

The Implicit Structure of Planck's Constant

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Abstract

Max Planck derived natural units of length, mass, and time on the assumption that each of the universal constants embodies natural units in its unit dimensions. The four natural units and dimensions comprising Planck's constant infuse more granular elements into the formulas enriching our understanding of the physical constants and the phenomena they represent. The natural units offer a consistent language for comparing classical and quantum mechanical formulas.

Index Terms

Constant of proportionality, natural units, Planck constant, quantum of action.

I. INTRODUCTION

Planck's constant plays a central role in quantum physics, appearing regularly in formulas describing the natural world on small scales. The initial motivation for the constant was to explain the spectral radiance of black bodies, but its importance has grown over more than a century as the constant has been linked with a wide variety of quantum phenomena.

While Planck's constant plays an unambiguous role in the mathematical formulations of quantum theory, it is not immediately clear what physical attribute or dynamic it represents [1] or why it plays such an important role in nature. Its meaning is derived from the output of equations yielding descriptions like the *constant of proportionality* and *quantum of action*. In this regard, Planck's constant behaves like a function, transforming formula inputs into outputs while the physical meaning of the function remains a mystery. A closer look at the constant in each of its unit dimensions gives a more detailed description of the physical relationships and transformations encoded in the formulas.

II. THE STRUCTURE OF PLANCK'S CONSTANT

Max Planck introduced the constant of proportionality h as a fundamental constant of nature with a single value and compound unit dimensions L^2MT^{-1} . By combining the new constant with values and dimensions of the gravitational constant and speed of light, Planck derived quantities for each unit dimension shared by the three constants.

TABLE I
PLANCK DERIVED VALUES FOR THE UNIT DIMENSIONS OF \hbar , G , AND c

Natural unit	Symbol	Planck formula
Planck length	l_P	$\sqrt{\frac{\hbar G}{c^3}}$
Planck mass	m_P	$\sqrt{\frac{\hbar c}{G}}$
Planck time	t_P	$\sqrt{\frac{\hbar G}{c^5}}$

These enigmatic ratios offer little or no *ontological* meaning to the concepts of length, mass, and time leaving many to question their usefulness. But an equivalent way of employing the universal constants is by expressing them in each unit dimension as shown in Table II. This approach has the considerable advantage of introducing more granular elements into the equations.

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TABLE II
 PLANCK’S CONSTANT, THE GRAVITATIONAL CONSTANT, AND THE SPEED OF LIGHT CAN BE EXPRESSED AS COMPOSITE QUANTITIES OF NATURAL UNITS

Physical constant	Symbol	Dimensions	Natural unit formula	Simple formula
Speed of light	c	LT^{-1}	$\frac{l_P}{t_P}$	c
Reduced Planck constant	\hbar	L^2MT^{-1}	$\frac{l_P^2 m_P}{t_P}$	$l_P m_P c$
Gravitational constant	G	$L^3M^{-1}T^{-2}$	$\frac{l_P^3}{m_P t_P^2}$	$\frac{l_P}{m_P} c^2$

That each of the universal constants may be reduced to natural units of the correct magnitude and dimensions suggests that perhaps Planck should have presented the natural units as fundamental and the universal constants as their compound derivatives. Natural units are more *elemental* in that each unit contains a single unit dimension. It requires only one Planck unit per unit dimension to construct the universal constants while it requires 16, 12, and 20 unit dimensions of universal constants to construct a single natural unit dimension of length, mass, and time respectively.

Table III gives a deeper look at Planck’s derivation of natural units. His formulas reflect the challenge of isolating a single unit dimension from a given set of compound quantities containing 2, 4, and 6 unit dimensions (c , \hbar , and G respectively). The particular ratios of universal constants in the formulas are simply what’s required to isolate an individual unit dimension; furthermore, it is difficult to conceive how these formulas represent anything fundamental about length, mass, and time.

TABLE III
 THE RATIOS OF UNIVERSAL CONSTANTS IN PLANCK’S FORMULAS REFLECT THE CHALLENGE OF ISOLATING A SINGLE UNIT DIMENSION FROM PARTICULAR SETS OF COMPOUND UNIT DIMENSIONS IN THE CONSTANTS

Natural unit	Planck formula	Natural unit formula
Planck length	$\sqrt{\frac{\hbar G}{c^3}}$	$\sqrt{\frac{l_P^2 m_P}{t_P} \frac{t_P}{m_P t_P} \frac{t_P}{l_P}} = l_P$
Planck mass	$\sqrt{\frac{\hbar c}{G}}$	$\sqrt{\frac{t_P^2 m_P}{t_P} \frac{t_P}{m_P} \frac{m_P t_P}{l_P}} = m_P$
Planck time	$\sqrt{\frac{\hbar G}{c^5}}$	$\sqrt{\frac{t_P^2 m_P}{t_P} \frac{t_P}{m_P} \frac{t_P^2}{l_P} \frac{t_P}{l_P}} = t_P$

Whether it is better to describe the physical universe in terms of the compound quantities of Planck’s constant and the gravitational constant, or by reducing these constants into their individual unit dimensions cannot be resolved mathematically—the computations are equivalent. However, the natural unit formulas provide more information about the physical transformations encoded in the mathematics. For example, it is well-established that the speed of light is the ratio of Planck length to Planck time; and it is the concepts of length and time that give physical meaning to velocity. It is similarly the case that the physical meaning of Planck’s constant is found in each of the unit dimensions it contributes to the formulas. These contributions are discussed in Section III.

III. NATURAL UNIT FORMULAS

The significance of natural units becomes clear when we investigate Planck’s constant in each unit dimension of length, mass, and time. One could imagine that the equations would form no meaningful pattern if the constant is fundamental and the natural units are derivatives; but this is not the case. Instead, the natural units reveal a consistent pattern by which physical attributes of a particle or system are proportional to a Planck scale basis or limit [2]. For example, replacing \hbar with $l_P m_P c$ gives the following momentum formula

$$p = \frac{\hbar}{\lambda} = \left(\frac{l_P}{\lambda} \right) m_P c \tag{1}$$

The important structural insight is that Planck's constant provides a computational basis of Planck momentum and a quantity of Planck length that produces a dimensionless ratio with the photon's wavelength. The formula shows that a photon's momentum is proportional to the Planck momentum and that the ratio between the Planck length and the photon's wavelength is the scaling factor.

The Planck scale plays a hidden role in the equations of physics, giving a quantitative basis for calculating physical phenomena like momentum and energy at all scales. However, it is not generally understood that the universal constants are injecting this basis into the formulas. It is an occasional practice to set the universal constants equal to one which gives proportionally meaningful insights, but this is an opaque form of the more granular insights contained in each unit dimension.

The dimensional structure of quantum formulas can be summarized in two parts:

- 1) A Planck scale limit or basis in the unit dimensions of a physical phenomenon (mass, momentum, energy, force, etc.).
- 2) One or more dimensionless ratios quantifying the physical phenomenon in relation to the Planck scale.

In each formula, the conversion of universal constants into natural units produces the correct Planck scale computational basis and the correct dimensionless ratio(s). This pattern is *necessary* for obtaining the correct unit dimensions in the result. The same pattern appears in the photon energy formula

$$E = \frac{\hbar c}{\lambda} = \left(\frac{l_P}{\lambda}\right) E_P \quad (2)$$

Equations 1 and 2 show that a photon's momentum and energy are not only proportional to Planck's constant; they are also proportional to the Planck momentum and Planck energy. This insight gives a more detailed explanation of quantization.

A. Quantization

Equations (1) and (2) illustrate why Planck's constant is associated with the quantization of physical quantities like momentum and energy. The natural unit formulas articulate two distinct levels of quantization that were described by Planck and Einstein in terms of Planck's constant [3] [4]. The quantization of energy for a single quantum of electromagnetic radiation is defined by equation (2) and is quantized in intervals of l_P/λ . The inverse wavelength ratio shows that intervals of a photon's energy become smaller as wavelength grows longer, asymptotically approaching zero energy. The range of possible photon energies extends from the asymptotic zero limit to the maximum Planck energy.

This natural energy scale shows that the possible energy states of a photon are not increments of a minimum unit in the same way that units of currency build up in multiples of a base unit. Photon energy can be arbitrarily small.

Another form of quantization is found in the separation of energy into individual packets of electromagnetic radiation which Einstein described in his seminal work on the photoelectric effect. The contribution of each discrete photon to radiant intensity is another level of quantization and one that more closely resembles the currency analogy.

These two forms of quantization give a structural description of the possible energy states represented by the formulas. However, an additional consideration is the state of the oscillator itself which may impose additional restrictions on the possible energy states as in the case of atomic radiation [5].

IV. PLANCK'S CONSTANT AND MATTER

While classical and quantum physics are considered separate domains in which different rules and formulas apply, it is reasonable to conjecture that classical physics is the large-scale manifestation of quantum phenomena. However, the relationship between these two scales, and the transitions from one to the other, are not well-defined.

Planck's constant establishes a connection between classical and quantum physics with the Compton and de Broglie wavelength formulas. The de Broglie formula relates Planck's constant with the mass and velocity of a matter particle, shown here in each unit dimension

$$\lambda = \frac{\hbar}{m_0 v} = \left(\frac{m_P}{m_0}\right) \left(\frac{c}{v}\right) l_P \quad (3)$$

The Compton wavelength formula gives the non-relativistic limit of the de Broglie wavelength as the particle's velocity approaches the speed of light

$$\lambda_C = \frac{\hbar}{m_0 c} = \left(\frac{m_P}{m_0}\right) \left(\frac{c}{c}\right) l_P. \quad (4)$$

According to the formulas, a matter particle's fixed rest mass limits the scale of its wavelength and momentum compared to a photon. While a photon approaches the Planck momentum $m_P c$ as its wavelength approaches the Planck length (shown in equation (1)), the electron approaches a maximum momentum $m_0 c$ and a minimum wavelength λ_C as its velocity approaches the speed of light. This physical characterization of the formulas and its consequences are generally overlooked. For example, the formulas explain why classical momentum is equivalent to quantum mechanical momentum given a massive particle's de Broglie wavelength. Re-arranging equation (3) shows why this is the case

$$\frac{m_0 v}{m_P c} = \frac{l_P}{\lambda} \quad (5)$$

The momentum of an electron can therefore be stated in each of the following forms

$$p = m_0v = \left(\frac{m_0v}{m_{pC}}\right)m_{pC} = \left(\frac{l_P}{\lambda}\right)m_{pC} = \frac{\hbar}{\lambda} \quad (6)$$

Furthermore, it is a philosophical choice as to which formula gives a proper physical representation of momentum. However, the quantum mechanical characterization of momentum may be applied to both matter and radiation—given the de Broglie hypothesis—whereas the rest mass and velocity terms of the classical formula do not pertain to a photon.

A fortuitous consequence of equation (6) is that classical systems are easily measured by their rest mass and velocity; but ease of measurement renders no insight into the physical meaning of the formulas.

Another meaningful explanation obtained from the formulas is the reason for a squared relationship between kinetic energy and velocity. Note from equations (3) and (4) that a matter particle's velocity and wavelength are inversely correlated

$$\frac{\lambda_C}{\lambda} = \frac{v}{c} \quad (7)$$

which presents another philosophical choice. If we choose the quantum mechanical description of momentum, we can explain a change in kinetic energy as the combined changes in wavelength and velocity, and not velocity squared. In other words, a $2x$ increase in velocity accompanies a $1/2x$ reduction in wavelength, giving a $4x$ increase in kinetic energy. The classical formula offers no justification for squaring the velocity.

The physical structures described by these formulas have practical applications as well. For example, equations (3) and (4) give a simple but overlooked method for measuring Planck's constant with the same precision as measurements of the electron mass and Compton wavelength, which are close to the precision of measurements taken with the Kibble Balance [6]

$$\hbar = l_P m_{pC} = \lambda_C m_0 c = 1.054\,571\,8176 \times 10^{-34} \text{ kgm}^2/\text{s} \quad (8)$$

Equation (4) shows that the ratio of a particle's mass to Planck mass is equal to the ratio of Planck length to its Compton wavelength

$$\frac{l_P}{\lambda_C} = \frac{m_0}{m_P} \quad (9)$$

which can also be expressed as the invariant product of Compton wavelength and rest mass

$$\lambda_C m_0 = l_P m_P \quad (10)$$

Substituting 10 into the natural unit formula for Planck's constant $\hbar = l_P m_{pC}$ gives equation (8). The reciprocal relationship between rest mass and Compton wavelength is an important structural insight that explains why Planck's constant *is* a constant.

V. CONCLUSION

More than a century after Max Planck introduced his constant of proportionality, the physical meaning of h is not widely appreciated. Equations with Planck's constant are black box functions that transform formula inputs into outputs while the inner workings of the functions remain a mystery.

Deconstructing the composite value and unit dimensions of Planck's constant into discrete units of length, mass, and time transforms black box functions into meaningful descriptions of physical phenomena. These formulas show how classical and quantum formulas are equivalent ways of representing a single, common structure underlying the mechanics of elementary particles of matter and radiation.

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