A Necessary Condition for the Geodynamo

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A necessary condition for the generation of magnetic fields by fluid motions in a sphere is derived in terms of the magnetic Reynolds number on the basis of the radial component of the velocity field. A second parameter entering the criterion is the ratio between the energy of the poloidal component of the magnetic field and the total magnetic energy. Since bounds on this ratio can be obtained from energetic considerations, the criterion can be used as a restriction on possible dynamo mechanisms. Several recent suggestions for the origin of the geodynamo in a stratified outer core are critically reviewed.

It is generally accepted that the earth's magnetic field is generated by motions within the liquid outer core of the earth. Yet in spite of a considerable research effort in the past decades, it has not been possible to find an unambiguous solution for the source of the energy dissipated by ohmic heating and viscous friction. The difficulty of this problem has been compounded recently by the suggestion of *Higgins and Kennedy* [1971] that the outer core is stably stratified. This proposal would eliminate or severely inhibit the traditional contenders for the energy source of the geodynamo, namely, convection and precession of the earth [*Bullard*, 1949; *Malkus*, 1968]. Stimulated by *Kennedy and Higgins'* [1973] 'core paradox,' a number of workers have proposed alternative sources for the earth's magnetic field [*Bullard and Gubbins*, 1971; *Won and Kuo*, 1973; *Mullan*, 1973]. In general, however, these proposals fail to take into account the rather stringent dynamic requirements for the geodynamo. This note will derive a simple necessary condition for the geodynamo that may help to restrict the class of feasible hypotheses.

In view of the complexities of actual solutions of the dynamo problem, necessary conditions for the generation of the earth's magnetic field have long been regarded as highly desirable. The only known quantitative condition of this kind is a lower bound on the magnetic Reynolds number Re_m. The existence of a lower bound was suggested originally by *Bullard and Gellman* [1954], and an explicit value applicable to the earth has been derived by *Backus* [1958]. According to this criterion, any magnetic field must decay unless

$$Re_m \equiv Ur_0/\eta \geq \pi \qquad (1)$$

where r_0 is the radius of the earth's core, which has been assumed as a homogeneous fluid sphere inside an insulating mantle, U is the maximum velocity with respect to an arbitrary system of coordinates rotating with a constant angular velocity, and η is the magnetic diffusivity. Condition (1) was derived by Backus with the maximum deformation rate in place of $U\pi/r_0$, which is advantageous in that it becomes obvious that a rigid rotation does not contribute to U. The form (1) of the criterion was given by *Childress* [1969]. We also refer to the discussion by *Roberts* [1971]. Neither the presence of the rigid inner core nor the inhomogeneities of the outer core and the finite conductivity of the mantle have been taken into account in (1) since their effects are of minor importance.

A disadvantage of (1) is that it does not distinguish between different components of the velocity field. Most theories of the geomagnetic field assume a large differential rotation in the earth's core and a smaller meridional circulation. Similarly, *Kahle et al.* [1967] found different orders of magnitude for the toroidal and poloidal components of the velocity field in their attempt to infer motions of the core from the observed secular variation. The poloidal component is generally smaller yet of particular importance since it can be shown that a purely toroidal velocity field cannot generate a magnetic field [*Bullard and Gellman*, 1954]. Only the poloidal part of the velocity field has a radial component, and it is desirable for this reason to find a condition similar to (1) involving the radial component of the velocity field. This will be the goal of the analysis described below. The importance of such a condition is emphasized in the case of a stably stratified core as proposed by *Higgins and Kennedy* [1971]. Although toroidal motions would remain unaffected in this case, any flow with a radial velocity component would be inhibited, with the possible exception of internal gravity waves.

MATHEMATICAL ANALYSIS

In order to derive our criterion, we consider an incompressible homogeneous fluid contained in the finite volume V. Since the first part of our derivation does not depend on the particular shape of V, we shall assume only later that V is a sphere. The magnetic flux density $\mathbf{B}$ is governed by the dynamo equation

$$(\partial/\partial t + \mathbf{v} \cdot \nabla)\mathbf{B} + \eta \nabla \times (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{v} \qquad (2)$$

which can be derived easily from Maxwell's equation and Ohm's law in the magnetohydrodynamic approximation. The magnetic diffusivity η is equal to $(\sigma\mu)^{-1}$, where σ is the electrical conductivity in V and μ is the magnetic permeability. We assume that the space outside V is insulating. Hence $\nabla \times \mathbf{B} = 0$ holds outside V, and $\mathbf{r} \cdot \mathbf{B}|\mathbf{r}|^2$ remains finite as the position vector $\mathbf{r}$ tends to infinity.

By multiplying (2) by $\mathbf{r}$ and using the vector identity $\mathbf{r} \cdot (\mathbf{b} \cdot \nabla\mathbf{a}) = \mathbf{b} \cdot \nabla\mathbf{r} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b}$, we obtain

$$(\partial/\partial t + \mathbf{v} \cdot \nabla)\mathbf{r} \cdot \mathbf{B} - \eta\nabla^2\mathbf{r} \cdot \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{v} \cdot \mathbf{r} \qquad (3)$$

in V. This equation appears in a slightly different form in *Backus'* [1968] paper, which also emphasizes the analogy to the heat equation, the right-hand side of (3) representing the heat source. Since diffusion ultimately balances the source term in the stationary case, (3) suggests an order of magnitude estimate for the radial velocity component.

$$v_r \sim \eta \frac{B_r}{|\mathbf{B}|\, r_0}$$

In the following we shall derive a relation of similar form by a rigorous analysis.

Multiplication of (3) by $\mathbf{r}\cdot\mathbf{B}$ and integration over V yield

$$\frac{1}{2}\frac{d}{dt}\int_V (\mathbf{B}\cdot\mathbf{r})^2\,dV = -\eta\int_{V+V'} |\nabla\mathbf{B}\cdot\mathbf{r}|^2\,dV + \int_V \mathbf{r}\cdot\mathbf{B}\,\mathbf{B}\cdot\nabla\,\mathbf{r}\cdot\mathbf{v}\,dV \qquad (4)$$

We have denoted the space outside V by V'. The surface separating V and V' is S with the outside normal $\mathbf{n}$. The integral over $V + V'$ in (4) has been obtained by partial integration and using the fact that $\nabla^2\mathbf{r}\cdot\mathbf{B}$ vanishes in V':

$$0 = \int_{V'} \mathbf{r}\cdot\mathbf{B}\,\nabla^2\mathbf{r}\cdot\mathbf{B}\,dV = -\oint_S \mathbf{B}\cdot\mathbf{r}\,\mathbf{n}\cdot\nabla\,\mathbf{r}\cdot\mathbf{B}\,dS - \int_{V'} |\nabla\mathbf{r}\cdot\mathbf{B}|^2\,dV$$

In deriving (4), the fact that the term

$$\int_V \mathbf{v}\cdot\nabla\tfrac{1}{2}\,|\mathbf{r}\cdot\mathbf{B}|^2\,dV = \oint_S \mathbf{n}\cdot\mathbf{v}\tfrac{1}{2}\,|\mathbf{r}\cdot\mathbf{B}|^2\,dS$$

vanishes since $\mathbf{n}\cdot\mathbf{v}$ vanishes on S has also been used. By further partial integration and by using $\nabla\cdot\mathbf{B} = 0$ and assuming that $\mathbf{v}\cdot\mathbf{r}$ vanishes on S, we find

$$\int_V \mathbf{r}\cdot\mathbf{B}\,\mathbf{B}\cdot\nabla\,\mathbf{r}\cdot\mathbf{v}\,dV = -\int_V \mathbf{v}\cdot\mathbf{r}\,\mathbf{B}\cdot\nabla\,\mathbf{B}\cdot\mathbf{r}\,dV$$

The latter term can be bounded from above,

$$-\int_V \mathbf{v}\cdot\mathbf{r}\,\mathbf{B}\cdot\nabla\,\mathbf{B}\cdot\mathbf{r}\,dV \le \max(\mathbf{v}\cdot\mathbf{r})\left(\int_V |\mathbf{B}|^2\,dV\int_V |\nabla\mathbf{r}\cdot\mathbf{B}|^2\,dV\right)^{1/2} \qquad (5)$$

where Schwarz's inequality has been used. Thus we obtain from (4) the inequality

$$\frac{1}{2}\frac{d}{dt}\int_V (\mathbf{B}\cdot\mathbf{r})^2\,dV \le \left[-\eta + \max(\mathbf{v}\cdot\mathbf{r})\cdot\left(\int_V |\mathbf{B}|^2\,dV\Big/\int_{V+V'} |\nabla\mathbf{r}\cdot\mathbf{B}|^2\,dV\right)^{1/2}\right]\cdot\int_{V+V'} |\nabla\mathbf{r}\cdot\mathbf{B}|^2\,dV \qquad (6)$$

Obviously, the radial component of $\mathbf{B}$ must decay when the quantity within the brackets is negative.

When V is a sphere, we can derive a condition that permits a physical interpretation. Assuming the origin at the center of the sphere, we use a representation of $\mathbf{B}$ in terms of poloidal and toroidal components:

$$\mathbf{B} = \nabla\times(\nabla\times\mathbf{r}h) + \nabla\times\mathbf{r}g \qquad (7)$$

It is evident that only the poloidal field h contributes to the radial component of $\mathbf{B}$,

$$\mathbf{r}\cdot\mathbf{B} = \mathbf{r}\cdot[\nabla\times(\nabla\times\mathbf{r}h)] = L^2h$$

where $-L^2$ is the two-dimensional Laplacian on the surface of the unit sphere. Since we can assume without losing generality that the average of h over any spherical surface $|\mathbf{r}| = \text{const}$ vanishes, we find $L^2h \ge 2h$, where the equality sign is assumed when the θ, φ dependence of h in a spherical system of coordinates corresponds to the lowest possible spherical harmonic $l = 1$.

$$\int_{V+V'} |\nabla\mathbf{B}\cdot\mathbf{r}|^2\,dV = -\int_V L^2h\nabla^2L^2h\cdot dV \ge -2\int_V h\mathbf{r}\cdot\nabla\times(\nabla\times\mathbf{r}\nabla^2h)\,dV \qquad (8)$$

The last term in (8) can be written in the form

$$2\int_V h\mathbf{r}\cdot\nabla\times\{\nabla\times[\nabla\times(\nabla\times\mathbf{r}h)]\}\,dV = 2\int_V (\nabla\times\mathbf{r}h)\cdot\nabla\times[\nabla\times(\nabla\times\mathbf{r}h)]\,dV = 2\int_{V+V'} |\nabla\times(\nabla\times\mathbf{r}h)|^2\,dV \qquad (9)$$

where the relation

$$\int_{V'} |\nabla\times(\nabla\times\mathbf{r}h)|^2\,dV = \oint_S (\nabla\times\mathbf{r}h)\times[\nabla\times(\nabla\times\mathbf{r}h)]\cdot\mathbf{n}\,dS$$

has been used. Apart from a factor 4μ, (9) gives the energy E_p of the poloidal part of the magnetic field. Hence (6) can be written in the case of a sphere as

$$\frac{1}{2}\frac{d}{dt}\int_V (B\cdot\mathbf{r})^2\,dV \le \left[-\eta + \max(\mathbf{v}\cdot\mathbf{r})\left(\frac{E_M}{2E_p}\right)^{1/2}\right]\cdot\int_{V+V'} |\nabla\mathbf{r}\cdot B|^2\,dV \qquad (10)$$

where E_M denotes the total energy of the magnetic field. Accordingly, we find as a necessary condition for the amplification of $\int_V(\mathbf{B}\cdot\mathbf{r})^2\,dV$

$$\max(\mathbf{v}\cdot\mathbf{r}) > \eta(2E_p/E_M)^{1/2} \qquad (11)$$

In the case of a nonstationary cyclic dynamo, this condition must be satisfied throughout only part of the cycle. In the case of a stationary dynamo, (11) provides a necessary condition for the existence of the dynamo. Since a lower limit for the value of E_p is available from the observed geomagnetic field and since an upper estimate for E_M can be obtained from energy considerations, (11) provides a useful test in addition to (1) for the feasibility of hypothetical geodynamos.

An analogous, though less useful, criterion can be derived by multiplying (2) by an arbitrary unit vector $\mathbf{k}$. Multiplication of the resulting equation by $\mathbf{k}\cdot\mathbf{B}$ and integration over V yield

$$\frac{1}{2}\frac{d}{dt}\int_V (\mathbf{B}\cdot\mathbf{k})^2\,dV \le \left[-\eta + \max(\mathbf{k}\cdot\mathbf{v})\cdot\left(\int_V |\mathbf{B}|^2\,dV\Big/\int_{V+V'} |\nabla\mathbf{B}\cdot\mathbf{k}|^2\,dV\right)^{1/2}\right]\cdot\int_{V+V'} |\nabla\mathbf{B}\cdot\mathbf{k}|^2\,dV \qquad (12)$$

after the same manipulations that led to (6) have been performed. Since the component of the velocity field in the direction of the axis of rotation is likely to be relatively small in the earth's core because of the approximate validity of the Taylor-Proudman theorem, (12) may serve as a useful constraint when k is identified with the direction of the rotation axis of the earth. Yet at this point we shall not pursue (12) further.

DISCUSSION

We begin the discussion by relating (11) to the toroidal theorem mentioned in the introduction, which states that toroidal motions cannot generate magnetic fields. Although theorems of this kind are highly significant from a mathematical point of view, their value for physical applications may be questionable unless it can be shown that they are not limited to singular cases with special symmetries. Criterion (11) is helpful in this respect since it demonstrates that the toroidal theorem also holds for sufficiently small deviations from a purely toroidal state of motion. In particular, in the case of the geodynamo a sizable radial velocity component is required for the maintenance of the geomagnetic field.

It is unlikely that the recent proposals for the energy source of the geodynamo to which we referred in the introduction provide for sufficiently high radial velocities if a diffusivity of the order of $2 \cdot 10^4$ cm^2/s is assumed, which corresponds to the frequently quoted value of $5 \cdot 10^5$ mhos m^{-1} for the conductivity of the earth's core. It should be noted that only the time average of the radial velocity component over periods of the order of the magnetic decay time r_0^2/η is relevant in (11), since the generation of magnetic flux cannot take place without diffusion. *Won and Kuo* [1973] proposed large earthquakes as a source of geomagnetism and point out the steady circulation induced by oscillations of the inner core of the earth. When Won and Kuo's values and the analysis by *Riley* [1966] to which they refer are used, an amplitude of the order of 10^{-9} cm/s is found for the steady flow, which is much too small to be significant, according to (11). The error made by Won and Kuo in the application of Riley's work has also been pointed out by *Smith* [1974]. Although the generation of magnetic fields by short-period oscillating velocity fields as envisioned by *Bullard and Gubbins* [1971] is feasible in principle, the required velocity amplitude increases with the parameter $\omega r_0^2/\eta$, where ω is a typical frequency of the velocity field. Thus the energy requirement for the possible source of the oscillatory velocity field becomes amplified. On the other hand, the dynamo proposals for a stably stratified core may not be necessary since in their second paper *Kennedy and Higgins* [1973] allow for a region of nearly 800 km outward from the inner core where convection may occur. The value of 800 km is taken from a graph in that paper since the value of 200 or 300 km quoted in the text appears to be in error.

It is interesting to note that the region close to the equator of the inner core is also the place where the critical Rayleigh number for the onset of convection is first reached either if the core is heated homogeneously or if heating takes place just at the boundary between the inner and outer cores owing to crystallization. This fact can be inferred from the approximate theory of *Busse* [1970], which we expect to hold even in the presence of a stratified outer part of the core in place of a rigid boundary. We conclude that convection remains the strongest contender as a source of the geodynamo if Higgins and Kennedy's proposal is accepted. Precession-induced turbulence would be less likely in this case since the shear layer from which the turbulence arises lies at a distance of about $(3)^{1/2}r_0/2$ from the earth's center in the strongly stratified region [*Malkus*, 1968; *Busse*, 1968]. On the other hand, the Grüneisen parameter appropriate for the conditions of the outer core and the possibility of slurry convection proposed by *Busse* [1972] and Elsasser need further investigation before the Higgins-Kennedy hypothesis can be accepted as a fact.

We close the discussion with a remark on a shortcoming of (10). Since the quantity within the brackets depends on the magnetic field, an asymptotic decay cannot be concluded when that quantity is negative at a particular point in time. This shortcoming is shared by (1) since the maximal velocity U in the core depends on the magnetic field in general. More appropriate criteria would involve the forces driving the motion or the heating rate in the case of convection, which can be assumed to be given independently of the magnetic field. To derive such criteria, the Navier-Stokes equations of motion have to be considered, and methods similar to those employed by *Payne* [1967] in the purely hydrodynamic case would have to be used. This will be the subject of future work.

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