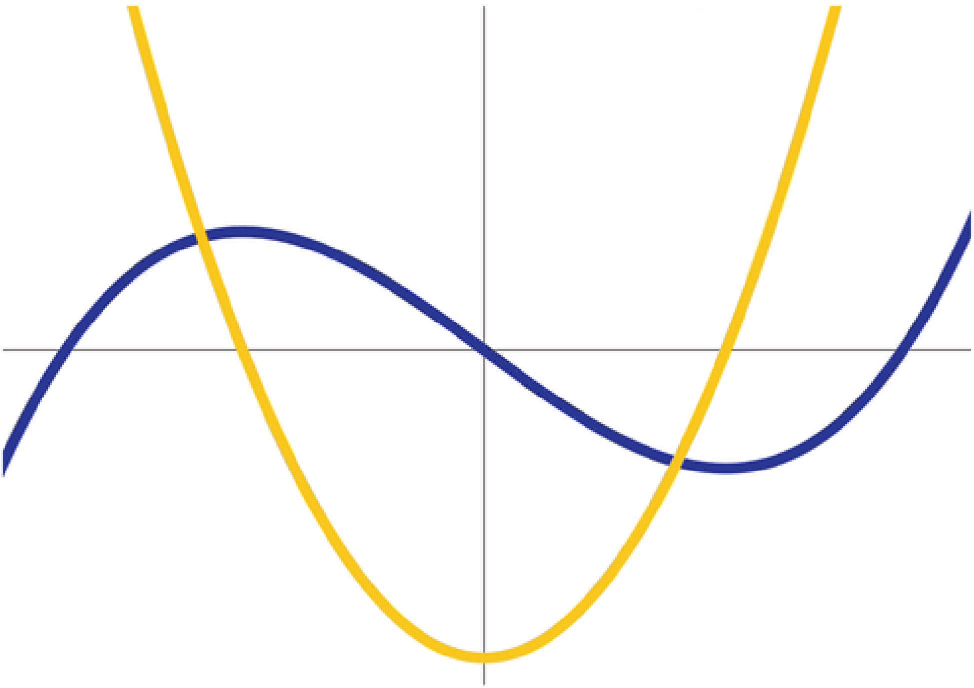
A SELF-TEACHING GUIDE

Quick Calculus

3RD EDITION



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Quick Calculus

A Self-Teaching Guide

Third Edition

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Preface

Quick Calculus is designed for you to learn the basic techniques of differential and integral calculus with a minimum of wasted effort, studying by yourself. It was created on a premise that is now widely accepted: in technical subjects such as calculus, students learn by *doing* rather than by *listening*. The book consists of a sequence of relatively short discussions, each followed by a problem whose solution is immediately available. One's path through the book is directed by the responses. The text is aimed at newcomers to calculus, but additional topics are discussed in the final chapter for those who wish to go further.

The initial audience for *Quick Calculus* was composed of students entering college who did not wish to postpone physics for a semester in order to take a prerequisite in calculus. In reality, the level of calculus needed to start out in physics is not high and could readily be mastered by self-study.

The readership for *Quick Calculus* has grown far beyond novice physics students, encompassing people at every stage of their career. The fundamental reason is that calculus is empowering, providing the language for every physical science and for engineering, as well as tools that are crucial for economics, the social sciences, medicine, genetics, and public health, to name a few. Anyone who learns the basics of calculus will think about how things change and influence each other with a new perspective. We hope that *Quick Calculus* will provide a firm launching point for helping the reader to achieve this perspective.

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Contents

CHAPTER ONE
Starting Out

In spite of its formidable name, calculus is not a particularly difficult subject. The fundamental concepts of calculus are straightforward. Your appreciation of their value will grow as you develop the skills to use them.

After working through *Quick Calculus* you will be able to handle many problems and be prepared to acquire more elaborate techniques if you need them. The important word here is *working,* though we hope that you find that the work is enjoyable.

Quick Calculus comprises four chapters that consist of sections and subsections. We refer to the subsections as *frames*. Each chapter concludes with a summary. Following these chapters there are two appendixes on supplementary material and a collection of review problems with solutions.

The frames are numbered sequentially throughout the text. Working *Quick Calculus* involves studying the frames and doing the problems. You can check your answers immediately: they will be located at the bottom of one of the following pages or, if the solutions are longer, in a separate frame. Also a summary of frame problems answers start on page 273.

Your path through *Quick Calculus* will depend on your answers. The reward for a correct answer is to go on to new material. If you have difficulty, the solution will usually be explained and you may be directed to another problem.

Go on to frame **1**.

1.1 A Few Preliminaries

1

Chapter 1 will review topics that are foundational for the discussions to come. These are:

- the definition of a mathematical function;
- graphs of functions;

(continued)

- the properties of the most widely used functions: linear and quadratic functions, trigonometric functions, exponentials, and logarithms.

A note about calculators: a few problems in *Quick Calculus* need a simple calculator. The calculator in a typical smartphone is more than adequate. If you do not happen to have access to a calculator, simply skip the problem: you can master the text without it.

Go on to frame **2**.

2

Here is what's ahead: this first chapter is a review, which will be useful later on; Chapter 2 is on differential calculus; and Chapter 3 introduces integral calculus. Chapter 4 presents some more advanced topics. At the end of each chapter there is a summary to help you review the material in that chapter. There are two appendixes—one gives proofs of a number of relations used in the book, and the other describes some supplementary topics. In addition, there is a list of extra problems with answers in the Review Problems on page 277, and a section of tables you may find useful.

First we review the definition of a function. If you are already familiar with this and with the idea of dependent and independent variables, skip to frame **14**. (In fact, in this chapter there is ample opportunity for skipping if you already know the material. On the other hand, some of the material may be new to you, and a little time spent on review can be a good thing.)

A word of caution about the next few frames. Because we start with some definitions, the first section must be somewhat more formal than most other parts of the book.

Go on to frame **3**.

1.2 Functions

3

The definition of a function makes use of the idea of a *set*. If you know what a set is, go to **4**. If not, read on.

A *set* is a collection of objects—not necessarily material objects—described in such a way that we have no doubt as to whether a particular object does or does not belong to it. A set may be described by listing its elements. Example: 23, 7, 5, 10 is a set of numbers. Another example: Reykjavik, Ottawa, and Rome is a set of capitals.

We can also describe a set by a rule, such as all the even positive integers (this set contains an infinite number of objects).

A particularly useful set is the set of all real numbers. This includes all numbers such as 5, −4, 0, ½, π, −3.482, $\sqrt{2}$. The set of real numbers does *not* include quantities involving the square root of negative numbers. Such quantities are called *complex numbers*; in this book we will be concerned only with real numbers.

The mathematical use of the word "set" is similar to the use of the same word in ordinary conversation, as "a set of chess pieces."

Go to **4**.

4

In the blank below, list the elements of the set that consists of all the odd integers between −10 and +10.

Elements: ____________________

Go to **5** for the correct answer.

5

Here are the elements of the set of all odd integers between −10 and +10:

$$-9, -7, -3, -5, -1,\ 1,\ 3,\ 5,\ 7,\ 9.$$

Go to **6**.

6

Now we are ready to talk about functions. Here is the definition.

A *function* is a rule that assigns to each element in a set A one and only one element in a set B.

The rule can be specified by a mathematical formula such as $y = x^2$, or by tables of associated numbers, for instance, the temperature at each hour of the day. If x is one of the elements in set A, then the element in set B that the function f associates with x is denoted by the symbol $f(x)$. This symbol $f(x)$ is the value of f evaluated at the element x. It is usually read as "f of x."

The set A is called the *domain* of the function. The set of all possible values of $f(x)$ as x varies over the domain is called the *range* of the function. The range of f need not be all of B.

(continued)

In general, A and B need not be restricted to sets of real numbers. However, as mentioned in frame **3**, in this book we will be concerned only with real numbers.

Go to **7**.

7

For example, for the function $f(x) = x^2$, with the domain being all real numbers, the range is ______________________________.

Go to **8**.

8

The range is *all nonnegative real numbers*. For an explanation, go to **9**.

Otherwise, skip to **10**.

9

Recall that the product of two negative numbers is positive. Thus for any real value of x positive or negative, x^2 is positive. When x is 0, x^2 is also 0. Therefore, the range of $f(x) = x^2$ is all nonnegative numbers.

Go to **10**.

10

Our chief interest will be in rules for evaluating functions defined by formulas. If the domain is not specified, it will be understood that the domain is the set of all real numbers for which the formula produces a real number, and for which it makes sense. For instance,

(a) $f(x) = \sqrt{x}$ Range = ____________________.

(b) $f(x) = \frac{1}{x}$ Range = ____________________.

For the answers go to **11**.

11

The function $\sqrt{x}$ is real for x nonnegative, so the answer to (a) is all nonnegative real numbers. The function $1/x$ is defined for all values of x except zero, so the range in (b) is all real numbers except zero.

Go to **12**.

12

When a function is defined by a formula such as $f(x) = ax^3 + b$, x is called the *independent variable* and $f(x)$ is called the *dependent variable*. One advantage of this notation is that the value of the dependent variable, say for $x = 3$, can be indicated by $f(3)$.

Often, a single letter is used to represent the dependent variable, as in

$$y = f(x).$$

Here x is the independent variable, and y is the dependent variable.

Go to **13**.

13

In mathematics the symbol x frequently represents an independent variable, f often represents the function, and $y = f(x)$ usually denotes the dependent variable. However, any other symbols may be used for the function, the independent variable, and the dependent variable. For example, we might have $z = H(r)$, which is read as "z equals H of r." Here r is the independent variable, z is the dependent variable, and H is the function.

Now that we know what a function means, let's describe a function with a graph instead of a formula.

Go to **14**.

1.3 Graphs

14

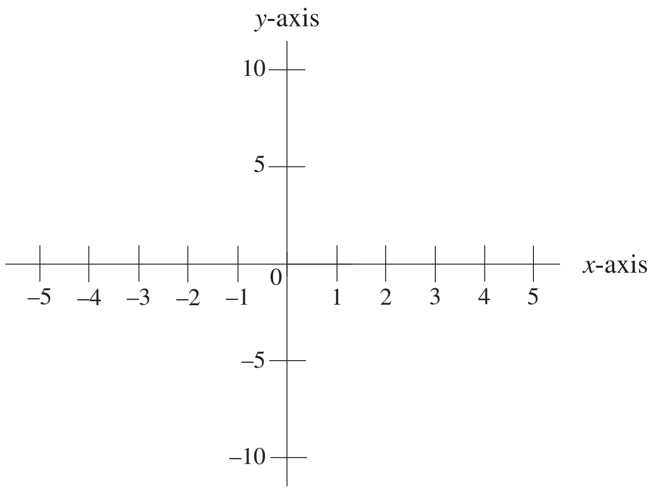
If you know how to plot graphs of functions, skip to frame **19**.

Otherwise, go to **15**.

15

We start by constructing coordinate axes. In the most common cases we construct a pair of mutually perpendicular intersecting lines, one horizontal, the other vertical. The horizontal line is often called the *x-axis* and the vertical line the *y-axis*. The point of intersection is the origin, and the axes together are called the *coordinate axes*.

(continued)

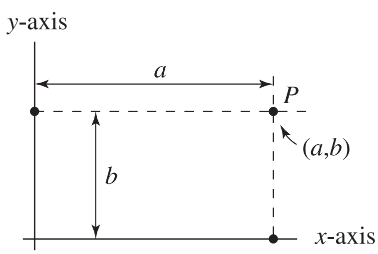


Next we select a convenient unit of length and, starting from the origin, mark off a number scale on the x-axis, positive to the right and negative to the left. In the same way, we mark off a scale along the y-axis with positive numbers going upward and negative downward. The scale of the y-axis does not need to be the same as that for the x-axis (as in the drawing). In fact, y and x can have different units, such as distance and time.

Go to **16**.

16

We can represent one specific pair of values associated by the function in the following way: let a represent some particular value for the independent variable x, and let b indicate the corresponding value of $y = f(x)$. Thus, $b = f(a)$.



We now draw a line parallel to the y-axis at distance a from the y-axis and another line parallel to the x-axis at distance b from that axis. The point P at which these two lines intersect is designated by the pair of values (a, b) for x and y, respectively.

The number a is called the x-coordinate of P, and the number b is called the y-coordinate of P. (Sometimes the x-coordinate is called the *abscissa*, and the y-coordinate is called the *ordinate*.) In the designation of a typical point by the notation (a, b) we will always designate the x-coordinate first and the y-coordinate second.

As a review of this terminology, encircle the correct answers below. For the point (5, −3):

x-coordinate: [−5 | −3 | 3 | 5]

y-coordinate: [−5 | −3 | 3 | 5]

Go to **17**.

17

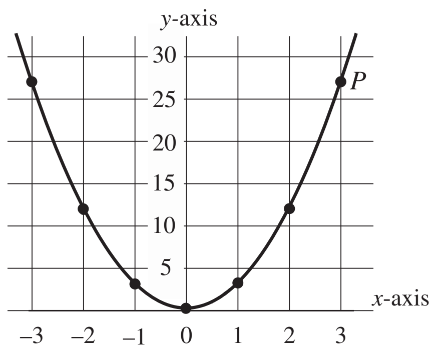
The most direct way to plot the graph of a function $y = f(x)$ is to make a table of reasonably spaced values of x and of the corresponding values of $y = f(x)$. Then each pair of values (x, y) can be represented by a point as in the previous frame. A graph of the function is obtained by connecting the points with a smooth curve. Of course, the points on the curve may be only approximate. If we want an accurate plot, we just have to be very careful and use many points. (On the other hand, crude plots are pretty good for many purposes.)

Go to **18**.

18

As an example, here is a plot of the function $y = 3x^2$. A table of values of x and y is shown, and these points are indicated on the graph.

x	y
–3	27
–2	12
–1	3
0	0
1	3
2	12
3	27



To test yourself, encircle the pair of coordinates that corresponds to the point P indicated in the figure.

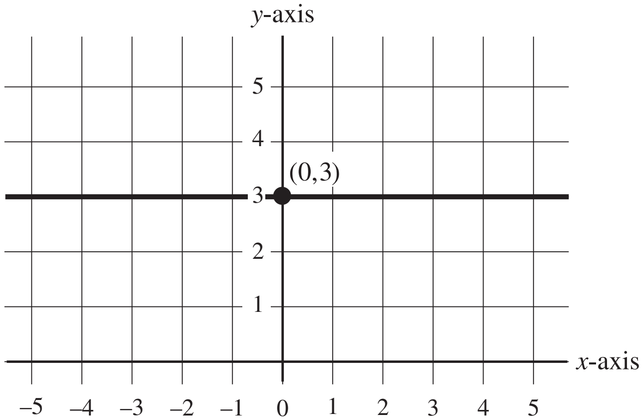
[(3, 27) | (27, 3) | none of these]

If incorrect, study frame **16** once again and then go to **19**. If correct,

Go to **19**.

19

Here is a rather special function. It is called a *constant function* and assigns a single fixed number c to every value of the independent variable, x. Hence, $f(x) = c$.



This is a peculiar function because the value of the dependent variable is the same for all values of the independent variable. Nevertheless, the relation $f(x) = c$ assigns exactly one value of $f(x)$ to each value of x as required in the definition of a function. All the values of $f(x)$ happen to be the same.

Try to convince yourself that the graph of the constant function $y = f(x) = 3$ is a straight line parallel to the x-axis passing through the point (0,3) as shown in the figure.

Go to **20**.

20

Another special function is the *absolute value function*. The absolute value of x is indicated by the symbol $|x|$. The absolute value of a number x determines the size, or magnitude, of the number without regard to its sign. For example,

$$|-3| = |3| = 3$$

Answers: Frame 16: 5, −3
Frame 18: (3, 27)

Now we will define $|x|$ in a general way. But first we need to recall the inequality symbols:

$a > b$ means a is greater than b.

$a \geq b$ means a is greater than or equal to b.

$a < b$ means a is less than b.

$a \leq b$ means a is less than or equal to b.

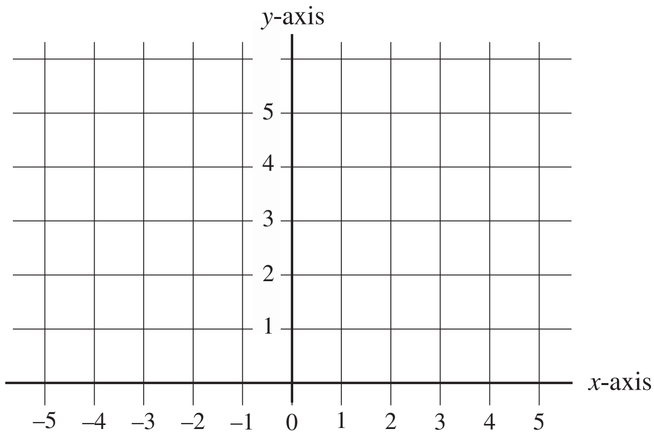
With this notation we can define the absolute value function, $|x|$, by the following two rules:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Go to **21**.

21

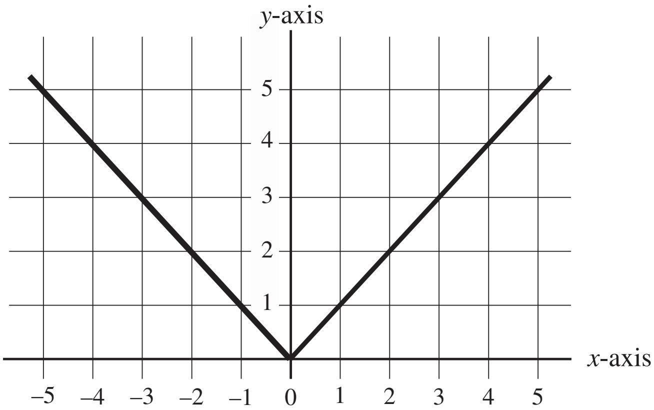
A good way to show the behavior of a function is to plot its graph. Therefore, as an exercise, plot a graph of the function $y = |x|$ in the accompanying figure.



To check your answer, go to **22**.

22

The graph for | x | is



This can be seen by preparing a table of x and y values as follows:

x	$y = \| x \|$
−4	+4
−2	+2
0	0
+2	+2
+4	+4

These points may be plotted as in frames **16** and **18** and the lines drawn with the results in the above figure.

The graph and x, y coordinates described here are known as a *Cartesian coordinate system*. There are other coordinate systems better suited to other geometries, such as cylindrical or spherical coordinate systems, but Cartesian coordinates are the best known.

With this introduction on functions and graphs, we are now going to familiarize ourselves with some important elementary functions.

These functions are the linear, quadratic, trigonometric, exponential, and logarithmic functions.

Go to **23**.

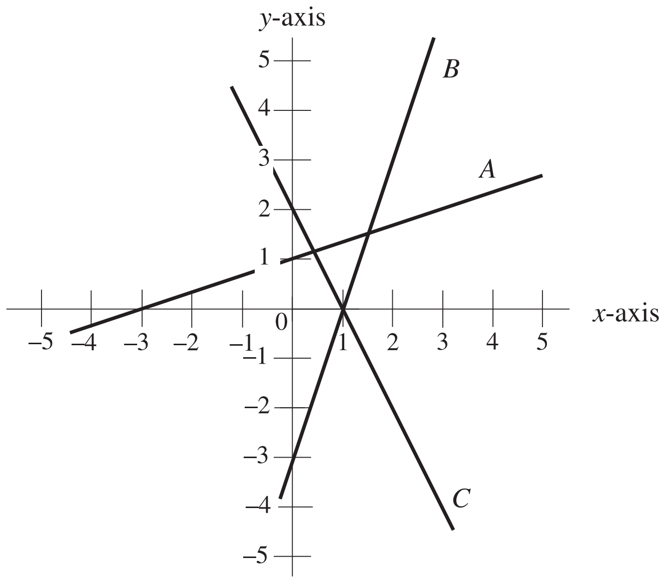
1.4 Linear and Quadratic Functions

23

A function defined by an equation in the form $y = mx + b$, where m and b are constants, is called a *linear function* because its graph is a straight line. This is a simple and useful function, and you need to become familiar with it.

Here is an example: Encircle the letter that identifies the graph (as labeled in the figure) of

$$y = 3x - 3. \qquad [A \mid B \mid C]$$



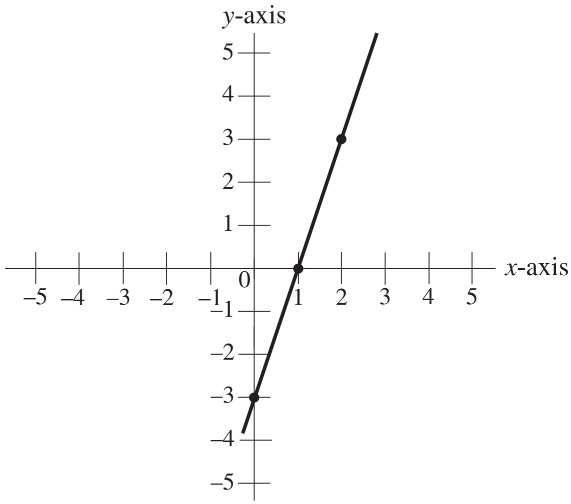
If you missed this or if you do not feel entirely sure of the answer, go to **24**.

Otherwise, go to **25**.

24

The table below gives a few values of x and y for the function $y = 3x - 3$.

x	y
–2	–9
–1	–6
0	–3
1	0
2	3



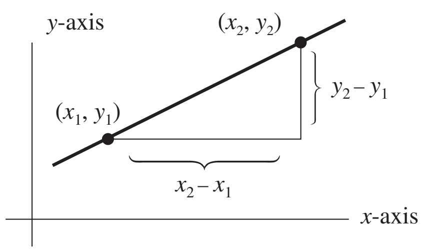
A few of these points are shown on the graph, and a straight line has been drawn through them. This is line B of the figure in frame **23**.

Go to **25**.

25

Here is the graph of a typical linear function. Let us take any two different points on the line, (x_2, y_2) and (x_1, y_1). We define the slope of the line in the following way:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$



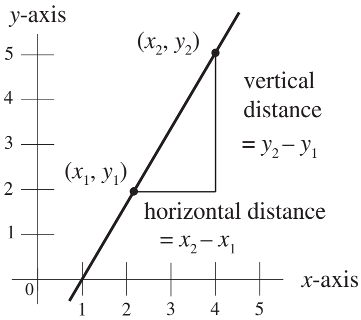
Answer: Frame 23: B

The idea of slope will be important in our later work, so let's spend a little time learning more about it.

Go to **26**.

26

If the x and y scales are the same, as in the figure, then the slope is the ratio of the vertical distance $y_2 - y_1$ to the horizontal distance $x_2 - x_1$ as we go from the point (x_1, y_1) on the line to (x_2, y_2). If the line is vertical, the slope is infinite (or, more strictly, undefined). Test for yourself that the slope is the same for any pair of two separate points on the line.



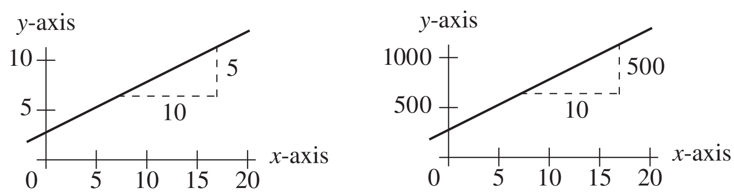
Go to **27**.

27

If the vertical and horizontal scales are not the same, the slope is still defined by

$$\text{slope} = \frac{\text{vertical distance}}{\text{horizontal distance}},$$

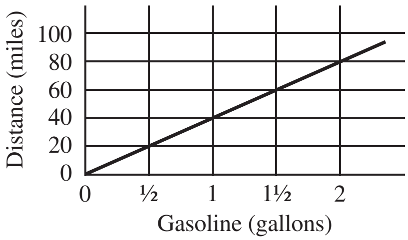
but now the distance is measured using the appropriate scale. For instance, the two figures below may look similar, but the slopes are quite different. In the first figure the x and y scales are identical, and the slope is ½. In the second figure the y scale has been changed by a factor of 100, and the slope is 50.



(continued)

Because the slope is the ratio of two lengths, the slope is a pure number if the lengths are pure numbers. However, if the variables have different dimensions, the slope will also have a dimension.

Below is a plot of the distance traveled by a car vs. the amount of gasoline consumed.



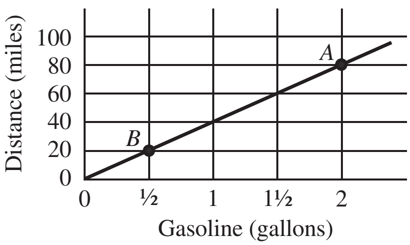
Here the slope has the units of miles per gallon (mpg). What is the slope of the line shown?

Slope = [20 | 40 | 60 | 80] mpg

If right, go to **29**.
Otherwise, go to **28**.

28

To evaluate the slope, let us find the coordinates of any two points on the line.



For instance, *A* has the coordinates (2 gallons, 80 miles) and *B* has the coordinates (½ gallon, 20 miles). Therefore, the slope is

$$\frac{(80 - 20)\text{ miles}}{(2 - 1/2)\text{ gallon}} = 40\,\frac{\text{miles}}{\text{gallon}} = 40\text{ mpg.}$$

We would have obtained the same value for the slope no matter which two points we used, because two points determine a straight line.

Go to **29**.

29

If the line is described by an equation of the form $y = mx + b$, then the slope is given by

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Substituting in the above expression for y, we have

$$\text{slope} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{mx_2 - mx_1}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m.$$

What is the slope of $y = 7x - 5$?

[5/7 | 7/5 | −5 | −7 | 5 | 7]

If right, go to **31**.
Otherwise, go to **30**.

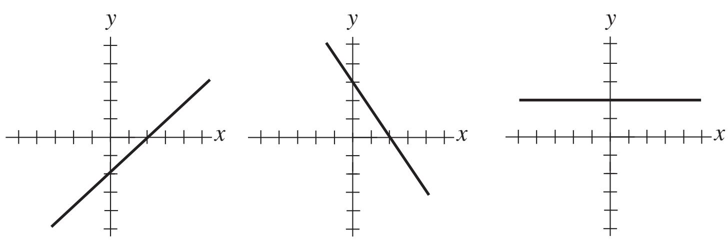
30

The equation $y = 7x - 5$ can be written in the form $y = mx + b$ if $m = 7$ and $b = -5$. Because slope $= m$, the line given has a slope of 7.

Go to **31**.

31

The slope of a line can be positive (greater than 0), negative (less than 0), or 0. An example of each is shown graphically below.



Positive slope Figure 1 — Negative slope Figure 2 — Zero slope Figure 3

Note how a line with positive slope rises in going from left to right, a line with negative slope falls in going from left to right, and a line of zero slope is horizontal. (It was pointed out in frame **26** that the slope of a vertical line is not defined.)

(continued)

Indicate whether the slope of the graph of each of the following equations is positive, negative, or zero by encircling your choice.

	Equation	Slope
1.	$y = 2x - 5$	{ + | − | 0 }
2.	$y = -3x$	{ + | − | 0 }
3.	$p = q - 2$	{ + | − | 0 }
4.	$y = 4$	{ + | − | 0 }

The answer is in the next frame.

32

Here are the answers to the questions in frame **31**.

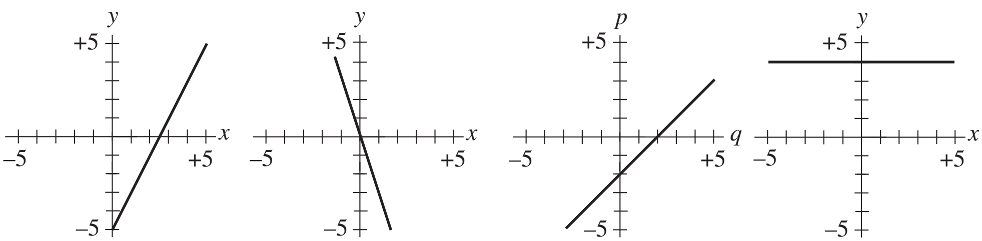
In frame **29** we saw that for a linear equation in the form $y = mx + b$ the slope is m.

1. $y = 2x - 5$. Here $m = 2$ and the slope is 2. Clearly this is a positive number. See Figure 1 below.

2. $y = -3x$. Here $m = -3$. The slope is -3, which is negative. See Figure 2 below.

3. $p = q - 2$. In this equation the variables are p and q, rather than y and x. Written in the form $p = mq + b$, it is evident that $m = 1$, which is positive. See Figure 3 below.

4. $y = 4$. This is an example of a constant function. Here $m = 0$, $b = 4$, and the slope is 0. See Figure 4 below.



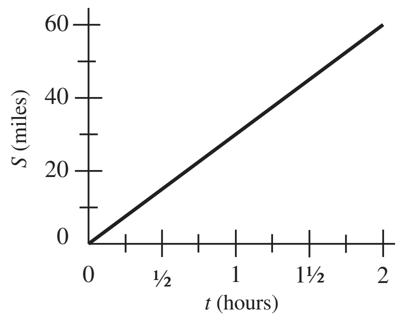
Positive slope $y = 2x - 5$ Figure 1; Negative slope $y = -3x$ Figure 2; Positive slope $p = q - 2$ Figure 3; Zero slope $y = 4$ Figure 4

Go to **33**.

Answers: Frame 27: 40 mpg
Frame 29: 7

33

Here is a linear equation in which the slope has a familiar meaning. The graph below shows the position S of a car on a straight road at different times. The position $S = 0$ means the car is at the starting point.



Try to guess the correct word to fill in the blank below:
The slope of the line has the same value as the car's ____________________.

Go to **34**.

34

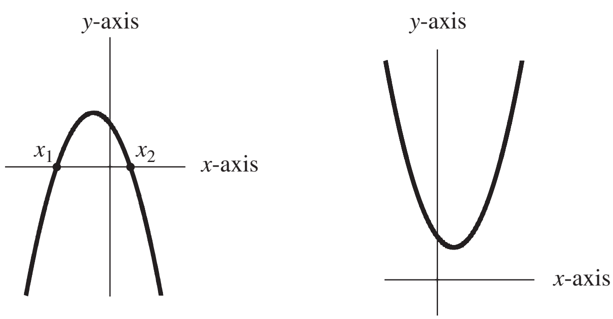
The slope of the line has the same value as the car's *speed* (or, for this one-dimensional motion *velocity*).

The slope is the ratio of the distance traveled to the time required. But, by definition, the speed is also the distance traveled divided by the time. Thus the value of the slope of the line is equal to the speed.

Go to **35**.

35

Now let's look at another type of equation. An equation in the form $y = ax^2 + bx + c$, where a, b, and c are constants ($a \neq 0$), is called a *quadratic function* and its graph is called a *parabola*. Two typical parabolas are shown in the figure.



Go to **36**.

36

Roots of an Equation:

The values of x for which $f(x) = 0$ are called the *roots* of the equation. The values at $y = 0$, shown by x_1 and x_2 in the figure on the left in frame **35**, correspond to values of x which satisfy $ax^2 + bx + c = 0$ and are thus the roots of the equation. Not all quadratic equations have real roots. For example, the curve on the right represents an equation with no real value of x when $y = 0$.

Although you will not need to find the roots of any quadratic equation later in this book, you may want to know the formula anyway. If you would like to see a discussion of this, go to frame **37**.

Otherwise, skip to frame **39**.

37

The equation $ax^2 + bx + c = 0$ has two roots. These are given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The subscripts 1 and 2 serve merely to identify the roots. The two roots can be summarized by a single equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We will not prove these results, though they can be checked by substituting the values for x in the original equation.

Here is a practice problem on finding roots: Which answer correctly gives the roots of $3x - 2x^2 = 1$?

(a) $\frac{1}{4}\left(3 + \sqrt{17}\right)$; $\frac{1}{4}\left(3 - \sqrt{17}\right)$

(b) -1; $-\frac{1}{2}$

(c) $\frac{1}{4}$; $-\frac{1}{4}$

(d) 1; $\frac{1}{2}$

Answers: Frame 31: +, −, +, 0

Encircle the letter of the correct answer.

$$[a \mid b \mid c \mid d]$$

If you got the right answer, go to **39**.
The answer is in the following frame.

38

Here is the solution to the problem in frame **37**.
The equation $3x - 2x^2 = 1$ can be written in the standard form

$$2x^2 - 3x + 1 = 0.$$

Here $a = 2$, $b = -3$, $c = 1$.

$$x = \frac{1}{2a}\left[-b \pm \sqrt{b^2 - 4ac}\right] = \frac{1}{4}\left[-(-3) \pm \sqrt{(-3)^2 - (4)(2)(1)}\right]$$

$$= \frac{1}{4}(3 \pm 1).$$

$$x_1 = \frac{1}{4}(3 + 1) = 1.$$

$$x_2 = \frac{1}{4}(3 - 1) = \frac{1}{2}.$$

Go to **39**.

39

This ends our brief discussion of linear and quadratic functions. Perhaps you would like some more practice on these topics before continuing. If so, try working Review Problems 1–5 on page 277. At the end of this chapter there is a concise summary of the material we have had so far, which you may find useful.

Whenever you are ready, go to **40**.

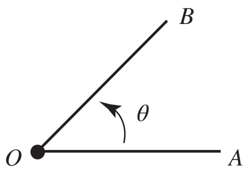
1.5 Angles and Their Measurements

40

Elementary features of rotations and angles:

If you are already familiar with rotations, angle, and degrees and radians, you can jump to frame **50**.

(continued)



The concept of the angle is the bedrock of trigonometry. Although the general idea of an angle is probably familiar, it is important to agree on the conventions and units for describing angles. For two straight-line segments *OA* and *OB* that intersect at a point *O*, the angle between them is a measure of how far the line segment *OA* must be rotated about the point *O* to coincide with the line segment *OB*.

If the two segments initially coincide, for instance, half a revolution of either segment will leave them pointing in opposite directions and a full revolution will bring them back to their original positions.

The Greek letter θ (theta) symbolizes the *rotation angle*. The sense of rotation is shown by the curved arrow. We will follow the convention that if the segment *OA* is rotated in the counterclockwise direction to coincide with the segment *OB*, then the rotation is positive. Conversely, a rotation in the clockwise direction is negative. The direction can be indicated by a small arrowhead on the curve between the two segments. If the sense of rotation is unimportant, the arrowhead is usually omitted.

Measuring the size of rotations:

There are two conventions used for measuring the size of rotation. The first divides one revolution into 360 parts called *degrees* (not to be confused with degrees of temperature). The number 360 has the attraction of being large—providing good angular resolution—and possessing many divisors. The symbol for a degree of rotation is °; hence, a quarter turn is 90°. The degree can be subdivided into 60 *minutes* (60′), and the minute subdivided into 60 *seconds* (60″). Until recent years degrees, minutes, and seconds (*DMS coordinates*) were commonly used in map-making and navigation. With advent of GPS and laser metrology, the convention for subdividing the degree has been changed: instead of minutes and seconds, the convention is to express the fraction of a degree by a decimal number, typically with four digits.

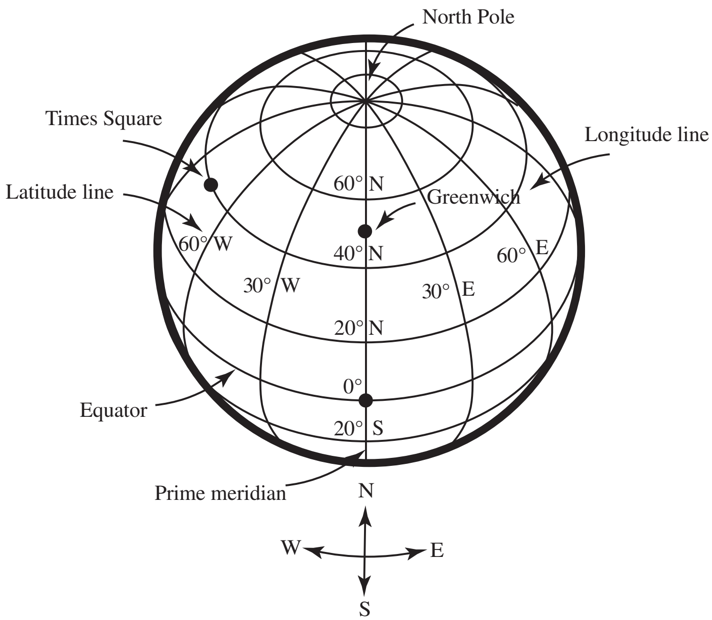
Location on Earth: latitude and longitude:

Two numbers are needed to describe positions on a surface. Because the Earth is spherical, Cartesian coordinates are not useful for specifying a position. Instead, a system based on coordinates known as *latitude* and *longitude* is employed. One of the coordinates is based on the idea of a *meridian*. This is an imaginary half circle on the Earth that connects the Earth's

---

Answers: Frame 37: d

South and North Poles. The *prime meridian* is the half circle passing through Earth's South and North Poles and Greenwich, England. Longitude is the angle between the prime meridian and the meridian through the point of interest. Points east of the prime meridian require a clockwise rotation around the polar axis and by convention are positive; those to the west are negative. The sign of longitude reverses when passing the meridian 180 degrees east or west of the prime meridian.



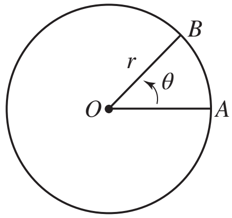
Instead of using positive and negative values of longitude, the convention is to assign east (E) for positive values and west (W) for negative values.

Latitude lines are parallel to the equator. Latitudes north of the equator use the symbol N and latitudes south of the equator use the symbol S. For example, the latitude and longitude of Times Square in New York City are 40.7580°N and 73.9855°W. In old-fashioned DMS units these are 77° 2′ 7.008″W and 38° 53′ 22.1424″N.

Go to **41**.

41

As we have seen, the full circle contains 360°, and so it follows that a semicircle contains 180°.



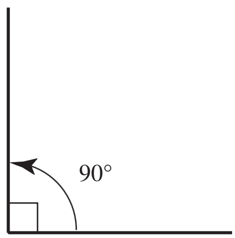
Which of the following angles is equal to the angle θ shown in the figure?

[25° | 45° | 90° | 180°]

If right, go to **43**.
Otherwise, go to **42**.

42

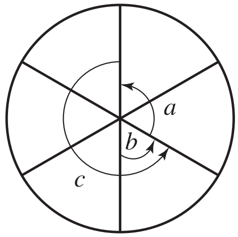
To find the angle θ, let's first look at a related example.



The angle shown is a *right angle*, constructed from two perpendicular lines. (The symbol between sides indicates a right angle.) Because there are four right angles in a full revolution, it is apparent that the angle equals

$$\frac{360°}{4} = 90°.$$

The angle θ shown in frame **41** is just half as big as the right angle; thus it is 45°.
Here is a circle divided into equal segments by three straight lines.



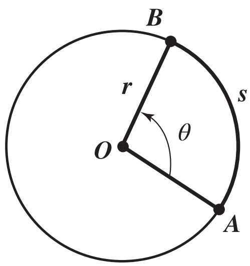
Which of the angles in the figure equals 240°?

[*a* | *b* | *c*]

Go to **43**.

43

The second convention for measuring the size of rotations is the *radian*. The symbol for the radian is *rad*.



To find the value of an angle in radians, we draw a circle of radius *r*, about the vertex, *O*, of the angle so that it intersects the sides of the angle at two points, shown in the figure as *A* and *B*. The length of the arc between *A* and *B* is designated by *s*. Then,

$$\theta \text{ (in radians)} = \frac{s}{r} = \frac{\text{length of arc}}{\text{radius}}.$$

Radians are used widely in scientific applications. For example, to calculate numerical values of trigonometric functions in this chapter naturally calls for radians. If no unit is given for the numerical value of an angle, the angle is in radians.

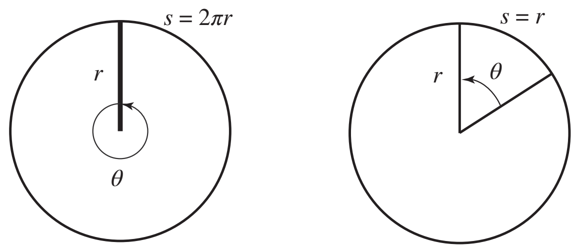
(continued)

To see whether you have caught on, answer this question: There are 360 degrees in a circle; how many radians are there?

$$[1 \mid 2 \mid \pi \mid 2\pi \mid 360/\pi]$$

If right, go to **45**.
Otherwise, go to **44**.

44



The circumference of a circle is πd or $2\pi r$, where d is the diameter and r is the radius. The length of an arc going completely around a circle is the circumference, $2\pi r$, so the angle enclosed is $2\pi r/r = 2\pi$ radians, as shown in the figure on the left. In the figure on the right the angle θ subtends an arc $s = r$.

Encircle the answer, which gives θ.

$$\left[1 \text{ rad} \mid {}^1\!/_4 \text{ rad} \mid {}^1\!/_2 \text{ rad} \mid \pi \text{ rad} \mid \text{none of these}\right]$$

Go to **45**.

45

Because 2π rad $= 360°$, the rule for converting angles from degrees to radians is

$$1 \text{ rad} = \frac{360°}{2\pi}.$$

Answers: Frame 41: 45°
Frame 42: c

Conversely,

$$1^\circ = \frac{2\pi \text{ rad}}{360}.$$

Try the following problems.

$$60^\circ = [2\pi/3 \mid \pi/3 \mid \pi/4 \mid \pi/6] \text{ rad}$$
$$\pi/4 = \left[22\tfrac{1}{2}^\circ \mid 45^\circ \mid 60^\circ \mid 90^\circ\right]$$

Which angle is closest to 1 rad? (Remember that $\pi = 3.14\ldots$)

$$[30^\circ \mid 45^\circ \mid 60^\circ \mid 90^\circ]$$

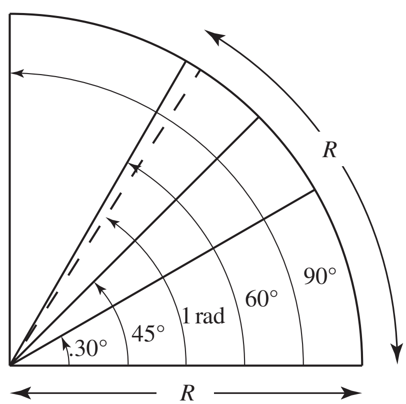
If correct, go to **47**.
If you made any mistake, go to **46**.

46

Here are the solutions to the problems in frame **45**. From the formulas in frame **45**, one obtains

$$60^\circ = 60 \times \frac{2\pi \text{ rad}}{360} = \frac{2\pi \text{ rad}}{6} = \frac{\pi}{3}\text{rad}.$$
$$\frac{\pi}{4}\text{rad} = \frac{\pi}{4} \times \frac{360^\circ}{2\pi} = \frac{360^\circ}{8} = 45^\circ.$$

Because 2π is just a little greater than 6, 1 rad is slightly less than $360^\circ/6 = 60^\circ$. (A closer approximation to the radian is 57.3°.) The figure below shows all the angles in this question.



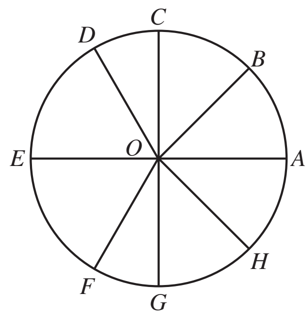
Go to **47**.

47

In the circle shown, CG is perpendicular to AE and

arc AB = arc BC = arc AH,

arc AD = arc DF = arc FA.



(Arc AB means the length of the arc along the circle between A and B, going the shortest way.)

We will designate angles by three letters. For example, $\angle AOB$ (read as "angle AOB") designates the angle between OA and OB.

Try the following:

$$\angle AOD = \{60° \mid 90° \mid 120° \mid 150° \mid 180°\}$$

$$\angle FOH = \{15° \mid 30° \mid 45° \mid 60° \mid 75° \mid 90 \text{ degrees}\}$$

$$\angle HOB = \{1/4 \mid 1 \mid \pi/2 \mid \pi/4 \mid \pi/8\}$$

If you did all these correctly, go to **49**.
If you made any mistakes, go to **48**.

48

Because *arc* AD = *arc* DF = *arc* FA, and the sum of their angles is 360°, $\angle AOD = 360°/3 = 120°$.

$$\angle FOA = 120°, \quad \angle GOA = 90°, \quad \angle GOH = 45°.$$

Answers: Frame 43: 2π
Frame 44: 1 rad
Frame 45: $\pi/3$, 45°, 60°

Thus

$$\angle FOH = \angle FOG + \angle GOH = 30^\circ + 45^\circ = 75^\circ.$$
$$\angle HOB = \angle HOA + \angle AOB = 45^\circ + 45^\circ = 90^\circ.$$

Now try the following:

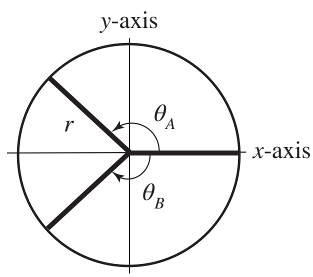
$$90^\circ = [2\pi \mid \pi/6 \mid \pi/2 \mid \pi/8 \mid 1/4]$$
$$3\pi = [240^\circ \mid 360^\circ \mid 540^\circ \mid 720^\circ]$$
$$\pi/6 = [15^\circ \mid 30^\circ \mid 45^\circ \mid 60^\circ \mid 90^\circ \mid 120^\circ]$$

Go to **49**.

49

Rotations can be clockwise or counterclockwise. By choosing a convention for the sign of an angle, we can indicate which direction is meant. As previously explained, an angle formed by rotating in a counterclockwise direction is positive; an angle formed by moving in a clockwise direction is negative.

Here is a circle of radius r drawn with x- and y-axes, as shown:



We will usually choose the positive x-axis as the initial side and, in general, we will measure angles from the initial to the final or terminal side, denoted by the curved arrow. For example, the angle θ_A measured in the counterclockwise direction is positive and θ_B is negative, as shown in the figure. If there is no curved arrow associated with the angle, then we shall assume that the angle is positive.

Go to **50**.

1.6 Trigonometry

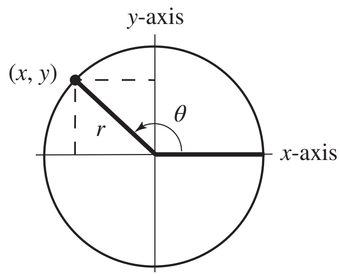
50

If you are not familiar with trigonometric functions, proceed with this frame. Otherwise, check yourself with frame **51**, or go right to frame **52**.

Our next task is to introduce the trigonometric functions. These functions relate the various sides and angles of triangles.

Do you know the general definitions of the trigonometric functions of angle θ? If you do, test yourself with the quiz below. If you don't, go right on to frame **51**.

The trigonometric functions of θ can be expressed in terms of the coordinates x and y and the radius of the circle, $r = \sqrt{x^2 + y^2}$.



These are shown in the figure. Try to fill in the blanks (the answers are in frame **51**):

$\sin\theta =$ _____ $\csc\theta =$ _____

$\cos\theta =$ _____ $\sec\theta =$ _____

$\tan\theta =$ _____ $\cot\theta =$ _____

Go to frame **51** to check your answers.

Answers: Frame 47: 120°, 75°, $\pi/2$
Frame 48: $\pi/2$, 540°, 30°

51

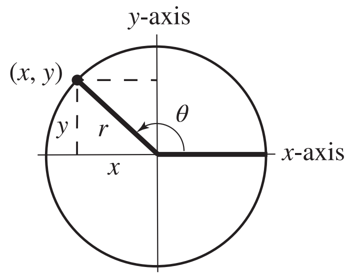
Here are the definitions of the trigonometric functions:

$$\text{sine}: \quad \sin\theta = \frac{y}{r}, \qquad \text{cotangent}: \quad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{\mathrm{y}},$$

$$\text{cosine}: \quad \cos\theta = \frac{x}{r}, \qquad \text{secant}: \quad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x},$$

$$\text{tangent}: \quad \tan\theta = \frac{y}{x}, \qquad \text{cosecant}: \quad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}.$$

Notice that the definitions in the right-hand equations are the reciprocal of those on the left.



For the angle shown in the figure, x is negative and y is positive ($r = \sqrt{x^2 + y^2}$ and is always positive) so that $\cos\theta$, $\tan\theta$, $\cot\theta$, and $\sec\theta$ are negative, while $\sin\theta$ and $\csc\theta$ are positive.

Go to **52**.

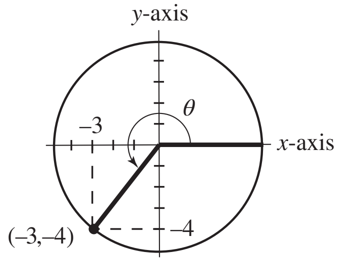
52

Below is a circle with a radius of 5. The point shown is (−3, −4). On the basis of the definition in the last frame, you should be able to answer the following:

$\sin\theta = [3/5 \mid 5/3 \mid 3/4 \mid -4/5 \mid -3/5 \mid 4/3]$

$\cos\theta = [3/5 \mid 5/3 \mid 3/4 \mid -4/5 \mid -3/5 \mid 4/3]$

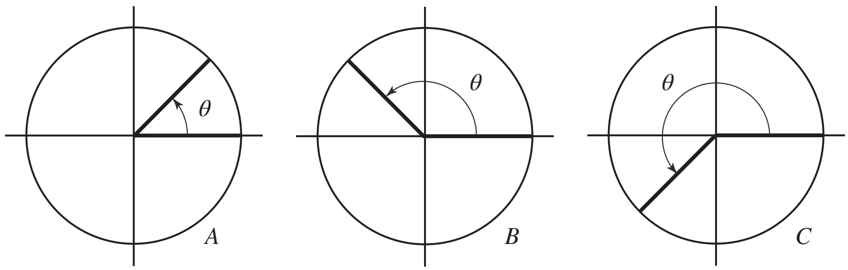
$\tan\theta = [3/5 \mid 5/3 \mid 3/4 \mid -4/5 \mid -3/5 \mid 4/3]$



If all right, go to **55**.
Otherwise, go to **53**.

53

Perhaps you had difficulty because you did not realize that x and y have different signs in different quadrants (quarters of the circle) while r, a radius, is always positive. Try this problem.



Indicate whether the function specified is positive or negative, for each of the figures, by checking the correct box.

	Figure A +	**Figure A** −	**Figure B** +	**Figure B** −	**Figure C** +	**Figure C** −
$\sin\theta$						
$\cos\theta$						
$\tan\theta$						

See frame **54** for the correct answers.

54

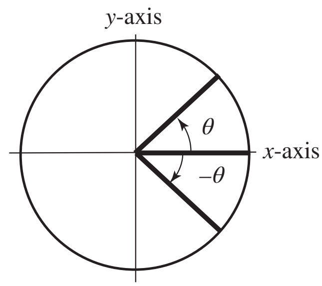
Here are the answers to the questions in frame **53**.

	Figure *A* +	Figure *A* −	Figure *B* +	Figure *B* −	Figure *C* +	Figure *C* −
$\sin\theta$	✓		✓			✓
$\cos\theta$	✓			✓		✓
$\tan\theta$	✓			✓	✓	

Go to **55**.

55

In the figure both θ and $-\theta$ are shown. The trigonometric functions for these two angles are simply related.



Can you do these problems? Encircle the correct sign.

$$\sin(-\theta) = [+ \mid -] \sin\theta$$

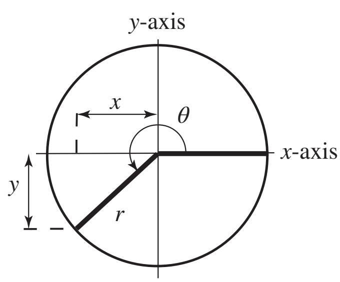
$$\cos(-\theta) = [+ \mid -] \cos\theta$$

$$\tan(-\theta) = [+ \mid -] \tan\theta$$

Go to **56**.

56

There are many relationships among the trigonometric functions.



(continued)

For instance, using $r^2 = x^2 + y^2$, we have

$$\sin^2\theta = \frac{y^2}{r^2} = \frac{r^2 - x^2}{r^2} = 1 - \left(\frac{x}{r}\right)^2 = 1 - \cos^2\theta.$$

Try these:

1. $\sin^2\theta + \cos^2 = \{\sec^2\theta \mid 1 \mid \tan^2\theta \mid \cot^2\theta\}$

2. $1 + \tan^2\theta = \{1 \mid \tan^2\theta \mid \cot^2\theta \mid \sec^2\theta\}$

3. $\sin^2\theta - \cos^2\theta = \{1 - 2\cos^2\theta \mid 1 - 2\sin^2\theta \mid \cot^2\theta \mid 1\}$

If any mistakes, go to **57**.
Otherwise, go to **58**.

57

Here are the solutions to the problems in frame **56**.

1. $\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1.$

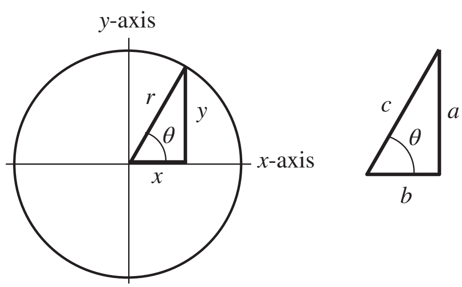
This is an important identity, which is worth remembering.
The other solutions are

2. $1 + \tan^2\theta = 1 + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} = \sec^2\theta.$

3. $\sin^2\theta - \cos^2 = (1 - \cos^2\theta) - \cos^2\theta = 1 - 2\cos^2\theta.$

Go to **58**.

58



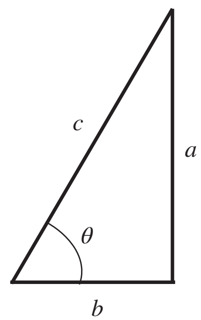
Answers: Frame 52: $-4/5, -3/5, 4/3$
Frame 55: $-, +, -$

The trigonometric functions are particularly useful when applied to right triangles (triangles with one 90°, or right angle). In this case θ is always acute (less than 90°, or $\pi/2$). You can then write the trigonometric functions in terms of the sides a and b of the right triangle shown, and its hypotenuse c. Fill in the blanks.

$$\sin\theta = \_\_\_\_\_\_\_\_ \qquad \csc\theta = \_\_\_\_\_\_\_\_$$
$$\cos\theta = \_\_\_\_\_\_\_\_ \qquad \sec\theta = \_\_\_\_\_\_\_\_$$
$$\tan\theta = \_\_\_\_\_\_\_\_ \qquad \cot\theta = \_\_\_\_\_\_\_\_$$

Check your answer in **59**.

59



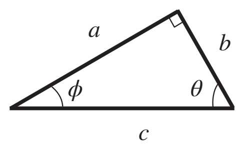
The answers are:

$$\sin\theta = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}, \csc\theta = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}},$$
$$\cos\theta = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}, \sec\theta = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}},$$
$$\tan\theta = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}, \cot\theta = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}}.$$

These results follow from the definitions in frame **51**, providing we let a, b, and c correspond to y, x, and r, respectively. (Remember that here θ is less than 90°.) If you are not familiar with the terms *opposite side*, *adjacent side*, and *hypotenuse*, they should be evident from the figure.

Go to **60**.

60



The following problems refer to the figure shown. (ϕ is the Greek letter "phi.")

$$\sin\theta = [b/c \mid a/c \mid c/a \mid c/b \mid b/a \mid a/b]$$

$$\tan\phi = [b/c \mid a/c \mid c/a \mid c/b \mid b/a \mid a/b]$$

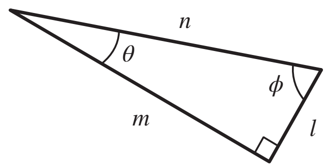
If all right, go to **62.**
Otherwise, go to **61.**

61

You may have become confused because the triangle was drawn in a new position. Review the definitions in **51**, and then do the following problems:

$$\cos\ \theta = [l/n \mid n/l \mid m/n \mid m/l \mid n/m \mid l/m]$$

$$\cot\ \phi = [l/n \mid n/l \mid m/n \mid m/l \mid n/m \mid l/m]$$

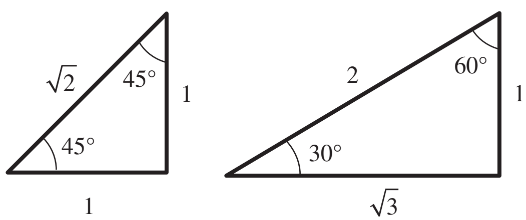


If you missed either of these, you will have to put in more work learning and memorizing the definitions.

Meanwhile go to **62.**

62

It is helpful to be familiar with the trigonometric functions of 30°,45°, and 60°. The triangles for these angles are particularly simple.



Answer: Frame 56: 1, $\sec^2\theta$, $1 - 2\cos^2\theta$

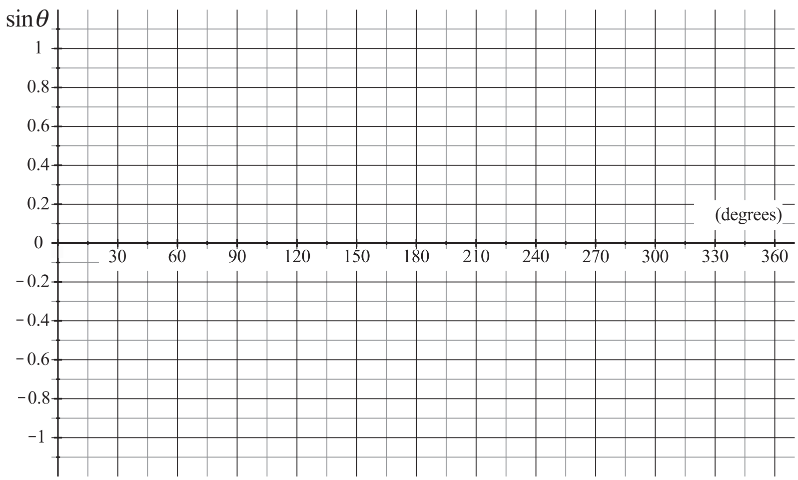
Try these problems:

$$\cos 45^\circ = [1/2 \mid 1/\sqrt{2} \mid 2\sqrt{2} \mid 2]$$
$$\sin 30^\circ = [3 \mid \sqrt{3/2} \mid 2/3 \mid 1/2]$$
$$\sin 45^\circ = [1/2 \mid 1/\sqrt{2} \mid \sqrt{2/2} \mid 2]$$
$$\tan 30^\circ = [1 \mid \sqrt{3} \mid 1/\sqrt{3} \mid 2]$$

Make sure you understand these problems. Then go to **63.**

63

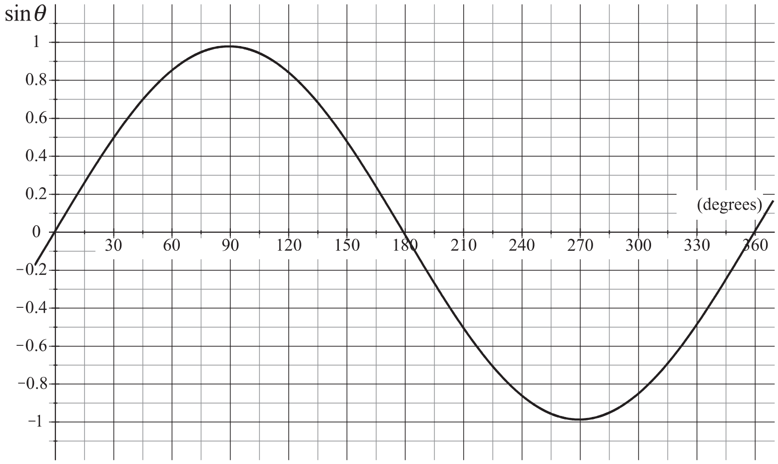
Many calculators provide values of trigonometric functions. With such a calculator, it is simple to plot enough points to make a good graph of the function. If you have a calculator, plot $\sin\theta$ for values between 0° and 360° on the coordinate axes below, and then compare your result with frame **64**. If you do not have a suitable calculator, go directly to **64** and check that $\sin\theta$ has the correct values for the angles you know.



Go to **64**.

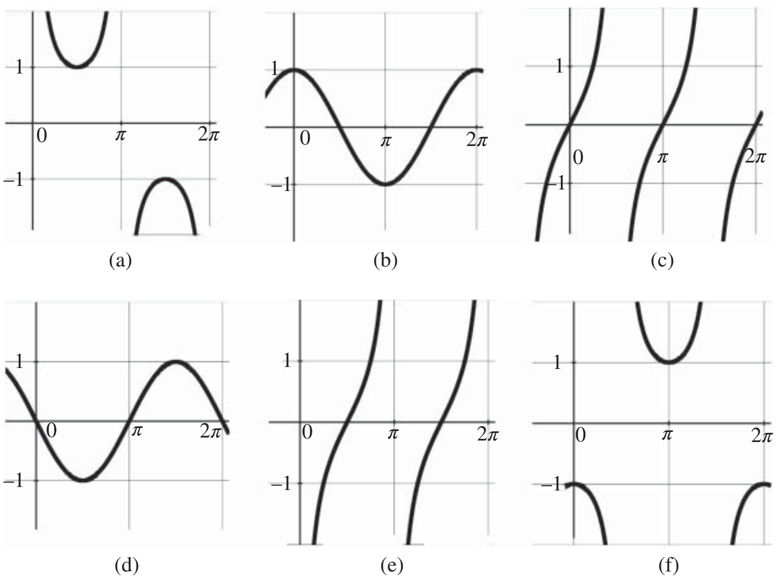
64

Here is the graph of the sine function.



Go to **65**.

65



Answers: Frame 60: a/c, b/a
Frame 61: m/n, l/m
Frame 62: $1/\sqrt{2}$, $1/2$, $1/\sqrt{2}$, $1/\sqrt{3}$

Try to decide which graph represents each function.

$\cos\theta$: [a | b | c | d | e | f | none of these]

$\tan\theta$: [a | b | c | d | e | f | none of these]

$\sin(-\theta)$: [a | b | c | d | e | f | none of these]

$\tan(-\theta)$: [a | b | c | d | e | f | none of these]

If you got these all right, go to **67**.
Otherwise go to **66**.

66

Knowing the values of the trigonometric functions at a few important points will help you identify them. Try these (∞ is the symbol for infinity, here meaning that the function is undefined):

$\sin 0° = [0 \mid 1 \mid -1 \mid -\infty \mid +\infty]$

$\cos 0° = [0 \mid 1 \mid -1 \mid -\infty \mid +\infty]$

$\cos 30° = [1 \mid 1/2 \mid \sqrt{3} \mid \sqrt{3}/2]$

$\tan 45° = [0 \mid 1 \mid -1 \mid -\infty \mid +\infty]$

$\cos 60° = [1 \mid 1/2 \mid \sqrt{3} \mid \sqrt{3}/2]$

$\sin 90° = [0 \mid 1 \mid -1 \mid -\infty \mid +\infty]$

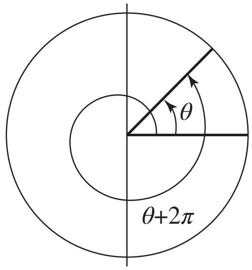
$\cos 90° = [0 \mid 1 \mid -1 \mid -\infty \mid +\infty]$

Go to **67**.

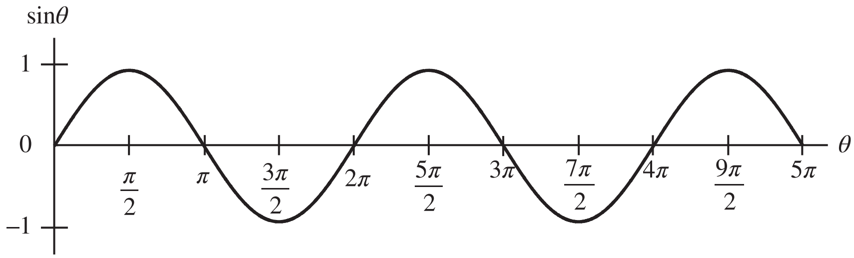
67

Because the angle $\theta + 2\pi n$, where n is any integer, is equivalent to θ as far as the trigonometric functions are concerned (i.e. for any trig function f, $f(\theta + 2\pi n) = f(\theta)$), we can add $2\pi n$ to any angle without changing the value of the trigonometric functions. Thus, the sine and cosine (as well as its reciprocals, csc and sec) functions repeat their values whenever θ increases by $2\pi n$ where n is an integer; we say that these functions are *periodic* in θ with a *fundamental period* of 2π, or 360°. (The fundamental period of the tangent and the cotangent is π.)

(continued)



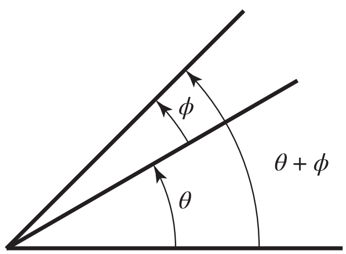
Using this property, you can extend the graph of sin θ in frame **64** to the following. (For variety, the angle here is in radians.)



Go to **68**.

68

It is helpful to know the sine and cosine of the sum and the difference of two angles.



Answers: Frame 65: b, c, d, none of these;
Frame 66: $\sin 0° = 0,\ \cos 0° = 1,\ \cos 30° = \sqrt{3}/2,\ \tan 45° = 1,$
$\cos 60° = 1/2,\quad \sin 90° = 1,\ \cos 90° = 0.$

Do you happen to remember the formulas from previous studies of trigonometry? If not, go to **69**. If you do, try the quiz below.

$$\sin(\theta + \phi) = \underline{\hspace{4cm}}.$$

$$\cos(\theta + \phi) = \underline{\hspace{4cm}}.$$

Go to **69** to see the correct answer.

69

Here are the formulas. They are derived in Appendix A1.

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi,$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

These formulas hold for both positive and negative values of the angles. (Note that $\tan(\theta + \phi)$ and $\cot(\theta + \phi)$ can be obtained from these formulas and the relation $\tan\theta = \sin\theta / \cos\theta$.

By using what you have already learned, circle the correct sign in each of the following:

(a) $\sin(\theta - \phi) = \{+ \mid -\} \sin\theta\cos\phi \; \{+ \mid -\} \cos\theta\sin\phi$

(b) $\cos(\theta - \phi) = \{+ \mid -\} \cos\theta\cos\phi \; \{+ \mid -\} \sin\theta\sin\phi$

If right, go to **71**.
If wrong, go to **70**.

70

If you made a mistake in problem **69,** recall from frame **55** that

$$\sin(-\phi) = -\sin\phi,$$

$$\cos(-\phi) = +\cos\phi.$$

Then

$$\begin{aligned}\sin(\theta - \phi) &= \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)\\ &= \sin\theta\cos\phi - \cos\theta\sin\phi,\\ \cos(\theta - \phi) &= \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi)\\ &= \cos\theta\cos\phi + \sin\theta\sin\phi.\end{aligned}$$

Go to **71**.

71

By using the expressions for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$, one can obtain the formulas for $\sin(2\theta)$ and $\cos(2\theta)$. Simply let $\theta = \phi$. Fill in the blanks.

$$\sin 2\theta = \underline{\hspace{3cm}}.$$

$$\cos 2\theta = \underline{\hspace{3cm}}.$$

See **72** for the correct answers.

72

Here are the answers:

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta,\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= 1 - 2\sin^2\theta\\ &= 2\cos^2\theta - 1.\end{aligned}$$

(Note, by convention, $(\sin\theta)^2$ is usually written $\sin^2\theta$, and $(\cos\theta)^2$is usually written $\cos^2\theta$.)

Go to **73**

73

It is often useful to use the *inverse trigonometric* function. This is the value of the angle for which the trigonometric function has a specified value. The inverse sine of x is denoted by $\sin^{-1}x$. (Warning: This notation is standard, but it can be confusing. $\sin^{-1}x$ always represents the inverse sine of x, not $1/\sin x$. The latter would be written $(\sin x)^{-1}$. An older notation for $\sin^{-1}x$ is arcsin x.)

For example, because the sine of 30°is 1/2, $\sin^{-1}(1/2) = 30°$. Note, however, that the sine of 150°is also 1/2. Furthermore, the trigonometric functions are periodic: there is an endless sequence of angles (all differing by 360°) having the same value for the sine, cosine, etc.

Answer: Frame 69 (a): +, −; (b): +, +

Because the definition of function (frame **6**) specifies the assignment of one and only one value of $f(x)$ for each value of x, the domain of the inverse trigonometric function must be suitably restricted.

The inverse functions are defined by

$$y = \sin^{-1}x \quad \text{Domain: } -1 \le x \le +1 \quad \text{Range: } -\frac{\pi}{2} \le y \le +\frac{\pi}{2}$$

$$y = \cos^{-1}x \quad \text{Domain: } -1 \le x \le +1 \quad \text{Range: } 0 \le y \le \pi$$

$$y = \tan^{-1}x \quad \text{Domain: } -\infty < x < +\infty \quad \text{Range: } -\frac{\pi}{2} < y < +\frac{\pi}{2}$$

Go to **74**.

74

Try these problems:

(a) $\sin^{-1}(1/\sqrt{2}) = [\pi/6 \mid \pi/4 \mid \pi/3 \mid \pi/2]$

(b) $\tan^{-1}(1) = [\pi/6 \mid \pi/4 \mid \pi/3 \mid \pi]$

(c) $\cos^{-1}(1/2) = [\pi/6 \mid \pi/4 \mid \pi/3 \mid \pi]$

If you have a calculator with inverse trigonometric functions, try the following:

(d) $\sin^{-1}(0.8) = [46.9 \mid 28.2 \mid 53.1 \mid 67.2]$ degrees

(e) $\tan^{-1}(12) = [0.82 \mid 1.49 \mid 1.62 \mid 1.83]$ radians

(f) $\cos^{-1}(0.05) = [4.3 \mid 12.6 \mid 77.2 \mid 87.1]$ degrees

Check your answers, and then go on to the next section, which is the last one in our reviews.

Go to **75**.

1.7 Exponentials and Logarithms

75

Are you already familiar with exponentials? If not, go to **76**. If you are, try this short quiz.

$$a^5 = [5^a \mid 5\log a \mid a\log 5 \mid \text{none of these}]$$
$$a^{b+c} = [a^b a^c \mid a^b + a^c \mid ca^b \mid (b+c)\log a]$$
$$a^f/a^g = [(f-g)\log a \mid a^{f/g} \mid a^{f-g} \mid \text{none of these}]$$
$$a^0 = [0 \mid 1 \mid a \mid \text{none of these}];\ a \neq 0$$
$$(a^b)^c = [a^b a^c \mid a^{b+c} \mid a^{bc} \mid \text{none of these}]$$

If any mistakes, go to **76**.
Otherwise, go to **77**.

76

By definition a^m, where m is a positive integer, is the product of m factors of a. Hence,

$$2^3 = (2)(2)(2) = 8 \text{ and } 10^2 = (10)(10) = 100.$$

Furthermore, by definition $a^{-m} = 1/a^m$. It is easy to see, then, that

$$a^m a^n = a^{m+n},$$
$$\frac{a^m}{a^n} = a^{m-n},$$
$$a^0 = \frac{a^m}{a^m} = 1 \qquad (\text{a} \neq 0,\ m \text{ can be any integer})$$
$$(a^m)^n = a^{mn},$$
$$(ab)^m = a^m b^m.$$

Note that a^{m+n} is evaluated as $a^{(m+n)}$; the expression in the exponential is always evaluated before any other operation is carried out.

If you have not yet tried the quiz in frame **75**, try it now. Otherwise,

Go to **77**.

Answer: Frame 74: (a) $\pi/4$, (b) $\pi/4$, (c) $\pi/3$, (d) 53.1°, (e) 1.49, (f) 87.1°

77

Here are a few problems:

$$3^2 = [6 \mid 8 \mid 9 \mid \text{none of these}]$$
$$1^3 = \left[1 \mid 3 \mid \frac{1}{3} \mid \text{none of these}\right]$$
$$2^{-3} = \left[-6 \mid \frac{1}{8} \mid -9 \mid \text{none of these}\right]$$
$$\frac{4^3}{4^5} = [4^8 \mid 4^{-8} \mid 16^{-1} \mid \text{none of these}]$$

If you did these all correctly, go to **79**.
If you made any mistakes, go to **78**.

78

Below are the solutions to problem **77**. Refer back to the rules in **76** if you have trouble understanding the solution.

$$3^2 = (3)(3) = 9,$$
$$1^3 = (1)(1)(1) = 1 \qquad (1^m = 1 \text{ for any } m),$$
$$2^{-3} = \frac{1}{2^3} = \frac{1}{8},$$
$$\frac{4^3}{4^5} = 4^{3-5} = 4^{-2} = \frac{1}{16} = 16^{-1}.$$

Now try these:

$$(3^{-3})^3 = [1 \mid 3^{-9} \mid 3^{-27} \mid \text{none of these}]$$
$$\frac{5^2}{3^2} = \left[\left(\frac{5}{3}\right)^2 \middle| \left(\frac{5}{3}\right)^{-1} \middle| 5^{-6} \middle| \text{none of these}\right]$$
$$4^3 = [12 \mid 16 \mid 2^6 \mid \text{none of these}]$$

Check your answers and try to track down any mistakes.

Then go to **79**.

79

Here are a few more problems.

$$10^0 = [0 \mid 1 \mid 10]$$

$$10^{-1} = [-1 \mid 1 \mid 0.1]$$

$$0.00003 = \left[\frac{1}{3} \times 10^{-3} \mid 10^{-3} \mid 3 \times 10^{-5}\right]$$

$$0.4 \times 10^{-4} = [4 \times 10^{-5} \mid 4 \times 10^{-3} \mid 2.5 \times 10^{-5}]$$

$$\frac{3 \times 10^{-7}}{6 \times 10^{-3}} = \left[\frac{1}{2} \times 10^{10} \,\middle|\, 5 \times 10^4 \,\middle|\, 0.5 \times 10^{-4}\right]$$

If these were all correct, go to **81**.
If you made any mistakes, go to **80**.

80

Here are the solutions to the problems in **79**:

$$10^0 = \frac{10}{10} = 1$$

$$10^{-1} = \frac{1}{10} = 0.1,$$

$$0.00003 = 0.00001 \times 3 = 3 \times 10^{-5},$$

$$0.4 \times 10^{-4} = (4 \times 10^{-1}) \times 10^{-4} = 4 \times 10^{-5},$$

$$\frac{3 \times 10^{-7}}{6 \times 10^{-3}} = \frac{3}{6} \times \frac{10^{-7}}{10^{-3}} = \frac{1}{2} \times 10^{-7+3} = 0.5 \times 10^{-4}.$$

Go to **81**.

81

Let's introduce the idea of *fractional exponents*. If $b^n = a$, then b is called the *nth root* of a and is written $b = a^{1/n}$. Hence $16^{1/4}$ = (fourth root of 16) = 2. That is, $2^4 = 16$.

Answers: Frame 75: a^5 = none of these, $a^{b+c} = a^b a^c$, $a^f/a^g = a^{f-g}$, $a^0 = 1$, $(a^b)^c = a^{bc}$

Frame 77: 9, 1, $1/8$, 16^{-1}

Frame 78: 3^{-9}, $(5/3)^2$, 2^6

If $y = a^{m/n}$, where m and n are integers, then $y = (a^{1/n})^m$. For instance

$$8^{2/3} = (8^{1/3})^2 = 2^2 = 4.$$

Try these:

$$27^{-2/3} = \left[\tfrac{1}{18} \mid \tfrac{1}{18} \mid \tfrac{1}{9} \mid -18 \mid \text{none of these}\right]$$

$$16^{3/4} = [12 \mid 8 \mid 6 \mid 64]$$

If right, go to **84**.
If wrong, go to **82**.

82

The answers are:

$$27^{-2/3} = (27^{1/3})^{-2} = 3^{-2} = \tfrac{1}{9},$$

$$16^{3/4} = (16^{1/4})^3 = 2^3 = 8.$$

Do these problems:

$$25^{3/2} = [125 \mid 5 \mid 15 \mid \text{none of these}]$$

$$(0.00001)^{-3/5} = [0.001 \mid 1000 \mid 10^{-15} \mid 10^{-25}]$$

If your answers were correct, go to **84**.
Otherwise, go to **83**.

83

Here are the solutions to the problems in **82**.

$$25^{3/2} = (25^{1/2})^3 = 5^3 = 125,$$

$$(0.00001)^{-3/5} = (10^{-5})^{-3/5} = 10^{15/5} = 10^3 = 1000.$$

Here are a few more problems. Encircle the correct answers.

$$\left(\tfrac{27}{64} \times 10^{-6}\right)^{1/3} = \left[\tfrac{3}{400} \,\middle|\, \tfrac{3}{16} \times 10^{-2} \,\middle|\, \tfrac{9}{64} \times 10^{-4}\right],$$

$$(49 \times 10^{-4})^{1/4} = \left[\sqrt{7}/10 \,\middle|\, (10 \times 7)^{-2} \,\middle|\, \sqrt{7}/1000\right].$$

Go to **84** after checking your answers.

84

Although our original definition of a^m applied only to integral values of m, we have also defined $(a^m)^{1/n} = a^{m/n}$, where both m and n are integers. Thus we have a meaning for a^p, where p is either an integer or a fraction (ratio of integers).

As yet we do not know how to evaluate a^p if p is an irrational number, such as π or $\sqrt{2}$. However, we can approximate an irrational number as closely as we desire by a fraction. For instance, π is approximately 31,416/10,000. This is in the form m/n, where m and n are integers, and we know how to evaluate it. Therefore, $y = a^x$, where x is any real number, is a meaningful expression in the sense that we can evaluate it as accurately as we please. (A more rigorous treatment of irrational exponents can be based on the properties of suitably defined logarithms.)

Try the following problem.

$$\frac{a^\pi a^x}{a^3} = \left[a^{\pi x/3} \mid a^{\pi+x-3} \mid a^{3\pi x} \mid a^{(\pi+x)/3}\right]$$

If right, go to **86**.
If wrong, go to **85**.

85

The rules given in frame **76** apply here as if all exponents were integers. Hence

$$\frac{a^\pi a^x}{a^3} = a^{\pi+x-3}.$$

Here is another problem:

$$(\pi^2)(2^\pi) = \left[1 \mid (2\pi)^{2\pi} \mid 2\pi^{2+\pi} \mid \text{none of these}\right]$$

If right, go to **87**.
If wrong, go to **86**.

Answers: Frame 79: 1, 0.1, 3×10^{-5}, 4×10^{-5}, 0.5×10^{-4}

Frame 81: 1/9, 8

Frame 82: 125, 1000

Frame 83: 3/400, $\sqrt{7}/10$

86

The quantity $(\pi^2)(2^\pi)$ is the product of two different numbers raised to two different exponents. None of our rules apply to this and, in fact, there is no way to simplify this expression.

Now go to **87**.

87

If you do not clearly remember logarithms, go to **88**.

If you do, try the following test. Let x be any positive number, and let $\log x$ represent the log of x to the *base* 10. Then:

$$10^{\log x} = \underline{\qquad\qquad}.$$

Go to **88** for the correct answer.

88

The answer to **87** is x; in fact we will take the logarithm of x to the *base* 10 to be defined by

$$\boxed{10^{\log x} = x.}$$

That is, the logarithm of a number x is the power to which 10 must be raised to produce the number x itself. This definition only applies for $x > 0$. Here are two examples:

$$100 = 10^2, \quad \text{therefore} \quad \log 100 = 2;$$
$$0.001 = 10^{-3}, \quad \text{therefore} \quad \log 0.001 = -3.$$

Now try these problems:

$$\log 1{,}000{,}000 = [1{,}000{,}000 \mid 6 \mid 60 \mid 600]$$

$$\log 1 = [0 \mid 1 \mid 10 \mid 100]$$

If right, go to **90**.
If wrong, go to **89**.

89

Here are the answers:

$$\log 1{,}000{,}000 = \log(10^6) = 6 \quad (\text{check, } 10^6 = 1{,}000{,}000),$$

$$\log 1 = \log(10^0) = 0 \quad (\text{check, } 10^0 = 1).$$

(continued)

Try the following:

$$\log(10^4/10^{-3}) = [10^7 \mid 1 \mid 10 \mid 7 \mid 70]$$
$$\log(10^n) = [10n \mid n \mid 10^n \mid 10/n]$$
$$\log(10^{-n}) = [-10n \mid -n \mid -10^n \mid -10/n]$$

If you had trouble with these, carefully review the material in this section.

Then go to **90**.

90

Here are three important relations for manipulating logarithms, a and b are any positive numbers:

$$\log(ab) = \log a + \log b,$$
$$\log(a/b) = \log a - \log b,$$
$$\log(a^n) = n \log a.$$

If you are familiar with these rules, go to **92**. If you want to see how they are derived,

Go to **91**.

91

We can derive the required rules as follows. From the definition of $\log x$, $a = 10^{\log a}$ and $b = 10^{\log b}$. Consequently, from the properties of exponentials,

$$ab = (10^{\log a})(10^{\log b}) = 10^{\log a + \log b}.$$

Taking the log of both sides, and again using $\log 10^x = x$ gives

$$\log(ab) = \log 10^{\log a + \log b} = \log a + \log b.$$

Similarly,

$$a/b = 10^{\log a} 10^{-\log b} = 10^{\log a - \log b}.$$
$$\log(a/b) = \log a - \log b$$

Answers: Frame 84: $a^{\pi + x - 3}$
Frame 85: None of these
Frame 88: 6, 0

Likewise,

$$a^n = (10^{\log a})^n = 10^{n \log a},$$

so that

$$\log(a^n) = n \log a.$$

Go to **92**.

92

Try these problems:

$$\text{If } \log n = -3, \quad n = \left[1/3 \mid 1/300 \mid 1/1000\right]$$

$$10^{\log 100} = [10^{10} \mid 20 \mid 100 \mid \text{none of these}]$$

$$\frac{\log 1000}{\log 100} = \left[3/2 \mid 1 \mid -1 \mid 10\right]$$

If right, go to **94**.
If wrong, go to **93**.

93

The answers are:

$$10^{\log n} = n, \text{so if } \log n = -3, n = 10^{-3} = 1/1000.$$

For the same reason,

$$10^{\log 100} = 100.$$

$$\frac{\log 1000}{\log 100} = \frac{\log 10^3}{\log 10^2} = \frac{3}{2}.$$

Try these problems:

$$1/2 \log 16 = [2 \mid 4 \mid 8 \mid \log 2 \mid \log 4]$$

$$\log(\log 10) = [10 \mid 1 \mid 0 \mid -1 \mid -10]$$

Go to **94**.

94

In this section we have discussed only logarithms to the base 10. However, any positive number except 1 can be used as a base. Bases other than 10 are usually indicated by a subscript.

(continued)

For instance, the logarithm of 8 to the base 2 is written $\log_2 8$, and is equal to 3 because $2^3 = 8$. If our base is denoted by r, then the defining equation for $\log_r x$ is

$$\boxed{r^{\log_r x} = x.}$$

All the relations explained in frame **91** are true for logarithms to any base (provided, of course, that the same base is used for all the logarithms in each equation).

There is a special base number

$$e = 2.71828\ldots,$$

called *Euler's number*, that is used to define *natural logarithms* that are usually designated by the symbol $\ln x = \log_e x$. Euler's number is an irrational number, and the three dots, known as an *ellipsis*, indicate the indefinite continuation of that number. The defining equation for natural logarithms is then

$$e^{\ln x} = x.$$

From the defining equation, set $x = e$, then $e^{\ln e} = e$, thus

$$\ln e = 1.$$

The significance of this special property will be described in Chapter 2.

Go to **95**.

95

From the definition of logarithm in the last frame we can obtain the rule for changing logarithms from one base to another, for instance from base 10 to the base e. (Many calculators give both $\log x$, i.e. $\log_{10} x$, and $\ln x$.) Take $\log_{10}$ of both sides of the defining equation $e^{\ln x} = x$,

$$\log(e^{\ln x}) = \log x.$$

Because $\log x^n = n \log x$ (frame **91**), this gives $\ln x \log e = \log x$ or

$$\ln x = \frac{\log x}{\log e}.$$

Answers: Frame 89: 7, n, $-n$
Frame 92: $1/1000$, 100, $3/2$
Frame 93: log 4, 0

The numerical value of $\log e$ is $1/(2.303\ \ldots)$ so

$$\ln x = (2.303)\log x.$$

If you have a calculator which evaluates both $\ln x$ and $\log x$, check this relation for a few values of x.

The $\ln x$ satisfies the same properties as $\log x$ as listed in frame **90**,

$$\ln(ab) = \ln a + \ln b,$$
$$\ln(a/b) = \ln a - \ln b,$$
$$\ln(a^n) = n \ln a.$$

Go to **96**.

96

Before concluding Chapter 1 it is worth commenting on how to find the values of the functions in this chapter. In former times one had to consult bulky books of tables. Today the values are essentially instantly generated on simple and inexpensive calculators. The technique for doing this will be explained in Chapter 4 in the section on Taylor's formula. This technique requires differential calculus, which is introduced in the next chapter.

On page 277, following the appendices and the solutions to the problems, there is a collection of review problems with answers, an index to the symbols, and an index to the text.

Before going on, here is a summary of Chapter 1 to help you review what you have learned. Take a look if you feel that this would be helpful.

As soon as you are ready, go to Chapter 2.

Summary of Chapter 1

1.2 Functions (frames 3–13)

A *set* is a collection of objects—not necessarily material objects—described in such a way that we have no doubt as to whether a particular object does or does not belong to the set. A set may be described by listing its elements or by a rule.

A *function* is a rule that assigns to each element in a set A one and only one element in a set B. The rule can be specified by a mathematical formula such as $y = x^2$, or by tables of associated numbers. If x is one of the elements of set A, then the element in set B that the function f associates with x is denoted by the symbol $f(x)$, which is usually read as "f of x."

The set A is called the *domain* of the function. The set of all possible values of $f(x)$ as x varies over the domain is called the *range* of the function. The range of f need not be all of B.

When a function is defined by a formula such as $f(x) = ax^3 + b$, then x is often called the *independent variable* and $f(x)$ is called the *dependent variable*. Often, however, a single letter is used to represent the single variable as in $y = f(x)$.

Here x is the independent variable and y is the dependent variable. In mathematics the symbol x frequently represents an independent variable, f often represents the function, and $y = f(x)$ usually denotes the dependent variable. However, any other symbols may be used for the function, the independent variable, and the dependent variable, for example, $x = H(r)$.

1.3 Graphs (frames 14–22)

A convenient way to represent a function is to plot a graph as described in frames **15–18**. The mutually perpendicular coordinate axes intersect at the origin. The axis that runs horizontally is called the horizontal axis, or x-axis. The axis that runs vertically is called the vertical axis, or y-axis. Sometimes the value of the x-coordinate of a point is called the *abscissa*, and the value of the y-coordinate is called the *ordinate*. In the designation of a typical point by the notation (a, b), we will always designate the x-coordinate first and the y-coordinate second.

The constant function assigns a single fixed number c to each value of the independent variable x. The absolute value function $|x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

1.4 Linear and Quadratic Functions (frames 23–39)

An equation of the form $y = mx + b$ where m and b are constants is called *linear* because its graph is a straight line. The slope of a linear function is defined by

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

From the definition it is easy to see (frame **29**) that the slope of the above linear equation is m.

An equation of the form $y = ax^2 + bx + c$, where a, b, and c, are constants (and $a \neq 0$), is called a *quadratic equation*. Its graph is called a *parabola*. The values of x at $y = 0$ satisfy $ax^2 + bx + c = 0$ and are called the *roots* of the equation. Not all quadratic equations have real roots. The equation $ax^2 + bx + c = 0$ has two roots given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1.5–1.6 Angles and Their Measurements; Trigonometry (frames 40–74)

Angles are measured in either *degrees* or *radians*. A circle is divided into 360 equal *degrees*. The number of *radians* in an angle is equal to the length of the subtending arc divided by the length of the radius (frame **42**). The relation between degrees and radians is

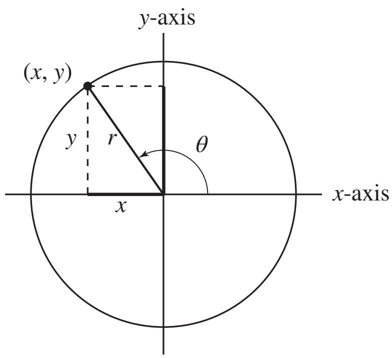
$$1 \text{ rad} = \frac{360^\circ}{2\pi}.$$

Rotations can be clockwise or counterclockwise. An angle formed by rotating in a counterclockwise direction is taken to be positive.

The trigonometric functions are defined in conjunction with the figure.

The definitions are

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r},$$
$$\tan\theta = \frac{y}{x}, \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y},$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}, \quad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}.$$



Although $r = \sqrt{x^2 + y^2}$ is always positive, x and y can be either positive or negative and the above quantities may be positive or negative depending on the value of θ. From the Pythagorean theorem it is easy to see (frame **56**) that

$$\sin^2\theta + \cos^2\theta = 1.$$

The sines and cosines for the sum of two angles are given by:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi,$$
$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

The inverse trigonometric function designates the angle for which the trigonometric function has the specified value. Thus the inverse trigonometric function to $x = \sin\theta$ is $\theta = \sin^{-1}x$ and similar definitions apply to $\cos^{-1}x$, $\tan^{-1}x$, etc. [Warning: This notation is standard, but it can be confusing: $\sin^{-1}x \neq (\sin x)^{-1}$. An older notation for $\sin^{-1}x$ is arc $\sin x$.]

1.7 Exponentials and Logarithms (frames 75–95)

If a is multiplied by itself as $aa \cdots$ with m factors, the product is written as a^m. Furthermore, by definition, $a^{-m} = 1/a^m$. From this it follows that

$$\begin{aligned} a^m a^n &= a^{m+n}, \\ \frac{a^m}{a^n} &= a^{m-n}, \\ a^0 &= \frac{a^m}{a^m} = 1, \\ (a^m)^n &= a^{mn}, \\ (ab)^m &= a^m b^m. \end{aligned}$$

If $b^n = a$, b is called the *nth root* of a and is written as $b = a^{1/n}$. If m and n are integers,

$$a^{m/n} = (a^{1/n})^m.$$

The meaning of exponents can be extended to irrational numbers (frame **84**) and the above relations also apply with irrational exponents, so $(a^x)^b = a^{xb}$, etc.

The definition of $\log x$ (the logarithm of x to the base 10) is

$$x = 10^{\log x}.$$

The following important relations can easily be seen to apply to logarithms (frame **91**):

$$\begin{aligned} \log(ab) &= \log a + \log b, \\ \log(a/b) &= \log a - \log b, \\ \log(a^n) &= n \log a. \end{aligned}$$

The logarithm of x to another base r is written as $\log_r x$ and is defined by

$$x = r^{\log_r x}.$$

The above three relations for logarithms of a and b are correct for logarithms to any base provided the same base is used for all the logarithms in each equation.

A particular important base is $r = e = 2.71828\ldots$ as defined in frame **109**. Logarithms to the base e are so important in calculus that they are given a different name; they are called *natural logarithms* and written as ln. With this notation the natural logarithm of x is defined by

$$e^{\ln x} = x.$$

If we take the logarithm to base 10 of both sides of the equation,

$$\log e^{\ln x} = \log x,$$
$$\ln x \log e = \log x,$$
$$\ln x = \frac{\log x}{\log e}.$$

Because the numerical value of $1/\log e = 2.303\ldots$,

$$\ln x = (2.303)\log x.$$

The special value of e and the importance that $\ln e = 1$ will be discussed in Chapter 2.

Continue to Chapter **2**.

CHAPTER TWO

Differential Calculus

In this chapter you will learn about

- The concept of the limit of a function;
- What is meant by the derivative of a function;
- Interpreting derivatives graphically;
- Shortcuts for finding derivatives;
- How to recognize derivatives of some common functions;
- Finding the maximum or minimum values of functions;
- Applying differential calculus to a variety of problems.

2.1 The Limit of a Function

97

Before diving into differential calculus, it is essential to understand the concept of the *limit* of a function. The idea of a limit may be new to you, but it is at the heart of calculus, and it is essential to understand the material in this section before going on. Once you understand the concept of limits, you should be able to grasp the ideas of differential calculus quite readily.

Limits are so important in calculus that we will discuss them from two different points of view. First, we will discuss limits from an intuitive point of view. Then, we will give a precise mathematical definition.

Go to **98**.

98

Here is a little bit of mathematical shorthand, which will be useful in this section. Suppose a variable x has values lying in an interval with the following properties:

1. The interval surrounds some number a.
2. The difference between x and a is less than another number B, where B is any number that you choose.
3. x does not take the particular value a. (We will see later why this point is excluded.)

The above three statements can be summarized by the following:

$|x - a| > 0$ (This statement means x cannot have the value a.)

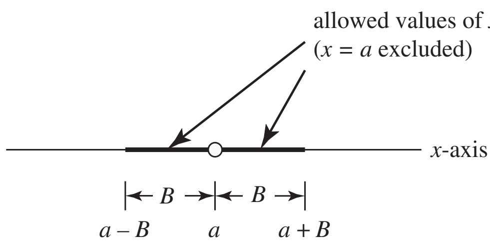
$|x - a| < B$ (The magnitude of the difference between x and a is less than the arbitrary number B.)

These relations can be combined in the single statement:

$$0 < |x - a| < B.$$

(If you need to review the symbols used here, see frame **20**.)

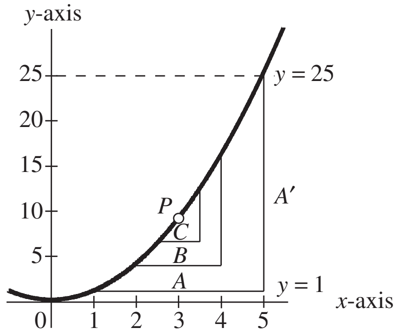
The values of x which satisfy $0 < | x - a | < B$ are indicated by the interval along the x-axis shown in the figure.



Go to **99**.

99

We begin our discussion of limits with an example. We are going to work with the equation $y = f(x) = x^2$, as shown in the graph. P is the point on the curve corresponding to $x = 3$, $y = 9$.



Let us concentrate on the behavior of y for values of x in an interval about $x = 3$. For reasons we shall see shortly, it is important to exclude the particular point of interest P, and to remind us of this, the point is encircled on the curve.

We start by considering values of y corresponding to values of x in an interval about $x = 3$, lying between $x = 1$ and $x = 5$. With the notation of the last frame, this can be written as $0 < |x - 3| < 2$. This interval for x is shown by line A in the figure. The corresponding interval for y is shown by line A' and includes points between $y = 1$ and $y = 25$, except $y = 9$.

A smaller interval for x is shown by line B. Here $0 < |x - 3| < 1$, and the corresponding interval for y is $4 < y < 16$, with $y = 9$ excluded.

The interval for x shown by the line C is given by $0 < |x - 3| < 0.5$. Write the corresponding interval for y in the blank below, assuming $y = 9$ is excluded.

____

In order to find the correct answer, go to **100**.

100

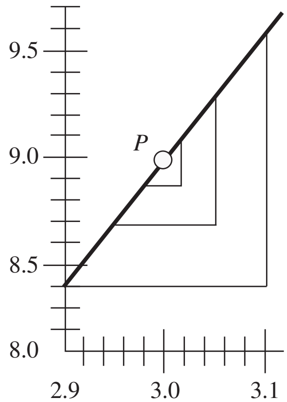
The interval for y which corresponds to $0 < |x - 3| < 0.5$ is

$$6.25 < y < 12.25,$$

(continued)

which you can check by substituting the values 2.5 and 3.5 for x in $y = x^2$ in order to find the values of y at either end point.

So far we have considered three successively smaller intervals of x about $x = 3$ and the corresponding intervals of y. Suppose we continue the process. The drawing shows the plot $y = x^2$ for values of x between 2.9 and 3.1. (This is an enlarged piece of the graph in frame **99**. Over the short distance shown the parabola looks practically straight.)



Three small intervals of x around $x = 3$ are shown along with the corresponding interval in y. The table below shows the values of y, corresponding to the boundaries of x at either end of the interval. (The last entry is for an interval too small to show on the drawing.)

Interval of x	**Corresponding interval of y**
1 – 5	1 – 25
2 – 4	4 – 16
2.5 – 3.5	6.25 – 12.25
2.9 – 3.1	8.41 – 9.61
2.95 – 3.05	8.70 – 9.30
2.99 – 3.01	8.94 – 9.06
2.999 – 3.001	8.994 – 9.006

Go to **101**.

101

We hope it is apparent from the discussion in the last two frames that as we diminish the interval for x around $x = 3$, the values for $y = x^2$ cluster more and more closely about $y = 9$. In fact, it appears that we can make the values for y cluster as closely as we please about $y = 9$ by merely limiting x to a sufficiently small interval about $x = 3$. Because this is true, we say that the *limit* of x^2, as x approaches 3, is 9. We write this as

$$\lim_{x \to 3} x^2 = 9.$$

Let's put this in more general terms. If a function $f(x)$ is defined for values of x about some fixed number a, and if, as x is confined to smaller and smaller intervals about a, the values of $f(x)$ cluster more and more closely about some specific number L, the number L is called the *limit* of $f(x)$ as x approaches a. The statement that "the limit of $f(x)$ as x approaches a is L" is customarily abbreviated by

$$\lim_{x \to a} f(x) = L.$$

In the example at the top of the page $f(x) = x^2$, $a = 3$, and $L = 9$.

The important idea in the definition is that the intervals we use lie on either side of the point of interest a, but that the point itself is not included. The value of the function $f(a)$ when $x = a$ may be different from $\lim_{x \to a} f(x)$, as we shall see.

To summarize the mathematical argument in more familiar language: in the spirit of "Anything you can do I can do better!" the challenge to an opponent is "Pick a point as close as you want to L as you please, and I can find a point close to a for which $f(x)$ will be closer to L than the point that you chose."

Go to **102**.

102

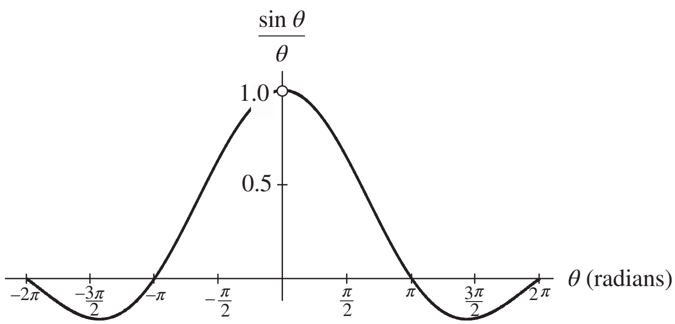
You may be wondering why we have been giving such a complicated discussion of an apparently simple problem. Why bother with $\lim_{x \to 3} x^2 = 9$ when it is obvious that $x^2 = 9$ for $x = 3$?

The reason is that the value of a function for a particular $x = a$ may not be defined, whereas the limit as x approaches a is perfectly well defined. For instance at $\theta = 0$ the function $\sin \theta/\theta$ has the value 0/0, which is not defined. Nevertheless

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

(continued)

You can see that this result is reasonable by graphing the function $\sin\theta/\theta$ as shown below. If you have a calculator, explore for yourself values of $\sin\theta/\theta$ as θ approaches zero. If you try to evaluate the function at $\theta = 0$, most calculators will indicate an error. This is as it should be because the function is not defined at $\theta = 0$. Nevertheless, its limit is well defined and has the value 1. (This is formally proved in Appendix **A12**.)



The actual procedure for finding a limit varies from problem to problem. For those interested in learning more, there are a number of theorems for finding the limits of simple functions in Appendix **A2**. As another illustration consider

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

For $x = 1, f(1) = \frac{1-1}{1-1} = \frac{0}{0}$, which is not defined. However, we can divide through by $x - 1$ provided x is not equal to 1, and we obtain

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1}. = x + 1.$$

Therefore, even though $f(1)$ is not defined,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x + 1) = 2.$$

Formal justification of these steps is given in Appendix **A2**, along with a number of rules for handling limits. (There is no need to read the appendix now unless you are really interested.)

We could also have obtained the above result graphically by studying the graph of the function in the neighborhood of $x = 1$ as we did in frame **99**.

Go to **103**.

103

To see whether you have caught on, find the limit of the following slightly more complicated functions by procedures similar to the ones described in frame **102**. (You will probably have to work these out on paper. Both of them involve a little algebraic manipulation.)

(a) $\lim_{x \to 0} \frac{(1+x)^2 - 1}{x} = \{1 \mid x \mid -1 \mid 2\}$

(b) $\lim_{x \to 0} \frac{1-(1+x)^3}{x} = \{1 \mid x \mid 3 \mid -3\}$

If right, go to **105**.
Otherwise, go to **104**.

104

Here are the solutions to the problems in frame **103**:

(a)
$$\lim_{x \to 0} \frac{(1+x)^2 - 1}{x} = \lim_{x \to 0} \frac{(1+2x+x^2)-1}{x}$$
$$= \lim_{x \to 0} \frac{2x+x^2}{x} = \lim_{x \to 0}(2+x) = \lim_{x \to 0} 2 + \lim_{x \to 0} x = 2+0=2.$$

(b)
$$\lim_{x \to 0} \frac{1-(1+x)^3}{x} = \lim_{x \to 0} \frac{1-(1+x)(1+x)(1+x)}{x}$$
$$= \lim_{x \to 0} \frac{1-(1+3x+3x^2+x^3)}{x} = \lim_{x \to 0}(-3-3x-x^2)$$
$$= \lim_{x \to 0}(-3) + \lim_{x \to 0}(-3x) + \lim_{x \to 0}(-x^2) = -3.$$

(These steps are formally justified in Appendix **A2**.)

Go to **105**.

105

So far we have discussed limits using expressions such as "confined to a smaller and smaller interval" and "clustering more and more closely." These expressions convey the intuitive meaning of a limit, but they are not precise mathematical statements. Now we are ready for a

(continued)

precise definition of a limit. (Because it is an almost universal custom, in the definition of a limit we will use the Greek letters δ (delta) and ε (epsilon).)

Definition of a Limit:

Let $f(x)$ be defined for all values of x in an interval centered about $x = a$ but not necessarily at $x = a$. If there is a number L such that to each positive number ε there corresponds a positive number δ such that

$$|f(x) - L| < \varepsilon \quad \text{provided} \quad 0 < |x - a| < \delta,$$

we say that L is the *limit* of $f(x)$ as x approaches a, and write

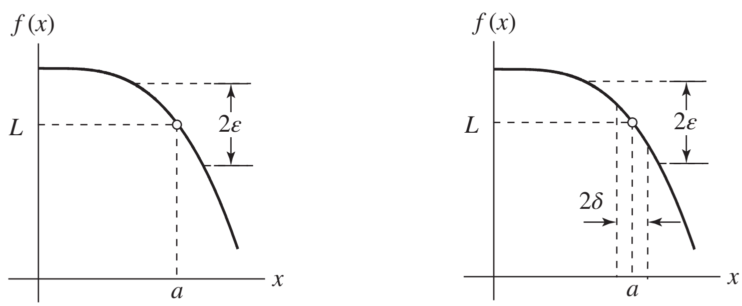
$$\lim_{x \to a} f(x) = L.$$

To see how to apply this definition,

Go to **106**.

106

Suppose we assert that $\lim_{x \to a} f(x) = L$, and an opponent disagrees. The formal definition of a limit in frame **105** provides a clear basis for settling the dispute as to whether the limit exists and is L. As a first step, we tell the opponent to pick any positive number ε no matter how small, say 0.001, or if the opponent wants to be difficult, 0.00001. Our task is to find some other number δ, such that for all x in the interval $0 < |x - a| < \delta$, the difference between $f(x)$ and L is smaller than ε. If we can always do this, we win the argument—the limit exists and is L. These steps are illustrated for a particular function in the drawings below.



Our opponent has challenged us to find a δ to fit this ε.

Here is one choice of δ. Obviously, for all values of x in the interval shown, $f(x)$ will satisfy $|f(x) - L| < \varepsilon$.

Answer: Frame 103: 2, −3

It may be that our opponent can find an ε such that we can never find a δ, no matter how small, that satisfies our requirement. In this case, she wins and $f(x)$ does not have the limit L. (In frame **114** we will come to an example of a function that does not have a limit.)

Go to **107**.

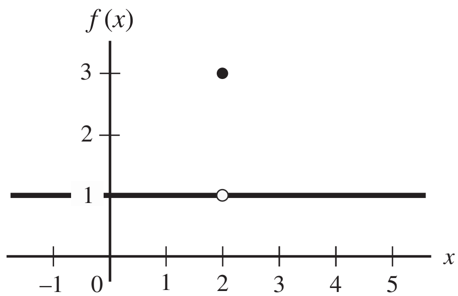
107

In the examples we have studied so far, the function has been expressed by a single equation. However, this is not necessarily the case. Here is an example:

$$f(x) = 1 \qquad \text{for } x \neq 2,$$
$$f(x) = 3 \qquad \text{for } x = 2.$$

(The symbol $\neq$ means "not equal.")

A sketch of this peculiar function is shown. You should be able to convince yourself that $\lim_{x\to 2} f(x) = 1$, whereas $f(2) = 3$.



If you would like further explanation of this, go to **108**; otherwise, go to **109**.

Go to **108**.

108

For every value of x except $x = 2$, the value of $f(x) = 1$. Consequently, $f(x) - 1 = 0$ for all x except $x = 2$. Because 0 is less than the smallest positive number ε that your opponent could select, it follows from the definition of a limit that $\lim_{x\to 2} f(x) = 1$, even though $f(2) = 3$.

Continue with **109**.

109

Calculator Problem

Here is another function which has a well-defined limit at a point but which can't be evaluated at that point: $f(x) = (1+x)^{1/x}$. Although the value of $f(x)$ at $x = 0$ is puzzling, it is possible to find $\lim_{x\to 0}(1+x)^{1/x}$.

Most calculators have the function y^x. If you have such a calculator, determine the values in Table 1:

Table 1

x	$f(x) = (1+x)^{1/x}$
1	
0.1	
0.01	
0.001	
0.0001	
0.00001	

The limit of $(1+x)^{1/x}$ as $x \to 0$ will play a key role in our study of logarithms. Its value is given a special symbol, e. Like π, e is an unending and unrepeating decimal; it is irrational. The value of e is 2.7182818 .... If you tried evaluating e with a calculator, the last entry in the table should give correct values for the first four digits after the decimal point. A proof that $\lim_{x\to 0}(1+x)^{1/x} = e$ is presented in Appendix **A4**.

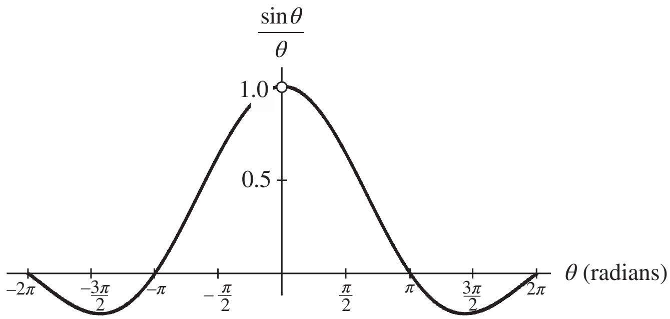
Go to **110**.

110

The actual procedure for finding a limit varies from problem to problem. For those interested in learning more, there are a number of theorems for finding the limits of simple functions in Appendix **A2**. The result mentioned earlier,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1,$$

is proved in Appendix **12**.



You can see that this result is reasonable by graphing the function $\sin\theta/\theta$ as shown above. If you have a calculator, explore for yourself values of $\sin\theta/\theta$ as θ approaches zero. The function is not defined at $\theta = 0$, but its limit is well defined and has the value 1.

Go to **111**.

111

Continuous Function

So far in most of our discussion of limits we have been careful to exclude the actual value of $f(x)$ at the point of interest, a. In fact, $f(a)$ does not even need to be defined for the limit to exist (as in the last frame). However, frequently $f(a)$ is defined. If this is so, and if in addition

$$\lim_{x\to a} f(x) = f(a),$$

then the function is said to be *continuous* at a. To summarize, fill in the blanks:

A function $f(x)$ is continuous at $x = a$ if

1. $f(a)$ is ________________________.

2. $\lim_{x\to a} f(x) =$ ________________.

Check your answers in frame **112**.

112

Here are the answers: A function $f(x)$ is continuous at $x = a$ if

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x) = f(a)$.

A more picturesque description of a continuous function is that it is a function you can graph without lifting your pencil from the paper in the region of interest.

Try to determine whether each of the following functions is continuous or discontinuous (not continuous) at the point indicated.

1. $f(x) = \dfrac{x^2 + 3}{9 - x^2}$.

   At $x = 3$, $f(x)$ is {continuous | discontinuous}

2. $f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$

   At $x = 1$, $f(x)$ is {continuous | discontinuous}

3. $f(x) = |x|$.

   At $x = 0$, $f(x)$ is {continuous | discontinuous}

4. $f(x) = \dfrac{\sin x}{x}$.

   At $x = 0$, $f(x)$ is {continuous | discontinuous}

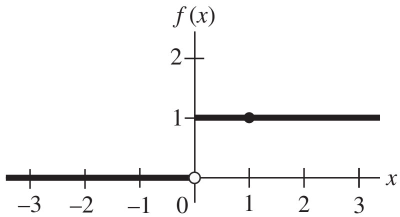
If you made any mistakes, or want more explanation, go to **113**.
Otherwise, skip on to **114**.

113

Here are the explanations of the problems in frame **112**.

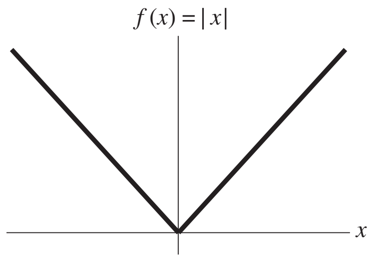
1. At $x = 3$, $f(x) = \frac{x^2+3}{9-x^2} = \frac{12}{0}$. This is an undefined expression and, therefore, the function is not continuous at $x = 3$.

2. Here is a plot of the function given.



This function satisfies both conditions for continuity at $x = 1$, and is thus continuous there. (It is, however, discontinuous at $x = 0$.)

3. Here is a plot of $f(x) = |x|$.



This function is continuous at $x = 0$ because it satisfies all the formal requirements.

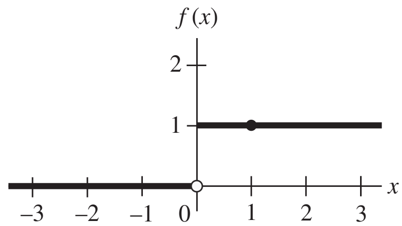
4. As discussed in frame **110**, $\sin x/x$ is not defined at $x = 0$, and so it is discontinuous at this point. (It is, however, continuous for all other values of x.)

Go to **114**.

114

Before leaving the subject of limits, it is worth looking at some examples of functions that somewhere have no limit. One such function is presented in problem 2 of the previous frame. The graph of the function is shown in the figure. We can prove that this function has no limit at $x = 0$ by following the procedure described in the definition of a limit.

(continued)



For purposes of illustration, suppose we guess that $\lim_{x \to 0} f(x) = 1$. Next, our opponent chooses a value for ε, say 1/4. Now, for $|x - 0| < \delta$, where δ is any positive number,

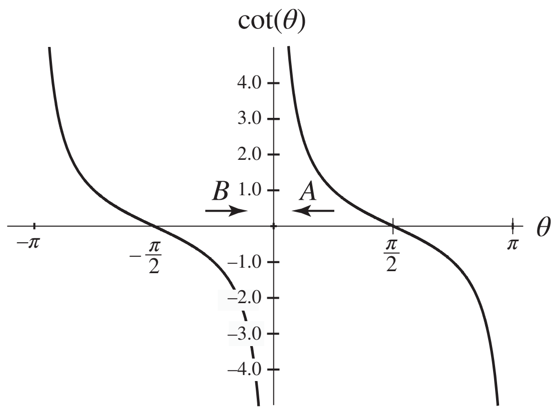
$$|f(x) - 1| = \begin{cases} |1 - 1| = 0 & \text{if } x > 0, \\ |0 - 1| = 1 & \text{if } x < 0. \end{cases}$$

Therefore, for all negative values of x in the interval, $|f(x) - 1| = 1$, which is greater than $\varepsilon = 1/4$. Thus 1 is not the limit. You should be able to convince yourself that there is *no* number L, which satisfies the criterion because $f(x)$ changes by 1 when x jumps from negative to positive values.

Go to **115**.

115

Here is another example of a function that has no limit at a particular point. From the graph it is obvious that $\cot \theta$ has no limit as $\theta \to 0$. Instead of clustering more and more closely to any number, L, the value of the function gets increasingly larger as $\theta \to 0$ in the direction shown by A, and increasingly more negative as $\theta \to 0$ in the direction shown by B.



Answer: Frame 112: (1) discontinuous, (2) continuous, (3) continuous, (4) discontinuous

This concludes our study of the limit of a function for the present. If you want some more practice with limits, see review problems 21–28 start on page 284. Now we are ready to go on to the next section, a discussion about velocity.

Go to **116**.

2.2 Velocity

116

Our discussion has become a little abstract, so before we go on to differential calculus, let's talk about something down to earth: motion. As a matter of fact, Leibniz and Newton invented calculus because they were concerned with problems of motion, so it is a good place to start. Besides, you already know quite a bit about motion.

Go to **117**.

117

In this chapter, we will only consider motion along a straight line. Here is a warm-up problem.

A train travels away at a velocity v mph (miles per hour). At $t = 0$, it is distance $S(0) = S_0$ from us. (The subscript on S_0 is to avoid confusion. S_0 is a particular distance and is a constant; $S(t)$ is a function that gives the distance the train is from us at time t.) Write the equation for $S(t)$ in terms of time t. (Take the unit of t to be hours.)

$$S(t) = \underline{\qquad\qquad}.$$

Go to **118** for the answer.

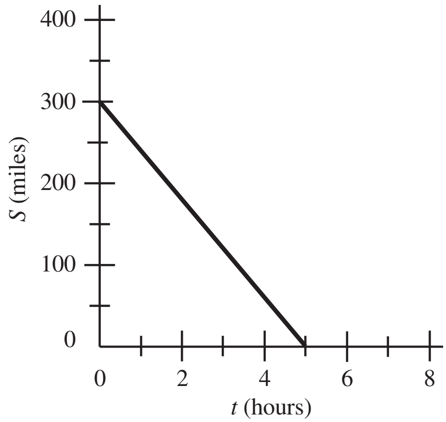
118

If you wrote $S(t) = S_0 + vt$, you are correct. Go on to frame **119**.

If your answer was not equivalent to the above, try to convince yourself that this answer is correct. Note that it yields S_0 when $t = 0$, as required. The equation is that of a straight line, and it might be worthwhile reviewing the section on linear functions, frames **23–39**, before continuing. Whenever you are satisfied with this result,

Go to **119**.

119



Here is a plot of the positions at different times of a train going in a straight line. Obviously, this represents a linear equation. Write the equation for the position of this train (in miles) in terms of time (in hours).

$$S(t) = \underline{\hspace{3cm}}.$$

Find the velocity of the train from your equation.

$$v = \underline{\hspace{3cm}}.$$

Go to **120** for the correct answers.

120

Here are the answers to the questions in frame **119**.

$$S(t) = -60t + 300 \text{ mi},$$

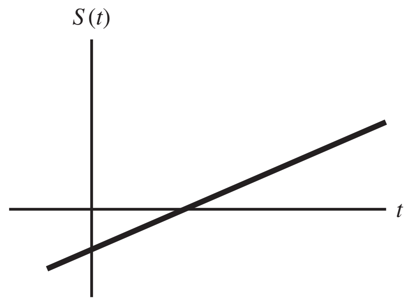
$$v = -60 \text{ mph}.$$

The velocity is negative because $S(t)$ decreases as time increases. (Note that the velocity along a straight line is positive or negative depending on the direction of motion. The *speed*, which is the magnitude of the velocity, is always positive.) If you would like further discussion, review frames **33 and 34**.

Go to **121**.

121

Here is another plot of position of a train traveling in a straight line.



The property of the line that represents the velocity of the train is the __________ of the line.

Go to **122** for the answer.

122

The property of the line that represents the velocity of the train is the *slope* of the line.

If you wrote this, go right on to **123**. If you wrote anything else, or nothing at all, then you may have forgotten what we reviewed back in frames **23–39**. It would be worthwhile going over that section once again (particularly frames **33** and **34**) and think about this problem before going on. At least convince yourself that the slope really represents the velocity.

Go to **123**.

123

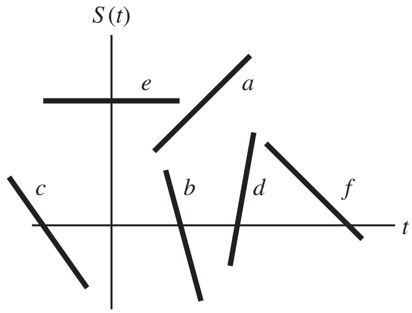
In the figure below are plots of the positions vs. time of six objects moving along straight lines. Which plot corresponds to the object that

Has the greatest velocity forward? $\{a \mid b \mid c \mid d \mid e \mid f\}$

Is moving backward most rapidly? $\{a \mid b \mid c \mid d \mid e \mid f\}$

Is at rest? $\{a \mid b \mid c \mid d \mid e \mid f\}$

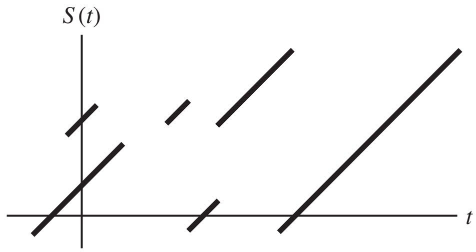
(continued)



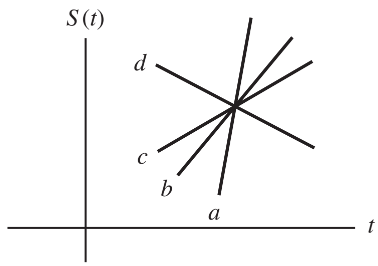
If all right, go to **125**.
If any wrong, go to **124**.

124

The velocity of the object is given by the slope of the plot of its position against time. Don't confuse the slope of a line with the line's location.



All the above lines have the same slope.



All these lines have different slopes.

A positive slope means that position is increasing with time, which corresponds to a positive velocity. Likewise, a negative slope means that position is decreasing in time, which means the velocity is negative. If you need to review the idea of slope, look at frames **25–27** before continuing.

Which line in the figure above on the right has

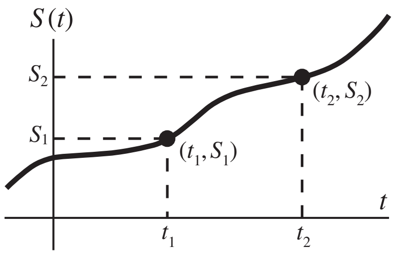
Negative slope? $\{a \mid b \mid c \mid d\}$

Greatest positive slope? $\{a \mid b \mid c \mid d\}$

Go to **125**.

125

So far, the velocities we have considered have all been constant in time. But what if the velocity changes?



Here is a plot of the position of a car whose velocity is varying while it moves along a straight line. In order to describe this, we introduce the *average velocity* $\bar{v}$ (read as "v bar"). This is the ratio of the change in position to the time taken. The change in position is called the *displacement*. For example, between the times t_1 and t_2 the displacement of the car is $S_2 - S_1$, so $(S_2 - S_1)/(t_2 - t_1)$ is its ____________________ during the time.

Go to **126**.

126

The answer to frame **125** is

$(S_2 - S_1)/(t_2 - t_1)$ is its average velocity during the time interval $t_2 - t_1$.

(The single word "velocity" is not a correct answer because the velocity was changing during this interval.)

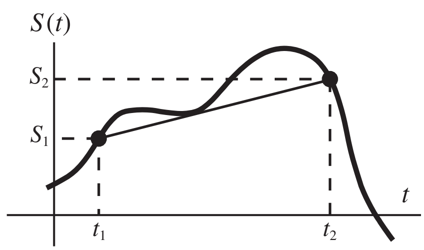
Go to **127**.

127

In addition to defining the average velocity $\bar{v}$ algebraically,

$$\bar{v} = \frac{S_2 - S_1}{t_2 - t_1},$$

we can interpret $\bar{v}$ graphically. If we draw a straight line between the points (t_1, S_1) and (t_2, S_2), then the average velocity is simply the slope of that line.

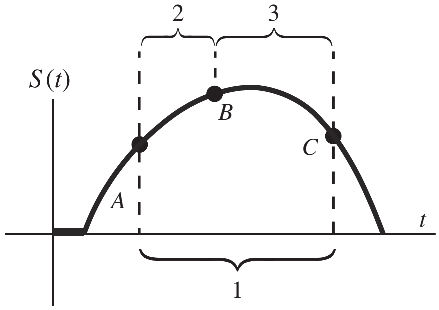


Go to **128**.

128

During which interval in the figure was the average velocity

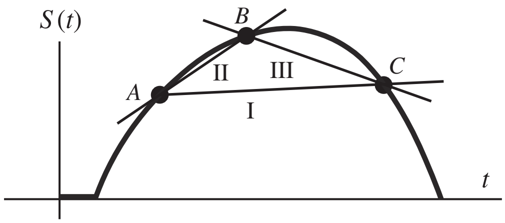
Closest to 0? {1 | 2 | 3}
Largest forward? {1 | 2 | 3}
Largest backward? {1 | 2 | 3}



If right, go to **130**.
If wrong, go to **129**.

129

Let us analyze the last problem in detail.



Here are straight lines drawn through the points *A*, *B*, *C*. Line I has a small positive slope and corresponds to almost 0 velocity. Line II has positive slope, and line III has negative slope, corresponding to positive and negative average velocities, respectively.

Go to **130**.

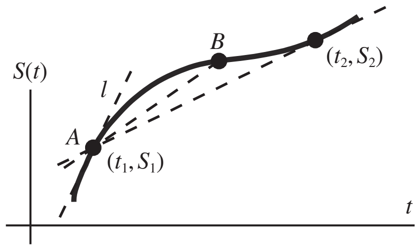
Answers: Frame 123: *d*, *b*, *e*
Frame 124: *d*, *a*

130

We now extend our idea of velocity in a very important manner: instead of asking, "What is the average velocity between time and t_1 and t_2?" let us ask, "What is the velocity at time t_1?" The velocity at a particular time is called the *instantaneous velocity*. This is a new term, and we will give it a precise definition shortly even though it may already be familiar to you.

Go to **131**.

131



We can give a graphical meaning to the idea of instantaneous velocity. The average velocity is the slope of a straight line joining two points on the curve, (t_1, S_1) and (t_2, S_2). To find the instantaneous velocity, we want t_2 to be very close to t_1. As we let point B on the curve approach point A (i.e. as we consider intervals of time starting at t_1 that become shorter and shorter), the slope of the line joining A and B approaches the slope of the line, which is labeled l. The instantaneous velocity is then the slope of line l. In a sense, then, the straight line l has the same slope as the curve at the point A. Line l is called a tangent to the curve at A.

Go to **132**.

132

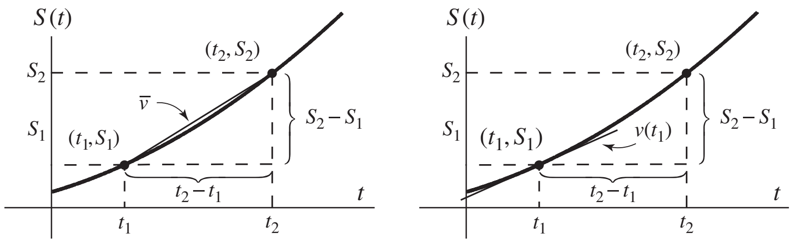
Here is where the idea of a limit becomes very important. If we draw a straight line through the given point A on the curve and some other point on the curve B and then let B get closer and closer to A, the slope of the straight line approaches a unique value and can be identified with the slope of the curve at A. What we must do is consider the limit of the slope of the line through A and B as $B \rightarrow A$.

Now, go to **133**.

133

We can now give a precise meaning to the intuitive idea of instantaneous velocity as the slope of a curve at a point. We start by considering the average velocity:

$$\bar{v} = (S_2 - S_1)/(t_2 - t_1) = \text{the slope of the line connecting points 1 and 2.}$$



As $t_2 \to t_1$, the average velocity approaches the instantaneous velocity, that is, $\bar{v} \to v(t_1)$ as $t_2 \to t_1$, or

$$\boxed{v(t_1) = \lim_{t_2 \to t_1} \frac{S_2 - S_1}{t_2 - t_1}.}$$

Go to **134**.

134

Because the ideas presented in the last few frames are important, let's review them.

If a point moves from S_1 to S_2 during the time t_1 to t_2, then

$$(S_2 - S_1)/(t_2 - t_1)$$

is the ____________________, $\bar{v}$.

If we consider the limit of the average velocity as the averaging time goes to zero, the result is called the ______________ ______________, v.

Now let's try to present these ideas in a neater form. If you can, write a formal definition of v in the blank space.

$$v = \underline{\hspace{3cm}}$$

Go to frame **135** for the answers.

Answer: Frame 128: 1, 2, 3

135

The correct answers to frame **134** are the following:

If a point moves from S_1 to S_2 during the time t_1 to t_2, then $(S_2 - S_1)/(t_2 - t_1)$ is the *average velocity*, $\bar{v}$.

If we consider the limit of the average velocity as the averaging time goes to zero, the result is called the *instantaneous velocity*, v.

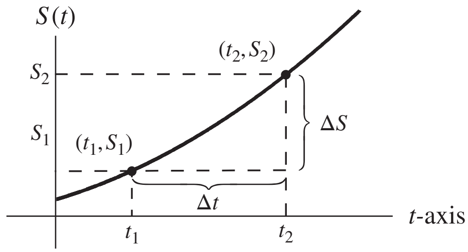
$$v = \lim_{t_2 \to t_1} \frac{S_2 - S_1}{t_2 - t_1}.$$

If you wrote this, congratulations! Go on to **136**. If you wrote something different, go back to frame **133** and work your way to this frame once more.

Then go on to **136**.

136

The Greek capital letter Δ ("delta") is often used to indicate the change in a variable. Thus, to make the notation more succinct, we can write $\Delta S = S_2 - S_1$, and $\Delta t = t_2 - t_1$. (ΔS is a single symbol read as "delta S"; it does not mean $\Delta \times S$.) Although this notation may be new, it saves lots of writing and is worth the effort to get used to.



With this notation, our definition of instantaneous velocity is

$$v = \underline{\hspace{6cm}}.$$

Go to **137** to find the correct answer.

137

If you wrote

$$v = \lim_{\Delta t \to 0} \frac{S_2 - S_1}{t_2 - t_1} \qquad \text{or} \qquad v = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t},$$

go ahead to frame **138**.

If you missed this, review frames **134–136** before going to **138**.

138

Now we are going to calculate an instantaneous velocity by analyzing an example step by step. Later on we will find shortcuts for doing this.

Suppose that we are given the following expression relating position and time:

$$S(t) = kt^2 \qquad (k \text{ is a constant}).$$

The goal is to find $\Delta S = S(t + \Delta t) - S(t)$, for any Δt, and then to evaluate the limit $\Delta S / \Delta t$ as $\Delta t \to 0$.

Here are the steps

$$\begin{aligned} \Delta S = S(t + \Delta t) - S(t) &= k(t + \Delta t)^2 - kt^2 \\ &= k[t^2 + 2t \;\; \Delta t + (\Delta t)^2] - kt^2 \\ &= k[2t \;\; \Delta t + (\Delta t)^2], \end{aligned}$$

$$\frac{\Delta S}{\Delta t} = \frac{k[2t \;\; \Delta t + (\Delta t)^2]}{\Delta t} = 2kt + k \;\; \Delta t,$$

$$v = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \lim_{\Delta t \to 0} (2kt + k \;\; \Delta t) = 2kt.$$

A simpler problem for you to try is in the next frame.

Go to **139**.

139

Suppose we are given that $S(t) = v_0 t + S_0$. The problem is to find the instantaneous velocity from our definition.

In time Δt the point moves distance ΔS.

$$\Delta S = \underline{\qquad\qquad\qquad}.$$

$$v = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \underline{\qquad\qquad\qquad}.$$

Write in the answers and go to **140**.

140

If you wrote $\Delta S = v_0\ \Delta t$ and $v = \lim_{\Delta t\to 0} \frac{\Delta S}{\Delta t} = \lim_{\Delta t\to 0} \frac{v_0 \Delta t}{\Delta t} = v_0$,

you are correct and can skip on to frame **142**.

If you wrote something different, study the detailed explanation in frame **141**.

141

Here is the correct procedure. Because $S(t) = v_0 t + S_0$,

$$\begin{aligned}\Delta S &= S(t + \Delta t) - S(t)\\ &= v_0(t + \Delta t) + S_0 - (v_0 t + S_0)\\ &= v_0\ \Delta t,\end{aligned}$$

$$\lim_{\Delta t\to 0} \frac{\Delta S}{\Delta t} = \lim_{\Delta t\to 0} \frac{v_0\ \Delta t}{\Delta t} = \lim_{\Delta t\to 0} v_0 = v_0.$$

The instantaneous velocity and the average velocity are the same in this case because the velocity is a constant, v_0.

Go to frame **142**.

142

Here is a problem for you to work out. Suppose the position of an object is given by

$$S(t) = kt^2 + bt + S_0,$$

where k, b, and S_0 are constants. Find $v(t)$.

$$v(t) = \lim_{\Delta t\to 0} \frac{\Delta S}{\Delta t} = \_\_\_\_\_\_\_\_\_\_.$$

To check your answer, go to **143**.

143

The answer is $v(t) = 2kt + b$. If you obtained this result and are ready to move on to the next section, go to frame **146**.

Otherwise, go to **144**.

144

Here is the solution to the problem in frame **142**.

$$\begin{aligned}
S(t) &= kt^2 + bt + S_0,\\
S(t+\Delta t) &= k(t+\Delta t)^2 + b(t+\Delta t) + S_0\\
&= k[t^2 + 2t\ \ \Delta t + (\Delta t)^2] + b(t+\Delta t) + S_0,\\
\Delta S &= S(t+\Delta t) - S(t) = k[2t\ \ \Delta t + (\Delta t)^2] + b\ \ \Delta t,\\
v(t) &= \lim_{\Delta t\to 0}\frac{\Delta S}{\Delta t} = \lim_{\Delta t\to 0}\left\{\frac{k[2t\ \ \Delta t + (\Delta t)^2] + b\ \ \Delta t}{\Delta t}\right\}\\
&= \lim_{\Delta t\to 0}[k(2t+\Delta t) + b] = 2kt + b.
\end{aligned}$$

Now try this problem:

If $S(t) = At^3$, where A is a constant, find $v(t)$.

Answer: ______________

To check your solution, go to **145**.

145

Here is the answer: $v(t) = 3At^2$. Go right on to frame **146** unless you would like to see the solution, in which case continue here.

$$\begin{aligned}
S(t) &= At^3,\\
\Delta S &= S(t+\Delta t)^3 - St^3\\
&= A[t^3 + 3t^2\ \ \Delta t + 3t(\Delta t)^2 + (\Delta t)^3] - At^3\\
&= 3At^2\ \ \Delta t + 3At(\Delta t)^2 + A(\Delta t)^3,\\
v(t) &= \lim_{\Delta t\to 0}\frac{\Delta S}{\Delta t} = \lim_{\Delta t\to 0}[3At^2 + 3At\ \ \Delta t + A(\Delta t)^2] = 3At^2.
\end{aligned}$$

Go to frame **146**.

2.3 Derivatives

146

In this section we will generalize our results on velocity. This will lead us to the idea of the *derivative* of a function, which is at the heart of differential calculus.

Go to **147**.

147

To launch the discussion, let's start with a couple of review questions.

When we write $S(t)$ we are stating that position depends on time.

Here position is the dependent variable and time is the __________ variable.

The velocity is the rate of change of position with respect to time. By this we mean that velocity is (give the formal definition again):

$$v(t) = \_\_\_\_\_\_\_\_\_\_\_\_$$

Go to frame **148** for the correct answers.

148

In the last frame you should have written ... time is the *independent* variable, and $v(t) = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}$.

Go on to **149**.

149

Let us consider any continuous function defined by $y = f(x)$. Here y is our dependent variable, and x is our independent variable. If we ask "At what rate does y change as x changes?," we can find the answer by taking the following limit:

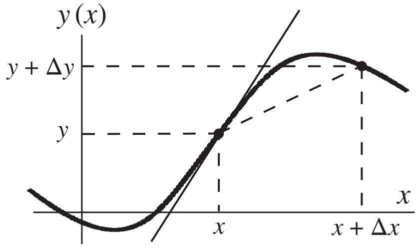
$$\text{Rate of change of } y \text{ with respect to } x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Go on to **150**.

150

You can give a geometrical meaning to $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, where $y = f(x)$. To do so, fill in the blanks:

Geometrically, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ can be found by drawing a straight line through the point (x, y) and the point (_____, _____) as shown. The slope of that line is given by $\frac{\Delta y}{\Delta x}$, and $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is the ____________________ of the tangent line to the curve at (x, y).



Go on to **151**.

151

The correct insertions for frame **150** are

$$(x + \Delta x, y + \Delta y) \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

For brevity, the slope of the tangent to a curve is usually called the *slope of the curve*. (If you would like to see a discussion of this, review frame **131** before continuing.)

Go on to **152**.

152

Another way of writing $\frac{\Delta y}{\Delta x}$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y(x_2) - y(x_1)}{x_2 - x_1}.$$

If the notation used here still seems unfamiliar, review frame **136** before proceeding.

Go on to **153**.

153

Let's review just once more. Fill in the blank below.

If we want to know how y changes as x changes, we find out by calculating the following limit:

____

Go on to **154**.

154

The correct answer to frame **153** is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{or} \quad \lim_{x_2 \to x_1} \frac{y_2 - y_1}{x_2 - x_1}.$$

If you were correct, go on to **155**.
If you missed this, go back to **149**.

155

Because the quantity $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is so useful, we give it a special name and a special symbol: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is called the *derivative* of y with respect to x, and it is often written with the symbol $\frac{dy}{dx}$,

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},}$$

where $\Delta y = y(x + \Delta x) - y(x)$.

Once again: $\frac{dy}{dx}$ is the __________ of ______ with respect to _____.

Go to **156** for the correct answer.

156

The answer is

$$\frac{dy}{dx} \text{ is the } \textit{derivative} \text{ of } y \text{ with respect to } x.$$

This symbol is read as "dee y by dee x." The derivative is frequently written in another form:

$$\frac{dy}{dx} = y'.$$

(continued)

(The symbol y' is read as "y prime.") y' and $\frac{dy}{dx}$ mean the same thing:

$$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

(Another symbol sometimes used for the derivative operator is D. Thus $Dy = y'$. However, we will not use the "D" symbol.)

Having two separate symbols for the derivative may look confusing at first, but they should both quickly become familiar. Each has its advantages. The symbol $\frac{dy}{dx}$ leaves no doubt that the independent variable is x, whereas y' might be ambiguous: because y could be a function of some other variable, z. (To avoid confusion, the "prime" form is sometimes written as $y'(x)$.) On the other hand, the symbol $\frac{dy}{dx}$ can be cumbersome to write. More seriously, in the form $\frac{dy}{dx}$ the derivative looks like the simple ratio of two quantities, dy and dx, which it is not.

We can apply the idea of a derivative to the notion of velocity, which we discussed earlier. Instantaneous velocity is the rate of change of position with respect to time, in other words, instantaneous velocity is the derivative of position with respect to time. Unless otherwise specified, velocity will be used as the common meaning of instantaneous velocity.

Go to **157**.

157

Let's state the definition of a derivative using different variables. Suppose z is some independent variable, and q depends on z. Then the derivative of q with respect to z is defined by

$$\frac{dq}{dz} = \underline{\hspace{3cm}}.$$

For the right answer, go to **158**.

158

The correct answer is

$$\frac{dq}{dz} = \lim_{\Delta z \to 0} \frac{\Delta q}{\Delta z}.$$

If correct, go to **159**.
If not, review to frame **155** and try again.

159

We have established the definition of df/dx, but there is more to explore. The symbol df/dx can be thought of as a *derivative operator* $\frac{d}{dx}$, operating on the function f.

If $f(x) = x^3 + 3$, then the derivative can be written in any of the following forms:

$$\frac{df}{dx} = \frac{d(x^3 + 3)}{dx} = \frac{d}{dx}(x^3 + 3).$$

Similarly, if $f(\theta) = \theta^2 \sin\theta$, then

$$\frac{d(\theta^2 \sin\theta)}{d\theta} = \frac{d}{d\theta}(\theta^2 \sin\theta).$$

(Here, θ is merely another variable.)

Thus $\frac{d}{dx}(\quad)$ means differentiate with respect to x whatever function $f(x)$ happens to be in the parentheses. For functions such as $\sin\theta$ that are not products, the derivative will be written as $\frac{d}{dx}\sin\theta$ with no parentheses. The symbol $\frac{df}{dx}$ means that one should obtain an expression for

$$\Delta f = f(x + \Delta x) - f(x),$$

and then use it to evaluate

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$

However, as we shall see, there are lots of shortcuts for calculating derivatives.

Go to **160**.

2.4 Graphs of Functions and Their Derivatives

160

We have just learned the formal definition of a derivative. Graphically, the derivative of a function $f(x)$ at some value of x is equivalent to the slope of the straight line that is tangent to the graph of the function at that point. Our chief concern in the rest of this chapter will

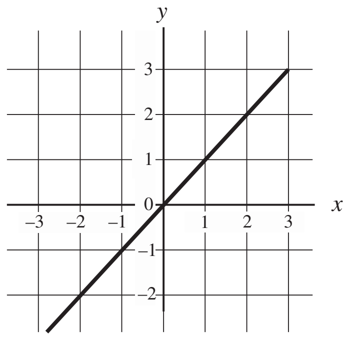
(continued)

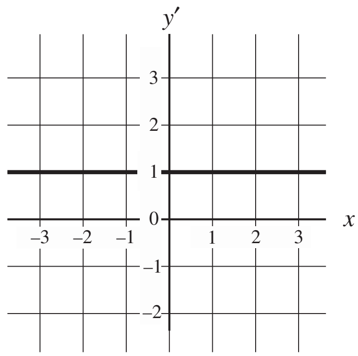
be to find methods for evaluating derivatives of different functions. In doing this it is helpful to have some intuitive idea of how the derivative behaves, and we can obtain this by looking at the graph of the function. If the graph has a steep positive slope, the derivative is large and positive. If the graph has a slight slope downward, the derivative is small and negative. In this section we will get some practice putting to use such qualitative ideas as these, and in the following sections we will learn how to obtain derivatives precisely.

Go to **161**.

161

Here is a plot of the simple function $y = x$. We have plotted $y' = \frac{dy}{dx}$. Because the slope of y is positive and constant, y' is a positive constant.





The graph indicates that $\frac{d}{dx}x = 1$. Can you prove this?

Go to **162**.

162

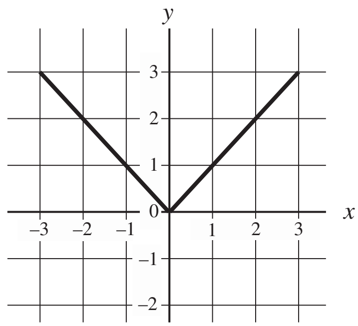
To prove that $\frac{d}{dx}x = 1$, let $y(x) = x$. Then

$$\Delta y = y(x + \Delta x) - y(x) = x + \Delta x - x = \Delta x.$$

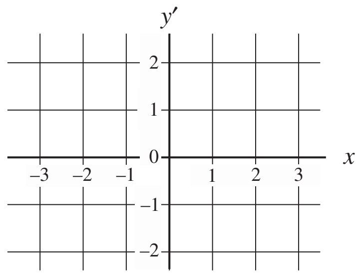
Hence,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

Here is a plot of $y = |x|$. (If you have forgotten the definition of $|x|$, see frame **20**.)



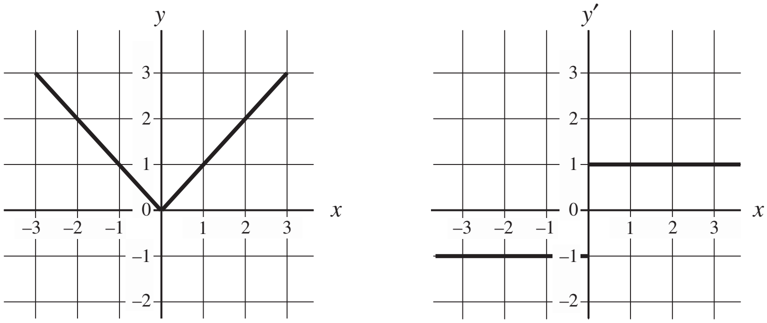
On the coordinates below, sketch y'.



For the correct answer, go to **163**.

163

Here are sketches of $y = |x|$ and y'. If you drew this correctly, go on to **164**. If you made a mistake or want further explanation, continue here.

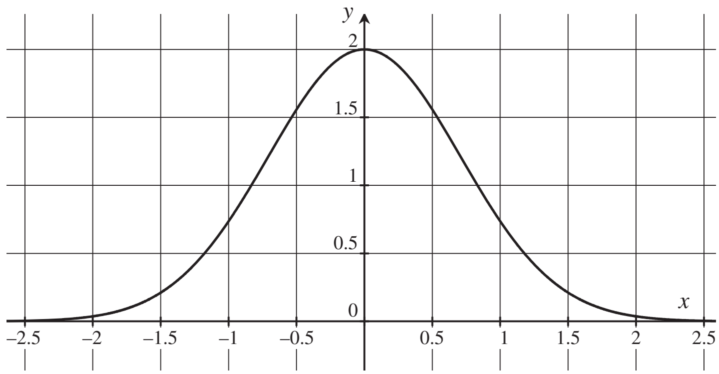


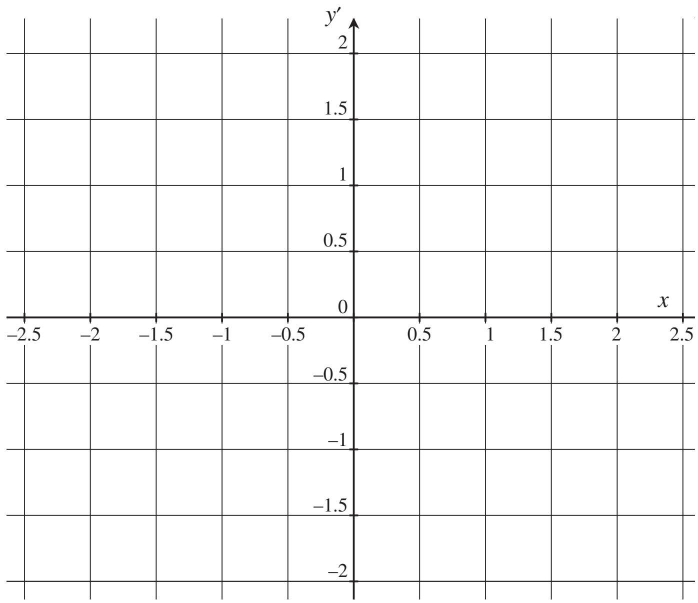
As you can see from the graph, $y = |x| = x$ for $x > 0$. So for $x > 0$ the problem is identical to that in frame **161**, and $y' = 1$. However, for $x < 0$, the slope of $|x|$ is negative and is easily seen to be -1. At $x = 0$, the slope is undefined, for it has the value $+1$ if we approach 0 along the positive x-axis and has the value -1 if we approach 0 along the negative x-axis. Therefore, $\frac{d}{dx}|x|$ is discontinuous at $x = 0$. (The function $|x|$ is continuous at this point, but the break in its slope at $x = 0$ causes a discontinuity in the derivative.)

Go to **164**.

164

Here is the graph of a function $y = f(x)$. Sketch its derivative in the space provided below. (The sketch does not need to be exact—just show the general features of y'.)

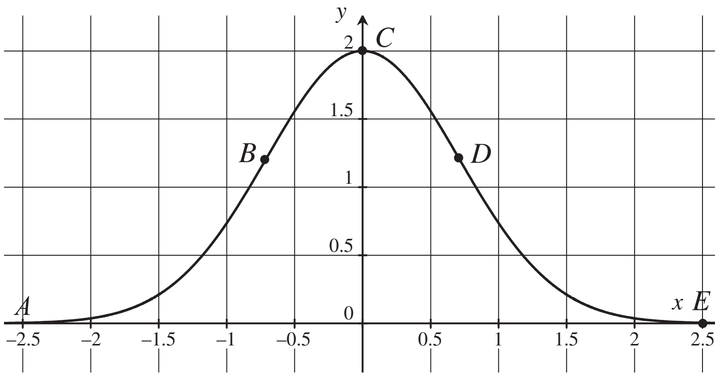


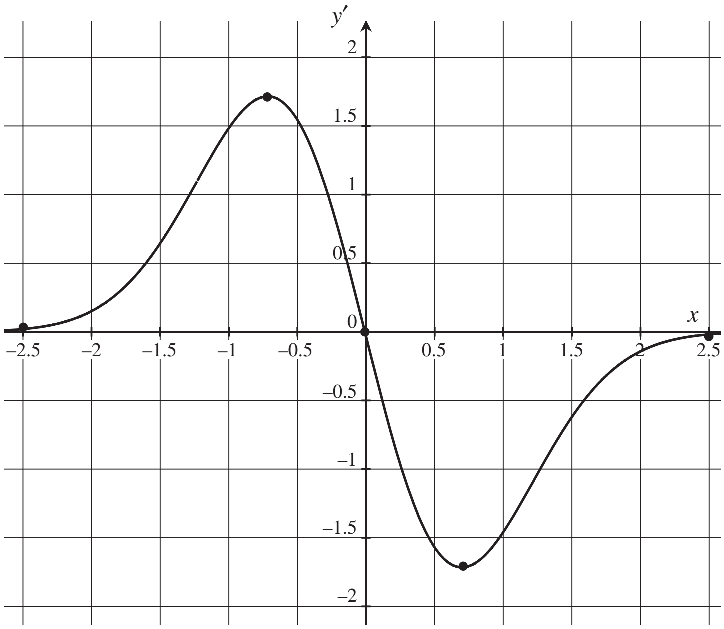


See **165** for the correct answer.

165

Here is the function and its derivative. If your sketch of y' is similar to that shown, go to **166**. Otherwise, read on.



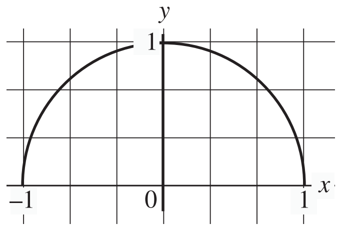


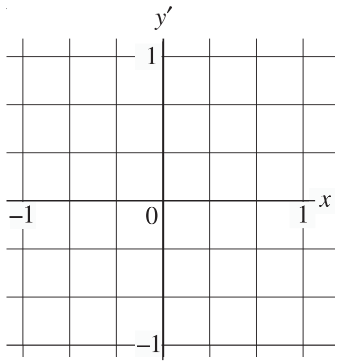
To see that the plot of y' is reasonable note that for $x < 0$, y increases with x so that y' is positive. The slope of y is greatest near point B, but it must abruptly decrease beyond B because it vanishes at $C(x = 0)$. At D, y is decreasing rapidly, so y' is negative. At the points, A and E, the slope of y is small and y' is close to zero.

Go to **166**.

166

Let's look at the behavior of y' graphically for one more function. Here the plot of y and x is a semicircle. In the space below, make a rough sketch of y' for the interval illustrated.

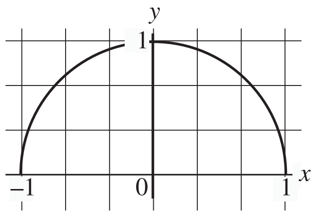


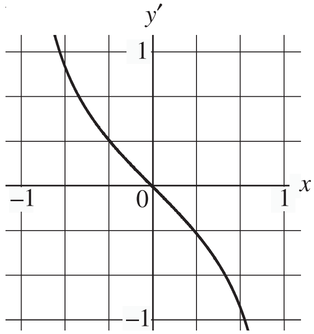


Go to **167** for the correct answer.

167

Here are the plots of y and y'. Read on if you would like further discussion of this. Otherwise, go to **168**.





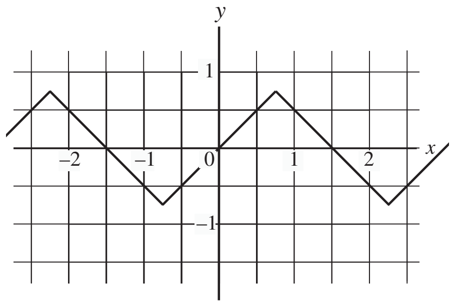
The slope of the semicircle does not behave nicely at the extreme values of x, so let's start by looking at $x = 0$. If we draw a line tangent to the curve at $x = 0$, it will be parallel to the x-axis, so the curve has 0 slope. Thus, $y' = 0$ at $x = 0$. For $x > 0$, a line tangent to the curve has negative slope, so $y' < 0$. As x approaches 1 the tangent becomes increasingly steep, and y' becomes increasingly negative. In fact, as $x \to 1$, $y' \to -\infty$.

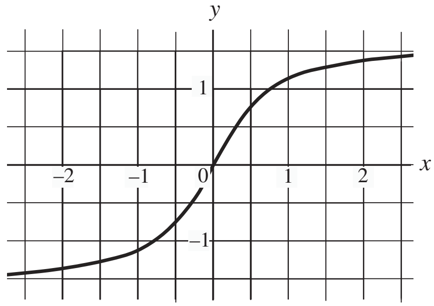
From this discussion it should be easy to find y' for $x < 0$.

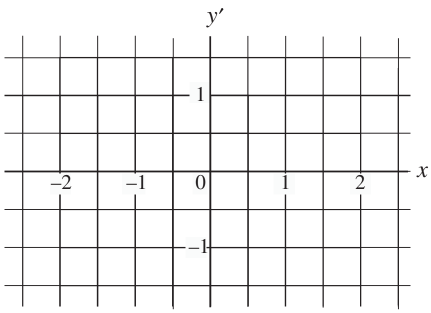
Go to **168**.

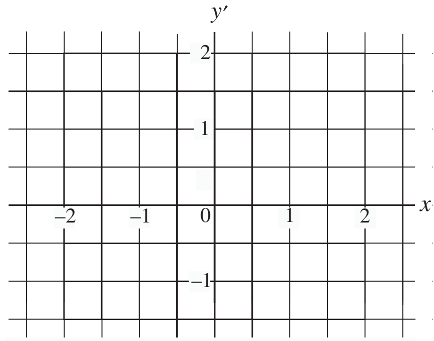
168

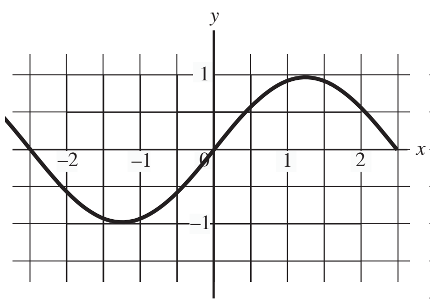
If you understand all the examples in this section, skip on to the next section. However, if you would like a little more practice, try sketching the derivatives for each function shown. The correct sketches are given in frame **169** without any discussion.

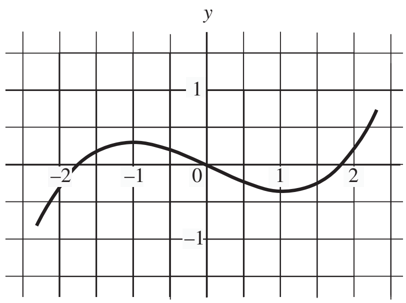


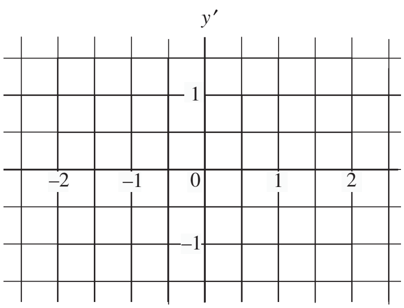


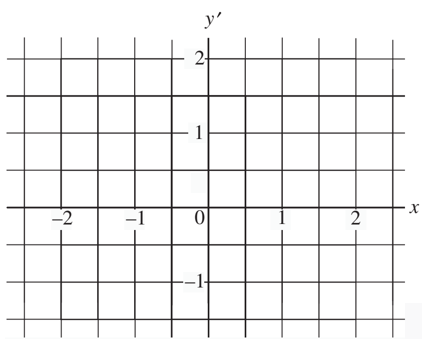








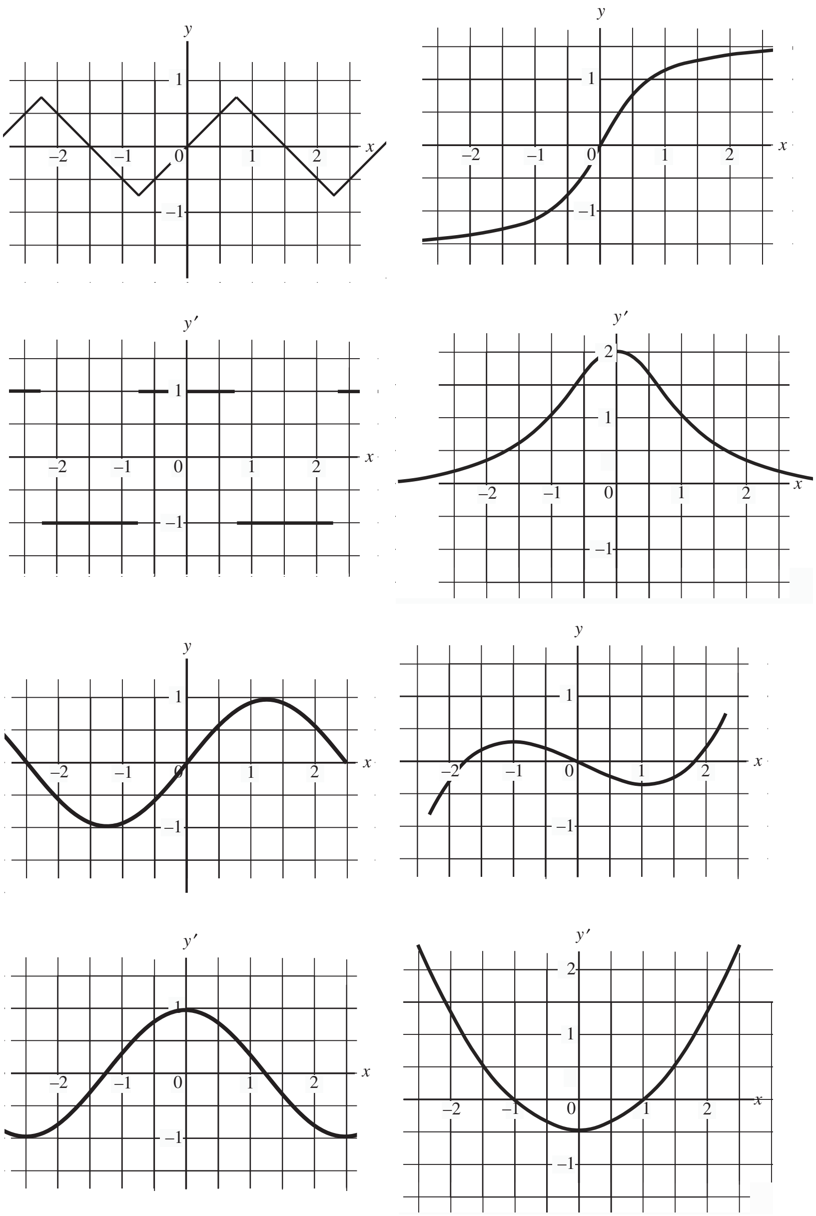




For the correct sketches, go to **169**.

169

Here are the solutions to the problems in frame **168**.



Convince yourself that the curves for y' have the general features we expect by comparing y' with the slope of a tangent to the graph of $y = f(x)$ at a few particular values of x.

2.5 Differentiation

170

We have accomplished a great deal in this chapter. In fact, all the really important new ideas involved in differential calculus have been introduced—limits, slopes of curves, and derivatives—and you are equipped in principle to apply these to solve a great variety of problems. However, using the fundamental definition to calculate the derivative in each problem as it comes along would be time-consuming. It would also be a great waste of time because there are numerous rules and tricks for differentiating apparently complicated functions in a few short steps.

You will learn the most important of these rules in the following sections. You will also learn how to differentiate a few functions that occur so often that it is useful to know and remember their derivatives. These include a few of the trigonometric functions, logarithms, and exponentials. The remaining sections cover some special topics, as well as applications of differential calculus to some problems. By the end of this chapter you should be able to use differential calculus for many applications. Well, let's get going!

Go to **171**.

171

Can you find the derivative of the following simple function?

$$y = a \quad (a \text{ is a constant}).$$

$$y' = \{1 \,|\, x \,|\, a \,|\, 0 \,|\text{none of these}\}$$

If right, go to **173**.
If wrong, go to **172**.

172

To find y', we go back to the definition $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. If $y = a$,

$$\frac{\Delta y}{\Delta x} = \frac{y(x + \Delta x) - y(x)}{\Delta x} = \frac{a - a}{\Delta x} = 0.$$

(Remember that the meaning of $y(x + \Delta x)$ is y evaluated at $x + \Delta x$.)

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0.$$

Go to **173**.

173

Because $y' = 0$, the plot of y in terms of x has 0 slope. (Figure 4 in frame **32** shows this graphically.)

You have just seen that the derivative of a constant is 0. Now, try to find the derivative of this function:

$$y = ax \quad (a = \text{constant}).$$

$$\frac{dy}{dx} = \{1 \mid x \mid a \mid 0 \mid ax \mid \text{none of these}\}$$

If right, skip to **175**.
If wrong, go to **174**.

174

Here is the formal procedure:

$$y(x) = ax,$$

$$y(x + \Delta x) - y(x) = a(x + \Delta x) - ax = (ax + a\Delta x) - ax = a\Delta x.$$

Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{a\,\Delta x}{\Delta x} = a.$$

Answer: Frame 171: 0

Now try to find the derivative of the function $f = -x$.

$$f' = \{1 \mid 0 \mid a \mid -1 \mid -x\}$$

If correct, go to **175**. If wrong, note that this problem is just a special case of **173**. Try again and then

Go to **175**.

175

Now we are going to find the derivative of a quadratic function. Suppose

$$y = f(x) = x^2.$$

What is y' ?

You should be able to work this out from the definition of the derivative. Choose the correct answer:

$$y' = \{1 \mid x \mid 0 \mid x^2 \mid 2x\}$$

If right, skip to **175**.
If wrong, go to **174**.

176

Let us again apply the definition of the derivative:

$$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}.$$

In this case, $y(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\ \Delta x + (\Delta x)^2$, so

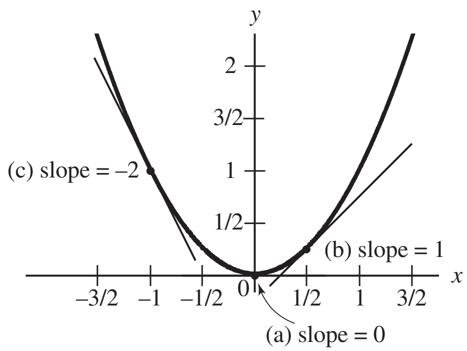
$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{[x^2 + 2x\ \ \Delta x + (\Delta x)^2] - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\ \ \Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x,
\end{aligned}$$

$$y' = \frac{dy}{dx} = 2x.$$

Go to **177**.

177

We have found the result that $\frac{d}{dx}x^2 = 2x$. To illustrate this, a graph of $y(x) = x^2$ is drawn in the figure. Because the slope of the curve at a point is simply the derivative at that point, each of the straight lines tangent to the curve has a slope equal to the derivative evaluated at the point of tangency.



The tangent through the origin has a slope of $(2)(0) = 0$. Line (b) passes through the point $x = ½$, and has slope $(2)(½) = 1$. Line (c) passes through the point $x = -1$, and has slope $(2)(-1) = -2$.

Go to **178**.

178

Here is a problem that summarizes the results we have had so far in this section (with a tiny bit of new material).

If $f(x) = 3x^2 + 7x + 2$, find f'.

$f' =$ ____________.

See frame **179** for the correct answer.

Answers: Frame 173: a

Frame 174: -1

Frame 175: $2x$

179

If $f(x) = 3x^2 + 7x + 2$, then $f' = 6x + 7$.

Congratulations if you obtained this answer. Go on to **180**. Otherwise, read below.

After you have finished this chapter, you will know several shortcuts for evaluating this derivative. However, right now we will use the basic definition:

$$f' = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x},$$

$$f(x) = 3x^2 + 7x + 2,$$

$$f(x + \Delta x) = 3[x^2 + 2x\Delta x + (\Delta x)^2] + 7(x + \Delta x) + 2,$$

$$\Delta f = f(x + \Delta x) - f(x) = 6x\ \ \Delta x + 3(\Delta x)^2 + 7\Delta x,$$

so

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \left(\frac{6x\Delta x + 3(\Delta x)^2 + 7\Delta x}{\Delta x} \right) = \lim_{\Delta x \to 0} (6x + 3\Delta x + 7)$$
$$= 6x + 7.$$

Go to **180**.

180

Now that we have found the derivatives $\frac{d}{dx}x = 1$ and $\frac{d}{dx}x^2 = 2x$, our next step is to find the derivative of x^n, where n is any number. We will state the rule here; you can find the proof in Appendix **A4**.

The result is

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}.}$$

This important result holds for all values of n: positive, negative, integral, fractional, irrational, etc. Note that our previous result, $\frac{d}{dx}x^2 = 2x$, is the particular case of this when $n = 2$, and $\frac{d}{dx}x = 1$ is the particular case when $n = 1$.

Go to **181**.

181

Now for a few problems.
Find $\frac{dy}{dx}$ for each of the following functions.

$$y = x^3, \qquad \frac{dy}{dx} = [3x^3 \mid 3x^2 \mid 2x^3 \mid x^2]$$

$$y = x^{-7}, \qquad \frac{dy}{dx} = [-7x^{-6} \mid 7x^{-7} \mid -7x^{-8} \mid -6x^{-7}]$$

$$y = \frac{1}{x^2}, \qquad \frac{dy}{dx} = \left[-2x \,\middle|\, \frac{2}{x} \,\middle|\, -\frac{2}{x^3}\right]$$

If all these were correct, go to **183**.
If you made any errors, go to **182**.

182

The solutions to these problems depend directly on the rule in frame **180**. Here are the details.

We use our general rule: $\frac{d}{dx}x^n = nx^{n-1}$.

$$y(x) = x^3; \quad \text{in this case } n = 3, \quad \text{so } \frac{d}{dx}x^3 = 3x^{3-1} = 3x^2.$$

$$y(x) = x^{-7}; \quad \text{here } n = -7, \quad \text{so } \frac{d}{dx}x^{-7} = -7x^{-7-1} = -7x^{-8}.$$

$$y(x) = 1/x^2 = x^{-2}; \quad \text{here } n = -2, \quad \text{so } \frac{d}{dx}\left(\frac{1}{x^2}\right) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}.$$

Now try these problems:

$$y = \frac{1}{x}, \qquad \frac{dy}{dx} = \left[1 + \frac{1}{x} \,\middle|\, -\frac{1}{x} \,\middle|\, -\frac{1}{x^2} \,\middle|\, 2\right]$$

$$y = \frac{-1}{3}x^{-3}, \qquad \frac{dy}{dx} = \left[x^{-4} \,\middle|\, -3x^{-4} \,\middle|\, \frac{-1}{4}x^{-2} \,\middle|\, +x^{-2}\right]$$

If right, go on to **183**.
If wrong, go back to **180** and continue from there.

183

Here is another problem.

If $y(x) = x^{1/2}$, find $\frac{dy}{dx}$.

The answer is $\left[x^{-1/2} \mid \frac{1}{2}x^{-1/2} \mid \frac{1}{2}x \mid \text{none of these}\right]$.

If right, go to **185**.
If wrong, go to **184**.

184

The rule $\frac{d}{dx}x^n = nx^{n-1}$ is true for any value of n. In this case, $n = 1/2$,

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{(1/2-1)} = \frac{1}{2}x^{-1/2}.$$

Try this problem:

$$\frac{d}{dx}x^{2/3} = \left[x^{-1/3} \;\middle|\; \frac{2}{3}x^{-2/3} \;\middle|\; \frac{2}{3}x^{-1/3} \;\middle|\; x^{5/3}\right]$$

Go to **185**.

2.6 Some Rules for Differentiation

185

In this section we are going to learn a number of shortcut rules for differentiation without having to go all the way back to the definition of the derivative each time. Some of these rules are derived here; others are derived in Appendix **A**.

For the rest of this section, we will let $u(x)$ and $v(x)$ stand for any two functions that depend on x.

Go to **186**.

186

Sum Rule:

Our first rule will let us evaluate the derivative of the sum of $u(x)$ and $v(x)$ in terms of their derivatives. We will derive the rule here. Let

$$y(x) = u(x) + v(x).$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x) - u(x)]}{\Delta x} + \lim_{\Delta x \to 0} \frac{[v(x+\Delta x) - v(x)]}{\Delta x} \\ &= \frac{du}{dx} + \frac{dv}{dx}.\end{aligned}$$

Hence the rule is

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

If you would like a rigorous justification of the manipulation of the limits in the above proof, see Appendix **A2**.

Go to **187**.

187

Now let's put the above rule to use by computing the derivative of the following function (you will also have to use some results from the last section):

$$y = x^4 + 8x^3.$$

$$\frac{dy}{dx} = \_\_\_\_\_\_\_\_\_\_.$$

For the correct answer, go to frame **188**.

Answers: Frame 181: $3x^2$, $-7x^{-8}$, $-2/x^3$

Frame 182: $-1/x^2$, x^{-4}

Frame 183: $\frac{1}{2}x^{-1/2}$

Frame 184: $\frac{2}{3}x^{-1/3}$

188

The answer to the question in frame **187** is

$$\frac{d}{dx}(x^4 + 8x^3) = 4x^3 + 24x^2.$$

If you obtained this answer, go to frame **189**. Otherwise, continue here to find your mistake.

Our problem is to find the derivative of the sum of two functions. To make use of the rule in frame **186** in the notation used there, suppose we let $u(x) = x^4$, $v(x) = 8x^3$.

Then

$$\frac{d}{dx}(u + v) = \frac{d}{dx}(x^4 + 8x^3) = \frac{d}{dx}x^4 + \frac{d(8x^3)}{dx}.$$

You can evaluate these two derivatives from the result of the last section:

$$\frac{d}{dx}x^4 = 4x^3, \quad \frac{d}{dx}(8x^3) = 24x^2.$$

Hence, $\frac{d}{dx}(x^4 + 8x^3) = 4x^3 + 24x^2$.

Go to **188**.

189

The Product Rule:

Now that we can differentiate the sum of two variables, our next task is to learn to differentiate the product of two functions, for instance, $f(x) = u(x)v(x)$. We want to express $\frac{d}{dx}(uv)$ in terms of $\frac{du}{dx}$ and $\frac{dv}{dx}$. The result, known as the *product rule*, will be stated here. Look in Appendix **A7** if you want to see how it is derived.

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} = uv' + vu'.}$$

Go to **190**.

190

Here is an example in which the *product rule* is used. Suppose

$$y(x) = (x^5 + 7)(x^3 + 17x).$$

(continued)

The problem is to find $\frac{dy}{dx}$. If we let $u(x) = x^5 + 7$ and $v(x) = x^3 + 17x$, then $y(x) = u(x)v(x)$.

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Because $\frac{du}{dx} = 5x^4$ and $\frac{dv}{dx} = 3x^2 + 17$, our result is

$$\frac{dy}{dx} = (x^5 + 7)(3x^2 + 17) + (x^3 + 17x)(5x^4).$$

Note that it is considered good practice to simplify expressions such as this by collecting terms in like powers of x. To save time in this chapter, you need not do so.

By using the product rule, we can derive in another way a result we have already found: $\frac{d}{dx}x^2 = 2x$. If we let $u(x) = x$ and $v(x) = x$, then the product rule tells us that

$$\frac{dx^2}{dx} = x\frac{dx}{dx} + x\frac{dx}{dx} = 2x.$$

Go to **191**.

191

Use the product rule to find the derivative $\frac{d}{dx}[(3x + 7)(4x^2 + 6x)]$.

Answer: ____________________

See **192** for the solution.

192

The answer is

$$(3x + 7)(8x + 6) + (4x^2 + 6x)(3).$$

If you obtained this or an equivalent result, go on to **194**. Otherwise, read below.

The problem is to differentiate the product of $3x + 73x + 7$ and $4x^2 + 6x$. Suppose we let $u(x) = 3x + 7$ and $v(x) = 4x^2 + 6x$. Then $u' = 3$ and $v' = 8x + 6$. Hence

$$\frac{d}{dx}(uv) = uv' + vu' = (3x + 7)(8x + 6) + (4x^2 + 6x)(3).$$

Try this problem:

What is $\frac{d}{dx}[(2x + 3)(x^5)]$?

Answer: ______________________________

The solution is in **193**.

193

The answer is

$$\frac{d}{dx}[(2x + 3)(x^5)] = (2x + 3)(5x^4) + (x^5)(2).$$

The method for obtaining this result is shown in frame **192**. You can use the rule in frame **180** for differentiating x^n in order to find $\frac{d}{dx}x^5 = 5x^4$.

Go to **194**.

194

The Quotient Rule:

Frame **189** stated the product rule: $(uv)' = uv' + vu'$. Sometimes one needs to differentiate the quotient of two functions, $u(x)/v(x)$. Here is the rule. It will be proven later in this section, in frame **206**.

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2} = \frac{vu' - uv'}{v^2}.}$$

Go to **195**.

195

Solve the following problem:

$$\frac{d}{dx}\left(\frac{1+x}{x^2}\right) = \_\_\_\_\_\_\_\_\_\_$$

The correct answer is in **196**.

196

The answer to the problem in **195** is

$$\frac{d}{dx}\left(\frac{1+x}{x^2}\right) = -\frac{1}{x^2} - \frac{2}{x^3}.$$

If right, go to **198**.
If wrong, go to **197** for help.

197

Let $u(x) = 1 + x$, $v(x) = x^2$. Then $\frac{du}{dx} = 1$, $\frac{dv}{dx} = 2x$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2},$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{x^2 - (1+x)(2x)}{x^4} = \frac{x^2 - 2x - 2x^2}{x^4} = -\frac{2}{x^3} - \frac{1}{x^2}.$$

Go to **198**.

198

The Chain Rule:

This frame describes a helpful rule for finding the derivative of a "*function of a function*." Suppose $f(u)$ is a function that depends on u, and $u(x)$ in turn depends on x. Then $f(u(x))$ also depends on x. The following rule is proved in Appendix **A7**.

$$\boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}.}$$

This formula is called the *chain rule* because it links together derivatives with related variables. It is one of the most frequently used rules in differential calculus.

Here is an example: suppose we want to differentiate $f(x) = (x + x^2)^2$. This is a complicated function. It looks much simpler if we let $u(x) = (x + x^2)$, in which case $f = u^2$. Then

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{du^2}{du}\frac{du}{dx} = 2u\frac{du}{dx}.$$

We now substitute $u = (x + x^2)$, and $\frac{du}{dx} = 1 + 2x$, to obtain

$$\frac{df}{dx} = 2(x + x^2)(1 + 2x).$$

(You can check that the chain rule gives the right answer in this case by multiplying out the expression for f, and then differentiating it. You will find that the answer is equivalent to df/dx found above.)

Caution: The chain rule would be a simple identity if df/dx and du/dx could be treated as ratios of independent quantities df, du, and dx. However, this is not the case; one cannot cancel du in the numerator and denominator. (Nevertheless, this fiction makes a handy way to remember the chain rule!)

Go to **199**.

199

Here are a few more examples of the use of the *chain rule*.

1. Find $\frac{d}{dt}(\sqrt{1 + t^2})$.

Suppose we let $w(t) = \sqrt{1 + t^2}$, and $u(t) = 1 + t^2$, so that $w(u) = \sqrt{u}$. Then

$$\frac{dw}{dt} = \frac{dw}{du}\frac{du}{dt} = \frac{1}{2\sqrt{u}}(2t)$$

$$= \frac{1}{2}\frac{1}{\sqrt{1 + t^2}}2t = \frac{t}{\sqrt{1 + t^2}}.$$

(continued)

2. Let $v = \left(q^3 + \frac{1}{q}\right)^{-3}$; find $\frac{dv}{dq}$.

This problem can be simplified by letting $p(q) = \left(q^3 + \frac{1}{q}\right)$ and $v(p) = p^{-3}$. With these symbols the chain rule is

$$\frac{dv}{dq} = \frac{dv}{dp}\frac{dp}{dq} = -3p^{-4}\frac{dp}{dq} = -3p^{-4}\left(3q^2 - \frac{1}{q^2}\right)$$
$$= -3\left(q^3 + \frac{1}{q}\right)^{-4}\left(3q^2 - \frac{1}{q^2}\right).$$

The following example will not be explained, because you should be able to work it by inspection.

3. $\frac{d}{dx}\left(1 + \frac{1}{x}\right)^2 = 2\left(1 + \frac{1}{x}\right)\left(-\frac{1}{x^2}\right).$

Go to **200**.

200

Now try the following problem:
Which expression correctly gives $\frac{d}{dx}(2x + 7x^2)^{-2}$?

(a) $(-2)(2 + 14x)^{-3}$

(b) $(-2)(2 + 14x)^{-2}(2x + 7x^2)$

(c) $(2x + 7x^2)^{-3}(2 + 14x)$

(d) $(-2)(2x + 7x^2)^{-3}(2 + 14x)$

The correct answer is [*a* | *b* | *c* | *d*]

If right, go to **203**.
Otherwise, go to **201**.

201

Here is how to work the problem in **200**. Suppose we let $u(x) = 2x + 7x^2$ and $w(u) = u^{-2}$. Then

$$\frac{du}{dx} = 2 + 14x.$$

Hence

$$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx} = \frac{du^{-2}}{du}\frac{du}{dx}$$
$$= -2u^{-3}\frac{du}{dx} = -2(2x + 7x^2)^{-3}(2 + 14x).$$

Try this problem:

Find $\frac{dw}{ds}$, where $q(s) = s^2 + 4$ and $w(q) = 12q^4 + 7q$

$$\frac{dw}{ds} = \_\_\_\_\_\_\_\_\_\_.$$

For the solution, go to **202**.

202

The problem in frame **201** can be solved by using the chain rule:

$$\frac{dw}{ds} = \frac{dw}{dq}\frac{dq}{ds}.$$

We are given that $w(q) = 12q^4 + 7q$ and $q(s) = s^2 + 4$, so

$$\frac{dw}{dq} = 48q^3 + 7 \quad \text{and} \quad \frac{dq}{ds} = 2s.$$

Substituting these, we have

$$\frac{dw}{ds} = (48q^3 + 7)(2s) = [48(s^2 + 4)^3 + 7](2s).$$

If you found this result, go on to **203**. Otherwise, study the last few frames to make sure you understand how to use the chain rule. Don't be confused by the names of variables.

Then go to **203**.

203

The next problem is to use the chain rule to derive $\frac{d}{dx}\left(\frac{1}{v}\right)$ in terms of v and $\frac{dv}{dx}$, where $v(x)$ depends on x. Which of the following answers correctly gives $\frac{d}{dx}\left(\frac{1}{v}\right)$?

$$\left[-\frac{1}{v^2}\frac{dv}{dx} \;\middle|\; \frac{1}{dv/dx} \;\middle|\; \frac{dx}{dv} \;\middle|\; -\frac{dv}{dx} \;\middle|\; \text{none of these}\right]$$

If right, go to **205**.
If wrong, go to **204**.

204

To find $\frac{d}{dx}\left(\frac{1}{v}\right)$, we apply the chain rule as follows. Suppose we let $f = \frac{1}{v} = v^{-1}$ with $\frac{df}{dx} = \frac{df}{dv}\frac{dv}{dx}$, where $\frac{df}{dv} = \frac{d}{dv}v^{-1} = -\frac{1}{v^2}$. Thus

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

Note that because we are not given explicitly $v(x)$, the derivative dv/dx in the answer is also not known.

Go to **205**.

205

By combining the result of the last frame with what you have learned previously, you should be able to derive the expression for the derivative of the quotient of two functions. This is an extremely important relation. Try to work it out for yourself without using the quotient rule.

Find $\frac{d}{dx}\left(\frac{u}{v}\right)$ in terms of u, v, $\frac{du}{dx}$, $\frac{dv}{dx}$.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \_\_\_\_\_\_\_\_\_\_\_\_.$$

To check your answer, go to **206**.

Answer: Frame 200: *d*

206

You should have obtained the following quotient rule, which was presented without proof in frame **194**, though possibly arranged differently,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

If you wrote this or an equivalent statement, go on to **207**. Otherwise, study the derivation below.

If we let $p = \frac{1}{v}$, then our derivative is that of the product of two variables

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}(up) = u\frac{dp}{dx} + p\frac{du}{dx}.$$

Now $\frac{dp}{dx} = \frac{dp}{dv}\frac{dv}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$, as in frame **194**, so

$$\frac{d}{dx}\left(\frac{u}{v}\right) = -\frac{u}{v^2}\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx} = \frac{v(du/dx) - u(dv/dx)}{v^2}.$$

Go to **207**.

207

Before going on to new material, let's summarize all the rules for differentiation we have used so far. Fill in the blanks. a and n are constants, u and v are variables that depend on x, w depends on u, which in turn depends on x.

$$\frac{da}{dx} = \underline{\qquad\qquad}. \qquad \frac{d}{dx}(u + v) = \underline{\qquad\qquad}.$$

$$\frac{d(ax)}{dx} = \underline{\qquad\qquad}. \qquad \frac{d(uv)}{dx} = \underline{\qquad\qquad}.$$

$$\frac{dx^2}{dx} = \underline{\qquad\qquad}. \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \underline{\qquad\qquad}.$$

$$\frac{dx^n}{dx} = \underline{\qquad\qquad}. \qquad \frac{d}{dx}[w(u)] = \underline{\qquad\qquad}.$$

To check your answers, go to **208**.

208

Here are the correct answers. The frame in which the relation was introduced is shown in parentheses.

$$\frac{da}{dx} = 0. \qquad (\mathbf{172})$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}. \qquad (\mathbf{186})$$

$$\frac{d(ax)}{dx} = a. \qquad (\mathbf{174})$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \qquad (\mathbf{189})$$

$$\frac{dx^2}{dx} = 2x. \qquad (\mathbf{176})$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}. \qquad (\mathbf{194})$$

$$\frac{dx^n}{dx} = nx^{n-1}. \qquad (\mathbf{180})$$

$$\frac{d}{dx}[w(u)] = \frac{dw}{du}\frac{du}{dx}. \qquad (\mathbf{198})$$

If you would like some more practice on problems similar to those in the last two sections, see review problems **34** through **38** on pages 279–280.

Go to **209**.

2.7 Differentiating Trigonometric Functions

209

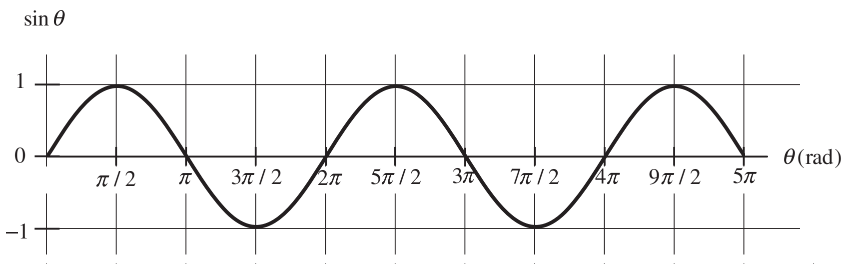
Trigonometric functions occur in so many applications that it is useful to know their derivatives. For instance, we would like to know $\frac{d}{d\theta}\sin\theta$. By definition,

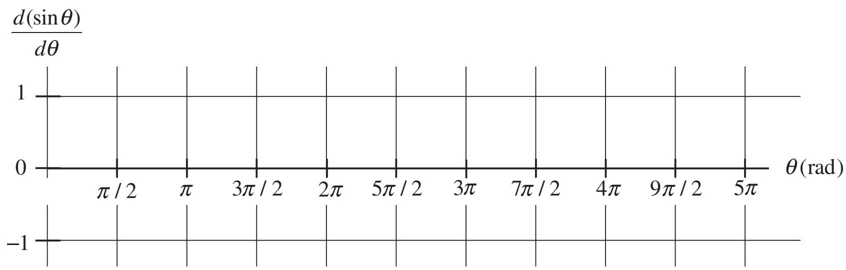
$$\frac{d}{d\theta}\sin\theta = \lim_{\Delta\theta\to 0}\frac{\sin(\theta+\Delta\theta) - \sin\theta}{\Delta\theta}.$$

Answer: Frame 203: $-\frac{1}{v^2}\frac{dv}{dx}$

It is not at all obvious how to evaluate this expression, so let's take another approach for a minute and try to guess geometrically what the result should be by looking at a plot of sin θ.

Here is a plot of sin θ vs. θ over the interval $0 \leq \theta \leq 2\pi$. (θ is measured in radians.)

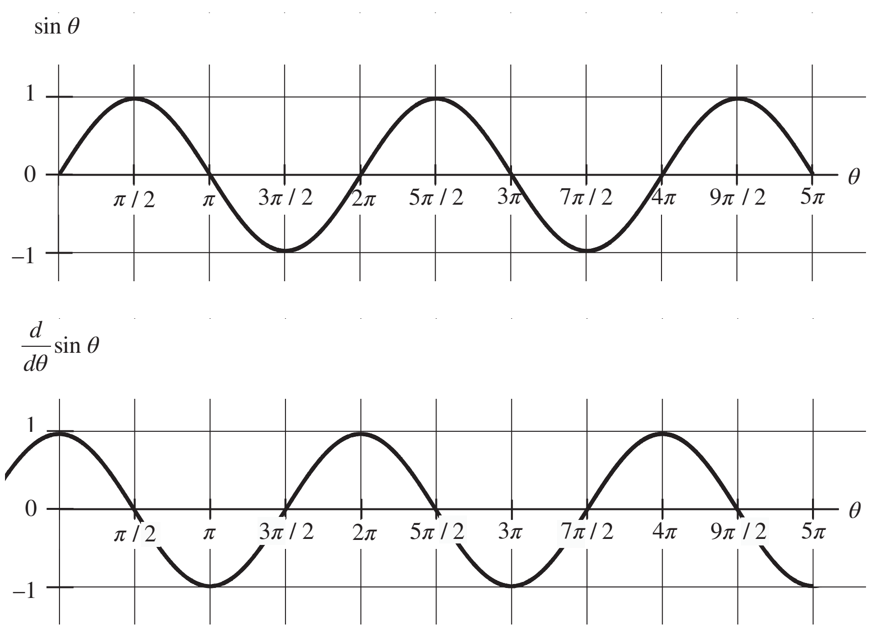




Draw a sketch of $\frac{d}{d\theta}\sin\theta$ in the space provided. To check your sketch,

Go to **210**.

210



Here are drawings of sin θ and $\frac{d}{d\theta}\sin\theta$. Note that where the slope of sin θ is greatest, at 0 and 2π, $\frac{d}{d\theta}\sin\theta$ has its greatest value, and that where the slope is 0, at $\theta = \pi/2$ and $\theta = 3\pi/2$, $\frac{d}{d\theta}\sin\theta$ is 0.

(If your sketch looked very different from the drawing shown above, you should review frames **160** and **169**. This problem is quite similar to problem (c) in frame **168**.)

Now, by looking at the graphs, you may be able to guess the correct answer for $\frac{d}{d\theta}\sin\theta$. Can you?

$$\frac{d}{d\theta}\sin\theta = \underline{\qquad\qquad}.$$

Go to frame **211** to see if your answer is right.

211

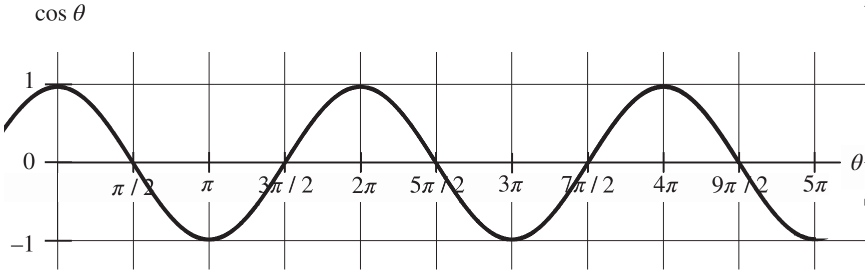
Here is the rule:

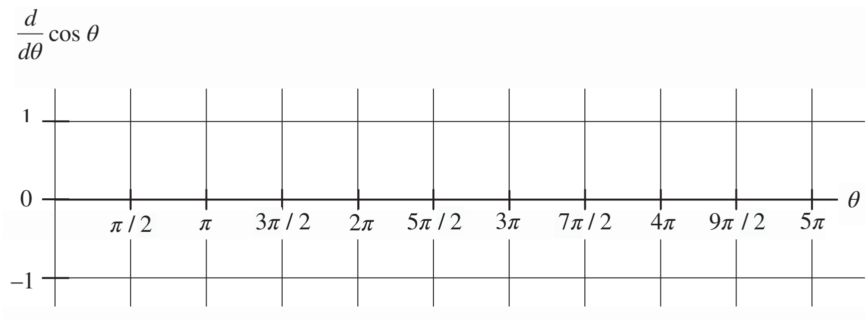
$$\frac{d}{d\theta}\sin\theta = \cos\theta.$$

Congratulations if you guessed this result in the last frame! If you arrived at some other result, study the drawings in frame **209** and compare the second one with the graph of cos θ shown below. (The formal proof that $\frac{d}{d\theta}\sin\theta = \cos\theta$ is given in Appendix **A6**.)

It is important to realize that this relation is true *only* when the angle is measured in radians—this is why the radian is such a useful unit.

Let's try to guess the result for $\frac{d}{d\theta}\cos\theta$ from a plot of cos θ.





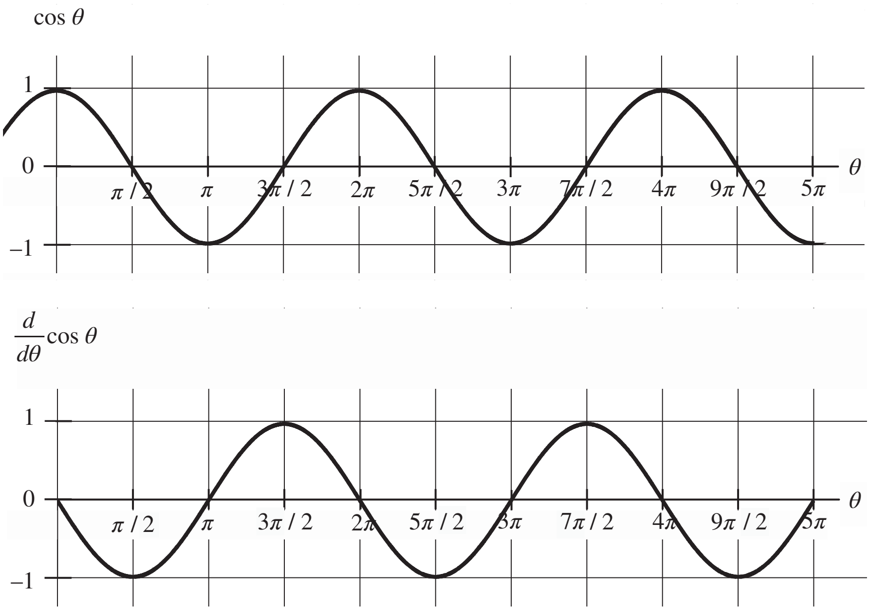
Draw a sketch of $\frac{d}{d\theta}$. cos θ in the space provided, and make a guess at the result.

$$\frac{d}{d\theta}\cos\theta = \underline{\hspace{3cm}}.$$

Go to **212**.

212

Here are plots of $\cos\theta$ and $\frac{d}{d\theta}\cos\theta$. The result is $\frac{d}{d\theta}\cos\theta = -\sin\theta$,



as should seem reasonable from the graph. This relation also is formally proved in Appendix **A6**.

To summarize:

$$\frac{d}{d\theta}\sin\theta = \cos\theta.$$
$$\frac{d}{d\theta}\cos\theta = -\sin\theta.$$

Go to **213**.

213

Using these results, find $\frac{d}{d\theta}\tan\theta$. (Hint: use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and apply the quotient rule, frame 194.)

$$\frac{d}{d\theta}\tan\theta = \underline{\hspace{4cm}}.$$

Using the hints in frame **212** we have

$$\begin{aligned}\frac{d}{d\theta}\tan\theta &= \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\cos\theta\dfrac{d(\sin\theta)}{d\theta} - \sin\theta\dfrac{d(\cos\theta)}{d\theta}}{\cos^2\theta}\\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.\end{aligned}$$

Now find the correct answer:

$$\frac{d}{d\theta}\sec\theta = [\sec\theta\tan\theta \mid -\sec\theta\tan\theta \mid \sec\theta]$$

If right, go to **215**.
If wrong, go to **214**.

214

Using the definition $\sec\theta = \frac{1}{\cos\theta}$, and the result in frame **204**, we have

$$\begin{aligned}\frac{d}{d\theta}\sec\theta &= \frac{d}{d\theta}\left(\frac{1}{\cos\theta}\right) = -\frac{1}{\cos^2\theta}\frac{d}{d\theta}\cos\theta\\ &= +\frac{1}{\cos^2\theta}\sin\theta = \frac{\tan\theta}{\cos\theta}\\ &= \sec\theta\tan\theta.\end{aligned}$$

(All three of these expressions are acceptable.)

Go to **215**.

215

Choose the correct answer:

$$\frac{d}{d\theta}(\sin\theta)^2 = [\sin\theta \mid 2\cos\theta \mid \cos\theta^2 \mid 2\sin\theta\cos\theta]$$

If right, go to **217**.
If wrong, go to **216**.

216

You could have analyzed the problem as follows:

Suppose we let $u(\theta) = \sin\theta$. Then $\frac{du}{d\theta} = \cos\theta$, and

$$\frac{d}{d\theta}(\sin\theta)^2 = \frac{d}{d\theta}(u^2) = \frac{d}{du}(u^2)\frac{du}{d\theta}$$
$$= 2u\frac{du}{d\theta} = 2\sin\theta\cos\theta.$$

If you did not get this result, where did you go wrong? Find your error and be sure you understand it. Then

Go to **217**.

217

Which of the following is $\frac{d}{d\theta}\cos(\theta^3)$?

$$[\cos\theta\sin(\theta^3) \mid -3\theta^2\sin(\theta^3) \mid 3\cos^2(\theta^3)\sin(\theta^3) \mid 3\cos^2\theta]$$

If right, skip on to frame **221**.
If wrong, go to frame **218**.

218

Did you forget how to use the *chain rule* to differentiate a function of a function? We can think of $\cos(\theta^3)$ as a function of a function. Suppose we write it this way:

$$f(u) = \cos u, \quad u = \theta^3.$$

Then

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta},$$
$$\frac{df}{du} = -\sin u = -\sin(\theta^3), \quad \frac{du}{d\theta} = 3\theta^2,$$

so

$$\frac{d}{d\theta}\cos(\theta^3) = -3\theta^2\sin(\theta^3).$$

Go to **219**.

Answer: Frame 213: $\sec\theta\tan\theta$

219

If ω (Greek letter omega) is a constant, which expression correctly gives $\frac{d}{dt}\sin(\omega t)$?

$$[\cos \omega t \mid \omega\cos \omega t \mid \sin \omega t \mid \text{none of these}]$$

If right, go to frame **221**.
Otherwise, go to **220**.

220

To solve problem in **219**, let $f(u) = \sin u$, $u = \omega t$,

$$\frac{df}{dt} = \frac{df}{du}\frac{du}{dt} = \cos u \frac{d(\omega t)}{dt} = \omega\cos(\omega t).$$

Go to frame **221**.

221

Before going on to the next section, let's state once more the important relations we have introduced in this section:

$$\boxed{\begin{aligned} \frac{d}{d\theta}\sin\theta &= \cos\theta, \\ \frac{d}{d\theta}\cos\theta &= -\sin\theta. \end{aligned}}$$

There are two other functions that are so common that it is worth knowing their derivatives by heart: logarithmic and exponential. To learn about them,

Go to **222**.

2.8 Differentiating Logarithms and Exponentials

222

Our next task is to learn how to differentiate logarithms. If you feel shaky about logarithms, review frames **75–95** of Chapter 1 before going on to the next frame.

Go to **223**.

223

In this section we are going to work with natural logarithms, $\ln x = \log_e x$, with $x > 0$. Natural logarithms were defined in frame **94**. The base $e = 2.71828\ldots$ was discussed in frame **109**.

Table 2 lists values of $\ln x$ for a few values of x.

Table 2.2

x	$\ln x$	x	$\ln x$
1	0	30	3.40
2	0.69	100	4.61
e	1	300	5.70
3	1.10	1000	6.91
10	2.30	3000	8.01

Using Table 2 and the rules for manipulating logarithms, find the answer that is most nearly correct for each of the following questions:

$$\ln 6 = [2.2 \mid 3.1 \mid 6/e \mid 1.79]$$

$$\ln \sqrt{10} = [1.15 \mid 2.35 \mid 2.25 \mid 1.10]$$

$$\ln 300^3 = [126 \mid 185 \mid 17.10 \mid 3.41]$$

If all your answers are correct, go to **225**.
If you made any mistakes, go to **224**.

Answers: Frame 215: $2\sin\theta\cos\theta$

Frame 217: $-3\theta^2\sin(\theta^3)$

Frame 219: $\omega\cos(\omega t)$

224

The rules for manipulating logarithms are summarized in frame **91**. These rules apply to logarithms to any bases, including the base *e*.

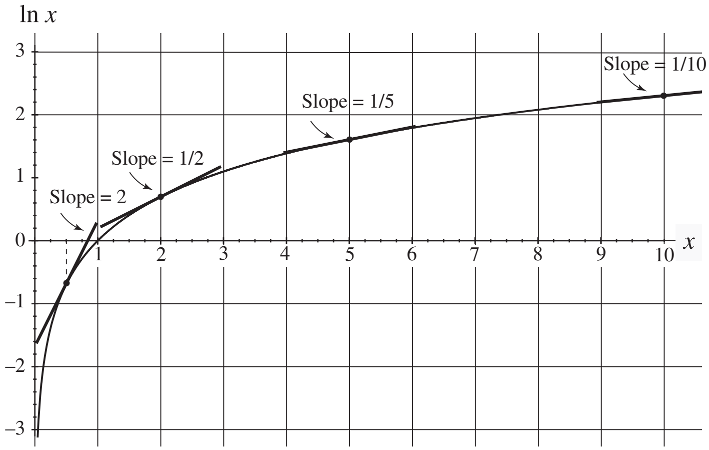
$$\ln 6 = \ln((2)(3)) = \ln 2 + \ln 3 = 0.69 + 1.10 = 1.79,$$
$$\ln\sqrt{10} = \ln 10^{1/2} = \frac{1}{2}\ln 10 = \frac{1}{2}2.30 = 1.15,$$
$$\ln 300^3 = 3\ln 300 = (3)(5.70) = 17.10.$$

Go on to **225**.

225

Calculator Problem:

Here is a plot of ln *x* vs. *x*. If your calculator provides ln *x*, check some of the points on this graph.



You can find the qualitative features of $\frac{d}{dx}\ln x$ by inspecting the graph. For small values of *x* the derivative is large, and for large values of *x* the derivative is small. In the figure above tangents are shown at a few points, and their slopes are listed in this table.

x	*Slope*
1/2	2
2	1/2
5	1/5
10	1/10

Perhaps you can guess the formula for $\frac{d}{dx}\ln x$. Try to fill in the blank.

$$\frac{d}{dx}\ln x = \_\_\_\_\_\_\_\_\_\_.$$

To see the correct expression, go to **226**.

226

Here is the formula for the derivative of a natural logarithm:

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}.}$$

If you did not guess this result, you can check that it agrees with the numerical values in the table in frame **225**.

The reason that e is so useful as a base for logarithms is that it leads to this simple expression. This relation is derived in Appendix **A9**, and it is so important that it is worth committing to memory.

Go to **227**.

227

Calculator Problem:

Using a calculator, you can numerically confirm that $\frac{d}{dx}\ln x = \frac{1}{x}$. The procedure is to calculate value of $\frac{\ln(x+\Delta x)-\ln x}{\Delta x}$ for successively smaller values of Δx. The result should approach $1/x$.

Try the following for $x = 5$, for instance, or any other value you may wish to choose. For $x = 5$, $\ln x = 1.6094$, and $d(\ln x)/dx = 1/5 = 0.2$.

Δx	$\ln(x+\Delta x)$	$\frac{\ln(x+\Delta x)-\ln x}{\Delta x}$
2		
1		
0.1		
0.01		

Go to **228**.

Answer: Frame 223: 1.79, 1.15, 17.10

228

Try this problem: Which of the following gives $\frac{d}{dx}\ln(x^2)$?

$$\left[2\ln x \,\middle|\, \frac{2}{x} \,\middle|\, \frac{1}{x^2} \,\middle|\, \frac{2}{x^2} \,\middle|\, \frac{2}{x}\ln x \right]$$

If right, go to **230**.
Otherwise, go to **229**.

229

The solution of this problem is straightforward if we use the chain rule. However, let's solve it a different way. Because $\ln(x^2) = 2\ln x$,

$$\frac{d}{dx}\ln(x^2) = \frac{d}{dx}(2\ln x) = \frac{2}{x}.$$

You should be able to do this one:

$$\frac{d}{dx}(\ln x)^2 = \left[2\ln x \,\middle|\, \frac{2\ln x}{x} \,\middle|\, \frac{2}{\ln x} \,\middle|\, \text{none of these} \right]$$

If right, go to **231**.
Otherwise, go to **230**.

230

Use the chain rule:

$$\frac{d}{dx}(\ln x)^2 = (2\ln x)\left(\frac{d}{dx}\ln x\right) = \frac{2\ln x}{x}.$$

Go to **231**.

231

$$\text{(a)} \quad \frac{d}{dr}\ln r = \underline{\hspace{3cm}}.$$

$$\text{(b)} \quad \frac{d}{dz}\ln(5z) = \underline{\hspace{3cm}}.$$

For the correct answers, go to **232**.

232

The correct answers are

$$\text{(a)}\frac{1}{r}; \qquad \text{(b)}\frac{1}{z}.$$

If you got both of these, you are doing fine: skip ahead to frame **234**. If you missed either or would like a worked solution.

Go to frame **233**.

233

(a) $\frac{d}{dr}\ln r = \frac{1}{r}$ for the same reason that $\frac{d}{dx}\ln x = \frac{1}{x}$. It makes no difference whether the variable is called r or x.

(b) The simplest way to find $\frac{d}{dx}\ln(5z)$ is to recall that $\ln(5z) = \ln 5 + \ln z$. Hence,

$$\frac{d}{dz}\ln(5z) = \frac{d}{dz}\ln 5 + \frac{d}{dz}\ln z = 0 + \frac{1}{z} = \frac{1}{z}.$$

Go to **234**.

Answers: Frame 228: $2/x$

Frame 229: $2\ln x/x$

234

Logarithmic Derivatives:

Here is a function involving exponentials that is interesting to differentiate:

$$y = f(x) = a^x \qquad (a \text{ is a positive constant}).$$

(Warning: Do not confuse a^x with x^a. We are sticking to the convention that x is a variable and a is a constant.) We can differentiate a^x by first taking the natural logarithm:

$$\ln y = \ln f(x) = \ln(a^x) = x \ln a.$$

The derivative of $\ln f(x)$ with respect to x is

$$\frac{d}{dx} \ln f = \frac{d}{dx}(x \ln a) = \ln a.$$

Recall that

$$\frac{d}{dx} \ln y = \frac{1}{y}\frac{dy}{dx}.$$

Now equate the two expressions for derivatives obtaining

$$\frac{1}{y}\frac{dy}{dx} = \ln a,$$

which we can solve for the derivative

$$\frac{dy}{dx} = (\ln a)y,$$

and so

$$\frac{d}{dx} a^x = a^x \ln a.$$

This example can be generalized to any function $y = f(x)$ for which $f(x) \neq 0$ for a range of values of x. This technique is useful for certain functions $y = f(x)$ in which $\ln f(x)$ can be simplified as much as possible using properties of the natural logarithm (frame **95**), and if the derivative $\frac{d}{dx} \ln f$ is fairly straightforward to calculate. Then the derivative we are seeking is

$$\frac{dy}{dx} = \left(\frac{d}{dx} \ln f\right) y.$$

Go to **235**.

235

The preceding frame gave the result

$$\frac{d}{dx}a^x = a^x \ln a.$$

A simple but important case occurs when $a = e$. Because $\ln e = 1$,

$$\boxed{\frac{d}{dx}e^x = e^x.}$$

Try finding the values for the following:

(a) $\frac{d}{dx}e^{cx} =$ ________________

(b) $\frac{d}{dx}e^{-x} =$ ________________

See **236** for the answers.

236

The answers are

(a) $\frac{d}{dx}e^{cx} = ce^{cx}$,

(b) $\frac{d}{dx}e^{-x} = -e^{-x}$.

If you did both of these correctly, go to **237**. Otherwise, continue here.

The result (a) is obtained by letting $u(x) = cx$ and following the usual procedure for a function of a function (i.e., using the chain rule, frame **194**). Thus

$$\frac{d}{dx}e^{cx} = \frac{d}{du}e^u \frac{du}{dx} = e^u c = ce^{cx}.$$

The result (b) is a special case of (a) with $c = -1$.

Go to **237**.

237

Calculator Problem:

You can confirm numerically that $\frac{d}{dx}e^x = e^x$ in the same way that you confirmed $\frac{d}{dx}\ln x = \frac{1}{x}$ in frame **227**. Calculate the following for some value of x, for instance, $x = 10$. See whether the last column approaches $e^{10} = 22\,026.46\ldots$.

Δx	$e^{x+\Delta x}$	$\dfrac{e^{x+\Delta x} - e^x}{\Delta x}$
1		
0.1		
0.01		

Go to **238**.

238

If $z = \frac{1}{\ln x}$, what is $\frac{dz}{dx}$?

Encircle the correct answer.

$$\left[\frac{1}{(\ln x)x} \;\middle|\; \frac{-x}{(\ln x)^2} \;\middle|\; \frac{-1}{(\ln x)^2 x} \;\middle|\; \frac{\ln x}{x^2}\right]$$

If right, go to **240**.
Otherwise, go to **239**.

239

One way to find the derivative of $\frac{1}{\ln x}$ is to use the chain rule. Let $u(x) = \ln x$. Then

$$\frac{d}{dx}\left(\frac{1}{\ln x}\right) = \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx} = -\frac{1}{u^2}\frac{1}{x} = -\frac{1}{(\ln x)^2 x}.$$

Go to **240**.

240

A number of relations have been used in this section, and you may want to give them a quick review before going on. Here is a list. The most important ones are in boxes.

$$e = 2.718\ldots,$$

$$\ln x = \log_e x,$$

$$\ln(x) = (2.303\ldots)\log_{10} x,$$

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x},}$$

$$\boxed{\frac{d}{dx}e^x = e^x,}$$

$$\frac{d}{dx}a^x = a^x \ln a.$$

Go to **241**.

241

We have learned how to differentiate the most useful common functions. The rest of this chapter will be devoted to special topics related to the use of derivatives. If you want a little more practice in differentiation before going on, see the review problems **34** through **58** on pages 279–281, and whenever you are ready,

Go to **242**.

2.9 Higher-Order Derivatives

242

Suppose a function f depends on x, and we have differentiated it to obtain df/dx. If we then differentiate df/dx with respect to x, the result is called the *second derivative* of f with respect to x. This is written $\frac{d^2f}{dx^2}$. Sometimes this is written as $f^{(2)}$, where the "(2)" superscript indicates the second derivative of f, not the square of f. The variable x is suppressed.

Answer: Frame 238: $\frac{-1}{(\ln x)^2 x}$

Try the following:

$$\text{If } f = 2x^3, \quad \text{then } f^{(2)} = \frac{d^2f}{dx^2} = [6x^2 \mid 12x \mid 0 \mid x^2 \mid x]$$

If right, go to **245**.
If wrong, go to **243**.

243

Here's how to do the problem in **242**.

$$f = 2x^3, \ \frac{df}{dx} = 6x^2,$$

$$f^{(2)} = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) \frac{d}{dx}(6x^2) = 12x.$$

Try this:

$$f(x) = x + \frac{1}{x}$$

$$f^{(2)} = \frac{d^2f}{dx^2} = \left[-\frac{1}{x^2} \,\middle|\, \frac{1}{x} \,\middle|\, +\frac{2}{x^3} \,\middle|\, \text{none of these}\right]$$

If right, go to **245**.
If wrong, go to **244**.

244

Here is the solution to **243**.

$$f(x) = x + \frac{1}{x}, \ \frac{df}{dx} = 1 - \frac{1}{x^2},$$

$$f^{(2)} = \frac{d^2f}{dx^2} = 0 - 1\left(\frac{-2}{x^3}\right) = \frac{2}{x^3}.$$

Go to **245**.

245

An example of a second derivative with which you may already be familiar is *acceleration*. Velocity is the rate of change of position with respect to time.

$$v = \frac{dS}{dt}.$$

Acceleration is the rate of change of velocity with respect to time and is commonly denoted by the symbol a. Hence

$$a = \frac{dv}{dt}.$$

It follows then that

$$a = \frac{d}{dt}\left(\frac{dS}{dt}\right) = \frac{d^2S}{dt^2}.$$

Go to **246**.

246

Let the position of a particle be given by

$$S(t) = A\sin(\omega t).$$

A and ω (omega) are constants. Find the acceleration.

Answer: $[0 \mid A\omega\cos(\omega t) \mid (A\omega\cos(\omega t))^2 \mid -A\omega^2\sin(\omega t)]$.

If right, go to **248**.
If wrong, go to **247**.

Answers: Frame 242: $12x$

Frame 243: $2/x^3$

247

$$\text{Acceleration} = \frac{d^2S}{dt^2} = \frac{d^2}{dt^2}A\sin(\omega t).$$

$$\frac{dS}{dt} = \frac{d}{dt}A\sin(\omega t) = A\omega\cos(\omega t),$$

$$\frac{d^2S}{dt^2} = \frac{d}{dt}\left(\frac{dS}{dt}\right) = \frac{d}{dt}A\omega\cos(\omega t) = -A\omega^2\sin(\omega t).$$

If this is not clear, see frame **219**.

Go to **248**.

248

There is really nothing essentially new about a second derivative. In fact, we can define derivatives of any order n, where n is a positive integer. Thus, $f^{(n)} = \frac{d^nf}{dx^n}$ is the nth derivative of f with respect to x. Try this problem:

If $f(x) = x^4$, find $f^{(4)} = \frac{d^4f}{dx^4}$.

$$f^{(4)} = \frac{d^4f}{dx^4} = [x^{16} \mid 4x^4 \mid 0 \mid 64 \mid (4)(3)(2)(1)]$$

Go to **249**.

249

$$f^{(4)} = \frac{d^4f}{dx^4} = \frac{d^4}{dx^4}(x^4) = \frac{d}{dx}\left(\frac{d}{dx}\left\{\frac{d}{dx}\left[\frac{d}{dx}(x^4)\right]\right\}\right)$$

$$= (4)\frac{d^3}{dx^3}(x^3) = (4)(3)\frac{d^2}{dx^2}(x^2) = (4)(3)(2)\frac{d}{dx}x$$

$$= (4)(3)(2)(1).$$

We can easily generalize this result:

$$\frac{d^n}{dx^n}x^n = (n)(n-1)(n-2)\cdots(1) = n!$$

($n!$ is called *n factorial* and is $(n)(n-1)(n-2)\cdots(1)$. By definition $0! = 1$.)

For more practice on higher-order derivatives, see review problems **59** through **63** on page 281.

Go to **250**.

2.10 Maxima and Minima

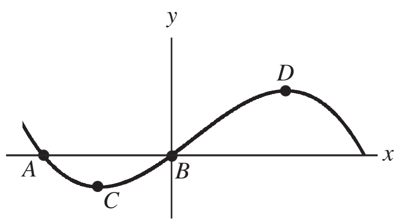
250

Now that we know how to differentiate simple functions, let's put our knowledge to use. Suppose we want to find the value of x and y at which $y = f(x)$ has a *minimum* or a *maximum* value in some given region. By the end of this section we will know how to solve this problem.

Go to **251**.

251

Here is the graph of a function. At which of the points indicated does y have a minimum value in the domain plotted?

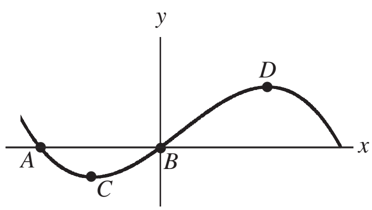


[A | B | C | D | A and B | C and D]

If correct, go to **253**.
If wrong, go to **252**.

252

The minimum value of y is at point C only, because y has its smallest value there, at least for the domain of x plotted.



Answers: Frame 246: $-A\omega^2 \sin(\omega t)$

Frame 248: (4)(3)(2)(1)

At A and B, y has the value 0, but this has nothing to do with whether or not it has a minimum value there.

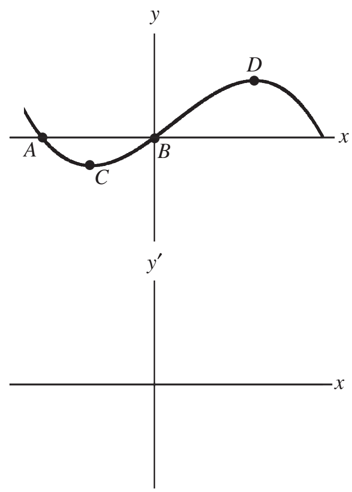
Point D is a maximum value of y.

Go to **253**.

253

We have shown that point C corresponds to a minimum value of y, at least insofar as nearby values are concerned, and that D corresponds similarly to a maximum value.

There is an interesting relation between the points of maximum or minimum values of y and the value of the derivative at those points. To help see this, sketch a plot of the derivative of the function shown, using the space provided.

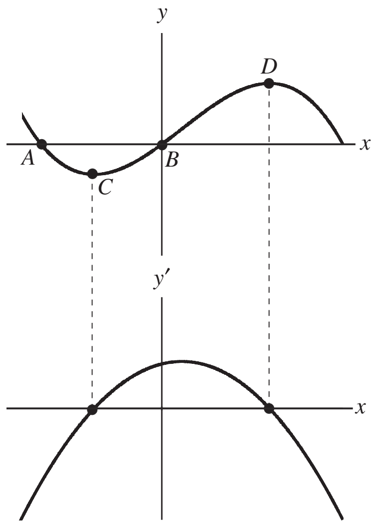


To check your sketch,

Go to **254**.

254

If you did not obtain a sketch substantially like this, review frames **160** to **169** before continuing.



This simple example should be enough to convince you that if $f(x)$ has a maximum or a minimum for some value of x within a given interval, then its derivative f' is zero for that x.

One way to tell whether it is a maximum or a minimum is to plot a few neighboring points. However, there is an even simpler method, as we shall soon see.

Go to **255**.

255

Test yourself with this problem:

Find the value of x for which the following has a minimum value.

$$f(x) = x^2 + 6x.$$

[−6 | −3 | 0 | +3 | none of these]

If right, go to **258**.
If wrong, go to **256**.

Answer: Frame 251: *C*

256

The problem is solved as follows:
The maximum or minimum occurs where x satisfies $f' = 0$.

$$f(x) = x^2 + 6x, \quad f' = 2x + 6.$$

Thus the equation for the value of x at the maximum or minimum is

$$2x + 6 = 0 \quad \text{or} \quad x = -3.$$

Here is another problem:
For which value(s) of x does the following $f(x)$ have a maximum or minimum value?

$$f(x) = 8x + \frac{2}{x}.$$

$$\left[\frac{1}{4} \;\middle|\; -\frac{1}{4} \;\middle|\; -4 \;\middle|\; 2 \;\middle|\; \frac{1}{2} \;\middle|\; -\frac{1}{2} \;\middle|\; 2 \text{ and } -4 \;\middle|\; \frac{1}{2} \text{ and } -\frac{1}{2}\right]$$

If you were right, go to **258**.
If you did not get the correct answer, go to **257**.

257

The problem in frame **256** can be solved as follows:
At the position of maximum or minimum, $f' = 0$. Because

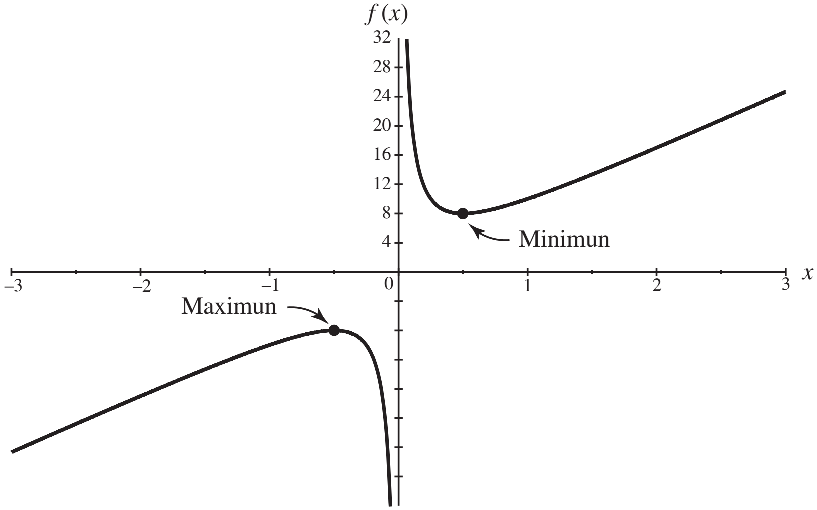
$$f(x) = 8x + \frac{2}{x}, \quad f' = 8 - \frac{2}{x^2}.$$

The desired points are solutions of

$$8 - \frac{2}{x^2} = 0 \quad \text{or} \quad x^2 = \frac{2}{8} = \frac{1}{4}.$$

Thus at $x = +1/2$ and $x = -1/2$, $f(x)$ has a maximum or a minimum value. A plot of $f(x)$ is shown in the figure, and, as you can see, $x = -1/2$ yields a maximum and $x = +1/2$ yields a minimum.

(continued)



Incidentally, as you can see from the drawing, the minimum falls above the maximum. This should not be paradoxical; we are talking about *local* minimum or maximum, that is, the minimum or maximum value of a function in some small region.

Go to **258**.

258

Now let's turn to an application of this idea in the real world that you are likely familiar with. You may have noticed that most metal cans look similar except for overall size. There is a reason for this that this problem will illustrate. Note this problem is more involved than most of the problems up to this point.

You want to design a cylinder of radius r and height h in order to minimize the surface area for a given fixed volume V. What should you choose for the ratio of the radius to the height, r_{min}/h that minimizes the surface area?

$$r_{\text{min}}/h = \_\_\_\_\_\_\_\_\_\_\_\_$$

Go to **259**.

Answers: Frame 255: −3

Frame 256: ½ and −½

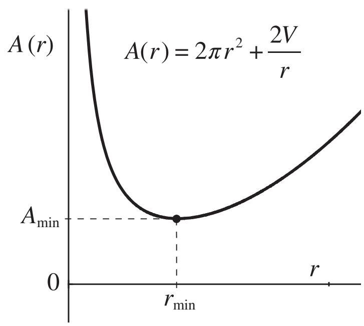
259

To find the value of r/h and the smallest surface area for a fixed volume, you need to find the value of r for which the surface area is minimum. The volume for a cylinder of radius r and height h is $V = \pi r^2 h$.

The total surface area is the sum of the surface area of the cylinder, plus the area of the two endcaps, each of area πr^2, hence $A = 2\pi r^2 + 2\pi rh$. Because the height and volume are related by $h = V/\pi r^2$, the total area can be expressed as a function of the radius r and the constant volume V according to

$$A = 2\pi r^2 + \frac{2V}{r}.$$

In the figure below, the area is plotted as a function of r. Note the radius r can only take positive values.



At the minimum, the first derivative is zero, $0 = \frac{dA}{dr} = 4\pi r - \frac{2V}{r^2}$, which you can solve for the radius $r_{\text{min}} = \left(\frac{V}{2\pi}\right)^{1/3}$. Because $V = \pi r^2 h$, $r_{\text{min}} = \left(\frac{\pi r_{\text{min}}^2 h}{2\pi}\right)^{1/3}$. Therefore, $r_{\text{min}}^3 = \frac{\pi r_{\text{min}}^2 h}{2\pi}$, which you can now solve for the ratio of the radius to the height

$$\frac{r_{\text{min}}}{h} = \frac{1}{2}.$$

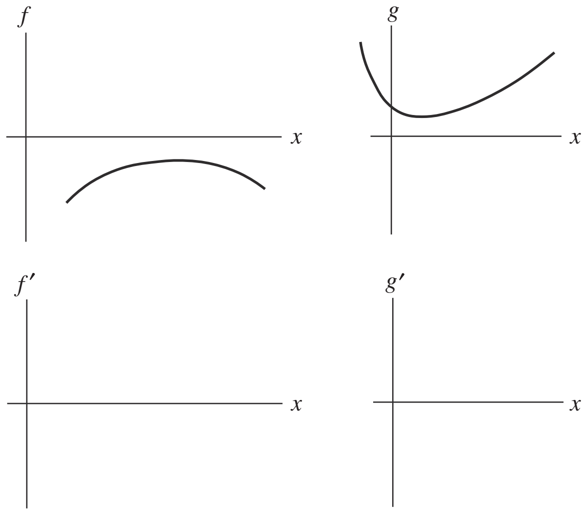
This means that the diameter and height are equal.

Go to **260**.

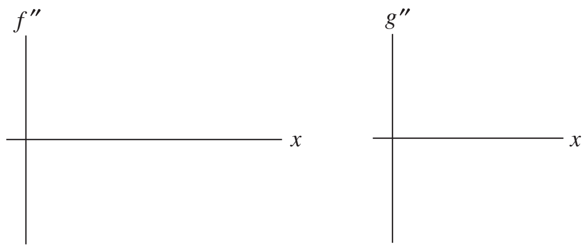
260

We mentioned earlier that there is a simple method for finding whether $f(x)$ has a maximum or a minimum value when $f' = 0$. Let's find the method by drawing a few graphs.

Below are graphs of two functions. On the left, $f(x)$ has a maximum value in the region shown. On the right, $g(x)$ has a minimum value. In the spaces provided, draw rough sketches of the derivatives of $f(x)$ and $g(x)$.



Now, let's repeat the process again. Make a rough sketch of the *second* derivative of each function (i.e. sketch the derivatives of the new functions you have just drawn).

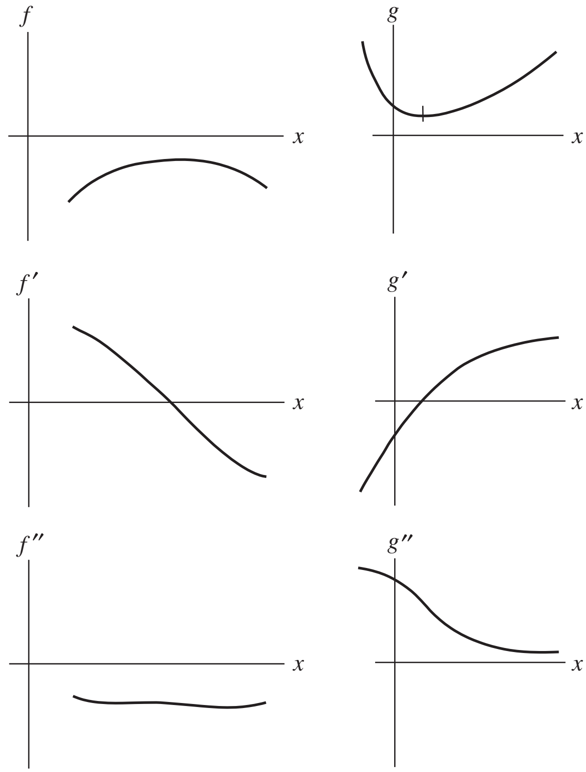


Perhaps from these sketches you can guess how to tell whether the function has a maximum or a minimum value when its derivative is 0. Whether you can or not,

Go to **261**.

261

The sketches should look approximately like this.



By studying these sketches, it should become apparent that wherever $f' = 0$,

$$f(x) \text{ has a maximum value if } f'' < 0,$$
$$\text{and } f(x) \text{ has a minimum value if } f'' > 0.$$

(If $f'' = 0$, this test is not helpful and we have to look further.)

If you are not convinced yet, go back and sketch the second derivatives of any of the functions shown in frames **164**, **166**, or **168** [(c) or (d)]. This should convince you that the rule is reasonable. Whenever you are ready,

Go to **262**.

262

Here is one last problem to try before we go on to another subject. Consider $f(x) = e^{-x^2}$. Find the value of x for which $f(x)$ has a maximum or minimum value, and determine which it is.

Answer: ____________________

To check your answer, go to **263**.

263

Let's solve the problem: $f(x) = e^{-x^2}$. Using the chain rule, we find

$$f' = -2xe^{-x^2}.$$

Maximum or minimum occurs at x given by

$$-2xe^{-x^2} = 0 \quad \text{or} \quad x = 0.$$

Now we use the product rule (frame **189**) to get

$$f'' = -2e^{-x^2} + 4x^2e^{-x^2} = (-2 + 4x^2)e^{-x^2}.$$

At $x = 0, f'' = (-2 + (4)(0))e^{-0} = -2$. Because f'' is negative where $f' = 0$, at $x = 0, f(x)$ has a maximum value there.

A word of caution—in evaluating a derivative, say f' at some value of x, $x = a$, you must always *first differentiate* $f(x)$ and *then* substitute $x = a$. If you reverse the procedure and first evaluate $f(a)$ and then try to differentiate it, the result will simply be 0 because $f(a)$ is a constant. Similar care must be taken with higher-order derivatives.

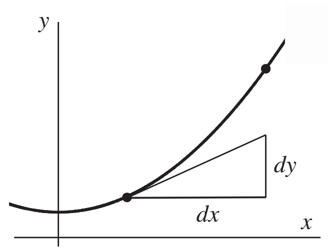
Go on to **the next section**.

2.11 Differentials

So far we have denoted the derivative by the symbol y' or dy/dx. Although either symbol stands for $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, the method of writing dy/dx suggests that the derivative might be regarded as the ratio of two quantities, dy and dx. This turns out to be the case. The new quantities that we now introduce are called *differentials*, which are defined in the next frame.

Go on to **264**.

264



Suppose that x is an independent variable, and that $y = f(x)$. Then the differential dx of x is defined as equal to any increment, $x_2 - x_1$, where x_1 is the point of interest. The differential dx can be positive or negative, large or small, as we please. We see that dx, like x, can be regarded as an independent variable.

The differential dy is defined by the following rule:

$$dy = y' \, dx,$$

where y' is the derivative of y with respect to x.

Go to **265**.

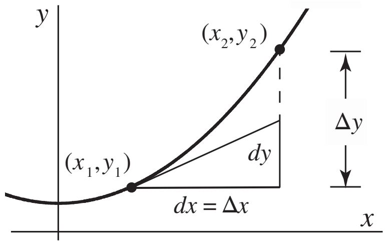
265

Although the meaning of the derivative y' is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, we can see from the preceding frame that it can now be interpreted as the ratio of the differentials dy and dx, where dx is any increment of x and dy is defined by the rule $dy = y' \, dx$.

Go to **266**.

266

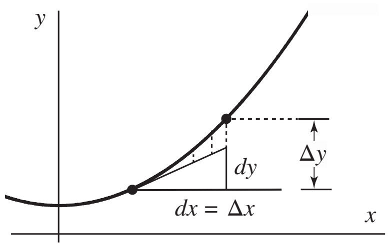
It is important not to confuse dy with Δy. As was pointed out in frame **136**, Δy stands for $y_2 - y_1 = f(x_2) - f(x_1)$ where x_2 and x_1 are two given values of x. Both dx and $\Delta x = x_2 - x_1$ are arbitrary intervals, dx is called a *differential* of x, and Δx is called an *increment* of x, but their meanings are similar here.



The diagram shows that dy and Δy are different quantities. Here we have set $dx = \Delta x$. The differential dy is then $dy = y'(x_1)dx$, where x_1 indicates that the derivative has been evaluated at the point x_1, while the increment Δy is given by $y_2 - y_1$. It is clear in this case that dy is not the same as Δy.

Go to **267**.

267



Although dy and Δy are different, you can see from the figure that for sufficiently small dx (with $dx = \Delta x$), dy is very close to Δy. We can write this symbolically as

$$\lim_{\Delta x \to 0} \frac{dy}{\Delta y} = 1.$$

Hence, if we intend to take the limit where $dx \to 0$, dy may be substituted for Δy. Furthermore, even if we don't take the limit, dy is almost the same as Δy, provided dx is sufficiently small. We, therefore, often use dy and Δy interchangeably when it is understood that the limit will be taken or that the result may be an approximation.

Go to frame **268**.

268

We can rewrite in differential form the various expressions for derivatives given earlier. Thus, if $y(x) = x^n$,

$$dy = d(x^n) = \frac{d}{dx}(x^n)dx = nx^{n-1} \ dx.$$

Find the following:

$$d(\sin x) = [-\sin x \ dx \mid -\sin x \mid -\cos x \ dx \mid \cos x \ dx]$$

$$d\left(\frac{1}{x}\right) = \left[\frac{dx}{x^2} \middle| -\frac{dx}{x^2} \middle| -\frac{dx}{x}\right]$$

$$d(e^x) = \left[xe^x dx \middle| dx \middle| e^x dx \middle| \frac{dx}{e^x}\right]$$

If you missed any of these go to **269**.
Otherwise, go to **270**.

269

Here are the solutions to the problems in frame **286**. The number of the frame in which each derivative is discussed is shown in parentheses.

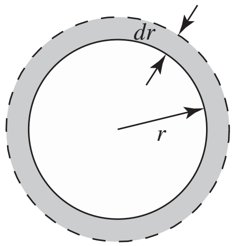
$$d(\sin x) = \left(\frac{d(\sin x)}{dx}\right) dx = \cos x \ dx \qquad \text{(frame } \mathbf{211}\text{)},$$

$$d\left(\frac{1}{x}\right) = \left[\frac{d}{dx}\left(\frac{1}{x}\right)\right] dx = -\frac{dx}{x^2} \qquad \text{(frame } \mathbf{180}\text{)},$$

$$de^x = \left(\frac{de^x}{dx}\right) dx = e^x dx \qquad \text{(frame } \mathbf{235}\text{)}.$$

Go to **270**.

270



Here is an example of the use of a differential. The diagram shows the surface of a disc to which a thin rim has been added. Suppose we want an approximate value for the change area ΔA which occurs when the radius is increased from r to $r + dr$.

$$dA = \left(\frac{dA}{dr}\right) dr = \frac{d}{dr}(\pi r^2) dr = 2\pi r\, dr.$$

Go to **271**.

271

The previous example can also be solved by taking the difference of the two areas:

$$\Delta A = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r\, \Delta r + \pi\, \Delta r^2.$$

When Δr is small compared with r, we can neglect the last term and we see that

$$\Delta A \approx 2\pi r\, \Delta r.$$

If we let $\Delta r = dr$ and assume that they are both small, then, as we know from frame **270**,

$$dA = \Delta A = 2\pi r\, dr.$$

Here is a more intuitive argument for the results. Because the rim is thin, its area dA is the approximate length, $2\pi r$, multiplied by its width, dr. Hence, $dA = 2\pi r\, dr$.

Go to **272**.

Answer: Frame 268: $\cos x dx$, $-dx/x^2$, $e^x dx$

272

Differentials are handy for remembering some important rules for differentiation. For instance, the chain rule

$$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx}$$

is almost an identity if we treat *dw*, *du*, and *dx* as differentials. Actually, it is not obvious that we can do so, because *w* and *u* both depend on a third quantity, *x*. Justification for using differentials to obtain the chain rule is given in Appendix **A10**.

Go to **273**.

273

Here is another relation, which is easy to remember with differentials, though the actual proof demands further explanation:

$$\boxed{\frac{dx}{dy} = \frac{1}{dy/dx}.}$$

This handy rule lets us reverse the role of dependent and independent variables, though it holds true only under certain conditions. If you want a further explanation, see Appendix **A4**.

2.12 A Short Review and Some Problems

Let's end the chapter by reviewing some of the ideas it introduced and then putting differential calculus to work.

Go to **274**.

274

Recall that the rate of change of position of a moving point with respect to time is called velocity.

(continued)

In other words, if position is related to time by a function $S(t)$, to find the velocity, we __________ $S(t)$ with respect to __________.

Go to **275**.

275

The answer is:

In other words, if the position and time are related by a function $S(t)$, in order to find the velocity, we *differentiate* $S(t)$ with respect to *time* (or t),

$$\frac{d}{dt}S(t) = v(t).$$

Go to **276**.

276

Try this problem.

The position of a particle along a straight line is given by the following expression,

$$S(t) = A\sin(\omega t),$$

where A and ω (omega) are constants.

Find the velocity of the particle.

$$v(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

For the answer, go to **277**.

277

The answer is

$$v(t) = \omega A\cos(\omega t).$$

The problem is to find the velocity, which is the rate of change of position with respect to time. In this problem, the position is $S(t) = A\sin(\omega t)$.

$$v(t) = \frac{dS}{dt} = \frac{d}{dt}A\sin(\omega t) = \omega A\cos(\omega t).$$

(If you are not sure of the procedure here, see frame **219**.)

Can you do this problem? The position of a point is given by

$$S(t) = A\sin(\omega t) + B\cos(2\omega t).$$

Find $v(t)$.

$$v(t) = \_\_\_\_\_\_\_\_\_\_\_\_.$$

See frame **278** for the answer.

278

$$v(t) = \frac{d}{dt}(A \sin(\omega t) + B\cos(2\omega t))$$
$$= \omega A \cos(\omega t) - 2\omega B\sin(2\omega t).$$

If you wrote this, go to frame **280**. If not, review frame **220** and then continue here. Try this problem: The position of a point is given by

$$S(t) = A \sin(\omega t)\cos(\omega t).$$

Find its velocity.

$$v(t) = \_\_\_\_\_\_\_\_\_\_.$$

Go to **279** for the answer.

279

The solution to problem **278** is:

$$\begin{aligned} v(t) = \frac{dS}{dt} &= \frac{d}{dt}(A \sin(\omega t)\cos(\omega t)) \\ &= A\sin(\omega t)\left(\frac{d}{dt}\cos(\omega t)\right) + \left(\frac{d}{dt}A\sin(\omega t)\right)\cos(\omega t) \\ &= -\omega A\sin^2(\omega t) + \omega A\ \cos^2(\omega t) \\ &= \omega A(\cos^2(\omega t) - \sin^2(\omega t)). \end{aligned}$$

As an alternative approach note that

$$\sin(\omega t)\cos(\omega t) = (1/2)\sin(2\omega t).$$

(See frame **71**.) Then, $v(t) = \frac{d}{dt}\left(\frac{A}{2}\sin(2\omega t)\right)$. If you feel energetic, show that this procedure yields the same result as above.

Go to **280**.

280

Suppose the height of a ball above the ground is given by $y(t) = a + bt + ct^2$ where a, b, c, are constants and $t \geq 0$. (Here we are using y rather than S to denote position. It makes no difference what we call our variable.) If c is negative, this type of equation describes the height of a freely falling body.

Find the velocity in the y direction.

$$v(t) = \_\_\_\_\_\_\_\_\_\_\_\_.$$

See **281** for the correct answer.

281

Here is the solution to the problem in frame **280**.

$$v(t) = \frac{dy}{dt} = \frac{d}{dt}(a + bt + ct^2) = b + 2ct.$$

If you found the correct answer, go to **283**. Otherwise, do the problem below. Let

$$S(t) = \frac{a}{(t+c)^2} + bt \qquad (a, b, \text{ and } c \text{ are constants, and } t \geq 0).$$

Find the velocity.

$$v(t) = \_\_\_\_\_\_\_\_\_\_\_\_$$

The answer is in frame **282**.

282

The answer is

$$v(t) = \frac{dS}{dt} = \frac{d}{dt}\left(\frac{a}{(t+c)^2} + bt\right) = -\frac{2a}{(t+c)^3} + b.$$

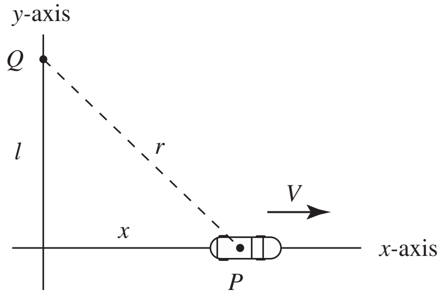
If this problem gave you any difficulty, you should review the beginning of this section before going on.

Otherwise, go to **283**.

283

Speed of a Car as Seen by a Stationary Observer:

Here is a more difficult problem.



A car P moves along a road in the x direction with a constant velocity V. The problem is to find how fast it is moving away from a man standing at point Q, distance l away from the road, as shown. In other words, if r is the distance between Q and P, find dr/dt. *Hint:* The chain rule is very useful here in the form $\frac{dr}{dt} = \frac{dr}{dx}\frac{dx}{dt}$.

$$\frac{dr}{dt} = \_\_\_\_\_\_\_\_\_\_.$$

Go to **284** for the solution.

284

From the diagram in frame **283** you can see that

$$r^2 = x^2 + l^2, \quad r = (x^2 + l^2)^{1/2}.$$

We can find dr/dt by the following procedure:

$$\begin{aligned}\frac{dr}{dt} = \frac{dr}{dx}\frac{dx}{dt} &= \frac{d}{dx}(x^2 + l^2)^{1/2}\frac{dx}{dt} \\ &= \frac{1}{2}\frac{2x}{(x^2 + l^2)^{1/2}}\frac{dx}{dt} \\ &= V\frac{x}{(x^2 + l^2)^{1/2}}.\end{aligned}$$

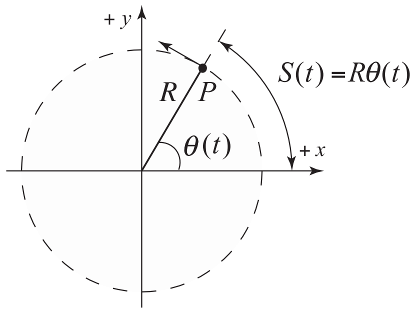
In the last step we used $V = dx/dt$.

Go to **285**.

285

Circular Motion:

So far we have been considering the velocity and acceleration for straight-line motion. Let's now consider a particle moving in a circle of radius R with constant speed v. At the instant shown in the diagram (time t), a line from the center of the circle to the particle makes an angle $\theta(t)$ with the x-axis.



The arc length subtended by the particle is given by the following expression:

$$S(t) = R\theta(t).$$

Find the speed of the particle.

$$v = \underline{\qquad\qquad}$$

Go to **286**.

286

Here is the solution to the problem in frame **285**.

$$v = \frac{d}{dt}S(t) = \frac{d}{dt}(R\theta(t)) = R\frac{d}{dt}\theta(t).$$

Go to **287**.

287

Tangential Acceleration in Circular Motion:

Suppose as the particle moves around the circle, it speeds up and thus the second derivative of the angle $\frac{d^2\theta}{dt^2}$ is non-zero. In this case, what is the change in magnitude of the speed? (This is called the *tangential acceleration* $a_{\tan}$.)

$$a_{\tan} = \underline{\qquad\qquad}$$

Go to **288**.

288

Here is how to do the problem in frame **287**.
The speed is given by

$$v = \frac{d}{dt}S(t) = R\frac{d}{dt}\theta(t).$$

The tangential acceleration is the rate of change of speed and is given by

$$a_{\tan} = \frac{dv}{dt} = \frac{d}{dt}\left(R\frac{d\theta(t)}{dt}\right) = R\frac{d^2\theta}{dt^2}.$$

Note that an object moving in a circle always has a radial inward acceleration called *centripetal acceleration* due to the fact that the direction of the velocity is changing. The magnitude of this acceleration is given by

$$a_{\text{rad}} = \frac{v^2}{R}.$$

For a derivation of this result see Appendix **B4**.

Go to **289**.

289

When to Sell a Car:

In deciding whether or not to keep an old automobile, an important consideration is the estimated cost per year of owning the car. The two major components of the cost are repairs and depreciation. We shall assume that the annual repairs cost $r(t)$, in dollars per year, is given by

$$r(t) = A + Bt,$$

(continued)

where A and B are constants. The repairs are lowest when the car is new, and we will assume that they increase linearly with time. The loss in value of the car in dollars per year—the *rate of depreciation*—is taken to be

$$d(t) = De^{-Ht},$$

where D and H are constants. The depreciation rate is highest when the car is new and most valuable; we will assume that it decreases exponentially in time, growing at a smaller rate as the car becomes less valuable.

The annual cost for repairs and depreciation is $c(t) = r(t) + d(t)$. Find an expression for the time t at which the cost is a minimum.

$t =$ ______________

Go to **290**.

290

The cost is

$$c(t) = r(t) + d(t) = A + Bt + De^{-Ht}.$$

An extremum occurs when

$$\frac{dc}{dt} = B - HDe^{-Ht} = 0.$$

This can be solved for t:

$$HDe^{-Ht} = B, \quad e^{-ht} = \frac{B}{HD},$$

$$-Ht = \ln \frac{B}{HD},$$

$$t = \frac{1}{H} \ln \frac{HD}{B}.$$

To determine whether the cost is a minimum or a maximum, we must examine d^2c/dt^2. (Recall from frame **261** that the second derivative is positive at a minimum.)

$$\frac{d^2c}{dt^2} = H^2De^{-Ht}.$$

This is always positive, so the extremum is a minimum. Note, however, that if $HD/B < 1$, then $\ln(HD/B) < 0$, and t is negative. What does this mean?

Consider dc/dt at $t = 0$.

$$\left.\frac{dc}{dt}\right|_{t=0} = B - HDe^{-H0} = B - HD.$$

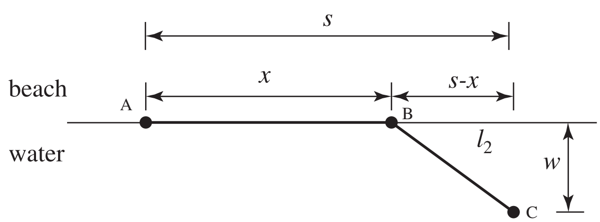
If $B < HD$, the slope is negative, and the cost, $c(t)$, initially decreases. It has a minimum at some later time just before it starts to increase. However, if $B > HD$, then $c(t)$ initially increases and keeps on increasing. This is the case for which the minimum in $c(t)$ occurs at a negative time. This solution has no meaning; you can't sell a car before you have bought it!

Go to **291**.

291

Path for the Shortest Time:

You are strolling along a straight section of beach when (at the point A in the figure) you spot a swimmer floundering offshore at the point C (see figure), a distance w from the shoreline and a distance s along the shore from where you stand. You can sprint at a speed of v_1 and swim at a slower speed of v_2. At point B (see figure), a distance x from where you stand, you dive into the water to rescue the person.



(continued)

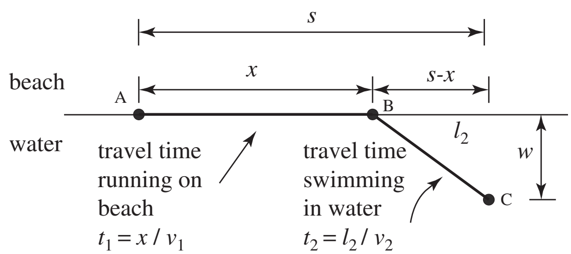
The problem is to determine the distance x you should run before entering the water in order to reach the swimmer in the shortest time.

$$x = \underline{\hspace{4cm}}$$

To check your solution, go to **292**.

292

You need to find the path that takes the shortest time to reach the swimmer.



It takes a time $t_1 = x/v_1$ to run distance x along the beach and time $t_2 = l_2/v_2$ to swim distance l_2. The total time to reach the swimmer is then

$$t = t_1 + t_2 = x/v_1 + l_2/v_2.$$

From the distances shown in the figure, you can see that

$$l_2 = ((s - x)^2 + w^2)^{1/2}.$$

The total time to reach the swimmer is then

$$t = t_1 + t_2 = x/v_1 + ((s - x)^2 + w^2)^{1/2}/v_2.$$

The minimal time occurs when

$$\frac{dt}{dx} = \frac{1}{v_1} - \frac{s - x}{v_2((s - x)^2 + w^2)^{1/2}} = 0,$$

which you can solve to find the distance x,

$$x = s - \frac{v_2 w}{({v_1}^2 - {v_2}^2)^{1/2}}.$$

The famous law for the refraction of light as it passes through different media (Snell's law), for example, through air and water, shows that the path light follows takes the shortest time.

Go to **293**.

293

Pricing a New Product:

You have just invented a new soccer helmet to better protect players from head injuries. You need to decide how many helmets you need to sell in order to maximize your total profit P. Your *revenue* r will be the product of the price p and the quantity q you sell,

$$r = pq.$$

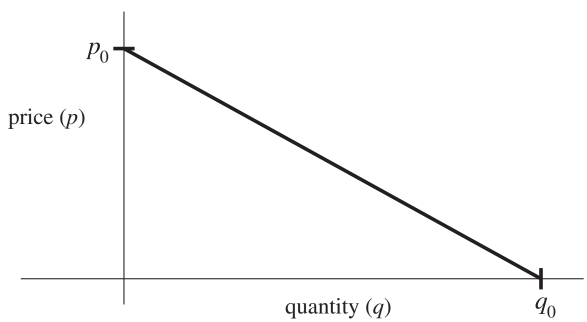
The number of helmets sold naturally depends on the price: as the price goes up, sales go down. The costs also depend on the quantity of helmets sold: the more helmets sold, the less the price per helmet but the more your total cost. Your *profit* will be the difference between your revenue and your costs

$$P = r - c.$$

If the price is low, you will sell many helmets, but the income may not be very large. If the price is high, you will sell very few. Somewhere between the extremes your profit will be greatest. For purposes of planning, you assume that the price of helmets sold depends on the number sold:

$$p = p_0[1 - (q/q_0)],$$

where p_0 is the maximum price that you can charge per helmet (above that price you will not sell any helmets), and q_0 is the maximum amount you can sell if you lower the price to zero.



(continued)

Suppose the cost depends on quantity sold according to

$$c = f + gq + hq^2,$$

where the constant f represents fixed costs, and g and h are positive constants.

The problem is to find the quantity q_m that you should sell in order to maximize the profit P.

$$q_m = \_\_\_\_\_\_\_\_\_\_\_\_$$

To check your solution, go to **294**.

294

Your revenue as a function of quantity sold is

$$r = p_0[1 - (q/q_0)]q.$$

Your profit is

$$P = r - c = p_0[1 - (q/q_0)]q - (f + gq + hq^2).$$

The maximum value of profit occurs $dP/dq = 0$,

$$0 = \left.\frac{dP}{dq}\right|_{q=q_m} = p_0 - 2p_0 q_m/q_0 - g - 2hq_m.$$

The quantity that maximizes profit is then

$$q_m = \frac{(p_0 - g)q_0}{2(p_0 + hq_0)}.$$

Go to **295**.

295

Growth of Single Cells:

A simple model for the growth of cells depends on the amount of nutrients in the cell. Nutrients are *absorbed* through the surface of the cell and then diffuse throughout the volume where they are *consumed* by processes in the cell.

If the cell radius is too small, there are not enough nutrients for the cell to grow. As the radius of the cell increases, the amount of materials that flow through the cell's surface is no longer able to sustain the cell's activities, and the growth rate of the cell slows.

For simplicity, we model the cell as a sphere with radius r, surface area $A(r) = 4\pi r^2$, and volume $V(r) = (4/3)\pi r^3$. Nutrients are absorbed on the surface of the cell so that the absorption rate is proportional to the area of the cell, $B(r) = a_1 4\pi r^2$. The nutrients are also consumed by directly reacting with material in the cell, and so the consumption rate of a cell is proportional to its volume, $C(r) = a_2(4/3)\pi r^3$. The proportionality constants a_1 and a_2 are positive and are known from measurements. The rate of increase of nutrients in the cell is then

$$G(r) = B(r) - C(r) = a_1 4\pi r^2 - a_2(4/3)\pi r^3,$$

The problem is to find (a) the radius r_m for which the rate of increase of nutrients in the cell is maximal, and (b) the maximum rate of increase of nutrients in the cell, $G(r_m)$.

$(a) \qquad r_m =$ ______________

$(b) \qquad G(r_m) =$ ______________

To see a solution, go to **296**.

296

(a) In order to find the radius such that the rate of increase of nutrients is maximum, set the derivative of the increase in rate of nutrients equal to zero:

$$0 = \left.\frac{dG(r)}{dr}\right|_{r=r_\mathrm{m}} = 8a_1\pi r_\mathrm{m} - 4a_2\pi r_\mathrm{m}^2 \Rightarrow r_\mathrm{m} = \frac{2a_1}{a_2}.$$

To verify that this radius gives a maximal rate of increase of nutrients, take the second derivative

$$0 = \left.\frac{d^2G(r)}{dr^2}\right|_{r=r_\mathrm{m}} = 8a_1\pi - 8a_2\pi\frac{2a_1}{a_2} = -8a_1\pi < 0,$$

which indicates that $G(r)$ is a local maximum when $r_m = 2a_1/a_2$.

(b) The maximal rate of increase of nutrients is then

$$G(r = r_\mathrm{m}) = a_1 4\pi\left(\frac{2a_1}{a_2}\right)^2 - a_2\frac{4}{3}\pi\left(\frac{2a_1}{a_2}\right)^3 = \frac{16\pi}{3}\frac{a_1{}^3}{a_2{}^2}.$$

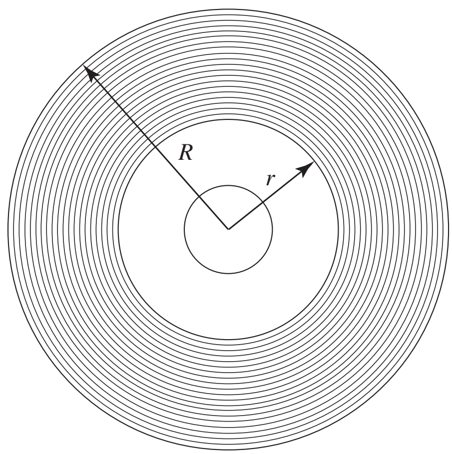
Go to **297**.

297

Maximize Storage on a Compact Disc:

Consider a compact disc with circular tracks that contain data. The innermost track is at a distance r from the center of the disc and the outermost track is a fixed distance R from the center of the disc. Each track contains the same number of bytes. The number of bytes per unit length on each track is a constant λ_1. The number of tracks per unit radial length is a constant λ_2.

In order to maximize the number of bytes on the disc, (a) what is the radius of the innermost track? (b) what is the maximum number of bytes that can be stored on this disc?



(*a*) r_{inner} = ______________

(*b*) $N_{\max}$ = ______________

To see a solution, go to **298**.

298

The number of bytes on each circular track is the maximum density λ_1 multiplied by the circumference of the innermost track, $\lambda_1 2\pi r$. The number of tracks is $\lambda_2(R-r)$. The total number of bytes is then

$$N = \lambda_2(R-r)\lambda_1 2\pi r.$$

To find an extremum, set

$$0 = \frac{dN}{dr} = \lambda_2 \lambda_1 2\pi (R - 2r).$$

Thus the innermost radius is

$$r_{\text{inner}} = R/2.$$

To check that this is a maximum take the second derivative

$$\frac{d^2N}{dr^2} = -4\pi \lambda_2 \lambda_1 < 0,$$

which is negative.
(b) The maximum number of stored bytes is

$$N_{\text{max}} = \frac{\pi \lambda_1 \lambda_2 R}{2}.$$

Go to **299**.

299

Compounded Interest:

Suppose that you have a sum of money p_0 to invest, called your *initial principal*. An institution guarantees a fixed yearly interest rate r that is compounded n times a year. This means that for each time interval your interest rate is r/n.

Find an expression for your principal balance p_j after j intervals.

$$p_j = \underline{\hspace{3cm}}$$

To see a solution, go to **300**.

300

After one interval, the value of the principal p_1 is

$$p_1 = p_0 + \frac{r}{n} p_0 = p_0 \left(1 + \frac{r}{n}\right).$$

After two intervals, the value is now

$$p_2 = p_1 \left(1 + \frac{r}{n}\right) = p_0 \left(1 + \frac{r}{n}\right)^2.$$

(continued)

Thus after j intervals, the value is

$$p_j = p_0\left(1 + \frac{r}{n}\right)^j.$$

Go to **301**.

301

If $r/n << 1$, after how many intervals j will the principal double for a given rate r? *Hint*: the approximation that $\ln(1 + r/n) \simeq r/n$, for $r/n << 1$, will be helpful. (We will prove this approximation in frame **414**.)

$$j = \underline{\hspace{6em}}$$

To see a solution go to **302**.

302

To determine the number of intervals for your principal to double, set $P(j) = 2p_0$ and solve for j.

$$p_j = 2p_0 = p_0\left(1 + \frac{r}{n}\right)^j \Rightarrow 2 = (1 + r/n)^j.$$

To solve for j, take natural logarithms of each side,

$$\ln 2 = j\ln(1 + r/n) \Rightarrow j = \frac{\ln 2}{\ln(1 + r/n)}.$$

If the interest is compounded monthly, then $n = 12$. For the case that $r/n << 1$, we use the approximation $\ln(1 + r/n) \simeq r/n$, and the principal will double after $j = (12/r)\ln 2$ intervals, or in terms of years,

$$y \simeq j/12 = \ln 2/r = 0.69/r.$$

To find the effect of compounding interest more frequently, we introduce a discrete time variable $t = j/n$ and then take the limit when $n \to \infty$. Set $x = n/r$. Then $j = nt = xrt$.

$$p(t) = p_0\left(1 + \frac{1}{x}\right)^{xrt}.$$

Now let $n \to \infty$, or equivalently $x \to \infty$. Then the value at time t is

$$p(t) = \lim_{x\to\infty} p_0 \left(1 + \frac{1}{x}\right)^{xrt} = p_0 e^{rt},$$

where we used the representation in Appendix **A3** that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x} = e^{x}.$$

Go to **303**.

303

Puzzle Problems:

These are tough problems, so don't feel bad if the solutions elude you. If you need a hint, see frame **234**.

(a) If $y = x^x$, what is $\frac{dy}{dx}$?

(b) If $y = \left(\frac{x+1}{x-1}\right)^{1/3}$, for $x > 1$, what is $\frac{dy}{dx}$?

To see a solution, go to **304**.

304

(a) To find the derivative of $y = f(x) = x^x$, consider $\ln f(x) = x \ln x$. We can differentiate this with respect to x using the result in frame **234**,

$$\frac{d}{dx}x^x = \frac{df}{dx} = \left(\frac{d}{dx}\ln f\right) y.$$

Because $\frac{d}{dx}\ln f = \frac{d}{dx}(x \ln x) = \ln x + 1$,

$$\frac{d}{dx}x^x = (\ln x + 1)x^x.$$

(b) The second problem can also be solved using the same technique. Set

$$y = f(x) = \left(\frac{x+1}{x-1}\right)^{1/3}.$$

Then

$$\ln f(x) = \ln \left(\frac{x+1}{x-1}\right)^{1/3} = \frac{1}{3}(\ln(x+1) - \ln(x-1)).$$

(continued)

Thus

$$\frac{d}{dx}\ln f = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) = -\frac{2}{3}\left(\frac{1}{(x+1)(x-1)}\right).$$

Therefore using the result $\frac{dy}{dx} = \left(\frac{d}{dx}\ln f\right) y$ yields

$$\frac{d}{dx}\left(\frac{x+1}{x-1}\right)^{1/3} = \left(\frac{d}{dx}\ln f\right) y = -\frac{2}{3}\left(\frac{1}{(x+1)(x-1)}\right)\left(\frac{x+1}{x-1}\right)^{1/3}$$

$$= -\frac{2}{3}\frac{1}{((x+1)^2(x-1)^4)^{1/3}} = -\frac{2}{3}((x^2-1)(x-1))^{-2/3}.$$

Go to **305**.

Conclusion to Chapter 2

305

The Appendixes contain additional material that may be helpful. For instance, sometimes one has an equation which relates two variables, y and x, but which cannot be written simply in the form $y = f(x)$. There is a straightforward method for evaluating y': it is called *implicit differentiation* and it is described in Appendix **B1**. Appendix **B2** shows how to differentiate the inverse trigonometric functions. In this chapter we have only discussed differentiation of functions of a single variable. The technique is not difficult to extend to functions of several variables. The technique for doing this is known as *partial differentiation*. This is explained in Appendix **B3**. In Appendix **B4**, you will find a derivation of the magnitude of centripetal acceleration for circular motion.

The important results of this chapter are summarized in the next section, which provides a quick review. In addition, a list of important derivatives is presented in Table 1 at the back of the book. If you want more practice, see the review problems on pages 278–281.

Summary of Chapter 2

2.1 Limit of a Function (frames 97–115)

Definition of a Limit:

Let $f(x)$ be defined for all x in an interval about $x = a$, but not necessarily at $x = a$. If there is a number L such that to each positive number ε there corresponds a positive number δ such that

$$|f(x) - L| < \varepsilon \text{ provided } 0 < |x - a| < \delta,$$

we say that L is the *limit* of $f(x)$ as x approaches a and write

$$\lim_{x \to a} f(x) = L.$$

The ordinary algebraic manipulations can be performed with limits as shown in Appendix **A2**; thus,

$$\lim_{x \to a} [F(x) + G(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} G(x).$$

Two trigonometric limits are of particular interest (Appendix **A3**):

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \text{ and } \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$$

The following limit is of such great interest in calculus that it is given the special name e, as discussed in frame **109** and Appendix **A3**:

$$e = \lim_{x \to 0} (1 + x)^{1/x} = 2.71828 \ldots$$

2.2 Velocity (frames 116–145)

If the function $S(t)$ represents the distance from a fixed location of a point moving at a varying speed along a straight line, the *average velocity* $\overline{v}$ between times t_1 and t_2 is given by

$$\overline{v} = \frac{S_2 - S_1}{t_2 - t_1},$$

whereas the *instantaneous velocity* v (frame **133**) at time t_1 is

$$v = \lim_{t_2 \to t_1} \frac{S_2 - S_1}{t_2 - t_1}.$$

In a plot of $S(t)$ vs. t, the instantaneous velocity is the slope of the curve at time t (frame **131**). It is often convenient to write $S_2 - S_1 = \Delta S$ and $t_2 - t_1 = \Delta t$, so

$$v = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}.$$

2.3 Derivatives (frames 146–159)

If $y = f(x)$, the instantaneous rate of change of y with respect to x is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. The expression $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is called the *derivative* of y with respect to x. It is often written as $\frac{dy}{dx}$ (but sometimes as y'). Thus

$$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{y_2 - y_1}{x_2 - x_1} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the derivative of y with respect to x. The derivative $\frac{dy}{dx}$ is equal to the slope of the curve of y plotted against x.

2.4 Graphs of Functions and Their Derivatives (frames 160–169)

From a graph of a function we can obtain the slope of the curve at different points, and by sketching a new curve of the slopes we can determine the general character and qualitative behavior of the derivative. See frames **160–169** for examples.

2.5–2.8 Differentiation (frames 170–241)

From the definition of the derivative, a number of formulas for differentiation can be derived. We will review just one example here: the method is typical. Let u and v be variables that depend on x.

$$\frac{d(uv)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(u + \Delta u)(v + \Delta v) - uv}{\Delta x}$$

$$\begin{aligned}
\frac{d(uv)}{dx} &= \lim_{\Delta x \to 0} \frac{uv + u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v - uv}{\Delta x} \\
&= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u\,\Delta v}{\Delta x} \\
&= u\frac{dv}{dx} + v\frac{du}{dx} + 0.
\end{aligned}$$

It is useful to be familiar with the relations listed here. There is a more complete list in Table 2.1, at the end of the text. In the following expressions u and v are variables that depend on x, w depends on u, which in turn depends on x, and a and n are constants. All angles are measured in radians.

	Frame
$\frac{da}{dx} = 0$	172
$\frac{d}{dx}(ax) = a$	174
$\frac{d}{dx}x^n = nx^{n-1}$	180
$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$	186
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	189
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)$	194
$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx}$	198
$\frac{d}{dx}\sin x = \cos x$	211
$\frac{d}{dx}\cos x = -\sin x$	212
$\frac{d}{dx}\ln x = \frac{1}{x}$	226
$\frac{d}{dx}e^x = e^x$	235

In the above list $e = 2.71828\ldots$, and $\ln x$ is the natural logarithm of x defined by $\ln x = \log_e x$.

More complicated functions can ordinarily be differentiated by applying several of the rules in Table 2.1 successively. Thus

$$\begin{aligned}\frac{d}{dx}(x^3 + 3x^2\sin 2x) &= \frac{d}{dx}x^3 + 3\left(\frac{d}{dx}x^2\right)\sin 2x + 3x^2\left(\frac{d}{dx}\sin 2x\right)\\ &= 3x^2 + 6x\sin 2x + 3x^2\frac{d\sin 2x}{d\,2x}\frac{d\,2x}{dx}\\ &= 3x^2 + 6x\sin 2x + 6x^2\cos 2x.\end{aligned}$$

2.9 Higher-Order Derivatives (frames 242–249)

If we differentiate df/dx with respect to x, the result is called the *second derivative* of f with respect to x. This is written $\frac{d^2f}{dx^2}$. Alternative symbols are $f^{(2)}$ and f'', where the "(2)" and " $''$ " superscripts indicate the second derivative. The variable x is suppressed. Likewise the nth derivative of f with respect to x is the result of differentiating f, n times successively, with respect to x and is written $\frac{d^nf}{dx^n}$ or as $f^{(n)}$.

2.10 Maxima and Minima (frames 250–263)

If $f(x)$ has a maximum or minimum value for some value of x in an interval, then its derivative $\frac{df}{dx}$ is zero for that x. If in addition $\frac{d^2f}{dx^2} < 0, f(x)$ has maximum value. If on the other hand $\frac{d^2f}{dx^2} > 0, f(x)$ has a minimum value there.

2.11 Differentials (frames 264–273)

If x is an independent variable and $y = f(x)$, the differential dx of x is defined as the increment, $x_2 - x_1$, where x_1 is the point of interest. The differential dx can be positive or negative, large or small, as we please. Then dx, like x, is an independent variable. The differential dy is then defined by the following rule: $dy = y'\ dx$ where y' is the derivative of y with respect to x. Although the meaning of the derivative, y', is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, we see that this can also be interpreted as the ratio of the differentials dy and dx. As discussed in frames **265** and **266**, dy is not the same as Δy, although

$$\lim_{dx=\Delta x \to 0} \frac{dy}{\Delta y} = 1.$$

$$\lim_{\Delta x \to 0} \Delta y = dy.$$

Differentiation formulas can easily be written in terms of differentials. Thus if $y = x^n$,

$$dy = d(x^n) = \left(\frac{d}{dx}x^n\right) dx = nx^{n-1}dx.$$

The relation, $\frac{dx}{dy} = \frac{1}{dy/dx}$, implied by the differential notation can be extremely useful. It's use is discussed in Appendix **A4**.

Ready for more? Take a deep breath and continue on to Chapter 3.

CHAPTER THREE

Integral Calculus

In this chapter you will learn about:

- Antiderivatives and the indefinite integral;
- Integrating a variety of functions;
- Some applications of integral calculus;
- Finding the area under a curve;
- Definite integrals with applications;
- Multiple integrals.

The previous chapter was devoted to the first major branch of calculus—differential calculus. This chapter is devoted to the second major branch—integral calculus. The two branches have different natures: differential calculus has procedures that make it possible to differentiate *any* continuous function; integral calculus has no such general procedures—every problem presents a fresh puzzle. Nevertheless, integral calculus is essential to all of the sciences, engineering, economics, and in fact to every discipline that deals with quantitative information.

There are two routes to introducing the concepts of integration. Although they start in different directions, they finally meld and create a single entity. If they were marked by road signs, the first would be "Antiderivatives and the indefinite integral" while the second would be "Area under a curve and the definite integral."

3.1 Antiderivative, Integration, and the Indefinite Integral

306

The Antiderivative:

The goal of this section is to learn some techniques for *integration*, sometimes called *antidifferentiation*.

In this section we generally designate a function by $f(x)$. The concept of the antiderivative is fundamental to the process of integration and is easily explained. When a function $F(x)$ is differentiated to give $f(x) = dF/dx$, then $F(x)$ is an *antiderivative* of $f(x)$, that is,

$$F'(x) = f(x).$$

This notation describes the defining property of the antiderivative, although only in terms of the derivative $F'(x)$, not $F(x)$ itself.

The antiderivative is usually written in the form:

$$F(x) = \int f(x)dx.$$

The expression $\int f(x)\,dx$ is also called the *integral* of $f(x)$. The symbol $\int$ is known as the *integration symbol*; it represents the inverse of differentiation.

To summarize the notation, if $F'(x) = f(x)$, then $F(x)$ is the ______________ or ___________________________ .

Go to **307**.

307

Indefinite Integral:

Often one can find the antiderivative simply by guesswork. For instance, if $f(x) = 1$, then $F(x) = \int f(x)\,dx = x$. To prove this, note that

$$F'(x) = \frac{d}{dx}(x) = 1 = f(x).$$

However, $f(x)$ is not the only antiderivative of $f(x) = 1$; $x + c$, where c is a constant, is also an antiderivative because

$$\frac{d}{dx}(x + c) = 1 + 0 = f(x).$$

In fact, a constant can always be added to a function without changing its derivative. If $y = F(x)$ is an antiderivative of $f(x)$, then all the antiderivatives of $f(x)$ can be denoted by writing

$$\int f(x)\,dx = F(x) + c,$$

where c is an arbitrary constant.

If it is useful, we can write the defining equation for the antiderivative in terms of a differential: $dF = f(x)dx$. (If you need to review differentials, see frame **264**.) Then we can describe all the antiderivatives by the notation

$$\int dF(x) = F(x) + c.$$

Because of the arbitrary constant, the definition is imprecise. For this reason this integral is called the *indefinite integral*.

In summary, the integral of the differential of a function is equal to the function plus a constant.

Go to **308**.

308

Because this first meaning of integration is the inverse of differentiation, for every differentiation formula in Chapter 2, there is a corresponding integration formula here. Thus from Chapter 2, frame **211**,

$$\frac{d}{dx}\sin x = \cos x,$$

so by the definition of the indefinite integral,

$$\int \cos x\,dx = \sin x + c.$$

Now you try one. What is $\int \sin x\,dx$? Choose the answer: [$\cos x + c$ | $-\cos x + c$ | $\sin x \cos x + c$ | none of these]

Make sure you understand the correct answer (you can check the result by differentiation).

Go to **309**.

309

Now try to find these integrals:

(a) $\int x^n\, dx = \left[\frac{1}{n}x^n + c \,\middle|\, \frac{1}{n}x^{n+1} + c \,\middle|\, \frac{1}{n+1}x^{n+1} + c \,\middle|\, \frac{1}{n-1}x^n + c\right]$, $n \neq 1$.

(b) $\int e^x\, dx = \left[\, e^x + c \,\middle|\, xe^x + c \,\middle|\, \frac{1}{x}e^x + c \,\middle|\, \text{none of these}\right]$

If you did both of these correctly, skip to frame **311**.
If not, go to frame **310**.

310

If you had difficulty with these problems, recall the definition of the indefinite integral. If $F = \int f(x)dx$, then $\frac{d}{dx}F(x) = f(x)$.

In order to find F, we must find an expression that when differentiated yields the given function $f(x)$. For instance, the derivative of $\frac{1}{n+1}x^{n+1}$ is

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\frac{dx^{n+1}}{dx} = \frac{1}{n+1}(n+1)\, x^n = x^n$$

by the formula for differentiating x^n in frame **180**. Hence, $\frac{1}{n+1}x^{n+1} + c$ is an integral of x^n. But the integral is indefinite because we could add any constant to it without changing its defining property: $\frac{d}{dx}(F + c) = \frac{dF}{dx} = f(x)$. Thus, including the integration constant c, we find $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$. (Note that this formula does not work for $n = -1$. That case will be discussed later.)

Likewise, by frame **235**,

$$\frac{d}{dx}e^x = e^x.$$

But the integral is indefinite because we could add any constant. Thus

$$\int e^x dx = e^x + c.$$

Go to **311**.

Answers: Frame 306: antiderivative, or $F(x) = \int f(x)\, dx$
Frame 308: $-\cos x + c$

311

A table with some common integrals is given in the next frame. You can check the truth of any of the equations $\int f(x)\,dx = F(x)$ by confirming that $\frac{d}{dx}F(x) = f(x)$. We will shortly use this method to verify some of the equations.

Go to **312**.

312

Table of Integrals:

The following *integral table* is a list of antiderivatives of some common functions. The integrals occur in many applications and are worth getting to know. For simplicity, in the integration table the arbitrary integration constants are omitted; *a* and *n* are constants.

Table of Integrals

1. $\int a\,dx = ax$
2. $\int af(x)\,dx = a\int f(x)\,dx$
3. $\int (u+v)\,dx = \int u\,dx + \int v\,dx$
4. $\int x^n\,dx, = \dfrac{x^{n+1}}{n+1}\quad n \neq -1$
5. $\int \dfrac{dx}{x} = \ln|x|$
6. $\int \dfrac{dx}{a+bx} = \dfrac{1}{b}\ln|a+bx|$
7. $\int e^x\,dx = e^x$
8. $\int e^{ax}dx = \dfrac{e^{ax}}{a}$
9. $\int b^{ax}\,dx = \dfrac{b^{ax}}{a\ln b}$
10. $\int \ln|x|\;dx = x\ln|x| - x$
11. $\int \sin x\,dx = -\cos x$
12. $\int \cos x\,dx = \sin x$
13. $\int \tan x\,dx = -\ln|\cos x|$
14. $\int \cot x\,dx = \ln|\sin x|$
15. $\int \sec x\,dx = \ln|\sec x + \tan x|$
16. $\int \sin x\cos x\,dx = \dfrac{1}{2}\sin^2 x$
17. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$
18. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a}$
19. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x - \sqrt{x^2 \pm a^2}|$
20. $\int w(u)\,dx = \int w(u)\dfrac{dx}{du}\,du$
21. $\int u\,dv = uv - \int v\,du$

For convenience this table is repeated as Table 2: Integrals on page 288.

Go to **313**.

313

Let's see if you can check some of the formulas in the table. Show that integral formulas 10 and 16 are correct.

If you have proved the formulas to your satisfaction, go to **315**.
If you want to see proofs of the formulas, go to **314**.

314

To prove that $F(x) = \int f(x)\, dx$, we must show that $\frac{d}{dx}F(x) = f(x)$.

10. $F(x) = x \ln x - x, f(x) = \ln x.$

$$\frac{dF}{dx} = \frac{d}{dx}(x \ln x - x) = x\left(\frac{1}{x}\right) + \ln x - 1 = \ln x = f.$$

16. $F(x) = \frac{1}{2}\sin^2 x, f(x) = \sin x \cos x.$

$$\frac{d}{dx}\left(\frac{1}{2}\sin^2 x\right) = \frac{1}{2}(2 \sin x)\left(\frac{d}{dx} \sin x\right) = \sin x \cos x.$$

Go to **315**.

3.2 Some Techniques of Integration

315

Change of Variable:

Often an unfamiliar function can be converted into a familiar function having a known integral by using a technique called *change of variable*. The method applies to integrating a "function of a function." (Differentiation of such a function was discussed in frame **198**. It is differentiated using the chain rule.) For example, e^{-x^2} can be written e^{-u}, where $u = x^2$. With the following rule, the integral with respect to the variable x can be converted into another integral, often simpler, depending on the variable u.

$$\boxed{\int w(x)\, dx = \int \left[w(u)\frac{dx}{du}\right]\, du.}$$

Let's see how this works by applying it to a few problems.

Go to **316**.

316

Consider the problem of evaluating the integral

$$\int xe^{-x^2}\, dx.$$

Let $u = x^2$, or $x = \sqrt{u}$, and $w(u) = \sqrt{u}e^{-u}$. Hence $\frac{dx}{du} = \frac{1}{2\sqrt{u}}$. Using the rule for change of variable, $\int w(x)\, dx = \int \left[w(u)\frac{dx}{du}\right]\, du$, the integral becomes

$$\int xe^{-u}\frac{1}{2x}du = \frac{1}{2}\int e^{-u}\, du = -\frac{1}{2}e^{-u} + c = -\frac{1}{2}e^{-x^2} + c.$$

To prove that this result is correct, note that

$$\frac{d}{dx}\left(-\frac{1}{2}e^{-x^2} + c\right) = xe^{-x^2},$$

as required.

Try the following somewhat tricky problem. If you need a hint, see frame **317**. Evaluate

$$\int \sin\theta\cos\theta\, d\theta = \underline{\qquad\qquad\qquad\qquad}$$

To check your answer, go to **317**.

317

Let $u = \sin\theta$. Then $\frac{du}{d\theta} = \cos\theta$, and by the rule for change of variable,

$$\int w(x)\, dx = \int \left[w(u)\frac{dx}{du}\right]\, du.$$

The integral becomes

$$\int \sin\theta\cos\theta\, d\theta = \int u\cos\theta\frac{1}{\cos\theta}\, du = \int u\, du = \frac{1}{2}u^2 + c = \frac{1}{2}\sin^2\theta + c.$$

Go to **318**.

318

Here is an example of a simple change of variable. The problem is to calculate $\int \sin 3x\, dx$. If we let $u = 3x$, then the integral is $\sin u$, which is easy to integrate. Using $dx = du/3$, we have

$$\int \sin 3x\, dx = \frac{1}{3}\int \sin u\, du = \frac{1}{3}(-\cos\ u + c) = \frac{1}{3}(-\cos 3x + c).$$

To see whether you have caught on, evaluate $\int \sin\frac{x}{2}\cos\frac{x}{2}\, dx$. (You may find the integral table in frame **312** helpful.)

$$\int \sin\frac{x}{2}\cos\frac{x}{2}\, dx = \underline{\hspace{6cm}}$$

To check your answer, go to **319**.

319

Here is the answer:

$$\int \sin\frac{x}{2}\cos\frac{x}{2} dx = \sin^2\frac{x}{2} + c.$$

If you obtained this result, go right on to **320**. Otherwise, continue. If we let $u = x/2$, then $dx = 2du$ and

$$\int \sin\frac{x}{2}\cos\frac{x}{2} dx = 2\int \sin u \cos u\, du.$$

From formula 16 of frame **312** we have

$$\int \sin u\ \cos u\, du = \frac{1}{2}\sin^2 u + c = \frac{1}{2}\sin^2\frac{x}{2} + c.$$

Hence

$$\int \sin\frac{x}{2}\cos\frac{x}{2}\, dx = 2\left(\frac{1}{2}\sin^2\frac{x}{2} + c\right) = \sin^2\frac{x}{2} + 2c.$$

Let's check this result:

$$\frac{d}{dx}\left(\sin^2\frac{x}{2} + 2c\right) = 2\left(\sin\frac{x}{2}\cos\frac{x}{2}\right)\left(\frac{1}{2}\right) = \sin\frac{x}{2}\cos\frac{x}{2}$$

as required. (We have used the chain rule here.)

Go to **320**.

320

Try to evaluate $\int \frac{dx}{a^2+b^2x^2}$, where a and b are constants. The integral table in frame **312** may be helpful.

$$\int \frac{dx}{a^2+b^2x^2} = \underline{\hspace{6cm}}.$$

Go to **321** for the solution.

321

If we let $u = bx$, then $dx = du/b$ and

$$\int \frac{dx}{a^2+b^2x^2} = \frac{1}{b}\int \frac{du}{a^2+u^2}$$

$$= \frac{1}{ab}\left(\tan^{-1}\frac{u}{a}\right) + c, \quad \text{(Frame } \mathbf{333}, \text{ formula 16)}$$

$$= \frac{1}{ab}\left(\tan^{-1}\frac{bx}{a}\right) + c.$$

Go to **322**.

322

We have seen how to evaluate an integral by changing the variable from x to $u = ax$, where a is some constant. Often it is possible to simplify an integral by substituting still other quantities for the variable.

Here is an example. Evaluate:

$$\int \frac{x\,dx}{x^2+4}.$$

Suppose we let $u^2 = x^2 + 4$. Then $2udu = 2xdx$, and

$$\int \frac{x\,dx}{x^2+4} = \int \frac{u\,du}{u^2} = \int \frac{du}{u} = \ln|\,u\,| + c = \ln\left|\sqrt{x^2+4}\right| + c.$$

Try to use this method for evaluating the integral: $\int x\sqrt{1+x^2}\,dx$.

$$\int x\sqrt{1+x^2}\,dx = \underline{\hspace{6cm}}$$

Go to **323** to check your answer.

323

Taking $u^2 = 1 + x^2$, then $2udu = 2xdx$ and

$$\int x\sqrt{1+x^2}dx = \int u(u\,du) = \int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}(1+x^2)^{3/2} + c.$$

Go to **324**.

324

Integration by Parts:

A technique known as *integration by parts* is sometimes helpful. Let u and v be any two functions of x. Then the rule for integration by parts is

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Here is the proof: using the product rule for differentiation,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Now integrate both sides of the equation with respect to x.

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx,$$

$$\int d(uv) = \int u\,dv + \int v\,du.$$

But $\int d(uv) = uv$, and after transposing, we have $\int u\,dv = uv - \int v\,du$.

Here is an example: Find $\int \theta \sin\theta\,d\theta$.

Let $u = \theta$, $dv = \sin\theta d\theta$. Then it is easy to see that $du = d\theta$, and $v = -\cos\theta = \int \sin\theta d\theta$. Note we have dropped the constant for simplicity. Thus

$$\int \theta\sin\theta\,d\theta = \int u\,dv = uv - \int v\,du$$

$$= -\theta\cos\theta - \int(-\cos\theta)\,d\theta$$

$$= -\theta\cos\theta + \sin\theta + c.$$

Go to **325**.

325

Try to use integration by parts to find $\int xe^x\,dx$.

$[(x-1)\,e^x + c \mid xe^x + c \mid e^x + c \mid xe^x + x + c \mid \text{none of these}]$

If right, go to **327**.
If you missed this, or want to see how to solve the problem, go to **326**.

326

To find $\int xe^x\,dx$ use the formula for integration by parts. Because we now know that $e^x = \int e^x\,dx$ (frame **312**, formula 7), we let $u = x$, $dv = e^x dx$, so that $du = dx$, $v = e^x$. Then,

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$
$$= xe^x - e^x + c = (x-1)e^x + c.$$

Go to **327**.

327

Use the method of integration by parts to find the integral: $\int x\cos x\,dx$.

$$\int x\cos x\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Check your answer in **328**.

328

Here is the answer:

$$\int x\cos x\,dx = x\sin x + \cos x + c.$$

If you want to see how this is derived, continue here. Otherwise, go on to **329**.

Let us make the following substitution $u = x$ and $dv = \cos x\,dx$, and integrate by parts. Thus $du = dx$, $v = \sin x$, and the integral is

$$\int x\cos x\,dx = \int u\,dv = uv - \int v\,du = x\sin x - \int \sin\ x\,dx$$
$$= x\sin x + \cos x + c.$$

Go to **329**.

329

In integration problems it is often necessary to use a number of different integration "tricks of the trade" in a single problem.

Try the following (b is a constant):

(a) $\int(\cos 5\theta + b)\, d\theta =$ __________

(b) $\int x \ln x^2\, dx =$ __________

Go to **330** for the answers.

330

The answers are

(a) $\int(\cos 5\theta + b)\, d\theta = \frac{1}{5}\sin 5\theta + b\theta + c$

(b) $\int x \ln x^2\, dx = \frac{1}{2}[x^2(\ln x^2 - 1) + c]$

If you did both of these correctly, you are doing fine—jump ahead to frame **332**. If you missed either problem, go to frame **331**.

331

If you missed (a), you may have been confused by the change in notation from x to θ. Remember x is merely a symbol for some variable. All the integration formulas could be written replacing the x with θ or z or whatever you wish. Now for (a) in detail:

$$\begin{aligned}\int(\cos 5\theta + b)\, d\theta &= \int \cos\ 5\theta\ d\theta + \int\ b\ d\theta \\ &= \frac{1}{5}\int \cos\ 5\theta\ d(5\theta) + \int\ b\ d\theta \\ &= \frac{1}{5}\sin\ 5\theta + b\theta + c.\end{aligned}$$

For problem (b), let $u = x^2$, $du = 2xdx$:

$$\int x \ln x^2\ dx = \frac{1}{2}\int \ln\ u\ du = \frac{1}{2}(u \ln u - u + c).$$

Answer: Frame 325: $(x - 1)e^x + c$

(The last step uses formula 10, frame **312**.) Therefore,

$$\int x \ln x^2\, dx = \frac{1}{2}(x^2 \ln x^2 - x^2 + c).$$

You could also have solved this problem using integration by parts.

Go to **332**.

332

Method of Partial Fractions:

A useful manipulation from elementary algebra is to combine two simple fractions into one function. The *method of partial fractions* involves reversing this process in which you split a function into a sum of fractions with simpler denominators.

To illustrate this method, here is a simple example. Consider the function

$$y(x) = \frac{1}{1 - x^2}.$$

Because $1 - x^2 = (1 - x)(1 + x)$, we can write

$$\frac{1}{1 - x^2} = \frac{a}{1 - x} + \frac{b}{1 + x},$$

where a and b, which are yet to be defined, are called *undetermined coefficients*. Combining terms yields

$$\frac{1}{1 - x^2} = \frac{a}{1 - x} + \frac{b}{1 + x} = \frac{a(1 + x) + b(1 - x)}{1 - x^2} = \frac{(a + b) + (a - b)x}{1 - x^2}.$$

Equate coefficient of like powers (this is comparable to solving a system of linear equations). Therefore $a + b = 1$ and $a - b = 0$. Thus $a = b = 1/2$.

Now integrate $1/(1 - x^2)$ in terms of simpler integrals that you already know how to evaluate see using formula 6, frame 312.

$$\int \frac{1}{1 - x^2} dx = \frac{1}{2}\int \frac{dx}{1 - x} + \frac{1}{2}\int \frac{dx}{1 + x} = \frac{1}{2}(-\ln|1 - x| + \ln|1 + x|) + c$$
$$= \ln\left(\left|\frac{1 + x}{1 - x}\right|\right) + c.$$

Go to **333**.

333

Now try this problem. Use the method of partial fractions to evaluate the integral $\int \frac{3x-4}{(x^2-2x-3)} dx$.

$$\int \frac{3x-4}{(x^2-2x-3)} dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Go to **334**.

334

First note that $x^2 - 2x - 3 = (x+1)(x-3)$. Then write

$$\frac{3x-4}{(x^2-2x-3)} = \frac{3x-4}{(x+1)(x-3)} = \frac{a}{x+1} + \frac{b}{x-3} = \frac{(-3a+b)+(a+b)x}{(x+1)(x-3)}.$$

Next, compare coefficients: $-3a + b = -4$ and $a + b = 3$. Solve these two equations with the result that $a = 7/4,\ b = 5/4$. Therefore

$$\frac{3x-4}{(x^2-2x-3)} = \frac{7}{4\,(x+1)} + \frac{5}{4(x-3)}.$$

Hence the integral is

$$\int \frac{3x-4}{(x^2-2x-3)} dx = \frac{7}{4}\int \frac{dx}{x+1} + \frac{5}{4}\int \frac{dx}{x-3}$$
$$= \frac{7}{4}\ln|1+x| + \frac{5}{4}\ln|x-3|) + c.$$

(The last step uses formula 6, frame **312**.)

3.3 Area Under a Curve and the Definite Integral

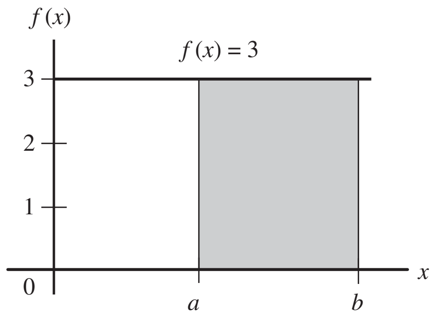
The first section of this chapter focused on the techniques of integration, i.e. finding antiderivatives, all of which are embodied in the term “indefinite integral.” Just as differentiation is useful for many applications besides finding slopes of curves—for instance, calculating rates of growth or finding maxima and minima—so integral calculus has many applications such as finding the volumes of solids or finding the distance traveled by a body moving with a velocity $v(t)$. These applications all rely on the second type of integration, called the *definite integral*. This originated in the geometric problem of finding the area under a curve. We start by explaining what "area under a curve" means.

Go to **335**.

335

Area under a Curve:

To illustrate what is meant by "the area under a curve," here is a graph of the simplest of all curves—a straight line given by $f(x) = 3$.



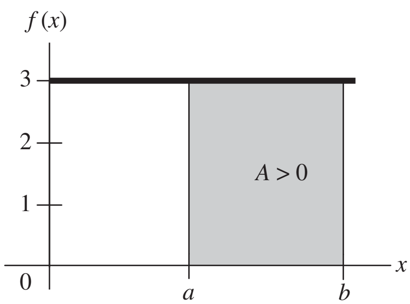
What is the area A between the line $f(x) = 3$, and the x-axis between the interval $x = a$ and $x = b$?

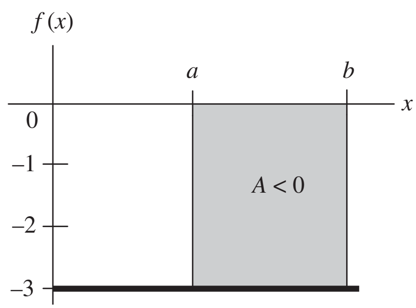
$$A = [3ab \mid 3(a + b) \mid 3(a - b) \mid 3(b - a)]$$

To check your answer, go to **336**.

336

The area in the rectangle is the product of the base, $(b - a)$, and the height, 3. Thus the area is $3(b - a)$.





(continued)

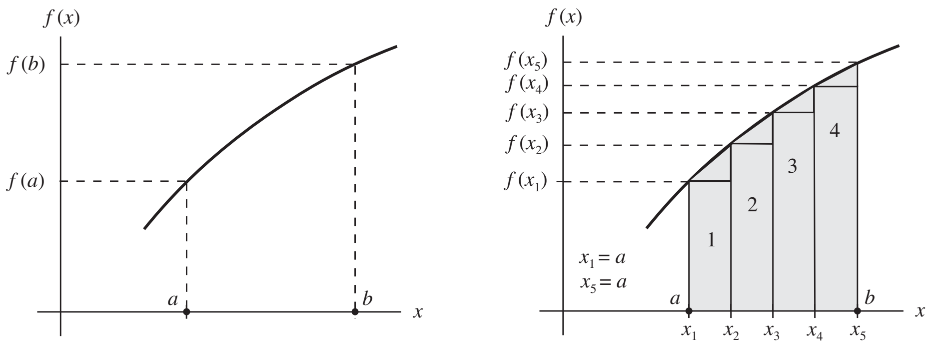
There is an important convention in the sign of the area. In the drawing at the left, because $f(x)$ is positive, the area is positive. However, the area under the graph at the right is negative because the height is $f(x)$, which is negative. Thus, areas can be positive or negative.

Go to **337**.

337

A key operation in integral calculus is to find the area under the graph of an arbitrary function $f(x)$ bordered by the x-axis between $x = a$ and $x = b$, and $f(a)$ and $f(b)$. Our procedure will be to divide the interval $b - a$ into N slices of width $(b - a)/N$, and then watch what happens as N increases and the width decreases.

Let's approximate the area under a curve shown in the figure below left using the above procedure.



First we divide the area into a number of strips of equal widths by drawing lines parallel to the vertical axis. The figure shows four such strips. Each strip has an irregular top: but we can divide each strip's area into two sections: a rectangular shape and an approximately triangular shape. Suppose we label the strips 1, 2, 3, 4. The width of each strip is

$$\Delta x = \frac{b - a}{4}.$$

The height of the first rectangular shape is $f(x_1)$, where $x_1 = a$ is the value of x at the beginning of the first strip. Similarly, the height of the second rectangular shape is $f(x_2)$ where $x_2 = x_1 + \Delta x$. The third and fourth rectangular shapes have heights $f(x_3)$ and $f(x_4)$, respectively, where $x_3 = x_1 + 2\Delta x$ and $x_4 = x_1 + 3\Delta x$.

The height of the first triangular shape is $\Delta h = f'(x_1)\Delta x$, where $f'(x_1)$ is the slope of $f(x_1)$ evaluated at x_1. The heights of the other triangles are similarly described.

You should be able to write an approximate expression for the area of any of the strips. Below write the approximate expression for the area of strip number 3, ΔA_3,

$$\Delta A_3 \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

The symbol $\approx$ means "approximately equal to."

For the correct answer, go to **338**.

338

The area of rectangular shape in strip number 3 is $f(x_3)\ \Delta x$. The triangular shape has area (1/2)(base)(height) = $(1/2)(\Delta x)(f'(x_3)\ \Delta x)$. The approximate area of strip 3 is then

$$\Delta A_3 \approx f(x_3)\ \Delta x + (1/2)(\Delta x)(f'(x_3)\ \Delta x).$$

Now try to write an approximate expression for A, the total area of all four strips.

$$A \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Try this, and then see **339** for the correct answer.

339

An approximate expression for the total area is simply the sum of the areas of all the strips. In symbols, because $A = \Delta A_1 + \Delta A_2 + \Delta A_3 + \Delta A_4$, we have

$$\begin{aligned} A \approx\ & f(x_1)\ \Delta x + f(x_2)\ \Delta x + f(x_3)\ \Delta x + f(x_4)\ \Delta x \\ & + (1/2)(\Delta x)(f'(x_1)\ \Delta x + f'(x_2)\ \Delta x + f'(x_3)\ \Delta x + f'(x_4)\ \Delta x). \end{aligned}$$

We could also write this as

$$A \approx \sum_{i=1}^{4} f(x_i)\ \Delta x + \frac{1}{2}\sum_{i=1}^{4} f'(x_i)\ (\Delta x)^2.$$

(continued)

$\sum$ is the Greek letter Sigma, which corresponds to the English letter S and stands here for the sum. The symbol $\sum_{i=1}^{N} g(x_i)$ means

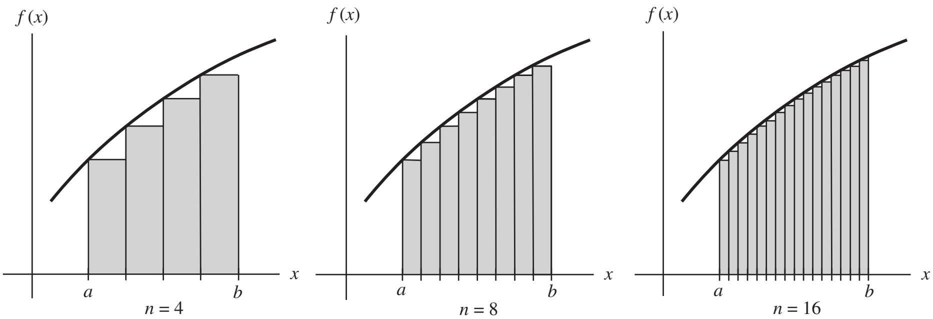
$$\sum_{i=1}^{N} g(x_i) = g(x_1) + g(x_2) + g(x_3) + \cdots + g(x_N),$$

where i is an *index* and the counting goes from $i = 1$ to $i = N$.

Go to **340**.

340

Suppose we divide the area into more strips each of which is narrower, as shown in the drawings. Evidently our approximation gets better and better.



If we divide the area into N strips, then

$$A \approx \sum_{i=1}^{N} f(x_i)\,\Delta x + \frac{1}{2}\sum_{i=1}^{N} f'(x_i)\,(\Delta x)^2,$$

where $N = \frac{b-a}{\Delta x}$. Now, if we take the limit where $\Delta x \to 0$, the approximation becomes an equality. Thus,

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\,\Delta x.$$

Such a limit is so important that it is given a special name and symbol. It is called the *definite integral* and is written $\int_a^b f(x)\,dx$. This expression looks similar to the antiderivative $\int f(x)\,dx$, from frame **327**, and as we shall see in the next frame, it is related. However, it is important to remember that the definite integral is defined by the limit described above:

$$\int_a^b f(x)\, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\, \Delta x.$$

(Incidentally, the integral symbol $\int$ also evolved from the letter S and, like sigma, it was chosen to stand for *sum*.)

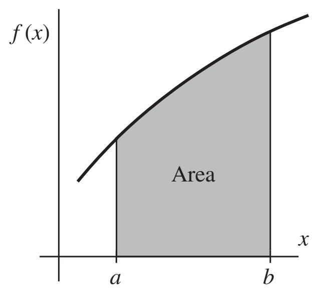
Go to **341**.

341

With this definition for the definite integral, the discussion in the last frame shows that the area A under the curve between $x = a$ and $x = b$ is equal to the definite integral.

$$A = \int_a^b f(x)\, dx.$$

The function $f(x)$ is called the *integrand*. The points a and b are called the *limits of the integral*. This usage has nothing to do with $\lim_{x \to a} f(x)$; here "limit" simply means the boundary.



The process of evaluating $\int_a^b f(x)\, dx$ is often spoken of as "integrating $f(x)$ from a to b," and the expression is called the "integral of $f(x)$ from a to b." Caution: the indefinite and definite integral both employ the integral symbol $\int$, and so they can easily be confused. They are entirely different: the definite integral is a *number* and is equal to the area under the curve between limits a and b. In contrast the indefinite integral is a *function* — the antiderivative of the integrand.

Go to **342**.

342

The Area Function:

For a given function $f(x)$, we can introduce a new *area function* $A(x)$ by introducing a variable x for the upper limit of the definite integral in frame **341**, and to avoid confusion we are renaming the integration variable u

$$A(x) = \int_a^x f(u)\, du.$$

The area function $A(x)$ and the function $f(x)$ are closely related; the derivative of the area is simply $f(x)$. Hence

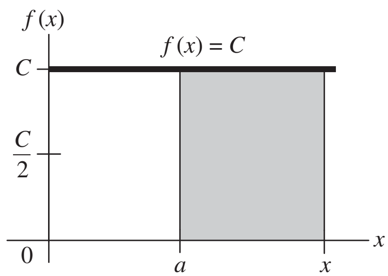
$$A'(x) = f(x).$$

Recall from frame **306** that this means that $A(x)$ is an antiderivative of the function $f(x)$ as we will now show.

Go to **343**.

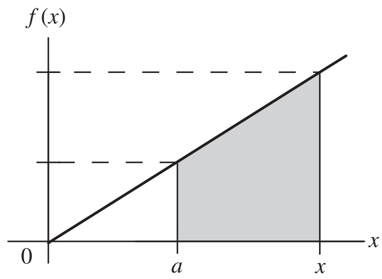
343

To illustrate that $A'(x) = f(x)$ let's look at some simple areas that one can calculate directly.



The area under the curve $f(x) = C$, where C is a constant, is $A(x) = C(x - a)$. Differentiating, $A'(x) = C = f(x)$.

Find the area $A(x)$ under $f(x) = Cx$ between a and x.



Prove to yourself that $A'(x) = f(x)$.

If you want to check your result, go to **344**.
Otherwise, skip to **345**.

344

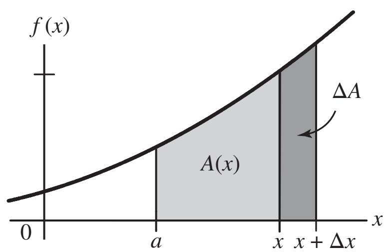
One way to calculate the area above is to think of it as the difference of the area of two right triangles. Using area = ½ base × height, we have

$$A(x) = \frac{1}{2}xf(x) - \frac{1}{2}af(a) = \frac{1}{2}Cx^2 - \frac{1}{2}Ca^2,$$
$$A'(x) = \frac{d}{dx}\left(\frac{1}{2}Cx^2 - \frac{1}{2}Ca^2\right) = Cx = f(x).$$

Go to **345**.

345

To see why $A'(x) = f(x)$, consider how the area $A(x)$ changes as x increases by an amount Δx. $A(x + \Delta x) = A(x) + \Delta A$, where ΔA is the narrow strip shown.



(continued)

Can you find an approximate expression for ΔA?

$$\Delta A \approx \underline{\hspace{6cm}}$$

This represents a major step in the development of integral calculus, so don't be disappointed if the result eludes you.

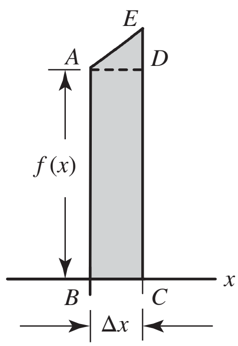
Go to **346**.

346

The answer we want is

$$\Delta A \approx f(x)\,\Delta x.$$

If you wrote this, go on to **347**. If you would like a more detailed discussion, read on.



The following argument may appear similar to our definition of the definite integral (frame **340**), but here we are proving that the area function is an antiderivative.

Let's take a close look at the area ΔA. As you can see, the area is a long narrow strip, which we divided into a rectangle and a triangle with a small piece remaining. In the figure above we just show the rectangle and triangle. As before (frame **338**) the area of the strip is then

$$\Delta A = f(x)\,\Delta x + (1/2)(f'(x)\,\Delta x)\Delta x + \textit{small term},$$

where the small term in the above expression represents the part of the area not included in the triangle or rectangle. For a sufficiently small value of Δx, the two last terms can be ignored compared to the first term. Therefore in the limit as $\Delta x \to 0$

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \lim_{\Delta x \to 0} f(x) + \lim_{\Delta x \to 0} (1/2)(f'(x)\,\Delta x) = f(x).$$

This supports the conjecture in frame **342** that the area function is an antiderivative, $A'(x) = f(x)$.

Go to **347**.

347

To summarize this section, we have found an expression for the area $A(x)$ under a curve defined by $y = f(x)$ that satisfies the equation $A'(x) = f(x)$. Thus, if we can find a function whose derivative is $f(x)$, we can find the area $A(x)$ up to a constant that remains to be determined to find the exact area.

Go to **348**.

348

The second meaning of integration, the definite integral, was introduced by the problem of finding the area A under a curve, $f(x)$, from $x = a$ to $x = b$; this involved finding the limit of an infinite sum. (Recall that this area is a number and not a function.)

Finding the limit of an infinite sum requires a tedious calculation. Fortunately, we can calculate a definite integral by a far simpler method using the techniques of integration for indefinite integrals. The result is that the area is

$$A = \int_a^b f(x)\, dx = F(b) - F(a),$$

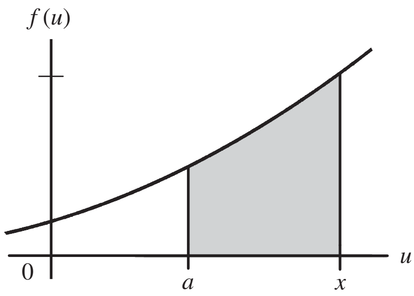
where F is any antiderivative of $f(x)$, i.e. $F(x) = \int f(x)dx + c$. What makes this possible is explicitly calculating the constant c. Instead of calculating a limit, we are now evaluating an antiderivative at the endpoints of the interval.

Let's begin with the area function associated with a function $f(u)$ from $u = a$ to $u = x$,

$$A(x) = \int_a^x f(u)\, du.$$

Note that our area function is a function of the variable x, which is the upper limit of the integral. That's why, to avoid confusion with the variable x, we chose a different variable, u, for integration. In frame **346** we showed that $A'(x) = f(x)$.

(continued)



When $x = a$, the area is zero

$$A(a) = \int_a^a f(u)\, du = 0.$$

Let $F(x)$ be any antiderivative of $f(x)$. Then the area function is equal to

$$A(x) = \int_a^x f(u)\, du = F(x) + c.$$

where c is an arbitrary constant. Set $x = a$. Then

$$0 = A(a) = \int_a^a f(u)\, du = F(a) + c.$$

Therefore $c = -F(a)$. Hence the area under the curve $f(x)$ is equal to

$$A(x) = \int_a^x f(u)\, du = F(x) - F(a).$$

Now set $x = b$. Then the definite integral $A = \int_a^b f(x)\, dx$ corresponds to the area under the curve of the function $f(x)$ for $a \le x \le b$. This can be determined by calculating the value of *any* antiderivative F of $f(x)$ at $x = b$ and subtracting the value of F at $x = a$.

The definite integral is commonly written as

$$\boxed{A = \int_a^b f(x)\, dx = F(b) - F(a)}.$$

Go to **349**.

349

Fundamental Theorem of Calculus:

The significance of the result in the last frame is that determining the area does not require calculating the limit of the summation in the definition of the definite integral (which can be a difficult calculation) but merely evaluating any antiderivative at the endpoints of the interval.

In frame **336**, we discussed the convention for determining the sign for the area. Note that a definite interval can be positive or negative for some interval. If a function $f(x)$ is negative in some interval, then the definite integral evaluated in that interval will be negative.

Summary: We have shown that for any continuous function $f(x)$,

$$\boxed{\lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\ \Delta x = \int_a^b f(x)\ dx = F(b) - F(a)}$$

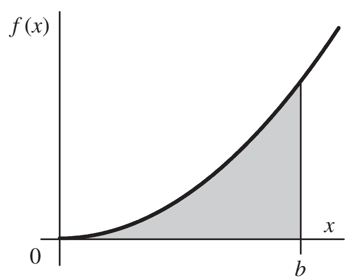
where $F'(x) = f(x)$.

This relation is so extraordinary that it is called *the fundamental theorem of calculus*.

Go to **350**.

350

To see how all this works, we will find the area under the curve $y = x^2$ between $x = 0$ and $x = b$.



In this example an antiderivative is $F(x) = x^3/3$ because $F'(x) = x^2$. Therefore

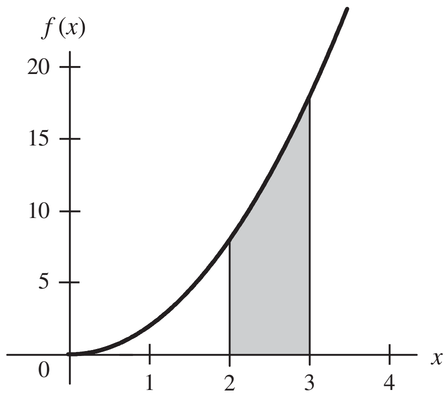
$$A = \int_0^b x^2 dx = F(b) - F(0) = \frac{1}{3}b^3 - \frac{1}{3}0^3 = \frac{1}{3}b^3.$$

(continued)

Note that the definite integral yields a number. Furthermore, we no longer need to introduce an undetermined constant c whenever we evaluate an expression such as $F(b) - F(0)$.

Go to **351**.

351



Can you find the area under the curve $y = 2x^2$, between the points $x = 2$ and $x = 3$?

$$A = [13 \mid 1/3 \mid 38/3 \mid 18]$$

If right, go to **353**.
Otherwise, go to **352**.

352

The solution is straightforward. An antiderivative for the function $f(x) = 2x^2$ is $F(x) = \frac{2}{3}x^3$. Therefore

$$A = F(3) - F(2) = \frac{2}{3}(3^3 - 2^3) = \frac{38}{3}.$$

Go to **353**.

353

Find the area under the curve $y = 4x^3$ between $x = -2$ and $x = 1$.

$$A = \left[17 \mid {}^{15}\!/_{4} \mid -15 \mid 16\right]$$

Go to **354**.

354

An antiderivative for the function $f(x) = 4x^3$ is $F(x) = x^4$. Therefore

$$A = F(1) - F(-2) = (1)^4 - (-2)^4 = 1 - 16 = -15.$$

Let's introduce a little labor-saving notation. Frequently we have to find the difference of an expression evaluated at two points, as $F(b) - F(a)$. This is often denoted by

$$F(b) - F(a) = F(x)\Big|_a^b.$$

For example, $x^2\Big|_a^b = b^2 - a^2$. Using this notation, the solution to this problem is written

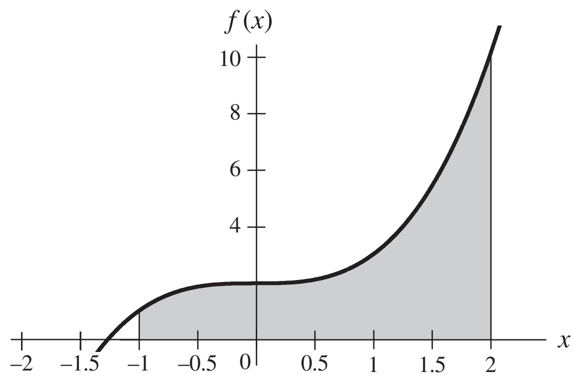
$$A = \int_{-2}^{1} 4x^3 \, dx = x^4\Big|_{-2}^{1} = 1^4 - (-2)^4 = 1 - 16 = -15.$$

Note that this area is negative.

Go to **355**.

355

Here is another practice problem:



(continued)

The graph shows a plot of $y = x^3 + 2$. Find the area between the curve and the x-axis from $x = -1$ and $x = 2$.

Answer: [5 | $\frac{1}{4}$ | 4 | $\frac{17}{4}$ | $\frac{39}{4}$ | none of these]

If right, go to **357**.
Otherwise, go to **356**.

356

Here is how to do the problem: one antiderivative is $F(x) = (x^4/4) + 2x$. You can check that $F'(x) = x^3 + 2$. Therefore

$$A = F(2) - F(-1) = F(x)\big|_{-1}^{2},$$

$$F = \int y\,dx = \int (x^3 + 2)\,dx = \frac{1}{4}x^4 + 2x,$$

$$A = \left(\frac{1}{4}x^4 + 2x\right)\bigg|_{-1}^{2} = \left(\frac{16}{4} + 4\right) - \left(\frac{1}{4} - 2\right) = \frac{39}{4}.$$

Go to **357**.

357

To help remember the definition of definite integral, try writing it yourself. Write an expression defining the definite integral of $f(x)$ between the limits a and b.

$$\int_a^b f(x)\,dx = \underline{\hspace{6cm}}$$

To check your answer, go to **358**.

358

The answer is

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\,\Delta x, \quad \text{where} \quad N = \frac{b-a}{\Delta x}.$$

Answers: Frame 351: $\frac{38}{3}$
Frame 353: -15

Congratulations if you wrote this or an equivalent expression. If you wrote

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{where } F(x) = \int f(x)\,dx,$$

your statement is true, but it is not the *definition* of a definite integral. The result is true because both sides represent the same thing—the area under the curve of $f(x)$ between $x = a$ and $x = b$. It is an important result because without it the definite integral would be much more difficult to calculate; however, it is not true by definition.

The definite integral appears merely to provide a second way to find the area under a curve. To compute the area, we were led back to the indefinite integral, but we could have found the area directly from the indefinite integral. The importance of the definite integral arises from its definition as the *limit* of a sum. The process of dividing a system into small elements and then adding them together is a powerful technique that is applicable to more problems than finding the area under a curve. These invariably lead to definite integrals, which we can evaluate in terms of indefinite integrals by using the fundamental theorem of calculus (frame **349**).

Go on to **359**.

359

Can you prove that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx?$$

After you have tried to prove this result, go to **360**.

360

The proof that $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ is as follows:

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{where } F(x) = \int f(x)\,dx.$$

Reversing the limits of the integral yield,

$$\int_b^a f(x)\,dx = F(a) - F(b) = -[F(b) - F(a)]$$
$$= -\int_a^b f(x)\,dx.$$

Go to **361**.

361

Which of the following expressions correctly gives $\int_0^{2\pi} \sin\theta\, d\theta$?

[1 | 0 | 2π | -2 | -2π | none of these]

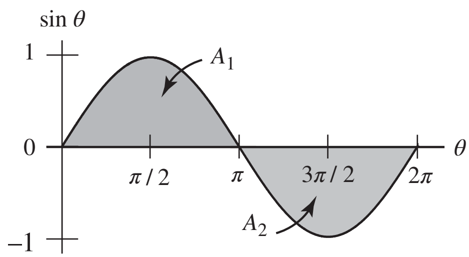
Go to **362**.

362

The answer is

$$\int_0^{2\pi} \sin\ \theta\ d\theta = -\cos\ \theta\Big|_0^{2\pi} = -(1-1) = 0.$$

It is easy to see why this result is true by inspecting the figure.



The integral yields the total area under the curve, from 0 to 2π, which is the sum of the area A_1 between 0 to π, and A_2 between π and 2π. But A_2 is negative, because $\sin\theta$ is negative in that region. By symmetry, the two areas just add to 0. However, you should be able to find A_1 or A_2 separately. Try this problem:

$$A_1 = \int_0^{\pi} \sin\ \theta\ d\theta = [1|\ |\ 2\ |\ -1|-2\ |\ \pi\ |\ 0]$$

If right, go to **364**.
Otherwise, go to **363**.

Answer: Frame 355: $^{39}/_{4}$

363

The answer is

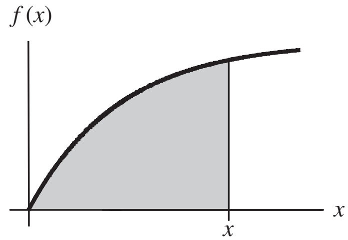
$$A_1 = \int \sin\ \theta\ d\theta = -\cos\theta\Big|_0^{\pi} = -[-1-(+1)] = 2.$$

If you forgot the integral, you can find it in frame **312**. In evaluating $\cos\theta$ at the limits, $\cos\pi = -1$, and $\cos 0 = 1$.

Go to **364**.

364

Here is a graph of the function $f(x) = 1 - e^{-x}$.



Can you find the shaded area under the curve between the origin and x?

Answer: [e^{-x} | $1 - e^{-x}$ | $x + e^{-x}$ | $x + e^{-x} - 1$]

Go to **366** if you did this correctly.
See **365** for the solution.

365

Here is the solution to **364**.

$$A(x) = \int_0^x f(x)\ dx = \int_0^x (1 - e^{-x})\ dx = \int_0^x dx - \int_0^x e^{-x}\ dx$$
$$= [x - (-e^{-x})]\Big|_0^x = [x + e^{-x}]\Big|_0^x = x + e^{-x} - 1,$$

The area is bounded by a vertical line through x. Our result gives $A(x)$ as a variable that depends on x. If we choose a specific value for x, we can substitute it into the above formula for $A(x)$ and obtain a specific value for $A(x)$. We have obtained a definite integral in which one of the boundary points is left as a variable.

Go to **366**.

366

Let's evaluate one more definite integral before going on. Find:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \underline{\hspace{6cm}}$$

(If you need to, use the integral tables, frame **312**.)

Answer: [0 | 1 | ∞ | π | $\pi/2$ | none of these]

If you got the right answer, go to **368**.
If you got the wrong answer, or no answer at all, go to **367**.

367

From the integral table, frame **312**, we see that

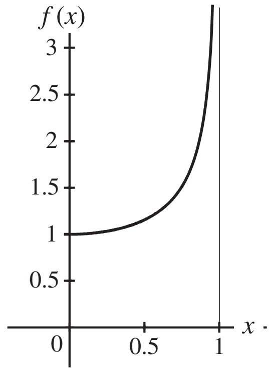
$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c.$$

Therefore,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\Big|_0^1 = \sin^{-1}1 - \sin^{-1}0.$$

Answers: Frame 361: 0
Frame 362: 2
Frame 364: $x + e^{-x} - 1$

Because $\sin \frac{\pi}{2} = 1$, we have that $\sin^{-1} 1 = \frac{\pi}{2}$. Similarly, $\sin^{-1} 0 = 0$. Thus, the integral has the value $\frac{\pi}{2} - 0 = \frac{\pi}{2}$.



A graph of $f(x) = \frac{1}{\sqrt{1-x^2}}$ is shown above. Although the function is discontinuous at $x = 1$, the area under the curve is perfectly well defined.

Go to **368**.

3.4 Some Applications of Integration

368

In this section we are going to apply integral calculus to a few problems.

In Chapter 2 we learned how to find the velocity of a particle if we know its position in terms of time. Now we can reverse the procedure and find the position from the velocity. For instance, we are in an automobile driving along a straight road through thick fog. To make matters worse, our mileage indicator is broken. Instead of watching the road all the time, let's keep an eye on the speedometer. We have a good watch along, and we make a continuous record of the speed starting from the time when we were at rest. The problem is to find how far we have gone. More specifically, given $v(t)$, how do we find the change in position of the automobile $S(t) - S(t_0)$, called the *displacement*, when we were at rest? Because the automobile is traveling in the same direction, the change of position is equal to the distance traveled.

Try to work out a method.

$S(t) =$ ______________

To check your result, go to **369**.

369

Because $v = dS/dt$, we must have $dS = v\,dt$ (as was shown in frame **275**). Now let us integrate both sides from the initial point $(t_0, S(t_0))$ to the final point in order to find the change in position of the automobile

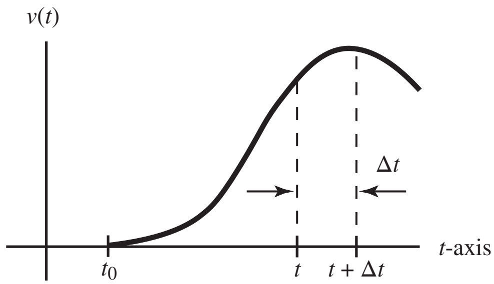
$$S(t) - S(t_0) = \int_{t_0}^{t} dS = \int_{t_0}^{t} v(t')\,dt',$$

where we have replaced the integration variable by t'.

If you did not get this result, or would like to see more explanation, go to **370**.
Otherwise, go to **371**.

370

Another way to understand this problem is to look at it graphically. Here is a plot of $v(t)$ as a function of t.



In time Δt the distance traveled is $\Delta S = v\Delta t$. The total distance traveled is thus equal to the area under the curve between the initial time and the time of interest, and this is $\int_{t_0}^{t} v(t)\,dt$. There may be some confusion because the same symbol t, which is the dependent variable of the function, appears both as an endpoint of the integral and as the integration variable in the integrand $v(t)$. The integration variable is what we call a dummy variable and we can denote it by any symbol, for example, u. Thus the integral is written as

$$S(t) = \int_{u=t_0}^{u=t} v(u)\,du = \int_{t_0}^{t} v(u)\,du.$$

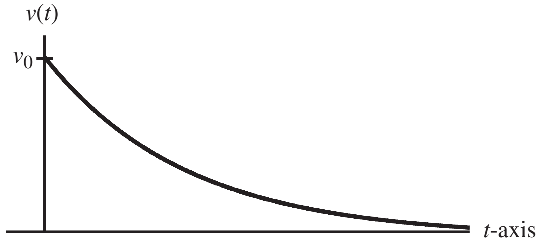
Often this distinction between the dependent variable of the function and the integration variable is not explicit but implicit, and the symbol t is used in both places, as we will do in a few frames.

Go to **371**.

Answer: Frame 366: $\pi/2$

371

Suppose an object moves with a velocity that continually decreases in the following way, $v(t) = v_0 e^{-bt}$ (v_0 and b are positive constants).



At $t = 0$ the object is at the origin; $S(0) = 0$. Which of the following is the distance, $S(t)$, the object will have moved after an infinite time (or, if you prefer, after a very long time)?

Answer: $\left[0 \mid v_0 \mid v_0 e^{-1} \mid \frac{v_0}{b} \mid \infty\right]$

If correct, go to **373**.
Otherwise, go to **372**.

372

Here is the solution to the problem of frame **371**.

$$S(t) - S(0) = \int_0^t v(t)\, dt = \int_0^t v_0\, e^{-bt}\, dt$$

$$S(t) - 0 = -\frac{v_0}{b} e^{-bt} \Big|_0^t = -\frac{v_0}{b}(e^{-bt} - 1).$$

We are interested in the $\lim_{t\to\infty} S(t)$. Because $e^{-bt} \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} S(t) = \lim_{t\to\infty} \left(-\frac{v_0}{b}(e^{-bt} - 1)\right) = -\lim_{t\to\infty} \frac{v_0}{b} e^{-bt} + \lim_{t\to\infty} \frac{v_0}{b} = \frac{v_0}{b}.$$

Although the object never comes completely to rest, its velocity gets so small that the total distance traveled is finite.

Go to **373**.

373

Not all integrals give finite results. For example, try this problem.

A particle starts from the origin at $t = 0$ with a velocity $v(t) = v_0/(b + t)$, where v_0 and b are constants. How far does it travel as $t \to \infty$?

Answer $\left[v_0 \ln \frac{1}{b} \;\middle|\; \frac{v_0}{b} \;\middle|\; \frac{v_0}{b^2} \;\middle|\; \text{none of these} \right]$

Go to **374**.

374

It is easy to see that problem **373** leads to an infinite integral.

$$S(t) - 0 = \int_0^t v_0 \frac{dt}{b + t} = v_0 \ln(b + t)\Big|_0^t$$
$$= v_0[\ln(b + t) - \ln b]$$
$$= v_0 \ln\left(1 + \frac{t}{b}\right).$$

(The last step uses formula 6, frame **312**.)

Because $\ln(1 + (t/b)) \to \infty$ as $t \to \infty$, we see that $S(t) \to \infty$ as $t \to \infty$. In this case, the particle is always moving fast enough so that its motion is unlimited. Or, alternatively, the area under the curve $v(t) = v_0/(b + t)$ increases without limit as $t \to \infty$.

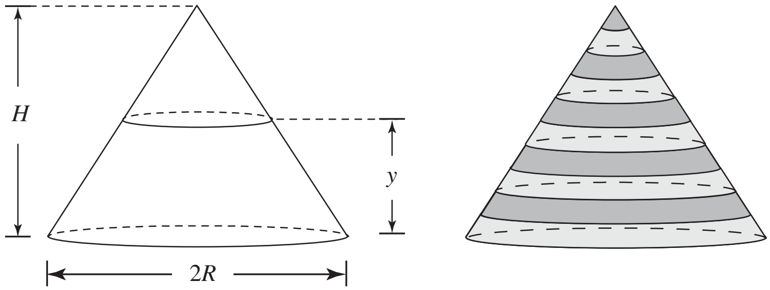
Go to **375**.

375

The Method of Slices:

Integration can be used for many tasks besides calculating the area under a curve. For example, it can be used to find the volumes of solids of known geometry. A general method for this is explained in frame **395**. However, one can calculate the volume of symmetric solids by a simple extension of methods we have learned already. In the next few frames we are going to find the volume of a right circular cone.

Answer: Frame 371: v_0/b



The height of the cone is H, and the radius of the base is R. We will let y represent distance vertically from the base.

Our method of attack, called the *method of slices,* is similar to that used in frame **378** to find the area under a curve. We will slice the body into a number of discs whose volume is approximately that of the cone in the figure (the cone has been approximated by ten circular discs). Then we have

$$V \approx \sum_{i=1}^{8} \Delta V_i,$$

where ΔV_i is the volume of one of the discs. In the limit where the height of each disc (and hence the volume) goes to 0, we have

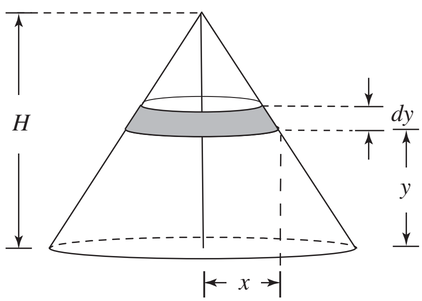
$$V = \int dV.$$

In order to evaluate this, we need an expression for dV.

Go to frame **376**.

376

Because we are going to take the limit where $\Delta V \to 0$, we will represent the volume element by dV from the start.



(continued)

A section of the cone is shown in the figure, which for our purposes is represented by a disc. The radius of the disc is x and its height is dy. Try to find an expression for dV in terms of y. (You will have to find x in terms of y.)

$$dV = \underline{\hspace{4cm}}$$

To check your result, or to see how to obtain it, go to **377**.

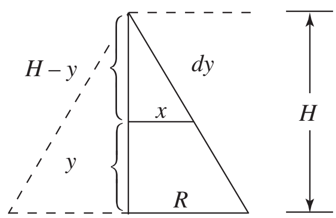
377

The answer is:

$$dV = \pi R^2 \left(1 - \frac{y}{H}\right)^2 dy.$$

If you got this answer, go on to **378**. If you want to see how to derive it, read on.

The volume of this disc is the product of the area and height. Thus, $dV = \pi x^2 dy$. Our remaining task is to express x in terms of y. The diagram shows a cross-section of the cone.



Because x and R are corresponding edges of similar triangles, it should be clear that

$$\frac{x}{R} = \frac{H - y}{H}, \text{ or } x = R\left(1 - \frac{y}{H}\right).$$

Thus,

$$dV = \pi x^2 dy = \pi R^2 \left(1 - \frac{y}{H}\right)^2 dy.$$

Go to **378**.

Answer: Frame 373: none of these

378

We now have an integral for V.

$$V = \int_0^H dV = \int_0^H \pi R^2 \left(1 - \frac{y}{H}\right)^2 dy.$$

Try to evaluate this.

$V =$ ______________________________

To check your answer, go to **379**.

379

The result is $V = \frac{1}{3}\pi R^2 H$. Congratulations, if you obtained the correct answer. Go on to **380**.

Otherwise, read below:

$$V = \int_0^H \pi R^2 \left(1 - \frac{y}{H}\right)^2 dy = \pi R^2 \int_0^H \left(1 - \frac{2y}{H} + \frac{y^2}{H^2}\right) dy$$

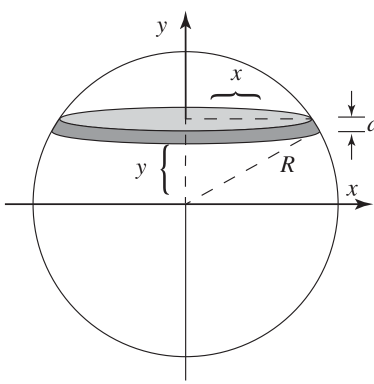
$$= \pi R^2 \left(y - \frac{y^2}{H} + \frac{1}{3}\frac{y^3}{H^2}\right)\Bigg|_0^H = \pi R^2 \left(H - H + \frac{1}{3}H\right)$$

$$= \frac{1}{3}\pi R^2 H.$$

Go to **380**.

380

Let's find the volume of a sphere. Can you write an integral that will give the volume of the hemisphere?



(The slice of the hemisphere shown in the drawing may help you in this.)

(continued)

$$V = \underline{\hspace{6cm}}$$

Go to **381** to check your formula.

381

The answer is

$$V = \int_{-R}^{R} \pi(R^2 - y^2)dy.$$

If you wrote this, go ahead to frame **382**. Otherwise, read on.

In order to calculate the volume of the sphere, consider a disk of thickness dy and radius x, located at a distance y from the center of the sphere as shown in the drawing in frame **280**. The differential volume dV of the disk between y and $y + dy$ is $dV = \pi x^2 dy$. By the Pythagorean theorem
$x = \sqrt{R^2 - y^2}$, therefore $dV = \pi(R^2 - y^2)dy$.

As you add up the disks, the limits of the variable y are $y = -R$ to $y = R$. The volume integral is

$$V = \int_{-R}^{R} \pi(R^2 - y^2)dys.$$

Go to **382**.

382

Now go ahead and evaluate the integral

$$V = \int_{-R}^{R} \pi(R^2 - y^2)dy = \underline{\hspace{5cm}}$$

To see the correct answer, go to **383**.

383

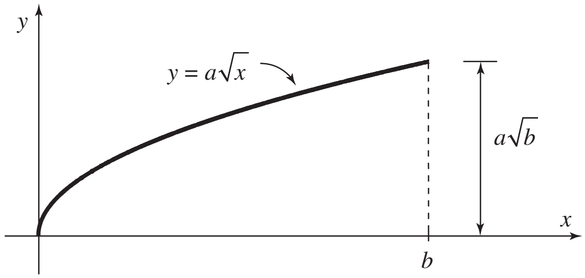
The answer is

$$V = \int_{-R}^{R} \pi \left(R^2 - y^2\right) dy = 2\int_{0}^{R} \pi \left(R^2 - y^2\right) dy = \left(2\pi R^2 y - 2\pi y^3/3\Big|_0^R\right)$$
$$V = 2\pi R^3 - 2\pi R^3/3 = 4\pi R^3/3.$$

Go to **384**.

384

Here's a problem that involves finding the area under a curve and the volume of a solid of revolution generated by the curve.



What is the area under the curve $y = a\sqrt{x}$, for the range $x = 0$ to $x = b$?

$A =$ ______________________

Go to **385**.

385

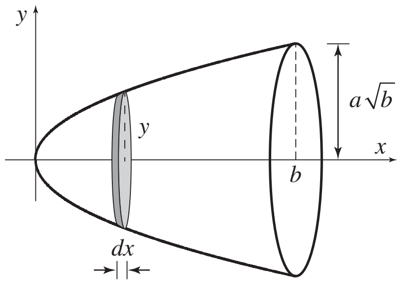
Here is how to solve the problem. The area under the curve is the integral

$$A = a\int_0^b x^{1/2} dx = \frac{2ax^{3/2}}{3}\Bigg|_0^b = \frac{2a}{3} b^{3/2}.$$

Go to **386**.

386

Now rotate that area about the x-axis to generate the volume shown in the diagram.



Can you use the method of slices to write an integral that will give the volume? (The slice in the drawing may help you visualize this.)

$V =$ ____________________________

Go to **387** to check your formula.

387

The answer is $V = \int_0^b \pi a^2 x dx$. If you wrote this, go ahead to frame **388**. Otherwise, read on.

Consider a disk of thickness dx and radius y, located at a distance x from the origin as shown in the diagram. The differential volume dV for the disk is

$$dV = \pi y^2 dx = \frac{\pi a^2}{h^2} x^2 dx.$$

The limits of the variable x are $x = 0$ to $x = h$. Now you can set up the volume integral

$$V = \int_0^b \pi a^2 x dx.$$

Go to **388**.

388

Now go ahead and evaluate the integral.

$$V = \int_0^b \pi a^2 x dx = \underline{\hspace{4cm}}$$

To see the answer, go to **389**.

389

The answer is

$$V = \int_0^b \pi a^2 x dx = \pi a^2 \left. \frac{x^2}{2} \right|_0^b = \frac{\pi a^2 b^2}{2}.$$

Go to **390**.

3.5 Multiple Integrals

390

The subject of this section—multiple integrals—introduces some new concepts and enables us to apply calculus to a world of problems that involve *multiple variable calculus*, in contrast to *single variable calculus*.

The integrals we have discussed so far, of the form $\int f(x)\, dx$, have had a single independent variable, usually called x. Double integrals are similarly defined for two independent variables, x and y. In general, multiple integrals are defined for an arbitrary number of independent variables, but we will only consider two. Note that up to now y has often been the dependent variable: $y = f(x)$. In this section, however, y along with x will always be independent variables, and $z = f(x, y)$ will be the dependent variable. Thus, z is a function of two variables.

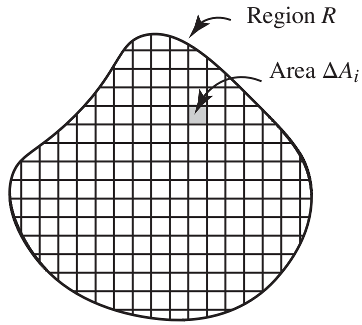
In frame **349** the definite integral of $f(x)$ between a and b was defined by

$$\int_a^b f(x)\, dx = \lim_{\Delta\ x \to 0} \sum_{i=1}^{N} f(x_i)\, \Delta x.$$

(continued)

The double integral is similarly defined, but with two independent variables. There are, however, some important differences. For a single definite integral the integration takes place over a closed interval between a and b on the x-axis. In contrast, the integration of $z = f(x, y)$ takes place over a closed region R in the $x - y$ plane.

To define the double integral, divide the region R into N smaller regions each of area ΔA_i.



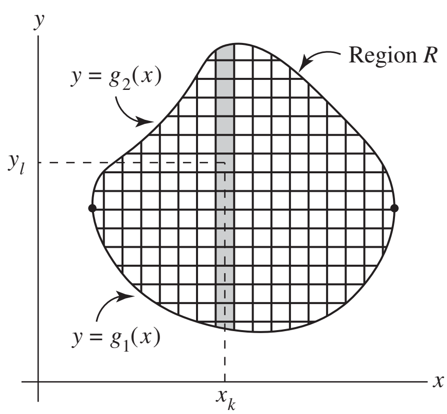
Let (x_i, y_i) be an arbitrary point inside the region ΔA_i. Then in analogy to the integral of a single variable, the double integral is defined as

$$\iint f(x, y)\, dA = \lim_{\Delta A_i \to 0} \sum_{i=1}^{N} f(x_i, y_i)\, \Delta A_i, \text{ the } \left(\iint\right) \text{ is called the } \textit{double integral symbol.}$$

Go to **391**.

391

The double integral is often evaluated by taking ΔA_i to be a small rectangle with sides parallel to the x and y axes. The procedure is first evaluate the sum and limit along one direction and then along the other. Consider the upper portion of the region R in the $x - y$ plane to be bounded by the curve $y = g_2(x)$, while the lower portion is bounded by $y = g_1(x)$, as in the diagram.



If we let $\Delta A_i = \Delta x_k \Delta y_l$, then

$$\iint_R f(x, y)\, dA = \lim_{\Delta A_i \to 0} \sum_{i=1}^{N} f(x_i, y_i)\, \Delta A_i$$
$$= \lim_{\Delta x_k \to 0} \lim_{\Delta y_l \to 0} \sum_{k=1}^{p} \sum_{l=1}^{q} f(x_k, y_l)\, \Delta y_l\, \Delta x_k.$$

This is a complicated expression, but it can be simplified by carrying it out in two separate steps. Let us insert some brackets to clarify the separate steps.

$$\iint_R f(x, y)\, dA = \lim_{\Delta x_k \to 0} \sum_{k=1}^{p} \left[\lim_{\Delta y_l \to 0} \sum_{l=1}^{q} f(x_k, y_l) \Delta y_l \right] \Delta x_k.$$

The first step is to carry out the operation within the brackets. Note that x_k is not altered as we sum over l in the brackets. This corresponds to summing over the crosshatched strip in the diagram with x_k treated as approximately a constant. The quantity in square brackets is then merely a definite integral of the variable y, with x treated as a constant. Note that although the limits of integration, $g_1(x)$ and $g_2(x)$, are constants for a particular value of x, they are in general non-constant functions of x. The quantity in square brackets can then be written as

$$\int_{g_1(x_k)}^{g_2(x_k)} f(x_k, y)\, dy.$$

This quantity will no longer depend on y, but it will depend on x_k both through the integrand $f(x_k, y)$ and the limits $g_1(x_k)$, $g_2(x_k)$. Consequently,

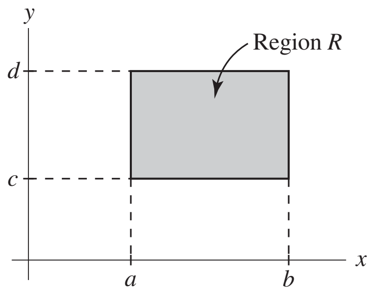
$$\iint_R f(x, y)\, dA = \lim_{\Delta x_k \to 0} \sum_{k=1}^{p} \left[\int_{g_1(x_k)}^{g_2(x_k)} f(x_k, y) dy \right] \Delta x_k$$
$$= \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right] dx.$$

In calculations it is essential that one first evaluate the integral in the square brackets while treating x as a constant. The result is some function, which depends only on x. The next step is to calculate the integral of this function with respect to x, treating x now as a variable.

Go to **392**.

392

Multiple integrals are most easily evaluated if the region R is a rectangle whose sides are parallel to the x and y coordinate axes, as shown in the drawing.



The double integral is

$$\iint_R f(x, y)\, dA = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx.$$

As an exercise to test your understanding, how would the above expression be written if the integration over x were to be carried over before the integration over y?

Go to **393**.

393

When x is chosen to be integrated first, the double integral becomes

$$\iint_R f(x, y)\, dA = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy.$$

This can be found merely by interchanging the y and x operations in evaluating the double integral. (Note that the integration limits must also be interchanged.)

To see how this works, let us evaluate the double integral of $f(x, y) = 3x^2 + 2y$ over the rectangle in the $x - y$ plane bounded by the lines $x = 0$, $x = 3$, $y = 2$, and $y = 4$. The double integral is equal to the iterated integral.

$$\iint_R \left(\frac{1}{3}x^2 + y \right) dA = \int_0^3 \left[\int_2^4 (3x^2 + 2y)\, dy \right] dx.$$

Alternatively we could have written

$$\iint_R \left(\frac{1}{3}x^2 + y\right)\ dA = \int_2^4 \left[\int_0^3 (3x^2 + 2y)\ dx\right] dy.$$

Evaluate each of the above expressions. The answers should be the same.

Integral = ________________________

If you made an error or want more explanation, go to **394**.
Otherwise, go to **395**.

394

Integrating first over y and then over x yields

$$\int_0^3 \left[\int_2^4 (3x^2 + 2y)\ dy\right]\ dx = \int_0^3 (3x^2y + y^2)\Big|_2^4\ dx$$

$$= \int_0^3 [3x^2(4-2) + (16-4)]\ dx = \int_0^3 (6x^2 + 12)\ dx$$

$$= \left(6\frac{x^3}{3} + 12x\right)\Bigg|_0^3 = 54 + 36 = 90.$$

Integrating first over x and then over y yields

$$\int_2^4 \left[\int_0^3 (3x^2 + 2y)\ dx\right]\ dy = \int_2^4 (x^3 + 2yx)\Big|_0^3\ dy$$

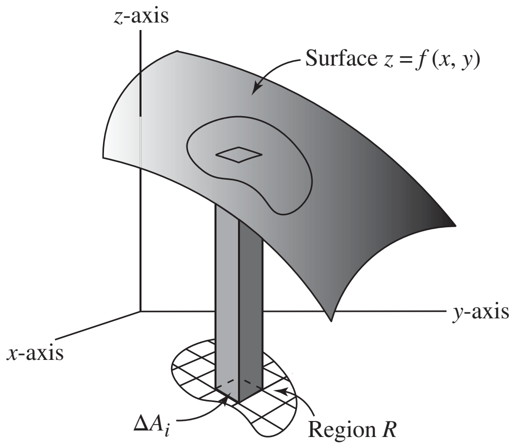
$$= \int_2^4 (27 + 6y)\ dy = (27y + 3y^2)\Big|_2^4$$

$$= 108 + 48 - (54 + 12) = 90.$$

Go to **395**.

395

Just as the equation $y = f(x)$ defines a curve in the two-dimensional $x - y$ plane, the equation $z = f(x, y)$ defines a surface in the three-dimensional space because that equation determines the value of z for any values assigned independently to x and y.



We can easily see from the above definition of the double integral that $\iint_R f(x, y)dA$ is equal to the volume V of space under the surface $z = f(x, y)$ and above the region R. In this case $f(x_i, y_i)$ is the height of column above ΔA_i. Therefore, $f(x_i, y_i)\Delta A_i$ is approximately equal to the volume of that column. The sum of all these columns is then approximately equal to the volume under the surface. In the limit as $\Delta A_i \to 0$, the sum defining the double integral becomes equal to the volume under the surface and above R, so

$$V = \iint_R z\, dA = \iint_R f(x, y)\, d\Delta A_i.$$

Calculate the volume under the surface defined by $z = x + y$ and above the rectangle whose sides are determined by the lines $x = 1$, $x = 4$, $y = 0$, and $y = 3$.

$V =$ ______________________

Go to **396**.

Answer: Frame 393: 90

396

The answer is 36. If you obtained this result, go to frame **397**. If not, study the following.

$$V = \iint_R (x + y)\, dA = \int_1^4 \left[\int_0^3 (x + y)\, dy\right] dx$$

$$= \int_1^4 \left(xy + \frac{y^2}{2}\right)\Bigg|_1^4 dx = \int_1^4 \left(3x + \frac{9}{2}\right) dx$$

$$= \left(\frac{3}{2}x^2 + \frac{9}{2}x\right)\Bigg|_1^4 = \frac{(3)(16)}{2} + \frac{(9)(4)}{2} - \frac{3}{2} - \frac{9}{2} = 36.$$

The iterated integral could just as well have been evaluated in the opposite order.

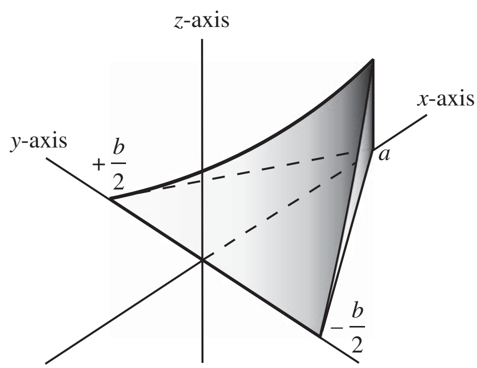
$$\int_0^3 \left[\int_1^4 (x + y)\, dx\right] dy = \int_0^3 \left(\frac{x^2}{2} + yx\right)\Bigg|_1^4 dy$$

$$= \int_0^3 \left(\frac{16}{2} + 4y - \frac{1}{2} - y\right) dy = \int_0^3 \left(3y + \frac{15}{2}\right) dx$$

$$= \left(\frac{3}{2}y^2 + \frac{15}{2}y\right)\Bigg|_0^3 = \frac{27}{2} + \frac{45}{2} = 36.$$

Go to **397**.

397



The bottom of this plow-shaped solid is in the form of an isosceles triangle with base b and height a. When oriented along the $x - y$ axes as shown, its thickness is given by $z = Cx^2$, where C is a constant. The problem is to find an expression for the volume.

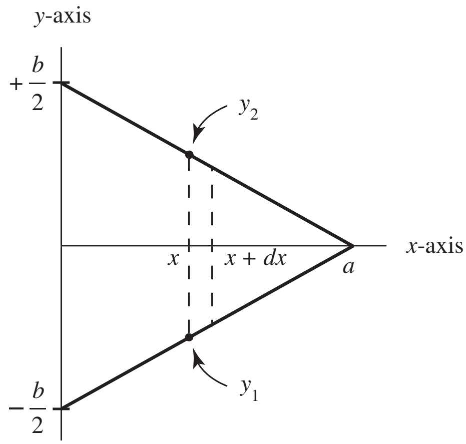
(continued)

$V =$ ______________________

To check your answer, go to **398**.

398

The volume is $\frac{1}{12}Cba^3$. Read on if you want an explanation; otherwise go to **399**.



The base of the object forms a triangle, as shown. The integral can be carried out with respect to x and y in either order. We shall integrate first over y.

$$V = \int_0^a \left[\int_{y_1}^{y_2} z\,dy\right] dx = \int_0^a \left[\int_{y_1}^{y_2} Cx^2\,dy\right] dx$$

$$= \int_0^a Cx^2 y\Big|_{y_1}^{y_2} dx.$$

From the drawing, $y_2 = \frac{b}{2}\left(1 - \frac{x}{a}\right) = -y_1$, so that $Cx^2y\Big|_{y_1}^{y_2} = Cx^2b\left(1 - \frac{x}{a}\right)$, and

$$V = Cb\int_0^2 x^2\left(1 - \frac{x}{\text{a}}\right)dx = Cb\left(\frac{1}{3}a^3 - \frac{1}{4}a^3\right) = \frac{1}{12}Cba^3.$$

The integral can also be evaluated in reverse order. The calculation is simplified by making use of symmetry; the volume is twice the volume over the upper triangle. Thus

$$V = 2\int_0^{b/2}\left[\int_0^{x(y)} Cx^2\,dx\right]dy,$$

where $x(y) = a\left(1 - \frac{2}{b}y\right)$. The answer is the same, $Cba^3/12$.

Go to **399**.

Conclusion to Chapter 3

399

Well, here you are at the last frame of Chapter 3. At this point you should understand the principles of integration and be able to do some integrals. With practice your repertoire will increase. Don't be afraid to use integral tables—everyone does. You can find them online, for instance, https://en.wikipedia.org/wiki/Lists_of_integrals.

Summary of Chapter 3

3.1 Antiderivative, Integration, and the Indefinite Integral (frames 306–314)

When a function $F(x)$ is differentiated to give $f(x) = dF/dx$, then $F(x)$ is an *antiderivative* of $f(x)$, that is, $F'(x) = f(x)$. If $F(x)$ is an antiderivative of $f(x)$, then all the antiderivatives of $f(x)$ can be denoted by writing

$$\int f(x)\,dx = F(x) + c,$$

where c is arbitrary constant. The expression $\int f(x)\,dx$is also called the *integral* of $f(x)$. The symbol $\int$ is known as the *integration symbol*, and represents the inverse of differentiation. It is important not to omit this constant. Otherwise the answer is incomplete.

Indefinite integrals are often found by hunting for an expression that when differentiated gives the integrand $f(x)$. Thus from the earlier result that

$$\frac{d}{dx}\cos x = -\sin x$$

we have that

$$\int \sin x\,dx = -\cos x + c.$$

By starting with known derivatives as in Table 1, a useful list of integrals can be found. Such a list is given in frame **312** and for convenience is repeated in Table 2. You can reconstruct the most important of these formulas from the differentiation expressions in Table 1. More complicated integrals can often be found in large integral tables.

3.2 Some Techniques of Integration (frames 315–334)

Often an unfamiliar function can be converted into a familiar function having a known integral by using a technique called *change of variable* which is related to the chain rule of differentiation and uses the relation

$$\int w(x)\,dx = \int \left[w(u)\frac{du}{dx}\right] dx.$$

Another valuable technique is *integration by parts*, as described by the relation proved in frame **324**.

$$\int u\,dv = uv - \int v\,du.$$

Frequently a number of different integration procedures are used in a single problem as illustrated in frames **329–331**.

The method of *partial fractions* (frame **332**) involves splitting a function into a sum of fractions with simpler denominators that can be integrated by other methods.

3.3 Area under a Curve and Definite Integrals (frames 335–367)

The area A under the curve of a function $f(x)$ between $x = a$ and $x = b$ can be found by dividing the area into N narrow strips parallel to the y-axis, each of area $f(x_i)\ \Delta x$, and summing the strips. In the limit as the width of each strip approaches zero, the limit of the sum approaches the area under the curve.

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\Delta x.$$

This infinite sum is called a *definite integral* and is denoted by

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\Delta x.$$

The *area function* $A(x)$ has the variable x for the upper limit of the definite integral in frame **341**, and the integration variable is renamed u.

$$A(x) = \int_a^x f(u)du.$$

The area function $A(x)$ is an antiderivative of the function $f(x)$ (frame 346), $A'(x) = f(x)$. Therefore, it is an indefinite integral.

$$A(x) = \int_a^x f(u)\ du = F(x) + c,$$

where $F(x)$ is any particular antiderivative of $f(x)$, and c is an arbitrary constant. If we want to know the area bounded by $x = a$ and some value x, the constant c can be evaluated by noting that if $x = a$, then the area is zero, so $A(a) = F(a) + c = 0$ and $c = -F(a)$. Therefore, $A(x) = F(x) - F(a)$. The area under the curve between $x = a$ and $x = b$ is then

$$A(b) = A = \int_a^b f(x)dx = F(b) - F(a).$$

This result is called the *fundamental theorem of integral calculus* (frame **349**). The significance of the result in the last frame is that determining the area does not require calculating the limit of the summation in the definition of the definite integral but merely evaluating any antiderivative at the endpoints of the interval.

3.4 Some Applications of Integration (frames 368–389)

If we know $v(t)$, the *velocity* of a particle as a function of t, we can obtain the *position* of the particle as a function of time by integration. We saw earlier that $v = dS/dt$, so $dS = v\ dt$, and if we integrate both sides of the equation from the initial point $(t_0, S(t_0))$ to the final point $(t, S(t))$, we have the change of position of the particle, called the *displacement*,

$$S(t) - S(t_0) = \int_{t_0}^{t} v(t')dt',$$

where we have replaced the integration variable by t'.

Applications of integration in finding volumes of symmetric solids are given in frames **375–389**.

3.5 Multiple Integrals (frames 390–399)

Multiple integrals may be defined for an arbitrary number of independent variables. We discuss two variables because the procedures for an arbitrary number are merely generalizations of those that apply to two independent variables. The double integral over a region R in the $x - y$ plane of the function $f(x, y)$ is defined as

$$\iint_R f(x, y)\ dA = \lim_{\Delta A_i \to 0} \sum_{i=1}^{N} f(x_i, y_i)\ \Delta A_i,$$

as discussed in frames **390–391**. The double integral can be evaluated by integrating over one variable while holding the other variable constant, in either order.

$$\iint_R f(x, y)\ dA = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\ dy \right] dx.$$

Continue to Chapter **4**.

CHAPTER FOUR

Advanced Topics: Taylor Series, Numerical Integration, and Differential Equations

Chapters 1–3 established the fundamental concepts of calculus. This chapter introduces three tools for putting calculus into action. Taylor's theorem provides a mathematical tool for representing any function that can be differentiated into a form that lends itself to computation, numerical integration describes the first step in evaluating integrals that have no analytical solution, and differential equations introduces the mathematical structure in which scientific problems are frequently posed.

Go to **400**.

4.1 Taylor Series

400

The mathematical functions reviewed in Chapter 1—exponentials, logarithms, and the trigonometric functions—have been employed for centuries in astronomy, navigation, and surveying, as well as engineering and finance. Whatever the application, using the functions requires knowing their values. Until the advent of computers in the 1960s, these values had to be looked up in books of tables. Each entry in the books had to be calculated by hand. The effort was enormous and errors were difficult to detect. A small mistake could result in a shipwreck.

Today, books of tables are museum pieces. A pocket calculator can give a value for any argument that is accurate to eight or more decimal places in less than the blink of an eye. The method for calculation is based on a simple and elegant formula for the function called a *Taylor series* based on a mathematical theorem, *Taylor's theorem*. This amazing theorem allows one to calculate the value of a continuous function everywhere in a given range provided one knows the value of the function and its derivatives at any point in that range.

(continued)

Taylor's theorem states that if a function $f(x)$ is finite at $x = 0$, and has finite derivatives of every order in an interval about $x = 0$, then the function can be written as an infinite polynomial, a power series of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots.$$

The *degree* of a term in the polynomial is the power that x is raised to. In a Taylor series, the constant coefficients a_n are given by the product of $(1/n!)$ with the nth derivative evaluated at $x = 0$,

$$a_n = \frac{f^{(n)}(0)}{n!},$$

where the nth derivative is written as $f^{(n)} = \dfrac{d^n f}{dx^n}$. In expressions such as $f^{(n)}(0)$, the function is first differentiated n times and the result is then evaluated at the argument—here 0.

Recall that the factorial function $n! = (n)(n-1)(n-2)\cdots(1)$, and $0! = (1)$ was introduced in frame **249**. The series is often written more compactly using the summation notation introduced in frame **339**.

$$f(x) = \sum_{n=0}^{n=\infty} a_n x^n = \sum_{n=0}^{n=\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Often one can achieve high accuracy in surprisingly few terms. Terminating the series introduces an error, known as the *truncation error*. It can be shown that the truncation error is no larger than the first term omitted, although that's beyond the scope of this book. This discussion is limited to understanding how to calculate a Taylor series.

Go to **401**.

401

Demonstrating a Taylor Series: Calculating $f(x) = \sin x$:

To demonstrate how to calculate a Taylor series, we shall apply it to find the numerical value of $\sin x$. The function $\sin x$ and its higher order derivatives are continuous everywhere.

What is the first non-zero term $p_1(x)$ in the Taylor series expansion of $f(x) = \sin x$ about the point $x = 0$?

$$p_1(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Go to **402**.

402

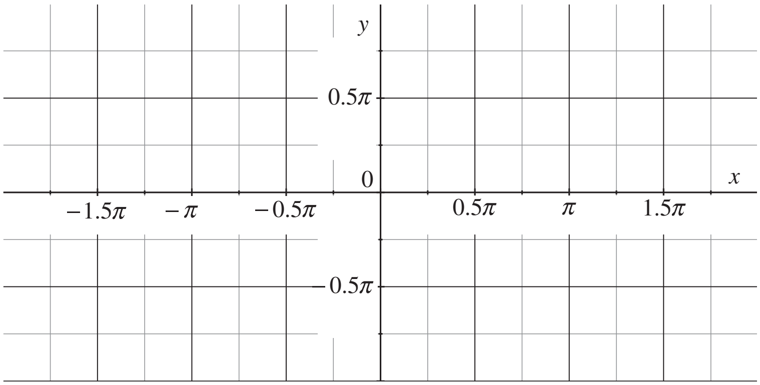
The value of $\sin(0) = 0$ and recall that the first derivative of $\sin x$ is $f^{(1)} = \cos x$. The first derivative evaluated $x = 0$ is $f^{(1)}(0) = \cos 0 = 1$. The first non-zero term of the Taylor series at $x = 0$ is then

$$p_1(x) = \sin 0 + \frac{f^{(1)}(0)}{1!}x = 0 + x = x.$$

Go to **403**.

403

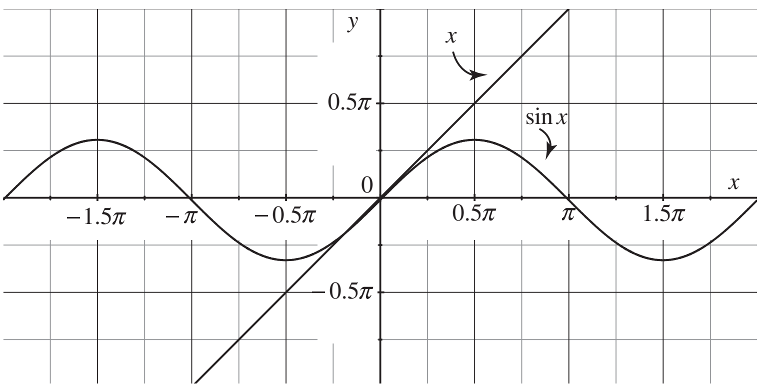
Make a sketch of $\sin x$ and x in the space provided.



Answer is in frame **404**.

404

Here are plots of $\sin x$ and x.



At $x = 0$, the values and first derivatives of $\sin x$ and x agree.

(continued)

Based on these plots, examine the accuracy of the approximation by estimating the largest possible value of x such that $\sin x$ is indistinguishable from x. For that value of x, calculate the difference $\sin x - x$.

$$x \simeq \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\sin x - x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Go to **405**.

405

Inspecting the plots in **404**, we estimate that when $x \approx \pi/10$ rad $= 18^\circ$, the polynomial $p_1(x) = x$ and $\sin x$ start to diverge. For the moment we note that for $x = \pi/10$ rad, $x - \sin x = .005142$, which shows that only one term in this series gives a 1% fractional error,

$$\frac{x - \sin x}{\sin x} = .01.$$

For sufficiently small values of x, the function and the polynomial agree well and we can make the approximation

$$\sin x \approx x.$$

We will use this approximation when describing the motion of a pendulum that is undergoing small angle oscillations (frame **442**).

Go to **406**.

406

We will now try to improve our approximation of $\sin x$. What are the first two non-zero terms $p_2(x)$ in the Taylor series expansion of $f(x) = \sin x$ about the point $x = 0$?

$$p_2(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Go to **407**.

407

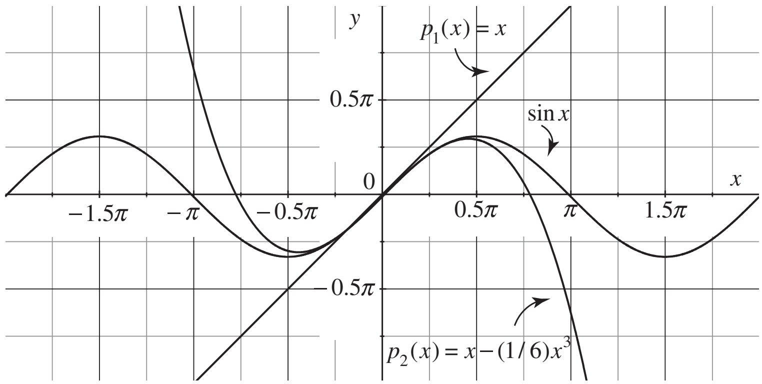
The second derivative of $\sin x$ vanishes at $x = 0$; $f^{(2)}(0) = -\sin 0 = 0$. Going on to the next term: the third derivative is non-zero, $f^{(3)}(0) = -\cos(0) = -1$. Therefore the first two non-zero terms are

$$p_2(x) = \frac{f^{(1)}(0)}{1!}x + \frac{f^{(3)}(0)}{3!}x^3 = x - \frac{1}{6}x^3.$$

Go to **408**.

408

Calculator Problem:
Here are plots of $\sin x$, $p_1(x) = x$ and $p_2(x) = x - (1/6)x^3$.



Based on these plots, the range of values that $\sin x$ agrees with $p_2(x) = x - (1/6)x^3$ has increased.

For $x = \pi/6$ rad $= 30°$, let's examine the difference between $p_1(x) = x$ and $f(x) = \sin x$, and compare that to the next non-vanishing term in the Taylor series, which is equal to $x^3/6$. Calculate the following quantities x, $\sin x$, $x - \sin x$, and $x^3/6$.

$x =$ ____________________

$\sin x =$ __________________

$x - \sin x =$ ________________

$x^3/6 =$ __________________

Go to **409**.

409

For $x = \pi/6$ rad $= 0.5236$, $\sin x = \sin(\pi/6) = 0.5$, $x - \sin x = 0.5236 - 0.5 = 0.0236$, and $x^3/6 = 0.0239$. So the difference between $p_1(x) - f(x) = x - \sin x$ is nearly equal to the next non-zero term in $p_2(x)$ term $x^3/6$.

Go to **410**.

410

We can continue this process and make better and better approximations to $\sin x$. The next polynomial is of degree 5 in powers of x given by

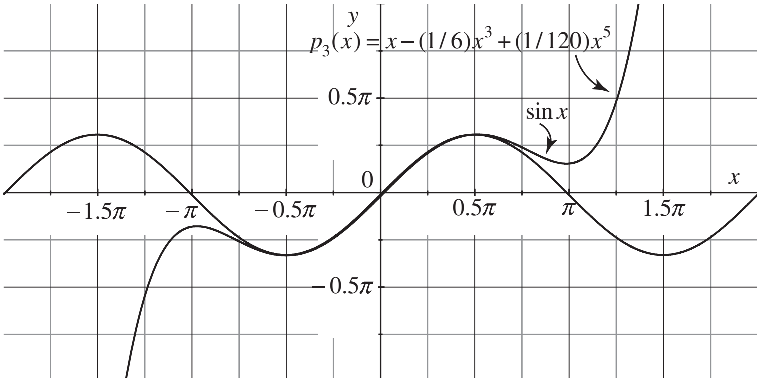
$$p_3(x) = x - (1/6)x^3 + (1/120)x^5.$$

(continued)

Note that the third term in $p_3(x)$ is

$$\frac{f^{(5)}(0)}{5!}x^5 = \frac{1}{120}x^5.$$

We show a plot of $\sin x$ and $p_3(x)$ in the figure below.



Go to **411**.

411

It would be extremely useful to have a general solution for the coefficient of the Taylor series for $\sin x$ because it would allow us to calculate values to any desired precision.

What is the $(2n-1)$-order term in the Taylor series for $\sin x$ about $x = 0$?

$$\frac{f^{(2n-1)}(0)}{(2n-1)!}x^{2n-1} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Go to **412**.

412

The $(2n-1)$-th derivative of $\sin x$ evaluated at $x = 0$ is

$$f^{(2n-1)}(0) = (-1)^{n-1}\cos 0 = (-1)^{n-1}.$$

Therefore the $(2n-1)$-order term in the Taylor series for $\sin x$ about $x = 0$ is

$$\frac{f^{(2n-1)}(0)}{(2n-1)!}x^{2n-1} = \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}.$$

Go to **413**.

413

Find the Taylor series, good to all orders, for $\sin x$ about $x = 0$, by calculating the first six terms:

$$\sin x = \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$+ \frac{f^{(5)}(0)}{5!}x^5 + \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1} + \cdots$$
$$\sin x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Go to **414**.

414

In order to find the coefficients of the Taylor series for $\sin(x)$ we need to take the function and all of the derivatives of $\sin(x)$ and evaluate them at $x = 0$. The first six terms are

$$\frac{f^{(0)}(0)}{0!} = \sin 0 = 0,$$
$$\frac{f^{(1)}(0)}{1!} = \cos 0 = 1,$$
$$\frac{f^{(2)}(0)}{2!} = -\frac{\sin 0}{2!} = 0,$$
$$\frac{f^{(3)}(0)}{3!} = -\frac{\cos 0}{6} = -\frac{1}{6},$$
$$\frac{f^{(4)}(0)}{4!} = \frac{\sin(0)}{4!} = 0,$$
$$\frac{f^{(5)}(0)}{5!} = +\frac{\cos(0)}{5!} = \frac{1}{120}.$$

Recall that by definition $0! = 1$. Higher order derivatives just replicate this pattern. Thus all even terms in powers of x are zero and all odd terms are non-zero and alternate in signs. The Taylor series is then

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots,$$

(continued)

where we used the result in **412** that

$$\frac{f^{(2n-1)}(0)}{(2n-1)!}x^{2n-1} = \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}.$$

Go to **415**.

415

Now let's try the same approach in order to find the Taylor series for logarthims. Because $\ln 0$ is not defined and $\ln 1 = 0$, we shall expand $\ln(1 + x)$ about $x = 0$.

Go to **416**.

416

We first evaluate the first five terms in the Taylor series:

$$\begin{aligned}
f^{(0)} &= \ln(1+x), & \frac{f^{(0)}(0)}{0!} &= \ln 1 = 0;\\
f^{(1)} &= \tfrac{1}{1+x}, & \frac{f^{(1)}(0)}{1!} &= 1;\\
f^{(2)} &= -\tfrac{1}{(1+x)^2}, & f^{(2)}(0) &= -1;\\
f^{(3)} &= (-2)(-1)\tfrac{1}{(1+x)^3}, & \frac{f^{(3)}(0)}{3!} &= 1;\\
f^{(4)} &= (-3)(-2)(-1)\tfrac{1}{(1+x)^4}, & \frac{f^{(4)}(0)}{3!} &= -1.
\end{aligned}$$

The derivatives alternate in sign. The nth derivative evaluated at $x = 0$ is proportional to $n!$ and that cancels the $n!$ term in the Taylor series. Therefore

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots.$$

Note that when $x << 1$, $\ln(1 + x) \approx x$. If you are in the mood, go to **417** to try to find the Taylor series for a couple of other well-known functions.

Go to **417**.

417

Find the Taylor series about $x = 0$ for the following functions:

$$\cos x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$e^x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Go to **418**.

418

Here are the Taylor series solutions for the functions in **417**.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots,$$
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots.$$

Go to **419**.

419

The Taylor series for a function $f(x)$ can be evaluated at any point where $f(x)$ is known, let's call that point $x = a$, provided $f(x)$ is infinitely differentiable in an interval about $x = a$, those derivatives are all finite at $x = a$, and $f(a)$ is also finite. Try to write down an expression for the Taylor series evaluated about the point $x = a$.

Go to **420**.

420

The Taylor series is most conveniently written in terms of the variable $u = x - a$. Then the point $u = 0$ corresponds to the point $x = a$, so that the known point for the function is $u = 0$. Writing the Taylor series for the variable u has the same form for the expansion about $x = 0$. Thus

$$\begin{aligned} f(x) &= \sum_{n=0}^{n=\infty} \left(\frac{f^{(n)}(a)}{n!}\right)(x-a)^n \\ &= f(a) + \left(\frac{f^{(1)}(a)}{1!}\right)(x-a) + \left(\frac{f^{(2)}(a)}{2!}\right)(x-a)^2 + \cdots. \end{aligned}$$

This procedure permits finding the Taylor series for a differentiable function provided its exact value is known somewhere.

Go to **421**.

4.2 Numerical Integration

421

In Chapter 3 we learned some techniques for integration, and the references listed in Appendix **B5** describe many others. There is, however, no general method for finding the indefinite integral of a function. Indefinite integrals of hundreds of functions are known and listed in integral tables. Many other functions can be integrated by a clever change of variable, which transforms them into one of the tabulated forms, but integrals for many other functions are simply not known. Nevertheless, definite integrals can always be evaluated numerically. With a computer, numerical integration is often so accurate and efficient that definite integrals can be calculated as easily as if they were already tabulated. Today, there are user-friendly software programs that will calculate definite integrals using methods of *numerical integration*. In this section, we describe an elementary and widely used technique of numerical integration called *Simpson's method*.

Go to **422**.

422

Recall from frame **340** that the area under a curve is given by definite integral, which is the limit of a sum

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\Delta x,$$

where $\Delta x = (b - a)/N$. As N increases and $\Delta x \to 0$, the area under the rectangles approaches the area under the curve. The problem is that the expression $\Delta x \to 0$ is a mathematical concept. We live in a world where the mathematical concept is unachievable. The result is that instead of equalities one has approximations. The challenge is to reduce the uncertainties and estimate the errors.

For a finite value of N, the area under the rectangle is not identical to the integral (unless $f(x)$ = constant), but it can be close. This is the basic idea of numerical integration, which follows the procedure we used to calculate the area under a curve:

1. Divide the interval $b - a$ into some convenient number N of equal intervals, $\Delta x = (b - a)/N$.
2. Evaluate $f_i = f(x_i)$ at each interval, where $i = 1,2, \ldots N$.
3. Multiply each f_i by Δx.
4. Add the results.

The final result is in an approximation to the integral. How good the approximation is depends on the choice of N and the precise method by which the sum is evaluated. The usual way to check the accuracy is simply to repeat the calculation taking smaller steps.

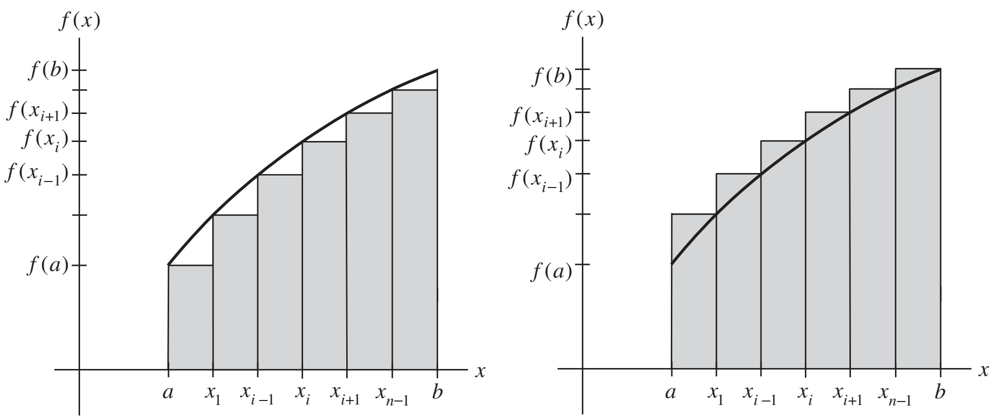
In carrying out the above steps, it may have already occurred to you that a great deal of multiplication is avoided if one first adds all the f_i's and then multiplies the final result by Δx. Thus,

$$S = \sum_{i=1}^{N} (f_i \Delta x) = \Delta x \sum_{i=1}^{N} f_i.$$

Go to **425**.

423

In evaluating the integral numerically, one could choose for f_i the value of f at either end of the interval, as in the drawings. For the function shown, it is evident that one choice underestimates the integral and the other overestimates it. Neither looks particularly accurate. Taking for f_i the value f at either end of the interval is clearly less good than taking it at the midpoint. However, an even better procedure would be to take a suitable weighted average of f at the ends and the middle.



An averaging process that is simple, accurate, and widely used considers the interval in pairs and weights the midpoint of each pair four times that of each end. In that case, for the interval $[x_{i-1}, x_{i+1}]$, the average value of the function is then

$$\bar{f}_i = \frac{1}{6}\left(f_{i-1} + 4f_i + f_{i+1}\right).$$

(continued)

(Note that the width of this pair of segments is $2\Delta x$, not Δx.) If the entire interval is divided into an even number of intervals, then

$$\int_a^b f(x)\,dx = \frac{2\Delta x}{6}\left((f_0 + 4f_1 + f_2) + (f_2 + 4f_3 + f_4) + \cdots + (f_{N-2} + 4f_{N-1} + f_N)\right)$$
$$= \frac{\Delta x}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 2f_{N-2} + 4f_{N-1} + f_N\right).$$

This method is called *Simpson's rule*. If you would like to know just why it works so well, go to **424**. Otherwise,

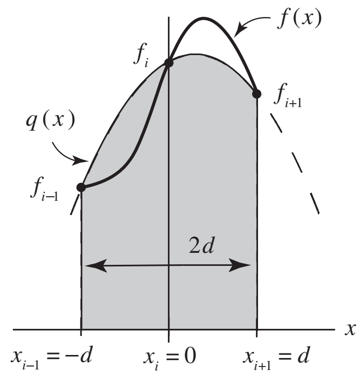
Go to **425**.

424

The reason that Simpson's rule works so well is based on the idea that the parabola is the simplest curve that can go through three arbitrary points:

$$q(x) = Ax^2 + Bx + C.$$

For simplicity, let $x_{i-1} = -d$, $x_i = 0$, and $x_{i+1} = d$ for the three adjacent points.



Integrate $q(x)$ to find the area

$$A_{area} = \int_{-d}^{d} q(x)dx = \int_{-d}^{d} (Ax^2 + Bx + C)dx$$
$$= \left(\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right)\Bigg|_{-d}^{d} = \frac{2Ad^3}{3} + 2Cd.$$

In order to solve for the constants A and C, demand that the parabolic curve agrees with the function at the three points $x_{i-1} = -d$, $x_i = 0$, and $x_{i+1} = d$. The three equations for the coefficients A, B, and C are then

$$f_{i-1} = Ad^2 - Bd + C, \tag{4.1}$$

$$f_i = C, \tag{4.2}$$

$$f_{i+1} = Ad^2 + Bd + C. \tag{4.3}$$

The algebra of solving for A, B, and C is straightforward. Solve for A by adding equations (4.1) and (4.3), and using equation (4.2):

$$A = \frac{1}{2d^2}(f_{i-1} + f_{i+1} - 2f_i). \tag{4.4}$$

Let $\bar{f}_i$ denote the average value of the function f such that the area under the curve for this double interval $2\Delta x = 2d$ is $A_{area} = \bar{f}_i 2d$. Then

$$\begin{aligned}\bar{f}_i 2d &= \frac{2Ad^3}{3} + 2Cd \\ &= \frac{2d^3}{3}\left(\frac{1}{2d^2}(f_{i-1} + f_{i+1} - 2f_i)\right) + 2f_i d \\ &= \frac{d}{3}(f_{i-1} + 4f_i + f_{i+1}).\end{aligned}$$

Therefore, by using these coefficients, we have the best fit of the parabola to the average value $\bar{f}_i$:

$$\bar{f}_i = \frac{1}{6}(f_{i-1} + 4f_i + f_{i+1}).$$

This is the rationale for the expression used in Simpson's rule. For an exercise demonstrating Simpson's rule, go to frame **442**.

Go to **425**.

4.3 Differential Equations

Any equation that involves a function and derivatives of that function is called a *differential equation*. Differential equations arise in practically every application of calculus to a real problem. Here are a few examples to show how a differential equation can occur and some methods for solving these equations. These examples, although elementary, arise again and again in science, engineering, and other disciplines.

Go to **425**.

425

Newton's Second Law and the Fundamental Theorem of Calculus:

When we apply a force to a particle that is moving in a straight line, the particle accelerates, according to Newton's second law, $F = ma$. Recall from **245** that acceleration is the derivative of velocity with respect to time, $a = dv/dt$. Thus, for a particle of mass m,

$$F = m\frac{dv}{dt}.$$

Go to **426**.

426

Here is an example of the motion of a particle in which a differential equation arises from Newton's second law and some methods for solving it, to find the velocity and position of the particle.

Consider the motion of a particle moving in a straight line with a *constant force F* acting on the particle along that line. Then the differential equation describing its motion, i.e. the rate of change of its velocity, can be written as

$$F = m\frac{dv}{dt} \qquad \text{or} \qquad \frac{dv}{dt} = \frac{F}{m}.$$

At time $t = 0$, the particle is located at distance x_0 from the origin and is traveling with velocity v_0. The problem is to find the velocity and position of the particle at time t.

One often sees Newton's second law written as $dv = (F/m)dt$, and then both sides of this equation are integrated to determine the change in velocity as a function of time. However, this is actually an application of the fundamental theorem of calculus (frame **349**) that

$$v(t) - v_0 = \int_{t'=0}^{t'=t} \frac{dv}{dt'}dt',$$

where $v_0 = v(0)$. We can use Newton's second law that $dv/dt = (F/m)dt$, and then perform the integration

$$v(t) - v(0) = \int_{t'=0}^{t'=t} \frac{dv}{dt'}dt' = \int_{t'=0}^{t'=t} \frac{F}{m}dt'.$$

Note that the time t' is treated as the integration variable and t is used as the upper limit of one of the integrals. Thus the velocity as a function of time is

$$v(t) = v_0 + \frac{F}{m}t.$$

Recall that velocity is the derivative of position

$$\frac{dx}{dt} = v_0 + \frac{F}{m}t.$$

Try to integrate this differential equation to find the position as a function of time, $x(t)$.

$$x(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Go to **427**.

427

The answer is

$$x(t) = x_0 + v_0 t + \frac{1}{2}\frac{F}{m}t^2.$$

If you answered this correctly, you have solved one of the most commonly used formulas for describing the motion of particle acted on by a constant force. Go on to frame **428**. Otherwise, read on.

The starting point is $v(t) = v_0 + (F/m)t$. (You may have forgotten the term v_0.) Now replace the velocity with $v = dx/dt$, which gives rise to the differential equation

$$\frac{dx}{dt} = v_0 + \frac{F}{m}t.$$

The definite integral can now be integrated:

$$x(t) - x_0 = \int_{t'=0}^{t'=t} \frac{dx}{dt'}dt' = \int_0^t \left(v_0 + \frac{F}{m}t'\right) dt' = v_0 t + \frac{1}{2}\frac{F}{m}t^2,$$

where $x_0 = x(0)$.

In summary, when the force F is constant, the position and velocity of the particle are

$$x(t) = x_0 + v_0 t + \frac{1}{2}\frac{F}{m}t^2,$$
$$v(t) = v_0 + \frac{F}{m}t.$$

Go to **428**.

428

Exponential Growth:

Suppose we let N represent the population (number of people) in a particular country. We assume that N is such a large number that we can neglect the fact that it must be an integer and treat it as a continuous positive number. (In any application we would eventually round off N to the nearest integer.)

The problem is this: assume the birthrate (the average number of babies born per year) is proportional to the number of people N, where A is the constant of proportionality. That is

$$\frac{dN}{dt} = AN.$$

If the initial population of the country is n_0 people, how many people are there at some later time, T? (In this problem we will neglect deaths.)

The above differential equation only involves first derivatives. We can solve it by first writing

$$\frac{dN}{N} = A \; dt.$$

At time $t = 0$, the population is $N = N_0$. Let $N(T)$ denote the number of people at time T. Integrate both sides of this equation,

$$\int_{N_0}^{N(T)} \frac{dN}{N} = \int_0^T A \; dt.$$

The integral on the left should be familiar (if not, see frame **333**, integral 5). Evaluating both integrals, we have

$$\ln \; N(T) - \ln \; N_0 = A(T - 0).$$

Thus

$$\ln\left(\frac{N(T)}{N_0}\right) = AT.$$

This equation has the form $\ln x = AT$, where we have set $x = N(T)/N_0$. We can solve this for x by taking the exponential of the two sides of this equation, using the relation $x = e^{\ln x}$. Thus, $x = e^{AT}$, and we have

$$N(T) = N_0 e^{AT}.$$

This expression describes the exponential increase of population.

Go to **429**.

429

Let's restate the differential equation in frame **428** using different variables. Suppose x is some independent variable, y depends on x, and the derivative of y with respect to x is

$$\frac{dy}{dx} = \pm bx,$$

where b is a positive constant of proportionality. The $\pm$ sign indicates that this differential equation describes two different "exponential" cases: the $+b$ is a rate of growth and $-b$ is a rate of diminishment. Let y_0 represent the value of y at $x = 0$. The solution of the differential equation is

$$y(x) = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Go to **430**.

430

The answer is

$$y(x) = y_0 e^{\pm bx}.$$

Expressions of similar form describe many processes, for instance, the growth of money in banks due to interest or the radioactive decay of atomic nuclei.

Check your answer by taking the derivative using $\dfrac{de^{\pm bx}}{dx} = \pm be^{\pm bx}$.

Go to **431**.

431

If you wrote

$$\frac{dy}{dx} = y_0 \frac{d}{dx} e^{\pm bx} = \pm b y_0 e^{\pm bx} = \pm by,$$

you are correct and you can now describe many different types of systems in which the derivative of the independent variable is proportional to plus or minus itself. The positive solution describes population growth, although other effects must be considered since population can't grow indefinitely. The negative sign indicates population decay, and this could go on indefinitely until the assumption that N is a continuous variable breaks down.

Go to **432**.

432

Radioactive Decay:

Our next example is *radioactive decay*. An unstable atom may spontaneously decay into a stable atom. The decay of individual atoms are random events; but, for a large number N of atoms, the atoms decay at a rate proportional to the total number,

$$\frac{dN}{dt} = -\lambda N.$$

The constant of proportionality λ is called the *decay constant*, and has units of $[\text{time}]^{-1}$. The negative sign indicates that atoms are being lost.

Let N_0 denote the number of unstable atoms at time $t = 0$. Find the number of unstable atoms $N(t)$ at time t.

$$N(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

[You may want to review frames **429–430**.] Go to **433**.

433

The answer is

$$N(t) = N_0 e^{-\lambda t}.$$

If you got this answer, go on to **434**. If you want to see how to derive it, read on.

To find $N(t)$ you need to integrate the governing equation, $dN/N = -\lambda dt$, from the initial time $t_0 = 0$ to a later time t, when the initial number N_0 changes to the final number $N(t)$:

$$\int_{N_0}^{N(t)} \frac{dN}{N} = -\int_0^t \lambda dt.$$

Integrating both sides yields

$$\ln N\Big|_{N_0}^{N(t)} = \ln N(t) - \ln N_0 = \ln\left(\frac{N(t)}{N_0}\right) = -\lambda t.$$

Taking exponentials of both sides of this equation gives

$$e^{\ln(N(t)/N_0)} = e^{-\lambda t}.$$

Applying the defining equation for the exponential (frame **94**) $e^{\ln x} = x$, which in this instance is $e^{\ln(N(t)/N_0)} = N(t)/N_0$, and solving for $N(t)$ yields

$$N(t) = N_0 e^{-\lambda t}.$$

Go to **434**.

434

The constant $\tau = 1/\lambda$ is called the *lifetime*. If N_0 denotes the number of unstable atoms at time $t = 0$, how many unstable atoms are present after one lifetime has passed, i.e. at $t = \tau$?

$$N(\tau) = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Go to **435**.

435

When $t = \tau$, the number of unstable atoms is

$$N(\tau) = N_0 e^{-\tau/\tau} = N_0 e^{-1} = 0.368 N_0.$$

Let $t = \tau_{1/2}$ denote the time in which half of the initial atoms decay. $\tau_{1/2}$ is called the *half-life*. How is the half-life related to the lifetime $\tau = 1/\lambda$?

$$\tau_{1/2} = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Go to **436**.

436

Set $N(\tau_{1/2}) = (1/2)N_0$. Then the condition that $\ln(N(\tau_{1/2})/N_0) = -\tau_{1/2}/\tau$ becomes $\ln(1/2) = -\tau_{1/2}/\tau$. Recall that $\ln(1/2) = -\ln(2)$. Therefore $\tau_{1/2} = \tau \ln(2)$. The value of $\ln(2)$ is 0.693, so the half-life, $\tau_{1/2}$, is given by $\tau_{1/2} = 0.693\tau$.

Go to **437**.

437

Simple Harmonic Motion:

Our next examples of a differential equation involve second derivatives.

Consider the motion of a particle in one dimension. For example, let x be the coordinate variable that describes the position of a particle relative to the origin $x = 0$, where x depends on the variable t. Suppose we require that the second derivative of x with respect to time is proportional to minus itself, where b is the constant of proportionality. Then the position of the particle satisfies the following differential equation:

$$\frac{d^2x}{dt^2} = -bx, \tag{4.5}$$

where b is a constant of proportionality. This equation is called the *simple harmonic oscillator equation*. It describes the motion of a pendulum, or a particle suspended by a spring, and many other kinds of physical phenomena.

(continued)

The problem is to find out how x varies with time when it obeys this equation, i.e. to solve the differential equation. One of the most fruitful means for solving differential equations is to guess a possible general form for the solution. Then this general form is substituted in the differential equation and one can determine whether this equation is satisfied and what restrictions apply to the solution.

First, what is a promising guess as to a solution? Note that x must depend upon time in such a way that when it is differentiated twice with respect to time it reverses sign. But this is exactly what happens to the sine function since $\frac{d}{dx}\sin x = \cos x$, and $\frac{d^2}{dx^2}\sin x = \frac{d}{dx}\cos x = -\sin x$ (frame **212**).

Because the argument of a trigonometric function must be dimensionless, let us introduce a constant ω that has the dimensions of $(\text{time})^{-1}$. So our trial solution becomes

$$x(t) = A \quad \sin(\omega t + c),$$

where A and c are undetermined constants. Note that if we had tried the solution

$$x(t) = A\sin(\omega t) + B\cos(\omega t),$$

we would have found that it was also a solution to our differential equation. However, for our present purposes the $A \quad \sin(\omega t + c)$ solution is adequate.

This may be differentiated twice with respect to time with the result

$$\frac{dx}{dt} = A\omega\cos(\omega t + c),$$

$$\frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t + c).$$

If these relations are substituted in the differential equation (4.1), above, we have

$$-A\omega^2\sin(\omega t + c) = -bA\sin(\omega t + c),$$

which is then satisfied for all t provided

$$\omega = \sqrt{b}.$$

(Alternatively, the equation is satisfied by $A = 0$. However, this leads to a trivial result, $x = 0$, so we disregard it.) Thus the solution is

$$x = A\sin(\omega t + c).$$

The constant ω is called the *angular frequency*, the constant A is called the *amplitude*, and the constant c is called the *phase constant*. If x and dx/dt are specified at some initial time, $t = 0$, the arbitrary constants A and c can be determined.

Note that the solution we have found corresponds to x oscillating back and forth indefinitely between $x = A$ and $x = -A$ with period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{b}}.$$

This type of oscillatory motion, *simple harmonic motion*, is characteristic of a particle suspended by a spring, or a pendulum swinging through a small angle. Many other systems are described by this differential equation, for example, an electrical circuit consisting of an inductor and a capacitor or a buoy bobbing on the surface of the sea.

Go to **438**.

438

Here is an example of simple harmonic motion.

An object on a smooth (frictionless) table is attached to a spring that is fixed. If the object is displaced, stretching the spring, and released, the object will oscillate abut the unstretched position, undergoing simple harmonic motion. Using Newton's second law we can determine the differential equation whose solution describes the motion. The spring exerts a restoring force that is proportional to the amount stretched, $F = -kx$ where k is the spring constant k and x is measured from the equilibrium point.

Newton's second law, $F = ma$, becomes $-kx = m(d^2x/dt^2)$. This can be written as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

What is the period of oscillation?

$T =$ ________________

Go to **439**.

439

Compare the differential equation describing the spring-object system with the general form of the simple harmonic oscillator equation in frame **437**, $\frac{d^2x}{dt^2} = -bx$. The constant $b = \frac{k}{m}$, so $\omega = \sqrt{\frac{k}{m}}$. Hence the period is $T = \frac{2\pi}{\sqrt{k/m}}$.

Go to **440**.

4.4 Additional Problems for Chapter 4

440

Taylor Series:

Find the first four terms in the Taylor series about $x = 0$ for the following function:

$$\frac{1}{(1+x)^{1/2}} = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Go to **441**.

441

The first four Taylor coefficients are

$$\frac{f^{(0)}}{0!} = \frac{1}{(1+x)^{1/2}}, \qquad a_0 = \frac{f^{(0)}(0)}{0!} = 1;$$

$$\frac{f^{(1)}}{1!} = -\frac{1}{2}\frac{1}{(1+x)^{3/2}}, \qquad a_1 = \frac{f^{(1)}(0)}{1!} = -\frac{1}{2};$$

$$\frac{f^{(2)}}{2!} = \left(\frac{1}{2!}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\frac{1}{(1+x)^{5/2}}, \qquad a_2 = \frac{f^{(2)}(0)}{2!} = \frac{3}{8};$$

$$\frac{f^{(3)}}{3!} = \left(\frac{1}{3!}\right)\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\frac{1}{(1+x)^{7/2}}, \qquad a_3 = \frac{f^{(3)}(0)}{3!} = -\frac{5}{16}.$$

The first four terms in the Taylor series for the function in **440** are

$$\frac{1}{(1+x)^{1/2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots.$$

Go to **442**.

442

Numerical Integration:

Here is an example of how Simpson's rule can be used to evaluate a definite integral.

The goal is to calculate $I = \int_0^{10} x^3 \;\; dx$. We can do this integral exactly, which makes it easy to check the accuracy of the numerical calculation.

$$I = \int_0^{10} x^4 \;\; dx = \frac{1}{5}x^5\Big|_0^{10} = \frac{1}{5}(100{,}000) = 20{,}000.$$

We shall take $N = 10$. Then, $\Delta x = \frac{10-0}{10} = 1$. $x_0 = 0$, $x_{10} = 10$, and in general, $x_i = i\Delta x = i$. If we denote the sum of the odd terms by

$$S_{\text{odd}} = f_1 + f_3 + f_5 + f_7 + f_9,$$

and the sum of the even terms within the interval by

$$S_{\text{even}} = f_2 + f_4 + f_6 + f_8,$$

then, by Simpson's rule (frame **423**), the approximation to the integral is

$$S = \frac{\Delta x}{3}(f_0 + 4S_{\text{odd}} + 2S_{\text{even}} + f_{10}).$$

S can be calculated using the tables below.

***i* odd**	x_i	$f_i = x_i^4$		***i* even**	x_i	$f_i = x_i^4$
1				2		
3				4		
5				6		
7				8		
9						

Then determine $S_{\text{odd}} =$ _______________, and $S_{even} =$ _____________.
Now tabulate the results.

$f_0 = x_0^4$	
$4S_{\text{odd}}$	
$2S_{\text{even}}$	
$f_{10} = x_{10}^4$	

The sum of the four terms is

$$f_0 + 4S_{\text{odd}} + 2S_{\text{even}} + f_{10} = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}.$$

Then the numerical integration result is

$$S = \frac{\Delta x}{3}(f_0 + 4S_{\text{odd}} + 2S_{\text{even}} + f_{10}) = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}.$$

Go to **443**.

443

S can be calculated using the tables below.

i odd	x_i	$f_i = x_i^4$		*i* even	x_i	$f_i = x_i^4$
1	1	1		2	2	16
3	3	81		4	4	256
5	5	625		6	6	1296
7	7	2401		8	8	4096
9	9	6561				

Then $S_{\text{odd}} = 9669$, and $S_{\text{even}} = 5664$, and

$f_0 = x_0^4$	0
$4S_{\text{odd}}$	38676
$2S_{\text{even}}$	11328
$f_{10} = x_{10}^4$	10000

The sum of the four terms is $f_0 + 4S_{\text{odd}} + 2S_{\text{even}} + f_{10} = 60004$, hence the answer is

$$S = \frac{\Delta x}{3}(f_0 + 4S_{\text{odd}} + 2S_{\text{even}} + f_{10}) = \frac{60{,}004}{3} = 20{,}001\,{}^1\!/_3.$$

This result is close to the exact value of the integral, $I = 20{,}000$. Considering the relatively small number of points used, this is remarkably accurate: one part in 2×10^4.

An interesting exercise in numerical integration you might want to try on your own is to evaluate π using the relation

$$\tan^{-1} A = \int_0^A \frac{dx}{1 + x^2},$$

which follows from formula 17 in frame **312**. Because $\pi/4 = \tan^{-1}(1)$, one has

$$\pi = 4 \int_0^1 \frac{dx}{1 + x^2}.$$

You could try your skill by integrating other functions whose integrals you know, for instance, $\sin\theta$ or e^{-x}.

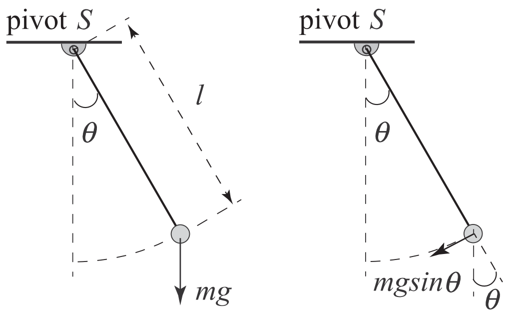
It is evident that by numerical integration you can find the definite integral of *any* differentiable function, and therein lies its power. With computers it is possible to integrate numerically at very high speed. One must have some criterion for choosing the interval size and be able to deal with problems such as singularities in the integral. Nevertheless, with the simple method described here you can often do surprisingly well.

Go to **444**.

444

The Simple Pendulum:

A simple pendulum consists of a massless string of length l attached to a pivot and a point-like object of mass m hanging from the other end. Suppose the string initially makes a small angle θ_0 from the vertical, and then the mass is released from rest.



Because the object moves in a circle, we can apply Newton's second law of motion, in the tangential direction,

$$F_{\tan} = ma_{\tan}.$$

Recall from **288**, the tangential acceleration is $a_{\tan} = l\frac{d^2\theta}{dt^2}$. When the object makes an angle θ with the vertical, the tangential force is the component of the gravitational force, mg in the tangential direction, $mg\sin\theta$. This is a restoring force in the sense that it always is directed to the equilibrium vertical position. Newton's second law is then

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta.$$

When the angle of oscillation is small ($\theta(t) << 1$, measured in radians), we can use the small angle approximation $\sin\theta \approx \theta$ (frame **405**). Then

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta.$$

(continued)

What is the period of oscillation of the object? The acceleration of gravity is about $9.8\ \text{m/s}^2$.

$T =$ ___________________

Go to **445**.

445

The differential equation has the same form as in **437** with x replaced by θ and $b = g/l$. Therefore $\omega = \sqrt{g/l}$. Hence the period is $T = \frac{2\pi}{\sqrt{g/l}}$.

Summary of Chapter 4

Let's conclude the chapter by reviewing some of the ideas it introduced.

4.1 Taylor's Theorem (frames 400–417)

Taylor's theorem states that if a function $f(x)$ is finite at $x = 0$, and has finite derivatives of every order in an interval about $x = 0$, then the function can be written as an infinite polynomial—a power series called the *Taylor series about* $x = 0$—with the form

$$f(x) = \sum_{n=0}^{n=\infty} a_n x^n = \sum_{n=0}^{n=\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

where the nth derivative is written as $f^{(n)} = \frac{d^n f}{dx^n}$. In expressions such as $f^{(n)}(0)$, the function is first differentiated n times and the result is then evaluated at the argument—here 0. Examples of Taylor series about $x = 0$ for some well-known functions are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots,$$

$$\frac{1}{(1+x)^{1/2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots.$$

4.2 Numerical Integration (frames 419–422)

When it is not possible to find an analytic expression for the definite integral, the definite integral is often evaluated by methods of numerical integration. A particularly effective method is Simpson's rule in which the entire interval over which the integral is to be evaluated is divided into an even number N of equal intervals of width Δx. By Simpson's rule the numerical value of the integral is approximately given by

$$\int_A^B y(x)\ dx = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{N-2} + 4y_{N-1} + y_N).$$

The accuracy of the approximation can be increased by increasing N with a corresponding decrease in Δx, but with a corresponding increase in numerical work.

4.3 Differential Equations (frames 423–437)

Any equation that involves a function and derivatives of that function is called a *differential equation*. Differential equations arise in practically every application of calculus to real-world problems.

Newton's second law is a differential equation:

$$F = m\frac{dv}{dt} \quad .$$

When the force F is constant, the position and velocity of the particle are

$$x(t) = x_0 + v_0 t + \frac{1}{2}\frac{F}{m}t^2,$$

$$v(t) = v_0 + \frac{F}{m}t.$$

The exponential decay or growth equation

$$\frac{dy}{dx} = \pm bx$$

has a solution of the form

$$y(x) = y_0 e^{\pm bx},$$

where b is a positive constant of proportionality, and y_0 represents the value of y at $x = 0$.

The *simple harmonic oscillator equation*

$$\frac{d^2x}{dt^2} = -bx$$

describes many kinds of physical phenomena, for instance, a pendulum swinging through a small angle or an object attached to a spring that oscillates. The constant b is determined by the particular application. The solution is

$$x = A \quad \sin(\omega t + c),$$

where $\omega = \sqrt{b}$ is the *angular frequency*. The constant A is called the *amplitude,* and the arbitrary constant c is called the *phase constant*. Both amplitude and phase are determined if x and dx/dt are specified at some time.

The solution corresponds to x oscillating back and forth indefinitely between $x = A$ and $x = -A$ with period.

Conclusion (frame 449)

Congratulations! You are now finished. However, if you skipped some of the proofs in Appendix **A**, you might want to read them now. You may also want to study some of the additional topics that are described in Appendix **B**. The appendixes are crammed with derivations of formulas, explanations of special topics, and the like.

Finally, if you would like a little more practice, try some of the review problems at the end of the book, starting on page 277.

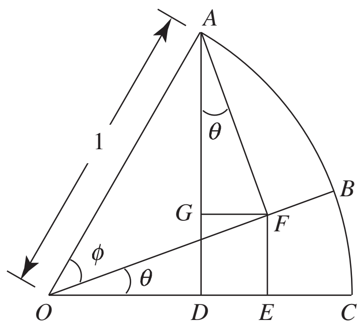
Good luck!

APPENDIX A
Derivations

This appendix presents derivations of certain formulas and theorems not derived earlier.

A.1 Trigonometric Functions of Sums of Angles

Formulas for the sine of the sum of two angles, θ and ϕ, can easily be derived with the aid of this drawing in which the radius of the circle is unity.



$$\begin{aligned}\sin(\theta+\phi) &= AD = FE + AG\\ &= OF\sin\theta + AF\cos\theta\\ &= \sin\theta\cos\phi + \cos\theta\sin\phi.\end{aligned}$$

In a similar fashion with the same figure,

$$\begin{aligned}\cos(\theta+\phi) &= OD = OE - DE\\ &= OF\cos\theta - AF\sin\theta\\ &= \cos\theta\cos\phi - \sin\theta\sin\phi.\end{aligned}$$

A.2 Some Theorems on Limits

In this appendix we prove several theorems on limits, which show that the usual algebraic manipulations can be carried out with expressions involving limits. We shall show, for example, that

$$\lim_{x\to a}[F(x) + G(x)] = \lim_{x\to a} F(x) + \lim_{x\to a} G(x).$$

Such results are intuitively reasonable but deserve a formal proof.

Before deriving these theorems, we need to note some general properties of the absolute value function introduced in frame **20**. These properties are

$$|a+b| \leq |a| + |b|, \tag{A.1}$$

$$|ab| = |a||b|. \tag{A.2}$$

These relations are easily verified by considering all the possible cases: a and b both negative, both positive, of opposite sign, and one or both equal to zero.

The following are theorems on limits that apply to any two functions F and G such that

$$\lim_{x\to a} F(x) = L \quad \text{and} \quad \lim_{x\to a} G(x) = M.$$

Theorem 1

$$\lim_{x\to a}[F(x) + G(x)] = \lim_{x\to a} F(x) + \lim_{x\to a} G(x).$$

Proof: By Equation (A.1),

$$\begin{aligned}|F(x) + G(x) - (L+M)| &= |[F(x) - L] + [G(x) - M]| \\ &\leq |F(x) - L| + |G(x) - M|.\end{aligned}$$

Using the definition of the limit (frame **105**) we see that for any positive number ε we can find a positive number δ such that

$$|F(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |G(x) - M| < \frac{\varepsilon}{2},$$

provided $0 < |x - a| < \delta$. (At first sight this may appear to differ from the definition of the limit since the symbol ε instead of $\varepsilon/2$ appeared there. However, the statements apply for *any* positive number and $\varepsilon/2$ is also a positive number.)

The above equations may be combined to give

$$|F(x) + G(x) - (L + M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, by the definition of the limit in frame **105**,

$$\lim_{x\to a}[F(x) + G(x)] = L + M = \lim_{x\to a} F(x) + \lim_{x\to a} G(x).$$

Theorem 2

$$\lim_{x\to a}[F(x)G(x)] = [\lim_{x\to a} F(x)][\lim_{x\to a} G(x)].$$

Proof: The proof is somewhat similar to the preceding. By writing out all the terms, we can see that the following is true identically:

$$F(x)\,G(x) - LM = [F(x) - L][G(x) - M] + L[G(x) - M] + M[F(x) - L].$$

Therefore, by Equation (A.1),

$$\begin{aligned}&|F(x)G(x) - LM|\\ &\quad \le |[F(x) - L][G(x) - M]| + |L[G(x) - M]| + |M[F(x) - L]|.\end{aligned}$$

Let ε be any positive number less than 1. Then by the meaning of limits we can find a positive number δ such that if $0 < |x - a| < \delta$,

$$|F(x) - L| < \frac{\varepsilon}{2}, \quad |L[G(x) - M]| < \frac{\varepsilon}{4}, \quad |M[F(x) - L]| < \frac{\varepsilon}{4},$$

and $|G(x) - M| < \frac{\varepsilon}{2}$. Then

$$|F(x)\,G(x) - LM| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \frac{3}{4}\varepsilon,$$

where the next to the last step arises as a result of our earlier restriction to $\varepsilon < 1$. Consequently, $|F(x)\,G(x) - LM| < \varepsilon$ so by the definition of the limit,

$$\lim_{x\to a}[F(x)\,G(x)] = LM = [\lim_{x\to a} F(x)]\;[\lim_{x\to a} G(x)].$$

Theorem 3

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \frac{\lim_{x\to a} F(x)}{\lim_{x\to a} G(x)} \quad \text{provided } \lim_{x\to a} G(x) \neq 0.$$

Proof: Since $\lim_{x\to a} G(x) \neq 0$, we can select a value of δ sufficiently small that $G(x) \neq 0$ for $0 < |x - a| < \delta$. Then we can write

$$\lim_{x\to a} F(x) = \lim_{x\to a}\left[G(x)\frac{F(x)}{G(x)}\right] = \lim_{x\to a} G(x) \lim_{x\to a}\frac{F(x)}{G(x)} = M \lim_{x\to a}\frac{F(x)}{G(x)},$$

where $M = \lim_{x\to a} G(x)$. Therefore, since $M \neq 0$, we have

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \frac{\lim_{x\to a} F(x)}{M} = \frac{\lim_{x\to a} F(x)}{\lim_{x\to a} G(x)}.$$

Note that if $M = 0$, this expression is meaningless, and we must evaluate $F(x)/G(x)$ before taking the limit.

A.3 Exponential Function

The examples of exponential growth and decay in several different scenarios that were described in Chapter 4, frames **428–436**, all display similar behavior: if a population of N "things" change—growing or decaying—at a steady rate γ (Greek letter gamma), then their number evolves in time as $N(t) = N_0 e^{\pm\gamma t}$. The result came from the solution of a differential equation and exemplifies a common approach to analyzing such behavior. Here we present a totally different approach to such problems that is based on a mathematical rather than a physical argument.

For concreteness we discussed the growth of money in a bank because of interest the bank pays (frames **299–302**). If the bank offers interest at a rate of b percent per year, then at the end of the year an initial deposit of $D(0)$ will be worth $D(1) = D(0)(1 + b)$. However, you might be able to get the bank to pay the interest *twice* a year in which case the money is $D(1) = D(0)(1 + b/2)^2$, which is a little bigger than for a single payment. Suppose you push your luck and ask for the interest to be compounded weekly, then daily, then hourly, and in fact go to the extreme of compounding it continually. We might expect the money to increase dramatically, but it does not. To see what happens, we can calculate the interest for N times a year and then find the limit as N goes to infinity ($N \to \infty$).

We will show that

$$\lim_{N\to\infty}\left(1 + \frac{b}{N}\right)^N = e^b,$$

where $f(b) = e^b$ is known as the *exponential function*.

Proof: Let's begin by proving that

$$\lim_{y \to 0} \frac{\ln(1 + by)}{y} = b.$$

We use the Taylor series for $\ln(1 + x)$ about $x = 0$ (frame **416**) where $x = by$,

$$\ln(1 + by) = by - \frac{1}{2}(by)^2 + \cdots.$$

Then

$$\lim_{y \to 0} \frac{\ln(1 + by)}{y} = \lim_{y \to 0} \frac{by - \frac{1}{2}(by)^2 + \cdots}{y} = b.$$

Define a^x for all $a > 0$ raised to any real number x by $a^x = e^{x \ln a}$. Let $a = 1 + by$ and $x = 1/y$. By our definition of a^x, $(1 + by)^{1/y} = e^{(1/y)\ln(1+by)}$, so $e^{\lim_{y \to 0}\left(\frac{\ln(1+by)}{y}\right)} = e^b$. Thus

$$\lim_{y \to 0}\left((1 + by)^{1/y}\right) = e^{\lim_{y \to 0}\left(\frac{\ln(1+by)}{y}\right)} = e^b.$$

Let $y = 1/N$. Then

$$\lim_{N \to \infty}\left(1 + \frac{b}{N}\right)^N = e^b.$$

A.4 Proof That $\frac{dy}{dx} = \frac{1}{dx/dy}$

If a function is specified by an equation $y = f(x)$, it is ordinarily possible, for at least limited intervals of x, to reverse the roles of the dependent and independent variables and to allow the equation to determine the value of x for a given value of y. (This cannot always be done, for instance as in the case of the equation $y = a$, where a is a constant.) When such an inversion is possible, the two derivatives are related by

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

This relation can be seen as follows:

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x/\Delta y} = \frac{1}{\lim\limits_{\Delta x\to 0}(\Delta x/\Delta y)}$$

by the limit theorems of Appendix **A2**. Furthermore, if $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} \neq 0$, then $\Delta y \to 0$ as $\Delta x \to 0$, so

$$\frac{dy}{dx} = \frac{1}{\lim\limits_{\Delta y\to 0}(\Delta x/\Delta y)} = \frac{1}{dx/dy}.$$

This result is a further justification of the use of differential notation because normal arithmetic manipulation with differential notation immediately gives

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

A.5 Differentiating x^n

Consider first the case of n a positive integer.

$$y = x^n. \tag{A.3}$$

Let $y(x + \Delta x) = (x + \Delta x)^n$. The right side can be expanded by the binomial theorem (this theorem is proved in any good algebra text):

$$\begin{aligned} y(x + \Delta x) &= (x + \Delta x)^n \\ &= x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{1\cdot 2}x^{n-2}\Delta x^2 + \cdots + \Delta x^n. \end{aligned} \tag{A.4}$$

Subtracting Equation (A.3) from Equation (A.4) and dividing by Δx, yields

$$\frac{y(x + \Delta x) - y(x)}{\Delta x} = \frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{1\cdot 2}x^{n-2}\Delta x + \cdots + \Delta x^{n-1}.$$

Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}.$$

This result has been proved only for n being a positive integer, but we can generally show that it holds for any positive number. We start by showing that it is true for $n = 1/q$ where q is a positive integer. Let $y = x^{1/q}$ so $x = y^q$. By the preceding theorem, then, $\frac{dx}{dy} = qy^{q-1}$. By the result in Appendix **A4**

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{qy^{q-1}} = \frac{1}{q}y^{1-q} = \frac{1}{q}(x^{1/q})^{1-q}$$

$$\frac{dy}{dx} = \frac{1}{q}x^{(1/q)-1} = nx^{n-1}.$$

We can further see that this theorem holds for $n = p/q$ where p and q are both positive integers. Let $y = x^n = x^{p/q}$ and $w = x^{1/q}$, hence $y = w^p$. Then

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx} = pw^{p-1}\left(\frac{1}{q}\right)x^{(1/q)-1} = px^{(p/q)-(1/q)}\left(\frac{1}{q}\right)x^{(1/q-1)}$$

$$= \left(\frac{p}{q}\right)x^{(p/q)-1} = nx^{n-1}.$$

So far, we have seen that the rule for differentiating x^n applies if n is any positive fraction. We will now see that it applies for negative fractions as well. Let $n = -m$, where m is a positive fraction. Then

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} = \frac{d}{dx}\left(\frac{1}{x^m}\right) = -\frac{1}{(x^m)^2}\frac{d}{dx}x^m$$

$$= -\frac{mx^{m-1}}{x^{2m}} = -mx^{-m-1} = nx^{n-1}.$$

Up to now our discussion applies if n is any rational number. However, the result may be extended to any irrational real number by the methods used in frame **84**. Therefore, for any real number n, whether rational or irrational, and regardless of sign,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

A.6 Differentiating Trigonometric Functions

From Appendix **A1**,

$$\begin{aligned}\frac{d}{d\theta}\sin\theta &= \lim_{\Delta\theta\to 0}\frac{\sin(\theta+\Delta\theta)-\sin\theta}{\Delta\theta}\\ &= \lim_{\Delta\theta\to 0}\frac{\sin\theta\cos\Delta\theta+\cos\theta\sin\Delta\theta-\sin\theta}{\Delta\theta}\\ &= \sin\theta\lim_{\Delta\theta\to 0}\frac{\cos\Delta\theta-1}{\Delta\theta}+\cos\theta\lim_{\Delta\theta\to 0}\frac{\sin\Delta\theta}{\Delta\theta}.\end{aligned}$$

The two limits were evaluated in Appendix **A3** as 0 and 1, respectively, so

$$\frac{d}{d\theta}\sin\theta = \cos\theta.$$

Likewise,

$$\begin{aligned}\frac{d}{d\theta}\cos\theta &= \lim_{\Delta\theta\to 0}\frac{\cos(\theta+\Delta\theta)-\cos\theta}{\Delta\theta}\\ &= \lim_{\Delta\theta\to 0}\frac{\cos\theta\cos\Delta\theta-\sin\theta\sin\Delta\theta-\cos\theta}{\Delta\theta}\\ &= \cos\theta\lim_{\Delta\theta\to 0}\frac{\cos\Delta\theta-1}{\Delta\theta}-\sin\theta\lim_{\Delta\theta\to 0}\frac{\sin\Delta\theta}{\Delta\theta}\\ &= -\sin\theta.\end{aligned}$$

Derivatives of other trigonometric functions can be found by expressing them in terms of sines and cosines, as in Chapter 2.

A.7 Differentiating the Product of Two Functions

Let $y = uv$, where u and v are variables that depend on x. Then

$$y + \Delta y = (u + \Delta u)(v + \Delta v) = uv + u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v.$$

Hence

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(y + \Delta y) - y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(uv + u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v) - uv}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x} \right).$$

But

$$\lim_{\Delta x \to 0} \Delta u \frac{\Delta v}{\Delta x} = (\lim_{\Delta x \to 0} \Delta u) \left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \right) = (0) \left(\frac{dv}{dx} \right) = 0,$$

where we have used Theorem 2 of Appendix **A2**. Thus

$$\frac{dy}{dx} = u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

A.8 Chain Rule for Differentiating

Let $w(u)$ depend on u, which in turn depends on x. Then $\Delta w = w(u + \Delta u) - w(u)$ so

$$\frac{\Delta w}{\Delta x} = \frac{\Delta w}{\Delta u} \frac{\Delta u}{\Delta x} = \frac{w(u + \Delta u) - w(u)}{\Delta u} \frac{\Delta u}{\Delta x}.$$

Therefore, using Theorem 2 of Appendix **A2**, we have

$$\frac{dw}{dx} = \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \left(\frac{dw}{du} \right) \left(\frac{du}{dx} \right).$$

A.9 Differentiating **ln** ***x***

Let $y = \ln x$, hence $y(x + \Delta x) = y + \Delta y = \ln(x + \Delta x)$. Then

$$\frac{y(x + \Delta x) - y(x)}{\Delta x} = \frac{\Delta y}{\Delta x} = \frac{\ln(x + \Delta x) - \ln x}{\Delta x}.$$

From frame **91**,

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{1}{\Delta x}\ln\left(\frac{x+\Delta x}{x}\right) = \frac{1}{x}\frac{x}{\Delta x}\ln\left(1+\frac{\Delta x}{x}\right)\\ &= \frac{1}{x}\ln\left(1+\frac{\Delta x}{x}\right)^{x/\Delta x} = \frac{1}{x}\ln(1+b)^{1/b},\end{aligned}$$

where we have written b for $\frac{\Delta x}{x}$. Note that as $\Delta x \to 0$, $b \to 0$. Therefore,

$$\begin{aligned}\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}\left[\frac{1}{x}\ln(1+b)^{1/b}\right]\\ &= \frac{1}{x}\ln\left[\lim_{b\to 0}(1+b)^{1/b}\right]\\ &= \frac{1}{x}\ln e = \frac{1}{x},\end{aligned}$$

because $\ln e = 1$.

A.10 Differentials When Both Variables Depend on a Third Variable

The relation $dw = \frac{dw}{du}du$ is true even when both w and u depend on a third variable. To prove this, let both w and u depend on x. Then

$$dw = \frac{dw}{dx}dx \quad \text{and} \quad du = \frac{du}{dx}dx. \tag{A.5}$$

By the chain rule for differentiating,

$$\frac{dw}{dx} = \left(\frac{dw}{du}\right)\left(\frac{du}{dx}\right),$$

and multiplying through by dx, we have

$$\frac{dw}{dx}dx = \left(\frac{dw}{du}\right)\left(\frac{du}{dx}\right)dx,$$

so by Equation (A.5),

$$dw = \frac{dw}{du}du.$$

This theorem justifies the use of the differential notation since it shows that with the differential notation the chain rule takes the form of an algebraic identity:

$$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx}.$$

A.11 Proof That if Two Functions Have the Same Derivative They Differ Only by a Constant

Let the functions be $f(x)$ and $g(x)$. Then $\frac{d}{dx}f(x) = \frac{d}{dx}g(x)$ so $\frac{d}{dx}[f(x) - g(x)] = 0$. Therefore $f(x) - g(x) = c$, where c is a constant.

This proof depends on the assumption that if $\frac{d}{dx}h(x)$, then $h(x)$ is a constant. This is indeed plausible since the graph of the function $h(x)$ must always have zero slope and hence it should be a straight line parallel to the origin, i.e. $h(x) = c$. A more complicated analytic proof of this theorem is given in advanced books on calculus.

A.12 Limits Involving Trigonometric Functions

1. Proof that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.
To prove this, use the Taylor series for $\sin\theta$ expanded about $\theta = 0$ (frame **412**), $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \cdots$. Then

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \lim_{\theta \to 0} \frac{1}{\theta}\left(\theta - \frac{1}{3!}\theta^3 + \cdots\right) = \lim_{\theta \to 0}\left(1 - \frac{1}{3!}\theta^2 + \cdots\right) = 0.$$

2. Proof that $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$.
To prove this, use the Taylor series for $\cos\theta$ expanded about $\theta = 0$ (frame **418**), $\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \cdots$. Then

$$\begin{aligned}\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} &= \lim_{\theta \to 0} \frac{1}{\theta}\left(1 - \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \cdots\right)\right) \\ &= \lim_{\theta \to 0}\left(\frac{1}{2!}\theta - \frac{1}{4!}\theta^3 + \cdots\right) = 0.\end{aligned}$$

APPENDIX B

Additional Topics in Differential Calculus

B.1 Implicit Differentiation

Most of the functions we use in this book can be written in the simple form $y = f(x)$, but this is not always the case. Sometimes we have two variables related by an equation of the form $f(x, y) = 0$. The function $f(x, y)$ means that the value of f depends on both x and y. Here is an example: $x^2 y + (y + x)^3 = 0$. We cannot easily solve this equation to yield a result of the form $y = g(x)$, or even $x = h(y)$. However, we can find y' by differentiating both sides of the equation with respect to x, remembering that y depends on x.

$$\frac{d}{dx}(x^2 y) + \frac{d}{dx}(y + x)^3 = \frac{d}{dx}(0) = 0,$$

$$x^2 \frac{dy}{dx} + 2xy + 3(y + x)^2 \left(\frac{dy}{dx} + 1 \right) = 0,$$

$$\frac{dy}{dx} \left[x^2 + 3(y + x)^2 \right] = -2xy - 3(y + x)^2,$$

$$\frac{dy}{dx} = -\frac{2xy + 3(y + x)^2}{x^2 + 3(y + x)^2}.$$

A function defined by $f(x, y) = 0$ is called an *implicit function* because it implicitly determines the dependence of y on x (or, for that matter, the dependence of x on y in case we need to regard y as the independent variable). The process we have just used, differentiating each term of the equation $f(x, y) = 0$ with respect to the variable of interest, is called implicit differentiation.

Here is another example of implicit differentiation. Let $x^2 + y^2 = 1$. Note that in this case $f(x, y) = x^2 + y^2 - 1 = 0$. The problem is to find y'. We will do this first by implicit differentiation, and then by solving the equation for y and using the normal procedure. By differentiating both sides of the equation with respect to x, we obtain $2x + 2y\ y' = 0$. Hence,

$$y' = -\frac{2x}{2y} = -\frac{x}{y}.$$

Alternatively, we can solve for y.

$$y^2 = 1 - x^2, \quad y = \pm\sqrt{1 - x^2},$$

$$y' = \pm\frac{1}{2}\left(\frac{-2x}{\sqrt{1 - x^2}}\right) = \mp\frac{x}{\sqrt{1 - x^2}} = -\frac{x}{y}.$$

We did not need to use implicit differentiation here because we could write the function in the form $y = f(x)$. Often, however, this cannot be done, as in the first example, and implicit differentiation is then necessary.

B.2 Differentiating the Inverse Trigonometric Functions

We can use the inversion formula for derivatives from frame **273** and Appendix **A4** to differentiate the inverse functions.

If $x = \ln y$, then define the inverse function $y = \ln^{-1} x$ by

$$x = \ln y = \ln(\ln^{-1} x).$$

This inverse function is just the familiar exponential function $y = e^x \equiv \ln^{-1}(x)$. Then the inversion formula for derivatives is given by

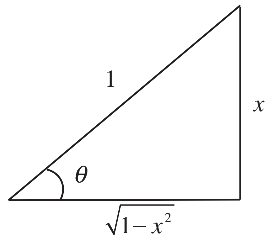
$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

Thus

$$\frac{dy}{dx} = \frac{d}{dx}e^x = \frac{1}{\frac{d}{dy}\ln y} = y = e^x.$$

Taking derivatives of the inverse trigonometric functions can be calculated in a similar way.

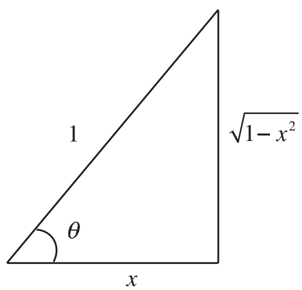
1. Evaluating $\frac{d}{dx}\sin^{-1}x$.



The angle θ is shown inscribed in a right triangle having unit hypotenuse, and an opposite side of length x. Therefore, $\sin\theta = x/1 = x$ and the inverse function is $\theta = \sin^{-1}x$. The inversion formula for derivatives is now

$$\frac{d}{dx}\sin^{-1}x = \frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}} = \frac{1}{\frac{d}{d\theta}\sin\theta} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-x^2}}.$$

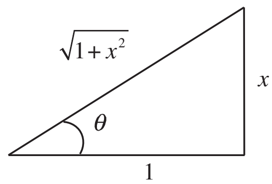
2. Evaluating $\dfrac{d}{dx}\cos^{-1}x$.



Let $x = \cos\theta$. Then $\theta = \cos^{-1}x$. The angle θ shown in the figure is inscribed in a right triangle having unit hypotenuse, and an adjacent side of length x. Therefore

$$\frac{d}{dx}\cos^{-1}x = \frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}} = \frac{1}{\frac{d}{d\theta}\cos\theta} = -\frac{1}{\sin\theta} = -\frac{1}{\sqrt{1-\cos^2\theta}} = -\frac{1}{\sqrt{1-x^2}}.$$

3. Evaluating $\dfrac{d}{dx}\tan^{-1}x$.



Let $x = \tan\theta$. Then $\theta = \tan^{-1}x$. The angle θ shown in the figure is inscribed in a right triangle having opposite side x, and an adjacent side of length 1. Therefore

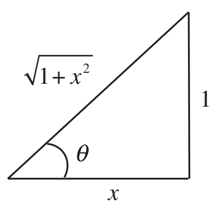
$$\frac{d}{dx}\tan^{-1}x = \frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}} = \frac{1}{\frac{d}{d\theta}\tan\theta} = \frac{1}{\sec^2\theta} = \cos^2\theta.$$

From the figure above $\cos\theta = 1/\sqrt{1+x^2}$. Therefore

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}.$$

(Alternatively, $\cos^2\theta = \dfrac{1}{1+\tan^2\theta} = \dfrac{1}{1+x^2}$.)

4. Evaluating $\dfrac{d}{dx}\cot^{-1}x$.



Let $x = \cot\theta$. Then $\theta = \cot^{-1}x$. The angle θ shown in the figure is inscribed in a right triangle having adjacent side x, and an opposite side of length 1. Therefore

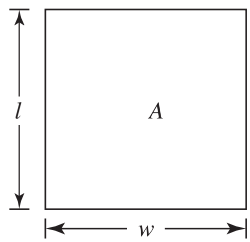
$$\frac{d}{dx}\cot^{-1}x = \frac{d\theta}{dx} = \frac{1}{\frac{dx}{d\theta}} = \frac{1}{\frac{d}{d\theta}\cot\theta} = -\frac{1}{\sec^2\theta} = -\sin^2\theta,$$

where we used the trigonometric identity $\sin^2\theta = \dfrac{1}{1 + \text{cotan}^2\theta} = \dfrac{1}{1 + x^2}$. (In the figure above $x = \cot\,\theta$, and $\sin\theta = \dfrac{1}{\sqrt{1 + x^2}}$.) Therefore

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1 + x^2}.$$

B.3 Partial Derivatives

In this book we have almost exclusively considered functions defined for a single independent variable. Often, however, two or more independent variables are required to define the function; in this case we have to modify the idea of a derivative. As a simple example, suppose we consider the area of a rectangle A, which is the product of its width w and length l. Thus, $A = f(l, w)$ (read "f of l and w"), where $f(l, w) = lw$. In this discussion we will let l and w vary independently, so they both can be treated as independent variables.



If one of the variables, say w, is temporarily kept constant, then A depends on a single variable, and the rate of change of A with respect to l is simply dA/dl. However, because A really depends on two variables, we must modify the definition of the derivative. The rate of change of A with respect to l is

$$\lim_{\Delta l \to 0}\frac{f(l + \Delta l, w) - f(l, w)}{\Delta l} = \lim_{\Delta l \to 0}\frac{(l + \Delta l)\,w - lw)}{\Delta l} = w,$$

where it is understood that w is held constant as the limit is taken. The above quantity is called the *partial derivative of A with respect to l* and is written $\dfrac{\partial A}{\partial l}$. In other words, the partial derivative is defined by

$$\frac{\partial A}{\partial l} = \frac{\partial f(l, w)}{\partial l} = \lim_{\Delta l \to 0}\frac{f(l + \Delta l, w) - f(l, w)}{\Delta l} = \lim_{\Delta l \to 0}\frac{(l + \Delta l)w - lw}{\Delta l} = w.$$

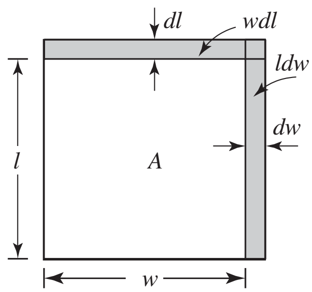
Similarly, the partial derivative of A with respect to l is

$$\frac{\partial A}{\partial w} = \lim_{\Delta w \to 0} \frac{f(l, w + \Delta w) - f(l, w)}{\Delta w} = \lim_{\Delta w \to 0} \frac{l(w + \Delta w) - lw}{\Delta w} = l.$$

The differential of A due to changes in l and w of dl and dw, respectively, is by definition

$$dA = \frac{\partial A}{\partial l} dl + \frac{\partial A}{\partial w} dw.$$

By analogy with the argument in **267** it should be plausible that as $dl \to 0$, the increment in A, $\Delta A = f(l + \Delta l, w + \Delta w) - f(l, w)$ approaches dA.



This result is shown by the figure. ΔA is the total increase in area due to dl and dw and comprises all the shaded areas.

$$dA = \frac{\partial A}{\partial l} dl + \frac{\partial A}{\partial w} dw = w\,dl + l\,dw.$$

$\Delta A = (dl)(dw)$ and dA differ by the area of the small rectangle in the upper-right-hand corner. As $dl \to 0$, $dw \to 0$, the difference becomes negligible compared with the area of each strip.

The above discussion can be generalized to functions depending on any number of variables. For instance, let p depend on q, r, s, … , then

$$dp = \frac{\partial p}{\partial q} dq + \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial s} ds + \dots .$$

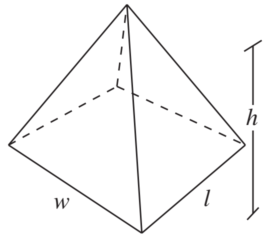
Here is an example: Let $p = q^2 r \sin z$. Then

$$\frac{\partial p}{\partial q} = 2qr \sin z, \quad \frac{\partial p}{\partial r} = q^2 \sin z, \quad \frac{\partial p}{\partial z} = q^2 r \cos z.$$

Therefore

$$dp = 2qr \sin z\, dq + q^2 \sin z\, dr + q^2 r \cos z\, dz.$$

Here is another example:



The volume of a pyramid with height h and a rectangular base with dimensions l and w is $V = \frac{1}{3}lwh$. Thus,

$$dV = \frac{1}{3}wh\,dl + \frac{1}{3}lh\,dw + \frac{1}{3}lw\,dh.$$

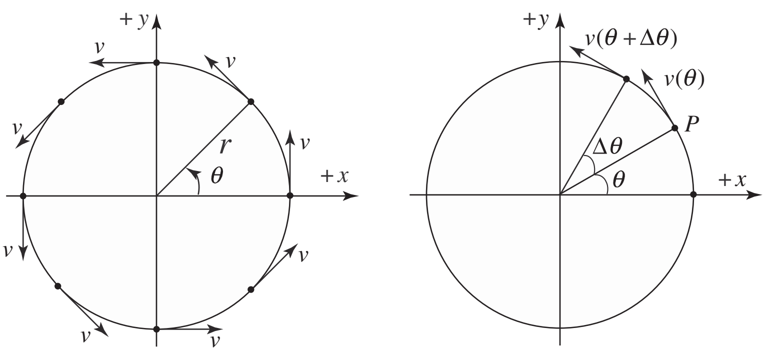
If the dimensions are changed by small amounts dl, dw, and dh, the volume changes by an amount $\Delta V \approx dV$, where dV is given by the expression above.

B.4 Radial Acceleration in Circular Motion

An object moving in a circle always has a radial inward acceleration, called *centripetal acceleration,* due to the continuous change in the direction of the velocity, and is given by

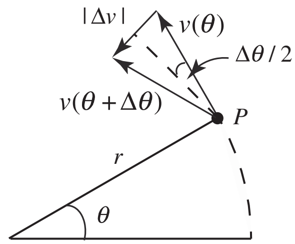
$$a_{\text{rad}} = \frac{v^2}{R}.$$

Let's consider the special case in which the speed is constant. The derivation of this result requires understanding how to calculate the derivative of a vector that is constant in length but changing direction.



The direction of the velocity is always tangent to the circle. Although the speed of the object is constant, the direction of the velocity is changing because the object is moving in a circle, as can be seen in the figures above.

In the figure below, where we have superimposed the directions of two velocities at the point P, the arrows indicate the change in direction but the speeds remain the same, i.e. $v(\theta + \Delta\theta) = v$ and $v(\theta) = v$.



The magnitude of the change in velocity $|\Delta v|$ is given by $|\Delta v| = 2v\sin(\Delta\theta/2)$. We use the small angle approximation (from the Taylor series), $\sin(\Delta\theta/2) \approx \Delta\theta/2$ (frame **405**) to approximate the magnitude of the change of velocity, $|\Delta v| \approx v\Delta\theta$, where we are assuming that $\Delta\theta > 0$. Then the magnitude of the radial acceleration is given by

$$|a_{\text{rad}}| = \lim_{\Delta t \to 0} \frac{|\Delta v|}{\Delta t} = v \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = v\frac{d\theta}{dt}.$$

We now use the result in frame **286** that the speed is $v = R\frac{d\theta}{dt}$, to write the magnitude of the radial acceleration as

$$|a_{\text{rad}}| = \frac{v^2}{R}.$$

In the limit $\Delta\theta \to 0$, the change in the direction of the velocity is perpendicular to the velocity, $\Delta v \perp v$ and so is directed radially inward.

B.5 Resources for Further Study

Online Calculus Content:

David Jerison. *18.01SC Single Variable Calculus*. Fall 2010. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu. License: Creative Commons BY-NC-SA.

University-Level Introductory Calculus Textbooks:

Robert E. Larson and Bruce Edwards, *Calculus, 10th ed.*, Cengage Learning, Boston, Mass., 2018.

Morris Kline, *An Intuitive and Physical Approach, 2nd Ed.*, Dover Publications, Mineola, N.Y., 1998.

James Stewart, *Calculus 8th Ed.,* Cengage Learning, Boston, Mass., 2015.

Introductory Calculus Textbooks:

George F. Simmons, *Calculus with Analytic Geometry. 2nd ed.,* McGraw-Hill, New York, NY, 1996.

C. H. Edwards, Jr., and David E. Penny, *Calculus and Analytic Geometry*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1982.

Ross Finney and George Thomas, *Elements of Calculus and Analytic Geometry, 9th ed.*, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1998.

Advanced Single Variable Calculus Textbook:

Tom M. Apostol, Calculus Volume 1, *One-Variable Calculus with an Introduction to Linear Algebra, 2nd ed.*, Blaisdell Publishing Company, Waltham, Mass., 1967.

Frame Problems Answers

Answers to Selected Problems from the Text

Chapter 1: A Few Preliminaries

Frame 16: 5, -3

Frame 18: (3, 27)

Frame 23: *B*

Frame 27: 40 mpg

Frame 29: 7

Frame 31: $+$, $-$, $+$, 0

Frame 37: d

Frame 41: 45°

Frame 42: c

Frame 43: 2π

Frame 44: 1 rad

Frame 45: $\pi/3$, 45 degrees, 60°

Frame 47: 120°, 75°, $\pi/2$

Frame 48: $\pi/2$, 540°, 30°

Frame 52: $-4/5$, $-3/5$, $4/3$

Frame 55: $-$, $+$, $-$

Frame 56: 1, $\sec^2\theta$, $1 - 2\cos^2\theta$

Frame 60: a/c, b/a

Frame 61: m/n, l/m

Frame 62: $1/\sqrt{2}$, $1/2$, $1/\sqrt{2}$, $1/\sqrt{3}$

Frame 65: b, c, d, none of these

Frame 66: $\sin 0° = 0$, $\cos 0° = 1$, $\cos 30° = \sqrt{3}/2$, $\tan 45° = 1$,
$\cos 60° = 1/2$, $\sin 90° = 1$, $\cos 90° = 0$

Frame 69 (a): $+$, $-$; (b): $+$, $+$

Frame 74: (a) $\pi/4$, (b) $\pi/4$, (c) $\pi/3$, (d) 53.1°, (e) 1.49, (f) 87.1°

Frame 75: a^5 = none of these, $a^{b+c} = a^b a^c$, $a^f/a^g = a^{f-g}$, $a^0 = 1$, $(a^b)^c = a^{bc}$

Frame 77: 9, 1, $\frac{1}{8}$, 16^{-1}

Frame 78: 3^{-9}, $(\frac{5}{3})^2$, 2^6

Frame 79: 1, 0.1, 3×10^{-5}, 4×10^{-5}, 0.5×10^{-4}

Frame 81: $\frac{1}{9}$, 8

Frame 82: 125, 1000

Frame 83: $3/400$, $\sqrt{7}/10$

Frame 84: $a^{\pi+x-3}$

Frame 85: None of these

Frame 88: 6, 0

Frame 89: 7, n, $-n$

Frame 92: $\frac{1}{1000}$, 100, $\frac{3}{2}$

Frame 93: log 4, 0

Chapter 2: Differential Calculus

Frame 103: 2, -3

Frame 112: (1) discontinuous, (2) continuous, (3) continuous, (4) discontinuous

Frame 123: d, b, e

Frame 124: d, a

Frame 128: 1, 2, 3

Frame 171: 0

Frame 173: a

Frame 174: -1

Frame 175: $2x$

Frame 181: $3x^2$, $-7x^{-8}$, $-2/x^3$

Frame 182: $-1/x^2$, x^{-4}

Frame 183: $\frac{1}{2}x^{-1/2}$

Frame 184: $\frac{2}{3}x^{-1/3}$

Frame 200: d

Frame 203: $-\frac{1}{v^2}\frac{dv}{dx}$

Frame 213: $\sec\theta\tan\theta$

Frame 215: $2\sin\theta\cos\theta$

Frame 217: $-3\theta^2\sin(\theta^3)$

Frame 219: $\omega\cos(\omega t)$

Frame 223: 1.79, 1.15, 17.10

Frame 228: $2/x$

Frame 229: $2\ln x/x$

Frame 238: $\frac{-1}{(\ln x)^2 x}$

Frame 242: $12x$

Frame 243: $2/x^3$

Frame 246: $-A\omega^2\sin(\omega t)$

Frame 248: (4)(3)(2)(1)

Frame 251: C

Frame 255: -3

Frame 256: ½ and $-$½

Frame 268: $\cos x dx$, $-dx/x^2$, $e^x dx$

Chapter 3: Integral Calculus

Frame 306: antiderivative, or $F(x) = \int f(x)\ dx$

Frame 308: $-\cos x + c$

Frame 309: (a) $\frac{1}{n+1}x^{n+1} + c$, (b) $e^x + c$

Frame 325: $(x-1)e^x + c$

Frame 351: $38/3$

Frame 353: -15

Frame 355: $39/4$

Frame 361: 0

Frame 362: 2

Frame 364: $x + e^{-x} - 1$

Frame 366: $\pi/2$

Frame 371: $\frac{v_0}{b}$

Frame 373: none of these

Frame 393: 90

Frame 395: 36

Frame 397: $Cba^3/12$

Review Problems

This list of problems is for benefit in case you want some additional practice. The problems are grouped according to chapter and section. Answers start on page **283**.

Chapter 1

Linear and Quadratic Functions

Find the slope of the graphs of the following equations:

1. $y = 5x - 5$

2. $4y - 7 = 5x + 2$

3. $3y + 7x = 2y - 5$

Find the roots of the following:

4. $4x^2 - 2x - 3 = 0$

5. $x^2 - 6x + 9 = 0$

Trigonometry

6. Show that $\sin\theta \cot\theta / \sqrt{1 - \sin^2\theta} = 1$, $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$.

7. Show that $\cos\theta \sin\left(\frac{\pi}{2} + \theta\right) - \sin\theta \cos\left(\frac{\pi}{2} + \theta\right) = 1$.

8. What is: (a) $\sin 135°$ (b) $\cos\frac{7\pi}{4}$, (c) $\sin\frac{7\pi}{6}$?

9. Show that $\cos^2\frac{\theta}{2} = \frac{1}{2}(1 + \cos\theta)$.

10. What is the cosine of the angle between any two sides of an equilateral triangle?

Exponentials and Logarithms

11. What is $(-1)^{13}$?

12. Find $[(0.01)^3]^{-1/2}$.

13. Express $\log (x^x)^x$ in terms of $\log x$.

14. If $\log(\log x) = 0$, find x.

15. Is there any number for which $x = \log x$?

In the following five questions, make use of the log table below and the rules for manipulating logarithms.

x	$\log x$	x	$\log x$
1	0.00	5	0.70
2	0.30	7	0.85
3	0.48	10	1.00

Find

16. $\log \sqrt{10}$

17. $\log 21$

18. $\log \sqrt{14}$

19. $\log 300$

20. $\log(7^{3/2})$

Chapter 2

Find the following limits, if they exist:

21. $\lim_{x \to 2} \dfrac{x^2 - 4x + 4}{x - 2}$

22. $\lim_{\theta \to \pi/2} \sin \theta$

23. $\lim_{x \to 0} \dfrac{x^2 + x + 1}{x}$

24. $\lim_{x \to 1} \left[1 + \dfrac{(x + 1)^2}{x - 1}\right]$

25. $\lim_{x \to 3} \left[(2 + x)\frac{(x-3)^2}{x-3} + 7 \right]$

26. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

27. $\lim_{x \to \infty} \frac{1}{x}$

28. $\lim_{x \to 0} \log x$

Velocity

29. What is the average velocity of a particle that goes forward 35 miles and backward for 72 miles, during the course of 1 hour?

30. A particle always moves in one direction. Can its average velocity exceed its maximum velocity?

31. A particle moves so that its position is given by $S(t) = S_0 \sin(2\pi t)$, where S_0 is in meters, t is in hours. Find its average velocity from $t = 0$ to

(a) $t = 1/4$ hour

(b) $t = 1/2$ hour

(c) $t = 3/4$ hour

(d) $t = 1$ hour

32. Write an expression for the average velocity of a particle, which leaves the origin at $t = 0$, whose position is given by $S(t) = at^3 + bt$, where a and b are constants. The average is from $t = 0$ to the present time t.

33. Find the instantaneous velocity of a particle when $t = 2$ whose position is given by $S(t) = bt^3$, where b is a constant.

Differentiation

Find the derivative of each of the following functions with respect to its appropriate variable, where a and b are constants.

34. $y = x + x^2 + x^3$

35. $y = (a + bx) + (a + bx)^2 + (a + bx)^3$

36. $y = (3x^2 + 7x)^{-3}$

37. $p = \sqrt{a^2 + q^2}$

38. $p = \dfrac{1}{\sqrt{a^2 + q^2}}$

39. $y = x^\pi$

40. $f = \theta^2 \sin\theta$

41. $f = \dfrac{\sin\theta}{\theta}$

42. $f = (\sin\theta)^{-1}$

43. $f = \left(\sqrt{1 + \cos^2\theta}\right)^{-1}$

44. $f = \sin^2\theta + \cos^2\theta$

45. $y = \sin(\ln x)$

46. $y = x \ln x$

47. $y = (\ln x)^{-2}$

48. $y = x^x$

(*Hint:* What is $\ln y$? Use implicit differentiation, Appendix **B1**.)

49. $y = a^{x^2}$

50. $f = \sin\sqrt{1 + \theta^2}$

51. $y = e^{-x^2}$

52. $y = \pi^x$

53. $y = \pi^{x^2}$

54. $f = \ln(\sin\theta)$

55. $f = \sin(\sin\theta)$

56. $f = \ln e^x$

57. $f = e^{\ln x}$

58. $y = \sqrt{1 - \sin^2\theta}$

Higher-Order Derivatives

Evaluate each of the following:

59. Find $\dfrac{d^2}{d\theta^2}(\cos a\theta)$.

60. Find $\dfrac{d^n}{dx^n}e^{ax} =$ (n is a positive integer).

61. $\dfrac{d^2}{dx^2}(\sqrt{1+x^2})$

62. $\dfrac{d^2}{d\theta^2}(\tan\theta)$

63. $\dfrac{d^3}{dx^3}(x^2e^x)$

Maxima and Minima

Find where the following functions have their maximum and/or minimum values. Either give the values of x explicitly, or find an equation for these values.

64. $y = e^{-x^2}$

65. $y = \dfrac{\sin x}{x}$

66. $y = e^{-x}\sin x$

67. $y = \dfrac{\ln x}{x}$

68. $y = e^{-x}\ln x$

69. Find whether y has a maximum or a minimum for the function given in question 64.

Differentials

Find the differential df of each of the following functions.

70. $f = x$

71. $f = \sqrt{x}$

72. $f = \sin(x^2)$

73. $f = e^{\sin x}$ (*Hint:* Use chain rule.)

Chapter 3

You may find Table 2 on page 288 helpful in doing the problems in this section.

Integration

Find the following indefinite integrals. (Omit the constants of integration.)

74. $\int \sin 2x \, dx$

75. $\int \frac{dx}{x+1}$

76. $\int x^2 e^x \, dx$ (Try integration by parts.)

77. $\int x e^{-x^2} dx$

78. $\int \sin^2\theta \cos\theta \, d\theta$

Some Techniques of Integration and Definite Integrals

Evaluate the following definite integrals.

79. $\int_{-1}^{+1} (e^x + e^{-x}) \, dx$

80. $\int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2}$

81. $\int_{-\infty}^{\infty} \frac{x \, dx}{\sqrt{a^2 + x^2}}$

82. $\int_{-\infty}^{0} x^2 e^x \, dx$ (Problem 76 may be helpful.)

83. $\int_{0}^{+\pi/2} \sin\theta \cos\theta \, d\theta$

84. $\int_{0}^{1} (x + a)^n \, dx$

85. $\int_{-1}^{+1} \frac{dx}{\sqrt{1 - x^2}}$

86. $\int_{-1}^{1} (x + x^2 + x^3) \, dx$

Answers to Review Problems

1. 5

2. $5/4$

3. −7

4. $\left(1 \pm \sqrt{13}\right)/4$

5. 3, 3 (roots are identical)

6. No answer

7. No answer

8. (a) $\frac{\sqrt{2}}{2}$, (b) $\frac{\sqrt{2}}{2}$, (c) $-\frac{1}{2}$

9. No answer

10. ½

11. −1

12. 1000

13. $x^2 \log x$

14. $x = 10$

15. No

16. 0.50

17. 1.33

18. 0.58

19. 2.48

20. 1.27

21. 0

22. 1

23. No limit

24. No limit

25. 7

26. 2

27. 0

28. No limit

29. −37 mph

30. No

31. (a) $4S_0$ m/hr, (b) 0 m/hr, (c) $-\frac{4}{3}S_0$ m/hr, (d) 0 m/hr

32. $at^2 + b$

33. $12b$

34. $1 + 2x + 3x^2$

35. $b + 2b(a + bx) + 3b(a + bx)^2$

36. $-3(3x^2 + 7x)^{-4}(6x + 7)$

37. $\frac{dp}{dq} = \frac{q}{\sqrt{a^2 + q^2}}$

38. $\frac{dp}{dq} = \frac{-q}{(a^2 + q^2)^{3/2}}$

39. $\frac{dy}{dx} = \pi x^{\pi - 1}$

40. $\frac{df}{d\theta} = 2\theta \sin\theta + \theta^2 \cos\theta$

41. $\frac{df}{d\theta} = \frac{\cos\theta}{\theta} - \frac{\sin\theta}{\theta^2}$

42. $\frac{df}{d\theta} = -\frac{\cos\theta}{\sin^2\theta}$

43. $\frac{df}{d\theta} = \frac{\cos\theta\sin\theta}{(1+\cos^2\theta)^{3/2}}$

44. $\frac{df}{d\theta} = 0$

45. $\frac{dy}{dx} = \frac{\cos(\ln x)}{x}$

46. $\frac{dy}{dx} = 1 + \ln x$

47. $\frac{dy}{dx} = \frac{-2}{x}(\ln x)^{-3}$

48. $\frac{dy}{dx} = x^x(1+\ln x)$

49. $\frac{dy}{dx} = 2xa^{x^2}\ln a$

50. $\frac{\theta}{\sqrt{1+\theta^2}}\cos\sqrt{1+\theta^2}$

51. $-2xe^{-x^2}$

52. $\pi^x \ln\pi$

53. $2x\pi^{x^2}\ln\pi$

54. $\cot\theta$

55. $[\cos(\sin\theta)]\cos\theta$

56. 1

57. 1

58. $-\sin\theta$

59. $-a^2\cos(a\theta)$

60. $a^n e^{ax}$

61. $\frac{1}{\sqrt{1+x^2}} - \frac{x^2}{(1+x^2)^{3/2}}$

62. $2\sec^2\theta\tan\theta$

63. $(6 + 6x + x^2)e^x$

64. $x = 0$

65. $x = \tan x$ $(x = 0, \dots)$

66. $x = \tan^{-1} 1 = \dfrac{\pi}{4} \pm n\pi,\ n = 0, 1, 2, \dots$

67. $x = ex = e$ $(\ln x = 1)$

68. $\dfrac{1}{x} = \ln x$

69. Maximum

70. $df = dx$

71. $df = \dfrac{dx}{2\sqrt{x}}$

72. $df = 2x \cos(x^2) dx$

73. $df = (\cos x)e^{\sin x} dx$

74. $\dfrac{-1}{2} \cos(2x)$

75. $\ln(x + 1)$

76. $x^2 e^x - 2xe^x + 2e^x$

77. $\dfrac{1}{2} e^{-x^2}$

78. $\dfrac{1}{3} \sin^3 \theta$

79. $2\left(e - \dfrac{1}{e}\right)$

80. $\dfrac{\pi}{a}$

81. 0

82. 2

83. ½

84. $\dfrac{(1 + a)^{n+1} - a^{n+1}}{n + 1}$

85. π

86. ⅔

Tables

Table 1: Derivatives

The differentiation formulas are listed below. References to the appropriate frames are given. In the following expressions $\ln x$ is the natural logarithm or the logarithm to the base e; u and v are variables that depend on x; w depends on u, which in turn depends on x; and a and n are constants. All angles are measured in radians.

1. $\dfrac{da}{dx} = 0$

2. $\dfrac{d}{dx}(ax) = a$

3. $\dfrac{dx^n}{dx} = nx^{n-1}$

4. $\dfrac{d}{dx}(u + v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$

5. $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

6. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{1}{v^2}\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)$

7. $\dfrac{dw}{dx} = \dfrac{dw}{du}\dfrac{du}{dx}$

8. $\dfrac{du^n}{dx} = nu^{n-1}\dfrac{du}{dx}$

9. $\dfrac{d\ln x}{dx} = \dfrac{1}{x}$

10. $\dfrac{de^x}{dx} = e^x$

11. $\dfrac{da^x}{dx} = a^x \ln a$

12. $\dfrac{du^v}{dx} = vu^{v-1}\dfrac{du}{dx} + u^v \ln u \dfrac{dv}{dx}$

13. $\dfrac{d \sin x}{dx} = \cos x$

14. $\dfrac{d \cos x}{dx} = -\sin x$

15. $\dfrac{d \tan x}{dx} = \sec^2 x$

16. $\dfrac{d \sec x}{dx} = \sec x \tan x$

17. $\dfrac{d \cot x}{dx} = -\csc^2 x$

18. $\dfrac{d \sin^{-1} x}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$

19. $\dfrac{d \cos^{-1} x}{dx} = \dfrac{-1}{\sqrt{1 - x^2}}$

20. $\dfrac{d \tan^{-1} x}{dx} = \dfrac{1}{1 + x^2}$

21. $\dfrac{d \cot^{-1} x}{dx} = \dfrac{-1}{1 + x^2}$

Table 2: Integrals

In the following u and v are variables that depend on x; w is a variable that depends on u, which in turn depends on x; a and n are constants; and the arbitrary integration constants are omitted for simplicity.

1. $\displaystyle\int a \, dx = ax$

2. $\displaystyle\int af(x) \, dx = a \int f(x) \, dx$

3. $\int (u+v)\,dx = \int u\,dx + \int v\,dx$

4. $\int x^n\,dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$

5. $\int \frac{dx}{x} = \ln|x|$

6. $\int \frac{dx}{a+bx} = \frac{1}{b}\ln|a+bx|$

7. $\int e^x\,dx = e^x$

8. $\int e^{ax}\,dx = \frac{e^{ax}}{a}$

9. $\int b^{ax}\,dx = \frac{b^{ax}}{a\ln b}$

10. $\int \ln|x|\,dx = x\ln|x| - x$

11. $\int \sin x\,dx = -\cos x$

12. $\int \cos x\,dx = \sin x$

13. $\int \tan x\,dx = -\ln|\cos x|$

14. $\int \cot x\,dx = \ln|\sin x|$

15. $\int \sec x\,dx = \ln|\sec x + \tan x|$

16. $\int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x$

17. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

19. $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x - \sqrt{x^2 \pm a^2}\right|$$

20. $$\int w(u)\,dx = \int w(u)\frac{dx}{du}\,du$$

21. $$\int u\,dv = uv - \int v\,du$$

Index

References are by page number.

Index of Symbols

References are by page number.