corresponding to the principal directions (12.116), and we finally obtain:

$$v_\gamma \approx 1 - \frac{3}{4}\omega^2 R^2 \sin^2\theta \left(1 - \frac{1}{2}\sin^2\theta\right) - a_2 \cos 2A \tag{12.119}$$

where R is the radius of the Earth, ω is its angular velocity and:

$$a_2 = \frac{3}{8}\omega^2 R^2 \sin^4\theta \tag{12.120}$$

Most of the experiments of the Michelson-Morley type include to improve its sensitivity a standard of length that rotates in the horizontal plane. And therefore although the purpose of the experiment is not to measure the parameter a_2 in fact they measure it as part of a raw result to be used to test what they claim would be violations of Special relativity had the experiment give a clear cut result. In the experiment of Brillet and Hall a_2 was measured to be 2.1 10^{-13} but this result was cited as being spurious without further comment. The predicted result calculated from (12.120) is $3.1\ 10^{-13}$. We believe that the work presented in this paper, as well as a few others that have preceded it, justifies that the result of Brillet and Hall be checked. A few recent experiments have improved the sensitivity of Brillet and Hall but unfortunately they do not include a rotating arm and therefore they are insensitive to the value of a_2.

Acknowledgments

I gratefully acknowledge the help provided by A. Hernández and A. Molina at different stages of the preparation of this manuscript as well as the stimulating discussions that followed a careful reading of it by J. M. M. Senovilla. I also gratefully acknowledge the position of visiting professor to the UPV/EHU that I have been holding since October 2000.

References

[1] J. W. Then, Am. J. Phys. **30** (1962)

[2] M. Trocheris, Phil. Mag., **40** (1949)

[3] H. Takeno, Prog. Theor. Phys., **7** (1952)

[4] L. Herrera and A. Di Prisco, arXiv:gr-qc/0201023

[5] B. Mashoon, Phys. Letters A, **145**, No. 4 (1990)

[6] B. Mashoon, arXiv:gr-qc/0301065

[7] Ll. Bel, Gen. Rel. and Grav. **28**, No. 9 (1996).

[8] J. M. Aguirregabiria and Ll. Bel, Gen. Rel. and Grav. **33**, No. 11 (2001)

[9] Ll. Bel, arXiv: gr-qc/0210057

[10] M. Wilson and H. A. Wilson, Proc. R. Soc. London Ser. A **89** (1913)

[11] G. N. Pellegrini and A. R. Swift, Am. J. Phys. **63**, 8, (1995)

[12] T. A. Weber, Am. J. Phys. **65**, 10, (1997)

[13] T. Ridgely, Am. J. Phys. **66**, 2, (1998)

[14] T. Ridgely, Am. J. Phys. **67**, 5, (1999)

[15] G. N. Pellegrini and A. R. Swift, Am. J. Phys. **65**, 9, (1997)

[16] R. D. Klauber, arXiv:gr-qc/0107035

[17] J. B. Hertzberg, S. R. Bickman, M. T. Hummon, D. Krause, Jr., S. K. Peck and L. R. Hunter, Am. J. Phys. **69**, 6, (2001)

[18] A. A. Michelson and E. W. Morley, Am. J. Sci. **34** (1887)

[19] A. Brillet and J. L. Hall, Phys. Rev. Letters **42**, 9 (1979)

[20] C. Braxmaier, H. Müller, O. Pradl, J. Mlynek, A. Peters and S. Schiller, Phys. Rev. Letters **88**, 1 (2002)

[21] P. Wolf, S. Bize, A. Clairon, A. N. Luiten, G. Santarelli, and M. E. Tobar, arXiv:gr-qc/0210049

[22] J. A. Lipa, J. A. Nissen, S. Wang, D. A. Stricker and D. Avaloff, Phys. Rev. Lett., **90**, 060403 (2003)

[23] H. Aspden, Phys. Letters **85A**, 8 (1981)

[24] Ching-Chuan Su, EuroPhys. Lett., **56**, 2 (2001)

[25] R. D. Klauber, Founds. Phys. Lett., **11**, 5 (1998) and arXiv:gr-qc/0103076; arXiv:gr-qc/0210106

[26] J. M. Aguirregabiria, Ll. Bel, J. Martín and A. Molina, in *Recent Developments in Gravitation*, Eds. A. Feinstein and J. Ibañez. World Scientific (1992)

[27] Ll. Bel, J. Martín and M. Molina, J. Phys. Soc. Japan, **63**, 12 (1994)

[28] Ll. Bel and A. Molina, Il Nuovo Cimento, **11B**, 6 (2000) and arXiv:gr-qc/9806099

[29] J. Stachel, in *Einstein and the History of General Relativity*, D. Howard, J. Stachel, Eds., *Einstein Studies*, **1** (1989)

Chapter 13

DOES ANYTHING HAPPEN ON A ROTATING DISK?

Angelo Tartaglia
Dipartimento di Fisica del Politecnico di Torino
INFN, Sezione di Torino, Torino, Italy
angelo.tartaglia@polito.it

Abstract The paper discusses the issue of time and length measurements on a rotating disk. Considering operational procedures it is shown that the only relevant phenomena related to the rotation of the disk are the ones affecting clocks. Direct measurements of length do not give results in contrast with the ones obtainable for a non-rotating disk. The space geometry on the disk is also discussed and again considering real triangles made of light rays an unambiguous result is obtained only calling into action the impossibility to uniquely synchronize clocks on the turntable.

1. Introduction

The problem of length and time measurements on a rotating disk is as old as the Special Relativity Theory. Many scientists have discussed it, reaching some times opposite conclusions with respect, for instance, to the length of the rim of the rotating disk. Extensive bibliographies may be found in other papers in this book, such as [1],[2], or in previous articles [3]. So I think I can avoid reproducing a long list of citations here and shall limit myself to what is strictly needed for the scope of the present discussion.

It can appear rather surprising that after almost one century there still is debate on the subject. Special and General Relativity (SR and GR) are by now well established theories, most of the fundamentals have been worked out in the early years soon after the outset, and now the research is focused on the problems of the compatibility between GR and quantum mechanics, on cosmology, on new experimental verifications beyond the classical experiments and observations.

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 261-273.

If the debate on the rotating disk is still there, it is because in rotations in general there is something fundamental and because the language we use to describe them is not exempt from ambiguities, whose effect sometimes is that people are saying the same thing being convinced to be on opposite positions, or, on the contrary, apparently innocent formulations hide underlying misunderstandings.

The questions on the floor, trivializing a bit the whole subject, are:

- does the rim of the disk lengthen or shorten as a consequence of rotation?

- is the space of the rotating disk curved, and is the curvature positive or negative?

- what is the space of the rotating disk?

I myself shall once again discuss the rotating disk in this paper, trying and adopting, as long as possible, an operational attitude. I shall consider possible actual measurements and the way they would be performed, were it not for the smallness of the effects. Then I shall look for the simplest interpretation of the data within the framework of SR. As we shall see, the conclusion will be that the only uncontroversial effect an observer will perceive and measure on a rotating disk regards time and clocks; in particular the relevant fact will be the lack of synchrony on board the disk. All other effects affecting lengths are pure consequences of the behaviour of proper time for the rotating observer, and are evidenced only when the length measurement is indirect and involves time.

2. Posing and defining the problem

Let us consider a solid disk initially at rest with respect to an inertial reference frame. The disk starts rotating with an increasing angular speed up to a predefined value ω. Once that value has been attained the movement continues steadily. We are interested in the final steady state, but would like to discuss the dynamic phase also, to see whether it can influence the ultimate result.

Of course we are assuming a rather idealized situation, even though we are excluding such things as rigid bodies that would contradict the bases of relativity. According to this approach we suppose: a) that the stresses induced in the disk during the acceleration phase never reach the threshold of permanent plastic deformation; b) that the elastic energy stored in the disk during the acceleration will in the end be totally dissipated in a way or another, leaving a state of simple steady rotation without any superimposed oscillation.

Pictorially we may think of two different procedures to speed up the disk: using an engine to apply a torque on the axle (stresses propagate from the middle towards the periphery); using symmetrically placed and identical rockets pushing tangentially along the rim of the disk (stresses propagate inward towards the center). In both situations we assume the axial symmetry is pre-

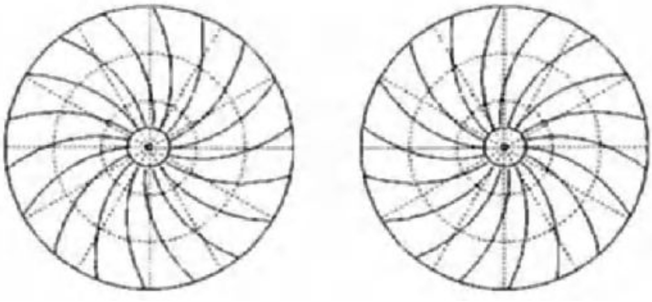

Figure 13.1. On the left the condition of a set of flexible rods during clockwise acceleration from the axel. On the right, same situation but with acceleration from the external end

served. Under the previous hypotheses and the symmetry condition we see that the physical continuity of the disk cannot be broken at any place, because there is no reason to privilege any particular place. To say better, if the acceleration were too violent the disk would break at a given radius still preserving the axial symmetry.

Should we replace the disk with a set of radial bars fixed to the central shaft, the two configurations we have depicted would lead, while accelerating, to the two patterns in figure 13.1. The bars would stay a bit apart one from the other according to lengthening induced by radial stresses. When reaching the steady state condition the system would oscillate back and forth between the two configurations until the total elastic energy is dissipated and the bars are radial again.

3. Local measurements

We are interested in measurements of lengths and time intervals on the disk in the final steady state. The measurements may be local or global. By local we mean over distances small with respect to the radius of the turntable and over times short as compared to the revolution period.

Let us start with local measurements of lengths. These may be performed, in principle, in two ways: either by comparing the distance to be measured with a standard 'rod', or using light and determining the proper time interval it takes to go from one end to the other of the distance, and back.

When available, the direct comparison method is the most appropriate and typical, because it does not involve anything else than length. Now, if we have a material meter whatever happens to the matter of the disk it will happen to the meter too, consequently the result of the measuring process will necessarily be the same as the one obtained for the non-rotating system. This point is not uncontroversial, since Rizzi and Ruggiero[5] (RR) have based their claim that the length of the circumference along the rim of the rotating disk is perceived as greater by a co-rotating observer precisely on the assumption that the solid disk behaves differently than a standard rod. Indeed in their paper they claim that a free ends meter on the rim of the rotating disk is not stressed tangentially, while matter of the rim feels a stress. This claim is logically inconsistent for at least two reasons.

a) The stress RR are considering has, in their view, a purely kinematic origin (they exclude, by hypothesis, any effect due to the very physical nature of the disk, and, as we do, are studying the final steady state condition). However if a stress is present different materials behave differently, so this 'kinematic' measurable effect would depend on the nature of the disk. In particular we could sometimes break the disk some others not, depending on the nature of it. Since a purely kinematic effect strictly preserves the symmetries, where should the disk start breaking? Suppose our rotating object, as suggested in the previous section, is a set of radial bars carrying at their extremity little transverse standard rods initially touching each other at the ends. According to RR, in the final steady state a gap would be present between each pair of rods, not considering elastic effects. However when considering kinematics, phenomena affecting the separation of two points stay the same irrespective of having matter or void interposed. This at least is the case of the typical kinematical effect: the Lorentz contraction. If it is so: why should the rods stay unchanged and the gaps increase? why should it not happen the inverse? If the same happens to both, nothing happens in the view of a comoving observer. On this respect sometimes standard rods are thought of as being something magic, not partaking the general properties of matter [4]: this view point is rather inconsistent; it cannot exist any absolute standard.

b) Suppose the rotating observer is sitting in a small black box fixed to the rim of the disk and performing purely local measurements. If RR's claim were true the observer would be able, by local measurements, to detect his proper rotation recognizing the 'true' nature of the 'gravitational field' he feels. However locally a steady rotation is not different from a translation, were it not for that 'gravitational field' we know is the centrifugal acceleration. This contention contradicts the principle of relativity.

Let us discuss the use of light. Here the physical properties of the light beam, such as wavelength and frequency, are unessential; what matters is the typical postulate of relativity i.e. the universal constancy of the speed of light c. Consider a portion of the rim of the disk whose length we wish to measure: at one end we have a source/receiver of light, at the other end there is a mirror. We must imagine we are sending a beam grazing the circle along the rim. Light of course does not spontaneously bend its rays, so we must think the trajectory is obtained by successive reflections and is consequently a polygon: infinitely frequent mirrors would produce a circle. Once this is accepted the rest goes on just as in the case of translational uniform motion: we simply measure a proper time interval. The result is independent from the actual velocity of the disk; it is the same as the one obtained when at rest.

RR in their paper [5] propose to use light as a meter, or, to say better, to compare the unknown length with the wavelength of a monochromatic light beam. Now, wavelength and frequency of light are no invariant per se, only the speed c is a real invariant. A light source on the rim of the rotating disk behaves as if it were in a static gravitational field, corresponding to the centrifugal potential. We can expect then a 'gravitational' red shift. However both the source and the final mirror are at the same distance from the center, i.e. at the same potential: they feel the same frequency. The frequency is a peculiarity of the source; once established that it appears to be unchanged for a corotating observer, since the speed of light is always the same, the wavelength too will be the same. Again the observer on the disk does not notice any change in the peripheral lengths. In fact light is travelling in empty and flat space time where its physical parameters do not undergo any kind of change. The actual difference between the emission of light in a real gravitational field and under the action of a centrifugal force is that in the latter case the effect is present in the source alone and is (kinematically) due to the acceleration of the clock represented by the source itself. The consequence is that an inertial observer at rest with the axle of the platform will perceive a redshifted frequency with respect to the one measured at the very source (the standard one). The frequency seen by the inertial observer is a combination of 'gravitational' redshift and of Doppler effect. In any case however the mirror on the rim senses the same as the source. Nothing can be expected from measuring the wavelength of light on board: what wavelength should we compare with what and how? As a paradox suppose, as RR do, that the length of a portion of the rim increases while the typical wavelength remains invariant. In a stationary situation and assuming perfect monochromatism the light sent to the distant mirror and the one reflected back would give an interference pattern. Changing the travelled length would lead to a different pattern, thus allowing for absolute detection of stationary motion, discriminating rotation from translation even locally. However locally and instantaneously a rotation is not different from a translation.

By the way, lengths measured along the radius are not affected by any change either, since the centrifugal forces act only on objects tied to the platform. If we compare the radius with standard rods tied to the disk we shall find exactly the same result when the disk stands still as when it rotates. Using light to determine the travel time back and forth from the periphery to the center, we must remember that the beam is freely moving in a perfectly flat background. The only device which is affected by rotation is the clock one uses. A clock on the rim would register a proper time interval for the flight time of light to the axis and back

$$\delta\tau = \sqrt{1 - \frac{\omega^2 R^2}{c^2}}\delta t \tag{13.1}$$

Of course δt in perfectly flat space-time is $\delta t = 2R/c$. If the observer decides to interpret the difference in proper times between the two conditions (no 'gravitational field'=no rotation; 'gravitational field'=rotation) as a difference in the length of the radius then he will conclude that the radius R' is now contracted according to

$$R' = R\sqrt{1 - \frac{\omega^2 R^2}{c^2}}$$

The discrepancy with respect to the direct measurement with meter sticks suggests that the problem is with the clock. We may also think that rotation is equivalent to introducing a 'refractive index' for light

$$n = \frac{1}{\sqrt{1 - \frac{\omega^2 R^2}{c^2}}} \tag{13.2}$$

In that case the relation between R and R' is the same we find between optical and geometric path in a transparent medium. Remarkably we see that apparently the result corresponds to a constant refractive index (13.2) along the path with a value depending on the position of the final observer. If we assumed a truly varying n as function of r the result would be different and would not correspond to the time of flight measured by our observer. This fact again stresses the dependency from the only clock one uses.

In general we can say that nothing strange happens on a rotating disk locally. Simply local observers feel the equivalent of a gravitational field with the usual implications of it.

4. Global measurements

As stated previously, by 'global' I mean a length or time measurement referring to extensions and spans comparable with the size of the platform and with the revolution period. A typical example of global measurement is when the path one considers encircles the axis of the turntable at least once.

Let us begin with the length measurement of the whole rim of the disk. Here I summarize and reproduce the procedure already expressed in [6]. We can imagine that the periphery of the disk carries a little vertical edge inserted in a narrow circular slit in a static basement. Suppose the gap in the slit is arbitrarily small. We can, by a sound measuring procedure, deduce the length of the rotating edge from the length of the non-rotating slit. Considering that also the radius R is not changing the conclusion of an inertial observer is of course that the length of the continuous rim of the rotating disk is exactly $2\pi R$.

This approach is really global and completely independent from the problem of synchronization.

Of course another typical procedure consists in using light and measuring times of flight. We may think of a cylindrical mirror placed at the rim of the rotating disk. A light beam is sent along the mirror and moving round along the circumference it arrives at the back of the source. Suppose the back of the source is in turn a mirror: the beam is reflected back until it returns to the starting point. Suppose you can measure the total time of flight; dividing by 2 and multiplying by the speed of light c you would expect to find the length of the rim. Note that this procedure introduces a sort of averaging between the co- and counter-rotation senses.

The result is

$$\delta\tau = \frac{2\pi R}{c\sqrt{1-\frac{\omega^2 R^2}{c^2}}}$$

The induced length would then be greater than for a static disk and in contradiction with the direct measurement. Of course we can again interpret what we see in terms of an effective refraction index like (13.2), which reduces the average speed of light along the rim of the disk due to the 'gravitational field' corresponding to the centrifugal forces of the rotation. These forces actually influence the mirrors, not the light as such, since no real gravitational field is present. Considering the specific configuration of the disk, however, it is now possible to measure one way times of flight separating clockwise motion from anticlockwise. If we distinguish the two cases the corotating and counter-rotating proper times of flight would be

$$\begin{aligned}\tau_{co} &= \frac{2\pi R}{c}\sqrt{\frac{c-\omega R}{c+\omega R}}\\ \tau_{counter} &= \frac{2\pi R}{c}\sqrt{\frac{c+\omega R}{c-\omega R}}\end{aligned}$$

What should the observer conclude with respect to the length of the rim? May be he would guess there is something with the clock and the global time measurement or simply that his reference frame is not inertial and in particular that it is steadily rotating.

4.1 Space curvature

Another kind of global measurement we may think of is to look for the space geometry on the surface of the disk using light rays and triangles. This is a very old idea which was also presented in a popularized version in the famous book on the adventures of Mr. Tompkins, by George Gamow[7].

Considering for instance the approach of Mœller[8] we may start from the metric for a rotating observer in Minkowski space-time:

$$ds^2 = c^2\left(1 - \frac{\omega^2 r^2}{c^2}\right)dt^2 - 2\omega r^2 d\phi dt - dr^2 - r^2 d\phi^2 - dz^2 \qquad (13.3)$$

Space-time is of course flat, no matters what coordinates we use, however geodesics do not in general appear to be straight. Geodesic proper time and the z coordinate are both proportional to coordinate time. Explicitly performing the calculation and considering motion in a z = constant plane, the meaningful geodesic equations are

$$\begin{cases} \frac{d^2r}{dt^2} - \omega^2 r - 2\omega r\frac{d\phi}{dt} - r\left(\frac{d\phi}{dt}\right)^2 = 0 \\ \frac{d^2\phi}{dt^2} + 2\frac{\omega}{r}\frac{dr}{dt} + \frac{2}{r}\frac{dr}{dt}\frac{d\phi}{dt} = 0 \end{cases}$$

A more compact form of the equations is

$$\begin{cases} \frac{d^2r}{dt^2} - r\left(\omega + \frac{d\phi}{dt}\right)^2 = 0 \\ \frac{d^2\phi}{dt^2} + \frac{2}{r}\frac{dr}{dt}\left(\omega + \frac{d\phi}{dt}\right) = 0 \end{cases} \qquad (13.4)$$

The second equation in (13.4) is easily solved giving

$$\frac{d\phi}{dt} = -\omega + \frac{A}{r^2} \qquad (13.5)$$

where A is a constant. Introducing this solution into the first equation in (13.4) transforms it into

$$\frac{d^2r}{dt^2} - \frac{A^2}{r^3} = 0 \qquad (13.6)$$

Only radial geodesics are straight lines [8], so in general geodesic triangles we may think to draw on the surface of the rotating disk will be non-Euclidean. Hence a first conclusion in favor of a curvature of the space of the disk. In fact however this is a sort of definition of the space of the disk. This definition is not at all trivial when dealing with metric tensors possessing off diagonal terms. Space and simultaneity are strictly connected and in a rotating system time appears to be polydromous [9]. In a sense, the properties of space depend on the way clocks tick.

To go a step further let us consider the case of light rays, which we expect to coincide with null geodesics. From (13.3) we deduce for any null world line:

$$\frac{d\phi}{dt} = -\omega \pm \frac{c}{r}\sqrt{1 - \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2} \qquad (13.7)$$

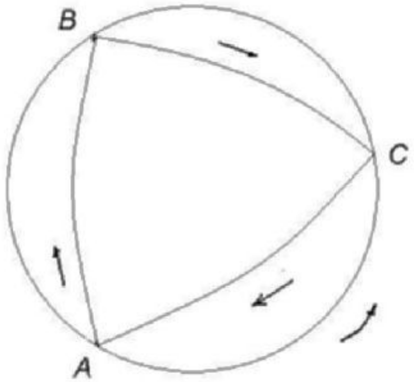

Figure 13.2. Light rays as seen by the rotating observer in A. The platform rotates counter-clockwise. Light goes around the 'triangle' clockwise.

Now introducing (13.7) into (13.5) and solving with respect to $r\left(t\right)$ we obtain

$$r\left(t\right) = \pm\frac{1}{c}\sqrt{C_1^2c^4 + A^2 + t^2c^4 + 2tc^4C_1} \tag{13.8}$$

C_1 is of course a constant. Equation (13.6) is identically satisfied.

Introducing (13.8) into (13.7) and solving for $\phi\left(t\right)$ we obtain

$$\phi\left(t\right) = -\omega t \pm \arctan\left(c^2\frac{t + C_1}{A}\right) + C_2 \tag{13.9}$$

C_2 is a new constant. Equations (13.9) and (13.8) represent the parametric equations for the space trajectories of null rays. Converting these equations to the coordinates of an inertial observer we would again find a straight line.

Letting aside for a moment any formal approach to the problem, we may think to a practical procedure that could directly show the geometric properties of the rotating disk. Suppose you are at some place on the rim and that actually we are inside a rotating cylinder whose atmosphere is rather smoky: this way any light ray will become visible along its whole extension. Let us send a photon across the disk towards appropriately positioned mirrors in such a way that it comes back after two reflections, drawing a triangle in the air. If our cylindric room is not rotating everything is as usual in flat space time and the space geometry is of course Euclidean. Suppose now we (the observer and source) and the mirrors start rotating counterclockwise. Light rays in space time continue of course to be straight lines, however, for a pure coordinate effect they will appear to the rotating observer as bending. If light is sent

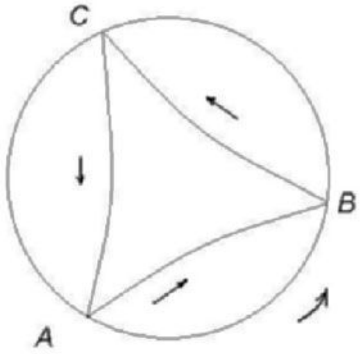

Figure 13.3. Light rays as seen by the rotating observer in A. The platform rotates counterclockwise. Light goes around the 'triangle' counterclockwise.

along the 'triangle' clockwise, the counterclockwise rotating observer will see something like figure 13.2

Apparently one can conclude that there is a positive space curvature. However look at the situation when light is sent counterclockwise (the same sense as the rotation of the platform): figure 13.3.

Now there is an apparent negative curvature. The latter is the conclusion that one finds in [8].

Coming back to space-time geometry, we know that the rotating disk is a cylinder with a timelike axis, where each point of the disk corresponds to a helix whose inclination is increasing with increasing radii. The 'space' of any observer on the disk will be a helicoid. Light rays across the cylinder are of course straight worldlines. Their projection onto the space of an observer are projections onto the helicoid, being the projection lines the helicoidal worldlines of the places in the disk. When the light ray is 'forwardly' directed (on one side of the central axis) its projection on the helicoid bends inwards; when the ray goes the other way the projection bends outwards.

The observer cannot interpret this behavior simply in terms of curvature on the disk. The light rays are telling him that he is rotating and in what direction he rotates.

When working with the line element (13.3) we proceed just as we could have been done in a space-time curved by gravity. Here however the situation is different. We are considering the viewpoint of a rotating observer in flat space-time. If the observer rotates there must exist appropriate constraints and forces that oblige him to do so. Our space-time is equivalent to a whole family of equally rotating observers, i.e. the whole solid rotating disk; geodesics are lines minimizing the travel time on the clocks of these observers. Here however not everything is rotating, like it is the case for instance of phonons in the bulk of the disk, or of any little animal crawling on its surface. What is not tied to the physical disk is indeed not accelerating, and the related effects seen by the rotating observer are just coordinate effects. If I turn on myself looking at the sky I could conclude that distant stars are moving much faster than the speed of light, but of course this contention is meaningless because distant stars are not rotating with me.

5. Conclusion

In this short discussion of the rotating disk I have shown what I am summarizing in the following.

1) Local measurements on the rotating platform only evidence the presence of a 'gravitational field', which a posteriori an observer performing both local and global measurements can interpret as being due to centrifugal forces.

2) Global measurements of length performed by direct comparison with a known length, be it the length of two circles in the static reference frame delimiting the rotating rim of the disk, or the length of a chain of standard roads, give the same results whether the turntable is moving or not.

3) Global measurements of time made on the rotating platform evidence the impossibility to uniquely synchronize clocks. In particular if we synchronize two identical clocks by direct comparison at some place, then slowly carry one of them around the axis of the platform until it can be compared again with the static (with respect to the platform) one, we see that the two clocks are no longer synchronous (see for instance [10]). This phenomenon is a manifestation of the Sagnac effect and the lack in synchrony after one complete turn is proportional to the angular velocity of the disk with respect to any inertial observer.

4) Global measurements of length performed through the measurements of times of flight induce peculiar results actually due to the non uniqueness of synchronization on revolving systems. These results are improperly used to conclude anything regarding lengths.

5) Global geometric properties deduced for instance from light rays triangles produce the same type of 'ambiguity' as the global measurements of time and may be used only to deduce the presence of the rotational motion, not to assess any unique 'curvature' of the space of the disk. Actually space has properties directly connected with 'simultaneity' surfaces for rotating observers. The 'synchrony' surface for a set of rotating observers is indeed a Riemann helicoid, whose 'global' curvature cannot be defined; the local curvature is of course absent.

All in one we see that (global) measurements on board the rotating turntable allow the observer to detect and interpret his condition as an absolute rotation at a given angular velocity in a flat space-time.

A remark we could consider now is that letting the radius of the platform go to infinity, while keeping the peripheral velocity constant should reproduce the known steady translation SR results. It is indeed so provided we consider that on an infinite radius disk no global measurement (in the sense stated above) is possible and locally everything is always as it should according to SR. Infinite radius and constant peripheral velocity means in fact no centrifugal force.

My personal conclusion is that no open problems or mysteries are left regarding the rotating disk and that this century long debate should not continue if not for pedagogical purposes. In fact the subtelties sometimes we need and the general discussion of the problem has a great educational value and can help to better understand this beautiful and solidly establish theory which is relativity.

Additional Comment

Some conversations with the editors and the reading of other contributions induces me to present this additional comment to my paper. Maybe it is a rather trivial remark, but I think it should explicitly be formulated. The point is that sometimes there is a bit of confusion between the physics on a rotating platform and the one in a rotating platform. What I mean is that the physics on a turntable (which is the one I dealt with in this paper) is the physics of flat space-time as seen by rotating observers. Sometimes however people are indeed considering a rotating space-time. This situation is what I call the physics in the turntable. Now everything, including the electromagnetic field, is indeed rotating; a line element like (13.3) is highly questionable because of its behaviour at space infinity, and of course the situation is entirely different. The latter is a peculiar general relativistic problem, which has scarce if any connection with turntables carrying instruments and observers. I have here discussed the special relativistic problem.

References

[1] G. Rizzi and M.L. Ruggiero in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series “Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)

[2] Ø. Grøn in *Relativity in Rotating Frames*, eds. Rizzi G. and Ruggiero M.L., in the series “Fundamental Theories of Physics", ed. Van der Merwe A., Kluwer Academic Publishers, Dordrecht, (2003)

[3] G. Rizzi, A. Tartaglia, Found. Phys. Lett., **12**, 181-188 (1999)

[4] Ø. Grøn, *Am. J. Phys.*, **43**, 869 (1975); Ø. Grøn, *Int. J. Theor. Phys.*, **16**, 603 (1977); Ø. Grøn, *Found. Phys.*, **9**, 353 (1979)

[5] G. Rizzi and M.L. Ruggiero, *Found. Phys.*, **32** 1525 (2002), gr-qc/0207104

[6] A. Tartaglia, Found. Phys. Lett., **12**, 17-28 (1999)

[7] G. Gamow, *Mr Tompkins in paperback*, Cambridge University Press, New York (1965)

[8] C. Møller, *The Theory of Relativity*, Oxford University Press, Oxford (1972)

[9] A. Tartaglia, Gen. Rel. and Grav., **32**, 1745 (2000)

[10] G. Rizzi, A. Tartaglia, Found. Phys., **28**, 1663 (1998)

Chapter 14

PROPER CO-ORDINATES OF NON-INERTIAL OBSERVERS AND ROTATION

Hrvoje Nikolić
Theoretical Physics Division, Rudjer Bošković Institute,
P.O.B. 180, HR-10002 Zagreb, Croatia
hrvoje@thphys.irb.hr

Abstract By proper co-ordinates of non-inertial observers (shortly - proper non-inertial co-ordinates) we understand the proper co-ordinates of an arbitrarily moving local observer. After a brief review of the theory of proper non-inertial co-ordinates, we apply these co-ordinates to discuss the relativistic effects seen by observers at different positions on a rotating ring. Although there is no relative motion among observers at different positions, they belong to different proper non-inertial frames. The relativistic length seen by an observer depends only on his instantaneous velocity, not on his acceleration or rotation. For any observer the velocity of light is isotropic and equal to c, provided that it is measured by propagating a light beam in a small neighbourhood of the observer.

1. Proper non-inertial co-ordinates

In physics, all dynamical equations of motion are certain differential equations that describe certain quantities as functions of space-time points. Space-time points are parameterized by their co-ordinates. It is convenient to write the equations of motion (as well as other related equations) in a form which is manifestly covariant with respect to general co-ordinate transformations. When one solves the equations, one must use some specific co-ordinates. The covariance provides that one can use any co-ordinates he wants, because later he can easily transform the results to any other co-ordinates. Therefore, it is convenient to choose co-ordinates that simplify the technicalities of the physical problem considered.

The general covariance is often interpreted as a statement that “physics does not depend on the co-ordinates chosen". However, this is not so. The choice

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 275-284.

of co-ordinates is more than a matter of convenience. The main purpose of theoretical physics is to predict what will be *observed* under given circumstances. The main lesson we have learned from Lorentz co-ordinates is the fact that what an observer observes (time intervals, space intervals, components of a tensor, ...) depends on how the observer moves. Lorentz co-ordinates are proper co-ordinates of an observer that moves inertially in flat space-time. Proper non-inertial co-ordinates are the generalization of Lorentz co-ordinates to arbitrary motion in arbitrary space-time. If one is interested in how a physical system looks like to a specific observer, one must transform the results to the corresponding proper non-inertial co-ordinates.

Proper non-inertial co-ordinates are determined by the (time-like) trajectory of the observer, by the rotation of the observer with respect to a local inertial observer and by the properties of space-time itself. The general geometrical construction of proper non-inertial co-ordinates is well established [1]. Here I present the most important properties of proper non-inertial co-ordinates:

1 Proper non-inertial co-ordinates are chosen such that the position of the observer is given by $x^\mu = (t, 0, 0, 0)$, where t is the time measured by a clock co-moving with the observer.

2 The metric expressed in proper non-inertial co-ordinates has the property

$$g_{\mu\nu}(t, 0, 0, 0) = \eta_{\mu\nu} \ . \tag{14.1}$$

3 The connections $\Gamma^\alpha_{\beta\gamma}$ vanish at $(t, 0, 0, 0)$ if and only if the trajectory is a geodesic and there is no rotation.

The general geometrical construction of proper non-inertial co-ordinates is not very useful for practical calculations. However, in flat space-time, proper non-inertial co-ordinates can be constructed in an alternative way, more useful for practical calculations [2]. Here I present an elegant form of this construction introduced in [3].

Let S be a Lorentz frame and let S' be the proper non-inertial frame of the observer whose 3-velocity is $u^i(t') \equiv \mathbf{u}(t')$, as seen by an observer in S. The co-ordinate transformation between these two frames can be obtained by summing the infinitesimal Lorentz transformations. Let

$$x^\mu = f^\mu(t', \mathbf{x}'; \mathbf{u}) \tag{14.2}$$

denotes the ordinary Lorentz transformation, i.e. the transformation between two Lorentz frames specified by the constant relative velocity $\mathbf{u}$. The quantities

$$f^\mu_{\ \nu} = \left(\frac{\partial f^\mu}{\partial x'^\nu} \right)_{\mathbf{u}=\text{const}} \tag{14.3}$$

have explicit values

$$f^0_{\ 0} = \gamma\ , \qquad f^0_{\ j} = -\gamma u_j\ , \qquad f^i_{\ 0} = \gamma u^i\ ,$$
$$f^i_{\ j} = \delta^i_{\ j} + \frac{1-\gamma}{\mathbf{u}^2} u^i u_j\ , \tag{14.4}$$

where $u^j = -u_j$, $\mathbf{u}^2 = u^i u^i$, $\gamma = 1/\sqrt{1-\mathbf{u}^2}$. The differential of (14.2) is

$$dx^\mu = f^\mu_{\ \nu}(t', \mathbf{x}'; \mathbf{u}) dx'^\nu. \tag{14.5}$$

The transition to a noninertial frame introduces a time-dependent velocity: $\mathbf{u} \to \mathbf{u}(t')$. The transformation between S and S' is given by the integration of (14.5) in the following way:

$$x^\mu = \int_0^{t'} f^\mu_{\ 0}(t', 0; \mathbf{u}(t')) dt' + \int_C f^\mu_{\ i}(t', \mathbf{x}'; \mathbf{u}(t')) dx'^i\ . \tag{14.6}$$

In (14.6), C is an arbitrary curve with constant t', starting from 0 and ending at x'^i.

In general, S' can also rotate i.e. change the orientation of the co-ordinate axes with respect to an inertial frame. The rotation can be described by the rotation matrix $A_{ji}(t') = -A_j^{\ i}(t')$. It satisfies the differential equation

$$\frac{dA_{ij}}{dt} = -A_i^{\ k} \omega_{kj}\ , \tag{14.7}$$

where $\omega_{ik} = \varepsilon_{ikl}\omega^l$, $\varepsilon_{123} = 1$ and $\omega^i(t')$ is the angular velocity as seen by an observer in S. In this more general case the transformation is also given by (14.6), but now C is an arbitrary curve with constant t', starting from 0 and ending at $-A_j^{\ i}(t')x'^j$. Note that the proper non-inertial co-ordinates x'^μ are constructed such that the space origins of S and S' coincide at $t = t' = 0$.

The metric tensor in S' is given by

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}\ , \tag{14.8}$$

where $g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric in S. Using (14.6), it is straightforward (but slightly tedious) to obtain

$$g'_{ij} = -\delta_{ij}\ , \qquad g'_{0j} = -(\boldsymbol{\omega}' \times \mathbf{x}')_j\ ,$$
$$g'_{00} = c^2 \left(1 + \frac{\mathbf{a}' \cdot \mathbf{x}'}{c^2}\right)^2 - (\boldsymbol{\omega}' \times \mathbf{x}')^2\ , \tag{14.9}$$

where

$$\omega'^i = \gamma(\omega^i - \Omega^i)\ , \qquad a'^i = \gamma^2 \left[a^i + \frac{1}{\mathbf{u}^2}(\gamma - 1)(\mathbf{u} \cdot \mathbf{a}) u^i\right]\ , \tag{14.10}$$

Ω^i is the time-dependent Thomas precession frequency

$$\Omega_i = \frac{1}{2\mathbf{u}^2}(\gamma - 1)\varepsilon_{ikj}(u^k a^j - u^j a^k) \ , \tag{14.11}$$

and $a^i = du^i/dt$ is the time-dependent acceleration.

From Property 2 we see that the space co-ordinates x^i are a generalization of Cartesian co-ordinates. However, this does not imply that an observer is not allowed to use polar co-ordinates, for example. The most general co-ordinate transformations that correspond merely to a redefinition of the co-ordinates of the same physical observer are the so-called restricted internal transformations [3], i.e. the transformations of the form

$$t' = f^0(t) \ , \quad x'^i = f^i(x^1, x^2, x^3) \ , \tag{14.12}$$

where t, x^i are proper non-inertial co-ordinates. The quantities $g_{00}dt^2$ and

$$dl^2 = -g_{ij}dx^i dx^j \tag{14.13}$$

do not change under restricted internal transformations. In order to describe physical effects as seen by a local observer, one must use proper non-inertial co-ordinates modulo restricted internal transformations. For simplicity, in the rest of the paper I use proper non-inertial co-ordinates.

Two observers with different trajectories have different proper non-inertial frames. In particular, this implies that *even if there is no relative motion between two observers, they belong to different frames if they do not have the same position.* This fact was not realized in many previous papers, which led to many misinterpretations. At first sight, this fact contradicts the well-known fact that two inertial observers in flat space-time may be regarded as belonging to the same Lorentz frame if there is no relative motion between them. However, this is because their proper non-inertial frames (with the space origins at their positions) are related by a translation of the space origin, which is a restricted internal transformation. In general, for practical purposes, two observers can be regarded as belonging to the same proper non-inertial frame if there is no relative motion between them and the other observer is close enough to the first one, in the sense that the metric expressed in proper non-inertial co-ordinates of the first observer does not depart significantly from $\eta_{\mu\nu}$ at the position of the second one. It is an exclusive property of Minkowski co-ordinates, among other proper non-inertial co-ordinates in flat or curved space-time, that the metric is equal to $\eta_{\mu\nu}$ *everywhere.* This implies that two observers at different positions but with zero relative velocity may be regarded as belonging to the same co-ordinate frame only if they move inertially in flat space-time.

2. Application to rotation

For motivation, let us first review the problems [3] of the standard resolution [4, 5] of the Ehrenfest paradox. We study a rotating ring in a rigid non-rotating circular gutter with radius r. One introduces the co-ordinates of the rotating frame S'

$$\varphi' = \varphi - \omega t \ , \quad r' = r \ , \quad z' = z \ , \quad t' = t \ , \tag{14.14}$$

where φ, r, z, t are cylindrical co-ordinates of the inertial frame S and ω is the angular velocity. The metric in S' is given by

$$ds^2 = (c^2 - \omega^2 r'^2)dt'^2 - 2\omega r'^2 \, d\varphi' dt' - dr'^2 - r'^2 \, d\varphi'^2 - dz'^2 \ . \tag{14.15}$$

It is generally accepted that the space line element should be calculated by the formula [6]

$$dl'^2 = \gamma'_{ij} dx'^i dx'^j \ , \quad i,j = 1,2,3 \ , \tag{14.16}$$

where

$$\gamma'_{ij} = \frac{g'_{0i} g'_{0j}}{g'_{00}} - g'_{ij} \ . \tag{14.17}$$

This leads to the circumference of the ring

$$L' = \int_0^{2\pi} \frac{r' d\varphi'}{\sqrt{1 - \omega^2 r'^2/c^2}} = \frac{2\pi r'}{\sqrt{1 - \omega^2 r'^2/c^2}} \equiv \gamma(r') 2\pi r' \ . \tag{14.18}$$

The circumference of the same ring as seen from S is $L = 2\pi r = 2\pi r'$. Since the ring is constrained to have the same radius r as the same ring when it does not rotate, L is not changed by the rotation, but the proper circumference L' is larger than the proper circumference of the non-rotating ring. This implies that there are tensile stresses in the rotating ring. The problem is that it is assumed here that (14.14) defines the proper frame of the whole ring. This implies that an observer on the ring sees that the circumference is $L' = \gamma L$. The circumference of the gutter seen by him cannot be different from the circumference of the ring seen by him, so the observer on the ring sees that the circumference of the relatively moving gutter is *larger* than the proper circumference of the gutter, whereas we expect, using our experience with the usual Lorentz contraction, that he should see that it is smaller. Of course, it depends on how the circumference is measured. Here we have in mind an experimental procedure that in principle can also be used to measure the usual Lorentz contraction, based on photographing with a very short exposition, such that the change of the photographed object position during the exposition can be neglected. The size of the object's picture on the photography corresponds to the measured size. Obviously, with such a measuring procedure, for any observer the apparent circumference of the *whole* ring must be equal to the apparent circumference of the *whole* gutter. This is a simple consequence of the fact that, at any instance of time, any part of the ring is somewhere inside the gutter and any part of the gutter has a part of the ring near it. (Note also that this is not so for a well known “paradox" of a car in a garage where different observers may disagree on whether a fast car can fit into an *open* garage at rest. This is because, for each part of the car, there are times for which that part is outside the garage as well as times for which it is inside the garage.)

The problem discussed above resolves when one realizes that (14.14) does not define the proper frame of the ring. Each point on the ring belongs to a different proper non-inertial frame. The co-ordinates (14.14) are actually proper non-inertial co-ordinates (modulo a restricted internal transformation) of an observer that rotates in the centre of the ring. However, this raises another problem. If (14.16) is the correct definition of the space line element, then the observer that rotates in the centre should see that the circumference of the gutter is larger than the proper circumference of the gutter by a factor $\gamma(r')$. However, $\omega r'/c$ can be arbitrarily large, so $\gamma(r')$ can be not only arbitrarily large, but also even imaginary. On the other hand, we know from everyday experience that the apparent velocity $\omega r'$ of stars, due to our rotation, can exceed the velocity of light, but we see neither a contraction nor an elongation of the stars observed. This implies that the definition of the space line element (14.16) is not always appropriate. In [6], (14.16) is derived by defining the space line element through a measuring procedure that lasts a finite time, so, in general, this formula is not appropriate for a definition of the *instantaneous* length. Of course, if rotation is stationary, then it is not necessary to use a definition appropriate for the instantaneous length. However, we want to have a definition that is appropriate for *any* case and to consider a stationary rotation only as a special case of arbitrary motion. In flat space-time, as shown in [3], if physics is described by proper non-inertial co-ordinates modulo restricted internal transformations, a more appropriate definition of the space line element is (14.13).

Using the co-ordinate transformation (14.6), one can study the relativistic contraction in the same way as in the conventional approach with Lorentz frames. One assumes that two ends of a body are seen to have the same time co-ordinate. From (14.6) and (14.3) one can easily see that the co-ordinate transformation is linear in x'^i. As demonstrated in more detail in [3, 7], it implies that *an arbitrarily accelerated and rotating observer sees equal lengths of other differently moving objects as an inertial observer whose instantaneous position and velocity are equal to that of the arbitrarily accelerated and rotating observer.*

Using (14.6), one can also study the rate of clocks as seen by various observers. In particular, one can study the twin paradox for various motions of the observers [8]. However, it is more interesting to study not only the time shift after the two differently moving observers eventually meet, but also the continuous changes of the time shifts during the motion. As can be seen from (14.6), it is the time dependence (not the space dependence) of the co-ordinate transformation that significantly differs from the ordinary Lorentz transformations. One cannot invert (14.6) simply by putting $u^i \to -u^i$. Therefore, inertial and non-inertial observers see quite different continuous changes of the time shift.

Following [3], let us discuss in more detail the case of uniform circular motion. Here we only present the results, while the technical details are delegated to the Appendix. Assume that there are three clocks. One is at rest in S, so it moves inertially. The other two are moving around a circle with the radius R and have the velocity ωR in the counter-clockwise direction, as seen by the observer in S. The relative angular distance between these two non-inertial clocks is $\Delta\varphi_0$, as seen in S. The inertial observer will see that the two non-inertial clocks lapse equally fast, so I choose that he sees that they show the same time. He will see the clock rate $t = \gamma t'$, where $\gamma = 1/\sqrt{1 - \omega^2 R^2/c^2}$. The observer co-moving with one of the non-inertial clocks will not see that the other non-inertial clock shows the same time as his clock. He will see the constant time shift given by the equation

$$\gamma\omega(t'' - t') = \beta^2 \sin(\gamma\omega(t'' - t') + \Delta\varphi_0) \ , \tag{14.19}$$

where $\beta^2 \equiv \omega^2 R^2/c^2$ and t'' is the time of the other non-inertial clock. Finally, let us see how the inertial clock looks like from the point of view of the observer co-moving with one of the non-inertial clocks. Let the position of the inertial clock be given by its co-ordinates (x, y). We choose the origin of S such that, at $t = t' = 0$, the space origins of S and S' coincide and the velocity of the non-inertial observer is in the y-direction, as seen in S. The rate of clocks as seen by the non-inertial observer is given by

$$t = \gamma t' + \frac{\omega R}{c^2}[y \cos \gamma\omega t' - (x + R) \sin \gamma\omega t'] \ . \tag{14.20}$$

The oscillatory functions in (14.20) vanish when they are averaged over time. This means that the observer in S' agrees with the observer in S that the clock in S' is slower, but only in a time-averaged sense. For example, when these two clocks are very close to each other, then, by expanding (14.20) for small t', one finds $t = t'/\gamma$, which is the result that one would obtain if the velocity of the non-inertial observer were constant. If the clock in S is in the centre, which corresponds to $x = -R$, $y = 0$, then (14.20) gives $t = \gamma t'$, so in this case there is no oscillatory behaviour.

Finally, let us shortly discuss the implications on the velocity of light. If (14.14) is interpreted as a proper frame of all observers on a rotating platform, then one can conclude that the observer on the rotating platform will see that the velocity of light depends on whether light is propagating in the clockwise or in the counter-clockwise direction (see, for example, [9]). This is related to the fact that the metric (14.15) is not time orthogonal. However, now we know that each observer belongs to a different proper non-inertial frame, and from Property 2 we see that in the *vicinity* of any observer the metric is equal to the Minkowski metric $\eta_{\mu\nu}$. This implies that *for any local observer the velocity of light is isotropic and is equal to c, provided that it is measured by propagating a light beam in a* **small** *neighbourhood of the observer.* In particular, this leads to a slightly different interpretation of the Sagnac effect [3], in which the velocity of light as seen by the observer on the rotating platform depends on the instantaneous relative position of the light beam with respect to the observer. The details are presented in the Appendix.

Acknowledgments

The author is grateful to G. Rizzi for his editorial critical remarks that significantly improved this paper. This work was supported by the Ministry of Science and Technology of the Republic of Croatia under the contract No. 0098002.

Appendix

In this Appendix, we derive Eqs. (14.19) and (14.20) and discuss how the velocity of light appears to an observer on the rim of a rotating platform. In these derivations, we follow [3].

Let the gutter be placed at the $z = 0$ plane. We put the space origin of S at a fixed point on the gutter, such that the y-axis is tangential to the gutter and the x-axis is perpendicular to the gutter at $\mathbf{x} = 0$. In the rest of the analysis the z-co-ordinate can be suppressed. A part of the ring in the gutter can be considered as a short rod initially placed at $\mathbf{x} = 0$ and uniformly moving along the gutter in the counterclockwise direction. The gutter causes a torque that provides that the rod is always directed tangentially to the gutter. Therefore, $\omega = u/R$, where $u = \sqrt{\mathbf{u}^2}$ is the time independent speed of rod. Now, $\gamma = 1/\sqrt{1 - \omega^2 R^2/c^2}$ is also time independent. Since a clock in S' is at $\mathbf{x}' = 0$, the clock rate between a clock in S and a clock in S' is given by $t = \gamma t'$, as seen by an observer in S. We assume that, initially, the axes x', y' are parallel to the axes x, y, respectively. Therefore the velocity

$$\mathbf{u}(t') = \omega R(-\sin\gamma\omega t', \cos\gamma\omega t') \tag{14.A.1}$$

is always in the y'-direction and the solution of (14.7) is

$$A_{ij}(t') = \begin{pmatrix} \cos\gamma\omega t' & \sin\gamma\omega t' \\ -\sin\gamma\omega t' & \cos\gamma\omega t' \end{pmatrix} . \tag{14.A.2}$$

The transformation (14.6) reduces to

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\gamma\omega t' & -\gamma\sin\gamma\omega t' \\ \sin\gamma\omega t' & \gamma\cos\gamma\omega t' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + R \begin{pmatrix} \cos\gamma\omega t' - 1 \\ \sin\gamma\omega t' \end{pmatrix} , \tag{14.A.3}$$

$$t = \gamma t' + \frac{\gamma}{c^2}\omega R y' . \tag{14.A.4}$$

In particular, at $t' = 0$ these transformations become

$$x = x' , \quad y = \gamma y' , \quad t = \frac{\gamma u}{c^2} y' , \tag{14.A.5}$$

which coincide with the ordinary Lorentz boost at $t' = 0$ for the velocity in the y-direction.

Let us now study how other parts of the ring appear to the observer on the ring. Since the rotation is uniform the result cannot depend on t', so without losing on generality, we evaluate this at $t' = 0$. We introduce polar co-ordinates r, φ, defined by

$$y = r \sin\varphi \ , \quad R + x = r\cos\varphi \ , \tag{14.A.6}$$

which are new space co-ordinates for S, with the origin in the center of the circular gutter. The angle φ is a good label of the position of any part of the ring even in S'. (To visualize this, one can draw angular marks on the gutter. The number of marks separating two points on the gutter or on the ring is a measure of the "angular distance" in any frame.) Let S'' be the frame of another part of the ring. The position of that part of the ring is $x'' = y'' = 0$. The relative position of the space origin of S'' with respect to that of S' is given by the constant relative angle $\Delta\varphi_0$, as seen by an observer in S. In analogy with (14.A.3)-(14.A.4), we find that S'' is determined by

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos(\gamma\omega t'' + \Delta\varphi_0) & -\gamma\sin(\gamma\omega t'' + \Delta\varphi_0) \\ \sin(\gamma\omega t'' + \Delta\varphi_0) & \gamma\cos(\gamma\omega t'' + \Delta\varphi_0) \end{pmatrix} \begin{pmatrix} x'' \\ y'' \end{pmatrix} \\ &\quad + R\begin{pmatrix} \cos(\gamma\omega t'' + \Delta\varphi_0) - 1 \\ \sin(\gamma\omega t'' + \Delta\varphi_0) \end{pmatrix} \ ,\end{aligned} \tag{14.A.7}$$

$$t = \gamma t'' + \frac{\gamma}{c^2}\omega R y'' \ . \tag{14.A.8}$$

Let the labels A, B denote the co-ordinates of the part of the ring that lie at S' and S'', respectively. Since the observer sees both parts of the ring at the same instant, we have $t'_A = t'_B = 0$. Since $x''_B = y''_B = 0$, from (14.A.6) we find

$$y_B = R\sin(\gamma\omega t''_B + \Delta\varphi_0) \ , \tag{14.A.9}$$

and from (14.A.8)

$$t_B = \gamma t''_B \ . \tag{14.A.10}$$

From $t'_B = 0$ and (14.A.5) it follows $t_B = \omega R y_B/c^2$, which, because of (14.A.10), can be written as $\gamma t''_B = \omega R y_B/c^2$. This, together with (14.A.9), leads to the equation that determines t''_B:

$$\gamma\omega t''_B = \beta^2 \sin(\gamma\omega t''_B + \Delta\varphi_0) \ , \tag{14.A.11}$$

where $\beta^2 \equiv \omega^2 R^2/c^2$. Eq. (14.A.11) together with the fact that the rotation is uniform implies (14.19).

To see how the inertial clock appears to the observer on the ring, the transformations (14.A.3) and (14.A.4) are sufficient. From (14.A.3) one expresses y' as a function of x, y and t' and puts this in (14.A.4). The result is given by (14.20).

Let us now discuss how the velocity of light appears to the observer on the ring. Let the light beam move along the gutter. The trajectory of the light beam expressed in S-co-ordinates is given by

$$y = R\sin\omega_L t \ , \quad x = R(-1 + \cos\omega_L t) \ , \tag{14.A.12}$$

where $\omega_L = \pm c/R$. The plus and minus signs refer to the counterclockwise and clockwise propagated beams, respectively. Using (14.A.3), (14.A.4), and (14.A.12), one can eliminate x, y, t and express x', y' as functions of t'. The speed of light as seen by the observer in S' is

$$v'_L = \sqrt{\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2} \ . \tag{14.A.13}$$

Expanding (14.A.3) and (14.A.12) for small t' and t, respectively, one can easily find $y' = \pm ct' + \mathcal{O}(t'^2)$, $x' = \mathcal{O}(t'^2)$, which means that the observer sees the velocity of light equal to c when the light is at the same position as the observer, just as expected.

References

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, New York 1995).

[2] R. A. Nelson, J. Math. Phys. **28**, 2379 (1987); Erratum: R. A. Nelson, J. Math. Phys. **35**, 6224 (1994).

[3] H. Nikolić, Phys. Rev. A **61**, 032109 (2000).

[4] Ø. Grøn, Am. J. Phys. **43**, 869 (1975).

[5] Ø. Grøn, Int. J. Theor. Phys. **16**, 603 (1977).

[6] D. Landau and E. M. Lifschitz, *The Classical Theory of Fields* (Pergamon, New York 1971).

[7] H. Nikolić, Am. J. Phys. **67**, 1007 (1999).

[8] H. Nikolić, Found. Phys. Lett. **13**, 595 (2000).

[9] R. D. Klauber, Found. Phys. Lett. **11**, 405 (1998).

Chapter 15

SPACE GEOMETRY IN ROTATING REFERENCE FRAMES: A HISTORICAL APPRAISAL

Øvind Grøn
Oslo College, Department of engineering, Cort Adelers gt. 30, 0254 Oslo, Norway
Institute of Physics, University of Oslo, P. O. Box 1048 Blindern, 0316 Oslo, Norway
oyvind.gron@iu.hio.no

Abstract The problem of giving a relativistic description of the geometry of a rotating disk has a history nearly as old as that of the theory of relativity itself. Already in 1909 Ehrenfest formulated his famous paradox in the context of the special theory of relativity. A few years later Einstein made heuristic use of this problem in order to motivate the introduction of non-Euclidean geometry in a relativistic theory of gravity. We shall here follow the conceptual evolution of this topic from Ehrenfest and Einstein to the present time. In particular we emphasize the importance of taking the relativity of simultaneity properly into account in order to obtain a full understanding of the issues connected with Ehrenfest's paradox.

1. Introduction

The relativistic description of the geometry of rotating bodies is more than 90 years long. It started with a short note by Paul Ehrenfest [1] who pointed out the contradictory conditions that the radius of a relativistically rotating cylinder has to fulfil: On the one hand the periphery is Lorentz contracted, and on the other hand a radial line on the cylinder is not. This problem was soon taken up by among others Max Planck [2] and Albert Einstein [3], and it has been discussed right up to the present day [4].

We shall here follow this discussion and try to see what conceptual difficulties made this topic so long lived.

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 285-333.

2. The discussion of 1910 and 1911 in Physikalische Zeitschrift

In 1909 Ehrenfest [1] wrote that he was trying to understand Max Born's [5] notion of relativistic rigidity. He first pointed out that the notion of rigid motion of a body can be formulated either with reference to an inertial frame in which the body moves, or with reference to the local rest frame of an element of the body. In the first case he writes (translated into English):

> To say that a body remains relativistically rigid means: It deforms continuously by arbitrary motion so that each of its infinitesimal elements Lorentz contract (relative to its rest length) all the time in accordance with the instantaneous velocity of each of its elements, as observed by an observer at rest.

In the second case:

> Relativistic rigidity means: As measured by a continuum of observers co-moving with each point of an arbitrarily moving body each element of the body remains undeformed.

He then writes:

> Consider a relativistically rigid cylinder with radius R and height H. It is given a rotating motion about its axis, which finally becomes constant. As measured by an observer at rest the radius of the rotating cylinder is R'. Then R' has to fulfill the following two contradictory requirements:
>
> 1 The circumference of the cylinder must obtain a contraction $2\pi R' < 2\pi R$ relative to its rest length, since each of its elements move with an instantaneous velocity $R'\omega$.
>
> 2 If one considers each element along a radius, then the instantaneous velocity of each element is directed perpendicular to the radius. Hence, the elements of a radius cannot show any contraction relative to their rest length. This means that: $R' = R$.

This contradiction was termed "Ehrenfest's paradox" ("Ehrenfestschen Paradoxon") by V. Varićak [6]. As pointed out by M. J. Klein [7] Ehrenfest's article was the first in a series of critical analyses which demonstrated that the concept of Born rigid motion cannot be applied to rotating motion in general.

Ehrenfest's paradox was discussed by Max Planck in 1910 [2]. He pointed out:

> The statement that the volume of a body with velocity v measured by an observer at rest is less by the ratio $\sqrt{c^2 - v^2} : c$ than its volume measured by a co-moving observer with velocity v, must not be mixed up with another statement, that the volume of a body which is brought from a state of rest to a velocity v is decreased by a factor $\sqrt{c^2 - v^2} : c$. The first statement is one of the fundamental requirements of the theory of relativity, while the last statement is not generally correct.

Planck then argued that the task of specifying the final state of a body set into rotation is a dynamical problem involving the theory of elastic media. It was, however, immediately clear that it is impossible to set a body into rotation while maintaining Born rigidity (see further references on this point in [8].

Then Theodor Kaluza [9] pointed out the necessity of considering non-Euclidean geometry in order to give a relativistic description of the geometry of rotating bodies. He wrote:

> According to the theory of relativity the "proper geometry" (corresponding to a fixed proper time) is in general the geometry of a surface orthogonal to the bundle of world lines of the body. Two events are "simultaneous" if they belong to this surface.

He further said:

> A closer investigation shows that the proper geometry of a rotating disc is a non-Euclidean special Lobachevskian geometry.

However, the details of the investigation were not included in the article.

In response to an article by W. von Ignatowski [10] on relativistic kinematics, Ehrenfest [11] offered a gedanken experiment: Consider a disk at rest with equally spaced circles about the origin of the rotational axis engraved on its surface. Let these circles be recorded on a piece of tracing paper by a stationary observer. Assume now that the disk could be put into rotation while remaining Born rigid and then rotate at constant angular velocity about an axis through its centre, while the observer remains at rest. If the observer *instantaneously* registered the rotating disk's markings on another piece of tracing paper, he would find upon comparison with the other piece of paper that the radial lengths are the same, but the circumference measured during rotation is less than before. This contradiction shows that the assumption of Born rigidity is not compatible with putting the disk into rotation.

In an article on relativistic theory of elasticity, Ignatowski [12] calculates the change of the radius and the periphery of a disk with given elastic properties that is put in rotational motion. In a critical comment to this work Ehrenfest [13] interprets Ignatowski to mean that the motion is Born rigid. Hence he indicates that the calculation, or at least its physical interpretation, contains an inconsistency.

In February 1911 V.Varićak [6] claimed that according to Einstein's theory the Lorentz contraction is a sort of observational illusion, and that in reality bodies are not contracted when moving. He thus concluded that there is no paradox. Einstein considered this misinterpretation of the theory of relativity to be rather serious and therefore gave an answer [3] where he explained the relativistic meaning of the Lorentz contraction.

3. Einstein's realization that the geometry on the rotating disk is non Euclidean

Although Einstein did not participate in the discussion of Ehrenfest's paradox, he was well aware of the problem with Born rigidity as applied to rotating motion, but he was more concerned with the purely geometrical aspect. Working on a relativistic theory of gravitation his discovery of the equivalence of being in a field of gravity and being in a non-inertial reference frame motivated him to search for a geometrical theory of gravity. Thinking about spatial measurements on a rotating disk he arrived at the conclusion that one needed to free one self from the restrictions of the Euclidean geometry.

He made several notes about spatial geometry in a rotating reference frame both in letters and in publications (see J. Stachel [14, 15].) In the great article in 1916 where he presented the general theory of relativity Einstein considered the disk as a rotating reference frame K and imagined this frame filled by radial and tangential standard rods. The definition of a standard rod is that it is Born rigid, so that it gets a Lorentz contraction when moving. Denoting the inertial rest frame of the axis by K he then wrote [16]:

> We suppose that the circumference and diameter of a circle have been measured with a standard measuring rod infinitely small compared with the radius, and that we have the quotient of the two results. If this experiment were performed with measuring rods at rest relatively to the Galilean system K', the quotient would be π. With measuring rods at rest relatively to K, the quotient would be greater than π. This is readily understood if we envisage the whole process of measuring from the "stationary" system K', and take into consideration that the measuring rods applied to the periphery undergoes a Lorentz contraction, while the ones applied along the radius do not. Hence Euclidean geometry does not apply to K.

Einstein has here for the first time made it clear that the length of the periphery of a rotating disk in *longer* than $2\pi r$ not shorter as stated in Ehrenfest's paradox. The reason for this difference is that Ehrenfest considered a hypothetical, but impossible situation where a disc had been put into rotational motion in a Born rigid way, while Einstein considered a situation in which the disk had been put into rotation in an arbitrary way, but the measuring rods were required to be Born rigid.

Einstein gave similar discussions of the geometry of the rotating disk in his semi popular introduction to the general theory of relativity [17] and in *The Meaning of Relativity* [18] based upon his Princeton lectures in 1921. The most complete treatment of this topic by Einstein is in fact found in a letter to Joseph Petzold dated August 19, 1919. This letter was translated by J. Stachel and published in 1989 [14]. In the part of this letter which concerns the geometry of the rotating disk, Einstein writes:

A rigid circular disk must break up if it is set into rotation, on account of the Lorentz contraction of the tangential fibres and the non-contraction of the radial ones. Similarly, a rigid disk in rotation must explode as a consequence of the inverse changes in length, if one attempts to bring it to the rest state.

Now you believe that a rigidly rotating circular line must have a circumference that is less than $2\pi r$ because of the Lorentz contraction. The basic error here is that you instinctively set the radius r of the rotating circular line equal to the radius r_0 that the circular line has in the case when it is at rest. This however, is not correct; because of the Lorentz contraction rather $2\pi r = 2\pi r_0 \sqrt{1 - v^2/c^2}$.

The treatment of the metric of the circular disk runs as follows in detail. Let U_0 be the circumference, r_0 the radius of the rotating disk, considered from the standpoint of K' [that is, the rest frame]; then, on account of ordinary Euclidean geometry,

$$U_0 = 2\pi r_0 \tag{15.1}$$

U_0 and r_0 naturally are to be thought of as measured with non rotating measuring rods, that is, at rest relative to K'.

Now let me imagine co-rotating measuring rods of rest length 1 laid out on the rotating disk, both along a radius as well as the circumference. How long are these considered from K'? Let us imagine, in order to make this clearer to ourselves, a "snapshot" taken from K' (definite time t_0). On this snapshot the radial measuring rods have length 1, the tangential ones, however, the length $\sqrt{1 - v^2/c^2}$. The "circumference" of the circular disk (considered from the rest frame of the disk, K) is nothing but the number of tangential measuring rods that are present in the snapshot along the circumference, whose length considered from K' is U_0. Therefore

$$U = U_0 / \sqrt{1 - v^2/c^2} \tag{15.2}$$

On the other hand, obviously

$$r = r_0 \tag{15.3}$$

(since the snapshot of the radial unit measuring rod is just as long as that of a measuring rod at rest relative to K').

Therefore, from (15.1)-(15.3),

$$\frac{U}{r} = \frac{U_0}{r_0\sqrt{1 - v^2/c^2}} = \frac{2\pi}{\sqrt{1 - v^2/c^2}} \tag{15.4}$$

Since $v = r\omega$ equation (15.4) implies that the ratio between the circumference and the diameter gets larger with increasing radius. The position dependence of this ratio can be measured by observers on the disk. Hence, they would conclude that the geometry on the disk is non-Euclidean.

These considerations by Einstein were soon made well known in the book by Born [19] and later by Einstein and Infeld [20]. The first edition of Max Born's popular book [19] on the theory of relativity came in 1920. Here he noted an interesting consequence of the non-Euclidean geometry on a rotating disk combined with the principle of equivalence: In a gravitational field a standard measuring rod is longer or shorter according to the position at which it is situated. A. Metz [21] later gave an illustration where he compared the measuring rods on a rotating disk with the wagons of a model train travelling around a circular path, and K. Kraus [22] considered the intervals between measuring rods attached to the spokes of a rotating wheel. However, he did not take the relativity of simultaneity properly into account.

Not everyone agreed with Einstein. J. Becquerel [23] argued in 1922 that the quotient between the periphery and diameter of a rotating disk is *less* than π. He arrived at this by saying that the measurements of the periphery in K is obtained by means of measuring rods at rest in the inertial frame K' as observed from K. The number of measuring rods around the periphery is invariant. If there are n measuring rods along the periphery in K', where the quotient between the circumference and the diameter is equal to π, then as observed in K the number of measuring rods is the same, but each rod is Lorentz contracted. Hence an observer in K would say that this measuring procedure leads to a quotient between the periphery and the diameter less than π. This argument is fallacious, however, because a measurement of the length of the circumference of the rotating disk must be performed with standard measuring rods at rest on the disk, not at rest in the inertial rest frame of the axis.

Shu [24] later came to the conclusion that the space geometry on a rotating disk is Euclidean. The report of Shu was not published in a scientific journal, but given to the library of Princeton University. His work is clearly that of an outsider. He came to his result by neglecting the reality of the Lorentz contraction as applied to measuring rods on a rotating disk, and concluded that the general theory of relativity is not correct.

W. Glaser [25] argued in 1934 that the line-element of flat spacetime in a rotating reference frame can not be separated in a temporal and a spatial part with no product terms between a time differential and a spatial differential, and concluded that space in a rotating frame is Euclidean. M. P. Langevin [26] reacted immediately and demonstrated explicitly that this is possible, by performing the following calculation.

Let marked co-ordinates refer to the inertial rest frame K' of the axis, and unmarked to co-moving frame K on the rotating disk. The transformation between these coordinate systems is

$$t' = t\,, \quad r' = r\,, \quad \theta' = \theta + \omega\, t\,, \quad z' = z \tag{15.5}$$

The spacetime line-element in the inertial frame has the form

$$ds^2 = -c^2\, dt'^2 + dr'^2 + r'^2\, d\theta'^2 + dz'^2 \tag{15.6}$$

The line-element in the rotating frame is

$$ds^2 = -\left(c^2 - r^2\omega^2\right) dt^2 + 2r^2\omega\, dt\, d\theta + dr^2 + r^2\, d\theta^2 + dz^2 \tag{15.7}$$

For $r < c/\omega$ this may be written as

$$ds^2 = -c^2\, d\tau^2 + d\sigma^2 \tag{15.8}$$

with

$$d\tau = \sqrt{1 - r^2\omega^2/c^2}\left(dt - \frac{r^2\omega}{c^2 - r^2\omega^2} d\theta\right) \tag{15.9}$$

and

$$d\sigma^2 = dr^2 + \frac{r^2\, d\theta^2}{1 - r^2\omega^2/c^2} + dz^2 \tag{15.10}$$

Here the spatial line-element $d\sigma$ represents the geometry on the space defined by $d\tau = 0$. Since $d\tau$ is not a perfect differential it cannot be integrated around a closed curve on the rotating disk. Hence the surface represented by $d\tau = 0$ has a discontinuity along a radial line (see figure 15.6). As seen from eq.(15.10) it is this surface that has the spatial geometry discussed by Einstein, with the ratio between the circumference and the diameter being $\pi/\sqrt{1 - r^2\omega^2/c^2}$.

The meaning of the time interval $d\tau$ was made clear by Rosen [27]. He introduced a local inertial coordinate system momentarily at rest relative to the rotating system by the relations

$$dx = dr\,, \quad dy = A d\theta\,, \quad d\tau = B dt + C d\theta \tag{15.11}$$

where A, B and C were determined so as to make

$$ds^2 = -c^2 d\tau^2 + dx^2 + dy^2 + dz^2 \tag{15.12}$$

This gives

$$A = \frac{r}{\sqrt{1 - r^2\omega^2/c^2}}\,, \quad B = \sqrt{1 - r^2\omega^2/c^2}\,, \quad C = -\frac{r^2\omega^2/c^2}{\sqrt{1 - r^2\omega^2/c^2}} \tag{15.13}$$

From this it follows that

$$d\sigma^2 \equiv dx^2 + dy^2 + dz^2 = dr^2 + \frac{r^2 d\theta^2}{1 - r^2\omega^2/c^2} + dz^2 \tag{15.14}$$

and that the transformation (15.11) for τ is identical to eq.(15.9). Hence τ is the time measured in a local inertial system momentarily at rest relative to a reference particle in the rotating system, and $d\tau = 0$ represents simultaneity in local inertial frames comoving with the rotating frame and positioned for example along a circle about the axis. The clocks of the local inertial frames instantaneously at rest relative to points along such circles are Einstein synchronized so that the velocity of light is isotropic as measured with these clocks.

The impossibility of Einstein synchronizing clocks around the circumference of a rotating disk has been perceived as a problem by F. Goy and F. Selleri [28]. They write:

> The existence of a synchronization is physically strange because if the whole disk is initially at rest in the laboratory (inertial) frame K', with clocks near its rim synchronized with the regular procedure used for all clocks of K', then when the disk moves, accelerates, and attains a constant angular velocity, the clocks must slow their rates but cannot desynchronize for symmetry reasons, since they have at all times the same speed.

The clocks along the rim of the rotating disk are assumed to have at all times the same speed. This means that the acceleration program of each clock is identical as observed from K'. Assume that the angular acceleration is due to a succession of blows at the rim. Then the blows are simultaneous in K'. Due to the relativity of simultaneity they are not simultaneous in the rest frame of an element on the rim. In such a frame the clocks at each end of the element get different velocities, which desynchronizes the clocks.

Goy and Selleri are of course right in saying that the clocks remain synchronized in K'. Another way of obtaining an identical synchronization to that which results from the procedure of Goy and Selleri, is to use a time signal emitted from the axis. This will reach all clocks at a circle with center at the axis simultaneously both as measured in K' and as measured in the rotating rest frame K of the disk. The clocks synchronized in this way are just the co-ordinate clocks on the rotating disk. They show the same time as the clock at the axis, and hence as the clocks in K'. However, these clocks are not Einstein synchronized.

If one includes the postulate that the velocity of light is isotropic as part of the special theory of relativity, and demands that special relativity is valid locally, then local physical measurements must be performed by means of Einstein synchronized clocks. Globally, however, it may be advantageous to use coordinate clocks.

The 3-space on a rotating disk is defined to be everywhere orthogonal to the world lines of fixed particles on the disk. This means that this space is defined by simultaneity on the Einstein synchronized clocks. The space defined by simultaneity on the coordinate clocks, on the other hand, is the 3-space of the inertial rest frame of the axis, which is flat.

Reichenbach [29] discussed the spatial geometry on a rotating disk in 1924. He distinguished between what he called "the spatial geometry of the circular disk" (SGD) and "the geometry of rigid rods on the disk" (GRD). He defined SGD as the geometry found by measurements made at a fixed co-ordinate time t =constant. Putting $dt = 0$ in eq.(15.7) he obtained for the line element of the surface z =constant: $d\sigma^2_{t=const} = dr^2 + r^2 d\theta^2$. Thus, he concluded that the

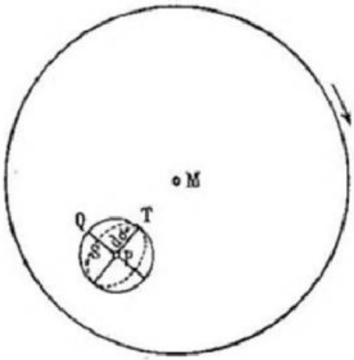

Figure 15.1. Rotating disk with measuring rod. The dashed ellipse is the curve followed by the end of the rod when it is rotated about P.

spatial geometry of the rotating disk is Euclidean. However, the space defined by t = constant is the simultaneity space of the inertial rest frame K' of the axis. It is not reasonable to identify the spatial geometry of the rotating disk with the geometry of this space. Reichenbach defined GRD by simultaneity in the local rest frame of a mass element on the disk. He then pointed out that this represents non-simultaneity in K', which cannot be defined globally in the rotating rest frame K of the disk. By considering rigid measuring rods on the disk he obtained the same line element as in eq.(15.14).

Also he gave a nice illustration of the difference between the two geometries. Imagine an arbitrary point P on the disk. Draw a circle around P with radius $d\sigma = 1$ (see figure 15.1).

If a rigid rod of length 1 is laid through P in the radial direction, its characteristic length is $d\sigma_r = 1$. If the rod lies tangentially, and its length is measured as the distance between events at its ends, that are simultaneous in K', one finds a Lorentz contracted length for the measuring rod. If the rod is rotated about an axis through P, its end describes the dotted curve in figure 15.1. This curve is an ellipse whose radial half axis = 1 and whose tangential half axis = $\sqrt{1 - r^2\omega^2/c^2}$.

GRD means that the same rigid rod shall have the spatial length 1 independently of its position and orientation. In other words, the half axes of the dotted curve are called equally long. SGD, on the other hand, means that all the radii of the circle are equally long. Note that figure 15.1 is drawn from the point of view of the non rotating frame K'.

In general an arbitrary spacetime line-element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{15.15}$$

may be separated in a temporal part $d\tau$ and a spatial part $d\sigma$ as in eq.(15.8) with

$$d\tau = \sqrt{-g_{tt}}\left(dt + \frac{g_{ti}}{g_{tt}}dx^i\right) \tag{15.16}$$

and

$$d\sigma^2 = \gamma_{i\,j}\,dx^i\,dx^j\,, \quad \gamma_{i\,j} = g_{i\,j} - \frac{g_{t\,i}g_{t\,j}}{g_{t\,t}} \tag{15.17}$$

J. W. Weysenhoff [30] used this separation and defined a local angular velocity vector by

$$\vec{\omega} = \frac{1}{2}\frac{c}{\sqrt{-g_{t\,t}}}\left(\nabla\times\vec{\alpha} + \vec{\alpha}\times\frac{\partial\vec{\alpha}}{\partial\,t}\right)\,, \quad \vec{\alpha} = \frac{g_{t\,i}}{g_{t\,t}}\vec{e}_i \tag{15.18}$$

He then showed that a condition for being able to Einstein synchronize clocks around a closed curve, or in other words, to introduce a coordinate system that is everywhere time orthogonal, is that $\vec{\omega} = 0$, and noted:

> We then see that for example on a rotating disk it is impossible everywhere to introduce a time orthogonal coordinate system. Expressed in another way: It is impossible to synchronize the clocks everywhere on the disk so that all light signals are symmetrical, that is, so that the velocity of light is the same in every direction.

In passing we note that this explains the result of Sagnac's experiment [31] from the point of view of observers on the rotating disk [32].

Einstein also noted that the properties of measuring rods and clocks on a rotating disk illustrates a general fact [16]:

> In the general theory of relativity space and time cannot be defined in such a way that differences of the spatial co-ordinates can be directly measured by the unit measuring rod, or differences in time co-ordinate by a standard clock.

This was illustrated very explicitly in connection with the rotating disk by H. Thirring, who also noted that there are tangential stresses in the material of a rotating disk, writing [33]:

> Man denke sich eine ebene Kreisscheibe in Rotation gegen ein Inertialsystem versetzt. Die radien r der Scheibe werden von der Lorentzkontraktion nicht betroffen, weil sie senkrecht zur Bewegungsrichtung stehen. Die Peripherie wird in ihrem Bestreben, sich zuzammenzuziehen, durch die Kohäsionskräfte gehindert; die Lorentzkontraktion wird durch die Dehnung kompensiert, die von den elastischen spannungen der in sich zusammenhängenden Peripherie bewirkt wird. Dagegen erleidet ein längs der Scheibenumfangs angelegter spannungsfreier Masstab die Lorentzkontraktion, was zur Folge hat, dass eine Ausmessung des Verhältnisses zwischen Umfang und Peripherie einen höheren Wert als 2π liefern muss. Es treten also Abweichungen von den Gesetzen der euklidischen Geometrie auf, in analoger Weise, wie sie sich bei entsprechenden Messungen auf gekrümmten Flächen ergeben mussen. Zeichnet man etwa auf

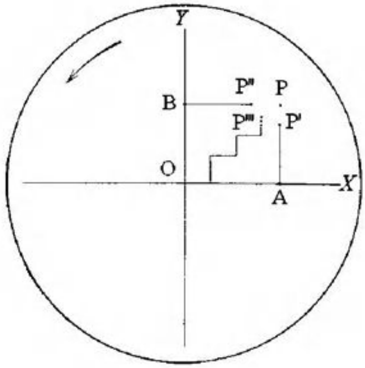

Figure 15.2. From Thirring's paper.

der Erdoberfläche einen Kreis und misst man das Verhältnis zwischen Kreisumfang und dem längs der Erdoberfläche selbst gemessenen Radius, so ergibt sich ebenfalls ein von 2π abweichender Wert und zwar, der positiven Krümmung der Kugel entsprechend, ein kleineren Wert als 2π, während in dem obenerwähnten Beispiel der rotierende Scheibe das Verhältnis grösser als 2π wird, was einer "negativen Krümmung" entspricht. Es zeigt sich ferner, dass die übliche Art der Koordinatenbestimmung von Punktereignissen mit hilfe von Uhrenangaben und kartesischen Koordinaten nicht mehr zu eindeutigen Ergebnissen führt, wie aus dem nachstehenden einfachen Beispiel hervorgeht: Es werde auf der rotierenden Scheibe ein rechtwinkliges Koordinatensystem *XY* gezeichnet, dessen uhrsprung mit dem Sheibenmittelpunkt zuzammenfällt, und es sei die Aufgabe gestellt, den punkt P mit den Koordinaten x, y zu finden. Das kan man nun zunächst so machen , dass man einen Einheitmasstab längs der X-Achse $x$mal aufträgt, dadurch gelangt man in den Punkt A; dort errichtet man eine Senkrechte, längs derer man den Einheitsmasstab y mal auträgt. Bei dieser letzteren Operation ist aber der Masstab gemäss dem oben Gesagten verkürzt; man gelangt also nicht in jenen Punkt P, der auf der ruhenden Scheibe die Koordinaten x und y hätte, sondern in einer näher an A gelegenen Punkt P'. – Würde man dagegen in den punkt Pgelangen wollen, indem man mit dem Auftragen eines Einheitsmasstabes längs der Y-Achse beginnt, so würde man aus dem gleichen Grunde in einen Punkt P'' gelangen, der näher zur Y-Achse liegt. Wenn man ferner einen dritten Weg, z.B. den in der Figur gezeichneten Treppenweg ginge, so würde man noch in einen anderen punkt P'''kommen usw.

A strange objection to Einstein's analysis of the spatial geometry in a rotating frame was given by Atwater [34]. He argued against the special relativistic assumption that the measuring rods along the circumference of a rotating disk are Lorentz contracted as observed from the inertial rest frame K' of the axis, writing:

Consider a circular disk, made possibly of transparent material, on the circumference of which a pattern of stripes is painted in four octants, as indicated in

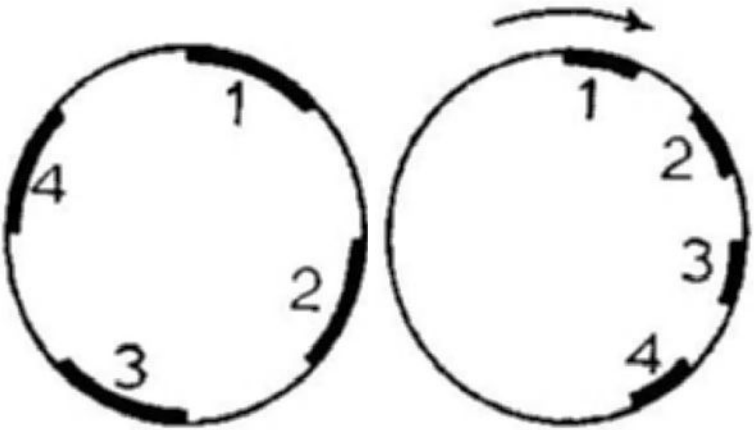

Figure 15.3. (a) On the left: stationary disk; (b) on the right: result of an arbitrary application of Lorentz contraction to periphery of a disk in rotation.

> figure 15.3 (a). The disk is then set into rotation at a speed such that the rim is travelling at 86,6 per cent of the velocity of light in free space, for a Lorentz contraction factor of $^1/_2$. The positions of the ends of the stripes on the rim can be observed at an instant of laboratory time. This is possible by means of a flash of light emitted above the plane of the disk which exposes the shadows of the stripes on a photographic film held stationary in the back of the disk; an alternative coincidence-detection observation may also be devised. An elementary application of the special relativistic hypothesis could then lead to the expectation of a pattern as in figure 15.3 (b), which is clearly impossible on the basis of the symmetry of the disk. The special relativistic assumption must therefore be discarded.

The argument of Atwater is not valid, however, because painted marks on the circumference of the disk will not appear Lorentz contracted unless no tensions appear on the disk, i.e. unless it is put into rotation in a Born rigid way. As will be shown below, this is not possible. Furthermore, as noted by Suzuki [35] the paradoxical, non-symmetric situation of figure 15.3 (b) is not predicted by the theory of relativity. If there were a contraction, the circumference of the disk would contract uniformly, and no asymmetry would result. Further replies to Atwater's letter are found in ref. [36].

4. Spatial geodesics on the rotating disk

C. Møller [37] and H. Arzeliès [38] have made some interesting observations concerning the spatial geometry of a rotating reference frame. For one thing they gave a nice illustration of the non-Euclidean character of this geometry by calculating spatial geodesics on the surface z =constant, $\hat{t}$ =constant.

We consider a geodesic curve between two points P_1 and P_2 on the periphery of the disk, as shown in figure 15.4. The Lagrangian function of the curve is

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{r^2\dot{\theta}^2}{1 - r^2\omega^2/c^2} \tag{15.19}$$

where $u^\mu = \dot{x}^\mu$ are the components of the unit tangent vector field of the curve. Hence $\mathbf{u} \cdot \mathbf{u} = 1$, which gives

$$\dot{r}^2 + \frac{r^2\dot{\theta}^2}{1 - r^2\omega^2/c^2} = 1 \tag{15.20}$$

Since θ is a cyclic coordinate

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{r^2\dot{\theta}}{1 - r^2\omega^2/c^2} = constant \tag{15.21}$$

or

$$\dot{\theta} = \left(1 - \frac{r^2\omega^2}{c^2}\right)\frac{p_\theta}{r^2} \tag{15.22}$$

Inserting eq.(15.22) into eq.(15.20) gives

$$\dot{r}^2 = 1 - \left(1 - \frac{r^2\omega^2}{c^2}\right)\frac{p_\theta^2}{r^2} \tag{15.23}$$

This leads to the equation of the geodesic curve between P_1 and P_2

$$\frac{dr}{d\theta} = \pm\frac{r^2\sqrt{1 - \left(1 - \frac{r^2\omega^2}{c^2}\right)\frac{p_\theta}{r^2}}}{\left(1 - \frac{r^2\omega^2}{c^2}\right)p_\theta} \tag{15.24}$$

Inserting the boundary conditions $\dot{r} = 0, \;\; r = r_0$ for $\theta = 0$ into eq.(15.23) we get

$$\frac{p_\theta}{r_0} = \sqrt{1 + \frac{p_\theta^2\omega^2}{c^2}} \tag{15.25}$$

Using this one can rearrange eq.(15.24) obtaining

$$\frac{dr}{r\sqrt{r^2 - r_0^2}} - \frac{\omega^2}{c^2}\frac{r\,dr}{\sqrt{r^2 - r_0^2}} = \frac{d\theta}{r_0} \tag{15.26}$$

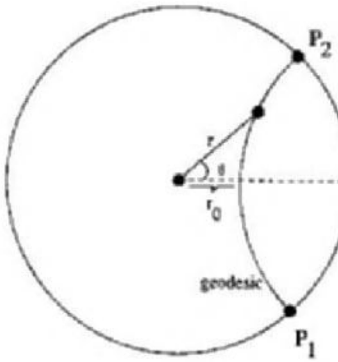

Figure 15.4. Spatial geodesic curve on a rotating disk.

Integration now yields

$$\theta = \pm\frac{r_0\omega^2}{c^2}\sqrt{r^2 - r_0^2} \mp arc\cos\frac{r_0}{r} \tag{15.27}$$

One such curve is plotted in figure 15.4. We see that the geodesic is curved inwards when the disk is rotating. This is intuitively reasonable since the tangential measuring rods in the rotating frame are longer, as observed from the non-rotating laboratory frame, the farther inwards they are. Hence, to have fewer measuring rods along the curve it should pass closer to the axis. On the other hand this bending makes the curve longer as observed in the inertial rest frame of the axis. The shape of the curve is represents a compromise between these two opposing effects.

Further properties of spacetime geodesics for material particles and photons and of spatial geodesics in a rotating reference frame have been discussed by Ashworth and Davies [39].

5. Relativity of simultaneity and coordinates in rotating frames

P. Franklin [40] argued in 1922 that the "Galilean" transformation (15.5) is not suitable in the context of the special theory of relativity. He suggested that one should apply a Lorentz like transformation

$$t = \frac{t' - v(r')\,r'\theta'/c^2}{\sqrt{1 - v(r')^2/c^2}}\,, \quad r = r'\,, \quad r\theta = \frac{r'\theta' - v(r')\,t'}{\sqrt{1 - v(r')^2/c^2}} \tag{15.28}$$

and calculated the velocity $v(r)$ as follows. Eq.(15.28) leads to the relativistic formula for velocity addition in the form

$$\omega\,\Delta r' = \frac{v(r' + \Delta r') - v(r')}{1 - v(r' + \Delta r')\,v(r')/c^2} \tag{15.29}$$

where ω is a constant angular velocity. Taking the limit $\Delta r' \to 0$ leads to

$$\omega \frac{dr'}{dv} = \frac{1}{1 - v^2/c^2} \tag{15.30}$$

Integrating with $v(0) = 0$ gives

$$v = c \tanh \frac{r'\omega}{c} \tag{15.31}$$

which is less than c for every finite value of r' and ω. Inserting this into eq.(15.28) finally leads to the transformation formula

$$t = t' \cosh \frac{r'\omega}{c} - \frac{r'\theta}{c} \sinh \frac{r'\omega}{c}\,, \quad r = r'\,, \quad \theta = \theta' \cosh \frac{r'\omega}{c} - \frac{ct'}{r'} \sinh \frac{r'\omega}{c} \tag{15.32}$$

The transformation (15.32) has later been discussed by Trocheris [41] and Takeno [42]. Trocheris noted that the coordinate clocks in the rotating system are Einstein synchronised. Hence, the coordinate velocity of light is isotropic in this system.

The rotating coordinate system obtained by this transformation has, however, some disadvantages compared to that obtained by the transformation (15.5). If one calculates the spatial line element of the space defined by putting t =constant in eq.(15.32), one obtains a time dependent spatial metric in spite of the fact that the system rotates with constant angular velocity. Also, the simultaneity, say $t = 0$, of the rotating coordinate system corresponds to

$$t' = \frac{r'\theta'}{c} \tanh \frac{r'\omega}{c} \tag{15.33}$$

in the inertial rest frame of the axis of rotation. Hence, going around a circle about the axis of rotation one arrives at a different point of time than at the start. This means that a certain event corresponds to different points of time in the rotating coordinate system. In other words there exists a time discontinuity along a radial line in this coordinate system.

L. Herrera [43] has recently discussed the above transformation and presented a modified form of it. A thorough discussion of the co-ordinate system above, and several other transformations to rotating frames, have been given by B. Chakraborty and S. Sarkar [44].

It should be noted that one may always introduce an orthonormal basis field with Minkowski metric at arbitrary points of a rotating disk. T. A. Weber [45] has explicitly demonstrated how this can be done, by giving the Lorentz transformation from the inertial rest frame K' of the axis to such a basis. As he pointed out, such a transformation has only a local geometrical significance, and must be given in terms of differentials that are not exact. The transformation is not integrable. This was emphasized also by J. F. Corum [46]. H. Nikolić [47] has recently described the relativistic kinematics with reference to a field of local Fermi frames comoving with a rotating disk.

A recent preprint by V. Bashkov and M. Malakhaltsev [48] contains a misunderstanding that should be clarified. They make a "Lorentz transformation" in differential form to "infinitesimal coordinates" on the rotating disk,

$$d\hat{r} = dr' \ , \quad \hat{r}\, d\hat{\theta} = \frac{r' d\theta' - r'\omega\, dt'}{\sqrt{1 - \frac{r'^2\omega^2}{c^2}}} \ , \quad d\hat{t} = \frac{dt' - \frac{r'^2\omega}{c^2} d\theta'}{\sqrt{1 - \frac{r'^2\omega^2}{c^2}}} \tag{15.34}$$

This transformation preserves the form of the line element. Hence,

$$ds^2 = -c^2 d\hat{t}^2 + d\hat{r}^2 + \hat{r}^2 d\hat{\theta}^2 \tag{15.35}$$

Bashkov and Malakhaltsev note that the transformation is not integrable and points out that it is therefore impossible to express $\hat{t},\ \hat{r},\ \hat{\theta}$ as functions of $t',\ r',\ \theta'$ in a finite way.

The clocks measuring the coordinate time $\hat{t}$ are Einstein synchronized. Hence the geometry on the rotating disk is defined by the simultaneity $d\hat{t} = 0$. Then putting $d\hat{t} = 0$ in eq.(15.35) one obtains the spatial line element

$$dl^2 = d\hat{r}^2 + \hat{r}^2 d\hat{\theta}^2 \tag{15.36}$$

From this Bashkov and Malakhaltsev concludes:

> Thus on the disk we get the Euclidean geometry, contrary to the conclusions of other researchers who obtained the spatial line element (15.10).

However, due to the local character of the non-coordinate basis field introduces by Bashkov and Malakhaltsev one cannot deduce the geometry of space just by inspecting the form of the line element. The curvature of the space must be calculated from the general formulae including the structure coefficients [49].

The comoving orthonormal basis of the rotating disk corresponding to the transformation (15.34) has spatial basis forms

$$\omega^{\hat{\mathbf{r}}} = dr' \ , \quad \omega^{\hat{\theta}} = \frac{r' d\theta' - r'\omega\, dt'}{\sqrt{1 - r'^{\,2}\omega^2/c^2}} \tag{15.37}$$

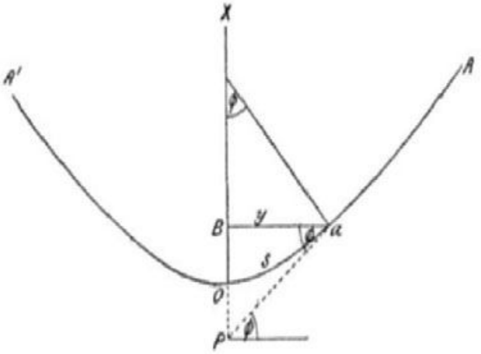

Figure 15.5. The shape of a relativistically rotating disk bent due to the Lorentz contraction in the tangential direction.

Using GRTensor to calculate the Ricci curvature scalar $R = R_{\hat{\imath}\hat{\jmath}}\, R^{\hat{\imath}\hat{\jmath}}$ and the Kretschmann curvature scalar $K = R_{\hat{\imath}\hat{\jmath}\hat{m}\hat{n}}\, R^{\hat{\imath}\hat{\jmath}\hat{m}\hat{n}}$ one obtains

$$R = -\frac{6\omega^2}{(1 - r'^{\,2}\omega^2/c^2)^2} \;, \quad K = \frac{36\omega^4}{(1 - r'^{\,2}\omega^2/c^2)^4} \tag{15.38}$$

showing that the simultaneity space $d\hat{t} = 0$ is curved.

6. What is the effect of the Lorentz contraction upon a disk that is put into rotation?

Having seen Einstein's explanation making it clear that one measures a longer circumference on a rotating disk the faster it rotates, let us go back once more to 1910 and see a consequence of taking seriously the supposition of Ehrenfest's paradox that the circumference of the rotating disk itself is Lorentz contracted. In an article published in 1910 G. Stead and H. Donaldson [50] gave an analysis of the geometrical properties of a rotating disk based upon the apprehension that the periphery of a rotating disk is contracted and the radius not. They treated the disk as an elastic membrane able to deform without any resistance. Hence the Lorentz contraction in the direction of motion of each element of the membrane forces it to bend, so that it gets the shape of a cup as shown in figure 15.5.

With reference to figure 15.5 their analysis is as follows: If $A'OA$ represents the vertical section of the final form of the disk with axis of rotation OX, then Oa measured along the arc is equal to r, while aB measured perpendicular to

OB will be $r\sqrt{1 - v^2/c^2}$. Writing $Oa = s$ and $aB = y$ we have

$$y = s\sqrt{1 - v^2/c^2} = s\sqrt{1 - y^2\omega^2/c^2} \tag{15.39}$$

ω being the angular velocity of the disk. Hence

$$y = \frac{s}{\sqrt{1 + s^2\omega^2/c^2}} \tag{15.40}$$

which leads to

$$\cos\phi \equiv \frac{dy}{ds} = \frac{1}{(1 + s^2\omega^2/c^2)^{3/2}} \tag{15.41}$$

The velocity of an arbitrary point on the disk due to the rotation is

$$v = y\,\omega = \frac{c}{\sqrt{1 + c^2/s^2\omega^2}} \tag{15.42}$$

The velocity is less than c for all values of y and ω. Equation (15.41) shows that when ω becomes very large, $\cos\phi$ is small and nearly independent of s. Hence, for large ω the former disk approaches the form of a right circular cone of small angle, except near the centre where it is curved. A similar description was given by M. Galli [52]

Stead and Donaldson also noted that if the disk were forced not to bend during the rotation, then its material would be strained.

In his popular introduction to the theory of relativity [53] A. S. Eddington comments on Ehrenfest's paradox. He considers a rapidly revolving wheel and writes:

> Each portion of the circumference is moving in the direction of its length, and might be expected to undergo the Fitzgerald contraction due to its velocity; each portion of a radius is moving transversely and would therefore have no longitudinal contraction. It looks as though the rim of the wheel should contract and the spokes remain the same length, when the wheel is set revolving. The conclusion is absurd, for a revolving wheel has no tendency to buckle – which would be the only way of reconciling these conditions. The point which the argument has overlooked is that the results here appealed to apply to unconstrained bodies, which have no acceleration relative to natural tracks in space. Each portion of the rim of the wheel has a radial acceleration, and this affects its extensional properties. When acceleration as well as velocities occur a more far-reaching theory is needed to determine the changes of length.

Comparing this statement with Einstein's considerations we see that they are concerned with different problems. Like Planck, Eddington talks about what happens to a disk that is set into rotation. Einstein, on the contrary, is not interested in this. He considers a disk rotating with constant angular velocity and does not make a comparison of the disk when it is at rest and when it rotates. He is interested in the results of measurements of the length of the circumference and a radial line as performed by means of standard measuring rods corotating with the disk. This is a purely kinematical problem, while Planck and Eddington consider a dynamical problem involving the relativistic theory of elastic media.

This latter problem was treated by H. A. Lorentz [54] and Eddington [55] in 1921 and 1923, respectively. Without giving details Lorentz reports that he has applied the mentioned theory to a thin disk and found a relativistic deformation:

> The result is that, if v is the velocity at the rim, the radius will be shortened in the ratio of 1 to $1 - (1/8)\left(v^2/c^2\right)$. The circumference changing to the same extent, its decrease is seen to be exactly one fourth of that of a rod moving with the same velocity in the direction of its length.

This is the same result as one of Ignatowsky's [12].

Lorentz adds a comment which seems to indicate that the principle of relativity is not valid for rotating motion:

> At first sight our problem seems to lead to a paradox. Let there be two equal disks A and B, mounted on the same axis, A revolving and B at rest. Then A will be smaller than B, and it must certainly appear so (the disks being assumed to be quite near each other) to any observer, whatever be the system of coordinates he chooses to use. However, we can introduce a system of coordinates K revolving with the disk A; with respect to these it will be B that rotates, and so one might think that now this latter disk will be the smaller of the two. The conclusion would be wrong because the system Kwould not be a normal one. If we leave S for it, we must at the same time change the potentials g_{ab}, and if this is done the fundamental equations will certainly again lead to the result that A is smaller than B.

A similar paradox of an electromagnetic nature was resolved by L. I. Schiff [56] in 1939, indicating how the principle of relativity may be extended to encompass rotating motion in the general theory of relativity. (A thorough discussion of this question is found in [57].)

Eddington arrived at the same result for the deformation as Lorentz. Let us follow his deduction. A disk made of homogeneous, incompressible material is caused to rotate with angular velocity ω. Eddington wants to calculate the alteration in radius due to the Lorentz contraction of its mass elements. The meaning of *incompressible* is that the particle density σ(referred to proper measure) is constant and equal to that of a non-rotating disk. But for a rotating disk the particle density σ' referred to axes fixed in space is different.

Since the number of particles in a comoving volume is invariant, then due to the Lorentz contraction,

$$\sigma' = \sigma \left(1 - r^2\omega^2/c^2\right)^{-1/2} \tag{15.43}$$

since $r\omega$ is the velocity of a mass element at a distance r from the axis of rotation. The proper volume element of the disk is

$$dV = \left(1 - r'^2\omega^2/c^2\right)^{-1/2} r'\, dr'\, d\theta'\, dz' \tag{15.44}$$

If the thickness of the disk is b, and its boundary is given by $r' = a'$, the total number of particles in the disk will be

$$N = 2\pi\sigma b \int_0^{a'} \left(1 - r^2\omega^2/c^2\right)^{-1/2} r'\, dr' \tag{15.45}$$

Since this number is unaltered by the rotation, a' must be a function of ω such that

$$\int_0^{a'} \left(1 - r^2\omega^2/c^2\right)^{-1/2} r'\, dr' = constant \tag{15.46}$$

or

$$\frac{1}{\omega}\left(1 - \sqrt{1 - a'^2\omega^2/c^2}\right) = constant \tag{15.47}$$

Expanding the square root this gives approximately

$$\frac{1}{2}a'^2\left(1 + \frac{1}{4}a'^2\omega^2/c^2\right) = constant \tag{15.48}$$

so that if a is the radius of the disk at rest

$$a'\left(1 + \frac{1}{8}a'^2\omega^2/c^2\right) = a \tag{15.49}$$

Hence to the same approximation

$$a' = a\left(1 - \frac{1}{8}a'^2\omega^2/c^2\right) \tag{15.50}$$

Here a' is the radius of the disk referred to its corotating rest frame, but a transformation to the stationary rest frame in which the axis is at rest does not change the radius. Hence, a' is equal to the radius of the rotating disk as measured in the inertial rest frame of its axis.

F. Winterberg [58] has recently deduced the result of Lorentz and Eddington from the theory of elastic media, and also generalized their result by taking account of different stresses in the radial and the tangential directions.

7. Curved space and discussion of Einstein's and Eddington's analysis of the rotating disk

In 1942 Berenda published a discussion of the spatial geometry of the surface of a rotating disk [59]. He came with some interesting observations, but also made a few misinterpretations of earlier works. Under the heading "Einstein's Geometry" he correctly noted that a standard measuring rod along the periphery of a rotating disk is Lorentz contracted. The inertial rest frame of the axis of the disk is called G. Berenda then wrote:

> If we add up the measurement results of the G observers all around the disk, we find
>
> $$\bar{C} = \sum \bar{P} = \sum P' \left(1 - \frac{v^2}{c^2}\right)^{1/2} = C' \left(1 - \frac{v^2}{c^2}\right)^{1/2} < C' \quad (15.51)$$
>
> where $\sum P' = C' = 2\pi r$ is the circumference of the disk when it is at rest in the G frame, and $\bar{C}$ is the circumference of the rotating disk relative to the G observers. Then
>
> $$\bar{C} = 2\pi r \left(1 - \frac{v^2}{c^2}\right)^{1/2} < 2\pi r \quad (15.52)$$

However, according to the Einstein citations above $\bar{C} > 2\pi r$. Einstein's point is that because each measuring rod along the periphery is Lorentz contracted there is place for more of them around the circumference the faster the disk rotates, and the length of the circumference is just the number of measuring rods around it.

Then, under the heading "Eddington's Geometry", Berenda claimed that there is an error in Eddington's treatment cited above. He noted that in the transition from eq.(15.44) to eq.(15.45) Eddington assumed that the angle around the periphery of the rotating disk is 2π just as for a disk at rest. Then he wrote:

> This is, however, the whole point at issue, since the latter assumption is equivalent to the postulate that angular measures are unaffected by rotation, i.e., that the geometry remains Euclidean.

However, Eddingtons's result is correct because the angle around a circle with an arbitrary centre on a curved surface is defined on the tangent plane of the centre of the circle, i.e. it is defined locally. Hence, in the case of a circle around the axis of a rotating disk one has to take the limit $r \to 0$ to find the angle. This implies that the angle around a circle *is* 2π even on a surface with non Euclidean geometry.

Berenda then deduced eq.(15.10) above and pointed out that this line element gives the geometry of a surface that is everywhere orthogonal to the world lines of points fixed on the disk. In the case of the rotating disk the surface is curved and described by the line element (15.10). He then found the non vanishing components of the Riemann curvature tensor for this surface

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} = -\frac{3r^2\omega^2}{c^2}\left(1 - \frac{r^2\omega^2}{c^2}\right)^{-3} \tag{15.53}$$

The Gaussian curvature of the surface is

$$K = \frac{R_{1212}}{g} = -\frac{3\omega^2}{c^2}\left(1 - \frac{r^2\omega^2}{c^2}\right)^{-2} \tag{15.54}$$

where $g = \det(g_{ij})$. Hence the surface orthogonal to the world lines of particles fixed on the rotating disk has negative curvature. The geometry is hyperbolic.

In a recent work Rizzi and Ruggiero [4] have given a thorough analysis of the space geometry of rotating platforms . The authors have applied Catteano's projection formalism to define precisely the concept of spatial geometry in a rotating frame in an operationally meaningful way. Their rigorous approach has reproduced what may be called the standard results leading to a hyperbolic spatial geometry with essentially the curvature given above.

8. Uniform contra rigid rotation

Hill [60] argued in 1946 that the limitation $r < c/\omega$ of the extension of a rotating disk coming from the Galilean form of the transformation (15.5) combined with the relativistic requirement that the velocity of each point of the rotating frame must be less than the velocity of light, seems unnatural. He therefore proposed to specify the motion of the disk so that close to the axis the rotational velocity is approximately proportional to the radius i.e. the angular velocity is approximately constant, and far from the axis the velocity approaches that of light. In addition he demanded that the disk consists of a fluid rotating uniformly, according to the following definition:

> The motion of the fluid will be considered to be uniform rotation with constant angular velocity ω_0 if the relative angular velocity of the material about any point P, as measured in a set of axes with respect to which P is momentarily at rest, has the value ω_0 independently of the choice of the point P.

The set of axes G' with respect to which P is momentarily at rest is defined by a Lorentz transformation from the rest frame of the axis, i.e. it is the co-moving inertial rest frame of the point P. The angular velocity is defined in G' by

$$\vec{\omega}' = \frac{1}{2}\nabla \times \vec{v}' \tag{15.55}$$

where $\vec{v}'$ is the velocity field of the disk material with respect to G'. The velocity of a point at a distance r from the axis with respect to the inertial rest frame G of the axis, has components

$$v_x = -y\,\omega\,(r)\ ,\quad v_y = x\,\omega\,(r)\quad \text{with}\quad r = \left(x^2 + y^2\right)^{1/2} \tag{15.56}$$

Hence

$$v\,(r) = r\,\omega\,(r) \tag{15.57}$$

Calculating $\vec{v}'$ from eq.(15.57) and the Lorentz transformation between G and G' and $\vec{\omega}'$ from eq.(15.55) and then demanding that $\vec{\omega}'$ =constant, Hill obtained a differential equation for $v\,(r)$. Solving this he found an expression involving Bessel functions, a result which was later found also by Prechtl [61]. In the limiting cases of large and small distances from the axis of rotation the expression of Hill gives

$$v\,(r) = r\omega'\left(1 - \frac{1}{2}\frac{r^2\omega'^2}{c^2}\right)\quad \text{for}\quad \frac{r\omega'}{c} << 1 \tag{15.58}$$

$$v\,(r) = c\left(1 - \frac{1}{4}\frac{c}{r\omega'} - \frac{1}{32}\frac{c^2}{r^2\omega'^2} - \cdots\right)\quad \text{for}\quad \frac{r\omega'}{c} >> 1 \tag{15.59}$$

As pointed out by Hill himself his discussion did not solve the problem posed in Ehrenfest's paradox. This would demand a theory of the generation of motion from a state of rest under suitable specifications.

Hill's discussion was followed up by N. Rosen [62]. He criticized Hill for making use of a Lorentz transformation in going over to a non inertial frame of reference. However, Hill only used the Lorentz transformation to go into an instantaneous rest frame of a point on the rotating disk, which is clearly a valid procedure.

9. Relativistically rigid motion and rotation

In a subsequent article [27] Rosen applied the concept of relativistic rigid motion as introduced by Born [5] to the case of uniform rotation in order to deduce the function $v\,(r)$ for this case. He argued as follows:

> In classical physics one can characterize the motion of a rigid body by the fact that the rate of strain vanishes. In a Cartesian coordinate system, the velocity components satisfy the relation
>
> $$\partial v_i/\partial x_k + \partial v_k/\partial x_i = 0 \quad (i,\, k = 1,\, 2,\, 3) \tag{15.60}$$
>
> In a relativistic treatment one looks for a covariant equation which reduces to (15.55) in a Galilean system if the velocity is small (compared to that of light). The obvious generalization is to introduce, in an arbitrary coordinate system, the symmetrical tensor
>
> $$P_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\nu} + u_{\nu;\mu}\right) \tag{15.61}$$

where the velocity 4-vector u^λ is given by $u^\lambda = dx^\lambda/ds$ and a semicolon denotes covariant derivation, and take as the condition for rigid-body motion $P_{\mu\nu} = 0$. However, this condition represents too severe a restriction. In view of the fact that the vector u^λ satisfies the identity $u^\lambda u_\lambda = -1$ it follows from $P_{\mu\nu} = 0$ on multiplication by u^ν that $u_{\mu;\nu}u^\nu = 0$. This is just the covariant form of the condition that the four acceleration vector vanishes, so that every particle of the body moves along a geodesic, or in a gravitation free space, along a straight line.

It is therefore necessary to weaken the condition imposed on the motion. For this purpose we replace $P_{\mu\nu}$ by

$$p_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\nu} + u_{\nu;\mu} - u_{\mu;\alpha}u^\alpha u_\nu - u_{\nu;\alpha}u^\alpha u_\mu\right) \tag{15.62}$$

since we then have the identity $p_{\mu\nu}u^\nu \equiv 0$. Let us now take as the condition for rigid-body motion

$$p_{\mu\nu} = 0 \tag{15.63}$$

In a Galilean coordinate system, at a point where the velocity 3-vector vanishes, this reduces to the classical condition (15.60). Therefore it is equivalent to the condition proposed by Born [5].

Since eq.(15.63) is a tensor equation it can be applied in any frame of reference. If we take an inertial system with Cartesian coordinates then we can describe rotation about an axis by setting

$$\begin{gathered} u_1 = u^1 = \sigma y\,, \quad u_2 = u^2 = \sigma x\,, \\ u_3 = u^3 = 0\,, \quad u_4 = -u^4 = -\left(1 + \sigma^2 r^2/c^2\right)^{1/2} \end{gathered} \tag{15.64}$$

with $\sigma = \sigma(r)$, $r^2 = x^2 + y^2$. From eq.(15.63) one gets the single equation

$$d\sigma/dr = \sigma^3 r \tag{15.65}$$

which has for its solution

$$\sigma = \omega/\left(1 - r^2\omega^2/c^2\right)^{1/2} \tag{15.66}$$

where ω is a constant. Going over to the three-dimensional velocity v by the relations

$$u = \left(u_1^2 + u_2^2\right)^{1/2} = \sigma r = v/\left(1 - v^2/c^2\right)^{1/2} \tag{15.67}$$

we get

$$v = r\omega \tag{15.68}$$

Hill [60] defined uniform rotation as a motion where the local angular velocity of the disk material as measured in local inertial frames in which the material instantaneously has no translational motion, is independent of the position. The calculation of Rosen [27] shows that uniform rotation in this sense is not Born rigid.

10. The theory of elastic media applied to the rotating disk

The results of Lorentz [54] and Eddington [55] were reviewed and carried further by G. L. Clark [63, 64]. According to the non-relativistic theory of elastic media an elastic disk will get an increase of its radius when it is put into rotation. For an approximately incompressible disk ($\lambda >> \mu$, where λ and μ are the Lame constants of isotropic stress and shear, respectively) with radius a the radial displacement is

$$\Delta r_N = \frac{a^3\omega^2}{8c_0^2} \tag{15.69}$$

where c_0 is the velocity of sound in the medium.

Taking into account the elastic effect of the tendency of the material to Lorentz contract in the tangential direction, there appears a tangential stress, which forces the disk to contract in the radial direction by an amount

$$\Delta r_R = -\frac{a^3\omega^2}{8c^2} \tag{15.70}$$

Hence, the change of radius of the disk is

$$\Delta r = \frac{a^3\omega^2}{8}\left(\frac{1}{c_0^2} - \frac{1}{c^2}\right) \tag{15.71}$$

By taking c_0 to be infinite for an incompressible medium one obtains the result of Lorentz and Eddington as given in eq.(15.50).

Clark notes, however, that according to the special theory of relativity the upper limit for the sound velocity is the velocity of light. Hence, relativistically a medium in which $c_0 = c$ is maximally rigid. For a disk consisting of such material $\Delta r = 0$. In this case the contraction of the disk due to the stresses induced by the efforts of the material to try to Lorentz contract is cancelled by the "classical elastic expansion" of the rotating disk.

Clark's analysis was followed up by Cavalleri [65] in 1968 in an interesting, although somewhat controversial paper. He first gave a thorough review of earlier work on this topic. Then he concluded that "Ehrenfest's paradox cannot be resolved from a purely kinematical point of view", and inferred that the relativistic kinematics for extended bodies is not generally self-consistent. Finally he noted that the analysis of Clark [63, 64] is valid only for small strains, and gave a more general analysis for material in which the sound velocity is equal to the velocity of light. While Clark found that the radius of such a disk is independent of the angular velocity, Cavalleri found that it increases with the angular velocity.

A few months later A. Brotas [66] followed up by calculating an explicite expression for the radius of a rotating ring consisting of the type of material considered by Cavalleri. McCrea [67] made the same calculation a few years later. Let us follow the main points of their calculation. Consider an element of the ring with proper length L_0^0 when the material is unstrained and proper length L_0 when it is strained. Let ρ_0^0 be the proper density of the unstrained material. Then the tension p is given by

$$p = \frac{1}{2}\rho_0^0 c^2 \left(\frac{1}{s^2} - 1\right) \ , \quad s = \frac{L_0}{L_0^0} \tag{15.72}$$

The density of the strained material is

$$\rho_0 = \frac{1}{2}\rho_0^0 \left(\frac{1}{s^2} + 1\right) \tag{15.73}$$

The radius and length of the ring fulfils the Euclidean relationship $L = 2\pi R$ whether it is strained or not. Hence the Lorentz contraction of the ring when it rotates will cause a decrease of its radius. Let $2\pi R$ be the Lorentz contracted length of the rotating ring, which is strained due to the centrifugal effect of the rotation, i.e. it is the length of the ring as measured in the inertial rest frame of the axis. Then its proper length is $\gamma 2\pi R$, where $\gamma = \left(1 - R^2\omega^2/c^2\right)^{-1/2}$. Thus,

$$s = \gamma \frac{R}{R_0} \tag{15.74}$$

where $2\pi R_0$ is the proper length of the ring when it is at rest so that it is not strained. Cavalleri showed that the tension of the material in the rotating ring is given by

$$p = -\rho_0 R^2 \omega^2 \tag{15.75}$$

From eqs.(15.72), (15.73) and (15.74) follow

$$s = \left(\frac{1 + R^2\omega^2/c^2}{1 - R^2\omega^2/c^2}\right)^{1/2} \tag{15.76}$$

Eqs.(15.74) and (15.76) finally lead to

$$R = \frac{R_0}{\sqrt{1 - R_0^2\omega^2/c^2}} \tag{15.77}$$

This expression shows how the ring is elongated with increasing angular velocity. The maximal radius is obtained when $R\omega = c$. Then $s = \infty$, $\rho_0 = (1/2)\,\rho_0^0 \quad p = -(1/2)\,\rho_0^0 c^2 = -\rho_0 c^2$, $R = R_0\sqrt{2}$. It may be noted that the equation of state is that of vacuum energy.

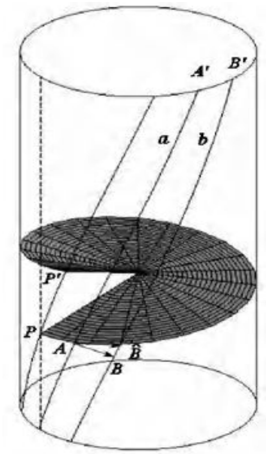

Figure 15.6. The helices such as a and b represent world lines of points at rest in the rotating frame. Points at rest in the inertial rest frame of the axis are represented by vertical world-lines. Cantoni's figure has here been modified by drawing not only the circumference defined by simultaneity in the local inertial rest frames of the disk material, but also the surface connecting the circumference with the axis.

R.G. Newburgh [68] presented a practical proposal for the experimental investigation of the type of motion studied by Cavalleri, Brotas and McCrea. It may also be mentioned that the results of applying radar measurements and triangulation to a rotating disk have been analyzed by R. C. Jennison, D. G. Ashworth and P. A. Davies [69-74]. Some of the results in these articles were earlier found by Arzeliès [38] in a comprehensible treatment of spacetime geodesics and spatial geodesics on a rotating disk. Furthermore, photographing of a rotating disk has been discussed by P. F. Browne [75]. These topics will, however, not be discussed in the present article. Further results and discussion of these matters are found in [76-79].

At about the same time V. Cantoni [80] presented a clarifying paper, writing:

> It is shown, *on purely kinematical grounds*, that one of the assumptions implicitly contained in the statement of Ehrenfest's paradox is not correct, the assumption being that the geometry of Minkowski space-time allows the passage of the disk from rest to rotation in such a fashion that both the length of the "radius" and the length of the periphery, measured with respect to the co-moving frame of reference, remain unchanged.
>
> The following discussion is believed to show that a careful definition of all quantities involved eliminates the paradox and, with it, the alleged inconsistency of relativistic kinematics.

At any rate, should relativistic kinematics not be self-consistent, it would seem hard, on logical grounds, to accept the view that the addition of dynamical arguments might improve the situation.

Cantoni then presented the following analysis with reference to figure 15.6.

One can give a consistent definition of the length of the whole "circumference" relative to the rotating reference frame K as the length of the curve *PP'* everywhere orthogonal to the world-lines of the reference points of K, starting from an event P and ending on an event P on the world-line of the same particle of the edge of the disk, and winding once around the axis of rotation. *Notice that such a curve is not closed* in space-time, and distant events on it, such as P and P', can in no sense be regarded as simultaneous. For the segment between two infinitely close particles with world-lines, say, a and b, denoting by dl the length relative to the inertial rest frame K of the axis (i.e., the length of $\overrightarrow{AB}$), and by $d\hat{l}$ the length relative to K (i.e., the length of $\overrightarrow{A\hat{B}}$), the latter being the same as the length of the segment with respect to its local inertial rest-frame, one has, according to the well known equation for the Lorentz contraction,

$$d\hat{l} = dl\left(1 - v^2/c^2\right)^{-1/2} \tag{15.78}$$

Since $\overrightarrow{AB} = \overrightarrow{A'B'}$, dl can also be interpreted as the length of the arc of circumference $A'B'$ at a fixed time, say $t = 0$ in F , and one has, with obvious notations,

$$d\hat{l} = \left(1 - v^2/c^2\right)^{-1/2} r\,d\theta \tag{15.79}$$

so that, integrating around the circumference, one gets for the length $\hat{L}$ of the edge relative to K

$$\hat{L} = 2\pi r\left(1 - v^2/c^2\right)^{-1/2} > 2\pi r \tag{15.80}$$

Clearly it was not justified to assume *a priori* that both the length of the radius of the disk and the length of its periphery, as measured from the co-moving frame of reference, could remain unchanged in the passage from rest to rotation.

In other words, in Minkowski space-time there exists no *rigid* family of world-lines describing the passage of a disk from rest to rotation: any family of world-lines describing a still disk during an initial stage and a rotating disk during a final stage is *necessarily non-rigid* in the intermediate stage. This fact only depends on the geometrical structure of Minkowski space and on the definition of rigidity, and is not a consequence of the dynamical properties of actual materials.

G. Rizzi and A. Tartaglia [81] stressed the significance for the analysis of the Sagnac effect of the fact that the periphery of the disk as defined by simultaneity in the instantaneous inertial rest frames of the elements of the periphery is discontinuous in spacetime, as shown in figure 15.6.

W. A. Rodrigues [82] did not accept the possibility of a purely kinematical solution to Ehrenfest's paradox. He returned to a pre-relativistic view of the ether,

arguing that the Lorentz contraction is a real phenomenon which results as a consequence of the interaction of material bodies with the ground-state vacuum of the universe.

This conception when applied to the Ehrenfest paradox implies that the circumference of the rotating disk gets a contraction due to the interaction of the periphery with the physical vacuum. Measuring rods along the periphery will also contract. Hence the measured length of the periphery is independent of the rotational velocity, and the geometry remains Euclidean.

According to Rodrigues there is no tangential strain, and hence no stress, in the rotating disk because its length has not been changed as the disk were put into rotation. A similar view was held by Phipps [83] which resulted in a discussion with Cantoni [84, 85] that did not, however, clarify the matter very much. Grøn [86] responded to Phipps' article by giving a covariant formulation of Hooke's law [87]. If this is accepted the existence of strain is invariant. The strain is defined with reference to the rest length of a mass element, so that a material becomes strained if the rest length changes. According to the covariant formulation of Hooke's law this leads to stress in the material. Thus, if a disk is put into rotation by accelerating all points identically as measured in K', then the length of the periphery remains unchanged in K', but its rest length increases and the material will get a tangential stress [88]. D. Dieks [89] pointed out that the existence of such stresses demonstrates the physical nature of the Lorentz contraction, even when interpreted within the special theory of relativity.

Winterberg [58] presented an "ether-interpretation" of the Lorentz contraction and "solved" the Ehrenfest paradox by calculating the deformation of the disk when it is put into rotation. He also suggested an experiment to test special relativity against the "dynamic Lorentz-Poincaré interpretation of the Lorentz contraction". E. M. Kelly [90] introduced a new sort of contraction (or expansion) for moving bodies to solve the Ehrenfest paradox. R. D. Klauber [91] gave an analysis of space and time on a rotating disk based upon the assumption that the special theory of relativity is not applicable to rotating frames. He replaced Einstein's postulate on the isotropy and invariance of the velocity of light by a postulate saying that "The speed of light is not invariant between the ground and the rotating frame, and in the rotating frame is found to first order by the velocity addition law $\left|v_{\text{tangential, light}}\right| \cong c \pm r\omega$." Then he deduced among other things an Euclidean spatial geometry on the rotating disk. A discussion about these topics between Klauber and Weber appeared in American Journal of Physics [92, 93]. Further discussion of the length of the circumference of a rotating disk with radius R, maintaining that it is $2\pi R$, was presented by A. Tartaglia [94].

11. The metric in a rotating frame as solution of Einstein's field equations

In 1951 there appeared a somewhat surprising article [95] by B. Kurşunoğlu titled *Space-time on the Rotating Disk.* There are 3 unusual properties of its contents.

I. He wrote the metric in the rotating frame in diagonal form

$$ds^2 = -e^{\nu} dt^2 + dr^2 + e^{\mu} d\theta^2 + dz^2 \tag{15.81}$$

where μ and ν are functions of r alone.

II. The field equations were solved with the energy-momentum of an elastic medium as source. But the density and pressure of the medium disappeared from the equations and are not present in the solution, which is given as follows

$$ds^2 = -\left(c^2 - r^2\omega^2\right) dt^2 + dr^2 + \frac{r^2 d\theta^2}{1 - r^2\omega^2/c^2} + dz^2 \tag{15.82}$$

Here ω is the angular velocity of the rotating frame.

III. Kurşunoğlu then calculated the curvature of space-time and found a non-vanishing value. Hence, he concluded that "for an observer on a rotating disk there is no way of escape from a curved space-time".

Concerning point I it was shown by Weysenhoff [30] that it is not possible to cover a rotating frame with an orthogonal coordinate system. Hence the metric cannot be diagonal in a rotating coordinate system.

The points II and III were investigated by Grøn [96].

Let us first consider the point II. It was shown that the energy-momentum tensor specified by Kurşunoğlu could be represented by a perfect fluid. For a static metric of the form (15.81) the field equations then imply the equation of state $p = -(1/3)\rho$ (in this section we use units so that $c = G = 1$). Solving the field equations with mass density ρ_0 at the axis, one finds the solution

$$ds^2 = -\left(1 + \frac{8\pi\rho_0}{3} r^2\right) dt^2 + dr^2 + \frac{r^2 d\theta^2}{1 + \frac{8\pi\rho_0}{3} r^2} + dz^2 \tag{15.83}$$

Introducing curvature coordinates by the transformation $\hat{r} = r(1 + \frac{8\pi\rho_0}{3} r^2)^{\frac{1}{2}}$ the line element takes the form

$$ds^2 = -\frac{dt^2}{1 - \frac{8\pi\rho_0}{3}\hat{r}^2} + \frac{d\hat{r}^2}{\left(1 - \frac{8\pi\rho_0}{3}\hat{r}^2\right)^3} + \hat{r}^2 d\theta^2 + dz^2 \tag{15.84}$$

The calculation shows that the interpretation of eq.(15.82) that it represents the metric of empty space in a rotating coordinate system is not viable.

On the other hand, if one starts with a stationary, cylindrically symmetric line-element of the form

$$ds^2 = f(r)\,dt^2 + dr^2 + l(r)\,d\theta^2 + dz^2 + 2m(r)\,dt\,d\theta \qquad (15.85)$$

and solves the vacuum field equations, one finds the line element (15.7).

We now go on to point III. As pointed out by Wilson [97] the spacetime described by the metric (15.7) is flat.

However, if one calculates the spacetime curvature from the line element (15.82) one finds $R_{\theta\theta} = -3r^2\omega^2\left(1 - r^2\omega^2\right)^{-3}$, which was the reason for the conclusion of Kurşunoğlu cited above. However, the correct interpretation of this expression is obtained by writing it as

$$R_{\theta\theta} = \frac{32\pi\rho_0}{3} r^2 \left(1 + \frac{8\pi\rho_0}{3} r^2\right)^{-3} \qquad (15.86)$$

This represents a component of the Ricci curvature tensor in a static spacetime filled with an elastic medium.

Finally it may be noted that one may obtain a line-element that is formally identical to eq.(15.82) by introducing a non coordinate basis of one-forms $\left(\boldsymbol{\omega}^t, \boldsymbol{\omega}^r, \boldsymbol{\omega}^\theta, \boldsymbol{\omega}^z\right)$ given by

$$\boldsymbol{\omega}^t = \mathbf{d}t' - r'^2\omega\left(1 - r'^2\omega^2\right)^{-1}\mathbf{d}\theta', \quad \boldsymbol{\omega}^{\boldsymbol{r}} = \mathbf{d}r', \quad \boldsymbol{\omega}^\theta = \mathbf{d}\theta - \omega\mathbf{d}t', \quad \boldsymbol{\omega}^z = \mathbf{d}z' \qquad (15.87)$$

where $(\mathbf{d}t', \mathbf{d}r', \mathbf{d}\theta', \mathbf{d}z')$ are the coordinate basis forms of the non-rotating cylindrical coordinate system. In this basis the line element takes the form

$$ds^2 = -\left(1 - r^2\omega^2\right)\left(\omega^t\right)^2 + \left(\omega^r\right)^2 + \frac{r^2\left(\omega^\theta\right)^2}{1 - r^2\omega^2} + \left(\omega^z\right)^2 \qquad (15.88)$$

Kurşunoğlu writes the metric in this form with dt instead of $\omega^{\mathbf{t}}$ and so forth. This metric has also been considered by Adler et al. [98] and essentially the same metric by Arzeliès [99] who also calculated a non-vanishing spacetime curvature. However, the vanishing of the curvature of the Minkowski spacetime is invariant, and cannot be changed by expressing the metric in a new basis. The reason for their result is that both Arzeliès and Kurşunoğlu treated the metric as if it was expressed in a coordinate basis. However, this is not the case since $\omega^{\mathbf{t}}$ is not an exact differential form. This means that the curvature tensor cannot be calculated by means of the usual expressions valid for a coordinate basis. One has to include the structure coefficients in the calculation. Making this one obtains zero curvature.

12. Kinematical solution of Ehrenfest's paradox

Sama [100] has argued that this paradox only arises from an ambiguous use of notation. However, there are deep and interesting points connected by the problem raised by Ehrenfest [1] that cannot be analyzed by restricting oneself to notational matters. One is the role of the relativity of simultaneity which is essential to obtain a kinematical solution of the paradox. This solution which will be reviewed in the present section, was given by Grøn several years ago [101].

Assume that there are n marks on the circumference of a disk which is rotating with an angular velocity ω. One wants to increase the rate of rotation by giving the marks small blows. In order to increase the angular velocity in a Born rigid way the blows must be given to the marks simultaneously in their instantaneous inertial rest frames.

We now consider this acceleration program from the point of view of the inertial rest frame of the axis. The marks are enumerated from 1 to n in the direction of the rotation. Performing Lorentz transformations from the instantaneous inertial rest frames of the marks to the laboratory frame, one finds that 2 gets a blow later than 1, 3 later than 2, and so forth. Going around the disk one finds that n happens later than n-1. Hence, n happens later than 1. But passing on from n to 1, the blow 1 should happen later than n since the events should be simultaneous as measured in the instantaneous rest frame of the element between the marks n and 1. P. Noerdlinger [102] has provided a graphical illustration of this fact, commenting:

> Observers following a rotating ring will not synchronize their clocks in the same way as inertial ones. Cumulating this result around the ring, leaves a time mismatch depending on the rotational speed and area enclosed.

This shows that due to the relativity of simultaneity the acceleration program that would represent a Born rigid increase of the angular velocity of the disk, define a set of kinematically self contradictory boundary conditions. The conclusion is that a Born rigid transition of the disk from rest to rotation is a *kinematic* impossibility. This was mentioned already in 1921 by Pauli [103] who wrote:

> A simple argument by Ehrenfest [1] shows however that such a body [Born rigid] cannot be set in rotation.

13. Energy associated with tangential stress in a rotating disk

M. H. Mac Gregor [104] has questioned the existence of relativistic stresses in a rotating ring. He argued as follows:

> Let us fasten a string between two points A and B on the circumference of a disk, and then give the disk an angular acceleration, so that it attains an angular velocity ω. An observer in the disk co-ordinate frame K who studies this event sees the distance *AB* increases with increasing angular velocity ω, as specified by eq.(15.10), and he concludes that the string has become stretched. An observer in the fixed inertial system K' who studies this same event sees the distance

> *AB* as remaining unchanged (since the points A and B has similar acceleration histories), but he knows that the string extending from A to B has undergone a relativistic contraction in length; hence he also concludes that the string has become stretched. This stretching is a purely kinematic result, and it gives rise to relativistic stress. The observers in K andK' each deduce a kinematic stretching of the string by a factor of $\left(1 - a^2\omega^2/c^2\right)^{-1/2}$, where a is the radius of the disk. Not only is the string stretched by a factor of $\left(1 - a^2\omega^2/c^2\right)^{-1/2}$ when the disk is accelerated to an angular velocity ω, but the mass of the string is also increased by this same factor. Hence the mass per unit length of the string remains unchanged. If we measure relativistic stresses in terms of changes in linear *density* (mass per unit length) rather than in terms of simple changes in *length*, then these stresses do not occur.

In this way Mac Gregor defines away the relativistic stress associated with the Lorentz contraction of the material. However the stress cannot be removed by a definition. It has physical effects. For example, by velocities of the periphery sufficiently close to the velocity of light, the disk material would crack. There is an energy associated with the relativistic stresses, and this implies that one must perform an extra amount of work in order to change the angular velocity of a disk due to these stresses. It was shown by Grøn [105] how this comes about due to the relativity of simultaneity.

We shall consider retardation of a rotating ring consisting of small springs with elastic constant k and rest length L_0. Initially the ring rotates so that the springs are close to each other, end to end, but without stress. We assume that the braking is performed by simultaneous blows around the ring, as measured in K'. Then the time difference of two blows at the ends of a spring, as measured in the instantaneous rest frame K_0 of the spring is

$$\Delta t = \gamma \left(a\omega/c^2\right) L \tag{15.89}$$

where $L = \gamma^{-1} L_0$ is the initial Lorentz contracted length of the spring. If the velocity change of the rear end of the spring is $dv' = a\, d\omega$ as observed in K', it follows from the Lorentz transformation that the velocity change in K_0 is

$$dv = -\gamma^2 a\, d\omega \tag{15.90}$$

During the time interval Δt the rear point moves towards the front point with this velocity as observed in K_0. So the spring gets a compression

$$ds = dv\, \Delta t = -\gamma^3 \left(a^2\omega/c^2\right) L\, d\omega \tag{15.91}$$

Integration gives

$$s = (\gamma_0 - \gamma)\, L \tag{15.92}$$

where $\gamma_0 = \left(1 - a^2\omega_0^2/c^2\right)^{-1/2}$.

The force acting on the spring in order to compress it is ks. Hence a work is performed on the spring,

$$W = \int_{\omega_0}^{0} ks\, ds = \frac{1}{2}k\, \left[s^2\right]_{\omega_0}^{0} = \frac{1}{2}kL^2\left(\gamma_0 - 1\right)^2 \tag{15.93}$$

where we have used eq.(15.92). As expressed by the initial uncompressed rest length of the spring the work is

$$W = \frac{1}{2}kL_0^2\left(1 - \gamma_0^{-1}\right)^2 \tag{15.94}$$

This work, which has been calculated in the instantaneous inertial rest frame of a spring, gives the contribution to the spring's rest mass, due to stresses developed when the angular velocity of the ring is changed in a Born rigid way. Expanding in powers of $a\omega/c$ and retaining only the first term gives

$$W \cong \left(a\omega_0/2c\right)^4 2kL_0^2 \tag{15.95}$$

This shows that the accumulated potential energy in a rotating ring due to relativistic stresses is a fourth order effect in v/c.

14. A rotating disk with angular acceleration

M. Strauss [106] claimed that:

> If the measuring rods laid along the circumference of the rotating disk are Lorentz contracted with respect to the inertial frame, so are the distances on the circumference they are supposed to measure; hence the two effects would cancel each other, and the ratio C/D (circumference/diameter) would turn out to equal π as in the Euclidean plane.

Grøn [107] has argued that this claim cannot be true. Firstly, as shown in Section 12, the disk cannot be put into rotation in a Born rigid way. Hence, the circumference will not be Lorentz contracted.

We now assume that the circumference of the disk is covered by n standard measuring rods. A *standard* measuring rod in a reference frame with arbitrary motion has by definition constant proper length. As observed from a reference frame where they move standard measuring rods are subject to Lorentz contractions only, which means that they perform Born rigid motions.

In order to obtain this, n rods are assumed to rest on the disk without friction, being kept in place by a frictionless rim on the circumference of the disk, each rod being fastened to the disk at one end only, at points k so that they just cover the circumference when the disk is not rotating, as shown in figure 15.7.

The only isotropic way of giving the disk an angular velocity is to accelerate all points of the disk simultaneously as measured in the laboratory frame K'.

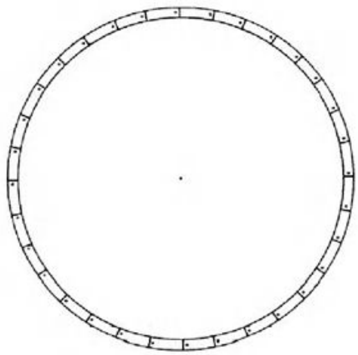

Figure 15.7. The disk and the measuring rods at rest.

In the rest frame K_k of a point k on the periphery of the disk one then measures that the point k is accelerated at a point of time

$$\Delta t_k = \left(1 - a^2\omega^2/c^2\right)^{-1/2} \left(a\omega/c^2\right) L_0 \tag{15.96}$$

earlier than the point k+1, where $L_0 = 2\pi a/n$ is the proper length of the standard measuring rod at k. Thus the distance between these points, that is the point at the front of one measuring rod and the front of the next, increases as observed in K_k. However, according to their definition, the standard measuring rods move rigidly. Their proper length remain unchanged. Accordingly, the rods separate from each other as the angular velocity of the disk increases. If the velocity of a rod is increased from $a\omega$ to $a\,(\omega + d\omega)$ as observed in K', its velocity change, as observed in K_k, is $\left(1 - a^2\omega^2/c^2\right)^{-1} a\, d\omega$. During this change the distance between two neighbouring rods increases with

$$ds_k = \left(1 - a^2\omega^2/c^2\right)^{-3/2} \left(a^2\omega/c^2\right) L_0\, d\omega \tag{15.97}$$

Integrating, one finds the distance between the rods, as measured in K_k, when the disk rotates with an angular velocity ω

$$s_k = \left[\left(1 - a^2\omega^2/c^2\right)^{-1/2} - 1\right] L_0 \tag{15.98}$$

Thus the distance as measured in K is

$$s = L_0 - L_0 \left(1 - a^2\omega^2/c^2\right)^{1/2} \tag{15.99}$$

in accordance with the fact that the measuring rods are Lorentz contracted, while the circumference of the disk is not. The observation in K' of the rotating disk and the measuring rods is shown in figure 15.8.

Figure 15.8. The rotating disk and the measuring rods.

The results of this analysis imply that the ratio of the length of the circumference of a rotating disk and the length of its diameter, measured by means of standard measuring rods at rest on the rotating disk is $\left(1 - a^2\omega^2/c^2\right)^{-1/2}\pi$. Since this statement is invariant under a transformation connecting two different coordinate systems inside the same system of reference, as it is based on the use of standard instruments, it is a statement characterizing the intrinsic spatial geometry of the rotating disk. It follows that this geometry is non-Euclidean.

A similar result has been found by considering a rotating disk with angular acceleration [108].

15. A rolling disk

We shall here follow the presentation of Kevin Brown [109]. A slightly different analysis of a rolling disk was given by Grøn [101].

A disk with radius a is rolling so that the axis moves with constant velocity $a\omega$ along the negative X-axis in the instantaneous inertial frame K_0 of that element of the disk which has contact with the ground. The co-moving coordinates of this frame are $(T,\, X,\, Y)$. A fixed point on the disk at the location $(r,\, \theta)$ has at the time t' the coordinates $(t',\, x',\, y')$ in the rest frame K' of the axis,

$$t'\left(r,\,\theta,\,t'\right),\;\; x'\left(r,\,\theta,\,t'\right) = r\,\cos\left(\omega t' + \theta\right),\;\; y'\left(r,\,\theta,\,t'\right) = r\,\sin\left(\omega t' + \theta\right) \tag{15.100}$$

Making a Lorentz transformation to the laboratory system leads to

$$\begin{aligned} T\left(r,\,\theta,\,t'\right) &= \gamma\left(t' - r\omega x'/c^2\right) = \gamma\left[t' - \left(r^2\omega/c^2\right)\cos\left(\omega\,t' + \theta\right)\right] \\ \text{where } \gamma &= \left(1 - r^2\omega^2/c^2\right)^{-1/2} \end{aligned} \tag{15.101}$$

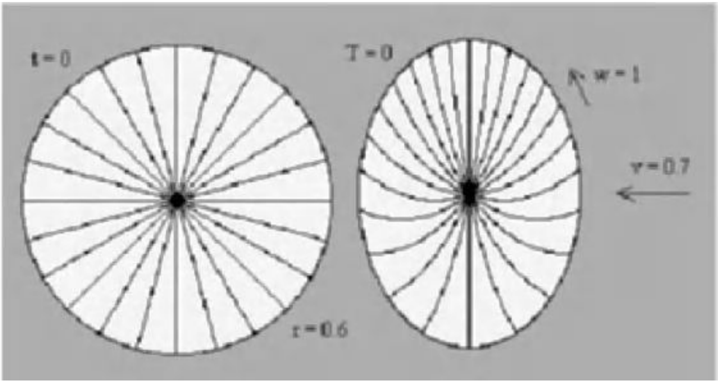

Figure 15.9. The non-rotating disk is drawn to the left and the rolling disk to the right. Radial lines of the rolling disk are curved as observed by simultaneity in the frame K_0.

$$\begin{aligned} X\left(r,\,\theta,\,t'\right) &= \gamma\left(x' - r\omega\,t'\right) = \gamma\left[r\,\cos\left(\omega\,t' + \theta\right) - r\omega\,t'\right] \\ Y\left(r,\,\theta,\,t'\right) &= r\,\sin\left(\omega\,t' + \theta\right) \end{aligned} \quad (15.102)$$

We shall determine the position of the point on the disk at a given point of time in the laboratory system, say $T = 0$. From the equation for $T\,(r,\,\theta,\,t')$ we see that the instant $T = 0$ in K_0 corresponds to different points of time in K' given by

$$t' = \left(r^2\omega/c^2\right)\cos\left(\omega t' + \theta\right) \quad (15.103)$$

Substituting this into eq. (15.102) gives

$$\gamma X\,(r,\,\theta)_{T=0} = r\,\cos\left[\omega\,t'\,(r,\,\theta) + \theta\right],\; Y\,(r,\,\theta)_{T=0} = r\,\sin\left[\omega\,t'\,(r,\,\theta) + \theta\right] \quad (15.104)$$

Hence the circumference $r = a$ is given by

$$(\gamma X)^2 + Y^2 = a^2 \quad (15.105)$$

which describes an ellipse with half-axis a in the Y-direction and half-axis $\gamma^{-1}a$ in the X-direction.

Radial lines of the disk, given by θ =constant, appear as curved by simultaneity in the laboratory frame. This is shown in figure 15.9.

Some complementary figures showing the appearance of the rotating disk in accordance with two operational procedures performed in the rotating rest frame of the disk, have been presented by K. MacFarlane [110].

The positions of points on a rolling ring at retarded points of time were calculated with reference to K_0 by Ø. Grøn [111]. The result is shown in figure 15.10. Part C of the figure shows the "optical appearance" of a rolling ring, i.e. the positions of emission events where the emitted light from all the

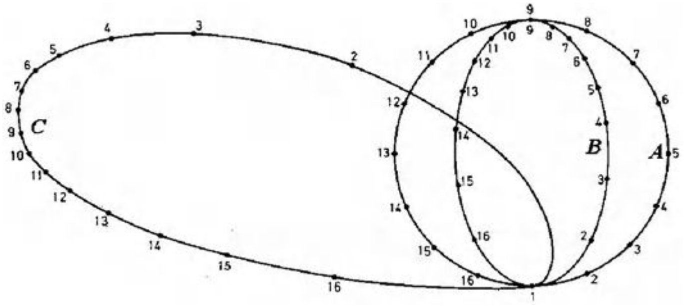

Figure 15.10. Points on a rolling ring. A: Observed by simultaneity in its rotating rest frame K; B: observed by simultaneity in the frame K_0; C: observed in K_0 at retarded points of time.

points arrives at a fixed point of time at the point of contact of the ring with the ground. In other words it is the position of the points when they emitted light that arrives at a camera on the ground just as the ring passes the camera.

An interesting observation concerning a rolling disk was made by K. Vøyenli [112]. He calculated the length of the circumference by considering the distance on the ground between marks of a fixed point 0 on the circumference, as the disk rolls with constant velocity.

> The period T of revolution relative to K_0 of a point P on the periphery is according to the time dilation formula given by $T = \gamma T_0$. We find accordingly that the point 0 in one revolution of the periphery moves a distance L relative to K_0 given by $L = vT$ or
>
> $$L = \gamma 2\pi a \tag{15.106}$$
>
> The same result has recently been obtained by Grøn [101] in a different connection and in a less simple way.
>
> The distance L may be called the "rolled out" circumference of the disk and may be identified with the proper circumference L' by the following argument. We consider a division of the periphery of the disk into infinitesimal line elements by points fixed on the periphery. These elements are matched one to one with line elements on the x-axis, such that corresponding elements coincide at the moment the element on the periphery touches the x-axis and is momentarily at rest in K_0. We make the usual assumption that an accelerated standard measuring rod, which momentarily is at rest in a given inertial frame, may coincide at this moment with an identical rod permanently at rest in the same inertial frame. It follows that corresponding elements on the periphery and the x-axis must have the same length relative to the rotating frame K and the inertial frame K_0, respectively, and that the proper circumference of the disk must be equal to the "rolled out" circumference.

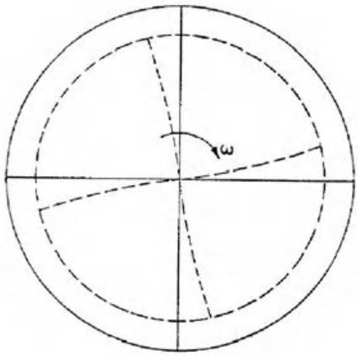

Figure 15.11. The rotating disk at rest is shown solid drawn and the rotating disk is shown with dashed line.

16. The rotating disk and the Thomas precession

D. H. Weinstein [113] claimed that the extension of a disk will be reduced, and radial lines deformed, as shown in figure 15.11, due to the Thomas precession when a disk is put into rotation.

The Thomas precession is a kinematical special relativistic effect appearing if a rod which is free to rotate and moves along a circular path normally to its axis of rotation. However, when a disk is put into rotation, say by giving all points on the circumference equal and simultaneous tangential blows, the motion of the disk material is determined mainly by the force acting on the disk and the elastic properties of the material, as was emphasized by G. Cavalleri [114]. Also, it was pointed out by Newburgh [115] that although the Thomas precession is clearly significant in the case of spinning elementary particles moving along curved paths, is by no means clear that the material of an extended body will undergo a Thomas precession.

The effect of the Thomas precession would be an accumulating strain which would cause a tension in the material. This tension would rapidly counteract further strain. Hence the effect of the Thomas precession would vanish practically immediately. It would therefore not be possible to measure any accumulated "Thomas strain" for a disk having rotated 1000 times per second for 30 days, as suggested by Weinstein.

This was shown very clearly by Whitmire [116] who wrote:

> Consider two elements S_1 and S_2 in a thin spinning disk. For definiteness we consider circular regions of the same size and the same radial position r. Let us assume that S_1 and S_2 undergo a Thomas rotation given by

$$\omega_T = (\gamma - 1)\,\frac{\mathbf{v}\times\mathbf{a}}{\mathbf{v}^2}\ , \quad \gamma = \left(1 - v^2/c^2\right)^{-1/2}\ , \quad \mathbf{v} = \omega\times\mathbf{r} \quad (15.107)$$

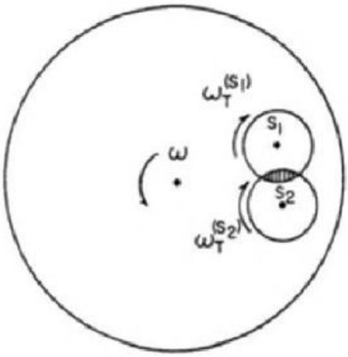

Figure 15.12. Thomas rotation of two overlapping circular regions

where ω is the angular velocity of the disk, and $\mathbf{a}$ is the centrifugal acceleration $|\mathbf{a}| = v^2/r$. The sense of ω_T is opposite to ω. Now consider the case where S_1 and S_2 are as in figure 15.12. Particles in the overlap region (shaded area in figure 15.12) must shear the motion of S_1 and S_2 simultaneously. In other words, the matter in the overlap region must move in two (opposite) directions at once if it is assumed that eq.(15.107) applies to an arbitrary section of mass in the disk. The problem of the Thomas rotation in a spinning macroscopic disk can be resolved by consideration of the elastic properties of the disk. Consider a small element of mass in the rotating disk. If it is uncoupled (or weakly coupled) to the rest of the disk then it will undergo the usual Thomas rotation. If it is strongly coupled to the rest of the disk, however, the mass elements will be constrained from rotating, thus introducing "Thomas shear stresses". In a laboratory experiment with a rotating disk there will be no Thomas rotation of segments in the disk.

17. Contracted rotating disk

Grünbaum and Janis [117] have considered a disk put into rotation in such a way that the radius contracts and no tangential stresses appear. This means that the rest length of tangential mass elements remains unchanged during the period of angular acceleration. At first moment one might think that this is not possible. Due to the relativity of simultaneity the special theory of relativity forbids, in the case of rotating motion with constant radius, to increase the angular velocity of a rotating disk in a Born rigid way. Hence, tangential stresses will appear, and the rest length of the periphery changes. There was a discussion of this in Foundations of Physics [118, 119] and it became clear that the type of motion considered by Grünbaum and Janis is indeed permitted by the theory of relativity.

The main result of Grünbaum and Janis' can be deduced as follows. Allowing for a changing radius of the disk the coordinate transformation between K and K' may be written

$$t' = t \ , \quad r' = f\,(r) \ \ , \quad \theta' = \theta + \omega t \tag{15.108}$$

Inserting this into the general expression (15.17) for the spatial line element leads to

$$d\sigma^2 = (df/dr)^2\, dr^2 + \left\{ f^2\,(r) \,/\, \left[1 - f^2\,(r)\,\omega^2/c^2\right]^{1/2} \right\} d\theta^2 \tag{15.109}$$

Consider a point on the disk. Before the rotation was started it had a radial coordinate $r_1' = r$ in K'. In the final state of uniform rotation it has a radial coordinate $r_2' = r'$ in K' and $r_2 = r$ in K.

The function f is determined from the condition that there are no tangential strains in the disk in the final state as compared to the non rotating state. This means that the rest length of a given circle on the disk is the same in the final state as in the non rotating state,

$$\sigma_\theta = (\gamma_{\theta\theta})_r^{1/2}\, 2\pi = 2\pi r \tag{15.110}$$

Hence $d\sigma_\theta = r\, d\theta$. This, together with eq.(15.109), gives

$$f\,(r) \,/\, \left[1 - f^2\,(r)\,\omega^2/c^2\right]^{1/2} = r \tag{15.111}$$

which leads to

$$f\,(r) = \frac{r}{\sqrt{1 + r^2\omega^2/c^2}} \tag{15.112}$$

Hence, from eq.(15.108) we have

$$r' = \frac{r}{\sqrt{1 + r^2\omega^2/c^2}} \tag{15.113}$$

This equation gives the radius r' of a circle on the rotating disk as compared to the radius of the same circle when the disk does not rotate. It shows that the disk contracts when the angular velocity increases.

Substituting eq.(15.112) for $f\,(r)$ into eq.(15.109) gives the spatial line element of the rotating disk

$$d\sigma^2 = \left(1 - r^2\omega^2/c^2\right)^{-3} dr^2 + r^2 d\theta^2 \tag{15.114}$$

This line-element shows that the circumference of the disk is $2\pi r$, independently of the angular velocity, and the spatial geometry is negatively curved since $circumference/diameter > \pi$. The proper radial distance is given by

$$d\sigma_r = (\gamma_{rr})^{1/2}\, dr = \left(1 + r^2\omega^2/c^2\right)^{-3/2} dr \tag{15.115}$$

Integration gives

$$\sigma_r = \frac{r}{\sqrt{1 + r^2\omega^2/c^2}} \tag{15.116}$$

which also exhibits the contraction of the rotating disk. Expressing the spatial line element of the rotating disk in terms of the proper radial distance one obtains

$$d\sigma^2 = d\sigma_r^2 + \frac{\sigma_r^2 d\theta^2}{1 - \sigma_r^2\omega^2/c^2} \tag{15.117}$$

which has the same form as the line element (15.10).

Since the proper length of the circumference does not change when the disk is put into rotation, one might think that the acceleration program that realizes this motion might represent self-contradictory boundary conditions due to the relativity of simultaneity, as is the case for purely tangential motion. However, the motion can be produced in the following way. The disk is initially compressed in accordance with eqs.(15.113) and (15.116). Then all points of the disk are accelerated in the tangential direction by a succession of blows, each blow being given to all points simultaneously in the inertial rest frame K' of the axis.

The disk motion introduced by Grünbaum and Janis has later been considered by Ziino [120] with a slightly different interpretation. He deduced the expression (15.116) and the line elements (15.114) and (15.117) and gave the following comment (using the notation above):

> What can essentially be gained is a more orthodox (and still relativistically consistent) geometrical definition of a rotating frame, in terms of a suitable "world" radial co-ordinate that may *naturally run to infinity,* with no need for values greater than c/ω to be ruled out. The new radial coordinate, r , differs from the standard one, σ_r, by the following: it is identically equal to the *Euclidean* radius, $\sigma_r\gamma\left(\omega\sigma_r\right)$, of a circumference of proper length $2\pi\sigma_r\gamma\left(\omega\sigma_r\right)$ which is described in the rotating frame at an actual radial distance σ_r from the rotation axis. A "new" metric could accordingly be assigned to a rotating frame, which can be obtained by just recasting the usual metric in terms of r. The most immediate physical application concerns the kinematics of a uniformly spinning disk (with presumably far-reaching effects on the physics of rotating black holes). The result is that a disk of *whatever* (original) radius r might be brought to spin with an *arbitrarily great* uniform angular velocity ω: its shape should not undergo any distortion with spinning, but should appear to be *globally contracted* by a scale factor $\gamma^{-1}\left(\omega\sigma_r\right)$, where $\omega\sigma_r = \omega r\left(1 + \sigma_r^2\omega^2/c^2\right)^{-1/2}$ and σ_r is the *new* radius that the disk would exhibit when it is seen rotating with an angular velocity ω.

18. Conclusion

There are several results of the long period with discussions on the geometry of a rotating disk.

1 Ehrenfest's paradox demonstrated that it is not possible to put a disk into rotating motion in a Born rigid way while remaining horizontal.

2 Einsteins's argument based on using standard measuring rods on a rotating disk to measure it geometrical properties shows that the periphery of a disk with radius r rotating with angular velocity ω has a length $2\pi r/\sqrt{1 - r^2\omega^2/c^2}$. Hence, it is longer than $2\pi r$, not shorter as in the formulation of Ehrenfest's paradox. With reference to the inertial rest frame of the axis this is explained as due to the Lorentz contraction of the measuring rods in the tangential direction and not in the radial direction. With reference to the rotating rest frame of the disk it is interpreted as a gravitational effect, i.e. the geometry of space is non-Euclidean in a gravitational field.

3 Due to the relativity of simultaneity Born rigid rotating motion of a ring with angular acceleration represents contradictory boundary conditions.

4 If the disk is regarded as a 2-dimensional surface it can be put into rotation in a Born rigid way, that is without any displacements in the tangential plane of the disk, by bending for example upwards so that it obtains the shape of a cup.

5 The surface orthogonal to the world lines of the disk particles is called the 3-space in the rotating rest frame of the disk. This space is defined by events that are simultaneous as measured by Einstein synchronized clocks on the disk. It has a discontinuity along a radial line as shown in figure 15.6, and is negatively curved.

6 Spatial geodesics curve inwards the 3-space of a rotating reference frame. This demonstrates the negative curvature of this space.

7 One may introduce local coordinates in the neighbourhood of arbitrary points on a rotating disk by means of differential transformations from coordinates in the inertial rest frame of the axis. These transformations may be chosen so that the spatial line element at constant time in the rotating system has Euclidean form. Also one may calculate a non-vanishing Riemann curvature tensor for spacetime in the rotating frame by employing the usual formulae valid in a coordinate basis. This does not mean, however, that the 3-space is flat and spacetime is curved in the rotating frame. Taking account of the non-vanishing structure coefficients in a non-coordinate basis one finds that the 3-space is curved and spacetime is flat in the rotating frame.

8 What actually happens when a disk is put into rotation depends upon its elastic properties. A maximally rigid disk, with sound velocity equal to

the velocity of light, will in fact contract when its angular velocity in increased.

References

[1] P. Ehrenfest, *Gleichförmige Rotation starrer Körper und Relativitätstheorie*, Physik Zeitschrift **10,** 918 (1909).

[2] M. Planck, *Gleichformige Rotation und Lorentz-Kontraktion*, Physik. Zeitschrift **11** 294 (1910).

[3] A. Einstein, *Zum Ehrenfestschen Paradoxon*, Physik. Zeitschrift, **12**, 509 (1911).

[4] G Rizzi and M. L. Ruggiero, *Space Geometry of Rotating Platforms: An Operational Approach*, Found. Phys. **32**, 1525 (2002).

[5] M. Born, *Die Theorie des starren Elektrons in der Kinematik des Relativität-Prinzipes*, Ann.d.Phys. **30**, 1 (1909).

[6] V. Varićak, *Zum Ehrenfestschen Paradoxon*, Physik. Zeitschrift, **12**, 169 (1911).

[7] M. J. Klein, *Paul Ehrenfest: volume 1: The making of a Theoretical Physicist*, North-Holland Publishing Company (1970), p.152-154.

[8]. A. I. Miller, *Albert Einstein's Special Theory of Relativity*, Addison-Wesley Publishing Co. (1981) section 2.4.8, p. 245.

[9] Th. Kaluza, *Zur Relativitätstheorie*, Physik.Zeitschrift, **11**, 977 (1910).

[10] W. von Ignatowsky, *Der starre Körper und das Relativitätsprinzip*, Ann.Phys. **33**, 607 (1910).

[11] P. Ehrenfest, *Zu Herrn v. Ignatowskys Behandlung der Bornschen Starrheits-definition*, Physik. Zeitschrift **11,** 1127 (1910).

[12] W. von Ignatowsky, *Zur Elastizitätstheorie vom Standpunkte des Relativitätsprinzips*, Physik. Zeitschrift **12**, 164 (1911).

[13] P. Ehrenfest, *Zu Herrn v. Ignatowskys Behandlung der Bornschen Starrheitsdefinition. II*, Physik. Zeitschrift **12**, 412 (1911),

[14] J. Stachel, *The Rigidly Rotating Disk as the "Missing Link" in the History of General Relativity*, in *Einstein and the History of General Relativity*, ed. By D. Howard and J. Stachel, Birkhäuser (1989) p.48.

[15] J. Stachel, *Einstein and the Rigidly Rotating Disk*, in *General Relativity and Gravitation*, ed. A. Held, Plenum (1980) p.1.

[16] A. Einstein, *Die Grunlage der allgemeinen Relativitätstheorie*, Annalen der Physik **49**, 769 (1916).

[17] A. Einstein, *Über die spezielle und die allgemeine Relativitätstheorie. (Gemeinverständlich).* Braunschweig: Friedrich Vieweg & Sohn. (1917).

[18] A. Einstein, *The Meaning of Relativity: Four Lectures Delivered at Princeton University*, Princeton University Press (1922).

[19] M. Born, *Einstein's Theory of Relativity*, first edition in German 1929, reference is to revised edition, Dover (1962), p.319.

[20] A. Einstein and L. Infeld, *The Evolution of Physics*, Cambridge Univ. Press, p.239-242 (1938).

[21] A. Metz, *Les Problèmes Relatifs a la Rotation Dans la Théorie de la Relativité*, Le Journal de Physique et le Radium, **13**, 224 (1952).

[22] K. Kraus, *Note on Lorentz Contractions and the Space Geometry of the Rotating Disk*, Zeitschrift für Naturforschung, **25a**, 792 (1970).

[23] J. Becquerel, *Le Principe de Relativité et la Théorie de la Gravitation*, Gauthier-Villars, Paris, 1922, page VIII.

[24] S. Shu, *Critical Studies on The Theory of Relativity*, Princeton, 1945.

[25] W. Glaser, *Gielt af der rotierende Scheibe die nichteuklidische Geometrie?* Physik. Zeitschrift **35**, 867 (1934).

[26] P. Langevin, *Remarques au sujet de la Note de Prunier*, Comptes rendus **200**, 48 (1935).

[27] N. Rosen, *Notes on Rotation and Rigid Bodies in Relativity Theory*, Phys.Rev. **71**, 54 (1947).

[28] F. Goy and F. Selleri, *Time on a rotating platform*, Found. Phys. Lett. **10**, 17 (1997).

[29] H. Reichenbach, *Axiomatization of the Theory of Relativity*, University of California Press 1969 p.172-178. Translated from *Axiomatik der relativistischen Raum-Zeit-Lehre* (1924).

[30] J. Weyssenhoff, *Metrisches field und Gravitatationsfield*, Bull. Acad. Polonaise Sc. Lett. Serie A, 252 (1937).

[31] G. Sagnac, *L'ether lumineux démontré par l'effect du vent relatif d'ether dans un intérferomètre an rotation uniforme*, Comptes rendus **157**, 708 (1913).

[32]. E. J. Post, *Sagnac Effect*, Rev. Mod. Phys. **39**, 475 (1967).

[33] H. Thirring, *Begriffsystem und Grundgezetze der Feldphysik*, Handbuch der Physik, **4,** Kapitel 2 (1929).

[34] H. A. Atwater, *Transforming to Rotating coordinates*, Nature Phys. Sci. **228**, 272 (1970).

[35] M. Suzuki, *Transformations to a Rotating Frame*, Nature Phys. Sci. **230**, 13 (1971).

[36] Letters by G. E. Marsh and T. W. Noonan together with reply from H. A. Atwater under the common heading *Transformation to Rotating Coordinates, reply to Atwater*, Nature Phys. Sci. **230**, 197 (1971).

[37] C. Møller, *The Theory of Relativity*, Clarendon Press, Oxford 1952, §90.

[38] H. Arzeliès, *Relativistic kinematics*, Pergamon Press 1966, p. 204 - 244.

[39] D. G. Ashworth and P. A. Davies, *Geodesics in Rotating Systems*, Int. J. Th. Phys. **16**, 845 (1977).

[40] P. Franklin, *The Meaning of Rotation in the Special Theory of Relativity*, Proc.Nat.Acad.Sc. **8**, 265 (1922).

[41] M.G.Trocheris, *Electrodynamics in a Rotating Frame of Reference*, Phil.Mag. **ser.7, vol.40**, 1143 (1949).

[42] H. Takeno, *On Relativistic Theory of Rotating Disk*, Progr.Th.Phys. **7**, 367 (1952).

[43] L. Herrera, *The relativistic transformation to rotating frames*, Il Nuovo Cimento **115B**, 307 (2000).

[44] B. Chakraborty and S. Sarkar, *Physics in Rotating Frames. I. On Uniform Rotation about a Fixd axis in Some Holonomic and Anholonomic Frames*, Ann. Phys. **163**, 167 (1985).

[45] T. A. Weber, *Measurements on a rotating frame in relativity, and the Wilson and Wilson experiment*, Am. J. Phys. **65**, 946 (1997).

[46] J. F. Corum, *Relativistic rotation and the anholonomic object*, J. Math. Phys. **18**, 770 (1977).

[47] H. Nikolić, *Relativistic contraction and related effects in noninertial frames*, Phys. Rev. **A61**, 032109 (2000).

[48] V. Bashkov and M. Malakhaltsev, *Geometry of rotating disk and the Sagnac effect*, gr-qc/0011061.

[49] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, New York (1973) p. 223, 224.

[50] G. Stead and H. Donaldson, *The Problem of Uniform Rotation treated on the Principle of Relativity*, Phil.Mag. **20**, 92 (1910).

[51] H. E. Ives, *Theory of the Double Fizeau Toothed Wheel*. J. Opt. Soc. Am. **29**, 472 (1939).

[52] M. Galli, *Le deformazioni relativistiche di un cilindro rotante*, Lincei – Rend. Sc. Fis. Mat. E nat. **12**, *Nota I*, p.86 and *Nota II*, p.569 (1952).

[53] A. S. Eddington, *Space, Time and Gravitation*, Cambridge University Press (1920) p.75.

[54] H. A. Lorentz, *The Michelson-Morley Experiment and the Dimensions of Moving Bodies*, Nature **17**, 793 (1921).

[55] A. S. Eddington, *The Mathematical Theory of Relativity*, Cambridge Univ. Press (1923) section 50, p.112.

[56] L. I. Schiff, *A question in general relativity*, Proceeding of National Academy of Sciences **25**, 391 (1939).

[57] Ø. Grøn and K. Vøyenli, *On the Foundation of the Principle of Relativity*, Found. Phys. **29**, 1695 (1999).

[58] F. Winterberg, *Resolution of the Ehrenfest Paradox in the Dynamic Interpretation of Lorentz Invariance*, Z. Naturforsch. **53a**, 751 (1998).

[59] C. W. Berenda, *The Problem of the Rotating Disk*, Physical Review **62**, 280 (1942).

[60] E. L. Hill, *A Note on the Relativistic Problem of Uniform Rotation*, Phys. Rev. **69**, 488 (1946).

[61] A. Prechtl, *Remarks on Steady Rotation About a Fixed Axis in Relativity*, Acta Mechanica **29**, 283 (1978).

[62] N. Rosen, *Note on the Problem of Uniform Rotation*, Phys. Rev. **70**, 93 (1946).

[63] G. L. Clark, *The Mechanics of Continuous Matter in the Relativity Theory*, Proc. Roy. Soc. Edinburgh **62A**, 434 (1947-8).

[64] G.L.Clark, *The Problem of a Rotating Incompressible Disk*, Proc. Cambr. Phil. Soc. **45**, 405 (1949).

[65] G. Cavalleri, *Solution of Ehrenfest's Paradox for a Relativistic Rotating Disk*, Il Nuovo Cimento, **53B**, 415 (1968).

[66] A. Brotas, *Sur le problème du disque tournant*, C.R. Acad. Sc. Paris, **267**, 57A (1968).

[67] W. H. McCrea, *Rotating Relativistic Ring*, Nature, **234**, 399 (1971).

[68] R. G. Newburgh, *A Practical Proposal for the Experimental Study of Born Rigid Motions*, Il Nuovo Cimento **23B**, 365 (1974).

[69] P. A. Davies and R. C. Jennison, *Experiments involving mirror transponders in rotating frames*, J. Phys. A. Math. Gen. **8**, 1390 (1975).

[70] D. G. Ashworth, *Surveying in rotating systems*, J. Phys. A. Math. Gen. **9**, 35 (1976).

[71] P. A. Davies, *Measurements in rotating systems*, J. Phys. A. Math. Gen. **9**, 951 (1976).

[72] R. C. Jennison and D. G. Ashworth, *Surveying in rotating systems II. Triangulation Radius.* J. Phys. A. Math. Gen. **9**, 1257 (1976).

[73] R. C. Jennison, *Relativity on a turntable*, New Scientist **73,** 283 (1977).

[74] D. G. Ashworth and P. A. Davies, *Transformations between inertial and rotating frames of reference*, J. Phys. A. Math. Gen. **12**, 1425 (1979).

[75] P. F. Browne, *Relativity of Rotation*, J. Phys. A. Math. Gen. **10**, 727 (1977).

[76] K. McFarlane and N. C. McGill, *Light ray and particle paths on a rotating disk*, J. Phys. A. Math. Gen. **11**, 2191 (1978).

[77] D. G. Ashworth, P. A. Davies and R. C. Jennison, *"Light ray and particle paths on a rotating disk": a reply to criticism therein*, J. Phys. A. Math. Gen. **11**, L259 (1978).

[78] K. McFarlane and N. C. McGill, *"Light ray and particle paths on a rotating disk": a reply to comments by Ashworth, Davies and Jennison*, J. Phys. A. Math. Gen. **12**, L71 (1979).

[79] K. McFarlane and N. C. McGill, *Comment on "Geodesics in Rotating Systems"*, Int. J. Th. Phys. **18**, 151 (1979).

[80] V. Cantoni, *What is Wrong with Relativistic Kinematics*, Il Nuovo Cimento, **57B,** 220 (1968).

[81] G. Rizzi and A. Tartaglia, *Speed of Light on Rotating Platforms*, Found. Phys. **28**, 1663 (1998).

[82] W. A. Rodrigues, *The Standard of Length in the Theory of Relativity and Ehrenfest Paradox*, Il Nuovo Cimento, **74B**, 199 (1983).

[83] T. E. Phipps, Jr. *Do Metric Standards Contract?* Found. Phys. **10**, 289 (1980).

[84] V. Cantoni, *Comments on "Do Metric Standards Contract?"*, Found. Phys. **10**, 809 (1980).

[85] T. E. Phipps, Jr. *Do Metric Standards Contract? – A Reply to Cantoni*, Found. Phys. **10**, 811 (1980).

[86] Ø. Grøn, *Special-Relativistic Resolution of Ehrenfest's Paradox: Comments on Some Recent Statements by T. E. Phipps, Jr.* Found. Phys. **11**, 623 (1981).

[87] Ø. Grøn, *Covariant formulation of Hooke's law*, Am. J. Phys. **49**, 28 (1981).

[88] T. A. Weber, *A note on rotating coordinates in relativity*, Am. J. Phys. **65**, 486 (1997).

[89] D. Dieks, *Time in special relativity and its philosophical significance*, Eur. J. Phys. **12**, 253 (1991).

[90] E. M. Kelly, *Resolution of the Ehrenfest Paradox by New Contraction (Expansion) Criteria*, Z. Naturforsch. **43a**, 865 (1988).

[91] R. D. Klauber, *New Perspectives on the Relativistically Rotating Disk and Non-Time-Orthogonal Reference Frames*, Found. Phys. Lett. **11**, 405 (1998).

[92] R. D. Klauber, *Comment regarding recent articles on relativistically rotating frames*, Am. J. Phys. **67**, 158 (1999).

[93] T. A. Weber, *Response to "Comment regarding recent articles on relativistic rotating frames"*, Am. J. Phys. **67**, 159 (1999).

[94] A. Tartaglia, *Lengths on Rotating Platforms*, Found. Phys. Lett. **12**, 17 (1999).

[95] B. Kurşunoğlu, *Spacetime on The Rotating Disk*, Proc. Cambr. Phil. Soc. **47**, 177 (1951).

[96] Ø. Grøn, *The Metric in a Static Cylindrical Elastic Medium and in an Empty Rotating Frame as Solutions of Einstein's Field Equations*, Found. Phys. **12,** 509 (1982).

[97] W. Wilson, *Space-Time Manifolds and corresponding Gravitational Fields*, Phil. Mag. **40**, 703 (1920).

[98] R. J. Adler, M. J. Bazin, and M. Schiffer, *General Relativity*, 2^{nd}. Ed. (McGraw Hill, 1975) §4.2.

[99] H. Arzeliès, *Relativitè gènèralisèe,* Fasc. I (Gauthier-Villars 1961), §198.

[100] N. Sama, *On the Ehrenfest Paradox*, Am. J. Phys. **40**, 415 (1972).

[101] Ø. Grøn, *Relativistic description of a rotating disk*, Am. J. Phys. **43,** 869 (1975).

[102] P. D. Noerdlinger, *Cylindrical cardboard modell for a rotating system in special relativity*, Am. J. Phys. **47**, 218 (1979).

[103] W. Pauli, *Theory of Relativity*, Pergamon Press (1958), p. 131. Translated from the article *Relativitätstheorie* in Encyklopädie der mathematischen Wissenschaften, **19** (1921).

[104] M. H. Mac Gregor, *Do Dewan-Beran Relativistic Stresses Actually Exist?*, Lett. Nuovo Cimento **30**, 417 (1981).

[105] Ø. Grøn, *Energy considerations in connection with a relativistic rotating ring*, Am. J. Phys. **50**, 1144 (1982).

[106] M. Strauss, *Rotating Frames in Special Relativity*, Int. J. Th. Phys. **11**, 107 (1974).

[107] Ø. Grøn, *Rotating Frames in Special Relativity Analyzed in Light of a Recent Article by M. Strauss*, Int. J. Th. Phys. **16**, 603 (1977).

[108] Ø. Grøn, *Relativistic Description of a Rotating Disk with Angular Acceleration*, Found. Phys. **9**, 353 (1979).

[109] K. Brown, *A Rotating Disk in Translation*, Math. Pages. Internet adress: http://www.mathpages.com/home/kmath197/kmath197.htm.

[110] K. MacFarlane, *Appearance of a Relativistically Rotating Disk*, Int. J. Th. Phys. **20**, 397 (1981).

[111] Ø. Grøn, *"Optical Appearance" of a Rolling Ring*, Int. J. Th. Phys. **22**, 821 (1983).

[112] K. Vøyenli, *Alternative derivation of the circumference of a relativistic, rotating disk*, Am. J. Phys. **9**, 875 (1977).

[113] D. H. Weinstein, *Ehrenfest's Paradox*, Nature Phys. Sci. **232**, 548 (1971).

[114] G. Cavalleri, *On Some Recent Papers Regarding the Ehrenfest's Paradox*, Lett. Nuovo Cimento **3**, 608 (1972).

[115] R. G. Newburgh, *Thomas Precession and Extended Structures*, Lett. Nuovo Cimento **5**, 387 (1972).

[116] D. P. Whitmire, *Thomas Precession and the Relativistic Disk*, Nature Phys. Sci. **235**, 175 (1972).

[117] A. Grünbaum and A. I. Janis, *The Geometry of the Rotating Disk in the Special Theory of Relativity*, Synthese **34**, 281 (1977).

[118] A. Grünbaum and A. I. Janis, *The Rotating Disk: Reply to Grøn*, Found. Phys. **10**, 495 (1980).

[119] Ø. Grøn, *The Rotating Disk: Reply to Grünbaum and Janis*, Found. Phys. **10**, 499 (1980).

[120] G. Ziino, *An Improved Approach to Relativistic Rotational Kinematics*, Helv. Phys. Acta **69**, 1 (1996).

Chapter 16

QUANTUM PHYSICS IN INERTIAL AND GRAVITATIONAL FIELDS

Giorgio Papini
Department of Physics, University of Regina, Regina, Sask. S4S 0A2, Canada.
International Institute for Advanced Scientific Studies, 84019 Vietri sul Mare (SA), Italy.
papini@uregina.ca

Abstract Covariant generalizations of well-known wave equations predict the existence of inertial-gravitational effects for a variety of quantum systems that range from Bose-Einstein condensates to particles in accelerators. Additional effects arise in models that incorporate Born reciprocity principle and the notion of a maximal acceleration. Some specific examples are discussed in detail.

1. Introduction

The interaction of quantum systems with external inertial and gravitational fields is of interest in studies regarding the ultimate structure of space-time. Covariant generalizations of well known wave equations provide examples of effects involving classes of quantum systems in conditions remote from the onset of quantum gravity, hence amenable, it is hoped, to observation. For this purpose, Schroedinger, Klein-Gordon, Maxwell-Proca and Dirac equations have been frequently discussed in the literature. The Landau-Ginzburg and Gross-Pitaevskii equations should also be added to this group because of the peculiar properties of charged and neutral Bose-Einstein condensates. As shown in Section 2, these equations can be solved exactly to first order in the weak field approximation (WFA), if the solutions of the corresponding field free equations are known. The same procedure can also be applied to de Rham, Maxwell-Proca and Dirac equations.

The interaction of quantum systems with external inertial and gravitational fields produces quantum phases. Though these are in general path-dependent, phase differences are observable, in principle, by means of interferometers. Section 2 refers to this first group of effects. An explicit calculation of the

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 335-359.

phase difference due to the Lense-Thirring (LT) effect is added for pedagogical reasons.

A second group of effects, considered in Sections 3, is derived from effective Hamiltonians for the motion of fermions in accelerators and storage rings. It deals, essentially, with spin-rotation coupling, its non-universal character and its invariance under parity and time reversal.

The problems considered in Section 4 stem from attempts to incorporate Born reciprocity theorem into the structure of space-time. They are related to the notion of a maximal acceleration (MA), whose presence, frequently discussed in both classical and quantum contexts and in string theory, plays the role of a field regulator while preserving the continuous structure of space-time. The MA corrections to the Lamb shift of one-electron atoms and ions, also discussed in Section 4, are comparable in magnitude with those of quantum electrodynamics of order seven in the fine structure constant and are not, therefore, negligible. Section 5 contains a summary.

2. Quantum phases

2.1 Landau-Ginzburg and Gross-Pitaevskii equations

In view of the wide variety of interferometers presently in use or under development, it is convenient to study systems whose wave functions satisfy the equation

$$\left[(\nabla_\mu + i\frac{e}{c}A_\mu)^2 + \frac{m^2c^2}{\hbar^2}\right]\Phi(x) = \beta \mid \Phi(x) \mid^2 \Phi(x), \tag{16.1}$$

where ∇_μ indicates covariant differentiation, β is a constant and $A_\mu(x)$ represents the total electromagnetic potential of all external and gravity induced fields present. Eq.(16.1) is the fully covariant version of the Landau-Ginzburg equation [1]. It reduces to the Gross-Pitaevskii equation when A_μ vanishes and to the Klein-Gordon equation when $\beta = 0$. It is therefore well suited to discuss a number of systems, from superfluids [2] and Bose-Einstein condensates [3], to scalar particles. If, in particular, heavy fermion systems admit minimal coupling [4], then Eq.(16.1) may be used in this case too with the added advantage of a much larger effective coupling in mixed gravity-electromagnetism interaction terms.

In the WFA $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the metric deviation, $|\gamma_{\mu\nu}| \ll 1$ and the signature of $\eta_{\mu\nu}$ is $(1, -1, -1, -1)$. To first order, Eq.(16.1) becomes ($\hbar = c = G = 1$)

$$[(\eta^{\mu\nu} - \gamma^{\mu\nu})\partial_\mu\partial_\nu - (\gamma^{\alpha\mu} - 1/2\gamma^\sigma_\sigma\eta^{\alpha\mu})_{,\mu}\,\partial_\alpha + m^2 - \beta \mid \Phi \mid^2]\Phi(x) = 0. \tag{16.2}$$

It is useful to start with the ansatz

$$\Phi(x) = exp\,(-i\chi)\,\phi_0(x) \simeq (1 - i\chi)\phi_0(x), \tag{16.3}$$

where $\phi_0(x)$ is a field quantity to be determined below and

$$\begin{aligned} i\chi\phi_0 &= \frac{1}{4}\int_P^x dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[(x^\alpha - z^\alpha)\partial^\beta - \\ &\quad (x^\beta - z^\beta)\partial^\alpha] - \frac{1}{2}\int_P^x dz^\lambda \gamma_{\alpha\lambda}(z)\partial^\alpha]\phi_0. \end{aligned} \tag{16.4}$$

Because coordinates play the role of parameters in relativity, phase (16.4) is sometimes referred to as the gravitational Berry phase [5].

It is easy to prove by differentiation that (2.4) leads to

$$\begin{aligned} i\partial_\mu(\chi\phi_0) &= \frac{1}{4}\int_P^x dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[\delta^\alpha_\mu\partial^\beta - \delta^\beta_\mu\partial^\alpha]\phi_0(x) + \\ &\quad \frac{1}{4}\int_P^x dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[(x^\alpha - z^\alpha)\partial^\beta - \\ &\quad (x^\beta - z^\beta)\partial^\alpha]\partial_\mu\phi_0(x) - \frac{1}{2}\int_P^x dz^\lambda\gamma_{\alpha\lambda}(z)\partial^\alpha\partial_\mu\phi_0(x) - \\ &\quad \frac{1}{2}\gamma_{\alpha\mu}(x)\partial^\alpha\phi_0(x), \end{aligned} \tag{16.5}$$

from which one gets

$$\begin{aligned} i\partial_\mu\partial^\mu(\chi\phi_0) &= -im^2\chi\phi_0 + i\chi(\beta \mid \phi_0 \mid^2 \phi_0) - \\ &\quad \gamma_{\mu\alpha}\partial^\mu\partial^\alpha\phi_0 - (\gamma^{\beta\mu} - \frac{1}{2}\gamma^\sigma_\sigma\eta^{\beta\mu})_{,\mu}\,\partial_\beta\phi_0. \end{aligned} \tag{16.6}$$

By substituting (16.6) and (16.3) into (16.2) one finds, to lowest order,

$$\begin{aligned} [(\eta^{\mu\nu} - \gamma^{\mu\nu})\partial_\mu\partial_\nu \quad &+ \quad m^2 - \beta \mid \Phi \mid^2]\Phi(x) = \\ &[\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2 - \beta \mid \phi_0 \mid^2]\phi_0(x) + \\ &\beta\left[\mid \phi_0 \mid^2 (i\chi\phi_0) - i\chi\left(\mid \phi_0 \mid^2 \phi_0\right)\right], \end{aligned} \tag{16.7}$$

where use has been made of the Lanczos-DeDonder gauge condition

$$\gamma^\nu_{\alpha,\nu} - \frac{1}{2}\gamma^\sigma_{\sigma,\alpha} = 0. \tag{16.8}$$

Eq.(16.3) therefore is a solution of (16.2) exact to first order if

$$\begin{aligned} &[\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2 - \beta \mid \phi_0 \mid^2]\phi_0(x) + \\ &\beta\left[\mid \phi_0 \mid^2 (i\chi\phi_0) - i\chi\left(\mid \phi_0 \mid^2 \phi_0\right)\right] = 0. \end{aligned} \tag{16.9}$$

In problems where $\mid \phi_0 \mid^2$ is constant, ϕ_0 satisfies the Ginzburg-Landau equation

$$[\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2 - \beta \mid \phi_0 \mid^2]\phi_0(x) = 0. \tag{16.10}$$

When $\beta = 0$, (16.1) becomes the covariant Klein-Gordon equation and (16.10) the Klein-Gordon equation in Minkowski space.

For a closed path in space-time one finds [1]

$$i\Delta\chi\phi = \frac{1}{4}\int_{\Sigma_p} R_{\mu\nu\alpha\beta}L^{\alpha\beta}d\tau^{\mu\nu}\phi_0, \tag{16.11}$$

where Σ_p is the surface bound by the closed path, $L^{\alpha\beta}$is the angular momentum of the particle of mass m, and $R_{\mu\nu\alpha\beta}$ is the linearized Riemann tensor

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left(\gamma_{\mu\beta,\nu\alpha} + \gamma_{\nu\alpha,\mu\beta} - \gamma_{\mu\alpha,\nu\beta} - \gamma_{\nu\beta,\mu\alpha}\right). \tag{16.12}$$

Result (16.11) is manifestly gauge invariant. The effect of the electromagnetic field can also be incorporated in the phase factor in a straight-forward way by adding to $i\chi$ the term $ie\int_P^x dz^\lambda A_\lambda(z)$. The additional phase difference is $e\int_{\Sigma_p} F_{\mu\nu}d\tau^{\mu\nu}$ where $F_{\mu\nu} = -A_{\mu,\nu} + A_{\nu,\mu}$.

2.2 de Rahm and Maxwell equations

The de Rahm wave equation

$$\nabla_\nu\nabla^\nu A_\mu - R_{\mu\sigma}A^\sigma = 0, \tag{16.13}$$

where $\nabla_\mu A^\mu = 0$, becomes, in the WFA and in the gauge (16.8),

$$\begin{aligned}\nabla_\nu\nabla^\nu A_\mu - R_{\mu\sigma}A^\sigma \;&\simeq\; (\eta^{\sigma\alpha} - \gamma^{\sigma\alpha})A_{\mu,\alpha\sigma} - \\ &\quad (\gamma_{\sigma\mu,\nu} + \gamma_{\sigma\nu,\mu} - \gamma_{\mu\nu,\sigma})A^{\sigma,\nu} = 0. \end{aligned}\tag{16.14}$$

This equation has the solution

$$A_\mu = exp(-i\xi) \simeq (1 - i\xi)a_\mu, \tag{16.15}$$

where

$$\begin{aligned} i\xi a_\mu(x) \;&=\; \frac{1}{4}\int_P^x dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[(x^\alpha - z^\alpha)\partial^\beta a_\mu(x) - \\ &\quad (x^\beta - z^\beta)\partial^\alpha a_\mu(x)] + \frac{1}{2}\int_P^x dz^\lambda(\gamma_{\mu\lambda,\sigma}(z) - \gamma_{\sigma\lambda,\mu}(z))a^\sigma - \\ &\quad \frac{1}{2}\int_P^x dz^\lambda\gamma_{\alpha\lambda}(z)\partial^\alpha a_\mu(x) - \int_P^x dz^\lambda\gamma_{\alpha\mu}(z)\partial_\lambda a^\alpha(x), \end{aligned}\tag{16.16}$$

$\partial_\nu\partial^\nu a_\mu = 0$ and $\partial^\nu a_\nu = 0$. If $R_{\mu\sigma}$ is negligible, then Eq.(16.13) becomes Maxwell wave equation and the phase operator ξ can also be written in the form [2]

$$\xi = \frac{1}{4}\int_P^x dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))J^{\alpha\beta} - \frac{1}{2}\int_P^x dz^\lambda\gamma_{\alpha\lambda}(z)\partial^\alpha - \frac{1}{2}\int_P^x dz^\lambda\gamma_{\alpha\beta,\lambda}(z)T^{\alpha\beta}, \quad (16.17)$$

where

$$J^{\alpha\beta} = L^{\alpha\beta} + S^{\alpha\beta}$$

is the total angular momentum,

$$(S^{\alpha\beta})^{\mu\nu} = -i(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})$$

is the spin-1 operator and

$$(T^{\alpha\beta})^{\mu\nu} \equiv -i\frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}).$$

All spin effects are therefore contained in the $S^{\alpha\beta}$ and $T^{\alpha\beta}$ terms. For a closed path one can again find a gauge invariant equation similar to (16.11).

The procedure discussed can be easily extended to massive vector particles.

2.3 Covariant Dirac equation

Some of the most precise experiments in physics involve spin-1/2 particles. They are very versatile tools that can be used in a variety of experimental situations and energy ranges while still retaining a non-classical behaviour. Within the context of general relativity, De Oliveira and Tiomno [6] and Peres [7] conducted comprehensive studies of the fully covariant Dirac equation which takes the form

$$[i\gamma^\mu(x)D_\mu - m]\Psi(x) = 0, \quad (16.18)$$

where $D_\mu = \nabla_\mu + i\Gamma_\mu$. The generalized matrices $\gamma^\mu(x)$ satisfy the relations $\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x)$, $D_\mu\gamma_\nu(x) = \nabla_\mu\gamma_\nu(x) + i[\Gamma_\mu(x), \gamma_\nu(x)] = 0$ and are related to the usual Dirac matrices $\gamma^{\hat{\alpha}}$ by means of the vierbeins $e^\mu_{\hat{\alpha}}(x)$. The spin connection Γ^μ is

$$\Gamma_\mu = \frac{i}{4}\gamma^\nu(\nabla_\mu\gamma_\nu) = -\frac{1}{4}\sigma^{\hat{\alpha}\hat{\beta}}e^\nu{}_{\hat{\alpha}}(\nabla_\mu e_{\nu\hat{\beta}}), \quad (16.19)$$

where $\sigma^{\hat{\alpha}\hat{\beta}} = \frac{i}{2}[\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$. Particularly interesting is the case of acceleration and rotation [8][9]. In this instance it is possible to define a local co-ordinate frame according to an orthonormal tetrad with three-acceleration $\vec{a}$ along a particle's world-line and three-rotation $\vec{\omega}$ of the spatial triad, subject to Fermi-Walker transport. This tetrad $\vec{e}_{\hat{\mu}}$, is related to the general co-ordinate tetrad $\vec{e}_\mu$ by

$$\vec{e}_{\hat{0}} = (1 + \vec{a}\cdot\vec{x})^{-1}\left[\vec{e}_0 - (\vec{\omega}\times\vec{x})^k\vec{e}_k\right], \vec{e}_{\hat{\imath}} = \vec{e}_i. \quad (16.20)$$

The corresponding vierbeins relating the two frames are then

$$\begin{aligned} e^0{}_{\hat{0}} &= (1+\vec{a}\cdot\vec{x})^{-1}\,, e^k{}_{\hat{0}} = -\,(1+\vec{a}\cdot\vec{x})^{-1}\,\epsilon^{ijk}\,\omega_i\,x_j, \\ e^0{}_{\hat{\imath}} &= 0, e^k{}_{\hat{\imath}} = \delta^k{}_i. \end{aligned} \quad (16.21)$$

Similarly, by inverting (16.20), we find the inverse vierbeins

$$\begin{aligned} e^{\hat{0}}{}_0 &= (1+\vec{a}\cdot\vec{x})\,, e^{\hat{k}}{}_0 = \epsilon^{ijk}\,\omega_i\,x_j, \\ e^{\hat{0}}{}_i &= 0, e^{\hat{k}}{}_i = \delta^k{}_i. \end{aligned} \quad (16.22)$$

The vierbeins satisfy the orthonormality conditions

$$\delta^{\hat{\alpha}}{}_{\hat{\mu}} = e^\nu{}_{\hat{\mu}} e^{\hat{\alpha}}{}_\nu, \delta^\alpha{}_\mu = e^{\hat{\nu}}{}_\mu e^\alpha{}_{\hat{\nu}}. \quad (16.23)$$

It follows that the metric tensor components are

$$\begin{aligned} g_{00} &= (1+\vec{a}\cdot\vec{x})^2 + \left[(\vec{\omega}\cdot\vec{\omega})\,(\vec{x}\cdot\vec{x}) - (\vec{\omega}\cdot\vec{x})^2\right], \\ g_{0j} &= -\,(\vec{\omega}\times\vec{x})_j\,, g_{jk} = \eta_{jk}. \end{aligned} \quad (16.24)$$

One also finds

$$\Gamma_0 = -\frac{i}{2}(\vec{a}\cdot\vec{\alpha}) - \vec{\omega}\cdot\vec{\sigma}, \Gamma_j = 0. \quad (16.25)$$

By using the definitions $\Psi(x) = S\tilde{\Psi}(x)$, $S = exp(-i\int_P^x dz^\lambda \Gamma_\lambda(z))$ and $\tilde{\gamma}^\mu(x) = S^{-1}\gamma^\mu(x)S$, in (16.18) one finds [9]

$$[i\tilde{\gamma}^\mu(x)\nabla_\mu - m]\tilde{\Psi} = 0. \quad (16.26)$$

By substituting $\tilde{\Psi} = [-i\tilde{\gamma}^\alpha(x)\nabla_\alpha - m]\psi'$ into (16.26) , one obtains

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)\psi' = 0 \quad (16.27)$$

which, as shown above, has the WFA solution $\psi' = exp(-i\chi)\psi_0$, where ψ_0 is a solution of the Klein-Gordon equation in Minkowski space. It is again possible to show that for a closed path the total phase difference experienced by the Dirac wave function is gauge invariant and is given by $\frac{1}{4}\int R_{\mu\nu\alpha\beta}J^{\alpha\beta}d\tau^{\mu\nu}$, where the total angular momentum is now

$$J^{\alpha\beta} = L^{\alpha\beta} + \sigma^{\alpha\beta}\,, \sigma^{\alpha\beta} = -\frac{1}{2}[\gamma^\alpha, \gamma^\beta]$$

and γ^β represents a usual, constant Dirac matrix [10]. It then follows that the Dirac Hamiltonian in the general co-ordinate frame is, to first-order in $\vec{a}$ and $\vec{\omega}$,

$$H \approx (\vec{\alpha}\cdot\vec{p}) + m\beta + V(\vec{x}), \quad (16.28)$$

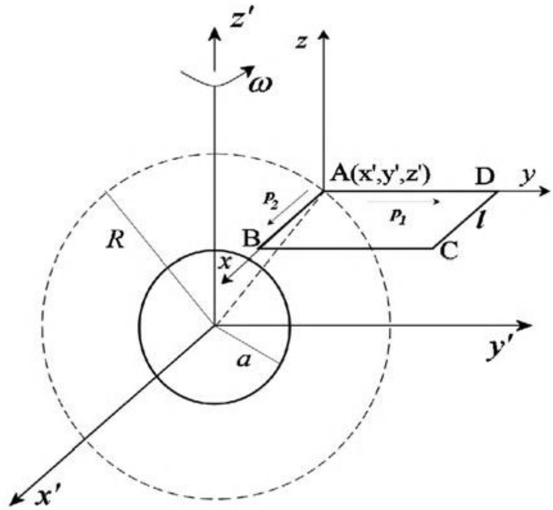

Figure 16.1. Rotating, homogeneous, solid sphere and useful coordinates.

where

$$V(\vec{x}) = \frac{1}{2}[(\vec{a}\cdot\vec{x})(\vec{\alpha}\cdot\vec{p}) + (\vec{\alpha}\cdot\vec{p})(\vec{a}\cdot\vec{x})] + m(\vec{a}\cdot\vec{x})\beta - \vec{\omega}\cdot(\vec{L}+\vec{S}) + \vec{\alpha}\cdot(\vec{\nabla}\Phi_{\rm G}) + \nabla_0\Phi_{\rm G}, \qquad (16.29)$$

the $\vec{\alpha}, \beta, \vec{\sigma}$ matrices are those of Minkowski space, $\vec{L} = \vec{x}\times\vec{p}$, $\vec{S} = \vec{\sigma}/2$ are the orbital and spin angular momenta, respectively, and

$$\nabla_\mu\Phi_{\rm G} = \frac{1}{2}\gamma_{\alpha\mu}(x)p^\alpha - \frac{1}{2}\int_X^x dz^\lambda(\gamma_{\mu\lambda,\beta}(z) - \gamma_{\beta\lambda,\mu}(z))p^\beta, \qquad (16.30)$$

where p^μ is the momentum eigenvalue of the free particle. The term $\vec{\omega}\cdot\vec{S}$ is the spin-rotation coupling term introduced by Mashhoon [11].

2.4 The Lense-Thirring effect for quantum systems

An example of how a gravity induced phase is calculated can best be given by applying (16.2)-(16.4), with $\beta = 0$, to the LT effect [12]. This requires knowledge of the particle paths and of the field $\gamma_{\mu\nu}$.

Consider the physical situation illustrated in Fig. 16.1. A square interferometer of side l is represented by the path *ABCD* in the (xy)-plane and a sphere of mass M and radius a is rotating about the z'-axis with angular velocity ω.

The spatial coordinates of the point A at which a coherent beam of particles is split are (x', y', z') in the coordinate system z'^{μ}. For the sake of generality A is taken a distance R from the center of the sphere. The beams interfere at C after describing the paths $p_1 \equiv ADC$ and $p_2 \equiv ABC$. Since the two coordinate systems z^{μ} and z'^{μ} are at rest relative to each other, one can choose $z^0 = z'^0$ and set the beam splitting time at A to be $z^0 = z'^0 = 0$. It is sufficient to take $\phi_0 \propto exp(ik_{\mu}x^{\mu})$, where k_{μ} is the momentum of the particles of mass m in the beams and $k_{\mu}k^{\mu} = m^2$. The only non-vanishing values of $\gamma_{\mu\nu}$ are [13]

$$\begin{aligned} \gamma_{00} &= \gamma_{ii} = -\frac{2M}{r}, \gamma_{01} = -\frac{4M\omega a^2 (y + y')}{5r^3}, \\ \gamma_{02} &= \frac{4M\omega a^2 (x + x')}{5r^3}, \end{aligned} \tag{16.31}$$

where $r^2 = (x + x')^2 + (y + y')^2 + (z + z')^2$ and $R^2 = x'^2 + y'^2 + z'^2$. The following expressions are also used below

$$\begin{aligned} \frac{1}{r} &= \frac{1}{R} - \frac{xx'}{R^3} - \frac{yy'}{R^3} - \frac{zz'}{R^3} - \frac{1}{2R^3}(x^2 + y^2 + z^2) + \\ & \frac{3x'^2x^2}{2R^5} + \frac{3y'^2y^2}{2R^5} + \frac{3z'^2z^2}{2R^5}, \\ \frac{x'^i}{r^3} &= \frac{x'^i}{R^3} - \frac{3x'^ix'x}{R^5} - \frac{3x'^iy'y}{R^5} - \frac{3x'^iz'z}{R^5}, \\ \frac{x'^ix'^j}{r^5} &= \frac{x'^ix'^j}{R^5}. \end{aligned} \tag{16.32}$$

The phase shift of the beams along the different arms of the interferometer is given by

$$\begin{aligned} \Delta\chi \equiv\ & \Delta\chi_1 + \Delta\chi_2 = \\ & \frac{1}{4}\int_{A,p_1}^{x} dz^{\lambda}(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[(x^{\alpha} - z^{\alpha})k^{\beta} - (x^{\beta} - z^{\beta})k^{\alpha}] - \\ & \frac{1}{4}\int_{A,p_2}^{x} dz^{\lambda}(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[(x^{\alpha} - z^{\alpha})k^{\beta} - (x^{\beta} - z^{\beta})k^{\alpha}] - \\ & \frac{1}{2}\int_{A,p_1}^{x} dz^{\lambda}\gamma_{\alpha\lambda}(z)k^{\alpha} + \frac{1}{2}\int_{A,p_2}^{x} dz^{\lambda}\gamma_{\alpha\lambda}(z)k^{\alpha}. \end{aligned} \tag{16.33}$$

The calculation can be simplified by taking $a = R$ and neglecting the contribution of gravity to the motion of the particles in the beams. The latter choice is certainly justified to first order in the WFA and for interferometers of laboratory dimensions. Then path p_1 is described by

$$\begin{array}{llll} 0 \leq z^0 \leq \frac{\ell}{v} & x = vz^0 & y = 0 & z = vz^0 \\ \frac{\ell}{v} \leq z^0 \leq \frac{2\ell}{v} & x = \ell & y = vz^0 - \ell & z = 0 \\ 0 \leq x \leq \ell & y = 0 & z = 0 & z^0 = \frac{x}{v} \\ x = \ell & 0 \leq y \leq \ell & z = 0 & z^0 = \frac{\ell}{v} + \frac{y}{v} \end{array}$$

and p_2 by

$$\begin{array}{llll} 0 \leq z^0 \leq \frac{\ell}{v} & x = 0 & y = vz^0 & z = 0 \\ \frac{\ell}{v} \leq z^0 \leq \frac{2\ell}{v} & x = vz^0 - \ell & y = \ell & z = 0 \\ x = 0 & 0 \leq y \leq \ell & z = 0 & z^0 = \frac{y}{v} \\ 0 \leq x \leq \ell & y = \ell & z = 0 & z^0 = \frac{\ell}{v} + \frac{x}{v}. \end{array}$$

In addition for p_1 one has:

$$\text{at } B: \quad x^\mu_{_B} = (\frac{\ell}{v}, \ell, 0, 0) \quad k^\mu_{_B} = (k^0, k, 0, 0)$$

$$\text{at } C: \quad x^\mu_{_C} = (\frac{2\ell}{v}, \ell, \ell, 0) \quad k^\mu_{_C} = (k^0, 0, k, 0)$$

and for p_2

$$\text{at } D: \quad x^\mu_{_D} = (\frac{\ell}{v}, 0, \ell, 0) \quad k^\mu_{_D} = (k^0, 0, k, 0)$$

$$\text{at } C: \quad x^\mu_{_C} = (\frac{2\ell}{v}, \ell, \ell, 0) \quad k^\mu_{_C} = (k^0, k, 0, 0).$$

Notice that the overall path described by the coherent beams is effectively closed in space-time, as required by (16.11). On using the expressions for $\gamma_{\mu\nu}$, one finds

$$\begin{aligned} \Delta\chi &= \frac{M\ell^2}{R^3}\frac{k^0}{v}\left(-x' + y' + \frac{3x'^2\ell}{2R^2} - \frac{3y'^2\ell}{2R^2}\right) + \\ &\quad \frac{M\ell^2}{R^3}k\left(-x' + y' + \frac{3x'^2\ell}{2R^2} - \frac{3y'^2\ell}{2R^2}\right) - \\ &\quad \frac{2M\ell^2\omega a^2}{5R^5}\left(\frac{k}{v} + k^0\right)\left(2R^2 - 3x'^2 - 3y'^2\right). \qquad (16.34) \end{aligned}$$

If the particles in the beam have speed v, then in the non-relativistic approximation $k^0 \simeq m(1+\frac{v^2}{2})$ and $k \simeq mv$ and $\Delta\chi$ represents the phase measured by an observer co-moving with the interferometer relative to which the sphere generating the LT field is spinning. The last term in $\Delta\chi$ depends on ω and represents the LT effect experienced by the quantum particles. It reaches its largest value when the interferometer is placed in the neighborhood of the poles of the sphere ($x' = y' = 0$). The remaining terms represent gravitational effects that are present even when $\omega = 0$. These terms vanish when the beam source is located at $x' = y'$ and, in particular at $x' = y' = 0$, at which positions the only contribution to the particle phase shift is that of the LT field. For the Earth the last term can also be written, in normal units, as

$$\Delta\chi_{LT} = \frac{2G}{c^2 R_\oplus^3} J_\oplus \frac{m\ell}{\hbar}[2R_\oplus^2 - 3(x'^2 + y'^2)], \tag{16.35}$$

where $J_\oplus = 2M_\oplus R_\oplus^2 \omega/5$ is the angular momentum of the Earth (assumed spherical and homogeneous) and $R_\oplus$ its radius. It is interesting to observe that $\Omega = \frac{G}{2c^2 R_\oplus^3} J_\oplus$ coincides with the effective LT precession frequency of a gyroscope [14, 15]. Since the precession frequency of a gyroscope in orbit is $\Omega = \frac{GJ_\oplus}{2c^2 R_\oplus^3}$, one can also write $\Delta\chi = \Omega\Pi$, where $\Pi = \frac{4ml^2}{\hbar}$ replaces the period of a satellite in the classical calculation. Its value, $\Pi \sim 1.4 \times 10^8 s$ for neutron interferometers with $l \sim 10^2 cm$, is rather high and yields $\Delta\chi \sim 10^{-7} rad$. This suggests that the development and use of large, heavy particle interferometers would be particularly advantageous in attempts to measure the LT effect.

3. Inertial fields in particle accelerators

3.1 Spin-rotation coupling in g-2 experiments

Prominent among the effects that can be derived from the covariant Dirac equation of Section 2.3 is the spin-rotation effect described by Mashhoon [11]. This effect is conceptually important. It extends our knowledge of rotational inertia to the quantum level. It also yields different potentials for different particles and for different spin states [10] and can not, therefore, be considered universal.

The relevance of spin-rotation coupling to physical [16] and astrophysical [10, 17] processes has already been pointed out.

It is shown below that the spin-rotation effect plays an essential role in precise measurements of the $g-2$ factor of the muon.

The experiment [18, 19] involves muons in a storage ring consisting of a vacuum tube, a few meters in diameter, in a uniform vertical magnetic field. Muons on equilibrium orbits within a small fraction of the maximum momentum are almost completely polarized with spin vectors pointing in the direction of motion. As the muons decay, those electrons projected forward in the muon rest frame are detected around the ring. Their angular distribution therefore reflects the precession of the muon spin along the cyclotron orbits.

The calculations are performed in the rotating frame of the muon and do not therefore require a relativistic treatment of inertial spin effects [20]. Then the vierbein formalism yields (16.25), or

$$\Gamma_0 = -\frac{1}{2}\, a_i \sigma^{0i} - \frac{1}{2}\, \omega_i \sigma^i \,, \tag{16.36}$$

where

$$\sigma^{0i} \equiv \frac{i}{2}\,[\gamma^0, \gamma^i] = i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

in the chiral representation of the usual Dirac matrices. The second term in (16.36) represents the Mashhoon effect. The first term drops out. The remaining contributions to the Dirac Hamiltonian, to first order in a_i and ω_i, add up to [8, 9]

$$\begin{aligned} H \;\; \approx \;\; & \vec{\alpha}\cdot\vec{p} + m\beta + \frac{1}{2}[(\vec{a}\cdot\vec{x})(\vec{p}\cdot\vec{\alpha}) + (\vec{p}\cdot\vec{\alpha})(\vec{a}\cdot\vec{x})] \\ & -\vec{\omega}\cdot\left(\vec{L} + \frac{\vec{\sigma}}{2}\right) . \end{aligned} \tag{16.37}$$

For simplicity all quantities in H are taken to be time-independent. They are referred to a left-handed tern of axes rotating about the x_2-axis in the clockwise direction of motion of the muons. The x_3-axis is tangent to the orbits and in the direction of the muon momentum. The magnetic field is $B_2 = -B$. Only the Mashhoon term then couples the helicity states of the muon. The remaining terms contribute to the overall energy E of the states, and H_0 is the corresponding part of the Hamiltonian.

Before decay the muon states can be represented as

$$|\psi(t) >= a(t)|\psi_+ > + b(t)|\psi_- > , \tag{16.38}$$

where $|\psi_+ >$ and $|\psi_- >$ are the right and left helicity states of the Hamiltonian H_0 and satisfy the equation

$$H_0|\psi_{+,-} >= E|\psi_{+,-} > \;. \tag{16.39}$$

The total effective Hamiltonian is $H_{eff} = H_0 + H'$, where

$$H' = -\frac{1}{2}\,\omega_2 \sigma^2 + \mu B \sigma^2 \,. \tag{16.40}$$

$\mu = (1 + a_\mu)\,\mu_0$ represents the total magnetic moment of the muon and μ_0 is the Bohr magneton. The effects of electric fields used to stabilize the orbits and of stray radial electric fields can be cancelled by choosing an appropriate muon momentum [19] and need not be considered.

The coefficients $a(t)$ and $b(t)$ in (16.38) evolve in time according to

$$i\frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = M \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \tag{16.41}$$

where M is the matrix

$$M = \begin{bmatrix} E - i\frac{\Gamma}{2} & i\left(\frac{\omega_2}{2} - \mu B\right) \\ -i\left(\frac{\omega_2}{2} - \mu B\right) & E - i\frac{\Gamma}{2} \end{bmatrix} \tag{16.42}$$

and Γ represents the width of the muon. The non-diagonal form of M (when $B = 0$) implies that rotation does not couple universally to matter.

M has eigenvalues

$$\begin{aligned} h_1 &= E - i\frac{\Gamma}{2} + \frac{\omega_2}{2} - \mu B\,, \\ h_2 &= E - i\frac{\Gamma}{2} - \frac{\omega_2}{2} + \mu B\,, \end{aligned} \tag{16.43}$$

and eigenstates

$$\begin{aligned} |\psi_1> &= \frac{1}{\sqrt{2}}\left[i|\psi_+> + |\psi_->\right], \\ |\psi_2> &= \frac{1}{\sqrt{2}}\left[-i|\psi_+> + |\psi_->\right]. \end{aligned} \tag{16.44}$$

The muon states that satisfy (16.38) and (16.41), and the condition $|\psi(0)> = |\psi_->$ at $t = 0$, are

$$\begin{aligned} |\psi(t)> &= \frac{e^{-\Gamma t/2}}{2} e^{-iEt}\left\{i\left[e^{-i\tilde{\omega}t} - e^{i\tilde{\omega}t}\right]|\psi_+> \right. \\ &\quad \left. + \left[e^{-i\tilde{\omega}t} + e^{i\tilde{\omega}t}\right]|\psi_->\right\}, \end{aligned} \tag{16.45}$$

where

$$\tilde{\omega} \equiv \frac{\omega_2}{2} - \mu B\,.$$

The spin-flip probability is therefore

$$\begin{aligned} P_{\psi_- \to \psi_+} &= |<\psi_+|\psi(t)>|^2 \\ &= \frac{e^{-\Gamma t}}{2}\left[1 - \cos(2\mu B - \omega_2)t\right]. \end{aligned} \tag{16.46}$$

The Γ-term in (16.46) accounts for the observed exponential decrease in electron counts due to the loss of muons by radioactive decay [19].

The spin-rotation contribution to $P_{\psi_- \to \psi_+}$ is represented by ω_2 which is the cyclotron angular velocity $\frac{eB}{m}$ [19]. The spin-flip angular frequency is then

$$\begin{aligned} \Omega &= 2\mu B - \omega_2 \qquad (16.47) \\ &= \left(1 + \frac{g-2}{2}\right)\frac{eB}{m} - \frac{eB}{m} \\ &= \frac{g-2}{2}\frac{eB}{m} = a_\mu \frac{eB}{m}\,, \qquad (16.48) \end{aligned}$$

which is precisely the observed modulation frequency of the electron counts [21]. This result is independent of the value of the anomalous magnetic moment of the particle. It is therefore the Mashhoon effect that evidences the $g-2$ term in Ω by exactly cancelling, in $2\mu B$, the much larger contribution μ_0 that relates to fermions with no anomalous magnetic moment [22]. The cancellation is made possible by the non-diagonal form of M and is therefore a direct consequence of the violation of the equivalence principle. It is significant that this effect is observed in an experiment that has already provided crucial tests of quantum electrodynamics and a test of Einstein's time-dilation formula to better than a 0.1 percent accuracy. Recent versions of the experiment [23, 24, 25] have improved the accuracy of the measurements from $270ppm$ to $1.3ppm$ and ultimately to $0.7ppm$ [26]. This, as well as measurements of the Mashhoon effect using the Global Positioning System [27], bode well for studies involving spin, inertia and electromagnetic fields, or inertial fields to higher order.

3.2 Tests of parity and time reversal invariance

The residual discrepancy $a_\mu(exp) - a_\mu(SM) = 26 \times 10^{-10}$ still existing [26] between the experimental and standard model values of the muon's a_μ can be used to set an upper limit on P and T invariance violations in spin-rotation coupling.

The possibility that discrete symmetries in gravitation be not conserved has been considered by some authors [28, 29, 30, 31]. Attention has in general focused on the potential

$$U(\vec{r}) = \frac{GM}{r}\left[\alpha_1 \vec{\sigma}\cdot\hat{r} + \alpha_2 \vec{\sigma}\cdot\vec{v} + \alpha_3 \hat{r}\cdot(\vec{v}\times\vec{\sigma})\right], \qquad (16.49)$$

which applies to a particle of generic spin $\vec{\sigma}$. The first term, introduced by Leitner and Okubo [29], violates the conservation of P and T. The same authors determined the upper limit $\alpha_1 \leq 10^{-11}$ from the hyperfine splitting of the ground state of hydrogen. The upper limit $\alpha_2 \leq 10^{-3}$ was determined in Ref.[31] from SN 1987A data. The corresponding potential violates the conservation of P and C. Conservation of C and T is violated by the last term, while (16.49), as a whole, conserves CPT. There is, as yet, no upper limit on α_3. These studies can be extended to the Mashhoon term.

Assume, in fact, that the coupling of rotation to $\mid \psi_+ >$ differs in strength from that to $\mid \psi_- >$ [32]. Then the Mashhoon term can be altered by means of a matrix $A = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$ that reflects the different coupling of rotation to the two helicity states. The total effective Hamiltonian is $H_{eff} = H_0 + H'$, where

$$H' = -\frac{1}{2} A \omega_2 \sigma_2 + \mu B \sigma_2. \tag{16.50}$$

A violation of P and T in (16.50) would arise through $\kappa_2 - \kappa_1 \neq 0$. The constants κ_1 and κ_2 are assumed to differ from unity by small amounts ϵ_1 and ϵ_2.

The muon states before decay are again as in (16.38) and the coefficients $a(t)$ and $b(t)$ evolve in time according to (16.41), but now the matrix M is replaced by

$$\tilde{M} = \begin{pmatrix} E - i\frac{\Gamma}{2} & i\left(\kappa_1 \frac{\omega_2}{2} - \mu B\right) \\ -i\left(\kappa_2 \frac{\omega_2}{2} - \mu B\right) & E - i\frac{\Gamma}{2} \end{pmatrix}. \tag{16.51}$$

The spin-rotation term, that is off-diagonal in (16.51), violates Hermiticity and T, P and PT, as shown in [32] and, in a general way, in [33], while nothing can be said about CPT conservation which requires H_{eff} to be Hermitian [34, 35]. Because of the non-Hermitian nature of (16.50), one expects Γ itself to be non-Hermitian. The resulting corrections to the width of the muon are, however, of second order in the ϵ's and are neglected.

$\tilde{M}$ has eigenvalues

$$\begin{aligned} h_1 &= E - i\frac{\Gamma}{2} + R \\ h_2 &= E - i\frac{\Gamma}{2} - R, \end{aligned} \tag{16.52}$$

where

$$R = \sqrt{\left(\kappa_1 \frac{\omega_2}{2} - \mu B\right)\left(\kappa_2 \frac{\omega_2}{2} - \mu B\right)}, \tag{16.53}$$

and eigenstates

$$\begin{aligned} |\psi_1 > &= b_1 \left[\eta_1 |\psi_+ > + |\psi_- >\right], \\ |\psi_2 > &= b_2 \left[\eta_2 |\psi_+ > + |\psi_- >\right]. \end{aligned} \tag{16.54}$$

One also finds

$$|b_1|^2 = \frac{1}{1+|\eta_1|^2}$$
$$|b_2|^2 = \frac{1}{1+|\eta_2|^2} \tag{16.55}$$

and

$$\eta_1 = -\eta_2 = \frac{i}{R}\left(\kappa_1 \frac{\omega_2}{2} - \mu B\right). \tag{16.56}$$

Then the muon states (16.38) are

$$|\psi(t)> = \frac{1}{2}e^{-iEt-\frac{\Gamma t}{2}}[-2i\eta_1 \sin Rt|\psi_+ > + 2\cos Rt|\psi_- >], \tag{16.57}$$

where the condition $|\psi(0) >= |\psi_- >$ has been applied. The spin-flip probability is therefore

$$P_{\psi_- \to \psi_+} = |<\psi_+|\psi(t)>|^2 = \frac{e^{-\Gamma t}}{2}\frac{\kappa_1\omega_2 - 2\mu B}{\kappa_2\omega_2 - 2\mu B}\left[1-\cos 2Rt\right]. \tag{16.58}$$

This equation and $\kappa_1 = \kappa_2 = 1$, yield (16.45) and (16.46) that provide the appropriate description of the spin-rotation contribution to the spin-flip transition probability. Notice that the case $\kappa_1 = \kappa_2 = 0$ (vanishing spin-rotation coupling) gives

$$P_{\psi_- \to \psi_+} = \frac{e^{-\Gamma t}}{2}\left[1 - \cos(1+a_\mu)\frac{eB}{m}\right] \tag{16.59}$$

and does not therefore agree with the results of the $g-2$ experiments. Hence the necessity of accounting for spin-rotation coupling whose contribution cancels the factor $\frac{eB}{m}$ in (16.59) [22].

Substituting $\kappa_1 = 1+\epsilon_1, \kappa_2 = 1+\epsilon_2$ into (16.57), one finds

$$P_{\psi_- \to \psi_+} \simeq \frac{e^{-\Gamma t}}{2}[1 - \cos\frac{eB}{m}(a_\mu - \epsilon)t], \tag{16.60}$$

where $\epsilon = \frac{1}{2}(\epsilon_1 + \epsilon_2)$. One may attribute the discrepancy between $a_\mu(exp)$ and $a_\mu(SM)$ to a violation of the conservation of the discrete symmetries by the spin-rotation coupling term in (16.50). The upper limit on the violation of P, T and PT is derived from (16.60) assuming that the deviation from the current value of $a_\mu(SM)$ is wholly due to ϵ, and therefore is 26×10^{-10}.

4. Maximal acceleration

In the 1980's, Caianiello and collaborators [36] developed a geometrical model of quantum mechanics in which quantization is interpreted as curvature of the eight-dimensional space-time tangent bundle $TM = M_4 \otimes TM_4$, where M_4 is the usual flat space–time manifold, of metric $\eta_{\mu\nu}$. In this space the standard operators of the Heisenberg algebra are represented as covariant derivatives and the quantum commutation relations are interpreted as components of the curvature tensor. The usual Minkowski line element is replaced in the model by the infinitesimal element of distance in the eight-dimensional space-time tangent bundle TM

$$d\tau^2 = \eta_{AB} dX^A dX^B \qquad A, B = 0, \dots, 7, \tag{16.61}$$

where, in normal units, $\eta_{AB} = \eta_{\mu\nu} \otimes \eta_{\mu\nu}$, $X^A = \left(x^\mu, \frac{c^2}{\mathcal{A}_m}\frac{dx^\mu}{ds}\right)$, $\mu = 0, \dots, 3$, $x^\mu = (ct, \vec{x})$, $dx^\mu/ds = \dot{x}^\mu$ is the relativistic four-velocity and $\mathcal{A}_m$ is a constant. In the model the symmetry between configuration and momentum space representations of field theory (Born reciprocity theorem) is automatically satisfied. The invariant line element (16.61) can be written in the form

$$\begin{aligned} d\tau^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{\mathcal{A}_m^2} \eta_{\mu\nu} d\dot{x}^\mu d\dot{x}^\nu = \\ &= \left[1 + \frac{\ddot{x}_\mu \ddot{x}^\mu}{\mathcal{A}_m^2}\right] ds^2 \equiv \sigma^2(x) ds^2, \end{aligned} \tag{16.62}$$

where all proper accelerations are normalized to $\mathcal{A}_m$, referred to as maximal acceleration, very much like velocities are normalized to their upper value c. Though $\mathcal{A}_m$ is, a priori, arbitrary, a value for it can be derived from quantum mechanics [37]. With some modifications and additions [38, 39], Caianiello's argument can be re-stated as follows.

If two observables $\hat{f}$ and $\hat{g}$ obey the commutation relation

$$\left[\hat{f}, \hat{g}\right] = -i\hbar\hat{\alpha}, \tag{16.63}$$

where $\hat{\alpha}$ is a Hermitian operator, then their uncertainties

$$\begin{aligned} (\Delta f)^2 &= \ < \Phi \mid \left(\hat{f} - < \hat{f} >\right)^2 \mid \Phi > \\ (\Delta g)^2 &= \ < \Phi \mid (\hat{g} - < \hat{g} >)^2 \mid \Phi > \end{aligned} \tag{16.64}$$

also satisfy the inequality

$$(\Delta f)^2 \cdot (\Delta g)^2 \geq \frac{\hbar^2}{4} < \Phi \mid \hat{\alpha} \mid \Phi >^2, \tag{16.65}$$

or

$$\Delta f \cdot \Delta g \geq \frac{\hbar}{2} \left| < \Phi \mid \hat{\alpha} \mid \Phi > \right| . \qquad (16.66)$$

Using Dirac's analogy between the classical Poisson bracket $\{f, g\}$ and the quantum commutator [40]

$$\{f, g\} \rightarrow \frac{1}{i\hbar} \left[\hat{f}, \hat{g} \right], \qquad (16.67)$$

one can take $\hat{\alpha} = \{f, g\}\, \hat{\mathbf{1}}$. With this substitution, Eq.(16.63) yields the usual momentum-position commutation relations. If in particular $\hat{f} = \hat{H}$, then Eq.(16.63) becomes

$$\left[\hat{H}, \hat{g} \right] = -i\hbar \{H, g\}\, \hat{\mathbf{1}}, \qquad (16.68)$$

Eq.(16.66) gives [40]

$$\Delta E \cdot \Delta g \geq \frac{\hbar}{2} \mid \{H, g\} \mid \qquad (16.69)$$

and

$$\Delta E \cdot \Delta g \geq \frac{\hbar}{2} \mid \frac{dg}{dt} \mid, \qquad (16.70)$$

when $\frac{\partial g}{\partial t} = 0$. Eqs.(16.69) is Ehrenfest theorem. Criteria for its validity are discussed at length in the literature [41, 40]. Eq.(16.70) implies that $\Delta E = 0$ when the quantum state of the system is an eigenstate of $\hat{H}$. In this case $\frac{dg}{dt} = 0$.

If $g \equiv v(t)$ is the (differentiable) velocity expectation value of a particle whose average energy is E, then Eq.(16.70) gives

$$\mid \frac{dv}{dt} \mid \leq \frac{2}{\hbar} \Delta E \cdot \Delta v(t). \qquad (16.71)$$

In general [42]

$$\Delta v = \left(< v^2 > - < v >^2 \right)^{\frac{1}{2}} \leq v_{max} \leq c. \qquad (16.72)$$

Caianiello's additional assumption, $\Delta E \leq E$, has so far remained unjustified. In fact, Heisenberg's uncertainty relation

$$\Delta E \cdot \Delta t \geq \hbar/2, \qquad (16.73)$$

that follows from (16.71) by writing $\Delta t = \Delta v / |dv/dt|$, seems to imply that, given a fixed energy E, a state can be constructed with arbitrarily large ΔE, contrary to Caianiello'assumption. An upper bound on ΔE can be found, however, if E is taken to represent the fixed *average* energy measured from an origin E_{min}. In what follows $E_{min} = 0$ for simplicity. Then the correct interpretation of (16.73) is that a quantum state with spread in energy ΔE

takes a time $\Delta t \geq \frac{\hbar}{2\Delta E}$ to evolve to a distinguishable (orthogonal) state. This evolution time must satisfy the more stringent limit [43]

$$\Delta t \geq \frac{\hbar}{2E}, \tag{16.74}$$

which determines a maximum speed of orthogonality evolution [44]. Obviously, both limits (16.73) and (16.74) can be achieved only for $\Delta E = E$, while spreads $\Delta E > E$, that would make Δt smaller, are precluded by (16.74). This effectively restricts ΔE to values $\Delta E \leq E$, as conjectured by Caianiello. One can now derive an upper limit on the value of the proper acceleration. In fact, in the instantaneous rest frame of the particle, where the acceleration is largest [38], $E = mc^2$ and (16.71) gives

$$\mid \frac{dv}{dt} \mid \leq 2\frac{mc^3}{\hbar} \equiv \mathcal{A}_m. \tag{16.75}$$

It also follows that in the rest frame of the particle, where $\frac{d^2x^0}{ds^2} = 0$, the absolute value of the proper acceleration is [38, 45]

$$\left(\mid \frac{d^2x^\mu}{ds^2}\frac{d^2x_\mu}{ds^2} \mid\right)^{\frac{1}{2}} = \left(\mid \frac{1}{c^4}\frac{d^2x^i}{dt^2} \mid\right)^{\frac{1}{2}} \leq \frac{\mathcal{A}_m}{c^2}. \tag{16.76}$$

Eq.(16.76) is a Lorentz invariant. The validity of (16.76) under Lorentz transformations is therefore assured.

Result (16.74) can also be used to extend (16.75) to include the average length of the acceleration $< a >$. If, in fact, $v(t)$ is differentiable, then fluctuations about its mean are given by

$$\Delta v \equiv v - < v > \simeq \left(\frac{dv}{dt}\right)_0 \Delta t + \left(\frac{d^2v}{dt^2}\right)_0 (\Delta t)^2 + ... \tag{16.77}$$

Eq.(16.77) reduces to $\Delta v \simeq \mid \frac{dv}{dt} \mid \Delta t = < a > \Delta t$ for sufficiently small values of Δt, or when $\mid \frac{dv}{dt} \mid$ remains constant over Δt. Eq.(16.74) then yields

$$< a > \leq \frac{2cE}{\hbar} \tag{16.78}$$

and again (16.75) follows [39].

Classical and quantum arguments supporting the existence of a maximal acceleration have long been discussed in the literature [46]. MA also appears in the context of Weyl space [47] and of a geometrical analogue of Vigier's stochastic theory [48].

MA has been used to obtain model independent limits on the mass of the Higgs boson [49] and on the stability of white dwarfs and neutron stars [50].

It is significant that a limit on the acceleration also occurs in string theory. Here the upper limit manifests itself through Jeans-like instabilities [51] which occur when the acceleration induced by the background gravitational field is larger than a critical value $a_c = (m\alpha)^{-1}$for which the string extremities become causally disconnected [52]. m is the string mass and α is the string tension. Frolov and Sanchez [53] have then found that a universal critical acceleration $a_c = (m\alpha)^{-1}$ must be a general property of strings.

Recently Castro [54] has derived the same MA limit (16.75) from Clifford algebras in phase space and Schuller [55] has rigorously shown that special relativity has a MA extension. Applications of the Caianiello model range from cosmology to particle physics. A sample of pertinent references can be found in [56]. Clearly (16.62) implies that the effective space-time metric experienced by accelerated particles is $\tilde{g}_{\mu\nu} = \sigma^2\eta_{\mu\nu}$ and is therefore altered by MA corrections that induce curvature, violate the equivalence principle and make the metric observer dependent as conjectured by Gibbons and Hawking [57]. These corrections vanish in the classical limit $(\mathcal{A}_m)^{-1} = \hbar/(2mc^3) \to 0$, as expected.

Recent advances in high resolution spectroscopy are now allowing Lamb shift measurements of unprecedented precision, leading in the case of simple atoms and ions to the most stringent tests of quantum electrodynamics (QED). MA corrections due to the metric (16.62) appear directly in the Dirac equation for the electron that must now be written in covariant form and referred to a local Minkowski frame by means of the vierbein field ${e_\mu}^a(x)$. From (16.62) one finds ${e_\mu}^a = \sigma(x){\delta_\mu}^a$, where Latin indices refer to the locally inertial frame and Greek indices to a generic non-inertial frame. The covariant matrices $\gamma^\mu(x)$ satisfy the anticommutation relations $\{\gamma^\mu(x), \gamma^\nu(x)\} = 2\tilde{g}^{\mu\nu}(x)$, while the covariant derivative $\mathcal{D}_\mu \equiv \partial_\mu + \omega_\mu$ contains the total connection $\omega_\mu = \frac{1}{2}\sigma^{ab}\omega_{\mu ab}$, where $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$, ${\omega_\mu}^a{}_b = (\Gamma^\lambda_{\mu\nu}\, {e_\lambda}^a - \partial_\mu {e_\nu}^a){e^\nu}_b$ and $\Gamma^\lambda_{\mu\nu}$ represent the usual Christoffel symbols. For conformally flat metrics ω_μ takes the form $\omega_\mu = \frac{1}{\sigma}\sigma^{ab}\eta_{a\mu}\sigma_{,b}$. By using the transformations $\gamma^\mu(x) = {e^\mu}_a(x)\gamma^a$ so that $\gamma^\mu(x) = \sigma^{-1}(x)\gamma^\mu$, where γ^μ are the usual constant Dirac matrices, the Dirac equation can be written in the form

$$\left[i\hbar\gamma^\mu\left(\partial_\mu + i\frac{e}{\hbar c}A_\mu\right) + i\frac{3\hbar}{2}\gamma^\mu(\ln\sigma)_{,\mu} - mc\sigma(x)\right]\psi(x) = 0\,. \quad (16.79)$$

From (16.79) one obtains the Hamiltonian

$$H = -i\hbar c\vec{\alpha}\cdot\vec{\nabla} + e\gamma^0\gamma^\mu A_\mu(x) - i\frac{3\hbar c}{2}\gamma^0\gamma^\mu(\ln\sigma)_{,\mu} + mc^2\sigma(x)\gamma^0\,, \quad (16.80)$$

which is in general non–Hermitian [58]. However, when one splits the Dirac spinor into large and small components, the only non-Hermitian term is $(\ln\sigma)_{,0}$. If σ varies slowly in time, or is time-independent, as in the present case, this term can be neglected and Hermiticity is recovered. Here the nucleus is considered to be point-like and its recoil is neglected.

In QED the Lamb shift corrections are usually calculated by means of a non–relativistic approximation [59] which is also followed here [60, 61]. For the electric field $E(r) = kZe/r^2 (k = 1/4\pi\epsilon_0)$, the conformal factor becomes $\sigma(r) = (1 - (\frac{r_0}{r})^4)^{1/2}$, where $r_0 \equiv (kZe^2/m\mathcal{A}_m)^{1/2} \sim \sqrt{Z}\, 2.3 \cdot 10^{-14}$m and $r > r_0$. The calculation of $\ddot{x}^\mu$ is performed classically in a non–relativistic approximation. This is justified because for the electron v/c is at most $\sim 10^{-3}$. Neglecting contributions of the order $O(\mathcal{A}_m^{-4})$, $\sigma(r) \sim 1 - (1/2)(r_0/r)^4$. This expansion requires that in the following only those values of r be chosen that are above a cut–off Λ, such that for $r > \Lambda > r_0$ the validity of the expansion is preserved. The actual value of Λ is chosen below. The length r_0 has no fundamental significance in QED and depends in general on the details of the acceleration mechanism. It is only the distance at which the electron would attain, classically, the acceleration $\mathcal{A}_m$ irrespective of the probability of getting there.

By using the expansion for $\sigma(r)$ in (16.80) one finds that all MA effects are contained in the perturbation terms

$$H_{r_0} = -\frac{mc^2}{2}\left(\frac{r_0}{r}\right)^4 \beta + i\frac{3\hbar c}{4} r_0^4 \vec{\alpha} \cdot \vec{\nabla}\frac{1}{r^4} \equiv \mathcal{H} + \mathcal{H}' \,. \tag{16.81}$$

By splitting $\psi(x)$ into large and small components φ and χ and using $\chi = -i(\hbar/2mc)\vec{\sigma} \cdot \vec{\nabla}\varphi \ll \varphi$ one obtains for the perturbation due to $\mathcal{H}$

$$\delta\mathcal{E}_{nlm} \simeq -\frac{mc^2}{2} r_0^4 \int d^3\vec{r} \frac{1}{r^4} \varphi^*_{nlm} \varphi_{nlm} \,. \tag{16.82}$$

The perturbation due to $\mathcal{H}'$ vanishes. In (16.82) φ_{nlm} are the well known eigenfunctions for one–electron atoms. The integrations over the angular variables in (16.82) can be performed immediately and yield

$$\delta\mathcal{E}_{20} = -\frac{mc^2}{16}\left(\frac{r_0}{a_0}\right)^4 \left\{\left[4\left(\frac{a_0}{\Lambda}\right) + 1\right] e^{-\Lambda/a_0} - 8E_1\left(\frac{\Lambda}{a_0}\right)\right\}, \tag{16.83}$$

$$\delta\mathcal{E}_{21} = -\frac{mc^2}{48}\left(\frac{r_0}{a_0}\right)^4 e^{-\Lambda/a_0} \,, \tag{16.84}$$

$$\delta\mathcal{E}_{10} = -2mc^2\left(\frac{r_0}{a_0}\right)^4 \left[\left(\frac{a_0}{\Lambda}\right) e^{-2\Lambda/a_0} - 2E_1\left(\frac{2\Lambda}{a_0}\right)\right] , \tag{16.85}$$

where $E_1(x) = \int_1^\infty dy\, e^{-xy}/y$ and a_0 is the Bohr radius divided by Z. In order to calculate the $2S - 2P$ Lamb shift corrections it is now necessary to choose the value of the cut–off Λ. While in QED Lamb shift and fine structure effects are cut–off independent, the values of the corresponding MA corrections increase when Λ decreases. This can be understood intuitively because the electron finds itself in regions of higher electric field at smaller values of r. Λ is a characteristic length of the system. It must also represent a distance from the nucleus that can be reached by the electron whose acceleration and relative perturbations depend on the position attained. One may tentatively choose $\Lambda \sim a_0$. According to the wave functions involved, the probability that the electron be at this distance ranges between 0.1 and 0.5. Smaller values of Λ lead to larger acceleration corrections, but are reached with much lower probabilities. This is the case of the Compton wavelength of the electron whose use as a cut–off is therefore ruled out in the present context. For $\Lambda \sim a_0$, Eqs. (16.83)-(16.85) give the corrections to the levels $2S, 2P$ and $1S$ ($Z = 1$) $\delta\mathcal{E}_{20} \sim -22.96$ kHz, $\delta\mathcal{E}_{21} \sim -33.42$ kHz, $\delta\mathcal{E}_{10} \sim -325.45$ kHz, yielding the Lamb shift correction $\delta\mathcal{E}_L = \delta\mathcal{E}_{20} - \delta\mathcal{E}_{21} \sim +10.46$ kHz. A fully relativistic calculation [62] gives $\delta\mathcal{E}_L \sim 11.37 kHz$. The MA corrections are comparable in magnitude with those of QED at order α^7, where α is the fine structure constant. The agreement between MA corrections and experiment [63, 64] is at present very good [61] for the $2S - 2P$ Lamb shift in hydrogen ($\sim$ 7kHz) and comparable with the agreement of experiments with standard QED with and without two-loop corrections [65]. The agreement is also good for the $\frac{1}{4}L_{1S} - \frac{5}{4}L_{2S} + L_{4S}$ Lamb shift in hydrogen and comparable, in some instances, with that between experiment and QED ($\sim$ 30kHz) [61, 66]. Finally, the MA corrections [61] improve the agreement between experiment [67] and theory by $\sim 50\%$ for the $2S - 2P$ shift in He^+.

5. Conclusions

Inertia and gravity induced quantum phases, helicity oscillations of particles in accelerators and storage rings and MA corrections in quantum processes are all effects that may occur well before the onset of quantum gravity. They represent research areas where both theoretical and experimental developments are possible.

The sensitivity of measurements in g-2 and Lamb shift experiments can respectively set upper limits on violations of P and T invariance in spin-rotation coupling and on the magnitude of MA corrections.

Further advances in these fields as well as in heavy particle interferometry, would greatly help in filling a gap of over forty orders of magnitude between planetary scales, over which Einstein's views on inertia and gravity are tested, and Planck length.

References

[1] Cai, Y.Q., Papini, G., *Class. Quantum Grav.*, **6**, 407 (1989)

[2] See also: Papini, G., in *Advances in the Interplay Between Quantum and Gravity Physics*, P.G. Bergmann and V. de Sabbata (eds.), Kluwer Aca-

demic Publishers, pp. 317-338 (2002)

[3] Chiao, Raymond Y., *arXiv:gr-qc/0211078 v4 22 Jan 2003*

[4] See, for instance: Varma, C.M., *Theoretical Survey of HFS, Proc. of the Adriatrico Research Conference on Heavy Fermion Systems*, I.C.T.P., Trieste, 1986

[5] Cai, Y.Q., Papini, G., *Modern Physics Letters A*, **4**, 1143 (1989); *Class. Quantum Grav.*, **7**, 269 (1990); *Gen. Rel. Grav.*, **22**, 259 (1990)

[6] De Oliveira, C.G. and Tiomno, J., *Nuovo Cimento*, **24**, 672 (1962)

[7] Peres, A., *Suppl. Nuovo Cimento*, **24**, 389 (1962)

[8] Hehl, F.W. and Ni, W.-T., *Phys. Rev. D*, **42**, 2045 (1990)

[9] Singh, D. and Papini, G. *Nuovo Cimento B*, **115**, 233 (2000)

[10] Cai, Y.Q., Papini, G., *Phys. Rev. Lett.*, **66**, 1259 (1991); **68**, 3811 (1992)

[11] Mashhoon, B., *Phys. Lett. A*, **143**, 176 (1990); **145**, 147 (1990); *Phys. Rev. Lett.*, **61**, 2639 (1988); **68**, 3812 (1992)

[12] Cai, Y.Q.and Papini, G., in *Modern Problems of Theoretical Physics*, P.I. Pronin and Yu. N. Obukhov (eds.), World Scientific, Singapore, pp. 131-142 (1991)

[13] Lense, J. and Thirring, H., *Z. Phys.*, **19**, 156 (1918); English translation: Mashhoon, B., Hehl, F.W. and Theiss, D.S., *Gen. Rel. Grav.* **16**, 711 (1984)

[14] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, New York (1972)

[15] Will, C.M., *Theory and Experiment in Gravitational Physics*, Cambridge University Press, Cambridge (1981)

[16] Cai, Y.Q., Lloyd, D.G. and Papini, G., *Phys. Lett. A*, **178**, 225 (1993)

[17] Papini, G., *Proc. of the* 5^{th} *Canadian Conference on General Relativity and Relativistic Astrophysics*, R.B. Mann and R.G. McLenaghan (eds.), World Scientific, Singapore, pp. 107-119 (1994)

[18] Bailey, J. *et al.*, *Nucl. Phys.*, **B150**, 1 (1979)

[19] Farley, F.J.M. and Picasso, E. *Advanced Series in High-Energy Physics*, vol. **7** *Quantum Electrodynamics*, T. Kinoshita (edt.), World Scientific, Singapore, p. 479 (1990)

[20] Ryder, Lewis, *J. Phys. A: Math. Gen.*, **31**, 2465 (1998)

[21] Farley, F.J. and Picasso, E., *Ann. Rev. Nucl. Part. Sci.*, **29**, 24 (1979)

[22] Papini, G., Lambiase, G., *Phys. Lett. A*, **294**, 175 (2002)

[23] Carey, R.M. *et al.*, Muon (g-2) Collaboration, *Phys. Rev. Lett.*, **82**, 1632 (1999)

[24] Brown, H.N. *et al.*, Muon (g-2) Collaboration, *Phys. Rev.D*, **62**, 091101 (2000)

[25] Brown, H.N. *et al.*, Muon (g-2) Collaboration, *Phys. Rev. Lett.*, **86**, 2227 (2001)

[26] Brown, H.N. *et al.*, Muon (g-2) Collaboration, *Phys. Rev. Lett.*, **89**, 101804 (2002), *arXiv:hep-ex/0208001 v2 13 Aug 2002*

[27] Ashby, Neil, in *Gravitation and Relativity at the Turn of the Millennium*, N. Dahdich and J. Narlikar (eds.), IUACC, Pune, pp.231-258 (1998)

[28] Schiff, L.I., *Phys. Rev. Lett.*, **1**, 254 (1958)

[29] Leitner, J. and Okubo, S., *Phys. Rev. B*, **136**, 1542 (1964)

[30] Hari Dass, N.D., *Phys. Rev. Lett.*, **36**, 393 (1976); *Ann. Phys. (NY)*, **107**, 337 (1977)

[31] Almeida, L.D., Matsas, G.E.A. and Natale, A.A., *Phys. Rev. D*, **39**, 677 (1989)

[32] Papini, G., *Phys. Rev. D*, **65**, 077901 (2002)

[33] Scolarici, G., Solombrino, L., *Phys. Lett. A*, **303**, 239 (2002); *arXiv:quant-ph/0211161 v1 25 Nov 2002*

[34] Kenny, Brian G. and Sachs, Robert G., *Phys. Rev. D*, **8**, 1605 (1973)

[35] Sachs, Robert G., *Phys. Rev. D*, **33**, 3283 (1986)

[36] Caianiello, E.R., *La Rivista del Nuovo Cimento*, **15**, No. 4 (1992) and references therein

[37] Caianiello, E.R., *Lett. Nuovo Cimento*, **41**, 370 (1984)

[38] Wood, W.R., Papini, G., Cai, Y.Q., *Nuovo Cimento B*, **104**, 361 and (errata corrige) 727 (1989)

[39] Papini, G., *arXiv:quant-ph/0301142 v1 26 Jan 2003*

[40] Landau, L.D. and Lifshitz, E.M., *Quantum Mechanics*, third edition, Pergamon Press, New York (1977), pp.27 and 49

[41] Messiah, Albert *Quantum Mechanics*, North-Holland, Amsterdam (1961), Vol.I, Chs. IV.10 and VIII.13

[42] Sharma, C.S., Srirankanathan, S., *Lett. Nuovo Cimento*, **44**, 275 (1985)

[43] Margolus, Norman, Levitin, Lev B., *Physica D*, **120**, 188 (1998)

[44] See also Bekenstein, Jacob D., *Phys. Rev. Lett.*, **46**, 623 (1981); Anandan, J., Aharonov, Y., *Phys. Rev. Lett.*, **65**, 1697 (1990)

[45] Stephenson, G. and Kilmister, C.W., *Special Relativity for Physicists*, Longmans, London, 1965

[46] Das, A., *J. Math. Phys.*, **21**, 1506 (1980); Gasperini, M., *Astrophys. Space Sci.*, **138**, 387 (1987) ; Toller, M., *Nuovo Cimento*, **B102**, 261

(1988); *Int. J. Theor. Phys.*, **29**, 963 (1990); *Phys. Lett. B*, **256**, 215 (1991); Mashhoon, B., *Phys. Lett. A*, **143**, 176 (1990); de Sabbata, V., Sivaram, C., *Astrophys. Space Sci.*, **176**, 145 (1991); *Spin and Torsion in Gravitation*, World Scientific, Singapore, (1994); Falla, D.F., Landsberg, P.T., *Nuovo Cimento B*, **106**, 669 (1991); Pati, A.K., *Nuovo Cimento B*, **107**, 895 (1992); *Europhys. Lett.*, **18**, 285 (1992); Brandt, H.E., *Lett. Nuovo Cimento*, **38**, 522 (1983); *Found. Phys. Lett.*, **2**, 39 (1989); Rama, S. Kalyana, *arXiv:hep-th/0209129 v2 20 Sept 2002*; Pardy, Miroslav, *arXiv:gr-qc/0302007 v1 3 Feb 2003*

[47] Papini, G. and Wood, W.R., *Phys. Lett. A*, **170**, 409 (1992); Wood, W.R. and Papini, G., *Phys. Rev. D*, **D45**, 3617 (1992); *Found. Phys. Lett.*, **6**, 409 (1993); Papini, G., *Mathematica Japonica*, **41**, 81 (1995)

[48] Vigier, J.P., *Found. Phys.*, **21**, 125 (1991)

[49] Kuwata, S., *Nuovo Cimento*, **111**, 893 (1996)

[50] Papini, G., *Phys. Lett. A*, **305**, 359 (2002)

[51] Sanchez, N. and Veneziano, G., *Nucl. Phys. B*, **333**, 253 (1990); Gasperini, M., Sanchez, N., Veneziano, G., *Nucl. Phys. B*, **364**, 365 (1991); *Int. J. Mod. Phys. A*, **6**, 3853 (1991)

[52] Gasperini, M., *Phys. Lett. B*, **258**, 70 (1991); *Gen. Rel. Grav.*, **24**, 219 (1992)

[53] Frolov, V.P. and Sanchez, N., *Nucl. Phys. B*, **349**, 815 (1991)

[54] Castro, Carlos, *arXiv:hep-th/0210061 v2 15 Oct 2002; hep-th/0211053 v1 7 Nov 2002*

[55] Schuller, Frederic P., *arXiv:hep-th/0203079 8Mar 2002*; *Ann. Phys. (NY)*, **299**, 174 (2002)

[56] Caianiello, E.R., Gasperini, M., Scarpetta, G., *Class. Quantum Grav.*, **8**, 659 (1981); *Nuovo Cimento B*, **105**, 259 (1990); Caianiello, E.R., Gasperini, M., Predazzi, E., Scarpetta, G., *Phys. Lett. A*, **132**, 83 (1988); Papini, G., Feoli, A., Scarpetta, G., *Phys. Lett. A*, **202**, 50 (1995); Lambiase, G., Papini, G., Scarpetta, G., *Phys. Lett. A*, **244**, 349 (1998); *Nuovo Cimento B*, **114**, 189 (1999); Feoli, A., Lambiase, G., Papini, G., Scarpetta, G., *Phys. Lett. A*, **263**, 147 (1999); Capozziello, S., Feoli, A., Lambiase, G., Papini, G., Scarpetta, G., *Phys. Lett. A*, **268**, 247 (2000); Bozza, V., Feoli, A., Papini, G., Scarpetta, G., *Phys. Lett. A*, **271**, 35 (2000); **279**, 163 (2001); **283**, 53 (2001); Papini, G., Scarpetta, G., Bozza, V., Feoli, A., Lambiase, G., *Phys. Lett. A*, **300**, 599 (2002)

[57] Gibbons, G.W. and Hawking, S.W., *Phys. Rev. D*, **15**, 2738 (1977)

[58] Parker, L., *Phys. Rev. D*, **22**, 1922 (1980)

[59] Itzykson, C. and Zuber, J.B., *Quantum Field Theory*, McGraw-Hill Inc., New York, (1980)

[60] Lambiase, G., Papini, G., Scarpetta, G., *Nuovo Cimento B*, **112**, 1003 (1997)

[61] Lambiase, G., Papini, G., Scarpetta, G., *Phys. Lett. A*, **244**, 349 (1998)

[62] Chen, C.X., Papini, G., Mobed, N., Lambiase, G., Scarpetta, G., *Nuovo Cimento B*, **114**, 199 (1999)

[63] Hand, L.N., Miller, D.J., Wilson, R., *Rev. Mod. Phys.*, **35**, 335 (1963)

[64] Simon, G.G., Schmitt, Ch., Borkonski, F., Waltheer, V.H, *Nucl. Phys. A*, **333**, 381 (1980)

[65] Pachucki, K., *Phys. Rev. Lett.*, **72**, 3154 (1994)

[66] Weitz, M., Schmidt-Kaler F. and Hänsch, T.W., *Phys. Rev. Lett.*, **68**, 1120 (1992); Weitz, M., Huber, A., Schmidt-Kaler, F., Leibfried, D. and Hänsch, T.W., *Phys. Rev. Lett.*, **72**, 328 (1994)

[67] van Vijngaarden, A., Kwela, J. and Drake, G.W.F., *Phys. Rev. A*, **43**, 3325 (1991); van Vijngaarden, A., Holuj, F. and Drake, G.W.F., *Can. J. Phys.*, **76**, 95 (1998)

Chapter 17

QUANTUM MECHANICS IN A ROTATING FRAME

Jeeva Anandan
and Jun Suzuki
Department of Physics and Astronomy
University of South Carolina.
jeeva@sc.edu
suzuki@physics.sc.edu

Abstract The rotating frame is considered in quantum mechanics on the basis of the position dependent boost relating this frame to the non rotating inertial frame. We derive the Sagnac phase shift and the spin coupling with the rotation in the non relativistic limit by a simple treatment. By taking the low energy limit of the Dirac equation with a spin connection, we obtain the Hamiltonian for the rotating frame, which gives rise to all the phase shifts discussed before. Furthermore, we obtain a new phase shift due to the spin-orbit coupling.

1. Introduction

The rotating frame has played an important role both in classical and quantum physics. One reason for this is that thermal equilibrium in a closed system can be realized when the system has uniform translation and rotation relative to an inertial frame [1]. So, the macroscopic properties of the system are not affected by the uniform rotation, apart from the influence of centrifugal and Coriolis fields. This result is significant because most experiments are done under the influence of the earth's rotation. In a quantum system, there are also global consequences of rotation, such as the phase shift in interferometry (Sagnac effect). An experiment to detect the Sagnac effect due to the earth's rotation in neutron interferometry by using a vertical incoming beam was proposed by one of us [2], which led to this experiment being performed subsequently by Werner, Staudenmann, and Colella [3]. There were many discussions of this effect in the past three decades [5], which we are in agreement with. However there still remain misconceptions, which may be a source of

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 361-370.

confusion for some people. Therefore we consider it as a good opportunity to clarify the consequences of quantum mechanics when it is applied to a rotating frame.

This paper continues as follows: in section 2, we begin with the Lorentz boost in special relativity, and by taking its non relativistic limit obtain all possible ways of implementing the Galilei boost in quantum mechanics. Then in section 3, we discuss a rotating frame in non relativistic quantum mechanics and obtain the Hamiltonian and derive phase shifts. We, furthermore, discuss relativistic aspects of the rotating frame to understand the limitation of non relativistic approach, in section 4 and obtain a Hamiltonian in the low energy limit of the Dirac equation. All the phase shifts in a rotating frame, including a new phase shift due to the spin-orbit coupling, are then obtained from this Hamiltonian. We use $\hbar = c = 1$ units throughout the paper unless we write them explicitly.

2. Lorentz and Galilei transformations in Quantum Mechanics

We shall consider a spinless particle to make our discussion clear and construct all possible Galilei transformations in the non relativistic limit.

The scalar field $\phi(x^\mu)$ is transformed under the infinitesimal Lorentz boost $\Lambda^\mu_{\ \nu} \simeq I^\mu_{\ \nu} + \omega^\mu_{\ \nu}$,

$$\delta\phi(x^\mu) = \phi(x'^\mu = \Lambda^\mu_{\ \nu} x^\nu) - \phi(x^\mu) \simeq \frac{i}{2}\omega_{\mu\nu} L^{\mu\nu}\phi(x^\mu) \tag{17.1}$$

where $L^{\mu\nu}$ are the generators of the Lorentz transformations defined by

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu. \tag{17.2}$$

Therefore the infinitesimal Lorentz boost for the direction x^i is given by

$$U \simeq I + i\omega_{0i} L^{0i}, \tag{17.3}$$

and hence, we obtain the Lorentz boost U as

$$U = \exp(i\omega_{0i}(x^0 p^i - x^i p^0)). \tag{17.4}$$

Now we take the non relativistic limit for the above boost and it becomes the Galilei boost with a velocity $\mathbf{V}$

$$U = \exp(it\mathbf{V}\cdot\hat{\boldsymbol{p}} - im\mathbf{V}\cdot\hat{\boldsymbol{x}}). \tag{17.5}$$

Using the Galilei boost (17.5), we can implement a boost from one inertial frame F_0 and another inertial frame F_0' which is related to F_0 by the velocity $\mathbf{V}$, i.e.

$$\boldsymbol{x}' = \boldsymbol{x} - \mathbf{V}t, \quad t' = t \tag{17.6}$$

where unprimed quantities refer to the frame F_0 and prime quantities refer to the frame F_0'. There are two natural and equivalent ways to do it as shown below.

i) U acting on the wave function;

$$\psi'(\boldsymbol{x'},t') = \exp(-im\mathbf{V}\cdot\boldsymbol{x}+i\frac{1}{2}m\mathrm{V}^2t)\psi(\boldsymbol{x},t) \tag{17.7}$$

$$\hat{\boldsymbol{p'}} = \hat{\boldsymbol{p}} \tag{17.8}$$

ii) U acting on the momentum operator;

$$\psi'(\boldsymbol{x'},t') = \psi(\boldsymbol{x},t) \tag{17.9}$$

$$\hat{\boldsymbol{p'}} = U^\dagger\hat{\boldsymbol{p}}U = \hat{\boldsymbol{p}} - m\mathbf{V} \tag{17.10}$$

The equivalence between i) and ii) is checked easily, if we notice that there exist a local gauge transformation between two pictures such as,

$$e^{if(\boldsymbol{x},t)} = \exp(-im\mathbf{V}\cdot\boldsymbol{x}+i\frac{1}{2}m\mathrm{V}^2t). \tag{17.11}$$

And we can immediately see the equivalence by rewriting i) as $\psi' = e^{if}\psi$, $\hat{\boldsymbol{p'}} = \hat{\boldsymbol{p}}$, and ii) as $\psi' = \psi$, $\hat{\boldsymbol{p'}} = e^{-if}\hat{\boldsymbol{p}}e^{if}$. This is like the difference between the Schrödinger picture i) and the Heisenberg picture ii). And it should be emphasized that observed quantities are neither operators nor the wave functions themselves but expectation values which are calculated from them, and (of course) two pictures yield same expectation values. Although above two methods seem natural to transform one frame to another, there are also an infinite number of ways which are related to above methods by some other gauge transformations. And this exhausts all possible ways of implementing the Galilei boost in quantum mechanics.

Before we discuss the Hamiltonian of the system, let us consider a non trivial example which helps us to understand the physics behind those two pictures. Suppose the wave function in the frame F_0 is given by a plane wave e^{ikx} ($k = 2\pi/\lambda$), and we examine the wave function seen from the frame F_0'. In the picture i) the wave length is changed due to the phase factor in front of (17.7), on the other hand in the picture ii) the wave length does not change since the wave function transforms as a scalar. However the momentum operator does change in ii) as (17.10), and this is consistent with the fact that the de Broglie relation $p = h/\lambda$ holds in all frames.

Next we shall examine the Schrödinger equation of the system and discuss energies measured in both frames. As we already saw, two methods i) and ii) are equivalent. Therefore it is enough to examine ii) only. Starting with the Schrödinger equation in the frame F_0

$$i\frac{\partial}{\partial t}\psi(\boldsymbol{x},t) = H\psi(\boldsymbol{x},t), \quad H = \frac{\hat{\boldsymbol{p}}^2}{2m}, \tag{17.12}$$

we can transform it to the frame F_0' as

$$i(\frac{\partial}{\partial t'} + \mathbf{V} \cdot \boldsymbol{\nabla'})\psi(\boldsymbol{x'}, t') = H'\psi(\boldsymbol{x'}, t'), \tag{17.13}$$

where the left hand side is obtained by the chain rule and H' is

$$H' = U^{\dagger}HU = \frac{(\hat{\boldsymbol{p}} - m\mathbf{V})^2}{2m} = \frac{\hat{\boldsymbol{p'}}^2}{2m}. \tag{17.14}$$

So we identify the energy operator $\hat{E}'$ in F' as $\hat{E}' = i(\frac{\partial}{\partial t'} + \mathbf{V} \cdot \boldsymbol{\nabla'})$ whose eigenvalue is positive. Notice that the energy measured in the frame F_0' is also obtained from the non relativistic limit of the Lorentz transformation for the energy, namely,

$$E' = (1 - \mathrm{V}^2)^{-\frac{1}{2}}(E - \mathbf{V} \cdot \boldsymbol{p}) = E - \mathbf{V} \cdot \boldsymbol{p} + \frac{1}{2}m\mathrm{V}^2 \tag{17.15}$$

which agrees with the result (17.14). Now we rewrite the equation (17.13) in the following form using $\hat{\boldsymbol{p'}} = \hat{\boldsymbol{\nabla'}} - m\mathbf{V}$,

$$(i\frac{\partial}{\partial t'} + \frac{1}{2}m\mathrm{V}^2)\psi(\boldsymbol{x'}, t') = \frac{1}{2m}(\hat{\boldsymbol{p'}} - m\mathbf{V})^2\psi(\boldsymbol{x'}, t'). \tag{17.16}$$

Therefore we find that this result is equivalent as the one obtained from the minimal coupling with a gauge field $\mathcal{A}^{\mu} = (-\frac{1}{2}\mathrm{V}^2, \mathbf{V})$ by a coupling constant m, namely,

$$\hat{p^{\mu}} \rightarrow \hat{p^{\mu}} - m\mathcal{A}^{\mu}. \tag{17.17}$$

3. Non Relativistic Aspects of the Rotating Frame

As is shown in the previous section there is no difficulty to implement the Galilei boost in quantum mechanics, next we shall extend the above method to a rotating frame F' whose angular velocity with respect to F_0 is $\boldsymbol{\Omega}$ (a constant of the time; we are dealing with a uniform rotating system throughout the paper). One may try to construct the boost for the rotating frame in the following way which leads a shortcoming. Since the velocity $\mathbf{V}$ is, now, given by $\boldsymbol{\Omega} \times \boldsymbol{x}$, the substitution it into (17.5) gives the boost $U = \exp(it(\boldsymbol{\Omega} \times \hat{\boldsymbol{x}}) \cdot \hat{\boldsymbol{p}}) = \exp(it\boldsymbol{\Omega} \cdot \hat{\boldsymbol{L}})$ where $\hat{\boldsymbol{L}} = \hat{\boldsymbol{x}} \times \hat{\boldsymbol{p}}$ is the orbital angular momentum operator of the particle. So one might conclude that the wave function transforms as a pure rotation from F_0 to F' in the picture i), or since U commutes with $\hat{\boldsymbol{L}}$ one might predict that the orbital angular momentum is the same in both frames. However those consequences are obviously wrong even classically, the reason is due to the fact that the boost $U = e^{it\boldsymbol{\Omega} \cdot \hat{\boldsymbol{L}}}$ transforms from F_0 to F_0', but not to F'.

To resolve this shortcoming we need to realize that the boost from F_0 to F' depends on the position and therefore it cannot be expressed as a single transformation. In general, two successive Lorentz transformations are written as the product of the Lorentz transformation and the rotation and hence, the boost, in this case, cannot be like a simple form as the one (17.5) obtained before. The easiest way to get the correct result in the non relativistic limit is the minimal coupling with the gauge field as is mentioned before. The gauge field $\mathcal{A}^\mu$ for the rotating frame is, now, defined by

$$\mathcal{A}^\mu(x^\mu) = (\mathcal{A}_0(\boldsymbol{x}), \boldsymbol{\mathcal{A}}(\boldsymbol{x})) = (-\frac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{x})^2, \boldsymbol{\Omega} \times \boldsymbol{x}). \tag{17.18}$$

And we obtain the Hamiltonian for a particle at rest with respect to the rotating frame F' as

$$H = \frac{1}{2m}(\hat{\boldsymbol{p}} - m\boldsymbol{\Omega} \times \boldsymbol{x})^2 - \frac{1}{2}m(\boldsymbol{\Omega} \times \boldsymbol{x})^2, \tag{17.19}$$

where we drop primes under the understanding. Notice that one can obtain the semiclassical equation of motion for the expectation value using the Heisenberg equation of motion, i.e.

$$m\frac{d^2\langle \boldsymbol{x} \rangle}{dt^2} = 2m\frac{d\langle \boldsymbol{x} \rangle}{dt} \times \boldsymbol{\Omega} + m\boldsymbol{\Omega} \times (\langle \boldsymbol{x} \rangle \times \boldsymbol{\Omega}) \tag{17.20}$$

which recover the Coriolis force and the centrifugal force correctly.

In order to take into account the spin of the particle (we consider the neutron, namely spin $\frac{1}{2}$ particle here), we need to realize the fact that the spin in F' rotates with the angular velocity $-\boldsymbol{\Omega}$ relative to the inertial frame F_0. So the interaction between the spin and the rotation is simply the same as the Thomas precession [7, 8], therefore the Hamiltoninan is obtained by adding the spin interaction term to (17.19),

$$H = \frac{1}{2m}(\hat{\boldsymbol{p}} - m\boldsymbol{\Omega} \times \boldsymbol{x})^2 - \frac{1}{2}m(\boldsymbol{\Omega} \times \boldsymbol{x})^2 - \boldsymbol{\Omega} \cdot \hat{\boldsymbol{S}}. \tag{17.21}$$

Thus we derive the Sagnac phase shift from (17.21) in the same manner as the Aharonov-Bohm effect [9, 10, 11];

$$\delta\phi_{\text{Sagnac}} = \frac{m}{\hbar}\oint d\boldsymbol{l} \cdot (\boldsymbol{\Omega} \times \boldsymbol{x}) = \frac{2m}{\hbar}\int d\boldsymbol{s} \cdot \boldsymbol{\Omega} = \frac{2m\boldsymbol{A} \cdot \boldsymbol{\Omega}}{\hbar}, \tag{17.22}$$

where $\boldsymbol{A}$ is the orientated area enclosed by the path of the neutron beam. The effect of coupling of the spin to the rotation (the last term in (17.21)) is to act on the initial wave function by the operator [6, 7]

$$\hat{\Phi}_{Spin} = \hat{T}[\exp(\frac{i}{\hbar}\int dt\hat{\boldsymbol{S}} \cdot \boldsymbol{\Omega})], \tag{17.23}$$

where $\hat{T}$ represents the time ordering operator. For the uniform rotational frame it is reduced to

$$\hat{\Phi}_{Spin} = \exp(\frac{i}{\hbar}\hat{\boldsymbol{S}} \cdot \boldsymbol{\Omega} t) = \hat{I}\cos(\frac{\Omega t}{2}) + i\frac{\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\Omega}}{|\boldsymbol{\Omega}|}\sin(\frac{\Omega t}{2}), \tag{17.24}$$

where $\hat{\boldsymbol{\sigma}} = (\sigma_x \ \sigma_y \ \sigma_z)$ are the Pauli spin matrices. This phase shift can be observed in nuclear or molecular beam resonance methods.

4. Relativistic Aspects of the Rotating Frame

In this section we shall discuss the relativistic aspects of the rotating frame in quantum mechanics, using the Dirac equation with a spin connection;

$$(i\gamma^{\mu}\nabla_{\mu} - m)\psi = 0 \tag{17.25}$$

$$\nabla_{\mu} = \partial_{\mu} - \frac{i}{4}\Gamma^{ab}_{\ \ \mu}M_{ab}, \ M^{ab} = \frac{i}{2}[\gamma^a, \gamma^b], \tag{17.26}$$

where we summarize conventions and notations in the appendix. The metric in a uniform rotating frame is

$$g_{00} = 1 - (\boldsymbol{\Omega} \times \boldsymbol{x})^2, \ g_{ii} = -1, \ g_{0i} = -(\boldsymbol{\Omega} \times \boldsymbol{x})^i, (i = 1, 2, 3), \tag{17.27}$$

and $g_{\mu\nu} = 0$ otherwise [14], then the problem becomes the simpler and solvable in the low energy limit. Using the properties of gamma matrices and the vierbein in the appendix, rewriting the spinors as $\psi \rightarrow e^{-imt}\psi$, and neglecting terms of order v^2/c^2, where v is the velocity *relative to the rotating frame* , we obtained the low energy limit of the Dirac equation (17.25) as

$$[\gamma^0(m + \hat{p}_0 - m\mathcal{A}_0 - \frac{1}{2}\mathcal{A}^i\hat{p}_i) + \gamma^i(\hat{p}_i - \frac{1}{2}m\mathcal{A}_i - \frac{i}{2}\mathcal{E}_i) - m]\psi = 0, \tag{17.28}$$

where $\boldsymbol{\mathcal{E}}$ is an analog of the electric field and is defined by

$$\boldsymbol{\mathcal{E}} = -\frac{1}{2}\boldsymbol{\nabla} h_{00} = -\boldsymbol{\nabla}\mathcal{A}_0. \tag{17.29}$$

Then, using the usual splitting of the four spinors into upper and lower components the Hamiltonian for the upper component two spinors in the low energy limit is

$$H = \frac{1}{2m}(\hat{\boldsymbol{p}} - m\boldsymbol{\mathcal{A}} - \hat{\boldsymbol{S}} \times \boldsymbol{\mathcal{E}})^2 + m\mathcal{A}_0 - \boldsymbol{\omega} \cdot \hat{\boldsymbol{S}}. \tag{17.30}$$

There is an additional term $(-\frac{1}{8m}\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}})$ which is an analogous to the Darwin term in the electromagnetic field case. In the present case this term is $-\frac{3\Omega^2}{8mc^2}$

which is a constant, therefore it can be subtracted away from the Hamiltonian (17.30). In the rotating frame $\boldsymbol{\mathcal{E}} = \Omega^2 \boldsymbol{x}/c^2$ so if we now neglect $\hat{\boldsymbol{S}} \times \boldsymbol{\mathcal{E}}$ term which is of order c^{-2} then we obtain the Hamiltonian (17.21) [16]. We obtain not only the phase shifts discussed in section 3, but also we get a new phase shift that is calculated by acting on the initial wave function by the following operator,

$$\hat{\Phi} = P[\exp(\frac{i}{\hbar} \oint d\boldsymbol{l} \cdot (\hat{\boldsymbol{S}} \times \boldsymbol{\mathcal{E}}))] \tag{17.31}$$

where P denotes the path ordering. As a special ideal case we consider, for a simplicity, a circular path. Then

$$\hat{\Phi} = \exp(\frac{i2\Omega^2}{\hbar c^2} \boldsymbol{A} \cdot \hat{\boldsymbol{S}}). \tag{17.32}$$

Moreover, if the spin is polarized perpendicular to the plane of the interferometry, the phase shift $\delta\phi$ due to this operator is,

$$\delta\phi = \frac{\Omega^2 A}{c^2}. \tag{17.33}$$

This phase shift is analogous to the phase shift due to the electric field in neutron interferometry found by Anandan [17], Aharonov and Casher [18]. Although this phase shift is very small compared to the dominant Sagnac term, it is interesting because it is due to a new spin-orbit coupling, and we hope that it would be experimentally tested in the future.

5. Conclusion

We have shown the significance of boosts in the treatment of the rotating frame in quantum mechanics. And also it is suggested [8, 9, 10, 11, 12] that the rotating frame is considered like a gauge field in the non relativistic limit. However, as we discussed in section 4, it should be remarked that there do exist several differences between the gauge field of the rotating frame and the electromagnetic field in the relativistic region [8, 12, 16]. Nevertheless, as we have seen, the rotating frame can be treated consistently within the usual framework of quantum mechanics, and it is shown that phase shifts to the first order are obtained *without any new hypothesis*.

Moreover, we directly obtained the Hamiltonian (17.30) for the uniform rotating frame from the Dirac equation with the spin connection in the low energy limit, which is analogous to the Hamiltonian of the magnetic dipole in the electric field [17], which gives rise to the Aharonov-Casher effect [18]. An analogous Hamiltonian ((3.18) in [8]) in a gravitational field was obtained by one of us by considering the parallel transport of the wave function [19]. We have in fact extended this Hamiltonian to the rotating frame. We shall discuss the further in a future paper.

Acknowledgments

This work was supported in part by an NSF grant and an ONR grant.

Appendix: Conventions and Notations

The metric is written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu} = \mathrm{diag}(+ - --)$. Both indices μ, ν and a, b run over $0, 1, 2, 3$, on the other hand i, j, k run over $1, 2, 3$. The vierbein $e^{\mu}_{\ a}$ and its inverse $e_{\mu}^{\ a}$ at each point which satisfy $g_{\mu\nu} = \eta_{ab} e_{\mu}^{\ a} e_{\nu}^{\ b}$ and $e_{\mu}^{\ a} e^{\mu}_{\ b} = \delta^{a}_{\ b}$. And they are used to connect latin indices and greek indices, for instance, $\gamma^{a} = e_{\mu}^{\ a}\gamma^{\mu}$ where γ^{a} and γ^{μ} satisfy $\{\gamma^{a}, \gamma^{b}\} = 2\eta^{ab}$ and $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$ respectively. The Minkowski metric η_{ab} and its inverse are used to lower and raise latin indices. In the weak field limit, the vierbein and its inverse can be written as $e^{\mu}_{\ a} = \delta^{\mu}_{\ a} - \frac{1}{2}h^{\mu}_{\ a}$ and $e^{a}_{\ \mu} = \delta^{a}_{\ \mu} + \frac{1}{2}h^{a}_{\ \mu}$, and we can check them to satisfy above properties to first order. $\Gamma^{ab}_{\ \ \mu}$ in (17.26) are the Ricci rotation coefficients, and in the weak field limit $\Gamma_{ab\mu} = \frac{1}{2}(\partial_a h_{\mu b} - \partial_b h_{\mu a}) = -\Gamma_{ba\mu}$.

References

[1] Landau L.D., Lifshitz E.M., and Pitaevskii L.P., *Statistical Physics (Course of Theoretical Physics, Volume 5)*, Butterworth-Heinemann; 3rd edition, (1999)

[2] Anandan J., *Phys. Rev. D*, **15**, 1448 (1977).

[3] Werner S. A., Staudenmann J-L., and Colella R. , *Phys. Rev. Lett.* **42**, 1103 (1979); Staudenmann J-L., Werner S. A., Colella R., and Overhauser A. W., *Phys. Rev. A*,**21**, 1419 (1980). See also S. A. Werner in the Proceedings of the Conference on "New Techniques and Ideas in Quantum Measurement Theory", New York, January 21-24, 1986, Annals of New York Academy of Sciences, **480**, 147 (1986) edited by D. Greenberger regarding the origin of this idea and some other ideas in neutron interferometry.

[4] Rauch H. and Werner S.A., *Neutron Inerferometry: Lessons in Experimental Quantum Mechanics*, Oxford (2000).

[5] See [4], [2], [12], and references therein.

[6] Mashhoon B., *Phys. Rev. Lett.*, **61**, 2639 (1988); *Phys. Rev. Lett.*, **68**, 3812 (1992).

[7] Anandan J., *Phys. Rev. Lett.*, **68**, 3809, (1992).

[8] Anandan J., in *QUANTUM COHERENCE AND REALITY; In Celebration of the 60th Birthday of Yakir Aharonov*, World Scientific, Singapore, (1994), gr-qc/9504002.

[9] The relationship between a rotating frame and an electromagnetic field was studied by Aharonov and Carmi before ref. [11]. Aharonov Y. and Carmi G., *Found. Phys.*, **3**, 493 (1973).

[10] Harris J.H. and Semon M.D., *Found. Phys.*, **10**, 151 (1980); Semon M.D., *Found. Phys.*, **12**, 49 (1982)

[11] Sakurai J.J., *Phys. Rev. D*, **21**, 2993 (1980).

[12] Anandan J., *Phys. Rev. D*, **24**, 338 (1981).

[13] Anandan J., *Nuov. Cim.*, **A 53**, 221 (1979); Anandan J. and Lesche B., *Lett. Nuov. Cim.*, **37**, 391 (1983).

[14] Landau L.D. and Lifshitz E.M., *The Classical Theory of Fields (Course of Theoretical Physics, Volume 2)*, Butterworth-Heinemann; 4th edition, (1997)

[15] Berestetskii V.B., Lifshitz E.M., and Pitaevskii L.P., *Quantum Electrodynamics (Course of Theoretical Physics, Volume 4)*, Butterworth-Heinemann; 2nd edition, (1999)

[16] It should be remarked that if we use the Pauli equation instead of the Dirac equation (17.28) and plug (17.18) into it, then we obtain $i\frac{\partial}{\partial t}\psi = \{\frac{1}{2m}(\hat{\boldsymbol{p}} - m\boldsymbol{\mathcal{A}})^2 + m\mathcal{A}_0 - 2\boldsymbol{\Omega}\cdot\hat{\boldsymbol{S}}\}\psi$ which has the twice factor in front of the spin term.

[17] Anandan J., *Phys. Rev. Lett.*, **48**, 1660, (1982) and *Phys. Lett. A*, **138**, 347 (1989).

[18] Aharonov Y. and Casher A., *Phys. Rev. Lett.*, **53**, 319 (1984).

[19] See section 3 in ref. [8].

Note that the gauge used in this reference is different from ours, which explains the difference between the Hamiltonian ((3.18) in [8]) and our Hamiltonian (17.30). The former gauge in the absence of gravity, as in the present case, would pick out an inertial Minkowskian coordinates system, which is different from the rotating coordinates system used present paper. But it is straightforward to obtain the Hamiltonian for our case by using the method described there.

Jeeva Anandan (1948-2003)

The distinguished physicist Professor Jeeva Satchith Anandan passed away in the morning on 29 July 2003 in Columbia, South Carolina.

Born in Sri Lanka on 10 June 1948, Prof. Anandan graduated from the University of Colombo in 1970 with a first class honors BS degree and received his PhD from the University of Pittsburgh in 1978. Working on his

PhD under Ralph Roskies he studied a gauge invariant formulation of the non-Abelian gauge theories using Wilson loops. At the same time he also worked on and gave a particularly elegant theory of the general relativistic Sagnac effect. Thus, while still in graduate school he had already gained recognition in this field. His research on this subject was, from the very early days, based on the application of the geometric methods to the study of quantum phases. Later on this led him to discover, with Yakir Aharonov, the non-adiabatic generalization of the Berry phase, now known as the Aharonov-Anandan phase.

As a postdoctoral fellow at the University of Maryland (1978-1980), the University of California, Berkeley (1980-1982), and the Max-Planck Institute, Munich (1982-1985), he continued to work on this subject and received recognition for his early work on the geometric phase and the Sagnac effect in general relativity. In particular, Prof. Anandan's work with Leo Stodolsky on quantum interference effects for charged particles in the context of general relativity was the basis for his essay which won first prize in the 1983 Gravity Research Foundation competition.

After a CNRS fellowship in Institut Henri Poincare, Paris (1985), Prof. Anandan became an Associate Professor (1986-1990) and a Professor in 1990 at the University of South Carolina (USC), where he spent the rest of his life. At USC, he was a member of the Foundations of Quantum Theory group and contributed significantly to this field. He was well known as one of pioneers of the geometric phase in quantum mechanics.

Prof. Anandan had profound insight in various areas of physics, mathematics, and philosophy as is evident from his many original works. Recently he had developed a new viewpoint on physics based on symmetries and the relational reality. He was also recognized as a philosopher of science and was awarded a DPhil in philosophy by the University of Oxford in 1997.

Prof. Anandan would ask the most fundamental questions in physics, and he had devoted his life to find answers to these questions. The spirit of his inquisitive intellect will remain with us.

After his sudden death, Yakir Aharonov wrote this reminiscence:

> It is hard for me to believe that Jeeva is no longer with us. We have been close collaborators and friends for many years. Jeeva was a deep thinker in both physics and philosophy. He had a very broad knowledge in many scientific and non-scientific subjects. He was also a dedicated and talented teacher. In recent years he had been developing a new approach to physics that demonstrated how bold and innovative his ideas were. It is indeed a great pity that he was taken from us at such an early stage of his life. I will greatly miss him.

Jun Suzuki

Chapter 18

ON ROTATING SPACETIMES

Fernando de Felice
Dipartimento di Fisica, Università di Padova, I–35131 Padova, Italy
I.N.F.N. Sezione di Padova, I–35131 Padova, Italy

Abstract I will outline the basic features of a rotating spacetime considering first the elementary measurements of time intervals and spatial lengths then deducing the properties of the radial motion. After a description of timelike geodesics I will deduce the light cone structure and show how the light behaves as one approaches the light cylinder.

1. Introduction

The spacetime seen by a rotating observer is very different from what would be seen in an inertial frame. Inertial fields act differentiably generating effects which can constrain physical laws in a significant way and closely resemble general relativistic situations. To illustrate that I first specify the metric on a rigidly rotating disk, then investigate the radial motion in the plane of the disk and calculate the acceleration needed to force a particle into a strictly radial motion. This problem may be relevant for example in pulsar's electrodynamics.

After a study of the timelike geodesics in the plane of the disk I analyze the null geodesics and show how the light cone *deforms* as one approaches the light cylinder limit.

In the paper, Greek indices run from 0 to 3 and units are such that $c = 1$, c being the velocity of light in vacuum; semicolon denotes covariant derivative with respect to the given metric and square brackets as $[\cdots]$ mean antisymmetrization.

G. Rizzi and M.L. Ruggiero (eds.), Relativity in Rotating Frames, 371-379.

2. A rotating spacetime

Consider an inertial frame parameterized by Cartesian spatial coordinates; the metric is Minkowski's and reads:

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \tag{18.1}$$

where $\eta \equiv \text{diag}\{-1, +1, +1, +1\}$, and $\{x^\alpha\} \equiv (t, x, y, z)$. Under the following coordinate transformation:

$$\begin{aligned} t' &= t \\ x' &= x\cos\omega t + y\sin\omega t \\ y' &= -x\sin\omega t + y\cos\omega t \\ z' &= z \end{aligned} \tag{18.2}$$

where ω is a constant, metric (18.1) becomes:

$$\begin{aligned} ds^2 &= -\left[1 - \omega^2({x'}^2 + {y'}^2)\right] {dt'}^2 + 2\omega(x'dy' - y'dx')dt' \\ &+ {dx'}^2 + {dy'}^2 + {dz'}^2. \end{aligned} \tag{18.3}$$

Performing a further transformation:

$$\begin{aligned} x' &= r\cos\phi \\ y' &= r\sin\phi, \end{aligned} \tag{18.4}$$

we obtain the space-time in a rotating frame and in cylindrical coordinates $\{\tilde{x}^\alpha\} \equiv (t, r, \phi, z')$, namely:

$$ds^2 = -(1 - \omega^2 r^2)dt^2 + 2\omega r^2 dt d\phi + dr^2 + r^2 d\phi^2 + {dz'}^2 \tag{18.5}$$

the inverse metric being:

$$\begin{aligned} \left(\frac{\partial}{\partial s}\right)^2 &= -\left(\frac{\partial}{\partial t}\right)^2 + 2\omega\frac{\partial}{\partial t}\frac{\partial}{\partial \phi} + \frac{1 - \omega^2 r^2}{r^2}\left(\frac{\partial}{\partial \phi}\right)^2 \\ &+ \left(\frac{\partial}{\partial r}\right)^2 + \left(\frac{\partial}{\partial z'}\right)^2. \end{aligned} \tag{18.6}$$

There are some interesting physical observers with respect to whom one can probe dynamics. Metric (18.5) admits two Killing vector fields, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$, corresponding respectively to stationarity and axisymmetry hence it is straightforward to see that the observer $\boldsymbol{\ell}$ with four velocity

$$\ell^\alpha = \eta^\alpha - \omega\xi^\alpha \tag{18.7}$$

where $\eta^\alpha = \delta_0^\alpha$ and $\xi^\alpha = \delta_0^\alpha$, is an inertial observer. It can be defined everywhere on the disk (on the axis ($r = 0$) it would be $\xi = 0$); clearly the observer (18.7) is at rest with respect to the spatial coordinates in space-time (18.1). We then deduce that $\omega = d\phi/dt$ is the angular velocity of the disk with respect to this inertial observer and the coordinate time t is just her/his proper-time.

Another most obvious observer is $\boldsymbol{u}$ which is locally comoving with the disk and has components:

$$u^\alpha = \mathrm{e}^\psi \eta^\alpha \tag{18.8}$$

where $\mathrm{e}^\psi = (1 - \omega^2 r^2)^{-1/2}$. This observer is at rest in spacetime (18.5) and obviously not inertial and can only exist where $\omega r < 1$.

The surface denoted as *light cylinder* where $\omega r = 1$ appears geometrically special since the time Killing vector $\boldsymbol{\eta}$ becomes null on it; in fact $(\eta_\alpha \eta^\alpha)_{\omega r=1} = 0$ from (18.5). The normal to the surface g_{00} =constant is the one-form:

$$n_\alpha = \frac{1}{2}\partial_\alpha g_{00} = \eta^\beta \eta_{\beta;\alpha}, \tag{18.9}$$

where semicolon means covariant derivative with respect to metric (18.5), is orthogonal to $\boldsymbol{\eta}$ hence is space-like everywhere $\boldsymbol{\eta}$ is time-like. On the light cylinder however $\boldsymbol{\eta}$ is null hence $\boldsymbol{n}$ may be null or space-like. We show that it is space-like. The Killing vector field $\boldsymbol{\eta}$ does not satisfy the Frobenious condition [2], in fact:

$$\eta_{[\alpha;\beta}\eta_{\gamma]} = g_{0\phi}\partial_r g_{0\phi} = 2\omega^2 r^3 \neq 0. \tag{18.10}$$

hence from Vishveshwara's theorem (see [3] [2] we deduce that $\boldsymbol{n}$ is not null on the light cylinder hence this surface is not a null surface. This has implications on the light cone structure as we shall later see.

3. Basic measurements

The measurements made by the observers $\boldsymbol{u}$ in their rest frame can be compared to those performed by the inertial observer $\boldsymbol{\ell}$ who instantaneously coincides with $\boldsymbol{u}$ at any given event on the disk. Consider then two events $P = (t, r, \phi, z' = 0)$ and $Q = (t + dt, r, \phi, z' = 0)$ taking place at the same spatial position. The time interval between these two events as measured by the observer $\boldsymbol{u}$ defined in P, say, is:

$$dT_u = -u_\alpha d\tilde{x}^\alpha = \sqrt{1 - \omega^2 r^2}dt. \tag{18.11}$$

Recalling, as stated, that $dt = dT_\ell$, the proper time of the observers at rest on a rotating disk runs slower than that of a locally inertial one. Clearly and as expected the larger is the distance of the static observer from the center of the disk the slower her/his proper time runs. The proper time of $\boldsymbol{u}$ *stops* at the light cylinder where $\omega r = 1$ and this means that the static observers *move* in that limit on a light-like trajectory and therefore they cannot be defined. This behaviour is much similar to what happens on the event horizon in the Schwarzschild spacetime or on the ergosphere in the Kerr metric, just to mention few examples. The behaviour of the proper time of $\boldsymbol{u}$ is partially due to the property of this observer of being not vorticity free since by definition we have:

$$\omega_{\rho\sigma} = h_\rho^\alpha h_\sigma^\beta u_{[\alpha;\beta]} \tag{18.12}$$

$$= (\delta_\rho^\phi \delta_\sigma^r - \delta_\sigma^\phi \delta_\rho^r)\frac{2\mathrm{e}^\psi \omega r}{(1-\omega^2 r^2)}. \tag{18.13}$$

Thus a rotating spacetime cannot be foliated into slices orthogonal to $\boldsymbol{u}$ and the surfaces t =cont. are *not* their rest-space. This implies that clock synchronization on a rotating disk is hardly a trivial problem.

Analogous considerations can be made for measurements of spatial distances. Consider two events $P = (t, r, \phi, z' = 0)$ and $Q = (t + dt, r + dr, \phi + d\phi, z' = 0)$ and valuate the spatial distance between them as measured by $\boldsymbol{u}$ located at P. From the general definition of infinitesimal displacements we have:

$$dL_u = \sqrt{h_{\alpha\beta}dx^\alpha dx^\beta} = \sqrt{dr^2 + \frac{r^2 d\phi^2}{(1-\omega^2 r^2)}}. \tag{18.14}$$

Interesting enough, the observer $\boldsymbol{u}$ will find a longer length with respect to the inertial observer $\boldsymbol{\ell}$ in the ϕ-direction; setting $dr = 0$, in fact, we have:

$$dL_u = \frac{r d\phi}{\sqrt{1-\omega^2 r^2}} \tag{18.15}$$

while she/he will measure no difference from the inertial observer along the radial direction:

$$dL_u = dr \tag{18.16}$$

4. The radial motion

As mentioned, metric form (18.5) describes the space-time from the point of view of a rotating observer. Let us now study the properties of a radial motion on the disk $z' = 0$. The four-velocity $\boldsymbol{v}$ of a test particle constrained to move along a radial rigid pipe is given by:

$$v^\alpha = \mathrm{e}^{\psi'}(\delta_0^\alpha + \beta(r)\delta_r^\alpha) \qquad u^\alpha u_\alpha = -1 \tag{18.17}$$

where $\beta(r) = dr/dt$ is a parameter whose significance will be clarified later and $\mathrm{e}^{\psi'}$ is the normalization factor which reads:

$$e^{\psi} = (1 - \omega^2 r^2 - \beta^2)^{-1/2}; \tag{18.18}$$

the reality condition on ψ' implies $\omega^2 r^2 + \beta^2 \leq 1$. Let us now clarify the physical significance of β and $e^{\psi'}$.

The spatial three-velocity of the test particle $\boldsymbol{v}$ in (18.17) relative to the locally inertial observer ℓ is given by:

$$\tilde{v} = -(v_\alpha \ell^\alpha)^{-1}\sqrt{(v_\alpha \ell^\alpha)^2 - 1} = \sqrt{\omega^2 r^2 + \beta^2}; \tag{18.19}$$

evidently ωr is the azimuthal velocity and β is the radial velocity of the moving particle with respect to $\boldsymbol{\ell}$. Similarly $e^{\psi'}$ just measures the ratio of the proper-times of $\boldsymbol{\ell}$ and $\boldsymbol{v}$ since $e^{\psi'} = -v_\alpha \ell^\alpha$. It is then obvious that if the radially moving particle reaches the light cylinder then necessarily $\beta = 0$.

Let us now consider the behaviour of this particle with respect to the static observers $\boldsymbol{u}$. Now the spatial velocity is locally given by:

$$v' = \pm\beta(r)(1 - \omega^2 r^2)^{-1/2}. \tag{18.20}$$

As expected the local spatial velocity is only radial but we see that $\beta(r)$ has to vanish as $(1 - \omega^2 r^2)^{1/2}$ when $\omega r \to 1$.

The radial motion (18.17) on a rotating disk is not geodesic; the four acceleration:

$$\dot{u}^\alpha \equiv u^\alpha{}_{;\beta} u^\beta \tag{18.21}$$

is easily calculated and reads:

$$\begin{aligned} \dot{u}^\alpha &= e^{4\psi'}\beta\{(\omega^2 r + \beta\partial_r\beta)\delta^\alpha_0 + [2\beta\omega^2 r + (1/\beta)(1 - \omega^2 r^2) \\ &\times (\beta\partial_r\beta - \omega^2 r)]\delta^\alpha_r + (2\omega/r)(1 - \omega^2 r^2 - \beta^2)\delta^\alpha_\phi\}. \end{aligned} \tag{18.22}$$

Since $\dot{u}^\alpha u_\alpha \equiv 0$, then the modulus of (18.21) gives directly the specific thrust which acts on the particle as it would be measured by the particle itself. From (18.21) and (18.5) we have:

$$\begin{aligned} f^2 &= \dot{u}^\alpha \dot{u}_\alpha \\ &= e^{4\psi'}\left[\Theta^2(1 - \omega^2 r^2)e^{2\psi'} - 4\omega^2 r\Theta + 4\omega^2(\beta^2 + \omega^2 r^2)\right] \end{aligned} \tag{18.23}$$

where $\Theta \equiv \omega^2 r + \beta\partial_r\beta$. The quantity $\beta\partial_r\beta = d\beta/dt$ measures the radial acceleration of the particle as seen by the inertial observer, hence the non relativistic limit of (18.22) ($\omega r \ll 1$, $\beta \ll 1$) yields as expected:

$$f^2_{\rm nr} \approx (\omega^2 r - d\beta/dt)^2 + 4\omega^2\beta^2. \tag{18.24}$$

In our case, the Coriolis force is always orthogonal to the radial direction moreover there is no real value of Θ for which $f = 0$ hence radial geodesic motion is not allowed on a rotating disk.

A more transparent interpretation of the specific thrust given by (18.22) is done in terms of a tetrad frame adapted to the radially moving particle. The most natural one is the following:

$$\begin{aligned}
\boldsymbol{\lambda}_{\hat{0}} &= \mathrm{e}^{\psi'}(\boldsymbol{\partial}_0 + \beta\boldsymbol{\partial}_r) \\
\boldsymbol{\lambda}_{\hat{r}} &= \mathrm{e}^{\psi'}\left[\frac{\beta}{(1-\omega^2r^2)^{1/2}}\boldsymbol{\partial}_0 + (1-\omega^2r^2)^{1/2}\boldsymbol{\partial}_r\right] \\
\boldsymbol{\lambda}_{\hat{\phi}} &= \frac{\omega r}{(1-\omega^2r^2)^{1/2}}\boldsymbol{\partial}_0 + \frac{(1-\omega^2r^2)^{1/2}}{r}\boldsymbol{\partial}_\phi \\
\boldsymbol{\lambda}_{\hat{z}'} &= \boldsymbol{\partial}_{z'}.
\end{aligned} \tag{18.25}$$

In terms of these, the non zero tetrad components of the acceleration read:

$$\begin{aligned}
\dot{u}_{\hat{r}} &= \frac{\mathrm{e}^{3\psi'}}{(1-\omega^2r^2)^{1/2}}\left[\left(\frac{d\beta}{dt} - \omega^2r^2\right)(1-\omega^2r^2) + 2\omega^2\beta^2r\right] \\
\dot{u}_{\hat{\phi}} &= \frac{2\omega\beta\mathrm{e}^{2\psi'}}{(1-\omega^2r^2)^{1/2}}.
\end{aligned} \tag{18.26}$$

Clearly $(\dot{u}_{\hat{r}})^2 + (\dot{u}_{\hat{\phi}})^2 = f^2$ as in (18.22). While $\dot{u}_{\hat{\phi}}$ directly describes the balance to the Coriolis force which acts on the particle, $\dot{u}_{\hat{r}}$ contains the parameter $d\beta/dt$ which allows for different types of radial motion. If we require the particle to be at rest, namely $\beta = d\beta/dt = 0$, then $\dot{u}_{\hat{r}}$ gives the balance to the relativistic centrifugal force which reads $\dot{u}_{\hat{r}}|_{\text{c.f.}} = -\omega^2r/(1-\omega^2r^2)$. If we require that the particle moves in the radial direction without physical constraints, then $\dot{u}_{\hat{r}} = 0$. Then:

$$\frac{d\beta}{dt} = -\frac{2\omega^2\beta^2r}{1-\omega^2r^2} + \omega^2r = \omega^2r\left[1 - \frac{2\beta^2}{1-\omega^2r^2}\right]. \tag{18.27}$$

Since $\beta \to 0$ as the particle approaches the light cylinder then we expect that the radial velocity reaches a maximum value at $\beta_{\max} = \sqrt{(1/2)(1-\omega^2r_o^2)}$, r_o being where the radial acceleration vanishes. Somehow the particle *knows* that its total velocity with respect to the inertial observer cannot exceed the velocity of light and so it *also* knows how to balance between the azimuthal and radial components of its velocity.

5. The time-like geodesic motion

Let us now examine the geodesics on the rigidly rotating disk. The two Killing vector fields $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ assure that there exist two constants of motion along the geodesics. Denoting as $\boldsymbol{k}$ the tangent to a time-like geodesic, we have, after standard analysis, the following set of components in the $z' = 0$ plane:

$$k^0 = E + \omega L \tag{18.28}$$

$$k^r = \pm\left[(E + \omega L)^2 - 1 - L^2/r^2\right]^{1/2} \tag{18.29}$$

$$k^\phi = -\omega E + \frac{L}{r^2}(1 - \omega^2 r^2). \tag{18.30}$$

Here E and L are the constants of motion mentioned that describe respectively the particle's total specific energy and its total specific angular momentum relative to the inertial observer $\boldsymbol{\ell}$ on the axis while $E + \omega L$ is the total specific energy relative to $\boldsymbol{\ell}$ anywhere else on the disk. Turning points in the radial direction are found where $k^r = 0$, namely when:

$$E = -\omega L + \frac{1}{\omega r}\sqrt{\omega^2 r^2 + \omega^2 L^2} \equiv E_+. \tag{18.31}$$

Function E_+ decreases monotonically from ∞ at $r = 0$ to $-\omega L + \sqrt{1 + \omega^2 L^2}$ at $\omega r = 1$. Since geodesic motion is allowed only where $E > E_+$ then no *free* particle can reach the center of the disk unless $L = 0$ or $E = \infty$.

The azimuthal orientation of a geodesic orbit, namely the sign of k^ϕ, changes during the motion only if $L > 0$, and that happens when:

$$E = \omega L\left(\frac{1 - \omega^2 r^2}{\omega^2 r^2}\right) \equiv E_\phi. \tag{18.32}$$

E_ϕ is a monotonic decreasing function of r and, since $E_\phi \approx 1/r^2$ as $r \to 0$ while $E_+ \approx 1/r$ in the same limit while $E_\phi \to 0$ as $\omega r \to 1$, then E_ϕ and E_+ always intersect. The locus where $E_\phi = E_+$ is given by the function:

$$\omega L = \frac{\omega^2 r^2}{(1 - \omega^2 r^2)^{1/2}}. \tag{18.33}$$

If particles were constrained to satisfy condition (18.33), then they would move radially on the disk with the acceleration law (18.27) (see [1]).

6. The behaviour of light

A general null geodesic in the space-time (18.5) is described by a null vector $\tilde{\boldsymbol{k}}$ which satisfies the light-like condition

$$\tilde{k}^\alpha \tilde{k}_\alpha = 0. \tag{18.34}$$

From the existence of the Killing vector fields $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ we deduce from (18.34) and (18.5)

$$\begin{aligned}\tilde{k}^0 &= \tilde{E} + \omega\tilde{l} \\ \tilde{k}^\phi &= \frac{\tilde{L}}{r^2} - \omega(\tilde{E} + \omega\tilde{L}) \\ \tilde{k}^r &= \pm\left(\tilde{E}^2 + 2\omega\tilde{E}\tilde{L} - \frac{1-\omega^2 r^2}{r^2}\tilde{L}^2\right)^{1/2}.\end{aligned} \quad (18.35)$$

Here $\tilde{E}$ and $\tilde{L}$ are constant of the motion having the same physical meaning as for time-like geodesics. From the expression of $\tilde{k}^r$ in (18.34) and denoting $\tilde{\mathcal{E}} \equiv \tilde{E} + \omega\tilde{L}$, we deduce that null geodesics are allowed if $\tilde{\mathcal{E}}^2 \geq \tilde{L}^2/r^2$; hence

i) no photon is allowed to leave or reach the axis at $r = 0$ unless $\tilde{L} = 0$;

ii) for any given set of parameters $\tilde{\mathcal{E}}$ and $\tilde{L}$ a photon moving towards the axis meets a turning point at $r = |\tilde{L}/\tilde{\mathcal{E}}|$.

It is now easy to deduce the light-cone structure in space-time (18.5). Solving (18.34) for dt/dr we obtain from (18.5) and (18.34):

$$\left(\frac{dt}{dr}\right)_\pm = \frac{\omega r^2\phi' \mp (\phi'^2 r^2 + 1 - \omega^2 r^2)^{1/2}}{1 - \omega^2 r^2} \quad (18.36)$$

where

$$\phi' \equiv \frac{d\phi}{dr} = \epsilon\frac{\tilde{L} - \omega r^2\tilde{\mathcal{E}}^2}{r^2(\tilde{\mathcal{E}}^2 - \tilde{L}^2/r^2)^{1/2}}, \qquad \epsilon = \pm 1. \quad (18.37)$$

It is clear that $(dt/dr)_+$, being < 0, describes ingoing photons while $(dt/dr)_-$, being > 0, describes outgoing ones. Moreover, from (18.34) we see that $\tilde{k}^\phi \to -\omega\tilde{E}$ as $\omega r \to 1$ so in the neighborhood of the light cylinder the photons are always counter-rotating with respect to the rotating frame. Hence outgoing photons $(dr/dt > 0)$ require $\phi'|_{\omega r\sim 1} < 0\,(\epsilon = +1)$ and ingoing ones $(dr/dt < 0)$ require $\phi'|_{\omega r\sim 1} > 0\,(\epsilon = -1)$. From (18.37), relation (18.36) becomes

$$\left(\frac{dt}{dr}\right)_\pm = \frac{\epsilon\omega\tilde{L}(1-\omega^2 r^2) - \tilde{E}(\epsilon\omega^2 r^2 \pm 1)}{(1-\omega^2 r^2)\sqrt{\tilde{\mathcal{E}}^2 - \tilde{l}^2/r^2}}, \qquad \epsilon = \pm 1. \quad (18.38)$$

At the light cylinder $(\omega r = 1)$ we have the following cases:

a) outgoing photons: $(dt/dr)_- > 0$ and $\phi' < 0\,(\epsilon = +1)$, hence

$$\left(\frac{dt}{dr}\right)_- = \frac{\tilde{\mathcal{E}}}{\sqrt{\tilde{\mathcal{E}}^2 - \omega^2\tilde{L}^2}}; \quad (18.39)$$

b) ingoing photons: $(dt/dr)_+ < 0$ and $\phi' > 0\,(\epsilon = -1)$, hence:

$$\left(\frac{dt}{dr}\right)_{+} \to -\frac{\tilde{\mathcal{E}}}{\sqrt{\tilde{\mathcal{E}}^2 - \omega^2 \tilde{L}^2}}. \tag{18.40}$$

Thus regular ingoing and outgoing photon trajectories exist at the light cylinder. This is consistent with the property of this surface of not being a null surface even if on it the coordinate time axis is a generator of the light cone. On the light cylinder then the light cone is tilted like a flag to the counter-rotating side without being tangent to that surface and this is what happens to the light cone on the ergosphere in the Kerr spacetime solution (see [2]).

7. Conclusions

Inertial forces may cause physical effects which mimic curvature effects as if we were in a gravitational field. In the case of a rotating spacetime an interesting issue is how one has to treat physics at the light cylinder. As stated this has an obvious relevance in pulsars's electrodynamics. An intriguing effect is the vanishing of the radial velocity when a particle constrained to move along a radial direction on the disk approaches the light cylinder. This means that *somewhere* the velocity reaches a maximum. How does the particle recognize that it should do so at the right position?

This and perhaps others issues in a rotating spacetime still need to be fully understood.

References

[1] de Felice F. 1995 *Dynamics on a rotating disk* Phys. Rev. A **52**, 3452

[2] de Felice F. and Clarke C.J.S. 1992 *Relativity on curved manifolds* Cambridge University Press, UK

[3] Vishveshwara C.V. 1968 *Generalization of the Schwarzschild surface to arbitrary static and statioanry metrics* J. Math. Phys. **9**, 1319

III

ROUND TABLE

THE VIRTUAL ROUND TABLE

The following Dialogues are based on the virtual round table that developed on-line, at our web site,[1] after the publication of the drafts of the contributing papers. However, the reader should be aware that these dialogues are not the outcome of an actual round-table discussion; actually, they have been composed by ourselves taking fragments from the papers and from contribution to the web discussion. Even though we did our best to quote accurately the opinions of the authors, and the authors themselves have read and approved the whole round table, we assume full responsibility for the these Dialogues, including the speeches of the Chairman, who coordinated the debate.

Guido Rizzi and Matteo Luca Ruggiero

I. Dialogue on the velocity of light in a rotating frame

Where two challenges pertaining to the one-way velocity of light in a rotating reference frame are issued by Klauber and Selleri. The operational meaning of the one-way velocity of light in a rotating frame and the purported local anisotropy of the propagation of light are examined, and their relevance for the challenges issued are discussed.

Chairman: I am pleased and honoured to be here chairing this debate, where many distinguished scholars are gathered to discuss ”Relativity in Rotating Frames”. Indeed, among these scholars, there are also some scientists whose opinions, somehow "heretical" if compared to the orthodox relativistic approach, will stimulate our discussion, compelling us to re-examine the foundations of the theory of relativity in a critical way. Their seminal contributions will be expedient in this debate, since it is well known that the distinctive feature of rotation, namely its absolute character inherited by Newtonian physics, causes many bewildering problems and paradoxes in relativistic physics.

For instance, Selleri and Klauber propose different approaches to the problem of rotation, but they have the same attitude towards the Special Theory of Relativity (SRT): both accept the theory only in the case of pure translation but they believe that the theory is not self-consistent when applied to rotating ref-

[1] `http://digilander.libero.it/solciclos`

erence frames. So, since the case of pure translation is a somewhat academic abstraction, they force us to re-examine the relativistic paradigm in depth.

To this end, I would like to start this round table by talking about the velocity of light on a rotating platform: Prof. Klauber, would you like to introduce this topic?

Klauber: Yes, of course. First of all, I want to define, from an operational viewpoint, what happens on the rotating platform to an observer who is willing to measure the velocity of light. The observer is at rest on the rim of the rotating platform, and she shines two short pulses of light tangent to the rim in opposite directions. A cylindrical mirror causes these light pulses to travel circular paths around the rim. In many papers of this book (my own included) it is shown that the two pulses return to the observer, after a complete round trip, at different times. For the observer at rest on the disk, both light rays travel the same number of meters around the circumference, but her clock readings tell her that the two pulses took different times to travel the same distance around the circumference, so she concludes that the speed of light as measured on the rotating disk is anisotropic. On the other hand, we know that the speed of light as measured in the laboratory frame S_0 (which is an inertial frame) is isotropic. As a consequence, the first relativity postulate, in the context of the hypothesis of locality, is violated for rotating frames.

Chairman: As I have just said, we shall be forced to critically examine the foundations of relativity, and this is the first challenge to those who believe in its orthodoxy!

Klauber: Let me be clear at the outset that I do not disagree with the SRT and its foundation in geometry. I do disagree with the traditional application of results derived for translation, rather than the direct application of fundamental geometric principles, to rotation. Specifically, in this context, I challenge the advocates of the traditional approach to derive, in a relativistic kinematics context, the result that the two counter-propagating beams come back to the emission point at different times: I challenge you to obtain this result assuming the local isotropy of the velocity of light.

Chairman: The terms of your claim are very clear Prof. Klauber, but I wonder whether Prof. Selleri shares your challenge...

Selleri: Yes I do, of course, and I would like to add another challenge to the traditionalists: let $\tilde{c}\,(0)$ and $\tilde{c}\,(\pi)$ be the light velocities, relative to the disk, for the beam propagating in the direction of the disk rotation and in the opposite direction, respectively. Now, let us consider the ratio $\tilde{c}\,(\pi)\,/\tilde{c}\,(0)$. In my paper

I show that this ratio turns out to be

$$\rho \equiv \frac{\tilde{c}(\pi)}{\tilde{c}(0)} = \frac{1 + \beta}{1 - \beta} \tag{1}$$

which expresses the anisotropy of the propagation of light along the rim, for any non vanishing value of the parameter $\beta \equiv \omega R/c$, where ω is the angular velocity of the rotating disk, and R is its radius.

Chairman: In order to be precise and as clear as possible, I would like to specify that in your demonstration $\tilde{c}\,(\pi)\,/\tilde{c}\,(0)$ is the ratio of the two *global* velocities for complete round trips around the platform; as a consequence, eq. (1) does not challenge the SRT because the rotating platform is not an inertial frame (IF).

Selleri: Well, you are right; however I want to stress that eq. (1) gives us not only the ratio of the two global light velocities, but *the ratio of the local instantaneous velocities as well.* In fact the isotropy of the IF S_0 ensures, by symmetry, that the instantaneous velocities of light are the same in all points of the rim of the rotating disk whose centre is at rest in S_0.

Chairman: This is a crucial issue that we shall discuss in depth later. Please go on Prof. Selleri.

Selleri: The result (1) holds with the same numerical value for platforms having different radius R and different angular velocity ω, but the same peripheral velocity $v \equiv \omega R$, that is, the same value of β. Then eq. (1) also applies to a platform with very large radius R and very small angular velocity ω, provided that $\beta \equiv \omega R/c$ is unchanged. In this case, a small part AB of the rim of a platform is completely equivalent (for a short time) to a small part of a local co-moving inertial reference frame (LCIF), endowed with the same velocity β. Let me stress that this "small part of the rim" approximates an IF better and better with increasing radius (and decreasing angular velocity). But the velocities of light in the two directions AB and BA must obey (1): it follows that the one-way velocity of light relative to the LCIF cannot be c, as required by the SRT.

Chairman: Your words sound intriguing. I would just comment that, if your argument works, there is no need to increase the radius and decrease the angular velocity: according to the hypothesis of locality, a LCIF must behave as an inertial frame regardless of the numerical values of R and ω, provided that the "small part AB of the rim" is small enough. So, I propose to simplify your argument disregarding the unnecessary limit $R \to \infty$, $\omega \to 0$, which, in my opinion, is just a rhetorical device.

Selleri: Maybe... but, according to me, this "rhetorical device" has a strong powers of persuasion! Anyway, the SRT predicts for ρ a discontinuity at zero acceleration. This is the crucial point. All the experiments are performed in the real physical world, where acceleration can be as small as we like, but not exactly null; in these cases the ratio ρ is always given by eq. (1). On the other hand, the prediction of the SRT is $\rho = 1$ when acceleration is exactly null (a case of purely academic interest, of course). Thus the SRT leads to an unacceptable discontinuity in the ratio ρ as a function of acceleration: an incontrovertible proof that the SRT has gone out of the physical world!

Chairman: You are saying that the SRT can be true only in an abstract platonic world, but not in the real physical world. This is puzzling and somewhat unpleasant. Actually, according to all textbooks, the SRT rests on the principle of relativity and on the principle of constancy of velocity of light, which states that the velocity of light is the same in all inertial systems. This means that any global or local inertial reference frame should be optically isotropic.

Selleri: My paper shows that among all inertial systems S there is only one isotropic system, that is S_0. One can easily recognize this fact. Actually, for every small region AB of each system S, it is possible to imagine a large rotating platform with center at rest in S_0 and rim locally co-moving with AB; then the previous considerations hold. As a consequence, the velocity of light is direction-dependent in all inertial systems, with the sole exception of the privileged one S_0.

Chairman: Just a moment, Prof. Selleri. You are talking about the one-way velocity of light, but this is a rather subtle issue. Actually, I read some papers of yours,[2], and also a related paper written by Rizzi and Serafini,[3] where you deal with the problem of the one-way velocity of light: there you maintain that the one-way velocity of light is not a measurable physical quantity (I suspect that probably you prefer a different terminology, but this is not important for me). Rather, you claim that the only measurable physical quantity is the two-ways velocity of light, which is c in any inertial frame. However, even if your arguments are rather compelling, I am not completely convinced: in fact, if things are this way, what about the principle of constancy of velocity of light? According to your theses, this principle should be expressed this way: "the two-ways velocity of light is the same in all inertial systems, in all direc-

[2] *Giornale di Fisica*, **38**, 67 (1997); *Giornale di Fisica*, **40**, 19 (1999)
[3] *Giornale di Fisica*, **44**, 3, (2003)

tions". My hesitation is due to the fact that, in doing so, we are abandoning the standard formulation of the SRT.

Selleri: There is no reason to hesitate, you would be in good company. Notice that many orthodox scientists (Poincaré, Reichenbach, Jammer, Mansouri-Sexl, Anderson, Stedman, Vetharaniam, Minguzzi, Rizzi, Ruggiero, Serafini and many others) think exactly the same. Let ask Serafini for details: I know he is very interested in this subject...

Serafini: Yes, I am! To start with, I would like to point out that the two-ways velocity of light is just a particular case of the round-trip velocity of light. In the paper written with Rizzi, I showed that the correct formulation of the principle of constancy of velocity of light should be: "the round-trip velocity of light is the same in all inertial systems for any round-trip". Viceversa, if the round-trip velocity of light is the same for any round-trip, in a given reference frame, the frame is inertial.

Chairman: Dr. Serafini, I have to point out that this is not the standard formulation of the principle of constancy of velocity of light...

Serafini: I am firmly convinced that this is the only operationally meaningful formulation of the principle of constancy of velocity of light. The standard formulation rests on a well defined synchrony choice which, in principle, can be freely chosen inside a suitable synchronization gauge. Selleri himself suggested a useful synchronization gauge, which we called ”Selleri gauge”.

Chairman: We shall discuss the role of the synchronization gauges later, Dr. Serafini. Now, I think that, for sake of clarity, it would be better to go on with Selleri. My question is: Prof. Selleri, if you claim that the one-way velocity of light is not a measurable physical quantity, how can your statement ”the velocity of light depends on direction in all inertial systems” (which of course refers to the one-way velocity of light) be a physical, i.e. testable, statement?

Selleri: The one-way velocity of light is not a measurable physical quantity if we confine ourselves to the class of inertial systems. In this case, and only in this case, the synchronization can be arbitrarily chosen. In particular, I showed that a suitable parameter e_1 can be introduced to allow for different synchronizations in the transformations of the space and time variables. The SRT, in its standard formulation, is obtained for a particular non-zero value of e_1. On the other hand, if accelerations (in particular rotations) are taken into account, the choice $e_1 = 0$ is the only one allowing the removal of the ”unacceptable discontinuity in the ratio ρ at zero acceleration”, which I have spoken of before. As a consequence, synchronization is somehow fixed by nature itself. Do you ask for an empirical trial of what I am saying? The best trial I can conceive of is the Sagnac effect!

Chairman: Well, according to you and also to Klauber, the Sagnac effect can be thought of as a ”subversive” tool against the SRT: we all know that often, in the past, it was used as a disproval of the theory.

Anyway, let me highlight that, in their speech, both Serafini and Selleri have introduced an important topic: the synchronization.

Serafini: Here it is: that is the issue! Actually, the first topic of the round table (the velocity of light on the rim of a rotating platform) is so entangled with the second one (the synchronization) that, in my opinion, the two topics should be discussed together.

Chairman: Maybe you are right, Dr. Serafini: I know that Rizzi, Ruggiero and yourself are deeply aware of the links among synchronization, velocity of light and Sagnac, and we shall discuss all these things in great detail later. For the time being, I would like to go on with the challenges of Klauber and Selleri: I believe that they deserve some replies. Prof. Sorge, would you like to start?

Sorge: Yes, of course. I shall try to sum up my viewpoint about some possible optical anisotropies consistent with the standard SRT. As pointed out by Rizzi and Tartaglia,[4] the anisotropy in the *measured* speed of light is not something heretical in the background of the SRT, if it is just a *global* effect in some non-inertial frame. The speed of light, c, appears in the Maxwell equations as a *constant* quantity, and the invariance of Maxwell equations with respect to Lorentz transformations is obviously related to such constant value. What is *measured* in any experiment *is not* the value of the universal constant c: all that we may infer from an experimental measurement is merely the speed of propagation of an electromagnetic field (or photons) in the observer’s reference frame.

Chairman: You are talking about the one-way velocity of light, aren’t you?

Sorge: Yes of course. Now, let me introduce the most important point. Any experimental apparatus has a typical size; moreover, any experiment has

[4] *Found. Phys. Lett.*, **12**, 179 (1999)

a typical time duration. As a consequence, any physical measurement is non-local. So, possible anisotropies in the observed speed of light (photons) may stem from a (non-local) measurement performed in a non-inertial reference frame. Such anisotropies simply reflect the inadequacy of the observer, who - so to say - "was changing" during the measurement. I stress that the result of a non-local measurement is not necessarily the constant c: c would be the result of an hypothetical local measurement. Since - strictly speaking - local measurements do not exist, for a non-inertial observer the result of a measurement turns to be *observer-dependent* (actually, it also encodes information about the observer's acceleration). Consequently, the *measured* value $c_{\rm meas}$ will usually differ from the universal constant c. Such a difference will depend on a parameter, say Δ, comparing the typical size of the experimental apparatus, L, with the length-scale Λ describing the non-inertial features of the employed reference frame. We may write:

$$c = \lim_{\Delta \to 0} c_{\rm meas}(\Delta) = \lim_{\Delta \to 0} [c + \delta c(\Delta)], \tag{2}$$

where δc represents a (usually small) correction, namely the *anisotropy*, vanishing in the limit $\Delta \to 0$. This may be the case of an *inertial* observer (in this case the length-scale Λ becomes infinite), or the case of an *ideal* local measurement ($L = 0$); so we look for something like $\Delta \propto L/\Lambda$. Since, as we pointed out above, any physical measurement is unavoidably *non-local*, non-inertial effects are expected to appear through small corrections affecting the universal constant value c. So, the experimental detection of any anisotropy in the speed of light is not in conflict with the foundations of the SRT: accordingly to the locality principle, it appears as a consequence of the unavoidable non-locality of the physical measurement. In this respect, any speculation (such as the claims of Klauber and Selleri) intended to introduce *local* anisotropies in the value of c are intrinsically inconsistent, in the context of the SRT.

Chairman: Thank you, Prof. Sorge. It seems to me that your approach is somewhat similar to Nikolić's.

Sorge: Well, there are actually some analogies, both in the mathematical approach and in some results. However the velocity of light is not the main issue of Nikolić's paper.

Chairman: Yes, you are right, and we shall discuss some issues of Nikolić's paper later. Now, I would like to continue the discussion about one-way velocity of light and synchronization. Prof. Weber, would you answer to Klauber and Selleri?

Weber: Actually, I did it already in my paper. In this round table, I am going to explain why, according to me, Klauber's challenge makes no sense: in brief, it is equivalent to demanding that two events are simultaneous when observed from different inertial frames. We know this is impossible in special relativity. Klauber is taking this observation from the inertial system in which the center of the disk is at rest to be an absolute. Let me show that using Einstein synchronization on the rotating disk, the pulses arrive back at the emission point at the same time. To see this use

$$ct^* = ct - \beta\gamma^2 r^* \varphi^*, \tag{3}$$

where t^* is the Einstein-synchronized time for a fixed radius r^* and t is the continuous coordinate time of the inertial frame and φ^* is the usual azimuthal angle. Then one gets

$$\gamma r^* \Delta\varphi^* / \Delta\tau^* = \pm c, \tag{4}$$

for the velocity of light constrained to move along the periphery of the disk of radius r^*. Here $\Delta\tau^*$ is $\Delta t^*/\gamma$. Note that this velocity applies to finite displacements.

Chairman: Excuse me, Prof. Weber, it seems to me that your formalism could sound unclear for those who are not familiar with it. For instance, eq. (3) looks like the special Lorentz transformation for time, but it is not exactly what one would expect. I would like all the listeners to be able to follow the discussion without too many technical difficulties.

Weber: I admit that the formalism is not so plain, but it is clearly explained in my paper. Actually, eq. (3) is just the special Lorentz transformation for time: but to see this you need some manipulations (Sec. 5 of my paper). Anyway, the formalism is just a tool. The point is that, in either case, using t or t^*, one must obtain the same result for physical measurements such as the number of fringe shifts caused from the rotation. The fringe shift comes about through the discontinuity in t^* for Einstein synchronization or, equivalently, through the difference in arrival time in t.

Klauber: This is not the point. You outline two approaches to obtaining fringe shifts in the Sagnac experiment, one with the "flash from center simultaneity" time coordinate t and one with the Einstein simultaneity time coordinate t^*. Moreover, you state: "Klauber is taking this observation from the inertial system in which the center of the disk is at rest to be an absolute". But I don't believe this is a correct interpretation of my challenge. So, let me rephrase my challenge. I am not talking about any particular time coordinate that one might use to set up a 4D coordinate grid: I am talking about the physical (standard) clock that is located at the emission point. There is no question that it shows different arrival times for the co-rotating and counter-rotating light pulses. My challenge is to derive the times on that clock by assuming that light speed is locally isotropic. Both pulses travel the same distance at the same speed: prove to the observer fixed at the emission point that these pulses do not arrive at the same time!

Weber: I have understood very well your challenge: deriving the difference in the times of return of the pulses on the clock by assuming that the light speed is *locally* isotropic. Let me try to answer. First of all, I see some problems in your statement. One problem is that physical clocks are usually set to read coordinate time, that is, the time that you decide to use. You use the "flash from the center" simultaneity: this gives a coordinate time on the periphery which is equivalent to the time on the inertial frame. But one of my points is that the elapsed time on the clock will depend on how the clock is programmed: a physical clock will read whatever time you demand. I will assume that Klauber's physical clock reads the continuous time according to his synchronization scheme.

Klauber: In my challenge, no synchronization scheme is needed. I just use a standard clock, say a Cesium atom, and I just let it run. One clock, no specific settings. Just measure the number of beats, i.e., effectively seconds, between arrival and departure on the same clock. There is no synchronization choice at all, and I demand no particular readings for the clock.

Weber: The main problem is that the exact conditions of the challenge cannot be met with the Einstein synchronization, since the time of return of the signals to the emission point must be done on two different clocks. This is due to the discontinuity in the Einstein-synchronized time.

Klauber: Discontinuity in the synchronized time? In operational terms, this means that in Einstein synchronization two different clocks, at the same point, read different times. Moreover, in this approach, we could actually have a plethora of times for a given event! Furthermore, we are free to change our measurement device settings in order to read whatever time we desire! I feel lost: physically, what does this mean?

Rizzi: Could I briefly intervene? I understand Klauber's disorientation, but it seems to me that this is not a correct interpretation of Weber's speech. Klauber is completely right on this point: a ”plethora of labels” for a single event is indeed a nonsense. In fact, the purpose of a coordinate system (a ”chart”) is to label events through a suitable *bijective* mapping. We can do this in infinitely different ways, but not globally: all of us know that any chart has a coordinate domain which in general is *local* (in the sense that it does not cover the whole space-time). On the other hand, until gravitation is not taken into account, a set of charts exists with a coordinate domain which is *global* (in the sense that it does cover the whole Minkowski space-time): we are speaking of the set of the affine charts. But of course no affine chart can be adapted to a rotating frame: in these cases, only local charts are available.

Chairman: As you have just said,”all of us know that”. What do you conclude, then?

Rizzi: The ”plethora of times” which troubles Klauber is quite similar to the ”plethora of azimuthal angles” which arises when the range of azimuthal angle is not properly restricted. Both these ”plethora” come from an improper parameterization: the one-to-one condition is not fulfilled because we have gone out of the coordinate domain of the chart adapted to the rotating frame. This is my first conclusion. My second conclusion is that, if we limit ourselves to parameterize a local domain of space-time, we are free to choose the coordinate time we like, and we can program the reading of the clocks as we like; this is just a matter of pragmatism, and I think Weber is completely right on this point. Do you agree, Prof. Dieks?

Dieks: Of course: the purpose of *coordinates* is to label events unambiguously, which can be done in infinitely many different ways. The choice between these different possibilities is a matter of pragmatism.

Klauber: I am glad that Rizzi agrees with me on some points, but I do insist: I don’t believe any theory can be valid that has, at its roots, a discontinuity in physical time.

Rizzi: Physical time is the proper time of a standard clock, and is fundamentally local (in the sense that it is associated with a word-line in space-time). Yet the relationship between two or more standard clocks is a matter of convention. When a convention is made, a ”coordinate time” is defined, in a given region of space-time. The discontinuity Weber is talking about does not refer to *physical* time, but to a suitable *coordinate* time, namely the Einstein-synchronized time.

Chairman: This is an important point...

Rizzi: Yes, I agree with you, and I believe that Serafini and Ruggiero would like to intervene about this. So I just emphasize that: (i) a coordinate time is needed in order to parameterize a finite region of space-time; (ii) any definition of ”coordinate time” rests on a suitable definition of synchronization of clocks located at different points of space, which unavoidably introduces some conventional elements. In general, any coordinate time is admissible, provided that it is mathematically and operationally well defined: of course this is the case Weber is speaking about.

Chairman: I was used to teaching my students that every physical frame has its own private space and time; but now Rizzi is teaching me that every physical frame can be endowed with a ”plethora” (thank you for the suggestion, Prof. Klauber) of times. To put it differently, a plethora of ways to spread time over space. Too many times for me, and I have just one watch! By the way: does my watch tick physical or coordinate time?

Rizzi: It depends on its setting, but, in principle, it ticks physical time - with some approximation, of course.

Chairman: Thank you, Prof. Rizzi. If my watch ticks a local physical time, I believe it is time for Prof. Klauber to get an answer to his challenge...

Weber: Here it is! Indeed, Klauber may say all this is nonsense, but the mathematics I am going to use seems compelling. Rizzi says that any coordinate time is admissible, if it is mathematically and operationally well defined. This is just the case in point: one can certainly go along the periphery of the rotating disk , synchronizing clocks according to the standard Einstein convention to obtain the coordinate time t^*.

Klauber: But only the “flash from center simultaneity” time coordinate t allows one to avoid bugs at large!

Rizzi: Right, but it is not the point. Although the expression “Einstein simultaneity” cannot be properly used at large, the procedure outlined by Weber is well defined on the operational ground, and establishes a strict equivalence relation between events in space-time. This is the point.

Weber: Thank you, Prof. Rizzi. Now, distances can be measured along the periphery using the speed of light c and the time of propagation in intervals of t^*. The difference in arrival times of the pulses referred to by Klauber is measured in the inertial frame, from which the rotation is observed. Therefore, one must have a relation between the time t^* and the time t of the inertial frame. This is given by eq. (3), where t is equivalent to the continuous time of Klauber. With the Einstein-synchronized time t^*, the speed is $\pm c$ for light constrained to travel along the periphery of the disk. Now consider light pulses travelling in opposite directions along the whole circumference of the disk. Since the distance is the same and the magnitude of the velocity is c in either directions, the elapsed time Δt^* is the same for both pulses. Then from eq. (3) we have

$$c\Delta t^* = c\Delta t_{cw} + 2\beta\gamma^2 r^* \pi \tag{5}$$

for light travelling clockwise and

$$c\Delta t^* = c\Delta t_{ccw} - 2\beta\gamma^2 r^* \pi \tag{6}$$

for light travelling counterclockwise. It should be noted that two different clocks reading time t^* are needed to record the end of the trip since it is required to continue the synchronization past the discontinuity onto a new "sheet" of the time function: differently speaking, the time t^* is mutivalued. Equating the two elapsed times gives

$$c(t_{ccw} - t_{cw}) = 4\beta\gamma^2 r^* \pi \tag{7}$$

for the time difference as measured in the inertial frame. This is the time difference as requested by Klauber, and precisely what is measured by an observer in the inertial frame.

Chairman: Thank you very much, Prof. Weber. Your answer to Prof. Klauber seems rather persuasive, but I presume that Prof. Klauber would like to counter-reply.

Klauber: I see that the math of Weber (and also of Rizzi, Ruggiero, Serafini, Dieks...) gives us the correct number. But I repeat: physically, what does it mean? As pointed out in my paper (Sec. 1.7.3), it means that, in a given reference frame and in a given synchronization gauge (in this case, in the Einstein synchrony gauge), we have a plethora of times to choose from for any given event. That is, every event in the same frame, and in the same synchrony gauge, has a multiplicity of clocks sitting at it, depending on the path used for synchronization. This never happens in translation, for any synchronization gauge. But here, there is an infinite number of such clocks, and we seem to be free to simply pick whichever ones we like for a given situation in order to get the answer we insist on having, i.e. an isotropic, invariant light speed. Moreover, we are free to change our measurement device settings to get the result we want. One can ask if this is really physics or not. I am not so arrogant as not to recognize that I may be wrong, but I remain deeply troubled by this methodology.

Chairman: And I must admit that I am deeply troubled by this discussion too, because it seems that both the opposing parties have sound arguments.

This debate is not closed at all, and it is still puzzling and fascinating. Let me stress also that the topic of synchronization, previously ubiquitous in the background, is now claiming our attention. But I wonder, with concern and mistrust, if this (rather philosophical) topic could really help us to solve such a physical conundrum, as Dr. Serafini maintains...

Serafini: Synchronizing clocks in a physical frame is a physical procedure which must be carefully defined on the operational ground. Very useful, for instance, in order to establish an accurate world-wide clock synchronization, as required by the Global Positioning System (GPS). Anything to do with philosophy?

Chairman: Well, perhaps you are right, anyway we shall discuss synchronization (and related topics) in the second Dialogue.

II. Dialogue on synchronization and Sagnac effect

Where the topic of the conventionality of synchronization is debated. The practical relevance of this topic with regard to the Global Positioning System is outlined, and its fundamental role in facing the previously issued challenges and in explaining the Sagnac effect is highlighted.

Chairman: Speaking about synchronization, with regard to the Sagnac effect, we shall be faced with a wide range of different opinions. So, let me introduce this topic by writing down something which is shared by everybody, namely the coordinate transformation

$$t' = t,\ r' = r,\ \varphi' = \varphi - \omega t,\ z' = z \tag{8}$$

where ω is the (constant) angular velocity of the rotating frame, (t, r, φ, z) is a cylindrical chart adapted to the central IF, and (t', r', φ', z') is a cylindrical chart adapted to the rotating frame. As a consequence, all of us agree that in the chart (t', r', φ', z') the line element is:

$$ds^2 = c^2\left(1 - \frac{\omega^2 r'^2}{c^2}\right)dt'^2 - 2\omega r^2 d\varphi' dt' - dr'^2 - r'^2 d\varphi'^2 - dz'^2 \tag{9}$$

Then the general agreement stops: to start with, I see that different authors give different interpretations of the line element (9). Actually, space and time are "fused" in the space-time interval (9) in a way that seems to be author-dependent, as if the separation between space and time were somehow "confused" by the cross term coming from rotation. So, I would like to examine your viewpoints, and I wish you could be as clear as possible, since there are many subtleties which your disagreement depends on. In this Dialogue we shall be concerned with the effects of synchronization in time-measurements; in the next Dialogue we shall consider the link between synchronization and space-measurements.

First of all, let me point out that, according to eq. (8), the clocks at rest on the rotating platform share the same synchronization as the clocks at rest in the central IF. Selleri refers to this synchronization as "absolute synchronization". As a relativist, I do not like the term "absolute", but I have no objections if this term is used merely to emphasize the following crucial fact: the only self-consistent synchronization on the rotating disk, at large, is the (Einstein) synchronization of the central IF. All of us accept this statement. To start our discussion, I would like to ask Prof. Ashby to begin. Indeed, I am afraid someone could think that what we are debating are just philosophical issues: therefore I think that Ashby's approach can show that this is not the case! Actually his interest in this topic is not philosophical but barely physical. His paper deals with the problem of the appropriate setting of the Global Positioning System (GPS), and shows that, in order to synchronize the clocks at rest on the rotating Earth (a very interesting rotating frame!), according to the ”absolute” time t of the central IF, a suitable ”Sagnac correction” is needed.

Selleri: This correction is needed just because on the rotating Earth a wrong velocity of light is assumed. A correct theory does not need any ”corrections” at all.

Chairman: Well, Prof. Selleri, this is your viewpoint, and you will have the chance to express it later: now I want to ask Prof. Ashby to introduce the topic of synchronization, and to explain why it is so important in the GPS.

Ashby: Thank you. Let me point out that, according to me, Selleri intervened rightly. Actually, in the GPS the Sagnac effect arises because the primary reference frame of interest for navigation is the rotating Earth frame, whereas the speed of light is constant in a locally inertial frame, the Earth-Centered IF.

Additional Sagnac-like effects arise because the satellite ephemerides are broadcast in a form allowing the receiver to compute satellite positions in the Earth-Centered, Earth-Fixed (ECEF) frame: but we must transform these positions into the satellite positions in the common Earth-Centered IF, in order to apply the principle of the constancy of c. In the rotating frame, the effect appears to arise from the Coriolis-like term $2\omega r^2 d\varphi' dt'$ in the scalar invariant (9). Whether synchronization procedures are performed by using electromagnetic signals travelling at the invariant velocity c, or slowly moving portable clock, which gives the same synchronization, Sagnac-like corrections are unavoidable.

Chairman: Could you explain what you mean by "Sagnac corrections" in the GPS?

Ashby: You can find everything in my paper. Anyway, I can summarize the key-points. First of all, I want emphasize that the purpose of the GPS is very practical indeed: accurate navigation on (or near) Earth's surface. Of course, most GPS users are interested in knowing their position on Earth; the developers of GPS have therefore adopted an ECEF rotating reference frame as the basis for navigation. However, such a frame does not allow an accurate world-wide clock synchronization. To this end, it is useful to introduce an IF with an axis fixed at the rotation axis of the Earth, which falls freely along with the Earth in the gravitational fields of the other solar system bodies. This is called an Earth-Centered IF (ECI). Clocks in the GPS (more explicitly, the entire system of ground-based and orbiting atomic clocks) are synchronized in the ECI, in which self-consistency can be achieved. In summary, *the reference frame for navigation is the rotating ECEF frame, but clocks are synchronized in the underlying ECI.* The time transformation $t = t'$ in eqs. (8) is a result of the convention to determine time t' in the rotating frame in terms of time in the underlying ECI frame. Now consider a process in which observers in the rotating frame attempt to use Einstein synchronization (that is, the principle of the constancy of the speed of light) to establish a network of synchronized clocks. Light travels along a null world-line, so I may set $ds^2 = 0$ in eq. (9). Keeping only terms of first order in the small parameter $\omega_E r'/c$, the total time required for light to traverse some path turns out to be

$$\int_{\text{path}} dt' = \int_{\text{path}} \frac{d\sigma'}{c} + \frac{2\omega_E}{c^2} \int_{\text{path}} dA'_z. \qquad [\,\text{light}\,] \tag{10}$$

where $d\sigma' \equiv \sqrt{dr'^2 + r'^2 d\varphi'^2 + dz'^2}$ and $dA'_z \equiv r'^2 d\phi'/2$ (for its geometrical interpretation, see my paper).

Chairman: Of course $d\sigma'$ is the space element on the rotating disk . In the next Dialogue, we shall confront different opinions about this topic in full theory; however, I don't expect any objections at first order approximation.

Ashby: Observers fixed on the Earth, who were unaware of Earth rotation, would use just $\int d\sigma'/c$ for synchronizing their clock network. But if rotation is taken into account, the last term in eq. (10) must be considered. This is a Sagnac-like correction term, which is crucial in GPS navigation. From the underlying Earth-Centered IF, this term can be regarded as the additional travel time required by light to catch up to the reference point moving together with the Earth. Therefore, simple-minded use of Einstein synchronization in the rotating frame would lead to a significant error. Eq. (10) can be reinterpreted as a means of realizing coordinate time $t' = t$ in the rotating frame, if - after performing an Einstein synchronization process, by using electromagnetic signals travelling at the invariant velocity c (or slowly moving portable clocks) - appropriate Sagnac-like corrections, of the form $+2\omega_E \int_{path} dA'_z/c^2$, are applied.

Selleri: From my perspective, there are two possibilities of synchronizing clocks on the rotating Earth: the right one and the wrong one. If you use the wrong one (i.e. Einstein synchronization in the rotating frame), you must correct the results with an *ad hoc* term: the so-called Sagnac-like correction. Do you really think this is a logical approach?

Ashby: Yes, if I retain full control of the procedure. Let me point out that the Sagnac corrections can be easily incorporated in the navigation software, which is the ultimate goal of the GPS.

Selleri: My viewpoint about Sagnac-like corrections is that they arise from the anisotropic propagation of light in the Earth rotating frame. These ”corrections” are fully accounted for if this anisotropy is properly taken into account, as I showed in my paper.

Ashby: Your viewpoint is admissible, but it is not the only one. For instance, the Sagnac effect can be regarded as arising from the relativity of simultaneity in a Lorentz transformation to a sequence of local inertial frames co-moving with points on the rotating Earth. As I said before, my viewpoint is that in the underlying Earth-Centered IF the reference clock, from which the synchronization process starts, is moving, requiring light to traverse a different path than it appears to traverse in the rotating ECEF frame.

Selleri: This amounts to saying that light propagates anisotropically in the rotating frame .

Rizzi: Yes, with respect to the time t of the Earth-Centered IF it is certainly so. However, let me point out that the time actually used by Selleri is not the the time t of the Earth-Centered IF, but the time t multiplied by a suitable Lorentz factor. Along a circular path the Lorentz factor is constant: so it has the role of an innocuous scale factor, and Selleri's approach is sound. But along a more general path the Lorentz factor turns out to be dependent on the radius: this could be a problem. Anyway, at least along a circular path, Selleri's approach seems to be consistent with Ashby's.

Chairman: This is surprising: I used to look at Selleri as a ”heretical”, since he rejects the principle of relativity. Anyway, I notice with pleasure this unexpected confluence of ideas. Now, I would like to know something more about Selleri's approach, with regard to synchronization. Prof. Selleri, once the principle of relativity is discarded, what's the end of the relativistic revolution?

Selleri: The end of relativistic revolution is Lorentz ether theory. Formally, the Lorentz transformations must be replaced by the Inertial Transformations (see my paper, Appendix B). In this way, it is easy to show that all inertial systems but one are optically anisotropic: the only one which is isotropic is the central IF S_0. If my theory describes correctly the physical reality, in this system simultaneity and time are not conventional but truly physical. As a consequence, the central inertial frame S_0 should be recognized as the system in which the Lorentz ether is at rest.

Chairman: Thank you, Prof. Selleri. Dr. Serafini, now you can extensively express your viewpoint on this topic, which you have longed for since the beginning of this discussion...

Serafini: Thank you very much. Well, all of us agree with Selleri on this point: if we look for a *global* synchronization on the rotating platform , Selleri's "absolute" synchronization is actually needed. Ashby was very clear on this issue. But my agreement with Selleri stops here. The so-called "mystery of the Sagnac correction" on the Earth surface is solved by Selleri on the basis of the ether hypothesis: the Earth rotates with respect to an IF S_0 with an axis fixed which coincides with the rotation axis of the Earth (Ashby's ECI frame), and S_0 must be recognized as the Lorentz ether at rest. Selleri's calculations are quite correct, but it is easy to realize that this ideological background is untenable. In fact, if we consider that the Earth rotates also around the Sun, we can pick out another IF S_1 which should be recognized as the Lorentz ether at rest. On the other hand, the solar system rotates around the galactic rotation axis, so we can find another "privileged" IF S_2 which should *et cetera*... And, of course, the galaxy in turn rotates and so on! Then we get a collection of privileged inertial frames: if we think carefully, we should admit that any IF S is privileged with respect to any frame rotating around any axis fixed in S.

Chairman: I see. You are saying that, if we admit that a given IF is privileged with respect to the frames rotating around some axis fixed in it, we are forced to conclude that *any* IF is privileged in the same way.

Serafini: Exactly. According to Selleri there is only one optically isotropic IF, but there is not a rule to single out this frame within the class of the IFs: as a matter of fact, *any* IF can play the role of "absolute (optically isotropic) IF". This shows that an arbitrary IF is not physically privileged with respect to the other IFs, but it can be *formally* privileged if an isotropic synchronization (that is Einstein synchronization) is assumed, *by stipulation,* in this IF.

Chairman: What do you mean by "formally privileged"? According to the SRT, within the class of the IFs no one is privileged at all: neither physically, nor formally.

Serafini: As far as the "formally", this holds as long as all IFs are Einstein-synchronized and the Lorentz transformations are used. However, Selleri does not use the Lorentz transformations but the Inertial transformations: in this case different IFs have different *formal* properties. Starting with an Einstein-synchronized IF S_0, any other IF borrows its synchronization from S_0; as a consequence, no IF turns out to be Einstein-synchronized except S_0.

Chairman: As far as I understand, we must choose the Lorentz transformations or the Inertial transformations...

Serafini: Indeed, both are admissible, in the SRT!

Chairman: Both? But, according to Selleri, they are in contention!

Serafini: In the same way as English is in contention with Chinese... In other words, they are just different ways to say the same things. Different conventions.

Chairman: Dr. Serafini, I am afraid your theses are not in agreement with the standard formulation of the SRT...

Serafini: But you yourself admitted that there is something unsatisfactory in the standard formulation of the SRT! Let me explain what I mean. Rizzi and I showed that in a local or global IF the synchronization is not ”given by God”, as some relativists tacitly assume, or ”given by Nature”, as Selleri explicitly states; but it can be arbitrarily chosen within the *synchronization gauge*

$$\begin{cases} t' = t'\,(t,\, x^1, x^2, x^3) \\ x'_i = x_i \end{cases} \tag{11}$$

which is a subset of the set of all possible parameterizations of the given IF. A suitable choice of the first equation in (11) yields the so-called ”Selleri gauge”. Inside this gauge, an admissible synchrony choice is the standard Einstein synchronization, which of course is ”relative”; another synchrony choice gives Selleri's synchronization, which is ”absolute”. In Selleri's formalism, any synchrony choice is determined by the ”synchronization parameter” e_1 mentioned by Selleri himself in the first Dialogue.

Chairman: You are using again the term "absolute", which is suitable for a pre-relativistic framework!

Serafini: I know that the term ”absolute” sounds rather eccentric in a relativistic context, but it simply means that Selleri's simultaneity hypersurfaces (contrary to Einstein's simultaneity hypersurfaces) define a frame-invariant foliation of space-time. The latter is just Einstein's foliation of the given IF assumed to be - by stipulation and once for all - optically isotropic.

Chairman: This is not Selleri's viewpoint: a plausible ether rest frame is not chosen ”by stipulation”!

Serafini: Of course! This is not Selleri's viewpoint: this is a strictly relativistic viewpoint. By the way, I think that Selleri's approach can survive and be useful only in this relativistic interpretation. Actually, Selleri's original viewpoint is simply untenable, unless we consider seriously the possibility that the ether rest frame is the Earth-Centered IF. As a matter of fact, the experiments performed on the Earth surface by Werner *et al.*[5] show that, in Selleri's interpretation, the anisotropy of the one-way speed of light (defined with respect to the time t of the Earth-Centered IF) depends on the velocity of the experimental device with respect to the Earth-Centered IF, not on the its velocity with respect to the Sun-Centered IF, neither on its velocity with respect to the IF where the cosmic microwave background radiation is isotropic. So, Selleri's original viewpoint explicitly requires that the ether rest frame must be the Earth-Centered IF.

Klauber: There is another, related, issue here. Let me first say that, as some contributors might guess, due to a shared disdain for time discontinuity, I have an affinity for Professor Selleri and his theory. I have long felt that he has a potentially viable approach to relativistic rotation. That said, I do have the following question. Consider a beam of relativistic muons in a storage ring, moving with velocity ωR with respect to the lab, and let v_e ($\sim$ 377 km/sec) be the velocity of the lab with respect to the cosmic microwave background, which should be recognized as the ether frame. Then the muon speed relative to the ether frame should vary between $v_{muon} = \omega R - v_e$ and $v_{muon} = \omega R + v_e$ (assuming the ring axis orthogonal to the direction of v_e, and using the classical law of velocities addition, for the sake of simplicity). As a consequence, in the Lorentz factor used by Selleri should appear on v_{muon} instead of ωR. So one would expect a difference in the decay times for the muons in different parts of the storage ring. However, in his paper (Sec. 3) Selleri refers to the 1977 CERN measurements, which showed muon decay time to be independent

[5] *Phys. Rev. Lett.*, **42**, 1103 (1979)

of location in the storage ring.

Chairman: This is a significant and thoughtful argument, I think. Of course this argument works in the same way if v_e is identified with the velocity of the lab with respect to the Earth-Centered IF, or with respect to the Sun-Centered IF, or with respect to anything else IF. So, Klauber's argument shows that Selleri's viewpoint seems to require that the ether rest frame is the lab frame itself; or rather, in more reasonable terms, it seems to require that the ether is dragged by the Earth.

Serafini: But this does not agree with the experiments performed by Werner *et al.* that I mentioned before...

Chairman: I think this could be a problem for Selleri's approach - at least in its original formulation, as you carefully pointed out. Anyway, Dr. Serafini, this is a digression with respect to your crucial point: in principle, you say, any parameterization - in particular, any synchronization - is admissible in any IF. I admit there is nothing strange in this statement. However, I don't like a synchronization which hides the principle of relativity and allows a formal privilege for an IF which is not physically privileged, but merely chosen "by stipulation". This seems to me a convoluted and misleading synchrony choice; I think that the only natural and meaningful synchronization, in the context of the SRT, is Einstein synchronization.

Dieks: I agree. As a matter of fact, it cannot be denied that in the IFs standard simultaneity has a special status: it allows a simple formulation of the laws, conforms to slow clock transport and agrees with Minkowski-orthogonality with respect to world-lines representing the state of rest. So the time coordinates t that correspond to this notion of Einstein's simultaneity (in the sense that $dt = 0$ expresses Einstein's simultaneity) may be said to be privileged. In non-inertial frames this is still so, even though now the argument applies only locally.

Serafini: As far as *global* IFs are concerned, I agree completely with you. Let me remind that the Chairman asked Selleri: "what's the end of relativistic revolution?" My tentative answer could be "all synchronizations are equal, but some synchronizations (Einstein synchronizations, of course) are more equal than others". I'm afraid this is the end of all revolutions...

Chairman: That is a weighty sentence, Dr. Serafini, but it could be a boomerang for you! In fact, it follows from such a sentence that Selleri's "absolute" synchrony choice is admissible, but useless and disadvantageous.

Serafini: Again, I agree as far as *global* IFs are concerned. But when *local* IFs enter the arena, things are quite different.

Chairman: Disadvantageous in a global IF, but perhaps advantageous in a local IF? Please, Dr. Serafini, leave out these philosophical speculations, and come back to physics.

Serafini: But this is physics! Let me clarify our point of view. We showed that the synchrony choice is not compelled by any empiric evidence; that is, also when rotation is taken into account, no physical effect can discriminate Selleri's synchrony choice from Einstein's synchrony choice. As a consequence, we have the opportunity of taking a very pragmatic view: both Selleri's ”absolute” synchronization and Einstein's relative synchronization can be used, depending on the aims and circumstances.

Chairman: Ok, ok, I believe you: but I do not know whether this ”very pragmatic view” you are speaking about is useful to face Klauber's thesis.

Serafini: Indeed, this is useful enough to give an adequate and definitive answer to both Klauber's and Selleri's challenges!

Chairman: Do you really think so? Let us wait and listen to some other speeches. Actually, it is time to give, at least if we can do it, adequate answers to the challenges issued by Profs. Klauber and Selleri. We have already got some answers (Weber, Ashby); so I guess that we just have to collect all answers, and see if they are consistent and sufficient to contradict their claims. Prof. Rizzi would you like to start?

Rizzi: Actually, a significant part of the paper written with Ruggiero could be considered as a direct answer to Klauber. With regard to consistency, our answer is consistent with Weber, Dieks and Ashby. Anyway, I think Ruggiero can condense in a few words this topic.

Chairman: Ok, so let us listen to Dr. Ruggiero.

Ruggiero: Yes, thank you. I am privileged to have the opportunity of explaining our viewpoint about this topic. Let me go back to the beginning of the round table, where the matter of the challenge has been formulated by Klauber in a very sharp way. Klauber deals with light beams, but I will deal with a more general case: I am going to consider both matter and light beams. Let

β, β_+, β_- the dimensionless velocities (with respect to the central IF) of the rim, the co-propagating beam and the counter-propagating beam, respectively. As correctly pointed out by Klauber, the two beams return to the starting point Σ at rest on the rim, after a complete round trip, at different times. On the basis of purely kinematical considerations, we calculated the proper time τ_+ (τ_-) elapsed between the emission and the absorption of the co-propagating (counter-propagating) beam, as read by a clock at rest in Σ. Then, the proper time difference $\Delta\tau \equiv \tau_+ - \tau_-$ turns out to be

$$\Delta\tau = \frac{2\pi\beta}{\omega}\sqrt{1-\beta^2}\frac{\beta_- - 2\beta + \beta_+}{(\beta_+ - \beta)(\beta_- - \beta)} \tag{12}$$

Without specifying any further conditions, the proper time difference (12) depends both on the velocity of rotation of the disk and on the velocities of the beams. Now, let us introduce the velocity of the beams with respect to the rim, namely with respect to a locally co-moving IF (LCIF). As pointed out by Serafini, such a velocity depends on the synchronization of the LCIF, which can be freely chosen within Selleri's gauge. In particular, we consider two synchrony choices: Einstein's synchrony choice and the "absolute" synchrony choice, used by Klauber and Selleri. According to Serafini, both of them can be used, depending on the aims and circumstances: so, let us carefully consider both choices.

Chairman: Ok, let us start with Einstein's synchronization.

Ruggiero: If the LCIF is Einstein-synchronized, eq. (12) takes the form

$$\Delta\tau = \frac{4\pi\beta^2}{\omega}\frac{1}{\sqrt{1-\beta^2}} + \frac{2\pi\beta}{\omega}\frac{1}{\sqrt{1-\beta^2}}\left(\frac{1}{\beta'_+} + \frac{1}{\beta'_-}\right) \tag{13}$$

where $\beta'_\pm$ are the velocities of the co-propagating and counter-propagating beams with respect to the Einstein-synchronized LCIF. Now, let us impose the condition "equal relative velocity in opposite directions":

$$\beta'_+ = -\beta'_- \tag{14}$$

If such a condition is imposed, the proper time difference (13) reduces to the relativistic Sagnac time difference

$$\Delta\tau = \frac{4\pi\beta^2}{\omega}\frac{1}{\sqrt{1-\beta^2}} \tag{15}$$

This shows that eq. (14) is really a vital condition in order to get the Sagnac effect for matter beams. I can summarize this result as follows: the beams take different times - as measured by the clock at rest on the starting-ending point Σ

on the platform - for a complete round trip, depending on their relative velocities $\beta'_\pm$. However, when condition (14) is imposed, the difference $\Delta\tau$ between these times does depend *only* on the angular velocity ω of the disk, and it does not depend on the velocities of propagation of the beams with respect the turnable. The Sagnac time difference applies to any couple of (physical or even mathematical) entities, as long as a velocity, with respect to the turnable, can be consistently defined. In particular, this result applies as well to photons (for which $|\beta'_\pm| = 1$) and to any kind of classical or quantum particles under the given conditions. This fact evidences, in a clear and straightforward way, the universality of the Sagnac effect.

Chairman: And what about the "absolute" synchronization?

Ruggiero: If the LCIF is absolute-synchronized, that is synchronized by means of the central inertial time t, eq. (12) takes the form

$$\Delta\tau = \frac{2\pi\beta}{\omega}\sqrt{1-\beta^2}\,\frac{\beta^r_- + \beta^r_+}{\beta^r_-\beta^r_+} \tag{16}$$

where

$$\beta^r_\pm \equiv \beta_\pm - \beta \tag{17}$$

is the the velocity of the co-propagating (counter-propagating) beam with respect to the absolute-synchronized LCIF. If the vital condition ”equal relative velocity in opposite directions” is expressed by eq.

$$\beta^r_+ = -\beta^r_- \tag{18}$$

instead of eq. (14), it is plain from eq. (16) that no time difference arises: $\Delta\tau = 0$. This calculation shows that the choice of the local Einstein synchronization is crucial even in non-relativistic motion.

Chairman: Excuse me, Dr. Ruggiero, but I wish you could make clear this point: actually, non-relativistic motion simply means classical motion; as a consequence, in this case the relativistic velocity addition law should reduce to the classical addition law (17). What is wrong?

Ruggiero: Nothing is wrong, but this classical approximation, which is the standard way to describe non-relativistic motion, simply cancels the Sagnac effect! Indeed, the Sagnac effect is not a classical effect, as claimed by Sagnac himself (for light beams), but a first order relativistic effect. As a consequence, its experimental detection, also in non-relativistic motion, depends on the first order approximation of the relativistic composition of velocities law [see f.i. eq. (1.5) in Ashby's paper], not on the classical composition law (17). This is true both for matter and light beams, but I admit that my argument is especially compelling for matter beams.

Chairman: In conclusion, what we can learn from the Sagnac effect?

Ruggiero: Well, we can say that the Sagnac effect is an experimental evidence of the SRT, and an experimental disproval of the classical (non-relativistic) ether.

Chairman: Is this a disproval of the Lorentz relativistic ether too?

Ruggiero: No, it is not. Lorentz ether is not a physical entity, but just an ideological assumption. Now, an ideological assumption can be useful or useless (and even misleading), but it cannot be proved or disproved.

Chairman: However, I think it should be possible to translate the condition (14) into a suitable condition for β_+^r, β_-^r.

Ruggiero: Of course it is possible, but it would result in a very artificial and convolute requirement; more specifically, it results in a totally *ad hoc* condition (see eq. (10.31) in our paper). An alternative condition - that is, an alternative synchrony choice - could be sensibly imposed only for light beams in vacuum. Actually, in this case, $\beta_\pm = \pm 1$, so that eq. (12) directly reduces to eq. (15): exploiting this fact, an (apparently) natural synchrony choice can be obtained by imposing that the global (round trip) velocities of the counter-propagating light beams must agree with their local velocities. So it is clear why both Klauber and Selleri favour this synchrony choice for light beams, but it seems to me rather preposterous to favour this synchrony choice for matter beams!

Chairman: Thank you, Dr. Ruggiero. But I'm not sure that everybody here agrees with your "very pragmatic view": the conventionality of the synchronization choice introduced by Serafini, which is the basis of your speech, does not belong to the standard formulation of the SRT. For instance, what do you think about the role of synchronization in the Sagnac effect, Prof. Sorge?

Sorge: The possibility of a re-parameterization (or re - synchronization) of the reference frames is indeed an interesting topic. However, the goal in my paper was to face the problem of the experimental observation of local anisotropy in the speed of light, just in the framework of the 'orthodox' theory of relativity. As I pointed out several times in my paper, the standard approach relies on the assumption that the 'speed of light' is a universal constant: it has the same value in any physical reference frame. So, in my opinion, any speculation intended to introduce 'local' anisotropies in such a value, still remaining in the framework of the theory of relativity, has to be considered intrinsically inconsistent.

Rizzi: Sorge's analysis rests on the Lorentz transformation, which in turn rests on the invariance of the one-way velocity of light. As a consequence, Sorge correctly states that, in a local inertial frame, any speculation intended to introduce local anisotropies in the values of c in the theory of relativity is intrinsically inconsistent. This is the traditional viewpoint in relativity. However, this approach neglects the possibility of a re-parameterization of the physical frame; in particular, it neglects the possibility of a re-synchronization. Let me point out that the local isotropy or anisotropy of the velocity of light in a LCIF is not a fact, with a well defined ontological meaning, but a convention depending on the synchronization chosen in the LCIF: the velocity of light has the invariant value c in any LCIF, in any direction, if and only if the LCIF are Einstein-synchronized. Anyway, my papers - the one with Serafini and the one with Ruggiero - are not the only papers in this book which suggest a relativistic explanation of the Sagnac effect for matter or light beams, based on the dialectic between local Einstein synchronization and "absolute" synchronization. Ruggiero's speech has thoroughly highlighted that the introduction of a synchronization gauge in a LCIF is not a philosophical speculation but a very important opportunity.

Dieks: Actually, I find that the approach of Rizzi and Ruggiero is in harmony with my own approach. First of all, I point out that $dt' = 0$ (i.e. $dt = 0$) does not automatically correspond to standard Einstein simultaneity in the rotating frame . As well known, Einstein simultaneity is realized through light signals; the procedure is summarized in my paper. Since a light signal follows a null-geodesic, the line element (9) must be equal to zero. A simple manipulation of the resulting equation leads to an interesting conclusion: *standard Einstein synchrony between infinitesimally close events corresponds to the following difference in* t*-coordinate:*

$$dt = (\omega r^2 d\varphi')/(c^2 - \omega^2 r^2). \tag{19}$$

(as was to be expected, it is only for events that differ in their φ-coordinates that $dt = 0$ is not equivalent to standard simultaneity). Expression (19) demonstrates that the standard simultaneity between neighboring events in the rotating frame corresponds to a non-zero difference dt. It follows that if we go along a circle with radius r, in the positive φ-direction, while establishing standard simultaneity along the way, we create a 'time gap' $\triangle t = 2\pi\omega r^2/(c^2 - \omega^2 r^2)$ upon completion of the circle. Doing the same thing in the opposite direction results in a time gap of the same absolute value but with opposite sign.

Now suppose that two light signals, emitted from a source fixed in the rotating frame, travel in opposite directions, along the same circle of constant r. We follow the two signals while locally using standard synchrony; this has the advantage that locally the standard constant velocity c can be attributed to the signals. We therefore conclude that the two signals, in order to complete their circles and return to their source, use the same amount of time, as calculated by integrating the elapsed time intervals measured in the successive LCIFs (the signals cover the same distances, with the same velocity c, as judged from these frames). However, because of the just-mentioned time gaps the two signals do not complete their circles simultaneously, in one event. There is a time difference $\triangle t = 4\pi\omega r^2/(c^2 - \omega^2 r^2)$ between their arrival times, *as measured in the coordinate* t. This is the celebrated Sagnac effect.

Chairman: Thank you, Prof. Dieks. It seems that you are giving convincing answers to Klauber.

Klauber: I've listened very carefully to the point by Dieks, but still I'm not satisfied by what he considers "convenient" choices of any of a multiplicity of possible clock times at a given event nor the discontinuity in time associated with those choices. So I guess I don't think his answer is "convincing".

Chairman: Thank you, Prof. Klauber, of course I cannot compel you to agree with Dieks. Anyway, I would like to listen to Dr. Serafini: would you finally give your answer to Selleri?

Serafini: On the basis of the conventionality of synchronization within Selleri's gauge, the answer is simple! The "unacceptable discontinuity" for the ratio (1) at zero acceleration is uniquely originated by the fact that Selleri compares two velocities resulting from different synchronization conventions. Let me remind that Selleri makes use of an isotropic synchronization in the central IF S_0, and of an anisotropic synchronization in any LCIF S along the rim. In other terms, the discontinuity found by Selleri is not a physical discontinuity, but merely expresses the difference between different synchrony choices. This discontinuity disappears if the same synchronization procedure is adopted in any (local or global) IF: this is a necessary condition (although not explicitly stated in most standard textbooks) for the validity of a proper formulation of the relativity principle.

Chairman: Does it mean that the "absolute" synchronization suggested by Selleri is ruled out?

Serafini: No, the ”absolute” synchronization belongs to Selleri’s gauge; therefore it is a legitimate choice. It doesn’t matter whether the “absolute” or Einstein’s synchronization is used: the only necessary condition is that in any (local or global) IF the same synchronization procedure is adopted. This sort of obviousness is, unfortunately, quite clouded by the standard formulation of the SRT. As a matter of fact, the optical isotropy of every IF is usually considered as a physical property of the IF itself, rather than a consequence of a “suitable” synchrony gauge choice.

Chairman: If I understand you correctly, the crucial *physical* property of any IF should take a testable, synchrony-independent form. Maybe something like this: ”any IF is isotropic on two-ways paths, regardless of the synchronization choice”. I guess this is what you have in mind.

Serafini: Yes, you are completely right. However, I prefer to rephrase this statement in a more general form, namely the one that I have suggested at the beginning of my speech: *the round-trip velocity of light takes the invariant value c in all IFs for any round-trip.* This is a testable synchrony-independent formulation of the principle of constancy of velocity of light.

Chairman: Well, thank you to all of you. I suppose we have gathered enough information about these topics, even if there are some other subtleties about Selleri’s paradox which we shall analyze later. In the next Dialogue we shall speak about ”measurements of lengths in a rotating frame”. A wide and controversial topic, which can no longer be procrastinated.

III. Dialogue on the measurement of lengths in a rotating frame

Where the problem of the measurement of lengths in a rotating reference frame is discussed on the basis of the elements gathered so far. On the formal point of view, different opinions about the concept of the "space of a rotating frame" (that is, different formal definitions of this concept) are thoroughly examined. On the operational point of view, the crucial role of the "standard rods" chosen to perform such measurements is highlighted. A general agreement is reached only on the (rather trivial) statement that different choices, both in the formal definition of the "space of a rotating frame" and in the choice of the "standard rods" chosen to perform length measurements, cause different results and conclusions.

Chairman: Now, let us introduce a very interesting and controversial topic, that is the measurements of lengths in a rotating frame, which is strictly related to the formal definition of what we call "the space of the rotating frame". This topic has been discussed since the early years of relativity: a very accurate review has been given by Grøn in his contribution to this book. However, I think that it could be useful to start from a "non traditional" approach, namely that of Klauber, who is always stimulating in posing his challenges to the "traditionalists". Klauber's approach gives us again the opportunity of thinking of the very foundations of relativity in a critical way. Prof. Grøn, since you are one of the most outstanding "traditionalists", as Klauber himself kindly recognizes, would you be so kind as to give us an outline of Klauber's point of view on this topic?

Grøn: Yes, with pleasure. Indeed, I believe that Klauber's analysis of rotation is unconventional and controversial, but not uninteresting. Klauber argues against Einstein's conclusion that the spatial geometry in a rotating reference frame is not Euclidean, and claims that the conventional relativistic analysis of rotating reference frames is not consistent. One can think of Klauber's paper as representing the point of view of "The Devil's Advocate", and it gives an opportunity to defend the point of view that is about to be "canonized".

Klauber: It is not without some trepidation that I rise to play "Devil's Advocate" with Professor Grøn, on whose articles I first cut my teeth on this issue years ago. I agree with him that my analysis is "controversial", but take issue, at least in part, with calling it "unconventional". It is certainly not traditional, but I believe it is a straightforward application of conventional differential geometry. If the world is truly described (at least in relativity theory) by geometry, then I submit my analysis is fully in accord with it.

Chairman: I presume that you can face the objections of the ”Devil’s Advocate” and explain why the conventional relativistic analysis is consistent, Prof. Grøn...

Grøn: I think so. I shall try to show that the relativistic analysis is indeed consistent, commenting Klauber’s claims. Let me recall what he says: ”According to SRT, an observer does not see his own lengths contracting. Only a second observer moving relative to him sees the first observer’s length dimension contracted. Hence, from the point of view of the disk observer, her own meter sticks are not contracted”. All of this is conventional wisdom, but then Klauber proceeds: ”and there can be no curvature of the rotating disk surface. The traditional analysis is thus inconsistent”. Well, according to me, the two last sentences seem to come out of the air!
The usual relativistic description is as follows. As the angular velocity of the disk increases while the elements of the disk material are constrained not to move in the radial direction, an observer on the disk will see that gaps between the standard measuring rods along the circumference of the disk are opening up. This may be interpreted within special relativity from the point of view of an inertial observer at rest relative to the axis, by invoking the Lorentz contraction of the standard measuring rods. However the interpretation of the same phenomenon as given by a co-moving observer in the rotating reference frame invokes the general theory of relativity. According to the principle of equivalence, he perceives the increasing Newtonian centrifugal field as a gravitational field, and interprets the increased length of the periphery as a gravitational effect.

Chairman: I would prefer to call them ”inertial” effects, since no curvature is present in a Minkowskian space-time.

Grøn: As you like; however, according to the equivalence principle, a local observer has no way to to distinguish gravitational from inertial effects.

Chairman: I agree, but ”the increased length of the periphery” of the disk is not a local effect. Anyway, please go on, Prof. Grøn.

Grøn: Klauber then says: “Consider further the disk observer looking out at the meter sticks at rest in the lab close to the disk’s rim. Via the hypothesis of locality (in which she is equivalent to a local co-moving Lorentz observer), she sees the lab meter sticks as having a velocity with respect to her. Hence, by traditional logic, she sees them as contracted in the circumferential direction. She must therefore conclude that the lab surface is curved. But those of us living in the lab know this is simply not true, and again the analysis is inconsistent". The problem raised here by Klauber is interesting. I will show how reciprocal Lorentz contraction is observed by the inertial observer and the disk observer. This will be demonstrated by means of the formulae (15.5)-(15.10) of my article, without invoking the Lorentz transformation. It will thereby become clear that the disk observer does not conclude that 3-space in the inertial frame is curved. Consider first a rod with rest length L_0 at rest in the rotating frame RF. This means that

$$\frac{r\Delta\varphi}{\sqrt{1 - r^2\omega^2/c^2}} = L_0 \tag{20}$$

The length of the rod in the inertial frame IF is the difference between the coordinates of its ends as measured simultaneously. Then $\Delta\varphi' = \Delta\varphi$. The length of the rod in IF is

$$L' = r'\Delta\varphi' = r\Delta\varphi = L_0\sqrt{1 - r^2\omega^2/c^2} \tag{21}$$

Consider then a rod with rest length L_0' at rest in IF. Then $r'\Delta\varphi' = L_0'$. At simultaneity in RF $d\tau = 0$ and from eq.(15.9) of my paper

$$\Delta t = \frac{r^2\omega/c^2}{1 - r^2\omega^2/c^2}\Delta\varphi \tag{22}$$

The transformation (15.5) then gives

$$\Delta\varphi = \Delta\varphi' - \omega\Delta t = \frac{L_0'}{r} - \frac{r^2\omega^2/c^2}{1 - r^2\omega^2/c^2}\Delta\varphi \tag{23}$$

This leads to

$$\frac{r\Delta\varphi}{1 - r^2\omega^2/c^2} = L_0' \tag{24}$$

Hence the length of the rods which are at rest in IF, as measured by the disk observer, is

$$L = \frac{r\Delta\varphi}{\sqrt{1 - r^2\omega^2/c^2}} = L_0'\sqrt{1 - r^2\omega^2/c^2} \tag{25}$$

It may be noted that due to the stretching of the tangential dimension in RF by the factor $\gamma = \left(1 - r^2\omega^2/c^2\right)^{-1/2}$ the Lorentz contraction of bodies at rest in IF is just what is needed to make the geometry in IF Euclidean an observed by a person at rest in the negatively curved space in RF, so no contradictions arise, provided that the correct measurements are performed, according to the principles of relativity. In particular, I want to stress the role of simultaneity, which is fundamental in order to define the measurements in the two frames.

Chairman: Thank you Prof. Grøn, your arguments are sound and clear enough to allow further replies. However, Klauber is not the only one in this book who claims that the length of the circumference, measured in the rotating frame, is given by the canonical Euclidean expression $2\pi r$: it seems to me that also Nikolić and Tartaglia have similar approaches, isn't it so?

Grøn: Yes, Nikolić takes up essentially the same problem, and also Tartaglia shares this conviction about the length of the rotating circumference. In particular, Nikolić says: "We study a rotating ring in a rigid non-rotating circular gutter with radius r. (...)An observer on the ring sees that the circumference is $L' = \gamma L$. The circumference of the gutter seen by him cannot be different from the circumference of the ring seen by him, so the observer on the ring sees that the circumference of the relatively moving gutter is *larger* than the proper circumference of the gutter, whereas we expect that he should see that it is smaller". The length of the gutter as seen in RF may be measured by measuring the time a point on the gutter uses to move around its circular path. According to the transformation (15.5) it moves with an angular velocity ω and hence with a velocity $v = r\omega$. As measured with a clock in IF the period is $T' = 2\pi r/v = 2\pi/\omega$. A clock in RF at the gutter goes slower. As measured with this clock the period is $T = (2\pi/\omega)\sqrt{1 - r^2\omega^2/c^2}$. During this time the point moves a distance $s = vT = 2\pi r\sqrt{1 - r^2\omega^2/c^2}$. This is the length of the gutter as measured in RF. It is a Lorentz contracted length. The result of the length measurement depends upon the measuring procedure. What has been shown is that the length measurement by measuring the time to pass one time around the circular path, gives the same result as measuring the length of the standard measuring rods that are at rest in IF by taking the difference of the coordinates of their end points simultaneously in RF and then adding the result around the gutter. Hence we arrive at the surprising result that the length of the stationary gutter is different from the length of the ring that rotates in it, as measured in the co-moving reference frame of the ring. A similar result is found in the well known paradox of the car entering a garage of the same rest length with relativistic velocity. The points of views of the chauffeur and a person at rest in the garage seem to contradict each other. The resolution of both these problems is obtained by taking into account the relativity of simultaneity which, as I said before, is essential in all length measurements.

Chairman: Your last comparison evidences, once again, the role of the relativity of simultaneity in measuring lengths in different frames.

Weber: I would like to support Grøn's statements with a physical example that appears equivalent to this problem. The dilation of time is closely associated to the contraction of length. Consider some unstable particle, for example mesons, travelling at high speed, so that v/c is close to one, in a circular orbit of radius r. Let the particles have a lifetime τ . Since clocks run slower in the rotating frame as observed from the lab, the effective lifetime of the particles is increased by the relativistic factor $\gamma \equiv 1/\sqrt{1 - r^2\omega^2/c^2}$. Most would agree with the result that in a lifetime a particle is able to make $\gamma\tau v/(2\pi r)$. This is a number which must be obtained by any observer, that is, it is an invariant. Now look at the situation as described by an observer at rest with respect to the particle. If there were no contraction of the laboratory path, the length of the orbit would be $2\pi r$ and the number of revolutions would be $\tau v/(2\pi r)$. The correct result is obtained with the contracted laboratory path of $2\pi r/\gamma$.

Chairman: Thank you, Prof. Weber. Please, Dr. Nikolić, would you reply to Prof. Grøn?

Nikolić: Yes, thank you. Let me start from Grøn conclusions: "... Hence we arrive at the surprising result that the length of the stationary gutter is different from the length of the ring that rotates in it, as measured in the co-moving reference frame of ring. A similar result is found in the well known paradox of the car entering a garage of the same rest length with relativistic velocity..." On the other hand, I still claim that the circumference of the gutter seen by the observer on the ring cannot be different from the circumference of the ring seen by him. Of course, it depends on how the circumference is measured. Here I have in mind an experimental procedure that in principle can also be used to measure the usual Lorentz contraction, based on photographing with a very short exposition, such that the change of the photographed object position during the exposition can be neglected. The size of the object's picture on the photography corresponds to the measured size. Obviously, with such a measuring procedure, for any observer the apparent circumference of the *whole* ring must be equal to the apparent circumference of the *whole* gutter. This is a simple consequence of the fact that, at any instance of time, any part of the ring is somewhere inside the gutter and any part of the gutter has a part of the ring near it. This is not so for a well known paradox of a car in a garage where different observers may disagree on whether a fast car can fit into an *open* garage at rest. This is because, for each part of the car, there are times for which that part is outside the garage as well as times for which it is inside the garage.

Ruggiero: Excuse me for this interruption, but I would like to add a brief remark. It seems to me that Nikolić's experimental procedure is rational, but it does not allow the claim that "the circumference of the gutter seen by the observer on the ring cannot be different from the circumference of the ring seen by him". In fact, as far as I understand, this measuring procedure, which consists in taking snapshots of the rotating ring and the gutter, must take place in the inertial frame: so, it does not provide any new element to distinguish what is measured in the rotating frame from what is measured in the inertial frame. Paraphrasing Grøn's garage paradox, it does not provide any new element to distinguish what the chauffeur sees from what the person in the garage sees.

Chairman: Thank you, Dr. Ruggiero. Now it is time for Prof. Klauber to reply to Grøn.

Klauber: Well, first I would like to congratulate Prof. Grøn on providing a mathematically based reply to my arguments, which has not always been the case when I have posed this question to others. I shall try to reply to his arguments. Grøn states that the rotating frame observer "perceives the increasing Newtonian centrifugal field as a gravitational field, and interprets the increased length of the periphery as a gravitational effect". I first note that in the rotating frame, the traditional analysis, based on the SRT Lorentz contraction along with the hypothesis of locality, posits that

$$C > 2\pi r, \tag{26}$$

where C is the circumference measured with meter sticks laid down sequentially by an observer in the rotating frame, and r is the radius measured the same way. This is negative curvature.

In gravitational systems like the Schwarzschild geometry, however, the curvature is positive, i.e.,

$$C < 2\pi r. \tag{27}$$

Chairman: Please, let me interrupt you for a while: I would like to make this point clear, which may be confusing or misleading. Grøn speaks of "gravitational" effects, but, as I said before, I prefer to call them "inertial" effects, since we must not forget that we confined ourselves to a flat space-time, where no curvature is present at all. So, the curvature which Grøn refers to is that of the 3-dimensional space, the so called "space of the disk", and as you have just pointed out, it is a negative curvature.

Klauber: Thank you, your remark is useful in order to make things as clear as possible. I will try to stick to the main point. If one uses the metric of eq. (15.8) in Grøn's paper, then he finds the negative curvature of the traditional analysis. However, that metric is based on a time coordinate that is discontinuous, as Grøn, myself, and others have noted. Those in the traditionalist camp don't seem to find that abhorrent, whereas I, and a few others like Selleri, do. As I emphasize in my paper, I don't believe any theory can be valid that has, at its roots, a discontinuity in physical time. By contrast, if one takes the metric of Grøn's eq. (15.7) [my eq (6.13)], which has continuous time, then the curvature of the surface is zero, and there is no Lorentz contraction in rotation. A key related point is that the metric of eq. (15.8) in Grøn is derived from a transformation on the time coordinate that results in simultaneity as found from the Lorentz transformation. In effect, that metric (15.8) reflects the Lorentz transformation, and hence Lorentz contraction. The traditional analysis starts by assuming Lorentz contraction exists, and then deduces a metric, i.e. (15.8), that ensures it occurs. The SRT (for translating systems), on the other hand, starts with two postulates, and derives Lorentz contraction. I submit we have to question whether those postulates hold for rotation. If they don't, then we have to question results deduced from them, such as Lorentz contraction (and not presume those results to be *a priori* true).

Chairman: I believe that this is your crucial claim, Prof. Klauber: you maintain that a fundamental difference exists between translating and rotating systems, and, as a consequence, you argue that the relativistic postulates valid for translation cannot be extended in a straightforward way to rotating systems. On the technical point of view, you claimed that your analysis "is a straightforward application of conventional differential geometry". This seems to be correct, but only at the cost of renouncing all fundamental principles of the SRT for rotating systems: the principle of invariance of the one-way velocity of light, the principle of relativity, and the hypothesis of locality. It should be recognized that you have the heart to renounce these principles in order to achieve a self-consistent handling of the matter on the basis of conventional differential geometry. I appreciate your audacity very much, but I think this is a very high price to pay: isn't it, Prof. Klauber?

Klauber: This is a price which must be payed by any theory that aims at truly describing the world. Firstly, the Sagnac effect shows that the local speed of light is anisotropic: this is a violation of the principle of invariance of the one-way velocity of light. Secondly, it is well known that an observer on the rim of a rotating disk can determine her angular velocity ω using a Foucault pendulum, and her distance from the center of rotation r using a spring mass system. From these, she can determine her circumferential speed $v = \omega r$ in an absolute way from local measurements made entirely within her own system. I use the classical limit formula for simplicity, though the same conclusion holds for the fully relativistic relation. Thus her velocity can be known in an absolute way, and this is a violation of the principle of relativity. Finally, the hypothesis of locality is only an assumption, which works very well for translating systems, but does not appear to hold for rotating systems, for reasons delineated in my paper.

Rizzi: Neither the principle of relativity nor the hypothesis of locality imply that a rotating observer cannot be aware of rotation: after Galileo, we are aware that the Earth rotates! It seems to me that you mix up the local and the global level. Actually, all the measurements you are talking about cannot be carried out in a LCIF, which should be infinitely small both in space (no gradient effects, which of course are vital for your spring mass system) and time (the period of a Foucault pendulum, in length units, is at least of the order of the Earth-Moon distance, which of course is much bigger, instead of much smaller, than the Earth radius). These measurements are normal laboratory experiences for students: the point is that these measurements are not ”local” in the sense used in relativity.

Chairman: Thank you for your useful remark, Prof. Rizzi. Now I would like to ask Prof. Klauber something more: it seems to me that you overlook the point of view outlined by Ruggiero about the Sagnac effect and light isotropy/anisotropy. What do you think about it?

Klauber: I think it interesting and ingenious, and certainly worthy of serious consideration, though it still seems to me to result in the same problematic issues I am concerned with. In this regard, note that I believe in the gauge synchronization philosophy for translation, but not for rotation, since only one gauge in rotation does not have a time discontinuity. By contrast, an infinite number of gauges in translation have fully continuous time. At any rate, let me tackle now, in more detail, the issue of the comparison between lengths measured by the two observers. I resume Grøn's statements: L_0= length of a rod at rest in RF as seen by an observer in RF.

L' = length of the same rod as seen in IF.

$$L' = L_0\sqrt{1 - r^2\omega^2/c^2} \tag{28}$$

So, the RF rod looks contracted to the IF observer.

L_0'= length of a rod at rest in IF as seen by an observer in IF.

L = length of the same rod as seen in RF.

$$L = L_0'\sqrt{1 - r^2\omega^2/c^2} \tag{29}$$

So, the IF rod looks contracted to the RF observer.

I believe this was my point. Each sees the other's meter sticks as contracted. Hence, if the RF observer measures his surface to be negatively curved, because there are more meter sticks around his circumference than $2\pi r$, then he must measure even more IF meter sticks around the same circumference. And he must thus find the IF surface to be even more negatively curved than his own. But it isn't. It is flat, and the analysis appears inconsistent. I think Grøn may be making the point that we have a given metric in the IF, with a flat surface; and we have given metric in the RF (his (15.8)) with a curved surface. Yet his analysis suggests that an observer in either frame sees the other's meter sticks as contracted.

Chairman: All of us agree that, according to the traditional wisdom in the SRT, an observer in either frame sees the other's meter sticks as contracted, provided that the meter sticks are infinitesimal ("local scale"). The general agreement stops when this statement is maintained at a global scale.

Klauber: But the metric (15.8) was deduced originally to satisfy the supposition that the IF observer sees the RF meter sticks as contracted and thus the RF surface must be curved. Physically, why doesn't the same logic hold for the RF observer seeing the IF meter sticks?

Chairman: Wait, what do you mean? You are going to and fro with the inertial and rotating frames, sometimes at local scale and sometimes at global scale, and I cannot understand clearly what you are saying.

Klauber: Well, I can try this way. A possible different traditional interpretation is this: the RF observer's meter stick contraction is absolute, and may be attributed to the effective gravitational, or if you want, inertial field. Thus, the RF observer would see, all else being equal, IF meter sticks as appearing longer than his own RF meter sticks (by the Lorentz factor.) But due to the speed of the IF meter sticks as seen in the RF frame, the IF meter sticks would appear shorter (by the Lorentz factor.) One effect is gravitational, or inertial (effective); the other is kinematic. The two would cancel each other, leaving a measurement by the RF observer of the IF meter sticks in which the RF observer would find the IF meter sticks to be equal in length to his own. Thus, again, the RF observer would conclude the IF surface is negatively curved. But again, it isn't.

Chairman: I see: according to you there are two competing effects, which cancel each other: your conclusion is that the rotating observer still concludes that the surface in the inertial frame is curved. Thank you Prof. Klauber. Now, I suppose that Prof. Grøn would like to counter-reply.

Grøn: Yes. I see from what Klauber says that we share many points of view concerning the relativistic analysis of kinematics and geometry in a rotating frame. However, there are still a few important points where we disagree. Such disagreement may be due to different preferences on how certain concepts should be defined. Let me distinguish and stress the key points:

1. Discontinuity of time in a rotating reference frame. Klauber says: ”I don’t believe that any theory can be valid that has, at it roots, a discontinuity in physical time”. If I understand Klauber correctly it is just such a time-discontinuity that disqualifies the theory to him. Since the conventional interpretation of special relativity implies such a time discontinuity as applied to a rotating reference frame, Klauber does not find the application of this theory to rotating frames valid. This is clearly an acceptable point of view. But it does not imply an inconsistency in the application of special relativity to rotating frames. It is more a matter of taste. I do not feel repelled by this time discontinuity. It means that in a rotating frame you cannot Einstein-synchronize clocks around closed paths. Hence, I accept as valid the application of the special theory of relativity to rotating frames.

2. Spatial geometry in a rotating reference frame. According to the conventional definition the *spatial geometry* in a reference frame is the geometry of a surface that is everywhere orthogonal to the world-lines of the reference particles of the frame. In a rotating frame such a surface has discontinuity which is essentially the one due to the impossibility of Einstein synchronizing clocks along a circle about the axis of rotation.

Chairman: Once again, as far as I can understand, synchronization (or simultaneity) and measurements of lengths prove to be topics deeply linked.

Grøn: Yes, I think so. However, one may dislike the discontinuity and choose another definition of spatial geometry. But then one does not talk about what has been called the three-space or simultaneity-space of a rotating frame. And, of course, if we talk about different surfaces or three-spaces, we will obtain different kinematical and geometrical results.

Chairman: Thank you Prof. Grøn for your clear explanation, and Prof. Klauber for your continued role as Devil’s advocate. Before turning to Dr. Ruggiero for his insights on this topic, I believe Prof. Klauber has one final remark.

Klauber: As I point out in my articles, and as I think we are all aware, Lorentz contraction is directly related to choice of simultaneity. If one chooses Einstein synchronization/simultaneity on the disk, one gets Lorentz contraction. If one chooses “flash from center” synchronization, there is no Lorentz contraction. If the hypothesis of locality is true for rotation, so a co-moving Lorentz meter stick and a RF meter stick have the same contraction, then clocks *must* be set as in Grøn’s metric (15.8). Different clock settings mean a different time transformation than Grøn's (15.9), and that means different length contraction. So in order to get what the traditionalists believe we must get for RF surface curvature, there can be no gauge freedom of simultaneity, i.e., there is an absolute simultaneity. That particular choice for time is made in order to get what has already been assumed to be true. Note that both the Non-Time-Orthogonal (NTO) and traditional methods yield the same results, in all regards, for translation. Simultaneity/synchronization for rotation, in either approach, must be absolute, but the NTO method has no discontinuity in physical time. We must make a choice. We can have continuous time and no Lorentz contraction, or discontinuous time with Lorentz contraction. I believe space-time continuity is more fundamental than Lorentz contraction and opt for the former.

Chairman: Thank you, Prof. Klauber. Now I would like to go on with this topic, and ask Dr. Ruggiero to give us a brief description of the approach to the space geometry of a rotating disk that he has outlined with Rizzi: it seems to me that yours is the latest contribution to this long-standing debate.

Ruggiero: Yes, I guess it is. I am going to explain our operational approach to the space geometry of a rotating disk, however let me start with a kind remark to Grøn. His paper gives a thorough and very well written review on the approaches to this topic, from the beginning up until today. However, in the Conclusions, where "several results of the long period with discussions on the geometry of a rotating disk" are listed, we cannot read any reference to the "relative space approach" to the study of the spatial geometry of a rotating disk, which we describe in our paper. On the contrary, Grøn states that: "The surface orthogonal to the world-lines of the disk particles is called the 3-space in the rotating rest frame of the disk. This space is defined by events that are simultaneous as measured by Einstein-synchronized clocks on the disk. It has a discontinuity along a radial line as shown in fig. 15.5, and is negatively curved".

Figure 1 (which is Fig. 15.5 in Grøn's paper) is very nice: indeed, Rizzi told me that, some years ago, when he was writing his first paper with Tartaglia[6] on this subject, he longed to draw a similar figure, but he was not able enough. Thereafter he discovered (together with Serafini, and thanks to Selleri) the synchronization gauge, and then together, Rizzi and I worked out the concept of relative space, which was only roughly outlined in the paper written with Tartaglia. Today, Rizzi believes (and I agree) that this figure is misleading because the "space of the disk" is incorrectly shown as a surface embedded in

[6] *Found. Phys.*, **28** 1663 (1998)

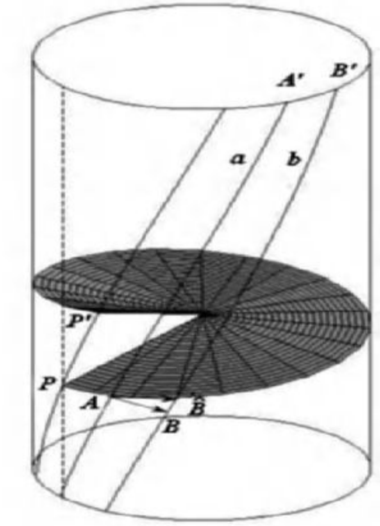

Figure 1. From Grøn's paper

space-time. As a matter of fact, the discontinuity appearing in figure depends on the embedding of the "space of the disk" in space-time, which in turn depends on the definition of the space of the disk as the locus of events "that are simultaneous as measured by Einstein-synchronized clocks on the disk". However, let me point out that such a discontinuity, that Klauber dislikes so much, is a mere "theoretical artefact" due to the use of Einstein synchronization arbitrarily extrapolated from local to global, and cannot be perceived by an actual observer who performs space measurements on the rotating disk (see the paper by Klauber, Sec. 1.7.2, and of course the one by Rizzi and Serafini). According to our "relative space approach", the space of the disk cannot be thought of as sub-manifold embedded in space-time, but it must be thought of as quotient space of the world tube of the disk with respect to a suitable equivalence relation. Roughly speaking, it is the Riemannian manifold whose points are the lines of the world-lines of the points of the disk.

Chairman: This is a standard concept in Riemannian geometry: where is the freshness?

Ruggiero: Yes, this is a standard concept; yet, in the standard textbooks, we couldn't find any reference to our crucial point, which is synchronization. We show in detail that synchronization is incorporated in the very definition of the relative space, and we stress the importance of this feature, which is sometimes overlooked.

Rizzi: Let me give an interesting example: using the quotient space without any reference to synchronization, Tartaglia claims that no Lorentz contraction takes place in case of rotation. This claim rests upon his assumption that ”no synchronization is needed” for space measurements on the platform.[7] But our approach shows that things are different. When synchronization is neglected, rotation itself is neglected; as a consequence, it is not surprising that no Lorentz contraction is found, and the space of the disk turns out to be Euclidean.

Ruggiero: Our approach is based, on the formal point of view, on a precise definition of the concept of ”space of the disk” and, on the operational point of view, on a precise choice of the ”standard rods” used by the observers on the platform. Once the space of the disk is formally identified with what we called the ”relative space”, its geometry turns out to be non-Euclidean and its metric coincides with the one which is found in classic textbooks of relativity:

$$d\sigma^2 = dr^2 + \gamma^2 r^2 d\varphi^2 + dz^2 \tag{30}$$

(where $\gamma = \gamma(\omega, r) \doteq 1/\sqrt{1 - \frac{\omega^2 r^2}{c^2}}$), and which is described in great detail in Grøn’s paper (see in particular sect.3).

Chairman: So, you too claim that the length of the circumference in the rotating frame is $2\pi\gamma r$... As far as I can see, your approach is the same as that of Grøn, or Weber...

Ruggiero: If you refer strictly to the measurement of the circumference, our result is not new, of course. However, the relative space approach introduces a kind of shift of context in the interpretation of the space of disk, which, according to us, can help to get rid of the many misunderstandings that arise in this field, and that have been highlighted also in this discussion. Let me explain what I refer to. To this end, first of all I want to stress that one of the crucial points is the definition of the standard rods used by rotating observers. Namely, when light rays are used *locally* as standard rods, the line element which allows measurements of lengths is given by eq. (30). In turn, the use of this ”local optical congruence” is based on the possibility of adopting *locally* Einstein’s synchronization. This is a vital point, although I know that some people do not agree.

The local spatial geometry, whose metric is given by eq. (30), is defined at each point of the rotating frame. However, in order to have the possibility of

[7] *Found. Phys. Lett.*, **12**, 17 (1999)

confronting measurements performed at different points in the frame, a procedure to extend all over the disk the local spatial geometry is required. But this cannot be done in a straightforward way, because a rotating frame is not time-orthogonal and hence it is not possible to choose an adapted chart in which, globally, the lines $x^0 = var$ are orthogonal to the 3-manifold $x^0 = const$.

Chairman: Excuse me if I interrupt you, but your mathematical language could be misinterpreted. Paraphrasing your statement, you are saying that it is not possible to obtain a *global* surface (that is a 3-D hypersurface embedded in space-time) which is everywhere orthogonal to the world-lines of the reference particles of the frame: is it so?

Ruggiero: Yes, it is. What I am trying to say is that a global space-time foliation, which would lead ”naturally” to the definition of the space of the disk, is not allowed. Nevertheless, if we shift the context, from the ill-defined notion of space of the disk thought of as a sub-manifold embedded in space-time, to a definition which has a well defined and operational meaning, we are lead to the concept of the ”relative space”; this is what we did in a previous paper,[8] and that we have recalled also in the paper in this volume. In other words, the relative space is a formal tool which allows a connection among all local optical geometries that are defined in the neighbourhood of each point of the space.

Rizzi: Excuse me for interrupting again, but I want to point out that the word ”connection” should be understood not only in descriptive sense, but also in a strictly technical sense: namely as ”Riemannian (Levi-Civita) connection”. Such a connection enables to parallel transport the direction of a light ray from a tangent space to another tangent space to the relative space.

Chairman: This is a more technical way to say the same things.

Ruggiero: As a consequence, space measurements globally defined in the relative space reduce, immediately, to standard measurements in any local frame co-moving with the disk. In this way, a natural procedure to make a comparison between observations performed by observers at different points is available. The physical context in which these distant observations are made is defined, both from a mathematical and operational point of view, by the relative space. I want to stress, again, that it is not possible to describe the relative space in terms of space-time foliation, i.e. in the form $x^0 = const$, where x^0 is an appropriate coordinate time, because the space of the disk, as

[8] *Found. Phys.*, **32,**1525 (2002)

I said before, is not time-orthogonal. Hence, thinking of the space of the disk as a sub-manifold or a subspace embedded in the space-time is misleading and leads to paradoxical (or simply unpleasant) statements, such as the discontinuity of the space that Klauber dislikes so much.

Chairman: Let me see if I have correctly understood your definition of the space of the disk. Firstly, the space of the disk is not a ”simultaneity space” *at large*, although it is an ”Einstein simultaneity space” *locally*; secondly, on the operational point of view the use of standard ”light” rods is fundamental.

Ruggiero: Yes it is; but let me remind you that the two points that you stressed are linked. Actually, the possibility of using standard ”light” rods (that is a ”local optical congruence”) is based on the possibility of adopting *locally* Einstein's synchronization. Some authors don't agree on some of these points (for instance Klauber, Tartaglia and, to some extent, Bel): this is the source of disagreement on the length of the circumference as measured in the rotating frame. As Nikolić has just said, ”[the length of the circumference] depends on how the circumference is measured”. We pointed out in our paper in *Foundations of Physics* that *locally* we can use light rays as standard rods, assuming as spatial geodesics the trajectories of light rays, and as unit of length the wavelength in vacuum of a given spectral line emitted, locally, by a source at rest. This congruence defines the local optical geometry that we used to obtain the metric (30).

Chairman: I have already noted that such a metric implies that the circumference, as measured in the rotating frame, is lengthened by a Lorentz factor. This is in agreement with Dieks, Grøn, Weber, Selleri, and many others; but not with Klauber, Tartaglia, Nikolić. What about your standard rod? Does it lengthen too, or is it something invariant?

Ruggiero: This is a rhetorical question: a standard rod is invariant by definition. As well known, this is a standard traditional assumption, both in special and general relativity.

Chairman: Let me remind you that your standard rod is not a sort of platonic entity, but the wavelength of a given spectral line; so it could be reasonably expected, on the basis of the Doppler effect, that it is a frame-dependent physical quantity.

Ruggiero: This is a misunderstanding. Let me repeat the definition of our optical standard rod: it is the wavelength λ_0 of a monochromatic radiation emitted, *locally,* by a source *at rest in the rotating frame.* It seems superfluous to say that any rest quantity is invariant by definition.

Chairman: What does "locally" exactly mean in this case?

Ruggiero: It means that the wavelength λ_0 should be so small, with respect to the length of the circumference, that it belongs entirely to the LCIF in which the source is at rest. So, both the source and the absorber belong to the same LCIF, at a distance λ_0.

Chairman: As a consequence, no relative motion between the source and the absorber takes place; so no Doppler effect arises. Maybe this is not practical (you need as many sources as there are the standard meters!), but clear and sound. Anyway, this point is not as self evident and shared by everyone as you may think; in particular, Tartaglia does not agree with you, and also his arguments seem clear and sound. Actually, in his paper for this book, he maintains that ”If we have a material meter, *whatever happens to the matter of the disk will happen to the meter too*; consequently the result of the measuring process will necessarily be the same as the one obtained for the non-rotating system” and also ”...the assumption that the solid disk behaves differently than a standard rod... is logically inconsistent”. Furthermore, he states clearly that ”...standard rods are thought of as being something magic, not partaking the general properties of matter: this viewpoint is rather inconsistent; *it cannot exist any absolute standard*”.

Rizzi: I do not want to argue with Tartaglia, but I must point out that the whole mathematical model of (both special and general) relativity, that is the Riemannian space-time, is based on the hypothesis that some absolute standard actually exists: in particular the proper time as measured by a standard clock, and the proper length as measured by a standard rod. It is universally accepted that, in the presence of a gravitational or inertial field, each rod maintains the same length that it would have in absence of fields.

Chairman: However, according to Tartaglia, the rim of the disk and the standard rod must behave likewise, because they must share the same material properties. So, if the rim stretches, the standard rod must stretch in the same way.

Rizzi: Let me make clear what we are speaking about, so that no misunderstandings arise. According to a suitable acceleration programme, the disk, initially at rest, is set into stationary rotation. During the acceleration period, each element of the periphery of the disk, initially of proper length λ_0, is stretched; at the end of this period, its length has become $\lambda_0\gamma$. All this is shown, with the aid of a figure, in our paper in *Foundations of Physics*. Also Bini and Jantzen point out this fact in a clear way in their figure 11.2 in the paper for this volume. This is a purely kinematical result of the acceleration program, and it is due to the change of simultaneity criterium in the LCIF where the rod is at rest.

Chairman: This is just what Tartaglia says: the length of the standard rod increases in the same way as the length of the whole rim.

Rizzi: No, this is just what happens to an infinitesimal part of the rim, which is a standard rod only before the acceleration period starts, but this is not the case during and after this acceleration period. A true standard rod is unaffected by any acceleration program: it is what Ruggiero has just defined.

Chairman: Okay, but I suspect that we are facing two different definitions of ”standard rod”.

Rizzi: I agree. Actually, we discussed this in detail in our paper in *Foundations of Physics*. Tartaglia introduces a congruence of material rods on the platform, whose lengths are not affected by rotation. Of course this choice is legitimate, but it should be realized that the rods used by Tartaglia, are nothing but parts of the circumference, whereas a true standard meter is a rod with free endpoints.

Chairman: I see. But this subtle distinction does not seem of vital importance.

Rizzi: On the contrary, it is fundamental! The rod used by Tartaglia cannot be transported in space without cutting its endpoints: because of the cuts, the tangential forces acting upon the endpoints, which compensated each other before the cuts for symmetry reasons, provoke the shortening of the rod, according to Hooke’s law. Only after the cuts, the rod becomes a true standard meter, whose length is not affected by the centripetal acceleration, according to the hypothesis of locality.

Chairman: And, of course, using different meters, the results are different! But your geometry makes me curious: in a sense, it seems to me that this geometry is the same as Reichenbach’s ”geometry of rigid rods”, that Grøn quoted in his paper. Is this correct?

Rizzi: Absolutely. First of all, let me recall that Reichenbach makes a very expedient distinction between *metrical (or universal) forces*, which are the forces which act in the same way on every physical body, and *physical (or* *differential) forces,* which are the forces which act in different way on bodies with different physical properties. According to relativistic wisdom, only the metrical forces invoke a geometrical interpretation in space-time: this is, of course, the case of gravitation. That having been said, Reichenbach's definition of rigid rod is as follows: (i) "A *rigid rod* is a solid rod on which no exterior physical forces act"; (ii) "The length of a rigid rod is that length which results when all physical forces are eliminated, either practically or by calculation; *metrical forces are disregarded*". It seems to me that our standard rod is the same as Reichenbach's.

Chairman: Thank you, Prof. Rizzi. But it seems to me that in his paper for this volume, Tartaglia uses also light meters; yet his results are somewhat different from yours...

Ruggiero: Wait a moment: in the second part of his paper, as you say, Tartaglia actually uses light meters, but in a very different way. In fact, while our relative space is the result of the union of the infinitesimal *local* optical geometry, Tartaglia uses a *global* optical geometry. To be exact, Tartaglia defines the space geodesics as the light rays as seen in the rotating frame; in this way he builds a global optical space. On the other hand, we define the local space geodesics as the local light rays; so we obtain the local optical geometry. Then we "glue" all local optical geometries according to the technical procedure outlined in our paper in *Foundations of Physics*: in this way we build the so-called "relative space", whose geometrical features are completely different from the ones of Tartaglia's global optical space.

Chairman: However, one could object that a global optical space should reduce, locally, to a local optical space.

Rizzi: Of course, but the way of connecting the many local optical spaces is quite different. For instance, in the relative space, which is endowed with a Riemannian geometrical structure, the tangent vector to a geodesic ζ is parallel transported along ζ from a tangent space to another tangent space, and the result is a bijective mapping between tangent spaces. Things are different in the optical space, where the geodesics between two points A,B, and the associated mappings between the tangent spaces $T(A)$, $T(B)$, do not depend only on A, B, but also on the direction: if we place a light source in A and a mirror in B, the outward trajectory of the photon is generally different from the return one! A non trivial consequence is that the optical space of a rotating frame cannot be endowed with a Riemannian geometrical structure: Prof. Tartaglia stated this fact very clearly, and I completely agree with him.

Chairman: But the space-time trajectory of a free photon is a geodesic in any case...

Rizzi: Yes, it is, of course. But this does not imply that the space trajectory of the photon is a geodesic too. For instance, it's easy to realize that the trajectory of a photon in a Schwarzschild field is not a space geodesic, except for the radial case.

Ruggiero: I agree. However, let me recall that Abramowicz *et al.*[9] showed that, introducing a suitable conformally adjusted space metric tensor, the physical space turns out to be an optical space with respect to such a metric. This mathematical technique, which is widely used in relativity, "straightens" the light rays (which are curved with respect to the standard space metric tensor) so that they become geodesics. Of course this applies to a static field only...

Rizzi: Yes, and it does not apply to a stationary field. For a generic stationary (in particular rotating) frame the space trajectory of a photon is not a geodesic of the quotient space, and no conformal transformation can "straighten" this space trajectory.

Chairman: This debate is very interesting indeed, but I'm afraid it's becoming too technical. Thank you, Dr. Ruggiero, for the thorough exposition of your point of view; thank you also to Prof. Rizzi for his enlightening remarks. Now, I would like to know something more about Bel's approach. I'm not sure I thoroughly understand this approach, but it seems to me that Bel also has something to object to your confidence in optical length in order to define the geometry of the space of the disk. The disagreement, I think, is not about your definition of the optical length, but about the way you manage it. Bel agrees with Rizzi and Ruggiero that, on the basis of the optical length, the quotient metric (30) is not Euclidean; on the other hand, he seems to agree with Tartaglia that the space of the disk *must* be Euclidean, if measured with a material - non optical - a physical rod, defined as "a stretched ideally inextensible thread". Am I expressing your viewpoint correctly, Prof. Bel?

Bel: Actually, the quotient metric (30) is not Euclidean, but the far-reaching consequences of this fact seem to be under-estimated by the relativity community, ever since the early years of relativity. This means that rigid bodies can not be compared in general if they are in different locations or have different orientations; another way of saying this is to say that a rigid body

---

[9] *Gen. Rel. Grav.,* **29**, 1173-1183 (1988)

can not be moved around. This shatters the very foundations of metrology and therefore of physics. In my opinion, this unsatisfactory situation stems from a misinterpretation of the line-element (30) as describing the geometry of space in a rotating frame of reference. The point of view that I develop in my paper consists in defining the geometry of space by the principal transform of (30), a concept that I introduced elsewhere,[10] and in re-interpreting (30) as defining an optical length, i.e. a length measured by a round trip transit time of light, instead of a physical length, i.e. measured for instance with an stretched ideally inextensible thread.

Chairman: It seems to me that many conclusions in your paper come from this distinction. In particular, the quotient metric (30) describes an hyperbolical Riemannian space, while the space described by (12.65) in your paper with the condition (12.66) is Euclidean by definition. Your Section 4 is entirely based on this distinction. So, I would like to ask you to make clear this distinction.

Bel: In my paper I use the expression "stretched rod" as a replacement for what most people call a "rigid rod" because I think that the quantity length based on the concept of un-extensible threads is more general and can be more easily axiomatized than the quantity distance. My claim is that "mechanical distance", as derived from the concept of "stretched rod" is different from optical length. Such claim can be negated by an experiment, i.e. it has scientific meaning, and the best experiment I can think of to check this claim is the Michelson-Morley experiment with a turning platform.

Chairman: Tartaglia agrees with you; as a consequence I suppose that Rizzi and Ruggiero disagree.

Rizzi: It seems to me that Bel's worry refers to rigid bodies of non negligible size. However, it is unquestionable that a *local* rod can be moved around everywhere in an hyperbolical Riemannian space. Of course, if we look for a *local* optical rod, the monochromatic radiation must be chosen in such a way that the rest wavelength λ_0 is very small when compared with the length of the circumference.

Chairman: What I can understand, summarizing, is that the misunderstandings that arise in the comparison of the various approaches to the study of the space geometry of a rotating frame, depend mainly on the choice and defi-

[10] *Gen. Rel. and Grav.* **28**, No. 9 (1996)

nition of the meters used to perform measurements. Then, it is also important how measurements at different points of the frame are related: namely a kind of ”connection” between the (many) local geometries is needed. Of course, it is plain that different choices yield different results and conclusions: I presume that all of us agree on this fundamental and somewhat obvious statement.

IV. Dialogue on the Brillet-Hall experiment

Where the Brillet and Hall experiment is discussed and the relevance of its result for the supporter of theories alternative to the special theory of relativity is explained.

Chairman: Since we have spoken a lot about the theoretical foundations of relativity, about which some of you raised doubts and questions, now I would like to know whether the criticisms you made have some experimental implications. Namely if, as someone maintains, the SRT is not fit to describe rotating reference frames, I ask whether there are some experiments, or some observations that can be performed to support your claims.

Klauber: Well, in my works I question the traditional analysis not only from a theoretical perspective, but, also, from an experimental one, which is very important! Indeed, I cited the Brillet and Hall result, which seems to be ignored repeatedly by traditionalists to whose attention I bring it. According to me, this experiment is a definitive empirical test between the traditional and Non-Time-Orthogonal(NTO) approaches to relativistic rotation. The result is not predicted by the traditional analysis, but it is predicted by NTO analysis. A repeat of this test would either send me into hiding, or the traditionalists to scouring my articles.

Chairman: It seems to me that you are not the only one who stresses the link between the Brillet and Hall experiment and the purported (local) anisotropy of the velocity of light.

Weber: Yes, indeed both Klauber and Bel suggest that the spurious signal of the Brillet and Hall experiment supports the hypothesis of the anisotropic velocity of light in rotating reference frames. But there is every reason to believe that the experimenters used great care in their interpretation of the data.

Chairman: I think that this is a very important issue, Prof. Weber: would you be so kind to describe this experiment and its implications?

Weber: The experiment, performed in 1978,[11] measures any change in the length of a Fabry-Perot cavity mounted horizontally on a table rotated at a rate f, about once every $10s$. A signal at frequency $2f$ would indicate an

[11] *Phys. Rev. Lett.* **42**, 549 (1979)

anisotropy in the velocity of light. Brillet and Hall state clearly that the major factor in limiting the sensitivity of the experiment is the change in length of the cavity caused by the variably gravitational stretching of the interferometer. The variation comes about because the axis of rotation of the interferometer is not perfectly vertical. This stretching of the interferometer gives a strong signal at frequency f while the signal of interest has a frequency $2f$. But the experimenters state that the spurious signal at frequency f is nearly sinusoidal and refer to the spurious signal at frequency $2f$ as a second harmonic. Brillet and Hall give a two dimensional plot of the orientation of the interferometer at the maximum amplitude of the signal at $2f$, but failed to give the direction of the axes with respect to the Earth. I trust that Brillet and Hall thoroughly tested their instrumentation and did preliminary runs of the experiment trying to minimize this spurious signal. A fixed orientation of the interferometer at maximum amplitude with changes in its axis of rotation in these preliminary runs would have indicated that the signal was not spurious. Such a test would have occurred to most of us. I would not suggest, as some do, that they discarded the signal "because it seemed inexplicable". Brillet and Hall average out this signal by using a sidereal reference frame. In any case I suppose the experiment should be repeated but I do not expect any difference in the results.

Bel: I would like to add my personal and direct testimony about this issue. Actually, about ten years ago I met several times A. Brillet and once J. Hall to discuss their experiment. They both told me that they were very puzzled about the "spurious" result that they mention in their paper, and that they tried very hard to figure out what could be the cause of what they thought was an accidental error. When I told them about it, they both agreed that a local origin to the anisotropy of the round trip speed of light due to the rotation of the Earth could not be excluded. The experimental protocol required that the orientation of the Fabry-Perot be always the same at the beginning of each run. To comply with this condition a "single hole pierced in a metal band under the table provided absolute re-synchronization each turn". But the corresponding geographical orientation is not given in the paper and it was not recorded in Brillet's laboratory notes. This is important because the signature of the effect, if it is of local origin, depends on this geographic orientation. In other words both the radius and angular direction of the cloud of circular dots of their Fig.2 are important to decide whether or not the result is spurious or not; or to discriminate between competing theories to explain it. For example, formula (12.119) of my paper is consistent with the data if this orientation is approximately 13° out from the East-West direction but not otherwise. In my opinion, Brillet and Hall, accepted with resignation their "spurious" result for two reasons: i) because it did not prevent them from improving the bound on the "cosmic directional anisotropy of space" which was what they really wanted to test; and ii) because they could not even imagine another cause to it that were not "accidental or cosmic".

Chairman: I would like to thank Prof. Bel for having told us of his personal and direct meetings with Brillet and Hall: I think this is important to enrich our discussion.

Weber: Me too, I would like to thank Prof. Bel for his informative comments on the Brillet and Hall experiment. I am rather surprised that the Brillet and Hall paper is silent on the lack of understanding the source of the spurious signal at frequency $2f$. Perhaps, as suggested, they were only interested in showing the cosmic isotropy of the speed of light.

Klauber: After years of trying to generate interest in Brillet and Hall's anomalous signal, it is gratifying to have it discussed openly in a public forum. Unless it is unequivocally proven one day to be a spurious result, I hope we will not again see widely quoted comments by distinguished physicists that local Lorentz invariance has been verified in every experiment ever performed to test it. If you do not mind, I would like to add further observations.

Chairman: Please go on, Prof. Klauber, the issue is intriguing.

Klauber: Certainly, as Weber notes, changes in length of Brillet and Hall's Fabry-Perot cavity would change the signal. But so would an anisotropic round trip speed of light. Gravitational stretching of the apparatus in appropriate measure would cause the first of these. Weber suggests the gravitational stretching at f has a second harmonic at $2f$, and the latter is the result of a non-exact sinusoidal wave form at f. This is a reasonable suggestion, though it immediately begs two questions. One, then shouldn't we also expect additional signals at $3f$, $4f$, etc.? Two, would it not be a truly remarkable coincidence that over 15 orders of magnitude, the $2f$ harmonic from gravitational stretching should appear at virtually the precise value predicted from an anisotropy in light speed due to the Earth surface speed? It is unfortunate that Brillet and Hall did not appear to have altered the axis of rotation at any time to test the gravitational stretching hypothesis.

Chairman: It is a pity that more direct and somewhat crucial data about this experiment are not available: they would certainly help us to understand the origin of the spurious signal. To this end, I think that it would be useful to perform again the experiment, of course with a greater accuracy...

Klauber: As I suggested in my paper (Section 3.6), Tobar *et al.* will be performing a modified Michelson-Morleyexperiment. He expects accuracy of three orders of magnitude better than Brillet and Hall, and should have results sometime in 2004. My preliminary analysis of his experiment suggests that it may be sensitive to effects predicted by non-time-orthogonal analysis and should not have a signal at f. However, Weber says that if the Brillet and Hall experiment were repeated, he would "not expect any difference in the results." This is perplexing, since he conjectures a gravitational stretching cause for the "spurious" signal. A change in apparatus would then most certainly mean, at the least, a change in magnitude of such a signal.

Chairman: It happened to me to come across a preprint of yours, Prof. Klauber, where you carry out a detailed analysis of the Brillet and Hall experiment...

Klauber: Yes, you probably refer to my preprint gr-qc/0210106, which will be submitted for publication shortly. The prediction depends on certain apparatus dimensions, which are not precisely known, but for suitable estimations of such dimensions, is in quite close agreement with the experimental result. In Appendix E of that preprint, I show that a second signal at f is also predicted by the NTO approach, and its magnitude depends on the (unknown and small) misalignment. In other words, the NTO analysis may conceivably predict both of the troubling signals observed by Brillet and Hall. Finally, I would like to point out that the NTO and Bel approaches, although predicting non-null Brillet and Hall/Michelson-Morley signals of similar magnitude, differ in the phase angle of the signal with respect to the East-West direction. As Prof. Bel has just stressed, his predictions are consistent with the data if the orientation "is approximately 13° out from the East-West direction but not otherwise". The NTO predicted angle in the Brillet and Hall experiment depends on certain unknown dimensions of the apparatus. However, the Tobar *et al.* experiment should have principle anisotropy, according to the NTO approach, directly aligned E-W. This should permit distinction between the Bel and Klauber predictions.

Weber: Prompted by Klauber's words, I would like to add the following remarks, which are based solely on the paper of Brillet and Hall published in the *Physical Review Letters*. I have noted that the spurious signal may be due to a second harmonic of the gravitational stretching of the interferometer. After all, the experimenters state that one of the limiting factors in their experiment is that the axis of rotation is not perfectly aligned to the true vertical. They claim the stretching is *nearly* sinusoidal which implies higher harmonics. Klauber says that one would also expect harmonics at $3f, 4f, 5f$, etc. That these are not reported as being seen is due to two reasons. First, the signal of interest is $2f$ and all other harmonics are easily discriminated against in the analysis of the spectrum. Second, one would expect the contributions of higher harmonics to be small. The fundamental has an amplitude of $200Hz$ while the second harmonic has amplitude of $17Hz$. If higher harmonics continue this drop off, then they are outside the resolution of the experiment. Klauber also claims that his theory gives a "virtually precise value" of the spurious signal. I'm not quite sure what that means since Klauber must estimate several lengths of the experimental apparatus. But even so, one should be impressed that his result is the same order of magnitude as the spurious signal. We know that Brillet and Hall were interested in testing the cosmic isotropy of the velocity of light. Since the Earth is in orbit about the sun, the theory of Klauber should also predict a large anisotropy due to this motion. After all, the Earth travels around the sun as if it were a point on a rotating disk (neglect eccentricity). Since the anisotropy predicted by Klauber depends on the velocity and the orbital velocity of the Earth is much larger than its rotational velocity, I would expect a huge effect. But this is not seen. Am I missing something here? I have suggested that a simple test, varying the axis of rotation and looking for any variation in direction of the maximum amplitude of the spurious signal, would have occurred to most people. I presume that Brillet and Hall would have noticed any fixed direction of the amplitude in their preliminary tests of their apparatus. Certainly the experiment should be repeated but I do not expect any difference in the results. Finally, Klauber suggests that his theory may also predict the fundamental signal that has been attributed to stretching of the interferometer and that such stretching may not be a factor at all. But Brillet and Hall mention observing the $24hr$ period of the floor tilt of about a μrad. They further state that they could improve their accuracy by an order of magnitude if they could stabilize the rotation axis to within $\pm 1''$. With such demanding sensitivity, one should be able to lesson the gravitational stretching but not eliminate it entirely.

Klauber: Indeed, I am tempted to simply defer further comment on Brillet and Hall until after the Tobar *et al.* results in 2004, which, though utilizing a different principle, is effectively a repeat of the Brillet and Hall experiment. Their apparatus should be impervious to gravitational stretching. However, there are subtleties involved, and as I have not yet studied it thoroughly enough, am at this point only about 85% convinced that it will be sensitive to the conjectured NTO effect. Weber makes a valid point that Brillet and Hall were filtering for the f and $2f$ signals. I wholeheartedly agree with Weber's suggestion of repeating the test and varying the alignment of the axis of rotation to prove or disprove the gravitational stretching hypothesis. Yet, I remained perplexed that if he attributes the signal in question to such stretching, he still would not expect any difference in the results. With a different alignment of the same apparatus or with different apparatus (and different structural stiffness), would we not get a different signal? Unless, of course, the true cause is light speed anisotropy... Finally, with respect to the cosmic isotropy question, the answer is subtle and is explained in my paper ”Non-time-orthogonality, gravitational orbits and Thomas precession”.[12] In my paper for this book I have summarized my results briefly and heuristically (Subsection 2.4.6), by saying that the objects in gravitational orbit are in free fall and thus obey Lorentzian mechanics. This is true, but there is more to it. However, as I said before, the subject is subtle and, since I do not want to take up too much room in this discussion, I suggest that you could read my paper quoted above.

Chairman: Thank you Profs. Klauber, Weber and Bel. I think that this discussion has shed new light on the Brillet and Hall experiment, which deserves greater attention. We also look forward to knowing the results of the new experiment which, as Klauber has just said, will be performed soon and, presumably, will represent a new test of the traditional relativistic approach against NTO analysis.

[12]gr-qc/0007018

V. Dialogue on quantum effects in rotating systems

Where the quantum effects in an inertial system, such as a rotating reference frame, are discussed and different viewpoints about the wave equations suitable for describing such systems are compared.

Chairman: Let us introduce now quantum effects in a rotating frame. This issue, that is halfway between the relativistic effects which are usually dealt with in the macrocosmos, and the quantum effects, whose arena is the microcosmos, has been studied thoroughly in the papers written by Papini, Anandan and Suzuki.

Papini: I would like to comment on the Anandan-Suzuki's paper. It has been known for some time (see for instance the papers by B.S. DeWitt[13] and myself[14]) that non-relativistic quantum particles can be described by a Schroedinger equation in which a *stationary* inertial field can be treated as a vector field. The method of infinitesimal transformations associated with boosts, rotations and time translations, applied by J. S. Bell and J. M. Leinaas[15] to the fully relativistic Dirac equation and here by Anandan and Suzuki to the non-relativistic Schroedinger equation with spin, leads to this same conclusion. However, while stationary inertial fields are important in many physical applications, the results obtained can not be automatically extrapolated to the time dependent case (e.g., pulsed rotations in accelerators and accumulation rings, plasma doughnuts, particle pulses in wakefield plasma waves) without resorting to a more complete theory of inertia. The issue here is gauge invariance. Let me explain more clearly: assume, in fact that inertia can be represented in the general case by a vector field (but let us also remind ourselves of Feynman's warning that "one consequence of the spin 1 is that likes repel, and unlikes attract"!). Then the corresponding gauge transformations would presumably look like those of an electromagnetic field and leave the Schroedinger equation (neglecting spin for simplicity)

$$i\frac{\partial\psi}{\partial t} = [\frac{1}{2m}(\vec{p} - m\vec{\mathcal{A}})^2 - \frac{1}{2}m\mathcal{A}_0^2]\psi \tag{31}$$

invariant. If, on the other hand, inertia is represented by the components of the metric tensor as in general relativity, then the Schroedinger equation becomes

[13] *Phys. Rev. Lett.* **16**, 1092 (1966)
[14] *Nuovo Cimento* **52B**, 136 (1967)
[15] *Nuclear Physics B* **284**, 488 (1987)

(to first order in the weak field approximation)

$$i\frac{\partial\psi}{\partial t} = [\frac{1}{2m}(p_i - mh_{0i})^2 - \frac{1}{2}mh_{00}]\psi. \tag{32}$$

Under a transformation of coordinates $x^\mu \to x^\mu + \xi^\mu$, still allowed by the weak field approximation (with ξ^μ small of first order), the "gauge" transformation of the metric deviation $h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$ does not leave (32) invariant, unless h_{0i} and h_{00} are time independent, as mentioned above. In other words, while eq.(31) is gauge invariant, eq.(32) is not so in the more general case and must be replaced by a covariant wave equation. Even in this simple case eqs.(31) and (32) lead to different results.

Anandan: Indeed, I discussed the issue of gauge invariance several years ago.[16] The Schrödinger equation in a weak gravitational field discussed by Papini

$$i\frac{\partial\psi}{\partial t} = [\frac{1}{2m}(p_i - mh_{0i})^2 - \frac{1}{2}mh_{00}]\psi \tag{33}$$

would not violate gauge invariance if it is interpreted to be valid only in coordinate systems at rest with respect to the apparatus, i.e. the 4-velocity field of the apparatus t^μ is proportional to $\delta_0^{\ \mu}$ in all such coordinate systems. Under the transformation between any two such coordinate systems

$$t'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^\nu} t^\nu \ , \tag{34}$$

with $t^\mu \propto \delta_0^{\ \mu}$ and $t'^{\mu} \propto \delta_0^{\ \mu}$. For an infinitesimal coordinate transformation $x'^{\mu} = x^\mu + \xi^\mu$, eq.(34) implies $\xi^i_{\ ,0} = 0, \ i = 1, 2, 3$.

Therefore the usual transformation $h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$ with this restriction transforms $G_\mu = (\frac{1}{2}h_{00,\mu} - h_{0\mu})$ (that is minimally coupled in (33)) according to

$$G_\mu \to G_\mu - \partial_\mu \xi_0$$

Thus the transformation of G_μ is entirely analogous to the gauge transformation of the electromagnetic 4-vector potential A_μ. This ensures that eq.(33) and the phase shift $\delta\phi = -\frac{m}{\hbar}\oint G_\mu dx^\mu$, which includes the Sagnac phase shift, are gauge invariant.

Therefore, it is not necessary to require that h_{00} and h_{0i} are time independent as mentioned by Papini. In fact, the latter requirement does not help to preserve gauge invariance because it is possible for $\xi^i_{\ ,0}$ to be time independent, yet non zero, in which case h_{00} and h_{0i} could be time independent in both coordinate systems; yet eq.(33) and the phase shift would not be gauge invariant.

[16]*Phys. Rev. Lett.* **47**, 463 (1981); Errata **52**, 401 (1984)

Papini: I also have a question: is the effect given in formula (17.33) of your paper a *bona fide* first order effect in view of the fact that it also is of order $(velocity)^2/c^2$?

Suzuki: Well, we take the opportunity to emphasize that, in our approximation, we neglect terms v^2/c^2, where v is the velocity *relative to the rotating frame*. But we keep terms ω^2/c^2, because in a rotating coordinate system, ω is a parameter determining the inertial fields (Coriolis and centrifugal fields). In other words, our low energy approximation is in the rotating coordinate system, and not in an inertial coordinate system.

Papini: Let me add a further remark on the effects of inertial fields on quantum particles, which is relevant to the case of the rotating frame. The phase shift $\Delta\chi$ due to the Sagnac effect, which is what one eventually measures, as Rizzi and Ruggiero rightly observe in their paper, can be readily calculated from the solution (see my paper in this volume)

$$\Phi(x) = exp\,(-i\chi)\,\phi_0(x) \simeq (1 - i\chi)\phi_0(x) \tag{35}$$

of the covariant Klein-Gordon equation, where

$$\begin{aligned} i\chi\phi_0 &= \frac{1}{4}\int_P^x dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[(x^\alpha - z^\alpha)\partial^\beta - \\ &\quad (x^\beta - z^\beta)\partial^\alpha] - \frac{1}{2}\int_P^x dz^\lambda\gamma_{\alpha\lambda}(z)\partial^\alpha]\phi_0, \end{aligned} \tag{36}$$

ϕ_0 is a solution of the free Klein-Gordon equation in Minkowski space and the metric deviation $\gamma_{\alpha\beta}$ follows from Møller's metric. For counter-rotating particle beams that originate and interfere at the same point, one obtains

$$\Delta\chi = \frac{4\kappa^0}{c}\vec{\omega}\cdot\vec{A}, \tag{37}$$

where $\kappa^0\kappa^0 - \kappa^2 = \frac{m^2c^2}{\hbar^2}$. This result is exact to first order in $\gamma_{\alpha\beta}$, completely "gauge invariant", applies to massive and massless particles, can be calculated in fast or slow particle motion approximations, does not require any dangerous (though useful) analogies with the electromagnetic field, avoids the pitfalls of Sakurai's approach and can be immediately extended to include spin-rotation coupling. It was published in a paper that I wrote with Y.Q. Cai.[17]

Bel: Let me add my personal point of view on the issue of the inertial fields. I understand what Papini means when he says that in non-relativistic quantum mechanics “a *stationary* inertial field can be treated as a vector field"

[17] *Class. Quantum Grav.* **6**, 407 (1989)

but I do not think that this is a fully correct geometrical identification. On the other hand I fully agree with him when he says that time dependent inertial fields need “resorting to a more complete theory of inertia". Here, I would like to give a short account of my personal view on this subject, first presented in my contribution to *Recent developments in gravitation.*[18] Let us consider the following family of functions in the space-time framework of non-relativistic classical mechanics:

$$x'^i = R^i_j(t)(x^j - S^j(t)), \quad t' = t \tag{38}$$

where $R^i_j(t)$ are time-dependent rotation matrices. If x^i are Cartesian coordinates in a Galilean frame of reference each member of the family can be considered as defining a rigid motion giving rise to an inertial field. But the family as a whole is an infinite dimensional group: the group of rigid motions.

The inertial field is actually a couple of two geometrical objects:

$$\Lambda^i(t, x^j) = -R^i_j(\ddot{S}^j + \ddot{R}_k^{-1j} x^k), \quad \Omega^i_j(t) = -2R^i_k \dot{R}_j^{-1k} \tag{39}$$

in the sense that any free test particle in the corresponding co-moving frame of reference moves according to the equations of motion:

$$\ddot{x}^i = \Lambda^i + \Omega^i_j \dot{x}^j \tag{40}$$

These fields are in fact components of a flat, symmetric and linear connection with non null components:

$$\Gamma^i_{00} = -\Lambda^i, \quad \Gamma^i_{j0} = -\frac{1}{2}\Omega^i_j \tag{41}$$

which means that these equations being invariant under the group of rigid motions (38).

We can associate to any such representation of this flat connection a 4-dimensional symmetric covariant tensor demanding the following invariant conditions:

$$\nabla_i g_{00} - 2\nabla_0 g_{0i} = 0, \quad \nabla_i g_{0j} - \nabla_j g_{0i} = 0, \quad g_{ij} = \delta_{i,j} \tag{42}$$

where ∇ is the symbol of covariant derivative with respect to Γ. This tensor is defined up to a gauge transformation:

$$g_{00} \to g_{00} + 2\partial_0 A(t, x^i), \quad g_{0j} \to g_{0j} + \partial_j A(t, x^i) \tag{43}$$

[18] *Recent developments in gravitation*, Eds. Verdaguer E., Garriga J. and Cespedes J. , World Scientific Pub. Co. (1990)

where $A(t, x^i)$ is an arbitrary function. One can think then of an inertial field as a connection of a particular class defined by (41), or as 4-dimensional Riemannian metric of a particular type defined by (42), to which it is connected through the formulas:

$$\Gamma^k_{00} = -\frac{1}{2}\delta^{ki}(\partial_i g_{00} - 2\partial_0 g_{0i}), \quad \Gamma^k_{0j} = -\frac{1}{2}\delta^{ki}(\partial_i g_{0j} - \partial_j g_{0i}) \tag{44}$$

So, the generalization of the Schrödinger equation for a free particle in an inertial field generated by one of the rigid motions (38) is:

$$(i\hbar\partial_t + \frac{m}{2})\Psi = \frac{1}{2m}\delta^{jk}(i\hbar\partial_j + mg_{0j})(i\hbar\partial_k + mg_{0k})\Psi \tag{45}$$

The equation remains invariant under any tensor transformation of the family (38) followed by:

i) a gauge transformation (43) where A depends on the particular transformation, and

ii) the phase change:

$$\Psi \to \exp(\frac{i}{\hbar}A)\Psi \tag{46}$$

Everything has been kept general and reduce to known results, in particular by G. Papini, in the stationary case.

Chairman: This topic is very interesting, and I know that a lot of work has been done, so I would like to thank you for these very introductory remarks, which help us to understand the foundations of this research field.

VI. Dialogue on non uniform motions and other details about Klauber's and Selleri's challenges

Where possible generalizations to the study of non uniform rotations are briefly discussed, and other details pertaining to the challenges issued by Klauber and Selleri are outlined.

Chairman: Up to now, we have spoken about stationary rotating systems. I wonder whether a more general case can be considered...

Nikolić: As far as I can see, among papers in the book, no one (except mine and that of Mashhoon) says anything about relativistic effects for motions more general than uniform rotations. For example, different parts of a rotating *soft elastic* ring may have different velocities. I want to stress that, in such a general case, it is not natural to introduce a proper frame unique for the whole ring. Instead, it is more natural to introduce a different non-inertial frame for each part of the ring. However, since this is the *general* approach, it should be used in all cases, including the case of uniform rotation. This is what I do in my paper. I would like to see what other contributors think about the problem of general motion and do they agree with my conclusions (drawn in my paper) related to this.

Ruggiero: According to our approach, the issue of rotating observers in non uniform motion is a very interesting one to deal with. As I said before, in our contribution to this book, and, also, in our paper in *Foundations of Physics*, we defined in a clear way the mathematical context in which physical measurements are performed: namely, the "space of the disk" which we have called *relative space*. Such a construction has a clear and operational meaning in the case of uniform rotation, and allows the possibility of confronting measurements performed at different points in the rotating frame. However, we believe that a similar construction cannot be done in a straightforward way in the case of non uniform rotations; furthermore, we wonder whether the notion itself of "space of the disk" is meaningful in that case. We think that this subject deserves further attention and work.

Papini: If we concern ourselves with the description of the motion of the particles, and not with the role of the observers (and, of course, with the peculiarities of their motion), we can say that motions of quantum particles and of some systems of quantum particles can be dealt with in great generality, within a weak field approximation, using appropriate wave equations in which the external inertial field is represented by the coefficients of the metric (see my contribution to this volume). Real material effects are at times very large and do not require the introduction of additional description tools. A typical example is that of some metals and of type I superconductors. If the lattice is incompressible, then it can be represented as a charged background relative to which the electrons of mass m_e flow in a viscous, or non viscous way. If the metal lattice is compressible, then the field experienced by the electrons is much larger (and of the opposite sign) because the mass of the ions is $M \gg m_e$. See, for instance the papers written by L.I. Schiff and M.V. Barnhill,[19] A.J. Dessler, F.C. Michel, H.E. Rorschach and G.T. Trammel,[20] C. Herring[21] and myself.[22]

Chairman: Thank you Prof. Papini. Now, it seems to me that Prof. Klauber wants to come back to the issue of the measurement of lengths in a rotating frame, to reply again to Prof. Grøn.

Klauber: Yes, let me delve a little into Prof. Grøn's position on the proposed contraction being due, from the rotating frame observer's position, as a "gravitational", rather than motional, effect. Imagine an effective gravity field set up on a disk as described by Grøn's metric (15.8), but the disk is not actually rotating. How this might be done is not the issue. Separating the speed effect from the effective gravitational effect is. Both the IF and the disk observer would agree that the meter sticks on the disk are contracted by the Lorentz factor and the disk surface is curved. Then, consider the disk rim moving at speed $v = \omega r$. Now, would the IF observer not see an additional contraction due to speed v? Would he not see contraction by the Lorentz factor squared? But the entire traditional analysis began from the initial assumption that he saw contraction by precisely the Lorentz factor. Once again, it seems to me, the traditional analysis is inconsistent.

Grøn: Klauber has just given a new example claiming that the traditional analysis is inconsistent. He first considers a gravity field set up in the disk as described by the metric (15.8) in my article, without putting the disk into rotation. Both an inertial observer and an observer fixed on the disk would agree that the meter sticks along circular paths about the centre of the disk are contracted by the Lorentz factor, and the disk surface is curved. Then the disk is put into rotation with angular velocity ω so that the disk rim moves with speed

[19] *Phys. Rev.*, **151**, 1067 (1966)
[20] *Phys. Rev.*, **168**, 737 (1968)
[21] *Phys. Rev.*, **171**, 1361 (1968)
[22] *Nuovo Cimento*, **63B**, 549 (1969)

$v = r\omega$. Then the inertial observer should measure an additional contraction due to the speed v. Hence, he should measure a contraction by the Lorentz factor squared. Klauber then says: "*But the entire traditional analysis began from the initial assumption that he saw contraction by precisely the Lorentz factor. Once again, it seems to me, the traditional analysis is inconsistent.*" The cited statements do not follow from the preceding description. There is no such assumption regarding meter sticks in the conventional relativistic description of a rotating system with a permanent gravitational field additionally. My point of view can be summarized as follows. In spite of certain unusual features, such as the impossibility of Einstein-synchronizing clocks along closed paths in a rotating reference frame, no inconsistency results from applying the special theory of relativity to rotating frames.

Klauber: I do know that Einstein's original thinking on this subject, as quoted in Held's book,[23] was with reference to the IF observer above the disk axis looking down on the rotating disk rim and seeing Lorentz contraction of meter sticks on the rim by precisely the Lorentz factor. I suggest also considering the observer in the disk frame on the rotation axis, rotating with the disk, and looking down at the lab meter sticks. By the same logic, she should see the lab meter sticks contracted in the circumferential direction, and therefore conclude that the lab surface is curved. Though we may all have exhausted our comments on this question for the time being, I do think it deserves more consideration in the future.

Chairman: Well, I thank you again, Prof. Klauber. The problem of the measurements of lengths in a rotating frame and the link with the spatial geometry has some interesting corollaries that are highlighted in some limit cases: they may appear surprising, isn't it so, Prof. Grøn?

Grøn: Yes, Klauber, for instance, considers the problem of the relations between lengths measured in the laboratory (inertial) frame and those measured in the rotating frame, in the limit case of low ω , high r, such that $a = \omega^2 r \simeq 0$, while $v = \omega r$ is close to the speed of light. He states: "In this case, each of the lab and disk observers must see the other's meter sticks as contracted and their own as normal. Yet, the non-inertial argument started with the assumption that the disk observer's meter sticks contracted in an absolute way, agreed to by all observers. Conclusion: Length contraction applied via the traditional analysis to rotating systems appears self-contradictory." As I said

[23] J. Stachel in *General Relativity and Gravitation* Ed. Held H., Plenum, NY (1980)

before, according to me there is no problem in obtaining reciprocal Lorentz contraction according to "the traditional analysis". The mentioned analysis is valid also in the limit considered by Klauber. In this connection it should be noted that the relativistic effects depend upon $r^2\omega^2/c^2$, not upon $a = r\omega^2$. The effects are absolute, but the interpretation of the effects depends upon the frame of reference. They are interpreted as velocity dependent effects in inertial frames, and as a combination of kinematical and gravitational or, if you want, inertial effects in rotating frames. These inertial effects depend upon the difference of inertial potential at the position observer and the object, not upon the acceleration. Hence, I think that there is no problem in going to the limit considered by Klauber.

Klauber: Well, I see that Grøn takes a different position regarding the limit case than I have seen from other traditionalists. He notes that the inertial (gravitational) potential is a function of $v = \omega r$, not $a = \omega^2 r$, and thus the traditional Lorentz contraction effects are not relative, but absolute. Indeed, my point was directed towards those traditionalists who argue the limit case is effectively equivalent to a Lorentz frame. I agree with Grøn that there is an absolute effect, dependent on ωr, or equivalently, the Newtonian potential $-\frac{1}{2}\omega^2 r^2$, which can be measured in experiments. For example, it would show up as a change in mass of known particle types like the electron or proton. Summarizing, as Grøn and I agree, we have an absolute velocity, $v = \omega r$, which can be determined from measurements made entirely within the rotating frame. As a consequence, a major point arises: we have a violation of a fundamental relativity postulate, which via the hypothesis of locality, is a cornerstone of the traditional rotating frame analysis. That is, Lorentz contraction is derived in the SRT, in part, from the postulate that implies velocity is relative (i.e., no preferred frame.) But in rotation, it is not. The traditional approach employs the hypothesis of locality, and thereby assumes Lorentz contraction is *a priori* true in rotation. But if one of the postulates from which Lorentz contraction is derived is not true, then it is hardly appropriate to simply assume that Lorentz contraction still holds. In my approach, using differential geometry, I show that such a contraction does not exist in rotation.

Chairman: As I said before, the consequences of the various approaches to the measurements of lengths in the rotating frame are far reaching: it is plain, and you state it clearly Prof. Klauber, that in your approach you question the hypothesis of locality. We have spoken before about Selleri's paradox, but now we can face it with a different approach, even though Serafini taught us that it can be solved, provided that the right synchronizations are taken into account.

Klauber: Yes, let me recall the terms of the paradox: Selleri states his thought-provoking point regarding the discontinuity in ρ, the ratio of co-propagating and counter-propagating lights speeds, in the limit case (low ω, high r, such that $a = \omega^2 r \approx 0$, while $v = \omega r$ is close to the speed of light.) The assumption is that the limit case approximates an inertial frame, since acceleration is negligible. However, as we have just discussed with Prof. Grøn, and I also wrote in my papers,[24] I believe that the limit case does not approximate an inertial frame, because the potential energy is not zero. The Newtonian inertial potential is $-\frac{1}{2}\omega^2 r^2$ and this is quite high in the limit case. The corresponding relativistic potential is given in eq. (26) of my paper in *Foundations of Physics* that I quoted above, but it is also a function of ωr. Hence, mass-energy of any known particle type, like the proton, will be different than its traditional rest value. And this can be measured via experiment. Thus, the limit case is not equivalent to an inertial frame, since speed v $(= \omega r)$ can be determined absolutely from inside the frame. I would like to stress again that this appears to me to negate the hypothesis of locality for rotation, as it permits a discrimination between the co-moving Lorentz frame and the local rotating frame that is not based on inertial "force". Furthermore, I submit that if one test, from within the rotating frame, can distinguish circumferential speed absolutely, then so must others, and the Brillet and Hall experiment is one of these tests.

Weber: Klauber claims the limit considered by Selleri does not approximate an inertial frame because the potential $-\omega^2 r^2/2$ can be quite large. But elementary physics tells us that only differences of potential have physical significance. Here, the potential is constant and so the associated force is zero. One can even reset the reference zero of the potential to the edge of the disk. But the potential given above is a difference between the point of interest and the arbitrary reference point taken to be the origin. Note that the origin is also part of the inertial frame. It appears that this potential is related to the difference in energy between a unit mass at rest in the inertial frame and one at rest in the rotating frame. With zero centripetal force, the rotating disk in this limit looks like an inertial frame to me.

Klauber: I agree with Weber that in elementary physics one can add any constant to the potential without affecting the force, and this is generally presumed to mean that the constant is meaningless, i.e., it simply changes gauge, not physical measurements. However, in relativity the mass of a particle depends on its total energy (kinetic plus potential.) For velocity of zero, the

[24] *Found. Phys. Lett.* **11** 405 (1998); gr-qc/0209025

baseline potential energy of the vacuum is taken as zero and one gets the rest mass of a particle like the electron or proton. Change the potential even by a constant (which it is not in gravitation or rotation) and you get a change in energy, and thus a change in mass. In rotation, the baseline potential should be taken as zero at the center of rotation in order to get the traditional value for the proton rest mass at that location. In this context, the choice of the baseline potential is not arbitrary. As one moves out radially from there, the potential energy decreases, and hence, so must the mass. As Weber says, ”this potential is related to the difference in energy between a unit mass at rest in the inertial frame and one at rest in the rotating frame”. Yes, I agree with him! So, I believe, if one is situated in a rotating frame at the limit location, there is a high negative potential and particles such as the proton would have lower than normal masses, and, measuring these masses, one could distinguish one’s local frame from an inertial frame (in which potential would be zero).

Chairman: Well, this is the end of our discussion, which, in my opinion, has been fruitful and enlightening for several reasons, and I have personally enjoyed it very much. I would like to thank all of you, for participating in this debate and for the stimulating and pleasant discussions we have had. I would like to express my special thanks to Prof. Klauber, for his passionate participation; as a matter of fact, he has deeply influenced the whole debate. Let me also say I too have learned many things, both reading your papers and chairing this discussion, which makes me now richer in knowledge and in doubts.

Weber: I would like to thank the organizers of this round table discussion and the editors of the book on rotating frames for helping to make this opportunity available.

Klauber: Tom Weber and I have been thrashing out the topic of rotation for years, and I would like to thank him for his insights, professionalism, and cordiality throughout. I join him in also thanking the editors and the round table chairman for providing this opportunity and for carrying out their responsibilities thereto so superbly.

Grøn: I would like to thank the editors of this volume, Guido Rizzi and Matteo Luca Ruggiero, for a fine job both in editing the book and in organizing the “round table conference”. It has been a pleasure to participate in this project.

Rizzi and Ruggiero: As for us, we want to thank you all for participating in this round table and, of course, in our book.

Index

Fundamental Theories of Physics

46. P.P.J.M. Schram: *Kinetic Theory of Gases and Plasmas.* 1991 ISBN 0-7923-1392-5
47. A. Micali, R. Boudet and J. Helmstetter (eds.): *Clifford Algebras and their Applications in Mathematical Physics.* 1992 ISBN 0-7923-1623-1
48. E. Prugovečki: *Quantum Geometry.* A Framework for Quantum General Relativity. 1992 ISBN 0-7923-1640-1
49. M.H. Mac Gregor: *The Enigmatic Electron.* 1992 ISBN 0-7923-1982-6
50. C.R. Smith, G.J. Erickson and P.O. Neudorfer (eds.): *Maximum Entropy and Bayesian Methods.* Proceedings of the 11th International Workshop (Seattle, 1991). 1993 ISBN 0-7923-2031-X
51. D.J. Hoekzema: *The Quantum Labyrinth.* 1993 ISBN 0-7923-2066-2
52. Z. Oziewicz, B. Jancewicz and A. Borowiec (eds.): *Spinors, Twistors, Clifford Algebras and Quantum Deformations.* Proceedings of the Second Max Born Symposium (Wrocław, Poland, 1992). 1993 ISBN 0-7923-2251-7
53. A. Mohammad-Djafari and G. Demoment (eds.): *Maximum Entropy and Bayesian Methods.* Proceedings of the 12th International Workshop (Paris, France, 1992). 1993 ISBN 0-7923-2280-0
54. M. Riesz: *Clifford Numbers and Spinors* with Riesz' Private Lectures to E. Folke Bolinder and a Historical Review by Pertti Lounesto. E.F. Bolinder and P. Lounesto (eds.). 1993 ISBN 0-7923-2299-1
55. F. Brackx, R. Delanghe and H. Serras (eds.): *Clifford Algebras and their Applications in Mathematical Physics.* Proceedings of the Third Conference (Deinze, 1993) 1993 ISBN 0-7923-2347-5
56. J.R. Fanchi: *Parametrized Relativistic Quantum Theory.* 1993 ISBN 0-7923-2376-9
57. A. Peres: *Quantum Theory: Concepts and Methods.* 1993 ISBN 0-7923-2549-4
58. P.L. Antonelli, R.S. Ingarden and M. Matsumoto: *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*. 1993 ISBN 0-7923-2577-X
59. R. Miron and M. Anastasiei: *The Geometry of Lagrange Spaces: Theory and Applications.* 1994 ISBN 0-7923-2591-5
60. G. Adomian: *Solving Frontier Problems of Physics: The Decomposition Method.* 1994 ISBN 0-7923-2644-X
61. B.S. Kerner and V.V. Osipov: *Autosolitons.* A New Approach to Problems of Self-Organization and Turbulence. 1994 ISBN 0-7923-2816-7
62. G.R. Heidbreder (ed.): *Maximum Entropy and Bayesian Methods.* Proceedings of the 13th International Workshop (Santa Barbara, USA, 1993) 1996 ISBN 0-7923-2851-5
63. J. Peřina, Z. Hradil and B. Jurčo: *Quantum Optics and Fundamentals of Physics*. 1994 ISBN 0-7923-3000-5
64. M. Evans and J.-P. Vigier: *The Enigmatic Photon.* Volume 1: The Field $\boldsymbol{B}^{(3)}$. 1994 ISBN 0-7923-3049-8
65. C.K. Raju: *Time: Towards a Constistent Theory*. 1994 ISBN 0-7923-3103-6
66. A.K.T. Assis: *Weber's Electrodynamics*. 1994 ISBN 0-7923-3137-0
67. Yu. L. Klimontovich: *Statistical Theory of Open Systems.* Volume 1: A Unified Approach to Kinetic Description of Processes in Active Systems. 1995 ISBN 0-7923-3199-0; Pb: ISBN 0-7923-3242-3
68. M. Evans and J.-P. Vigier: *The Enigmatic Photon.* Volume 2: Non-Abelian Electrodynamics. 1995 ISBN 0-7923-3288-1
69. G. Esposito: *Complex General Relativity.* 1995 ISBN 0-7923-3340-3

Fundamental Theories of Physics

70. J. Skilling and S. Sibisi (eds.): *Maximum Entropy and Bayesian Methods.* Proceedings of the Fourteenth International Workshop on Maximum Entropy and Bayesian Methods. 1996
ISBN 0-7923-3452-3
71. C. Garola and A. Rossi (eds.): *The Foundations of Quantum Mechanics Historical Analysis and Open Questions.* 1995 ISBN 0-7923-3480-9
72. A. Peres: *Quantum Theory: Concepts and Methods.* 1995 (see for hardback edition, Vol. 57)
ISBN Pb 0-7923-3632-1
73. M. Ferrero and A. van der Merwe (eds.): *Fundamental Problems in Quantum Physics*. 1995
ISBN 0-7923-3670-4
74. F.E. Schroeck, Jr.: *Quantum Mechanics on Phase Space.* 1996 ISBN 0-7923-3794-8
75. L. de la Peña and A.M. Cetto: *The Quantum Dice.* An Introduction to Stochastic Electrodynamics. 1996 ISBN 0-7923-3818-9
76. P.L. Antonelli and R. Miron (eds.): *Lagrange and Finsler Geometry.* Applications to Physics and Biology. 1996 ISBN 0-7923-3873-1
77. M.W. Evans, J.-P. Vigier, S. Roy and S. Jeffers: *The Enigmatic Photon.* Volume 3: Theory and Practice of the $\boldsymbol{B}^{(3)}$ Field. 1996 ISBN 0-7923-4044-2
78. W.G.V. Rosser: *Interpretation of Classical Electromagnetism.* 1996 ISBN 0-7923-4187-2
79. K.M. Hanson and R.N. Silver (eds.): *Maximum Entropy and Bayesian Methods.* 1996
ISBN 0-7923-4311-5
80. S. Jeffers, S. Roy, J.-P. Vigier and G. Hunter (eds.): *The Present Status of the Quantum Theory of Light.* Proceedings of a Symposium in Honour of Jean-Pierre Vigier. 1997
ISBN 0-7923-4337-9
81. M. Ferrero and A. van der Merwe (eds.): *New Developments on Fundamental Problems in Quantum Physics.* 1997 ISBN 0-7923-4374-3
82. R. Miron: *The Geometry of Higher-Order Lagrange Spaces.* Applications to Mechanics and Physics. 1997 ISBN 0-7923-4393-X
83. T. Hakioğlu and A.S. Shumovsky (eds.): *Quantum Optics and the Spectroscopy of Solids.* Concepts and Advances. 1997 ISBN 0-7923-4414-6
84. A. Sitenko and V. Tartakovskii: *Theory of Nucleus.* Nuclear Structure and Nuclear Interaction. 1997 ISBN 0-7923-4423-5
85. G. Esposito, A.Yu. Kamenshchik and G. Pollifrone: *Euclidean Quantum Gravity on Manifolds with Boundary.* 1997 ISBN 0-7923-4472-3
86. R.S. Ingarden, A. Kossakowski and M. Ohya: *Information Dynamics and Open Systems.* Classical and Quantum Approach. 1997 ISBN 0-7923-4473-1
87. K. Nakamura: *Quantum versus Chaos*. Questions Emerging from Mesoscopic Cosmos. 1997
ISBN 0-7923-4557-6
88. B.R. Iyer and C.V. Vishveshwara (eds.): *Geometry, Fields and Cosmology.* Techniques and Applications. 1997 ISBN 0-7923-4725-0
89. G.A. Martynov: *Classical Statistical Mechanics.* 1997 ISBN 0-7923-4774-9
90. M.W. Evans, J.-P. Vigier, S. Roy and G. Hunter (eds.): *The Enigmatic Photon.* Volume 4: New Directions. 1998 ISBN 0-7923-4826-5
91. M. Rédei: *Quantum Logic in Algebraic Approach.* 1998 ISBN 0-7923-4903-2
92. S. Roy: *Statistical Geometry and Applications to Microphysics and Cosmology*. 1998
ISBN 0-7923-4907-5
93. B.C. Eu: *Nonequilibrium Statistical Mechanics.* Ensembled Method. 1998
ISBN 0-7923-4980-6

Fundamental Theories of Physics

94. V. Dietrich, K. Habetha and G. Jank (eds.): *Clifford Algebras and Their Application in Mathematical Physics.* Aachen 1996. 1998 ISBN 0-7923-5037-5
95. J.P. Blaizot, X. Campi and M. Ploszajczak (eds.): *Nuclear Matter in Different Phases and Transitions.* 1999 ISBN 0-7923-5660-8
96. V.P. Frolov and I.D. Novikov: *Black Hole Physics*. Basic Concepts and New Developments. 1998 ISBN 0-7923-5145-2; Pb 0-7923-5146
97. G. Hunter, S. Jeffers and J-P. Vigier (eds.): *Causality and Locality in Modern Physics.* 1998 ISBN 0-7923-5227-0
98. G.J. Erickson, J.T. Rychert and C.R. Smith (eds.): *Maximum Entropy and Bayesian Methods.* 1998 ISBN 0-7923-5047-2
99. D. Hestenes: *New Foundations for Classical Mechanics (Second Edition)*. 1999 ISBN 0-7923-5302-1; Pb ISBN 0-7923-5514-8
100. B.R. Iyer and B. Bhawal (eds.): *Black Holes, Gravitational Radiation and the Universe*. Essays in Honor of C. V. Vishveshwara. 1999 ISBN 0-7923-5308-0
101. P.L. Antonelli and T.J. Zastawniak: *Fundamentals of Finslerian Diffusion with Applications.* 1998 ISBN 0-7923-5511-3
102. H. Atmanspacher, A. Amann and U. Müller-Herold: *On Quanta, Mind and Matter Hans Primas in Context.* 1999 ISBN 0-7923-5696-9
103. M.A. Trump and W.C. Schieve: *Classical Relativistic Many-Body Dynamics.* 1999 ISBN 0-7923-5737-X
104. A.I. Maimistov and A.M. Basharov: *Nonlinear Optical Waves.* 1999 ISBN 0-7923-5752-3
105. W. von der Linden, V. Dose, R. Fischer and R. Preuss (eds.): *Maximum Entropy and Bayesian Methods Garching, Germany 1998.* 1999 ISBN 0-7923-5766-3
106. M.W. Evans: *The Enigmatic Photon Volume 5: O(3) Electrodynamics.* 1999 ISBN 0-7923-5792-2
107. G.N. Afanasiev: *Topological Effects in Quantum Mecvhanics.* 1999 ISBN 0-7923-5800-7
108. V. Devanathan: *Angular Momentum Techniques in Quantum Mechanics.* 1999 ISBN 0-7923-5866-X
109. P.L. Antonelli (ed.): *Finslerian Geometries A Meeting of Minds.* 1999 ISBN 0-7923-6115-6
110. M.B. Mensky: *Quantum Measurements and Decoherence Models and Phenomenology.* 2000 ISBN 0-7923-6227-6
111. B. Coecke, D. Moore and A. Wilce (eds.): *Current Research in Operation Quantum Logic*. Algebras, Categories, Languages. 2000 ISBN 0-7923-6258-6
112. G. Jumarie: *Maximum Entropy, Information Without Probability and Complex Fractals.* Classical and Quantum Approach. 2000 ISBN 0-7923-6330-2
113. B. Fain: *Irreversibilities in Quantum Mechanics*. 2000 ISBN 0-7923-6581-X
114. T. Borne, G. Lochak and H. Stumpf: *Nonperturbative Quantum Field Theory and the Structure of Matter*. 2001 ISBN 0-7923-6803-7
115. J. Keller: *Theory of the Electron*. A Theory of Matter from START. 2001 ISBN 0-7923-6819-3
116. M. Rivas: *Kinematical Theory of Spinning Particles*. Classical and Quantum Mechanical Formalism of Elementary Particles. 2001 ISBN 0-7923-6824-X
117. A.A. Ungar: *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession.* The Theory of Gyrogroups and Gyrovector Spaces. 2001 ISBN 0-7923-6909-2
118. R. Miron, D. Hrimiuc, H. Shimada and S.V. Sabau: *The Geometry of Hamilton and Lagrange Spaces*. 2001 ISBN 0-7923-6926-2

Fundamental Theories of Physics

119. M. Pavšič: *The Landscape of Theoretical Physics: A Global View*. From Point Particles to the Brane World and Beyond in Search of a Unifying Principle. 2001 ISBN 0-7923-7006-6
120. R.M. Santilli: *Foundations of Hadronic Chemistry*. With Applications to New Clean Energies and Fuels. 2001 ISBN 1-4020-0087-1
121. S. Fujita and S. Godoy: *Theory of High Temperature Superconductivity*. 2001 ISBN 1-4020-0149-5
122. R. Luzzi, A.R. Vasconcellos and J. Galvão Ramos: *Predictive Statitical Mechanics*. A Nonequilibrium Ensemble Formalism. 2002 ISBN 1-4020-0482-6
123. V.V. Kulish: *Hierarchical Methods*. Hierarchy and Hierarchical Asymptotic Methods in Electrodynamics, Volume 1. 2002 ISBN 1-4020-0757-4; Set: 1-4020-0758-2
124. B.C. Eu: *Generalized Thermodynamics*. Thermodynamics of Irreversible Processes and Generalized Hydrodynamics. 2002 ISBN 1-4020-0788-4
125. A. Mourachkine: *High-Temperature Superconductivity in Cuprates.* The Nonlinear Mechanism and Tunneling Measurements. 2002 ISBN 1-4020-0810-4
126. R.L. Amoroso, G. Hunter, M. Kafatos and J.-P. Vigier (eds.): *Gravitation and Cosmology: From the Hubble Radius to the Planck Scale*. Proceedings of a Symposium in Honour of the 80th Birthday of Jean-Pierre Vigier. 2002 ISBN 1-4020-0885-6
127. W.M. de Muynck: *Foundations of Quantum Mechanics, an Empiricist Approach*. 2002 ISBN 1-4020-0932-1
128. V.V. Kulish: *Hierarchical Methods.* Undulative Electrodynamical Systems, Volume 2. 2002 ISBN 1-4020-0968-2; Set: 1-4020-0758-2
129. M. Mugur-Schächter and A. van der Merwe (eds.): *Quantum Mechanics, Mathematics, Cognition and Action*. Proposals for a Formalized Epistemology. 2002 ISBN 1-4020-1120-2
130. P. Bandyopadhyay: *Geometry, Topology and Quantum Field Theory*. 2003 ISBN 1-4020-1414-7
131. V. Garzó and A. Santos: *Kinetic Theory of Gases in Shear Flows*. Nonlinear Transport. 2003 ISBN 1-4020-1436-8
132. R. Miron: *The Geometry of Higher-Order Hamilton Spaces*. Applications to Hamiltonian Mechanics. 2003 ISBN 1-4020-1574-7
133. S. Esposito, E. Majorana Jr., A. van der Merwe and E. Recami (eds.): *Ettore Majorana: Notes on Theoretical Physics.* 2003 ISBN 1-4020-1649-2
134. J. Hamhalter. *Quantum Measure Theory.* 2003 ISBN 1-4020-1714-6
135. G. Rizzi and M.L. Ruggiero: *Relativity in Rotating Frames.* Relativistic Physics in Rotating Reference Frames. 2004 ISBN 1-4020-1805-3

KLUWER ACADEMIC PUBLISHERS – DORDRECHT / BOSTON / LONDON