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THE PROPAGATION OF LIGHT IN ROTATING SYSTEMS*

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The purpose of the present paper† is to investigate some questions concerning light propagation in a uniformly rotating rigid system, such as the Earth, on both the aether theory and the relativity theory.

On both theories we shall have to understand by "rotation" a rotation of our rigid system[1] S with uniform angular velocity *relatively to the fixed stars*, or to any other inertial system, which will be shortly referred to as the reference system S*. That such a specification of rotation is still necessary even in the relativity theory, in spite of appearances to the contrary, will become clear in the sequel where we shall also have the opportunity to point out some outstanding difficulties of the relativistic gravitation theory with respect to the concept of rotation.

With regard to the rotating system S itself, it will perhaps be well to have in mind our own Earth. The more so as the most interesting experiment in connection with our subject will be a purely terrestrial one.

1. To begin with the aether-theory treatment, let $\tilde{\omega}$ be the (scalar) angular velocity of the Earth (S) relatively to the fixed

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[1] It is well known that among the possible motions of a relativistically rigid body (as defined by Born and Herglotz) there is uniform rotation, such as is familiar to us from the kinematics of ordinary classical rigid bodies.

stars (S^*) and let κ-1 be the rotatory dragging coefficient at and near the surface of the Earth, in other words, let $\kappa\tilde{\omega}$ be the relative angular velocity of the aether and the Earth. If the unit vector $\mathbf{k}$ is taken along the positive axis of rotation, the vector velocity of the aether stream past a point P of S will be

$$\mathbf{u}=\kappa\tilde{\omega}\mathrm{V}\mathbf{rk},$$

$\mathbf{r}$ being the vector drawn from the center of the Earth (or from any point fixed on the axis) to the point P; in Cartesians, with $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$,

$$u_x=\kappa\tilde{\omega}y, \quad u_y=-\kappa\tilde{\omega}x, \; u_z=0,$$

and the resultant velocity, $u=\kappa\tilde{\omega}r$, where r is distance from the axis. As to the factor κ, it is not our intention to prejudice its value, which may be any fraction from zero to unity, corresponding to a full drag and to no drag, respectively.

The propagation of light in S^* being isotropic, of constant velocity c, the velocity of light in S, always in vacuo, along the wave-normal $\mathbf{n}$ (unit vector) will be

$$v=c+\mathbf{un}=c\left[1+\frac{\kappa\tilde{\omega}r}{c}\cos(\mathbf{u},\mathbf{n})\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1a)$$

Notice that, in the case of the Earth, $\tilde{\omega}=2\pi/86164$, the reciprocal length $\tilde{\omega}/c$ amounts only to about $2.43\times10^{-5}\mathrm{cm.}^{-1}$, so that even if r be of the order of the Earth's radius, the factor $\frac{\kappa\tilde{\omega}r}{c}$ is a very small fraction. Such being the case it will be enough to retain in all our formulae $\kappa\tilde{\omega}r/c$ itself, rejecting its square and higher powers. Now, rigorously speaking, formula (1) is valid for the *wave-normal* and not for the ray or the tangent to the "light path." But $\mathbf{n}$ appears in (1a) only in the term multiplied by $\kappa\tilde{\omega}r/c$, and since $\mathbf{n}$ differs from the light ray only by small terms, we have up to higher order terms, simply

$$\frac{v}{c}=1+\frac{1}{c}(\mathbf{up})=1+\frac{\kappa\tilde{\omega}r}{c}\cos\gamma,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $\mathbf{p}$ is a unit vector along the optical ray, and γ the angle between $\mathbf{p}$ and $\mathbf{u}$.

Such being the expression for the velocity of light along the ray, we can at once find the shape of the ray or light path in S by means of Fermat's law, which it will be enough to work out in detail for the case of a light ray contained in a plane parallel to the equatorial plane.[2] In fact let $d\sigma$ be a line-element of the ray; then Fermat's principle is $\delta\int\frac{d\sigma}{v}=o$, the limits of the integral being fixed. Now, introducing in the said plane polar co-ordinates r, θ, the latter measured positively in the sense of the rotation of S, we have

$$\cos\gamma=-r\,d\theta/d\sigma=r\,\theta'\sin\gamma,$$

so that, by (1),

$$\frac{v}{c}=1+\frac{r^2\bar{\omega}\kappa\theta'}{c}\sin\gamma,$$

and Fermat's principle becomes, after easy reductions, and considering as corresponding points those having the same r,

$$\int\left\{\frac{\partial}{\partial\theta'}\left[\frac{c}{v\sin\gamma}\right]\right\}\delta\theta'.\,dr=0.$$

Thus, $\frac{d}{dr}\{\}=o$, and the required equation of the light path becomes

$$\frac{\partial}{\partial\theta'}\left[\frac{c}{v\sin\gamma}\right]=\text{const.}$$

Now $\frac{c}{v\sin\gamma}=-\frac{1+r^2\theta'^2}{\sqrt{1+r^2\theta'^2-\frac{\kappa\bar{\omega}r^2}{c}\theta'}}$. Thus after simple reductions and putting $\gamma=\eta+\frac{\pi}{2}$,

$$\sin\eta+\frac{r\omega(1-2\sin^2\eta)}{(1-\omega r\sin\eta)^2}=\frac{A}{r},\text{ where }\omega=\frac{\kappa\bar{\omega}}{c},\ A=\text{const.}$$

Rejecting second order terms, the left hand member of this equation can be written $\sin\eta+\omega r$.

Ultimately, therefore, the equation of the light path in S becomes

$$r\left(\sin\eta+\frac{r\kappa\bar{\omega}}{c}\right)=A,\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

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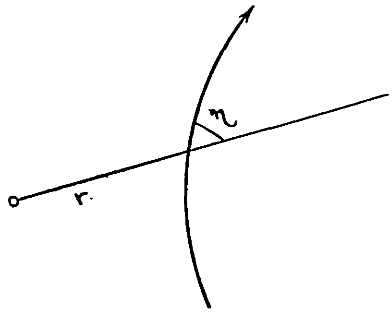
[2] If the end-points (any two points) of a light path be in such a plane, the whole light path is contained in that plane.

where η is the angle under which the light path cuts the radius vector (Fig. 1).

Fig. 1



Notice that in absence of rotation (2) reduces, as it should, to r sin η = const, this being the equation of a straight line. There is thus, due to rotation, a first-order deviation of the light rays from straight lines.

In order to introduce into the ray equation the polar coordinates r, θ instead of r, η, it is enough to remember that

$$\sin\eta = r\frac{d\theta}{dr} : \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}.$$

Thus (2) becomes, with $\frac{1}{r}=\rho$, $B=1+2A\omega$, $\omega=\kappa\bar{\omega}/c$,

$$-d\theta = \frac{\omega r dr}{\sqrt{Br^2-A^2}} + \frac{A d\rho}{\sqrt{B-A^2\rho^2}},$$

whence, integrating, putting $A/\sqrt{B}=r_0$, and counting $\theta-\theta_0$ from the radius vector $r=r_0$ (in the sense of rotation of S),

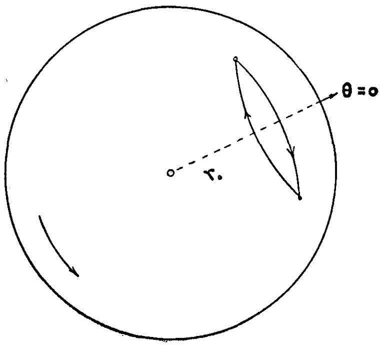
$$\cos\left[\theta-\theta_0\pm\omega\sqrt{\frac{r^2-r_0^2}{B}}\right]=\frac{r_0}{r}.$$

Here $B=1+2\omega A$ and $A=r_0(1+r_0\omega)$, so that rejecting second order terms and with the same right also replacing $\sqrt{r^2-r_0^2}$ by $r_0 \tan\theta$, we have ultimately

$$\frac{r_0}{r}=\cos\left[\theta-\theta_0\pm\frac{\kappa\bar{\omega}}{c}\sqrt{r^2-r_0^2}\right]=\cos\left[\theta-\theta_0\pm\frac{r_0\kappa\bar{\omega}}{c}\tan(\theta-\theta_0)\right], \quad (3)$$

the upper or the lower sign to be taken for light travelling in the sense of rotation of S or in the opposite sense. Or, to put the sign

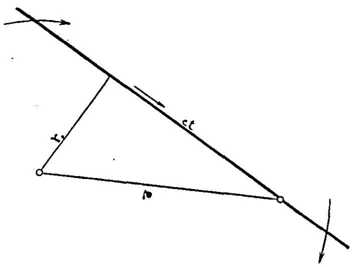
Fig. 2



rule in a more convenient form, the light path is convex (or bulges out) always to the left of a person[3] walking in the direction of propagation (Fig. 2).

This equation of the light ray can also be obtained more directly by transforming the light rays of S^* to our system S by means of the substitution $r = r'$, $\theta' = \theta + \kappa\bar{\omega}t$ (and $t' = t$). In fact, the equa-

Fig. 3



tion of a straight line, and such is in S^* every optical ray, can be written

$$r' \cos\theta' = r_0' = \text{const.},$$

and this becomes, through the said transformation,

$$\frac{r_0}{r} = \cos\,(\theta + \kappa\bar{\omega}t).$$

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[3] For whom the rotation of S is clockwise.

In other words the light path is the path of a particle moving with uniform velocity c along a straight line which is itself spinning uniformly around O (Fig. 3). If $t=o$ for $r=r_o$ we have $ct=ct'=\sqrt{r^2-r_o^2}$, which gives again

$$\frac{r_0}{r}=\cos\left[\theta-\theta_0\pm\frac{\kappa\bar{\omega}}{c}\sqrt{r^2-r_0^2}\right],$$

as under (3).

Developing the cosine, up to the second order, we can write

$$\frac{r_0}{r}=\cos(\theta-\theta_0).\left[1\mp\frac{\kappa\bar{\omega}r_0}{c}\tan^2(\theta-\theta_0)\right]\ldots(3a)$$

The constants θ_0, r_o can be determined if either the initial direction or any two points, A, B of the light path, say the sending and the receiving stations, are given through two pairs of r, θ. It will be noted that the light path or ray BA (i.e., with B as sending and A as receiving station) does not coincide with AB. The two optical rays AB and BA enclose between them a certain area, having the shape of a biconvex lens. In other words light propagation in S is irreversible. Under appropriate circumstances A will see B without being seen by B.

The last ray-equation can still be simplified. Putting the x-axis along the radius vector $r_o(\theta=\theta_o)$, so that

$$x=r\cos(\theta-\theta_0),\ y=r\sin(\theta-\theta_0),$$

we have $x(1\mp\epsilon y^2/x^2)=r_o$, whence

$$x=r_0\left(1\pm\frac{\epsilon y^2}{x^2}\right),\qquad \epsilon=\frac{r_0\kappa\bar{\omega}}{c}\ldots\ldots\ldots\ldots\ldots(4)$$

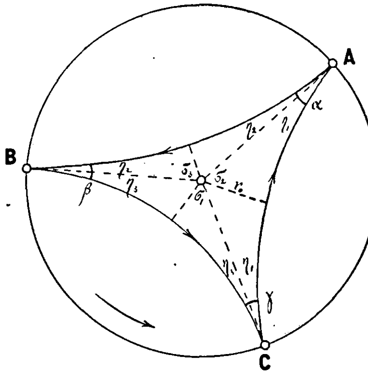
In the small term we can write, with the same approximation $x=r_o$, so that the ray equation becomes

$$x=r_0\pm\frac{\kappa\bar{\omega}}{c}y^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4a)$$

Thus, up to the second order, the optical ray (which more rigorously is a complicated spiral) becomes simply a parabola, with apex in shortest distance (r_o) from O. If the propagation is in the sense of rotation of S, the parabolic ray turns its convex side towards the axis of rotation. In the opposite case it will turn its concave side towards O. Various problems of what may be called optical trigonometry of the rotating system, i.e., problems concerning triangles or polygons built up of optical rays, can now be easily dealt with, namely by constructing the said parabolic arcs as light paths and keeping well in mind the sense of propagation.

By way of illustration consider an optical triangle traversed, say, in the sense of rotation, and having its corners A, B, C on the circle $r=a$. Given the angles $2\sigma_1=BOC$, $2\sigma_2=COA$, $2\sigma_3=AOB$ $(\sigma_1+\sigma_2+\sigma_3=\pi)$, find the sum of the angles α, β, γ of the optical triangle ABCA, the order of the letters giving the sense of propagation.

Fig. 4



Denote by η_1 the angle between the optical ray CA and the radius vector at A, and let η_2, η_3 have analogous meanings for B and C, as in Fig. 4. Then, since the angle OCA is also equal η_1, and similarly for the remaining angles, we shall have

$$\alpha=\eta_1+\eta_2,\ \beta=\eta_2+\eta_3,\ \gamma=\eta_3+\eta_1, \text{ and}$$
$$\alpha+\beta+\gamma=2(\eta_1+\eta_2+\eta_3)\ldots\ldots\ldots\ldots(5)$$

Now, by (2) and remembering that $A=r_o(1+r_o\omega)$, $\omega=\kappa\bar{\omega}/c$,

$$r(\sin\eta+r\omega)=r_o(1+r_o\omega).$$

Apply this to the ray CA at the corner A. Then

$$\sin\eta_1=\frac{r_o}{a}(1+\omega r_o)-a\omega\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

and it remains only to find r_o. Now, the radius vector r_o bisects the angle COA $=\sigma_2$; thus, applying (3a), with $\theta_A-\theta_o=\sigma_2$, we have

$$\frac{r_o}{a}=\cos\sigma_2.\left(1-a\omega\frac{\sin^2\sigma_2}{\cos\sigma_2}\right).$$

Substitute this in (6) and reject second order terms; then the result will be

$$\sin\eta_1=\cos\sigma_2-2\,a\omega\sin^2\sigma_2.$$

Since $\eta_1+\sigma_2$ differs but little from a right angle, put here $\eta_1=\frac{\pi}{2}-(\sigma_2+\delta_1)$; then the last equation will give for the small angle

$$\delta_1=2\,a\omega\,\sin\sigma_2.$$

Similarly the defects of $\eta_2+\sigma_3$ and $\eta_3+\sigma_1$ will be $\delta_2=2\,a\omega\,\sin\,\sigma_3$, $\delta_3=2a\omega\,\sin\sigma_1$. Thus by (5), and since $\sigma_1+\sigma_2+\sigma_3=\pi$,

$$\alpha+\beta+\gamma=\pi-2(\delta_1+\delta_2+\delta_3),$$

i.e., the *defect* of the optical triangle ABCA will be

$$4a\frac{\kappa\bar{\omega}}{c}\,(\sin\sigma_1+\sin\sigma_2+\sin\sigma_3)\,\ldots\ldots(8)$$

For the optical triangle (or triangular circuit) ACBA we shall have an equal *excess* of the angle sum.

Thus, for instance, for an equilateral triangle, $\sigma_1=\sigma_2=\sigma_3=60°$, the defect, or the excess, will be $4a\frac{\kappa\bar{\omega}}{c}.\ 3\frac{\sqrt{3}}{2}=6\sqrt{3}a\frac{\kappa\bar{\omega}}{c}$. For the Earth (and a triangle parallel to the equatorial plane) even if $\kappa=1$ (no drag) this would amount to $0.''00052$ per kilometer of a, and the difference between the angle sum of ACBA and ABCA would be the double of this. Thus, even for $a=10$ or 20 km. the difference would certainly be too small to be measured directly.

The experimental possibilities with regard to the optical effects of the rotation of the Earth lie in another direction, to wit in the phase retardation in an optical circuit (i.e., closed light path) as in the well-known laboratory experiment of Sagnac[4] with a small spinning interferometer as our system S. The corresponding formula used by Sagnac and before him by Michelson (Phil. Mag. vol. 2, 1904, p. 716–719) who actually proposed but never carried out a terrestrial experiment[5] of the kind here aimed at, can be

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[4] G. Sagnac, "L'ether lumineux démontré par l'effet du vent relatif d'ether dans un intérferomètre en rotation uniforme." C. R. Paris, 157 (1913), p. 708. Ibidem, p. 1410, "Sur la preuve de la réalité de l'ether lumineux par l'experience de l'interférographe tournant." The titles seemed interesting enough to be quoted in full. But Sagnac's experiments (even apart from the question of the reliability of his measurements) by no means decide for the aether as against relativity.

[5] Such a terrestrial experiment was already hinted at by Oliver Lodge, in 1897, Phil. Trans. Roy. Soc. A, vol. 189, p. 151, where also the experiment carried out by Sagnac twenty-five years later is suggested, only with "telescope and observer" instead of a photo camera mounted on a rotating "turn-table."

most simply deduced in the following way. Consider any optical circuit s, traversed by light in the sense of rotation or "positive" circuit, say.[6] By formula (1), the time taken to traverse an element ds of the circuit is, with u_s written for **up**,

$$ds.(c+u_s)^{-1}=\frac{ds}{c}\left(1-\frac{u_s}{c}\right),$$

giving a retardation $-\frac{1}{c^2}u_s$ ds per line-element, and thus for the whole circuit the phase retardation

$$\Delta\tau=-\frac{1}{c^2}\int \mathbf{u}ds,$$

as already noticed by Lorentz (Wiss. Abhandlungen, vol. I). Now, by Stokes' theorem, this can be written, if σ be any surface laid through s, and **n** its normal,

$$\Delta\tau=-\frac{1}{c^2}\int \mathbf{n}\ \mathrm{curl}\ \mathbf{u}.\ d\sigma.$$

This holds for any circuit, plane or not, and for any distribution of velocity. Since $-$curl **u** is the double angular velocity, $2\kappa\bar{\omega}.\mathbf{k}$, we have, writing $\bar{\omega}_n=\bar{\omega}\mathbf{kn}$ for the normal component of the spin,

$$\Delta\tau=\frac{2}{c^2}\int \kappa\bar{\omega}_n d\sigma \qquad (9)$$

This, the required formula, is valid for any, not necessarily constant value of $\kappa\bar{\omega}$ throughout the surface of integration. In our case, $\bar{\omega}$, and therefore $\bar{\omega}_n$ for a plane circuit, are manifestly constant, but the drag of the aether, if any, may vary from point to point, thus giving rise to a variable coefficient κ.

If the optical circuit is plane and small compared with the dimensions of the Earth, we have simply $\Delta\tau=2\kappa\bar{\omega}_n\sigma/c^2$, or if T be the period of oscillation and $\lambda=cT$,

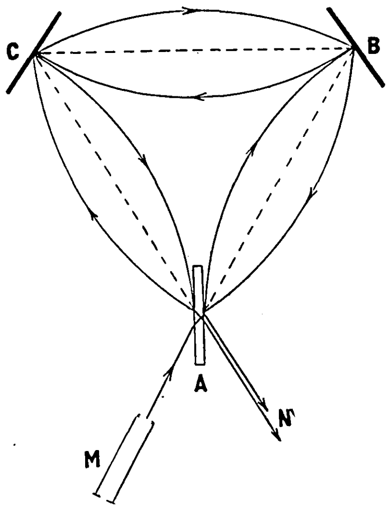
$$\frac{\Delta\tau}{T}=\frac{2\kappa\bar{\omega}_n}{c\lambda}\sigma \qquad (10)$$

where σ is the total area embraced by the circuit. This gives the retardation, in parts of the period, for a positive circuit. As was already mentioned, the same path cannot, rigorously, serve for light propagation in the opposite sense [or, which is the same

[6] As we already know, the same light path cannot be described in the opposite sense.

thing, with inverted normal $\mathbf{n}$ in (10)]. Thus, for instance, in the case of a triangle (Fig. 5) we have to take for the positive circuit the area σ_1 of the inner, and for the negative circuit the area σ_2 of the outer, convex triangle, so that rigorously we would

Fig. 5



have for the phase difference of the two beams (say, separated at A by a semi-transparent plate and reflected at B, C) in parts of the period, or for the corresponding shift in fringe widths,

$$\epsilon = \frac{2\kappa\bar{\omega}}{c\lambda}(\sigma_1+\sigma_2) \qquad (11)$$

But the difference of σ_1 and σ_2 is itself small of the first order. Moreover, up to higher terms, the paths AB and BA, etc., are symmetrical with respect to the corresponding (dotted) straights, so that even up to terms of an order higher than the second we can replace $\sigma_1+\sigma_2$ by trice the area (σ) of the rectilinear (dotted) triangle, and similarly in the case of any polygons. Thus:[7]

$$\epsilon = \frac{4\kappa\bar{\omega}_n\sigma}{c\lambda} \qquad (12)$$

[7] Prof. Michelson's paper of 1904, (l.c.) has by a manifest slip the factor 2 instead of our 4.

For horizontally placed circuits we have $\tilde{\omega}_n = \tilde{\omega} \sin \varphi$, if φ be the geographic latitude at which the experiment is performed. For a latitude $\varphi = 45°$ and the wave-length $\lambda = 5000$ Å. U., formula (12) gives

$$\epsilon = 1.38\kappa \frac{\sigma}{\text{km}^2} \quad \text{(12a)}$$

i.e., about 1.4κ fringe widths per each square kilometer embraced by the circuit. If the drag is complete ($\kappa = 0$) there should be no shift; if there is no drag, we should have the total amount of 1.4 per km², and intermediate values if the Earth in its spinning motion drags the aether, even at its surface, only partially.

In Sagnac's experiment the spinning motion of the (disc-shaped) table bearing the interferometer, the light source, as well as the photographic camera, was reversed, and thus the double of (12) was observed[8] as the shift of the system of interference fringes. No such reversal, of course, is possible in the case of the Earth as the rotating system. But as Prof. Michelson has already pointed out in 1904 (loc. cit.) there is an easy way out of this difficulty, to wit by silvering heavily one, e.g., the upper, half of the dividing glass plate (such as A in the case of Fig. 5 corresponding to three stations; B, C, being mirrors) and leaving the lower part clear or but lightly silvered. A beam of parallel rays from the collimator *M* impinging upon the plate A is here divided into ACBA and ABCA by reflection and transmission respectively. Now, according to Michelson's suggestion, cover the lower half and observe first by reflection from the upper half only (ACBA) when simply the image of the slit is seen. Place the cross-hair of the eyepiece in the center of this image. Next, covering the upper half and leaving the lower half of the plate clear, observe the interference fringes. Then, if there is an effect ϵ, the mid-point of the central fringe will be displaced from the crosshair by ϵ fringe widths. The effect sought for will be easily discernible from undesired accessorial shifts by being proportional to the area σ. Instead of comparing the position of the interference

[8] In Sagnac's case $\kappa = 1$, since such small masses as was his table certainly do not drag the aether, as follows from the widely known older experiments of Sir Oliver Lodge.

fringes with that of the slit image Prof. Michelson contemplates also the comparison of the two fringe systems given by two considerably differing values of the area σ.

It is estimated that the terrestrial experiment could be carried out with the required precision on comparatively small areas, such as 1/10 of a square kilometer. The beams can, of course, be sent around twice or more times making ϵ as many times larger; as many times, however, would the light path be increased which may not be convenient. A more radical way is to leave the circuits simple but to extend the linear dimensions of the circuits; for then the value of ϵ will be increased in the squared ratio. Again the more stations (arranged in a regular polygon) the greater σ for the same light paths. Yet, to avoid too many reflections and other inconvenients, the triangular arrangement as suggested in Fig. 5 or a quadrangular one may turn out to be preferable: But details of a technical kind need not detain us in the present paper.

2. Let us now try to find out what aspect the same problem assumes from the standpoint of the theory of relativity, the special and the generalized one. It goes without saying that with neither of these theories can there be any question of an aether and its being dragged by the Earth in its daily rotation around its axis, or in its annual motion around the sun.

First of all, then, the special relativity theory is wholly incompetent to deal with the problem of light propagation in a rotating system rigorously, simply because it has nothing to do with any reference systems other than the galilean or inertial ones, i.e. the fixed-stars system S^* and those moving relatively to it with uniform translational velocity. The only thing the special theory can do is to treat our problem approximately up to the second order, i.e., rejecting the square (and higher powers) of the ratio β of the velocity of motion[9] to the light velocity. Now up to β^2 the relativistic addition theorem of velocities does not differ at all from that of ordinary, Newtonian kinematics. This amounts simply to a neglect of the Lorentz-Fitgerald contraction. Under

[9] Of any point of S relatively to S^* say.

these circumstances the propagation of light in S becomes simply a superposition of that in S* and of the reversed spinning motion of S relatively to S* in much the same way as on the aether theory. The only difference is that our previous $\kappa\tilde{\omega}$ has now to be replaced by *the full angular velocity* $\tilde{\omega}$ of S relatively to S* (fixed stars). Thus formula (1) for the light velocity v will be replaced by

$$\frac{v}{c}=1+\frac{1}{c}u_s=1+\frac{\tilde{\omega}r}{c}\cos\gamma,$$

and the shift formula (12) by

$$\epsilon=\frac{4\tilde{\omega}_n\sigma}{c\lambda}\qquad\text{(12r)}$$

In fact v. Laue, the chief exponent of Einstein's older theory in Germany, gives in his well-known book, in a section on Sagnac's experiment, a formula identical with (12r) at which he arrives by a rather roundabout way, instead of using the simple relation (9), valid for a circuit of any shape.[10]

Similarly all other formulae given above will continue to hold on the special relativity theory with unity written for κ.

From the standpoint of the general relativity theory all questions concerning light propagation (in vacuo) relatively to any system whatever and therefore also to our terrestrial system S will be answered if the four-dimensional line-element $ds^2=\Sigma g_{\iota\kappa}\,dx_\iota dx_\kappa$ belonging to that system be known, to wit, by putting

$$ds=o\qquad\text{(13)}$$

Thus the problem is reduced to building up ds in terms of terrestrial or S-co-ordinates, say $x_1, x_2, x_3, x_4=r, \theta, z, ct$, respectively, keeping in mind that in the fixed-star system S* (disregarding in our present connection the extremely minute terms due to the Earth's gravitation) the line-element is

$$ds^2=c^2dt'^2-dr'^2-r'^2d\theta'^2-dz'^2\qquad\text{(14*)}$$

To be faithful to its own leading principles, the relativity theory ought to deduce the terrestrial ds or its coefficients $g_{\iota\kappa}$ as a gravi-

[10] M. v. Laue, Relativitätstheorie, vol. I, 3rd ed. Braunschweig, 1919, p. 125–127. Laue seems, by the way, to be under the misapprehension that the light rays relative to the rotating table are straight lines, which they are not. As we saw before, their departure from straight lines is a first order effect and does, therefore, by no means disappear though "the Lorentz contraction" be neglected. The influence of the curvature of the rays on (12) or (12r) remains, of course, negligible.

tational effect of the stars and all the remaining matter of the universe rotating around the Earth. This, however, it proved unable to do. As a matter of fact Einstein himself never entered into the details of this important problem of rotation. Thirring[11] tried to solve it by considering a huge massive spherical shell in uniform rotation and evaluating by means of Einstein's approximate integrals of his field-equations the $g_{\iota\kappa}$ thus produced at comparatively small distances from the centre of that gigantic sphere. But the result, though mathematically interesting, was a complete failure,[12] although the treatment of the same problem on Einstein's newest cosmological views (elliptic space and so on) seems more promising. At any rate, the relativity theory is unable to construct the required line-element on its own great principles, and is content to transcribe it from the galilean line-element (14*) by putting simply $r', z', t' = r, z, t$, and

$$\theta' = \theta + at,$$

where a is an appropriate constant. In fact, de Sitter, one of the chief exponents of Einstein's theory, and even Weyl in his interesting book[13] write down without much discussion this simple transformation in order to pass from one to the other system. This gives for the latter,

$$ds^2 = \left(1 - \frac{a^2r^2}{c^2}\right)c^2dt^2 - 2r^2\frac{a}{c}d\theta cdt - dr^2 - r^2d\theta^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

The constant a is to be chosen so as to describe correctly the terrestrial laws of physical phenomena, approximately, at least, in our present connection up to the second order only.

Now, if but such an approximation is required, the fundamental principles of general relativity offer us a good test of whether a chosen value of the constant a is or is not the appro-

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[11] H. Thirring, Ueber die Wirkung rotierender ferner Massen in der Einsteinschen Gravitationstheorie. Phys. Zeitschrift, Vol. 19, 1918, p. 33–39.

[12] Not only that Thirring's "centrifugal force" had also a component along the axis of rotation, but the coefficients of the centrifugal and the Coriolis force, apart from being very unsatisfactory in their structure, bore a wrong numerical ratio to one another.

[13] H. Weyl, Zeit-Raum-Materie, 3rd ed., Berlin, J. Springer, 1920.

priate one. In fact, according to one of these principles $ds=0$ represents light propagation, and according to the other the geodesics of the world, determined by the same line-element, i.e.,

$$\delta\int ds=0 \qquad (15)$$

give the equations of motion of a free particle. And since the terrestrial laws of motion are certainly known to that degree of approximation, we can derive from (15) the correct value to be attributed to the constant a.

Now, using (14) in (15) the terrestrial equations of motion of a free particle (always apart from gravitation proper) follow at once. Of these equations it will be enough to write down only that which most interests us here, i.e., the equation corresponding to the variation δr. This is

$$\frac{d^2r}{ds^2}=a^2r\left(\frac{dt}{ds}\right)^2+2ar\frac{d\theta}{ds}\frac{dt}{ds}+r\left(\frac{dt}{ds}\right)^2,$$

and since, up to the second order, $ds^2=c^2\,dt^2$, we can write

$$\frac{d^2r}{dt^2}=r\left(a+\frac{d\theta}{dt}\right)^2.$$

The right-hand member will represent the correct value of the centrifugal acceleration (or "centrifugal force" per unit mass), in size and direction, provided that a stands for the *full* angular velocity of the Earth relatively to the fixed stars, i.e.

$$a=\bar{\omega}.$$

This then is the value to be substituted into (14) to suit our dynamical experience, and at the same time, as already explained, into the light equation $ds=0$. In short, on the relativity theory the same ds and therefore the same $\bar{\omega}$ is required for light as for mechanics, while on the aether theory we may have any fraction $\kappa\bar{\omega}$ of $\bar{\omega}$ for light, though the full angular velocity $\bar{\omega}$ is required for mechanics.

Thus, on the relativity theory the terrestrial velocity of light for any direction of the ray will be determined by

$$\left(1-\frac{\bar{\omega}r^2}{c^2}\right)c^2dt^2-2\frac{\bar{\omega}}{c}r^2d\theta c dt-dr^2-r^2d\theta^2=0 \qquad (16)$$

whence also the form of the optical rays and the circuitous shift formula can be deduced.[14] Or, equivalently and much more simply, we may deduce all these terrestrial optical laws from those of S* (uniform propagation with velocity c) by means of the transformation

$$\theta'=\theta+at=\theta+\bar{\omega}t.$$

Consequently, all our previous formulae for the light velocity, for the shape of rays and the phase retardation in a terrestrial optical circuit, will reappear, with the only difference that the unknown fraction κ will be replaced by the special and definite value 1, i.e., our $\kappa\bar{\omega}$ will be replaced by $\bar{\omega}$. Thus also the shift formula, which from the experimental point of view is the only important thing, will again be

$$\epsilon=\frac{4\bar{\omega}n\sigma}{c\lambda},$$

as on the special relativity theory.

The planned terrestrial experiments with the optical circuit might thus enable us to decide between relativity and aether theory. That is to say, if the result of such experiments will be a full effect (a shift of 1.38 fringe widths under the stated conditions, per square kilometer), there will be no discrimination between the two theories. But should there be either no shift at all or only a fraction κ of the full effect, sensibly different from unity, the relativity theory, special or general, will be irremediably disproved, while on the aether theory we would have only to assume a complete or partial rotational dragging of the aether.

In fine, the optical circuit experiment may easily become crucial and fatal for Einstein's theory especially if it gives a nil-effect. Should it, on the other hand, give a full effect, it will certainly cease to be decisive either way but will even then (as every new positive experiment) be certainly a valuable contribution

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[14] The only effect of terrestrial gravitation on the optical-circuit experiment would according to relativistic gravitation theory, be represented by the term $\frac{8}{5}\cos^2\varphi\frac{M}{c^2R}$ to be subtracted from the factor 2 of the second term in (16), φ being the geographic latitude, M the mass and R the radius of the Earth. But since $M/c^2=0.45$ cm., the correction term due to gravitation amounts, even at the equator, only to 1.2×10^{-9} which is entirely negligible in presence of 2.

to our experimental knowledge of the Earth as a physical reference system.

Speculations on a possible rotational drag of the aether by the Earth were made, in connection with the diurnal aberration, by Christian Doppler (1845) and after him by v. Oppolzer.[15] But owing to an almost complete absence of direct observational data on daily aberration, a state of things prevailing even at the present time, Doppler's formula, generalized by Oppolzer, could neither be proved nor disproved. More recently the question of a rotational drag was taken up by Lorentz[16] but it had again to be left unsettled in absence of daily aberration data. If there is anything on the experimental side to make the rotational drag unlikely it is the absence of shift of a star near occultation by Jupiter, as mentioned by Lorentz (loc. cit. p. 413). On the theoretical side the question of the influence of a *spinning* planet on the adjacent aether could not be answered definitely without some special and more or less artificial hypotheses about the properties of that medium in addition to those given to it already by Stokes-Planck. In this respect we can say only that even if there is an almost complete translational (annual) drag of the aether due to its condensation around the planet, there may yet be no appreciable rotational (daily) drag. Again, from Lodge's experiments (loc. cit) one can judge only that there is no such drag by comparatively small spinning masses, such as can be used in a laboratory, but not by such massive bodies as the Earth or other planets. In fine, the only sound way of settling the question would be to carry out the terrestrial experiment with optical circuits embracing as large an area as is technically possible.

ROCHESTER, N. Y.
March, 24, 1921.

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[15] E. v. Oppolzer, Erdbewegung and Aether, Sitzber. Akad. Wien, vol. CXI, IIa, Febr. 1902, pp. 244–254.

[16] H. A. Lorentz, Abhandlungen, Vol. I.